Università degli Studi di Napoli *Federico II*

DOTTORATO DI RICERCA IN

FISICA FONDAMENTALE ED APPLICATA

Ciclo XXVII

Coordinatore: prof. Raffaele Velotta

Particle Physics and Symmetries in Noncommutative Geometry

Settore Scientifico Disciplinare Fis/02

Dottorando
Agostino Devastato

Tutore
Prof. Fedele Lizzi

Anni 2012/2015

Contents

Introduction

The purpose of this thesis is the description of some important physical mechanisms of the four foundamental interactions and how they emerge in the context of Connes' noncommutative geometry. This is a modern mathematical approach that generalizes the usual geometric concepts such as point, distance, integration in a totally algebraic language. Noncommutative geometry $-$ or also spectral geometry $-$ is a bottom-up approach describing the well-established standard model of high-energy physics in a gravitational background given by Einstein's general relativity. Its basic ideas have been nicely organized in a mathematical framework by Connes in the 1980s [26]. The key elements of this new mathematical structure are three algebraic objects - a C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a generalization of the Dirac operator D - naturally appearing in some elementary applications of quantum field theory. In fact, the set of quantum observables (position, time, energy) forms an associative algebra; spinors, which describe the system, belong to an Hilbert space and the dynamic is determined by an operator $\partial\!\!\!/$, said Dirac operator. As we will see further in the course of this dissertation, the roots of noncommutative geometry are, therefore, inherent to quantum mechanics.

One of the outstanding issues in modern theoretical physics is the unification of the four fundamental interactions in a single framework capable to describe all interactions as different aspects of a single theory. This unification is currently being attempted in a variety of ways, among which the one described here, where the Lagrangian of the proper classical Yang-Mills model coupled with gravity is a natural choice in a generalize geometrical framework.

The noncommutative description of the standard model could recall the original Kaluza-Klein theories [58, 59]. In fact, one starts with a product of ordinary four-dimensional space-time $\mathcal{M}$ with an internal space F describing the gauge content of the theory

$$\mathcal{M} \times F$$

in which, of course, space-time itself still describes the gravitational part. The main difference with Kaluza-Klein theories is that the additional space F is a discrete zero dimensional space whose structure is described by a noncommutative algebra, therefore, noncommutative geometry does not require the introduction of extra space-time dimensions. We describe F by matrices yielding an algebraic structure with multiplication given by ordinary matrix multiplication:

$$(AB)_{ij}(x) = \sum_k A_{ik}(x)B_{kj}(x),\ \forall A, B \in \mathcal{M} \times F$$

and the entries of matrices are functions of the space-time $\mathcal{M}$. The corresponding matrix algebra of coordinates on F is typically $\mathbb{M}_N(\mathbb{C})$ or direct sums of its subgroups, like quaternions $\mathbb{H}$. It turns out, as we will see below, that a metric on F can also be described in terms of algebraic data. In this way, we can fully describe the geometrical structure of $\mathcal{M} \times F$. We call this type of noncommutative manifolds *almost-commutative manifolds.*

Let we point out the difference of these spaces with another type of noncommutative spaces, such as the Moyal plane, for which the noncommutative structure is implemented on the space-time itself:

$$[x_\mu, x_\nu] = i\hbar\theta_{\mu\nu}.$$

We stress that although such spaces fit in the framework of noncommutative geometry - see for instance [29, 30] for the compact and [46, 47] for the non-compact case - this is not the type of noncommutativity that we are dealing in the following.

In this dissertation, we will give several examples of almost-commutative manifolds of interest in physics, showing how they can improve the knowledge of some fundamental elementary particle properties. Why the Higgs mass has an experimental value so low? Its potential is stable? The gauge interactions can be unified into a single point? These are some questions that we will try to answer in the course of this thesis. In some cases, like the gauge unification problem, it will be sufficient to consider simple extensions of the existing models. However, in other cases, like the correct prediction of the Higgs mass, we will need to build a larger mathematical model that could reveal more structure and more properties than the existing one.

In particular, after a general summary, in chapter 1, on the foundamental interactions and some related open questions, in chapter 2 we derive the Glashow-Weinberg-Salam electroweak theory, and the full standard model including the Higgs mechanism from noncommutative spaces. Naturally, to do this, we first need to develop the mathematical framework of spectral geometry, based on the concept of spectral triple, leading to the construction of noncommutative manifolds. Then, we show how, in spectral geometry, the three standard model interactions - electromagnetic, weak and strong - are coupled to the gravitational force. This is accomplished by the spectral action principle, that is a simple counting of the eigenvalues of a Dirac operator on $\mathcal{M} \times F$ lower than a cutoff Λ; the spectral action allows to construct a Lagrangian from the geometry of $\mathcal{M} \times F$, i.e. the right one for the standard model, in addition minimally coupled to gravity.

Chapter 3 contains the main part of the dissertation: in the attempt to derive the correct mass of the Higgs boson, and at the same time to solve one of the open questions of the standard model, i.e. the instability of the electroweak vacuum, we introduce the *grand symmetry* model. It is an almost commutative model, derived from an algebra larger than the one of the standard model, which allows to obtain an additional scalar field, usually called σ, coupled to the Higgs field in the action. This new field σ both stabilizes the electroweak vacuum and makes the computation of the mass of the Higgs compatible with its experimental value. However, in the usual *spectral triple* approach, the breaking of this grand symmetry to the standard model is accomplished by a mathematical requirement. To overcome this limit and to cure a technical problem of the grand symmetry, that is the appearance together with the extra scalar field σ of unbounded vectorial terms, in chapter 4, we introduce *Connes-Moscovici twisted spectral triples*. The twist makes these terms bounded, and also permits to understand the breaking to the standard model as a dynamical process induced by the spectral action. This is a spontaneous breaking from a pre-geometric Pati-Salam model to the almost-commutative geometry of the standard model, with two Higgs-like fields: scalar and vector.

In the last two chapters we focus other two problems of the spectral action: in its present form it requires the unification of the three gauge couplings at a single scale, Λ, and physical predictions are based on the value of this scale. It is known experimentally that in the absence of new physics the three constants do not meet in a single point, but the three lines form an elongated triangle spanning nearly four orders of magnitude, between $10^{13} - 10^{17} GeV$. Therefore, in chapter 5, we want to investigate whether the presence of higher dimensional terms in the standard model Lagrangian - coming from the spectral action expansion up to dimension six terms - may cause the unification of the coupling constants. This chapter may be read in two contexts: as an application of the spectral action, or independently on it, from a purely phenomenologically point of view.

On the other side in the spectral action is not clear what would happen after the unification scale, if one considers scales higher than Λ, i.e. earlier epochs. For a theory dealing with the unification of gauge theory and gravity a more natural scale is the Planck scale. For this reason, in chapter 6, we study the interaction between the gravitational force and the other three foundamental interactions. In particular we show how the gravitational effects change the main running coupling constants and if they lead to a restriction on the free parameters of the theory still compatible with the Higgs, top and neutrino mass predictions.

In the end, a conclusion section contains some comments and future possible research topics.

Chapter 1

Standard Model, Gravity and open questions

The Standard Model of particle physics is a gauge quantum field theory based on the unitary group $U(1) \times SU(2) \times SU(3)$, concerning the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the known subatomic particles. At present, it describes all known fundamental forces, excluding gravity. The standard model is certainly not a complete theory for fundamental interactions. Although it has demonstrated huge and continued successes in providing experimental predictions, it does leave some phenomena unexplained: it does not incorporate the full theory of gravitation as described by general relativity. It also does not contain any viable dark matter particle that possesses all of the required properties deduced from observational cosmology. In its original formulation, it does not incorporate neutrino oscillations and their non-zero masses, although one can simply extend the model by addressing this lack, [48]

From a theoretical point of view, the standard model is an excellent laboratory of study, since it is a paradigm of a quantum field theory which exhibits a wide range of physics including spontaneous symmetry breaking, anomalies, non-perturbative behavior, etc. It is used as a basis for building more exotic models that incorporate hypothetical particles, extra dimensions, and elaborate symmetries (such as supersymmetry) in an attempt to explain experimental results at variance with it.

In the following sections, we will discuss the standard model starting by its mathematical base, i.e. the Yang-Mills gauge theory. Then we give the standard model action, explaining the electroweak model of Weinberg and Salam and the QCD lagrangian. In §1.4, we discuss the consequences of the renormalization group flow on the main coupling constants of the model without discuss on how obtain this, since the calculation is standard and can be found in any usual textbook on quantum field theory. Then, in §1.5 we introduce the gravitational force, giving the basic ingredients of General Relativity and formulating the gravitational action. Finally, in §1.6 we show how is it possible the coupling of the gravitational field to the other fundamental fields. In the second chapter, with the spectral action approach, we can show how to put gravitation and standard model on the same footing, deriving them from the same principle.

1.1 Yang-Mills theories

We begin with the fermion field ψ_i transforming in some representation of $SU(N)$, not necessarily the fundamental one. It transforms as[1] $SU(N)$,

$$\psi_i(x) \longrightarrow \Omega_{ij}(x)\psi_j(x)\,, \tag{1.1}$$

where Ω_{ij} is an element of $SU(N)$. Let τ^a be the generators on its Lie algebra:

$$[\tau^a, \tau^b] = if^{abc}\tau^c, \tag{1.2}$$

where the parameters f_{abc} are called *structure constants* of the Lie group and determine the Lie brackets of elements of the Lie algebra, and consequently nearly completely determine the group structure of the group.

Essential point is that the group element Ω is now a function of space-time; that is, it changes at every point in the space. It can be parametrized as:

$$\Omega_{ij}(x) = \left(e^{i\theta^a(x)\tau^a}\right)_{ij} \Rightarrow \psi_i^{'} = \left(e^{i\theta^a(x)\tau^a}\right)_{ij}\psi_j \tag{1.3}$$

where the parameters $\theta^a(x)$ are local variables, and where τ^a is defined in whatever representation we are analyzing.

The problem with this construction is that derivatives of the fermion field are not covariant under this transformation. A naive transformation of the derivatives of these fields picks up terms like $\partial_\mu\Omega_{ij}$,

$$(\partial_\mu\psi_i) \longrightarrow (\partial_\mu\psi_i') = \Omega_{ij}\left(\partial_\mu\psi_j\right) + \left(\partial_\mu\Omega_{ij}\right)\psi_j\,. \tag{1.4}$$

In order to cancel the second unwanted term, we would like to introduce a new derivative operator D_μ that is truly covariant under the group. To construct such an operator D_μ, let us introduce a new field, called the *connection* A_μ:

$$D_\mu = \partial_\mu - igA_\mu \tag{1.5}$$

[1]‡A practical summary of the foundamental elements of group theory can be found in [60, chap.2]

with

$$A_\mu(x) \equiv A^a_\mu(x)\tau^a. \tag{1.6}$$

If we set $A_\mu(x)$ to transform according the rule

$$A'_\mu(x) = \Omega A_\mu(x)\Omega^{-1} - \frac{i}{g}(\partial_\mu\Omega)\Omega^{-1} \tag{1.7}$$

then the covariant derivative of the ψ field is gauge covariant:

$$\begin{aligned}(D_\mu\psi)' &= \partial_\mu\psi' - igA'_\mu\psi' \\ &= \Omega(\partial_\mu\psi) + (\partial_\mu\Omega)\psi - igA'_\mu\Omega\psi \\ &= \Omega(D_\mu\psi).\end{aligned} \tag{1.8}$$

Infinitesimally, this becomes

$$\begin{cases}\delta\psi = ig\theta^a\tau^a\psi \\ \delta A^a_\mu = -\frac{i}{g}\partial_\mu\theta^a + f^{abc}\theta^b A^c_\mu\end{cases} \tag{1.9}$$

Example, $U(1)$**,** $SU(2)$**,** $SU(3)$

1. As first example, let us consider the unitary group $U(1)$. From this group we recover the field transformation for QED. It has a unique generator, τ, that in this case we set equal to -1. This means $\Omega = e^{-i\theta(x)}$, and the fermion field transforms as

   $$U(1):\ \psi' = e^{-i\theta(x)}\psi \tag{1.10}$$

   By setting the gauge coupling g equal to the usual electromagnetic coupling constant e, $g \equiv -e$, we obtain for the covariant derivative,

   $$D_\mu = \partial_\mu + ieA_\mu\,. \tag{1.11}$$

   Then we have $\partial_\mu\Omega = i(\partial_\mu\theta)e^{i\theta}$ and the connection $A_\mu$becomes,

   $$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta\,, \tag{1.12}$$

   recovering the field transformations for QED.

2. Now we analyze the case $SU(N) = SU(2)$. The generators of this group are the 2×2 Pauli matrices $\tau^a = \sigma^a/2$ whose Lie algebra is given by the commutation rules:

   $$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2}\ \text{, con } i,j,k = 1,2,3. \tag{1.13}$$

Therefore, in this case[2] $\Omega = e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}$,

$$SU(2):\ \psi' = e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\psi \tag{1.14}$$

and the covariant derivative takes the form

$$D_\mu\psi = \partial_\mu\psi - \frac{i}{2}g\mathbf{A}_\mu\psi \tag{1.15}$$

with the connection $\boldsymbol{A_\mu}$ in matrix form that means

$$\boldsymbol{A}_\mu = \sigma^i A^i_\mu = \begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix} \tag{1.16}$$

Since we have $\partial_\mu\Omega = \frac{i}{2}\left(\sigma\cdot\partial_\mu\theta\right)\Omega$, it is easy to show that, for infinitesimal θ^i, the transformation (1.7) becomes

$$\mathbf{A}^{'}_\mu = \mathbf{A}_\mu - \boldsymbol{\theta}\times\mathbf{A}_\mu + \frac{1}{g}\partial_\mu\boldsymbol{\theta}. \tag{1.17}$$

3. Finally, we can see the *color* group $SU(3)$. Its generators are the eight 3×3 Gell-Mann matrices $\tau^a = \lambda^a/2$, whose Lie algebra obeys the commutation rules:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2}\ ,\ \text{for} a, b, c = 1, 2, ...8 \tag{1.18}$$

the structure constants f_{abc} are totally antisymmetric in their indices and the only non zero components are

$$\begin{cases} f_{123} = 1 \\ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2 \\ f_{458} = f_{678} = \frac{\sqrt{3}}{2} \end{cases} \tag{1.19}$$

Therefore, $\Omega = e^{i\frac{\lambda^a}{2}\theta^a}$ and the covariant derivative takes the form,

$$D_\mu\psi = \partial_\mu\psi - \frac{i}{2}g\lambda^a\mathcal{A}^a_\mu\psi \tag{1.20}$$

The connection A_μ in matrix form can be written

$$\boldsymbol{\mathcal{A}}_\mu = \mathcal{A}^a_\mu\frac{\lambda^a}{2} = \begin{pmatrix} A^3_\mu + \frac{1}{\sqrt{3}}A^8_\mu & A^1_\mu - iA^2_\mu & A^4_\mu - iA^5_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu + \frac{1}{\sqrt{3}}A^8_\mu & A^6_\mu - iA^7_\mu \\ A^4_\mu + iA^5_\mu & A^6_\mu + iA^7_\mu & -\frac{2}{\sqrt{3}}A^8_\mu \end{pmatrix} \tag{1.21}$$

[2] The scalar product between the bold symbols means by definition $\boldsymbol{\theta}\cdot\boldsymbol{\sigma} = \sum_{j=1}^{3}\theta^j\sigma^j$

1.2 Field strength and Yang-Mills action

It is also possible to construct the invariant action for the connection field itself. Since D_μ is covariant, then the commutator of two covariant derivatives is also covariant. We define this commutator as *field strength (or gauge field)* $G_{\mu\nu}$:

$$\begin{aligned} G_{\mu\nu} &\equiv \frac{i}{g}[D_\mu,\, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)\tau^a \,. \end{aligned} \tag{1.22}$$

Because D_μ is genuinely covariant this means that the $G^a_{\mu\nu}$ tensor is also covariant:

$$G_{\mu\nu} \to \Omega\, G_{\mu\nu} \Omega^{-1} \tag{1.23}$$

Now we can construct an invariant action out of this tensor. We want an action that only has two derivatives (since actions with three or higher derivatives are not unitary, i.e. they have ghosts). The simplest invariant is given by the trace of the commutator. This is invariant because:

$$\mathrm{Tr}(\Omega\, G_{\mu\nu} S^{-1} \Omega\, G^{\mu\nu} \Omega^{-1}) = \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}) \tag{1.24}$$

The unique action with only two derivatives is therefore given by:

$$\mathcal{S}_B = \int d^4x \left(-\frac{1}{4} \mathrm{Tr} G_{\mu\nu} G^{\mu\nu} \right) = \int d^4x \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right) . \tag{1.25}$$

This is the action for the Yang-Mills theory, which is the starting point for all discussion of gauge theory.

The field tensor $G_{\mu\nu}$, we want point out, obeys the Bianchi identities. We know, by the Jacobi identity, that certain multiple commutators vanish identically. Therefore we have:

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] \equiv 0 \tag{1.26}$$

This is easily checked by explicitly writing out the terms in the commutators. Written in terms of the field tensor, this becomes:

$$[D_\mu, G_{\nu\rho}] + [D_\nu, G_{\rho\mu}] + [D_\rho, G_{\mu\nu}] \equiv 0 \tag{1.27}$$

It is important to stress that these are exact identities. They are not equations of motion, nor are they new constraints on the field tensor.

Lastly, since $\bar\psi \longrightarrow \bar\psi \Omega^\dagger(x)$ and $D_\mu \psi \to \Omega D_\mu \psi$, we can define the invariant fermion action, coupled to the gauge field, as:

$$\mathcal{S}_F = \int d^4x\, \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{1.28}$$

therefore the final action will be

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_B + \mathcal{S}_F \\ &= \int d^4x \left[\overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} \right] \end{aligned} \tag{1.29}$$

For example, a gauge field theory containing the internal symmetries of the unitary product group $SU(3) \times SU(2) \times U(1)$ will take the form

$$\begin{aligned} S &= \int d^4x \left[i\bar{\psi}\gamma^\mu D_\mu \psi \right] + \\ & \int d^4x \left[-\frac{1}{4}\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu} - \frac{1}{4}\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \end{aligned} \tag{1.30}$$

with $D_\mu\psi = \left(\partial_\mu + ig_1 A_\mu - i\frac{g_2}{2}\,\boldsymbol{\sigma}\boldsymbol{\cdot}\,\boldsymbol{W}_\mu - i\frac{g_3}{2}\,\boldsymbol{\lambda}\boldsymbol{\cdot}\,\boldsymbol{G}_\mu\right)\psi$. The standard model of particle physics is just a gauge quantum field theory containing the internal symmetries of this unitary product group. The full standard model action will be given in the next section, together with an explanation of the form of the Hilbert space containing the spinor ψ.

1.3 Standard model action

The standard model is the gauge quantum field theory based on the unitary group $SU(3) \times SU(2) \times\ U(1)$. It is one of the greatest successes of the gauge revolution. At present, this model can describe all known fundamental forces, excluding gravity.

It is given by crudely splicing the electroweak theory and the theory of quantum chromodynamic (QCD), taking the sum of their respective actions,

$$S_{sm} = S_{e.w.} + S_{QCD} \tag{1.31}$$

where

$$S_{QCD} = \int d^4x \left[-\frac{1}{4}G^{a\,\mu\nu}G^a_{\mu\nu} + \sum_{\alpha=1}^{6} \bar{Q}^\alpha_i (i\gamma^\mu D_\mu - m) Q^\alpha_i \right] \tag{1.32}$$

$$\begin{aligned} S_{e.w.} &= \int d^4x \left[-\frac{1}{4}W^{a\,\mu\nu}W^a_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{R}\gamma^\mu D_\mu R + i\bar{L}\gamma^\mu D_\mu L \right] + \\ & \int d^4x \left[D_\mu H^\dagger D^\mu H - m^2 H^\dagger H - \lambda \left(H^\dagger H\right)^2 \right] + \\ & \int d^4x \left[G_e(\overline{L}\phi R + \overline{R}\phi^\dagger L) \right] . \end{aligned} \tag{1.33}$$

In (1.32), $G^a_{\mu\nu}$ are the massless Yang-Mills field associated to the unbroken “color” gauge group $SU(3)$, describing the strong interaction. The spinors Q^α_i

describe the quarks having two indices. The α index is taken over the flavors, which labels the *up, down, strange, charm, top, and bottom quarks.* The flavor index is not gauged; it represents a global symmetry. However, the quarks also carry the important local color $SU(3)$ index, $i = 1, 2, 3$. In other words, quarks come in six flavors and three colors, but only the color index participates in the local gauge symmetry. Synthetically,

$$\begin{pmatrix} u^1 & u^2 & u^3 \\ d^1 & d^2 & d^3 \\ s^1 & s^2 & s^3 \\ c^1 & c^2 & c^3 \\ t^1 & t^2 & t^3 \\ b^1 & b^2 & b^3 \end{pmatrix} \tag{1.34}$$

where 1, 2, 3 are the color indeces, sometimes also labeled as R, W, B.

1.3.1 Electroweak action

The electroweak model (or Weinberg-Salam model), one of the most successful quantum theories besides the original QED, is a curious amalgam of the weak and electromagnetic interactions. Strictly speaking, it is not a "unified field theory" of the weak and electromagnetic forces, since we must introduce two distinct coupling constants g and g' for the $SU(2)$ and $U(1)$ interactions. Nonetheless, it represents the one of the most important extensions of QED in the past century. The model we explain here, differently from the original work of Weinberg and Salam, takes into account also the right-handed neutrino, since we now know they are massive particle too.

Let us consider only one leptonic generation, i.e. the doublet $\begin{pmatrix} \nu_e \\ e \end{pmatrix}$; the same considerations hold for $\mu-$ and $\tau-$families.
As already mentioned in (1.33), the electroweak model is described by an action invariant under the group $SU(2) \times U(1)$,

$$\begin{aligned} \mathcal{L}_{e.w.} \;\; = \;\; & -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ & +i\overline{R}_e\gamma \cdot D\, R_e + i\overline{R}_\nu\gamma \cdot D\, R_\nu + i\overline{L}\gamma \cdot D\, L \\ & +D_\mu H^\dagger D^\mu H - m^2 H^\dagger H - \lambda \left(H^\dagger H\right)^2 + \\ & +G_e(\overline{L}\phi R + \overline{R}\phi^\dagger L)\,. \end{aligned} \tag{1.35}$$

with

$$\begin{aligned}
W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu \\
F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\
D_\mu R_e &= (\partial_\mu + ig' B_\mu) R_e \\
D_\mu R_\nu &= \partial_\mu R_\nu \\
D_\mu L &= (\partial_\mu + \frac{i}{2} g' B_\mu - \frac{i}{2} g \sigma_i W^i_\mu) L \,,
\end{aligned} \tag{1.36}$$

the isospinor $L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, containing the Weyl left-handed neutrino and electron, forms an $SU(2)$ isodoublet with non-abelian charge $I_w = \frac{1}{2}$; $I^3_w(\nu_e) = \frac{1}{2}$ e $I^3_w(e_L) = -\frac{1}{2}$. This quantum number is called *weak isospin charge*. On the other hand, $R_e \equiv (e_R)$ and $R_\nu \equiv (\nu_R)$, containing the Weyl right-handed electron and neutrino, are both isospin singlet with charge $I_w = 0$ e ${I^3}_w = 0$. Therefore, these two sectors transform in a different way under $SU(2)$:

$$SU(2) : \begin{cases} L \to e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} L \\ R_e \to R_e \\ R_\nu \to R_\nu \end{cases} \tag{1.37}$$

The quantum number associated to the abelian group $U(1)$ is the *hypercharge* Y_w , which takes the values $Y_w = -1$ for the isospinor L, $Y_w = -2$ for R_e, e $Y_w = 0$ for R_ν, so that the experimental Gell-Mann-Nishijima relation

$$Q = I^3_w + \frac{Y_w}{2} \tag{1.38}$$

holds. Therefore, the correct transformation for L, R_e,R_ν under $U(1)$ will be

$$U(1) : \begin{cases} L \to e^{\frac{i}{2}\beta} L \\ R_e \to e^{i\beta} R_e \\ R_\nu \to R_\nu \end{cases} \tag{1.39}$$

The third row in 1.35 contains the scalar Higgs sector. The scalar multiplet H is a complex isodoublet given by,

$$H \equiv \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \tag{1.40}$$

where the doublet has charge $(1,0)$ which can be given by $Q = I^3_w + Y_w/2$ such that $\left(I_w = \pm\frac{1}{2}, Y_w = 1\right)$

Symmetry breaking is induced by the vacuum expectation value v

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{1.41}$$

After symmetry breaking, the fields W_μ^a and B_μ recombine and emerges the physical photon A_μ, a neutral massive vector particle Z_μ, and a charged doublet of massive vector particles $W_\mu^\pm$:

$$\begin{aligned} Z_\mu &= \frac{gW_\mu^3 + g'B_\mu}{(g^2+g'^2)^{1/2}} \equiv \cos\theta_W W_\mu^3 + \sin\theta_W B_\mu \\ A_\mu &= \frac{gB_\mu - g'W_\mu^3}{(g^2+g'^2)^{1/2}} \equiv -\sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \\ W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2) \end{aligned} \tag{1.42}$$

where the Weinberg angle θ_W is defined via:

$$\cos\theta_W \equiv \frac{g}{\sqrt{g^2+g'^2}}\,. \tag{1.43}$$

By examining the mass sector, we can read off the masses of the resulting vector particles:

$$\begin{aligned} M_{W_1}^2 &= M_{W_2}^2 = \frac{g^2v^2}{2} \\ M_Z^2 &= \frac{M_W^2}{\cos^2\theta_W} \\ M_A &= 0 \\ M_H^2 &= 2\lambda v^2 \end{aligned} \tag{1.44}$$

After the symmetry breaking, the Yukawa couplings of the fourth row in 1.35 give mass to the fermions:

$$\begin{aligned} M_e &= y_e v/\sqrt{2} \\ M_\nu &= y_\nu v/\sqrt{2} \end{aligned} \tag{1.45}$$

finally, the electric charge emerges as

$$e = g\, sin\theta_W \tag{1.46}$$

1.4 Unification point and instability problem

The standard model successfully incorporates all the known properties of the strong, weak and electromagnetic forces. In 2012 there has been a great success for it, i.e. the discovery of the Higgs boson at the LHC, giving validity to the electro-weak theory, that is one of the most important mechanism of the standard model. However, if on the one hand this discovery gave great

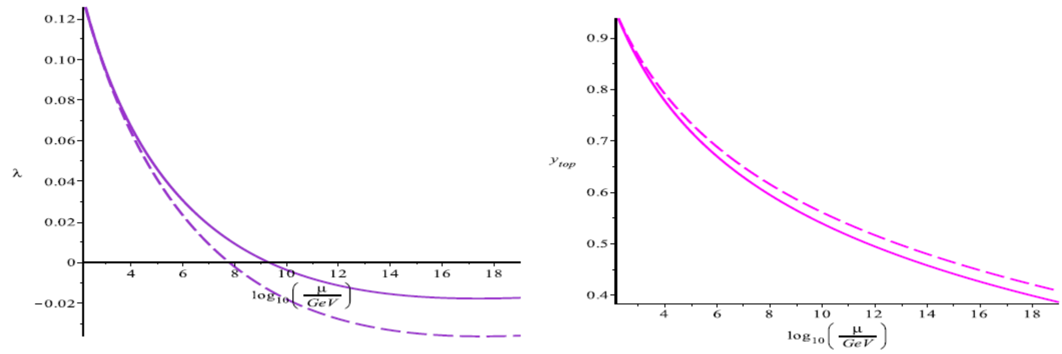

Figure 1.1: The running of the autointeraction parameter λ and of the top Yukawa coupling. The dashed and solid lines represent the one and two loop respectively.

enthusiasm to the scientific community, on the other hand the value of the Higgs boson mass $\simeq 125GeV$ has raised some questions about the completeness of the model: it leads to an instability of the electroweak vaccum [54, 81], as explained in the following.

The various couplings of the standard model run with the energy [12], as dictated by the renormalization group. In particular let us discuss the evolution of the auto-interaction coefficient λ of the quartic term $(\bar{H}H)^2$ in the Higgs sector. We skip the discussion on how to obtain this, since the calculation is standard and can be found in any usual textbook on Quantum Field Theory (see for example [60, 80]).

At one-loop its running is given by:

$$\mu\frac{d}{d\mu}\lambda = \frac{1}{16\pi^2}\left(24\lambda^2 - \left(3g_1^2 + 9g_2^2\right)\lambda + \frac{3}{8}\left(g_1^4 + 2g_1^2g_2^2 + 3g_2^4\right) + 12y_{top}^2\lambda - 6y_{top}^4\right) \tag{1.47}$$

and it depends on the gauge couplings (whose running is described below), as well as y_{top}, the Yukawa coupling of the top quark that, in turn, runs with equation,

$$\mu\frac{d}{d\mu}y_{top} = \frac{y_{top}}{16\pi^2}\left(\frac{9}{2}y_{top}^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right) \tag{1.48}$$

The solution of these running with the boundary conditions given by experimental values are in fig. 1.1; we have used

$$y_{top} = 0.937\ ,\ \ \lambda = 0.126 \tag{1.49}$$

for $M_H = 125GeV$ at the scale of the top mass $M_{top} = 172.9GeV$. The important aspect is the fact that λ becomes negative at a scale of the order of $10^{10}GeV$. Two loop calculations make the situation slightly worse. A negative

λ means an instability of the vacuum, that could make the whole model inconsistent, if it was not that the vacuum lifetime is longer than the age of the Universe. Therefore, one says that with a Higgs mass in the range 124−126GeV and the current central value of the top mass (173GeV), the Higgs potential develops a *metastability* around $10^{10} - 10^{11}$ GeV, i.e. an electroweak vacuum with a lifetime much longer than the age of the Universe.

Another important aspect of the standard model is the expectation of the unification of the three foundamental forces. In fact, at extremal high energy the renormalization flow shows that the three coupling constants of the strong, weak, and electromagnetic interactions begin to converge, leading us to suspect that all the three interactions become part of the same interaction at a very high energy. The running of the three gauge coupling constants g_1, g_2 and g_3 is given by the following equations [10, appendix A],

$$\begin{aligned}
\mu\frac{dg_1}{d\mu} &= -\frac{g_1^3}{16\pi^2}\left(-\frac{1}{10}-\frac{2}{3}n_f\right) = \frac{g_1^3}{16\pi^2}\frac{41}{10} \\
\mu\frac{dg_2}{d\mu} &= -\frac{g_2^3}{16\pi^2}\left(\frac{43}{6}-\frac{2}{3}n_f\right) = -\frac{g_2^3}{16\pi^2}\frac{19}{6} \\
\mu\frac{dg_3}{d\mu} &= -\frac{g_3^3}{16\pi^2}\left(11-\frac{2}{3}n_f\right) = -\frac{g_3^3}{16\pi^2}7
\end{aligned} \tag{1.50}$$

The values of the b_i's are given by the number of charged fermions $n_f = 6$. The two nonabelian interactions have a different sign from the $U(1)$ coupling. At high energy they become asymptotically free. The abelian interaction has instead a Landau pole at very high energy, well above the Planck scale. At higher loops the functions will depend in a nonlinear way from the other couplings, including the parameters of the Higgs and the Yukawa couplings. In order to establish the running, low energy boundary conditions are necessary; they are experimental value and for fig. 1.2 we have taken

$$g_1 = 0.3575\,,\ g_2 = 0.6514\,,\ g_3 = 1.1221 \tag{1.51}$$

at the m_Z scale. Let us note that in fig. 1.2 we run $\sqrt{5/3}g_1$ in place of g_1 according to conventional $SU(5)$ normalization. Of relevance for us is the fact that the three coupling constants almost coincide at a single scale. The three lines create a triangle: the values of g_i go from approximatively 0.5 to 0.6. These are pure numbers and therefore the total span is about 25% of the values. But the scale at which this happens goes from 10^{14}GeV to 10^{17}GeV. A span of more than three orders of magnitude. Two and three loops calculations do not significantly alter these numbers. The presence of "new physics", in the form of new particles or new interactions could improve this aspect [5, 35, 72]. We discuss this point, in more detail, in chapter 6 studying the interaction

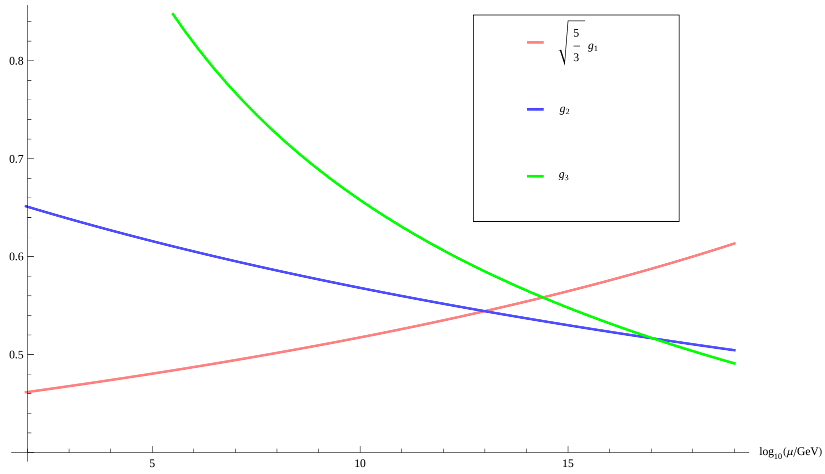

Figure 1.2: The running of g_i in the standard model picture.

with gravity at energies near the Planck scale and in chapter 5 considering the contribution of new interactions in the form of high-order operators in the action.

1.5 Gravitation

The basic elements of the theory of general relativity can be deduced following a path parallel to the gauge theories of Yang-Mills. The transformation group for which the action must be invariant is no longer a unitary group $SU(N)$, but the group of generic transformations of coordinates.

Given a coordinate system x^μ describing the space-time, a general coordinate transformation is an arbitrary reparametrization of kind

$$\overline{x}^\mu = \overline{x}^\mu(x^\nu)\,, \qquad \mu,\nu = 1,...,4 \tag{1.52}$$

Unlike Lorentz transformations, which are global space-time transformations, general coordinate transformations are local and they were one of the original inspirations leading Yang and Mills to postulate local gauge theories explained above.

Under reparametrizations, a scalar field transforms simply as follows:

$$\overline{\phi}(\overline{x}) = \phi(x) \tag{1.53}$$

Vectors transform like $\partial/\partial x^\mu$ or dx^μ. Using ordinary calculus, we can construct two types of vectors under general coordinate transformations: covariant vectors, like ∂_μ, and controvariant vectors, like dx^μ:

$$dx^\mu \longrightarrow d\overline{x}^\mu = \frac{\partial\overline{x}^\mu}{\partial x^\nu}dx^\nu \equiv \left(G^{-1}\right)^\mu_\nu dx^\nu \tag{1.54}$$

$$\frac{\partial}{\partial x^\mu} \longrightarrow \frac{\partial}{\partial\overline{x}^\mu} = \frac{\partial x^\nu}{\partial\overline{x}^\mu}\frac{\partial}{\partial x^\nu} \equiv G^\nu_\mu\frac{\partial}{\partial x^\nu} \tag{1.55}$$

Given these transformation laws, we can now give the abstract definition of covariant tensors, with lower indeces, and controvariant tensors, with upper indeces, depending on their transformation properties:

$$\begin{aligned}
\overline{\phi}(\overline{x}) &= \phi(x) \quad \text{scalar invariant}\\
\overline{B}_\mu(\overline{x}) &= \frac{\partial x^\nu}{\partial\overline{x}^\mu}B_\nu(x) \text{ covariant vector}\\
\overline{A}^\mu(\overline{x}) &= \frac{\partial\overline{x}^\mu}{\partial x^\nu}A^\nu(x) \text{ controvariant vector}
\end{aligned}$$

Since we have arbitrary coordinate transformations, these vectors transform under the group $GL(4)$, that is, arbitrary real 4×4 matrices.

Similarly we can construct tensors of arbitrary rank or indeces. They are objects transforming as the product of a series of first-rank tensors (vector):

$$\overline{T}^{\nu_1\nu_2\dots}_{\mu_1\mu_2\dots}(\overline{x}) = \prod_{i=1}^{m}\left(\frac{\partial x^{\mu_i}}{\partial\overline{x}^{\bar{\mu}_i}}\right)\prod_{j=1}^{n}\left(\frac{\partial\overline{x}^{\bar{\nu}_j}}{\partial x^{\nu_j}}\right)T^{\nu_1\nu_2\dots}_{\mu_1\mu_2\dots}(x) \tag{1.56}$$

For example, a controvariant tensor of rank 2, $T^{\mu\nu}$, transforms as

$$\overline{T}^{\mu\nu}(\overline{x}) = \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial\overline{x}^\nu}{\partial x^\beta}T^{\alpha\beta}(x) \tag{1.57}$$

We can also construct an invariant under general coordinate transformations by contracting controvariant tensors with covariant ones

$$A^\mu B_\mu = \overline{A}^\mu\overline{B}_\mu = \text{invariant} \tag{1.58}$$

We now introduce a metric tensor $g_{\mu\nu}$ that allow us to calculate distances on the space-time. Given two points separated by the infinitesimal distance dx^μ, the invariant scalar associated to this distance is given by

$$ds^2 = dx^\mu g_{\mu\nu}dx^\nu\,, \tag{1.59}$$

if $g_{\mu\nu}$ is defined to be a second-rank covariant tensor transforming as

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\alpha}{\partial\overline{x}^\mu}\frac{\partial x^\beta}{\partial\overline{x}^\nu}g_{\alpha\beta}(x)\,, \tag{1.60}$$

then the distance ds^2 is a genuine invariant.

Now that we have defined how scalar, vector, and tensor fields transform under reparametrizations, the next step is to write down derivatives of these fields that are also covariant. The derivative of a scalar field is a genuine tensor under general coordinate transformations:

$$\frac{\partial}{\partial \overline{x}^{\mu}}\bar{\phi}(\overline{x}) = \frac{\partial x^{\nu}}{\partial \overline{x}^{\mu}}\frac{\partial}{\partial x^{\nu}}\phi(x) \tag{1.61}$$

However, as in the case of gauge transformation §1.1 , we have to define a covariant derivative, introducing new fields, called connections which absorb the unwanted terms of the usual derivation. The connection field for general relativity is called the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$. We introduce the symbol ∇_{μ}, which is a covariant derivative

$$\begin{aligned} \nabla_{\mu}A_{\nu} &\equiv \partial_{\mu}A_{\nu} + \Gamma^{\lambda}_{\mu\nu}A_{\lambda} &(1.62)\\ \nabla_{\mu}A^{\nu} &\equiv \partial_{\mu}A^{\nu} - \Gamma^{\nu}_{\mu\lambda}A^{\lambda} &(1.63)\end{aligned}$$

We will define the transformation properties of the connection such that the derivative of a vector becomes a genuine tensor, paralleling the situation in gauge theories:

$$\overline{(\nabla_{\mu}A_{\nu})} = \left(\frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}\right)\nabla_{\alpha}A_{\beta} \tag{1.64}$$

Given the transformation law, we can, as in gauge theories, extract the transformation law for the Christoffel symbol:

$$\overline{\Gamma}^{\lambda}_{\mu\nu}(\overline{x}) = \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}\frac{\partial \overline{x}^{\lambda}}{\partial x^{\gamma}}\Gamma^{\gamma}_{\alpha\beta} + \frac{\partial x^{\sigma}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\rho}}{\partial \overline{x}^{\nu}}\frac{\partial^2 \overline{x}^{\lambda}}{\partial \overline{x}^{\sigma}\partial x^{\rho}} \ . \tag{1.65}$$

We see that the Christoffel symbol is not a genuine tensor, but has an inhomogeneous piece (we recall that the gauge field A^a_{μ} also has an inhomogeneous piece in its transformation under $SU(N)$).

1.5.1 General Relativity action

Now that we have defined the transformation properties of the fields and constructed covariant derivatives, the last step is to write down the action for general relativity and couple it to other fields. To construct the action, we will need to take the commutator between two covariant derivatives. In flat space, this commutator vanishes. However, for general coordinate transformations, we find that this commutator does not vanish. By explicit construction, we find

$$\begin{aligned} [\nabla_{\mu}, \nabla_{\nu}]\, A_{\lambda} &= R^{\rho}_{\mu\nu\lambda}A_{\rho} \\ R^{\rho}_{\mu\nu\lambda} &\equiv \partial_{\mu}\Gamma^{\rho}_{\nu\lambda} - \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} &(1.66)\end{aligned}$$

Yang-Mills gauge theory	***Einstein general relativity theory***
Gauge transformations on internal space $\psi' = (e^{i\theta^a(x)\tau^a})\psi$	Trasformazioni generiche di coordinate sullo spazio-tempo, $\overline{x}^\mu = \overline{x}^\mu(x^\nu)$
Symmetry group: unitary group $SU(N)$, $\Omega(x) = (e^{i\theta^a(x)\tau^a})$	Symmetry group: general linear group $GL(4)$, $G(x) = \frac{\partial \overline{x}^\nu}{\partial x^\mu}$
Connection: gauge potential, A_μ $A_\mu \to \overline{A}_\mu$as in (1.7)	Connection: Christoffel connection, $\Gamma^\lambda_{\mu\nu}$ $\Gamma^\lambda_{\mu\nu} \to \overline{\Gamma}^\lambda_{\mu\nu}$as in(1.65)
Field strength, gauge field $G_{\mu\nu}$ $G_{\mu\nu} = [D_\mu, D_\nu]$	Field strength, Riemann curvature tensor $R^\rho_{\mu\nu\lambda}$ $R^\rho_{\mu\nu\lambda} A_\rho = [\nabla_\mu, \nabla_\nu] A_\lambda$

Table 1.1: Parallels between gauge theory and general relativity

We call $R^\rho_{\mu\nu\lambda}$ the Riemann curvature tensor. From this we can see the close analogy between the elements of gauge theory and general relativity. This close correspondence can be synthesized in tab. 1.1.

By suitably contracting the indices in the curvature tensor, we can reduce it to tensors of smaller rank. Contracting ρ and ν gives us a second-rank curvature tensor called Ricci curvature tensor:

$$R_{\mu\lambda} = R^\rho_{\mu\nu\lambda}\delta^\nu_\rho \,. \tag{1.67}$$

Finally, we can construct a genuine invariant by contracting all the indeces:

$$R^\rho_{\mu\nu\lambda}\delta^\nu_\rho g^{\mu\lambda} = R \tag{1.68}$$

this is called Ricci scalar.

It is also easy to calculate the transformation properties of the determinant of the metric tensor, g. Because $\det(ABC) = \det A \, \det B \, \det C$, one can easly shows:

$$\sqrt{-\overline{g}(\overline{x})} = \det\left(\frac{\partial x^\mu}{\partial \overline{x}^\nu}\right)\sqrt{-g(x)} \tag{1.69}$$

An object that transforms like this is not a scalar in the usual sense, but we call it *scalar density.*

The point is that now the product of these two is a genuine invariant:

$$\sqrt{-g}d^4x = \text{invariante} \tag{1.70}$$

From this we can construct actions, fulfilling a few key condition:

1. The action must contain no more than two derivatives of $g_{\mu\nu}$, or else there are ghosts in the theory that threaten unitary.

2. The action must be invariant under general coordinate transformations.

3. It has to give equations of motion that, in the low energy limit, reduce to the Newton gravitational field equation.

The solution to these constrains is given by the celebrated Einstein-Hilbert action, which is the starting point for all calculations in general relativity:

$$S = -\frac{1}{2k^2} \int d^4x \sqrt{-g} R \tag{1.71}$$

(one can also add the cosmological term, which is proportional to $\sqrt{-g}\Lambda_{cosm}$, also experimentally Λ_{cosm} is very close to zero). Taking the variation of the action, we find the equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{1.72}$$

In the presence of matter fields, we must alter this equation. We know that scalar matter couples to gravity via the interaction

$$S_{matter} = \int d^4x \sqrt{-g} g^{\mu\nu} T_{\mu\nu} \sim \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{1.73}$$

therefore the right-hand side of the previous equation should contain the energy-momentum tensor:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi k}{c^2} T_{\mu\nu} \tag{1.74}$$

As required by the third point above, this equation reduces to the usual Newtonian potential equation in the limit $c \to \infty$. In this limit, the metric tensor becomes the Lorentz metric, except for the term g_{00}:

$$g_{00} \sim 1 + \phi \tag{1.75}$$

then the ϕ field becomes the scalar potential, and Einstein's equation reduces to Poisson's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi k}{c^2} T_{\mu\nu} \to \nabla^2 \phi = 4\pi k \rho \tag{1.76}$$

where ρ is the source term. From this, one can derive the Newton's original universal law of gravitation, that the gravitational force is proportional to the product of the masses and inversely proportional to the distance of separation squared.

1.6 Gravity coupled with other fields

The coupling of the gravitational field to the other fields is possible with the introduction of the so called *vierbein*.

The generally covariant action for scalar and Yang-Mills fields is very simple to write and is given by:

$$\begin{aligned} \mathcal{L}_{scalar} &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) \\ \mathcal{L}_{YM} &= -\frac{1}{4}\sqrt{-g}g^{\mu\sigma}g^{\nu\rho}F^a_{\mu\nu}F^a_{\rho\sigma} \end{aligned} \tag{1.77}$$

On the other hand, the coupling of gravity to spinor fields leads to an immediate difficulty: *there are no finite dimensional spinorial representation of* $GL(4)$. This prevents a naive incorporation of spinors into general relativity. There is fortunately, a trick that we may use to circumvent this problem. Although spinor representations do not exist for general covariance, there are, of course, spinorial representations of the Lorentz group. We utilize this fact and construct a flat tangent space at every point in the space. Imagine space-time as a rolling-hill. Then the tangent space would correspond to placing a flat plane on each point of the hill. Spinors can then be defined at any point on the curved manifold only if they transform within the flat tangent space.

We will label the flat tangent space indices with latin letters $a, b, c...$, while tensors under general coordinate transformations are labeled by greek letters $\alpha, \beta, \mu, \nu, ...$ The vierbein are introduced in order to marry the two sets of indices

$$\text{Vierbein: } e^a_\mu(x) \tag{1.78}$$

which is a mixed tensor whose inverse is $e^\mu_a(x)$.

The vierbien can be viewed as the square root of the metric tensor via the following

$$\begin{aligned} e^a_\mu e^a_\nu &= g_{\mu\nu} \\ e^{a\mu} &= g^{\mu\nu}e^a_\nu \\ e^a_\mu e^{b\mu} &= \delta^{ab} \end{aligned} \tag{1.79}$$

Since the Lorentz group acts on the tangent space indices, we can define spinors on the tangent space. The Dirac matrices γ^a do not depend by coordinates and respect the usual commutation relation

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{1.80}$$

These matrices can now be contracted onto vierbeins, defining the coordinates dependent Dirac matrices $\gamma^\mu(x)$,

$$\gamma^a e^{a\mu} = \gamma^\mu(x) \tag{1.81}$$

It is easy to show that the anti-commutator between two of these matrices yields the metric tensor:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x) \tag{1.82}$$

To construct the generally covariant Dirac equation we introduce a spinor $\psi(x)$ that is defined to be a scalar under general coordinate transformations and an ordinary spinor under flat tangent space Lorentz transformations:

$$\begin{array}{ll} \text{Coordinate transformations} & \psi \to \psi \\ \text{Lorentz transformations :} & \psi \to e^{i\epsilon^{ab}(x)\sigma_{ab}}\psi \end{array} \tag{1.83}$$

It is important to note that we have introduced local Lorentz transformations on the flat tangent space, so ϵ_{ab} is a function of space-time.

This, of course, means that the derivative of a spinor is no longer a genuine tensor. As before, we must introduce a connection field ω_μ^{ab} that allows us to *gauge the Lorentz group*. The covariant derivative for gauging the Lorentz group is therefore

$$\nabla_\mu\psi = (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\sigma_{ab})\psi \tag{1.84}$$

The generally covariant Dirac equation is therefore given by:

$$(i\gamma^\mu\nabla_\mu - m)\psi = 0 \tag{1.85}$$

and hence the action for a Dirac particle interacting with gravity is

$$\mathcal{L} = -\frac{1}{2k^2}\sqrt{-g}\,R + e\overline{\psi}(i\gamma^\mu\nabla_\mu - m)\psi \tag{1.86}$$

where $e \equiv \det e_\mu^a = \sqrt{-g}$.

This new connection field gives us an alternative way to construct the Riemann curvature tensor. By taking the commutator of two covariant derivatives, we can construct a new version of the curvature tensor:

$$[\nabla_\mu, \nabla_\nu]\,\psi = -\frac{i}{4}R_{\mu\nu}^{ab}\sigma^{ab}\psi \tag{1.87}$$

Written out, this curvature tensor is generally covariant in μ, ν but flat in a, b :

$$R_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\nu^{ac}\omega_\mu^{cb}\,. \tag{1.88}$$

At this point, the connection field ω_μ^{ab} is still an indipendent field. We can eliminate in favor of the vierbein by placing an external constraint on the theory:

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_\mu^{ab}e_\nu^b = 0 \tag{1.89}$$

The number of independent equations in the constraint $(4 \times 6 = 24)$ equals the number of independent components of the connection field, so we have eliminated the connection field entirely as ab independent field.

The connection field can be calculated by rotating the various indices and then adding and subtracting them. The final result is:

$$\omega_\mu^{ab} = \frac{1}{2} e^{a\nu} \left(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b \right) + \frac{1}{4} e^{a\rho} e^{b\sigma} \left(\partial_\sigma e_\rho^c - \partial_\rho e_\sigma^c \right) e_\mu^c - (a \leftrightarrow b) \; . \tag{1.90}$$

Chapter 2

Spectral Geometry

In this chapter we introduce the noncommutative geometry formalism based on Connes' spectral geometry. It is the generalization of the usual spin geometry of Riemannian manifolds to the noncommutative case via the notion of Spectral Triple. Its main element is a generalized Dirac operator which gives the metric structure of the space, as well as containing information on dynamic and gauge interactions of particles. Therefore, we give the definition of spectral triple and with it we show how is it possible to build gauge theories, as example the standard model of particle physics coupled to general relativity. All the standard model properties, conservation laws and symmetry breaking, will be resumed in the *spectral action principle* that represents the powerful predictive tool of this theory.

2.1 Spectral triples

The basic device in the construction of noncommutative geometry is the *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consisting of a *-algebra $\mathcal{A}$ of bounded operators in a Hilbert space $\mathcal{H}$ - containing the identity operator - and a non-necessarily bounded self-adjoint operator D on $\mathcal{H}$ with the following properties,

1. Resolvent $(D-\lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator on $\mathcal{H}$;
2. $[D, a] \equiv Da - aD \in \mathcal{B}(\mathcal{H}), \forall a \in \mathcal{A}$.

The *spectral triple* is said to be *even* if there is a grading $\mathbb{Z}_2$, i.e. an operator γ on $\mathcal{H}$, $\gamma = \gamma^*$, $\gamma^2 = 1$, such that

$$\begin{aligned} &\gamma D + D\gamma = 0, \\ &\gamma a - a\gamma = 0, \forall a \in \mathcal{A} \end{aligned} \tag{2.1}$$

If $\mathcal{H}$ is even then it is possible to separate

$$\mathcal{H} = \frac{(1+\gamma)}{2}\mathcal{H} \oplus \frac{(1-\gamma)}{2}\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \tag{2.2}$$

In general, one could ask that condition 2 is satisfied only for a dense subalgebra of $\mathcal{A}$. Then, by the two assumptions in the definition above, the self-adjoint operator D has a real discrete spectrum made of eigenvalues, i.e. the collection λ_n forms a discrete subset of $\mathbb{R}$ and each eigenvalue has finite multiplicity. Furthermore, for infinite dimensional algebras, $\lambda_n \to \infty$ as $n \to 0$. Indeed, $(D-\lambda)^{-1}$ being compact, its characteristic values $\mu_k(D-\lambda)^{-1} \to 0$ from which $|\lambda_k| = \mu_k(D) \to \infty$.

Finally, the *spectral triple* is said to be *real* if there is an antilinear isometry J (called real structure) which implements an action of the opposite algebra[1] $\mathcal{A}^\circ$ obtained by identifying $b^\circ = Jb^*J^{-1}$, and which commutes with the action of $\mathcal{A}$:

$$[a, JbJ^{-1}] = 0 \quad \forall\, a, b \in \mathcal{A}. \tag{2.3}$$

The operator J must obey 1) $J^2 = \pm\mathbb{I}$; 2) $JD = \pm DJ$; 3) $J\Gamma = \pm\Gamma J$, with choice of signs dictated by the KO-dimension of the spectral triple [19, §2.3].

These three elements satisfy a set of properties allowing to prove *Connes reconstruction theorem*: given any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with commutative $\mathcal{A}$ satisfying the required conditions, then $\mathcal{A} \simeq C^\infty(\mathcal{M})$ for some Riemannian spin manifold $\mathcal{M}$. Although we are going to discuss these conditions in the next section, more detailed informations can be found in [28], and their noncommutative generalization in [27].

2.2 Canonical triple over a Manifold

The first example of spectral triple is the canonical triple over a closed n-dimensional Riemannian spin manifold $(\mathcal{M}, g)$. As elements of this triple $(\mathcal{A}, \mathcal{H}, D)$ one takes,

1. $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$ is the algebra of complex valued smooth functions on $\mathcal{M}$.
2. $\mathcal{H} = L^2(\mathcal{M}, S)$ is the Hilbert space of square integrable sections of the irreducible spinor bundle over $\mathcal{M}$, whose rank being equal to $2^{n/2}$. The scalar product in $L^2(\mathcal{M}, S)$ is the usual one of the measure associated with the metric g,

$$(\psi, \phi) = \int d\mu(g)\psi(x)^*\phi(x) = \int dx\sqrt{-g}\,\psi(x)^*\phi(x) \tag{2.4}$$

[1] Identical to $\mathcal{A}$ as a vector space, but with reversed product: $a^\circ b^\circ = (ba)^\circ$.

with $\cdot^*$ indicating complex conjugation and scalar product, in the spinor space, being the natural one on $\mathbb{C}^{2^{n/2}}$.

3. $D = D_{\mathcal{M}} := \gamma^\mu \nabla_\mu := \gamma^\mu(\partial_\mu + \omega_\mu)$ is the Dirac operator associated with the Levi-Civita connection $\omega = dx^\mu \omega_\mu$ of the metric g.

The elements $f \in \mathcal{A}$ act as multiplicative operators on $\mathcal{H}$,

$$(f\,\psi)(x) \equiv f(x)\psi(x)\,,\ \forall f \in \mathcal{A}, \psi \in \mathcal{H}. \tag{2.5}$$

Next, let $(e^\mu_a,\ a = 1, 2, ...n)$ be an orthonormal basis of vector fields which is related to the natural basis $(\partial_\mu,\ \mu = 1, 2, ...n)$ via the n-beins components e^μ_a, (1.78), so that the components $\{g^{\mu\nu}\}$ of the curved metric and $\{\eta^{ab}\}$ of the flat metric are related by

$$g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}\ \ ,\ \eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}. \tag{2.6}$$

The coefficients $(\omega_{\mu b}^{\ \ a})$ of the Levi-Civita connection of the metric g defined by $\nabla_\mu e_a = \omega_{\mu a}^{\ \ b} e_b$, are the solutions of the equations

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \omega_{\mu b}^{\ \ a} e^b_\nu + \omega_{\nu b}^{\ \ a} e^b_\mu = 0. \tag{2.7}$$

The curved gamma matrices $\{\gamma^\mu(x)\}$ are defined in terms of the flat gamma matrices $\{\gamma^a\}$ by

$$\gamma^\mu(x) = \gamma^a e^\mu_a \tag{2.8}$$

and obey the relations

$$\begin{array}{ll} \{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x) & \mu, \nu = 1, 2, ..., n \\ \{\gamma^a, \gamma^b\} = 2\eta^{ab} & a, b = 1, 2, ..., n \end{array} \tag{2.9}$$

The lift ∇^S of the Levi-Civita connection to the bundle of spinors is then

$$\nabla^s_\mu = \partial_\mu + \omega^s_\mu = \partial_\mu + \frac{1}{4}\omega^{ab}_\mu \gamma^a \gamma^b \tag{2.10}$$

The Dirac operator can be locally written as

$$D_{\mathcal{M}} = \gamma^\mu(x)\,(\partial_\mu + \omega^S_\mu) = \gamma^a e^\mu_a\,(\partial_\mu + \omega^S_\mu) \tag{2.11}$$

Finally, we mention the Lichnérowicz formula for the square of the Dirac operator,

$$D^2_{\mathcal{M}} = \nabla^s + \frac{1}{4}R \tag{2.12}$$

where R is the scalar curvature of the metric and ∇^S is the Laplacian operator lifted to the bundle of spinors,

$$\nabla^s = -g^{\mu\nu}(\nabla^s_\mu \nabla^s_\nu - \Gamma^\rho_{\mu\nu}\nabla^s_\rho) \tag{2.13}$$

with $\Gamma^{\rho}_{\mu\nu}$ the Christoffel symbols of the connection.

If the dimension of the manifold $\mathcal{M}$ is even, the previous spectral triple is even by taking for grading operator just the product of all flat gamma matrices,

$$\gamma_{\mathcal{M}} = i^{-n/2}\gamma^1 \cdots \gamma^n \tag{2.14}$$

with n being even that anti-commutes with the Dirac operator $D_{\mathcal{M}}$,

$$\gamma_{\mathcal{M}} D_{\mathcal{M}} + D_{\mathcal{M}}\gamma_{\mathcal{M}} = 0 \tag{2.15}$$

Furthermore, we have

$$\gamma_{\mathcal{M}}^2 = 1, \quad \gamma_{\mathcal{M}}^* = \gamma_{\mathcal{M}}. \tag{2.16}$$

For the canonical triple of dimension $n = 4$ we can also define as real structure operator the charge conjugation that permutes particles and antiparticles:

$$J_M \equiv C = i\gamma^0\gamma^2 \circ \text{complex conjugation} = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ cc \tag{2.17}$$

Theorem 2.2.1. *The canonical triple allows to characterize the manifold $\mathcal{M}$, in fact it is possible to show that:*

1. *The space $\mathcal{M}$ is the structure space of the algebra $\bar{\mathcal{A}}$ of continuous functions on $\mathcal{M}$, which is the norm closure of $\mathcal{A}$.*

2. *The geodesic distance between any two points on $\mathcal{M}$ is given by*

$$d(p, q) = \underset{f\in\mathcal{A}}{sup}\{|f(p) - f(q)| \,:\, ||\,[D_{\mathcal{M}}, f]\,|| \leq 1\}, \ \forall p, q \in \mathcal{M}\,. \tag{2.18}$$

3. *The Riemannian measure on $\mathcal{M}$ is given by*[2]

$$\begin{aligned} \int_{\mathcal{M}} f \quad &\equiv \quad c(n)\,\mathrm{tr}_{\omega}(f\,|D_{\mathcal{M}}|^{-n})\,, \ \forall f \in \mathcal{A} \\ &c(n) \equiv 2^{(n-[n/2]-1)}\pi^{n/2} n\Gamma\left(\frac{n}{2}\right). \end{aligned} \tag{2.19}$$

For the proof of the theorem we refer to [65, §5.5].

[2] tr_{ω} is the Diximier trace defined as $\mathrm{tr}_{\omega}(T) := \lim\limits_{N\to\infty} \frac{1}{\ln N}\sum_{n=0}^{N-1}\mu_n(T)$, for all compact operators T.

In particular, we want to stress that the canonical triple gives an algebraic formulation of the notion of Manifold, with the commutative algebra $C^\infty(\mathcal{M})$ giving the points of the space and the Dirac operator defining the metric structure on it:

$$\begin{array}{ccc} Canonical\,spectral\,triple & \longrightarrow & Riemannian\,spin\,manifold \\ (C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), D_{\mathcal{M}}) & & (\mathcal{M}, g) \end{array} \tag{2.20}$$

with the five items of the canonical triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), D_{\mathcal{M}}; \gamma_{\mathcal{M}}, J_M)$ enjoy the following properties:

1. $J_M^2 = -1$ in dimensions 4 ($J_M^2 = 1$ in zero dimensions)
   $DJ_M = J_M D$ i.e. particles and antiparticles have the same dynamics,
   $J_M\,\gamma_{\mathcal{M}} = \gamma_{\mathcal{M}}\,J_M$.

2. The so called *order-zero condition*

$$[a, J_M b^* J_M^{-1}] = 0\ \forall a, b \in \mathcal{A}. \tag{2.21}$$

3. $[D, a]$ is bounded $\forall a \in \mathcal{A}$ and

$$[[D, a], J_M b^* J_M] = 0\,\forall a, b \in \mathcal{A}, \tag{2.22}$$

   this property is called order one condition because it states that the Dirac operator is a first order differential operator.

4. $\gamma_{\mathcal{M}}^2 = 1$ and $[\gamma_{\mathcal{M}}, a] = 0\ \forall\, a \in \mathcal{A}$, allowing the decomposition $\mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_L$.

5. $D\gamma_{\mathcal{M}} = -\gamma_{\mathcal{M}} D$, chirality does not change under time evolution,

6. The chirality can be written as a finite sum $\gamma_{\mathcal{M}} = \sum_i a_i J a_i' J^{-1}$ which is a 0-dim Hochschild cycle. This condition is called orientability.

7. The intersection form $\cap_{ij} = \mathrm{tr}(\gamma_{\mathcal{M}} p_i J p_j J^{-1})$ is not degenerate, $\det\cap \neq 0$. The p_i are minimal rank projections in $\mathcal{A}$. This condition is called *Poincarè* duality. Demanding the *Poincarè* duality to hold requires an even number of summands in the matrix algebra [91]

2.3 Almost commutative manifold

In the previous section we showed how it is possible to reconstruct a Riemannian spin manifold starting from the canonical triple satisfying the seven conditions enumerated above. These conditions are promoted to axioms for a noncommutative spectral triple defining a noncommutative space, schematically

$$\boxed{\begin{array}{ccc} \text{Canonical spectral triple} & \longrightarrow & \text{Riemannian spin manifold} \\ (C^\infty(\mathcal{M}), L^2(\mathcal{M},S), D_{\mathcal{M}}; J_M, \gamma_{\mathcal{M}}) & & (\mathcal{M}, g) \\ \downdownarrows & & \\ \text{Noncommutative spectral triple} & \longrightarrow & \text{Noncommutative space} \\ (\mathcal{A},\ \mathcal{H},\ D; J, \Gamma) & & \end{array}}$$

Therefore, analogously to the description of a spin manifold $\mathcal{M}$, we can describe a generally noncommutative finite space F by a triple

$$F = (\mathcal{A}_F, \mathcal{H}_F, D_F) \tag{2.23}$$

here we have a finite-dimensional complex Hilbert space $\mathcal{H}_F$ say of dimension N, for example $\mathcal{H}_F = \mathbb{C}^N$; the algebra $\mathcal{A}_F$ is a (real or complex) matrix algebra, which acts on the Hilbert space via matrix multiplication, $\mathcal{A}_F = \mathbb{M}_N(\mathbb{C})$; the operator D_F is given by a complex $N \times N$ matrix acting on $\mathcal{H}_F$. Moreover, we can make this noncommutative spectral triple a *real* and *even* spectral triple by adding a chirality operator γ_F, that is a $N \times N$ matrix, and a anti-unitary operator J_F, that is a $N \times N$ matrix times a complex conjugation, both satisfying conditions similar to the noncommutative spectral triple axioms 1 - 4 - 5 of §2.2.

Technically the simplest noncommutative examples are almost commutative. To construct the latter, we need a natural property of spectral triples, commutative or not: the tensor product of two even spectral triples is an even spectral triple. If both are commutative, i.e. describing two manifolds, then their tensor product simply describes the direct product of the two manifolds.

Definition. Let M be a Riemannian spin manifold with canonical triple $(C^\infty(\mathcal{M}),\ L^2(S,\mathcal{M}),\ D_{\mathcal{M}};\ J_M,\gamma_{\mathcal{M}})$ and let $(\mathcal{A}_F,\ \mathcal{H}_F,\ D_F; J_F, \gamma_F)$ be a finite real spectral triple. The almost commutative manifold $\mathcal{M} \times F$ is defined by the real spectral triple:

$$\mathcal{M}\times F = \big(C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, L^2(S,\mathcal{M}) \otimes \mathcal{H}_F, D_{\mathcal{M}} \otimes 1 + \gamma_{\mathcal{M}} \otimes D_F; J_M \otimes J_F, \gamma_{\mathcal{M}} \otimes \gamma_F\big) \tag{2.24}$$

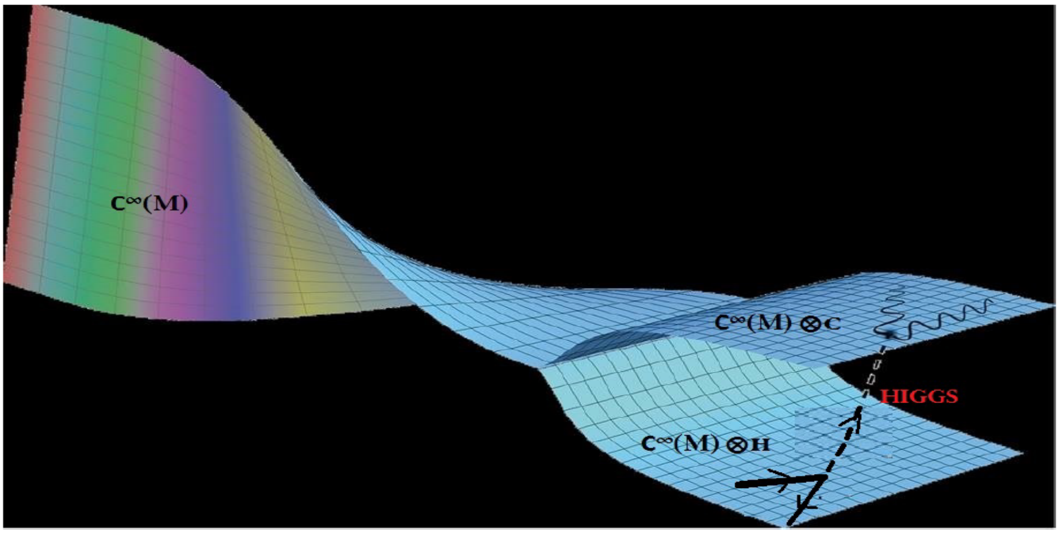

Figure 2.1: *Intuitive graphic representation of an almost-commutative manifold in which the space-time sheet – represented by $C^\infty(\mathcal{M})$ – is split in the two sheets of the internal space F, given by the noncommutative algebra $\mathbb{C}\oplus\mathbb{H}$*

In the following, we explain in more details the elements of the almost commutative manifold of the standard model coupled to the general relativity given by, [27]

$$\mathcal{M}\times F_{sm} = \left(C^\infty(\mathcal{M})\otimes\mathcal{A}_{sm}, L^2(S,\mathcal{M})\otimes\mathcal{H}_{sm}, D_{\mathcal{M}}\otimes 1+\gamma_{\mathcal{M}}\otimes D_{sm}; J_M\otimes J_{sm}, \gamma_{\mathcal{M}}\otimes\gamma_{sm}\right). \tag{2.25}$$

2.3.1 The algebra of the standard model

Under assumptions on the representation – irreducibility and existence of a separating vector – it is possible to show [16] that the most general finite algebra in (2.23) satisfying all conditions for the noncommutative space to be a manifold is

$$\mathcal{A}_F = \mathbb{M}_a(\mathbb{H})\oplus\mathbb{M}_{2a}(\mathbb{C}) \qquad a\in\mathbb{N}^*. \tag{2.26}$$

This algebra acts on an Hilbert space of dimension $2(2a)^2$ [15,16] presented in the next paragraph explicitly.

To have a non trivial grading on $\mathbb{M}_a(\mathbb{H})$ the integer a must be at least 2, meaning the simplest possibility is

$$\mathcal{A}_F = \mathbb{M}_2(\mathbb{H})\oplus\mathbb{M}_4(\mathbb{C}). \tag{2.27}$$

The grading condition $[a,\Gamma]=0$, with Γ given in (2.40), reduces the algebra to the left-right algebra:

$$\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}). \tag{2.28}$$

This is basically a Pati-Salam model [79], one of the not many models allowed by the spectral action [66]. The order one condition reduces further the algebra to [19] (for a review see also [96])

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}), \tag{2.29}$$

where $\mathbb{H}$ are the quaternions, which we represent as 2×2 matrices, and $\mathbb{M}_3(\mathbb{C})$ are 3×3 complex valued matrices. $\mathcal{A}_{sm}$ is the algebra of the standard model, that is the one whose unimodular group is U(1)×SU(2)×U(3). The details of these reductions can be found in [36, appendix A].

This algebra - tensorized by $C^\infty(\mathcal{M})$ - has to be represented on a Hilbert space of dimension $2(2\cdot 2)^2 = 32$, that is the dimension of the standard model Hilbert space for one particle generation, as explained below.

2.4 The Hilbert space for the standard model

The standard model algebra $C^\infty(M)\otimes(\mathbb{C}\oplus\mathbb{H}\oplus\mathbb{M}_3(\mathbb{C}))$ is represented on the Hilbert space $L^2(S,\mathcal{M}) \otimes \mathcal{H}_{sm}$ whose elements are 384 components vectors given by the 4 degrees of the Lorentz spinors times the 128 quantum numbers of the standard model gauge group:

$$\begin{array}{cccccc}
\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, & \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}, & \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix} & \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}, & \begin{pmatrix} c_L^i \\ s_L^i \end{pmatrix}, & \begin{pmatrix} t_L^i \\ b_L^i \end{pmatrix} \\
(e_R), & (\mu_R), & (\tau_R) & (u_R^i), & (c_R^i), & (t_R) \\
(\nu_R^e), & (\nu_R^\mu), & (\nu_R^\tau) & (d_R^i), & (s_R^i), & (b_R^i)
\end{array}$$

We denote a generic fermion, i.e. an element of the Hilbert space, by

$$\Psi^{\mathsf{CI}m}_{s\dot{s}\alpha}(x) \in L^2(S,\mathcal{M}) \otimes \mathsf{H}_F = sp(L^2(M)) \otimes \mathcal{H}_F. \tag{2.30}$$

The position of the indices, whose meaning is described below, is a matter of convention, Ψ is a $\mathbb{C}^{384}$-vector valued function on $\mathcal{M}$, we write some of them as upper indices and some as lower to avoid having six indices in a row. Note the difference between H_F and $\mathcal{H}_F$: the latter is a 96 dimensional space and its vectors are to be multiplied by spinors, while the former is the larger 384 dimensional space which exhibits explicitly the fermion doubling over-counting. Until the end of this chapter the Hilbert space will be considered in its factorized form involving $\mathcal{H}_F$, while in the next two chapters - when we need larger symmetries - we will use the factorized form involving H_F, since it allows to consider algebras which do not act separately on spinors and the internal part. This means that in addition of the internal degrees of freedom used in [17], our tensorial notation also includes spin indices $s, \dot{s}$.

The meaning and range of the various indices of $\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$ is the following:

$s = r, l$
$\dot{s} = \dot{0}, \dot{1}$ are the spinor indices. They are not internal indices in the sense that the algebra $\mathcal{A}_F$ acts diagonally on it. They take two values each, and together they make the four indices on an ordinary Dirac spinor. The index $s = r, l$ indicates chirality and runs over the right, left part of the spinor, while $\dot{s}$ differentiates particles from antiparticles. In the chiral basis one thus has[3]

$$\gamma^\mu = \begin{pmatrix} 0_2 & \sigma^{\mu}{}_{\dot{s}}^{\dot{t}} \\ \overline{\sigma}^{\mu}{}_{\dot{s}}^{\dot{t}} & 0_2 \end{pmatrix}_{st}, \quad \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix}_{st}, \tag{2.31}$$

where for $\mu = 0, 1, 2, 3$ one defines

$$\sigma^\mu = \{\mathbb{I}_2, -i\sigma_i\}, \quad \bar{\sigma}^\mu = \{\mathbb{I}_2, i\sigma_i\} \tag{2.32}$$

with σ_i, $i = 1, 2, 3$ the Pauli matrices, namely $\sigma^0 = \mathbb{I}_2$,

$$\sigma^1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}_{\dot{s}\dot{t}} \sigma^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}\dot{t}} \sigma^3 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}_{\dot{s}\dot{t}}.$$

$\mathrm{I} = 0, \dots 3$ indicates a "lepto-colour" index. The zeroth "colour" actually identifies leptons while $\mathrm{I} = 1, 2, 3$ are the usual three colours of QCD.

$\alpha = 1 \dots 4$ is the flavour index. It runs over the set u_R, d_R, u_L, d_L when $\mathrm{I} = 1, 2, 3$, and ν_R, e_R, ν_L, e_L when $\mathrm{I} = 0$. It repeats in the obvious way for the other generations.

$\mathsf{C} = 0, 1$ indicates whether we are considering "particles" ($\mathsf{C} = 0$) or "antiparticles" ($\mathsf{C} = 1$).

$m = 1, 2, 3$ is the generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters.

---

[3]The multi-index st after the closing parenthesis is to recall that the entries of the γ's matrices are labelled by indices s, t taking values in the set $\{l, r\}$. For instance the l-row, l-column block of γ^5 is $\mathbb{I}_2$. Similarly the entries of the σ's matrices are labelled by $\dot{s}, \dot{t}$ indices taking value in the set $\{\dot{0}, \dot{1}\}$: for instance $\sigma^2{}_{\dot{0}}^{\dot{0}} = \sigma^2{}_{\dot{1}}^{\dot{1}} = 0$.

For the remainder of this work the generation index m does not play any role. We will therefore suppress it and work with one generation, thus effectively considering $\mathcal{H}_F$ and H_F having dimension 32 and 128 respectively.

A generic element $A = \{Q, M\}$ in $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$ (with $Q \in C^\infty(\mathcal{M}) \otimes \mathbb{M}_2(\mathbb{H})$ and $M \in C^\infty(\mathcal{M}) \otimes \mathbb{M}_4(\mathbb{C})$) acts as a matrix on vectors of the finite Hilbert space H_F with index structure (2.30), it is therefore a matrix with twice as many indices[4]:

$$A^{ti\,\mathsf{C}\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathrm{J}\alpha} = \delta^t_s \delta^{i}_{\dot{s}} \left(\delta^{\mathsf{C}}_0 \delta^{\mathrm{I}}_{\mathrm{J}} Q^\beta_\alpha + \delta^{\mathsf{C}}_1 M^{\mathrm{I}}_{\mathrm{J}} \delta^\beta_\alpha \right). \tag{2.33}$$

Here Q^β_α evaluated at $x \in M$ denotes the entries $Q^\beta_\alpha(x) \in \mathbb{C}$ of the matrix $Q(x) \in \mathbb{M}_2(\mathbb{H})$, viewed as a 4×4 complex matrix with components labelled by the α, β flavour indices. Similarily $M^{\mathrm{I}}_{\mathrm{J}}$ evaluated at x stands for the components of the matrix $M(x) \in \mathbb{M}_4(\mathbb{C})$, whose entries are labelled by the I, J lepto-colour indices.

The two Kronecker δ at the beginning of the expression for A show that the algebra acts in a trivial way (i.e. as the identity operator) on the spin indices. In other words the finite dimensional algebra $\mathcal{A}_F$ acts only on the internal indices. The two terms in the bracket act only on particles and antiparticles respectively, as signified by δ^{C}_0 and δ^{C}_1. They are such that the order zero condition holds. Note in fact that for particles the action is trivial on the I, J indices, and for antiparticles is trivial on the α, β indices. Since the real structure J exchanges particles with antiparticles the two A and JBJ^{-1} will commute. There is no room for the representation of a larger algebra satisfying the order 0 condition, unless more fermions are added, or one renounces to the trivial action on the spin indices. The second possibility is the one we will use for the grand algebra in the next chapter, when we introduce the Grand Symmetry model.

2.4.1 The Dirac operator for the standard model

The Dirac operator D for the spectral triple of the standard model is

$$D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^5 \otimes D_F \tag{2.34}$$

[4] D, J, β, have the same range as C, I, α and serve as contracting indices.

the finite dimensional Dirac operator $D_F = D_0 + D_R$ is a 96×96 matrix where[5]

$$D_0 := \begin{pmatrix} 0_{8N} & \mathcal{M}_0 & 0_{8N} & 0_{8N} \\ \mathcal{M}_0^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & \bar{\mathcal{M}}_0 \\ 0_{8N} & 0_{8N} & \mathcal{M}_0^T & 0_{8N} \end{pmatrix} \text{ and } D_R := \begin{pmatrix} 0_{8N} & 0_{8N} & \mathcal{M}_R & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \end{pmatrix}. \tag{2.35}$$

The matrix $\mathcal{M}_0$ contains the Yukawa couplings of the fermions and the mixing matrices (CKM for quarks and NPMS for neutrinos). It couples left with right particles. The matrix $\mathcal{M}_R = \mathcal{M}_R^T$ contains Majorana masses and couples right particles with right antiparticles. The Dirac operator is a datum of the problem, i.e. the fermion masses (Yukawa couplings) are known quantities. The matrix D_F, is "sparse", meaning than most of its entries are zeros. Supersymmetry may be described "filling" some of these voids with the bosonic superpartners of the fermions [95].

In tensorial notation, the charge conjugation operator $J = J_M \otimes J_F$ is

$$J_M = i\gamma^0\gamma^2 cc = i \begin{pmatrix} \overline{\sigma}^{2\,\dot{t}}_{\ \dot{s}} & 0_2 \\ 0_2 & \sigma^{2\,\dot{t}}_{\ \dot{s}} \end{pmatrix}_{st} cc = -i\eta^t_s \tau^{\dot{t}}_{\dot{s}}\, cc, \tag{2.36}$$

while

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_{\mathsf{CD}} cc, \tag{2.37}$$

hence

$$(J\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = -i\eta^t_s\, \tau^{\dot{t}}_{\dot{s}}\, \xi^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathsf{I}\beta}_{\mathsf{J}\alpha}\, \bar{\Psi}^{\mathsf{DJ}}_{t\dot{t}\beta} \tag{2.38}$$

where for any pair of indices $x, y \in [1, ..., n]$ one defines

$$\xi^x_y = \begin{pmatrix} 0_n & \mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}, \quad \eta^x_y = \begin{pmatrix} \mathbb{I}_n & 0_n \\ 0_n & -\mathbb{I}_n \end{pmatrix}, \quad \tau^x_y = \begin{pmatrix} 0_n & -\mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}. \tag{2.39}$$

The chirality $\Gamma = \gamma_{\mathcal{M}} \otimes \gamma_F$ acts as $\gamma^5 = \eta^t_s \delta^{\dot{t}}_{\dot{s}}$ on the spin indices, and as $\gamma_F = \eta^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathsf{I}}_{\mathsf{J}}\, \eta^\beta_\alpha$ on the internal indices:

$$(\Gamma\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = \eta^t_s \delta^{\dot{t}}_{\dot{s}}\, \eta^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathsf{I}}_{\mathsf{J}}\, \eta^\beta_\alpha\, \Psi^{\mathsf{DJ}}_{t\dot{t}\beta}. \tag{2.40}$$

The operators γ_F, J_F and D_F are such that

$$J_F^2 = \mathbb{I}, \quad J_F D_F = D_F J_F, \quad J\gamma_F = -\gamma_F J_F, \tag{2.41}$$

meaning that the finite part of the spectral triple has KO-dimension 6 [11,19]. The commutative part has KO-dimension 4, and the full spectral triple has KO-dimension $6 + 4 = 10 \bmod 8 = 2$.

[5] Here $^-$ denotes the complex conjugation, $\dagger$ the adjoint, ^{T} the transpose.

2.5 The spectral action principle

Given an almost commutative geometry $(\mathcal{A}, \mathcal{H}, D; J, \Gamma)$, a *fluctuation of the metric*[6] [27] means the substitution of D by the gauge Dirac operator [84]

$$D_A \equiv D + \mathbb{A} + J\mathbb{A}J^{-1} \tag{2.42}$$

where $\mathbb{A} = \sum_i a_i[D, b_i]$, with $a_i, b_i \in \mathcal{A}$, is a generalized gauge potential. It is made of two parts: a scalar field on $\mathcal{M}$ with value in $\mathcal{A}_F$, and 1-form field on $\mathcal{M}$ with value in the group of unitaries of $\mathcal{A}_F$. In case $\mathcal{A}_F = \mathcal{A}_{sm}$ is the algebra of the standard model (discussed in §2.3.1), the 1-form fields yield the vector bosons mediating the three fundamental interactions, and the scalar field is the Higgs field H.

Now, let us consider the spectral action principle [14] to derive the bosonic action of the Yang-Mills gauge fields coupled to the General Relativity. One cannot construct too many invariants by using the spectral triple data. One obvious choice is the ordinary fermionic action

$$S_F = \langle J\psi, D_A\psi \rangle \,. \tag{2.43}$$

As well, one can use the operator trace Tr in $\mathcal{H}$ to construct invariants from the Dirac operator alone. In this way one obtains the *spectral action*

$$S_\Lambda(D_A) = \mathrm{Tr}\left[f\left(\frac{D_A}{\Lambda}\right)\right], \tag{2.44}$$

where f is a function restricted only by the requirement that trace in (2.44) exists. The function f is usually taken as a cutoff function since it has to regularize (2.44) at large eigenvalues of D. Λ is a cutoff scale.

One can use the heat kernel expansion of an operator[7] D^2

$$\mathrm{Tr}\left[f\left(\frac{D_A}{\Lambda}\right)\right] = \sum_{n=0}^{\infty} F_{4-n}\Lambda^{4-n}a_n \tag{2.45}$$

where F is defined by $F(u) = f(v)$ with $u = v^2$, thus $F(D^2) = f(D)$. If we define

$$f_k = \int_0^\infty f(v)v^{k-1}dv, \quad k > 0 \tag{2.46}$$

[6] The name comes from the fact that the substitution $D \to D_A$ modifies the metric associated to the spectral triple. See [75] for a detailed account on this point.

[7] See [49, 98] for a detailed overview of the heat trace asymptotics.

then

$$F_4 = \int_0^\infty F(u)udu = 2\int_0^\infty F(v)v^3 dv = 2f_4$$
$$F_2 = \int_0^\infty F(u)du = 2\int_0^\infty F(v)vdv = 2f_2$$
$$F_0 = F(0) = f(0) = f_0$$
$$F_{-2n} = (-1)^n F^{(n)}(0) = \left[(-1)^n \left(\frac{1}{2v}\frac{d}{dv}\right)^2 f\right](0) \quad n \geq 1 \tag{2.47}$$

In four dimensions the first four terms of the asymptotic expansion are

$$S_\Lambda\left(D_A\right) = \Lambda^4 F_4 a_0\left(D_A^2\right) + \Lambda^2 F_2 a_2\left(D_A^2\right) + F_0 a_4\left(D_A^2\right) + \frac{1}{\Lambda^2}F_{-2}a_6\left(D_A^2\right) + \ldots \tag{2.48}$$

D_A^2 is an operator of Laplace type, that means it can be represented as

$$D_A^2 = -(\nabla^2 + E)\,, \tag{2.49}$$

where ∇ is a covariant derivative, $\nabla_\mu = \partial_\mu + \omega_\mu$, E is a zeroth order term. Denoting by tr the usual matrix trace one may write

$$\begin{aligned}
a_0 &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}\mathrm{tr}\ (1)\,,\\
a_2 &= \frac{1}{16\pi^2}\frac{1}{6}\int d^4x\sqrt{g}\mathrm{tr}\ [6E + R\cdot 1]\,,\\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int d^4x\sqrt{g}\mathrm{tr}\ \left[\left(12R;_\mu^\mu + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma}\right)\cdot 1\right]\\
&+ \frac{1}{16\pi^2}\frac{1}{360}\int d^4x\sqrt{g}\mathrm{tr}\ \left[60E;_\mu^\mu + 60ER + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\right].
\end{aligned} \tag{2.50}$$

Here $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R are the Riemann tensor, the Ricci tensor and the curvature scalar, respectively. The semicolon denotes covariant derivatives, and $\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]$.

The expression for a_6 is rather long (see [98]), but it simplifies if one considers a flat-space time

$$a_6^{flat} = \frac{1}{16\pi^2}\int d^4x\left(\Sigma_\Omega + \Sigma_E + \Sigma_{E\Omega}\right), \tag{2.51}$$

where

$$\Sigma_\Omega = \mathrm{tr}\left[-\frac{1}{90}\Omega_{\mu\nu;\tau}\Omega^{\mu\nu;\tau} + \frac{1}{180}\Omega^{\mu\nu}{}_{;\nu}\Omega_{\mu\rho}{}^{;\rho} - \frac{1}{30}\Omega_{\mu\nu}\Omega^{\nu\tau}\Omega_\tau{}^\mu\right], \tag{2.52}$$

$$\Sigma_{E\Omega} = \mathrm{tr}\left[\frac{1}{12}E\Omega_{\mu\nu}\Omega^{\mu\nu}\right], \tag{2.53}$$

$$\Sigma_E = \mathrm{tr}\left[-\frac{1}{12}E^{;\mu}E_{;\mu} + \frac{1}{6}E^3\right], \tag{2.54}$$

Taking just the contributions from a_0, a_2 and a_4 to the expansion (2.48) one reproduces quite well the bosonic part of the standard model action,

$$\begin{aligned}
S_\Lambda(D_A) &= \int d^4x\sqrt{g}\frac{1}{2\pi^2}F_0\left[a(\nabla_\nu H)^2 + 2e(\overline{H}H) + b(\overline{H}H)^2 + \frac{d}{2}\right] + \\
&+ \int d^4x\sqrt{g}\frac{1}{2\pi^2}F_0\left[\frac{5}{3}g_1^2B_{\mu\nu}^2 + g_2^2W_{\mu\nu}^2 + g_3^2V_{\mu\nu}^2\right] + \mathit{Gravity\ Terms} + \\
&+ \int d^4x\sqrt{g}\left[\frac{24}{\pi^2}F_4\Lambda^4 - \frac{2}{\pi^2}F_2\Lambda^2\left(\frac{1}{2}a\overline{H}H + \frac{c}{4}\right)\right], \qquad (2.55)
\end{aligned}$$

where $B_{\mu\nu}$, $\mathbf{W}_{\mu\nu}$ and $\mathbf{V}_{\mu\nu}$ are respectively the field strength associated with the gauge groups $U(1)$, $SU(2)$ and $SU(3)$; H is identified with the Higgs field. The three momenta F_0, F_2 and F_4 can be used to specify the initial conditions of the gauge couplings, the Newton constant and the cosmological constant. The coefficients a, b, c, d, and e are related to the fermionic Yukawa couplings and Majorana mass matrix and will be written in the crude approximation where the Yukawa couplings of the top quark y_{top} and the neutrino (both Majorana y_{ν_R} and Dirac y_ν) are dominant; in addition, we introduce the dimensionless constant ρ defined by the ratio between the Dirac Yukawa couplings $y_\nu = \rho y_{top}$:

$$\begin{aligned}
a &= \mathrm{tr}\left[y_\nu^* y_\nu + y_e^* y_e + 3\left(y_{top}^* y_{top} + y_d^* y_d\right)\right] \simeq (3+\rho^2)y_{top}^2 \\
b &= \mathrm{tr}\left[(y_\nu^* y_\nu)^2 + (y_e^* y_e)^2 + 3\left(y_{top}^* y_{top} + y_d^* y_d\right)^2\right] \simeq (3+\rho^4)y_{top}^4 \\
c &= \mathrm{tr}\left[y_{\nu_R}^* y_{\nu_R}\right] \simeq y_{\nu_R}^2 \\
d &= \mathrm{tr}\left[\left(y_{\nu_R}^* y_{\nu_R}\right)^2\right] \simeq y_{\nu_R}^4 \\
e &= \mathrm{tr}\left[y_\nu^* y_\nu y_{\nu_R}^* y_{\nu_R}\right] \simeq \rho^2 y_{top}^2 y_{\nu_R}^2 \qquad (2.56)
\end{aligned}$$

Finally, the gravity terms, including $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R, collected from (2.50) are

$$\begin{aligned}
\text{Gravity terms} &= \frac{1}{2\pi^2}F_0\int d^4x\sqrt{g}\left[\frac{1}{30}(-18C_{\mu\nu\rho\sigma}^2 + 11R^*R^*) + \frac{1}{6}aR\bar{H}H + \frac{1}{12}cR\right] + \\
&- \frac{2}{\pi^2}F_2\Lambda^2\int d^4x\sqrt{g}R. \qquad (2.57)
\end{aligned}$$

In order to have the classical form for the bosonic standard model action we require

$$\begin{aligned}
\frac{5}{3}g_1^2 = g_2^2 = g_3^2 &\equiv g \\
\frac{aF_2\Lambda^2 - e\,F_0}{a\,F_0} &\equiv m_0^2 \\
\frac{F_0 g^2}{2\pi^2} &= \frac{1}{4} \qquad (2.58)
\end{aligned}$$

Moreover, the normalization of the kinetic term $\frac{1}{2}\left|\nabla_\mu H\right|^2$ in (2.55), leads to a rescaling

$$H \to H = \sqrt{\frac{2}{3+\rho^2}}\frac{g}{y_t}H \tag{2.59}$$

where g is the gauge coupling to the unification scale. Defining the auto-interaction parameter λ_0 as

$$\frac{F_0(3+\rho^4)}{2\pi^2}y_t^4 \equiv \lambda_0 \tag{2.60}$$

and gathering together the definitions (2.58),(2.60) the action becomes,

$$\begin{aligned} S_\Lambda(D_A) &= \int d^4x\sqrt{g}\left[\frac{1}{2}(\nabla_\nu H)^2 - \frac{m_0^2}{2}(\overline{H}H) + \lambda(\overline{H}H)^2\right] + \\ &+ \int d^4x\sqrt{g}\left[+\frac{1}{4}B_{\mu\nu}^2 + \frac{1}{4}W_{\mu\nu}^2 + \frac{1}{4}V_{\mu\nu}^2\right] + \\ &+ \int d^4x\sqrt{g}\left[-\frac{24}{\pi^2}F_4\Lambda^4 + \frac{d}{4\pi^2}F_0 - \frac{c}{\pi^2}F_2\Lambda^2\right] + \textit{Gravity Terms} \end{aligned} \tag{2.61}$$

The fermionic action of the standard model is given by the usual definition

$$\langle J\psi, D_A\psi\rangle = \langle J\psi, D\psi\rangle + \langle J\psi, \mathbb{A}\psi\rangle + \langle J\psi, J\mathbb{A}J^{-1}\psi\rangle \tag{2.62}$$

the first term gives kinetic terms while the remaining ones return radiation-matter interaction and Yukawa couplings (see [96, §6.4]). Moreover, we obtain a relation between the fermion mass matrices $\mathcal{M}$ and the gauge coupling unification g: in fact together with the rescaling of the Higgs field (2.59) we have the rescatation of its v.e.v. v_0, which gives mass to the fermions ,

$$\bar{\psi}H\psi \underset{\text{(symmetry breaking)}}{\longrightarrow} \bar{\psi}\,(h+v_0)\,\psi \underset{\text{(rescaling (2.59))}}{\longrightarrow} \sqrt{\tfrac{2}{a}}g\,\bar{\psi}\,(h+v_0)\,\psi \tag{2.63}$$

since the fermion masses are given by the coefficients in front of $\bar{\psi}\psi$ and Yukawa couplings are defined as $y_{fermion} = m_{fermion}\sqrt{2}/v_0$, we have for the top mass

$$\sqrt{\frac{2}{a}}gv_0 = m_{top} \equiv \frac{1}{\sqrt{2}}y_t v_0 \Rightarrow y_t = \frac{2}{\sqrt{a}}g \tag{2.64}$$

since the top mass dominates the relation, it becomes

$$y_t = \frac{2}{\sqrt{3+\rho^2}}g \tag{2.65}$$

Some years ago the data were compatible with the presence of a single unification point Λ. This was one of the motivations behind the building of grand unified theories. Such a feature is however desirable even without the presence of a larger gauge symmetry group which breaks to the standard model with the usual mechanisms. In particular, the approach to field theory, based on noncommutative geometry and spectral physics [14], needs a scale to regularize the theory. In this respect, the finite mode regularization [3, 4, 45] is ideally suited. In this case Λ is also the field theory cutoff. In fact using this regularization it is possible to generate the bosonic action starting form the fermionic one [6, 7, 61, 62], or describe induced gravity on an equal footing with the anomaly-induced effective action [64].

2.6 Running coupling constants from the spectral action principle

The usual strategy is to use the spectral action as an effective action at a fixed scale, of the order of the unification scale, and to impose the additional relations (2.60, 2.65) between the independent parameters of the standard model as a boundary condition at that scale. One can then let these parameters run down using the RG equations to their value at ordinary scale.

The SM running coupling constants at one loop, associated to (2.61), are ruled by the following equations, where we defined the dot derivations as $16\pi^2\mu\frac{d}{d\mu}$:

$$\begin{aligned}
\dot{g}_i &= \left(b_i g_i^3\right) \text{ with } \left(b_1, b_2, b_3 = \frac{41}{6}, -\frac{19}{6}, -7\right) \\
\dot{\lambda} &= \left(24\lambda^2 - \left(3g_1^2 + 9g_2^2\right)\lambda + \frac{3}{8}\left(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4\right) + \left(12y_t^2 + 4y_\nu^2\right)\lambda - 6y_t^4 - 2y_\nu^4\right) \\
\dot{y}_t &= \left(\frac{9}{2}y_t^2 + y_\nu^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right) \\
\dot{y}_\nu &= \left(\frac{5}{2}y_\nu^2 + 3y_t^2 - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right)
\end{aligned} \tag{2.66}$$

For our purpose one loop is sufficient, the running up to three loops can be found in [25, 69–71] and references therein. In the present case, one separately solves the equations for the gauge coupling constants and the other couplings; for the former, the boundary conditions are given at the electro-weak scale by the experimental values [12],

$$g_1(m_Z) = 0.358,\ \ g_2(m_Z) = 0.651,\ \ g_3(m_Z) = 1.221 \tag{2.67}$$

while for the other coupling constants λ, y_t and y_ν we take the relations (2.60, 2.65) as boundary conditions at the cut-off scale Λ that is the scale at which the spectral action lives. These boundary conditions use two free parameters: the value of the unified gauge coupling constants, g, and the ratio between the top and neutrino Yukawa couplings $\rho \equiv y_\nu(\Lambda)/y_t(\Lambda)$,

$$\lambda(\Lambda) = \frac{4\left(\rho^2+3\right)}{\left(\rho+3\right)^2} g^2$$
$$y_t(\Lambda) = \sqrt{\frac{4}{\rho^2+3}} g$$
$$y_\nu(\Lambda) = \sqrt{\frac{4\rho^2}{\rho^2+3}} g \tag{2.68}$$

Since the coupling constants g_i do not meet exactly, forming a triangle, one takes for the unification energy Λ a range of values beetwen the extremal points of the triangle. The results, for a particular set of values of the free parameters $(g, \rho, \Lambda) = (0.530,\ 1.25, 10^{16} GeV)$, are plotted in fig. 2.2.

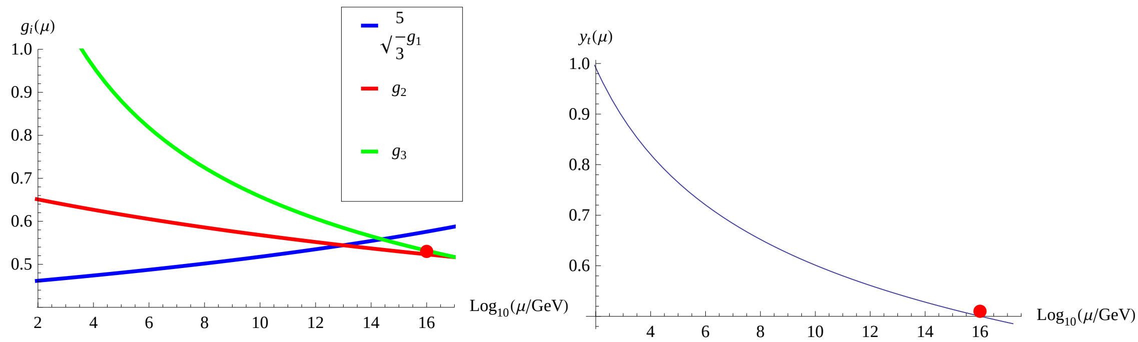

Figure 2.2: *The standard model running for the gauge coupling constant (left) and the Yukawa coupling (right) in the spectral action approach for* $(g, \rho, \Lambda) = (0.530,\ 1.25,\ 10^{16} GeV)$*. The red dot indicates the starting value of the parameters* $(Log_{10}(\frac{\Lambda}{GeV}), g)$ *and* $(Log_{10}(\frac{\Lambda}{GeV}), \lambda(\Lambda))$*.*

After running these couplings from unification energy Λ to low energy M_Z, we compare the values of $y_t(M_Z)$ and $\lambda(M_Z)$ with their experimental values

$$y_t^{exp}(M_Z) = 0.997, \qquad \lambda^{exp}(M_Z) = 0.130 \tag{2.69}$$

In fig. 2.2 we can see the good agreement between $y_t(M_Z)$ predicted by the spectral action and its experimental value. Very different is the case for the Higgs self-coupling λ, fig. 2.3, whose predicted value, in the spectral action approach, is around 0.240 with a resulting Higgs mass $M_H = \sqrt{2\lambda v^2} \simeq 170$ GeV. On the other hand, the experimental value for the Higgs mass ($\simeq$125 GeV) leads to the instability problem for the autointeraction parameter λ, which becomes

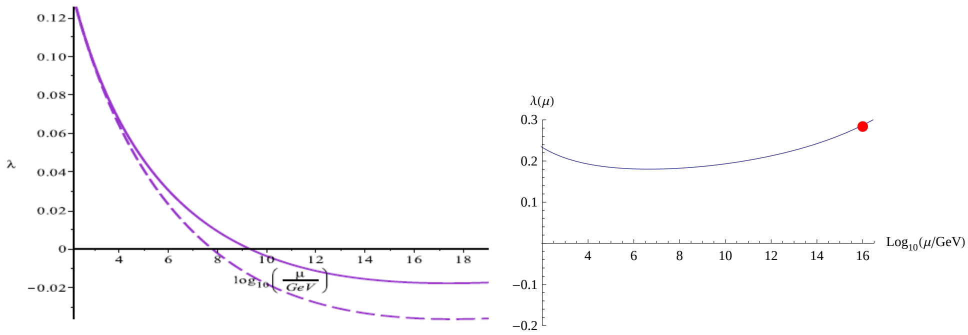

Figure 2.3: *On the left, the standard model running for the coupling constant λ starting from $\lambda(m_Z) = 0.130$ corresponding to $M_H = 125GeV$. The dashed and solid lines represent the one and two loop respectively. On the right, the λ behaviour starting from the red point of the spectral action and culminating in the prediction $\lambda(m_Z) = 0.240$.*

negative at a scale of the order of $10^8 GeV$; two loop calculations make the situation slightly worse as one can see on the left side of fig. 2.3.

A negative λ means an instability and renders the model inconsistent, although it may just mean the presence of a long lived metastable state [33, 38]. However, the spectral action model can be fixed [18,21,36,42,93] with the introduction of a scalar field, σ, possibly coming from a larger symmetry, connected with the fluctuations of a Majorana neutrino mass term in the action, [24, 39], as we show in next chapter with the introduction of the Grand Symmetry model.

Chapter 3

The grand symmetry model

In the previous chapter we showed how the coupling constants of the standard model come out to be function of the gauge coupling constants to the unification scale $g(\Lambda)$ and of the parameters in D_F, in particular the ratio between the top and neutrino Yukawa couplings $\rho = y_t(\Lambda)/y_\nu(\Lambda)$. In this sense the model predicts the Higgs and the top mass as functions of the gauge couplings and of the unification scale at which the three gauge couplings constants coincide. This last point is known to be true only in an approximate sense. If one takes the unification scale to be $\Lambda = 10^{17}$GeV then one finds - assuming the big desert hypothesis - a Higgs mass of the order of 170 GeV. This value is not in agreement with the recent LHC experiments [1, 23] and it was already excluded by Tevatron [83] in 2008.

One can think of extending the model to solve this. There have been several proposals in this sense, and some of them are reviewed in [86]. In particular C. Stephan has proposed in [93] that the presence of an extra scalar field, corresponding to the breaking of a extra U(1) symmetry, can bring down the mass of the Higgs to 126 GeV. This model however contains extra fermions. Earlier examples of extensions are in [77, 82, 87, 89, 90, 92, 94].

Recently, in [18] the noncommutative geometry model was enhanced to also overcome the high energies instability of a Higgs boson with mass around 126 GeV, in addition to predicting the correct mass. This is done ruling out the hypothesis of the "big desert" and considering an additional scalar field σ that lives at high energies and gives mass to the Majorana neutrinos. Explicitly σ is obtained in [18] by turning (inside the finite dimensional part D_F of the Dirac operator) the constant-entry y_R of the Majorana matrix $\mathcal{M}_R$ into a field:

$$y_R \to y_R\sigma(x) \tag{3.1}$$

However, the origin of the field σ is quite different from the Higgs. The latter, like the other bosons, are components of the gauge potential $\mathbb{A}$. They are obtained from the commutator of D_F with the algebra. D_F has constant components, that is without manifold dependence, but when these numbers are commuted with elements of the algebra they give rise to the desired bosonic fields. One could hope to obtain σ in a similar way, by considering y_R as a Yukawa coupling. As explained in [36, appendix B] the problem is that in taking the commutator with elements of the algebra $\mathcal{A}_{sm}$, the coefficient y_R does not contribute to the potential because of the first order condition. This forced the authors of [18] to "promote to a field" only the entry y_R, in a somewhat arbitrary way. Indeed the components of D_F cannot all be fields to start with, otherwise the model would loose its predictive power, in that all Yukawa couplings would be fields, and the masses of all fermions would run independently, thus making any prediction impossible. In the following (§3.1 and §3.2) we show that there is a way to obtain the field σ from y_R by a fluctuation of the metric, provided one starts with an algebra larger than the one of the standard model.

3.1 The grand algebra

Let us restart from the most general finite algebra that satisfies all the conditions for the noncommutative space to be a manifold, eq. (2.26). The standard model coupled with gravity is described by the case $a = 2$. The case $a = 3$ would require a 72-dimensional Hilbert space, and there is no obvious way to build it from the particle content of the standard model. The next case, $a = 4$, requires the Hilbert space to have dimension 128, which is the dimension of H_F, as defined in (2.30). Said in an other way, considering together the spin and internal degrees of freedom as part of the "grand Hilbert space" H_F gives precisely the number of dimension to represent the *grand algebra*

$$\mathcal{A}_G = \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C}). \tag{3.2}$$

This means that $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ can be represented on the same Hilbert space $\mathcal{H}$ as $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$. The only difference is that one needs to factorize $\mathcal{H}$ in (2.30) as $L^2(\mathcal{M}) \otimes \mathsf{H}_F$ instead of $sp(L^2(\mathcal{M})) \otimes \mathcal{H}_F$, that is possible at least in a local trivialization. It is a remarkable "coincidence" that the passage from the standard model to the grand algebra, namely from $a = 2$ to $a' = 4 = 2a$, requires to multiply the dimension of the internal Hilbert space by 4 (for $2(2a')^2 = 2(4a)^2 = 4(2(2a)^2)$) which is precisely the dimension of spinors in a space-time of dimension 4. Once more we stress that no new particles are introduced: $\mathcal{A}_F$ acts on $\mathcal{H}_F = \mathbb{C}^{32}$, $\mathcal{A}_G$ acts on $\mathsf{H}_F = \mathbb{C}^{128}$ but $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ and $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$ acts on the same Hilbert space $\mathcal{H}$. Since the Hilbert space is not changed, the Dirac operator will remain the same as in the standard model case, eq. (2.34).

The representation of the grand algebra $\mathcal{A}_G$ on H_F is more involved than the one of $\mathcal{A}_F$ on $\mathcal{H}_F$, given in section 2.4. In analogy with what was done earlier we consider an element of $\mathcal{A}_G$ as two 8×8 matrices, and see both of them having a block structure of four 4×4 matrices. Thus the component $Q\in M_4(\mathbb{H})$ of the grand algebra gets two new extra indices with respect to the quaternionic component of $\mathcal{A}_F$, and the same is true for $M\in\mathbb{M}_8(\mathbb{C})$. For the quaternions we choose to identify these two new indices with the spinor (anti)-particles indices $\dot{0},\dot{1}$; and for the complex matrices with the spinor left-right indices r,l introduced in §2.4. The choice is not unique: the other alternative, characterized by indices r,l for quaternions and $\dot{0},\dot{1}$ for complex matrices, is almost equivalent to the first one, since leads to the same final result. However in the next chapter we prefer changing representation, dealing with this last representation for technical details explained in §4.4.

In all the cases, having both sectors diagonal on different indices ensures that the order zero condition is satisfied, as explained below.

We have

$$Q=\begin{pmatrix} Q^{\dot{0}\beta}_{\dot{0}\alpha} & Q^{\dot{1}\beta}_{\dot{0}\alpha} \\ Q^{\dot{0}\beta}_{\dot{1}\alpha} & Q^{\dot{1}\beta}_{\dot{1}\alpha}\end{pmatrix}_{\dot{s}\dot{t}}\in\mathbb{M}_4(\mathbb{H}),\quad M=\begin{pmatrix} M^{r\mathrm{I}}_{r\mathrm{J}} & M^{l\mathrm{I}}_{r\mathrm{J}} \\ M^{r\mathrm{I}}_{l\mathrm{J}} & M^{l\mathrm{I}}_{l\mathrm{J}}\end{pmatrix}_{st}\in\mathbb{M}_8(\mathbb{C}) \tag{3.3}$$

where, for any $\dot{s},\dot{t}\in\{\dot{0},\dot{1}\}$ and $s,t\in\{l,r\}$, the matrices

$$Q^{\dot{t}\beta}_{\dot{s}\alpha}\in\mathbb{M}_2(\mathbb{H}),\quad M^{t\mathrm{I}}_{s\mathrm{J}}\in\mathbb{M}_4(\mathbb{C}) \tag{3.4}$$

and the representation of the element $A=(Q,M)\in\mathcal{A}_G$ is[1]:

$$A^{\mathsf{C}\,t\,\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}=\left(\delta^{\mathsf{C}}_0\delta^t_s\delta^{\mathrm{I}}_{\mathrm{J}}\,Q^{\dot{t}\beta}_{\dot{s}\alpha}+\delta^{\mathsf{C}}_1 M^{t\mathrm{I}}_{s\mathrm{J}}\delta^{\dot{t}}_{\dot{s}}\delta^{\beta}_{\alpha}\right). \tag{3.5}$$

This representation is to be compared with (2.33). As before the quaternionic part acts on the particle sector of the internal indices (δ^{C}_0) and the complex part on the antiparticle sector (δ^{C}_1). The difference is that the grand algebra acts in a non-diagonal way not only on the flavour and lepto-colour indices α,I, but also on the s and $\dot{s}$ indices. The novelty is in this mixing of internal and space-time indices: at the grand algebra level, the spin structure is somehow hidden. Specifically, the representation (3.5) is not invariant under the action of the Lorentz group (or rather of $Spin(4)$ since we are dealing with spin representation, in euclidean signature) as addressed in §3.3.2.

The representation of $C^\infty(\mathcal{M})\otimes\mathcal{A}_G$ is given by (3.5) where the entries of Q and M are now functions on $\mathcal{M}$. Since the total Hilbert space $\mathcal{H}$ is unchanged,

[1] To take into account the non-diagonal action of Q and M, it is convenient to change the order of the indices with respect to (2.33). We now adopt the order: $C,s,I,\dot{s},\alpha$.

there is not reason to change the real structure and the grading. In particular one easily checks that the order zero condition holds true for the grand algebra

$$\left[A, JBJ^{-1}\right] = 0 \quad \forall A, B \in \mathcal{A}_G. \tag{3.6}$$

This is because the real structure J in (2.36) acts as the charge conjugation operator

$$J_M = i\gamma^0\gamma^2 cc = i \begin{pmatrix} \overline{\sigma}^{2}{}_{\dot{s}}^{\dot{t}} & 0_2 \\ 0_2 & \sigma^{2}{}_{\dot{s}}^{\dot{t}} \end{pmatrix}_{st} cc \tag{3.7}$$

on the spinor indices, and as J_F, eq. (2.37), in $\mathcal{H}_F$ (where it exchanges the two blocks corresponding to particles and antiparticles). In tensorial notations one has

$$(J\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = -i\eta^t_s\,\tau^{\dot{t}}_{\dot{s}}\,\xi^{\mathsf{C}}_{\mathsf{D}}\,\delta^{\mathrm{I}}_{\mathsf{J}}\,\delta^{\beta}_{\alpha}\,\bar{\Psi}^{\mathsf{DJ}}_{t\dot{t}\beta} \tag{3.8}$$

where we use Einstein summation and define

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathsf{CD}}, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{st}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}\dot{t}}. \tag{3.9}$$

Hence J preserves the indices structure in (3.5), apart from the exchange $\delta^{\mathsf{C}}_0 \leftrightarrow \delta^{\mathsf{C}}_1$: since Q and M act on different indices, the commutation (3.6) is assured. Notice that without the enlargement of the action of the finite dimensional algebra to the spinorial indices, it would have been impossible to find a representation of $\mathcal{A}_G$ which satisfies the order zero condition, unless one adds more fermions. In this respect the grand algebra is not anymore an internal algebra.

3.2 The Majorana coupling and the σ field

In the following we see how the grand algebra makes possible to have a Majorana mass giving rise to the field σ. Although the calculations are quite involved, the principle is quite simple. Since we have a larger algebra, the Majorana Dirac operator needs not be diagonal in the spin indices. This added degree of freedom enables the possibility to satisfy the order one condition in a non trivial way, namely to still have a one form which commutes with the opposite algebra, but that at the same time gives rise to a field. In the following we will show this analytically, all calculations have also been performed with a symbolic manipulation program, leading to the same results.

We first show in §3.2.1 the effects of the grading condition on the grand algebra, leading to a first reduced algebra. Then we work out in §3.2.2 the most general Dirac operator D_M with Majorana coupling compatible with the grading condition and the KO dimension of the spectral triple of the standard model. In §3.2.3 we study the first order condition induced by D_M and the subsequent reduction $\mathcal{A}_G \to \mathcal{A}''_G$ of the grand algebra. Finally we show in §3.2.4 that D_M can be fluctuated by $\mathcal{A}''_G$ so that to generate the field σ as required by (3.1).

3.2.1 Reduction due to grading

In a way similar to the reduction $\mathcal{A}_F \to \mathcal{A}_{LR}$ of §2.3.1, the grading condition imposes a reduction $\mathcal{A}_G \to \mathcal{A}'_G$ where

$$\mathcal{A}'_G = (\mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R) \oplus (\mathbb{M}_4(\mathbb{C})_l \oplus \mathbb{M}_4(\mathbb{C})_r)\,. \tag{3.10}$$

To see it, recall that the chirality Γ in (2.40) acts as $\gamma^5 = \eta^t_s \delta^{\dot t}_{\dot s}$ on the spin indices, and as γ_F on the internal indices:

$$(\Gamma\Psi)^{\mathsf{CI}}_{s\dot s\alpha} = \eta^t_s \delta^{\dot t}_{\dot s}\; \eta^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathrm{I}}_{\mathrm{J}}\, \eta^{\beta}_{\alpha}\; \Psi^{\mathsf{DJ}}_{t\dot t\beta} \tag{3.11}$$

where $\eta^{\mathsf{C}}_{\mathsf{D}}$ and η^{β}_{α} are defined as in (2.39). Changing the order of the indices – so that to match (3.5) – one has

$$\Gamma = \eta^{\mathsf{C}}_{\mathsf{D}}\, \eta^t_s\, \delta^{\mathrm{I}}_{\mathrm{J}}\, \delta^{\dot t}_{\dot s}\, \eta^{\beta}_{\alpha}. \tag{3.12}$$

Since the representation of $\mathcal{A}_G$ is diagonal in the C index, the grading condition is satisfied if and only if it is satisfied by both sectors - quaternionic and complex - independently.

For quaternions, one asks $[\eta^t_s\delta^{\mathrm{I}}_{\mathrm{J}}\delta^{\dot t}_{\dot s}\eta^{\beta}_{\alpha}, \delta^t_s\delta^{\mathrm{I}}_{\mathrm{J}}Q^{\dot t\beta}_{\dot s\alpha}] = 0$, that is $[\delta^{\dot t}_{\dot s}\eta^{\beta}_{\alpha}, Q^{\dot t\beta}_{\dot s\alpha}] = 0$. This imposes

$$Q = \begin{pmatrix} \mathsf{Q}^{\dot 0\beta}_{\dot 0\alpha} & \mathsf{Q}^{\dot 1\beta}_{\dot 0\alpha} \\ \mathsf{Q}^{\dot 0\beta}_{\dot 1\alpha} & \mathsf{Q}^{\dot 1\beta}_{\dot 1\alpha} \end{pmatrix}_{\dot s\dot t} \tag{3.13}$$

where for any $\dot s, \dot t \in \{\dot 0, \dot 1\}$ one has

$$\mathsf{Q}^{\dot s\beta}_{\dot t\alpha} = \begin{pmatrix} q_R{}^{\dot s}_{\dot t} & 0_2 \\ 0_2 & q_L{}^{\dot s}_{\dot t} \end{pmatrix}_{\alpha\beta} \quad \text{with} \quad q_R{}^{\dot s}_{\dot t}, q_L{}^{\dot s}_{\dot t} \in \mathbb{H}. \tag{3.14}$$

Elements of the kind (3.13) generates $\mathbb{M}_2(\mathbb{H})_R \oplus \mathbb{M}_2(\mathbb{H})_L$. Hence the reduction

$$\mathbb{M}_4(\mathbb{H}) \to \mathbb{M}_2(\mathbb{H})_R \oplus \mathbb{M}_2(\mathbb{H})_L. \tag{3.15}$$

For matrices, one asks $[\eta^t_s\delta^{\mathrm{I}}_{\mathrm{J}}\delta^{\dot t}_{\dot s}\eta^{\beta}_{\alpha}, M^{t\mathrm{I}}_{s\mathrm{J}}\delta^{\dot t}_{\dot s}\,\delta^{\beta}_{\alpha}] = 0$, that is $[\eta^t_s\delta^{\mathrm{I}}_{\mathrm{J}}, M^{t\mathrm{I}}_{s\mathrm{J}}] = 0$. This forces

$$M = \begin{pmatrix} M^{r\mathrm{I}}_{r\mathrm{J}} & 0_4 \\ 0_4 & M^{l\mathrm{I}}_{l\mathrm{J}} \end{pmatrix}_{st}, \tag{3.16}$$

meaning the reduction

$$\mathbb{M}_8(\mathbb{C}) \to \mathbb{M}_4(\mathbb{C})_r \oplus \mathbb{M}_4(\mathbb{C})_l. \tag{3.17}$$

Hence the reduction of the grand algebra to $\mathcal{A}'_G$. Notice that the grading causes a reduction not only in the quaternionic sector, as in the case of $\mathcal{A}_F$, but also in the complex matrix part. This is because $\mathcal{A}_G$ is not anymore acting only on internal indices.

3.2.2 Dirac operator with Majorana mass term

We will consider a Majorana-like mass only for the right handed neutrinos. The natural mass scale of this matrix is very high, so that it provides a natural see-saw mechanism (although in realistic scheme the right handed neutrino mass is somewhat lower than the Planck scale). The standard model can be considered as a low energy limit of the theory we present in this section. We will assume therefore that all the quantities involved in the internal Dirac operator D_F, except the Majorana coupling, are small compared to the scale of the breaking described here. We take advantage of the flexibility introduced by the grand algebra and we do *not* assume a priori that the Majorana coupling is diagonal on the spin indices. This means that instead of $\gamma^5 \otimes D_F$ as in (2.34) we consider a finite dimensional matrix D_M containing a Majorana mass term with non trivial action on the spin indices. Right handed neutrinos have indices $\mathrm{I} = 0$ and $\alpha = 1$, so that the most general Majorana coupling matrix is

$$D_M = \mathcal{R} \otimes D_R = \begin{pmatrix} 0_{64} & \mathsf{D}_M \\ \mathsf{D}_M^\dagger & 0_{64} \end{pmatrix}_{\mathsf{CD}} \quad \text{with} \quad \mathsf{D}_M = \mathcal{R}^{t\dot{t}}_{s\dot{s}} \;\; \Xi^{\mathrm{I}}_{\mathrm{J}} \; \Xi^{\beta}_{\alpha} \tag{3.18}$$

where $\mathcal{R}$ is - at this stage - an arbitrary 4×4 complex matrix while Ξ is the projector on the first component

$$\Xi = \begin{pmatrix} 1 & 0 \\ 0 & 0_3 \end{pmatrix}. \tag{3.19}$$

The constraints on the matrix $\mathcal{R}$ come from the grading condition and the real structure. Remembering (3.12), one has that $\Xi^{\mathrm{I}}_{\mathrm{J}}$ and Ξ^{β}_{α} commute with $\delta^{\mathrm{I}}_{\mathrm{J}}$ and η^{β}_{α}, while the r.h.s. of (3.18) as a matrix in CD anticommutes with $\eta^{\mathsf{C}}_{\mathsf{D}}$. So D_M anticommutes with Γ if and only if $\mathcal{R}$ commutes with γ^5, meaning that $\mathcal{R}$ is block diagonal

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}^{r\dot{t}}_{r\dot{s}} & 0_2 \\ 0_2 & \mathcal{R}^{l\dot{t}}_{l\dot{s}} \end{pmatrix}_{st} =: \begin{pmatrix} \mathsf{r}^{\dot{t}}_{\dot{s}} & 0_2 \\ 0_2 & \mathsf{l}^{\dot{t}}_{\dot{s}} \end{pmatrix}_{st}. \tag{3.20}$$

The requirement to have KO-dimension $2 \bmod 8$ means that $JD_M = D_M J$. Remembering (2.38), this is equivalent to

$$\left[-i \begin{pmatrix} 0_4 & \eta^t_s \tau^{\dot t}_{\dot s} \\ \eta^t_s \tau^{\dot t}_{\dot s} & 0_4 \end{pmatrix}_{CD} cc\,, \begin{pmatrix} 0_4 & \mathcal{R}^{t\dot t}_{s\dot s} \\ \mathcal{R}^{\dagger\, t\dot t}_{\ s\dot s} & 0_4 \end{pmatrix}_{CD} \right] = 0, \tag{3.21}$$

that is

$$(\tau \otimes \eta)\mathcal{R}^T - \mathcal{R}(\tau \otimes \eta) = 0, \quad (\tau \otimes \eta)\bar{\mathcal{R}} - \mathcal{R}^\dagger(\tau \otimes \eta) = 0. \tag{3.22}$$

By (3.20), the first equation above yields (omitting the st and $\dot s\dot t$ indices)

$$\begin{pmatrix} \tau & 0_2 \\ 0_2 & -\tau \end{pmatrix} \begin{pmatrix} \mathsf{r}^T & 0_2 \\ 0_2 & \mathsf{l}^T \end{pmatrix} - \begin{pmatrix} \mathsf{r} & 0_2 \\ 0_2 & \mathsf{l} \end{pmatrix} \begin{pmatrix} \tau & 0_2 \\ 0_2 & -\tau \end{pmatrix} = 0 \tag{3.23}$$

i.e. $\mathsf{r}\tau = \tau \mathsf{r}^T$ and $\mathsf{l}\tau = \tau \mathsf{l}^T$, whose solution is

$$\mathsf{r}^{\dot t}_{\dot s} = y_R \delta^{\dot t}_{\dot s}, \ \mathsf{l}^{\dot t}_{\dot s} = y_l \delta^{\dot t}_{\dot s} \qquad y_R, y_l \in \mathbb{C}. \tag{3.24}$$

The second equation in (3.22) is then satisfied as well.

Relations (3.18), (3.20) and (3.24) give the most general Dirac operator D_M on $L^2(\mathbb{R}^4) \otimes \mathsf{H}_F$, with Majorana mass term, coupling the right neutrino with its anti-particle. In tensorial notations, one has

$$\mathsf{D}_M = \kappa^t_s \, \Xi^{\mathrm{I}}_{\mathrm{J}} \, \delta^{\dot t}_{\dot s} \, \Xi^\beta_\alpha \quad \text{where} \quad \kappa = \begin{pmatrix} y_R & 0 \\ 0 & y_l \end{pmatrix}_{st}. \tag{3.25}$$

By choosing $y_R = -y_l = 1$, one gets $\mathcal{R} = \gamma^5$ and one retrieves the Majorana coupling

$$D_M = \gamma^5 \otimes D_R \tag{3.26}$$

of the standard model. However, at this stage nothing forces us to make this choice.

3.2.3 First order condition for Majorana Dirac operator

We aim at obtaining the field σ as a fluctuation of D_M, compatible with the first order condition. By (3.5) a generic element (Q, M) of $\mathcal{A}_G$ acts as[2]

$$A = \begin{pmatrix} \delta^{t\mathrm{I}}_{s\mathrm{J}} \, Q^{\dot t\beta}_{\dot s\alpha} & 0_{64} \\ 0_{64} & M^{t\mathrm{I}}_{s\mathrm{J}} \, \delta^{\dot t\alpha}_{\dot s\beta} \end{pmatrix}_{\mathsf{CD}} =: \begin{pmatrix} \mathsf{Q} & 0_{64} \\ 0_{64} & \mathsf{M} \end{pmatrix}_{\mathsf{CD}}. \tag{3.27}$$

[2]To lighten notation, for any pairs of indices x, y and u, v we write $\delta^{xu}_{yv} = \delta^x_y \delta^u_v$.

As well, with $B = (R, N) \in \mathcal{A}_G$, a generic element of the opposite algebra is

$$JBJ^{-1} = -JBJ = -\begin{pmatrix} \tilde{N}^{t\mathrm{I}}_{s\mathrm{J}}\,\delta^{t\alpha}_{\dot{s}\beta} & 0_{64} \\ 0_{64} & \delta^{t\mathrm{I}}_{s\mathrm{J}}\tilde{R}^{t\beta}_{\dot{s}\beta} \end{pmatrix}_{\mathsf{CD}} = -\begin{pmatrix} \tilde{\mathsf{N}} & 0_{64} \\ 0_{64} & \tilde{\mathsf{R}} \end{pmatrix}_{\mathsf{CD}} \tag{3.28}$$

where we define

$$\tilde{R}^{t\beta}_{\dot{s}\alpha} = (\tau\bar{R}\tau)^{t\beta}_{\dot{s}\alpha}, \quad \tilde{N}^{t\mathrm{I}}_{s\mathrm{J}} = -(\eta\bar{N}\eta)^{t\mathrm{I}}_{s\mathrm{J}} = -\bar{N}^{t\mathrm{I}}_{s\mathrm{J}}. \tag{3.29}$$

The first order condition for D_M means that

$$\begin{aligned} 0 = \quad & \left[\left[D_M, A\right], JBJ^{-1}\right] = \left[\left[\begin{pmatrix} 0_{64} & \mathsf{D}_M \\ \mathsf{D}_M^\dagger & 0_{64} \end{pmatrix}_{\mathsf{CD}}, \begin{pmatrix} \mathsf{Q} & 0_{64} \\ 0_{64} & \mathsf{M} \end{pmatrix}_{\mathsf{CD}}\right], \begin{pmatrix} \tilde{\mathsf{N}} & 0_{64} \\ 0_{64} & \tilde{\mathsf{R}} \end{pmatrix}_{\mathsf{CD}}\right] \\ = \quad & \begin{pmatrix} 0_{64} & \mathsf{D}_M\mathsf{M}\tilde{\mathsf{R}} - \mathsf{Q}\mathsf{D}_M\tilde{\mathsf{R}} - \tilde{\mathsf{N}}\mathsf{D}_M\mathsf{M} + \tilde{\mathsf{N}}\mathsf{Q}\mathsf{D}_M \\ \mathsf{D}_M^\dagger\mathsf{Q}\tilde{\mathsf{N}} - \mathsf{M}\mathsf{D}_M^\dagger\tilde{\mathsf{N}} - \tilde{\mathsf{R}}\mathsf{D}_M^\dagger\mathsf{Q} + \tilde{\mathsf{R}}\mathsf{M}\mathsf{D}_M^\dagger & 0_{64} \end{pmatrix}_{\mathsf{CD}}. \end{aligned}$$

We look for solutions that satisfy the grading condition, i.e. in $\mathcal{A}'_G$. Inspired by the first order condition for $\mathcal{A}_{LR}$ and D_F described in [36, appendix A], we also impose the reductions

$$\mathbb{M}_4(\mathbb{C})_r \to \mathbb{C}_r \oplus \mathbb{M}_3(\mathbb{C})_r, \quad \mathbb{M}_4(\mathbb{C})_l \to \mathbb{C}_l \oplus \mathbb{M}_3(\mathbb{C})_l \tag{3.30}$$

as well as

$$\mathbb{M}_2(\mathbb{H})_R \to \mathbb{H}_R \oplus \mathbb{H}'_R, \quad \mathbb{M}_2(\mathbb{H})_L \to \mathbb{H}_L \oplus \mathbb{H}'_L. \tag{3.31}$$

The reduction (3.30) is obtained assuming that the components in (3.16) are $(i, j = 1, 2, 3)$

$$\begin{aligned} M^{r\mathrm{I}}_{r\mathrm{J}} &= \begin{pmatrix} M^{r0}_{r0} & 0 \\ 0 & M^{ri}_{rj} \end{pmatrix}_{\mathrm{IJ}} =: \begin{pmatrix} m_r & 0 \\ 0 & M^{ri}_{rj} \end{pmatrix}_{\mathrm{IJ}} \quad m_r \in \mathbb{C}_r, \\ M^{l\mathrm{I}}_{l\mathrm{J}} &= \begin{pmatrix} M^{l0}_{l0} & 0 \\ 0 & M^{li}_{lj} \end{pmatrix}_{\mathrm{IJ}} =: \begin{pmatrix} m_l & 0 \\ 0 & M^{li}_{lj} \end{pmatrix}_{\mathrm{IJ}} \quad m_l \in \mathbb{C}_l. \end{aligned} \tag{3.32}$$

The reduction (3.31) is obtained imposing that the off-diagonal part of Q in (3.13) is zero:

$$Q = \begin{pmatrix} \mathsf{Q}^{\dot{0}\beta}_{\dot{0}\alpha} & 0_4 \\ 0_4 & \mathsf{Q}^{\mathrm{i}\beta}_{\mathrm{i}\alpha} \end{pmatrix}_{\dot{s}t} \tag{3.33}$$

where

$$\mathsf{Q}^{\dot{0}\beta}_{\dot{0}\alpha} = \begin{pmatrix} q_R & 0_2 \\ 0_2 & q_L \end{pmatrix}_{\alpha\beta}, \quad \mathsf{Q}^{\mathrm{i}\beta}_{\mathrm{i}\alpha} = \begin{pmatrix} q'_R & 0_2 \\ 0_2 & q'_L \end{pmatrix}_{\alpha\beta} \quad q_{R,L} \in \mathbb{H}_{R,L}, \quad q'_{R,L} \in \mathbb{H}'_{R,L}. \tag{3.34}$$

Finally we impose that q_R and q'_R are diagonal quaternions, that is

$$q_R = \begin{pmatrix} c_R & 0 \\ 0 & \bar{c}_R \end{pmatrix}_{\dot{s}t}, \quad q'_R = \begin{pmatrix} c'_R & 0 \\ 0 & \bar{c}'_R \end{pmatrix}_{\dot{s}t} \quad \text{with } c_R, c'_R \in \mathbb{C}, \tag{3.35}$$

meaning the reduction $\mathbb{H}_R \oplus \mathbb{H}'_R \to \mathbb{C}_R \oplus \mathbb{C}'_R$. We thus look for solutions of (3.30) in

$$\mathcal{A}''_G = (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}_l \oplus \mathbb{M}_3(\mathbb{C})_l \oplus \mathbb{C}_r \oplus \mathbb{M}_3(\mathbb{C})_r)\,. \tag{3.36}$$

Notice that we do *not* claim there is no solution of (3.30) outside $\mathcal{A}''_G$. But for our purposes, it turns out that it is sufficient to work with $\mathcal{A}''_G$.

Under these conditions, the first equation coming from (3.30), namely

$$\mathsf{D}_M\mathsf{M}\tilde{\mathsf{R}} - \mathsf{Q}\mathsf{D}_M\tilde{\mathsf{R}} - \tilde{\mathsf{N}}\mathsf{D}_M\mathsf{M} + \tilde{\mathsf{N}}\mathsf{Q}\mathsf{D}_M = 0, \tag{3.37}$$

has explicit components

$$\begin{aligned}
\mathsf{D}_M\mathsf{M}\tilde{\mathsf{R}} = (\kappa^t_s\,\Xi^{\mathrm{I}}_{\mathrm{J}}\,\delta^{\dot t}_{\dot s}\,\Xi^{\beta}_{\alpha})(M^{t\mathrm{I}}_{s\mathrm{J}}\,\delta^{\dot t\alpha}_{\dot s\beta})(\delta^{t\mathrm{J}}_{s\mathrm{I}}\tilde{R}^{\dot t\beta}_{\dot s\alpha}) &= (\kappa\Xi M)^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi\tilde{R})^{\dot t\beta}_{\dot s\alpha}\\
&= \begin{pmatrix} y_R\,\mathsf{m}_r & 0_4\\ 0_4 & y_l\mathsf{m}_l\end{pmatrix}_{st} \otimes \begin{pmatrix} -\bar{\mathsf{d}}'_R & 0_4\\ 0_4 & -\bar{\mathsf{d}}_R\end{pmatrix}_{\dot s\dot t};\\
\mathsf{Q}\mathsf{D}_M\tilde{\mathsf{R}} = (\delta^{t\mathrm{J}}_{s\mathrm{I}}Q^{\dot t\beta}_{\dot s\alpha})(\kappa^t_s\,\Xi^{\mathrm{I}}_{\mathrm{J}}\,\delta^{\dot t}_{\dot s}\,\Xi^{\beta}_{\alpha})(\delta^{t\mathrm{J}}_{s\mathrm{I}}\tilde{R}^{\dot t\beta}_{\dot s\alpha}) &= (\kappa\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(Q\Xi\tilde{R})^{\dot t\beta}_{\dot s\alpha}\\
&= \begin{pmatrix} y_R\,\Xi & 0_4\\ 0_4 & y_l\,\Xi\end{pmatrix}_{st} \otimes \begin{pmatrix} -\mathsf{c}_R\bar{\mathsf{d}}'_R & 0_4\\ 0_4 & -\mathsf{c}'_R\bar{\mathsf{d}}_R\end{pmatrix}_{\dot s\dot t};\\
\tilde{\mathsf{N}}\mathsf{D}_M\mathsf{M} = (\tilde{\mathsf{N}}^{t\mathrm{I}}_{s\mathrm{J}}\delta^{\dot t\beta}_{\dot s\alpha})(\kappa^t_s\,\Xi^{\mathrm{I}}_{\mathrm{J}}\,\delta^{\dot t}_{\dot s}\,\Xi^{\beta}_{\alpha})(\mathsf{M}^{t\mathrm{I}}_{s\mathrm{J}}\delta^{\dot t\beta}_{\dot s\alpha}) &= ((\tilde{\mathsf{N}}\kappa\Xi M)^{t\mathrm{I}}_{s\mathrm{J}}\,(\delta\Xi)^{\dot t\beta}_{\dot s\alpha} =\\
&= \begin{pmatrix} -y_R\bar{\mathsf{n}}_r\mathsf{m}_r & 0_4\\ 0_4 & -y_l\bar{\mathsf{n}}_l\mathsf{m}_l\end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4\\ 0_4 & \Xi\end{pmatrix}_{\dot s\dot t};\\
\tilde{\mathsf{N}}\mathsf{Q}\mathsf{D}_M = (\tilde{\mathsf{N}}^{t\,\mathrm{I}}_{s\mathrm{J}}\,\delta^{\dot t\alpha}_{\dot s\beta})(\delta^{t\mathrm{J}}_{s\mathrm{I}}\mathsf{Q}^{\dot t\beta}_{\dot s\alpha})(\kappa^t_s\,\Xi^{\mathrm{I}}_{\mathrm{J}}\,\delta^{\dot t}_{\dot s}\,\Xi^{\beta}_{\alpha}) &= (\tilde{\mathsf{N}}\kappa\Xi)^{t\,\mathrm{I}}_{s\mathrm{J}}(\mathsf{Q}\Xi)^{\dot t\beta}_{\dot s\alpha} =\\
&= \begin{pmatrix} -y_R\bar{\mathsf{n}}_r & 0_4\\ 0_4 & -y_l\bar{\mathsf{n}}_l\end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4\\ 0_4 & \mathsf{c}'_R\end{pmatrix}_{\dot s\dot t}
\end{aligned} \tag{3.38}$$

where we defined the 4×4 complex matrices

$$\mathsf{m}_{r,l} = \begin{pmatrix} m_{r,l} & 0\\ 0 & 0_3\end{pmatrix}_{\mathrm{IJ}} \quad \mathsf{c}_{R,L} = \begin{pmatrix} c_{R,L} & 0\\ 0 & 0_3\end{pmatrix}_{\alpha\beta} \quad \mathsf{c}'_{R,L} = \begin{pmatrix} c'_{R,L} & 0\\ 0 & 0_3\end{pmatrix}_{\alpha\beta} \tag{3.39}$$

with m_r, m_l the components of M and c_R, c'_R the one of Q. Similarly we define the matrices $\mathsf{n}_{r,l}$ from the components $n_{l,r}$ of N, and the matrices $\mathsf{d}, \mathsf{d}'_R$ from the components d_R, d'_R of R. The matrix Ξ carries the indices I, J in the second equation, and α, β in the third. In each equation, to pass from the first to the second lines one uses (3.29).

Collecting the components and assuming that both y_R and y_l are non zero, one finds that (3.37) is equivalent to

$$\begin{aligned}
(c_R - m_r)(\bar{n}_r - \bar{d}'_R) = 0, \quad (\bar{d}_R - \bar{n}_r)(m_r - c'_R) = 0\\
(c_R - m_l)(\bar{n}_l - \bar{d}'_R) = 0, \quad (\bar{d}_R - \bar{n}_l)(m_l - c'_R) = 0.
\end{aligned} \tag{3.40}$$

A similar calculation for the second components of (3.30) yields the same system of equations. Thus one solution to the first order condition induced by D_M is to impose

$$c_R = m_r = m_l \text{ and } d_R = n_r = n_l, \tag{3.41}$$

meaning the reduction of $\mathcal{A}''_G$ to

$$\mathcal{A}'''_G = \mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}'_R \oplus \mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})_l \oplus \mathbb{M}_3(\mathbb{C})_r. \tag{3.42}$$

3.2.4 The σ field as a 1-form

We now consider the set of 1-forms $\sum_i B_i[D_M, A_i]$ generated by the Majorana Dirac operator and the algebra $\mathcal{A}'''_G$ above. We are interested in showing that this set is non empty, and it is enough to consider the simplest 1-form

$$[D_M, A] = \begin{pmatrix} 0_{64} & \mathsf{D}_M\mathsf{M} - \mathsf{Q}\mathsf{D}_M \\ \mathsf{D}_M^\dagger\mathsf{Q} - \mathsf{M}\mathsf{D}_M^\dagger & 0_{64} \end{pmatrix}. \tag{3.43}$$

We begin with $A = (Q, M)$ in $\mathcal{A}'_G$. With notations of the precedent section, one has

$$\begin{aligned}
\mathsf{D}_M\mathsf{M} - \mathsf{Q}\mathsf{D}_M &= (\kappa_s^t\, \Xi_{\mathrm{J}}^{\mathrm{I}}\, \delta_{\dot s}^{\dot t}\, \Xi_\alpha^\beta)(M_{s\mathrm{J}}^{t\mathrm{I}}\delta_{\dot s\alpha}^{\dot t\beta}) - (\delta_{s\mathrm{I}}^{t\mathrm{J}} Q_{\dot s\alpha}^{\dot t\beta})(\kappa_s^t \Xi_{\mathrm{J}}^{\mathrm{I}} \delta_{\dot s}^{\dot t} \Xi_\alpha^\beta) \\
&= (\kappa\Xi M)_{s\mathrm{J}}^{t\mathrm{I}} (\Xi\delta)_{\dot s\alpha}^{\dot t\beta} - (\kappa\Xi)_{s\mathrm{I}}^{t\mathrm{J}} (Q\Xi)_{\dot s\alpha}^{\dot t\beta} \\
&= \begin{pmatrix} y_R\mathsf{m}_r & 0_4 \\ 0_4 & y_l\mathsf{m}_l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot s\dot t} - \begin{pmatrix} y_R\,\Xi & 0_4 \\ 0_4 & y_l\,\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}'_R \end{pmatrix}_{\dot s\dot t} \\
&= \begin{pmatrix} \begin{pmatrix} y_R(m_r - c_R)\Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta} & 0 \\ 0 & y_R(m_r - c'_R)\Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta} \end{pmatrix}_{\dot s\dot t} & 0_{32} \\ 0_{32} & \begin{pmatrix} y_l(m_l - c_R)\Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta} & 0_{32} \\ 0_{32} & y_l(m_l - c'_R)\Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta} \end{pmatrix}_{\dot s\dot t} \end{pmatrix}_{st}.
\end{aligned}$$

By the reduction $\mathcal{A}'_G \to \mathcal{A}'''_G$, the component $y_R(m_r - c_R)$ vanishes, but the component $y_l(m_l - c'_R)$ does not. This is the crucial difference with the algebra of the standard model: the grand algebra allows to generate a non-vanishing 1-form associated to the Majorana Dirac operator D_M, which satisfies the first order condition.

Restoring the order $s\dot s\mathrm{I}\alpha$ of the indices, the matrix above is $\mathcal{R} = \mathcal{R}_{s\dot s}^{t\dot t}\, \Xi_{\mathrm{I}}^{\mathrm{J}}\, \Xi_\alpha^\beta$ with

$$\mathcal{R}_{s\dot s}^{t\dot t} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y_R(m_r - c'_R) \end{pmatrix}_{\dot s\dot t} & 0_2 \\ 0_2 & \begin{pmatrix} 0 & 0 \\ 0 & y_l(m_r - c'_R) \end{pmatrix}_{\dot s\dot t} \end{pmatrix}_{st}. \tag{3.44}$$

For anti-selfadjoint A (that is $\mathsf{M} = -\mathsf{M}^\dagger, \mathsf{Q} = -\mathsf{Q}^\dagger$), one obtains the self-adjoint 1-form

$$[D_M, A] = \begin{pmatrix} 0_{64} & \mathcal{R} \\ \mathcal{R}^\dagger & 0_{64} \end{pmatrix}. \tag{3.45}$$

The conjugate action of the real structure J yields

$$J[D_M, A]J^{-1} = -J[D_M, A]J = -\begin{pmatrix} 0_{64} & \mathcal{J}\mathcal{R}^\dagger\mathcal{J} \\ \mathcal{J}\mathcal{R}\mathcal{J} & 0_{64} \end{pmatrix} \tag{3.46}$$

where the charge conjugation $\mathcal{J}$ acts only on the spin indices. Explicitly, omitting the factor $\Xi^{\mathrm{J}}_{\mathrm{I}}\,\Xi^{\beta}_{\alpha}$ in the expression of $\mathcal{R}$, one gets

$$\mathcal{J}\mathcal{R}^\dagger\mathcal{J} = \eta^t_s\,\tau^{\dot{t}}_{\dot{s}}\,(\mathcal{R}^T)^{t\dot{t}}_{s\dot{s}}\,\eta^t_s\,\tau^{\dot{t}}_{\dot{s}} = \begin{pmatrix} \tau^{\dot{t}}_{\dot{s}}\,\mathcal{R}^{r\dot{t}}_{r\dot{s}}\,\tau^{\dot{t}}_{\dot{s}} & 0_4 \\ 0_4 & \tau^{\dot{t}}_{\dot{s}}\,\mathcal{R}^{l\dot{t}}_{l\dot{s}}\,\tau^{\dot{t}}_{\dot{s}} \end{pmatrix}_{st} \tag{3.47}$$

$$= -\begin{pmatrix} \begin{pmatrix} y_R(m_r - c'_R) & 0 \\ 0 & 0 \end{pmatrix}_{\dot{s}\dot{t}} & 0_2 \\ 0_2 & \begin{pmatrix} y_l(m_r - c'_R) & 0 \\ 0 & 0 \end{pmatrix}_{\dot{s}\dot{t}} \end{pmatrix}_{st}, \tag{3.48}$$

that is $-\mathcal{J}\mathcal{R}^\dagger\mathcal{J}$ is obtained by permuting the components in the blocks $\dot{s}\dot{t}$ of $\mathcal{R}$. As well

$$\mathcal{J}\mathcal{R}\mathcal{J} = \eta^t_s\,\tau^{\dot{t}}_{\dot{s}}\,\bar{\mathcal{R}}^{t\dot{t}}_{s\dot{s}}\,\eta^t_s\,\tau^{\dot{t}}_{\dot{s}} = \eta^t_s\,\tau^{\dot{t}}_{\dot{s}}\,(\mathcal{R}^\dagger)^{t\dot{t}}_{s\dot{s}}\,\eta^t_s\,\tau^{\dot{t}}_{\dot{s}} \tag{3.49}$$

is obtained from $-\mathcal{R}^\dagger$ by permuting the components in $\dot{s}\dot{t}$. Consequently,

$$D_M + [D_M, A] + J[D_M, A]J^{-1} = \begin{pmatrix} 0_{64} & \mathcal{M}_\nu \\ \mathcal{M}_\nu^\dagger & 0_{64} \end{pmatrix} \tag{3.50}$$

where $\mathcal{M}_\nu = \mathsf{R}^{t\dot{t}}_{s\dot{s}}\Xi^{\mathrm{J}}_{\mathrm{I}}\Xi^{\beta}_{\alpha}$ with

$$\mathsf{R} = \begin{pmatrix} y_R(1 + (m_r - c'_R))\delta^{\dot{t}}_{\dot{s}} & 0_2 \\ 0_2 & y_l(1 + (m_r - c'_R))\delta^{\dot{t}}_{\dot{s}} \end{pmatrix}_{st}. \tag{3.51}$$

Now, considering that A is in $C^\infty(\mathcal{M}) \otimes \mathcal{A}''_G$, the coefficients m_r and c'_R becomes functions on the manifold $\mathcal{M}$. Taking $y_l = -y_R = y_R$, one obtains $\mathsf{R}^{t\dot{t}}_{s\dot{s}} = y_R\sigma\gamma^5$ where

$$\sigma = (1 + (m_r - c'_R)) \tag{3.52}$$

is now a field on $\mathcal{M}$. In other terms, the fluctuation of D_M by $\mathcal{A}_G$ yields the substitution (3.1). The grand algebra gives a justification for the presence of the field σ, necessary to obtain the mass of the Higgs in agreement with experiment.

3.3 Reduction to the standard model

Starting with the grand algebra $\mathcal{A}_G$ reduced to $\mathcal{A}'_G$ by the grading condition, we have shown how to generate the field σ by a fluctuation of the Majorana-Dirac operator D_M, in a way satisfying the first order condition imposed by D_M. As explained below (3.25), one can choose in particular $D_M = \gamma^5 \otimes D_R$, where D_R is the internal Dirac operator D_F of the standard model in which only the dominant term (i.e. the Majorana mass) is taken into account. In other words, the field σ is generated by fluctuating the second term in the Dirac operator (2.34) of the standard model. We now show that the first order condition of the first term in (2.34), i.e. the free Dirac operator, yields the reduction of the grand algebra to the standard model.

3.3.1 First order condition for the free Dirac operator

The first term in (2.34) is the Euclidean free Dirac operator, extended trivially to the internal space of one generation. In tensorial notation it reads

$$\not{D} := \not{\partial} \otimes \mathbb{I}_{32} = -i\, \delta^{\mathsf{CI}\beta}_{\mathsf{DJ}\alpha}\, \gamma^\mu \partial_\mu. \tag{3.53}$$

For $A = (Q, M) \in C^\infty(\mathcal{M}) \otimes \mathcal{A}'_G$, the commutator

$$[\not{D}, A] = -i \begin{pmatrix} \delta^{\mathrm{I}}_{\mathrm{J}}[\gamma^\mu \partial_\mu \delta^\beta_\alpha, \delta^t_s Q^{i\beta}_{\dot{s}\alpha}] & 0_{64} \\ 0_{64} & [\gamma^\mu \partial_\mu \delta^{\mathrm{I}}_{\mathrm{J}}, M^{t\mathrm{I}}_{s\mathrm{J}}]\delta^\beta_\alpha \end{pmatrix}_{\mathsf{CD}} \tag{3.54}$$

has components (omitting the non relevant δ)

$$\begin{aligned} [\gamma^\mu \partial_\mu \delta^\beta_\alpha, \delta^t_s Q^{i\beta}_{\dot{s}\alpha}] &= \left[\begin{pmatrix} 0_8 & \sigma^{\mu}{}^{\dot{t}}_{\dot{s}} \partial_\mu \delta^\beta_\alpha \\ \bar{\sigma}^{\mu}{}^{\dot{t}}_{\dot{s}} \partial_\mu \delta^\beta_\alpha & 0_8 \end{pmatrix}_{st}, \begin{pmatrix} Q^{i\beta}_{\dot{s}\alpha} & 0_8 \\ 0_8 & Q^{i\beta}_{\dot{s}\alpha} \end{pmatrix}_{st} \right] \\ &= \begin{pmatrix} 0_8 & P^{i\beta}_{\dot{s}\alpha} + T^{i\beta,\mu}_{\dot{s}\alpha}\partial_\mu \\ \bar{P}^{i\beta}_{\dot{s}\alpha} + \bar{T}^{i\beta,\mu}_{\dot{s}\alpha}\partial_\mu & 0_8 \end{pmatrix}_{st} \end{aligned} \tag{3.55}$$

where

$$P^{i\beta}_{\dot{s}\alpha} = (\sigma^{\mu}{}^{\dot{u}}_{\dot{s}} \partial_\mu Q^{i\beta}_{\dot{u}\alpha}), \quad T^{i\beta,\mu}_{\dot{s}\alpha} = \left[\sigma^{\mu}{}^{\dot{t}}_{\dot{s}}, Q^{i\beta}_{\dot{s}\alpha}\right] \tag{3.56}$$

and similar definitions for $\bar{P}$ and $\bar{T}$ with $\bar{\sigma}$ instead of σ; and

$$\begin{aligned} [\gamma^\mu \partial_\mu \delta^{\mathrm{I}}_{\mathrm{J}}, M^{t\mathrm{I}}_{s\mathrm{J}} \delta^{\dot{t}}_{\dot{s}}] &= \left[\begin{pmatrix} 0_8 & \sigma^{\mu}{}^{\dot{t}}_{\dot{s}} \partial_\mu \delta^{\mathrm{I}}_{\mathrm{J}} \\ \bar{\sigma}^{\mu}{}^{\dot{t}}_{\dot{s}} \partial_\mu \delta^{\mathrm{I}}_{\mathrm{J}} & 0_8 \end{pmatrix}_{st}, \begin{pmatrix} M^{r\mathrm{I}}_{r\mathrm{J}} \delta^{\dot{t}}_{\dot{s}} & 0_8 \\ 0_8 & M^{l\mathrm{I}}_{l\mathrm{J}} \delta^{\dot{t}}_{\dot{s}} \end{pmatrix}_{st} \right] \\ &= \begin{pmatrix} 0_8 & L^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} + K^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}}\partial_\mu \\ \bar{L}^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} + \bar{K}^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}}\partial_\mu & 0_8 \end{pmatrix}_{st} \end{aligned} \tag{3.57}$$

where

$$
\begin{aligned}
L^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} &= \left(\sigma^{\mu\dot{t}}{}_{\dot{s}}\partial_\mu M^{l\mathrm{I}}_{l\mathrm{J}}\right), \quad K^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}} = \left(M^{l\mathrm{I}}_{l\mathrm{J}} - M^{r\mathrm{I}}_{r\mathrm{J}}\right)\sigma^{\mu\dot{t}}{}_{\dot{s}},\\
\bar{L}^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} &= \left(\overline{\sigma}^{\mu\dot{t}}{}_{\dot{s}}\partial_\mu M^{r\mathrm{I}}_{r\mathrm{J}}\right), \quad \bar{K}^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}} = \left(M^{r\mathrm{I}}_{r\mathrm{J}} - M^{l\mathrm{I}}_{l\mathrm{J}}\right)\overline{\sigma}^{\mu\dot{t}}{}_{\dot{s}}.
\end{aligned}
\tag{3.58}
$$

For $B = (R, N) \in \mathcal{A}'_G$, the commutator of $[\not{D}, A]$ with JBJ given in (3.28) is a block diagonal matrix in CD with components

$$
\begin{aligned}
&\left[\delta^{\mathrm{I}}_{\mathrm{J}}\,[\gamma^\mu\partial_\mu\delta^\beta_\alpha, \delta^t_s Q^{\dot{t}\beta}_{\dot{s}\alpha}], \tilde{N}^{t\mathrm{I}}_{s\mathrm{J}}\,\delta^{\dot{t}\beta}_{\dot{s}\alpha}\right] =\\
&= \left[\begin{pmatrix} 0_{32} & \delta^{\mathrm{I}}_{\mathrm{J}}(P^{\dot{t}\beta}_{\dot{s}\alpha} + T^{\dot{t}\beta,\mu}_{\dot{s}\alpha}\partial_\mu) \\ \delta^{\mathrm{I}}_{\mathrm{J}}(\bar{P}^{\dot{t}\beta}_{\dot{s}\alpha} + \bar{T}^{\dot{t}\beta,\mu}_{\dot{s}\alpha}\partial_\mu) & 0_{32}\end{pmatrix}_{st}, \begin{pmatrix} \tilde{N}^{r\mathrm{I}}_{r\mathrm{J}}\delta^{\dot{t}\beta}_{\dot{s}\alpha} & 0_{32} \\ 0_{32} & \tilde{N}^{l\mathrm{I}}_{l\mathrm{J}}\delta^{\dot{t}\beta}_{\dot{s}\alpha}\end{pmatrix}_{st}\right];
\end{aligned}
$$

$$
\begin{aligned}
&\left[[\gamma^\mu\partial_\mu\delta^{\mathrm{I}}_{\mathrm{J}}, M^{t\mathrm{I}}_{s\mathrm{J}}\delta^{\dot{t}}_{\dot{s}}]\,\delta^\beta_\alpha,\ \delta^{t\mathrm{I}}_{s\mathrm{J}}\,\tilde{R}^{\dot{t}\beta}_{\dot{s}\alpha}\right] =\\
&= \left[\begin{pmatrix} 0_{32} & (L^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} + K^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}}\partial_\mu)\delta^\beta_\alpha \\ (\bar{L}^{\mathrm{I}\dot{t}}_{\mathrm{J}\dot{s}} + \bar{K}^{\mathrm{I}\dot{t},\mu}_{\mathrm{J}\dot{s}}\partial_\mu)\delta^\beta_\alpha & 0_{32}\end{pmatrix}_{st}, \begin{pmatrix} \delta^{\mathrm{J}}_{\mathrm{I}}\,\tilde{R}^{\dot{t}\beta}_{\dot{s}\alpha} & 0_{32} \\ 0_{32} & \delta^{\mathrm{J}}_{\mathrm{I}}\,\tilde{R}^{\dot{t}\beta}_{\dot{s}\alpha}\end{pmatrix}_{st}\right].
\end{aligned}
\tag{3.59}
$$

Omitting the indices (and noticing that the $P, T, \bar{P}, \bar{T}$ all commute with $\tilde{N}^r_r, \tilde{N}^l_l$), the first components is a diagonal matrix with first entry

$$
(\tilde{N}^l_l - \tilde{N}^r_r)(P + T^\mu\partial_\mu) + T^\mu(\partial_\mu\tilde{N}^l_l). \tag{3.60}
$$

The vanishing of the differential operator part implies either $T^\mu = 0$ or $\tilde{N}^l_l = \tilde{N}^r_r$. But the expression should be zero in particular for non-constant fields, that is for $P \neq 0$. So in case one imposes $T^\mu = 0$, the vanishing of the term in P implies $\tilde{N}^l = \tilde{N}^r$. In case one imposes $\tilde{N}^l_l = \tilde{N}^r_r$, the vanishing of the remaining term implies either $T^\mu = 0$, or $\tilde{N}^l_l = \tilde{N}^r_r = \text{cst}$. The last solution is unacceptable, it would mean that space-time is reduced to a point, hence in any case one has both conditions: $T^\mu = 0$ and $\tilde{N}^l_l = \tilde{N}^r_r$. One then checks that the other components of (3.59) vanish as well.

The only matrix that commutes with all the Pauli matrices is the identity, therefore

$$
T^\mu = 0\ \forall\mu \Longleftrightarrow Q^{\dot{0}\beta}_{\dot{0}\alpha} = Q^{\dot{1}\beta}_{\dot{1}\alpha} \text{ and } Q^{\dot{0}\beta}_{\dot{1}\alpha} = Q^{\dot{1}\beta}_{\dot{0}\alpha} = 0, \tag{3.61}
$$

meaning the breaking

$$
\mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R \to \mathbb{H}_L \oplus \mathbb{H}_R. \tag{3.62}
$$

Meanwhile $\tilde{N}^l_l = \tilde{N}^r_r$ means that

$$
\mathbb{M}_4(\mathbb{C})_l \oplus \mathbb{M}_4(\mathbb{C})_r \to \mathbb{M}_4(\mathbb{C}). \tag{3.63}
$$

Thus
$$\mathcal{A}'_G \to \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}) \tag{3.64}$$
where representation of the r.h.s. algebra is now diagonal on the spinorial indices $\dot{s}, s$.

To summarize, the reduction of $\mathcal{A}_G$ to the algebra of the standard model is obtained as follows

$$\boxed{\begin{array}{rcl} \mathcal{A}_G &=& M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \\ && \Downarrow \text{ grading condition} \\ \mathcal{A}'_G &=& \mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}) \\ && \Downarrow \text{ 1}^{\text{st}}\text{-order for the Majorana-Dirac operator } D_M \\ \mathcal{A}''_G &=& (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}^l \oplus M_3^l(\mathbb{C}) \oplus \mathbb{C}^r \oplus M_3^r(\mathbb{C}))\text{with } \mathbb{C}_R = \mathbb{C}^r = \mathbb{C}^l \\ && \Downarrow \text{ 1}^{\text{st}}\text{-order for the free Dirac operator } \not{\partial} \\ \mathcal{A}_{sm} &=& \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \end{array}} \tag{3.65}$$

3.3.2 Emergence of spin

In noncommutative geometry the topological information is encoded in the algebra, while the geometry (e.g. the metric[3]) is in the Dirac operator. In particular the Riemann-spin structure is encoded in the way this operator, which contains the gamma matrices, acts on the Hilbert space. Without this operator there is just an algebra which acts in an highly reducible way on a 128 dimensional Hilbert space. This is conceptually what distinguishes $\mathcal{H}_F$ from H_F in (2.30): on $C^\infty(\mathcal{M}) \otimes \mathcal{H}_F$, the free Dirac operator (trivially extended to the internal indices) is
$$\not{\partial} = -i\gamma^\mu \partial_\mu \otimes \delta^{\mathsf{CI}\beta}_{\mathsf{DJ}\alpha}. \tag{3.66}$$
On $C^\infty(\mathcal{M}) \otimes \mathsf{H}_F$ the same operator writes
$$\not{\partial} = -i\partial_\mu \otimes \delta^{\mathsf{CI}\beta}_{\mathsf{DJ}\alpha}\gamma^\mu \tag{3.67}$$
and the spin structure, carried by the γ matrices, is hidden among the internal degrees of freedom. In this sense the first order condition, which governs

[3]The metric aspects of the almost commutative geometry of the standard model have been investigated in [14, 76]

the passage from (3.67) to (3.66), corresponds to the emergence of the spin structure.

Alternatively, a spin structure means that the vectors in the Hilbert space transform in a particular representation under the "Lorentz" group. Since we are dealing with spinors in the Euclidean case, the group is actually Spin(4). It is generated by the commutators of the Dirac matrices, that act on $\mathcal{H}$ as

$$S^{\mu\nu} := [\gamma^\mu, \gamma^\nu] \otimes \mathbb{I}_{32}^{(\mathrm{CI}\alpha)}. \tag{3.68}$$

Let us distinguish between an element a of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ and its representation $\pi(a) := A$ given in (3.5). For any $\Lambda = \lambda_{\mu\nu} S^{\mu\nu} \in \mathrm{Spin}(4)$ and $A \in \pi(C^\infty(\mathcal{M}) \otimes \mathcal{A}_G)$, let

$$\alpha_\Lambda A = U(\Lambda) A U(\Lambda)^*. \tag{3.69}$$

The representation (3.5) of the grand algebra is not invariant under the adjoint action (3.69) of Spin(4) since $\alpha_\Lambda \pi(a)$ is not in $\pi(C^\infty(\mathcal{M}) \otimes \mathcal{A}_G)$. In this sense the representation of the grand algebra is not Lorentz invariant, unlike its reduction to $\mathcal{A}_{LR}$ which is diagonal in the spin indices. However, at the abstract level the algebra is preserved under Lorentz transformations since the latter are implemented by unitary operators: for any Λ one has that $\alpha_\Lambda(\pi(C^\infty(\mathcal{M}) \otimes \mathcal{A}_G))$ is isomorphic to $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. This suggests to view the grand algebra as a phase of the universe in which the spin and rotation structure of space-time has not yet emerged, only the topology (i.e. the abstract algebra) is fixed.

Let us finally note that although the Grand Symmetry mixes space-time and internal symmetries combining them in a non trivial way, the model does not fall in the forbidden case of Coleman-Mandula theorem, for two reasons: on the one hand the Grand Symmetry separates the two components of the Lorentz symmetry and on the other hand the symmetries we are considering are not at the S-matrix level, dealing with a spontaneously broken symmetry.

3.3.3 Fiat neutrino

The grand algebra together with the Majorana Dirac operator D_M generates the field σ at the right position (i.e. as required in (3.1)), respecting the first order condition induced by D_M. However, by (3.52) one has that σ becomes constant when one takes into account the first order condition imposed by the free Dirac operator, because (3.61) implies that $c'_R = c_R = m_r$. This suggests a scenario in which the neutrino Majorana mass is the first field to appear and fluctuate, before the geometric structure of space-time emerges through the breaking described in §3.3.2. In this picture, the field σ is viewed as a fluctuation of a vacuum that satisfies the first order condition of the free Dirac operator.

This scenario is supported by the calculations of chapter 4 which indicate that the first order condition of the free Dirac operator can be equivalently obtained as a minimum of the spectral action. In this way, the geometrical breaking imposed by the mathematical requirement of the theory becomes a dynamical breaking, and the field σ appears as the "Higgs field" associated to it. This idea has been also investigated, in the case of the standard model algebra, in the papers [21, 22]. The case of the grand algebra has been developed in [37] and will be presented in the next chapter.

3.4 Comments

In this chapter we have shown the presence of a "next level" in noncommutative geometry, that it is intertwined with the Riemannian and spin structure of space-time, and therefore naturally appearing at a high scale. The added degrees of freedom of the Grand Symmetry model are related to the Riemann-spin structure of space-time, which emerges as a symmetry breaking very similar in nature to the Higgs mechanism. In addition, this higher symmetry explains the presence of the σ field necessary for a correct fit of the mass of the Higgs and to cure the instability problem of the electro-weak vacuum. To summarize, the grand algebra transfers the problem of generating σ from the noncommutative to the commutative part of the geometry: with the algebra of the standard model, $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$, the first order condition is always satisfied by the free Dirac operator, the problem is all in D_M. Using the grand algebra, we have that $D_M = \gamma^5 \otimes D_R$ both generates the field ÏČ and satisfies the first-order condition. But the free Dirac does not satisfies this condition (neither the bounded commutator one). Of course this is not satisfactory but this suggests interesting path to explore. Another question is whether the reduction to the SM imposed by the first order condition can be understood dynamically, i.e. by a minimization of the spectral action, and it will be the discussion topic of the next chapter.

Chapter 4

Twisting the Grand Symmetry

In [21, 22] it was shown how to obtain σ by an inner fluctuation that does not satisfy the first-order condition, but in such a way that the latter is retrieved dynamically, as a minimum of the spectral action. The field σ is then interpreted as an excitation around this minimum.

In the previous chapter, from a different approach, we have shown how to generate σ in agreement with the first-order condition, taking advantage of the fermion doubling in the Hilbert space $\mathcal{H}$ of the spectral triple of the SM [50, 67, 68]. We said that the algebra of the standard model

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{4.1}$$

is obtained from $\mathcal{A}_F$ for $a = 2$, by the grading and the first-order conditions. Starting instead with the "grand algebra" ($a = 4$)

$$\mathcal{A}_G := M_4(\mathbb{H}) \oplus M_8(\mathbb{C}), \tag{4.2}$$

one generates the field σ by a inner fluctuation which respects the first-order condition imposed by the part D_M of the Dirac operator that contains the Majorana coupling y_R. The breaking to $\mathcal{A}_{sm}$ is then obtained by the first-order condition imposed by the free Dirac operator

$$\not{D} = \not{\partial} \otimes \mathbb{I}_F \tag{4.3}$$

where $\mathbb{I}_F$ is the identity operator on the finite dimensional Hilbert space $\mathcal{H}_F$ on which acts $\mathcal{A}_G$.

Unfortunately, before this breaking not only is the first-order condition not satisfied, but the commutator

$$[\not{D}, A] \quad A \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G \tag{4.4}$$

is never bounded. This is problematic both for physics, because the connection 1-form describing the gauge bosons is unbounded; and from a mathematical point of view, because the construction of a Fredholm module over $\mathcal{A}$ and Hochschild character cocycle depends on the boundedness of the commutator (4.4).

In this chapter, we show how is it possible to solve this problem by using instead a *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D, \rho)$, [32][1]. Rather than requiring the boundedness of the commutator, one asks that there exists a automorphism ρ of $\mathcal{A}$ such that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \tag{4.5}$$

is bounded for any $a \in \mathcal{A}$. Accordingly, we introduce in Def. 4.1.1 a *twisted first-order condition*

$$[[D, a]_\rho, Jb^*J^{-1}]_\rho := [D, a]_\rho Jb^*J^{-1} - J\rho(b^*)J^{-1}[D, a]_\rho = 0 \quad \forall a, b \in \mathcal{A}. \tag{4.6}$$

We then show that for a suitable choice of a subalgebra $\mathcal{B}$ of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$, a *twisted fluctuation* of $\not{\partial} + D_M$ that satisfies (4.6) generates a field $\boldsymbol{\sigma}$ - slightly different from the one of chapter 3 - together with an additional vector field X_μ.

Furthermore, the breaking to the standard model is now spontaneous, as conjectured in the previous part. Namely the reduction of the grand algebra $\mathcal{A}_G$ to $\mathcal{A}_{sm}$ is obtained dynamically, as a minimum of the spectral action. The scalar field σ then play a role similar as the one of the Higgs in the electroweak symmetry breaking.

Mathematically, twists make sense as explained in [32], for the Chern character of finitely summable spectral triples extends to the twisted case, and lands in ordinary (untwisted) cyclic cohomology. Twisted spectral triples have been introduced to deal with type III examples, such as those arising from transverse geometry of codimension one foliation, and have been used in various context like quantum statistical dynamical systems [51]. It is quite surprising that the same tool allows a rigorous implementation in NCG of the idea of a “bigger symmetry beyond the SM”.

The main results of this chapter are summarized in the following theorem [37]:

Theorem 4.0.1. *Let $\mathcal{H}$ be the Hilbert space of the standard model described in §2.4. There exists a sub-algebra $\mathcal{B}$ of the grand algebra $\mathcal{A}_G$ containing $\mathcal{A}_{sm}$ together with an automorphism ρ of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ such that*

i) $T := (C^\infty(\mathcal{M}) \otimes \mathcal{B}, \mathcal{H}, \not{\partial} + D_M; \rho)$ is a twisted spectral triple satisfying the twisted 1^{st}-order condition (4.6);

[1] Also called σ-triple, but to avoid confusion with the field σ, we denote by ρ the automorphism called σ in [32].

ii) twisted fluctuations of $\partial\!\!\!/ + D_M$ *by* $\mathcal{B}$ *are parametrized by a scalar field* $\boldsymbol{\sigma}$ *and a vector field* X_μ*;*

iii) the spectral triple of the standard model is obtained from T *by minimizing the potential of the vector field* X_μ *induced by the spectral action coming from a twisted fluctuation of* $\partial\!\!\!/$*.*

iv) the spectral triple of the standard model is also obtained by minimizing the potential induced by the spectral action of a twisted fluctuation of the whole Dirac operator $\partial\!\!\!/ + D_M$*. The potential of the scalar field* $\boldsymbol{\sigma}$ *is minimum for the standard model, and corresponds to the constant field* y_R*.*

Explicitly, $\mathcal{B}$ is a sub-algebra $\mathbb{H}^2 \oplus \mathbb{C}^2 \oplus M_3(\mathbb{C})$ of $\mathcal{A}_G$. Labelling the two copies of the quaternions and complex algebras by the left/right spinorial indices l, r and the left/right internal indices L/R, that is

$$\mathcal{B} = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus M_3(\mathbb{C}), \tag{4.7}$$

the automorphism ρ is the exchange of the left/right spinorial indices:

$$\rho\,(\,q_L^l,\, q_L^r,\, c_R^l,\, c_R^r,\, m) \to (q_L^r,\, q_L^l,\, c_R^r,\, c_R^l,\, m) \tag{4.8}$$

where $m \in M_3(\mathbb{C})$ while the q's and c's are quaternions and complex numbers belonging to their respective copy of $\mathbb{H}$ and $\mathbb{C}$.

The chapter is organized as follows. The section 4.1 deals with the twist. It begins with the definition of the twisted first-order condition in definition 4.1.1. In §4.1.1 we fix the representation of the grand algebra, which differs from the one used in [36]. It is used in §4.1.2 to build a twisted spectral triple with the free Dirac operator. In §4.1.3 the twisted first-order condition for D_M yields the reduction to the algebra $\mathcal{B}$ and the construction of the spectral triple T. This proves the first point of theorem 4.0.1. In section 4.2 we compute the twisted fluctuations D_X of the free Dirac operator $\partial\!\!\!/$ (§4.2.2), and D_σ of the Majorana-Dirac operator D_M (§4.2.3). This yields the additional vector field in eq. (4.64), and the extra scalar field $\boldsymbol{\sigma}$ in eq. (4.81), proving the second point of theorem 4.0.1. In section 4.3, we compute the generalized Lichnerowicz formula for the twisted-fluctuated Dirac operator in §4.3.1. The dynamical reduction of $\mathcal{B}$ to the standard model by minimizing the potential of the additional vector field is obtained in §4.3.2. The potential of the scalar field is treated in §4.3.3, and the interaction potential between the vector and the scalar field in §4.3.4. These results are discussed in the final section 4.5.

4.1 Twisting the standard model

A *twisted spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting on a Hilbert space $\mathcal{H}$ and D a selfadjoint operator on $\mathcal{H}$ with compact resolvent, together with an automorphism ρ of $\mathcal{A}$ such that

$$[D, a]_\rho = Da - \rho(a)D \tag{4.9}$$

is bounded for any $a \in \mathcal{A}$. It is graded if, in addition, there is a selfadjoint operator Γ of square $\mathbb{I}$ which commutes the algebra and anticommutes with D.

Up to now the other conditions satisfied by a spectral triple have not been adapted to the twisted case yet. As long as the commutator between the algebra and the Dirac operator is not involved, one can keep the definitions of an ordinary spectral triple, for instance the order-zero condition. In the 1^{st}-order condition (2.22) it is natural to substitute $[D, a]$ with the twisted commutator $[D, a]_\rho$. The question is whether to twist the commutator with Jb^*J^{-1} as well. As explained in [32, Prop. 3.4], the set Ω^1_D of twisted 1-forms, that is all the operators of the form

$$\mathbb{A} = \sum_i b^i [D, a_i]_\rho, \tag{4.10}$$

is a $\mathcal{A}$-bimodule for the left and right actions

$$a \cdot \omega \cdot b := \rho(a)\omega b \quad \forall a, b \in \mathcal{A}, \omega \in \Omega^1_D. \tag{4.11}$$

Therefore it is natural to twist the commutator with JbJ^{-1}. As pointed out below in §4.1.2 and §4.1.3, this choice is also the one which is efficient for our purposes. Furthermore we assume that ρ is a $*$-automorphism that commutes with the real structure J, which permits us to define the twisted version of the 1^{st}-order condition as follows.

Definition 4.1.1. *A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D, \rho)$ with real structure J satisfies the twisted 1^{st}-order condition if and only if*

$$[[D, a]_\rho,\, JbJ^{-1}]_\rho = [D, a]_\rho\, JbJ^{-1} - J\rho(b)J^{-1}[D, a]_\rho = 0 \qquad \forall a, b \in \mathcal{A}. \tag{4.12}$$

4.1.1 Representation

For reasons discussed in §4.4 it is convenient to work with the other natural representation of the grand algebra than we used in chapter 3. Namely instead of (3.4) one asks that quaternions carry the chiral index s of spinors while the complex matrices carry the (anti)-particle index:

$$Q = Q^{t\beta}_{s\alpha}, \quad M = M^{\dot{t}\mathrm{J}}_{\dot{s}\mathrm{I}}. \tag{4.13}$$

Explicitly, the representation of the grand algebra $\mathcal{A}_G$ is

$$Q = \begin{pmatrix} Q_r^r & Q_r^l \\ Q_l^r & Q_l^l \end{pmatrix}_{st} \in M_4(\mathbb{H}), \quad M = \begin{pmatrix} M_{\dot{0}}^{\dot{0}} & M_{\dot{0}}^{\dot{1}} \\ M_{\dot{1}}^{\dot{0}} & M_{\dot{1}}^{\dot{1}} \end{pmatrix}_{\dot{s}\dot{t}} \in M_8(\mathbb{C}), \tag{4.14}$$

where for any $s, t \in \{l, r\}$ and $\dot{s}, \dot{t} \in \{\dot{0}, \dot{1}\}$ one defines

$$Q_s^t = \begin{pmatrix} Q_{sa}^{ta} & Q_{sa}^{tb} & Q_{sa}^{tc} & Q_{sa}^{td} \\ Q_{sb}^{ta} & Q_{sb}^{tb} & Q_{sb}^{tc} & Q_{sb}^{td} \\ Q_{sc}^{ta} & Q_{sc}^{tb} & Q_{sc}^{tc} & Q_{sc}^{td} \\ Q_{sd}^{ta} & Q_{sd}^{tb} & Q_{sd}^{tc} & Q_{sd}^{td} \end{pmatrix}_{\alpha\beta} \in M_2(\mathbb{H}), \quad M_{\dot{s}}^{\dot{t}} = \begin{pmatrix} M_{\dot{s}0}^{\dot{t}0} & M_{\dot{s}0}^{\dot{t}1} & M_{\dot{s}0}^{\dot{t}2} & M_{\dot{s}0}^{\dot{t}3} \\ M_{\dot{s}1}^{\dot{t}0} & M_{\dot{s}1}^{\dot{t}1} & M_{\dot{s}1}^{\dot{t}2} & M_{\dot{s}1}^{\dot{t}3} \\ M_{\dot{s}2}^{\dot{t}0} & M_{\dot{s}2}^{\dot{t}1} & M_{\dot{s}2}^{\dot{t}2} & M_{\dot{s}2}^{\dot{t}3} \\ M_{\dot{s}3}^{\dot{t}0} & M_{\dot{s}3}^{\dot{t}1} & M_{\dot{s}3}^{\dot{t}2} & M_{\dot{s}3}^{\dot{t}3} \end{pmatrix}_{\mathrm{IJ}} \in M_4(\mathbb{C}).$$

Here we use a, b, c, d to denote the value of the flavor index α. On the remaining indices, Q and M act trivially, that is as the identity operator. The representation of $A = (Q, M) \in \mathcal{A}_G$ on H_F is thus

$$A_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}^{\mathsf{C}t\mathrm{I}\dot{t}\beta} = \left(\delta_{0\dot{s}\mathrm{J}}^{\mathsf{C}\dot{t}\mathrm{I}}\, Q_{s\alpha}^{t\beta} + \delta_1^{\mathsf{C}} M_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}} \delta_{s\alpha}^{t\beta}\right) = \begin{pmatrix} \delta_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\, Q_{s\alpha}^{t\beta} & 0_{64} \\ 0_{64} & M_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\, \delta_{s\alpha}^{t\beta} \end{pmatrix}_{\mathsf{CD}}. \tag{4.15}$$

One easily checks the order-zero condition (2.21): with $A = (R, N) \in \mathcal{A}_G$, a generic element of the opposite algebra is

$$JAJ^{-1} = -JAJ = \begin{pmatrix} -\delta_{s\alpha}^{t\beta}\,(\tau\bar{N}\tau)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}} & 0_{64} \\ 0_{64} & \delta_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\,(\eta\bar{R}\eta)_{s\alpha}^{t\beta} \end{pmatrix}_{\mathsf{CD}} \tag{4.16}$$

where the bar denotes the complex conjugate and we used

$$\mathcal{J}R\mathcal{J} := (\tau^2)_{\dot{s}}^{\dot{t}}\,(\eta\bar{R}\eta)_{s\alpha}^{t\beta} = -\delta_{\dot{s}}^{\dot{t}}\,(\eta\bar{R}\eta)_{s\alpha}^{t\beta}, \quad \mathcal{J}N\mathcal{J} := (\eta^2)_s^t\,(\tau\bar{N}\tau)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}} = \delta_s^t(\tau\bar{N}\tau)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}. \tag{4.17}$$

Obviously (4.15) commutes with (4.16).

The first statement we want to show is that the biggest subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ that satisfies the grading condition $[\Gamma, A] = 0$ with $A \in \mathcal{A}_G$ and has bounded commutator with $\not{D}$ is the left-right algebra $\mathcal{A}_{LR}$ given in (2.28).

In fact, by (2.40), for the quaternion sector $[\Gamma, A] = 0$ amounts to asking $[\eta_s^t \eta_\alpha^\beta, Q_{s\alpha}^{t\beta}] = 0$. This imposes

$$Q = \begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st} \tag{4.18}$$

where

$$Q_r^r = \begin{pmatrix} q_R^r & 0_2 \\ 0_2 & q_L^r \end{pmatrix}_{\alpha\beta}, \; Q_l^l = \begin{pmatrix} q_R^l & 0_2 \\ 0_2 & q_L^l \end{pmatrix}_{\alpha\beta} \quad \text{with } q_R^r, q_L^r, q_R^l, q_L^l \in \mathbb{H}. \tag{4.19}$$

For matrices, one asks $[\delta^{t\mathrm{I}}_{\dot{s}\mathrm{J}}, M^{t\mathrm{I}}_{\dot{s}\mathrm{J}}] = 0$ which is trivially satisfied. So the grading condition $[\Gamma, A] = 0$ imposes the reduction of $\mathcal{A}_G$ to

$$\mathcal{B}_{LR} := (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R) \oplus M_8(\mathbb{C}). \tag{4.20}$$

For $A = (Q, M) \in C^\infty(\mathcal{M}) \otimes \mathcal{B}_{LR}$, the boundedness of the commutator[2]

$$[\not{D}, A] = \begin{pmatrix} \delta^I_J\,[\not{\partial}, Q] & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\,[\not{\partial}, M] \end{pmatrix}_{\mathsf{CD}} \tag{4.21}$$

means that

$$[\not{\partial}, Q] = -i\gamma^\mu(\nabla^S_\mu Q) - i[\gamma^\mu, Q]\nabla^S_\mu \quad \text{and} \quad [\not{\partial}, M] = -i\gamma^\mu(\nabla^S_\mu M) - i[\gamma^\mu, M]\nabla^S_\mu \tag{4.22}$$

are bounded. This is obtained if and only if Q and M commute with all the Dirac matrices, i.e. are proportional to $\delta^{ti}_{s\dot{s}}$. For Q this means $Q^r_r = Q^l_l$ in (4.18), hence the reductions

$$\mathbb{H}^r_R \oplus \mathbb{H}^l_R \to \mathbb{H}_R, \quad \mathbb{H}^r_L \oplus \mathbb{H}^l_L \to \mathbb{H}_L. \tag{4.23}$$

For M, this means that all the components $M^{\dot{t}}_{\dot{s}}$ in (4.14) are equal, that is the reduction

$$M_8(\mathbb{C}) \to M_4(\mathbb{C}). \tag{4.24}$$

Therefore $\mathcal{B}_{LR}$ is reduced to $\mathcal{A}_{LR}$, acting diagonally on spinors. ■

This result is nothing but a restatement in the peculiar representation (4.15) of the fact that in order to have bounded commutators the action of $\mathcal{A}_G$ on spinors has to be trivial, as shown with eq. (3.64). Nevertheless, it is useful to have it explicitly, in order to understand how to get rid of the unboundedness of the commutator. It is also worth stressing the difference with the representation (3.4), for which the grading breaks both matrices and quaternions and reduces $\mathcal{A}_G$ to $\mathcal{A}'_G$. Here only quaternions are broken by the grading.

To cure the unboundedness of the commutator, the idea proposed in [37] is to impose the reduction (4.24) by hand, and deal with the unboundedness of $[\not{\partial}, Q]$ thanks to a twist. This is a "middle term solution": imposing by hand both reductions (4.24) and (4.23) is not interesting from the grand algebra point of view, since it brings us back to an almost commutative geometry where spinorial and internal indices are not mixed; solving both the unboundedness of $[\not{\partial}, Q]$ and $[\not{\partial}, M]$ by a twist yields some complications discussed in §4.4. The remarkable point is that this middle term solution is sufficient to obtain the σ-field by a fluctuation that respects the twisted first-order condition of definition 4.1.1.

[2] To lighten notation, we omit the trivial indices in the product (hence in the commutators) of operators. From (4.13) one knows that Q carries the indices $s\alpha$ while γ^μ carries $s\dot{s}$, hence $[\not{\partial}, Q]$ carries indices $s\dot{s}\alpha$ and should be written $[\delta^\beta_\alpha \not{\partial}, \delta^{\dot{t}}_{\dot{s}} Q]$. As well, $[\not{\partial}, M]$ carries indices $s\dot{s}I$ and holds for $[\delta^J_I \not{\partial}, \delta^t_s M]$.

4.1.2 The twist and the first-order condition for the free Dirac

Imposing (4.24) on the grand algebra $\mathcal{A}_G$ reduced by the grading to $\mathcal{B}_{LR}$ yields

$$\mathcal{B}' := (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_4(\mathbb{C}). \tag{4.25}$$

An element $A = (Q, M)$ of $\mathcal{B}'$ is given by (4.15) where Q is as in (4.18) while M in (4.13) is proportional to $\delta^{\dot{t}}_{\dot{s}}$:

$$M = \delta^{\dot{t}}_{\dot{s}} M^{\mathrm{I}}_{\mathrm{J}} \in M_4(\mathbb{C}). \tag{4.26}$$

The algebra $\mathcal{B}'$ contains the algebra of the standard model $\mathcal{A}_{sm}$, and still has a part (the quaternion) that acts in a non-trivial way on the spin degrees of freedom. In this sense $\mathcal{B}'$ is still from the grand algebra side, even if it is "not so grand".

Let ρ be the automorphism of $(\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r)$ that exchanges Q_r^r and Q_l^l in (4.18), that is the exchange

$$\mathbb{H}_R^r \leftrightarrow \mathbb{H}_R^l, \quad \mathbb{H}_L^r \leftrightarrow \mathbb{H}_L^l. \tag{4.27}$$

This means in components

$$\rho\left(\begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st}\right) = \begin{pmatrix} Q_l^l & 0_4 \\ 0_4 & Q_r^r \end{pmatrix}_{st}. \tag{4.28}$$

Denoting by the same letter the extension of ρ to $C^\infty(\mathcal{M}) \otimes (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r)$. For any μ one has

$$\gamma^\mu Q = \rho(Q)\gamma^\mu, \quad \gamma^\mu \rho(Q) = Q\gamma^\mu, \tag{4.29}$$

so that

$$[\eth, Q]_\rho = -i\gamma^\mu(\nabla_\mu^S Q). \tag{4.30}$$

In fact, writing explicitly the δ's, one gets

$$\begin{aligned}\gamma^\mu Q &= \left(\delta_\alpha^\beta \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0_2 \end{pmatrix}_{st}\right)\left(\begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st} \delta^{\dot{t}}_{\dot{s}}\right) = \begin{pmatrix} 0_8 & \sigma^\mu Q_l^l \\ \bar{\sigma}^\mu Q_r^r & 0_8 \end{pmatrix}_{st} \\ &= \left(\begin{pmatrix} Q_l^l & 0_4 \\ 0_4 & Q_r^r \end{pmatrix}_{st} \delta^{\dot{t}}_{\dot{s}}\right)\left(\delta_\alpha^\beta \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0_2 \end{pmatrix}_{st}\right) = \rho(Q)\gamma^\mu. \end{aligned} \tag{4.31}$$

The second part of (4.29) follows because $\rho^2 = \mathbb{I}$ while eq. (4.30) comes from

$$[\eth, Q]_\rho = -i\gamma^\mu(\nabla_\mu^S Q) - i[\gamma^\mu, Q]_\rho \nabla_\mu^S,$$

where the second term is zero by (4.29). $\blacksquare$

We still denote by the same letter the extension of ρ to $C^\infty(\mathcal{M})\otimes\mathcal{B}'$:

$$\rho((Q,M)) := ((\rho(Q), M). \tag{4.32}$$

Is it easy to show that $(C^\infty(\mathcal{M})\otimes\mathcal{B}', \mathcal{H}, D\!\!\!/, \rho)$ *together with the graduation* Γ *in* (2.40) *and the real structure* J *in* (2.38) *is a graded twisted spectral triple which satisfies the twisted first-order condition of definition 4.1.1.*

To prove this, let we take $A=(Q,M)\in C^\infty(\mathcal{M})\otimes\mathcal{B}'$. The twisted version of (4.21) is

$$[D\!\!\!/, A]_\rho = \begin{pmatrix} \delta^I_J\,[\eth\!\!\!/, Q]_\rho & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\,[\eth\!\!\!/, M] \end{pmatrix}_{\mathsf{CD}}. \tag{4.33}$$

From (4.26) and (4.15) M commutes with γ^μ, so that the second equation in (4.22) reduces to

$$[\eth\!\!\!/, M] = -i\gamma^\mu(\nabla^S_\mu M), \tag{4.34}$$

which is a bounded operator. By eq. (4.30), $[\eth\!\!\!/, Q]_\rho = -i\gamma^\mu(\partial_\mu Q)$ is bounded as well. Hence $(C^\infty(\mathcal{M})\otimes\mathcal{B})', \mathcal{H}, D\!\!\!/, \rho)$ together with Γ form a graded twisted spectral triple.

We now examine the twisted first-order condition (4.9). Let $B=(R,N)\in C^\infty(\mathcal{M})\otimes\mathcal{B}'$. A generic element of the algebra opposite to $C^\infty(\mathcal{M})\otimes\mathcal{B}'$ is

$$JBJ^{-1} = -JBJ = \begin{pmatrix} \delta^{t\beta}_{s\alpha}\bar{N} & 0_{64} \\ 0_{64} & \delta^{t\mathrm{J}}_{\dot{s}\mathrm{I}}\bar{R} \end{pmatrix}_{\mathsf{CD}} \tag{4.35}$$

where we used (4.16) and noticed that for R as in (4.18) and N as in (4.26) one has

$$(\eta\bar{R}\eta)^{t\beta}_{s\alpha} = \bar{R}^{t\beta}_{s\alpha}, \qquad (\tau\bar{N}\tau)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} = -\bar{N}^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}. \tag{4.36}$$

As well, one has

$$J\rho(B)J^{-1} = -J\rho(B)J = \begin{pmatrix} \delta^{t\beta}_{s\alpha}\bar{N} & 0_{64} \\ 0_{64} & \delta^{t\mathrm{J}}_{\dot{s}\mathrm{I}}\rho(\bar{R}) \end{pmatrix}_{\mathsf{CD}}. \tag{4.37}$$

Thus $[D\!\!\!/, A]_\rho JBJ^{-1} - J\rho(B)J^{-1}[D\!\!\!/, A]_\rho$ is a diagonal matrix with components

$$[\delta^I_J\,[\eth\!\!\!/, Q]_\rho,\, \delta^{t\beta}_{s\alpha}\bar{N}], \qquad \delta^\beta_\alpha\,[\eth\!\!\!/, M]\,\delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\bar{R} - \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\rho(\bar{R})\,\delta^\beta_\alpha\,[\eth\!\!\!/, M]. \tag{4.38}$$

The first term vanishes because the only non-trivial index carries by $\bar{N}$ is IJ. The second term is (omitting the deltas and a global $-i$ factor)

$$\begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st} \begin{pmatrix} \bar{R}^r_r & 0_8 \\ 0_8 & \bar{R}^l_l \end{pmatrix}_{st} - \begin{pmatrix} \bar{R}^l_l & 0_8 \\ 0_8 & \bar{R}^r_r \end{pmatrix}_{st} \begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st}$$

$$= \begin{pmatrix} 0_8 & [\sigma^\mu(\partial_\mu M), \bar{R}^l_l] \\ [\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r] & 0_8 \end{pmatrix}_{st} \tag{4.39}$$

which vanishes because R only non-trivial index is $\alpha\beta$ while $[\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r]$ is proportional to δ^β_α. $\blacksquare$

4.1.3 Twisted first-order condition for the Majorana-Dirac operator

We individuate a subalgebra $\mathcal{B}$ of $\mathcal{B}'$ such that a twisted fluctuation of the *Majorana-Dirac operator* D_M in (3.26) by $\mathcal{B}$ satisfies the twisted first-order condition. Since we are working with one generation of fermions only, in (2.35) the Majorana mass matrix $\mathcal{M}_R$ in D_R is $\Xi_\alpha^\beta y_R$, where

$$\Xi = \begin{pmatrix} 1 & 0 \\ 0 & 0_3 \end{pmatrix} \tag{4.40}$$

denotes the projection on the first component. Therefore

$$D_M = \gamma^5 D_R = \eta_s^t\, \delta_{\dot{s}}^{\dot{t}} \Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta} \begin{pmatrix} 0 & y_R \\ \bar{y}_R & 0 \end{pmatrix}_{\mathsf{CD}}. \tag{4.41}$$

In this equation the product $\gamma^5 D_R$ is intended with the convention of the footnote p.66, namely this is the tensorial notation $\gamma^{5}{}_{s\dot{s}}^{t\dot{t}} D_R{}_{\mathrm{J}\alpha\mathsf{C}}^{\beta\mathrm{I}\mathsf{D}}$ in which we omit the indices. In practice, this amounts to omit the tensor product symbol in $\gamma^5 \otimes D_R$, which makes sense because of our choice of viewing the total Hilbert space no longer as the tensor product of spinors by $\mathcal{H}_\mathcal{F}$. These distinctions may seem pedantic here, but they will be important later on, when writing the product $\gamma^\mu X_\mu$ for a vector field X_μ that no longer commutes with the Dirac matrices: $\gamma^\mu X_\mu$ will holds for $\gamma^{\mu}{}_{s\dot{s}}^{t\dot{t}} X_\mu{}_{\mathrm{J}\alpha\mathsf{C}}^{\beta\mathrm{I}\mathsf{D}}$, while $\gamma^\mu \otimes X_\mu$ no longer makes sense.

Now we want to show that a subalgebra of $\mathcal{B}'$ which satisfies the twisted first-order condition

$$[[D_M, A]_\rho, JBJ^{-1}]_\rho = 0 \tag{4.42}$$

is

$$\mathcal{B} := \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus M_3(\mathbb{C}). \tag{4.43}$$

To show this, let us consider first the subalgebra

$$\tilde{\mathcal{B}} := (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r) \oplus (M_3(\mathbb{C}) \oplus \mathbb{C}) \tag{4.44}$$

of $\mathcal{B}'$ obtained by asking that q_R^l, q_R^r in (4.19) are diagonal quaternions, namely

$$q_R^l = \begin{pmatrix} c_R^l & 0 \\ 0 & \bar{c}_R^l \end{pmatrix},\ q_R^r = \begin{pmatrix} c_R^r & 0 \\ 0 & \bar{c}_R^r \end{pmatrix} \text{ with } c_R^l, c_R^r \in \mathbb{C}; \tag{4.45}$$

while M in (4.26) is of the form

$$M = \delta_{\dot{s}}^{\dot{t}} \begin{pmatrix} m & 0 \\ 0 & \mathbf{M} \end{pmatrix}_{\mathrm{IJ}} \text{ with } m \in \mathbb{C}, \mathbf{M} \in M_3(\mathbb{C}). \tag{4.46}$$

This means that while Q carries non-trivial indices $\dot{s}, \alpha$, the action of M is non-trivial only in the I index. Define similarly $B = (R, N) \in \tilde{\mathcal{B}}$ with components $d_R^l, d_R^r \in \mathbb{C}$, $n \in \mathbb{C}$, $\mathbf{N} \in M_3(\mathbb{C})$. For any $A, B \in \tilde{\mathcal{B}}$, one has from (4.41) where we write

$$\mathsf{D}_M := \eta_s^t \, \delta_{\dot{s}}^{\dot{t}} \, \Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta}, \tag{4.47}$$

and (4.15) (omitting the deltas)

$$[D_M, A]_\rho = \begin{pmatrix} 0_{64} & y_R(\mathsf{D}_M M - \rho(Q)\mathsf{D}_M) \\ \bar{y}_R(\mathsf{D}_M Q - M\mathsf{D}_M) & 0_{64} \end{pmatrix}_{\mathsf{CD}}. \tag{4.48}$$

By (4.35), (4.37) one obtains

$$[[D_M, A]_\rho, JBJ^{-1}]_\rho = \begin{pmatrix} 0_{64} & y_R\left((\mathsf{D}_M M - \rho(Q)\mathsf{D}_M)\bar{R} - \bar{N}(\mathsf{D}_M M - \rho(Q)\mathsf{D}_M)\right) \\ \bar{y}_R\left((\mathsf{D}_M Q - M\mathsf{D}_M)\bar{N} - \rho(\bar{R})(\mathsf{D}_M Q - M\mathsf{D}_M)\right) & 0_{64} \end{pmatrix}_{\mathsf{CD}}.$$

The terms entering the upper-right components of this matrix are (omitting a global y_R factor)

$$\begin{aligned}
\bar{N}\mathsf{D_M}M &= (\bar{N}\Xi M)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}(\eta\Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st}, \\
\bar{N}\rho(Q)\mathsf{D_M} &= (\bar{N}\Xi)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\,(\rho(Q)\eta\Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}_R^l & 0_4 \\ 0_4 & -\mathsf{c}_R^r \end{pmatrix}_{st}, \\
\mathsf{D_M}M\bar{R} &= (\Xi M)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\,(\eta\Xi\bar{R})_{s\alpha}^{t\beta} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R^r & 0_4 \\ 0_4 & -\bar{\mathsf{d}}_R^l \end{pmatrix}_{st}, \\
\rho(Q)\mathsf{D}_M\bar{R} &= (\Xi\delta)_{\dot{s}\mathrm{J}}^{\dot{t}\mathrm{I}}\,(\rho(Q)\eta\Xi R)_{s\alpha}^{t\beta} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}_R^l\,\bar{\mathsf{d}}_R^r & 0_4 \\ 0_4 & -\mathsf{c}_R^r\,\bar{\mathsf{d}}_R^l \end{pmatrix}_{st},
\end{aligned} \tag{4.49}$$

where we defined

$$\mathsf{m} := \begin{pmatrix} m & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}}, \quad \mathsf{c}_R^r = \begin{pmatrix} c_R^r & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}_R^l = \begin{pmatrix} c_R^l & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta} \tag{4.50}$$

and similarly for d_R^r, d_R^l and $\mathbf{n}$. Collecting the various terms, one finds that the upper-right component of $[[D_M, A]_\rho, JBJ^{-1}]_\rho$ vanishes if and only if

$$(c_R^l - m)(\bar{d}_R^r - \bar{n}) = 0, \quad (c_R^r - m)(\bar{d}_R^l - \bar{n}) = 0. \tag{4.51}$$

Similarly, for the lower-left component of $[[D_M, A]_\rho, JBJ^{-1}]_\rho$ one has

$$\rho(\bar{R})M\mathsf{D}_\mathsf{M} = (\Xi M)^{t\mathrm{I}}_{\dot{s}\mathrm{J}}(\rho(\bar{R})\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{d}}^l_R & 0_4 \\ 0_4 & -\bar{\mathsf{d}}^r_R \end{pmatrix}_{st},$$

$$\rho(\bar{R})\mathsf{D}_M Q = (\Xi\delta)^{t\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\rho(\bar{R})\,\eta\Xi\,Q)^{t\beta}_{s\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^r_R\bar{\mathsf{d}}^l_R & 0_4 \\ 0_4 & -\mathsf{c}^l_R\bar{\mathsf{d}}^r_R \end{pmatrix}_{st},$$

$$M\mathsf{D}_M\bar{N} = (M\Xi\bar{N})^{t\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st},$$

$$\mathsf{D}_M Q\bar{N} = (\Xi\bar{N})^{t\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\eta\Xi Q)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^r_R & 0_4 \\ 0_4 & -\mathsf{c}^l_R \end{pmatrix}_{st}, \tag{4.52}$$

yielding the same condition (4.51). Hence the twisted first-order condition is satisfied as soon as

$$c^r_R = m,\; d^r_R = n, \tag{4.53}$$

which amounts to identify $\mathbb{C}^r_R$ with $\mathbb{C}$. Hence the reduction of $\mathcal{B}'$ to $\mathcal{B}$ as defined in (4.43). ■

One could identify $\mathbb{C}^l_R$ with $\mathbb{C}$, instead of $\mathbb{C}^r_R$, without changing the result. As discussed before definition 4.1.1, one might also consider a first-order condition where only the commutator with D is twisted, that is

$$[[D_M, A]_\rho, JBJ^{-1}] = 0. \tag{4.54}$$

This is not pertinent in our case however, for this amounts to permuting $\bar{R}^l_l$ with $\bar{R}^r_r$ in - and only in - the second term in (4.39), which then no longer vanishes as soon as $R^r_r \neq R^l_l$.

Let us note that relation (4.42) deals only with the finite dimensional part of the spectral triple. However it is still satisfied with $A, B \in C^\infty(\mathcal{M}) \otimes \mathcal{B}$ (though, strictly speaking, one can no longer talk of "twisted first-order condition for D_M", for on $L^2(\mathcal{M}) \otimes \mathbb{C}^{128}$ the operator D_M does not have a compact resolvent). On the other hand, the same is true for the subalgebra $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ and the free Dirac operator $\not{\partial}$, as shown in §4.1.2. Therefore the twisted first-order condition (4.12) with $\mathcal{B}$ is true for $\not{\partial} + D_M$ since it is true for $\not{\partial}$ and D_M independently. This proves the first statement of theorem 4.0.1.

4.2 Twisted-covariant Dirac operators

The twisted spectral triple

$$(C^\infty(\mathcal{M}) \otimes \mathcal{B},\, L^2(\mathcal{M}) \otimes \mathbb{C}^{128},\, \not{\partial} + D_M;\, \rho) \tag{4.55}$$

of theorem 4.0.1 solves the problem of the non-boundedness of the commutators $[\not{\partial}, A]$ raised by the non-trivial action of the grand algebra on spinors. But to be of interest, this spectral triple should preserve the property the grand algebra has been invented for, that is generating the field σ by a fluctuation of D_M, or a twisted version of it. As shown in this section this is indeed the case, because although $\mathcal{B}$ is not so grand (it is smaller than $\mathcal{A}_G$), it is neither too small ($C^\infty(\mathcal{M}) \otimes \mathcal{B}$ still has non trivial action on spinors).

4.2.1 Twisted fluctuation

In analogy with gauge fluctuation of almost commutative geometries described in §2.5, we call *twisted fluctuation of D by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$* the substitution of $D = \not{\partial} + D_M$ with

$$D_{\mathbb{A}} = D + \mathbb{A} + J\,\mathbb{A}\,J^{-1} \tag{4.56}$$

where $\mathbb{A}$ is twisted 1-form

$$\mathbb{A} = B^i[D, A_i]_\rho \quad A_i, B^i \in C^\infty(\mathcal{M}) \otimes \mathcal{B}. \tag{4.57}$$

We do not require $\mathbb{A}$ to be selfadjoint, we only ask that $D_{\mathbb{A}}$ is selfadjoint and called it *twisted-covariant Dirac operator*. It is the sum $D_{\mathbb{A}} = D_X + D_\sigma$ of the twisted-covariant free Dirac operator

$$D_X := \not{\partial} + \not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} \qquad \not{\mathbb{A}} := B^i[\not{\partial}, A_i]_\rho \tag{4.58}$$

with the twisted-covariant Majorana-Dirac operator

$$D_\sigma := D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} \qquad \mathbb{A}_M := B^i[D_M, A_i]_\rho. \tag{4.59}$$

In this section, we compute explicitly D_X and D_σ, and show that they are parametrized by a vector field X_μ and a scalar field $\boldsymbol{\sigma}$.

In the following, $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ are arbitrary elements of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$, where i a summation index and

$$Q_i = \begin{pmatrix} Q^r_{ri} & 0_4 \\ 0_4 & Q^l_{li} \end{pmatrix}_{st}, \quad M_i = \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} c^r_i & 0 \\ 0 & \mathsf{M}_i \end{pmatrix}_{\mathrm{IJ}} \tag{4.60}$$

with[3] $\mathsf{M}_i \in M_3(\mathbb{C})$ and

$$Q^r_{ri} = \begin{pmatrix} q^r_{Ri} & 0_2 \\ 0_2 & q^r_{Li} \end{pmatrix}_{\alpha\beta}, \; Q^l_{li} = \begin{pmatrix} q^l_{Ri} & 0_2 \\ 0_2 & q^l_{Li} \end{pmatrix}_{\alpha\beta} \tag{4.61}$$

[3]In all this section, the components of the matrices are functions on $\mathcal{M}$. To lighten notation we write $M_3(\mathbb{C})$ instead of $C^\infty(\mathcal{M}) \otimes M_3(\mathbb{C})$. The same is true for the various copies of $\mathbb{H}$ and $\mathbb{C}$.

with $q^l_{Li} \in \mathbb{H}^l_L$, $q^r_{Li} \in \mathbb{H}^r_L$ and

$$q^r_{Ri} = \text{diag}\,(c^r_i, \bar{c}^r_i), \quad q^l_{Ri} = \text{diag}\,(c^l_i, \bar{c}^l_i) \quad \text{with} \quad c^r_i \in \mathbb{C}^r_R,\ c^l_i \in \mathbb{C}^l_R. \tag{4.62}$$

The components R^i, N^i of B^i are defined similarly, with

$$d^{ri} \in \mathbb{C}^r_R,\ d^{li} \in \mathbb{C}^l_R, \quad r^{ri}_L \in \mathbb{H}^r_L,\ r^{ri}_L \in \mathbb{H}^l_L \quad \text{and} \quad \mathsf{N}_i \in M_3(\mathbb{C}). \tag{4.63}$$

4.2.2 Twisted-covariant free Dirac operator D_X

The twisted fluctuations (4.58) of the free Dirac operator $\partial\!\!\!/$ in eq. (3.53) by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ are parametrized by a vector field, $\mathbb{X}\!\!\!/$.

In particular, one has

$$D_X = \partial\!\!\!/ + \mathbb{X}\!\!\!/ \tag{4.64}$$

with

$$\mathbb{X}\!\!\!/ := -i\gamma^\mu \mathbb{X}_\mu\,, \qquad \mathbb{X}_\mu := \begin{pmatrix} X_\mu & 0_{64} \\ 0_{64} & -\bar{X}_\mu \end{pmatrix}_{\mathsf{CD}}, \tag{4.65}$$

where we define the bounded-operator valued vector field[4]

$$X_\mu := \delta^I_J\, \rho(R^i)\, \nabla^S_\mu Q_i - \delta^\beta_\alpha\, \bar{N}^i \nabla^S_\mu \bar{M}_i \tag{4.66}$$

which commutes with γ^5 and twisted-commutes with γ^ν, that is for all μ, ν one has

$$\gamma^\mu X_\nu = \rho(X_\nu)\gamma^\mu, \quad \gamma^\mu \rho(X_\nu) = X_\nu \gamma^\mu. \tag{4.67}$$

The last equation is direct consequences of definition (4.66) and eq. (4.30), while the commutation of X_μ with γ^5 is a consequence of the breaking of $\mathcal{A}_G$ by the grading condition and can be checked explicitly using (4.61) and (4.15).

To check eq. (4.64), let we take $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ in $\mathcal{B}$, one gets from (4.33), (4.34) and (4.29)

$$\mathbb{A}\!\!\!/ = -iB^i[\partial\!\!\!/, A_i]_\rho = -i \begin{pmatrix} \delta^I_J\, \gamma^\mu \rho(R^i) \nabla^S_\mu Q_i & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\, \gamma^\mu N^i \nabla^S_\mu M_i \end{pmatrix}_{\mathsf{CD}} \tag{4.68}$$

where we used that N^i commutes with γ^μ and $R^i\gamma^\mu = \gamma^\mu\rho(R^i)$ (see eq. (4.29)). By eq. (2.38) one gets

$$J\mathbb{A}\!\!\!/ J^{-1} = -J\mathbb{A}\!\!\!/ J = i \begin{pmatrix} \delta^\beta_\alpha\, \gamma^\mu \bar{N}^i \nabla^S_\mu \bar{M}_i & 0_{64} \\ 0_{64} & \delta^I_J \gamma^\mu \rho(\bar{R}^i) \nabla^S_\mu \bar{Q}_i \end{pmatrix}_{\mathsf{CD}} \tag{4.69}$$

[4] To lighten notations we omit the parenthesis around $\partial_\mu Q_i$ and $\partial_\mu \bar{M}_i$: the latter are bounded operators and act as matrices, not as differential operators.

where we used that $\mathcal{J}$ anti-commutes with the γ's matrices and commute[5] with ∇^S_μ so that, inserting $\mathcal{J}^2 = -\mathbb{I}$ before ∇^S_μ, one obtains

$$\begin{aligned} &\mathcal{J}\,(\gamma^\mu N^i \nabla^S_\mu M_i)\,\mathcal{J} = \gamma^\mu (\mathcal{J}N^i\mathcal{J})\,\nabla^S_\mu(\mathcal{J}M_i\mathcal{J}) = \gamma^\mu \bar{N}^i \nabla^S_\mu \bar{M}_i, \\ &\mathcal{J}(\gamma^\mu \rho(R^i)\partial_\mu Q_i)\mathcal{J} = \gamma^\mu(\mathcal{J}\rho(R^i)\mathcal{J})\nabla^S_\mu(\mathcal{J}Q_i\mathcal{J}) = \gamma^\mu \rho(\bar{R}^i)\nabla^S_\mu \bar{Q}_i. \end{aligned} \tag{4.70}$$

In both equations above the last term comes from (4.17), noticing that $\rho(R_i)$ and Q_i are now diagonal in the st index and so commute with η, while N_i, M_i are proportional to $\delta^{\dot t}_{\dot s}$, hence commute with τ. Summing up (4.68) and (4.69), one obtains

$$\not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} = -i\gamma^\mu \mathbb{X}_\mu \tag{4.71}$$

with $\mathbb{X}_\mu$ as in (4.66). ∎

At the end of this section we show that D_X is selfadjoint if and only if for any $\mu = 0,1,2,3$ one has

$$\rho(X_\mu) = -X^\dagger_\mu. \tag{4.72}$$

and we called it twisted-covariant free Dirac operator.

In fact, in the st indices, X_μ is a block diagonal matrix which is proportional to $\delta^{\dot t}_{\dot s}$,

$$X_\mu = \delta^{\mathrm{I}\dot t}_{\mathrm{J}\dot s} \begin{pmatrix} R^{il}_{\;l}\nabla^S_\mu Q^{\;r}_{ir} & 0_4 \\ 0_4 & R^{ir}_{\;r}\nabla^S_\mu Q^{\;l}_{il} \end{pmatrix}_{st} - \delta^{\beta t \dot t}_{\alpha s \dot s}\bar{N}^i \nabla^S_\mu \bar{M}_i =: \delta^{\dot t}_{\dot s} \begin{pmatrix} X^r_\mu & 0_{32} \\ 0_{32} & X^l_\mu \end{pmatrix}_{st}, \tag{4.73}$$

thus

$$\gamma^\mu X_\mu = \begin{pmatrix} 0_{32} & \sigma^\mu X^l_\mu \\ \tilde\sigma^\mu X^r_\mu & 0_{32} \end{pmatrix}_{st}, \quad (\gamma^\mu X_\mu)^\dagger = \begin{pmatrix} 0_{32} & \sigma^\mu (X^r_\mu)^\dagger \\ \tilde\sigma^\mu (X^l_\mu)^\dagger & 0_{32} \end{pmatrix}_{st} = \gamma^\mu \rho(X^\dagger_\mu), \tag{4.74}$$

where we used that X_μ commutes with the σ's matrices and $(\sigma^\mu)^\dagger = \tilde\sigma^\mu$. Therefore $\gamma^\mu X_\mu$ is selfadjoint iff

$$\sigma^\mu (X^r_\mu)^\dagger = \sigma^\mu X^l_\mu. \tag{4.75}$$

Since $\mathrm{Tr}\,\bar\sigma^\nu\sigma^\mu = 2\delta^\mu_\nu$ and both X^r_μ and X^l_μ are proportional to $\delta^{\dot t}_{\dot s}$, the partial trace on the $\dot s\dot t$ indices of the above equation, where both side have been multiplied by $\bar\sigma^\lambda$, yields $(X^r_\mu)^\dagger = X^l_\mu$ for any μ, that is

$$X^\dagger_\mu = \rho(X_\mu). \tag{4.76}$$

Eq. (4.72) is obtained noticing that D_X is selfadjoint if and only if $i\gamma^\mu X^\mu$ is selfadjoint, that is $\gamma^\mu X^\mu$ is anti-selfadjoint. ∎

[5] $\{\mathcal{J}, \gamma^\mu\} = i(\gamma^0\gamma^2\bar\gamma^\mu + \gamma^\mu\gamma^0\gamma^2) = 0$ because $\bar\gamma^\mu = -\gamma^\mu$ for $\mu = 1,3$, $\bar\gamma^\mu = \gamma^\mu$ for $\mu = 0,2$. That $\mathcal{J}$ commutes with the spin covariant derivative ∇^S_μ is a classical result, see e.g. [97, Prop. 4.18].

4.2.3 Twisted-covariant Majorana-Dirac operator D_σ

Twisted fluctuations of the Majorana-Dirac operator D_M are parametrized by a scalar field $\boldsymbol{\sigma}$. To show that, we begin by a short calculation in tensorial notations.

For $A = (Q, M) \in \mathcal{B}$ with components $c^r, c^l \in \mathbb{C}$ as in (4.62), one has

$$[D_M, A]_\rho = \begin{pmatrix} 0_2 & y_R(c^r - c^l)\mathcal{S} \\ \bar{y}_R(c^r - c^l)\mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} \tag{4.77}$$

where

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{st}, \quad \mathcal{S}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{st}. \tag{4.78}$$

The above twisted commutator is obtained computing explicitly (4.48) with notations (4.50) and omitting y_R and $\bar{y}_R$:

$$\begin{aligned} \mathsf{D_M} M - \rho(Q)\mathsf{D_M} &= (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} (\eta\Xi)^{t\beta}_{s\alpha} - (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, (\rho(Q)\eta\Xi)^{t\beta}_{s\alpha} \\ &= \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi^{\mathrm{I}}_{\mathrm{J}} & 0_4 \\ 0_4 & -\Xi^{\mathrm{I}}_{\mathrm{J}} \end{pmatrix}_{st} - \begin{pmatrix} \Xi^{\beta}_{\alpha} & 0_4 \\ 0_4 & \Xi^{\beta}_{\alpha} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^l_R & 0_4 \\ 0_4 & -\mathsf{c}^r_R \end{pmatrix}_{st} \\ &= \begin{pmatrix} \begin{pmatrix} (m - c^l_R)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} & 0 \\ 0 & (m - c^l_R)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} & 0_{32} \\ 0_{32} & \begin{pmatrix} -(m - c^r_R)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} & 0 \\ 0 & -(m - c^r_R)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} \end{pmatrix}_{st} \end{aligned} \tag{4.79}$$

$$\begin{aligned} \mathsf{D_M}^\dagger Q - M\mathsf{D}^\dagger_\nu &= (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, (\eta\Xi Q)^{t\beta}_{s\alpha} - (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} (\eta\Xi)^{t\beta}_{s\alpha} \\ &= \begin{pmatrix} \Xi^{\beta}_{\alpha} & 0_4 \\ 0_4 & \Xi^{\beta}_{\alpha} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{c}}^r_R & 0_4 \\ 0_4 & -\bar{y}_R \mathsf{c}^l_R \end{pmatrix}_{st} - \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\Xi}^{\mathrm{I}}_{\mathrm{J}} & 0_4 \\ 0_4 & -\bar{\Xi}^{\mathrm{I}}_{\mathrm{J}} \end{pmatrix}_{st} \\ &= \begin{pmatrix} \begin{pmatrix} \bar{(}c^r_R - m)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} & 0 \\ 0 & \bar{R}(c^r_R - m)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} & 0_{32} \\ 0_{32} & \begin{pmatrix} -\bar{(}c^l_R - m)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} & 0 \\ 0 & -\bar{(}c^l_R - m)\, \Xi^{\beta \mathrm{J}}_{\alpha \mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} \end{pmatrix}_{s} \end{aligned} \tag{4.80}$$

Identifying c^r_R with m following (4.53) yields the result in (4.77), where we drop the index R to match notation (4.62). ■

Now let us derive the selfadjoint twisted fluctuation (4.59) of the Majorana-Dirac operator $D_M = \gamma^5 D_R$ by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$. We call it *twisted-covariant Majorana-Dirac operator* having the form

$$D_\sigma = \boldsymbol{\sigma}\gamma^5 D_R \tag{4.81}$$

where

$$\boldsymbol{\sigma} = (\mathbb{I} + \gamma^5\phi) \tag{4.82}$$

with ϕ a real scalar field.

To find the result (4.81), let $B^i = (R^i, N^i)$ as in (4.63). From eq. (4.77) one gets

$$\mathbb{A}_M = B^i[D_M, A_i]_\rho = \phi \begin{pmatrix} 0_2 & y_R\,\mathcal{S} \\ \bar{y}_R\,\mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\; \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.83}$$

where

$$\phi := d^{ir}(c_i^r - c_i^l). \tag{4.84}$$

One has $\mathcal{J}(\mathcal{S}\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}\delta^{\dot{t}}_{\dot{s}}$ and $\mathcal{J}(\mathcal{S}'\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}'\delta^{\dot{t}}_{\dot{s}}$. Hence

$$J\mathbb{A}_M J^{-1} = -J\mathbb{A}_M J = \bar{\phi} \begin{pmatrix} 0_2 & y_R\mathcal{S}' \\ \bar{y}_R\mathcal{S} & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\; \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.85}$$

so that

$$D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} = \begin{pmatrix} 0_2 & y_R\,(\eta^t_s + \phi\,\mathcal{S} + \bar{\phi}\,\mathcal{S}') \\ \bar{y}_R\,(\eta^t_s + \phi\,\mathcal{S}' + \bar{\phi}\mathcal{S}) & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\; \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}. \tag{4.86}$$

It is selfadjoint if and only if $\phi = \bar{\phi}$. Then

$$D_\sigma := D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} = \begin{pmatrix} 0_4 & y_R(\gamma^5 + \phi\mathbb{I}_4) \\ \bar{y}_R(\gamma^5 + \phi\mathbb{I}_4) & 0_4 \end{pmatrix}_{\mathsf{CD}} \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}, \tag{4.87}$$

$$= (\gamma^5 + \phi\mathbb{I})D_R.$$

Factorizing by γ^5, one gets the form of (4.81). ■

The results expressed by eq. (4.64) and (4.81) prove the second statement of theorem 4.0.1. The field $\boldsymbol{\sigma}$ in (4.82) is slightly different from the one obtained in (3.52) by a non-twisted fluctuation of D_M by $\mathcal{A}_{sm} \otimes C^\infty(\mathcal{M})$, that is in the form:

$$\sigma = (1 + \phi)\mathbb{I}. \tag{4.88}$$

4.3 Breaking of the grand symmetry to the standard model

We give a justification of the third and fourth point of theorem 4.0.1 by computing the spectral action for the twisted-covariant Dirac operator

$$D_{\mathbb{A}} = D_X + D_{\boldsymbol{\sigma}}, \tag{4.89}$$

where D_X and $D_{\boldsymbol{\sigma}}$ have been obtained by twisted fluctuation of $\not{\partial}$ and D_M in (4.64) and (4.81). The full proof can be found in [37] where it is shown that the potential part of this action is minimum when the Dirac operator $\not{\partial} + D_M$ of the twisted spectral triple is fluctuated by a subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ which is invariant under the automorphism ρ. The maximal such sub-algebra is precisely the algebra $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$ of the standard model. Indeed by (4.32) an element (Q, M) of $\mathcal{B}$ is invariant by the automorphism ρ if and only if

$$\rho(Q) = Q, \tag{4.90}$$

which means $\mathbb{H}_R^r = \mathbb{H}_R^l$ and $\mathbb{C}_L^r = \mathbb{C}_L^l$, that is $(Q, M) \in \mathcal{A}_{sm}$.

We begin establishing the generalized Lichnerowicz formula for $D_{\mathbb{A}}$ and then we study the potential for the vector field, the scalar field, and their interaction.

4.3.1 Lichnerowicz formula for the twisted-covariant Dirac operator

We define

$$\not{X} := -i\gamma^\mu X_\mu, \quad \not{\rho}(X) := -i\gamma^\mu \rho(X_\mu). \tag{4.91}$$

These are selfadjoint operators since by (4.67) and (4.72) one has

$$\not{X}^\dagger = i\, X_\mu^\dagger \gamma^\mu = -i\rho(X_\mu)\gamma^\mu = -i\gamma^\mu X_\mu = \not{X}, \tag{4.92}$$

and similarly for $\not{\rho}(X)$. The same is true for

$$\bar{\not{X}} := -i\gamma^\mu \bar{X}_\mu, \quad \not{\rho}(\bar{X}) := -i\gamma^\mu \rho(\bar{X}_\mu). \tag{4.93}$$

Similar equations hold for the field $\boldsymbol{\sigma}$, by extending the automorphism ρ to $\mathcal{B}(\mathcal{H})$ as the conjugate action of the unitary operator that exchanges the indices l and r in the basis of H. Doing so, one gets $\rho(\gamma^5) = -\gamma^5$, that is

$$\rho(\boldsymbol{\sigma}) = \mathbb{I} - \gamma^5 \phi. \tag{4.94}$$

Thus $\boldsymbol{\sigma}$ twisted-commutes with γ^μ - as X_μ in (4.67) - for the anti-commutativity of γ^μ and γ^5 yields

$$\gamma^\mu \boldsymbol{\sigma} = \rho(\boldsymbol{\sigma})\gamma^\mu, \qquad \gamma^\mu \rho(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\gamma^\mu. \tag{4.95}$$

The standard model algebra $\mathcal{A}_{sm}$ is the subalgebra of $\mathcal{B}$ invariant under the twist. To measure how far the grand symmetry is from the SM, we introduce as physical degrees of freedom the fields

$$\Delta(X)_\mu := X_\mu - \rho(X_\mu), \quad \Delta(\boldsymbol{\sigma}) := (\boldsymbol{\sigma} - \rho(\boldsymbol{\sigma}))D_R. \tag{4.96}$$

Both are selfadjoint, $\Delta(X)_\mu$ by eq. (4.72), $\Delta(\boldsymbol{\sigma})$ because $\boldsymbol{\sigma}$ and D_R are selfadjoint and commute. Moreover, by (4.67) and (4.95) one has

$$\{\gamma^\mu, \Delta(X)_\nu\} = \{\gamma^\mu, \Delta(\boldsymbol{\sigma})\} = 0, \tag{4.97}$$

while γ^5 commuting with X_μ and $\boldsymbol{\sigma}$ guarantee that

$$\left[\gamma^5, \Delta(X)_\nu\right] = \left[\gamma^5, \Delta(\boldsymbol{\sigma})\right] = 0. \tag{4.98}$$

We write

$$\rho(\mathbb{X}_\mu) := \begin{pmatrix} \not{\rho}(X) & 0_{64} \\ 0_{64} & -\not{\rho}(\bar{X}) \end{pmatrix}_{\mathsf{CD}}, \qquad \Delta(\mathbb{X})_\mu := \mathbb{X}_\mu - \rho(\mathbb{X}_\mu), \tag{4.99}$$

and in agreement with (4.64) and (4.65) written as

$$\not{\mathbb{X}} = -i\gamma^\mu \mathbb{X}_\mu = \begin{pmatrix} \not{X} & 0_{64} \\ 0_{64} & -\not{\bar{X}} \end{pmatrix}_{\mathsf{CD}}, \tag{4.100}$$

we also define the selfadjont operators

$$\not{\rho}(\mathbb{X}) := -i\gamma^\mu \rho(\mathbb{X}_\mu), \quad \not{\Delta}(\mathbb{X}) := \not{\mathbb{X}} - \not{\rho}(\mathbb{X}). \tag{4.101}$$

Finally, we let

$$D_\mu := \partial_\mu + \text{ad } \mathbb{X}_\mu \tag{4.102}$$

denotes the covariant derivative associated with the connection $\mathbb{X}_\mu$.

Using the above definitions, we want to show that the square of the twisted-covariant Dirac operator (4.89) is

$$D^2_{\mathbb{A}} = -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + (\alpha^\mu_X + \alpha^\mu_{\boldsymbol{\sigma}})\partial_\mu + \beta_X + \beta_{X\boldsymbol{\sigma}} + \beta_{\boldsymbol{\sigma}}\right) \tag{4.103}$$

where

$$\alpha^\mu_X \;\; := i\left\{\not{\mathbb{X}}, \gamma^\mu\right\}, \quad \beta_X = i\gamma^\mu(\partial_\mu \not{\mathbb{X}}) - \not{\mathbb{X}}\not{\mathbb{X}}, \tag{4.104}$$

while

$$\alpha^\mu_{\boldsymbol{\sigma}} := i\gamma^\mu\gamma^5\Delta(\boldsymbol{\sigma}), \qquad \beta_{\boldsymbol{\sigma}} := -\boldsymbol{\sigma}^2 D_R^2, \tag{4.105}$$

and

$$\beta_{X\boldsymbol{\sigma}} := i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) + \Delta(\boldsymbol{\sigma})\,\mathbb{X}_\mu\right). \tag{4.106}$$

To prove eq. (4.103) we can start from $D^2_{\mathbb{A}} = D^2_X + D^2_\sigma + \{D_X, D_\sigma\}$. By (4.64), the first term is

$$\begin{aligned} D^2_X &= -\gamma^\mu(\partial_\mu + \mathbb{X}_\mu)\,\gamma^\nu(\partial_\nu + \mathbb{X}_\nu) \\ &= -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - i\left\{\not{\mathbb{X}}, \gamma^\mu\right\}\partial_\mu - i\gamma^\mu(\partial_\mu\not{\mathbb{X}}) + \not{\mathbb{X}}\not{\mathbb{X}} \\ &= -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \alpha^\mu_X\partial_\mu + \beta_X\right). \end{aligned} \tag{4.107}$$

By using eq. (4.64) one has

$$\{D_X, D_{\boldsymbol{\sigma}}\} = -i\,\{\gamma^\mu\partial_\mu, D_{\boldsymbol{\sigma}}\} - i\,\{\gamma^\mu\mathbb{X}_\mu, D_{\boldsymbol{\sigma}}\}\,. \tag{4.108}$$

From the expression of $D_{\boldsymbol{\sigma}}$ given in (4.81), using (4.95) and $\{\gamma^5, \gamma^\mu\} = [\gamma^5, \boldsymbol{\sigma}] = 0$, one gets

$$\begin{aligned}\{\gamma^\mu\partial_\mu, D_{\boldsymbol{\sigma}}\} = \left\{\gamma^\mu\partial_\mu, \gamma^5\boldsymbol{\sigma}D_R\right\} &= \gamma^\mu\gamma^5\partial_\mu\boldsymbol{\sigma}D_R - \gamma^\mu\gamma^5\rho(\boldsymbol{\sigma})D_R\,\partial_\mu,\\ &= \gamma^\mu\gamma^5\left(\partial_\mu\boldsymbol{\sigma}D_R\right) + \gamma^\mu\gamma^5\Delta(\boldsymbol{\sigma})D_R\,\partial_\mu.\end{aligned} \tag{4.109}$$

Similarly, using that γ^5 commutes with X_μ, hence with $\mathbb{X}_\mu$, one has

$$\begin{aligned}\{\gamma^\mu\mathbb{X}_\mu, D_{\boldsymbol{\sigma}}\} = \left\{\gamma^\mu\mathbb{X}_\mu, \boldsymbol{\sigma}\gamma^5 D_R\right\} &= \gamma^\mu\mathbb{X}_\mu\boldsymbol{\sigma}\gamma^5 D_R + \boldsymbol{\sigma}\gamma^5 D_R\gamma^\mu\mathbb{X}_\mu,\\ &= \gamma^\mu\gamma^5[\mathbb{X}_\mu, \boldsymbol{\sigma}D_R]_\rho = \gamma^\mu\gamma^5\left([\mathbb{X}_\mu, \boldsymbol{\sigma}D_R] + \Delta(\boldsymbol{\sigma})\,\mathbb{X}_\mu\right).\end{aligned} \tag{4.110}$$

Summing (4.110) and (4.109), and using the definition (4.102) of D_μ, one rewrites (4.108) as $\{D_X, D_{\boldsymbol{\sigma}}\} = -(\alpha^\mu_{\boldsymbol{\sigma}}\partial_\mu + \beta_{X\boldsymbol{\sigma}})$. Finally from (4.81) one has $D^2_{\boldsymbol{\sigma}} = -\beta_{\boldsymbol{\sigma}}$. ■

Remarkably the contributions $\alpha^\mu_{\boldsymbol{\sigma}}$ of the anti-commutator of D_X and $D_{\boldsymbol{\sigma}}$ to the order one part of $D^2_{\mathbb{A}}$ depends on $\boldsymbol{\sigma}$ only, and not on X. The same is true for $\beta_{\boldsymbol{\sigma}}$. The contributions α^μ_X and β_X of D_X depend on X only, and not on $\boldsymbol{\sigma}$. Thus in the Lichnerowicz formula for $D^2_{\mathbb{A}}$, that is

$$D^2_{\mathbb{A}} = -\nabla_\mu\nabla^\mu - E \tag{4.111}$$

with

$$\nabla_\mu = \partial_\mu + \frac{1}{2}g_{\mu\nu}(\alpha^\nu_X + \alpha^\nu_{\boldsymbol{\sigma}}), \tag{4.112}$$

the bounded endormorphism E is the sum

$$E = E_X + E_{\boldsymbol{\sigma}} + E_{X\boldsymbol{\sigma}} \tag{4.113}$$

of three terms:

$$E_X := \beta_X - \frac{1}{4}\alpha_X\cdot\alpha_X - \frac{1}{2}\partial_\mu\alpha^\mu_X, \tag{4.114}$$

which depends only on X,

$$E_{\boldsymbol{\sigma}} := \beta_{\boldsymbol{\sigma}} - \frac{1}{4}\alpha_{\boldsymbol{\sigma}}\cdot\alpha_{\boldsymbol{\sigma}} - \frac{1}{2}\partial_\mu\alpha_{\boldsymbol{\sigma}}, \tag{4.115}$$

that depends only on $\boldsymbol{\sigma}$, and an interaction term

$$E_{X\boldsymbol{\sigma}} := \beta_{X\boldsymbol{\sigma}} - \frac{1}{4}\left(\alpha_X\cdot\alpha_{\boldsymbol{\sigma}} + \alpha_{\boldsymbol{\sigma}}\cdot\alpha_X\right). \tag{4.116}$$

The endomorphisms $E_X, E_{\boldsymbol{\sigma}}$ and $E_{X\boldsymbol{\sigma}}$ can be written in terms of the physical degrees of freedom $\Delta(\boldsymbol{\sigma})$, $\Delta(X)_\mu$ defined in (4.96) which permit to measure how far the twisted spectral triple (4.55) is from the spectral triple of the SM, (the complete calculation can be found in [37, §5.3])

$$E_X = \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{F}_{\mu\nu} + D_\nu\,\Delta(\mathbb{X}_\mu) + \Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu\right), \tag{4.117}$$

$$E_{X\boldsymbol{\sigma}} = i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) - \frac{1}{2}[\mathbb{X}_\mu, \Delta(\boldsymbol{\sigma})] + \frac{1}{2}\left\{3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\right\}\right) \tag{4.118}$$

$$E_{\boldsymbol{\sigma}} = \Delta(\boldsymbol{\sigma})^2 - \boldsymbol{\sigma}^2 D_R^2 - \frac{i}{2}\gamma^\mu\gamma^5\partial_\mu\Delta(\boldsymbol{\sigma}). \tag{4.119}$$

where

$$\mathbb{F}_{\mu\nu} := (\partial_\mu\mathbb{X}_\nu) - (\partial_\nu\mathbb{X}_\mu) + [\mathbb{X}_\mu, \mathbb{X}_\nu] \tag{4.120}$$

is the field strength of $\mathbb{X}_\mu$

Summing up these terms, one obtains from (4.111)

$$\begin{aligned} D_{\mathbb{A}}^2 = &-g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{F}_{\mu\nu} + D_\nu\,\Delta(\mathbb{X})_\mu + \Delta(\mathbb{X})_\mu\Delta(\mathbb{X}_\nu)\right) \\ &+ \boldsymbol{\sigma}^2 D_R^2 - \Delta(\boldsymbol{\sigma})^2 - i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) - \frac{1}{2}D_\mu\Delta(\boldsymbol{\sigma})\right) \\ &- \frac{i}{2}\gamma^\mu\gamma^5\left\{3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\right\}. \end{aligned} \tag{4.121}$$

One may be puzzled by the presence of two distincts covariant derivatives in the Lichnerowicz formula for $D_{\mathbb{A}}$: ∇_μ in the Laplacian and D_μ that encodes the dynamics of $\Delta(\mathbb{X})_\mu$ and $\Delta(\boldsymbol{\sigma})$. In the non-twisted case this is the same covariant derivative which play both role. However, because we switch gravitation off[6] and consider the flat case, in the heat kernel expansion of the spectral action the covariant derivative ∇_μ only appears through the term $\nabla^\mu\nabla_\mu E$ (in a_4). The latter is interpreted as a boundary term (see [31, Remark 1.155]) and we shall not take it into account here. Doing so, only one covariant derivative remains, D_μ. This makes sense from our perspective: the fields $\Delta(\mathbb{X})_\mu$ and $\Delta(\boldsymbol{\sigma})$ are viewed as "excitations" generated by the twist, living on a background gauge theory with connection 1-form X_μ; so their dynamics is encoded by D_μ, not by ∇_μ.

The remaining Seeley-de Witt coefficients are a_0, which is not affected by the twist and is interpreted as the cosmological constant (which recently turns out to be quantized, see [20]) and the integral of the trace of E (in a_2) and E^2 (in a_4) for E given in (4.113). In other terms the potential is the part of

$$V := \Lambda^2 f_2\,\mathrm{Tr}\,E \;+\; \frac{1}{2}\,f_0\mathrm{Tr}\,E^2 \tag{4.122}$$

[6]Our aim in this chapter is to understand how the twist allows to generate the field $\boldsymbol{\sigma}$. That is why for simplicity we consider the flat case. The curved case, which should be similar, will be studied elsewhere.

that does not depend on the covariant derivative D_μ. We analyze it below, dividing it into three pieces: the potential $V(X)$ of the vector field, $V(\boldsymbol{\sigma})$ of the scalar field, and a interaction potential $V(X, \boldsymbol{\sigma})$.

4.3.2 The vector field and the breaking to the standard model

The potential $V(X)$ is the part of V that depends on $\Delta(\mathbb{X})_\mu$ and no on its derivative, that is

$$V(X) = \Lambda^2 f_2 \operatorname{Tr} E^0_X + \frac{1}{2} f_0 \operatorname{Tr} (E^0_X)^2, \tag{4.123}$$

where $E^0_X := \frac{1}{2}\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu$ is read in (4.117). One rewrites it as

$$E^0_X = \frac{1}{2}\not{\Delta}^2(\mathbb{X}), \tag{4.124}$$

thanks to (4.97) which guarantees that γ^ν anti-commutes with $\Delta(\mathbb{X})_\mu$ for all μ.

In the following we want to show that the potential $V(X)$ is never negative and vanishes iff $\Delta(\mathbb{X})_\mu = 0$ for any μ.

In fact, since $\not{\Delta}(\mathbb{X})$ is selfadjoint, E^0_X and $(E^0_X)^2$ are positive. Thus their trace is never negative, and vanishes if and only if $E^0_X = (E^0_X)^2 = 0$. This condition is equivalent to

$$\Delta(\mathbb{X})_\mu = 0 \quad \forall\mu. \tag{4.125}$$

Indeed, since $\{\gamma^\nu, \Delta(\mathbb{X})_\mu\} = 0$ one has

$$\operatorname{Tr}\,(\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) = \operatorname{Tr}\,(\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu\gamma^\nu) = \operatorname{Tr}\,(\gamma^\nu\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) \tag{4.126}$$

where the last equality comes from the tracial property. Therefore

$$\begin{aligned}\operatorname{Tr} E^0_X &= \frac{1}{4}(\operatorname{Tr}\,(\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) + \operatorname{Tr}\,(\gamma^\nu\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu)),\\ &= \frac{1}{2}\, g^{\mu\nu} \operatorname{Tr}(\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) = \frac{1}{2}\sum_\mu \operatorname{Tr}\left(\Delta^2(\mathbb{X})_\mu\right).\end{aligned} \tag{4.127}$$

Since $\Delta(\mathbb{X})_\mu$ is selfadjoint, $\Delta^2(\mathbb{X})_\mu$ is positive. Its trace is never negative and vanishes if and only if $\Delta(\mathbb{X})_\mu$ is zero. The same is true for the sum in (4.127), meaning that Tr E^0_X - hence E^0_X - vanishes if and only if $\Delta(\mathbb{X})_\mu = 0$ for all μ.

In conclusion, since f_0 and f_2 are positive numbers we can say that the potential $V(X)$ is never negative. ∎

Condition (4.125) is equivalent to $\Delta(X)_\mu = 0$ for any μ. To obtain the breaking to the standard model, one can check that the vanishing of $\Delta(X)_\mu$, that is the invariance of X_μ under the twist, implies the invariance of its components R^i, Q_i [37, §5.4]

4.3.3 The scalar field

The part of the potential containing only the extra scalar field and not the vector field is

$$V(\boldsymbol{\sigma}) := \Lambda^2 f_2 \operatorname{Tr} E^0_{\boldsymbol{\sigma}} + \frac{1}{2} f_0 \operatorname{Tr} (E^0_{\boldsymbol{\sigma}})^2, \tag{4.128}$$

where

$$E^0_{\boldsymbol{\sigma}} := \Delta^2(\boldsymbol{\sigma}) - \sigma^2 D_R^2 \tag{4.129}$$

is read in (4.119). Compared to $V(X)$ which contains only $\Delta(\mathbb{X})_\mu$ and not $\mathbb{X}_\mu$, the potential $V(\boldsymbol{\sigma})$ contains both $\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma})$. This gives two possibilites for minimizing:

Either one considers only $\Delta(\boldsymbol{\sigma})$ as degree of freedom. The potential then reduces to

$$V(\Delta(\boldsymbol{\sigma})) := \Lambda^2 f_2 \operatorname{Tr}(\Delta^2(\boldsymbol{\sigma})) + \frac{1}{2} f_0 \operatorname{Tr}(\Delta^4(\boldsymbol{\sigma})). \tag{4.130}$$

Since $\Delta(\boldsymbol{\sigma})$ is selfadjoint, this potential is positive and vanishes if and only if $\Delta(\boldsymbol{\sigma}) = 0$. Going back to the the definition 4.96 of $\Delta(\boldsymbol{\sigma})$, this means

$$\boldsymbol{\sigma} = \rho(\boldsymbol{\sigma}) = \mathbb{I}. \tag{4.131}$$

Or one may prefer to take into account the whole potential (4.128) taking as degree of freedom the field ϕ but we omit the proof that can be found in [37, §5.5]

The invariance (4.131) of $\boldsymbol{\sigma}$ under the twist implies that $D_{\boldsymbol{\sigma}} = D_M$, so that one is back to the Dirac operator of the standard model. However this does not imply the reduction of the algebra to the one of the standard model. Indeed, from (4.84) the vanishing of ϕ means $c_R^r = c_R^l$, so that the bigger subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ for which any fluctuation yields a ρ-invariant $\boldsymbol{\sigma}$ is

$$C^\infty(\mathcal{M}) \otimes (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C} \oplus M_3(\mathbb{C})), \tag{4.132}$$

which contains, but is different from $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$.

4.3.4 Interaction potential

The interaction term $V(X, \boldsymbol{\sigma})$ between the scalar and the vector fields, as calculated in [37, §5.6], has the following form

$$V(X, \boldsymbol{\sigma}) := \frac{1}{2} f_0 \operatorname{Tr} (E^0_{X\boldsymbol{\sigma}})^2 + f_0 \operatorname{Tr} E^0_X E^0_{\boldsymbol{\sigma}}. \tag{4.133}$$

with

$$E^0_{X\boldsymbol{\sigma}} := \frac{i}{2}\gamma^\mu\gamma^5\left\{\mathbb{H}_\mu, \Delta(\boldsymbol{\sigma})\right\} \quad \text{with} \quad \mathbb{H}_\mu := 3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu \tag{4.134}$$

In [37] it is shown that the whole potential $V(X)+V(\boldsymbol{\sigma})+V(X,\boldsymbol{\sigma})$ is never negative and it is zero if and only if both the scalar field $\boldsymbol{\sigma}$ and the vector field $\Delta(X)_\mu$ are zero. This proves the point iii) in theorem 4.0.1.

4.4 Twist and representations

We discuss the choices made in the construction of the twisted spectral triple of the standard model: the intermediate solution consisting in imposing by hand the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, and the representation of $\mathcal{A}_G$.

4.4.1 Global twist

Instead of reducing by hand $\mathcal{B}_{LR}$ to $\mathcal{B}'$ by imposing the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, one could twist $\mathcal{B}_{LR}$ as well. This means finding an automorphism ρ of $M_8(\mathbb{C})$ such that

$$\sigma^\mu M\,\partial_\mu - \sigma(M)\sigma^\mu\partial_\mu = 0, \qquad \bar{\sigma}^\mu M\,\partial_\mu - \bar{\sigma}(M)\bar{\sigma}^\mu\partial_\mu = 0. \tag{4.135}$$

Using $\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu = \nabla^2$, the first expression yields

$$\sigma(M) = \sigma^\mu M\bar{\sigma}^\nu\frac{1}{\nabla^2}\partial_\mu\partial_\nu. \tag{4.136}$$

The right-hand side of (4.136) has to be understood in the context of the pseudo differential operator theory of which a very sketchy overlook can be found in [65, appendix F]. This does not define an automorphism of $C^\infty(\mathcal{M})\otimes\mathcal{A}_G$. Indeed, writing $T_{\mu\nu} \equiv \frac{1}{\nabla^2}\partial_\mu\partial_\nu$ and $M_1^{\mu\nu} \equiv \sigma^\mu M_1\bar{\sigma}^\nu$, one gets

$$\begin{aligned}
\sigma(M_1)\sigma(M_2) &= \left(M_1^{\mu\nu}T_{\mu\nu}\right)\left(M_2^{\alpha\beta}T_{\alpha\beta}\right) && (4.137)\\
&= M_1^{\mu\nu}\left[T_{\mu\nu}, M_2^{\alpha\beta}\right]T_{\alpha\beta} + M_1^{\mu\nu}M_2^{\alpha\beta}T_{\mu\nu}T_{\alpha\beta}, && (4.138)\\
&= \sigma(M_1M_2) + M_1^{\mu\nu}\left[T_{\mu\nu}, M_2^{\alpha\beta}\right]T_{\alpha\beta} && (4.139)
\end{aligned}$$

where we compute

$$\begin{aligned}
M_1^{\mu\nu}M_2^{\alpha\beta}T_{\mu\nu}T_{\alpha\beta} &= \sigma^\mu M_1\bar{\sigma}^\nu\sigma^\alpha M_2\bar{\sigma}^\beta\frac{1}{\nabla^2}\frac{1}{\nabla^2}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\\
&= \sigma^\mu M_1M_2\bar{\sigma}^\beta\frac{1}{\nabla^2}\partial_\mu\partial_\beta\\
&= \sigma(M_1M_2). && (4.140)
\end{aligned}$$

A possible solution is to look for a $\star$ product such that

$$\sigma(M_1) \star \sigma(M_2) = \sigma(M_1 \star M_2), \tag{4.141}$$

that would encode the intrinsic mixing between the manifold (space-time) and the matrix part (gauge sector) that is the core of the Grand Symmetry. This would also force us to consider an algebra $\mathcal{A}_0$ of pseudo-differential operators bigger than $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. This point is particularly interesting if one believes that almost commutative geometries are an effective low energy description of a more fundamental theory, based on a "truly" non-commutative algebra (that is with a finite dimensional center). This idea has been often advertised by D. Kastler, and it could be that $\mathcal{A}_0$ is not so far from the "noncommutative salmon" he aims at fishing. All this will be investigated in future works.

The reason why we choose the representation (4.13) instead of (3.4) as in §3.1 is that while it is right that (4.136) is still in $\mathbb{M}_4(\mathbb{C})$, it would not be true for an element $Q = Q^{\dot{t}\beta}_{\dot{s}\alpha} \in M_2(\mathbb{H})$ that $\sigma^\mu Q \bar{\sigma}^\nu$ is still in $M_2(\mathbb{H})$. However, all the results presented in this work would also be true with the representation (3.4), as explained in the next paragraph.

4.4.2 Invariance of the constraints

The grand algebra in the representation (4.13) is broken by the grading to (3.10)

$$\mathcal{A}'_G = M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}). \tag{4.142}$$

To have bounded commutators with $\not{D}$, we impose by hand that quaternions act trivially on the $\dot{s}$ index, yielding the reduction to

$$\mathcal{A}' := \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}) \tag{4.143}$$

whose elements are (Q, M) where

$$Q = \delta^{t\dot{t}}_{s\dot{s}} \begin{pmatrix} q_R & 0_2 \\ 0_2 & q_L \end{pmatrix}_{\alpha\beta}, \quad M = \begin{pmatrix} M_l^l & 0_4 \\ 0_4 & M_r^r \end{pmatrix}_{st} \quad \text{with } q_r \in \mathbb{H}, M_l^l, M_r^r \in M_4(\mathbb{C}). \tag{4.144}$$

The twist ρ is still defined as the exchange of the left and right part of spinors, but it now acts on the matrix part

$$\rho(M) = \begin{pmatrix} M_r^r & 0_4 \\ 0_4 & M_l^l \end{pmatrix}_{st}. \tag{4.145}$$

This guarantees that

$$[\not{D}, M]_\rho = (\not{\partial} M) + [\gamma^\mu, M]_\rho = (\not{\partial} M) \tag{4.146}$$

is bounded, so that $(C^\infty(\mathcal{M})\otimes\mathcal{A}', \mathcal{H}, \partial\!\!\!/ + D_M; \rho)$ is a twisted spectral triple. The twisted first-order condition for $\partial\!\!\!/$ is checked as in §4.1.2.

For the twisted first-order condition imposed by D_M, one first consider the subalgebra of $\mathcal{A}'$

$$\tilde{\mathcal{A}} := \mathbb{H}_L \oplus \mathbb{C}_R \oplus M_3^l(\mathbb{C}) \oplus \mathbb{C}^l \oplus M_3^r(\mathbb{C}) \oplus \mathbb{C}^r \tag{4.147}$$

obtained by asking

$$q_R = \begin{pmatrix} c_R & 0 \\ 0 & \bar{c}_R \end{pmatrix} \quad \text{with } c_R \in \mathbb{C} \tag{4.148}$$

in (4.144) and

$$M_r^r = \begin{pmatrix} m^r & 0_2 \\ 0_2 & \mathbf{M}^r \end{pmatrix}_{\mathrm{IJ}}, \quad M_l^l = \begin{pmatrix} m^l & 0_2 \\ 0_2 & \mathbf{M}^l \end{pmatrix}_{\mathrm{IJ}} \text{ with } \mathbf{M}^r, \mathbf{M}^l \in M_3(\mathbb{C}),\ m^r, m^l \in \mathbb{C}. \tag{4.149}$$

Let $B = (R, N) \in \tilde{\mathcal{B}}$ be another element of $\tilde{\mathcal{A}}$, with components $d_r, n^r, n^l \in \mathbb{C}$ and $\mathbf{N}^r, \mathbf{N}^l \in M_3(\mathbb{C})$. The double twisted commutator $[[D_M, A]_\rho,\ JBJ^{-1}]_\rho$ is an off-diagonal matrix with components

$$(\mathsf{D}_M M - Q\mathsf{D}_M)\bar{R} - \rho(\bar{N})(\mathsf{D}_M M - Q\mathsf{D}_M), \tag{4.150}$$
$$(\mathsf{D}_M Q - \rho(M)\mathsf{D}_M)\bar{N} - \bar{R}(\mathsf{D}_M Q - \rho(M)\mathsf{D}_M). \tag{4.151}$$

One has

$$\begin{aligned}
\rho(\bar{N})\mathsf{D}_\nu M &= (\rho(\bar{N})\eta\Xi M)_{s\mathrm{J}}^{t\mathrm{I}}(\Xi\delta)_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \bar{\mathsf{n}}^l\mathsf{m}^\mathsf{r} & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r\mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}}, \\
\rho(\bar{N})Q\mathsf{D}_\nu &= (\rho(\bar{N})\eta\Xi)_{s\mathrm{J}}^{t\mathrm{I}}\,(Q\Xi)_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \bar{\mathsf{n}}^l & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}, \\
\mathsf{D}_\nu M\bar{R} &= (\eta\Xi M)_{s\mathrm{J}}^{t\mathrm{I}}\,(\Xi\bar{R})_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \mathsf{m}^r & 0_4 \\ 0_4 & -\mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \\
Q\mathsf{D}_\nu\bar{R} &= (\eta\Xi)_{s\mathrm{J}}^{t\mathrm{I}}\,(Q\Xi\bar{R})_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}},
\end{aligned} \tag{4.152}$$

where we defined

$$\mathsf{m}^\mathsf{r} := \begin{pmatrix} m^r & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{m}^\mathsf{l} := \begin{pmatrix} m^l & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}_R = \begin{pmatrix} c_R & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}} \tag{4.153}$$

and similarly for n^r, n^l and d_R. Collecting the various terms, one finds that (4.150) is zero if and only if

$$(c_R - m^r)(\bar{d}_R - \bar{n}^l) = 0, \quad (c_R - m^l)(\bar{d}_R - \bar{n}^r) = 0 \tag{4.154}$$

which are the same constraints (4.51) coming from the other representation. The same is true for (4.151), using

$$\bar{R}\rho(M)\mathsf{D}_\nu = (\rho(M)\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi\bar{R})^{i\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l & 0_4 \\ 0_4 & -\mathsf{m}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}},$$

$$\bar{R}\mathsf{D}_\nu Q = (\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\bar{R}\Xi Q)^{i\beta}_{\dot{s}\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}},$$

$$\rho(M)\mathsf{D}_\nu\bar{N} = (\rho(M)\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}(\Xi)^{i\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l\,\bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\mathsf{m}^r\,\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}},$$

$$\mathsf{D}_\nu Q\bar{N} = (\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi Q)^{i\beta}_{\dot{s}\alpha} = \begin{pmatrix} \bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}. \qquad (4.155)$$

Solving (4.51) by asking $m^r = c_R$, that is identifying $\mathbb{C}^r$ and $\mathbb{C}_R$ with a single copy $\mathbb{C}^r_R$ of the complex numbers, one reduces $\tilde{A}$ to

$$\mathcal{A} := \mathbb{H}_L \oplus \mathbb{C}^r_R \oplus \mathbb{C}^l \oplus M^l_3(\mathbb{C}) \oplus M^r_3(\mathbb{C}). \qquad (4.156)$$

This algebra plays for the representation (3.4) the same role as the algebra $\mathcal{B}$ for the representation (4.13). Repeating the computation of §4.2.3, one finds a scalar field similar to $\boldsymbol{\sigma}$. Thus, except for the hope of a global twist described in §4.4.1, there is at the moment no motivation to prefer one or the other of the two natural representations of the grand algebra.

4.5 Comments

Let us summarize our results by the following chain of breaking, to be compared with (3.65):

$$\begin{array}{rcl}
\mathcal{A}_G & = & M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{grading condition} \\
\mathcal{B}_{LR} & = & (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{ bounded commutator for } M_8(\mathbb{C}) \\
\mathcal{B}' & = & (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_4(\mathbb{C}) \\
& \Downarrow & 1^{\text{st}}\text{-order for the Majorana-Dirac operator} D_M \\
\mathcal{B} & = & (\mathbb{H}_L^l \oplus \mathbb{C}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{C}_R^r) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \text{ with } \mathbb{C} = \mathbb{C}_R^r \\
& \Downarrow & \text{minimum of the spectral action} \\
\mathcal{A}_{sm} & = & \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{array}$$

Starting with the "not so grand algebra" $\mathcal{B}$, one builds a twisted spectral triple whose fluctuations generate both an extra scalar field $\boldsymbol{\sigma}$ and an additional vector field X_μ. This is a Pati-Salam like model - the unitary of $\mathcal{B}$ yields both an $SU(2)_R$ and an $SU(2)_L$, together with an extra $U(1)$ - but in a pre-geometric phase since the Lorentz symmetry (in our case: the Euclidean $SO(n)$ symmetry) is not explicit. The spectral action spontaneously breaks this model to the standard model, in which the Lorentz symmetry is explicit, with the scalar and the vector fields playing a role similar as the one of Higgs field. We thus have a dynamical model of emergent geometry.

The idea that the scalar field σ is associated to the spontaneous breaking of a bigger symmetry to the standard model had been also implemented in [21], where the standard model symmetry does not come from a bigger algebra, but follows from relaxing the first-order condition. It would be interesting to understand to what extend the twisted fluctuations presented here are a particular case of those inner fluctuation without first oder condition. More generally, the structure of the set of twisted fluctuations and of the associated twisted-gauge transformations of $\mathbb{A}$ needs to be worked out.

The twist ρ is remarkably simple, and its mathematical significance should be studied more in details, in particular how it should be incorporated in the axioms of noncommutative geometry, like the orientability condition where the commutator with the Dirac operator plays a crucial role. Also, the physical meaning of the twist is intriguing: the un-twisting of $\mathcal{B}$ forces the action of the algebra to be the same on the left and right components of spinors. In this sense the breaking of the grand algebra to the standard model is a sort of "primordial" chiral symmetry breaking.

Chapter 5

Gauge unification

The coupling constants of the three gauge interactions run with energy, as seen in §1.4. The ones relating to the nonabelian symmetries are relatively strong at low energy, but decrease, while the abelian interaction increases. As already explained in chapter 1, at an energy comprised between $10^{13} - 10^{17}$ GeV their values are very similar, around 0.52, but, in view of present data, and *in absence of new physics*, they fail to meet at a single scale. Here by absence of new physics we mean extra terms in the Lagrangian of the model. The extra terms may be due for example to the presence of new particles, or new interaction, like the supersymmetric models which can alter the running and cause the presence of the unification point [74].

Since the gauge unification is a foundamental statement of the spectral action approach, in this chapter we want to investigate whether the presence of higher dimensional terms in the standard model action – dimension six in particular – may cause the unification of the coupling constants. This work may be read in two contexts: as an application of the spectral action, or independently on it, from a purely phenomenologically point of view.

From the spectral point of view, as shown in §2.5, the spectral action is solved as a heath kernel expansion in powers in the inverse of an energy scale. The terms up to dimension four reproduce the standard model qualitatively, as shown in §2.6, but the theory is valid at a scale in which the couplings are equal. The expansion gives, however, also higher dimensional terms, suppressed by the power of the scale, and depending on the details of the cutoff. This fixes relations among the coefficients of the new terms. The analysis in this chapter gives the conditions under which the spectral action can predict the unification of the three gauge coupling constants.

On the other side, it is also possible to read this part at a purely phenomenological level, using the spectral action as input only for the choice of the subset of all possible higher dimensions terms in the action, and as a guide for the setting of the low energy values for the couplings of the coefficients of the extra terms. We show that the presence of these terms enables the possibility of a unification.

In both cases the scale of unification Λ is considered the cutoff, and we run the theory below it. We assume, therefore, that perturbation theory is valid. There appears a hierarchy problem. From the point of view of the spectral action this implies a rather strange (though admissible) cutoff function. From a phenomenological point of view this entails either unnaturally large dimensionless quantities, or the presence of a new intermediate scale, Γ, which is not however the limit of validity of the effective theory, but it gives a guide to the magnitude of the coefficients of the new terms. The latter option is, of course, more desirable and we will discuss it below.

In section 5.1, we present the new dimension six operators, in addition to the standard model action, coming from the Seely-deWitt coefficient (2.51) of spectral action expansion (the complete calculations can be found in [35]). In section 5.2 we give the new renormalization group equations at one loop, due to the dim-6 operators; then, we show how these new operators affect the SM phenomenology. In section 5.3 we run the renormalization group equations to study the new coupling constants behavior, checking the possibility to improve the gauge unification point. A final section contains some comments and open questions.

5.1 Dimension six operators

As explained in chapter 2 the Lagrangian of the standard model can be obtained from first principles using the spectral action, which is a regularized trace, with Λ appearing as the cutoff. The relevant point is the fact that the spectral action requires the coupling constants of the three gauge groups to be equal at a scale Λ, which is also the cutoff of the theory. There is no need for a unified gauge group at the scale Λ, which in fact may signify a phase transition to a pre geometric phase [63], although larger symmetries are also possible, for example the grand symmetry of chapter 3 or other ones [21].

As seen in §2.5, the spectral action is an expansion in inverse powers of Λ^2, and it enables the presence of a set of new dimension six operators. Dimension five operators, which violate lepton number, and do not change the properties of the Higgs boson are not present in the expansion. The spectral action also gives relations among the coefficients of the required dimension six operators, which are described in [35, appendix A].

A complete classification of the dimension-six operators in the standard model is given in [53]. There, it is shown that there are 59 independent operators, preserving baryon number, after eliminating redundant operators using the equations of motion. Here we consider only the following dimension-six operators, mixing the gauge field strength and the Higgs field. They are the ones coming from the spectral action expansion[1]:

$$\begin{aligned}\mathcal{L}^{(6)} &= C_{HB} H\overline{H}\, B_{\mu\nu}B^{\mu\nu} + C_{HW} H\overline{H}\,\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu} + C_{HV} H\overline{H}\,\mathbf{V}_{\mu\nu}\mathbf{V}^{\mu\nu} + \\ &\quad + C_W \mathbf{W}_{\mu\nu}\mathbf{W}^{\nu\alpha}\mathbf{W}^{\mu}_{\alpha} + C_V \mathbf{V}_{\mu\nu}\mathbf{V}^{\nu\alpha}\mathbf{V}^{\mu}_{\alpha} + C_H\left(\overline{H}H\right)^3 \end{aligned} \tag{5.1}$$

The coefficients C_i have the dimension of an inverse energy square. The spectral action fixes their value at the cutoff Λ. To these terms we have to add a coupling between the Higgs, the W and the B which is absent in the spectral action at scale Λ, but is dynamically created. With the couplings considered here no other term is induced.

5.2 Coupling Constants RGEs

In this section we give the new renormalization group equations (RGEs) at one loop due to the dimensions-6 operators in the Lagrangian (5.1). Although the choice of the dimension six operators and some of characteristics of the Lagrangian are coming from the spectral action, this section can be read independently of it.

The full one-loop contributions to the SM running for dimension six operators have been calculated in [2, 52, 55, 56]. The modifications to the standard model RGEs are given by the following new terms to be added to the rhs of (2.66):

$$\begin{aligned}\delta\dot{g}_3 &= -4m_H^2 g_3 C_{HV} \\ \delta\dot{g}_2 &= -4m_H^2 g_2 C_{HW} \\ \delta\dot{g}_1 &= -4m_H^2 g_1 C_{HB} \\ \delta\dot{\lambda} &= m_H^2\left(9g_2^2 C_{HW} + 3g_1^2 C_{HB} + 12C_H + 3g_1g_2C_{HWB}\right) \\ \delta\dot{y}_{t,\nu} &= 0\end{aligned} \tag{5.1}$$

[1] See [35, appendix A] for the complete derivation.

and the RGEs for the dim-6 coupling constants are given by

$$
\begin{aligned}
\dot{C}_{HB} &= C_{HB}\left(12\lambda + 2\left(3y_t^2 + y_\nu^2\right) + \frac{85}{6}g_1^2 - \frac{9}{2}g_2^2\right) + 3C_{HWB}g_1g_2 \\
\dot{C}_{HW} &= C_{HW}\left(12\lambda + 2\left(3y_t^2 + y_\nu^2\right) - \frac{47}{6}g_1^2 - \frac{5}{2}g_2^2\right) + C_{HWB}g_1g_2 - 15C_W g_2^3 \\
\dot{C}_{HV} &= C_{HV}\left(12\lambda + 2\left(3y_t^2 + y_\nu^2\right) - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2 - 14g_3^2\right) \\
\dot{C}_{HWB} &= C_{HWB}\left(4\lambda + 2\left(3y_t^2 + y_\nu^2\right) + \frac{19}{3}g_1^2 + \frac{4}{3}g_2^2\right) + 2g_1g_2\left(C_{HB} + C_{HW}\right) + 3C_W g_1 g_2^2 \\
\dot{C}_W &= \frac{29}{2}C_W g_2^2 \\
\dot{C}_V &= 15C_V g_3^2 \\
\dot{C}_H &= C_H\left(108\lambda + 6\left(3y_t^2 + y_\nu^2\right) - \frac{9}{2}g_1^2 - \frac{27}{2}g_2^2\right) - 3C_B g_1^2\left(g_1^2 + g_2^2 - 4\lambda\right) + \\
&\quad + 3C_W g_2^2\left(12\lambda - 3g_2^2 - g_1^2\right) + C_{HWB}\left(12\lambda g_1g_2 - 3g_1^3g_2 - 3g_1g_2^3\right) \qquad (5.2)
\end{aligned}
$$

Although the spectral action does not contain explicitly the term $C_{HWB}H^2W_{\mu\nu}B^{\mu\nu}$, due to the unimodular condition, the coupling constant C_{HWB} is however induced by the running of C_{HB}, C_{HW} and C_W.

In the framework of the spectral action these equations are solved with boundary conditions at the cut-off scale Λ given by the coefficients appearing in the sixth Seeley-De Witt coefficient [35, appendix A]:

$$
\begin{aligned}
&C_{HB}(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{4\left(3\rho^2+17\right)}{9\left(\rho^2+3\right)}g^4 \ , \ C_{HW}(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{4}{3}g^4 \ , \ C_{HV}(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{16}{3\left(\rho^2+3\right)}g^4 \ ; \\
&C_H(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{512(\rho^6+3)}{3\left(\rho^2+3\right)^3}g^6 \ , \quad C_W(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{26}{15}g^3 \ , \quad C_V(\Lambda) = -\frac{f_6}{16\pi^2\Lambda^2}\frac{26}{15}g^3 \ .
\end{aligned}
\qquad (5.3)
$$

The coupling C_{HWB} is set to zero at the cut-off scale $C_{HWB}(\Lambda) = 0$ since it does not appear in the spectral action.

In (5.3) $g \equiv g_3(\Lambda) = g_2(\Lambda) = \frac{5}{3}g_1(\Lambda)$ is the value of the gauge coupling constants at the cut-off scale which, therefore, is identified with the unification scale. These two constants, g and Λ, together with the ratio ρ and the parameter f_6 appearing in the spectral action, will be the four free parameters of this model.

There are also constraints at low energy to satisfy. The values of the g_i's are known at the scale of the top mass with very high precision, and the parameters λ and the y_t are related to the Higgs and top mass. As we said earlier, the spectral action requires a positive value of λ at the cutoff scale Λ, (2.68), and without the field σ, it predicts a mass of the Higgs at 170 GeV.

However, the presence of higher-order operators in the action alters the form of the usual coupling constants, leading to a new phenomenology which we outline in the following section.

5.2.1 New phenomenology

In this section, following [2, §5] we give the main modifications to the SM phenomenology due to the dim-6 Lagrangian, i.e. the new form of the observables measured at the electroweak scale. The new operators, in fact, alter the definition of the SM parameters at tree level in several ways.

First of all, we focus on the effects of the dimension-six Lagrangian on the Higgs mass m_H and the self-coupling λ. The dim-6 operator $C_H(H^\dagger H)^3$ changes the shape of the scalar doublet potential at order $C_H v^2$ to

$$V(H) = -\frac{m^2}{2}H^\dagger H + \lambda \left(H^\dagger H\right)^2 - C_H \left(H^\dagger H\right)^3 \tag{5.4}$$

generating the new minimum

$$\begin{aligned} \langle H^\dagger H \rangle &= \frac{1}{3C_H}\left(\lambda - \sqrt{\lambda^2 - 3C_H\lambda v^2}\right) \\ &\simeq \frac{v^2}{2}\left(1 + \frac{3C_H v^2}{4\lambda}\right) \equiv \frac{v_T^2}{2} \end{aligned} \tag{5.5}$$

in the second line we have expanded the exact solution to first order in C_H. Therefore the shift in the vacuum expectation value is proportional to $C_H v^2$, which is of order $f_6 \frac{v^2}{\Lambda^2}$. On expanding the potential (5.4) around the minimum and neglecting kinetics corrections,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h + v_T \end{pmatrix}, \tag{5.6}$$

we find for the Higgs boson mass

$$m_H^2 = 2\lambda v_T^2 \left(1 - \frac{3C_H v^2}{2\lambda}\right) \tag{5.7}$$

At the same time the gauge fields and the gauge couplings are also affected by the dim-6 couplings.

In the broken theory the X^2H^2 operators (with X being any field strength) contribute to the gauge kinetic energies, through the Lagrangian terms

$$\begin{aligned} (\mathcal{L}_{SM} + \mathcal{L}_6)_{kin} &= -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\mathbf{V}_{\mu\nu}\mathbf{V}^{\mu\nu} - \frac{1}{4}\mathbf{W}^3_{\mu\nu}\mathbf{W}_3^{\mu\nu} - \frac{1}{2}W^+_{\mu\nu}W_-^{\mu\nu} + \\ &\quad + \frac{1}{2}v_T^2\left(C_{HB}B_{\mu\nu}B^{\mu\nu} + C_{HW}\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu} + C_{HV}\mathbf{V}_{\mu\nu}\mathbf{V}^{\mu\nu} - C_{HWB}W^3_{\mu\nu}B^{\mu\nu}\right) \end{aligned} \tag{5.8}$$

while for the mass terms of the gauge bosons, arising from $(D_\mu H)^\dagger (D^\mu H)$, we have

$$(\mathcal{L}_{SM} + \mathcal{L}_6)_{mass} = \frac{1}{4} g_2^2 v_T^2 W^+_{\mu\nu} W^{\mu\nu}_- + \frac{1}{8} v_T^2 \left(g_2 W^3_\mu - g_1 B_\mu \right)^2 \tag{5.9}$$

The mass terms coming from the kinetic term of the Higgs has been ignored since while the operators H^6 produces a shift of order $C_H v^2/\lambda$ (see eq. (5.7)), the kinetic operators produce a shift of order $C_{kin} v^2$ (see [2]) which will be negligible if C_H and C_{kin} are of the same order.

The gauge fields have to be redefined, so that the kinetic terms are properly normalized and diagonal,

$$G_\mu = \mathcal{G}_\mu \left(1 + C_{HV} v_T^2\right), \; W_\mu = \mathcal{W}_\mu \left(1 + C_{HW} v_T^2\right), \; B_\mu = \mathcal{B}_\mu \left(1 + C_{HB} v_T^2\right), \tag{5.10}$$

so that the modified coupling constants become

$$\bar{g}_3 = g_3 \left(1 + C_{HV} v_T^2\right), \; \bar{g}_2 = g_2 \left(1 + C_{HW} v_T^2\right), \; \bar{g}_1 = g_1 \left(1 + C_{HB} v_T^2\right), \tag{5.11}$$

and the products $g_1 B_\mu = \bar{g}_1 \mathcal{B}_\mu$ etc. are unchanged. Therefore, the electroweak Lagrangian is

$$\begin{aligned} \mathcal{L} \;\; = \;\; & -\frac{1}{4}\mathcal{B}_{\mu\nu}\mathcal{B}^{\mu\nu} - \frac{1}{4}\mathcal{W}^3_{\mu\nu}\mathcal{W}^{\mu\nu}_3 - \frac{1}{2}\mathcal{W}^+_{\mu\nu}\mathcal{W}^{\mu\nu}_- - \frac{1}{2}\left(v_T^2 C_{HWB}\right)\mathcal{W}^3_{\mu\nu}\mathcal{B}^{\mu\nu} \\ & + \frac{1}{4}\bar{g}_2^2 v_T^2 \mathcal{W}^+_{\mu\nu}\mathcal{W}^{\mu\nu}_- + \frac{1}{8} v_T^2 \left(\bar{g}_2 \mathcal{W}^3_\mu - \bar{g}_1 \mathcal{B}_\mu\right)^2 \end{aligned} \tag{5.12}$$

The mass eigenstate basis is given by, [2, eq. 5.21],

$$\begin{bmatrix} \mathcal{W}^3_\mu \\ \mathcal{B}_\mu \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} v_T^2 C_{HWB} \\ -\frac{1}{2} v_T^2 C_{HWB} & 1 \end{bmatrix} \begin{bmatrix} \cos\bar{\theta} & \sin\bar{\theta} \\ \text{-}\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix} \begin{bmatrix} \mathcal{Z}^3_\mu \\ \mathcal{A}_\mu \end{bmatrix}, \tag{5.13}$$

with $\bar{\theta}$, rotation angle, given by

$$\tan\bar{\theta} = \frac{\bar{g}_1}{\bar{g}_2} + \frac{v_T^2}{2} C_{HWB} \left[1 - \frac{\bar{g}_1^2}{\bar{g}_2^2}\right]. \tag{5.14}$$

The photon remains massless and the W and Z masses are

$$\begin{aligned} M_W^2 &= \frac{\bar{g}_2^2 v_T^2}{4}, \\ M_Z^2 &= \frac{(\bar{g}_1^2 + \bar{g}_2^2) v_T^2}{4} + \frac{1}{2} v_T^4 \bar{g}_1 \bar{g}_2 C_{HWB} \end{aligned} \tag{5.15}$$

The covariant derivative has the form

$$D_\mu = \partial_\mu + i\frac{\bar{g}_2}{\sqrt{2}} \left[\mathcal{W}^+_\mu T^+ + \mathcal{W}^-_\mu T^-\right] + i g_Z \left[T_3 - \sin\bar{\theta}^2 Q\right] \mathcal{Z}_\mu + i\bar{e} Q \mathcal{A}_\mu, \tag{5.16}$$

where $Q = T_3 + Y$ and the effective couplings become,

$$\begin{aligned}
\bar{e} &= \frac{\bar{g}_1\bar{g}_2}{\sqrt{\bar{g}_1^2+\bar{g}_2^2}}\left[1-\frac{\bar{g}_1\bar{g}_2}{\bar{g}_1^2+\bar{g}_2^2}v_T^2C_{HWB}\right] = \bar{g}_2\mathrm{sin}\bar{\theta}-\frac{1}{2}\mathrm{cos}\bar{\theta}\bar{g}_2v_T^2C_{HWB}\,,\\
\bar{g}_Z &= \sqrt{\bar{g}_1^2+\bar{g}_2^2}+\frac{\bar{g}_1\bar{g}_2}{\sqrt{\bar{g}_1^2+\bar{g}_2^2}}v_T^2C_{HWB} = \frac{\bar{e}}{\mathrm{sin}\bar{\theta}\mathrm{cos}\bar{\theta}}\left[1+\frac{\bar{g}_1^2+\bar{g}_2^2}{2\bar{g}_1\bar{g}_2}v_T^2C_{HWB}\right]\,,\\
\mathrm{sin}\bar{\theta}^2 &= \frac{\bar{g}_1^2}{\bar{g}_1^2+\bar{g}_2^2}+\frac{\bar{g}_1\bar{g}_2\left(\bar{g}_2^2-\bar{g}_1^2\right)}{\bar{g}_1^2+\bar{g}_2^2}v_T^2C_{HWB}\,. \qquad (5.17)
\end{aligned}$$

Considering (5.17) and (5.15), the experimental values for the W and Z masses and couplings fix $\bar{g}_1$,$\bar{g}_2$,v_T, C_{HWB}.This procedure consists of solving 4 equations in 4 variables: the unique solution of this system is given by the classical values for $\bar{g}_1$,$\bar{g}_2$ and v_T, i.e.

$$\bar{g}_1 = 0.358,\ \ \bar{g}_2 = 0.651,\ \ v_T = 246\,GeV \qquad (5.18)$$

while the dim-6 parameter C_{HWB} must give negligble corrections to the standard results. This means the product $v_T^2C_{HWB}$ has to be, at least, of the order 10^{-3}, i.e. $C_{HWB} \lesssim 10^{-7}GeV^{-2}$.

5.3 Running of the constants

In the following section we run the renormalization group equations, presented in § 5.2, to study the modification of the coupling constants behavior, due to the dim-6 operators. We check the possibility, for these new terms, to give a gauge unification point and to return values for the coupling constants compatible with the spectral action predictions.

5.3.1 Renormalization group flow

One can run the equations of the renormalization group in two directions. A "bottom-up" running assumes boundary values for the various constants at low energy (usually the Z or top mass) and runs toward higher energies. This is the way fig. 2.2 has been obtained. On the contrary the spectral action is defined at the high energy scale Λ, and its strength lies in the fact that it specifies the boundary conditions of all constants there. Therefore a "top-down" approach is more natural. Here, we follow a combined approach.

We start at the scale Λ in the range 10^{13-17} GeV. At this energy we give the boundary values given by the spectral action. In particular we use for the dimension six terms the values we have presented in eq. (5.3). The top-down running depends on four other parameters (described below) and gives a set of values for all of physical parameters at low energy. The parameters we find are not too distinct from the experimentally known ones, but there are discrepancies. As it should be: the heat kernel expansion is akin to a one loop calculation and, apart form any other incomplete aspect of the theory, it would be unreasonable to find the correct values for all parameters. The values one finds are however close to the experimental ones for the three g_i and y_t, while as remarked earlier λ, which is the parameter appearing in the Higgs mass, is off by nearly a factor two. The top-down running gives a set of values of the dimension six couplings C_i at M_Z .

We then performed a bottom-up running to see if the presence of the new terms could give a unification point, and we found that in several cases it does. As boundary conditions we used the experimental values for the g_i's and y_t and the low energy values of the C_i's obtained in the top-down running. The case of λ deserves a little discussion. Since the experimental and spectral action values are quite different, the qualitative behavior in the two cases are different. On the other side, it is known that the problem is fixed by the presence of another field (σ), which we do not discuss in this thesis. We have therefore performed our analysis in the two cases, i.e. the value of λ obtained by the spectral action, and the experimental one. The strategy we followed is synthesized in Table 5.1.

$$\left[\begin{array}{cc} \underbrace{\begin{array}{c}\Lambda\text{ scale}\overset{\text{RGEs}}{\longrightarrow}M_Z\,\text{scale}\\ \text{In:}\{\text{eq. (5.3)}\}\\ \text{Out:}\{C_i(M_Z)\}\\ \end{array}}_{\text{top-down running}} & \underbrace{\begin{array}{c}M_Z\,\text{scale}\overset{\text{RGEs}}{\longrightarrow}\Lambda\text{scale}\\ \text{In:}\{C_i(M_Z),\,g_i^{exp}\}\\ \text{Out:}\{C_i(\Lambda),g_i(\Lambda)\}\\ \end{array}}_{\text{bottom-up running}} \end{array}\right] \tag{5.1}$$

Table 5.1: *Resolution scheme adopted for the renormalization group flow. Varying* Λ, g, ρ *and* f_6 *we solve the RGEs, starting from the unification scale* Λ *down to the* M_Z *scale, and we use the resulting values for the dim-6 parameters together with the experimental values for the usual dim-4 couplings* g_i^{exp}, λ^{exp}, y_{top}^{exp} *to run again toward high energies.*

The second case, in which we used the experimental values as initial conditions, can be considered on a purely phenomenological basis, to show that higher dimension operators may cause unifications of the constants at one loop.

5.3.2 Top-Down running

In the spectral action model we have four free parameters: the value of the gauge coupling constants at the unification, g. The value of the cut-off and unification scale, Λ. The ratio between top and neutrino Yukawa couplings, ρ. The momentum f_6 which will fix the new physics scale Γ. This last parameter appears as coefficient to the dimension six operators with the combination f_6/Λ^2, and therefore effectively defines a new energy scale. This scale is an artifact of the spectral expansion, and does not signal the onset of new physics. It does however give a measure of the scale at which the new term will play a role.

All parameters have a particular range in which we expect they could be chosen. From the SM running of the gauge coupling constants we know g is expected around 0.55 ± 0.03, while Λ has a more significant range between 10^{13} GeV and 10^{17} GeV. The ratio ρ between the top and neutrino Yukawa couplings should be expected of $O(1)$. The value of the parameter f_6 requires a separate discussion. From the internal logic of the spectral action its "natural" value would be of order unity, or not much larger. Such a value would however make the corrections to the running totally irrelevant. The parameter appears with a denominator in Λ^2, and the corrections are often quadratic in this ratio. On the other side, from the phenomenology of electroweak processes it can be expected the effects of these new physics terms on the measured signal strength for $H \to \gamma\gamma$ decay, whose measured value is given by ATLAS and CMS [1, 23]. To obtain comparable data the new physics scale has to be fixed around $\Gamma \sim 1 - 10$ TeV. This leads to expected values for the dim-6 coefficients C_i around $10^{-6} - 10^{-8}\text{GeV}^{-2}$. The range for f_6 will be $\sim \Lambda^2/\Gamma^2$, i.e. 10^{20-28}. Given the fact that the cutoff function is undetermined in the scheme, such numbers are allowed, although a more physical explanation of their size would be preferable. The spectral action, given by an expansion valid below the unification scale, gives a framework to use a perturbative expansion valid beyond the scale of new physics, although it does not explain it. From the spectral point of view this is a weak point, the presence of such a high value for f_6 is very strange and creates an unnatural hierarchy with the other coefficients.

Since the point of the calculation was to verify the possibility of unification, the top-down calculation has been performed with the aim of obtaining values which would be a good starting point for the bottom up calculation. We did search for the best solutions for the range of parameters above. We performed first a coarse search to restrict the range, and then optimized the input parameters to find a good unification point. For the scope of this chapter, i.e. to show that dimension six operators could give unification, this is sufficient.

The boundary conditions at M_Z for the subsequent bottom-up run approach are the experimental values for the g_i and y_t, and the values obtained from the top-down for the C_i's. In the case of λ we have the two choices: either the values obtained from the top down, or the one from experiment. Since these two are different, in the following we present both cases.

Spectral action value for λ

In the following table we describe the values of the free parameters we used which will enable the best unification.

Table 5.2 shows, for various values of Λ, the parameters used for the top-down running, and the value of the couplings at low energy, shown as ratio with respect to the experimental value, corrected as described in the previous section: $\gamma_i = \frac{g_i(M_Z)}{\bar{g}_i^{exp}}$ and $\gamma_t = \frac{y_t(M_Z)}{y_t^{exp}}$. The values for λ are not shown since, for the reasons described above, they are not significant.

Λ GeV	$g(\Lambda)$	$\rho(\Lambda)$	$\frac{f_6}{16\pi^2\Lambda^2}$Gev^{-2}	γ_1	γ_2	γ_3	γ_t
10^{14}	0.580	1.6	4.8 10^{-6}	1.0	1.0	1.0	1.0
10^{15}	0.570	1.9	7.3 10^{-6}	0.98	1.0	1.0	1.0
10^{16}	0.550	1.9	6.9 10^{-6}	0.95	0.99	1.0	1.0
10^{17}	0.540	2.0	8.3 10^{-6}	0.93	0.97	1.1	1.0

Table 5.2: *The values of the coupling constants at M_Z compared with the experimental values for the top-down running. The values of the free parameters are optimized for the subsequent bottom-up run.*

Note that the choice of parameters has been made to optimize the subsequent bottom-up running. The amount of variations with respect to the experimental values for the couplings could be made smaller with a different choice of g, f_6 and ρ. This top-down running gives values for the C_i's, which are shown in Table 5.3.

One can see that with the choice of parameters, mainly f_6, the C_i's are in the range expected by a new physics scale of the order of 1 TeV.

Experimental value for λ

The values described above are made with parameters which are natural in the framework of the spectral action, but from the phenomenological point of view, since we now have the mass of the Higgs, and therefore the value

Λ	C_{HWB}	C_W	C_V	C_{HV}	C_H	C_{HB}	C_{HW}
10^{14}	$1.1\,10^{-7}$	$-5.8\,10^{-7}$	$-2.7\,10^{-7}$	$-1.1\,10^{-6}$	$3.8\,10^{-8}$	$-1.7\,10^{-7}$	$-7.5\,10^{-7}$
10^{15}	$1.4\,10^{-7}$	$-8.1\,10^{-7}$	$-3.3\,10^{-7}$	$-1.4\,10^{-6}$	$5.6\,10^{-8}$	$-2.1\,10^{-7}$	$-9.9\,10^{-7}$
10^{16}	$1.2\,10^{-7}$	$-6.7\,10^{-7}$	$-2.6\,10^{-7}$	$-1.3\,10^{-6}$	$4.2\,10^{-8}$	$-1.7\,10^{-7}$	$-8.2\,10^{-7}$
10^{17}	$1.3\,10^{-7}$	$-7.4\,10^{-7}$	$-2.5\,10^{-7}$	$-1.4\,10^{-6}$	$4.6\,10^{-8}$	$-1.7\,10^{-7}$	$-8.8\,10^{-7}$

Table 5.3: *The values of the coefficients of the dimension six operators at M_Z. The values of the free parameters are the ones in Table 5.2. All C_i's are in GeV^{-2}.*

of $\lambda(M_Z)$, we can also perform the analysis using as boundary condition the experimental value. As in the previous subsection the parameters are chosen in such a way to optimize the subsequent bottom-up run. Tables 5.4 and 5.5 are the counterparts of 5.2 and 5.3 for the case optimized for unification using as input the experimental value of λ at M_Z. Of course some principle like the spectral action must be operating in the background, to make sense of the fact that we are running the theory above the scale Γ all the way to the unification point.

Λ GeV	$g(\Lambda)$	$\rho(\Lambda)$	$\frac{f_6}{16\pi^2\Lambda^2}$Gev^{-2}	γ_1	γ_2	γ_3	γ_t
10^{14}	0.580	1.1	1.1 10^{-5}	0.98	0.95	0.80	1.0
10^{15}	0.560	0.7	8.3 10^{-6}	0.98	0.96	0.85	1.1
10^{16}	0.550	1.0	9.6 10^{-6}	0.98	0.96	0.89	1.1
10^{17}	0.540	0.9	8.3 10^{-6}	0.99	0.98	0.95	1.2

Table 5.4: *The values of the coupling constants at M_Z compared with the experimental values for the top-down running. The values of the free parameters are optimized for the subsequent bottom-up run. The initial value of $\lambda(M_Z)$ is the experimental one.*

One can see that with respect to the previous case the values of the γ's are slightly worse, showing that in this case the result of the top-down running spectral action "predictions" are off. This is not surprising because for the subsequent running (for which these values are optimized) the connections with the spectral action are weaker. One can notice that the values for the couplings in the two cases are not drastically different.

Λ	C_{HWB}	C_W	C_V	C_{HV}	C_H	C_{HB}	C_{HW}
10^{14}	$2.3\,10^{-7}$	$-1.4\,10^{-6}$	$-7.3\,10^{-7}$	$-2.9\,10^{-6}$	$9.4\,10^{-8}$	$-4.4\,10^{-7}$	$-1.6\,10^{-6}$
10^{15}	$1.5\,10^{-7}$	$-9.0\,10^{-7}$	$-4.5\,10^{-7}$	$-2.4\,10^{-6}$	$6.1\,10^{-8}$	$-5.7\,10^{-8}$	$-1.3\,10^{-6}$
10^{16}	$1.6\,10^{-7}$	$-9.3\,10^{-7}$	$-4.0\,10^{-7}$	$-2.6\,10^{-6}$	$6.1\,10^{-8}$	$3.7\,10^{-7}$	$-1.1\,10^{-6}$
10^{17}	$1.3\,10^{-7}$	$-7.3\,10^{-7}$	$-2.6\,10^{-7}$	$-3.1\,10^{-6}$	$4.9\,10^{-8}$	$8.6\,10^{-7}$	$-8.6\,10^{-7}$

Table 5.5: *The values of the coefficients of the dimension six operators at M_Z. The values of the free parameters are the ones in Table 5.2. All C_i's are in GeV^{-2}.*

5.3.3 Bottom-up running

In this section we present the result of the running from low to high energy, with the parameters chosen to have the three coupling constants meet near a common value in the range $10^{14} - 10^{17}$ GeV. As in the previous subsection we first discuss the case in which the boundary condition for λ is the one obtained from the running of the spectral action.

Spectral action value for λ

A good solution is one for which the common intersection is the starting point for the top-down running, and the C_i come back to the original values given by the spectral action. We optimized our search for the unification, therefore the fact that the values of the C_i "come back" to the same order within a factor of two or so, and are not off by an order a magnitude, is a check. The coefficient C_{HWB} is not present at the Λ scale in the spectral action, in this case one should expect it to be smaller than the other. A further check is the value of the top Yukawa at Λ which should be close to the value determined by the spectral action. The results for the coupling constants are in Table 5.6. The quantities $\delta g_i(\%)$ indicate (in percent) how different is the value of the runned constants (g_i^{run}) with respect to the original spectral action value $g(\Lambda)$ we started with, as shown in Table 5.2.

$$\delta g_i\% = \frac{|g_i^{run}(\Lambda - g(\Lambda)|}{g(\Lambda)} \times 100 \tag{5.2}$$

with an analogous definition for δy_t.

One can see that for the smaller values of $\Lambda \simeq 10^{14} - 10^{15} GeV$, one finds a good unification point, while for higher values the unification is worse. This can also be seen in Figs. 5.1 and 5.2 for the two extreme cases of 10^{14} and

Λ	$\delta g_1\%$	$\delta g_2\%$	$\delta g_3\%$	$\delta y_t\%$	$\delta\lambda\%$
10^{14}	1.4	2.1	0.17	0.30	4.1
10^{15}	3.3	0.02	0.54	3.6	4.7
10^{16}	7.8	0.078	0.97	6.4	2.3
10^{17}	13	1.7	1.1	6.6	3.9

Table 5.6: *The percent variation of the values of the three coupling constants and the top Yukawa coupling compared with the initial values of the top-down run.*

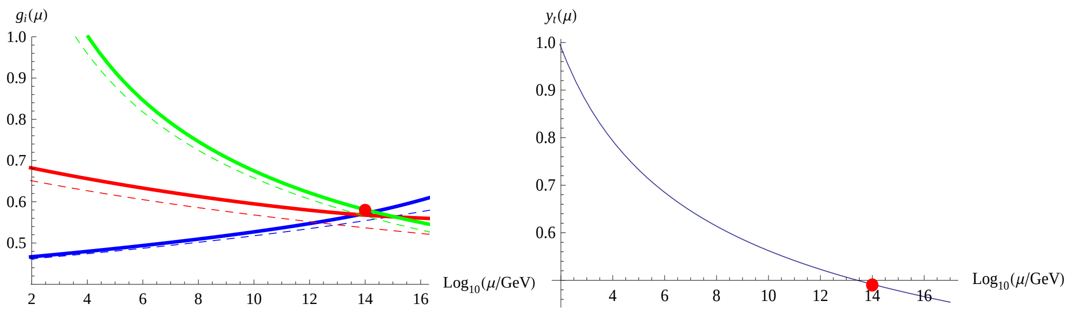

Figure 5.1: *Running of the auto-interaction parameter λ(on the rigth) and gauge coupling constants (on the left) in the presence of dimension six operators (thick lines) and their standard behaviour (dashed lines) for $\Lambda = 10^{14}GeV$. The values of the parameters are discussed in the text. The red dot indicates the starting value of the parameter. The dashed lines are the values of the g_i's in the standard model.*

$10^{17}GeV$ respectively, compared with the standard model running. In the first case there is a good unification, while in the second case the point at which the constants meet is some way off the initial energy. The values of the C_i's at the scale Λ are usually close to the one we started with in the top-down running, checking the consistency of the model. In particular C_{HWB}, which was zero, is constantly about one order of magnitude smaller than the other. We show this in Table 5.7 for the two extreme values of Λ. Also in this case, the lower value for Λ fares slightly better.

Experimental value for λ

If one ignores the spectral action, and trusts it only in that it gives some boundary values for the dimension six operator coefficients, then the bottom-up running can be performed independently. In this subsection we present, therefore, the running of the coupling constants using as boundary conditions

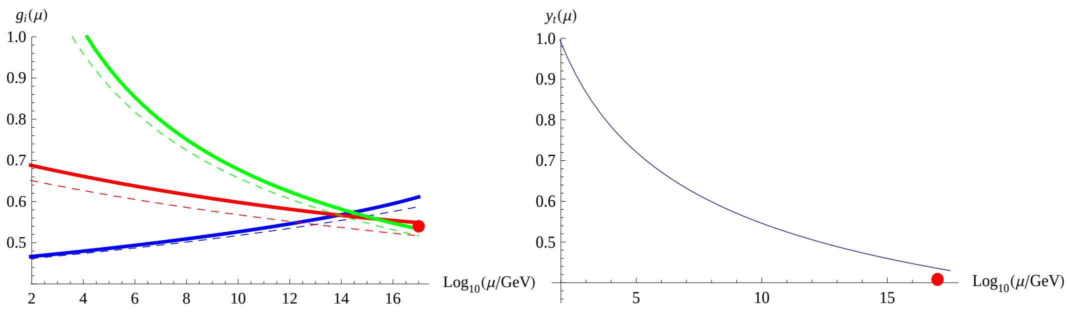

Figure 5.2: *Same as in fig. 5.1 for $\Lambda = 10^{17}$GeV.*

$\Lambda = 10^{14}$	C_{HWB}	C_W	C_V	C_{HV}	C_H	C_{HB}	C_{HW}
Spec. Act.	0	$-3.0\,10^{-6}$	$-3.0\,10^{-6}$	$-5.2\,10^{-7}$	$-3.7\,10^{-6}$	$-8.1\,10^{-7}$	$-7.3\,10^{-7}$
Run	$1.3\,10^{-8}$	$-1.5\,10^{-6}$	$-1.6\,10^{-6}$	$-5.5\,10^{-7}$	$-6.8\,10^{-6}$	$-6.7\,10^{-7}$	$-9.2\,10^{-7}$
$\Lambda = 10^{17}$	C_{HWB}	C_W	C_V	C_{HV}	C_H	C_{HB}	C_{HW}
Spec. Act.	0	$-4.2\,10^{-6}$	$-4.2\,10^{-6}$	$-5.4\,10^{-7}$	$6.9\,10^{-6}$	$-1.0\,10^{-6}$	$-9.4\,10^{-7}$
Run	$5.0\,10^{-8}$	$-2.4\,10^{-6}$	$-1.9\,10^{-6}$	$-6.5\,10^{-7}$	$7.5\,10^{-6}$	$-8.7\,10^{-7}$	$-7.2\,10^{-7}$

Table 5.7: *Comparison of the values of the coefficients of the dimension six operators at Λ. The second and fifth line are the initial values of the top-down running, as predicted by the spectral action for $\Lambda = 10^{14}$. The third and the last lines refer to the 10^{17} case. All C_i's are in GeV^{-2}.*

at M_Z the experimental values for the g_i, y_t, λ, (eq. 2.67, 2.69), and the values of Table 5.5 for C_i's and we check if the unification is possible. As we can see from fig. 5.3, for two different unification scales, the answer is positive if one relaxes the values of the dim-6 coefficients with respect to that suggested by the spectral action. In fact, in this case, the value of the γ's are slightly different from 1, as shown in table 5.4, but these allow to correct the unification point within an error of 1%, as summarized in Table 5.8.

$\Lambda(GeV)$	$\delta g_1\%$	$\delta g_2\%$	$\delta g_3\%$
10^{14}	0.62	0.74	1.0
10^{15}	1.4	0.38	0.56
10^{16}	1.2	0.50	0.50
10^{17}	0.14	0.98	1.1

Table 5.8: *The percent variation of the values of the there coupling constants compared with the initial value of the unification point.*

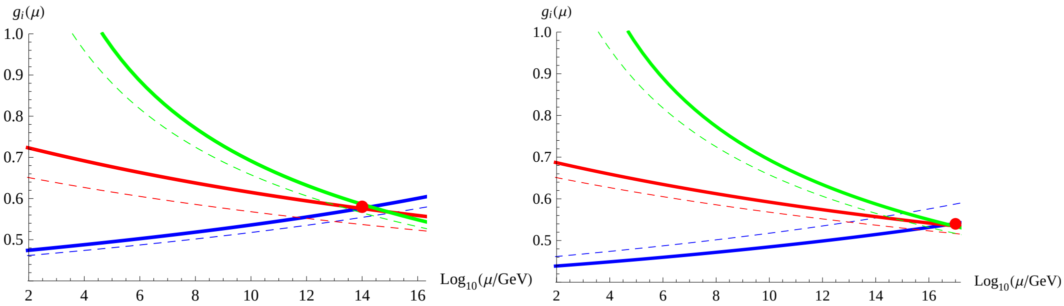

Figure 5.3: *Gauge couplings unification for two different unification scale $\Lambda = 10^{14}GeV$ (left) and $\Lambda = 10^{17}GeV$ (right) if one relaxes the spectral action boundaries.*

5.4 Comments

In this chapter we have studied the effects of the sixth order terms appearing in the spectral action Lagrangian. We have verified that the presence of these terms, with a proper choice of the free parameters, could cause the unification of the three constants at a high energy scale. Although the motivation for this investigation lies in the spectral noncommutative geometry approach to the standard model, the result can be read independently on it, showing that if the current Lagrangian describes an effective theory valid below the unification point, then the dimension six operator would play the proper role of facilitating the unification. In order for the new terms to have an effect it is however necessary to introduce a scale of the order of the TeV, which for the spectral action results in a very large second momentum of the cutoff function.

We note that we did not require a modification of the standard model spectral triple, although such a modification, and in particular the presence of the scale field σ, could actually improve the analysis. From the spectral action point of view the next challenge is to include the ideas currently come from the extensions of the standard model currently being investigated. From the purely phenomenological side instead a further analysis of the effects of the dimension six operators for phenomenology at large, using the parameters suggested by this work, can be a useful pointer to new physics.

In the next and final chapter we will see another approach to solve the gauge unification problem, that is by considering the effects of the gravitational terms on the renormalization flow of the gauge coupling constants

Chapter 6

Foundamental forces and Gravity

In chapter 3, with the Grand Symmetry model, we have shown it is possible to extend the standard model by including an additional singlet scalar field that stabilizes the running coupling constants of the Higgs field. This singlet scalar field is closely related to the right-handed Majorana neutrinos, conferring them mass, and leading to the prediction of the seesaw mechanism which explains the large difference between the masses of neutrinos and those of the other fermions. The model presented in §3 suggests the possibility of a further extension in the construction of a noncommutative manifold, going one step higher, in a sort of noncommutative geometry grand unification: it is pointed out that there could be a "next level" in noncommutative geometry, intertwined with the Riemannian and spin structure of space-time, where the singlet-scalar field arises. Accordingly, it naturally appears at high scale, near to the Planck scale.

A possible framework for describing interactions at energies and momenta below the Planck scale is given in [85, 88]. In this part of the thesis we check the possibility to extend the unification scale up to the Planck scale $M_P \equiv \sqrt{\hbar c/G_N} \simeq 10^{19}$GeV, including not negligible gravitational effects. For a theory dealing with the unification of gauge theory and gravity, a more natural scale is the Planck scale. The usual strategy is to use the spectral action as an effective action at a fixed scale, of the order of the unification scale, and to impose the additional relations between the independent parameters of the standard model. Then, using the renormalization group (RG) equations, one can let these parameters run to their value at low scales and evaluate the Higgs, the top and neutrino masses. The question here is: what is the predictive power of this extended model with exchange of gravitons at the Planck scale? We want to see how the gravitational effects change the main running coupling constants and if they lead to a restriction on the free parameters of the theory still compatible with the Higgs, top and neutrino mass predictions.

In [40] Marcolli and Estrada carried out a similar analysis within the asymptotic safety scenario with Gaussian matter fixed point; differently from us, they have not considered the effect of the scalar field σ introduced in [18] , which is necessary to reproduce the seesaw mechanism and to have the correct value for the Higgs mass.

This chapter is organized as follows. In section 6.1 we extend the derivation, from the spectral action principle, of the full standard model bosonic action coupled to gravity to a model containing also the singlet scalar field σ. In section 6.2, the gravitational contributions to the three gauge couplings, not negligible at the Planck scale, are presented. In section 6.3, it is shown how the gravitational effects change the RG equations of the Yukawa and self-interaction Higgs couplings leading to a restriction of the free parameters of the theory compatible with the Higgs and top mass. Section 6.4 contains a summary of these results and some comments.

6.1 Higgs-singlet scalar potential and Gravity

After the breaking due to the grand symmetry, the Majorana coupling y_R in the fluctuated Dirac operator (2.42) is multiplied by a singlet scalar field σ. This change means the turn of the constant-entry y_R of the Majorana matrix $\mathcal{M}_R$ into a field, eq. (3.1). In this way, the relations for the Seeley-DeWitt coefficients (2.50) are slightly modified leading to the standard model action plus a new singlet scalar field coupled to gravity [17, eq. (5.49)]:

$$\begin{aligned} S_B = {} & \frac{24}{\pi^2}F_4\Lambda^4\int d^4x\sqrt{g} - \frac{2}{\pi^2}F_2\Lambda^2\int d^4x\sqrt{g}\left[R+\frac{1}{2}a\overline{H}H+\frac{1}{4}c\sigma^2\right]+ \qquad (6.1)\\ & +\frac{1}{2\pi^2}F_0\int d^4x\sqrt{g}[\frac{1}{30}\left(-18C^2_{\mu\nu\rho\sigma}+11R^*R^*\right)+\frac{5}{3}g_1^2B^2_{\mu\nu}+g_2^2\mathbf{W}^2_{\mu\nu}+g_3^2\mathbf{V}^2_{\mu\nu}\\ & +\frac{1}{6}aR\overline{H}H+b\left(\overline{H}H\right)^2+a(\nabla_\mu H)^2+2e\overline{H}H\sigma^2+\frac{1}{2}d\sigma^4+\frac{1}{12}cR\sigma^2+\frac{1}{2}c\left(\partial_\mu\sigma\right)^2]+... \end{aligned}$$

in addition to eq. (2.61), here we have the singlet-scalar field σ, related to the neutrino Majorana mass which allows to reproduce a seesaw mechanism of type I as described in [19]. Furthermore, this σ field lowers the standard model Higgs mass to its experimental value.

As already shown in §2.5, it is more transparent to work with the rescaled fields

$$H \to \left(\sqrt{\frac{2}{3+\rho^2}}g\right)\frac{H}{y_{top}} \quad ; \quad \sigma \to (2g)\frac{\sigma}{y_{\nu_R}} \qquad (6.2)$$

so that the spectral action for scalar fields and gravity reduces to

$$S_B = \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x\sqrt{g} - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x\sqrt{g}\left[R + g^2 H^2 + g^2\sigma^2\right] + \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g}\left[\left(\frac{4}{3+\rho^2}\right) g^4 H^4 + \right.$$
$$\left. +2(\nabla_\mu H)^2 + 8g^4 \frac{2\rho^2}{3+\rho^2} H^2\sigma^2 + 8g^4\sigma^4 + 2g^2\left(\partial_\mu\sigma\right)^2 + \frac{1}{3} g^2 R\left(H^2+\sigma^2\right)\right]. \quad (6.3)$$

In the action above, we have neglected the additional gravitational term given by the Weyl curvature. This term is sub dominant to the Einstein-Hilbert term at the unification scale [73]. It could be shown [19] that the running of this term changes by at most an order of magnitude at lower scales, so we can assume that it remains sub dominant and neglect it in the first approximation. Moreover, we are neglecting the quadratic term in R.

As in (2.58), we set the coefficient F_0 to be $\frac{1}{2\pi^2}F_0 = \frac{1}{4g^2}$ to obtain the normalization of the gauge fields kinetic terms and the Higgs-singlet potential plus gravity reduces to

$$\begin{aligned} V(H,\sigma;R) = &\frac{1}{4}\left(\lambda_H H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2\sigma^2\right) - \frac{2g^2}{\pi^2} f_2 \Lambda^2 \left(H^2+\sigma^2\right) \\ &+ \frac{1}{12} R\left(H^2+\sigma^2\right) - \frac{2}{\pi^2} f_2 \Lambda^2 R + \frac{24}{\pi^2} f_4 \Lambda^4 \end{aligned} \quad (6.4)$$

where λ_H, λ_σ, $\lambda_{H\sigma}$ are defined in terms of g, that is the value of the three coupling constants at the unification scale,

$$\lambda_H \equiv \frac{\rho^4+3}{\left(3+\rho^2\right)^2} 4g^2\,; \quad \lambda_{H\sigma} \equiv \frac{2\rho^2}{\rho^2+3} 4g^2\,; \quad \lambda_\sigma \equiv 8g^2\,. \quad (6.5)$$

The usual strategy, at this point, is to use the spectral action as an effective action at a fixed scale, of the order of the GUT scale $\simeq 10^{17}$GeV, and to impose the additional relations (6.5) between the independent parameters of the standard model as a boundary condition at that scale. In this case we will use a different strategy by shifting the unification scale to the Planck scale M_P. Hence, we want to study the framework in which general relativity is quantized for small fluctuations around a flat space-time, and the Planck scale becomes the real unification scale of all physical interactions. In this extension of the spectral action to higher energy scales, we will include the contribution of gravitons exchange in the running coupling constants. Of course, these contributions will not be significant for low energies and they will be only important near the Planck scale. By using these new RG equations, we can let the standard model parameters run to their value at low scale and test the predictive power of the model: we will obtain a constrain of the free parameters of the theory still compatible with the Higgs and top mass prediction.

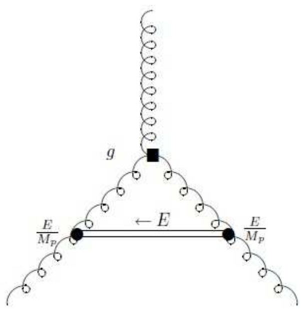

Figure 6.1: *A typical Feynman diagram at one-loop for a gravitational process contributing to the gauge coupling renormalization. Double lines represent gravitons. Curly lines represent gluons. The three-gluon vertex ■ is proportional to g_i, while the gluon-graviton vertex • is proportional to E/M_P.*

6.2 Gravitational correction to the gauge coupling constants

A possible framework for describing interactions at energies and momenta below the Planck scale is given in [85]. The dynamics for a non-Abelian gauge field coupled to gravity is given by the action,

$$\int d^4x\sqrt{g}\left[\frac{1}{k_{pl}^2}R - \frac{1}{4g^2}\left(\frac{5}{3}g_1^2B_{\mu\nu}^2 + g_2^2\mathbf{W}_{\mu\nu}^2 + g_3^2\mathbf{V}_{\mu\nu}^2\right)\right] \tag{6.6}$$

where the momentum F_2 is used to specify the initial conditions of the Planck constant, $\frac{2}{\pi^2}F_2\Lambda^2 \equiv \frac{1}{k_{pl}^2} \equiv M_P^2/16\pi$. The form of the gravitational correction can be determined on a general basis, involving in the one-loop Feynman diagrams of interest a gluon vertex dressed by the exchange of gravitons (see fig. 6.1).

Since the gauge boson vertex has strength g_i and gravitons couple to the energy-momentum tensor with a dimensional coupling $\propto 1/M_P$, dimensional analysis implies that the running of couplings in four dimensions will be governed by a Callan-Symanzik β function of the form [85, eq. 19]

$$\beta(g_i, E) = \frac{b_i}{16\pi^2}g_i^3 + a_g\frac{E^2}{M_P^2}g_i \text{ , con } b_i = \left(\frac{41}{6}, -\frac{19}{6}, -7\right) \tag{6.7}$$

where the first term represents the usual standard model contribution, and the second includes the gravitational correction. Initial values of g_i are set with the experimental values at $M_Z \simeq 91\,\text{GeV}$: $g_1(M_Z) = 0.3575$, $g_2(M_Z) = 0.6514$, $g_3(M_Z) = 1.221$. The numerical value of a_g, also called *anomalous dimension*, is determined by a detailed calculation described in [85] leading to

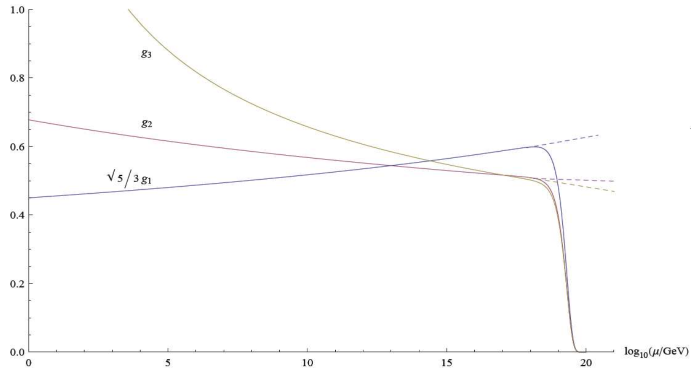

Figure 6.2: *Including gravity at one-loop, the couplings remain unified near 10^{17} GeV, but evolve rapidly to zero at high E.*

$a_g = -3/\pi$, which we can rewrite $a_g = -\frac{3}{16\pi^2}k_{pl}^2 M_P^2$. The negative sign of this coefficient means that the gravitational correction works in the direction of asymptotic freedom: it forces the couplings to decrease at large energy, as it is shown in fig. 6.2. At one-loop order, when gravity is ignored, the three gauge couplings evolve like the inverse logarithm of E (dashed curves); when gravity is included, see the solid lines, the couplings evolve rapidly towards weaker coupling at high E. Of course, its effect only becomes quantitatively important when the energy approaches the Planck scale, and gravitons exchanges are no longer negligible. We finally note that the three gauge coupling constants approximately assume the same value, about zero, from $E \geq 3 \times 10^{19}$GeV. Near the Planck scale $E \simeq 10^{19}$GeV the three gauge couplings are not exactly equal: we have $g_1(\Lambda) = 0.372$, $g_3(\Lambda) = 0.386$ and $g_2(\Lambda) = 0.396$.

6.3 Renormalization group equations with gravitational corrections

The running of the Higgs mass with the presence of a scalar field has been studied in [18]. However, the RG equations for the matter sector have to be adapted via the addition of the anomalous dimensions of the running parameters, that take into account the contribution of gravity [88],

$$\frac{dx_i}{dt} = \beta_{x_i}^{SM} + \beta_{x_i}^{grav} \tag{6.8}$$

where x_i are the running parameters, $\beta_{x_i}^{SM}$ is the Standard Model beta function for x_i and $\beta_{x_i}^{grav}$ is the gravitational correction. The latter is of the general form,

$$\beta_{x_i}^{grav} = a_{x_i}\frac{E^2}{8\pi M_P^2}x_i(t) \tag{6.9}$$

In our analysis, we use an estimate of the anomalous dimensions as suggested in [88]: a_{x_j} are fixed to 1 for the Yukawa couplings and to 3.1 for the self-interaction couplings of the scalar fields.

For the analysis of the renormalization group flow we shall expand the approach presented in [57, 96] with the presence of gravitational contributions. Let M_R be the Majorana mass for the right-handed tau-neutrino. By the Appequist-Carazzone decoupling theorem [9], we can distinguish two different energy domains: $E > M_R$ and $E < M_R$.

For high energies $E > M_R$, the renormalization group equations are given by [8, eq. 15], [70, eq. B.4] and [71, eq. B.3], adapted via the addition of the gravitational contributions described above

$$\begin{aligned}
\frac{dy_{top}}{dt} &= \frac{y_{top}}{16\pi^2}\left(\frac{9}{2}y_{top}^2 + y_\nu^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right) - a_{y_{top}}\frac{E^2}{8\pi M_P^2}y_{top}\\
\frac{dy_\nu}{dt} &= \frac{y_\nu}{16\pi^2}\left(3y_{top}^2 + \frac{5}{2}y_\nu^2 - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right) - a_{y_\nu}\frac{E^2}{8\pi M_P^2}y_\nu\\
\frac{d\lambda_H}{dt} &= \frac{1}{16\pi^2}\left(24\lambda_H^2 - \left(3g_1^2 + 9g_2^2\right)\lambda_H + 2\lambda_{H\sigma}^2 + \frac{6}{16}\left(g_1^4 + 2g_1^2g_2^2 + 3g_2^4\right)\right) + a_{\lambda_H}\frac{E^2}{8\pi M_P^2}\lambda_H.\\
\frac{d\lambda_{H\sigma}}{dt} &= \frac{1}{16\pi^2}\left(6y_{top}^2 + 2y_\nu^2 - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2 + 12\lambda_H + 6\lambda_\sigma + 8\lambda_{H\sigma}\right)\lambda_{H\sigma} + a_{\lambda_{H\sigma}}\frac{E^2}{8\pi M_P^2}\lambda_{H\sigma}\\
\frac{d\lambda_\sigma}{dt} &= \frac{1}{16\pi^2}\left(8\lambda_{H\sigma}^2 + 18\lambda_\sigma^2\right) + a_{\lambda_\sigma}\frac{E^2}{8\pi M_P^2}\lambda_\sigma
\end{aligned} \tag{6.10}$$

with $E = E(t) = m_Z e^t$. Below the threshold $E = M_R$, the tau-neutrino Yukawa coupling is replaced by an effective coupling [8, eq. 14]

$$\kappa = 2\frac{y_\nu^2}{M_R}\,, \tag{6.11}$$

which gives an effective mass $m_l = \frac{1}{4}\kappa v_0^2$ to the light tau-neutrino. In the range $0 < E < M_R$ the renormalization group equations for λ_σ and $\lambda_{H\sigma}$ are the same, whereas the ones for y_{top},y_ν, and λ_H are replaced by

$$\begin{aligned}
\frac{dy_{top}}{dt} &= \frac{1}{16\pi^2}\left(\frac{9}{2}y_{top}^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right) - \frac{a_y E^2}{8\pi M_P^2}y_{top}\\
\frac{d\kappa}{dt} &= \frac{1}{16\pi^2}\left(6y_{top}^2 + \frac{1}{36}\lambda_H - 3g_2^2\right)\kappa - \frac{a_y E^2}{8\pi M_P^2}\kappa\\
\frac{d\lambda_H}{dt} &= \frac{1}{16\pi^2}\Big(24\lambda_H^2 - \left(3g_1^2 + 9g_2^2\right)\lambda_H + 2\lambda_{H\sigma}^2 + \frac{6}{16}\left(g_1^4 + 2g_1^2g_2^2 + 3g_2^4\right) +\\
&\quad +12y_{top}^2\lambda - 3y_{top}^4\Big) + \frac{a_{\lambda_H}E^2}{8\pi M_P^2}\lambda_H
\end{aligned} \tag{6.12}$$

The numerical solutions to the coupled differential equations (6.10) to (6.12) depend on three input parameters: (1) the unification scale Λ; (2) the Majorana mass M_R which produces the threshold in the renormalization group flow; (3) the ratio ρ between the Dirac Yukawa couplings of the top quark and neutrino.

The scale Λ, usually taken at the unification $\Lambda_{12} = 10^{13} GeV$ or $\Lambda_{23} = 10^{17} GeV$ i.e. the two extreme points in which $g_1 = g_2$ and $g_2 = g_3$, is now shifted to the Planck scale where, due to the gravitational corrections, the three gauge couplings come together asymptotically free. We will determine the numerical solution from (6.10) to (6.12) for a range of values of ρ, Λ and M_R. The initial conditions of the running parameters at the scale Λ are given by (6.5) plus those for y_{top} and y_ν :

$$y_{top}(\Lambda) = \frac{2}{\sqrt{3+\rho^2}} g_2(\Lambda), \quad y_\nu(\Lambda) = \frac{2\rho}{\sqrt{3+\rho^2}} g_2(\Lambda)\,. \tag{6.13}$$

The effective mass of the light neutrino is determined by the effective coupling κ and we choose to evaluate this mass at the scale M_Z. Moreover, the running mass of the top quark to the ordinary energies is given by

$$M_{top} = \frac{1}{\sqrt{2}} y_{top} v_0 \tag{6.14}$$

where $v_0 \simeq 246\, GeV$ is the vacuum expectation value of the Higgs field.

For the Higgs mass, we have to use the new relation due to the presence of the new scalar field [18, eq. 35],

$$M_H(M_H) = v_0 \sqrt{2\lambda_H(M_H)\left(1 - \frac{\lambda_{H\sigma}^2(M_H)}{\lambda_H(M_H)\lambda_\sigma(M_H)}\right)} \tag{6.15}$$

while the scalar-singlet σ mass is proportional to its vacuum expectation value w_0, near the Planck scale according to us, through [18, eq. 34], $M_\sigma^2 = 2\lambda_\sigma w_0^2 + 2v_0^2\lambda_{H\sigma}^2/\lambda_\sigma$.

The results of the renormalization procedure for the Higgs and top mass in terms of the three parameters ρ, Λ, M_R are shown in fig. 6.3 and 6.4. In fig. 6.3 we see the Higgs and top mass values in terms of ρ for seven different values of Λ and M_R fixed: the Higgs mass around 125GeV and the top mass around 173GeV suggest a consistent choice of Λ not over $1.0\, 10^{19} GeV$. In fig. 6.4 it is shown the behavior of the two masses in function of Λ for eight different values of ρ with M_R fixed: also in this case, we can see that the Higgs mass around 125GeV suggests an appropriate choice of ρ not over 1.0 whereas the top mass does not impose any constrain. Moreover, both M_H and M_{top} behaviors become ρ independent for $\rho \leq 0.1$. Furthermore, it is possible to

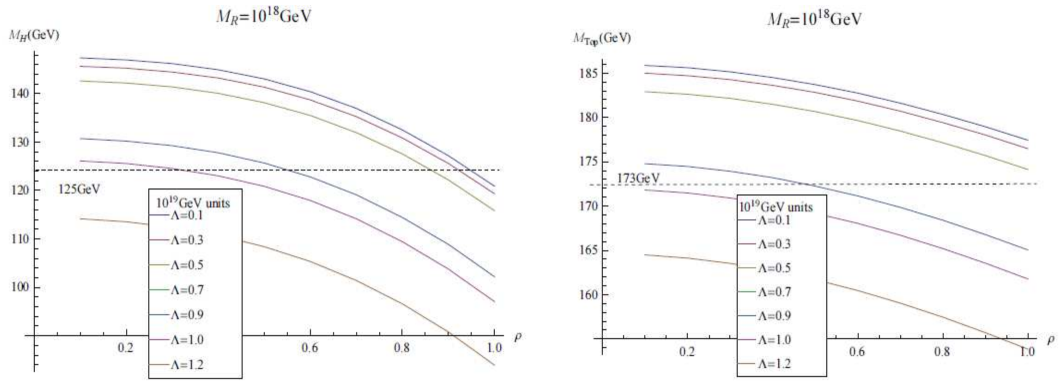

Figure 6.3: *Higgs and top mass in function of the parameter ρ for seven different values of Λ. We can see that the Higgs mass around 125GeV and the top mass around 173GeV constrain Λ not over* $1.0\,10^{19}$*GeV*.

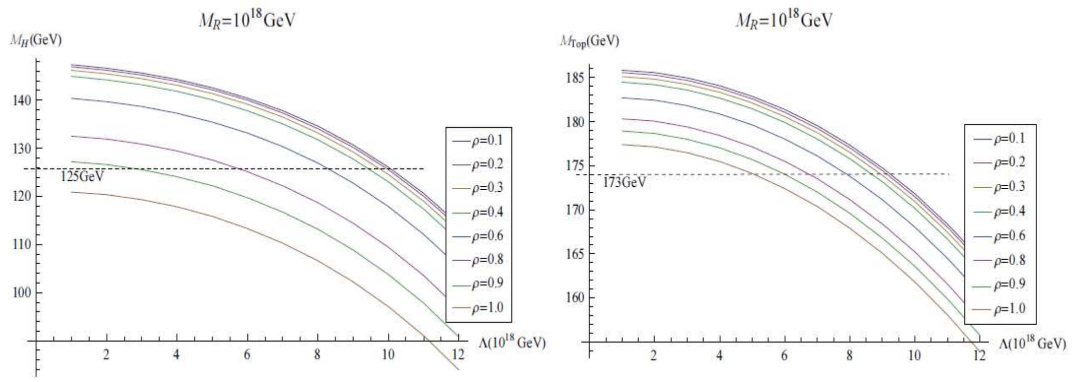

Figure 6.4: *Higgs and top mass, changing the unification parameter Λ for eight different values of ρ. Also in this case we can see that the Higgs mass around 125GeV suggests an appropriate choice of ρ not over* 1.0 *whereas the top mass does not impose any constrain. Moreover both M_H and M_{top} behaviours become ρ-independent for* $\rho \leq 0.1$

verify that the parameter M_R is not important for the mass prediction since M_H and M_{top} grow very slowly for its changes. Therefore, in the end, we have a sensible reduction on the choice of the three parameters values.

6.4 Comments

In chapter 3 and 4, the new singlet-scalar field σ, responsible for the stability of the Higgs boson, has been derived spontaneously from an high symmetry breaking that occurs at the Planck scale (that means $w_0 \simeq M_P$), mixing

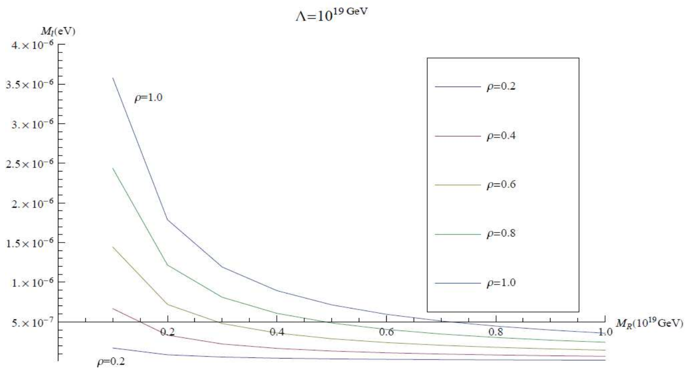

Figure 6.5: *Neutrino light mass, changing the Majorana right mass value in the range* $10^{18}Gev - 10^{19}Gev$*, for five different values of* ρ *and the unification scale* Λ *fixed. We can see that the neutrino mass has a very low value, of the order of* μeV*. Its value increases for increasing* ρ *and for decreasing* M_R*.*

space-time spin and gauge degrees of freedom. In this chapter we checked the possibility to extend the unification scale up to the Planck scale with the presence of the new scalar field non-minimal coupled to gravity.

Then, we deduced a restriction of the free parameters of the theory compatible with the Higgs and top mass: in particular, we have to take the parameters $\rho < 1$ and Λ not over 10^{19}GeV. However, this constrain leaves some open questions: for $\Lambda \lesssim 10^{19}GeV$ the three coupling constants are not exactly the same, although very close: e.g. for $\Lambda = 10^{19}$GeV we have $g_1(\Lambda)^2 = 0.138$, $g_3(\Lambda)^2 = 0.148$ and $g_2(\Lambda)^2 = 0.156$. Actually, we shall take at least $\Lambda \gtrsim 3.0 \times 10^{19}$GeV to have $g_2(\Lambda)^2 = g_3(\Lambda)^2 = g_1(\Lambda)^2 = 0.003$ and then consistently use the spectral action at the fixed unification scale.

Moreover, we have a neutrino mass problem which now becomes too small since its light mass $m_l = \frac{1}{4}\kappa v_0^2$ is influenced by M_R in the denominator of κ as in (6.11); as shown in fig. 6.5 for $M_R \simeq 10^{18}$GeV the neutrino mass has a very low value of the order of μeV. In order to rise the neutrino mass to few electron-volt, just two actions are possible: (1) increasing the ρ value, but nevertheless there is an upper limit imposed by the Higgs and top mass; (2) lowering the value of the Majorana right mass M_R to $10^{14}GeV$. This second possibility seems to indicate that the Majorana right mass (proportional to the σ v.e.v. w_0) responsible for the seesaw mechanism, cannot live at too high energy scales. This observation suggests that we cannot naively identify the scalar field σ of the grand symmetry breaking [36] with the field that gives mass to the Majorana right neutrino; otherwise, there may be some mechanisms that contribute to lower its mass, as in the case of neutrinos. Furthermore, a more punctual analysis is required to investigate the phenomenological consequences of this new and intriguing picture.

Conclusions

In this dissertation, we have pointed out there is a “next level” in the context of the noncommutative geometry approach to the physical fundamental interactions, that is intertwined with the Riemannian and spin structure of space-time. In fact, the added degrees of freedom of the Grand Symmetry model are related to the Riemann-spin structure of the manifold, which emerges as a symmetry breaking very similar in nature to the Higgs mechanism. Moreover, we have shown that this higher symmetry has important physical consequences since it explains the presence of the σ field necessary for a correct fit of the mass of the Higgs and to cure the instability problem of the electro-weak vacuum.

The presence of the Grand Symmetry will have also other phenomenological consequences which should be investigated. The breaking mechanisms we described in chapter 3 and 4 are just barely sketched, we only looked at the group properties. A more punctual analysis should reveal more structure, as already partially emerged by the Twisted Grand Symmetry model in which, for example, the new vector fields X_μ emerged. The study of these additional degrees at very high energies should alter the running of the coupling constants and open a new scenario for a sort of a "primordial" chiral symmetry breaking.

It is known that, although the spectral action requires the unification of interactions at a single scale, the usual grand unified theories, such as SU(5) or SO(10), do not fit in the noncommutative geometry framework, and are possible only renouncing to associativity [41, 99]. However, we have shown that is possible to improve the unification scheme considering new interactions in the spectral action expansion, as explained in chapter 5, or taking into account gravitational contributions up to the Planck scale. The framework is certainly not complete since some open questions remain, for example the fine tuning problem for the dim-six coefficients or the energy scale of the σ field with its relation to the Majorana neutrino mass.

The results presented in all this work, as is common in this model, are crucially depending on the Euclidean structure of the theory. This is particularly important as far as the role of chirality and the doubling of the degrees of freedom is concerned. A Wick rotation is far from simple in this context, and the construction of a Minkowskian noncommutative geometry is yet to come (for recent works see [13, 43, 44, 78]).

Acknowledgments

Special thank to my tutor F. Lizzi for the many attentions and advices that allowed to expand my professional and human experience. I want to thank also W. v. Suijlekom, that during my stay at the university of Nijmegen, suggested the idea of the twist presented in chapter 4. Thanks also to P. Martinetti for discussions and helps in several collaborations.

Bibliography

[1] G. Aad *et al.* [ATLAS Collaboration], "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," Phys. Lett. B **716** (2012) 1 [arXiv:1207.7214 [hep-ex]].

[2] R. Alonso, E. E. Jenkins, A. V. Manohar and M. Trott, "Renormalization Group Evolution of the Standard Model Dimension Six Operators III: Gauge Coupling Dependence and Phenomenology," JHEP **1404** (2014) 159 [arXiv:1312.2014 [hep-ph]].

[3] A. A. Andrianov and L. Bonora, "Finite-Mode Regularization Of The Fermion Functional Integral," Nucl. Phys. B **233**, 232 (1984).

[4] A. A. Andrianov and L. Bonora, "Finite Mode Regularization Of The Fermion Functional Integral. 2," Nucl. Phys. B **233**, 247 (1984).

[5] A. A. Andrianov, D. Espriu, M. A. Kurkov and F. Lizzi, "Universal Landau Pole," Phys. Rev. Lett. **111** (2013) 011601 [arXiv:1302.4321 [hep-th]].

[6] A. A. Andrianov, M. A. Kurkov and F. Lizzi, "Spectral action, Weyl anomaly and the Higgs-Dilaton potential," JHEP **1110** (2011) 001 [arXiv:1106.3263 [hep-th]].

[7] A. A. Andrianov and F. Lizzi, "Bosonic Spectral Action Induced from Anomaly Cancelation," JHEP **1005** (2010) 057 [arXiv:1001.2036 [hep-th]].

[8] S. Antusch, J. Kersten, M. Lindner and M. Ratz, "Neutrino mass matrix running for nondegenerate seesaw scales," Phys. Lett. B **538** (2002) 87 [hep-ph/0203233].

[9] T. Appelquist and J. Carazzone, "Infrared Singularities and Massive Fields" Phys. Rev. D **11** (1975) 2856.

[10] H. Arason, D. J. Castano, B. Keszthelyi, S. Mikaelian, E. J. Piard, P. Ramond and B. D. Wright, "Renormalization group study of the standard model and its extensions. 1. The Standard model," Phys. Rev. D **46** (1992) 3945.

[11] J. W. Barrett, "A Lorentzian version of the non-commutative geometry of the standard model of particle physics," J. Math. Phys. **48** (2007) 012303 [hep-th/0608221].

[12] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012).

[13] F. Besnard, "A Noncommutative view on topology and order," J. Geom. Phys. **59** (2009) 861 [arXiv:0804.3551 [math.OA]].

[14] A. H. Chamseddine and A. Connes, "The spectral action principle," Commun. Math. Phys. **186**, 731 (1997) [arXiv:hep-th/9606001].

[15] A. H. Chamseddine and A. Connes. "A dress for sm the beggar", Phys. Rev. Lett. **99** (2007) 191601

[16] A. H. Chamseddine and A. Connes. "Why the standard model ?", J. Geom. Phys. **58** (2008) 38–47.

[17] A. H. Chamseddine and A. Connes, "Noncommutative Geometry as a Framework for Unification of all Fundamental Interactions including Gravity. Part I," Fortsch. Phys. **58** (2010) 553 [arXiv:1004.0464 [hep-th]].

[18] A. H. Chamseddine and A. Connes, "Resilience of the Spectral Standard Model," arXiv:1208.1030 [hep-ph].

[19] A. H. Chamseddine, A. Connes and M. Marcolli, "Gravity and the standard model with neutrino mixing," Adv. Theor. Math. Phys. **11** (2007) 991 [arXiv:hep-th/0610241].

[20] A. H. Chamseddine, A. Connes, and V. Mukhanov, *Quanta of geometry*, arXiv 1409.2471; *Geometry and the quantum: basics*, arXiv 1411.0977 (2014).

[21] A. H. Chamseddine, A. Connes and W. D. van Suijlekom, "Inner Fluctuations in Noncommutative Geometry without the first order condition," arXiv:1304.7583 [math-ph].

[22] A. H. Chamseddine, A. Connes and W. D. van Suijlekom, "Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification," arXiv:1304.8050 [hep-th].

[23] S. Chatrchyan *et al.* [CMS Collaboration], "Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC," Phys. Lett. B **716** (2012) 30 [arXiv:1207.7235 [hep-ex]].

[24] C.-S. Chen and Y. Tang, *Vacuum stability, neutrinos, and dark matter*, JHEP **1204** (2012), no. 019.

[25] K. G. Chetyrkin and M. F. Zoller, "Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model," JHEP **1206**, 033 (2012) [arXiv:1205.2892 [hep-ph]].

[26] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[27] A. Connes. "Gravity coupled with matter and the foundations of noncommutative geometry", Commun. Math. Phys. **182** (1996) 155–176.

[28] A. Connes, "On the spectral characterization of manifolds," J. Noncommutative Geometry, **7** 1, 2013. arXiv:0810.2088 [math.OA].

[29] A. Connes and M. Dubois-Violette, "Non commutative finite dimensional manifolds. II. Moduli space and structure of non commutative 3-spheres," math/0511337 [math-qa].

[30] A. Connes and G. Landi, "Noncommutative manifolds: The Instanton algebra and isospectral deformations," Commun. Math. Phys. **221** (2001) 141 [math/0011194 [math.QA]].

[31] A. Connes, M. Marcolli, "Noncommutative Geometry, Quantum Fields and Motives", AMS 2007;

[32] A. Connes and H. Moscovici, *Type III and spectral triples*, Traces in number theory, geoemtry and quantum fields, Aspects Math. **E38** (2008), no. Friedt. Vieweg, Wiesbaden, 57–71.

[33] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori and A. Strumia, "Higgs mass and vacuum stability in the Standard Model at NNLO," JHEP **1208** (2012) 098 [arXiv:1205.6497 [hep-ph]].

[34] A. Devastato, "Spectral Action and Gravitational effects at the Planck scale," Phys. Lett. B **730** (2014) 36 [arXiv:1309.5973 [hep-th]].

[35] A. Devastato, F. Lizzi, C. V. Flores and D. Vassilevich, "Unification of Coupling Constants, Dimension six Operators and the Spectral Action," arXiv:1410.6624 [hep-ph].

[36] A. Devastato, F. Lizzi and P. Martinetti, "Grand Symmetry, Spectral Action, and the Higgs mass," JHEP **1401** 042 (2014) [arXiv:1304.0415 [hep-ph]];
A. Devastato, F. Lizzi and P. Martinetti, "Higgs mass in Noncommutative Geometry," Fortschr. Phys. **62**, 863 (2014). arXiv:1403.7567 [hep-th].

[37] A. Devastato and P. Martinetti, "Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry," arXiv:1411.1320 [hep-th].

[38] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto and A. Strumia, "Higgs mass implications on the stability of the electroweak vacuum," Phys. Lett. B **709** (2012) 222 [arXiv:1112.3022 [hep-ph]].

[39] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, H. M. Lee and A. Strumia, "Stabilization of the Electroweak Vacuum by a Scalar Threshold Effect," JHEP **1206** (2012) 031 [arXiv:1203.0237 [hep-ph]].

[40] C. Estrada and M. Marcolli, "Asymptotic safety, hypergeometric functions, and the Higgs mass in spectral action models" arXiv:1208.5023 [hep-th].

[41] S. Farnsworth and L. Boyle, "Non-Associative Geometry and the Spectral Action Principle," arXiv:1303.1782 [hep-th].

[42] S. Farnsworth and L. Boyle, "Rethinking Connes' approach to the standard model of particle physics via non-commutative geometry," arXiv:1408.5367 [hep-th].

[43] N. Franco, "Lorentzian approach to noncommutative geometry," arXiv:1108.0592 [math-ph].

[44] N. Franco and M. Eckstein, "An algebraic formulation of causality for noncommutative geometry," Class. Quant. Grav. **30** (2013) 135007 [arXiv:1212.5171 [math-ph]].

[45] K. Fujikawa, "Comment On Chiral And Conformal Anomalies," Phys. Rev. Lett. **44**, 1733 (1980).

[46] V. Gayral, J. M. Gracia-Bondia, B. Iochum, T. Schucker and J. C. Varilly, "Moyal planes are spectral triples," Commun. Math. Phys. **246** (2004) 569 [hep-th/0307241].

[47] V. Gayral and B. Iochum, "The Spectral action for Moyal planes," J. Math. Phys. **46** (2005) 043503 [hep-th/0402147].

[48] M. V. Gargiulo, M. Sakellariadou and G. Vitiello, "Doubling of the Algebra and Neutrino Mixing within Noncommutative Spectral Geometry," Eur. Phys. J. C **74** (2014) 2695 [arXiv:1305.0659 [hep-th]].

[49] P. Gilkey, *Invariance Theory, the Heat Equation and the Athiya-Singer Index Theorem*, Publish or Perish, 1984.

[50] J. M. Gracia-Bondia, B. Iochum and T. Schucker, "The Standard model in noncommutative geometry and fermion doubling," Phys. Lett. B **416** (1998) 123 [hep-th/9709145].

[51] M. Greenfield, M. Marcolli, and K. Teh, *Type III spectral triples and quantum statistical mechanical systems*, (XXXXX).

[52] C. Grojean, E. E. Jenkins, A. V. Manohar and M. Trott, "Renormalization Group Scaling of Higgs Operators and $\Gamma(h->\gamma\gamma)$," JHEP **1304** (2013) 016 [arXiv:1301.2588 [hep-ph]].

[53] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, "Dimension-Six Terms in the Standard Model Lagrangian," JHEP **1010**, 085 (2010) [arXiv:1008.4884 [hep-ph]].

[54] P. Q. Hung, "Vacuum Instability and New Constraints on Fermion Masses," Phys. Rev. Lett. **42** (1979) 873.

[55] E. E. Jenkins, A. V. Manohar and M. Trott, "Renormalization Group Evolution of the Standard Model Dimension Six Operators I: Formalism and lambda Dependence," JHEP **1310** (2013) 087 [arXiv:1308.2627 [hep-ph]].

[56] E. E. Jenkins, A. V. Manohar and M. Trott, "Renormalization Group Evolution of the Standard Model Dimension Six Operators II: Yukawa Dependence," JHEP **1401** (2014) 035 [arXiv:1310.4838 [hep-ph], arXiv:1310.4838].

[57] J. H. Jureit, T. Krajewski, T. Schucker and C. A. Stephan, "See-saw and noncommutative geometry" Phys. Lett. B **654** (2007) 127 [arXiv:0801.3731 [hep-th]].

[58] T. Kaluza, "On the Problem of Unity in Physics," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921** (1921) 966.

[59] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English)," Z. Phys. **37** (1926) 895 [Surveys High Energ. Phys. **5** (1986) 241].

[60] M. Kaku, "Quantum field theory: A Modern introduction," New York, USA: Oxford Univ. Pr. (1993) 785 p

[61] M. A. Kurkov and F. Lizzi, "Higgs-Dilaton Lagrangian from Spectral Regularization," Mod. Phys. Lett. A **27** (2012) 1250203 [arXiv:1210.2663 [hep-th]].

[62] M. A. Kurkov, F. Lizzi, M. Sakellariadou and A. Watcharangkool, "Spectral action with zeta function regularization," Phys. Rev. D **91** (2015) 6, 065013 [arXiv:1412.4669 [hep-th]].

[63] M. A. Kurkov, F. Lizzi and D. Vassilevich, "High energy bosons do not propagate," Phys. Lett. B **731**, 311 (2014) [arXiv:1312.2235 [hep-th]].

[64] M. A. Kurkov and M. Sakellariadou, "Spectral Regularisation: Induced Gravity and the Onset of Inflation," JCAP **1401** (2014) 035 [arXiv:1311.6979 [hep-th]].

[65] G. Landi, *An Introduction to Noncommutative Spaces and their Geometries, Springer Lecture Notes in Physics 51*, Springer Verlag (Berlin Heidelberg) 1997. arXiv:hep-th/9701078.

[66] F. Lizzi, G. Mangano, G. Miele and G. Sparano, "Constraints on unified gauge theories from noncommutative geometry," Mod. Phys. Lett. A **11** (1996) 2561 [hep-th/9603095].

[67] F. Lizzi, G. Mangano, G. Miele and G. Sparano, "Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories," Phys. Rev. D **55** (1997) 6357 [hep-th/9610035].

[68] F. Lizzi, G. Mangano, G. Miele and G. Sparano, "Mirror fermions in noncommutative geometry," Mod. Phys. Lett. A **13** (1998) 231 [hep-th/9704184].

[69] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization," Nucl. Phys. B **222** (1983) 83.

[70] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings" Nucl. Phys. B **236** (1984) 221.

[71] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings," Nucl. Phys. B **249** (1985) 70.

[72] L. Maiani, G. Parisi and R. Petronzio, "Bounds on the Number and Masses of Quarks and Leptons" Nucl. Phys. B **136** (1978) 115.

[73] M. Marcolli and E. Pierpaoli, "Early Universe models from Noncommutative Geometry," Adv. Theor. Math. Phys. **14** (2010) [arXiv:0908.3683 [hep-th]].

[74] S. P. Martin, "A Supersymmetry primer," In Kane, G.L. (ed.): Perspectives on supersymmetry II, 1-153 [hep-ph/9709356].

[75] P. Martinetti, "Carnot-Carathéodory metric and gauge fluctuation in noncommutative geometry", Commun. Math. Phys. **265** (2006) 585–616.

[76] P. Martinetti and R. Wulkenhaar, "Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry," J. Math. Phys. **43** (2002) 182 [hep-th/0104108].

[77] M. Paschke, F. Scheck and A. Sitarz, "Can (noncommutative) geometry accommodate leptoquarks?," Phys. Rev. D **59** (1999) 035003 [hep-th/9709009].

[78] M. Paschke and R. Verch, "Local covariant quantum field theory over spectral geometries," Class. Quant. Grav. **21** (2004) 5299 [gr-qc/0405057].

[79] J. C. Pati and A. Salam, "Lepton Number as the Fourth Color," Phys. Rev. D **10** (1974) 275 [Erratum-ibid. D **11** (1975) 703].

[80] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p

[81] H. D. Politzer and S. Wolfram, "Bounds on Particle Masses in the Weinberg-Salam Model," Phys. Lett. B **82** (1979) 242 [Erratum-ibid. **83B** (1979) 421].

[82] I. Pris and T. Schucker, "Noncommutative geometry beyond the standard model," J. Math. Phys. **38** (1997) 2255 [hep-th/9604115].

[83] J. Qian [CDF and D0 Collaborations], "Higgs Searches at the Fermilab Tevatron $p\bar{p}$ Collider," PoS **2008LHC** (2008) 005 [arXiv:0812.3979 [hep-ex]].

[84] M.A. Rieffel, "Morita equivalence for operator algebras"", in *Operator Algebras and Applications* (R. V. Kadison, ed.), 285-298, Proc. Symp. Pure Math. 38, Amer. Math. Soc., Providence, 1982.

[85] S. P. Robinson and F. Wilczek, "Gravitational correction to running of gauge couplings" Phys. Rev. Lett. **96** (2006) 231601 [hep-th/0509050].

[86] T. Schucker, "Forces from Connes' geometry," Lect. Notes Phys. **659** (2005) 285 [hep-th/0111236].

[87] T. Schucker and S. Zouzou, "Spectral action beyond the standard model," hep-th/0109124.

[88] M. Shaposhnikov and C. Wetterich, "Asymptotic safety of gravity and the Higgs boson mass," Phys. Lett. B **683** (2010) 196 [arXiv:0912.0208 [hep-th]].

[89] R. Squellari and C. A. Stephan, "Almost-Commutative Geometries Beyond the Standard Model. III. Vector Doublets," J. Phys. A **40** (2007) 10685 [arXiv:0706.3112 [hep-th]].

[90] C. A. Stephan, "Almost-commutative geometries beyond the standard model," J. Phys. A **39** (2006) 9657 [hep-th/0509213].

[91] C. A. Stephan, "Almost-commutative geometry, massive neutrinos and the orientability axiom in KO-dimension 6," hep-th/0610097.

[92] C. A. Stephan, "Almost-commutative geometries beyond the standard model. II. New Colours," J. Phys. A **40** (2007) 9941 [arXiv:0706.0595 [hep-th]].

[93] C. A. Stephan, "New Scalar Fields in Noncommutative Geometry," Phys. Rev. D **79** (2009) 065013 [arXiv:0901.4676 [hep-th]].

[94] C. A. Stephan, "Beyond the Standard Model: A Noncommutative Approach," arXiv:0905.0997 [hep-ph].

[95] T. van den Broek and W. D. van Suijlekom, "Supersymmetric QCD from noncommutative geometry," Phys. Lett. B **699** (2011) 119.

[96] K. van den Dungen and W. D. van Suijlekom, "Particle Physics from Almost Commutative Spacetimes," arXiv:1204.0328 [hep-th].

[97] W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*, (Springer, Dordrecht, 2015).

[98] D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388** (2003) 279 [hep-th/0306138].

[99] R. Wulkenhaar, “The Standard model within nonassociative geometry,” Phys. Lett. B **390** (1997) 119 [hep-th/9607096].