Balancing cyclic pursuit by means of set-theoretic techniques

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This thesis is dedicated to Stefania
for her patience.

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Abstract

We propose a decentralized and non cooperative algorithm for estimation and control in a multi-agent system of oscillators to achieve a balanced circular formation. Each agent gathers an uncertain measurement of its phase distance from other agents only when they are in its proximity. Based on this uncertain and intermittent data and on the a priori knowledge of the nominal (e.g. uncontrolled) agent's velocities, we employ an estimation algorithm to reconstruct the relative angular positions. The algorithm combines the information coming from the collected measures with the information on the agents' dynamics, and its convergence is proved by means of Interval Analysis. Interesting connection are highlighted with contractions and fractals. Then, we develop a bang-bang controller to achieve a balanced circular formation. The novelty of the approach is that the balanced formation is achieved by using proximity sensors rather than distance transducers. Moreover, the bang-bang control strategy is designed so that the control goal is achieved even when the range of the sensors is lower than the desired spacing distance. The effectiveness of the approach is illustrated through extensive numerical simulations.

Contents

List of Figures

Chapter 1

Introduction and motivation

1.1 Proximity in coordination of multi-agent systems

Coordination of multi-agent systems is becoming research topic of increasing popularity, and with a good reason: there exist numerous and relevant potential engineering applications [37], ranging from terrestrial, space, and oceanic exploration [4] performed by teams of coordinated mobile robots, to military surveillance and rescue missions, or even automated highway systems [11], [1], to mention only a few.

From an engineering standpoint, the main purpose of using multi-agent systems, is to collectively reach goals that are difficult or impossible to achieve by an individual agent or a monolithic system, hence the question of how to prescribe them desired global behaviors, through the application of only simple and local interactions' rules, is of significant and practical interest. The ultimate objective is that of synthesis, i.e. is it possible to construct decentralized controllers, with severe constraints on information available to each agent on the collective state of the system, that generate a predictable and stable behavior if the multi-agent system as a whole?

As outlined, it is implicitly postulated that the properties of the system are, in a some meaningful way, better or more desiderable that the sum of individuals' ones. This important property of multi-agent systems is termed *emergence*. Usually, global properties of the system, as motion coordination for example, is a property that *emerges* through the local interactions between an agent and the ones that are in its neighborhood. Unfortunately, it is difficult to study emergence from both modeling and design perspectives. In other words, given a desired emergent property or behavior, usually it is very difficult to deduce the local interaction rules that will lead to that desired property or behavior. There has not been much work on modeling of emergence in the multi-agent dynamic systems literature and it remains an important open problem.

The problem is made more difficult by the constraints imposed on the system: each agent has limited sensing, computation, motion, and communication capabilities, moreover the performance of the group should be insensitive to variation of the number of agents. Because of the intrinsic complexity of the problem and the presence of the constraints, the global outcome of these behavior-based systems is often difficult to predict analytically. Thus, corresponding mathematical results are rare, as noted in [28] and [46]. Some researchers have argued that rigorous analysis of even the most simple cases can be an impractical task [5].

Despite its complexity, there has been a lot of research interest in analyzing nearest-neighbor or "proximity" strategies, where simple coordination rules are employed locally to generate global behaviors. When the emergent property of the system is expressed in terms of state of motion (position, velocity, acceleration, and so on) of the agents, the problem is usually referred as *formation control*. In this field of application, we can found the major examples of proximity strategies. For example, Wang [51] proposed a strategy where agents are controlled to move based on the motions of their nearest neighbors. Certain formation stability properties were then analyzed for the case when one individual agent is provided a reference trajectory and designated group leader. The cited work by [46] investigated a set of heuristic algorithms for the generation of geometric patterns in the plane (e.g., lines, circles, or polygons). Other works have stressed the need for rigorous proof of the correctness of these types of algorithms [47]. Justh and Krishnaprasad [23], [22] have developed steering laws for achieving both rectilinear and circular formations in the plane. In their approach, each vehicles control input, which is based on the pose of all other vehicles in the group, uses alignment and separation terms to determine the formation. Jadbabaie et al. [19] proved convergence results for a nearest-neighbor type problem, guaranteeing that all agents eventually move in an identical fashion, despite the distributed nature of the coordination law.

The majority of local interactions' rules and algorithms based on proximity of agents are biologically inspired. Indeed, in networks of biological agents collective phenomena such as synchronization and consensus ([37, 12, 35, 29]) are observed, and global collective behavior is achieved on the basis of proximity rules. This is the case, for instance, of migration phenomena, where the system moves towards a given target or when toroidal behaviors around a common center are observed ([10, 49]). Patterns of this sort seem to appear in nature. For example, Bruckstein's curiosity

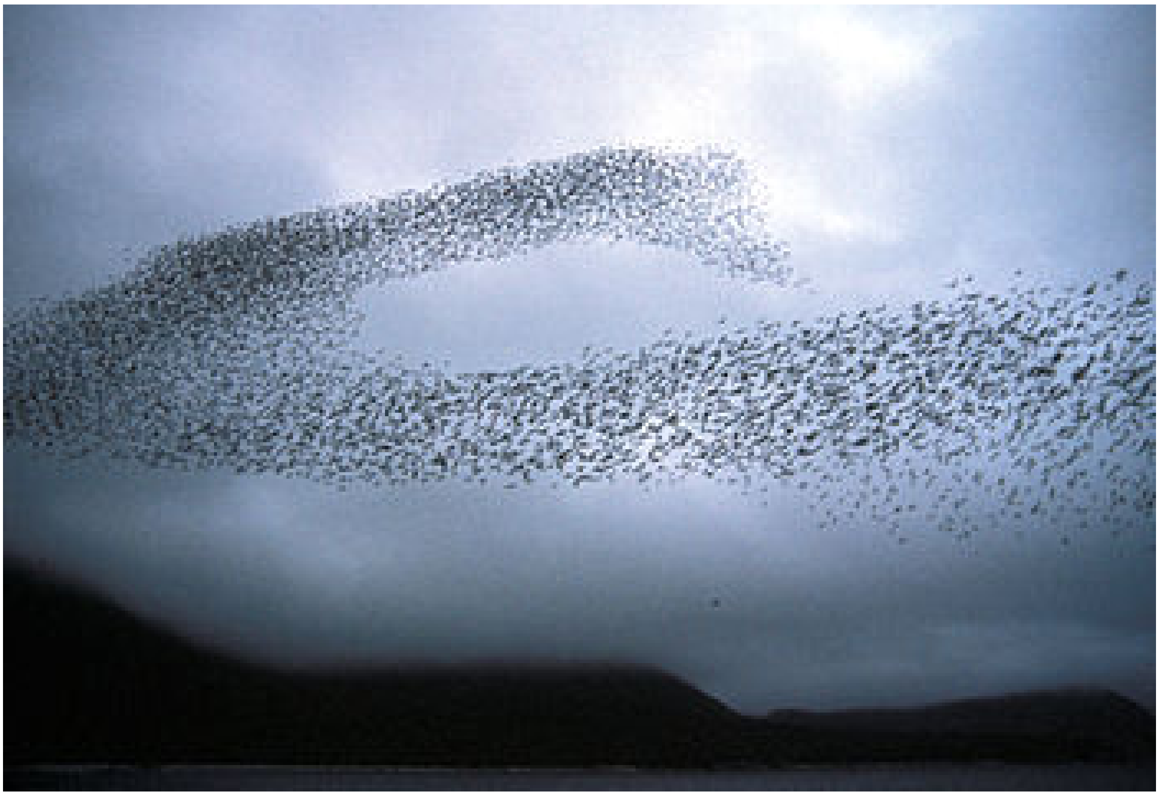


Figure 1.1: Birds' formation.

with regards to the evolution of ant trails led him to an interesting mathematical discovery in [6]. Others have studied aggregate behavior in swarms of organisms (e.g., birds, fish, mammals, and bacteria), where operational models are analyzed for the purpose of potential engineering application (e.g., see [15] and the references therein).

These biological phenomena constitute a source of inspiration to design the interaction rules for coordinating artificial multi-agent systems ([50, 14, 2, 53, 27]). On this subject, perhaps the most widely recognized artificially created example is the distributed behavioral model of Reynolds [39]. Reynolds bird-oids each obey a set of local control rules, which together result in a natural and appealing steady-state flocking behavior for the group.

In fact, much of the multi-agent robotics research has focused on the use of similar proximity and behavior based techniques. For example, Balch and Arkin [3] evaluated proximity control rules designed to implement multi-vehicle formations in combination with rules for collision avoidance and other navigational goals.

Not unlike Reynolds boids, the behavior-based approach to coordinate a group of agents is often to mimic biological systems, where emergent behaviors result from

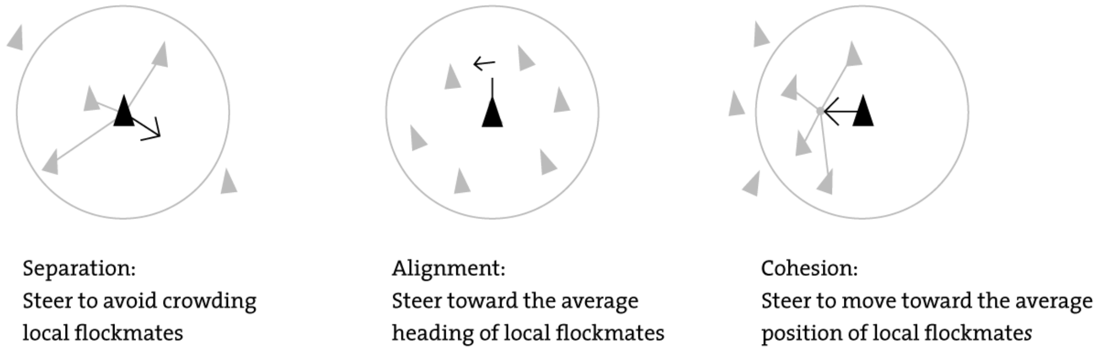


Figure 1.2: Local control law of Reynold's boids.

agents that appear to act autonomously. For a more complete review, see [4], [7], [28], and the references therein. Still, many of these distributed coordination ideas have yet to be explored for agents subject to motion constraints.

The main elements of a multi-agents systems, relevant to the problem we tackle in this thesis, are - obviously - the agents and the information (i.e. sensing, control, and communication) links among these agents, assuming that the individual dynamics of the agents are uncoupled or loosely coupled. A useful and thus increasingly common approach to the analysis of algorithms similar to Reynolds distributed behavioral model is to employ techniques from mathematical graph theory. Typically, a graph is used to track the influence of neighboring agents on an individual. The vertices of the graph usually represent agents, while the edges, which are possibly directed and/or dynamic, represent the existence (or non-existence if no edge is present) of explicit or implicit communication between agents. For example, Fax and Murray (2004) investigated the effect of information flow topology between agents, modeled using algebraic graph theory, on formation stability. Inspired by Reynolds approach, Jadbabaie, Lin, and Morse [19] proved convergence results for a nearest-neighbour type problem, guaranteeing that all n agents eventually move in an identical fashion, despite the distributed nature of the coordination law. In the cooperative control systems literature, this result has become commonly known as consensus or agreement and is analogous to synchronization. Jadbabaie et al. employed a family of graphs with n vertices to track the interactions between neighboring agents. Olfati-Saber and Murray [40] developed a graph theoretic framework for generating similar flocking behaviours in the presence of obstacles.

In fact, Reynolds model has spawned a flurry of papers, too numerous to list, claiming rigorous analysis.

Ih the following sections we illustrate the main difficulties we encountered in approaching the problem of balanced circular formations.

1.2 Noise and uncertainty

As stated, the problem is to have the vehicle formation evolve as much as possible along a certain desired trajectory, and the question is to find control laws based on proximity that achieve this goal. An excellent overview of the literature can be found in [33]. As a general rule, it is very difficult to solve, because is not simple at all to deduce local control laws from the global goal of the multi-agent system. Even if the problem can be satisfactory solved, and reachability and stability of the formation can be analytically proved, usually this is done under ideal assumption of perfect knowledge of the dynamics and of the state of the neighbors, since the local interactions' rules usually are assumed to depend on these quantities.

Real-life multi-agent systems typically don't conform to these hypotheses: their dynamics is usually complex and affected by noise and uncertainty, moreover the state of the neighbors can be measured through low cost and noisy sensors, that are not able to measure some relevant quantities, as *directional* state difference, and can only measure *undirectional* state distance.

Obviously, the formation control problem for a real multi-agent system, as opposed to ideal, is even more complicated, rendering solution designed for ideal systems not directly applicable.

The problem we tackle in this thesis is exactly that of formation control of low cost multi-agent systems, with limited sensing capabilities, able to measure only noisy relative distances, related in a non-linear fashion to relative positions of the agents.

To solve the formation control problem we adapt the classical approach of closed-loop estimator-based regulator to our problem. Each agent, on the basis of intermittent measures of relative distance from its neighbors, estimates the relative positions, that are fed-in to generate a local control law, capable of rendering emergent the desired formation.

In the literature, there are only a few results on the consensus of multi-agent systems with measurement noises. In [38], Ren, Beard and Kingston make use of a Kalman filter structure to design a time-varying consensus protocol, however, the consensus is only proved to be achieved in the noise-free case. In [8], it is shown that the consensus problem of discrete-time first-order integral multi-agent systems with measurement noises cannot be solved in the average by the the conventional control laws used in the noise-free case. Several heuristic techniques were proposed to deal with this problem. Furthermore, in [54], Sun and Ruan imposed several constraints on the communication topology and the noise intensity of continuous-time first-order integral multi-agent systems to ensure the average consensus with probability one. It is noted that, to ensure the consensus of multi-agent systems in the noisy framework, a great majority of methods proposed in the literature require the communication topology to satisfy certain conditions. Therefore, it is necessary to asses a methodology to study how to design a proper formation control strategy that is independent from these topology constraints.

Furthermore papers that address the problem of consensus of multi-agent systems, usually assume the implicit hypotheses of observability and controllability of the system as a whole. But, in the case we study in this thesis, because of the low cost nature of the multi-agent system, the individuals have few informations on the neighbors, and the question of how this lack of information affects the observability and controllability of the whole multi-agent system is an open question.

Ultimately, the problem we pose is that of *robust formation control*, with respect to noisy and non-linear measures. For work on robustness in the context of consensus with agents given scalar systems we refer to [44]. The paper [17] deals with robust stability analysis of multi-agent systems.

In the literature of synchronization of complex network, that can be a source of inspiration for control of multi-agent systems, the problem of achieving consensus with uncertainty and noise on agents' measurements and dynamics that provide robustness under perturbations of the coupling strengths in the network graph have been studied for example in [48]. Robustness against communication delays in the network was studied in [30]. The paper [55] deals with consensus protocols that remain to achieve consensus under quantization of the relative state information, thus providing a robustness result under information quantization.

1.3 Limited and intermittent sensors

To make things more complex, inspired by common applications, we suppose that the agents are equipped with low cost sensors of limited range. This circumstance means that two agents can measure their relative distance only if that distance is below a given threshold. So we have to cope not only with the problem of measurement noise, but also with the absence of measurements in selected time instants.

Some applications are so low cost, that only *proximity sensors* are viable. Nowadays, proximity sensors are common devices in aeronautical, automotive, and manufacturing industry but also in civil applications as in garage and elevator doors, gates, vending machines, parking lots, ATMs. The widespread use of this kind of devices is motivated by their compactness, high reliability, and their suitability for harsh environments. Based on different technologies (e.g. eddy currents, Hall effect), they share the same functioning mechanism: when the distance of the target from the sensor's head is lower than the so-called detecting distance, a trigger signal is produced and is then passed through the output conditioning circuitry to give a high or low output, depending on the sensor application.

In control technology, proximity sensors are mainly used as proximity switches, in combination with position control loops which verify if the desired positions is reached. In fact, their use in a feedback loop is limited as the data are gathered by the sensors only when their distance from the target is less than the detecting distance.

When the agents are equipped with proximity or limited range sensors, the information flow becomes intermittent and noisy, rendering the estimation and control problem difficult to solve. The problem of controlling a system with intermittent data flow has motivated several researchers to investigate the paradigmatic problem of state estimation based on fleeting data, see ([25]) and references therein. Intermittent data flow renders the information graph possibly disconnected making useless the control strategies based on connectivity.

In the recent literature on the problem of coordinating a multi-agent system of oscillators to achieve a balanced circular formation (see [9, 26] and references therein), a discontinuous control law is proposed to solve the control problem when the relative angular position between the agents is perfectly known. But to analytically guarantee the achievement of a balanced circular formation, it is typically assumed that the graph describing the information exchange among the agents is connected or jointly connected. In this thesis, we rely neither on the measurement of the relative angular position, nor on the agent's connectivity, as we model the case in which proximity sensors are used, whose range is lower than the desired distance.

Specifically, to overcome the limitation of absence of jointly connectedness and keep the loop closed even when the measurements are intermittent, we propose a model based estimator of the distance between the pursuer and his target, inspired by the estimation algorithm proposed in [13]. When no data from proximity sensors are available, the estimation is made only on the basis of the model, while the model predicted estimation is corrected when a measurement is available. Then, a control action is exerted based on the estimated distance among the agents. Specifically, we propose a bang-bang controller: based on the estimation algorithm, each agent tries to identify its closest follower; then, the bang-bang controller is activated and the distance from the follower adjusted to obtain the desired spacing. We emphasize that the controller is designed to be effective also when the range of the sensors is lower than the desired spacing, so no more measurements are collected.

1.4 Limited knowledge of the neighbors' behavior

The estimation and control strategy is typically based on the fact that each agent knows, with a certain degree of uncertainty, the nominal (i.e. the uncontrolled) angular velocity of the neighbors. But, as usual, the estimator is based on the knowledge of the input, that is the total relative angular velocity. Since an agent can't know how the neighbors are exerting their control action, if a non-cooperative scheme is assumed, the input is not totally known. Two approach are possible to overcome this limitation. One is that of estimating also the input, resorting to an augmented state estimator. The other is to treat the unknown velocity as uncertainty. We follow the first one.

1.5 Thesis overview

In this thesis we tackle the problem of balancing a circular formation of agents in motion along a circle. We suppose that each agent is equipped with a limited range sensor, able of capturing the (undirectional) relative distance between the neighbors. This work is principally motivated by the study of how the noise and uncertainty affect applicability and performance of control law designed for nominal multi-agent systems, moreover we analyze the problem of how to cope with intermittent and limited range measures. We first state the problem in formal terms, and introduce the framework in which it will be solved. Next we pose the estimation problem, and we resort to set-theoretic techniques (Interval Analysis) for its solution. We observe that the estimation problem is made complex by the non-linear equation that relates measurements (angular distances) to state (relative phase). Nevertheless we are able to prove the convergence of the estimator, with a mathematical technique that is related to complex contraction properties of the whole system. Then we introduce the bang-bang controller, and we show how it can simplify the estimator. At last but not at least, results of extensive numerical simulation are showed, that prove the validity of our control strategy.

Chapter 2

Problem Formulation

2.1 Biological inspiration for a circular pursuit problem

Inspired by the so-called bugs problem from mathematics and by potential interesting applications, in this thesis we study the circular formations of multi-vehicle systems under cyclic pursuit, equipped with low cost and noisy proximity sensors.

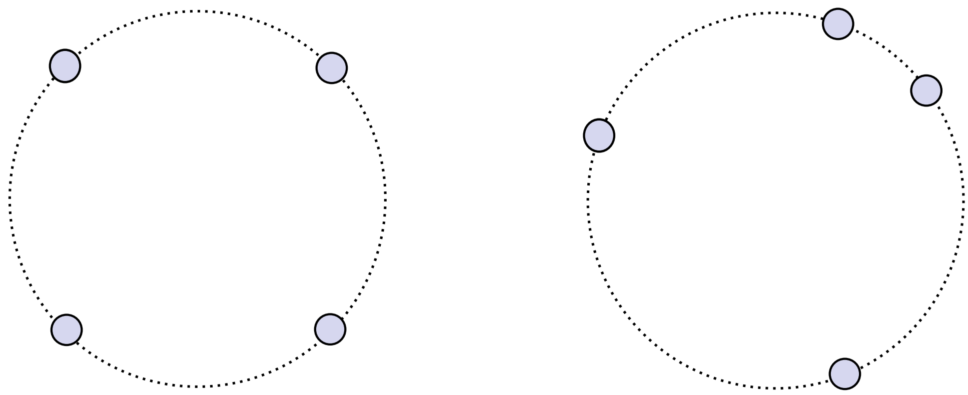
The bugs problem refers to what is also variously known as the dogs, mice, ants, or beetles problem, and originally stems from the mathematics of pursuit curves, first studied by French scientist Pierre Bouguer. In 1877, Edouard Lucas asked, what trajectories would be generated if three dogs, initially placed at the vertices of an equilateral triangle, were to run one after the other? Three years later, Henri Brocard replied with the answer that each dogs pursuit curve would be that of a logarithmic spiral and that the dogs would meet at a common point, known now as the Brocard point of a triangle. Bernhart reports that Gordon Peterson extended this problem to ordered bugs that start at the vertices of a regular-polygon, illustrating his results for the square using four cannibalistic spiders. If each bug pursues the next modulo (i.e., cyclic pursuit) at fixed speed, the bugs will trace out logarithmic spirals and eventually meet at the polygons centre. In 1969, Watton and Kydon provided an elegant solution to this regular -bugs problem, further noting that the constant-speed assumption taken by previous investigators is not necessary.

What happens if our bugs do not start at the vertices of a regular -polygon? In 1971, Klamkin and Newman showed that, for three bugs, so long as the initial triangle formed by the bugs is not degenerate (i.e., the bugs are not collinear), they will meet at a point and this meeting will be mutual. For bugs, this notion was later examined by Behroozi and Gagnon, who proved that a bug cannot capture a bug which is not capturing another bug (i.e., mutual capture), except by head-on collision. They used this result to show that, for the general four-bugs problem, the capture is indeed mutual. Quite recently, Richardson resolved this issue for the general-bugs problem, showing that it is possible for bugs to capture their prey without all bugs simultaneously doing so, even for non-collinear initial positions. However, he proved that for randomly chosen initial positions, the probability of a nonmutual capture is exactly zero.

Variations on this traditional cyclic pursuit problem have also been studied. Some researchers investigated both continuous and discrete pursuit problems, as well as both constant and varying speed scenarios. Consider now a particular cyclic pursuit scheme where each bug is additionally subject to a single non-holonomic constraint, or equivalently, modeled as a kinematic unicycle. In this case, the unicycles will not generally be able to head toward their designated prey at each instant. Instead, depending on the allowed control energy, each vehicle will require some finite time to steer itself toward its preassigned target.

From a practical viewpoint, cyclic pursuit may turn out to be a feasible strategy for multi-vehicle systems since it is distributed and relatively simple in that each agent is required to sense information from only one other agent, hence it is clear that the graph representing agent interconnections is cyclic in nature.

In this thesis, we assume that the agents doesn't know the identity of the neighbors, and we study the problem of reaching anyway a balanced circular formation, where *balanced* means that the agents are equispaced on the circle. See figure 2.1.



(a) Balanced formation

(b) Unbalanced formation.

Figure 2.1: Balanced VS unbalanced circular formations.

2.2 Model of the multiagent system

To state the problem in more formal terms, let us consider N agents moving on a circle at different time varying angular velocities. At time k, each agent i is characterized by its angular position $\theta_i(k)$ and its mean angular velocity $\omega_i(k)$ in the sampling period. Its motion is described by

$$\theta_i(k+1) = \theta_i(k) + \omega_i(k), \tag{2.1}$$

$$\theta_i(0) = \theta_{i0}. \tag{2.2}$$

where θ_{i0} is the unknown initial condition for the i-th agent, and, without loss of generality, a unitary sampling period is selected.

We introduce the relative angular position matrix $\Theta(k)$ at time k, whose element (i, j) is given by the relative angular position $\theta_{ij}(k) = \theta_i(k) - \theta_j(k)$.

The time evolution of $\Theta(k)$ can be written as

$$\Theta(k+1) = \Theta(k) + \Omega(k), \tag{2.3}$$

$$\Theta(0) = \Theta_0, \tag{2.4}$$

where $\Omega(k) := \{\omega_{ij}(k)\}_{i,j=1}^{N} = \{\omega_i(k) - \omega_j(k)\}_{i,j=1}^{N}$, and $\Theta(k) := \{\theta_{ij}(k)\}_{i,j=1}^{N} = \{\theta_i(k) - \theta_j(k)\}_{i,j=1}^{N}$.

Without loss of generality, and for the sake of clarity, we refer to the case in which $\theta_{ij}(0)$ belongs to the interval $] - \pi, \pi[$.

We assume that each pair of agents, i and j,when they are in proximity can measure their phase distance α_{ij}, defined as

$$\begin{aligned} \alpha_{ij}(k) = d(\theta_{ij}(k)) &:= \min\{\mathrm{mod}(\theta_{ij}(k)), \mathrm{mod}(-\theta_{ij}(k))\} \\ &= |\mathrm{rem}(\theta_{ij}(k))|, \end{aligned} \tag{2.5}$$

where $\mathrm{mod}(a) := b$ is the remainder of a modulo 2π, with b being the unique solution of $b = a - 2q\pi$, with $0 \leq c < 2\pi$, and $q \in \mathbb{Z}$. It is easy to show that $0 \leq \mathrm{mod}(a) < 2\pi$ and $\mathrm{mod}(-a) = 2\pi - \mathrm{mod}(a)$, for all $a \neq 2k\pi$, $k \in \mathbb{Z}$. Also, we define $\mathrm{rem}(a) = \mathrm{mod}(a - \pi) - \pi$. To better illustrate the behavior of functions $\mathrm{mod}(\cdot)$ and $\mathrm{rem}(\cdot)$ we report their graph in Fig. 2.2 and 2.3. The graph of function $d(\cdot)$, instead, is reported in Fig. 2.4.

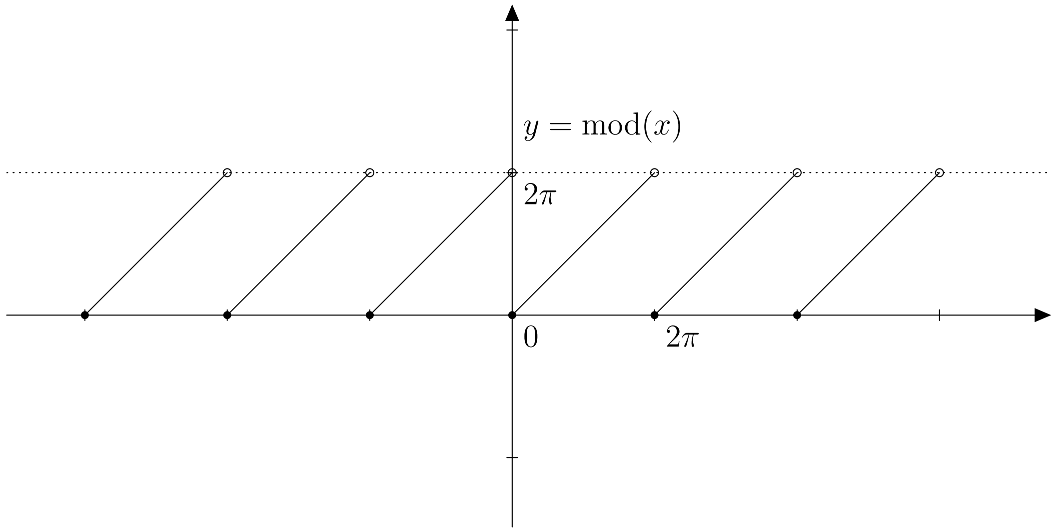


Figure 2.2: Graph of function $\mathrm{mod}(\cdot)$.

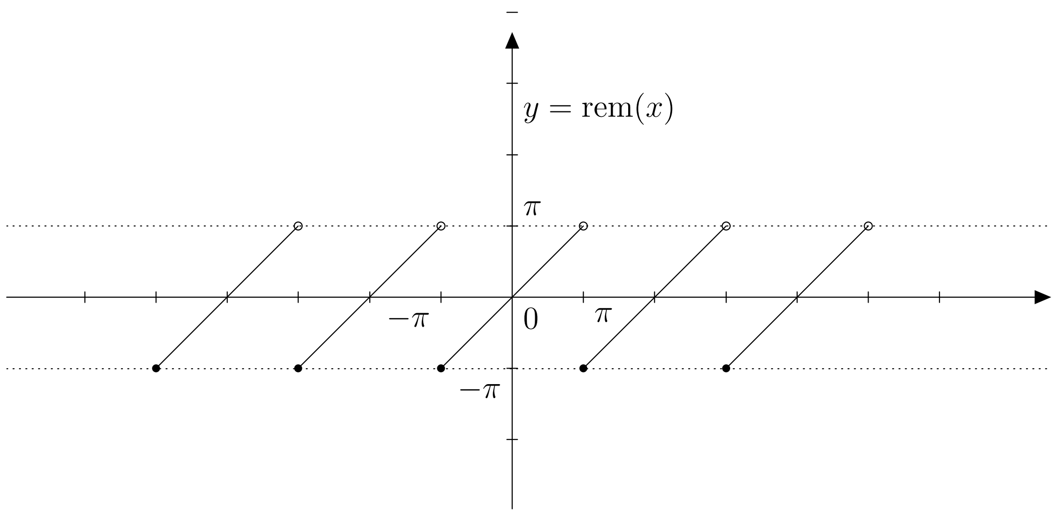


Figure 2.3: Graph of $\mathrm{rem}(\cdot)$ function

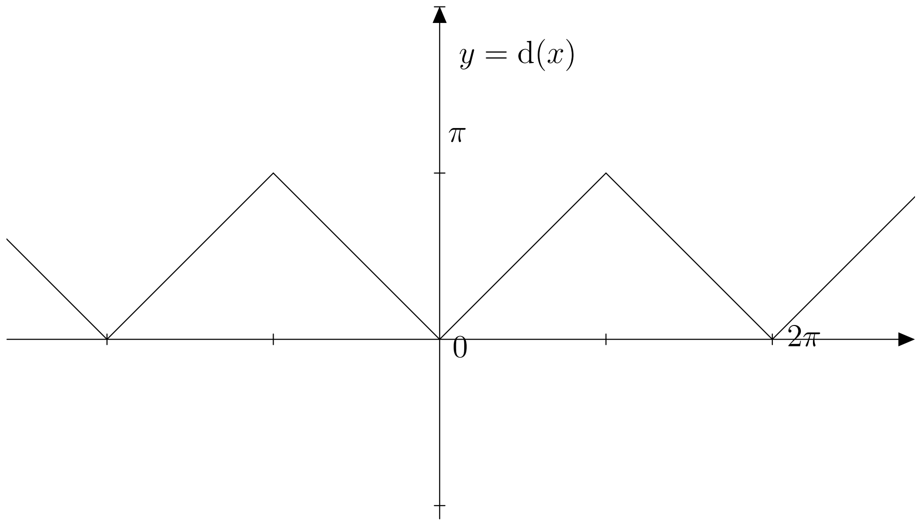


Figure 2.4: Graph of function $d(\cdot)$

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Moreover, we assume that, according to a proximity communication rule, the measurement is performed only if their phase distance is lower than $\theta_{\max}$, where $\theta_{\max} \leq \pi/2$, and that it is affected by an uncertainty of amplitude lower than a scalar φ.

Let us summarize the assumptions we made on the measurement capabilities of the agents

(a) Each couple of agents can measure the phase distance $\alpha_{ij}(k)$ instead of $\theta_{ij}(k)$.

(b) For each pair of agents, the measurement $y_{ij}(k)$ of $\alpha_{ij}(k)$ is only available if $\alpha_{ij}(k)$ is lower than the detecting distance $\theta_{\max} > 0$[1].

(c) The measurement y_{ij}, when available, is affected by a bounded uncertainty $\nu_{ij}(k)$.

These assumptions can be formalized introducing the measurement matrix $Y(k)$, whose element (i, j) is defined as

$$y_{ij}(k) = \begin{cases} \alpha_{ij}(k) + \nu_{ij}(k) & \text{if } \alpha_{ij}(k) \in [0, \theta_{\max}] := I, \\ \text{no measure} & \text{otherwise,} \end{cases} \tag{2.6}$$

In simple words, as illustrated in Fig. 2.5, this means that any agent can only perceive the proximity but not the relative orientation of its neighbors.

[1]Note that the phase distance is biunivocally related to the magnitude of the linear distance, and to the absolute value of the relative orientation. Hence, different sensors can be employed in different application areas.

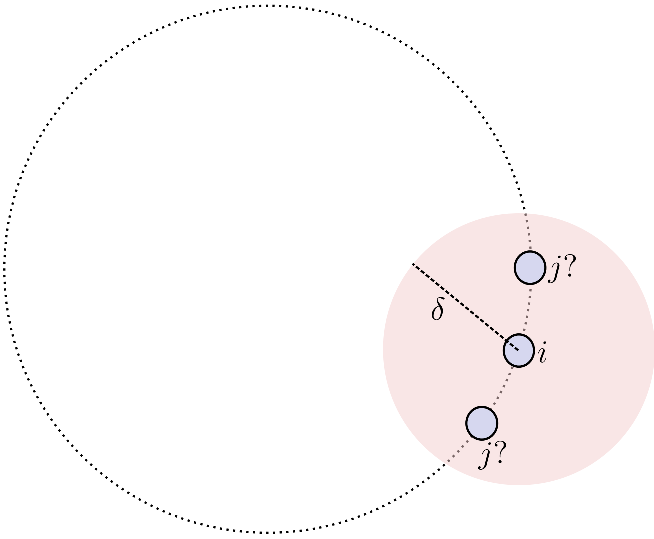

Figure 2.5: When an agent i sense an agent j, since their distance is less than the range of visibility δ, i doesn't know if j is actually onward or backward.

The problem we face is the estimation of the true distance and relative orientation of the agents on the circle, based on repeated intermittent phase distance measurements. Then a local control law capable of equispacing the agents along the circular trajectory should be synthesized.

In formal terms, we seek for a state estimator for system (2.3) that, based on the measurement matrix $Y_{ij}(k)$, is capable of asymptotically nullifying the effects of the uncertainty on the mutual distance, and for a set of local control law u_i, $i = 1, .., N$ that, based on the estimated state, asymptotically leads the multi-agent system (2.3) towards a balanced circular formation, i.e. for all $\theta_{ij}(0)$, $i, j = 1, \dots, N, i \neq j$,

$$\lim_{k\to\infty} \theta_{ij}(k) = \frac{2\pi}{N} := \psi, \tag{2.7}$$

for all $(i, j) \in \{(1, 2), \dots, (N-1, N), (N, 1)\}$, and ψ is the desired phase spacing distance.

We also assume that the detecting distance $\theta_{\max}$ is lower than the desired spacing distance ψ.

2.3 The underneath complexity

This apparently simple problem is made complex by three factors. The first and most obvious is the presence of measurement uncertainty. The second is the limited visual range of each agent, that renders the information flow coming from measurements intermittent with a fleeting topology. The third and less obvious factor is the inherent non-linear nature of the relation between $\alpha_{ij}(k)$ and $\theta_{ij}(k)$. Indeed, α_{ij} and θ_{ij} are related through the function $d(\cdot)$ defined in (2.5), whose domain is the real line, and whose range is the interval $[0, \pi]$: its inverse $d^{-1}(\cdot)$ is a multi-valued function that associates a countable infinite set of angles to the angle y, so the following lemmas can be stated

Lemma 1. *Given a scalar y and a closed interval Y, we have that (1) $d^{-1}(y)$ is the set of degenerate intervals $\{(-y + z2\pi) \cup (y + z2\pi), z \in \mathbb{Z}\}$; (2) $d^{-1}(Y)$ is the set of infinite intervals $\{[\underline{Y}, \overline{Y}] \cup [-\overline{Y}, -\underline{Y}] + z2\pi, z \in \mathbb{Z}\}$* [2]

Lemma 2. *If Y is a closed interval such that $\overline{Y} = \pi$, then $A = d^{-1}(Y) = \{[\underline{Y}, 2\pi - \underline{Y}] + z2\pi,\ z \in \mathbb{Z}\}$, and its complement is $B = \{] - \underline{Y}, \underline{Y}[+z2\pi,\ z \in\ \mathbb{Z}\}$.*[3]

The particular structure of function $d(\cdot)$ implies, as already noticed, that agents can only sense an undirectional distance from the neighbors, so when an agents sense another agent it doesn't know if its neighbor is onward or backward. That circumstance renders the problem very complex because not only we have to cope with *uncertainty* due to measurement noise, but also with *ambiguity*, namely with the possibility that an agent can be located in different disjoint regions of the state space.

Let's discuss more deeply the problem of ambiguity. Assume that at a time instant k a couple (i, j) of agents measure their relative phase. Agent i, on the base of that measurement, can't know uniquely where agent j is located. This is because the measurement available is compatible with two different position of agent j: one is backward agent i, the other is onward. As time passes through, agent i can update its knowledge of position of agents j resorting to dynamics, that is assumed known. Now suppose that the relative phase between i and j becomes again less that $\theta_{\max}$, so a new measurement can be gathered. On the base of this new measurement, what agent i think about the position of agent j? Since agent i doesn't know its position, it thinks that two more positions of agent j are compatible with the new measurement.

The described phenomenon could lead, in principle, to a combinatorial explosion of number of position that agent i assign to agent j. But, are all positions where i think that j can be compatible with all measurement? And are all the supposed state trajectory compatible with all output trajectory? These are the kind of questions we aim to answer.

[2] If Y is open, then $d^{-1}(Y)$ is a set of infinite open intervals $\{]\underline{Y}, \overline{Y}[\cup] - \overline{Y}, -\underline{Y}[+z2\pi, z \in \mathbb{Z}$.

[3] If Y is right-closed, but left-open, then the intervals composing A are open and the ones in B are open.

2.4 The information graph

As stated in the introduction, a multi-agent system can be characterized by its information graph, e.g. the time varying graph that represents available measurements or information exchange at time k. Under some hypotheses on the information graph, a particular set of local control laws can be proved able to make the system reach and maintain the desired formation. In the case of circular balanced formation, the relevant property of the graph is the *jointly connectedness*.

In more formal terms, from the definition of α_{ij}, it is possible to give the following definition of proximity graph.

Definition 1. *Let*

$$\mathcal{E}(\theta_{\max}, k) := \{(i,j) : i,j \in \mathcal{V}, \alpha_{ij}(k) \leq \theta_{\max}\}.$$

Then, the pair $\mathcal{G}(k) = \{\mathcal{V}, \mathcal{E}(\theta_{\max}, k)\}$ *is the proximity graph associated to multi-agent system* (2.3) *at time* k*.*

Notice that $\theta_{\max} > 0$ in Definition 1 can be viewed as the detecting distance of a proximity sensor. Notice that, as the multi-agent systems evolves, the proximity graph changes over time.

Le us denote $\mathcal{G}(k, k+r)$, $r \in \mathbb{N}$, the union of all proximity graph across a nonempty finite time interval $\{k, k+1, \ldots, k+r\}$, whose edges are the union of the edges of the proximity graphs at every discrete-time instants over this time interval, that is,

$$\mathcal{G}(k, k+\delta k) := \left\{\mathcal{V}, \cup_{\tau \in \{k, k+1, \ldots, k+\delta k\}} \mathcal{E}(\theta_{\max}, \tau)\right\}.$$

Now, we can give the following definition.

Definition 2. *The multi-agent system* (2.3) *is jointly connected over* $\{k, k+1, \ldots, k+r\}$ *if* $\mathcal{G}(k, k+r)$ *is connected.*

In a continuous time setting, and assuming that the multi-agent system was jointly connected for any k and δk, [9] proved analytically that a balanced circular formation is achieved through the following class of controllers:

$$u_i(k) = \omega_0 + \sum_{j \in \mathcal{N}_i,\ j \neq i} \beta(\alpha_{ij}(k)) \mathrm{sgn}^+(\sin(\theta_{ij}(k))), \tag{2.8}$$

where $\beta(\cdot)$ belongs to the so called class $\mathcal{S}$ functions (see [9] for details) defined as

$$\mathcal{S}(x) = \Bigg\{ f : [0,\infty) \rightarrow [0,a] \mid f \text{ is Lispchitz continuous}$$

$$\text{and } f(\tau) = \begin{cases} 0, & \tau \geq x \\ > 0, & \tau < x \end{cases} \Bigg\}, \tag{2.9}$$

the function $\mathrm{sgn}^+(x)$ is defined as

$$\mathrm{sgn}^+(\sin(\mathrm{x})) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{2.10}$$

and $\mathcal{N}_i$ is the set of neighbors of agent i.

The control law in (2.8) stabilizes the system towards the balanced formation (2.7) moving at the reference speed $\omega_r = \omega + \omega_0$.

Differently from [9], here we consider a discrete time setting. Furthermore, we assume that only cheap proximity sensors with limited range are available. Since we assumed that the detecting distance $\theta_{\max}$ is lower than the desired spacing distance ψ, we can observe that when the desired spacing ψ is achieved, the proximity graph is not connected. Therefore, in our estimation and control design we are not going to rely on connectivity.

For the multi-agent system presented, we propose a decentralized estimation and control strategy capable of achieving a balanced circular formation.

As distance measurements do not provide any information on the signs of the relative angular positions ($d^{-1}(\cdot)$ is multi-valued and non-smooth), and the measurement noise is bounded, we employ an estimation algorithm inspired to that proposed in [13], which is based on a technique known as Interval State estimation ([36]). Such strategy is then complemented with a bang-bang control law capable of equispacing the agents on the circle.

Chapter 3

Estimation Strategy

3.1 Set theoretic state estimator

The estimation strategy we employ in our problem is based on a deterministic treatment of uncertainty, e.g. no hypotheses are made on the probabilistic distribution of the measurement noise, that is assumed bounded. The goal of estimation strategy is to recursively characterize the set of all possible state that are compatible at time k with system dynamics and available measurements. If the model is assumed linear, ellipsoids that are guaranteed to contain the present state can be used. For non-linear models, as the one we study in this thesis, the problem is much more difficult, since the set that contains the state in usually non convex, non connected, and relatively few theoretical results exist. To illustrate the concept underneath the set theoretic state estimator, consider a non-linear and possibly time-varying system defined by

$$x_{k+1} = f_k(x_k, u_k) \tag{3.1}$$

$$y_k = g_k(x_k) + v_k \tag{3.2}$$

and assume that $v_k \in V$ and that X_k is some set guaranteed to contain the state x_k. In principle, we can define the *projected* set $X_{k+1|k}$ as

$$X_{k+1|k} = f_k(X_k, u_k), \tag{3.3}$$

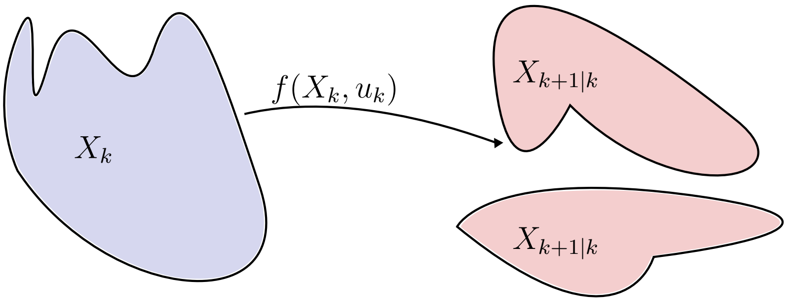

Figure 3.1: Projection step. Properties (e.g. connectedness, convexity and so on) are not preserved.

namely the set of images through $f(\cdot, u_k)$ of present set X_k. The projected set is guaranteed to contain x_{k+1}, and represents the information on the state we can extract from the model. Now, let Y_k be the set of all possible output, when the value of the measured one is y_k

$$Y_k = \{y_k - v | v \in V\} \tag{3.4}$$

obviously the set of all values of the state at time k that could have led to an observation y in Y_k, e.g. the information on the state coming from a measure, is the set $h_k^{-1}(Y_k)$.

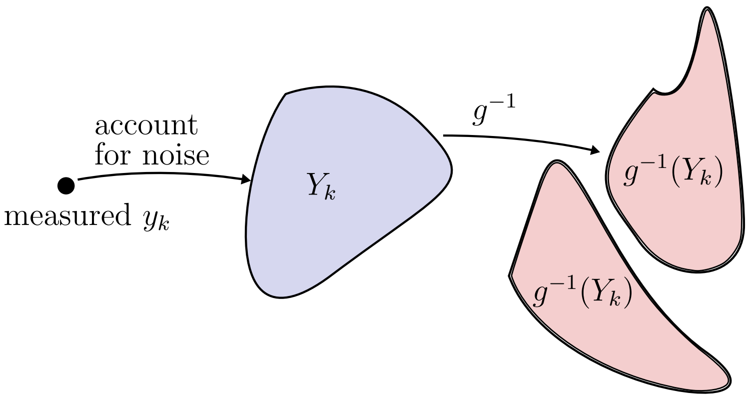

Figure 3.2: Extracting info from measurement. A non convex nor connected set can be obtained.

Then, to combine information coming from measurements and information coming from model the following *corrected* set can be computed

$$X_{k+1} = X_{k+1|k} \cap h_k^{-1}(Y_{k+1}) \tag{3.5}$$

that is also guaranteed to contain x_{k+1}.

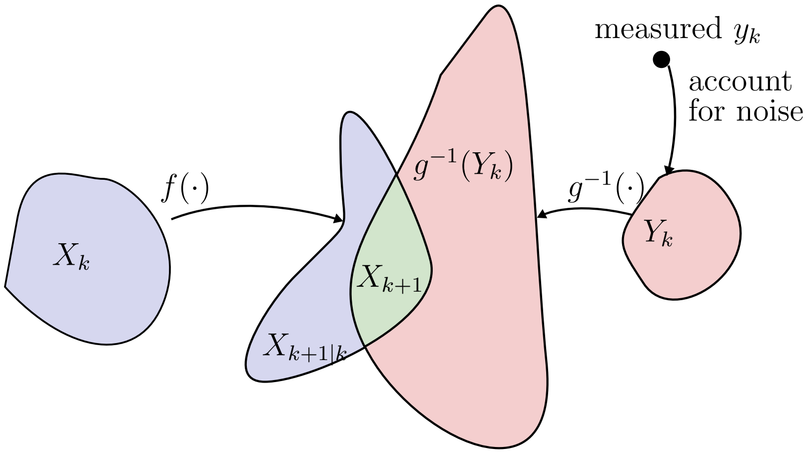

Figure 3.3: Correction step by intersection of sets representing informations coming from model, and information coming from measurement.

The described procedure can be applied recursively, starting from a certain set X_0, assumed to be available a-priori. It can be shown that the corrected set is the smallest one (at time k) that is guaranteed to contain x_k, but except in very particular cases, it cannot be evaluated exactly. An outer approximation can be computed using Interval Analysis, as illustrated in [24]. In our problem, we impose a particular structure on X_0, and study how this structure propagates in time under particular hypotheses on $f(\cdot)$ and $g(\cdot)$, to prove a convergence result of the whole multi-agent system to the desired formation. In particular we assume that X_0 is a closed box of $\mathbb{R}^N$, that f is isometric, and that g is non injective, as can be easily deduced from the proposed problem formulation. The box X_0, since $g^{-1}(Y_k)$ is a union of disjoint closed boxes, because of intersections, splits over time, becoming a finite union of disjoint (smaller) boxes. Given the structure of the set X_k, it can be observed that each box can be projected on the orthonormal basis of $\mathbb{R}^N$, allowing to decouple the dynamics, in such a way that X_k can be viewed, at each time instant, as the product of unions of disjoint closed intervals of the coordinated axes. See Figure 3.4.

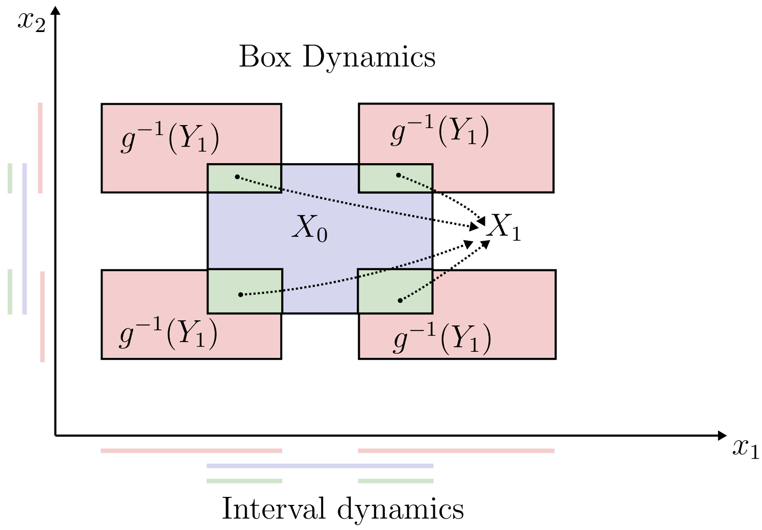

Figure 3.4: Decoupling uncertainty boxes dynamics to uncertainty interval dynamics.

Estimation of the state of the whole system is thus equivalently reduced to estimation of the state of each couple of agents, and uncertainty boxes are reduced to uncertainty intervals, whose dynamics led to a convergent and very efficient estimator.

To study the dynamics of intervals we resort on techniques from Interval Analysis, that we briefly introduce.

3.2 Interval Analysis

In this section we give some relevant operations and notation on intervals [31]:

- given an interval $J \subset \mathbb{R}$, we denote its infimum $\underline{J}$, its supremum $\bar{J}$, and $w(J) \in \mathbb{R}$ denotes its width, i.e. $\bar{J} - \underline{J}$; if $J = \{x\}$ is a singleton, we call it *degenerate*, and we identify the interval with x. In this case, $\underline{J} = \bar{J} = x$;

- a generic binary operation $\odot$ between intervals X and Y is defined as the set $\{x \odot y \in \mathbb{R} | x \in X, y \in Y\}$.

- given λ intervals $X_1, \dots, X_\lambda$, the infimum and the supremum of the interval hull $H = \text{hull}_l \{X_l\}$ are given by $\underline{H} = \inf_l \{\underline{X_l}\}$ and $\bar{H} = \sup_l \{\bar{X}_l\}$, respectively. Note that, if both $\underline{H}$ and $\bar{H}$ belong to $\cup_l X_l$, then the hull is a closed interval.

- the algebraic sum of a closed interval X and a scalar z is $X+z = [\underline{X}+z, \bar{X}+z]$.

Lemma 3. *Given three intervals A_1, A_2, and J, if $\underline{A}_2 > \bar{A}_1, J \cap A_1 \neq \emptyset$, and $J \cap A_2 \neq \emptyset$, one of the following conditions is satisfied: (1) $w(J) > \underline{A}_2 - \bar{A}_1$; (2) $w(J) = \underline{A}_2 - \bar{A}_1$, where J is closed, A_1 is right-closed and A_2 is left-closed.*

Lemma 4. *Given three intervals A_1, A_2, and J, if $\underline{A}_2 > \bar{A}_1$, $\underline{J} \geq \underline{A}_1$, $\bar{J} \leq \bar{A}_2$, and J has a non empty intersection with both A_1 and A_2, then the set $J \cap (A_1 \cup A_2)$ is the union of two intervals J_1 and J_2 such that $w(J_1) + w(J_2) = w(J) - (\underline{A}_2 - \bar{A}_1)$.*

3.3 Uncertainty intervals

As outlined, the dynamics of boxes in $\mathbb{R}^N$, can be equivalently reduced to dynamics of uncertainty intervals in $\mathbb{R}$. As the oscillators we consider have, in general, different angular velocities, the proximity-based communications between the agents imply alternate activations and deactivations of the measurement system, and therefore an irregular and intermittent flow of positional data. However, we observe that, at every time instant, we can compute an uncertainty interval for the relative agents position. In fact, as illustrated in Figure 3.5, when a distance measure is available, we build an uncertainty interval assuming a bounded measurement error. On the other hand, when a pair of agents do not perform any measurement, we can still extract an information from the absence of communications: each agent of the pair is outside the visual cone of the other, and a (wider) uncertainty interval can be computed.

Is it possible to recursively estimate the relative angular position of all the agents moving on the circle (i.e. the state of the system), on the basis of the sequence of these uncertainty intervals? This is the problem we tackle in this chapter.

We emphasize that, even though oscillatory dynamics are widely studied in different fields of sciences and engineering [16, 34, 42, 45], the answer to this question is non-trivial. This is mainly due to the limited visual cone of the agents, due to proximity measurement rule, and the lack of orientation in the intermittent measurements.

We point out that the problem is strongly related to the so called "state bounding" in the framework of Interval Analysis [21, 20, 25]. which has already been applied to state estimation problems to cope with the problem of measurement uncertainty, see [43] and references therein. Recently, Interval Analysis was also used in localization problems, see [32].

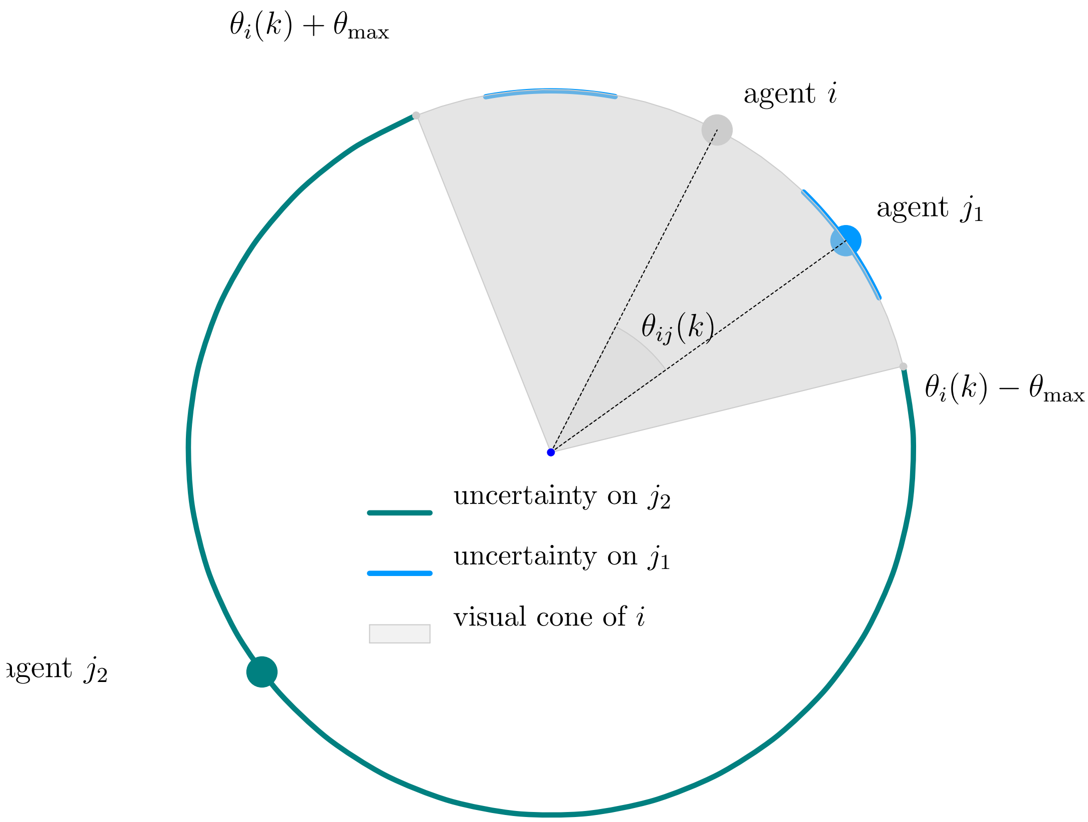

Figure 3.5: Various uncertainty intervals that may arise during estimation.

For instance, in [25], the authors deals with offline nonlinear state estimation where measurements are available only when some given equality conditions are satisfied, and its effectiveness is illustrated when directive measurements are available.

In what follows, we demonstrate some interesting properties of the uncertainty intervals, whose computation is necessary to solve the state estimation we undertake. Then, we propose a recursive algorithm for state estimation which exploits these properties and strongly limits the computational burden, by combining the information coming from the measurement (or from the absence of the measurement) with that coming from the knowledge of the agents' dynamics, thus avoiding an exponential increase of the number of bounding intervals.

In what follows, we show that the information available at each time instant can be exploited to recursively reduce the measurement uncertainty. We establish a set of theorems and corollaries on the properties of the uncertainty intervals that arise in our estimation problem. These theorems are used to develop an estimation algorithm which can be run in a completely decentralized fashion by each agent of the system. Therefore, for the sake of clarity, we illustrate the estimation method with reference to a single pair (i, j) of agents. Our approach can be easily adapted to scenarios in which a wider set of information is available, e.g. in centralized or cooperative scenarios.

3.4 Properties of uncertainty intervals

We start by observing that, if at time k two agents i and j perceive each other, as the measurement noise is bounded, then $\alpha_{ij}(k)$ belongs to the interval

$$\Upsilon_{ij}(k) := [\max\{y_{ij}(k) - \varphi, 0\}, \min\{y_{ij}(k) + \varphi, \theta_{\max}\}] \subseteq I, \tag{3.6}$$

where the max and min operators are used to take into account that a distance cannot be negative and that, when $\alpha_{ij}(k) > \theta_{\max}$, the agents do not perceive each other.

On the other hand, if, at time k, the agents do not sense each other, then $\alpha_{ij}(k)$ belongs to $I^c := (\theta_{\max}, \pi]$. In both cases $\theta_{ij}(k)$ belongs to an *unbounded* set of intervals

$$J^{\Upsilon}(k) := \begin{cases} d^{-1}(\Upsilon_{ij}(k)) & \text{if } \alpha_{ij}(k) \in I, \\ d^{-1}(I^c) & \text{otherwise.} \end{cases} \tag{3.7}$$

Now, consider the time evolution of $\theta_{ij}(k)$ given by the linear system (2.3). As its eigenvalues are unitary and $\omega_{ij}(k)$ is known, by means of the operations on intervals, we can say that (2.3) maps uncertainty intervals at time k into uncertainty intervals of identical width at time $k+1$, (i.e. without adding further uncertainty). The congruence between state and measurement equations enables us to conclude that, at each time instant k, $\theta_{ij}(k)$ belongs to the *uncertainty set*

$$J(k) = (J(k-1) + \omega_{ij}(k-1)) \cap J^{\Upsilon}(k), \tag{3.8}$$

$$J(0) = J^{\Upsilon}(0) \cap]-\pi, +\pi[. \tag{3.9}$$

From now on, we define $J(k|k-1) := J(k-1) + \omega_{ij}(k-1)$. Notice, that $J(k)$ is a multi-interval and we denote with $J_l(k)$ its l-th element. Moreover, we emphasize that a conservative measure of the uncertainty on $\theta_{ij}(k)$ is given by $w(\text{hull}_l(J_l(k)))$. Hence, in what follows, we characterize the uncertainty on $\theta_{ij}(k)$ by illustrating some properties of $J(k)$ and of the width of its hull.

Theorem 1. *For all $k \in \mathbb{N}$, (1) there exists a finite l_k such that $J(k) = \cup_l J_l(k)$, $l = 1, ..., l_k$; (2) $w(J_l(k)) \leq \pi - \theta_{\max}, \forall l$; (3) $w(\text{hull}_l(J_l(k))) \leq 2\pi$.*

Proof. **(1)** At time $k = 0$, the statement is a direct implication of (3.9). Moreover, as $J(k|k-1)$ is finite and bounded, it may have a non-empty intersection with a finite number of elements of $J_{\Upsilon}(k)$. Hence, the first statement follows. **(2)** As the maximum possible width of $\Upsilon_{ij}(k)$ is $\min\{2\varphi, \theta_{\max}\}$, and from equation (3.9), we have that $\max_l\{w(J_l(0))\} = w(I^c) = \pi - \theta_{\max}$. As all the eigenvalues of the dynamical system defined in (2.3) are unitary, we conclude that $w(J_l(k)) \leq \pi - \theta_{\max}$, for all $k \in \mathbb{N}$; **(3)** From (3.9), we have that $w(\text{hull}_l J_l(0)) \leq 2\pi$. Then, using the same argument as above, we can state that $w(\text{hull}_l J_l(k)) \leq 2\pi$, for all $k \in \mathbb{N}$. □

As we intersect the information coming from the intermittent and uncertain measurements with that on the dynamics, the intervals of the uncertainty set may dynamically shrink, split or disappear. Hence, we now discuss some remarkable properties of these intersections.

Denote with $\tilde{k}_{ij}$ the first time instant in which two agents i and j perceive each other, and with $J_l(k+n|k)$ the projection of the interval $J_l(k)$ through the dynamic equation (2.3) n steps ahead. The following theorem and related corollaries hold:

Theorem 2. *For all k, n such that $k + n < \tilde{k}_{ij}$, the set $S^l_{ij} := J_l(k+n|k) \cap d^{-1}(I^c)$ can be (1) the empty set; (2) an interval; (3) the union of two disjoint intervals, if $\theta_{\max} < \pi/3$.*

Proof. The first two points are trivially true. Hence, we focus on the third point and show that S_{ij}^l may also be the union of two disjoint intervals. From Lemmas 1 and 2, we know that $d^{-1}(I^c)$ is the union of an infinite number of open intervals A_z of width $2\pi - 2\theta_{\max}$ and separated by closed intervals of width $2\theta_{\max}$. From Lemma 3, we know that if $J_l(k+n|k)$ intersects both A_{z+1} and A_z, then $w(J_l(k+n|k)) > \underline{A}_{z+1} - \bar{A}_z = 2\theta_{\max}$. From Theorem 1, this is possible if $\theta_{\max} < \pi/3$. Now, we show that the set S_{ij}^l may not be the union of three or more disjoint intervals. Indeed, from Lemma 3, if $J_l(k+n|k)$ intersected A_z, A_{z+1} and A_{z+2}, then its width would be greater than $\underline{A}_{z+2} - \bar{A}_z = 2\pi + 2\theta_{\max}$. Hence, we would get a contradiction, as from Theorem 1 the maximum width of $J_l(k+n|k)$ is $\pi - \theta_{\max}$. □

Corollary 1. *For all k, n such that $k+n < \tilde{k}_{ij}$, if the set S_{ij}^l is made of two disjoint intervals, say J_{l_1} and J_{l_2}, then $w(J_{l_1}) + w(J_{l_2}) = w(J_l(k+n|k)) - 2\theta_{\max}$.*

Proof. As S_{ij}^l is made of two disjoint intervals, we have $\underline{A}_{z+1}(k+n) - \underline{A}_z(k+n) = \bar{A}_{z+1}(k+n) - \bar{A}_z(k+n) = 2\pi \geq \max(w(J))$[1]. Hence, the hypotheses of Lemma 4 are fulfilled, and therefore the thesis follows. □

Remark 1. *Corollary 1 implies that, for all $k < \tilde{k}_{ij}$, each of the $J_l(k)$ are separated by intervals whose width is greater or equal to $2\theta_{\max}$. Such statement is obviously true at time $k = 0$. Corollary 1 proves that it is still true until the agents first perceive each other. Indeed, when an intersection gives rise to two intervals $J_l(k)$ and $J_{l+1}(k)$, we have $\underline{J}_{l+1} - \bar{J}_l = 2\theta_{\max}$.*

Corollary 2. *For all $k < \tilde{k}_{ij}$, the maximum number of intervals composing the uncertainty set is $2 + 2m$, where $m = \lfloor(\pi - \theta_{\max})/2\theta_{\max}\rfloor$.*

Proof. If two agents do not perceive each other at time $k = 0$, the uncertainty set is made of two intervals $J_1(0)$ and $J_2(0)$, such that $w(J_1(0)) = w(J_2(0)) = \pi - \theta_{\max}$. Moreover, from Corollary 1, every time a new interval is generated, the total widths of the uncertainty intervals is reduced by $2\theta_{\max}$. Hence, the thesis follows. □

Theorem 2 and the related corollaries encompass all the possible cases related to the intersection between $J_l(k+n|k)$ and $d^{-1}(I^c)$, thus completing the analysis of the properties of the intersections when $k < \tilde{k}_{ij}$. Now, let us consider the case in which $|\mathrm{rem}(\theta_{ij}(k+n))| \leq \theta_{\max}$, and a distance measurement is then available at time $k+n$. According to Lemma 1, we know that $\theta_{ij}(k+n)$ belongs to the

[1] We remind the reader that $d^{-1}(I^c) = \cup_z A_z(k+n)$

unbounded multi-interval set defined by $d^{-1}(\Upsilon_{ij}(k+n))$. By intersecting it with the a priori uncertainty set $J(k+n|k)$, we obtain the a posteriori uncertainty set, which is denoted by $J(k+n)$. The following two theorems state two relevant properties of the a posteriori uncertainty set:

Theorem 3. *For all k, n such that $k+n = \tilde{k}_{ij}$, $\tilde{S}_{ij} := \cup_l \left(J_l(k+n|k) \cap d^{-1}(\Upsilon_{ij}(k+n))\right)$ can be made of no more than three intervals.*

Proof. Trivially, if $k+n=0$, then the uncertainty set is given by $J(0) = \Upsilon_{ij}(0) \cup -\Upsilon_{ij}(0)^2$ and thus it is made of two intervals. Next, we consider the case $k+n \neq 0$. To prove our statement, we show that $\tilde{S}_{ij}$ can be made of no more than three intervals. In fact, as the maximum possible width of $\Upsilon_{ij}(k)$ is $\min\{2\varphi, \theta_{\max}\}$, from Equation (3.6) and Lemma 1, we know that, for any l, $d^{-1}(\Upsilon_{ij}(k+n)) := \cup_l(B_l)$, and $w(B_l) = \tilde{w} \leq \min\{2\varphi, \theta_{\max}\}$. Moreover, again from Equation (3.6) and Lemma 1, we have that the minimum width of $[\bar{B}_l, \underline{B}_{l+3}]$ is 2π. Thus, from Theorem 1 and Lemma 3, $\text{hull}_l(J_l(k+n|k))$, and the multi-interval set $\cup_l(J_l(k+n|k))$ cannot intersect all the four intervals $B_l, ..., B_{l+3}$. As the maximum possible width of $\Upsilon_{ij}(k)$ is $\min\{2\varphi, \theta_{\max}\}$, from Remark 1 and Lemma 3, we have that if $B_{l_1} \cap J_{l_2}(k+n|k) \neq \emptyset$, then $B_{l_1} \cap J_l(k+n|k) = \emptyset$, $\forall\, l \neq l_2$. Hence, the thesis follows. □

Remark 2. *After the first measurement is gathered, that is, for all $k > \tilde{k}_{ij}$,* $\max_l\{w(J_l(k))\} \leq \min\{2\varphi, \theta_{\max}\}$.

Theorem 4. *For all k, n such that $k+n \geq \tilde{k}_{ij}$, $J_l(k+n|k) \cap d^{-1}(\Upsilon_{ij}(k+n))$ can be (1) the empty set; (2) an interval; (3) the union of two intervals.*

Proof. The first two points are trivially true. Thus, we focus on point 3. As showed in the proof of Theorem 3, $J_l(k+n|k)$ can intersect B_l and B_{l+1}. Hence, we only need to show that it cannot intersect B_l, B_{l+1} and B_{l+2}. If it were possible, from Lemma 4, we would have $w(J_l(k+n|k)) \geq \underline{B}_{l+2} - \bar{B}_l \geq 2\pi - \theta_{\max}$. As we know that $0 < \theta_{\max} \leq \pi/2$, from Theorem 1, we would get a contradiction, as $w(J_l(k+n|k)) \leq \pi - \theta_{\max} < \pi$, and $\underline{B}_{l+2} - \bar{B}_l \geq 2\pi - \theta_{\max} > \pi$. Hence, the thesis follows. □

Theorem 5. *$\forall k \geq \tilde{k}_{ij}$, $n \in \mathbb{N}$, $J_l(k+n|k) \cap d^{-1}(I^c)$ is either the empty set or an interval.*

[2] $-\Upsilon_{ij}(k) \triangleq [\max\{-\bar{\Upsilon}_{ij}(k+n), -\theta_{\max}\}, \min\{-\underline{\Upsilon}_{ij}(k+n), 0\}]$

Proof. Given the definition of I^c, and from Lemmas 1 and 2, we know that $d^{-1}(I^c)$ is the union of an infinite number of open intervals A_z of width $2\pi - 2\theta_{\max}$ separated by closed intervals of width $2\theta_{\max}$. Moreover, as emphasized in Remark 2, for all $k \geq \tilde{k}_{ij}$, $J_l(k+n|k)$ is $\min\{2\varphi, \theta_{\max}\}$. Thus, none of the $J_l(k+n|k)$ fulfils the requirements given in Lemma 3 necessary to intersect any two A_z and A_{z+1}. □

Algorithm 1 Estimation Algorithm based on Interval Analysis

```
  k = 0
Require: k̃_ij < 0
  λ(0) = 2                                            ▷ from Th. 1
  if |rem(θ_ij(0)) ≤ θ_max| then
      J(0) = −Υ_ij(0) ∪ Υ_ij(0)
      k̃_ij = 0
  else
      J(0) = −I^c ∪ I^c
  end if
  k = 1
  while w(H(k)) ≥ δ do                                ▷ δ is the tolerance on the convergence
      if y_ij(k) ∈ ℝ then
          if k̃_ij < 0 then                            ▷ i.e. a measure is available for the first time
              Routine A                               ▷ Th. 3 and 4 are leveraged
              k̃_ij = k
          else                                        ▷ i.e. a measure is available but not for the first time
              Routine B                               ▷ Th. 4 is leveraged
          end if
      else                                            ▷ i.e. the agents do not perceive each other
          if k̃_ij < 0 then                            ▷ the agents do not perceived each other yet
              Routine C                               ▷ Th. 2 is leveraged, see the Appendix
          else                                        ▷ the agents have already perceived each other
              Routine D                               ▷ Th. 5 is leveraged
          end if
      end if
      k = k + 1
      w(H(k)) = max_l{J̄_l(k)} − min_l{J_l(k)}
  end while
  return θ_ij(k) = (max_l{J̄_l(k)} + min_l{J_l(k)})/2
```

```
Routine A
  λ(k) = 0, l = 1
  while l < λ(k − 1) and λ(k) ≤ 3 do                  ▷ λ(k) ≤ 3, see Th. 3
      J_l(k|k − 1) ∩ d^−1(Υ_ij(k))                    ▷ Three only alternatives, see Th. 4
      if J_l(k|k − 1) ∩ d^−1(Υ_ij(k)) is an interval then
          λ(k) = λ(k) + 1
      else
          if J_l(k|k − 1) ∩ d^−1(Υ_ij(k)) is the union of two intervals then
              λ(k) = λ(k) + 2
          end if
      end if
      l = l + 1
  end while
```

```
Routine B
  λ(k) = 0
  for l = 1 : λ(k − 1) do
      J_l(k|k − 1) ∩ d^−1(Υ_ij(k))                    ▷ Three alternatives, see Th. 4
      if J_l(k|k − 1) ∩ d^−1(Υ_ij(k)) is an interval then
          λ(k) = λ(k) + 1
      else
          if J_l(k|k − 1) ∩ d^−1(Υ_ij(k)) is the union of two intervals then
              λ(k) = λ(k) + 2
          end if
      end if
  end for
```

Routine C
$\lambda(k) = 0$
for $l = 1 : \lambda(k-1)$ **do**
  $J_l(k|k-1) \cap d^{-1}(I^c)$ ▷ Three alternatives, see Th. 2
  if $J_l(k|k-1) \cap d^{-1}(I^c)$ is an interval **then**
    $\lambda(k) = \lambda(k) + 1$
  else
    if $J_l(k|k-1) \cap d^{-1}(I^c)$ is the union of two intervals **then**
      $\lambda(k) = \lambda(k) + 2$ ▷ only possible if $\theta_{\max} < \pi/3$ from Th. 2
    end if
  end if
end for

Routine D
$\lambda(k) = 0$
for $l = 1 : \lambda(k-1)$ **do**
  $J_l(k|k-1) \cap d^{-1}(I^c)$
  if $J_l(k|k-1) \cap d^{-1}(I^c)$ is an interval **then** ▷ can be only the empty set or an interval thanks to Th. 5
    $\lambda(k) = \lambda(k) + 1$
  end if
end for

The theorems presented above are the foundations of our estimation technique, illustrated in Algorithm 1. We remark that, when a wider set of information is available, our estimation method may be easily adapted. This is the case, for instance, of a centralized estimation, where every agent communicates with a central base station, or of cooperative estimation, in which the agents can collaborate to improve the accuracy of their estimates. In these scenarios, the additional information can be exploited by considering that the elements of matrix $\Theta(k)$ must satisfy the following concatenation property:

$$\sum_{\substack{i=l\\ j=i+1}}^{p-1} \theta_{ij} = \theta_{lp},\ l = 1, \ldots, N-1,\ p = l+2, \ldots, N. \tag{3.10}$$

These constraints can be used to further restrict the uncertainty on the elements of matrix $\Theta(k)$. To clarify this point, we refer to the simplest case of $N = 3$ agents, where (3.10) becomes $\theta_{12}(k) + \theta_{23}(k) = \theta_{13}(k)$. In this case, at each time instant, it is possible to intersect the multi-interval $J^{12}(k) + J^{23}(k)$ with the multi-interval $J^{13}(k)$[3], thus possibly reducing the width of the uncertainty sets on $\theta_{12}(k)$, $\theta_{13}(k)$ and $\theta_{23}(k)$. The obvious advantage of considering a larger amount of information entails an increase in the computational burden. We remark that all the results presented in this section refer to the non-cooperative scenario.

3.5 Convergence by a density argument

Here, we show under which conditions the estimate provided by the proposed algorithm converge to the true value $\theta_{ij}(k)$. Let us denote with $H(k) = [\min_l\{\underline{J}_l(k)\}, \max_l\{\bar{J}_l(k)\}]$

[3] $J^{ij}(k) = \bigcup_l J_l^{ij}(k)$ is the uncertainty of the pair (i, j)

the hull of all intervals $J_l(k)$ of the uncertainty set $J(k)$.

Definition 3. *We say that Algorithm 1 converges if* $\lim_{k\to\infty} w(H(k)) = 0$.

The following theorem establishes a sufficient condition for convergence.

Theorem 6. *If*

$$\{\mathrm{mod}\,(\theta_{ij}(k) - \theta_{ij}(0))\}_{k=0}^{\infty} = \left\{\mathrm{mod}\left(\sum_{h=0}^{k-1} \omega_{ij}(h)\right)\right\}_{k=0}^{\infty} \tag{3.11}$$

is dense in $[0, 2\pi]$*, then Algorithm 1 converges.*

Proof. To prove our statement, we must show that

$$\forall \epsilon > 0,\ \exists h : \forall k > h,\ w(H(k)) < \epsilon.$$

As from Theorem 1 we know that $\max\{w(H(k))\} = 2\pi$, and that in such case $H(k)$ is open, then

$$2\pi - (\theta_{ij}(k) - \underline{H}(k)) = \kappa_1 > 0,$$
$$2\pi - (\bar{H}(k) - \theta_{ij}(k)) = \kappa_2 > 0.$$

Moreover, if (3.11) holds, then

$$\forall \epsilon > 0,\ \exists\ h_1, h_2 \in \mathbb{N} : \begin{cases} 0 < \theta_{\max} - \mathrm{mod}(\theta_{ij}(h_1)) < \epsilon/2, \\ 0 < \mathrm{mod}(\theta_{ij}(h_2)) - \theta_{\max} < \epsilon/2. \end{cases} \tag{3.12}$$

In Equation (3.12), the first condition implies that, for $k \geq h_1$, $]\theta_{ij}(k) - 2\pi + \epsilon/2, \theta_{ij}(k) - 2\theta_{\max}[\cap J(k) = \emptyset$. Taking $\epsilon < 2\kappa_1$ we have that, for $k \geq h_1$, $(\theta_{ij}(k) - \underline{H}(k)) \leq 2\theta_{\max}$. Moreover, the second condition in (3.12) assures that $]\theta_{ij}(k) - 2\pi + 2\theta_{\max}, \theta_{ij}(k) - \epsilon/2[\cap J(k) = \emptyset$. Hence, we have that, for $k \geq \max\{h_1, h_2\}$, $(\theta_{ij}(k) - \underline{H}(k)) \leq \epsilon/2$. Following the same line of arguments, we obtain that, for $k \geq \max\{h_1, h_2\}$, $\bar{H}(k) - \theta_{ij}(k) \leq \epsilon/2$ which proves our statement.

□

Obviously, the fulfillment of the hypothesis (3.11) depends on $\omega_{ij}(k)$, $i, j = 1, \ldots, N$, $k \in \mathbb{R}$. It is possible to prove that the range of sequence (3.11) is dense for a specific, but relevant, class of angular velocities.

Corollary 3. *If* $\omega_{ij}(k) = \tilde{\omega}_{ij}$ *for all* $k \in \mathbb{N}$*, with* $\frac{\tilde{\omega}_{ij}}{2\pi} \notin \mathbb{Q}$*, then Algorithm 1 converges.*

Proof. From the hypothesis, the range of sequence (3.11) can be rewritten as $\{\mathrm{mod}\,(k \cdot \tilde{\omega}_{ij})\}_{k=0}^{\infty}$. Thanks to the well known Kronecker's theorem [18], such sequence is dense in $[0, 2\pi]$. From Theorem 6, the thesis follows. □

We extend this result to periodic angular velocities.

Corollary 4. *If $\omega_{ij}(k)$ is periodic with period d, and*

$$\frac{1}{2\pi}\sum_{h=k}^{k+d}\omega_{ij}(h) \notin \mathbb{Q}, \tag{3.13}$$

then Algorithm 1 converges.

Proof. From the hypotheses, the range of the sequence (3.11) is dense in $[0, 2\pi]$. From Theorem 6, the thesis follows. □

Note, that all convergence results are derived without any assumption on the probability distribution of the bounded measurement noise. This is a key feature of the proposed strategy, in it paves the way for the extension of our results to any other closed curve that may be described in polar coordinates. In fact, the circle is the only curve that allows to unequivocally associate a phase distance to a Euclidean distance, unless the absolute position of one of the agents is known. In any other scenario, each Euclidean distance is mapped into an interval of phase distances, generating a bounded deterministic uncertainty.

3.6 Convergence and contraction in a metric space

In this section our goal is to introduce a theoretical framework in which convergence of the (set theoretic) estimator-based controller can be analyzed. This framework is based on contraction properties of Lipschitz maps, when they are interpreted as maps between metric spaces whose elements are particular sets.

The main point, here, is that the map f that characterize the dynamics of the overall multi-agent system can be *lifted* in a particular metric space, where it associates sets to sets. In that space, convergence results can be obtained by means of fixed point theorems.

To do so, let us consider the set $\mathcal{K}$ of compact subsets of R^N and suppose that the system is regulated by a control function $u(x)$ that renders the compound map $f(x, u(x)) = F(x)$ Lipschitz with Lipschitz coefficient k. Since F is Lipschitz, it satisfies the following metric condition

$$d(F(x_2), F(x_1) \leq k \cdot d(x_2, x_1), \forall x_2, x_1 \in \mathbb{R}^N. \tag{3.14}$$

If $0 < k < 1$, $F(x)$ is termed *contraction.* Contractions are very important functions in non-linear analysis, because they satisfy the well known Banach's Contraction Principle, that states that contractions have unique fixed point.

In problems that involve deterministic treatment of noise and uncertainty, as the one we study in this thesis, one wish to study how uncertainty set propagates in time. We can observe that $F(x)$ defines a transformation on the space set $\mathcal{K}$ of subsets of $\mathbb{R}^{\mathbb{N}}$

$$F(A) = \{F(x)|x \in A\} \tag{3.15}$$

and since $F(\cdot)$ is continuous, it carries $\mathcal{K}$ into itself.

Now suppose that $\mathcal{K}$ is equipped with the Hausdorff metric $d_H(\cdot)$, that can be defined as

$$d_H(A, B) = \max\{d_o(A, B), d_o(B, A)\} \tag{3.16}$$

where $d_o(\cdot)$ is the oriented distance between sets A and B, defined as

$$d_o(A, B) = \max_{x \in A} \min_{y \in B} d(x, y) \tag{3.17}$$

where $d(\cdot)$ is a distance in R^N. A very important property of the Hausdorff distance to our scope is the following

$$d_H(A \cap C, B \cap C) \leq d_H(A, B), A \cap C \neq \varnothing, B \cap C \neq \varnothing. \tag{3.18}$$

Equipped with the Hausdorff distance, the set $\mathcal{K}$ becomes a complete metric space. When $F(\cdot)$ is considered as a function from $\mathcal{K}$ to $\mathcal{K}$ one may ask if the contraction property (that involves only the Hausdorff metric defined in $\mathcal{K}$) is preserved. By simple calculations, it is possible to prove that a contraction $F(x)$ on $\mathbb{R}^N$ induces a contraction with the same Lipschitz constant k on the space $\mathcal{K}$. Indeed results

$$\begin{aligned} d_o(F(A), F(B)) &= \max_{a \in A} \min_{b \in B} d(F(a), F(b)) \\ &\leq \max_{a \in A} \min_{b \in B} k \cdot d(a, b) \\ &= k \cdot d(A, B). \end{aligned}$$

Since $F(\cdot)$ is a contraction in the space $\mathcal{K}$ of compacts of R^N it admits a fixed point, hence the recurrent equation $X_{k+1} = F(X_k)$ converges to it. This result is one of the key of the Hutchinson's remarkable construction of fractals. We remark that the fixed point is a compact set, non necessarily a singleton.

Let us now consider the set theoretic state estimator to analyze its convergence properties.

$$X_{k+1} = f(X_k, u_k) \cap h_k^{-1}(Y_{k+1}) = F(X_k) \cap h_k^{-1}(Y_{k+1}). \tag{3.19}$$

Property 3.18 implies that

$$\begin{aligned} d_H(F(A) \cap h_k^{-1}(Y_{k+1}), F(B) \cap h_k^{-1}(Y_{k+1})) &\leq d_H(F(A), F(B)) && (3.20) \\ &\leq k \cdot d_H(A, B). && (3.21) \end{aligned}$$

We remark that the highlighted contraction properties of the set theoretic state estimator allow to conclude that

- if the control system is designed to be contractive in absence of noise and uncertainty, it remains contractive when the uncertainty is described by compact sets;
- correction step in the set theoretic state estimator don't make worse the contraction property of the control system.

These results require further investigations to assess whether exist particular class of control inputs that, as done in Theorem 6 for our paradigmatic multi-agent system, allow to render the convex hull of the fixed point of the set theoretic state estimator contained in a set that represents the control goals.

3.7 Numerical simulation of estimation algorithm

In this section, we present extensive numerical simulations to complement the theoretical analysis and to test the performance of the algorithm. To this aim, we considered two scenarios for our simulations:

1. $\omega_{ij}(k)$ is randomly selected in $[0, 2\pi]$, $k = 1, ..., T$;
2. the relative angular velocity between each pair of agents is constant, that is, $\omega_{ij}(k) = \tilde{\omega}_{ij}$, $k = 1, ..., T$, with $\tilde{\omega}_{ij}$ randomly selected in $[0, 2\pi[$.

	Scenario (1)	Scenario (2)
p	100.00	99.85
$\langle k_c \rangle$	8.40	11.40
$\langle w(H(T)) \rangle$	6.00×10^{-3}	5.90×10^{-3}

Table 1. Percentage of convergent estimates p, average convergence time $\langle k_c \rangle$, and average final uncertainty $\langle w(H(T)) \rangle$.

For both scenarios, we perform 30000 simulations each involving a pair of agents, and we select the simulation time $T = 300$, $\varphi = \theta_{\max}/5$, and, for each simulation, we take $\theta_{\max}$ randomly in $[0.11\pi, 0.30\pi]$. To test the performance of the algorithm, we introduce the definition of practical convergence. Namely, we say that the algorithm practically converges if there exists a time instant $k_c \leq T$ such that $w(H(k_c)) \leq 2\varphi$, and we say that k_c is the convergence time. In Table 1, we report the percentage of convergent estimates p, the average convergence time $\langle k_c \rangle$, and the average final uncertainty $\langle w(H(T)) \rangle$ for the two numerical scenarios. We observe that, in Scenario (1), as $\omega_{ij}(k)$ is randomly selected from a uniform distribution, then (3.11) is verified and, from Theorem 6, asymptotic convergence is guaranteed. Accordingly, we find that the estimation practically converges in all simulations, see Table 1. In Scenario (2), the hypothesis of Corollary 3 is fulfilled. In fact, the probability of randomly picking an $\tilde{\omega}_{ij}$ such that $\tilde{\omega}_{ij}/2\pi \in \mathbb{Q}$ is zero. However, from Table 1, we observe that practical convergence is achieved in 99.85% of simulations. This is due to our convergence criterion, which is defined on a finite time horizon. This implies that, when $\tilde{\omega}_{ij}/2\pi$ is sufficiently close to a rational number q, the algorithm may not practically converge.

Chapter 4

Control Strategy

The objective in this chapter is to select a control law that allows the estimation algorithm to be executed and that is proved to converge in the nominal case (e.g. without noise and uncertainty).

4.1 Adaptation of a control law designed for the nominal case

In the application considered in this thesis, the estimation strategy would require the knowledge of $u_{ij}(k)$ to compute the a priori uncertainty set $J^{ij}(k+1|k)$. In what follows, we show that our selection of the control law facilitate the estimation algorithm. In fact, in the assumption that all the agents share the same type of control law, it is possible to define an interval in which $u_{ij}(k)$ falls, allowing to perform an interval prediction of $\theta_{ij}(k)$.

Furthermore, as our control strategy only requires information on phase differences $\vartheta_{ij}(k) := \mathrm{rem}(\theta_{ij}(k))$ and not relative angular positions, we define the interval

$$H^{ij}(k) : \begin{cases} \inf_x\{x \in \mathrm{rem}(J^{ij}(k))\}, \\ \sup_x\{x \in \mathrm{rem}(J^{ij}(k))\}, \end{cases} \tag{4.1}$$

which represents an overestimate of the uncertainty on $\vartheta^{ij}(k)$.

Finally, as a scalar estimate is required by our control law, we define it as

$$\hat{\vartheta}_{ij}(k) = \frac{\bar{H}^{ij}(k) - \underline{H}^{ij}(k)}{2}. \tag{4.2}$$

As in [9], we design our control strategy so that each agent is *pushed* by its followers. Specifically, in our case, each agent i is only influenced by its nearest follower $i-1$, defined as

$$i-1:\begin{cases}\underline{H}^{i,i-1}>0\\ \bar{H}^{i,i-1}<\underline{H}^{ij}_l\ \forall j,l|\underline{H}^{ij}_l>0, j\neq i-1\end{cases} \tag{4.3}$$

Furthermore, each agents is labeled, as is the case when, for instance, proximity measurements are made by means of RFID technology ([52, 41]). The agent L, which is randomly picked, which adopts the following control law:

$$u_L(k) = \omega_0, \tag{4.4}$$

while the control law of the remaining agents is described by

$$u_i(k) = \Big(\omega_0 + K\mathrm{sgn}^+(\psi - \mathrm{mod}(\hat{\vartheta}_{i,i-1}(k)))\Big)\mathcal{I}(i), \tag{4.5}$$

where $\mathcal{I}(i)$ is the following indicator function:

$$\mathcal{I}(i) = \begin{cases}1 & \text{if } i-1 \text{ is univocally determined,}\\ 0 & \text{otherwise,}\end{cases} \tag{4.6}$$

for all $i = 1, \ldots, N$, $i \neq L$.

4.2 How the control law facilitates the estimator

Notice that the selected control strategy is totally decentralized and non-cooperative as the action exerted by each agent depends only on the estimated distance from its closest follower. The key point of the proposed control law is that the bang-bang controller (4.5) is triggered only once the ambiguity on the identity of the follower has been solved by the estimator. However, the selected control law facilitates the estimator, as it implies that

$$u_{ij}(k) \in \{-\omega_0 - K, -\omega_0, -K, 0, K, \omega_0, \omega_0 + K\}, \tag{4.7}$$

allowing to compute $J^{ij}(k+1|k)$ from $J^{ij}(k)$ as

$$\bigcup_{\lambda}[\underline{J}^{ij}_{\lambda}(k) - \omega_0 - K, \bar{J}^{ij}_{\lambda}(k) + \omega_0 + K]. \tag{4.8}$$

Nevertheless, to improve the performance of the estimator, it is possible to further restrict the set of allowed values for $u_{ij}(k)$ for selected cases of interest. To clarify this point, let us consider a generic agent $i \neq L$ and its follower $i-1$, as the control law u_i only requires information on the relative angular position of this pair of agents. In this case, we know that, before that agent i identifies its follower,

$$u_{i,i-1} \in \{-\omega_0 - K, -\omega_0, -K, 0\} \tag{4.9}$$

Furthermore, we observe that agent i may identify its follower only at a time instant $\bar{k}_i$ in which a measurement is available, and therefore $u_{i-1}(\bar{k}_i)$ cannot be zero. Given the previous considerations, as at time $\bar{k}_i$ the two agents perceive each other, and as $\theta_{\max} < \psi$, we know that $u_i(\bar{k}_i) = \omega_0 + K$. Furthermore, as we pointed out that $u_{i-1}(\bar{k}_i) \neq 0$, we know that the control law of agent $i-1$ has already been triggered. Hence, we have

$$u_{i,i-1} \in \{0, K\}, \ \forall k \geq \bar{k}_i. \tag{4.10}$$

Finally, if at time $\tilde{k}_i > \bar{k}_i$ agent i is able to push agent $i-1$ outside of its visual cone, then $u_{i,i-1}(\tilde{k}_i) = K$. This would imply that, at time $\tilde{k}_i$, agent $i-1$ has already reached the desired spacing with agent $i-2$, and its estimate $\hat{\vartheta}_{i-1,i-2}(k)$ of the angle $\vartheta_{i-1,i-2}(k)$ is greater than or equal to ψ. Therefore, $u_{i-1}(k) = \omega_0$ for all $k \geq \tilde{k}_i$. Then, for all $k \geq \tilde{k}_i$, agent i can estimate $\vartheta_{i,i-1}(k+1)$ on the basis of a scalar and unambiguous $u_{i,i-1}(k)$.

4.3 Block diagram of the closed loop

Summing up, when the detecting distance is lower than the desired spacing distance, and therefore, after the transient, the controller push the pairs of consecutive agents outside their mutual detecting distance, and the estimate must rely only on the predictive component of our estimator. For that reason, we choose an extremely simple control law, as the bang-bang action described in equation (4.5), so that the estimation of $\vartheta_{ij}(k)$ is complemented with a simple estimator of $u_{ij}(k)$, see the estimation and control scheme depicted in Figure 4.1.

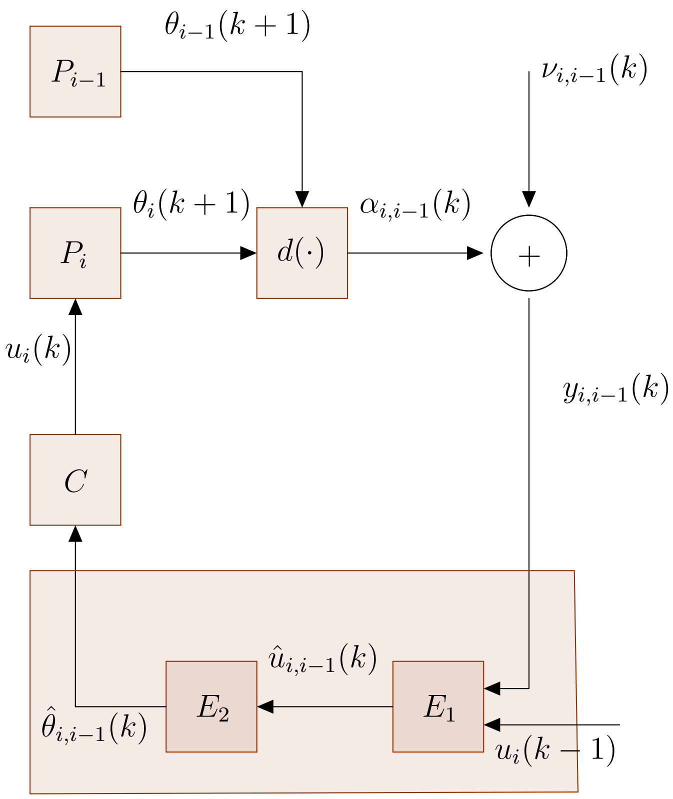

Figure 4.1: Schematic of the estimation and control strategy: P_i is the i-th agent and P_{i-1} its follower; E_1 is the estimator of the relative control input $u_{i,i-1}$; E_2 is the estimator of the relative angular position $\theta_{i,i-1}$; C is the bang-bang controller.

Chapter 5

Numerical Results

5.1 Simulation conditions

To validate our estimation and control strategy, we performed extensive numerical simulations. In all experimental conditions, we set

(a) the number of agents $N = 6$. Hence, the target spacing between consecutive agents is $\psi = 2\pi/N$;

(b) $\omega_0 = 0.01$;

(c) the simulation time $T = 3000$.

We test the effectiveness of our approach for different values of the detecting distance $\theta_{\max}$, the bound of the modulus of the measurement noise φ, and the control gain K. Specifically,

(a) $\theta_{\max}$ is varied between 0.18ψ and 0.9ψ with step 0.18ψ;

(b) for each value of $\theta_{\max}$, φ is varied between $0.04\theta_{\max}$ and $0.20\theta_{\max}$ with step $0.04\theta_{\max}$;

(c) for each combination of $\theta_{\max}$ the control gain K is varied between 0.002 and 0.010 with step 0.002.

The result of this scheme is a total of 125 parameter combinations. For each parameter combination, we consider the same set of $R = 100$ randomly selected initial conditions for the angular positions. We remark that, in the parameter selection, we take $\psi < \theta_{\max}$ to remove the assumption of jointed connectivity. Moreover, we select φ as a function of $\theta_{\max}$ as it is typically related to the sensor's range. For the sake of clarity, we restrict the analysis to the case where the agents cannot overtake each other.

Accordingly, the initial conditions for θ_{ij}, $i, j = 1, \ldots, N$, are selected in the interval $[2\varphi, \pi]$. As we took the same set of R initial conditions for each of the 125 simulated scenarios, the previous condition must be fulfilled considering the maximum value of φ, that is, 0.036ψ.

To test the performance of the algorithm, we introduce the definition of practical convergence. Namely, we say that the algorithm practically converges if there exists a time instant $k_c \leq T$ such that $(1/N)\sum_i |\vartheta_{i,i-1}(k) - \psi| \leq \delta$ for all $k \geq k_c$[1], and we say that k_c is the convergence time. For our simulations, we set $\delta = 0.05\psi = 0.0524$rad, and we say that $k_c(K_i, \varphi_j, \theta_{\max,m}, s)$ is the convergence time of the s-th repetition of parameter combination corresponding to the i-th values of K, the j-th value of φ, and the m-th value of $\theta_{\max}$, for $i, j, m = 1, \ldots, 5$, and $s = 1, \ldots, R$.

5.2 Statistics and comments on simulation results

Let us now describe the numerical results. Firstly, we underline that only in two simulations the specified tolerance was not fulfilled, and practical convergence is achieved in the 99.9984% of the runs. As for the convergence time k_c, its average

$$\langle k_c \rangle = \frac{1}{125R} \sum_{i,j,m=1}^{5} \sum_{s=1}^{R} k_c(K_i, \varphi_j, \theta_{\max,m}, s),$$

computed on the basis of all 12500 simulations, is 735 time instants. To have a fist insight on the effect of K, φ, and $\theta_{\max}$, we evaluated

$$\langle k_c(K) \rangle = \frac{1}{25R} \sum_{j,m=1}^{5} \sum_{s=1}^{R} k_c(K, \varphi_j, \theta_{\max,m}, s),$$

$$\langle k_c(\varphi) \rangle = \frac{1}{25R} \sum_{i,m=1}^{5} \sum_{s=1}^{R} k_c(K_i, \varphi, \theta_{\max,m}, s),$$

and

$$\langle k_c(\theta_{\max}) \rangle = \frac{1}{25R} \sum_{i,j=1}^{5} \sum_{s=1}^{R} k_c(K_i, \varphi_j, \theta_{\max}, s).$$

As expected, increasing the control gain K the convergence time decreases, see Fig. 5.1, while an increase of the measurement noise φ produces a slight increase in the convergence time, see Fig. 5.2. Finally, increasing $\theta_{\max}$, we experienced a steep increase in $\langle k_c(\theta_{\max}) \rangle$, as depicted in Fig. 5.3.

[1] We remind the reader that the follower of agent i is labeled as $i - 1$. The follower of agent 1 is obviously agent 6.

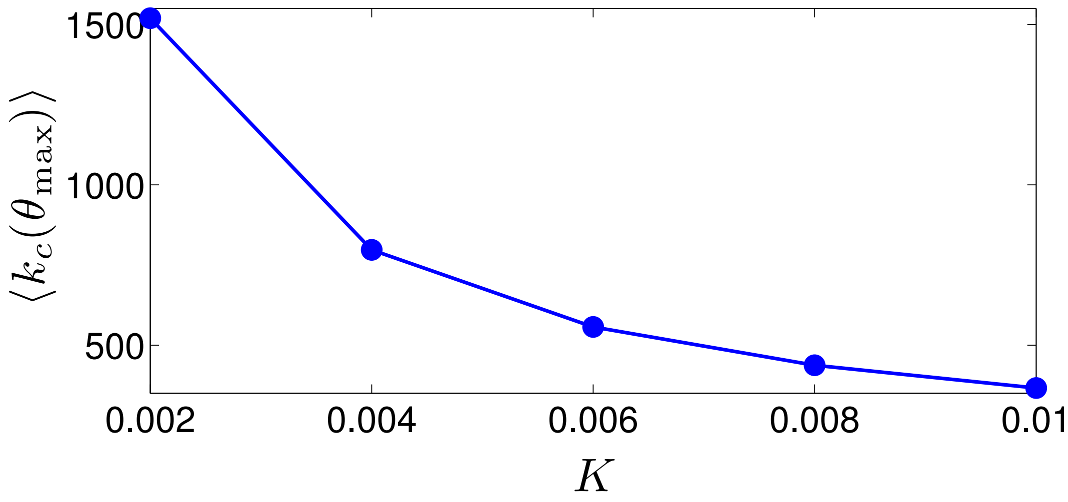

Figure 5.1: Plot of $\langle k_c(K)\rangle$ as a function of K.

To delve into the statistical significance of the observed variations, we performed a three-way ANOVA (analysis of variance), whose results are reported in Table 5.1. Such analysis tests the null hypothesis that each factor has no influence on the convergence time k_c. Hence, a low p-value implies that the null hypothesis must be rejected. The lowest p-value of 0 is obtained for the detecting distance $\theta_{\max}$ and the control gain K, while the null hypothesis may not be rejected for the effect of φ on k_c, as the p-value is 0.62. To further discuss the lack of significance of the variation of the convergence time as a function of φ, in Fig. 5.4 we display a box plot for a representative simulation scenario: as a result, the variability induced by φ, which is represented by the difference between the medians of the distributions (red horizontal lines) is negligible if compared to the natural variability of k_c (the width of the blue boxes). Therefore, we can conclude that the inter-class sampled variance is much smaller than the intra-class sampled variance. This means that the effect of measurement noise φ on k_c is too small to be statistically significant, if compared to the effect of the other parameters, and to the natural variability of k_c.

5.3 Simulation for agents with not negligible inertia

Finally, to further test the effectiveness of our control strategy, we consider the case in which the inertia is not negligible and the approximation of instantaneously switching the angular velocities is not acceptable. To model this scenario, we modify equation

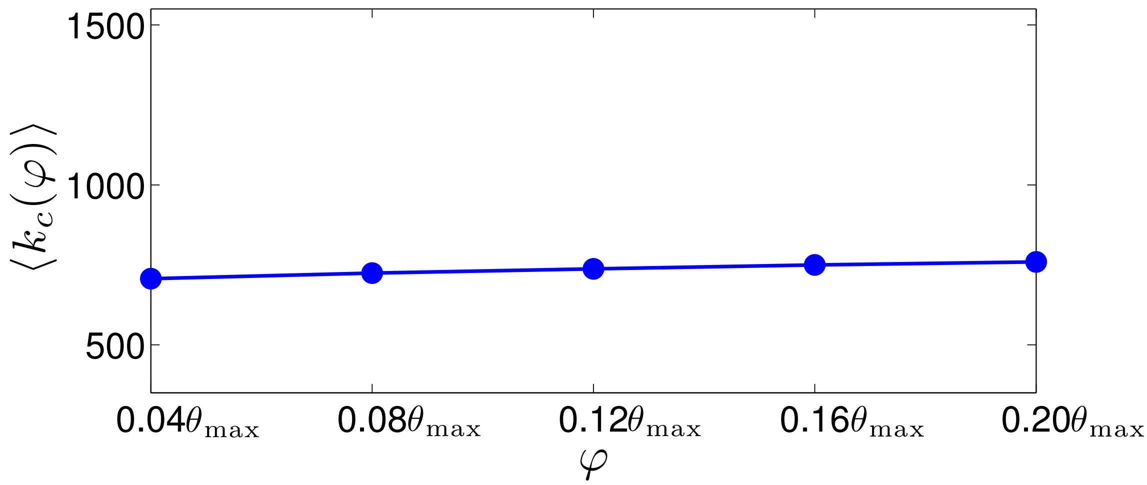

Figure 5.2: Plot of $\langle k_c(\varphi)\rangle$ as a function of φ.

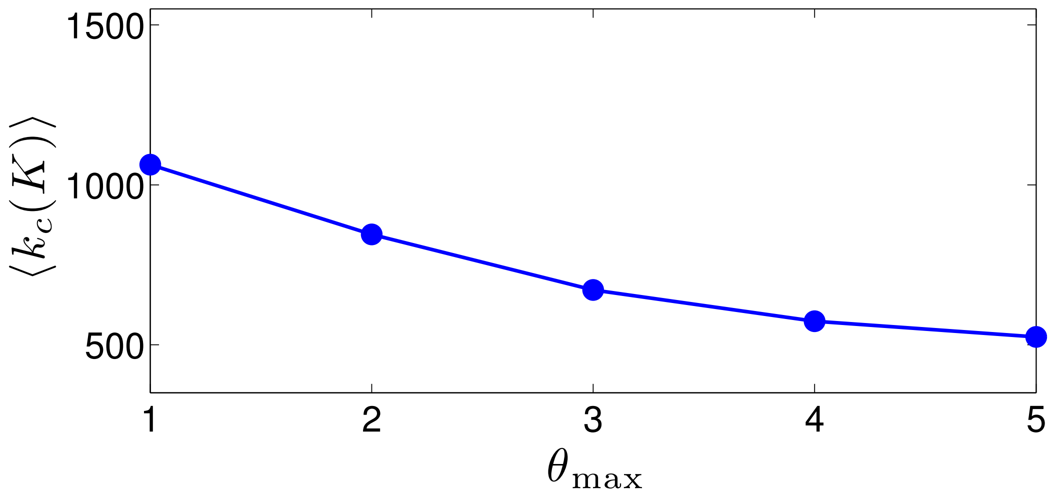

Figure 5.3: Plot of $\langle k_c(\theta_{\max})\rangle$ as a function of $\theta_{\max}$.

Factor	degrees of freedom	p-value
K	4	0
φ	4	0.62
$\theta_{\max}$	4	0

Table 5.1: Three-way ANOVA to test the influence on k_c of the factors K, φ, and $\theta_{\max}$.

(2.3) and obtain the following expression for the dynamics of each agent:

$$\theta_i(k+1) = \theta_i(k) + \omega_i(k) \tag{5.1}$$

where $\omega_i(k)$ is

$$\begin{cases} \omega_0 + \min\{\omega_i(k-1) + \alpha_1, u_i(k)\} & \text{if } u_i(k) \geq \omega_i(k-1) \\ \omega_0 + \max\{\omega_i(k-1) - \alpha_2, u_i(k)\} & \text{if } u_i(k) < \omega_i(k-1), \end{cases} \tag{5.2}$$

and $\omega_i(0) = \omega$. The parameters α_1 and α_2, possibly different, account for the inertias of the multi-agent system. In our preliminary analysis, based on a set of 100 simulations, with the same set of initial conditions considered above, and where we set $\theta_{\max} = 0.5864$ and $K = 0.01$, our approach successfully achieved practical convergence in all the repetitions.

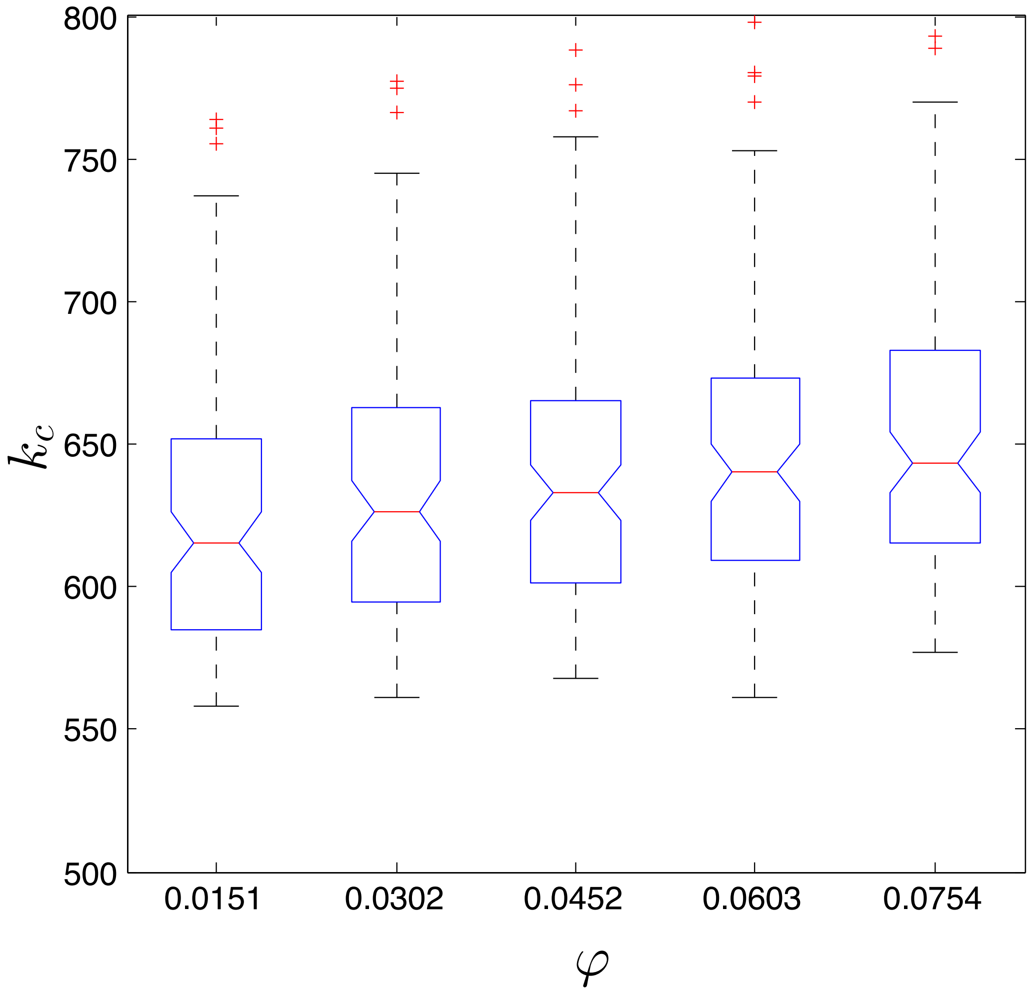

Figure 5.4: Box plot of k_c as a function of φ for $\theta_{\max} = 0.3770$, $K = 0.008$. The central mark are the medians, the edges of the boxes are the 25-th and 75-th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.

Chapter 6

Conclusions

In this thesis, we tackled the problem of coordinating a multi-agent system of oscillators to achieve a balanced circular formation. Differently from the existing literature, we did not rely on the exact knowledge of the relative angular positions between the agents. This is motivated by the fact that, in many fields of application, the accuracy of the measurements is limited due to physical or economical constraints. Hence, we considered intermittent, uncertain and ambiguous measurements as those performed by cheap proximity sensors. Furthermore, we assumed that the sensor have a limited detecting distance, lower than the desired spacing among the agents. To cope with this reduced level of information, we developed a decentralized strategy for estimation and control. Inspired by the algorithm presented in [13], we proposed an estimator capable of reconstructing the relative angular position from the intermittent distance measurements. Implementing an appropriately designed bang-bang control law, each agent is capable of univocally identifying its closest follower and achieves an appropriate spacing from it. Extensive numerical simulations illustrated the effectiveness of the approach: the desired equispaced configuration is achieved and the convergence speed can be regulated with the control gain and is not significantly affected by the measurement noise. Moreover, preliminary results show how the approach can be successfully applied when the inertia is not negligible and the switches prescribed by the bang-bang control law cannot be instantaneous. A formal proof of convergence of the estimation and control strategy is subject of ongoing research. Here we remark that the problem of convergence can't be approached with a separation principle, but that the most promising direction, at the best of our knowledge, seems to be the exploration of contraction properties of the set theoretic estimator, that has proved deep connections with fractals' theory. It is hoped that the work of this thesis will serve as a basis for continuing research in this direction, and that the presented ideas and techniques will augment the set of tools available to scientists and engineers studying interconnected systems and problems of coordinated agents.

Bibliography

[1] J. B. de Sousa A. R. Girard and J. K. Hedrick. An overview of emerging results in networked multi-vehicle systems. In *40th IEEE Conf. Decision and Control*, pages 1485–1490, Orlando, FL, 2001.

[2] T. Balch and R.C. Arkin. Behavior-based formation control for multirobot teams. *IEEE Transactions on Robotics and Automation*, 4(6):1–23, 1997.

[3] Tucker R. Balch and Ronald C. Arkin. Behavior-based formation control for multirobot teams. *IEEE T. Robotics and Automation*, 14(6):926–939, 1998.

[4] T. D. Barfoot, E. J. P. Earon, and G. M. T. DEleuterio. Controlling the masses: Control concepts for multi-agent mobile robotics. In *2nd Canadian Space Exploration Workshop, Calgary*, New York, 1999.

[5] V Braitenberg. *Vehicles: experiments in synthetic psychology.* MIT Press, Cambridge, MA, USA, 1984.

[6] AlfredM. Bruckstein. Why the ant trails look so straight and nice. *The Mathematical Intelligencer*, 15(2):59–62, 1993.

[7] Y.U. Cao, A.S. Fukunaga, A.B. Kahng, and F. Meng. Cooperative mobile robotics: antecedents and directions. In *Intelligent Robots and Systems 95. 'Human Robot Interaction and Cooperative Robots', Proceedings. 1995 IEEE/RSJ International Conference on*, volume 1, pages 226–234 vol.1, Aug 1995.

[8] Ruggero Carli, Fabio Fagnani, Paolo Frasca, Tom Taylor, and Ro Zampieri. Average consensus on networks with transmission noise or quantization. In *Proceedings of European Control Conference*, 2007.

[9] Z. Chen and H.-T. Zhang. No-beacon collective circular motion of jointly connected multi-agents. *Automatica*, 47(9):1929–1937, 2011.

[10] I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks. Collective memory and spatial sorting in animal groups. *Journal of Theoretical Biology*, 218(1):1–11, 2002.

[11] C. Canudas de Wit and A. D. NDoudi-Likoho. Nonlinear control for a convoy-like vehicle. *Automatica*, 36:457–462, 2000.

[12] P. DeLellis, M. di Bernardo, F. Garofalo, and D. Liuzza. Analysis and stability of consensus in networked control systems. *Applied Mathematics and Computation*, 217(3):988–1000, 2010.

[13] P. DeLellis, F. Garofalo, F. Lo Iudice, and G. Mancini. State estimation of heterogeneus oscillators by mean of proximity measurements. submitted to Automatica, 2013.

[14] P. DeLellis, M. Porfiri, and E. M. Bollt. Topological analysis of group fragmentation in multi-agent systems. *Phys. Rev. E*, 87(2):022818, 2013.

[15] V. Gazi and K.M. Passino. Stability analysis of swarms. *Automatic Control, IEEE Transactions on*, 48(4):692–697, April 2003.

[16] A. Hajimiri. A general theory of phase noise in electrical oscillators. *IEEE J. Solid-St. Circ.*, 33(2):179–194, 2002.

[17] S. Hara and H. Tanaka. D-stability and robust stability conditions for lti systems with generalized frequency variables. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 5738–5743, Dec 2010.

[18] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*, chapter XXIII. Kronecker's Theorem, pages 501–522. Oxford University Press, New York, 2008.

[19] A. Jadbabaie, Jie Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Automatic Control, IEEE Transactions on*, 48(6):988–1001, June 2003.

[20] L. Jaulin, M. Kieffer, I. Braems, and E. Walter. Guaranteed non-linear estimation using constraint propagation on sets. *Int. J. Control*, 74(18):1772–1782, 2010.

[21] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis*. Springer, 2001.

[22] E. W. Justh and P. S. Krishnaprasad. Steering laws and continuum models for planar formations. In *in IEEE Conf. on Decision and Control, (Maui, Hawaii*, pages 3609–3615, 2003.

[23] E. W. Justh, P. S. Krishnaprasad, E. W. Justh, and P. S. Krishnaprasad. A simple control law for uav formation flying. Technical report.

[24] M. Kieffer, L. Jaulin, and E. Walter. Guaranteed recursive nonlinear state estimation using interval analysis. In *Decision and Control, 1998. Proceedings of the 37th IEEE Conference on*, volume 4, pages 3966–3971 vol.4, Dec 1998.

[25] F. Le Bars, J. Sliwka, L. Jaulin, and O. Reynet. Set-membership state estimation with fleeting data. *Automatica*, 48(2):381–387, 2012.

[26] J. A. Marshall, M. E. Broucke, and B. A. Francis. Formations of vehicles in cyclic pursuit. *IEEE Transactions on Automatic Control*, 49(11):1963–1974, 2004.

[27] S. Martinez, J. Cortes, and F. Bullo. Motion coordination with distributed information. *Control Systems, IEEE*, 27(4):75–88, 2007.

[28] Maja J. Mataric. Issues and approaches in the design of collective autonomous agents. *Robotics and Autonomous Systems*, 16:321–331, 1995.

[29] R. E. Mirollo and S. H. Strogatz. Synchronization of pulse-coupled biological oscillators. *SIAM Journal on Applied Mathematics*, 50(6):1645–1662, 1990.

[30] Ulrich Mnz, Antonis Papachristodoulou, and Frank Allgwer. Delay robustness in consensus problems. *Automatica*, 46(8):1252 – 1265, 2010.

[31] R. E. Moore, R. B. Kearfott, and M. J. Cloud. *Introduction to Interval Analysis*, chapter 2. The Interval Number System. Society for Industrial and Applied Mathematics, 2009.

[32] G. Nassreddine, F. Abdallah, and T. Denoeux. State estimation using interval analysis and belief-function theory: Application to dynamic vehicle localization. *IEEE Trans. Sys. Man Cyb. B*, 40(5):1205–1218, 2009.

[33] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, Jan 2007.

[34] D. A. Paley, N. E. Leonard, R. Sepulchre, and D. Grünbaum. Oscillator models and collective motion. *IEEE Cont. Syst. Mag.*, 27(4):89–105, 2007.

[35] A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization: A Unified Approach to Nonlinear Science*. Cambridge University Press, Cambridge, 2001.

[36] T. Raissi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEE Transactions on Automatic Control*, 57(1):260–265, 2012.

[37] W. Ren, R. W. Beards, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. In *Proceedings of the 2005 American Control Conference*, volume 3, pages 1859–1864, 2005.

[38] Wei Ren, R.W. Beard, and D.B. Kingston. Multi-agent kalman consensus with relative uncertainty. In *American Control Conference, 2005. Proceedings of the 2005*, pages 1865–1870 vol. 3, June 2005.

[39] C. W. Reynolds. Flocks, herds, and schools: a distributed behavioral model. In *SIGGRAPH '87 Proceedings of the 14th annual conference on Computer graphics and interactive techniques*, volume 21, pages 25–34, New York, 1987.

[40] R.O. Saber and R.M. Murray. Flocking with obstacle avoidance: cooperation with limited communication in mobile networks. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, volume 2, pages 2022–2028 Vol.2, Dec 2003.

[41] T. Sanpechuda and L. Kovavisaruch. A review of RFID localization: Applications and techniques. In *5th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Infomation Technology*, pages 769–772, 2008.

[42] R. Sepulchre, D. A. Paley, and N. E. Leonard. Stabilization of planar collective motion with limited communication. *IEEE Trans. Automatic Control*, 53(3):706–719, 2008.

[43] G. Shan, Y. Baoming, and Ye Qiang. Interval analysis and its applications to control problems. In *Int. Conf. on Educ. and Inform. Tech.*, volume 1, pages 213–218, Chongqing, 2010.

[44] Guodong Shi and Karl Henrik Johansson. Multi-agent robust consensus-part i: Convergence analysis. In *CDC-ECE*, pages 5744–5749. IEEE, 2011.

[45] S. H. Strogatz. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143(1-4):1–20, 2000.

[46] K. Sugihara and I. Suzuki. Distributed motion coordination of multiple mobile robots. In *Intelligent Control, 1990. Proceedings., 5th IEEE International Symposium on*, pages 138–143 vol.1, Sep 1990.

[47] Ichiro Suzuki and Masafumi Yamashita. Distributed anonymous mobile robots: Formation of geometric patterns. *SIAM Journal on Computing*, 28:1347–1363, 1999.

[48] Yu-Ping Tian and Cheng-Lin Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 45(5):1347 – 1353, 2009.

[49] T. Vicsek. Universal patterns of collective motion from minimal models of flocking. In *Second IEEE International Conference on Self-Adaptive and Self-Organizing Systems, SASO '08.*, pages 3–11, 2008.

[50] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.*, 75(6):1226–1229, 1995.

[51] P. K C Wang. Navigation strategies for multiple autonomous mobile robots moving in formation. In *Intelligent Robots and Systems '89. The Autonomous Mobile Robots and Its Applications. IROS '89. Proceedings., IEEE/RSJ International Workshop on*, pages 486–493, Sep 1989.

[52] R. Want. An introduction to RFID technology. *IEEE Pervas. Comput.*, 5(1):25–33, 2006.

[53] Kevin Warburton and John Lazarus. Tendency-distance models of social cohesion in animal groups. *Journal of Theoretical Biology*, 150(4):473–488, 1991.

[54] Sun Yong-Zheng and Ruan Jiong. Consensus problems of multi-agent systems with noise perturbation. *Chinese Physics B*, 17(11):4137–05, 2008.

[55] Keyou You and Lihua Xie. Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *Automatic Control, IEEE Transactions on*, 56(10):2262–2275, Oct 2011.