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INFINITE GROUPS WITH RESTRICTIONS ON PROPER SUBGROUPS

Tesi di Dottorato

di

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Contents

Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of groups. A group G is said to be an $\mathcal{X}$-by-$\mathcal{Y}$ group if there exists a normal subgroup N which is an $\mathcal{X}$-group such that G/N belongs to the class $\mathcal{Y}$. Furthermore, a group G is said to be a minimal non-$\mathcal{X}$ group if G is not an $\mathcal{X}$-group but all its proper subgroups belong to $\mathcal{X}$. Minimal non-$\mathcal{X}$ groups have been studied for various choices of $\mathcal{X}$, and the results obtained show that the behaviour of subgroups of a given group has a strong influence on the structure of the whole group.

A group G is called a T-group if normality in G is a transitive relation, or equivalently if all subnormal subgroups of G are normal. The structure of soluble T-groups was described by W. Gaschütz and D. J. S. Robinson. A further contribution to this kind of investigation was given by C. Casolo who studied soluble groups with finite conjugacy classes of subnormal subgroups (the so-called V-groups). The second chapter of this thesis deals with generalized soluble groups whose proper subgroups belong to the class V.

A group G is said to be a CF-group if all its subgroups are normal-by-finite, i.e. if the index $|X : X_G|$ is finite for every subgroup X of G. This class was considered by Buckley, Lennox, Neumann, Smith, and Wiegold [16],[29]. They proved that if G is a CF-group such that every periodic image of G is locally finite, then G is abelian-by-finite. In chapter 3 we obtain an extension of this result to the class of groups whose proper subgroups have the

CF-property.

In recent years, the subject of the research has moved on groups of infinite rank and, in particular, on groups whose proper subgroups of infinite rank have some suitable property. Recall that a group G is said to have finite rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. If there is no such r, the group G has infinite rank. A classical result of A.I. Mal'cev [19] states that a locally nilpotent group of infinite rank contains an abelian subgroup of infinite rank, and V. P. Šunkov [30] shows that a similar result holds for locally finite groups. On the other hand, Y. I. Merzljakov [20] has shown that there exist locally soluble groups of infinite rank in which every abelian subgroup has finite rank. These results suggest that also the behaviour of subgroups of infinite rank in a generalized soluble group can influence the whole group. Many authors have studied groups in which all subgroups of infinite rank have a given property, and in Chapter 4 we give a further contribution to this topic investigating the structure of groups in which every proper subgroup of infinite rank contains an abelian (or a nilpotent) subgroup of finite index.

Observe that B. Bruno and R. E. Phillips [3], [5], proved that, within the universe of imperfect groups, any minimal non-(abelian-by-finite) group is periodic and this result is still true if we consider minimal non-(nilpotent-by-finite) groups.

Most of our notation is standard, and we refer to [26].

Chapter 1

Preliminary results

1.1 Groups with finite conjugacy classes of subnormal subgroups

A group G is said to be a T-group if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of infinite soluble T-groups has been described by D.J.S. Robinson [24]. Obviously, this class is closed by homomorphic images and by subnormal subgroups. However the class of T-groups is not subgroup closed. For example, the alternating group of degree 5 A_5 is a T-group, but it has a subgroup isomorphic to A_4 which does not have the T-property.

Since a subgroup is normal if and only if its conjugacy class is trivial, one of the most natural generalization of normality, in the case of infinite group, is the requirement that the subgroup has only finitely many conjugates. This approach was suggested by a famous theorem of B.H. Neumann [22], who proved that a group G has finite conjugacy classes of subgroups if and only if its center $Z(G)$ has finite index.

Let us introduce two classes of groups that can be viewed as generalizations of the class of T-groups.

Following C. Casolo [7] we will say that a group G is a V-group if each subnormal subgroup of G has only finitely many conjugates, i.e. if the normalizer of any subnormal subgroup of G has finite index. Of course, all finite groups and all groups satisfying the minimal condition on subnormal subgroups have the V-property. By result of Roseblade and Robinson (see, for instance, [26] Theorem 5.49), if G is a group satisfying the minimal condition on subnormal subgroups, then the Wielandt subgroup $\omega(G)$, which is the intersection of the normalizers of all subnormal subgroups, has finite index. Hence, in this case, it's clear that $|G : N_G(H)|$ is finite, for every subnormal subgroup H of G.

A group G is said to be a T^*-group if every subnormal subgroup of G has finite index in its normal closure, i.e. if $|H^G : H|$ is finite for every subnormal subgroup H of G. Obviously, subnormal subgroups and homomorphic images of a group belonging to any of the classes under consideration, belong to the same class.

Let Γ be a group of operators on the group G, and $H \leq G$; we write

$$N_\Gamma(H) = \{a \in \Gamma | H^a = H\} \leq H;$$

moreover, we put

$$H^\Gamma = \langle H^a | a \in \Gamma \rangle \text{ and } H_\Gamma = \bigcap_{a\in\Gamma} H^a.$$

In particular if $\Gamma = G$ in its action by conjugation then H^G and H_G are, respectively, the normal closure and the normal core of H in G. Finally, we define

$$Paut_\Gamma(G) = \{a \in \Gamma | H^a = H, \forall H \leq G\}$$

and, obviously,

$$Paut(G) = \{a \in Aut(G) | H^a = H, \forall H \leq G\}$$

is the group of power automorphism of G.
Now we observe some results on the action of those particular types of automorphism groups of abelian groups, which are relevant in our discussion.

Lemma 1.1.1. *Let A be a periodic abelian group, $\Gamma \leq Aut(A)$ such that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leq A$. Then $|\Gamma : Paut_\Gamma(A)|$ is finite.*

Lemma 1.1.2. *Let A be a torsion free abelian group, $\Gamma \leq Aut(A)$ such that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leq A$. Then Γ is finite.*

We recall that a Baer group is a group in which all finitely generated subgroups are subnormal, while the FC-center $FC(G)$ of a group G is the subgroup consisting of all elements of G with finitely many conjugates. A group G is called FC-group if it coincides with its FC-center, or equivalently if the index $|G : C_G(x)|$ is finite for each element x of G. Finite groups and abelian groups are obvious examples of groups with FC-property. If x is any element of a group G, we have $x^g = x[x, g]$ for each element g of G, and hence the conjugacy class of x is contained in the coset xG'. Thus groups with finite commutator subgroup are FC-groups. Moreover, it is also clear that all central-by-finite groups belong to the class of FC-groups. The theory of FC-groups had a strong development in the second half of last century. A special mention is due here to R. Baer, Y.M. Gorcakov, P. Hall, L.A. Kurdachenko, B.H. Neumann, M.J. Tomkinson for the particular relevance of their works. In this section we want to recall some classical results about FC-groups.

Lemma 1.1.3. *If G is an FC-group, then $G/Z(G)$ is a residually finite torsion group.*

The following lemma is crucial in the study of FC-property. It is known as Dietzmann's Lemma.

Lemma 1.1.4. *(Dietzmann's Lemma) In any group G a finite normal subset consisting of elements of finite order generates a finite normal subgroup.*

This allows us to describe FC-torsion groups in a different manner.

Theorem 1.1.5. *A torsion group G is an FC-group if and only if each finite subset is contained in a finite normal subgroup.*

Now, we are able to show some results on V-groups.

Lemma 1.1.6. *Every Baer group belonging either to the class of V-groups or to the class of T^*-groups is nilpotent.*

Proof. We first show that a Baer V-group is a T^*-group. Let G be a Baer V-group, and let $x \in G$. The subgroup $\langle x \rangle$ is subnormal in G, hence $|G : N_G(x)|$ is finite, and so the centralizer $C_G(x)$ has finite index in G. Therefore, G is an FC-group. Let H be a subnormal subgroup of G and let $H = H_1, H_2, \dots, H_n$ be the conjugates of H in G, with $H_i = H^{x_i}$, $x_i \in G$ for $i = 1, 2, \dots, n$. If h is an element of $C_H(x_i)$, then $h = h^{x_i} \in H \cap H_i$, hence $C_H(x_i) \leq H \cap H_i$, for any $i = 1, 2, \dots, n$. Consequently $|H : H \cap H_i| \leq |H : C_H(x_i)|$ is finite. If $R = H_G$, then H/R is finite and , as G/R is an FC-group, this implies that H^G/R is finite and, in particular, $|H^G : H| < \infty$. This holds for each subnormal subgroup of G and so G is a T^*-group.

Now we want to prove that a Baer T^*-group is nilpotent. Let G be such a group. Then G is an FC-group and so $G/Z(G)$ is a residually finite torsion group. Without loss of generality we may assume that G is a residually finite torsion group.

Assume, by contradiction, that G is not nilpotent. Since G is a Baer group, it is also locally nilpotent, so G does not have nilpotent subgroups of finite index. Now, we construct, by induction on i, a group R which is direct product of non-Dedekind, finite normal subgroups of G. G is not nilpotent, hence there exists a finitely generated subgroup A_1 of G, such that A_1 is a non-Dedekind group. A_1 is also finite because G is periodic. We put $R_1 = A_1^G$; then R_1 is finite, since both A_1 and $|R_1 : A_1|$ are finite. Assume now that we have already constructed non-Dedekind finite normal subgroups $R_1, \ldots, R_{i-1}$ such that $B_{i-1} = \langle R_1, \ldots, R_{i-1}\rangle \simeq R_1 \times \ldots \times R_{i-1}$. B_{i-1} is a finite subgroup of the residually finite group G, so there exists a normal subgroup N of finite index in G, such that $N \cap B_{i-1} = \{1\}$. By what we observed above, N is not nilpotent and so it has a finite subgroup A_i that is non-Dedekind. Put $R_i = A_i^G$, then R_i is finite and $R_i \cap B_{i-1} \leq N \cap B_{i-1} = \{1\}$, whence $B_i = \langle R_1, \ldots, R_i\rangle \simeq R_1 \times \ldots \times R_i$.

By construction, for every $i \in \mathbb{N}$, there exists a subgroup H_i of R_i which is subnormal of defect exactly 2 in R_i. Put

$$R = \langle R_i : i \in \mathbb{N}\rangle \simeq \bigotimes_{i\in\mathbb{N}} R_i$$

and

$$H = \langle H_i : i \in \mathbb{N}\rangle.$$

Then H is subnormal of defect 2 in R, and, in particular, it is subnormal in G. Thus $|H^G : H|$ is finite, because G is a T^*-group. But $H^G \leq R$, and so there exists an $n \in \mathbb{N}$ such that $H^G \leq HB_n$, where $B_n = \langle R_i, \ldots, R_n\rangle$.

Then, if $j > n$ and π_j is the canonical projection of R on R_i, we get $\pi_j(H^G) \leq \pi_j(HB_n) = \pi_j(H) = H_j$. Whence $\pi_j(H^G) = H_j$ and so $H_j \lhd R_j$ contradicting our choice of H_j. This contradiction implies that G is nilpotent.

□

We recall that a Fitting subgroup of a group G is the subgroup generated by all nilpotent normal subgroups of G.

Casolo proved also this important result on soluble V-groups.

Theorem 1.1.7. *Let G be a soluble V-group. Then there exists a normal subgroup N, of finite index in G, such that N' is periodic and every subgroup of N' is normal in N.*

Proof. Let G be a soluble V-group and let F be the Fitting subgroup of G. Then by Lemma 1.1.6 F is nilpotent. Let $A = Z(F)$. By result of B.H. Neumann quoted at the beginning of this chapter, A has finite index in F, thus $|G : C_G(F/A)|$ is finite. Let T be the torsion subgroup of A. By Lemma 1.1.1 and Lemma 1.1.2, respectively, $|G : Paut_G(T)|$ and $|G : C_G(A/T)|$ are finite. Put

$$L = C_G(F/A) \cap C_G(A/T) \cap Paut_G(T);$$

then L is normal in G and has finite index. Since the group of power automorphisms of any group is abelian, we have

$$L' \leq C_G(F/A) \cap C_G(A/T) \cap C_G(T).$$

Hence

$$[F, L', L', L'] \leq [A, L', L'] \leq [T, L'] \leq [T, C_G(T)] = 1.$$

Then, since $C_{L'}(F) \leq F$,

$$[C_{L'}(F), L', L', L'] = \{1\}$$

so that $C_{L'}(F) \leq Z_3(L')$. Moreover, $L'/C_{L'}(F)$ is nilpotent and this implies $L' \leq F$. Now

$$\gamma_4(L) = [L', L, L] \leq [F, L, L] \leq [A, L] \leq T,$$

thus L/T is a nilpotent V-group. If $N/T = Z(L/T)$, then $|L : N|$ is finite, which implies that $|G : N|$ is finite and moreover N is normal in G. Now $N' \leq T$, so N' is a periodic abelian group; futhermore $N \leq L \leq Paut_G(T)$, hence every subgroup of N' is normal in N.

□

Corollary 1.1.8. *Every soluble V-group is metabelian-by-finite.*

Corollary 1.1.9. *A finitely generated soluble V-group is abelian-by-finite.*

Proof. Let G be a finitely generated soluble V-group, and let N be a normal subgroup of finite index in G, such that N' is periodic and every subgroup of N' is normal in N. Then N is also finitely generated. Put $N = \langle x_1, \dots, x_n \rangle$. If $d_{ij} = [x_i, x_j]$, for $i, j = 1, \dots, n$, then $\langle d_{ij} \rangle$ is normal in N. N' is an abelian periodic group generated by the set of d_{ij} elements, so it is finite. Since N is finitely generated, this implies that $Z(N)$ has finite index in N, and so the corollary is true. □

1.2 Groups whose proper subgroups are abelian-by-finite

Let $\mathcal{X}$ be a class of groups. A group G is said to be a minimal non-$\mathcal{X}$ group if G is not an $\mathcal{X}$-group but all its proper subgroups belong to $\mathcal{X}$. Many results have been obtained on minimal non-$\mathcal{X}$ groups, for various choices

of $\mathcal{X}$. In particular, in 1984, B. Bruno [4] studied minimal non-(abelian-by-finite) groups. Clearly, Tarski groups, i.e. infinite simple groups whose proper non-trivial subgroups have prime order, are minimal non-abelian, and this example suggests that some further restriction is necessary in order to investigate the behavior of minimal non-(abelian-by-finite) groups. A group G is called locally graded if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Locally graded groups form a wide class, containing in particular all locally (soluble-by-finite) groups, and the assumption for a group to be locally graded is sufficient to avoid Tarski groups and other similar pathologies. In fact, it is easy to show that any locally (soluble-by-finite) group whose proper subgroups are abelian is either abelian or finite.
We record the following obvious result, which will be frequently used.

Lemma 1.2.1. *If G is a minimal non-(abelian-by-finite) group, G does not have proper subgroups of finite index, hence $FC(G) = Z(G)$ and, if N/K is a finite normal subgroup of G/K, then N/K is a central in G/K.*

Lemma 1.2.2. *Let G be a minimal non-(abelian-by-finite) group and N a proper normal subgroup of G. If $F = FC(N)$, then $F \lhd G$ and N/F is finite. Moreover F' is finite and central in G and $[G, N] \leq F$. Thus N is a soluble group of derived lenght at most three and F is nilpotent of nilpotency class at most two.*

Proof. If $F = FC(N)$, clearly $F \lhd G$, and since N is abelian-by-finite we have that F has finite index in N. Moreover F is central-by-finite because it's an abelian-by-finite FC-group, and F' is finite (see [26], part I, pag. 102) and normal in G. By Lemma 1.2.1 $F' \leq Z(G)$ and N/F is central in G/F. Thus

$[G, N] \leq F$ and $N'' \leq N' \leq Z(G)$. Hence $N''' = \{1\}$ and $\gamma_3(F) = \{1\}$. □

A group G is called 3-decomposable if there exist three normal subgroups A_1, A_2, A_3, such that $G = A_1A_2A_3$, and $A_iA_j \lneq G$, for $i, j = 1, 2, 3$.

B. Bruno also proved that such groups are also abelian-by-finite provided that all proper subgroups are abelian-by-finite.

Lemma 1.2.3. *Let G be a group such that all proper subgroups of G are abelian-by-finite. If G is a 3-decomposable group, then G is abelian-by-finite.*

Proof. Suppose, by contradiction, that G is not abelian-by-finite. Since G is 3-decomposable, there are three normal subgroups A_1, A_2, A_3, such that $G = A_1A_2A_3$. Since A_iA_j is a proper normal subgroup of G, the FC-center $FC(A_iA_j) = F_{ij}$ has finite index in A_iA_j, for every $i, j \in \{1, 2, 3\}$, with $i \neq j$. Since $|G : F_{ij}F_{kj}|$ is finite and G has no proper subgroups of finite index, it follows that $G = F_{ij}F_{jk}$. Let $x \in F_{ij} \cap F_{jk}$, then the set of conjugates of x is finite. Thus $F_{ij} \cap F_{jk} \leq FC(G) = Z(G)$. Since $F_{ij} \cap F_{jk} \cap A_j$ has finite index in A_j, then $Z(G) \cap A_j$ has finite index in A_j, for every $j \in \{1, 2, 3\}$. Hence $G/Z(G)$ is a product of finite normal subgroups and so it is finite. This contradiction proves the lemma. □

Using this result we can prove that a minimal non-(abelian-by-finite) group cannot be the product of two proper normal subgroups.

Lemma 1.2.4. *Let G be a group such that all proper subgroups of G are abelian-by-finite. If G is the product of two proper normal subgroups, then G is abelian-by-finite.*

Proof. Let $G = N_1N_2$, with $N_1, N_2 \lhd G$ and, for $i = 1, 2$, put $F_i = FC(N_i)$. Assume, by contradiction, that G is not abelian-by-finite. Then the product F_1F_2 has finite index in G, so by Lemma 1.2.1 we have $G = F_1F_2$. Now Lemma 1.2.2 implies that F_1 and F_2 are nilpotent normal subgroups of G and so G is nilpotent. The quotient G/G' is divisible and abelian and, since G is non-abelian, it follows that G/G' is not periodic. This implies that G is 3-decomposable and for the result mentioned above we have the conclusion. □

To fully understand the theory of groups with abelian-by-finite proper subgroups we need to enunciate the following theorem, due to B. Bruno and R. E. Phillips [5], which state, essentially, that a non perfect group, in which all proper subgroups are abelian-by-finite is either abelian-by-finite or periodic.

Theorem 1.2.5. *A non-perfect locally graded minimal non-(abelian-by-finite) group is periodic.*

Moreover, B. Bruno and R. E. Phillips, in the same paper, proved the following lemma which is useful for our purpose (see [5], Lemma 2.3).
Recall that, for any group G, the symbol $\pi(G)$ denotes the set of all prime numbers p such that G has elements of order p.

Lemma 1.2.6. *Let G be a group, and let A be a torsion-free abelian non trivial normal subgroup of G such that G/A is periodic. Then for any finite set p of prime numbers, there exists a G-invariant subgroup N of A such that A/N is periodic and π is contained in $\pi(A/N)$.*

After this discussion, it's clear that if we want to study locally graded minimal non-(abelian-by-finite) groups, we have to consider periodic groups with such properties.

First of all, we want record the following proposition, based essentially on a result of G. Shute [28] which state that "if G is an infinite locally finite simple group, which all proper subgroups soluble-by-finite, then either $G \simeq PSL(2, F)$ or $G \simeq S_Z(F)$, where F is an infinite locally finite field with no infinite proper subfields".

Proposition 1.2.7. *If G is a locally graded periodic minimal non-(abelian-by-finite) group, then G is not simple.*

Using this result we can now prove the following proposition:

Proposition 1.2.8. *If G is a locally graded periodic minimal non-(abelian-by-finite) group, then the derived subgroup G' is a proper subgroup of G.*

Proof. Let F be a subgroup maximal with respect to $F \lhd G$ and $F''' = \{1\}$. Suppose that F is a proper subgroup of G. Since G has no proper subgroups of finite index, G/F is infinite and also G/F is a locally graded periodic minimal non-(abelian-by-finite) group. From 1.2.7, there is a normal subgroup N of G with $F \lneq N \lneq G$. By 1.2.2, $N''' = \{1\}$ and this contradicts the maximality of F. Thus we have $F = G$ and $G''' = \{1\}$. □

Now we give some further results about locally graded periodic minimal non-(abelian-by-finite) groups.

Lemma 1.2.9. *If G is a locally graded minimal non-(abelian-by-finite) group and $G' < G$. Then G' has no proper G-invariant subgroups of finite index and G' is abelian.*

Proof. Let N be a proper subgroup of G' such that $N \lhd G$ and $|G' : N|$ is finite. Since $(G/N)' = G'/N$ is finite we have that G/N is an FC-group. By

Lemma 1.2.1 G/N must be abelian and we have a contradicition.
Now, if $F = FC(G')$, by Lemma 1.2.2 and by the first part of this lemma, we have $G' = F$. Thus G' is an abelian-by-finite FC-group and this implies that G' is central-by-finite, and again the first part of this lemma gives G' abelian. □

Observe, obviuosly, that the previous Lemma holds also for the class of locally graded periodic minimal non-(abelian-by-finite) groups.

Proposition 1.2.10. *If G is a locally graded periodic minimal non-(abelian-by-finite) group, then $G/G' \simeq Z(p^\infty)$, for some prime p, and , if G is not a p-group, then G' is a p'-group, and there is a subgroup $A \simeq Z(p^\infty)$ such that $G = AG'$*

Proof. The derived group G' is a proper subgroup by Proposition 1.2.8 and the quotient G/G' is divisible for Lemma 1.2.1. Moreover Lemma 1.2.4 implies that $G/G' \simeq Z(p^\infty)$, for some prime p, whence there is a maximal p-subgroup P of G, such that $G = PG'$ (see [18] 1.D.4). Now, since G' is abelian by Lemma 1.2.9, we have $G = PO_{p'}(G')$, where $O_{p'}(G')$ is the p'-component of G'.
Obviously, if G is not a p-group, we have that P is a proper subgroup of G and again, using the fact that G has no proper subgroups of finite index, we have $G = AOp'(G')$, where A is an abelian p-subgroup of G. Thus $G' = [AO_{p'}(G'), AO_{p'}(G')] \leq O_{p'}(G')$ and so $G' = O_{p'}(G')$. Finally $A \simeq G/O_{p'}(G') = G/G' \simeq Z(p^\infty)$. □

Lemma 1.2.11. *If G is a locally graded periodic minimal non-(abelian-by-finite) group and there exists a subgroup $A \simeq Z(p^\infty)$ such that $G = AG'$. Then G' cannot be decomposed into the product of two proper normal subgroups of G.*

Proof. Let $G' = N_1N_2$, with $N_1, N_2 \lhd G$ and $N_1, N_2 < G'$. Then N_iA is a proper subgroup of G, for $i = 1, 2$. Since N_iA is abelian-by-finite and $A \simeq Z(p^\infty)$, we have $A \leq FC(N_iA)$, for $i = 1, 2$, and this implies that every element of A has just a finite number of conjugates in $G = AN_1N_2$. Thus $A \leq FC(G) = Z(G)$ and so G is abelian. This contradiction completes the proof. □

As a consequence of previous lemma we have:

Lemma 1.2.12. *If G is a locally graded periodic minimal non-(abelian-by-finite) group and there exists a subgroup $A \simeq Z(p^\infty)$ such that $G = AG'$, then G' is an abelian q-group, for some prime q.*

The following lemma is a general result about automorphisms of torsion abelian groups.

Lemma 1.2.13. *[17] Let A be a torsion abelian group and let α be an automorphism of A which has prime order $q \in \pi(A)$. Then $A = C_A(\alpha) \times [A, \alpha]$.*

Proposition 1.2.14. *Let G be a locally graded periodic minimal non-(abelian-by-finite) group and suppose that G is not a p-group, for some prime p. Then, for all element $a \in A$, either $a \in Z(G)$ or $C_{G'}(a) = \{1\}$. In particular $C_{G'}(A) = \{1\}$.*

Proof. Since G is not a p-group, by Lemma 1.2.13 we have

$$G' = C_{G'}(a) \times [\langle a \rangle, G'].$$

Moreover $C_{G'}(a)$ and $[\langle a \rangle, G']$ are normal subgroups of G, thus, by Lemma 1.2.11, either $G' = C_{G'}(a)$ and $a \in Z(G)$, or $G' = [\langle a \rangle, G']$ and $C_{G'}(a) = \{1\}$. Finally, since G' is not central in G, there exists an element a of A such that

$$C_{G'}(A) \leq C_{G'}(a) = \{1\}.$$

□

Lemma 1.2.15. *Let G be a locally graded periodic minimal non-(abelian-by-finite) group and suppose that G is not a p-group, for some prime p. Then G' is a minimal normal subgroup of G and A is a maximal subgroup of G. In particular G' is an elementary abelian q-group, with $q \neq p$.*

Proof. Let $N \lhd G$ and $N < G'$. Since AN is a proper subgroup of G we have that AN is abelian-by-finite. If B is an abelian subgroup of finite index in AN, $A \leq B$ and

$$B \cap N \leq C_N(A) \leq C_{G'}(A) = \{1\};$$

so N is finite.
Now Lemma 1.2.1 gives

$$N \leq Z(G) \cap G' \leq C_{G'}(A) = \{1\};$$

thus G' is a minimal normal subgroup of G and A is maximal in G, since $G = A \ltimes G'$. □

Now we are able to prove the following theorem:

Theorem 1.2.16. *Let G be a locally graded periodic group which is not an r-group, for any prime r. Then G has all of its proper subgroups abelian-by-finite if and only if either*

1. *G is abelian-by-finite,*

   or

2. *$G = A \ltimes G'$, where*

   - *$A \simeq Z(p^\infty)$, for some prime p;*

- *G' is an elementary abelian minimal normal q-subgroup of G, for some prime $q \neq p$.*

Proof. To prove the necessary condition we may suppose that G is not abelian-by-finite and then use Proposition 1.2.10 and Lemma 1.2.15.
For the sufficient condition, we will prove that if $H < G$, then H is abelian-by-finite. For this suppose H non-abelian. If $HG'/G' < G/G'$, we have $H \cap G'$ is an abelian subgroup of finite index in H, and then H is abelian-by-finite. Hence, let $HG' = G$, then $H \cap G' \lhd G$ and since G' is minimal normal subgroup of G, either $G' < H$ and then $H = G$, or $H \cap G' = \{1\}$ and so H is abelian. The theorem is proved. □

In this second part we analyze locally graded p-groups with abelian-by-finite proper subgroups.

Theorem 1.2.17. *A locally graded p-group G has all its proper subgroups abelian-by-finite, if and only if, either G is abelian-by-finite, or G' is abelian, the quotient $G/G' \simeq Z(p^\infty)$, and for all proper subgroups H of G, $HG' < G$.*

Proof. The necessary condition has already been proved in Lemma 1.2.9 and in Proposition 1.2.10. The sufficient condition is quite obvious. Note that for every $H < G$, HG' is a proper subgroup of G and HG'/G' is finite. Moreover $H \cap G'$ is an abelian subgroup of finite index in H. Thus H is abelian-by-finite and the theorem is proved. □

Proposition 1.2.18. *If G is a non-abelian periodic group, such that $G/G' \simeq Z(p^\infty)$, and G' is abelian, then G is not hypercentral. Moreover, if G is a locally graded minimal non-(abelian-by-finite) p-group, then all proper subgroups of G are hypercentral and ascendant.*

Proof. Suppose that G is hypercentral, and let N be a G-invariant subgroup of G' of finite exponent. Since G' is abelian, then it is the product of a family of G-invariant subgroups of finite exponent. Moreover $G' \leq C_G(Z_2(G) \cap N)$. Now let $x \in G$, $y \in Z_2(G) \cap N$. Then, if t is the exponent of N, we have

$$[y, x^t] = [y^t, x] = 1.$$

Thus for every element x of G, $x^t \in C_G(Z_2(G) \cap N)$, so $G/C_G(Z_2(G) \cap N)$ has finite exponent and it is a quotient of $Z(p^\infty)$. Therefore $Z_2(G) \cap N$ is a subgroup of the centre $Z(G)$.

By induction on α, we will prove that $Z_\alpha(G) \cap N$ is contained in $Z(G)$ for all terms of the upper central series of G $Z_\alpha(G)$. For, if we suppose that $Z_\beta(G) \cap N \leq Z(G)$, $\forall \beta < \alpha$, and α is a limit ordinal, then, obviously, $Z_\alpha(G) \cap N \leq Z(G)$.

If α is not limit and $y \in Z_\alpha(G) \cap N$, then, $\forall x \in G$, $[y, x] \in Z_{\alpha-1} \cap N$, and hence $[y, x] \in Z(G)$. The same argument used above gives $Z_\alpha(G) \cap N \leq Z(G)$. Since G is hypercentral, $N \leq Z(G)$ and then $G' \leq Z(G)$. This implies that G is abelian and this is a contradiction.

Let G be a locally graded minimal non-(abelian-by-finite) p-group. Then, for all proper subgroup H of G, $HG' < G$. Thus $HG' = H_i = \langle a_i, G' \rangle$, where $a_i G'$ is one of the generators of G/G'. We will prove that H_i is hypercentral. In order to do this, let $\bar{H}_i$ be any homomorphic image of H_i, $\bar{H}_i = \langle \bar{a}_i, \bar{G'} \rangle$, and let $\bar{x}$ be a non trivial element of $\bar{G'}$. Since $\bar{H}_i$ is locally nilpotent, the subgroup $\bar{V} = \langle \bar{a}_i, \bar{x} \rangle$ is nilpotent and $\bar{x} \in \bar{V} \cap \bar{G'}$. Therefore $\bar{V} \cap \bar{G'} \neq \{1\}$ is a normal subgroup of $\bar{V}$ and $\bar{G'} \cap Z(\bar{V}) \neq \{1\}$. Hence there exists a non trivial element $\bar{y} \in \bar{G'} \cap Z(\bar{V})$ such that $[\bar{y}, \bar{a}_i] = 1$. This implies $C_{\bar{G'}}(\bar{a}_i) \neq 1$ and then $Z(\bar{H}_i) \neq 1$. Thus H_i and H are hypercentral and hence H is also ascendant for all $H < G$. □

1.3 Groups whose proper subgroups are nilpotent-by-finite

In this section we'll study locally graded minimal non-(nilpotent-by-finite) groups. We denote with $\overline{NF}$ this class. Such groups were studied by B. Bruno in papers [2], [3].
Moreover, in 1995, B. Bruno and R. E. Phillips proved that Theorem 1.2.5 holds also for groups with nilpotent-by-finite proper subgroups. In fact, they proved the following theorem.

Theorem 1.3.1. *A non-perfect locally graded minimal non-(nilpotent-by-finite) group is periodic.*

We become with the study of locally graded periodic minimal non-(nilpotent-by-finite) groups which are not locally nilpotent. Obviously such groups are locally finite and contain no proper subgroups of finite index. Moreover if G is a periodic non-(locally nilpotent) $\overline{NF}$-group and N/K is a finite normal subgroup of G/K, then N/K is central in G/K.
The next proposition shows that a periodic non-(locally nilpotent) $\overline{NF}$-group is not simple.

Proposition 1.3.2. *Let G be a periodic non-(locally nilpotent) $\overline{NF}$-group. Then G is not simple.*

Proof. By hypotheses it follows that G is a locally finite and it's known, by the quoted result of Schute, that a simple locally finite group whose proper subgroups contain a soluble subgroup of finite index is isomorphic to PSL(2,F) or to $S_Z(F)$, where F is an infinite locally finite field with no infinite proper subfields. Nor PSL(2,F) neither $S_Z(F)$ are $\overline{NF}$-groups, because they contain a proper subgroup which does not have a nilpotent subgroup of finite index. From this contradictions follows the theorem.

□

If G is a group, the product of all the normal locally nilpotent subgroups of G is called Hirsch-Plotkin radical HP(G). Clearly, HP(G) is the unique largest maximal normal locally nilpotent subgroup of G, and if G is not locally nilpotent, then HP(G) is a proper subgroup.

Proposition 1.3.3. *Let G be a periodic non-(locally nilpotent)* $\overline{NF}$*-group. Then* $G' \leq HP(G) < G$ *and the commutator subgroup* G' *is soluble.*

Proof. Obviously, $G/HP(G)$ is infinite. Let N be a normal subgroup of G such that $HP(G) < N < G$. If $G/HP(G)$ is locally nilpotent such group clearly exists, otherwise $G/HP(G)$ is a periodic non-(locally nilpotent) $\overline{NF}$-group and the existence of N is assured by Proposition 1.3.2. Moreover N isn't locally nilpotent, so $HP(N) < N$ and $HP(N) \lhd G$. Therefore $HP(N) \leq HP(G)$ and $|N : HP(N)|$ is finite, since N is nilpotent-by-finite. This implies that $N/HP(G)$ is finite and hence central in $G/HP(G)$.
Since G/N is infinite, for the same resons given at the beginning of the proof, there exists a subgroup N_1 such that $HP(G) < N < N_1 \lhd G$ and $N_1/HP(G) \leq Z(G/HP(G))$. Repeating the reasoning, $G/HP(G)$ is union of an ascending chain of finite abelian subgroups. Hence $G/HP(G)$ is abelian and $G' \leq HP(G)$. Finally, G' is locally nilpotent and nilpotent-by-finite, so G' is soluble.

□

Proposition 1.3.4. *Let G be a periodic non-(locally nilpotent)* $\overline{NF}$*-group. Then G cannot be the product of two proper normal subgroups.*

Proof. Suppose, by contradiction, that $G = N_1N_2$, with $N_1, N_2 \lhd G$. Since $|N_i : HP(N_i)|$ is finite, for $i = 1, 2$, and G cannot contains proper subgroups of finite index, we have $G = HP(N_1)HP(N_2)$. This means that G is locally nilpotent, a contradiction.

□

Proposition 1.3.5. *Let G be a periodic non-(locally nilpotent) $\overline{NF}$-group. Then G/G' is isomorphic to $Z(p^\infty)$, for some prime number p.*

Proof. The quotient G/G' is a periodic divisible abelian group. By Proposition 1.3.4, G can't be decomposed into the product of two proper normal subgroups, so, necessarily, $G/G' \simeq Z(p^\infty)$. □

Finally, we have the following characterization:

Theorem 1.3.6. *Let G be a periodic non-(locally nilpotent) $\overline{NF}$-group. Then, there exists a subgroup $A \simeq Z(p^\infty)$, such that G is the semidirect product $G = A \ltimes G'$, and G' is a p'-group. Moreover A centralizes every proper A-invariant subgroup of G'.*

Proof. G is a locally finite group, and by Proposition 1.3.5 it follows that there exists a Sylow p-subgroup P of G such that $G = PG'$. Since G is not locally nilpotent, P is a proper subgroup of G and hence P is nilpotent-by-finite. Let $O_p(G')$ be a Sylow p-subgroup of G'. G' is locally nilpotent by Proposition 1.3.3, and it implies that $O_p(G') \lhd G$ and $G' = O_p(G') \times O_{p'}(G')$, where $O_{p'}(G')$ is a p'-subgroup of G'. Furthermore $O_p(G') \leq P$ and hence $G = PO_{p'}(G')$. Let A be a nilpotent subgroup of finite index in P. Then $G = AO_{p'}(G')$ and $A \simeq G/O_{p'}(G')$. It follows

$$A/A' \simeq (G/O_{p'}(G'))/(G/O_{p'}(G'))' \simeq G/G' \simeq Z(p^\infty).$$

A is a nilpotent subgroup, so A is abelian and we can conclude that $A \simeq Z(p^\infty)$ and $G' = O_{p'}(G')$.

Now, let D be a proper subgroup of G' such that $D^A = D$. Then $DA < G$ and DA is nilpotent-by-finite. Let N be a nilpotent normal subgroup of DA such that $|DA : N|$ is finite. Since $A \simeq Z(p^\infty)$, $A \leq N$ and $DA = DN$ is locally nilpotent. Now A is a p-group, D is a subgroup of G' that is a p'-group, therefore every element of A commutes with every element of D, and the theorem is proved. □

Corollary 1.3.7. *Let G be a periodic non-(locally nilpotent) $\overline{NF}$-group. Then G' cannot be decomposed into the product of two proper normal subgroup of G and G' is a q-group, for some prime $q \neq p$.*

Now we give a description of locally graded p-groups which are minimal non-(nilpotent-by-finite). We will see that such groups are exactly the locally nilpotent periodic minimal non-(nilpotent-by-finite) groups.

First of all we show a general property of $\overline{NF}$-groups.

Proposition 1.3.8. *Let G be an $\overline{NF}$-group and H a proper normal subgroup of G. Then there exists a nilpotent normal subgroup N of G, such that N has finite index in H.*

Proof. Let

$$\mathcal{A} = \{K \lhd H | K \text{ is nilpotent of finite index in } H\}.$$

Obviously, $\mathcal{A}$ is non empty since H is nilpotent-by-finite. Let

$$A = \{n \in \mathbb{N} | n = |H/K|, K \in \mathcal{A}\}.$$

$A \neq \emptyset$ and it's a subgroup of the set of naturals numbers, so there exists

$\lambda = minA$. If $\lambda = 1$, H is nilpotent and normal in G , thus the proposition is proved. Otherwise, if H is not nilpotent, let $K \in \mathcal{A}$ be such that $|H/K| = \lambda$. Let g be any element of G, then $K^g \lhd H^g = H$ and $KK^g \in \mathcal{A}$. KK^g is a proper subgroup of H, since KK^g is nilpotent, and

$$\mu = |(KK^g)/K| < |H/K| = \lambda \leq |H/KK^g|,$$

moreover

$$|H/(KK^g)| = |(H/K)/(KK^g/K)| = \lambda/\mu.$$

Thus we get $\lambda \leq \lambda/\mu$ and this gives a contradiction if $\mu \neq 1$. Hence $\mu = 1$ and $KK^g = K$, this means $K^g \leq K$, $\forall g \in G$. Thus $K \lhd G$ and the theorem is proved. □

From the definition and from previous porposition it follows that an $\overline{NF}$-group have no proper subgroups of finite index and cannot be the join of two proper normal subgroups.

Proposition 1.3.9. *A periodic $\overline{NF}$-group G is locally nilpotent if and only if G is a p-group, for some prime p.*

Proof. Suppose that G is locally nilpotent. Since G cannot be the product of two proper normal subgroups, then G is a p-group. Suppose, now, that G is a p-group. In order to prove that G is locally nilpotent we will prove that G is locally finite. Let H be a finitely generated subgroup of G. Then, since G is locally graded and not nilpotent-by-finite, H is a proper subgroup of G. Thus H is nilpotent-by-finite and this implies that H is finite. □

Lemma 1.3.10. *Every proper normal subgroup of an $\overline{NF}$-group G is soluble.*

Proof. Let H be a proper normal subgroup of G. By Proposition 1.3.8 there exists a nilpotent normal subgroup N of G, such that H/N is finite. Since G does not have proper subgroups of finite index, then H/N is central in G/N. Thus H is soluble. □

Chapter 2

Groups whose proper subgroups have finite conjugacy classes of subnormal subgroups

In this section we will focuse our attention on the class of V-groups. In particular we will see what happens when all proper subgroups of a group G have the V-property.
Recall here that a group G is said to have finite (Prüfer) rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. If there is no such r, the group G has infinite rank. Clearly, groups of rank 1 are just the locally cyclic groups.
Our results are proved in the class of strongly locally graded groups, it is a class of generalized soluble groups, containing, in particular, all locally (soluble-by-finite) groups. It was introduced in [10]. We will see its construction.
Let $\mathscr{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathscr{D}}$ be the

closure of $\mathscr{D}$ by the operators $\acute{P}$, $\grave{P}$, R, L (we are using the first chapter of the monograph [26] as a general reference for definitions and properties of closure operations on group classes). It is easy to prove that any $\bar{\mathscr{D}}$-group is locally graded and that the class $\bar{\mathscr{D}}$ is closed with respect to forming subgroups. Moreover, N.S. Černikov [8] proved that any $\bar{\mathscr{D}}$-group of finite rank is a finite extension of a locally soluble subgroup. Obviously, all residually finite groups belong to $\bar{\mathscr{D}}$, and hence the consideration of any free non-abelian group shows that the class $\bar{\mathscr{D}}$ is not closed with respect to homomorphic images. We shall say that a group G is strongly locally graded if every section of G is a $\bar{\mathscr{D}}$-group. Thus strongly locally graded groups form a large class of generalized soluble groups, which is, obvious, closed with respect to subgroups and homomorphic images.

The following lemma shows that strongly locally graded group admitting an ascending normal series whose factors have finite rank must be hyper-(abelian or finite).

Lemma 2.1.1. *Let G be a strongly locally graded group admitting an ascending normal series whose factors have finite rank. Then G also has an ascending normal series in which every factor is either finite or torsion-free abelian of finite rank.*

Proof. Let $\bar{H} = H/K$ be any normal section of G which has finite rank. Then $\bar{H}$ is a $\bar{\mathscr{D}}$-group, so that it follows from a result of N.S. Černikov [8] that $\bar{H}$ contains a locally soluble subgroup $\bar{L}$ of finite index, and of course $\bar{L}$ can be chosen to be characteristic in $\bar{H}$, and so also G-invariant. Moreover, it is known that there exists a positive integer n such that $\bar{L}^{(n)}$ is hypercentral (see [26] Part 2, Lemma 10.39), and hence $\bar{L}$ has a characteristic ascending series with abelian factors. As the primary components of any abelian group of finite rank satisfy the minimal condition, it follows that G has an ascending normal series whose factors are either finite or torsion-free abelian groups of finite rank. □

We need also to prove that hyperabelian V-groups are soluble. This is actually an easy consequence of the results proved by Casolo on groups of generalized power automorphisms of abelian groups.

Lemma 2.1.2. *Let G be a hyperabelian V-group. Then G is soluble.*

Proof. Let F be the Fitting subgroup of G. Then F is nilpotent by Lemma 1.1.6, and so every subgroup of F has finitely many conjugates in G. In particular, it follows from Neumann's Theorem that the centre $A = Z(F)$ has finite index in F, so that the factor group $G/C_G(F/A)$ is finite, and hence also soluble. If T is the subgroup consisting of all elements of finite order of A, the abelian group A/T is torsion-free and so $G/C_G(A/T)$ is finite by Lemma 1.1.2. Moreover, the subgroup $K = Paut_G(T)$ is normal in G, the index $|G : K|$ is finite and K normalizes all subgroups of T, see Lemma 1.1.1. In this situation it is well-known that $[T, K'] = 1$. Put

$$N = K' \cap C_G(A/T) \cap C_G(F/A).$$

Of course, G/K' is a soluble group, so that also G/N is soluble. On the other hand, N stabilizes the series

$$\{1\} \leq T \leq A \leq F,$$

and hence $N/C_N(F)$ is nilpotent. As $C_N(F)$ is contained in F, it follows that the group G is soluble. □

Lemma 2.1.3. *Let G be a group admitting an ascending normal series whose factors have finite rank. If the factor group G/G' is periodic and divisible, then $G' = G''$*

Proof. Replacing G by the factor group G/G', it can be assumed without loss of generality that G is metabelian. Let X/Y be any G-invariant section of G' which is torsion-free abelian of finite rank. Then G' is contained in the centralizer $C_G(X/Y)$, so that $G/C_G(X/Y)$ is a periodic linear group over the field of rational numbers, and hence it is finite (see [26] Part 1, p.85). It follows from the hypothesis that G' admits an ascending G-invariant series whose factors are either finite or torsion-free abelian of finite rank, and by the above argument G acts trivially on all such factors, because G/G' is divisible. Then G' is contained in the hypercentre of G, and hence G is hypercentral. In this situation it is known that G is periodic (see [27], Lemma 4), and so it is abelian (see [26] Part 2, Theorem 9.23). Therefore $G' = G''$ and the statement is proved. □

Corollary 2.1.4. *Let G be a soluble group admitting an ascending normal series whose factors have finite rank. If the factor group G/G' is periodic and divisible, then G is abelian.*

Recall that a finite residual of a group G is the intersection of all (normal) subgroups of finite index of G. Of course, the finite residual J of a group G may have infinite index in G, but if the index $|G : J|$ is finite, then J cannot contain proper subgroups of finite index.

Now me can prove the main theorem.

Theorem 2.1.5. *Let G be a strongly locally graded group admitting an ascending normal series whose factors have finite rank. If all proper subgroups* *of G have the V-property, then G itself is a V-group.*

Proof. Assume for a contradiction that the statement is false, and let X be a subnormal subgroup of G with infinitely many conjugates, so that the index $|G : N_G(X)|$ is infinite. If X_G is the largest normal subgroup of G contained in X, the subnormal subgroup X/X_G of G/X_G has likewise infinitely many conjugates, and hence replacing G by the factor group G/X_G it can be assumed without loss of generality that X does not contain non-trivial normal subgroups of G. Moreover, since every proper subgroup of G has the V-property, we have that X is contained in no proper subgroup of finite index of G.

By Lemma 2.1.1 the group G has an ascending normal series whose factors are either finite or torsion-free abelian of finite rank, and such property is also inherited by all subgroups of G. In particular, X contains a non-trivial normal subgroup Y which is either finite or torsion-free abelian. Then Y is subnormal in G and the normalizer $N_G(Y)$ is a proper subgroup of G containing X. It follows that the index $|G : N_G(Y)|$ is infinite, and so replacing X by Y we may also suppose that X is either finite or torsion-free abelian. Moreover, if X is infinite and x is a non-trivial element of X, the cyclic subgroup $\langle x \rangle$ is subnormal in G and X is contained in $N_G(\langle x \rangle)$, so that $|G : N_G(\langle x \rangle)|$ is infinite, and in this case we may even assume that X is infinite cyclic.

Let N be any proper normal subgroup of finite index of G. Then $X \cap N$ is a subnormal subgroup of G having finitely many conjugates in N, and so also in G. As X is contained in $N_G(X \cap N)$, it follows that $X \cap N$ is normal in G, and hence $X \cap N = \{1\}$. Since X is contained in no proper subgroups of finite index, it follows that $G = NX$, thus X is finite and $|G : N| \leq |X|$. This argument shows that there is a bound for the indices of (normal) subgroups of finite index of G, and so G has only finitely many subgroups of finite index.

Therefore the finite residual J of G has finite index in G, so that $G = XJ$ and J contains no proper subgroups of finite index.
Let Σ be an ascending normal series of J whose factors are either finite or torsion-free abelian of finite rank. Let H/K be a finite factor of Σ. The index $|J : C_J(H/K)|$ is finite, hence $J = C_J(H/K)$, that is H/K is central. From this observation, it follows that J is a hyperabelian group. Assume that J is not soluble. Since Lemma 2.1.2 shows that all hyperabelian groups with the V-property are soluble, it follows that J is perfect. If U/V is any torsion-free factor of Σ, the group $J/C_J(U/V)$ is isomorphic to a linear group of finite degree over the field of rational numbers and it is soluble (see for instance [26] Part 1, p.78). Thus $C_J(U/V) = J$, so that Σ is a central series of J and hence J is hypercentral, a contradiction since any hypercentral non-trivial group has proper commutator subgroup. Therefore J is soluble.
Suppose now that the divisible abelian group J/J' is not periodic. Then the additive group of rational numbers occurs as a homomorphic image of J, and so J contains a normal subgroup L such that J/L is a periodic abelian group with infinitely many non-trivial primary components. Clearly, L has finitely many conjugates in G, so that J/L_G is likewise periodic and hence J has a decomposition $J = J_1J_2$, where J_1 and J_2 are proper G-invariant subgroups. If $G = XJ_i$, with $i = 1, 2$ it follows from Dedekind's modular law that

$$J = J \cap G = J \cap XJ_i = J_i(X \cap J)$$

and

$$|J : J_i| = |J_i(X \cap J) : J_i| = |X \cap J : X \cap J_i| < \infty.$$

This is impossibile, hence XJ_1 and XJ_2 are proper subgroups of G, so that they have the V-property. Then the index $|XJ_i : N_{XJi}(X)|$ is finite for $i = 1, 2$, and hence $N_G(X)$ has finite index in G. This contradiction proves

that J/J' is periodic, so that J is abelian by Corollary 2.1.4, and of course it is the direct product of a collection of Prüfer subgroups. Observe also that the intersection $X \cap J$ is a normal subgroup of G, so that $X \cap J = \{1\}$ and X is finite. If P is any subgroup of J of type p^∞, where p is a prime number, the product XP^G is a Černikov group and hence it has finite conjugacy classes of subnormal subgroups. Therefore P normalizes X, $J \leq N_G(X)$, and so X is normal in $G = XJ$. This is a contradiction.
Suppose now that the group G has no proper subgroups of finite index, then arguing as for J, we can prove that G is soluble and G/G' is periodic. Corollary 2.1.4 implies G abelian and this last contradiction completes the proof of the theorem.

□

Using a group construction due to V.S. Čarin (see [26] Part 1, p.152), we show an example of a periodic metabelian group which does not have the V-property while all its proper subgroups are V-groups.
Let p be a prime number, and let GF(p) be the field with p-elements. If F is the algebraic closure of GF(p), the multiplicative group F^* of F is a direct product of Prüfer subgroups, one for each prime $q \neq p$; consider one of such subgroups $Q \simeq Z(q^\infty)$, and let A be the additive group of the subfield of F generated by Q. Consider the semidirect product $G = Q \ltimes A$, then A is a minimal normal subgroup of G of exponent p and the elements of Q act by multiplication on A so that $C_G(a) = A$ for all $a \in A \backslash \{0\}$. Clearly, G is a metabelian group of infinite rank, and it does not have the V-property, since every proper non-trivial subgroup of A has infinitely many conjugates in G. On the other hand all proper subgroups of G are V-groups. Let H be a non-abelian proper subgroup of G such that $\pi(H) = \{p, q\}$. Let X be a subnormal subgroup of H and put $X_p = X \cap A$. Obviously X/X_p is a q-group

and it follows that

$$[X/X_p, A/X_p] = \{1\}.$$

Therefore $[X, A] \leq X_p \leq X$ so that $A \leq N_G(X)$ and $X \lhd XA$ and this implies that X is a subnormal subgroup of G. Note that $[A, X] = [A, AX]$ is normal in G, but A is minimal normal in G and necessarily $[A, X] = A$. It follows that $A \leq X$ and hence $X \lhd G$.

We have seen that Casolo considered also the class T^*, consisting of all groups in which every subnormal subgroup has finite index in its normal closure. We can observe that the statement obtained replacing the property V by the property T^* in our theorem is false.
To see this, consider the standard wreath product

$$G = Z(2^\infty) \wr Z_2,$$

and let $B = B_1 \times B_2$ be its base group, where

$$B_1 \simeq B_2 \simeq Z(2^\infty).$$

Of course, G is a metabelian group of finite rank, and all its proper subgroups have the T^*-property. Let H be a non-abelian infinite proper subgroup of G, then $H \cap B \neq \{1\}$. If there exist two distincts subgroups K_1, K_2 of H such that $K_1 \simeq K_2 \simeq Z(2^\infty)$, then $H = B$, that is H abelian. So, let K be the only subgroup of H isomorphic to $Z(2^\infty)$ and take an element h in $H \backslash K$. It follows that $G = \langle B, h \rangle$, and K is normal in G. Then $K \leq Z(G)$ and this contradiction shows that every infinite proper subgroup is abelian. Moreover G is not a T^*-group, since

$$B_i^G / B_i \simeq B/B_i \simeq Z(2^\infty),$$

for $i = 1, 2$.

Chapter 3

Infinite groups whose subgroups are close to be normal-by-finite

As already mentioned, in 1955 Neumann began a systematic study of finiteness conditions on group theory defined by restrictions on the conjugacy classes of subgroups. Among other results he proved that every subgroup H of a group G has finite index in its normal closure H^G if and only if G is finite-by-abelian. In this chapter we consider a dual condition. A subgroup X of a group G is said to be normal-by-finite if the core X_G of X in G has finite index in X . Finite subgroups and normal subgroups of an arbitrary group are obvious examples of normal-by-finite subgroups. We shall say that a group G is a CF-group, which is the abbreviation of core-finite, if each of its subgroups is normal-by-finite, that is, if H/H_G is finite for all subgroups H of G. Such groups need not even be abelian-by-finite due to the existence of Tarski groups. The class of CF-groups is obviously subgroup-closed, but we can prove that there exists a periodic metabelian group that is not a CF-group but all its proper subgroups have the CF-property.

Let K be a group of type p^∞ for some prime number p, and consider the standard wreath product

$$G = K \wr \langle x \rangle,$$

of K by a group $\langle x \rangle$ of order 2. Since K is not normal in G, the group G does not have the CF-property. On the other hand, every proper subgroup of G has the CF-property. Infact, let B be the base group of G and let H be a proper subgroup of G. If $H \leq B$, then H is abelian. So let suppose that H is not contained in B. Obviously, the intersection $H \cap B \neq B$. Thus $H \cap B$ is either finite or finite extension of $Z(p^\infty)$. Since the index $|H : H \cap B|$ is finite, we have that H is either finite or finite extension of $Z(p^\infty)$, so that H is a CF-group.

We shall say that a CF-group is BCF, which is the abbreviation of boundedly core-finite group, if there is an integer n such that H/H_G has order at most n for all $H \leq G$.
CF-groups were studied, first, by Buckley, Lennox, Neumann, Smith and Wiegold in [6] and later by H. Smith and J. Wiegold in [29].
In particular they proved that:

Theorem 3.1.1. *(see [6]) Every locally finite CF-group is abelian-by-finite and BCF.*

Theorem 3.1.2. *(see [29]) Let G be a CF-group such that every periodic image of G is locally finite. Then G is abelian-by-finite.*

It is still an open question whether an arbitrary locally graded CF-group G is abelian-by-finite. However the class of groups whose periodic homomorphic images are locally finite form a very large class, containing in particular the class of locally (generalized radical) groups. Recall that a group G is called generalized radical if it has an ascending series whose factors are either locally nilpotent or locally finite.

About BCF-groups Smith and Wiegold proved the following result.

Theorem 3.1.3. *(see [29]) Every locally graded BCF-group is abelian-by-finite.*

The aim of this section is to show that Theorem 3.1.2 can be extended to the case of groups whose proper subgroups have the CF-property. Of course, a group G satisfies this condition if and only if the index $|X : X_Y|$ is finite, whenever $X \leq Y < G$, which means that the subgroup X is close to be normal-by-finite in G.

Our first lemma describes a general property of CF-groups, which is of independent interest.

Lemma 3.1.4. *Let G be a CF-group whose periodic homomorphic images are locally finite. Then the subgroup G'' is locally finite.*

Proof. Let a be any element of infinite order of G. As the subgroup $\langle a \rangle$ is normal-by-finite, there exists a positive integer $n(a)$ (depending of course on a) such that the cyclic subgroup $\langle a^{n(a)} \rangle$ is normal in G, and hence $[G', a^{n(a)}] = \{1\}$. Let A be the subgroup of G generated by the powers $a^{n(a)}$, where a runs over all elements of infinite order of G. Clearly, A is an abelian normal subgroup of G, and the factor group G/A is periodic, and so even locally finite. Moreover, $A \cap G'$ is contained in the center of G', so that $G'/Z(G')$ is locally finite, and it follows from a classical result of Schur that also the subgroup G'' is locally finite (see for instance [26] Part 1, Theorem 4.12). □

We need the following case of Maschke's Theorem for the proof of the main theorem.

Theorem 3.1.5. *(Maschke's Theorem) Let K be a finite group that acts via automorphisms on an elementary abelian p-group V, and assume that p does not divide $|K|$. Suppose that $U \subseteq V$ is a K-invariant subgroup. Then there exists a K-invariant subgroup $N \subseteq V$ such that $V = U \times N$.*

Now we can prove our main result on this topic.

Theorem 3.1.6. *Let G be a locally graded group whose periodic sections are locally finite. If every proper subgroup of G has the CF-property, then G is abelian-by-finite.*

Proof. Assume for a contradiction that the group G is not abelian-by-finite. It follows from Theorem 3.1.2 that every proper subgroup of G is abelian-by-finite, so that G cannot contain proper subgroups of finite index, and in particular it is not finitely generated. Then every finitely generated subgroup E of G has the CF-property, and so E'' is locally finite by Lemma 3.1.4. Therefore, the subgroup G'' is likewise locally finite. On the other hand, an application of Theorem 1.2.5 yields that G is either perfect or locally finite, and hence G must be locally finite. By Proposition 1.2.8 it follows that the commutator subgroup G' is properly contained in G, and by Proposition 1.2.10, we have that G/G' is a group of type p^∞ and G' is a q-group, where p and q are prime numbers.
Suppose first that $p \neq q$, so that Lemma 1.2.15 yields that G' is an infinite abelian minimal normal subgroup of G, and there exists in G a subgroup P of type p^∞ such that $G = PG'$. Then $Z(G) = C_P(G')$ is finite, and

$$[Z_2(G), G] \leq Z(G) \cap G' \leq P \cap G' = \{1\},$$

so that $\overline{Z}(G) = Z(G)$ and the factor group $G/Z(G)$ has trivial center. Clearly, $G/Z(G)$ is likewise a counterexample, and hence replacing G by $G/Z(G)$ it can be assumed without loss of generality that $Z(G) = \{1\}$.
Since G' is a minimal normal subgroup of G, there exist elements a of G' and y of P such that $\langle a\rangle^y \neq \langle a\rangle$. Let b be any element of G' such that $\langle b\rangle^y = \langle b\rangle$, so that $b^y = b^i$ for some positive integer $i < q$. The set

$$K = \{c \in G' | c^y = c^i\}$$

is a proper G-invariant subgroup of G' , so that $K = \{1\}$, since G' is a minimal normal subgroup of G. Thus, $b = 1$, and hence y moves all cyclic non-trivial subgroups of G', which means that

$$\langle u\rangle \cap \langle u\rangle^y = \{1\}$$

for every element u of G'.
For the sake of simplicity, suppose that y has order p, and let x_{11} be an element of G'. Suppose that y maps x_{11} in an element x_{12}, with $x_{11} \neq x_{12}$, x_{12} will be mapped into x_{13}, with $x_{13} \neq x_{11}, x_{12}$, and so on up to x_{1p-1} that will be mapped into x_{11}. Therefore

$$\langle x_{11}, x_{12}, ..., x_{1p-1}\rangle^y = \langle x_{11}, x_{12}, ..., x_{1p-1}\rangle$$

and, by Theorem 3.1.5, we have

$$G' = \langle x_{11}, x_{12}, ..., x_{1p-1}\rangle \times X_1,$$

where X_1 is an $\langle y\rangle$-invariant subgroup.
Now, let x_{21} be an element of X_1 and suppose that y maps x_{21} in an element x_{22}, with $x_{21} \neq x_{22}$, x_{22} will be mapped into x_{23}, with $x_{23} \neq x_{21}, x_{22}$, and so on up to x_{2p-1} that will be mapped into x_{21}, with the same reasoning as

before. Hence

$$\langle x_{21}, x_{22}, ..., x_{2p-1}\rangle^y = \langle x_{21}, x_{22}, ..., x_{2p-1}\rangle,$$

and

$$X_1 = \langle x_{21}, x_{22}, ..., x_{2p-1}\rangle \times X_2.$$

Iterating the procedure the commutator G' can be decomposed into the following direct product

$$G' = \langle x_{11}, x_{12}, ..., x_{1p-1}\rangle \times \langle x_{21}, x_{22}, ..., x_{2p-1}\rangle \times \ldots \times \langle x_{n1}, x_{n2}, ..., x_{np-1}\rangle \times \ldots$$

Put

$$Y_n = \langle x_{n1}, x_{n2}, \ldots, x_{np-1}\rangle,$$

for every $n \in \mathbb{N}$, and let x_n be a non-trivial element of Y_n, for each positive integer n. Consider the infinite group

$$X = \langle x_n | n \in \mathbb{N}\rangle = Dr_{n\in\mathbb{N}}\langle x_n\rangle.$$

Then $X \cap X^y = \{1\}$, and so in particular the core of X in $\langle y, G'\rangle$ is trivial. Therefore, the CF-property does not hold for the proper subgroup $\langle y, G'\rangle$ of G, and this contradiction shows that $p = q$, so that G is a p-group.
By Proposition 1.2.18, the group G cannot be hypercentral, so that its hypercenter $\overline{Z}(G)$ is a proper subgroup and the factor group $G/\overline{Z}(G)$ is likewise a counterexample. Thus, it can be assumed without loss of generality that $Z(G) = \{1\}$. Suppose that G' contains a subgroup Q of type p^∞, and let H/G' be any proper subgroup of G/G'. Since H has the CF-property, Q is normal in H, and so its unique subgroup Q_1 of order p is normalized by H. On the other hand, G/G' is the join of its proper subgroups, and hence the subgroup Q_1 is normal in G, contradicting the assumption $Z(G) = \{1\}$.

Therefore, G' is reduced, and so $G'/(G')^p$ is infinite. Clearly, the factor group $G/(G')^p$ is also a counterexample, and replacing G by $G/(G')^p$ we may also suppose that G' has exponent p. Consider again any proper subgroup H/G' of G/G'. Then H is nilpotent (see [26] Part 2, Lemma 6.34), and hence $Z(H) \cap G'$ is a non-trivial normal subgroup of G'. Let V be a subgroup of G' such that

$$G' = (Z(H) \cap G') \times V.$$

Since H is a CF-group, the index $|V : V_H|$ is finite. On the other hand, V_H has obviously trivial intersection with Z(H), so that $V_H = \{1\}$ and V is finite. It follows that $Z(H) \cap G'$ has finite index in G' , so that also the index $|H : Z(H)|$ is finite and hence the commutator subgroup H' of H is finite by the already quoted theorem of Schur. However, G has no finite non-trivial normal subgroups, so that $H' = \{1\}$ and H is abelian. As

$$G = \bigcup_{G' \leq H < G} H,$$

it follows that G itself is abelian, and this last contradiction completes the proof of the theorem. □

We have seen that in [29] it was proved that if G is a locally graded group for which there exists a positive integer k such that $|X : X_G| \leq k$ for every proper subgroup X, then G is abelian-by-finite. Since all periodic sections of these groups are locally finite, our theorem has the following consequence.

Corollary 3.1.7. *Let G be a locally graded group whose proper subgroups have the BCF-property. Then G is abelian-by-finite.*

We have already seen that the behavior of subgroups of infinite rank of a given group has a strong influence on the structure of the whole group. If we require the CF-property only for subgroups of infinite rank, we can see that the group is still abelian-by-finite, just in case the group belongs to the class of locally (soluble-by-finite) groups.

First of all, we need the following lemma.

Lemma 3.1.8. *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G have the CF-property, then G is either abelian-by-finite or periodic.*

Proof. Assume for a contradiction that the group G neither is abelian-by-finite nor periodic. Since any locally (soluble-by-finite) CF-group is abelian-by-finite, it follows from Lemma 4.1.10 that all proper subgroups of G are abelian-by-finite. Thus, $G = G'$ by Lemma 1.2.5. It is also clear that G has no proper subgroups of finite index, so that in particular every finitely generated subgroup of G is abelian-by-finite, and so has finite rank. Therefore, G contains an abelian normal subgroup A such that G/A has finite rank by Theorem 4.1.5. Let E be any finitely generated subgroup of G. As G/A cannot be finitely generated, the product EA is a proper subgroup of infinite rank of G, and hence it has the CF-property. Then $(EA)''$ is locally finite by Lemma 3.1.4, and so E'' is finite. Therefore $G = G''$ is locally finite, and this contradiction completes the proof.

□

Theorem 3.1.9. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank have the CF-property. Then G is abelian-by-finite.*

Proof. Assume for a contradiction that the group G is not abelian-by-finite. Then G is periodic by Lemma 3.1.8, and a contradiction can be obtained arguing as in the proof of Theorem 3.1.6. To this aim, it is enough to observe that in the second part of that proof the hypotheses are applied only to subgroups of infinite rank. □

It is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank, and this problem seems to be the main obstacle in order to extend Theorem 3.1.9 to the case of locally graded groups.

Chapter 4

Groups whose proper subgroups of inifinite rank have an $\mathcal{X}$-group of finite index

4.1 Groups whose proper subgroups of inifinite rank are abelian-by-finite

In recent years a series of papers has been published on the behavior of groups of infinite rank in which every proper subgroup of infinite rank has a suitable property (see for instance [9], [11], [15]). In this section we give a contribution to this topic, investigating the structure of groups in which every proper subgroup of infinite rank contains an abelian subgroup of finite index.

To fully understand our result, it is necessary to consider groups in which every proper subgroup is nilpotent-by-(finite rank). This class was considered by M. R. Dixon, M. J. Evans and H. Smith in [16].

First of all, we note the following fact.

Lemma 4.1.1. *Suppose that G is an infinite simple locally finite group. Then G has all proper subgroups nilpotent-by-(finite rank) if and only if $G \simeq PSL(2, F)$ or $G \simeq S_Z(F)$, for some infinite locally finite field F containing no infinite proper subfield.*

Theorem 4.1.2. *Let G be a locally (soluble-by-finite) group in which every proper subgroup is nilpotent-by-(finite rank). Suppose that G is not a perfect p-group. If G has no infinite simple images then G is nilpotent-by-(finite rank).*

Theorem 4.1.3. *Let G be a group with all proper subgroups nilpotent-by-(finite rank) and suppose that G is locally (soluble-by-finite) and locally of finite rank. Suppose that G is not locally finite, then G is nilpotent-by-(finite rank).*

A similar result holds also for groups in which every proper subgroup is abelian-by-(finite rank).

Theorem 4.1.4. *Let G be a group whose proper subgroups are abelian-by-(finite rank). If G is locally nilpotent, or locally finite with no infinite simple images, then G is abelian-by-(finite rank).*

Finally, we have a corresponding result for groups in which every proper subgroup is (nilpotent of class c)-by-(finite rank).

Theorem 4.1.5. *Suppose that G is a group whose proper subgroups are (nilpotent of class c)-by-(finite rank).*

1. *If G is locally soluble then G is nilpotent-by-(finite rank) and either G is (nilpotent of class c)-by-(finite rank) or G is soluble.*

2. *If G is locally (soluble-by-finite) and locally of finite rank, but not locally finite, then G is nilpotent-by-(finite rank) and either G is (nilpotent of class c)-by-(finite rank) or G is soluble.*

Now, we can return to the study of groups of infinite rank in which all proper subgroups of infinite rank have an abelian subgroup of finite index.

Lemma 4.1.6. *Let G be a group whose proper subgroups of infinite rank are abelian-by-finite. If the factor group G/G' has infinite rank, then every proper subgroup of G is abelian-by-finite.*

Proof. Let X be any subgroup of finite rank of G. Then XG' is properly contained in G, and so there exists a proper subgroup Y of G of infinite rank such that $XG' \leq Y$. Therefore X is abelian-by-finite. □

Our next result allows us to restrict considerations to the case of groups whose lower central series terminates with the commutator subgroup.

Lemma 4.1.7. *Let G be a group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. If G contains a proper subgroup which is not abelian-by-finite, then $G' = [G', G]$ and G is either perfect or metabelian.*

Proof. Since G/G' has finite rank by Lemma 4.1.6 and $G'/[G', G]$ is a homomorphic image of the tensor product

$$(G/G') \otimes (G/G'),$$

it follows that also the nilpotent group $G/[G', G]$ has finite rank. Then $[G', G]$ must have infinite rank, and hence all proper subgroups of $G/[G', G]$

are abelian-by-finite. Assume for a contradiction that $[G', G]$ is properly contained in G'. Then $G/[G', G]$ cannot be abelian-by-finite, since G has no proper subgroups of finite index, and hence Theorem 1.2.5 and Proposition 1.2.10 show that

$$(G/[G', G])/(G/[G', G])' \simeq G/G'$$

is a group of type p^∞ for some prime number p. It follows that the nilpotent group $G/[G', G]$ is abelian, and this contradiction shows that $G' = [G', G]$. Suppose now that G' is a proper subgroup of G. Then G' has infinite rank, thus it is abelian-by-finite, and so it contains an abelian characteristic subgroup A of finite index. As the factor group G/A has finite commutator subgroup, it is nilpotent-by-finite (see [26], Part 1, Theorem 4.25). On the other hand, G has no proper subgroups of finite index, so that G/A is nilpotent and hence even abelian by the first part of the proof. Therefore $G' = A$ is abelian. □

Now we give a general result on locally (soluble-by-finite) groups.

Lemma 4.1.8. *Let G be a locally (soluble-by-finite) group containing a proper normal subgroup N such that G/N has finite rank. Then G has a non-trivial homomorphic image which is either finite or abelian.*

Proof. Assume that G has no proper subgroups of finite index. As the factor group G/N is (locally soluble)-by-finite by a result of N.S. Černikov, it is even locally soluble. Then there exists a positive integer k such that $G^{(k)}N/N$ is hypercentral (see [26], Part 2, Lemma 10.39), and it follows that the commutator subgroup of G is a proper subgroup. □

The next lemma improves the result obtained by Bruno and Phillips on periodicity of non-perfect minimal non-(abelian-by-finite) groups.

Lemma 4.1.9. *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G are abelian-by-finite, then G is either abelian-by-finite or periodic.*

Proof. Assume for a contradiction that the group G is neither abelian-by-finite nor periodic. In particular, G has no proper subgroups of finite index and so it cannot be finitely generated. As finitely generated abelian-by-finite groups have obviously finite rank, it follows from the hypotheses that every finitely generated subgroup of G has finite rank. Moreover, it is also clear that each proper subgroup of G is abelian-by-(finite rank). Then either G is soluble or it contains an abelian normal subgroup A such that G/A has finite rank by Theorem 4.1.5, so that an application of Lemma 4.1.8 shows that G' is a proper subgroup of G.

It follows from Theorem 1.2.5 that G contains proper subgroups which are not abelian-by-finite. Then G/G' must have finite rank by Lemma 4.1.6, and G' is an abelian subgroup of infinite rank by Lemma 4.1.7. Suppose first that the group G/G' is not periodic. Since G/G' is divisible, G can be decomposed into the product of two proper normal subgroups of infinite rank. Then G is nilpotent-by-finite, and so nilpotent. Again Lemma 4.1.7 gives now the contradiction that G is abelian. Therefore G/G' must be periodic, and hence G' is not periodic. Let T be the subgroup consisting of all elements of finite order of G'. If T has infinite rank, then all proper subgroups of G/T are abelian-by-finite, and hence the non-periodic group G/T is abelian-by-finite by Theorem 1.2.5. Then G/T is abelian, a contradiction, because T is properly contained in G'. Therefore T has finite rank, and the factor group G/T is likewise a counterexample, so that without loss of generality it can be assumed that G' is a torsion-free abelian group. If q_1 and q_2 are distinct prime numbers, by Lemma 1.2.6 there exists a G-invariant subgroup N of G'

such that G'/N is periodic and q_1,q_2 belong to $\pi(G'/N)$. As G' is torsion-free, the subgroup N has infinite rank, and hence all proper subgroups of G/N are abelian-by-finite. On the other hand, G/N is not abelian-by-finite, and its commutator subgroup is not a primary group, contradicting Theorem 1.2.5. The statement is proved. □

We are now in a position to prove our main result on this topic.

Theorem 4.1.10. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. Then all proper subgroups of G are abelian-by-finite.*

Proof. Assume for a contradiction that the statement is false. In particular, G is not abelian-by-finite, and so it has no proper subgroups of finite index. Moreover, G is periodic by Lemma 4.1.9, and hence locally finite. Suppose that G contains a proper normal subgroup K such that G/K is simple. Then G/K must have infinite rank by Lemma 4.1.8 and hence it is isomorphic either to PSL(2,F) or to Sz(F) for some infinite locally finite field F containing no infinite proper subfields as a consequence of Lemma 4.1.1. On the other hand, each of the last two groups contains a proper subgroup of infinite rank which is not abelian-by-finite (see for instance [23]). This contradiction shows that G has no infinite simple homomorphic images, so that Theorem 4.1.4 ensures that it contains an abelian normal subgroup N such that G/N has finite rank, and hence G' is a proper subgroup of G by Lemma 4.1.8. It follows from Lemmas 4.1.6 and 4.1.7 that G' is an abelian group of infinite rank. Let X be a subgroup of finite rank of G which is not abelian-by-finite. Clearly, X cannot be contained in any proper subgroup of infinite rank of G, and so $XG' = G$. Then $X/X \cap G' \simeq G/G'$ has no proper subgroups of finite index, and hence it is divisible. Moreover, the abelian normal subgroup $X \cap G'$ of

G has finite rank, so it is the direct product of its primary components, and hence it is covered by finite characteristic subgroups. Then $X/C_X(X \cap G')$ is residually finite, since $X/X \cap G'$ does not contain proper subgroups of finite index, it follows that $X \cap G'$ is contained in the center of X, and so X is nilpotent. Consider now a maximal abelian normal subgroup A of X containing $X \cap G'$. Then $C_X(A) = A$, and hence the divisible group X/A is isomorphic to a group of automorphisms of A. On the other hand, A is covered by finite characteristic subgroups, so that its full automorphism group is residually finite. Therefore $X = A$ is abelian, and this contradiction proves the statement. □

It is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank. This seems to be the main obstacle to extending our theorem to the case of locally graded groups. However, we will point out that this is at least possible for a special class of locally graded groups, we can prove that our main theorem remains true in the class of strongly locally graded groups.
We need the following result.

Lemma 4.1.11. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are locally (soluble-by-finite), then G is locally (soluble-by-finite).*

Proof. Suppose first that G is a finitely generated non-trivial group, so that it contains a proper subgroup H of finite index. Then H is a finitely generated subgroup of infinite rank, hence it is soluble-by-finite and therefore G itself is soluble-by-finite. Assume now that G is not finitely generated, so that all its finitely generated subgroups of infinite rank are soluble-by-finite. On the other hand, it follows from Černikov's result on $\bar{\mathscr{D}}$-groups that any finitely generated subgroup X of G of finite rank must be soluble-by-finite. Therefore G is locally (soluble-by-finite). □

Corollary 4.1.12. *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. Then all proper subgroups of G are abelian-by-finite.*

Proof. The group G is locally (soluble-by-finite) by Lemma 4.1.11, and so the statement is a direct consequence of our Theorem 4.1.10. □

4.2 Groups whose proper subgroups of infinite rank are nilpotent-by-finite

We can prove that results obtained, in the last paragraph, for groups with abelian-by-finite proper subgroups can be extended to groups with nilpotent-by-finite proper subgroups.

Lemma 4.2.1. *Let G be a group whose proper subgroups of infinite rank are nilpotent-by-finite. If the factor group G/G' has infinite rank, then every proper subgroup of G is nilpotent-by-finite.*

Proof. Let X be any subgroup of finite rank of G. Then XG' is properly contained in G, and so there exists a proper subgroup Y of G of infinite rank such that $XG' \leq Y$. Therefore X is nilpotent-by-finite. □

Lemma 4.2.2. *Let G be a group of infinite rank whose proper subgroups of infinite rank are nilpotent-by-finite. If G contains a proper subgroup which is not nilpotent-by-finite, then either G is perfect or G' is nilpotent.*

Proof. Assume that G is not perfect. Since G/G' has finite rank by Lemma 4.2.1, G' has infinite rank. Thus, there exists a proper characteristic subgroup N of G' such that N is nilpotent and $|G' : N|$ is finite. Clearly, G/N is an FC-group and has no proper subgroups of finite index. Then G/N is abelian and G' is nilpotent. □

The next lemma is the analogue of the Lemma 4.1.9.

Lemma 4.2.3. *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G are nilpotent-by-finite, then G is either nilpotent-by-finite or periodic.*

Proof. Assume for a contradiction that the group G is neither nilpotent-by-finite nor periodic. In particular, G has no proper subgroups of finite index and so it cannot be finitely generated. As finitely generated nilpotent-by-finite groups have obviously finite rank, it follows from the hypotheses that every finitely generated subgroup of G has finite rank. Moreover, it is also clear that each proper subgroup of G is nilpotent-by-(finite rank). Then, by Theorem 4.1.3 there exists a nilpotent normal subgroup K such that G/K has finite rank, so that an application of Lemma 4.1.8 yields that G is not perfect. Lemma 1.3.1 implies that G contains a proper subgroup which is not nilpotent-by-finite. Then G/G' must have finite rank by Lemma 4.2.1, and G' is a nilpotent group of infinite rank by Lemma 4.2.2.

Suppose first that the group G/G' is not periodic. Since G/G' is divisible, G can be decomposed into the product of two proper normal subgroups of infinite rank. Then G is nilpotent-by-finite, and so nilpotent. Therefore G/G' must be periodic, and hence G' is not periodic. Let T be the subgroup consisting of all elements of finite order of G'. If T has infinite rank, then all proper subgroups of G/T are nilpotent-by-finite, and hence the non-periodic group G/T is nilpotent-by-finite. Then G/T is nilpotent and $(G/T)/(G/T)' \simeq G/G'$ is periodic, hence G/T is periodic. Therefore T has finite rank, and the factor group G/T is likewise a counterexample, so that without loss of generality it can be assumed that G' is a torsion-free group. Let L be the numerator of the torsion subgroup of G'/G'', then G'/L is abelian and torsion-free. If q_1 and q_2 are distinct prime numbers, by Lemma 1.2.6 there exists a G-invariant subgroup N/L of G'/L such that $(G'/L)/(N/L) \simeq G'/N$ is periodic and q_1,q_2 belong to $\pi(G'/N)$. Since G'/L has infinite rank, N/L has infinite rank and all proper subgroups of G/N are nilpotent-by-finite. If G/N is nilpotent-by-finite, then G/N is nilpotent and $(G/N)/(G/N)' \simeq G/G'$ is periodic and hence G/N is periodic too. It follows that

$$G/N = G_1/N \times G_2/N,$$

where G_1/N is a p-group and G_2/N is a p'-group, for some prime number p. Since N has infinite rank, G_1 and G_2 have inifinite rank. This implies that G is nilpotent-by-finite. Hence G/N is a periodic minimal non-(nilpotent-by-finite) group. If G/N is not locally nilpotent, by Corollary 1.3.7 G'/N is a q-group, with q prime number. So G/N is locally nilpotent, and by Proposition 1.3.9 follows that G/N is a p-group, with p prime number. This last contradiction completes the proof. □

Now we are able to prove the main theorem of this paragraph.

Theorem 4.2.4. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are nilpotent-by-finite. Then all proper subgroups of G are nilpotent-by-finite.*

Proof. Assume for a contradiction that the statement is false. In particular, G is not nilpotent-by-finite, and so it has no proper subgroups of finite index.

Moreover, G is periodic by Lemma 4.2.3, and hence locally finite. Suppose that G contains a proper normal subgroup K such that G/K is simple. Then G/K must have infinite rank by Lemma 4.1.8 and hence it is isomorphic either to PSL(2,F) or to Sz(F), for some infinite locally finite field F containing no infinite proper subfields by Lemma 4.1.1. On the other hand, each of the last two groups contains a proper subgroup of infinite rank which is not nilpotent-by-finite (see for instance [23]). This contradiction shows that G has no infinite simple homomorphic images, so that either G contains a nilpotent normal subgroup N such that G/N has finite rank or G is a perfect p-group by Theorem 4.1.2. If G is a perfect p-group, every proper subgroup of G is nilpotent-by-Černikov , so G itself is nilpotent-by-Černikov (see [1], Teorema 1.3) . Therefore G is nilpotent-by-finite rank and by Lemma 4.1.8 it follows that G' is a proper subgroup. Clearly, from Lemmas 4.2.1 and 4.2.2 we have that G/G' has finite rank and G' is a nilpotent subgroup of infinite rank.

Let's prove that G' is abelian.

Assume, by contradiction, that $G'' \neq \{1\}$.

If G'' has infinite rank, let X be a subgroup of finite rank of G'' which is not nilpotent-by-finite. Since X cannot be contained in any proper subgroup of infinite rank, it follows that $G = XG''$. G/G'' is a group of finite rank and has no proper subgroup of finite index, so that it is nilpotent (see [26] part 2, Corollary 3 pag. 129), and hence it is abelian.

Now, let's suppose that G'' has finite rank, and let k be the derived lenght of G. Let A be the last term of the derived series of G. A is an abelian subgroup of finite rank, so it is the direct product of its Černikov primary components A_p. Every $C_G(A_p)$ has finite index in G, hence $A_p \leq Z(G)$, for every component A_p. Thus A is contained in the centre of G. Note that the quotient G/A does not have all proper subgroups nilpotent-by-finite. In fact, if X is a subgroup of finite rank which is not nilpotent-by-finite, XA is a proper subgroup of G and XA/A is not nilpotent-by-finite. Then G/A is a counterexemple, and we may suppose that G has derived lenght $k-1$. Repeating the reasoning finitely many times we have that the derived series of G'' is G-central, that is $G'' \leq Z_n(G)$. Obviously, G/G'' is a counterexemple, and we may suppose that $G'' = \{1\}$. Let X be a subgroup of finite rank of G which is not nilpotent-by-finite. Clearly, X cannot be contained in any proper subgroup of infinite rank of G, and so $G = XG'$. Then $X/X \cap G' \simeq G/G'$ has no proper subgroups of finite index, and hence it is divisible. Moreover, the abelian normal subgroup $X \cap G'$ of G has finite rank, so it is the direct product of its primary components, and hence it is covered by finite characteristic subgroups. Then $X/C_X(X \cap G')$ is residually finite, since $X/X \cap G'$ contains no proper subgroups of finite index, it follows that $X \cap G'$ is contained in the center of X, and so X is nilpotent. □

Also in this case is not clear if our main theorem can be extended to arbitrary locally graded groups. In any case we are able to prove our result in the class of strongly locally graded groups.

Corollary 4.2.5. *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are nilpotent-by-finite. Then all proper subgroups of G are nilpotent-by-finite.*

Proof. The group G is locally (soluble-by-finite) by Lemma 4.1.11, and so the statement is a direct consequence of our Theorem 4.2.4. □

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