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Nash equilibrium selection in multi–leader multi–follower games with vertical separation

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Contents

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Chapter 1

Introduction

In this thesis a selection concept of Nash equilibria of a two–stage game with vertical separation and partially observed actions is investigated. The structure of this game is similar to the structure of a classic multi–leader multi–follower game, a generalization of the Stackelberg game introduced in [91] for a duopoly model: the leaders — in the first stage — and the followers — in the second stage — choose simultaneously an action. So, two kinds of competition arise in this situation, each in a separate stage: between the leaders in the first stage and between the followers in the second stage. The main difference with the classic multi–leader multi–follower game is that the action chosen by each leader is not observed by all followers. In particular, it is assumed that the action taken by a leader is observed only by one follower, that is there is an exclusivity between a single leader and a single follower. The exclusivity is also embodied in the model assuming that the action of any leader does not affect the payoffs to other players but the exclusive follower. These assumptions bring to an extensive form game without proper subgames and all the previous results for multi–leader multi–follower games with observed actions are no longer applicable in our context.

Many real–world situations can be modeled as a two–stage game with vertical separation and partially observed actions. An example is in games of multilateral vertical contracts, as in Pagnozzi and Piccolo [70]: competing manufacturers (the leaders) delegate retail decisions to exclusive retailers (the followers), offering a private contract. In this case the two kinds of competition can be interpreted as *interbrand* competition among the manufacturers, who are the producers of a substitute good, and *intrabrand* competition among the retailers, who are the dealers of the good to the final consumers. The action of delegation is known in Industrial Organization literature as *vertical separation*. Studies on vertical separation in an oligopolistic framework show that the incentive conflicts derived from a vertical separation between a manufacturer and external retailers can be used to leaders' advantage (see, for example, Bonanno and Vickers [8], Rey and Stiglitz [72], McAfee and Schwartz [59], Dobson and Waterson [23], Pagnozzi and Piccolo [70]). Furthermore, many papers in the field of multilateral vertical contracting are interested in the consequences of private contracting between manufacturers and retailers (Rey and Tirole [73], O'Brien and Shaffer [68], McAfee and Schwartz [59], Caillaud et al. [14], Pagnozzi and Piccolo [70],...). The fact that the contract is private could represent situations in which commitment is difficult because of the possibility of private renegotiation or of secret discount, as pointed out in McAfee and Schwartz [59].

In the light of the previous motivations, the thesis generalizes a multi–leader multi–follower two–stage game that follows the interaction scheme described previously. This game will be called *two–stage game with vertical separation and partially observed actions* (or else for short, *two–stage game with vertical separation*), in analogy with the industrial organization literature. Possibly discontinuous payoff functions are considered and it is assumed that each leader's payoff function could also depend explicitly on his exclusive follower's *optimal value* function (that is the payoff function of the follower when he is reacting optimally to any given action of his corresponding leader). This assumption can be viewed as an altruistic/spiteful behaviour, depending on the way the optimal value function of a follower affects his leader's payoff. It is compatible with the fact that if a follower is an exclusive retailer of the good produced by the leader, the latter can take into account the profit of his retailer (possibly in a percentage term) when he has to decide which strategy to play. Moreover, in engineering applications, the class of problems that can be modeled in this way are the so–called *parameter design problem and the resource allocation problem for decentralized systems* (Shimizu and Ishizuka [81], Shimizu et al. [82],. . .).

On the one hand a two–stage game with vertical separation is a particular type of multi–leader multi–follower problem, on the other hand the partial unobservability of the actions is a novelty in this type of models and makes an ineffective refinement the concept of subgame perfect Nash equilibrium because of the absence of proper subgames. Moreover, the concept of perfect Bayesian equilibrium could be not useful in that it does not prescribe limitations on the beliefs out of the equilibrium path. So, one way to overcome multiplicity of equilibria is to restrict attention only to specific beliefs that each follower has about the strategy chosen by the rival leaders. Taking into account the specificity of the structure, the case of *passive beliefs* is considered, in line with the economic literature from which the model comes from. Passive beliefs are quite a common assumption mainly in the multilateral vertical contracting (see Crémer and Riordan [18], Hart et al. [31], O'Brien and Shaffer [68], McAfee and Schwartz [59], Pagnozzi and Piccolo [70]) but also in games of electoral competition (see Gavazza and Lizzeri [27]) and consumer search literature (see Bar-Isaac et al. [3],. . .), as pointed out in Eguia et al. [24].

The aim of this thesis is to investigate the existence of equilibria under passive beliefs of a two–stage game with vertical separation under conditions of minimal character, mainly when the optimal reaction of any follower is single–valued. Chapter 4 is entirely devoted to the analysis of these results. An equilibrium under passive beliefs is proved to correspond to a fixed point of particular set–valued maps related to the sets of solutions of parametric Bilevel Optimization problems. Therefore, the existence for an equilibrium under passive beliefs is guaranteed once the existence of fixed points of the

considered set–valued map is proved. Given the peculiar structure of such set–valued maps, the fixed point theorems that fit well in these situations generally satisfy two main conditions: the closedness of the considered set–valued maps and the convexity of their values.
Previous results (Loridan and Morgan [50], Morgan [62], Lignola and Morgan [45]) help to find conditions of minimal character that ensure the closedness of the set–valued map we examine and the continuity of the optimal value functions of the followers.
Answer to the second issue requires more effort; also for multi–leader multi–follower games with observed actions this condition represents a main problem for both theoretical and computational approaches. Chapter 2 and Chapter 3 are mainly devoted to answer to these questions.

More precisely, Chapter 2 deals with Generalized Concave and Set–Valued Analysis and related Optimization problems. Besides the notion of quasiconcavity, the notion of pseudoconcavity is presented in a generalized version, for possibly discontinuous functions, in terms of Dini derivatives (see Diewert [21]). Then, some results on the concavity and the generalized concavity of composite functions are recalled, since a central problem when proving existence results for multi–leader multi–follower games is to find conditions that guarantee the quasiconcavity of the leaders' payoffs which are expressed as composite functions. Furthermore, basic notions of Set–Valued Analysis, including semicontinuity and convexity of set–valued maps and their applications, are recalled. Moreover, properties of the *optimal value function* and the so–called *optimal solutions set–valued map* are emphasized. Finally, a brief review on the Stackelberg problems is presented, in particular when any follower's reaction is single–valued.

In Chapter 3, following Ceparano and Quartieri [16], new results on the concavity and the isotonicity of an optimal reaction function are proved. The concavity and isotonicity of the optimal reactions of the followers are used for proving in Chapter 4 the existence of an equilibrium under passive beliefs for a special class of two–stage games with vertical separation and partially observed actions. Furthermore, a fixed point theorem for set–valued maps, used in Chapter 4, is proved.

Chapter 4 deals with the analysis of two–stage games with vertical separation and partially observed actions. Once the definition of an equilibrium for such a game is provided and existence of Nash equilibria is obtained for possibly discontinuous payoff functions, the problem of multiplicity of equilibria is illustrated through an example in the context of multilateral vertical contracting. This motivates the introduction of *selections of Nash equilibria* based on the beliefs that each follower has about the action chosen by the other leaders. So, a solution concept based on passive beliefs is defined when the optimal reaction of any follower is single–valued and it is shown that in the above–mentioned example it leads to the selection of a unique Nash equilibrium.
When the optimal reaction of any follower is a linear function, a first existence result is given together with sufficient conditions on the data in order to obtain such property.
With the aim of obtaining existence of equilibria without the condition of linearity, particular classes of two–stage games with vertical separation are considered.
First, the optimal reaction of any leader is assumed to be single–valued; existence is obtained without concavity assumptions.
When the action sets of the followers are subsets of $\mathbb{R}$, existence is proved assuming concavity of the optimal reaction of any follower and sufficient conditions for the concavity of the optimal reactions are given.
When the action sets of both leaders and followers are subsets of $\mathbb{R}$ and the payoff function of any leader does not depend directly on his own action, existence is proved assuming the isotonicity of the optimal reaction of any follower. Sufficient conditions for the isotonicity of the optimal reactions are provided.
Finally, an existence result is proved when the optimal reaction of any follower is not single–valued but the payoff function of any leader depends on the

action of the corresponding follower only through the optimal value function.

In the Appendix basic concepts and results on Euclidean spaces, Real and Concave Analysis used in the thesis are recalled.

Important properties of the set of the solutions, in particular, closedness and stability with respect to perturbations on the data, could be also investigated. These, together with applications to other economic models and the extension of the results to other types of beliefs, are the purposes of future studies.

Chapter 2

Preliminary tools

The scope of this thesis is to give existence results for a selection of Nash equilibria for a multi–leader multi–follower game with vertical separation. The precise description of such a game will be given in Chapter 4; here it is worth mentioning that the definition of a solution for this game involves parametric Bilevel Optimization problems. Thus, in this chapter we recall some mathematical tools that fit for the purpose. In particular, in the first section basic notions and preliminary results on Generalized Concave Analysis are recalled.
In Section 2.2 basic notions of Set–Valued Analysis, including upper and lower semicontinuity and their applications, are recalled.
In Section 2.3 some results on the so–called *optimal value* function and on the *optimal solutions set–valued* map are given. In the last section a brief review of works on leader–follower interactions is presented. In line with the purpose of this thesis, we mainly report results where the optimal reaction of any follower is single–valued. This chapter is the starting point for the analysis of Chapter 3.

The reader is referred to the Appendix for basic notions on Euclidean spaces and Real and Concave Analysis.

Note that all notions are given in $\mathbb{R}^n$ but they would be also valid in a more general context.

The material in this chapter is based mainly on Berge [6], Mangasarian [57], Avriel et al. [2], Aubin and Frankowska [1], Kyparisis and Fiacco [40, 41], Loridan and Morgan [49, 50], Lignola and Morgan [45, 44, 46], Rockafellar [75], Cambini and Martein [15], Bazaraa et al. [5].

2.1 Generalized Concave Analysis

Concave functions are very useful for applications in economics, since a local maximum point for a concave function is also a global one and also since a point that satisfies the first–order necessary conditions of a concave differentiable function is a global maximum point of the function.

Nonetheless, in many applications it is sufficient to handle with convex upper level sets, where we recall that an *upper level set at height* α, with $\alpha \in \mathbb{R}$ (also called *upper contour set at height* α) of a real–valued function f defined on a set $X \subseteq \mathbb{R}^n$ is the set $\Lambda_\alpha = \{x \in X\colon f(x) \geq \alpha\}$. The following notion exploits this property within the field of Generalized Concave Analysis.

Definition 2.1.1 ([2] Definition 3.1)
Let X be a convex subset of $\mathbb{R}^n$. An (extended) real–valued function f defined on X is said to be *quasiconcave* on X if its upper level sets Λ_α at height α are convex sets, for any $\alpha \in \mathbb{R}$; f is said to be *quasiconvex* if $-f$ is quasiconcave.

A very useful characterization of quasiconcavity is given in the next theorem.

Theorem 2.1.2 ([2] Theorem 3.1)**.** *Let f be a real–valued function defined on a convex subset X of $\mathbb{R}^n$. Then, f is quasiconcave if and only*

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\},$$

for any $x, y \in X$ and, for any $\lambda \in [0,1]$.

From an application of the definition of quasiconcavity it follows:

Proposition 2.1.3 ([15] Theorem 2.2.4)**.** *Let f be a real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Let B be the set of the set of the global maximum points of f on X, that is $B = \operatorname{Arg}\max_{x \in X} f(x)$. If f is quasiconcave on X then B is a convex set.*

Differently from what happens for concave functions, the sum of quasiconcave functions is not necessarily quasiconcave. For example, consider $X = [0, +\infty[$ and f and g defined by $f(x) = -x^2$ and $g(x) = x^3$ for any $x \in X$. Both functions are quasiconcave on X but $f + g$ is not quasiconcave on X. Moreover, even if f is linear and g is quasiconcave the sum is not necessarily quasiconcave. Consider, for example, $f(x) = -x$ and $g(x) = e^x$.

A function that is both quasiconcave and quasiconvex is said to be *quasimonotonic.*

Remark 2.1.4 As pointed out in [2], for real–valued functions defined on $\mathbb{R}$ the notion of monotonicity and quasimonotonicity are equivalent.

Proposition 2.1.5 ([2] Theorem 3.11)**.** *Let f be a differentiable real–valued function defined on the open convex set $X \subseteq \mathbb{R}^n$. Then f is quasiconcave if and only if the following implication holds:*

$$x_1, x_2, \in X \quad f(x_2) \geq f(x_1) \quad \textit{implies} \quad \langle \nabla f(x_1), (x_2 - x_1) \rangle \geq 0.$$

Definition 2.1.6 ([2] Definition 3.8)

A real–valued function f defined on a convex set $X \subseteq \mathbb{R}^n$ is said to be *strictly quasiconcave* if

$$f(\lambda x_1 + (1 - \lambda) x_2) > \min\{f(x_1), f(x_2)\},$$

for any $x_1, x_2 \in X$, with $x_1 \neq x_2$ and $\lambda \in]0, 1[$; f is said to be *strictly quasiconvex* if $-f$ is strictly quasiconcave.

A (strictly) concave function is also (strictly) quasiconcave. The reverse is not true. In fact, consider the real–valued function f defined on $\mathbb{R}$ by $f(x) = e^x$: f is (strictly) quasiconcave on $\mathbb{R}$ but it is not (strictly) concave on $\mathbb{R}$.

A strictly quasiconcave function is quasiconcave. In general, the reverse is not true: a function which is constant over an interval in its domain of definition is quasiconcave but not strictly quasiconcave. The examples above can also be used to show that there is not an inclusion relationship between the class of strictly quasiconcave functions and the class of concave functions.

Definition 2.1.7 ([2] Definition 3.10)
A convex set $X \subseteq \mathbb{R}^n$ is said to be *strictly convex* if for any two points $x_1, x_2 \in X$ on its boundary and for any $\lambda \in]0,1[$ the point $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ is an interior point of X.

A necessary condition for strict quasiconcavity on $\mathbb{R}^n$ is given by the following definition.

Proposition 2.1.8 ([2] Proposition 3.28). *Let f be a continuous function defined on $\mathbb{R}^n$. If f is strictly quasiconcave then its upper level sets are strictly convex.*

The reverse of the above proposition is not true, as shown in Example 3.3 in [2].

The importance of the strict quasiconcavity in optimization is due to the following property:

Proposition 2.1.9 ([2] Proposition 3.29). *A strictly quasiconcave function f defined on the convex set $X \subseteq \mathbb{R}^n$ attains its maximum on X at no more than one point.*

Another notion related to quasiconcavity is the following:

Definition 2.1.10 ([2] Definition 3.11)
A real–valued function f defined on a convex set $X \subseteq \mathbb{R}^n$ is said to be

semistrictly quasiconcave if

$$f(x_2) > f(x_1) \quad \text{implies} \quad f(\lambda x_1 + (1-\lambda)x_2) > f(x_1).$$

for any $x_1, x_2 \in X$ and any $\lambda \in]0,1[$; f is said to be *semistrictly quasiconvex* if $-f$ is semistrictly quasiconcave.

Note that we use the definition given in Avriel et al. [2]. This definition differs from the one given by Mangasarian in [57] who calls strict quasiconcavity what Avriel calls semistrict quasiconcavity.

A strictly quasiconcave function is characterized by:

$$f(x_2) \geq f(x_1) \quad \text{implies} \quad f(\lambda x_1 + (1-\lambda)x_2) > f(x_1)$$

for any $x_1, x_2 \in$ and any $\lambda \in]0,1[$. Then, every strictly quasiconcave function is semistrictly quasiconcave.

A semistrictly quasiconcave function is not necessarily quasiconcave as we can easily see in the following example.

Example 2.1.1 Let f be defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{2.1}$$

This function is semistrictly quasiconcave on $\mathbb{R}$ but not quasiconcave on $\mathbb{R}$ and hence f is also not strictly quasiconcave. ◇

Nevertheless, we have:

Proposition 2.1.11 ([34] or [2] Proposition 3.30). *If f is an upper semicontinuous semistrictly quasiconcave function on a convex set $X \subseteq \mathbb{R}^n$, then it is also quasiconcave on X.*

If f is upper semicontinuous, semistrict quasiconcavity is stronger than quasiconcavity, as we can see in the following example.

Example 2.1.2 Let f be a real–valued function defined on $[0,1]$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in]1,2]. \end{cases}$$

f is a continuous quasiconcave function but it is not semistrictly quasiconcave. ◇

The semistrict quasiconcavity does not guarantee the uniqueness of the optimal solution of an optimization problem. Nevertheless, an important property is expressed in the following proposition.

Proposition 2.1.12 ([2] Theorem 3.37)**.** *Let f be a continuous quasiconcave real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Then f is semistrictly quasiconcave on X if and only if every local maximum point $\bar{x}$ of f on X is a global maximum point of f on X.*

Another way to generalize the concept of concavity is provided by the concept of pseudoconcave function.

Definition 2.1.13 ([2] Definition 3.13)
Let f be a differentiable real–valued function defined on an open convex set $X \subseteq \mathbb{R}^n$. Then f is said to be *pseudoconcave* on X if:

$$x_1, x_2 \in X \text{ and } \quad f(x_1) > f(x_2) \quad \text{implies} \quad \langle \nabla f(x_2), (x_1 - x_2) \rangle > 0; \quad (2.2)$$

f is said to be *strictly pseudoconcave* on X if:

$$x_1, x_2 \in X \text{ with } x_1 \neq x_2 \text{ and } f(x_1) \geq f(x_2) \text{ implies } \langle \nabla f(x_2), (x_1 - x_2) \rangle > 0;$$

f is said to be *(strictly) pseudoconvex* on X if $-f$ is (strictly) pseudoconcave on X.

A strictly pseudoconcave function is pseudoconcave. The reverse is not true. Indeed, a constant function is pseudoconcave but not strictly pseudoconcave.

Proposition 2.1.14 ([57] Theorem 9.3.5)**.** *Let f be a differentiable real–valued function defined on an open convex set $X \subseteq \mathbb{R}^n$. If f is pseudoconcave on X then f is semistrictly quasiconcave on X and hence also quasiconcave on X. The converse is not true.*

Proposition 2.1.15 ([15] Theorem 3.2.12)**.** *Let f be a differentiable real–valued function defined on an open convex set $X \subseteq \mathbb{R}^n$. If f is strictly pseudoconcave on X then f is strictly quasiconcave on X. The converse is not true.*

Then, from Proposition 2.1.9, a strictly pseudoconcave function has at most one maximum point.
Finally, there is no an inclusion relation between the class of pseudoconcave functions and the class of strictly quasiconcave functions. Indeed, a constant function over an open convex subset of $\mathbb{R}^n$ is pseudoconcave but not necessarily strictly quasiconcave. Vice versa, if we consider the function $f(x) = x^3$ on $\mathbb{R}$, f is strictly pseudoconcave but not pseudoconcave (condition (2.2) does not hold if $x_2 = 0$)[15].

Furthermore, it holds:

Proposition 2.1.16 ([2] Theorem 3.39)**.** *Let f be a differentiable real–valued function defined on an open convex set $X \subseteq \mathbb{R}^n$. If f is (strictly) pseudoconcave on X and $\nabla f(\bar{x}) = 0$ for some $\bar{x} \in X$, then $\bar{x}$ is a (unique) global maximum point of f on X.*

When f is not differentiable, we consider a generalization of pseudoconcavity due to Diewert [21] which uses the so–called Dini Derivative.

Let us first recall the definition of Dini derivatives of a function.

Definition 2.1.17 ([22])
Let f be a real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. The *right–hand upper Dini derivative* $D^+f(x,u)$ of f at $x \in X$ in the direction $u \in X$ is defined by:

$$D^+f(x;u) = \limsup_{h \to 0^+} \frac{f(x+hu) - f(x)}{h},$$

provided that $x + hu \in X$. Note that $D^+ f(x, u)$ can be infinite.
Analogously, the *right–hand lower Dini derivative* of f at $x \in X$ in the direction $u \in X$ is:

$$D_+ f(x; u) = \liminf_{h \to 0^+} \frac{f(x + hu) - f(x)}{h},$$

provided that $x + hu \in X$; the *left–hand upper Dini derivative* of f at $x \in X$ in the direction $u \in X$ is:

$$D^- f(x; u) = \limsup_{h \to 0^-} \frac{f(x + hu) - f(x)}{h},$$

provided that $x + hu \in X$; the *left-hand lower Dini derivative* of f at $x \in X$ in the direction $u \in X$ is:

$$D_- f(x; u) = \liminf_{h \to 0^-} \frac{f(x + hu) - f(x)}{h}$$

provided that $x + hu \in X$, where the $\liminf$ and $\limsup$ are defined as in (A.5) and (A.6) in the Appendix.

If the function f is defined on $X \subseteq \mathbb{R}$, we indicate the right–hand upper Dini derivative with $D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ (without specifying the direction).

Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}^m$ be convex sets. If g is a real–valued function defined on $X \times Y$, we denote with $D_- g(\cdot, y)$ the left-hand lower Dini derivative of the function $g(\cdot, y)$ defined on X, for any $y \in Y$. Therefore

$$D_- g(\cdot, y)(x) = \liminf_{h \to 0^-} \frac{g(x + h, y) - g(x, y)}{h}. \tag{2.3}$$

for any $y \in Y$.

A detailed investigation of the Dini derivative with applications to optimization problems can be found in [28]. Let us report a result that will be useful in the future:

Lemma 2.1.18 ([28] Theorem 1.13)**.** *Let f be an upper semicontinuous real–valued function defined on an interval $[a, b] \subseteq \mathbb{R}$. If $D^+ f(t) \geq 0$ ($D^+ f(t) > 0$) for any $t \in [a, b[$ then f is nondecreasing (strictly increasing) on $[a, b[$.*

Now, we can recall the concept of pseudoconcavity in terms of the Dini derivative.

Definition 2.1.19 ([21])
A function f defined on a convex set $X \subseteq \mathbb{R}^n$ is said to be *pseudoconcave in terms of the upper Dini derivative* (in short *D–pseudoconcave*) if

$$x, y \in X, \quad f(x) < f(y) \quad \text{implies} \quad D^+ f(x; y - x) > 0;$$

a function is strictly D–pseudoconcave if

$$x, y \in X, \quad f(x) \leq f(y) \quad \text{implies} \quad D^+ f(x; y - x) > 0;$$

f is said to be (strictly) D–pseudoconvex if the function $g = -f$ is (strictly) D–pseudoconcave.

Then, if f is a real–valued function defined on $I \subseteq \mathbb{R}$, f is strictly D–pseudoconcave if and only if

$$\begin{cases} x, y \in I, x < y \text{ and } f(x) \leq f(y) \quad \text{implies} \quad D^+ f(x) > 0 \\ x, y \in I, x < y \text{ and } f(x) \geq f(y) \quad \text{implies} \quad D_- f(y) < 0. \end{cases} \tag{2.4}$$

A relation between D–pseudoconcavity and quasiconcavity is given in [29]. Recall the following definition.

Definition 2.1.20 ([29])
A real–valued function f defined on a convex set $X \subset \mathbb{R}^n$ is said to be *radially upper semicontinuous* on X (also known as *upper semicontinuous on the segments*) if the function ϕ defined by:

$$\phi(t) = f(tx + (1 - t)y)$$

is upper semicontinuous on $[0, 1]$, for any $x, y \in X$.

Then, it holds:

Proposition 2.1.21 ([29] Theorem 3.5)**.** *Let f be a real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$. If f is radially upper semicontinuous and D–pseudoconcave on X, then f is quasiconcave on X.*

Other results on connections between the class of quasiconcave functions and the class of D–pseudoconcave functions can be found in [32].

A strict D–pseudoconcave function has at most one global maximum point. Indeed, the following proposition holds.

Proposition 2.1.22. *Let f be a real–valued strictly D–pseudoconcave function defined on a convex set $X \subseteq \mathbb{R}^n$. Then, f attains its maximum on X at no more than one point.*

Proof. Let $\bar{x}, \bar{\bar{x}} \in \operatorname{Arg\,max}_{x \in X} f(x)$ with $\bar{x} \neq \bar{\bar{x}}$. From the definition of strict D–pseudoconcavity of f we have

$$\limsup_{t \to 0^+} \frac{f(\bar{x} + t(\bar{\bar{x}} - \bar{x})) - f(\bar{x})}{t} > 0,$$

that is, there exists a sequence $(t_k)_k$ that converges to 0^+ and $\bar{k} \in \mathbb{N}$ such that $t_k \in]0,1[$ and

$$\frac{f(\bar{x} + t_k(\bar{\bar{x}} - \bar{x})) - f(\bar{x})}{t_k} > 0, \quad \text{for any } k > \bar{k}. \tag{2.5}$$

From the convexity of X it follows that $\bar{x} + t_k(\bar{\bar{x}} - \bar{x}) \in X$.
So, $f(\bar{x} + t_k(\bar{\bar{x}} - \bar{x})) > f(\bar{x})$, for some k, in contradiction with the fact that $\bar{x}$ is a maximum point of f on X. ◇

In order to prove existence results for multi–leader multi–follower games, a central problem is to find conditions that guarantee the quasiconcavity of leaders' payoff functions which are expressed as composite functions. Hence, we conclude the section recalling some results on the concavity and the generalized concavity of composite functions.

Proposition 2.1.23 ([2] Proposition 2.16)**.** *Let $\phi_1, \ldots, \phi_m$ be concave (convex) real–valued functions defined on a convex set $X \subseteq \mathbb{R}^n$ and f be a* *concave real–valued componentwise nondecreasing (nonincreasing) function defined on a concave set* $Y \subseteq \mathbb{R}^m$*. Assume that* Y *contains the range of* $(\phi_1, \dots, \phi_m)$*. Then the composite function* $\tilde{f} = f(\phi_1, \dots, \phi_m)$ *is concave on* X*.*

Proof. We prove the concave case, the convex case could be obtained analogously. Let ϕ_i be concave on X, for any $i = 1, \dots, m$. Then for any $x_1, x_2 \in X$ and $\lambda \in]0, 1[$ we have

$$\phi_i(\lambda x_1 + (1-\lambda)x_2) \geq \lambda\phi_i(x_1) + (1-\lambda)\phi_i(x_2), \tag{2.6}$$

for any $i = 1, \dots, m$. Since f is concave and componentwise nondecreasing, using (2.6) we have:

$$\begin{aligned}
\tilde{f}(\lambda x_1 + (1-\lambda)x_2) &= f(\phi_1(\lambda x_1 + (1-\lambda)x_2), \dots, \phi_m(\lambda x_1 + (1-\lambda)x_2)) \\
&\geq f(\lambda\phi_1(x_1) + (1-\lambda)\phi_1(x_2), \dots, \lambda\phi_m(x_1) + (1-\lambda)\phi_m(x_2)) \\
&\geq \lambda f(\phi_1(x_1), \dots, \phi_m(x_1)) + (1-\lambda) f(\phi_1(x_2), \dots, \phi_m(x_2)) \\
&= \lambda\tilde{f}(x_1) + (1-\lambda)\tilde{f}(x_2),
\end{aligned}$$

that is $\tilde{f}$ is concave on X. ◇

An analogous result can be proved if f is quasiconcave:

Proposition 2.1.24 ([2] Proposition 5.3)**.** *Let* $\phi_1, \dots, \phi_m$ *be concave (convex) real–valued functions defined on a convex set* $X \subseteq \mathbb{R}^n$ *and* f *be a quasiconcave real–valued componentwise nondecreasing (nonincreasing) function defined on a convex set* $Y \subseteq \mathbb{R}^m$*. Assume that* Y *contains the range of* $(\phi_1, \dots, \phi_m)$*. Then the composite function* $\tilde{f} = f(\phi_1, \dots, \phi_m)$ *is quasiconcave on* X*.*

Proof. We prove the concave case, the convex case could be obtained analogously. Let $x_1, x_2 \in X$ and $\lambda \in]0, 1[$. By definition:

$$\phi_i(\lambda x_1 + (1-\lambda)x_2) \geq \lambda\phi_i(x_1) + (1-\lambda)\phi_i(x_2),$$

for any $i = 1, \ldots, m$. If f is componentwise nondecreasing and quasiconcave we have:

$$\begin{aligned}
\tilde{f}(\lambda x_1 + (1-\lambda)x_2) &= f(\phi_1(\lambda x_1 + (1-\lambda)x_2), \ldots, \phi_m(\lambda x_1 + (1-\lambda)x_2)) \\
&\geq f(\lambda\phi_1(x_1) + (1-\lambda)\phi_1(x_2), \ldots, \lambda\phi_m(x_1) + (1-\lambda)\phi_m(x_2)) \\
&\geq \min\{f(\phi_1(x_1), \ldots, \phi_m(x_1)), f(\phi_1(x_2), \ldots, \phi_m(x_2))\} \\
&= \min\{\tilde{f}(x_1), \tilde{f}(x_2)\},
\end{aligned}$$

that is $\tilde{f}$ is quasiconcave on X. ◇

If $m > 1$, the previous proposition cannot be weakened assuming some ϕ_i quasiconcave. Indeed, if f is defined on $\mathbb{R}^2$ by $f(y_1, y_2) = y_1 + y_2$, we know that the sum of two quasiconcave functions could be not quasiconcave, as shown previously.

We can generalize the result in Proposition 2.1.24 when we compose a quasiconcave function with both concave and convex functions. The proof is straightforward.

Proposition 2.1.25 ([2] Proposition 5.4)**.** *Let $\phi_1, \ldots, \phi_{m_1}$ be concave real–valued functions defined on a convex set $X \subseteq \mathbb{R}^n$ and $\psi_1, \ldots, \psi_{m_2}$ be convex real–valued functions defined on X. Let Y be a convex subset of $\mathbb{R}^m$, with $m = m_1 + m_2$. Denote $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)$ with y_{-i} and the set $\{y_i \in \mathbb{R} \colon (y_i, y_{-i}) \in Y\}$ with $Y_{y_{-i}}$. Let f be a quasiconcave real–valued function such that $f(\cdot, y_{-i})$ is nondecreasing on $Y_{y_{-i}}$, for any $i = 1, \ldots, m_1$, and $f(\cdot, y_{-i})$ is nonincreasing on $Y_{y_{-i}}$, for any $i = m_1 + 1, \ldots, m$. Assume that Y contains the range of $(\phi_1, \ldots, \phi_{m_1}, \psi_1, \ldots, \psi_{m_2})$.*
Then the composite function $\tilde{f} = f(\phi_1, \ldots, \phi_{m_1}, \psi_1, \ldots, \psi_{m_2})$ is quasiconcave on X.

When $m = 1$ in Proposition 2.1.24 we can weaken the hypothesis on both f and ϕ.

Proposition 2.1.26 ([2] Proposition 3.2)**.** *Let ϕ be a quasiconcave function real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$ and f be a nondecreasing* *real–valued function defined on a convex set $Y \subseteq \mathbb{R}$. Assume that Y contains the range of ϕ. Then, the composite function $\tilde{f} = f \circ \phi$ is quasiconcave on X.*

Proof. Let $x_1, x_2 \in X$ and $\lambda \in]0,1[$. By definition:

$$\phi(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\phi(x_1), \phi(x_2)\}.$$

If f is nondecreasing on Y we have:

$$\begin{aligned} f(\phi(\lambda x_1 + (1-\lambda)x_2)) &\geq f(\min\{\phi(x_1), \phi(x_2)\}) \\ &= \min\{f(\phi(x_1)), f(\phi(x_1))\}, \end{aligned}$$

that is $f \circ \phi$ is quasiconcave on X. ◇

Finally, we consider the case in which the functions $(\phi_i)_{i=1}^m$ are linear.

Proposition 2.1.27. *Let X be a subset of $\mathbb{R}^n$ and Y_i be a convex subset of $\mathbb{R}^{r_i}$, for $i = 1, \ldots, m$. Let ψ_i be a linear function from X to Y_i, for $i = 1, \ldots, m$, and let ϕ_i be a concave real–valued function defined on X, for $i = 1, \ldots, q$. Let Z be a convex subset of $\mathbb{R}^q$ and $Y = \prod_{i=1}^m Y_i \times Z$ and assume that Y contains the range of $(\psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_q)$. If f is a quasiconcave real–valued function on Y and $f(\psi_1, \ldots, \psi_m, \cdot, \ldots, \cdot)$ is componentwise nondecreasing on Z, then the composite function $\tilde{f} = f(\psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_q)$ is quasiconcave on X.*

Proof. Let $x_1, x_2 \in X$ and $\lambda \in]0,1[$. Let $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Assume for the sake of simplicity, that $m = q = 1$. Using the linearity of ψ_1, the quasiconcavity of ϕ_1 and the nondecreasingness of $f(\psi_1, \cdot)$ on Z, we obtain:

$$\begin{aligned} \tilde{f}(x_\lambda) &= f(\psi_1(x_\lambda), \phi_1(x_\lambda)) = f(\lambda\psi_1(x_1) + (1 - \lambda\psi_1(x_2)), \phi_1(x_\lambda)) \\ &\geq f(\lambda\psi_1(x_1) + (1-\lambda)\psi_1(x_2), \lambda\phi_1(x_1) + (1-\lambda)\phi_1(x_2)) \\ &\geq \min\{f(\psi_1(x_1), \phi_1(x_1)), f(\psi_1(x_2), \phi_1(x_1))\}, \end{aligned}$$

where the last inequality follows from the quasiconcavity of f. Then $\tilde{f}$ is quasiconcave. ◇

2.2 Set–Valued Analysis

In this section on *Set–Valued Maps* we recall some notions that have many applications in theoretical economics.

This section is based, mainly, on Berge [6], Aubin and Frankowska [1], Ok [69]. In the first part of the section we recall the definitions of limits of sets as formulated by Painlevé–Kuratowski [39]. The second part is devoted to basic notions of Set–Valued Analysis.

Let H be a subset of $\mathbb{R}^n$ and $(H_k)_k$ be a sequence of subsets of $\mathbb{R}^n$.

Definition 2.2.1 ([1] Definition 1.1.1)
The set

$$\operatorname*{Lim\,sup}_{k\to+\infty} H_k = \{x \in X\colon \liminf_{k\to+\infty} d(x, H_k) = 0\} \tag{2.7}$$

is said to be the *upper limit (in the sense of Kuratowski)* of the sequence $(H_k)_k$, where $d(x, H) = \inf_{y\in H} d(x, y)$, is the distance from a point $x \in \mathbb{R}^n$ to the set H and d is the Euclidean metric.

Equivalently, $\operatorname{Lim\,sup}_{k\to+\infty} H_k$ is the set of cluster points of sequences $(x_k)_k$ in $(H_k)_k$, that is:

$$\begin{aligned}\operatorname*{Lim\,sup}_{k\to+\infty} H_k = \{x \in X\colon \ &\text{there exists a sequence } (x_k)_{k\in N_1}, \text{ with } N_1 \subseteq \mathbb{N} \\ &\text{countably infinite, converging to } x \text{ s.t. } x_k \in H_k \text{ for any } k \in N_1\}.\end{aligned}$$

Definition 2.2.2 ([1] Definition 1.1.1)
The set

$$\operatorname*{Lim\,inf}_{k\to+\infty} H_k = \{x \in X\colon \lim_{k\to+\infty} d(x, H_k) = 0\} \tag{2.8}$$

is the *lower limit (in the sense of Kuratowski)* of the sequence $(H_k)_k$.

Equivalently, $\operatorname{Lim\,inf}_{k\to+\infty} H_k$ is the set of limits of sequences $(x_k)_k$ in

$(H_k)_k$, that is:

$$\operatorname*{Lim\,inf}_{k\to+\infty} H_k\{x \in X \colon \exists\,(x_k)_k \text{ converging to } x \text{ s.t. } x_k \in H_k \text{ for } k \text{ large}\}. \tag{2.9}$$

So, both the lower and the upper limits are closed sets and $\operatorname{Lim\,inf}_{k\to+\infty} H_k \subseteq \operatorname{Lim\,sup}_{k\to+\infty} K_k$.

Definition 2.2.3 ([1] Definition 1.1.1)
A subset H is said to be the *limit (in the sense of Kuratowski)* of the sequence $(H_k)_k$ if

$$H = \operatorname*{Lim\,inf}_{k\to+\infty} H_k = \operatorname*{Lim\,sup}_{k\to+\infty} H_k.$$

Example 2.2.1 Consider the sequence $(H_k)_k$ of subsets of $\mathbb{R}^2$ defined, for every k in $\mathbb{N}$, by:

$$H_k = \begin{cases} \left\{\frac{1}{k}\right\} \times [-1, 0] & \text{if } n \text{ is odd,} \\ \left\{\frac{1}{k}\right\} \times [0, 1] & \text{if } n \text{ is even.} \end{cases}$$

Then, $\operatorname{Lim\,inf}_{k\to+\infty} H_k = \{(0,0)\}$ and $\operatorname{Lim\,sup}_{k\to+\infty} H_k = \{0\} \times [-1,1]$ (see [1]). ◇

From now on in this section we will denote with X and Y two subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.
A *set–valued map* $F\colon X \rightrightarrows Y$, also called *correspondence*, *multifunction*, or *point to set map*, is a map such that $F(x)$ is a subset of Y, for any $x \in X$.
The set $F(x)$ is called *image* or *value* of F at x.
A set–valued map F is *strict* if all images are not empty. The *domain* of F is the set $\operatorname{Dom} F = \{x \in X \colon F(x) \neq \emptyset\}$.
The *image* of F is the set $\operatorname{Im} F = \bigcup_{x\in X} F(x)$.
For any $S \subseteq X$, the image of S under F is the set $F(S) = \bigcup_{x\in S} F(x)$.
The *graph* of F is the set $\operatorname{Graph} F = \{(x,y) \in X \times Y \colon y \in F(x)\}$.

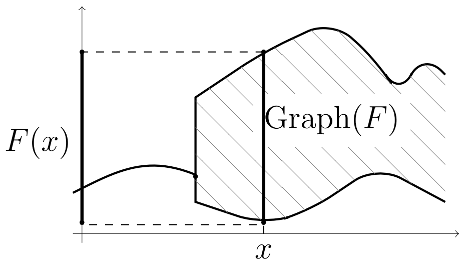

Figure 2.1: Graph of set–valued map F.

An example of a set–valued map is in Figure 2.1.

A set–valued map $F\colon X \rightrightarrows Y$ is said to be *closed*, or F has the *closed graph property*, if Graph F is closed in the product space $X \times Y$, that is if

$$\operatorname*{Lim\,sup}_{k \to +\infty} F(x_k) \subseteq F(x),$$

for any $x \in X$ and for any sequence $(x_k)_k$ converging to x in X.

Analogously, F is said to be *open graph* if Graph F is open.

The set–valued map F is said to be *closed–valued* (resp. *open–valued*) if $F(x)$ is said to be closed (resp. open) subset of Y, for any $x \in X$; F is said to be *compact–valued* if $F(x)$ is compact, for any $x \in X$.

If F is a closed set–valued map, then F is closed–valued; the vice versa is not always true.

Now, let Z and Y be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A set–valued map $F\colon Z \rightrightarrows T$ is said to be *convex* if Graph F is a convex subset of $\mathbb{R}^n \times \mathbb{R}^m$, or, equivalently, if for each $x, y \in Z$, and for each $\lambda \in]0,1[$ we have:

$$\lambda F(x) + (1-\lambda)F(y) \subseteq F(\lambda x + (1-\lambda)y).$$

F is said to be *convex–valued* if the set $F(x)$ is convex, for any $x \in Z$.

Definition 2.2.4

The set–valued map $F\colon Z \rightrightarrows T$ is said to be *quasiconvex* (see [41]) on a convex set $Z \subseteq \mathbb{R}^n$ if $F^{-1}(M)$ is a convex subset of $\mathbb{R}^n$ for any convex subset M of $\mathbb{R}^m$, where $F^{-1}(M) = \{x \in \mathbb{R}^m \colon F(x) \cap M \neq \emptyset\}$ is the *inverse image* of M by F.

A convex set–valued map is quasiconvex, the converse is not true, as shown in the following example.

Example 2.2.2 Consider the set–valued map $F\colon Z = [0,1] \rightrightarrows T = [0,1]$ defined for any $x \in Z$ by:

$$F(x) = \begin{cases} \{0\}, & \text{if } x \in [0, \frac{1}{2}[; \\ [0,1], & \text{if } x = \frac{1}{2}; \\ \{1\}, & \text{if } x \in]\frac{1}{2}, 1]; \end{cases}$$

F is quasiconvex on Z but not convex. ◇

Each function f from a set X to a set Y can be seen as a particular set–valued map F from X to Y defined by $F(x) = \{f(x)\}$, for any $x \in X$. On the other hand, if $F\colon X \rightrightarrows Y$ is single–valued for any $x \in X$, it can be defined the function $f\colon X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$.

Remark 2.2.5 Let us emphasize that, if T is a convex subset of $\mathbb{R}$ and F is a quasiconvex single–valued map then, considered as a function, it is both a quasiconvex and a quasiconcave function. So, if F, considered as a set–valued map, is quasiconvex and also single–valued, then, considered as a function, it is quasimonotone.

The two concepts of semicontinuity for set–valued maps that we will use in the following were introduced independently by Kuratowski and Bouligand in 1932. Both concepts of semicontinuity for set–valued maps are an extension of the notion of continuity for functions.

Definition 2.2.6 ([1] Definition 1.4.1)
The set–valued function F is said to be *upper semicontinuous at* $x \in \operatorname{Dom} F$ if, for any sequence $(x_k)_k$ converging to x in X and for any open set O of Y such that $F(x) \subseteq O$ we have $F(x_k) \subseteq O$ for k large.
F is said to be *upper semicontinuous* if it is upper semicontinuous at x, for

any $x \in \operatorname{Dom} F$.

Assume that F is a single–valued map. Then F is upper semicontinuous on X if and only if, considered as function, it is continuous on X.

Proposition 2.2.7 ([69]). *Let F be a set–valued map from X to Y. Assume that for any sequence $(x_k)_k$ in X and $(y_k)_k$ in Y such that $y_k \in F(x_k)$, for all $k \in \mathbb{N}$, there exists a subsequence $(y_k)_{k \in N_1}$, with $N_1 \subseteq \mathbb{N}$ countably infinite, converging to a point $y \in F(x)$. Then F is upper semicontinuous at $x \in X$. If F is compact–valued, then the converse is also true.*

Furthermore, the closed graph property is related, but not equivalent, to the property of upper semicontinuity.

Example 2.2.3 Consider, for example, the set–valued map:

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{\frac{1}{x}\} & \text{if } x > 0. \end{cases}$$

F is a closed but not an upper semicontinuous set–valued map. ◇

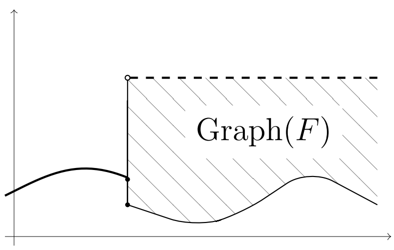

Figure 2.2: Graph of a set–valued map F upper semicontinuous which is not closed.

Conversely, the upper semicontinuity is not sufficient to ensure the closed graph property. Nevertheless, the relation between closedness and upper semicontinuity is given in the following proposition.

Proposition 2.2.8 ([1] Proposition 1.4.8)**.** *Assume that* $F\colon X \rightrightarrows Y$ *is a set–valued map. Then the following statements hold:*

(i) If F *is closed and* Y *is compact, then* F *is upper semicontinuous.*

(ii) If F *is an upper semicontinuous closed–valued map, then* F *is closed.*

Then, if Y is compact and F is closed–valued, we have that F is closed if and only if F is upper semicontinuous.

Definition 2.2.9 ([1] Definition 1.4.2)
A set–valued map F is *lower semicontinuous at* $x \in \mathrm{Dom}(F)$ if for any sequence $(x_k)_k$ converging to x in X we have

$$F(x) \subseteq \operatorname*{Lim\,inf}_{k\to+\infty} F(x_k),$$

that is for any sequence of elements $(x_k)_k$ in $\mathrm{Dom}(F)$ that converges to x in X and for any $y \in F(x)$, there exists a sequence of elements $(y_k)_k$ converging to y such that $y_k \in F(x_k)$ for k large.

F is *lower semicontinuous* if F is lower semicontinuous at x, for any $x \in \mathrm{Dom}(F)$.

If F is a single–valued map, then F is a lower semicontinuous map if and only if, considered as function, it is continuous. So, for a single–valued map the concepts of lower semicontinuity and upper semicontinuity coincide.

This is not longer true for set–valued maps, as we can see in the examples in Figure 2.3 – 2.4 – 2.5.

Definition 2.2.10 ([1] Definition 1.4.3)
A set–valued map F is *continuous at* $x \in X$ if it is upper and lower semicontinuous at $x \in X$. A set–valued map F is *continuous* on X if it is continuous at $x \in X$, for any $x \in X$.

In the following, we recall well–known theorems of Set–Valued Analysis. The last theorem of the section is the Kakutani's Theorem, a fixed–point theorem that has a wide range of applications in economic problems.

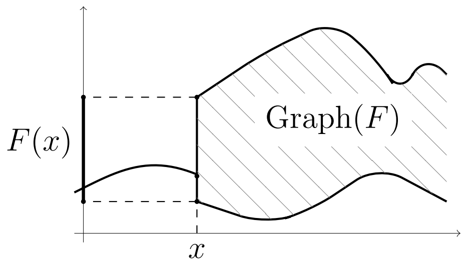

Figure 2.3: Graph of a set–valued map upper (but not lower) semi–continuous at x.

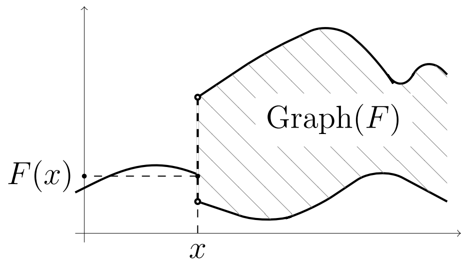

Figure 2.4: Graph of a set–valued map lower (but not upper) semicontinuous at x.

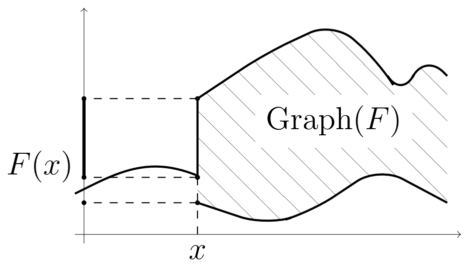

Figure 2.5: Graph of a set–valued map neither lower nor upper semicontinuous at x.

Proposition 2.2.11 (Border [9] Michael selection Theorem)**.** *Let X be a compact subset of $\mathbb{R}^n$ and $F\colon X \rightrightarrows \mathbb{R}^m$ be a lower semicontinuous set–valued map with closed and convex values. Then, there is a continuous function $f\colon X \to \mathbb{R}^m$ such that $f(x) \in F(x)$ for any $x \in X$.*

Proposition 2.2.12 ([9] von Neumann's Approximation Lemma)**.** *Let X be a compact subset of $\mathbb{R}^n$ and Y be a convex subset of $\mathbb{R}^m$. Let $F\colon X \rightrightarrows Y$ be an upper semicontinuous set–valued map with nonempty compact convex values. Then, for any $\epsilon > 0$ there exists a continuous function f_ϵ such that* $\operatorname{Graph} f_\epsilon \subseteq U_\epsilon(\operatorname{Graph} F)$, *where* $U_\epsilon(\operatorname{Graph} F) = \cup_{(x,y)\in \operatorname{Graph} F} B((x,y);\epsilon)$.

Definition 2.2.13

A point $x^* \in X$ is said to be a *fixed point* on X of a set–valued map

$F : X \rightrightarrows X$ if $x^* \in F(x^*)$.

Graphically, if $X = \mathbb{R}$, the set of the fixed points of a set–valued map corresponds to the intersection between the graph of F and the 45° line in the space $\mathbb{R}^2$, as in Figure 2.6.

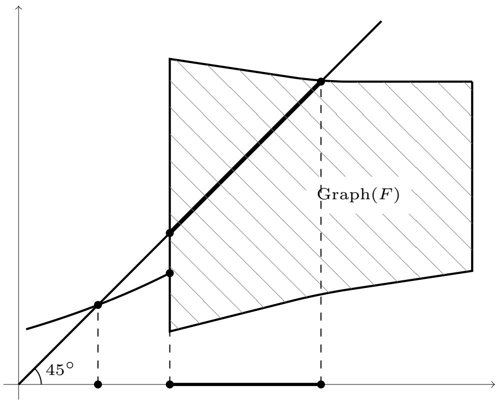

Figure 2.6: Set of fixed points of F

Theorem 2.2.14 ([33] Kakutani's Fixed Point Theorem)**.** *Let X be a non-empty, convex, compact subset of $\mathbb{R}^n$ and $F\colon X \rightrightarrows X$ be a set–valued map. Let F be convex–valued and closed. Then, there exists at least a fixed point of F on X.*

2.3 Parametric Optimization

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let f be a real–valued function defined on $X \times Y$ and F be a set–valued map from X to Y. The function defined for any $x \in X$ by:

$$v(x) = \sup_{y \in F(x)} f(x, y) \tag{2.10}$$

is called *optimal value function* (or *marginal function*) and it will play an important role in Chapter 4. Furthermore, the set–valued map B from X to

Y defined for any $x \in X$ by:

$$B(x) = \operatorname*{Arg\,max}_{y \in F(x)} f(x, y), \tag{2.11}$$

that is called *optimal solutions set–valued map*, is a very useful concept in an economic setting. For example, in Game Theory it corresponds to the *best response* of a player and the analysis of its properties is central in order to prove the existence of an equilibrium of a game. In case in which $B(x)$ is a singleton, for any $x \in X$, we call it *optimal solution function* (or else *optimal reaction*).

Remark 2.3.1 When F is compact–valued and $f(x, \cdot)$ is upper semicontinuous on Y, for any $x \in X$, we have $B(x) \neq \emptyset$ and $v(x) = \max_{y \in F(x)} f(x, y)$.

In the first part of this section let us summarize some properties about the continuity of the optimal value function. The first result is the well–known theorem due to Berge that gives also information on the optimal solutions set–valued map.

Theorem 2.3.2 (Berge's Theorem (see, for example, Border [9])). *Let X and Y be two subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let F be a compact–valued map from X to Y and f be a real–valued continuous function defined on* Graph F. *If F is continuous at $\bar{x} \in X$ then the optimal value function v defined in* (2.10) *is continuous at $\bar{x}$ and the optimal solutions set–valued map B defined in* (2.11) *is nonempty compact–valued and upper semicontinuous at $\bar{x}$.*

A generalized version of Berge's Theorem that uses conditions of minimal character in order to obtain the continuity of v is investigated in Lignola and Morgan [45]. In the next propositions we we limit ourself to the case in which the constraint of the optimization problem is constant, that is $F(x) = Y$ for any $x \in X$. Then, the optimal value function and the optimal solutions set–valued map are defined for any $x \in X$ by:

$$\bar{v}(x) = \sup_{y \in Y} f(x, y), \tag{2.12}$$

and

$$\overline{B}(x) = \operatorname*{Arg\,max}_{y \in Y} f(x, y). \tag{2.13}$$

Proposition 2.3.3 ([45] Proposition 3.1.1)**.** *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and f be a real–valued function defined on $X \times Y$. The optimal value function $\bar{v}$ defined in (2.12) is lower semicontinuous at $\bar{x} \in X$ if and only if for any $y \in Y$ and for any sequence $(x_k)_k$ converging to $\bar{x}$ in X there exists a sequence $(y_k)_k$ such that:*

$$\liminf_{k \to +\infty} f(x_k, y_k) \geq f(\bar{x}, y).$$

Furthermore:

Proposition 2.3.4 ([45] Proposition 4.1.1)**.** *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and f be a real–valued function defined on $X \times Y$. If Y is compact and f is upper semicontinuous at $(\bar{x}, y)$, for any $y \in Y$, then the optimal value function $\bar{v}$ defined in (2.12) is sequentially upper semicontinuous at $\bar{x}$.*

Furthermore, for the optimal solutions set–valued map we have:

Proposition 2.3.5 (Loridan and Morgan [51] Proposition 4.1)**.** *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and f be a real–valued function defined on $X \times Y$. If Y is compact and:*

(i) f is upper semicontinuous on $X \times Y$,

(ii) for any $(x, y) \in X \times Y$, for any sequence $(x_k)_k$ converging to $x \in X$, there exists a sequence $(y_k)_k$ in Y such that:

$$\liminf_{k \to +\infty} f(x_k, y_k) \geq f(x, y)$$

then the optimal solutions set–valued map $\overline{B}$ defined in (2.13) is closed.

Now, let us recall some results on convexity and concavity of the optimal value function. They are investigated in Kyparisis and Fiacco [40] and Rockafellar [74], among others.

Proposition 2.3.6 ([74, 40]). *Let X and Y be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If f is real–valued concave function on $X \times Y$, then the optimal value function $\bar{v}$ defined in defined in (2.12) is concave on X.*

Proof. It is a well known result. For the sake of completeness we recall the proof. Let $x_1, x_2 \in X$ such that $\bar{v}(x_i) > -\infty$, for $i = 1, 2$, and $\lambda \in]0, 1[$. Then

$$\bar{v}(\lambda x_1 + (1-\lambda)x_2) \geq f(\lambda x_1 + (1-\lambda)x_2, y) \quad \text{for any } y \in Y. \tag{2.14}$$

Assume, preliminarily, that both $\bar{v}(x_1)$ and $\bar{v}(x_2)$ are finite.
By definition of supremum, for any $\epsilon > 0$ there exists $y_i \in Y$ such that

$$f(x_i, y_i) > \bar{v}(x_i) - \epsilon, \tag{2.15}$$

for $i = 1, 2$. Then

$$\lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2) > \lambda \bar{v}(x_1) + (1-\lambda)\bar{v}(x_2) - \epsilon.$$

Thus, let $y = \lambda y_1 + (1-\lambda) y_2$ in (2.14). From the concavity of f we have:

$$\bar{v}(\lambda x_1 + (1-\lambda)x_2) > \lambda \bar{v}(x_1) + (1-\lambda)\bar{v}(x_2) - \epsilon.$$

Taking the limit when ϵ goes to 0 we have the thesis.
Assume, now that, for example, $\bar{v}(x_1)$ is infinite and $\bar{v}(x_2)$ is finite. Then, also $\bar{v}(\lambda x_1 + (1-\lambda)x_2)$ will be infinite. Indeed, for any $M > 0$ there exists $y_1 \in Y$ such that $f(x_1, y_1) > M$ and for any $\epsilon > 0$ there exists $y_2 \in Y$ such that $f(x_2, y_2) > \bar{v}(x_2) - \frac{\epsilon}{1-\lambda}$. Again, from the concavity of f we have:

$$\bar{v}(\lambda x_1 + (1-\lambda)x_2) > \lambda M + (1-\lambda)\bar{v}(x_2) - \epsilon.$$

Taking the limit for ϵ that goes to 0 and M that goes to $+\infty$, we have $\bar{v}(\lambda x_1 + (1-\lambda)x_2) = +\infty$.
Analogously if both $\bar{v}(x_1)$ and $\bar{v}(x_2)$ are infinite. ◇

Proposition 2.3.7 ([40] Proposition 3.5). *Let X and Y be convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If f is a real–valued convex function on X, then the optimal value function $\bar{v}$ is convex on X.*

Proof. It follows from Proposition A.3.4 (viii) in the Appendix. ◇

Let us conclude the section with some results on the quasiconcavity of the optimal value function and on the quasiconvexity of the optimal solutions set–valued map that will be used in the Bilevel Optimization problems in Chapter 4.

Proposition 2.3.8 (Kyparisis and Fiacco [41] Proposition 2.9)**.** *Let X and Y be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let f be a real–valued function defined on Y and F be a set–valued map from X to Y. If f is upper semicontinuous and quasiconcave on the set $F(X)$ and F is compact–valued and quasiconvex on X then the optimal value function v defined in* (2.10) *is quasiconcave on X.*

Proof. Let $\alpha \in \mathbb{R}$ and consider the upper level set $\Lambda_\alpha = \{x \in X : v(x) \geq \alpha\}$. Let $x_1, x_2 \in \Lambda_\alpha$ and $\lambda \in]0,1[$. Then $v(x_1) \geq \alpha$ and $v(x_2) \geq \alpha$, that is $\max_{y \in F(x_i)} f(y) \geq \alpha$, for $i = 1, 2$, where we have applied the considerations in Remark 2.3.1. So, there exists $y_i \in F(x_i)$ such that $f(x_i) \geq \alpha$, for $i = 1, 2$. Denoted by $[y_1, y_2]$ the line segment between y_1 and y_2, we have that 1 $x_1, x_2 \in F^{-1}([y_1, y_2])$. Therefore, from the quasiconvexity of F, we have that $x_\lambda = \lambda x_1 + (1-\lambda)x_2 \in F^{-1}([y_1, y_2])$, that is there exists $y_\lambda \in [y_1, y_2]$ such that $y_\lambda \in F(x_\lambda)$. f is quasiconcave on Y, thus $f(y_\lambda) \geq \min\{f(y_1), f(y_2)\} \geq \alpha$. Then $\max_{y \in F(x_\lambda)} f(y) \geq \alpha$, that is $v(x_\lambda) \geq \alpha$ and $x_\lambda \in \Lambda_\alpha$. ◇

In the following propositions, results on the optimal solutions set–valued map are proved when the constraint of the optimization problem F is constant, that is $F(x) = Y$ for any $x \in X$.

Proposition 2.3.9. *Let X and Y be convex subsets of $\mathbb{R}$. Assume that $\overline{B}$, the optimal solutions set–valued map from X to Y defined in* (2.13)*, is nonempty single–valued. If $\overline{B}$, considered as a function, is monotone then, considered as a set–valued map, it is quasiconvex.*

Proof. Suppose $\overline{B}(x) = \{b(x)\}$, and assume, without loss of generality, that b is a nondecreasing function. Let M be a convex subset of Y. Let $x_1, x_2 \in$

$b^{-1}(M)$, that is $b(x_1), b(x_2) \in M$. Let $\lambda \in]0,1[$ and suppose that $x_1 < x_2$. Then $\lambda x_1 + (1-\lambda)x_2 \in]x_1, x_2[$. So, we have $b(\lambda x_1 + (1-\lambda)x_2) \in [b(x_1), b(x_2)]$, that is $b(\lambda x_1 + (1-\lambda)x_2) \in M$, given the convexity of M. $\diamond$

Proposition 2.3.10. *Let X and Y be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let f be a real–valued function defined on $X \times Y$. Let $\overline{B}$ be the optimal solutions set–valued map defined in (2.13) and assume that $\overline{B}$ is nonempty–valued. If f is concave on $X \times Y$ and $f(\cdot, y)$ is convex on X, for any $y \in Y$, then $\overline{B}$ is a convex set–valued map on X.*

Proof. Let $y_1 \in \overline{B}(x_1)$, $y_2 \in \overline{B}(x_2)$ and $\lambda \in]0,1[$. From the concavity of f and from the definition of $\overline{B}(x)$ we have:

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) &\geq \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2) \\ &\geq \lambda f(x_1, y) + (1-\lambda) f(x_2, y) \quad \text{for any } y \in Y. \end{aligned}$$

That is, from the convexity of $f(\cdot, y)$ on X:

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \geq f(\lambda x_1 + (1-\lambda)x_2, y) \quad \text{for any } y \in Y;$$

that is $\lambda y_1 + (1-\lambda)y_2 \in \overline{B}(\lambda x_1 + (1-\lambda)x_2)$ and $\overline{B}$ is a convex set–valued map. $\diamond$

Next corollary follows directly from Proposition 2.3.10.

Corollary 2.3.11. *Let X and Y be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let f be a real–valued function defined on $X \times Y$. If f is concave on $X \times Y$, $f(\cdot, y)$ is convex on X, for each $y \in Y$, and the problem $\max_{y \in Y} f(x, y)$ has a unique solution $b(x)$, for any $x \in X$, then the function b is linear on X.*

2.4 Stackelberg Games

In this section a brief review on Stackelberg games is presented. This kind of game was introduced by von Stackelberg in [91], where a duopoly model was presented: two firms compete in quantities, the one who moves first (the *leader*) chooses an action and the other (the *follower*) replies taking into account the action of the leader that he observes.

Stackelberg games have been widely examined in literature from many points of views, in particular in Simaan and Cruz Jr [83], Simaan and Cruz Jr [84] and Basar and Olsder [4]. For basic introductions, in case of one–leader one–follower games, see also, for example, Tirole [86], Fudenberg and Tirole [26], Mas-Colell et al. [58]. In this following subsection, in line with the scope of the thesis, we consider mainly the case of uniqueness of the optimal reaction of the follower.

Moreover, although the games we investigate are multi–leader multi–follower games, results concerning one–leader one–follower games will be useful for our purpose, as we will see in Chapter 4.

2.4.1 One–leader one–follower

A Stackelberg game is an extensive game between two players: the leader L and the follower F. Let us denote with X (respectively Y) the action set and with $l\colon X \times Y \to \mathbb{R}$ (resp. $f\colon X \times Y \to \mathbb{R}$) the payoff function of L (resp. F). A strategy for the follower F is a complete contingent plan that specifies an action in every possible distinguishable circumstance in which the follower might be called upon to move. So, given the interaction scheme explained before, a strategy for F is a function s from X to Y. Let S be the set of the strategies of the follower, that is $S = X^Y$. A classic equilibrium concept in Game Theory for extensive form games is the concept of *subgame perfect Nash equilibrium* (introduced by Selten [78]):

Definition 2.4.1
A couple $(\bar{x}, \bar{s}) \in X \times S$ is a *subgame perfect Nash equilibrium* of the Stackelberg game presented above if it satisfies the following conditions:

(i) $\bar{x} \in \operatorname{Arg\,min}_{x\in X} l(x, \bar{s}(x))$, i.e. $\bar{x}$ solves the problem $\min_{x\in X} f(x, \bar{s}(x))$;
(ii) $\bar{s}(x) \in \operatorname{Arg\,min}_{y\in Y} f(x, y)$, for any $x \in X$.

Basically, in order to find a subgame perfect Nash equilibrium, the follower determines an optimal choice for each situation he might face (that is for each choice of the leader) and, then, the leader chooses an optimal action working backward and taking into account the reaction of the follower. Let us notice that in case there exists at least an action of the leader for which the follower has more than one optimal reaction, the leader could be not able to correctly anticipate what action the follower will play. Then, the predictive power of the Stackelberg model may fall. We briefly discuss this case at the end of this section.
Now assume in this section, unless otherwise specified, that the follower's reaction is single–valued, that is:

($\mathcal{I}$) *for any* $x \in X$*, the problem* $\min_{y\in Y} f(x, y)$ *has a unique solution.*

Then, a Stackelberg problem, also called *Bilevel Optimization problem* (see, for example, Morgan [62], Colson et al. [17]), can be formalized as follows:

$$\mathcal{S} \qquad \begin{cases} \text{find } \bar{x} \in X \text{ that solves } \min_{x\in X} l(x, \bar{s}(x)) \\ \text{where, for any } x \in X,\ \bar{s}(x) \text{ is the solution of} \\ \mathcal{P}(x) : \quad \min_{y\in Y} f(x, y). \end{cases} \tag{2.16}$$

The function $\bar{s}$ is called the *reaction function* of the follower and the action $\bar{x}$ is called *Stackelberg solution* of $\mathcal{S}$.

Existence of solutions and well-posedness of the Stackelberg problem defined in (2.16) are investigated in [62], where well-posed means that a solution to the problem $\mathcal{S}$ exists, and is unique and any method which constructs "minimizing sequence" (in some sense to define) automatically allows to approach a solution ([62, p.307]). Let us report a result that will be used in Chapter 4.

Proposition 2.4.2 ([62] Corollary 5.1)**.** *Under the following assumptions:*

(i) X *and* Y *are two sequentially compact spaces;*

(ii) l *and* f *are sequentially lower semicontinuous on* $X \times Y$*;*

(iii) for any $(x, y) \in X \times Y$*, for any sequence* $(x_n)_n$ *converging to* x *in* X*, there exists a sequence* $(y_n)_n$ *in* Y *such that* $\limsup_{n\to+\infty} f(x_n, y_n) \le f(x, y)$*;*

there exists a Stackelberg solution to $\mathcal{S}$*.*

In Loridan and Morgan [49, 50] a general approximation of the Stackelberg problem defined in (2.16) by a sequence of Stackelberg problems is considered. Such a technique, as motivated in [49], is useful for various reasons: on the one hand it offers some results about stability and continuous dependence of the optimal solutions under perturbations on the data of the problem; on the other hand it can be used to approximate the problem by using a sequence of problems that can be easier to solve with respect to the original problem and that converge to it in some sense. As the authors say, the main purpose is to give a general framework, for a stability analysis of Stackelberg problems under data perturbations, using appropriate notions of convergence (see [50]). They also assume that the follower's optimization problem (lower level) has a unique solution.

Starting from the problem defined in (2.16), they consider the following sequence of Stackelberg problems $\mathcal{S}_n$, for $n \in \mathbb{N}$:

$$\mathcal{S}_n \qquad \begin{cases} \text{find } \bar{x}_n \in X \text{ that solves } \min_{x\in X} l_n(x, \bar{s}_n(x)) \\ \text{where, for any } x \in X,\ \bar{s}_n(x) \text{ is the (unique) solution of} \\ \mathcal{P}_n(x): \quad \min_{y\in Y} f_n(x, y), \end{cases} \tag{2.17}$$

where l_n and f_n are extended real–valued functions defined on $X \times Y$, for every $n \in \mathbb{N}$.

The main results, that will be useful in the next chapters, are summarized in the next two propositions.

Proposition 2.4.3 ([50] Proposition 3.1)**.** *If the two conditions hold:*

(i) X *and* Y *are two sequentially compact spaces;*

(ii) *for any* $(x, y) \in X \times Y$, *for any sequence* $((x_n, y_n))_n$ *converging to* (x, y) *in* $X \times Y$, *we have* $\liminf_{n\to+\infty} f_n(x_n, y_n) \geq f(x, y)$;

(iii) *for any* $(x, y) \in X \times Y$, *for any sequence* $(x_n)_n$ *converging to* x *in* X, *there exists a sequence* $(y_n)_n$ *in* Y *such that* $\limsup_{n\to+\infty} f_n(x_n, y_n) \leq f(x, y)$;

then, for any sequence $(x_n)_n$ *converging to* x *in* X, *we have*

$$\lim_{n\to+\infty} \bar{s}_n(x_n) = \bar{s}(x) \quad \text{and} \quad \lim_{n\to+\infty} f_n(x_n, \bar{s}_n(x_n)) = f(x, \bar{s}(x)).$$

Furthermore, denoted the set $\{\bar{x}_n \in X : l_n(\bar{x}_n, \bar{s}_n(\bar{x}_n)) = \inf_{x\in X} l_n(x, \bar{s}_n(x))\}$ with N_n and the set $\{\bar{x} \in X : l(\bar{x}, \bar{s}(\bar{x})) = \inf_{x\in X} l(x, \bar{s}(x))\}$ with N, they prove the following proposition:

Proposition 2.4.4 ([50] Proposition 3.2)**.** *Under the assumptions in Proposition 2.4.3 and the following:*

(i) *for any* $(x, y) \in X \times Y$, *for any sequence* $((x_n, y_n))_n$ *converging to* (x, y) *in* $X \times Y$, *we have* $\liminf_{n\to+\infty} l_n(x_n, y_n) \geq l(x, y)$;

(ii) *for any* $x \in X$, *there exists a sequence* $(x_n)_n$ *converging to* x *in* X, *such that, for any* $y \in Y$ *and for any sequence* $(y_n)_n$ *converging to* $y \in Y$, *we have* $\limsup_{n\to+\infty} l_n(x_n, y_n) \leq l(x, y)$;

then $\limsup N_n \subseteq N$.

The last result means that each cluster point of a sequence of solutions to the approximate problems is a solution to problem $\mathcal{S}$.
Let us emphasize that assumptions (ii) and (iii) in Proposition 2.4.3 and assumptions (i) and (ii) in Proposition 2.4.4 are weaker than continuous convergence of f_n and l_n to f and l, respectively, as illustrated in Remark 2.2 in [50].

Other interesting issues about a Stackelberg game concern the description of necessary and sufficient conditions for optimality and algorithms that solve Bilevel Optimization problems. See, for example, Shimizu and Ishizuka [81], Vicente and Calamai [88], Luo et al. [54], Dempe [19], Colson et al. [17]

where bibliographic reviews can be found.

In case of nonuniqueness of the follower's problem there could be different ways to define a "rational" response of the follower. So, different models can be defined, depending on the way the leader "approaches" the follower's problem. In the *pessimistic* approach the leader is assumed to provide himself against the possible worst choice of the follower. This approach leads to the definition of the so–called *weak Stackelberg problem* also called *pessimistic Bilevel Optimization problem* (Leitmann [43], Breton et al. [12], Morgan [62], Lignola and Morgan [48], Dempe et al. [20],...).
In the *optimistic* approach the leader is assumed to be able to force the follower to choose the best strategy for himself. In this case we obtain the so–called *strong Stackelberg problem* (Leitmann [43], Breton et al. [12],...) also called *optimistic Bilevel Optimization problem*, but other models can be defined such as the so–called *intermediate Stackelberg problems* (Mallozzi and Morgan [55], Lignola and Morgan [48],...). Let us emphasize that the predicted choice for the leader can be different for each different approach, as illustrated in Morgan and Patrone [63]. Existence of a solution to a weak Stackelberg problem and continuous dependence on the data could be not guaranteed, even under very mild assumptions. Strong Stackelberg problem is easier for what concerns the existence of solutions but it does not display continuous dependence on the data of the problem. Using approximate solutions to the follower's problem help to overcome this drawback, even for more than one follower, as it can be also found in Loridan and Morgan [51, 52], Lignola and Morgan [47, 46], Morgan and Raucci [64, 66, 65], Mallozzi and Morgan [56].

In Chapter 4 we mainly limit ourself to the case of single–valued followers' reactions.

2.4.2 M–leaders N–followers

Now, we briefly present some papers on multi–leader multi–follower games.

Denote with L_i, for $i = 1, \dots, M$, the leaders and with F_i, for $i = 1 \dots, N$, the followers. Let us denote with X_i the action set of leader L_i, for $i = 1, \dots, M$, and with Y_i the action set of follower F_i, for $i = 1, \dots, N$. Let $X = \prod_{i=1}^{M} X_i$ and $Y = \prod_{i=1}^{N} Y_i$ and denote with $l_i \colon X \times Y \to \mathbb{R}$ the payoff function of L_i, for $i = 1, \dots, M$, and with $f_i \colon X \times Y \to \mathbb{R}$ the payoff function of F_i, for $i = 1, \dots, N$. A strategy for follower F_i is a function $s_i \colon X \to Y_i$. Let S_i be the set of the strategies of F_i and let $S = \prod_{i=1}^{N} S_i$.
A subgame begins after the choice of any action profile $x = (x_1 \dots, x_M) \in X$ of the leaders and in any subgame only the followers are involved and play simultaneously. Let $\mathcal{N}$ be the set–valued map from X to Y such that $\mathcal{N}(x)$ is the set of Nash equilibria of the subgame that begins after the action profile $x = (x_1, \dots, x_M)$ of the leaders.

Definition 2.4.5
A strategy profile $(\bar{x}, \bar{s}) = (\bar{x}_1, \dots, \bar{x}_M, \bar{s}_1, \dots, \bar{s}_N) \in X \times S$ is a *subgame perfect Nash equilibrium* of the multi–leader multi–follower game if it satisfies the following conditions:

(i) $\bar{x}$ is a Nash equilibrium of the game $\Gamma^{\bar{s}} = \{(L_i)_{i=1}^{M}; (X_i)_{i=1}^{M}; (l_i^{\bar{s}})_{i=1}^{M}\}$;
(ii) $\bar{s}(x) \in \mathcal{N}(x)$, for any $x \in X$;

where $l_i^{\bar{s}}$ is the function defined by $l_i^{\bar{s}}(x_1, \dots, x_M) = l_i(x_1, \dots, x_M, \bar{s}_1, \dots, \bar{s}_N)$, for any $x \in X$.

As in the case of one follower, the predictive power of the Stackelberg model may fall if $\mathcal{N}$ is not single–valued.

Let us first consider a Stackelberg game in which one leader compete with N followers. In this simplified model, the leader solves an Optimization problem.

In oligopolistic models, when the payoff functions of the players correspond to the profits (note that in Definition 2.4.5 we deal with a maximization problem), a leader can correctly anticipate the reaction of the followers even if the aggregate reaction R of the followers is single–valued, where R is defined for any action x of the leader by $R(x) = \{\sum_{i=1}^{N} \bar{y}_i \colon (\bar{y}_1, \ldots, \bar{y}_N) \in \mathcal{N}(x)\}$.

In Sherali et al. [80] a model of quantity competition between firms is investigated: first, the leader announces his output and then the followers, simultaneously, announce their respective production levels. The existence and the uniqueness of the subgame perfect Nash equilibrium when the aggregate reaction R is single–valued is proved together with sufficient conditions on the data in order to obtain such property. Moreover, they investigate the properties of the aggregate reaction of the Cournot firms and they find an algorithm to determine such a solution.

An oligopolistic problem where the inverse demand function can be set–valued is presented in Flåm et al. [25]. A way to face this problem is to take as market price for every quantity the maximum price compatible with that quantity. An existence result is given, under appropriate hypotheses on the inverse demand function.

In Morgan and Raucci [64, 65, 66] existence theorems and convergence results for games with one leader and two followers are proved, under sufficient conditions of minimal character.

Multi–leader multi–follower games have not been widely investigated in the literature, maybe for the difficulties that may arise, although they have many real–world applications (see for example Pang and Fukushima [71]).

A first example of a multi–leader multi–follower game in an oligopolistic context can be found in Sherali [79], where an extension of the model in [80] is presented when a few leader–firms supply a homogeneous good noncooperatively. In the paper it is assumed that the actions taken by the leaders are observable in the lower level by any follower. The authors prove the existence of an equilibrium, assuming that the aggregate reaction of the followers is single–valued and, considered as a function, is convex.

In [71] a 2–leader multi–follower noncooperative game is investigated and it is assumed that the actions taken by the leaders are observable by any follower. The set–valued map $\mathcal{N}$ defined above is not assumed to be single–

valued and the authors use an optimistic approach:

Definition 2.4.6 ([71])
A strategy profile $(x_1^*, x_2^*) \in X$ is said to be a *L/F Nash equilibrium* if there exist two action profiles of the followers $y^{*,1}, y^{*,2} \in \mathcal{N}(x_1^*, x_2^*)$ such that $(x_1^*, y^{*,1})$ is an optimal solution of leader L_1's problem and $(x_2^*, y^{*,2})$ is an optimal solution of leader L_2's problem, being the problem of leader L_i, for $i = 1, 2$, defined by:

$$\begin{aligned} &\min l_i(x_i, x_{3-i}^*, y^i) \\ &\text{subject to } x_1 \in X_i \\ &\text{and } y^i \in \mathcal{N}(x_i, x_{3-i}^*). \end{aligned}$$

Then, when $\mathcal{N}$ is not single–valued, in the above definition any leader can have different anticipations of the reaction of the followers. The main problem is the lack of convexity of the set $\mathcal{N}(x_1, x_2)$. For this reason they introduce a new class of models, the so–called *remedial models*, that are noncooperative multi–leader multi–follower games with convexified strategy sets. For that convexified multi–leader multi–follower game they show an existence result. The drawback is that there is not a unique way to convexify. A remedial model is associated to any different convexification: "The remedial models are not all desirable; while they always have equilibria, a careful choice of a remedial model leads to a sensible equilibrium solution, whereas a not-so-careful choice could lead to an equilibrium solution where the leaders have entirely different expectations on the follower's behavior". Finally, applications in oligopolistic competition on the electricity power markets are considered.

In Kulkarni and Shanbhag [38] there is another recent example of multi–leader multi–follower interaction, that is formulated similarly to [71]. The authors also assume that the actions taken by the leaders are observable in the second stage by any follower. They propose two different approaches to overcome the problem of the lack of convexity of the lower level problem. In the first they introduce a new class of multi–leader multi–follower games in which the leaders' payoff functions exhibit a particular structure, the so–called *quasipotential games*. The other approach is based on a modified formulation of the multi–leader multi–follower game in which in the original problem they add a particular *shared constraint*. They show that if the payoff function of any leader admits a potential function, then there exists an equilibrium of the original game. Then, in both the approaches they relate the equilibrium of the game to the solution of an appropriate optimization problem.

When the action of a leader is not observed by all the followers, the situation is completely different. As we will see in Chapter 4, a problem that may arise is the multiplicity of Nash equilibria. Unfortunately, the concept of subgame perfect equilibrium coincides with the concept of Nash equilibrium since there are no proper subgames. Then, a selection criterion that can be used is the so–called *perfect Bayesian equilibrium*(see [26]), that prescribes beliefs on the information sets of the followers. Unfortunately, such a game may also have multiple perfect Bayesian equilibria. So, in the literature particular restrictions on beliefs are considered.

The example of multi–leader multi–follower interaction in multilateral vertical contracting given in Pagnozzi and Piccolo [70] will provide a motivating example for the analysis in Chapter 4. It will be discussed in more details in Section 4.1.

Finally, in the very recent working paper Eguia et al. [24] is addressed the question of the existence of a selection criterion for a multi–leader multi–follower game in which players in the second stage are imperfectly informed on the actions played in the first stage. The authors define a new selection criterion that is tested by experiments in game with multilateral vertical contracting and in game of electoral competition. The selection used in [24] is based on the idea that the equilibrium action profile most likely to emerge is the action profile that can be supported in equilibrium by a larger set of different equilibrium beliefs. The criterion, however, does not select the specific beliefs that accompany these strategy profiles in equilibrium. This approach differs from the one in Chapter 4 in which we consider, as said, a perfect Bayesian equilibrium with particular type of beliefs.

Chapter 3

Some new results in Generalized Convexity Analysis and in Parametric Optimization

In this chapter we introduce new results that will be used in Chapter 4 in order to prove existence theorems for a multi–leader multi–follower game with vertical separation under passive beliefs. As we will see, one of the main problems for the existence of this kind of equilibrium is related to the quasiconcavity of some composite functions. From Proposition 2.1.24, we know that if we compose a quasiconcave function with m concave functions we will obtain a quasiconcave function. One of these m functions in the problem in Chapter 4 is the optimal solution function defined in 2.13. So, in the first section we give a new concavity result on the optimal solution function of a parametric optimization problem.
Another interesting problem concerns the isotonicity of the optimal solution function. This problem is widely studied in the literature and it is related to the concept of supermodularity of the functions involved in the optimization problem. For a detailed discussion a reader can be referred, for example, to Milgrom and Shannon [61], Topkis [87]. The property of isotonicity is also central in the theory of supermodular games (see, for example, Bulow

et al. [13], Milgrom and Roberts [60], Vives [89]), where the best responses exhibit strategic complementarity, i.e. increasing best responses. For these classes of games existence results are proved using the Tarski Fixed Point Theorem (Tarski et al. [85]), in a lattice–theoretic context. A new result that guarantees the isotonicity of the optimal solution function when the strategy space is a real interval is given in the first section.
Both the announced results are used in Ceparano and Quartieri [16] to prove uniqueness of Nash equilibrium.
In Section 3.2 we present two generalizations of Cellina Fixed Point Theorem which will be used in Chapter 4 in order to prove existence of equilibria under passive beliefs of particular classes of two–stage games with vertical separation.

3.1 Concavity and isotonicity of the optimal solution function

In this section, following [16], we present new results on concavity and isotonicity of the optimal solutions function.

Let X and Y be nonempty convex subsets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively, and let f be a real–valued function defined on $X \times Y$. We denote with $\mathcal{D}_f$ the function from $X \times \operatorname{int}(Y)$ to $\overline{\mathbb{R}}$ defined by

$$\mathcal{D}_f(x, y) = D_{-}f(x, \cdot)(y), \tag{3.1}$$

where, for any $x \in X$, $D_{-}f(x, \cdot)$ is the left–hand lower Dini derivative of the function $f(x, \cdot)$, defined in (2.3).

Theorem 3.1.1. *Let X be a nonempty convex subset of $\mathbb{R}^n$ and let Y be a proper compact real interval. Let f be a real–valued function defined on $X \times Y$. Assume that the function $f(x, \cdot)$ is upper semicontinuous and strictly D–pseudoconcave on Y, for any $x \in X$. Let $\mathcal{D}_f$ be as in (3.1) and b be the optimal solutions function from X to Y defined by $\{b(x)\} = \operatorname{Arg\,max}_{y \in Y} f(x, y)$,* *for any $x \in X$. Assume that $\mathcal{D}_f$ is quasiconcave on $X \times Y$. Then, the set $\overline{X} = \{x \in X \colon b(x) > \min Y\}$ is convex and the function b is concave on $\overline{X}$.*

Proof. Let us first observe that, from proposition A.2.5 and 2.1.22, f attains its maximum at a unique point, for any $x \in X$, then b is a function on X. Suppose that there are $t, z \in \overline{X}$ with $t \neq z$. Let

$$\xi = b(t); \quad \zeta = b(z).$$

Then $\min\{\xi, \zeta\} > \min Y$.
Let $\lambda \in]0, 1[$ and put $x = \lambda t + (1 - \lambda)z$. We have to prove that

$$\min Y < \overline{v} = \lambda\xi + (1 - \lambda)\zeta \leq b(x) = v. \tag{3.2}$$

If $\min\{\xi, \zeta\} < \max Y$, suppose, to the contrary, that

$$v < \overline{v}.$$

First of all we have $\min Y \leq v < \overline{v} < \max Y$.
From the strict D–pseudoconcavity of f and from the fact that $f(x, \cdot)$ is maximized at v, we have that

$$D_{-}f(x, \cdot)(\overline{v}) < 0. \tag{3.3}$$

Furthermore, $f(t, \cdot)$ is maximized at ξ, that is $f(t, \xi) \geq f(t, y)$, for any $y \in Y$. So, for any $y < \xi$ we have that

$$\frac{f(t, y) - f(t, \xi)}{y - \xi} \geq 0,$$

that is

$$D_{-}f(t, \cdot)(\xi) \geq 0; \qquad \text{and, analogously} \qquad D_{-}f(z, \cdot)(\zeta) \geq 0. \tag{3.4}$$

Then, from (3.1), we have $\min\{\mathcal{D}_f(\xi, t), \mathcal{D}_f(\zeta, z)\} \geq 0$. Therefore, (ξ, t) and (ζ, z) belong to the upper level set at height 0 of the quasiconcave function $\mathcal{D}_f$ and then also $\mathcal{D}_f(\bar{v}, x) \geq 0$, that is $D_{-}f(x, \cdot)(\bar{v}) \geq 0$ in contradiction with (3.3).

If $\min\{\xi,\zeta\} = \max Y$, that is $\xi = \zeta = \overline{v} = \max Y$, from the definition of maximum point and from the strict D–pseudoconcavity of f, we have

$$D^+ f(t,\cdot)(y) > 0, \quad D^+ f(z,\cdot)(y) > 0 \quad \text{for all } y \in \text{int}(Y).$$

By Lemma 2.1.18 we have that $f(t,\cdot)$ and $f(z,\cdot)$ are increasing on $\text{int}(Y)$; then:

$$D_- f(t,\cdot)(y) \geq 0; \quad D_- f(z,\cdot)(y) \geq 0 \quad \text{for all } y \in \text{int}(Y).$$

Consequently, by the quasiconcavity of $\mathcal{D}_f$ on $X \times \text{int}(Y)$, we must have that

$$D_- f(x,\cdot)(y) \geq 0 \quad \text{for all } y \in \text{int}(Y). \tag{3.5}$$

Therefore $\overline{v} = \max Y = v = b(x)$ by the strict D–pseudoconcavity of $f(x,\cdot)$. Indeed, if $v < \max Y$ from (3.5) we can find some $y > v$ such that we have $f(x,y) \leq f(x,v)$ and $D_- f(x,\cdot)(y) \geq 0$ that is in contradiction with the definition of a strictly D–pseudoconcave function. ◇

Corollary 3.1.2. *In addition to the assumptions in Theorem 3.1.1, let* $\underline{y} = \min Y$ *and* $D^+ f(x,\cdot)(\underline{y}) > 0$*, for all* $x \in X$*. Then* b *is concave on* X*.*

Proof. Let $x \in X$. Directly from the definition, if $D^+ f(x,\cdot)(\underline{y}) > 0$ there exists some $y > \underline{y}$ such that

$$\frac{f(y) - f(\underline{y})}{y - \underline{y}} > 0,$$

which implies that $f(y) > f(\underline{y})$. So f cannot attain its maximum in $\underline{y}$. ◇

Now, we will present a result on the isotonicity of the solution of a parametric optimization problem. Let us recall the following definition.

Definition 3.1.3 (Gratzer [30])
A real–valued function g defined on a set $X \subseteq \mathbb{R}^n$ is called *isotone* (or *order–preserving*) if $x \leq_X y$ implies $f(x) \leq f(y)$, for any $x, y \in X$, where $\leq_X$ is the usual partial order relation on X (that is if $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in X$, then $x \leq y$ if $x_i \leq y_i$, for all $i = 1,\ldots,n$).

Proposition 3.1.4. *Let X be a nonempty subset of $\mathbb{R}^n$ and Y be a proper real interval closed on the left. Let f be a real–valued function defined on $X \times Y$ such that $f(x, \cdot)$ is upper semicontinuous and strictly D–pseudoconcave on Y, for every $x \in X$. Defined $\mathcal{D}_f$ as in (3.1), let $\mathcal{D}_f(\cdot, y)$ be isotone on X, for all $y \in Y$. Then, the optimal solutions function b is isotone.*

Proof. Take any t and z in X such that $t \neq z$ and $t_i \leq z_i$ for all $i = 1, \ldots, m$. If $b(t) = \min Y$ then

$$b(t) \leq b(z).$$

Henceforth suppose $b(t) > \min Y$. By the strict D–pseudoconcavity of f in its second argument (and by the definition of b) we have that

$$D^+ f(t, \cdot)(y) > 0 \text{ for all } y \in [\min Y, b(t)[,$$

and consequently, by Lemma 2.1.18, $f(t, \cdot)$ is increasing on $]\min Y, b(t)[$, that is:

$$D_- f(t, \cdot)(y) \geq 0 \text{ for all } y \in]0, b(t)[.$$

Hence, from the isotonicity of $\mathcal{D}_f(\cdot, y)$, we have:

$$D_- f(z, \cdot)(y) \geq 0 \text{ for all } y \in]0, b(t)[. \tag{3.6}$$

Therefore, (3.6) and the strict D–pseudoconcavity of f in its second argument imply that

$$b(t) \leq b(z).$$

Indeed, if $b(z) < b(t)$, from (3.6), it follows that $D_- f(z, \cdot)(y) \geq 0$ for some $y \in]b(z), b(t)[$ in contradiction with the strict D–pseudoconcavity of $f(z, \cdot)$ on $\text{int}(Y)$.
Therefore, b is an isotone function. ◇

3.2 Fixed point theorems

Let us conclude this chapter proving two fixed point theorems that are a generalization of the following fixed point theorem.

Theorem 3.2.1 ([9] Cellina Fixed Point Theorem)**.** *Let X be a nonempty compact convex subset of $\mathbb{R}^n$ and Y be a compact convex subset of $\mathbb{R}^m$. Let N be a set–valued map from X to X and G be a closed set–valued map defined from X to Y with nonempty compact convex values. Let f be a continuous function defined on $X \times Y$ to X such that for any $x \in X$*

$$N(x) = \{f(x, y) \colon y \in G(x)\}.$$

Then, N has a fixed point on X, i.e. there exists $x^* \in X$ satisfying $x^* \in N(x^*)$.

Now, let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, let G be a set–valued map from X to Y and F be a set–valued map from $X \times Y$ to X. A natural extension of the Cellina Fixed Point Theorem is to determine when the set–valued map M from X to X defined by $M(x) = \cup_{y\in G(x)} F(x, y) = \{x \in F(x, y) \colon y \in G(x)\}$ has a fixed point on X.

Two results have been obtained. The first uses the Michael selection Theorem (Proposition 2.2.11) but it requires the lower semicontinuity of the set–valued map F. The second is based on the Von Neumann's Approximation Lemma and requires the upper semicontinuity of F.

Theorem 3.2.2. *Let X be a nonempty compact convex subset of $\mathbb{R}^n$ and Y be a compact convex subset of $\mathbb{R}^m$. let M be a set–valued map from X to X, G be a closed set–valued map from X to Y with nonempty compact convex values and F be a lower semicontinuous set–valued map from $X \times Y$ to X with nonempty closed convex values. Then, the set–valued map on X defined by:*

$$M(x) = \cup_{y\in G(x)} F(x, y) \tag{3.7}$$

has a fixed point on X, i.e. there exists $x^* \in X$ satisfying $x^* \in M(x^*)$.

Proof. It is a consequence of Theorem 3.2.1 and of the Michael Selection Theorem (Proposition 2.2.11).

Indeed, if F is lower semicontinuous, there exists a continuous function $f \colon X \times Y \to X$ such that $f(x, y) \in F(x, y)$, for any $(x, y) \in X \times Y$. Consider the set–valued map $M'(x) = \{f(x,y) \colon y \in G(x)\}$. From Theorem 3.2.1, there exists a point $x^* \in X$ such that $x^* \in M'(x^*)$, that is $x^* = f(x^*, y) \subseteq F(x^*, y)$, for some $y \in G(x)$; that is x^* is a fixed point of M on X. ⋄

A problem with the above theorem arises if one wants to prove existence of a selection (of Nash equilibria) for a two–stage game under vertical separation via Theorem 3.2.2. Indeed, an equilibrium of such a game can be stated as a fixed point of a set–valued map defined as M in (3.7) where F will be the optimal solutions set–valued map of the lower level problem. In this case, requiring the lower semicontinuity of F may be a too strong assumption (see, for example, Loridan and Morgan [53]). Next theorem gives sufficient conditions easier to verify.

Theorem 3.2.3. *Let X be a nonempty compact convex subset of $\mathbb{R}^n$ and Y be a compact convex subset of $\mathbb{R}^m$. Let M be a set–valued map from X to X, G be a closed set–valued map from X to Y with nonempty compact convex values and F be a closed set–valued map from $X \times Y$ to X with nonempty convex compact values. Then the set–valued map defined on X by:*

$$M(x) = \cup_{y \in G(x)} F(x, y)$$

has a fixed point on X, i.e. there exists $x^* \in X$ satisfying $x^* \in M(x^*)$.

Proof. From Proposition 2.2.8(i) we have that F is an upper semicontinuous set–valued map. So, from the von Neumann's Approximation Lemma (Proposition 2.2.12) for any $\epsilon > 0$ there exists a continuous function f_ϵ such that

$$\text{Graph}\, f_\epsilon \subseteq U_\epsilon(\text{Graph}\, F). \tag{3.8}$$

Consider the set–valued function $M_\epsilon \colon X \rightrightarrows X$ defined by:

$$M_\epsilon(x) = \{f_\epsilon(x, y) \colon y \in G(x)\},$$

for any $x \in X$. From Theorem 3.2.1 M_ϵ has a fixed point, for any $\epsilon > 0$. Consider the sequence $(\epsilon_k)_k = \left(\frac{1}{k}\right)_k$ and let $\bar{x}_k \in M_{\epsilon_k}(\bar{x}_k)$, for any $k \in \mathbb{N}$.

Then, there exists $\bar{y}_k \in G(\bar{x}_k)$ such that $\bar{x}_k = f_{\epsilon_k}(\bar{x}_k, \bar{y}_k)$ or, equivalently, $(\bar{x}_k, \bar{y}_k, \bar{x}_k) \in \operatorname{Graph} f_{\epsilon_k}$, for any $k \in \mathbb{N}$. From the compactness of X and Y, there exists a subsequence $(\bar{x}_k, \bar{y}_k)_{k \in K_1}$, with $K_1 \subseteq \mathbb{N}$ countably infinite, that converges to (x^*, y^*) in $X \times Y$. So, from the closedness of G, we have $y^* \in G(x^*)$. We claim that $x^* \in F(x^*, y^*)$.
In fact, from (3.8), for any $k \in K_1$ there exists a point $(x_k, y_k, z_k) \in \operatorname{Graph} F$ such that $d((\bar{x}_k, \bar{y}_k, \bar{x}_k), (x_k, y_k, z_k)) < \frac{1}{k}$.
Moreover, $\operatorname{Graph} F \subseteq X \times Y \times X$ is a closed subset of a compact set, that is $\operatorname{Graph} F$ is compact. Hence, there exists a subsequence $(x_k, y_k, z_k)_{k \in K_2}$, with $K_2 \subseteq K_1$ countably infinite, and a point $(\tilde{x}, \tilde{y}, \tilde{z}) \in \operatorname{Graph} F$ such that (x_k, y_k, z_k) converges to $(\tilde{x}, \tilde{y}, \tilde{z})$. Using the triangular inequality, for any $k \in K_2$ we have:

$$\begin{aligned} d((\bar{x}_k, \bar{y}_k, \bar{x}_k), (\tilde{x}, \tilde{y}, \tilde{z})) &< d((\bar{x}_k, \bar{y}_k, \bar{x}_k), (x_k, y_k, z_k)) \\ &\quad + d((x_k, y_k, z_k), (\tilde{x}, \tilde{y}, \tilde{z})) \\ &< \frac{1}{k} + d((x_k, y_k, z_k), (\tilde{x}, \tilde{y}, \tilde{z})). \end{aligned}$$

Taking the limit, we have $d((x^*, y^*, x^*), (\tilde{x}, \tilde{y}, \tilde{z})) = 0$, being the distance continuous, that is $x^* = \tilde{x} = \tilde{z}$ and $y^* = \tilde{y}$. Thus $(x^*, y^*, x^*) \in \operatorname{Graph} F$, so $x^* \in M(x^*)$ and x^* is a fixed point of M on X.

◇

Chapter 4

Selection of equilibria for two-stage possibly discontinuous games with vertical separation: existence results

In this chapter we prove some existence results for a selection of Nash equilibria of a two–stage game with vertical separation. This particular interaction scheme is inspired by an economic problem, developed by Pagnozzi and Piccolo [70], in which two competing manufacturers delegate retail decisions to independent retailers (such a situation is called *vertical separation*) and each producer has to decide a wholesale price (the *contract*) that their exclusive retailers have to pay assuming that each retailer sells the good produced by only one manufacturer. The case in which the contract is private (that is each retailer observes only the wholesale price chosen by his manufacturer) brings to an extensive form game without proper subgames. Then, the concept of subgame perfect equilibrium, that is usually used for two–stage games, do not allow to select among Nash equilibria (that can be infinitely many). So, Pagnozzi and Piccolo consider solution concepts depending on conjectures (or *beliefs*) that each retailer has about the contract offered to the other retailer. Starting from this problem, we analyze in a more general context this kind of interaction that, by analogy, we will call *vertical separation*.

First, in Section 4.1 we consider an example of oligopoly with vertical separation and private contract and we show that the associate normal form game has an infinite number of Nash equilibria. This motivates the introduction of selections of Nash equilibria based on the beliefs that each retailer has about the contract offered to the other retailer. We consider the case of *passive beliefs* (or *market–by–market conjectures*) assumed also, for example, in Crémer and Riordan [18], Hart et al. [31], McAfee and Schwartz [59], Laffont and Martimort [42], Gavazza and Lizzeri [27], Bar-Isaac et al. [3], Pagnozzi and Piccolo [70] and we show that in the example it brings to the selection of a unique Nash equilibrium. Then, in Section 4.2 we give the formal definitions of a two–stage game with vertical separation in a general context and in Section 4.3 we examine the existence of Nash Equilibria of this game. In Section 4.4, when the optimal reactions of the followers are single–valued, we present a solution concept for a general multi–leader multi–follower game with vertical separation based on passive beliefs which corresponds to a selection in the set of Nash equilibria. For this solution concept we prove existence results using conditions of minimal character for possibly discontinuous functions. A first existence result is given when the optimal reaction of any follower is a linear function. Conditions on data are given in order to obtain the linearity of the optimal reactions of the followers. With the aim of obtaining existence of equilibria without the condition of linearity, In Section 4.5 we prove existence results for particular classes of two–stage games with vertical separation, via characterizations of the concept of equilibrium under passive beliefs. More in detail, when the optimal reaction of the leader is single–valued we prove an existence result without concavity assumptions.
Another particular class of problem is the one in which the action sets of the followers are subsets of $\mathbb{R}$. In this case we prove an existence result using the concavity of the optimal reactions of the followers. Sufficient conditions

for the concavity of the optimal reactions are given as application of Theorem 3.1.1. Furthermore, if also the action sets of the leaders are subsets of $\mathbb{R}$ and the payoff of the leaders do not depend directly from their actions, we prove an existence result using the isotonicity of the the optimal reactions of the followers.
Finally, we consider the case in which the optimal reactions of the followers are not single–valued and the payoff of any leader depends on the action of the corresponding follower only through the optimal value function.

4.1 Vertical Separation in competitive markets with private contracts: an illustrative example

First, as in Pagnozzi and Piccolo [70], we consider a delegation problem, in which two competing manufacturers (the *leaders*), producers of substitute goods, choose vertical separation as their organizational structure, that is the producers delegate the sale of the good they produce through an exclusive retailer. The manufacturers offer a contract to the retailers that decide in a competitive setting the retail price after observing the contract. The contract is private so each retailer observes only the contract offered by his corresponding manufacturer. The situation is modeled as a two–stage game in the following way:

Stage 1 any manufacturer M_i offers a contract a_i that specifies the condition of the delegation to retailer R_i,

Stage 2 all R_i choose, simultaneously, the downstream market price b_i.

A strategy for M_i, $i = 1, 2$, specifies the contract a_i that is offered to retailer R_i. A strategy for R_i corresponds to a downstream market price for every contract the retailer can observe, that is a function from the space of the contracts to the set of possible prices. A solution concept considered in [70]

is based on the fact that, under private contract, the market price chosen by a retailer depends on his beliefs about the contract offered to the rival retailer. The authors combine a backward induction framework and the compatibility with these beliefs in a way that is illustrated forward. In particular, they considered a kind of beliefs — called *passive beliefs* — that is plausible in the context of two competing vertical separate structures.
Passive beliefs, also known as *market–by–market conjecture*(Hart et al. [31]), means that if a retailer observes a contract different from the one he expects in equilibrium then he believes that the rival retailer still observes the corresponding equilibrium contract; so, under passive beliefs, each retailer does not revise his beliefs about the contract offered to the rival retailer, even if his corresponding manufacturer is deviating. In other words, if a_i^* and a_j^* are the equilibrium strategies for M_i and M_j, where $i, j = 1, 2$ and $j \neq i$, and if R_i observes a contract different from a_i^*, then R_i believes that M_j is still offering the contract a_j^*. So, under passive beliefs, a manufacturer's strategy does not influence the strategy of the rival retailer nor his conjecture about these strategies. In order to better illustrate the interest of the model presented in [70], we formalize now a simplified version.
Suppose that, for $i = 1, 2$, all manufacturers M_i offer a contract that is the wholesale price (for unit of good) a_i to the corresponding retailer R_i. Let b_i the retail price chosen by retailer R_i. Denoted with A_i (resp. B_i) the set of the wholesale price (resp. retailer price), we assume $A_i = \mathbb{R}^+$ (resp. $B_i = \mathbb{R}^+$). For $i, j = 1, 2$ and $i \neq j$, denote with $D^i(b_i, b_j)$ the market demand function for the good produced by M_i if the retail price is b_i and the other retailer chooses a price b_j. Let the profit functions be given by the functions:

$$l_i(a_i, b_i, b_j) = D^i(b_i, b_j)a_i,$$

$$f_i(a_i, b_i, b_j) = D^i(b_i, b_j)(b_i - a_i).$$

In order to determine the set of Nash equilibria of the corresponding two–stage game consider the associate normal form game. Since each leader has only one information set, each leader's strategy set is the set A_i. For $i = 1, 2$, the strategy set of the follower R_i is the space $S_i = A_i^{B_i}$ of functions β_i from A_i to B_i ($\beta_i : A_i \to B_i$). Therefore, the strategy profile $(a_1^*, a_2^*, \beta_1^*, \beta_2^*)$ is a *Nash Equilibrium* (Nash et al. [67]) of the two–stage game previously defined if, for $i = 1, 2$ and $j \neq i$:

$$a_i^* \in \operatorname*{Arg\,max}_{a_i \in A_i} l_i(a_i, \beta_i^*(a_i), \beta_j^*(a_j^*)) \tag{4.1a}$$

$$\beta_i^* \in \operatorname*{Arg\,max}_{\beta_i \in S_i} f_i(a_i^*, \beta_i(a_i^*), \beta_j^*(a_j^*)). \tag{4.1b}$$

Equation (4.1b) can be simplified observing that the only dependence of the maximizing function from the variable β_i is through the value that β_i assumes in a_i^*. Denoting

$$b_i^* = \beta_i^*(a_i^*), \qquad \text{for } i = 1, 2,$$

the system in (4.1) is satisfied if:

$$a_i^* \in \operatorname*{Arg\,max}_{a_i \in A_i} l_i(a_i, \beta_i^*(a_i), b_j^*) \tag{4.2a}$$

$$\beta_i^* \text{ s.t. } b_i^* \in \operatorname*{Arg\,max}_{b_i \in B_i} f_i(a_i^*, b_i, b_j^*). \tag{4.2b}$$

Consider the special case in which the demand function is linear in both retail prices (See, for example, Vives [90]) and in particular

$$D^i(b_i, b_j) = 1 - 2b_i + b_j. \tag{4.3}$$

Imposing the first order necessary and sufficient conditions ($f_i(a_i^*, \cdot, b_j^*)$ is concave on $\mathbb{R}^+$) for the follower R_i, we obtain that b_i^* has to satisfy the following variational inequality:

$$(1 + 2a_i^* + b_i^* - 4b_i^*)(b_i - b_i^*) \leq 0, \quad \text{for any } b_i \geq 0. \tag{4.4}$$

But, b_i^* cannot be zero. Indeed, we should have $1 + 2a_i^* + b_j^* \leq 0$ in contradiction with the fact that $A_i = B_j = \mathbb{R}^+$. So $b_i^* > 0$. Then, (4.4) is satisfied for any $b_i \geq 0$ if and only if $1 + 2a_i^* + b_j^* - 4b_i^* = 0$, that is $b_i^* = \frac{1}{4} + \frac{a_i^*}{2} + \frac{b_j^*}{4}$. Therefore, b_i^* is given by:

$$b_i^* = \frac{1}{3} + \frac{8}{15}a_i^* + \frac{2}{15}a_j^*. \tag{4.5}$$

- For example, one can verify that the strategy profile given by

$$\boxed{\check{s} = \left(\frac{2}{5}, \frac{2}{5}, \check{\beta}_1, \check{\beta}_2\right) \qquad \text{s.t.} \qquad \check{\beta}_i(a_i) = \begin{cases} \frac{3}{5} & \text{if } a_i = \frac{2}{5}, \\ \nu & \text{otherwise,} \end{cases}} \tag{4.6}$$

is a Nash equilibrium of the game when $\nu > \frac{18}{25}$. Note that $\check{\beta}_i$ is discontinuous on A_i, $i = 1, 2$.

In fact:

  - the strategies $\check{\beta}_i$ of R_i, $i = 1, 2$, satisfy condition (4.5) for $a_i^* = a_j^* = \frac{2}{5}$, $j \neq i$;
  - the strategy of M_i, $i = 1, 2$, satisfies (4.2a), being the payoff of M_i when the other players play in accordance with $\check{s}$:

$$\check{l}_i(a_i) = \begin{cases} \frac{4}{25} & \text{if } a_i = \frac{2}{5}, \\ \left(\frac{8}{5} - 2\nu\right) a_i & \text{otherwise.} \end{cases}$$

  So, $a_i^* = \frac{2}{5}$ is the unique solution of the problem in (4.2a) if and only if $\frac{8}{5} - 2\nu < \frac{4}{25}$, that is $\nu > \frac{18}{25}$.

- Another Nash equilibria, such that the strategies of all the retailers are continuous functions in the relative manufacturer strategy, is the strategy profile:

$$\boxed{\left(\frac{2}{5}, \frac{2}{5}, \tilde{\beta}_1, \tilde{\beta}_2\right), \qquad \text{s.t.} \qquad \tilde{\beta}_i(a_i) = \frac{2}{5} + \frac{a_i}{2}} \tag{4.7}$$

  - $\tilde{\beta}_i$ solves (4.5) when $a_i^* = a_j^* = \frac{2}{5}$, $i, j = 1, 2$ and $j \neq i$.
  - (4.2a). We have:

$$\bar{l}_i(a_i) = l_i(a_i, \tilde{\beta}_i(a_i), \tilde{\beta}_j(\frac{2}{5})) = \frac{4}{5}a_i - a_i^2$$

  that assumes its maximum value for $a_i = \frac{2}{5}$.

As this simple example shows, it is possible that we have an infinity of equilibria. We cannot refine using the concept of subgame perfect equilibrium because the two–stage game has no proper subgames. So, we consider a selection criterion that use the concept of *passive beliefs* that is a particular type of perfect Bayesian equilibrium, as we will see in the next section.

As emphasized in McAfee and Schwartz [59], the concept of passive beliefs is invoked implicitly or explicitly by many authors (see, for example, Crémer and Riordan [18], Hart et al. [31], O'Brien and Shaffer [68], McAfee and Schwartz [59], Laffont and Martimort [42], Gavazza and Lizzeri [27], Bar-Isaac et al. [3]).

Definition 4.1.1

A strategy profile $(a_1^*, a_2^*, \beta_1^*, \beta_2^*)$ is an *equilibrium with passive beliefs* of the two–stage game previously defined if

(i) for $i, j = 1, 2$ and $j \neq i$, the strategy of R_i is a optimal reaction to the strategy a_i of M_i, assuming that leader M_j always chooses the equilibrium strategy; that is, for any a_i, $\beta_i^*(a_i)$ is the solution of the maximization problem:

$$\operatorname*{Arg\,max}_{b_i \in B_i} f_i(a_i, b_i, b_j^*). \tag{4.8}$$

(ii) The strategy a_i^* maximizes the payoff function of M_i assuming that R_i replies in accordance with β_i^* and that M_j always chooses the equilibrium strategy; that is:

$$a_i^* \in \operatorname*{Arg\,max}_{a_i \in A_i} \left[l_i(a_i, \beta_i^*(a_i), b_j^*) \right].$$

Going back to the example, the maximization problem in (4.8) admits a unique solution $\beta_i^*(a_i)$ given by:

$$\beta_i(a_i) = \frac{1}{4} + \frac{1}{4} b_j^* + \frac{1}{2} a_i, \tag{4.9}$$

for any $a_i \in A_i$. Then, substituting (4.9) into the leader's problem, we have:

$$a_i^* \in \operatorname*{Arg\,max}_{a_i} \left[\left(\frac{1}{2} + \frac{b_j^*}{2} \right) a_i - a_1^2 \right],$$

that is:

$$a_i^* = \frac{1}{4} + \frac{b_j^*}{4}. \tag{4.10}$$

The system:

$$\begin{cases} a_1^* = \dfrac{1}{4} + \dfrac{b_2^*}{4} \\ a_2^* = \dfrac{1}{4} + \dfrac{b_1^*}{4} \\ b_1^* = \dfrac{1}{4} + \dfrac{1}{4}b_2^* + \dfrac{1}{2}a_1^* \\ b_2^* = \dfrac{1}{4} + \dfrac{1}{4}b_1^* + \dfrac{1}{2}a_2^*. \end{cases}$$

has a unique solution $(a_1^*, a_2^*, b_1^*, b_2^*) = (2/5, 2/5, 3/5, 3/5)$ and the unique equilibrium with passive beliefs is the strategy profile

$$\boxed{\left(\frac{2}{5}, \frac{2}{5}, \beta_1^*, \beta_2^*\right), \qquad \text{with } \beta_i^*(a_i) = \frac{2}{5} + \frac{a_i}{2}, \text{ for } i = 1, 2.} \tag{4.11}$$

In the previous example, the assumption of passive beliefs allows to refine the Nash equilibria concept of the two–stage game with continuous action spaces when the two–stage game has no proper subgame.

4.2 General two–stage games with vertical separation

First, in order to extend the concept considered in Section 4.1, we define a general two–stage game Γ with vertical separation in which at any stage a finite number of agents competes. In the first stage k players, called leaders L_i, $i = 1, \ldots, k$, choose simultaneously an action a_i in A_i, nonempty subset of the finite–dimensional Euclidean space $\mathbb{R}^{n_i}$; in the second stage k players, called followers F_i $i = 1, \ldots, k$, choose simultaneously an action b_i in the set B_i, nonempty subset of the finite–dimensional Euclidean space $\mathbb{R}^{m_i}$. Let us denote $A = \prod_{i=1}^k A_i$ and $A_{-i} = \prod_{\substack{r=1 \\ r\neq i}}^k A_r$. An element of A_{-i} is denoted by

$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$. Analogously, we adopt the same notations for the sets associated to the follower's problem.

In a two–stage game *with observed actions* at the end of the first stage any player observes the action chosen by all the leaders. Such games are also called *games of almost–perfect information* (see Fudenberg and Tirole [26]) meaning that there is imperfect information if there are agents that play simultaneously within a stage. So the followers know all the past history at the moment they have to play: an action of a follower may depend on the actions taken by all the leaders. Differently, we will consider a two–stage game in which the actions taken by any leader is not observable by all the followers in stage two. First, let us specify the definition of some crucial concepts.

Definition 4.2.1
A two–stage game is said to be *with partially observed actions* if the action chosen by each leader L_i is observed only by the corresponding follower F_i, for $i = 1, \ldots, k$.

So, in the case of partially observed actions, each follower's choice depends only on the action chosen by his corresponding leader, that is a follower has as many information sets as the number of actions of his corresponding leader. Then, a strategy of follower F_i is a function β_i from A_i to B_i, that is $\beta_i \in S_i = (B_i)^{A_i}$. Let $S = \prod_{i=1}^{k} S_i$ be the set of all (pure) strategy profiles of the followers.

Definition 4.2.2
A two–stage game is said to be *with vertical separation* if the action of any leader L_i affects only the payoff of L_i and F_i, for $i = 1, \ldots, k$.

Vertical separation expresses the idea that a leader can have an exclusive follower. So, for the remainder of the chapter we will consider a game with vertical separation and partially observed action. For sake of brevity, we just call it a two–stage game with vertical separation.

Furthermore, we assume that the leaders' payoff functions can depend explicitly on the follower optimal value function, that is the optimal payoff of the follower when he is reacting optimally to any given action of his corresponding leader. This can be viewed as an altruistic/spiteful behaviour, depending on the way the optimal value function of a follower affects his leader's payoff. This assumption is compatible with the fact that if a follower is an exclusive retailer of the good produced by the leader, the latter can take into account the profit of his retailer (possibly in a percentage term) when he has to decide about the strategy to play. Moreover, in an engineering context, the class of problems that can be modeled in this way are the so–called *parameter design problem and the resource allocation problem for decentralized systems* (Shimizu and Ishizuka [81], Shimizu et al. [82]): a central system (the leader) decides about the value of a parameter which is assigned to a subsystem or regional systems (the followers) and optimizes the central objective function which depends from the value of optimized subsystems' performances ([81]).

The objective of each player is to maximize his own payoff function, taking into account that it depends also on the strategies of the other players. The payoff function of follower F_i is a real–valued function f_i defined on $A_i \times B$. Let v_i the optimal value function defined of follower F_i defined on $A_i \times B_{-i}$ by:

$$v_i(a_i, b_{-i}) = \sup_{b_i \in B_i} f_i(a_i, b_i, b_{-i}) \tag{4.12}$$

For the sake of simplicity, we assume that $v_i(a_i, b_{-i})$ is finite, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$. For any $i = 1, \dots, k$, leader L_i's payoff when he plays the action $a_i \in A_i$, given an action profile (b_i, b_{-i}) of the followers, is $l_i(a_i, b_i, b_{-i}, v_i(a_i, b_{-i}))$. Then, the payoff function of leader L_i is a function l_i from the set $A_i \times B \times \mathbb{R}$ to $\mathbb{R}$.

When each player has only the choice between two actions the game can be represented by the extensive form given in Figure 4.1.

Now, let $(a, \beta) = (a_1, \dots, a_k, \beta_1, \dots, \beta_k) \in A \times S$. Denote with $\beta(a)$ the

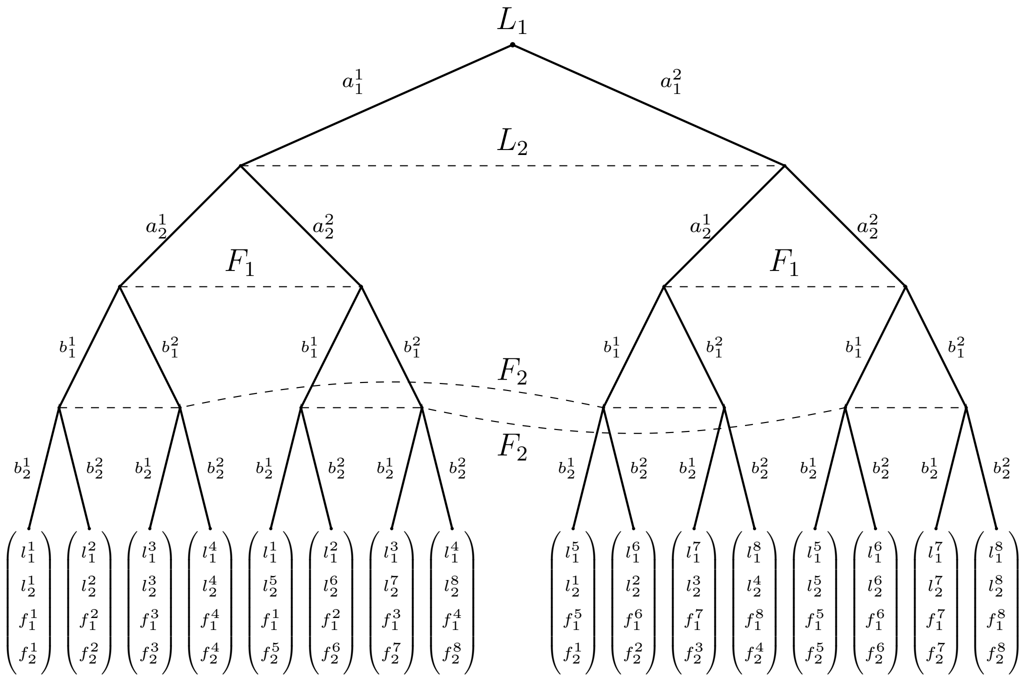

Figure 4.1: Representation of the extensive form game with a finite set of actions in all the information sets

action profile $(\beta_1(a_1), \dots, \beta_k(a_k))$ and denote with $\beta_{-i}(a_{-i})$ the action profile $(\beta_1(a_1), \dots, \beta_{i-1}(a_{i-1}), \beta_{i+1}(a_{i+1}), \dots, \beta_k(a_k))$. The payoff functions in the associate normal form game are:

$$\begin{aligned} \hat{l}_i(a, \beta) &= l_i(a_i, \beta(a), v_i(a_i, \beta_{-i}(a_{-i}))), \text{ for leader } L_i, \\ \hat{f}_i(a, \beta) &= f_i(a_i, \beta(a)), \text{ for follower } F_i, \end{aligned} \tag{4.13}$$

for $i = 1, \dots, k$. Let Γ be the normal form of the two–stage game with vertical separation

$$\Gamma = \{(L_i)_{i=1,\dots,k}\,, (F_i)_{i=1,\dots,k}\,; (A_i)_{i=1,\dots,k}\,, (S_i)_{i=1,\dots,k}\,; (\hat{l}_i)_{i=1,\dots,k}, (\hat{f}_i)_{i=1,\dots,k}\}. \tag{4.14}$$

Then, applying the well–known concept of Nash equilibrium ([67]) to game Γ, we have:

Definition 4.2.3

A *Nash equilibrium* of the game Γ is a $2k$-tuple $(a^*, \beta^*) \in A \times S$ satisfying, for any $i = 1, \dots, k$, the following inequalities:

$$\hat{l}_i(a^*, \beta^*) \geq \hat{l}_i(a_i, a^*_{-i}, \beta^*) \quad \text{for any } a_i \in A_i \tag{4.15a}$$

$$\hat{f}_i(a^*, \beta^*) \geq \hat{f}_i(a^*, \beta_i, \beta^*_{-i}) \quad \text{for any } \beta_i \in S_i. \tag{4.15b}$$

Let $i = 1, \dots, k$. From (4.13), conditions (4.15) are equivalent to

$$l_i(a^*_i, \beta^*_i(a^*_i), \beta^*_{-i}(a^*_{-i}), v_i(a^*_i, \beta^*_{-i}(a^*_{-i}))) \geq \\ l_i(a_i, \beta^*_i(a_i), \beta^*_{-i}(a^*_{-i}), v_i(a_i, \beta^*_{-i}(a^*_{-i}))) \quad \text{for any } a_i \in A_i \tag{4.16a}$$

and

$$f_i(a^*_i, \beta^*_i(a^*_i), \beta^*_{-i}(a^*_{-i})) \geq f_i(a^*_i, \beta_i(a^*_i), \beta^*_{-i}(a^*_{-i})) \quad \text{for any } \beta_i \in S_i;$$

where the last condition is satisfied if:

$$f_i(a^*_i, \beta^*_i(a^*_i), \beta^*_{-i}(a^*_{-i})) \geq f_i(a^*_i, b_i, \beta^*_{-i}(a^*_{-i})) \quad \text{for any } b_i \in B_i. \tag{4.16b}$$

Conditions (4.16) could generate an infinity of equilibria, provided that an equilibrium exists. So, let us first give an existence result for a Nash equilibrium of the game Γ.

4.3 Existence of Nash Equilibria

Let us observe that the existence of a Nash equilibrium of the game

$$\Gamma = \{(L_i)_{i=1,\dots,k}\,, (F_i)_{i=1,\dots,k}\,; (A_i)_{i=1,\dots,k}\,, (S_i)_{i=1,\dots,k}\,; (\hat{l}_i)_{i=1,\dots,k}, (\hat{f}_i)_{i=1,\dots,k}\}$$

is ensured when there exists an equilibrium of the game

$$\Gamma' = \{(L_i)_{i=1,\dots,k}\,, (F_i)_{i=1,\dots,k}\,; (A_i)_{i=1,\dots,k}\,, (B_i)_{i=1,\dots,k}\,; (l_i)_{i=1,\dots,k}\,, (f_i)_{i=1,\dots,k}\}.$$

Indeed, let $(\bar{a}, \bar{b}) = (\bar{a}_1, \dots, \bar{a}_k, \bar{b}_1, \dots, \bar{b}_k)$ be a Nash equilibrium of Γ' and let $(\bar{a}, \bar{\beta}) = (\bar{a}_1, \dots, \bar{a}_k, \bar{\beta}_1, \dots, \bar{\beta}_k)$ such that $\bar{\beta}_i(a_i) = \bar{b}_i$, for any $a_i \in A_i$, for $i = 1, \dots, k$. Rewriting conditions (4.16) for $(\bar{a}, \bar{\beta})$ we obtain the two conditions:

$$l_i(\bar{a}_i, \bar{b}_i, \bar{b}_{-i}, v_i(\bar{a}_i, \bar{b}_{-i})) \geq l_i(a_i, \bar{b}_i, \bar{b}_{-i}, v_i(a_i, \bar{b}_{-i})) \quad \text{for any } a_i \in A_i$$
$$f_i(\bar{a}_i, \bar{b}_i, \bar{b}_{-i}) \geq f_i(\bar{a}_i, b_i, \bar{b}_{-i}) \quad \text{for any } b_i \in B_i,$$

that are satisfied being $(\bar{a}, \bar{b})$ a Nash equilibrium of Γ'. Then, $(\bar{a}, \bar{\beta})$ is a Nash equilibrium of Γ.

Sufficient conditions for the existence of an equilibrium of Γ' are given in the following theorem.

Theorem 4.3.1. *Assume, for $i = 1, \dots, k$:*

($\mathcal{A}_{L_i}$ 1) A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 2) l_i is a real–valued upper semicontinuous function on $A_i \times B \times \mathbb{R}$;

($\mathcal{A}_{L_i}$ 3) for any $(a_i, b_i, b_{-i}, t) \in A_i \times B \times \mathbb{R}$, for any sequence $((b_{i,n}, b_{-i,n}, t_n))_n$ converging to (b_i, b_{-i}, t) in $B_i \times B_{-i} \times \mathbb{R}$, there exists a sequence $(\breve{a}_{i,n})_n$ in A_i such that:

$$\liminf_{n \to \infty} l_i(\breve{a}_{i,n}, b_{i,n}, b_{-i,n}, t_n) \geq l_i(a_i, b_i, b_{-i}, t);$$

($\mathcal{A}_{L_i}$ 4) $l_i(\cdot, b, \cdot)$ is concave on $A_i \times \mathbb{R}$ for any $b \in B$ and $l_i(a_i, b, \cdot)$ is nondecreasing on $\mathbb{R}$, for any $(a_i, b) \in A_i \times B$;

($\mathcal{A}_{F_i}$ 1) B_i is a nonempty compact convex subset of a $\mathbb{R}^{m_i}$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $(a_{i,n}, b_{-i,n})_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n \to \infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i});$$

($\mathcal{A}_{F_i}$ 4) $f_i(a_i, \cdot, b_{-i})$ is concave, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$;

Then there exists a Nash equilibrium of Γ' and, therefore, an equilibrium of Γ.

Proof. Let $i = 1, \dots, k$. Given a strategy profile (a, b), we define the set–valued map $\tilde{B}_i$ as the optimal reaction of follower F_i, that is:

$$\tilde{B}_i(a_i, b_{-i}) = \operatorname*{Arg\,max}_{b_i \in B_i} f_i(a_i, b_i, b_{-i}) \tag{4.17}$$

for any $(a_i, b_{-i}) \in A_i \times B_{-i}$ and the set–valued map $\tilde{C}_i$ as the optimal reaction of leader L_i, that is:

$$\tilde{C}_i(b) = \operatorname*{Arg\,max}_{a_i \in A_i} l_i(a_i, b_i, b_{-i}, v_i(a_i, b_{-i}))$$

for any $b \in B$.

- From $(\mathcal{A}_{F_i}\ 1)$ and $(\mathcal{A}_{F_i}\ 2)$, we have that $\tilde{B}_i$ is nonempty–valued and the optimal value function v_i is defined on $A_i \times B_{-i}$.
- The optimal reaction of follower F_i is a closed set–valued map as a consequence of Proposition 2.3.5. For the sake of completeness, let us detail here the proof.

  Let $(a_i, b_{-i}) \in A_i \times B_{-i}$ and let $\left((a_{i,n}, b_{-i,n})\right)_n$ a sequence converging to (a_i, b_{-i}) in $A_i \times B_{-i}$. Let $\bar{b}_i \in \operatorname{Lim\,sup}_{n \to +\infty} \tilde{B}_i(a_{i,n}, b_{-i,n})$. Then, there exists a subsequence $(\bar{b}_{i,n})_{n \in N_1}$, with $N_1 \subseteq \mathbb{N}$ countably infinite, such that $\bar{b}_{i,n} \in \tilde{B}_i(a_{i,n}, b_{-i,n})$ for any $n \in N_1$ and such that $\lim_{\substack{n \to +\infty \\ n \in N_1}} \bar{b}_{i,n} = \bar{b}_i$.
  We claim that $\bar{b}_i \in \tilde{B}_i(a_i, b_{-i})$. Indeed, let $b_i \in B_i$. From $(\mathcal{A}_{F_i}\ 3)$ we have that there exists a sequence $(\hat{b}_{i,n})_n$ such that

  $$\liminf_{n \to +\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i}). \tag{4.18}$$

  Using $(\mathcal{A}_{F_i}\ 2)$ with the subsequence $\left(\bar{b}_{i,n}\right)_n$ we have:

  $$\limsup_{\substack{n \to +\infty \\ n \in N_1}} f_i(a_{i,n}, \bar{b}_{i,n}, b_{-i,n}) \leq f_i(a_i, \bar{b}_i, b_{-i}). \tag{4.19}$$

  Furthermore, from the definition of $\tilde{B}_i$ we get:

  $$f_i(a_{i,n}, \bar{b}_{i,n}, b_{-i,n}) \geq f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \qquad \text{for any } n \in N_1. \tag{4.20}$$

  Thus, from (4.18)–(4.20) we have $f_i(a_i, b_i, b_{-i}) \leq f_i(a_i, \bar{b}_i, b_{-i})$ for all $b_i \in B_i$; that is $\bar{b}_i \in B_i$.

- v_i is a continuous function on $A_i \times B_{-i}$. It is a consequence of Proposition 2.3.3– 2.3.4. For the sake of completeness, we report here the proof. Let $(a_i, b_{-i}) \in A_i \times B_{-i}$ and a sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$. From the definition of maximum, for any $\epsilon > 0$ there exists $b_i^\epsilon \in B_i$ such that
$$f_i(a_i, b_i^\epsilon, b_{-i}) \geq v_i(a_i, b_{-i}) - \epsilon.$$
From ($\mathcal{A}_{F_i}$ *3*), there exists a sequence $\left(b_{i,n}^\epsilon\right)_n$ in B_i such that
$$\liminf_{n\to+\infty} f_i(a_{i,n}, b_{i,n}^\epsilon, b_{-i,n}) \geq f_i(a_i, b_i^\epsilon, b_{-i}).$$
Therefore, using the fact that $v_i(a_{i,n}, b_{-i,n}) \geq f_i(a_{i,n}, b_{i,n}^\epsilon, b_{-i,n})$ for any $n \in \mathbb{N}$, we obtain that, for any $\epsilon > 0$, $\liminf_{n\to+\infty} v_i(a_{i,n}, b_{-i,n}) \geq v_i(a_i, b_{-i}) - \epsilon$, that is, taking the limit on ϵ:
$$\liminf_{n\to+\infty} v_i(a_{i,n}, b_{-i,n}) \geq v_i(a_i, b_{-i}); \tag{4.21}$$
that is, the sequence $(v_i(a_{i,n}, b_{-i,n}))_n$ is bounded from below. Then, there exists a subsequence $(v_i(a_{i,n}, b_{-i,n}))_{n\in N_1}$ such that $N_1 \subseteq \mathbb{N}$ countably infinite and
$$\lim_{\substack{n\to+\infty \\ n\in N_1}} v_i(a_{i,n}, b_{-i,n}) = v_i^* \in \mathbb{R} \cup \{+\infty\}. \tag{4.22}$$
From the fact that $\tilde{B}_i$ is nonempty valued, there exists a sequence $(\tilde{b}_{i,n})_{n\in N_1}$ such that $\tilde{b}_i \in \tilde{B}_i(a_{i,n}, b_{-i,n})$, for any $n \in N_1$, that is
$$f_i(a_{i,n}, \tilde{b}_{i,n}, b_{-i,n}) = v_i(a_{i,n}, b_{-i,n}), \quad \text{for any } n \in N_1. \tag{4.23}$$
From the compactness of B_i there exists a subsequence $(\tilde{b}_{i,n})_{n\in N_2}$, with $N_2 \subseteq N_1$ countably infinite, such that $\tilde{b}_{i,n}$ converges to a point $\tilde{b}_i$ in B_i. Hence, $\tilde{b}_i \in \operatorname{Lim\,sup}_{n\to+\infty} \tilde{B}_i(a_{i,n}, b_{-i,n})$ and, from the closedness of $\tilde{B}_i$, it follows that $\tilde{b}_i \in \tilde{B}_i(a_i, b_{-i})$, that is:
$$f_i(a_i, \tilde{b}_i, b_{-i}) = v_i(a_i, b_{-i}). \tag{4.24}$$
Therefore, from ($\mathcal{A}_{F_i}$ *2*), we have
$$\limsup_{\substack{n\to+\infty \\ n\in N_2}} f_i(a_{i,n}, \tilde{b}_{i,n}, b_{-i,n}) \leq f_i(a_i, \tilde{b}_i, b_{-i})$$

that is, using (4.23)–(4.24):

$$\limsup_{\substack{n\to+\infty \\ n\in N_2}} v_i(a_{i,n}, b_{-i,n}) \le v_i(a_i, b_{-i}).$$

So, from (4.22), we have

$$v_i^* \le v_i(a_i, b_{-i}),$$

that is $v_i^* \in \mathbb{R}$.

Conversely, from (4.22) and the fact that

$$\liminf_{\substack{n\to+\infty \\ n\in N_2}} v_i(a_{i,n}, b_{-,n}) \ge \liminf_{n\to+\infty} v_i(a_{i,n}, b_{-,n}),$$

we obtain $v_i^* \ge v_i(a_i, b_{-i})$, that is

$$v_i^* = v_i(a_i, b_{-i}). \tag{4.25}$$

Condition (4.25) is true for any converging subsequence of $(v_i(a_{i,n}, b_{-i,n}))_n$, then it holds for the entire sequence, that is

$$\lim_{n\to+\infty} v_i(a_{i,n}, b_{-i,n}) = v_i(a_i, b_{-i})$$

that is v_i is a continuous function.

- $\tilde{B}_i$ is convex–valued. Indeed, it follows from ($\mathcal{A}_{F_i}$ *4*) and Proposition 2.1.3.
- The function $\check{l}_i$ defined on $A_i \times B$ by:

$$\check{l}_i(a_i, b_i, b_{-i}) = l_i(a_i, b_i, b_{-i}, v_i(a_i, b_{-i})), \tag{4.26}$$

is upper semicontinuous on $A_i \times B_i \times B_{-i}$ as a consequence of ($\mathcal{A}_{L_i}$ *2*) and the continuity of v_i on $A_i \times B_{-i}$ previously proved. So, from ($\mathcal{A}_{L_i}$ *1*), we deduce that $\tilde{C}_i$ is nonempty–valued. Furthermore, from the continuity of v_i on $A_i \times B_{-i}$ and ($\mathcal{A}_{L_i}$ *3*), we have that for any $(a_i, b) \in A_i \times B$ and for any sequence $(b_{i,n})_n$ converging to b_i there exists a sequence $(a_{i,n})_n$ such that

$$\liminf_{n\to+\infty} \check{l}_i(a_{i,n}, b_{i,n}, b_{-i,n}) \ge \check{l}_i(a_i, b_i, b_{-i}).$$

Then, replicating the proof in the first point, we have that $\tilde{C}_i$ is a closed set–valued map.

- $\tilde{C}_i$ is convex–valued. Indeed, from Theorem 2.3.6 the function $v_i(\cdot, b_{-i})$ is concave on A_i. Then, from Proposition 2.1.25 we have $\check{l}_i$ quasiconcave on A_i and, from Proposition 2.1.3, $\tilde{C}_i(b_{-i})$ is a convex set.

- Finally, the set–valued map D defined on $A \times B$ by

$$D(a_1, \dots, a_k, b_1, \dots, b_k) = \prod_{i=1}^{k} C_i(b_1 \dots, b_k) \times \prod_{i=1}^{k} \tilde{B}_i(a_i, b_{-i})$$

  satisfies the hypothesis of Kakutani Fixed Point Theorem (Theorem 2.2.14). So, there exists a fixed point of D on $A \times B$, that is an equilibrium of Γ'.

$\diamond$

Remark 4.3.2 Let $i = 1, \dots, k$. If f_i is a continuous function, then assumptions ($\mathcal{A}_{F_i}$ *2*), ($\mathcal{A}_{F_i}$ *3*) are satisfied. The vice versa is not true, as illustrated in the following example. The same arguments hold for leaders' payoff functions.

Example 4.3.1 Let $A_i = B_i = [0, 1]$, for $i = 1, 2$. Let f_i be a real–valued function defined on $[0, 1]^3$ by:

$$f_i(a_i, b_i, b_j) = \begin{cases} -[b_i^2 + 1 + a_i b_j] & \text{if } a_i \in]0, 1], b_i \in [0, 1] \setminus \{\frac{a_i}{4} + \frac{b_j+1}{4}\}, \\ 0 & \text{if } a_i = 0, b_i \in [0, \frac{b_j+1}{4}[, \\ 1 - \frac{1}{2} b_i & \text{if } a_i = 0, b_i \in]\frac{b_j+1}{4}, 1], \\ 2 & \text{if } a_i \in [0, 1], b_i = \frac{a_i}{4} + \frac{b_j+1}{4} \end{cases}$$

with $i, j = 1, 2$, $j \neq i$. f_i is not continuous but it satisfies ($\mathcal{A}_{F_i}$ *2*)–($\mathcal{A}_{F_i}$ *3*).

($\mathcal{A}_{F_i}$ *2*) - Let $a_i = 0$, $b_i \in [0, \frac{b_j+1}{4}[$, $b_j \in [0, 1]$ and let $((a_{i,n}, b_{i,n}, b_{j,n}))_n$ a sequence in $[0, 1]^3$ converging to (a_i, b_i, b_j). Then, any converging subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}))_n$ can converge only to $-(b_i^2 + 1)$ or to $0 = f((a_i, b_i, b_j))$.

- Let $a_i = 0$, $b_i \in]\frac{b_j+1}{4}, 1]$, $b_2 \in [0, 1]$ and let $((a_{i,n}, b_{i,n}, b_{j,n}))_n$ a sequence in $[0, 1]^3$ converging to (a_i, b_i, b_j). Then, any converging subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}))_n$ can converge only to $-(b_i^2 + 1)$ or to $1 - \frac{1}{2} b_i = f((a_i, b_i, b_j))$.

- Let $a_i = 0$, $b_i = \frac{b_j+1}{4}$, $b_j \in [0,1]$ and let $((a_{i,n}, b_{i,n}, b_{j,n}))_n$ a sequence in $[0,1]^3$ converging to (a_i, b_i, b_j). Then, any converging subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}))_n$ can converge to $-(b_i^2+1)$, 0, $1-\frac{1}{2}b_j$, or to $2 = f((a_i, b_i, b_j))$.
- Let $a_i \in]0,1]$, $b_i = \frac{a_i}{4} + \frac{b_j+1}{4}$, $b_j \in [0,1]$ and let $((a_{i,n}, b_{i,n}, b_{j,n}))_n$ a sequence in $[0,1]^3$ converging to (a_i, b_i, b_j). Then, any converging subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}))_n$ can converge only to $-(b_i^2+1+a_i b_j)$ or to $2 = f((a_i, b_i, b_j))$.

In any case ($\mathcal{A}_{F_i}$ *2*) is satisfied.

($\mathcal{A}_{F_i}$ *3*) Let $(a_i, b_i, b_j) \in [0,1]^3$ and let $((a_{i,n}, b_{j,n}))_n$ a sequence converging to (a_i, b_j) in $[0,1]^2$. Then, taken $b_{i,n} = \frac{a_{i,n}}{4} + \frac{b_{j,n}+1}{4}$, for any $n \in \mathbb{N}$, we have that

$$\liminf_{n \to +\infty} f_i(a_{i,n}, b_{i,n}, b_{j,n}) = f_i(a_i, b_i, b_j);$$

that is ($\mathcal{A}_{F_i}$ *3*) holds with the equality sign.

$\diamond$

Remark 4.3.3 Let $i = 1, \ldots, k$. When l_i does not depend from v_i, Theorem 4.3.1 still holds if we require the quasiconcavity of $f_i(a_i, \cdot, b_{-i})$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$, instead of assumption ($\mathcal{A}_{F_i}$ *4*). Moreover, assumption ($\mathcal{A}_{L_i}$ *4*) can be weaken requiring $l_i(\cdot, b)$ quasiconcave on A_i, for any $b \in B$.

In order to reduce the set of Nash equilibria, we will consider in the next section a selection criterion generalizing the concept considered in Section 4.1.

4.4 Equilibria under passive beliefs: definition and existence result

As anticipated, the unique subgame of the two–stage game with vertical separation presented in Section 4.2 is the whole game, so the set of subgame perfect Nash equilibria coincides with the set of Nash equilibria. A method for selecting among the Nash equilibria is suggested by the concept of *perfect Bayesian equilibrium* (see for example Fudenberg and Tirole [26] or Mas-Colell et al. [58]). Let us extend the definition of perfect Bayesian equilibrium

given in [58] for a multi–stage game with partially observed actions when the sets of actions of the players have cardinality infinite.
Note that we aim to give a definition that is independent from the specific extensive form representation of the two–stage game. Let $\mathfrak{P}_i$ be the set of all the probability measures on A_{-i}, for $i = 1, \dots, k$.

Definition 4.4.1
A *system of beliefs* of F_i is a family of probability measures $\mu_i = (\mu_i^{a_i})_{a_i \in A_i}$ in $\mathfrak{P}_i$ where $\mu_i^{a_i}$ represents the beliefs that F_i has about the actions chosen by leaders L_{-i} after observing an action a_i of leader L_i.
A system of beliefs is a profile $\mu = (\mu_1, \dots, \mu_k)$, such that μ_i is a system of beliefs of F_i, for any $i = 1, \dots, k$.

The expected payoff (see, for example, Billingsley [7]) of follower F_i, if he plays the action b_i as response to the observed action a_i of L_i and to the strategy profile $\beta_{-i} \in S^{-i}$ of followers F_{-i}, given the system of beliefs μ_i of F_i, is:

$$f_i^{\mu_i}(a_i, b_i, \beta_{-i}) = \int_{A_{-i}} f_i(a_i, b_i, \beta_{-i}(a_{-i}))\, d\mu_i^{a_i}(a_{-i}). \tag{4.27}$$

So, we can give the following definition.

Definition 4.4.2
A strategy profile $s^* = (a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ with the system of beliefs $\mu = (\mu_1, \dots, \mu_k)$ is a *perfect Bayesian equilibrium* of the two–stage game with vertical separation if, for any $i = 1, \dots, k$:

(i) any follower F_i is playing optimally, assuming that he is in the second stage and he has the system of beliefs μ_i; that is we have:

$$f_i^{\mu_i}(a_i, \beta_i^*(a_i), \beta_{-i}^*) = \max_{b_i \in B_i} f_i^{\mu_i}(a_i, b_i, \beta_{-i}^*) \quad \text{for any } a_i \in A_i;$$

(ii) the leader's equilibrium action is such that:

$$\check{l}_i(a_i^*, \beta_i^*(a_i^*), \beta_{-i}^*(a_{-i}^*)) = \max_{a_i \in A_i} \check{l}_i(a_i, \beta_i^*(a_i), \beta_{-i}^*(a_{-i}^*)),$$

where $\check{l}_i$ is defined as in (4.26);

(iii) the system of beliefs μ satisfies the following consistency hypothesis:

$$\mu_i^{a_i^*}(a_{-i}) = \begin{cases} 1, & \text{if } a_{-i} = a_{-i}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{4.28}$$

The consistency requirement corresponds to the requirement *(ii)* in the definition of *weak perfect Bayesian equilibrium* in [58, Definition 9.C.3] and it ensures that the followers' beliefs are compatible with the equilibrium strategy profile along the equilibrium path, that is the followers have correct beliefs in equilibrium. Then, a perfect Bayesian equilibrium is a Nash equilibrium. Furthermore, the compatibility requirement does not impose any restriction out of the equilibrium path. So, the concept of perfect Bayesian equilibrium may be not sufficient to select among all Nash equilibria. For this reason we restrict our attention only on the equilibria that are supported by the system of beliefs called *passive beliefs*. A follower of a two–stage game with vertical separation has *passive beliefs* about the action chosen by the leaders L_{-i} if, when he observes an action of leader L_i different from the one he expects in equilibrium, he does not revise his beliefs about the action offered to the rival follower even if his corresponding leader is deviating. That is, given an equilibrium strategy profile $a^* = (a_i^*, a_{-i}^*)$ of leaders, if F_i observes an action different from a_i^*, he believes that the other leaders are still playing a_{-i}^*. Formally:

Definition 4.4.3

The strategy profile $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ is an *equilibrium under passive beliefs* of a two–stage game with vertical separation if it is a perfect Bayesian equilibrium with the system of beliefs $\mu = (\mu_1, \dots, \mu_k)$ defined by:

$$\mu_i^{a_i}(a_{-i}) = \begin{cases} 1, & \text{if } a_{-i} = a_{-i}^* \\ 0, & \text{otherwise,} \end{cases}$$

for any $a_i \in A_i$, for any $i = 1 \dots, k$; that is $\mu_i^{a_i}$ is a *unit mass* at a_{-i}^*, for any

$a_i \in A_i$, for $i = 1, \dots, k$.

Then, a system of passive beliefs for player F_i is independent on the action a_{-i} of leaders L_{-i}.
Since condition (4.27) in case of passive beliefs is equivalent to:

$$f_i^{\mu_i}(a_i, b_i, \beta_{-i}) = f_i(a_i, b_i, \beta_{-i}(a_{-i}^*)), \tag{4.29}$$

that is, the strategies of the followers F_{-i} affect the payoff of follower F_i only through the actions they choose as response to their leaders' equilibrium strategies, a strategy profile $s^* = (a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ is an equilibrium under passive beliefs of the two–stage game with vertical separation if for any $i = 1, \dots, k$:

(i) the strategy β_i^* of F_i solves:

$$f_i(a_i, \beta_i^*(a_i^*), \beta_{-i}^*(a_{-i}^*)) = \max_{b_i \in B_i} f_i(a_i, b_i, \beta_{-i}^*(a_{-i}^*)) \quad \text{for any } a_i \in A_i;$$

(ii) the action a_i^* of L_i solves:

$$\check{l}_i(a_i^*, \beta_i^*(a_i^*), \beta_{-i}^*(a_{-i}^*)) = \max_{a_i \in A_i} \check{l}_i(a_i, \beta_i^*(a_i), \beta_{-i}^*(a_{-i}^*)),$$

where $\check{l}_i$ is defined in (4.26).

In Figure 4.2 is represented the interaction scheme between 2 leaders and 2 followers with vertical separation under passive beliefs.
Taken as given the action of the other followers, leader L_i and the corresponding follower F_i act as a team that solves a parametric Bilevel Optimization problem. In other words, the strategic interaction is between the two teams.

Then, taken as given the action b_{-i} in B_{-i} of followers F_{-i}, a possible strategy profile for team L_i–F_i is a couple that maximizes payoff functions of both L_i and F_i taking into account the hierarchical structure between L_i and F_i. As in (4.17), denote with $\tilde{B}_i(a_i, b_{-i})$ the *follower reaction set* to

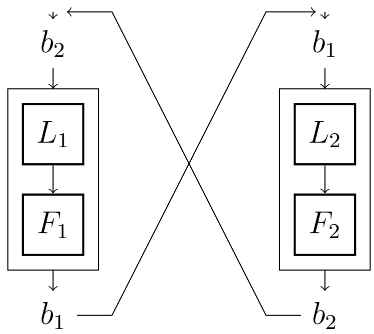

Figure 4.2: Interaction scheme in passive beliefs

the action $a_i \in A_i$ of leader L_i, for any fixed b_{-i} in B_{-i}. That is $\tilde{B}_i(a_i, b_{-i})$ is the set of solutions to the optimization problem:

$$\mathcal{P}_i(a_i, b_{-i}) \qquad \begin{cases} \text{find } b_i \in B_i \text{ such that} \\ f_i(a_i, b_i, b_{-i}) = \max_{b'_i \in B_i} f_i(a_i, b'_i, b_{-i}). \end{cases} \tag{4.30}$$

In the following, for any $i = 1, \ldots, k$, a very useful condition will be:

($\mathcal{I}_i$) *for any* $(a_i, b_{-i}) \in A_i \times B_{-i}$, *the reaction set* $\tilde{B}_i(a_i, b_{-i})$ *of* F_i *is a singleton, that is*

$$\tilde{B}_i(a_i, b_{-i}) = \{\tilde{b}_i(a_i, b_{-i})\}. \tag{4.31}$$

Sufficient conditions for assumption ($\mathcal{I}_i$) are given, for example, by:

Lemma 4.4.4. *Let* $i = 1, \ldots, k$. *Assume that* ($\mathcal{A}_{F_i}$ 1) *and* ($\mathcal{A}_{F_i}$ 2) *hold and* $f_i(a_i, \cdot, b_{-i})$ *is a strictly D–pseudoconcave function on* B_i. *Then* $\mathcal{P}_i(a_i, b_{-i})$ *has a unique solution.*

Proof. See Proposition 2.1.22 and Theorem A.2.5. ◇

Then, we can define the function $\tilde{l}_i$ from $A_i \times B_{-i}$ to $\mathbb{R}$ by:

$$\tilde{l}_i(a_i, b_{-i}) = l_i(a_i, \tilde{b}_i(a_i, b_{-i}), b_{-i}, v_i(a_i, b_{-i})). \tag{4.32}$$

So, the problem of leader L_i is described by:

$$\mathcal{S}_i(b_{-i}) \qquad \begin{cases} \text{find } a_i \in A_i \text{ such that} \\ \tilde{l}_i(a_i, b_{-i}) = \max_{a'_i \in A_i} \tilde{l}_i(a'_i, b_{-i}). \end{cases} \tag{4.33}$$

Definition 4.4.5

A strategy profile $(a^*, \beta^*) = (a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ is an *equilibrium under passive beliefs* of the two–stage game with vertical separation if it solves, for $i = 1, \dots, k$, the problems:

$$a_i^* \text{ solves } S(b_{-i}^*) \tag{4.34a}$$

and

$$\beta_i^*(a_i) = \tilde{b}_i(a_i, b_{-i}^*) \quad \text{for any } a_i \in A_i. \tag{4.34b}$$

where

$$b_{-i}^* = \beta_{-i}^*(a_{-i}^*). \tag{4.34c}$$

In an equivalent way, we can describe a vertical separated equilibrium as a fixed point of an appropriate set–valued map.

Let $X_i = A_i \times B_i$, $X_{-i} = A_{-i} \times B_{-i}$ and $X = A \times B$. Define the set–valued map M_i from B_{-i} to A_i such that

$$M_i(b_{-i}) = \operatorname*{Arg\,max}_{a_i \in A_i} \tilde{l}_i(a_i, b_{-i}), \tag{4.35}$$

for any $b_{-i} \in B_{-i}$. Define the set–valued map N_i from X_{-i} to X_i such that

$$N_i(x_{-i}) = N_i(a_{-i}, b_{-i}) = \{(a_i, b_i) \in X_i \colon a_i \in M_i(b_{-i}), b_i = \tilde{b}_i(a_i, b_{-i})\}, \tag{4.36}$$

for any $x_{-i} = (a_{-i}, b_{-i}) \in X_{-i}$, and the set–valued map

$$N \colon X \rightrightarrows X, \tag{4.37}$$

such that $N(x) = \prod_{i=1}^{k} N_i(x_{-i})$, for any $x \in X$.

It follows that:

Lemma 4.4.6. *Any fixed point of N on X can be associated to an equilibrium under passive beliefs of the two–stage game Γ with vertical separation, and vice versa.*

Proof. Let $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ be an equilibrium under passive beliefs. Then, $\beta_i^*(a_i) = \tilde{b}_i(a_i, b_{-i}^*)$ for any $a_i \in A_i$ and, denoted $\beta_i^*(a_i^*)$ by b_i^*, we have that $x^* = (a_1^*, \dots, a_k^*, b_1^*, \dots, b_k^*)$ is a fixed point of N on X, that is $x^* \in N(x^*)$.
Vice versa, if $(a_1^*, \dots, a_k^*, b_1^*, \dots, b_k^*)$ is a fixed point of N on X, then one can easily verify that the strategy profile $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ such that $\beta_i^*(a_i) = \tilde{b}_i(a_i, \beta_{-i}^*(a_i^*))$, for any $a_i \in A_i$ and $i = 1 \dots, k$ is an equilibrium under passive beliefs of Γ. ◇

Let us study the properties of the set–valued map N.

First, we prove that $\tilde{b}_i$ is a continuous function.

Proposition 4.4.7. *Let $i = 1, \dots, k$. Assume ($\mathcal{I}_i$), ($\mathcal{A}_{F_i}$ 2)–($\mathcal{A}_{F_i}$ 3) and*

($\mathcal{A}_{F_i}$ 5) B_i is a nonempty compact subset of $\mathbb{R}^{m_i}$.

Then, $\tilde{b}_i$ and v_i are continuous functions on $A_i \times B_{-i}$.

Proof. In the proof of Theorem 4.3.1 it is proved that if $(a_{i,n}, b_{-i,n})$ converges to (a_i, b_{-i}) in $A_i \times B_{-i}$ then $\tilde{b}_i$, considered as a set–valued map, is closed. This property, together with the compactness of B_i, gives the continuity of $\tilde{b}_i$ (Proposition 2.2.8–(i)).
The continuity of the optimal value function v_i is proved in Theorem 4.3.1. ◇

Now, we prove that $N(x)$ is nonempty and closed, for any $x \in X$.

Proposition 4.4.8. *Let $i = 1, \dots, k$. Assume ($\mathcal{I}_i$), ($\mathcal{A}_{F_i}$ 5) and*

($\mathcal{A}_{L_i}$ 5) A_i is a nonempty compact subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 6) $l_i(\cdot, \cdot, b_{-i}, \cdot)$ is upper semicontinuous on $A_i \times B_i \times \mathbb{R}$, for any $b_{-i} \in B_{-i}$;

($\mathcal{A}_{F_i}$ 6) $f_i(\cdot, \cdot, b_{-i})$ is upper semicontinuous on $A_i \times B_i$, for any $b_{-i} \in B_{-i}$;

($\mathcal{A}_{F_i}$ 7) for any $(a_i, b_i) \in A_i \times B_i$, for any sequence $(a_{i,n})_n$ converging to a_i in A_i, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i}) \geq f_i(a_i, b_i, b_{-i}),$$

for any $b_{-i} \in B_{-i}$.

Then, $N_i(x_{-i})$ defined in (4.36) is nonempty, for any $x_{-i} \in X_{-i}$.

Proof. It can be proved using Proposition 2.4.2. For the sake of completeness let us prove it directly.
Let $i = 1, \ldots, k$ and $x_{-i} = (a_{-i}, b_{-i}) \in X$. Let us observe that substituting assumptions ($\mathcal{A}_{F_i}$ 2)–($\mathcal{A}_{F_i}$ 3) in Proposition 4.4.7 with weaker assumptions ($\mathcal{A}_{F_i}$ 6)–($\mathcal{A}_{F_i}$ 7) and replicating the proof we can prove that $\tilde{b}_i(\cdot, b_{-i})$ and $v_i(\cdot, b_{-i})$ are continuous on A_i.

Let $(a_{i,n})_n$ be a sequence converging to a_i. From assumption ($\mathcal{A}_{L_i}$ 6) and from the continuity of $\tilde{b}_i(\cdot, b_{-i})$ and $v_i(\cdot, b_{-i})$ we have:

$$\begin{aligned}\limsup_{n\to+\infty} \tilde{l}_i(a_{i,n}, b_{-i}) &= \limsup_{n\to+\infty} l_i(a_{i,n}, \tilde{b}_i(a_{i,n}, b_{-i}), b_{-i}, v_i(a_{i,n}, b_{-i})) \\ &\leq l_i(a_i, \tilde{b}_i(a_i, b_{-i}), b_{-i}, v_i(a_i, b_{-i})) = \tilde{l}_i(a_i, b_{-i}), \qquad (4.38)\end{aligned}$$

that is $\tilde{l}_i(\cdot, b_{-i})$ is upper semicontinuous on A_i. Then, $M_i(b_{-i})$ is nonempty since A_i is compact. So, $N_i(x_{-i})$ is nonempty. ◇

In the next proposition we prove that any set–valued map N_i is closed.

Proposition 4.4.9. *Let $i = 1, \ldots, k$. Assume ($\mathcal{I}_i$), ($\mathcal{A}_{L_i}$ 2), ($\mathcal{A}_{L_i}$ 5), ($\mathcal{A}_{F_i}$ 2), ($\mathcal{A}_{F_i}$ 3), ($\mathcal{A}_{F_i}$ 5), and*

($\mathcal{A}_{L_i}$ 7) $l_i(a_i, \cdot, \cdot, \cdot)$ is lower semicontinuous on $B_i \times B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$.

Then, the set–valued map N_i is closed.

Proof. Let $b_{-i} \in B_{-i}$ and let $(b_{-i,n})_n$ be a sequence converging to b_{-i} in B_{-i}. Let $\bar{a}_i \in \operatorname{Lim\,sup}_{n\to+\infty} M_i(b_{-i,n})$. By definition, there exists a set $K_1 \subseteq \mathbb{N}$

countably infinite and a sequence $(\bar{a}_{i,n})_{n\in K_1}$ such that $\bar{a}_{i,n} \in M_i(b_{-i,n})$ for all $n \in K_1$ and such that $\bar{a}_{i,n}$ converges to $\bar{a}_i$.
Let a'_i be in A_i. From ($\mathcal{A}_{L_i}$ 7) and from the continuity of $\tilde{b}_i(a'_i,\cdot)$ and $v_i(a'_i,\cdot)$ on B_{-i}, we have

$$\tilde{l}_i(a'_i, b_{-i}) \leq \liminf_{n\to+\infty} \tilde{l}_i(a'_i, b_{-i,n}). \tag{4.39}$$

On the other hand, from the upper semicontinuity of l_i and the continuity of both $\tilde{b}_i$ and v_i, we have:

$$\limsup_{\substack{n\to+\infty\\ n\in K_1}} \tilde{l}_i(\bar{a}_{i,n}, b_{-i,n}) \leq \tilde{l}_i(\bar{a}_i, b_{-i}). \tag{4.40}$$

But $\bar{a}_{i,n} \in M_i(b_{-i,n})$ for any $n \in K_1$. Then, we have

$$\tilde{l}_i(\bar{a}_{i,n}, b_{-i,n}) \geq l_i(z_i, b_{-i,n}) \quad \text{for any } z_i \in A_i. \tag{4.41}$$

Putting $z_i = a'_i$ in (4.41), from (4.39) and (4.40) we obtain

$$\begin{aligned} \tilde{l}_i(a'_i, b_{-i}) &= \lim_{n\to+\infty} \tilde{l}_i(a'_i, b_{-i,n}) = \limsup_{\substack{n\to+\infty\\ n\in K_1}} \tilde{l}_i(a'_i, b_{-i,n}) \\ &\leq \limsup_{\substack{n\to+\infty\\ n\in K_1}} \tilde{l}_i(\bar{a}_{i,n}, b_{-i,n}) \leq \tilde{l}_i(\bar{a}_i, b_{-i}), \end{aligned} \tag{4.42}$$

that is $\bar{a}_i$ is in $M_i(b_{-i})$. Thus, it is proved that

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_i(b_{-i,n}) \subseteq M_i(b_{-i}), \tag{4.43}$$

that is M_i is a closed set–valued map. Then, from the continuity of $\tilde{b}_i$, we obtain that N_i, is a closed set–valued map. ◇

Remark 4.4.10 If l_i and f_i are continuous in $A_i \times B_i$, then the assumptions on the payoff functions in Proposition 4.4.9 are satisfied. Vice versa there are functions that satisfy the hypothesis in Proposition 4.4.9 that are not continuous. An example for the problem of the followers is in Example 4.3.1. In the following example we discuss the problem of the leaders.

Example 4.4.1 Let $A_i = B_i = [0,1]$, for $i = 1,2$. Let l_i be a real–valued function defined on $[0,1]^3 \times \mathbb{R}$ by:

$$l_i(a_i, b_i, b_j, t) = \begin{cases} (b_i + b_j - a_i + t)^2 & \text{if } a_i \in]0,1], \\ (b_i + b_j + t)^2 + 1 & \text{if } a_i = 0. \end{cases}$$

with $i, j = 1,2$, $j \neq i$. l_i is not continuous but it satisfies ($\mathcal{A}_{L_i}$ 2)–($\mathcal{A}_{L_i}$ 7)

($\mathcal{A}_{L_i}$ 2) l_i is discontinuous at any point $(a_i, b_i, b_j, t) \in \{0\} \times [0,1]^2 \times \mathbb{R}$. Then, let $b_i, b_j, t \in [0,1]^2 \times \mathbb{R}$ and $(a_{i,n}, b_{i,n}, b_{j,n}, t_n)_n$ be a sequence converging to $(0, b_i, b_j, t)$ in $[0,1]^3 \times \mathbb{R}$. Any converging subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}, t_n))_n$ converges to $(b_i + b_j + t)^2$ or to $(b_i + b_j + t)^2 + 1 = f(0, b_i, b_j)$. Then, l_i is upper semicontinuous on $[0,1]^3$.

($\mathcal{A}_{L_i}$ 7) It is straightforward.

$\diamond$

The next theorem gives a sufficient condition for the existence of an equilibrium under passive beliefs.

Theorem 4.4.11. *Assume, for $i = 1, \dots, k$:*

($\mathcal{I}_i$) the reaction set $\tilde{B}_i(a_i, b_{-i})$ of F_i is a singleton, that is $\tilde{B}_i(a_i, b_{-i}) = \{\tilde{b}_i(a_i, b_{-i})\}$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$;

($\mathcal{A}_{L_i}$ 1) A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 2) l_i is a real–valued upper semicontinuous function on $A_i \times B \times \mathbb{R}$;

($\mathcal{A}_{L_i}$ 7) $l_i(a_i, \cdot, \cdot, \cdot)$ is lower semicontinuous on $B_i \times B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$;

($\mathcal{A}_{F_i}$ 1) B_i is a nonempty compact convex subset of $\mathbb{R}^{m_i}$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such

that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i}).$$

Furthermore, assume that

($\mathcal{A}$ 1) the set valued function N is convex–valued.

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ with vertical separation.

Proof. The set–valued map N satisfies the hypothesis of Kakutani Fixed Point Theorem (Theorem 2.2.14). Indeed, from Proposition 4.4.8 and assumption ($\mathcal{A}$ 1), N is nonempty convex–valued and, from Proposition 4.4.9, N is closed. Then, from Lemma 4.4.6 we have the thesis. ◇

The main problem for the existence of an equilibrium under passive beliefs is associated to the possible lack of convexity of the values of the set–valued map N. The next proposition gives sufficient conditions for the convexity of the values of N.

Proposition 4.4.12. *Let $i = 1, \ldots, k$. Let A_i and B_i be convex subset of $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively. Assume ($\mathcal{I}_i$) and:*

($\mathcal{A}_{L_i}$ 8) $l_i(\cdot, \cdot, b_{-i}, \cdot)$ is quasiconcave on $A_i \times B_i \times \mathbb{R}$, for any $b_{-i} \in B_{-i}$ and $l_i(a_i, b_i, b_{-i}, \cdot)$ is nondecreasing (nonincreasing) on $\mathbb{R}$;

($\mathcal{A}_{F_i}$ 8) $\tilde{b}_i(\cdot, b_{-i})$ is linear on A_i, for any $b_{-i} \in B_{-i}$, and the optimal value function $v_i(\cdot, b_{-i})$ is concave (convex) on A_i, for any $b_{-i} \in B_{-i}$.

Then the set–valued map N_i is convex–valued.

Proof. Let $x_{-i} \in X_{-i}$ and $(a_i', b_i'), (a_i'', b_i'') \in N_i(x_{-i})$. By definition, we have a_i', $a_i'' \in M_i(b_{-i})$ and $b_i' = \tilde{b}_i(a_i', b_{-i})$, $b_i'' = \tilde{b}_i(a_i'', b_{-i})$. Let $\lambda \in]0,1[$. From the linearity of $\tilde{b}_i(\cdot, b_{-i})$ on A_i it follows that $\lambda b_i' + (1-\lambda) b_i'' = \tilde{b}_i(\lambda a_i' + (1-\lambda) a_i'', b_{-i})$.
Furthermore, from Proposition 2.1.27 we have that $\tilde{l}_i(\cdot, b_{-i})$ is quasiconcave

on A_i. So, from Proposition 2.1.3, we have $\lambda a'_i+(1-\lambda)a''_i \in M_i(b_{-i})$. Finally, from the linearity of $\tilde{b}_i(\cdot, b_{-i})$, we have that $N_i(x_{-i})$ is a convex set. Then, N is a convex–valued map. $\diamond$

A sufficient condition in order to have $\tilde{b}_i$ linear and v_i concave is given by:

Proposition 4.4.13. *Let $i = 1, \dots, k$. Let A_i and B_i be strictly convex (resp. convex) subset of $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively. Assume ($\mathcal{I}_i$) and:*

($\mathcal{A}_{F_i}$ 9) $f_i(\cdot,\cdot, b_{-i})$ is concave on int(A_i) $\times$ int(B_i) (resp. $A_i \times B_i$), for any $b_{-i} \in B_{-i}$, and $f_i(\cdot, b_i, b_{-i})$ is convex on A_i, for any $(b_i, b_{-i}) \in B_i \times B_{-i}$.

Then, $\tilde{b}_i(\cdot, b_{-i})$ is linear on A_i and $v_i(\cdot, b_{-i})$ is concave on A_i, for any $b_{-i} \in B_{-i}$, that is ($\mathcal{A}_{F_i}$ 8) is verified.

Proof. We prove only the case in which f_i is concave on $\text{int}(A_i)\times\text{int}(B_i)$ and A_i, B_i are strictly convex. The other case is straightforward.
Let $b_{-i} \in B_{-i}$, $a'_i, a''_i \in A_i$ and $\lambda \in]0,1[$. Observe that $\lambda a'_i+(1-\lambda)\, a''_i \in \text{int}(A_i)$ and $\lambda\tilde{b}_i(a'_i, b_{-i}) + (1-\lambda)\tilde{b}_i(a''_i, b_{-i}) \in \text{int}(B_i)$, for any $\lambda \in]0,1[$, being A_i and B_i strictly convex sets.

Then, from the concavity of f_i on $\text{int}(A_i)\times\text{int}(B_i)$ and from the definition of $\tilde{b}_i$ we have:

$$\begin{aligned} & f_i(\lambda a'_i + (1-\lambda)a''_i, \lambda\tilde{b}_i(a'_i, b_{-i}) + (1-\lambda)\tilde{b}_i(a''_i, b_{-i}), b_{-i}) \geq \\ & \geq \lambda f_i(a'_i, \tilde{b}_i(a'_i, b_{-i}), b_{-i}) + (1-\lambda) f_i(a''_i, \tilde{b}_i(a''_i, b_{-i}), b_{-i}) \\ & \geq \lambda f_i(a'_i, b_i, b_{-i}) + (1-\lambda) f_i(a''_i, b_i, b_{-i}) \quad \text{for any } b_i \in B_i. \end{aligned}$$

That is, from the convexity of $f_i(\cdot, b_i)$ on A_i:

$$\begin{aligned} & f_i(\lambda a'_i + (1-\lambda)a''_i, \lambda\tilde{b}_i(a'_i, b_{-i}) + (1-\lambda)\tilde{b}_i(a''_i, b_{-i}), b_{-i}) \geq \\ & f_i(\lambda a'_i + (1-\lambda)a''_i, b_i, b_{-i}) \quad \text{for any } b_i \in B_i; \end{aligned}$$

and then $\lambda\tilde{b}_i(a'_i, b_{-i}) + (1-\lambda)\tilde{b}_i(a''_i, b_{-i}) = \tilde{b}_i(\lambda a'_i + (1-\lambda)a''_i, b_{-i})$, that is $\tilde{b}_i$ is linear on A_i.

The concavity of $v_i(\cdot, b_{-i})$ follows from Proposition 2.3.6 if we observe again that if A_i (B_i) is a strictly convex set each combination of points of A_i (B_i) is in the interior of the set. $\diamond$

Example 4.4.2 We present an example where both leaders' and followers' payoff functions that satisfy all the conditions required for the existence of an equilibrium under passive beliefs.

Let $A_i = B_i = B_j = [0, 1]$ and

$$f_i(a_i, b_i, b_j) = \begin{cases} (1 - \frac{b_j}{2})(a_i - b_i^j) + 2 & \text{if } b_i \neq 0 \\ 3 + a_i b_j & \text{otherwise.} \end{cases}$$

for $i, j = 1, 2$ and $j \neq i$.

Then f_i satisfies ($\mathcal{A}_{F_i}$ *2*), ($\mathcal{A}_{F_i}$ *3*) and ($\mathcal{A}_{F_i}$ *9*) (being A_i a strictly convex subset of $\mathbb{R}$).

Indeed:

($\mathcal{A}_{F_i}$ ***2***) f is discontinuous at any point $(a_i, b_i, b_j) \in [0, 1] \times \{0\} \times [0, 1]$. So, let $(a_i, b_j) \in [0, 1]^2$ and $(a_{i,n}, b_{i,n}, b_{j,n})_n$ be a sequence that converges to $(a_i, 0, b_j)$ in $[0, 1]^3$. Any subsequence of $(f_i(a_{i,n}, b_{i,n}, b_{j,n}))_n$ can converge only to $3+a_i b_j$ or $(1-\frac{b_j}{2})a_i+2$. Then ($\mathcal{A}_{F_i}$ *2*) is satisfied being $f_i(a_i, 0, b_j) = 3 + a_i b_j > (1 - \frac{b_j}{2})a_i + 2$, for any $(a_i, b_j) \in [0, 1]^2$.

($\mathcal{A}_{F_i}$ ***3***) Let $(a_i, b_j) \in [0, 1]^2$ and $(a_{i,n}, b_{j,n})_n$ be a sequence that converges to (a_i, b_j) in $[0, 1]^2$. Take $b_{i,n} = 0$, for any $n \in \mathbb{N}$. Then $f_i(a_{i,n}, 0, b_{j,n}) = 3 + a_{i,n} b_{j,n}$ that converges to $3 + a_i b_j$. Hence, condition in ($\mathcal{A}_{F_i}$ *3*) holds with the equality sign.

($\mathcal{A}_{F_i}$ ***9***) It is quite immediate if we observe that A_i, B_i are strictly convex subsets of $\mathbb{R}$, the discontinuity is on the boundary and $f_i(\cdot, \cdot, b_j)$ is concave on $]0, 1[^2$, for any b_j.

Moreover,

$$l_i(a_i, b_i, b_j) = \begin{cases} b_j^2(1 - a_i^2) & \text{if } a_i \neq 0, \\ 1 + b_i + b_j + t & \text{if } a_i = 0,\ t \geq 0, \\ 1 + b_i + b_j & \text{otherwise,} \end{cases}$$

satisfies ($\mathcal{A}_{L_i}$ *2*), ($\mathcal{A}_{L_i}$ *7*) and and ($\mathcal{A}_{L_i}$ *8*).

($\mathcal{A}_{L_i}$ *2*) l_i is discontinuous is discontinuous at any point $(a_i, b_i, b_j, t) \in \{0\} \times [0,1]^2 \times \mathbb{R}$. So, let $(b_i, b_j, t) \in [0,1]^2 \times \mathbb{R}$ and $(a_{i,n}, b_{i,n}, b_{j,n}, t_n)_n$ be a sequence that converges to $(0, b_i, b_j, t)$ in $[0,1]^3 \times \mathbb{R}$. If $t > 0$, a subsequence of $(l_i(a_{i,n}, b_{i,n}, b_{j,n}, t_n))_n$ can converge only to $b_j^2(1 - a_i^2)$ or to $1 + b_i + b_j + t$. If $t \le 0$, a subsequence of $(l_i(a_{i,n}, b_{i,n}, b_{j,n}, t_n))_n$ can converge only to $b_j^2(1-a_i^2)$ or to $1+b_i+b_j$. In both cases ($\mathcal{A}_{L_i}$ *2*) is satisfied, being $b_j^2(1 - a_i^2) < l_i(0, b_i, b_j, t)$, for any $(b_i, b_j, t) \in [0,1]^2 \times \mathbb{R}$.

($\mathcal{A}_{L_i}$ *7*) It is straightforward.

($\mathcal{A}_{L_i}$ *8*) let $b_j \in [0,1]$. Denoted with Λ_α the upper level set at height α of $l_i(\cdot, \cdot, b_j, \cdot)$, that is $\Lambda_\alpha = \{(a_i, b_i, t) \colon l_i(a_i, b_i, b_j, t) \ge \alpha\}$, we have:

$$\Lambda_\alpha = \begin{cases} [0,1] \times [0, 1R] \times \mathbb{R} & \text{if } \alpha < 0 \\ [0, \sqrt{1 - \frac{\alpha}{b_j^2}}] \times [0,1] \times \mathbb{R} & \text{if } 0 \le \alpha < b_j^2 \\ \{0\} \times [0,1] \times \mathbb{R} & \text{if } b_j^2 \le \alpha \le 1 + b_j \\ \{(0, b_i, t) \colon t + b_i \ge \alpha - 1 - b_j, t \ge 0, b_i \in [0,1]\} & \text{if } \alpha > 1 + b_j \end{cases} \tag{4.44}$$

that are convex sets, for any α.

◇

With the characterization of an equilibrium under passive beliefs given in Lemma 4.4.6, the hypothesis of linearity of $\tilde{b}_i$ is necessary to prove an existence result. So, the scope of the next section is to reformulate the problem in order to find weaker conditions for the existence of an equilibrium under passive beliefs.

4.5 Particular cases: existence results under weaker conditions

In this section we consider particular classes of two–stage games with vertical separation. For each one of these classes we give a characterization of the concept of equilibrium under passive beliefs that simplifies the analysis of the existence of equilibria.

First, when the solution of any leader's problem is a singleton for any action profile b_{-i} of the followers F_{-i}, we prove an existence result without concavity assumptions.
Then, we assume that the followers' actions spaces are subsets of $\mathbb{R}$. Existence results are proved assuming that the optimal reaction of any follower is first concave and then isotone.
In the last case we consider the situation in which the problem of any follower may have more than one solution but the payoff function of any leader depends on the action of the corresponding follower only through the optimal value function.

4.5.1 In case of uniqueness of the leader's optimal reaction

Let $\tilde{b}_i$ be the optimal reaction of the follower F_i defined in (4.31). In this section we assume that M_i defined in (4.35) is a single–valued map.

More precisely, assume for any $i = 1, \dots, k$:

($\mathcal{I}_i$) *the reaction set* $\tilde{B}_i(a_i, b_{-i}) = \operatorname{Arg\,max}_{b_i \in B_i} f_i(a_i, b_i, b_{-i})$ *of* F_i *is a singleton, that is* $\tilde{B}_i(a_i, b_{-i}) = \{\tilde{b}_i(a_i, b_{-i})\}$*, for any* $(a_i, b_{-i}) \in A_i \times B_{-i}$*;*

($\mathcal{J}_i$) $M_i(b_{-i}) = \operatorname{Arg\,max}_{a_i \in A_i} \tilde{l}_i(a_i, \tilde{b}_i(a_i, b_{-i}), b_{-i}, v_i(a_i, b_{-i}))$ *is a singleton, that is* $M_i(b_{-i}) = \{\bar{a}_i(b_{-i})\}$*, for any* $b_{-i} \in B_{-i}$*.*

Define the following vector function

$$\bar{b}(b) = (\tilde{b}_1(\bar{a}_1(b_{-1}), b_{-1}), \dots, \tilde{b}_k(\bar{a}_k(b_{-k}), b_{-k})), \tag{4.45}$$

for any $b \in B$, where the i-th component of $\bar{b}$ corresponds to the optimal reaction of the follower F_i to the optimal reaction of the leader L_i given the parameter b_{-i}.

Lemma 4.5.1. *Any fixed point of* $\bar{b}$ *on* B *can be associated to an equilibrium under passive beliefs of the two–stage game* Γ *with vertical separation, and vice versa.*

Proof. Let $b^* = (b_1^*, \dots, b_k^*)$ be a fixed point of $\bar{b}$ on B. Then, according to the definition, we have $b_i^* = \tilde{b}_i(\bar{a}_i(b_{-i}^*), b_{-i}^*)$, for $i = 1, \dots, k$. So, taken $a_i^* = \bar{a}_i(b_{-i}^*)$, the strategy profile $(a_1^*, \dots, a_k^*, \beta_1{}^*, \dots, \beta_k^*)$ is an equilibrium under passive beliefs, where β_i^* is a function from A_i to B_i such that $\beta_i{}^*(a_i) = \tilde{b}_i(a_i, b_{-i}^*)$ for any $a_i \in A_i$ and $i = 1, \dots, k$.
Vice versa, one can easily verify that if $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ is an equilibrium under passive beliefs, then $(\beta_1^*(a_i^*), \dots, \beta_k^*(a_k^*))$ is a fixed point of $\bar{b}$ on B. ◇

Now, let us give a preliminary result on the function $\bar{a}_i$.

Proposition 4.5.2. *Let $i = 1, \dots, k$. Assume ($\mathcal{I}_i$), ($\mathcal{J}_i$), ($\mathcal{A}_{L_i}$ 2), ($\mathcal{A}_{L_i}$ 5), ($\mathcal{A}_{L_i}$ 7), ($\mathcal{A}_{F_i}$ 2), ($\mathcal{A}_{F_i}$ 3), ($\mathcal{A}_{F_i}$ 5). Then the function $\bar{a}_i$ is continuous on B_{-i}.*

Proof. From Proposition 4.4.7, $\tilde{b}_i$ is continuous on $A_i \times B_{-i}$ and, from the proof of Proposition 4.4.9, M_i is a closed single–valued map. This fact, together with the compactness of A_i, gives the thesis. ◇

Finally, we can state:

Theorem 4.5.3. *Assume, for $i = 1, \dots, k$:*

($\mathcal{I}_i$) *the reaction set $\tilde{B}_i(a_i, b_{-i})$ of F_i is a singleton, that is $\tilde{B}_i(a_i, b_{-i}) = \{\tilde{b}_i(a_i, b_{-i})\}$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$;*

($\mathcal{J}_i$) *$M_i(b_{-i})$ is a singleton, that is $M_i(b_{-i}) = \{\bar{a}_i(b_{-i})\}$, for any $b_{-i} \in B_{-i}$;*

($\mathcal{A}_{L_i}$ 1) *A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;*

($\mathcal{A}_{L_i}$ 2) *l_i is a real–valued upper semicontinuous function on $A_i \times B \times \mathbb{R}$;*

($\mathcal{A}_{L_i}$ 7) *$l_i(a_i, \cdot, \cdot, \cdot)$ is lower semicontinuous on $B_i \times B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$;*

($\mathcal{A}_{F_i}$ 1) *B_i is a nonempty compact convex subset of $\mathbb{R}^{m_i}$;*

($\mathcal{A}_{F_i}$ 2) *f_i is a real–valued upper semicontinuous function on $A_i \times B$;*

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i}).$$

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ with vertical separation.

Proof. The function $\bar{b}$ satisfies the hypothesis of the Brouwer Theorem (Theorem A.2.2 in the Appendix). Indeed, from propositions 4.4.7 and 4.5.2, $\bar{b}$ is continuous on B, with B compact and convex, so there exists a fixed point of $\bar{b}$ on B. Then, from Lemma 4.5.1, there exists an equilibrium under passive beliefs of the two–stage game with vertical separation. ◇

4.5.2 In case of nonuniqueness of the leader's optimal reaction

Assume that $(\mathcal{I}_i)$ is satified for any $i = 1, \dots, k$, where:

($\mathcal{I}_i$) the reaction set $\tilde{B}_i(a_i, b_{-i}) = \operatorname{Arg\,max}_{b_i \in B_i} f_i(a_i, b_i, b_{-i})$ of F_i is a singleton, that is $\tilde{B}_i(a_i, b_{-i}) = \{\tilde{b}_i(a_i, b_{-i})\}$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$.

Let M_i be the set–valued map from B_{-i} to A_i defined in (4.35); that is

$$M_i(b_{-i}) = \operatorname*{Arg\,max}_{a_i \in A_i} l_i(a_i, \tilde{b}_i(a_i, b_{-i}), b_{-i}, v_i(a_i, b_{-i})),$$

for any $b_{-i} \in B_{-i}$ and for $i = 1, \dots, k$.
We define the set–valued map E_i from B_{-i} to B_i such that:

$$E_i(b_{-i}) = \{\tilde{b}_i(a_i, b_{-i}) \mid a_i \in M_i(b_{-i})\}. \tag{4.46}$$

Let

$$E(b) = \prod_{i=1}^{k} E_i(b_{-i}). \tag{4.47}$$

Lemma 4.5.4. *Any fixed point of the set valued–map E on B can be associated to an equilibrium under passive beliefs of the two–stage game Γ with vertical separation, and vice versa.*

Proof. Let $b^* = (b_1^*, \dots, b_k^*)$ be a fixed point of E on B. According to the definition of E, there exists $a_i^* \in M_i(b_{-i}^*)$ such that $b_i^* = \tilde{b}_i(a_i^*, b_{-i}^*)$, for $i = 1, \dots, k$. Then, the strategy profile $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$, where $\beta_i^*(a_i) = \tilde{b}_i(a_i, b_{-i}^*)$ for any $a_i \in A_i$, for $i = 1, \dots, k$, is a vertical separated equilibrium under passive beliefs. Vice versa, if $(a_1^*, \dots, a_k^*, \beta_1^*, \dots, \beta_k^*)$ is a vertical separated equilibrium under passive beliefs, then one can easily verify that $(\beta_1^*(a_i^*), \dots, \beta_k^*(a_k^*))$ is a fixed point of E on B. ◇

The next proposition shows that if M_i is a convex–valued and closed set–valued map then there exists an equilibrium under passive beliefs of Γ.

Proposition 4.5.5. *Let $i = 1, \dots, k$. Assume, for $i = 1, \dots, k$, $(\mathcal{I}_i)$, $(\mathcal{A}_{L_i}$ 1), $(\mathcal{A}_{L_i}$ 2), $(\mathcal{A}_{L_i}$ 7), $(\mathcal{A}_{F_i}$ 1)–$(\mathcal{A}_{F_i}$ 3). If M_i is convex–valued, for any $i = 1, \dots, k$, then the set–valued map E has a fixed point on B.*

Proof. It is a direct consequence of Theorem 3.2.1. Indeed, from Proposition 4.4.7 we have that $\tilde{b}_i$ is continuous on $A_i \times B_{-i}$, for $i = 1, \dots, k$. So, also the function $\tilde{b} \colon A \times B \to B$ defined by $\tilde{b}(a, b) = (\tilde{b}_1(a_1, b_{-1}), \dots, \tilde{b}_k(a_k, b_{-k}))$ is continuous on $A \times B$. From a result in Proposition 4.4.9, we have that M_i is a closed set–valued map and, being A_i compact, it is also compact–valued, for $i = 1, \dots, k$. Defined the set–valued map $M \colon B \rightrightarrows A$ as $M(b) = \prod_{i=1}^k M_i(b_{-i})$, we obtain that M is closed and with nonempty compact convex values. So, we have the thesis if we observe that:

$$
\begin{aligned}
E(b) &= \{b' \in B \colon b_i' \in E_i(b_{-i}), \text{ for } i = 1, \dots, k\} \\
&= \{b' \in B \colon \exists a_i \in M_i(b_{-i}) \text{ s.t. } b_i' = \tilde{b}_i(a_i, b_{-i}), \text{ for } i = 1, \dots, k\} \\
&= \{\tilde{b}(a, b) \colon a \in M(b)\}.
\end{aligned}
$$

◇

So, in the following we consider explicit classes of problems and we investigate conditions under which M_i is convex–valued.

Assume in this section that the actions sets of the followers are subsets of $\mathbb{R}$.

Proposition 4.5.6. *Let $i = 1, \ldots, k$. Assume that ($\mathcal{I}_i$) is satisfied. Let A_i be a nonempty compact convex subset of $\mathbb{R}^{n_i}$ and B_i be a nonempty compact convex subset of $\mathbb{R}$. Let $b_{-i} \in B_{-i}$ and assume that $l_i(\cdot, \cdot, b_{-i}, \cdot)$ is quasiconcave on $A_i \times B_i \times \mathbb{R}$ and $l_i(a_i, \cdot, b_{-i}, \cdot)$ is componentwise nondecreasing (nonincreasing) on $B_i \times \mathbb{R}$. Moreover, assume that $\tilde{b}_i(\cdot, b_{-i})$ and $v_i(\cdot, b_{-i})$ are concave (convex) on A_i.*
Then, the function $\tilde{l}_i(\cdot, b_{-i})$ defined in (4.32) is quasiconcave on A_i and $M_i(b_{-i})$ is a convex set.

Proof. The quasiconcavity of $\tilde{l}_i(\cdot, b_{-i})$ follows from Proposition 2.1.27. Then, from Proposition 2.1.3, $M_i(b_{-i})$ is convex. ◇

By application of the new result Theorem 3.1.1, the following theorem gives an existence result using the concavity of $\tilde{b}_i(\cdot, b_{-i})$.

Theorem 4.5.7. *Assume, for $i = 1, \ldots, k$:*

($\mathcal{A}_{L_i}$ 1) A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 2) l_i is a real–valued upper semicontinuous function on $A_i \times B \times \mathbb{R}$;

($\mathcal{A}_{L_i}$ 7) $l_i(a_i, \cdot, \cdot, \cdot)$ is lower semicontinuous on $B_i \times B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$;

($\mathcal{A}_{L_i}$ 9) $l_i(\cdot, \cdot, b_{-i}, \cdot)$ is quasiconcave on $A_i \times B_i \times \mathbb{R}$, for any $b_{-i} \in B_{-i}$; $l_i(a_i, \cdot, b_{-i}, \cdot)$ is componentwise nondecreasing on $B_i \times \mathbb{R}$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such

that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i});$$

($\mathcal{A}_{F_i}$ *10)* B_i *is a nonempty compact interval of* $\mathbb{R}$*;*

($\mathcal{A}_{F_i}$ *11)* $f_i(\cdot,\cdot,b_{-i})$ *is concave on* $A_i \times B_{-i}$*, for any* $b_{-i} \in B_{-i}$*;*

($\mathcal{A}_{F_i}$ *12)*

$$\begin{cases} f_i(a_i,\cdot,b_{-i}) \text{ is strictly D–pseudoconcave on } B_i, \\ \qquad\qquad \text{for any } (a_i,b_{-i}) \in int(A_i) \times B_{-i}; \\ \mathcal{D}_{f_i}(\cdot,\cdot,b_{-i}) \text{ is quasiconcave on } A_i \times B_i \text{ for any } b_{-i} \in B_{-i}; \\ D^+ f_i(a_i,\cdot,b_{-i})(\min B_i) > 0, \text{ for any } a_i \in A_i. \end{cases}$$

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ *with vertical separation.*

Proof. First, observe that assumption $(\mathcal{I}_i)$ on the uniqueness of values of $\tilde{B}_i$ is not necessary. Indeed, it follows from Proposition 2.1.22.
Then, the result follows from Proposition 4.5.5 and the fact that, by application of Proposition 2.1.24, M_i is convex–valued. Indeed, from Theorem 3.1.1, we have that $\tilde{b}_i(\cdot, b_{-i})$ is concave on A_i, for any $b_{-i} \in B_{-i}$. From Proposition 2.3.6 we have that $v_i(\cdot, b_{-i})$ is concave on A_i, for any $b_{-i} \in B_{-i}$.
Then, from Proposition 4.5.6 we have the thesis.

◇

Remark 4.5.8 As shown in Proposition 2.14, the assumption ($\mathcal{A}_{F_i}$ *11*) is sufficient for the concavity of the optimal value function $v_i(\cdot, b_{-i})$ on A_i. If the sets A_i and B_i are strictly convex, we can have the concavity of $v_i(\cdot, b_{-i})$ if we require $f_i(\cdot,\cdot,b_{-i})$ concave on $\text{int}(A_i) \times \text{int}(B_i)$, as in Proposition 4.4.13.

We can obtain an alternative existence theorem changing assumptions ($\mathcal{A}_{L_i}$ *9*) and ($\mathcal{A}_{F_i}$ *11*).

Theorem 4.5.9. *Assume, for $i = 1, \ldots, k$:*

($\mathcal{A}_{L_i}$ 1) A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 2) l_i is a real–valued upper semicontinuous function on $A_i \times B \times \mathbb{R}$;

($\mathcal{A}_{L_i}$ 7) $l_i(a_i, \cdot, \cdot, \cdot)$ is lower semicontinuous on $B_i \times B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$;

($\mathcal{A}_{L_i}$ 10) $l_i(\cdot, \cdot, b_{-i}, \cdot)$ is quasiconcave on $A_i \times B_i \times \mathbb{R}$, for any $b_{-i} \in B_{-i}$; $l_i(a_i, \cdot, b_{-i}, t)$ is nondecreasing on B_i, for any $(a_i, b_{-i}, t) \in A_i \times B_{-i} \times \mathbb{R}$ and $l_i(a_i, b_i, b_{-i}, \cdot)$ is nonincreasing on $\mathbb{R}$, for any $(a_i, b_i, b_{-i}) \in A_i \times B$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i});$$

($\mathcal{A}_{F_i}$ 10) B_i is a nonempty compact interval of $\mathbb{R}$;

($\mathcal{A}_{F_i}$ 12)

$$\begin{cases} f_i(a_i, \cdot, b_{-i}) \text{ is strictly } D\text{–pseudoconcave on } B_i, \\ \qquad\qquad \text{for any } (a_i, b_{-i}) \in int(A_i) \times B_{-i}; \\ \mathcal{D}_{f_i}(\cdot, \cdot, b_{-i}) \text{ is quasiconcave on } A_i \times B_i \text{ for any } b_{-i} \in B_{-i}, \\ D^+ f_i(a_i, \cdot, b_{-i})(\min B_i) > 0, \text{ for any } a_i \in A_i; \end{cases}$$

($\mathcal{A}_{F_i}$ 13) $f_i(\cdot, b_i, b_{-i})$ is convex on A_i, for any $(b_i, b_{-i}) \in B$.

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ with vertical separation.

Proof. It is similar to the proof of Theorem 4.5.7. The only difference is that from Proposition 2.3.7, $v_i(\cdot, b_{-i})$ is convex on A_i, for any $b_{-i} \in B_{-i}$. Then the result follows from Proposition 2.1.25. ◇

Remark 4.5.10 Note that condition ($\mathcal{A}_{F_i}$ *13*) is not incompatible with ($\mathcal{A}_{F_i}$ *12*). Indeed, a function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ may be both strictly D–pseudoconcave and convex on I. Consider, for example, the exponential function $f(x) = e^x$.

Example 4.5.1 In the following we give an example of follower's payoff function which satisfies the assumption in Theorem 4.5.7. Let $A_i = B_i = B_j = [0, 1]$ and

$$f_i(a_i, b_i, b_j) = \begin{cases} -b_i^2 + 2b_i & \text{if } b_i \le \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2 \\ -(a_i + b_j)b_i & \text{otherwise.} \end{cases}$$

for $i, j = 1, 2$ and $j \neq i$. Then, f satisfies assumptions ($\mathcal{A}_{F_i}$ *2*), ($\mathcal{A}_{F_i}$ *3*) and ($\mathcal{A}_{F_i}$ *12*).

($\mathcal{A}_{F_i}$ *2*) for any (a_i, b_i, b_j) such that $b_i > \frac{1}{2}(\frac{a_i+b_j}{2} - 1)^2$ or $b_i < \frac{1}{2}(\frac{a_i+b_j}{2} - 1)^2$ it is obvious. Let $b_i = \frac{1}{2}(\frac{a_i+b_j}{2} - 1)^2$ and $((a_{i,n}, b_{i,n}, b_{j,n}))_n$ a sequence converging to (a_i, b_i, b_j). Then if there exists a set of index $N_1 \subseteq \mathbb{N}$ with cardinality infinite such that $b_{i,n} \le \frac{1}{2}(\frac{a_{i,n}+b_{j,n}}{2} - 1)^2$, then $f_i(a_{i,n}, b_{i,n}, b_{j,n}) = -b_{i,n}^2 + 2b_{i,n}$, for each $n \in N_1$ that is $\lim_{\substack{n\to+\infty \\ n\in\mathbb{N}_1}} f(a_{i,n}, b_{i,n}, b_{j,n}) = f(a_i, b_i, b_j)$; otherwise $\limsup_{n\to+\infty} f(a_{i,n}, b_{i,n}, b_{j,n}) < f(a_i, b_i, b_j)$.

($\mathcal{A}_{F_i}$ *3*) Let (a_i, b_i, b_j) and $((a_{i,n}, b_{j,n}))_n$ a sequence converging to (a_i, b_j). Again, it is sufficient to discuss only the case $b_i = \frac{1}{2}(\frac{a_i+b_j}{2} - 1)^2$. So, let $b_i = \frac{1}{2}(\frac{a_i+b_j}{2} - 1)^2$ and take $b_{i,n} = \frac{1}{2}(\frac{a_{i,n}+b_{j,n}}{2} - 1)^2 \in [0, \frac{1}{2}]$. Then, $f_i(a_{i,n}, b_{i,n}, b_{j,n}) = -b_{i,n}^2 + 2b_{i,n}$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} f_i(a_{i,n}, b_{i,n}, b_{j,n}) = -b_i^2 + 2b_i = f_i(a_i, b_i, b_j)$.

($\mathcal{A}_{F_i}$ ***12***)

$$\overline{D}_1^+ f_i(a_i, b_i, b_j) = \begin{cases} -2b_i + 2 & \text{if } 0 \le b_i < \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2, \\ -\infty & \text{if } b_i = \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2, \\ -(a_i + b_j) & \text{if } \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2 < b_i < 1. \end{cases}$$

$$\underline{D}_1^- f_i(a_i, b_i, b_j) = \begin{cases} -2b_i + 2 & \text{if } 0 < b_i \le \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2, \\ -(a_i + b_j) & \text{if } b_i > \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2. \end{cases} \tag{4.48}$$

Let $(a_i, b_j) \in A_i \times B_j$. Let $\bar{b}_i, \tilde{b}_i \in B_i$ and assume, for sake of simplicity, $\bar{b}_i < \tilde{b}_i$. If $\tilde{b}_i \le \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2$, then $f_i(a_i, \bar{b}_i, b_j) < f_i(a_i, \tilde{b}_i, b_j)$ and $\overline{D}_1^+ f_i(a_i, \bar{b}_i, b_j) > 0$. If $\tilde{b}_i > \frac{1}{2}(1 - \frac{a_i+b_j}{2})^2$ then $f_i(a_i, \bar{b}_i, b_j) > f_i(a_i, \tilde{b}_i, b_j)$ and $\underline{D}_1^- f_i(a_i, \tilde{b}_i, b_j) < 0$, that is f_i strictly D–pseudoconcave on B_i, for each $(a_i, b_j) \in A_i \times B_j$.

From (4.48), it follows that $\overline{D}_1^- f_i(a_i, b_i, b_j)$ is quasiconcave on B_i. Instead of ($\mathcal{A}_{F_i}$ *11*), $f_i(\cdot, b_i, b_j)$ is convex on A_i, as required in Remark 4.5.8. ⋄

Moreover, consider the case in which the payoff function of a leader depends only on the strategies of the followers, that is, for $i = 1, \dots, k$:

($\mathcal{K}_i$) *the payoff function l_i of L_i is defined on $B_i \times B_{-i}$.*

Furthermore, suppose that the actions sets of both leaders and followers are subsets of $\mathbb{R}$. In this simplified case, mild assumptions guaranties that an equilibrium exists.

Theorem 4.5.11. *Assume, for $i = 1, \dots, k$:*

($\mathcal{A}_{L_i}$ 11) A_i is a nonempty compact convex subset of $\mathbb{R}$;

($\mathcal{A}_{L_i}$ 12) l_i is a real–valued continuous function on $B_i \times B_{-i}$;

($\mathcal{A}_{L_i}$ 13) $l_i(\cdot, b_{-i})$ is quasiconcave on B_i, for any $b_{-i} \in B_{-i}$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $\left((a_{i,n}, b_{-i,n})\right)_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i});$$

($\mathcal{A}_{F_i}$ 10) B_i is a nonempty compact interval of $\mathbb{R}$;

($\mathcal{A}_{F_i}$ 14) $f_i(a_i, \cdot, b_i)$ is strictly D–pseudoconcave on B_i, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$ and $\mathcal{D}_{f_i}(\cdot, b_i, b_{-i})$ is isotone on A_i, for all $b \in B$;

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ with vertical separation.

Proof. As in Theorem 4.5.7, assumption ($\mathcal{I}_i$) on the uniqueness of values of $\tilde{B}_i$ is not necessary. From Proposition 3.1.4 we have that the function $\tilde{b}_i(\cdot, b_{-i})$

is nondecreasing on A_i, for any $b_{-i} \in B_{-i}$. Then, from Proposition 2.3.9 it follows that $\tilde{b}_i$, considered as a set–valued map, is quasiconvex. Thus, from Proposition 2.3.8, we obtain that $\tilde{l}_i$ is a quasiconcave function on B_i. So, M_i is convex–valued. Then, from Proposition 4.5.5, we have the thesis. $\diamond$

4.5.3 In case of nonuniqueness of the follower's optimal reaction

In the last part of this chapter we do not require the assumption $(\mathcal{I}_i)$ about the uniqueness of solutions to followers' parametric Optimization problems but we consider the case in which the payoff function of any leader depends on the action of the corresponding follower only through the optimal value function, that is we have $l_i(a_i, b_{-i}, v_i(a_i, b_{-i}))$.
So, in this section we assume for $i = 1, \ldots, k$:

$(\mathcal{H}_i)$ *the payoff function* l_i *of* L_i *is defined on* $A_i \times B_{-i} \times \mathbb{R}$.

In this case we can extend in a natural way the characterization given in the previous subsection when the optimal reaction $\tilde{B}_i$ defined on $A_i \times B_{-i}$ by $\tilde{B}_i(a_i, b_{-i}) = \operatorname{Arg\,max}_{b_i \in B_i} f_i(a_i, b_i, b_{-i})$ is not single–valued.
For any $i = 1, \ldots, k$ we define the set–valued map E_i from B_{-i} to B_i by:

$$E_i(b_{-i}) = \cup_{a_i \in M_i(b_{-i})} \tilde{B}_i(a_i, b_{-i}),$$

where $M_i(b_{-i}) = \operatorname{Arg\,max}_{a_i \in A_i} l_i(a_i, b_{-i}, v_i(a_i, b_{-i}))$. So, with the same arguments given before, an equilibrium under passive beliefs is associated to a fixed point of the set–valued map

$$E = \prod_{i=1}^{k} E_i, \tag{4.49}$$

and vice versa.

In the following theorem we prove an existence result using Theorem 3.2.3, a new fixed point theorem proved in Section 3.2.

Theorem 4.5.12. *Assume, for* $i = 1, \dots, k$*:*

($\mathcal{A}_{L_i}$ 1) A_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$;

($\mathcal{A}_{L_i}$ 14) l_i is a real–valued upper semicontinuous function on $A_i \times B_{-i} \times \mathbb{R}$;

($\mathcal{A}_{L_i}$ 15) $l_i(a_i, \cdot, \cdot)$ is lower semicontinuous on $B_{-i} \times \mathbb{R}$, for any $a_i \in A_i$;

($\mathcal{A}_{L_i}$ 16) $l_i(\cdot, b_{-i}, \cdot)$ is quasiconcave on $A_i \times \mathbb{R}$, for any $b_{-i} \in B_{-i}$ and $l_i(a_i, b_{-i}, \cdot)$ is nondecreasing on $\mathbb{R}$, for any $(a_i, b_{-i}) \in A_i \times B_{-i}$;

($\mathcal{A}_{F_i}$ 1) B_i is a nonempty compact convex subset of $\mathbb{R}^{m_i}$;

($\mathcal{A}_{F_i}$ 2) f_i is a real–valued upper semicontinuous function on $A_i \times B$;

($\mathcal{A}_{F_i}$ 3) for any $(a_i, b_i, b_{-i}) \in A_i \times B$, for any sequence $((a_{i,n}, b_{-i,n}))_n$ converging to (a_i, b_{-i}) in $A_i \times B_{-i}$, there exists a sequence $(\hat{b}_{i,n})_n$ in B_i such that:

$$\liminf_{n\to\infty} f_i(a_{i,n}, \hat{b}_{i,n}, b_{-i,n}) \geq f_i(a_i, b_i, b_{-i});$$

($\mathcal{A}_{F_i}$ 11) $f_i(\cdot, \cdot, b_{-i})$ is concave on $A_i \times B_{-i}$, for any $b_{-i} \in B_{-i}$;

Then, there exists an equilibrium under passive beliefs of the two–stage game Γ with vertical separation.

Proof. From a result in Theorem 4.3.1 we have that $\tilde{B}_i$ is a closed set–valued map and the optimal value function v_i is continuous on $A_i \times B_{-i}$. Define the set–valued map $\tilde{B}\colon A \times B \rightrightarrows B$ such that $\tilde{B}(a, b) = \prod_{i=1}^{k} \tilde{B}_i(a_i, b_{-i})$. Then, $\tilde{B}$ is a closed set–valued map. With the same arguments used in the proof of Proposition 4.4.9 we obtain that the set–valued map M_i is closed. Then the set–valued map M defined by $M(b) = \prod_{i=1}^{k} M_i(b_{-i})$, for any $b \in B$, is closed. From Proposition 2.3.6, $v_i(\cdot, b_{-i})$ is concave on A_i, for any $b_{-i} \in B_{-i}$. From Proposition 2.1.27, it follows that $\tilde{l}_i(\cdot, b_{-i})$ is quasiconcave on A_i, for any $b_{-i} \in B_{-i}$, for $i = 1, \dots, k$, that is M is convex–valued. Then, from Theorem 3.2.3, we obtain that the set–valued map E defined in (4.49) has a fixed point on B. Then there exists an equilibrium under passive beliefs of Γ. $\diamond$

Appendix A

Basic Preliminaries

In this appendix we recall basic concepts and results on Euclidean spaces and Real and Concave analysis that are used in this thesis. All the material in this chapter is based mainly on Berge [6], Kelley and Namioka [36], Kelley [35], Mangasarian [57], Bourbaki [10, 11], Klein [37], Rudin [77, 76].Cambini and Martein [15].

A.1 The Euclidean Space $\mathbb{R}^n$

Having in mind applications to economic problems, we will restrict our attention to finite dimension spaces. In particular, we limit the attention to the space $\mathbb{R}^n$, where with $\mathbb{R}$ we indicate the set of real numbers.

The set $\mathbb{R}^n$ can be endowed with the addition operator $+\colon \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$x + y = (x_1 + y_1, \dots, x_n + y_n).$$

Denoted with $\cdot$ the function $\cdot\colon \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that for any $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$a{\cdot}x = (a \cdot x_1, \dots, a \cdot x_n)$$

$\mathbb{R}^n$ with these two operations is a *vector space* (or else *linear space*) on $\mathbb{R}$.

Let us denote with $\langle\cdot,\cdot\rangle$ a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, called *inner scalar product*, defined by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, for any $x, y \in \mathbb{R}^n$.

Denoted with d the real–valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that $d(x,y) = (\langle x-y, x-y\rangle)^{1/2}$, d is called *Euclidean distance*. The Euclidean distance satisfies the following properties:

(i) $d(x,y) = d(y,x)$;
(ii) (triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$;
(iii) $d(x,y) = 0$ if and only if $x = y$;

for all points $x, y, z \in \mathbb{R}^n$.
The function $\|\cdot\| \colon \mathbb{R}^n \to \mathbb{R}$ defined by $\|x\| = (\langle x, x\rangle)^{1/2} = d(x,x)$ is called *Euclidean norm*. The norm function satisfies the following properties:

(i) $\|x+y\| \leq \|x\| + \|y\|$,
(ii) $\|a \cdot x\| = |a|\|x\|$,
(iii) $\|x\| = 0$ if and only if $x = 0$,

for all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$. A real coordinate space together with this Euclidean structure is called *Euclidean space*.

The *open ball* of radius $r > 0$ and centered in $x_0 \in \mathbb{R}^n$ is the subset of $\mathbb{R}^n$ defined by:

$$B(x_0; r) = \{x \in \mathbb{R}^n \colon d(x, x_0) < r\}. \tag{A.1}$$

Given $E \subseteq \mathbb{R}^n$ and $\epsilon > 0$, we define the set:

$$U_\epsilon(E) = \cup_{x \in E} B(x; \epsilon). \tag{A.2}$$

Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We denote by

$$d(x, E) = \inf_{y \in E} d(x, y) \tag{A.3}$$

the *distance from the point x to the set E* and, also, if $F \subseteq \mathbb{R}^n$ we define the *distance between the two sets E and F* as

$$d(E, F) = \inf_{x \in E} d(x, F). \tag{A.4}$$

A set $A \subseteq \mathbb{R}^n$ is said to be *open* if for any $x \in A$ there exists an open ball $B(x; r)$ centered in x such that $B(x; r) \subseteq A$. The sets $\mathbb{R}^n$ and $\emptyset$ are open sets. Furthermore, the union of members of an arbitrary family of open sets is open and the intersection of a finite family of open sets is open.

A subset U of $\mathbb{R}^n$ is a *neighborhood* of a point $x \in \mathbb{R}^n$ if U contains an open set that contains x. A set X is open if and only if it contains a neighborhood of any of its point.

A subset C of $\mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \setminus C$ is open, where $\mathbb{R}^n \setminus C$ is the set of all elements of $\mathbb{R}^n$ that are not in C. The sets $\mathbb{R}^n$ and $\emptyset$ are closed sets. Furthermore, the union of a finite number of closed sets is closed an the intersection of the members of an arbitrary family of closed sets is closed.

A point x is an *accumulation point* (or *cluster point*) of a set $E \subseteq \mathbb{R}^n$ if every neighborhood of x contains points of E besides x.

A subset of $\mathbb{R}^n$ is closed if and only if it contains the set of its accumulation points.

The *closure* of a set $E \subseteq \mathbb{R}^n$, denoted with $\overline{E}$, is the union of the set and the set of its accumulation points. So, for any $E \subseteq \mathbb{R}^n$, we have $E \subseteq \overline{E}$ and

$$C \text{ closed} \quad \Leftrightarrow \quad C = \overline{C}.$$

Equivalently, the closure of E is the intersection of the members of the family of all closed sets containing E.

A point x of a set $E \subseteq \mathbb{R}^n$ is an *interior* point of E if E is a neighborhood of x; the set of all the interior points of E is the *interior* of E and it is denoted by $\text{int}(E)$. So, for any $E \subseteq \mathbb{R}^n$, we have $E \supseteq \text{int}(E)$ and

$$A \text{ open} \quad \Leftrightarrow \quad A = \text{int}(A).$$

Equivalently, the interior of E is the union of the members of the family of all open sets contained in E.

The set of the points of E which are interior to neither E nor $\mathbb{R}^n \setminus E$ are in the *boundary* of E, denoted by ∂E.

A set is closed if and only if it contains its boundary; it is open if and only if it is disjoint from its boundary.

A *sequence* of points of $\mathbb{R}^n$ is a function $\tilde{x}\colon \mathbb{N} \to \mathbb{R}^n$. Placing $x_n = \tilde{x}(n)$, we denote the sequence with $(x_n)_n$. Let $(x_n)_n$ be a sequence in $\mathbb{R}^n$ and N_1 be a countably infinite subset of $\mathbb{N}$. Then, the sequence $(x_n)_{n\in N_1}$ is said to be a *subsequence* of $(x_n)_n$.

A sequence $(x_n)_n$ *converges* to a point x in $\mathbb{R}^n$ if for any neighborhood U of x there exists a $n_0 \in \mathbb{N}^+$ such that

$$x_n \in U, \quad \text{for any } n > n_0.$$

Equivalently, sequence $(x_n)_n$ in $\mathbb{R}^n$ converges to a point x in X if for every $\epsilon > 0$ there exists a $\bar{n} \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $d(x_n, x) < \epsilon$, that is the sequence of real numbers $(d(x_n, x))_n$ converges to zero.
Any sequence $(x_n)_n$ converging in $\mathbb{R}^n$ has a unique limit. The limit of a convergent sequence $(x_n)_n$ in $\mathbb{R}^n$ coincides with the limit of any subsequence extract from $(x_n)_n$.
Convergence in the product space is called *coordinatewise convergence*: a sequence $((x_n, y_n))_n$ converges to (x, y) in $\mathbb{R}^n \times \mathbb{R}^m$ if and only if $(x_n)_n$ converges to x in $\mathbb{R}^n$ and $(y_n)_n$ converges to y in $\mathbb{R}^m$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Let E be the set defined as:

$$E = \{x \in \overline{\mathbb{R}} \colon \exists\, (x_n)_{n\in N_1}\,,\ \text{with } N_1 \text{ countably infinite}\,,\ \text{s. t.} \lim_{\substack{n\to+\infty \\ n\in N_1}} x_n = x\}$$

. We define the *upper limit* $\limsup_{n\to+\infty} x_n$ of $(x_n)_{n\in\mathbb{N}}$ as:

$$\limsup_{n\to+\infty} x_n = \sup E, \tag{A.5}$$

where with $\sup E$ we denote the *supremum* of E, that is $\bar{x} = \sup E$, if $\bar{x} \geq x$, for any $x \in E$, and for any $\epsilon > 0$ there exists a $x \in E$ such that $x \geq \bar{x} - \epsilon$ ($\tilde{x}$ is said to be the *maximum* of E, that is $\tilde{x} = \max E$, if $\tilde{x} \in E$ and $\tilde{x} \geq y$, for any $y \in E$).

In analogous way, we define the *lower limit* $\liminf_{n\to+\infty} x_n$ of $(x_n)_{n\in\mathbb{N}}$ as:

$$\liminf_{n\to+\infty} x_n = \inf E, \tag{A.6}$$

where with $\inf E$ we denote the *infimum* of E, that is $\bar{x} = \inf E$, if $\bar{x} \leq y$, for any $y \in E$, and for any $\epsilon > 0$ there exists a $x \in E$ such that $x \leq \bar{x} + \epsilon$ ($\tilde{x}$ is said to be the *minimum* of E, that is $\tilde{x} = \min E$, if $\tilde{x} \in E$ and $\tilde{x} \leq y$, for any $y \in E$).

It can be proved that the following characterization holds:

Proposition A.1.1. *Let* $(x_n)_n$ *be a sequence of real numbers. Then* $\bar{x}$ *is the upper limit of* $(x_n)_n$ *if and only if the following properties hold:*

(i) for any $\epsilon > 0$ *there exists an* $\bar{n} > 0$ *such that* $x_n < \bar{x} + \epsilon$ *for any* $n \geq \bar{n}$*;*

(ii) for any $\epsilon > 0$ *and* $n \in \mathbb{N}$*, there exists an integer* $k > n$ *such that* $x_k > \bar{x} - \epsilon$*.*

Analogously for the lower limit.

For every sequence of real numbers $(x_n)_n$ the following properties are satisfied:

(i) $\liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n$;

(ii) $\limsup_{n\to+\infty} (-x_n) = -\liminf_{n\to+\infty} x_n$;

(iii) (x_n) converges if and only if:

$$\liminf_{n\to+\infty} x_n = \limsup_{n\to+\infty} x_n = \lim_{n\to+\infty} x_n;$$

(iv) if (y_n) is a sequence of real numbers we have:

$$\limsup_{n\to+\infty} (x_n + y_n) \leq \limsup_{n\to+\infty} x_n + \limsup_{n\to+\infty} y_n$$
$$\liminf_{n\to+\infty} (x_n + y_n) \geq \liminf_{n\to+\infty} x_n + \liminf_{n\to+\infty} x_n.$$

If $E \subseteq \mathbb{R}^n$, a point x is an accumulation point for E if and only if there exists a sequence $(x_n)_n$ of points in E converging to x. Then, a subset C of X is closed if and only if the limit of any converging sequence (x_n) of points of C is in C; that is if $(x_n)_n$ is a sequence in C that converges to x then $x \in C$.

A subset K of $\mathbb{R}^n$ is *compact* if any cover by open set has a finite subcover, that is, for every arbitrary collection of open sets $(U_i)_{i\in I}$ such that $K \subseteq \cup_{i\in I} U_i$, there exists a finite set $J \subseteq I$ such that $K \subseteq \cup_{j\in J} U_j$.

If K is compact, then it can be shown that every infinite subset of K has an accumulation point.

Let K be a compact subset of $\mathbb{R}^n$ and C a closed subset of K. Then C is compact.

A set K is *sequentially compact* if any sequence of points in K has a subsequence which converges to a point in K.

In $\mathbb{R}^n$ the notions of compactness and sequential compactness are equivalent.

The product of compact subsets of Euclidean spaces is compact sets.

A subset X of $\mathbb{R}^n$ is said to be *bounded* if there exists a $r > 0$ and $x \in \mathbb{R}^n$ such that $X \subseteq B(x; r)$.

A subset X of $\mathbb{R}^n$ is a *compact* set if it is a closed and bounded set.

A.2 On Continuous and Semicontinuous Functions

Let X and Y be subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then a function f from X to Y ($f: X \to Y$) is *continuous at a point* x of X if the inverse image of any neighborhood of $f(x)$ is a neighborhood of x. The function f is *continuous* if f is continuous at any point x in X, or, equivalently, if and only if the inverse of any open subset A of Y is a open subset of X, that is if $f^{-1}(A) = \{x \in X : f(x) \in A\}$ is open.

Equivalently, the continuity property can be stated in sequential terms.

Proposition A.2.1. *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A function $f: X \to Y$ is continuous at $x \in X$ if for any sequence $(x_n)_n$ in X converging to x, the sequence $(f(x_n))_n$ converges to $f(x)$ in Y.*

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and f be a function defined on the product space $X \times Y$ to $Z \subseteq \mathbb{R}^r$. Then f is *continuous* at x (resp. y) if for any $y \in Y$ (resp. $x \in X$) the function $f(\cdot, y)$ (resp. $f(x, \cdot)$), whose value at x (resp. y) is $f(x, y)$, is continuous at x (resp. y). If f is continuous on the product space, then f is continuous both in x and y, but the converse is no longer true. For example, consider the function f defined on $\mathbb{R}^2$ such that $f(x, y) = \frac{xy}{x^2+y^2}$ for any $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

The *graph* of a function $f : X \to Y$ is the subset of $X \times Y$ of all pairs $(x, f(x))$ such that $x \in X$:

$$\operatorname{Graph} f = \{(x, y) \in X \times Y : y = f(x)\}.$$

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $f : X \to Y$ be continuous. Then, $\operatorname{Graph} f$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}^m$. The vice versa is not always true. For example consider the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us report a result on continuous functions that will be used in Chapter 4.

Theorem A.2.2 (Brouwer Fixed Point Theorem). *Let f be a continuous function from a nonempty compact convex set $X \subseteq \mathbb{R}^n$ to itself. Then f has a fixed point, that there exists a point $x^* \in X$ such that $x^* = f(x^*)$.*

A function f from $X \subseteq \mathbb{R}^m$ to $\mathbb{R}$ is also said to be a *real–valued* function. A function f from $X \subseteq \mathbb{R}^m$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, is said to be an *extended real–valued* function.
The Euclidean norm, the Euclidean distance and the inner scalar product are real–valued continuous functions.
A real–valued function f is continuous at $x \in X$ if and only if the two conditions holds:

(i) for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ such that $d(x, y) < \delta$ we have $f(x) - f(y) < \epsilon$;

(ii) for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ such that $d(x, y) < \delta$ we have $f(x) - f(y) > -\epsilon$;

Requiring only one of the above conditions at a time we obtain two concepts weaker than continuity that are called in literature lower and upper semicontinuity.
More precisely:

Definition A.2.3
A real–valued function f defined on $X \subseteq \mathbb{R}^n$ is *upper semicontinuous at* $x \in X$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that, for any $y \in X$ such that $d(x, y) < \delta$ we have $f(x) - f(y) < \epsilon$. The function f is *upper semicontinuous* if f is upper semicontinuous at every point of X.
A real–valued function g defined on $X \subseteq \mathbb{R}^n$ is *lower semicontinuous* at $x \in X$ if $-g$ is upper semicontinuous at x, that is if for any $\epsilon > 0$ there exists a $\delta > 0$ such that, for any $y \in X$ such that $d(x, y) < \delta$ we have $f(x) - f(y) > -\epsilon$; g is *lower semicontinuous* in X if $-g$ is upper semicontinuous.

So, a function f is continuous if and only if it is upper and lower semicontinuous.

The notion of semicontinuity can be formulated in a sequential way, using the upper and the lower limit.

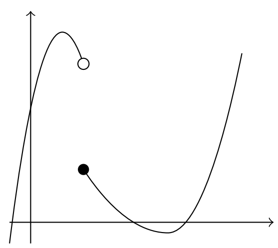

Figure A.1: A lower semi–continuous function on $\mathbb{R}$

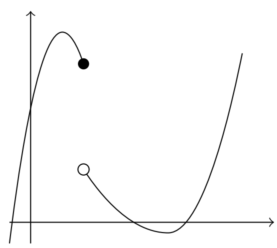

Figure A.2: An upper semi–continuous function on $\mathbb{R}$

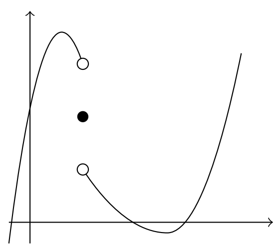

Figure A.3: A neither upper nor lower semi–continuous function on $\mathbb{R}$

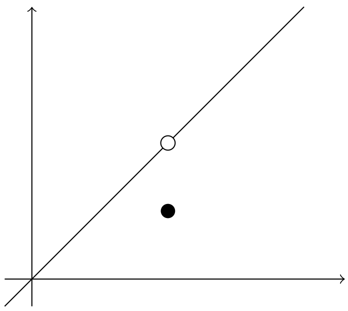

Figure A.4: $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 1/2 & \text{if } x = 1 \end{cases}$ is a lower semi–continuous function on $\mathbb{R}$

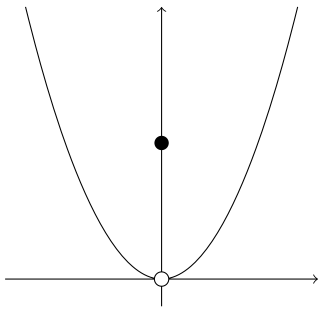

Figure A.5: $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1/2 & \text{if } x = 0 \end{cases}$ is a upper semi–continuous function on $\mathbb{R}$

Proposition A.2.4. *A real–valued function f defined on $X \subseteq \mathbb{R}^n$ is* upper semicontinuous *(resp.* lower semicontinuous*) at x if for any sequence $(x_n)_n$ in X converging to x in X, we have:*

$$\limsup_{n\to+\infty} f(x_n) \leq f(x)$$

$$\left(\textit{resp. } \liminf_{n\to+\infty} f(x_n) \geq f(x) \right).$$

Let f be a real–valued function defined on $X \subseteq \mathbb{R}$. Then, f is *bounded from below* (resp. *bounded from above*) if there exists a number α such that $f(x) \geq \alpha$ (resp. $f(x) \leq \alpha$) for any $x \in X$. The number α is said to be an *upper bound* (resp. *lower bound*) of f on X.
If there exists a lower bound (resp. upper bound) $\bar{\alpha}$ of f on X such that for any $\epsilon > 0$ there exists an element $x \in X$ such that $f(x) \leq \bar{\alpha} + \epsilon$ (resp. $f(x) \geq \bar{\alpha} - \epsilon$), then $\bar{\alpha}$ is said to be the *infimum* (resp. *supremum*) of f on X and we write $\bar{\alpha} = \inf_{x\in X} f(x)$ (resp. $\bar{\alpha} = \sup_{x\in X} f(x)$). If the function f is not bounded from below (resp. from above) then $\inf_{x\in X} f(x) = -\infty$ (resp. $\sup_{x\in X} f(x) = +\infty$).

Let f be a real–valued function defined on $X \subseteq \mathbb{R}^n$. If there exists an $\bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$ (resp. $f(\bar{x}) \geq f(x)$), for any $x \in X$ then $\bar{x}$ is said to be a *global minimum point* (resp. global maximum point) of f on X

and $f(\bar{x})$ is said to be the *minimum value*, or else for short *minimum*, (resp. *maximum value*) of f on X. We write:

$$f(\bar{x}) = \min_{x \in X} f(x) \qquad \text{(resp. } f(\bar{x}) = \max_{x \in X} f(x)) \tag{A.7}$$

and

$$\bar{x} \in \operatorname{Arg}\min_{x \in X} f(x) \qquad \text{(resp. } \bar{x} \in \operatorname{Arg}\max_{x \in X} f(x)). \tag{A.8}$$

Let us emphasize that not every real–valued function has a minimum (resp. maximum) point. Moreover, a minimum (resp. maximum) value, if it exists, must be finite and coincide with the infimum (resp. supremum) of the function.

Furthermore, a point $\bar{x} \in X$ is said to be a *local minimum point* (resp. *local maximum point*) for f on X if there exists a open ball $B(\bar{x}; r)$ such that $f(\bar{x}) \leq f(x)$ (resp. $f(\bar{x}) \geq f(x)$), for any $x \in B(\bar{x}; r) \cap X$.
A global minimum point is also a local one, the vice versa is not true.

Semicontinuous functions are useful in optimization for the existence of a minimum and a maximum value. That is:

Theorem A.2.5 (Generalized Weierstrass Theorem). *A lower (resp. upper) semicontinuous function f defined on a nonempty compact set $X \subseteq \mathbb{R}^n$ has a minimum (resp. maximum); that is, there exists $\bar{x} \in X$ such that, for any $x \in X$, $f(\bar{x}) \leq f(x)$ (resp. $f(\bar{x}) \geq f(x)$).*

Differentiable real–valued functions

Let f be a real–valued function defined on an open set $X \subseteq \mathbb{R}^n$. The i–th *partial derivative* at $\bar{x} \in X$ is:

$$\frac{\partial f}{x_i}(\bar{x}) = \lim_{t \to +\infty} \frac{f(\bar{x} + te^i) - f(\bar{x})}{t}$$

if the limit exists, where e^i is the n–tuple that has all components equal to zero, except the i–th that is equal to 1. Assuming that all the partial derivatives exist at $x \in X$, then the *gradient* of f at $\bar{x}$ is the n-tuple $\nabla f(\bar{x}) = \left(\frac{\partial f}{x_1}(\bar{x}), \ldots, \frac{\partial f}{x_n}(\bar{x})\right)$.

The function f is said to be *differentiable* at $\bar{x} \in X$ if there exists a linear function $A\colon X \to \mathbb{R}$ such that

$$\lim_{\substack{h \to o \\ h \in \mathbb{R}^n}} \frac{\|f(\bar{x}+h) - f(\bar{x}) + A(h)\|}{\|h\|} = 0$$

A function f is differentiable on X if it is differentiable at x, for any $x \in X$.

A.3 On Concave Functions

Convexity and concavity of a function play an important role in economic problems, in particular in problems that are formulated in terms of optimization tools.

Definition A.3.1
A subset X of $\mathbb{R}^n$ is said to be *convex* if $\lambda x + (1-\lambda)y \in X$, for any x, y in X and for any $\lambda \in [0,1]$; that is the closed line segment joining two points of X is included in X.

Definition A.3.2
Let X be a convex subset of $\mathbb{R}^n$. A real–valued function (resp. extended real–valued function) f defined on X is said to be *concave* if for any x, y in X (resp. $x, y \in X$ such that $f(x) > -\infty$, $f(y) > -\infty$) and any $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y). \tag{A.9}$$

A real–valued function (resp. extended real–valued function) f defined on X is said to be *strictly concave* if for any x, y in X (resp. $x, y \in X$ such that $f(x) > -\infty$, $f(y) > -\infty$) with $x \neq y$ and any $\lambda \in]0,1[$

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda) f(y). \tag{A.10}$$

(A.11)

A real–valued function f, defined on X, is *convex* (resp. *strictly convex*) in S if $-f$ is concave (resp. strictly concave) in S.
An extended real–valued function f, defined on X, is *convex* (resp. *strictly convex*) in S if $-f$ is concave (resp. strictly concave) in S.

Definition A.3.3
A real–valued function $f\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \langle a, x\rangle + \alpha$, where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, is a *linear affine* function. It is both convex and concave.

Let us summarize some properties.

Proposition A.3.4. *Let f be a real–valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Then, the following results hold:*

(i) *If f is concave then every local maximum point for f on X is a global maximum point of f on X.*

(ii) *If f is concave then the set $\operatorname{Arg\,max}_{x\in X} f(x) = \{\bar{x} \in X\colon f(\bar{x}) = \max_{x\in X} f(x)\}$ is convex.*

(iii) *Let $\bar{x}$ be a global maximum point of f on X. If f is strictly concave, then $\bar{x}$ is the unique global maximum point.*

(iv) *f is concave (resp. convex) if and only if the hypograph*

$$hyp(f) = \{(x, y) \in X \times Y \times \mathbb{R}\colon f(x) \geq y\} \tag{A.12}$$

(resp. epigraph $epi(f) = \{(x, y) \in X \times Y \times \mathbb{R}\colon f(x) \leq y\}$) is a convex subset of $\mathbb{R}^{n+1}$.

(v) *f is concave if and only if for any $x_1, x_2 \in X$ the function $\phi(t) = f(tx_1 + (1-t)x_2)$ is concave on the segment $[0, 1]$.*

(vi) *If X is open and f is concave on X, then f is continuous on X.*

(vii) *If f is a concave function and $\lambda \geq 0$, then the function λf is also a concave function. If f_1 and f_2 are concave functions defined on X, then the function $f_1 + f_2$ is also concave.*

(viii) Let $(f_i)_{i\in I}$ be a collection of concave functions defined on X, with I finite or infinite. Let

$$\tilde{f}(x) = \inf_{i\in I} f_i(x)$$

for any $x \in X$. Then, the function $\tilde{f}$ is concave on X.

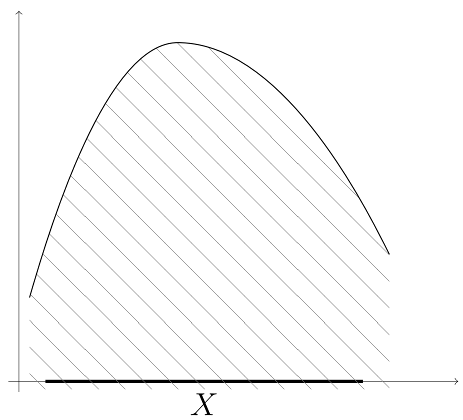

Figure A.6: The hypograph of a concave function

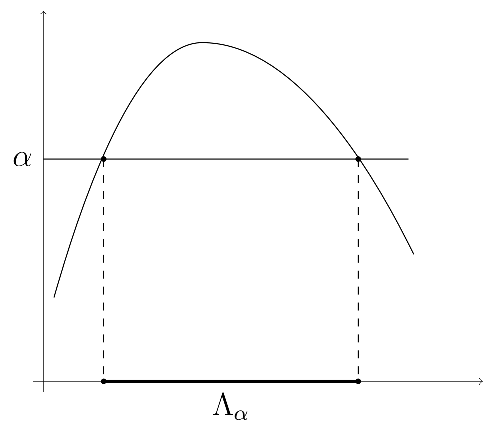

Figure A.7: The upper level set Λ_α of a concave function

Corollary A.3.5. *If f is concave on a convex set $X \subseteq \mathbb{R}^n$, then, the set $\Lambda_\alpha = \{x \in \mathbb{R}^n \colon f(x) \geq \alpha\}$ called* upper level set at height α *(or else* upper contour set at height α*) is convex, for any real number α.*
If f is convex on X, then, the set $\Gamma_\alpha = \{z \in \mathbb{R}^n \colon f(x) \leq \alpha\}$ called lower level set at height α *(or else* lower contour set at height α*) is convex, for any real number α.*

The last corollary give a necessary, but not sufficient condition for concavity, as we see in Figure A.8, in which is represented the graph of a concave function with convex lower level sets.

A differentiable function satisfies the following property:

Proposition A.3.6. *Let f be a differentiable real–valued function defined on an open convex set $X \subseteq \mathbb{R}^n$. Then f is concave on X if and only if*

$$f(x_2) - f(x_1) \leq \langle \nabla f(x_1), x_2 - x_1 \rangle \qquad \text{for any } x_1, x_2 \in X. \tag{A.13}$$

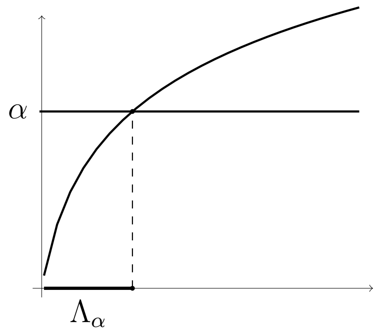

Figure A.8: Each lower contour set of f is convex but f is not a convex function

f is strictly concave if and only if the inequality in A.13 is strict for $x_1 \neq x_2$.

As pointed out in [57], the above proposition can be interpreted geometrically saying that for a differentiable concave function the linearization $f(\bar{x}) + \langle \nabla f(\bar{x})(x - \bar{x}) \rangle$ at $\bar{x}$ never underestimates $f(x)$ for any $x \in X$.

Proposition A.3.7. *Let f be a real–valued differentiable function on an open convex set $X \subseteq \mathbb{R}^n$. Then f is concave on X if and only if for any $x_1, x_2 \in X$ we have*

$$\langle \nabla f(x_2) - \nabla f(x_1), x_2 - x_1 \rangle \leq 0; \tag{A.14}$$

that is the operator ∇f is monotone on X, where a n-dimensional function g defined on $X \subseteq \mathbb{R}^n$ is said to be monotone on X if $\langle g(x_2) - g(x_1), x_2 - x_1) \rangle \leq 0$ for all $x_1, x_2 \in X$. f is strictly concave if and only if the inequality in A.14 is strict for $x_1 \neq x_2$.

Finally, the following theorem states that for a concave function the first–order necessary conditions for optimality are also sufficient:

Theorem A.3.8. *Let f a differentiable concave function defined on a convex set $X \subseteq \mathbb{R}$. If $\nabla f(x^*) = 0$ at a point $x^* \in X$, then x^* is a global maximum point of f on X.*

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