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VARIATIONAL METHODS FOR A PSEUDO-RELATIVISTIC SCHRÖDINGER EQUATION

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"All our dreams can come true,
if we have the courage to pursue them"

Walter Elias Disney

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Abstract

This work concerns the study of the solutions to elliptic nonlinear fractional equations involving the pseudo-differential operator $\sqrt{-\Delta + m^2} - m$. This operator plays an important role in quantum mechanics since it corresponds to the kinetic energy of a free relativistic particle of mass m. Physical models related to this operator have been widely studied over the past 30 years and there exists an important literature on its properties. Most of them have been strongly influenced by Lieb's investigations on the stability of matter.

There is also a deep connection between $\sqrt{-\Delta + m^2} - m$ and the theory of stochastic processes: l'operator in question is an infinitesimal generator of a Lévy process, called 1-stable relativistic process.

This thesis is divided into two parts.

The first part is devoted to the existence of bounded monotone heteroclinic solutions to the problem

$$\begin{cases} \left(\sqrt{-\frac{d^2}{dx^2} + m^2} - m\right)u = -G'(u) \text{ in } \mathbb{R} \\ u(\pm\infty) = \pm 1 \end{cases}$$

where $m > 0$, $G \in C^{2,\alpha}(\mathbb{R})$ is even and it has two and only two absolute minima localized in ± 1. The same problem with $m = 0$ has been already treated by Cabré and Solà-Morales. Our goal is to extend their results to a more general operator. The above equation can be realized as a local elliptic equation in $\mathbb{R}^2_+$ together with a nonlinear Neumann boundary condition on $\partial\mathbb{R}^2_+$. We exploit this fact, and by using variational methods, we prove the existence of a solution to

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\} \\ \frac{\partial v}{\partial \nu} = mv - G'(v) & \text{on } \partial\mathbb{R}^2_+ = \{(x,0) : x \in \mathbb{R}\} \end{cases},$$

odd and monotone increasing in x, with limits $\pm e^{-my}$ as $x \to \pm\infty$. Under the additional assumption $G''(\pm 1) > 0$, we show the uniqueness of such solution. By a limit procedure as $m \to 0$, we obtain the existence of a nontrivial heteroclinic solution to the problem

$$\begin{cases} \sqrt{-\frac{d^2}{dx^2}}u = -G'(u) \text{ in } \mathbb{R} \\ u(\pm\infty) = \pm 1 \end{cases}.$$

The second part of the thesis is concerned with the existence of T-periodic solutions to the problem

$$\begin{cases} (\sqrt{-\Delta_x + m^2} - m)u = f(x,u) & \text{in } (0,T)^N \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, \quad i = 1,\dots,N \end{cases},$$

where (e_i) is the standard basis in $\mathbb{R}^N$, the nonlinearity $f(x,u)$ is a locally Lipschitz function, T-periodic in x, satisfying the Ambrosetti -Rabinowitz condition and a polynomial growth at rate p for some subcritical exponent $1 < p < 2^\sharp - 1$.

To our knowledge, these are the first results on the periodic problems involving fractional operators of elliptic type.

Similarly to the first part, in order to prove the existence of T-periodic solutions, we exploit the fact that the square root of the operator $-\Delta + m^2$ in $(0,T)^N$ with periodic boundary conditions, can be realized as through a local problem in $(0,T)^N \times (0,\infty)$. More precisely, we consider the corresponding elliptic problem in the half-cylinder $(0,T)^N \times (0,\infty)$ with a nonlinear Neumann boundary condition

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } (0,T)^N \times (0,\infty) \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) \\ \frac{\partial v}{\partial \nu} = mv + f(x,v) & \text{on } (0,T)^N \times \{0\} \end{cases}.$$

This problem has a variational nature and its solutions can be found as critical points of a suitable functional associated to the elliptic boundary problem. We get such critical points using the Linking Theorem and we study also their regularity. Taking the limit as $m \to 0$ in the weak formulation of the problem in the half-cylinder, we are able to obtain the existence of a nontrivial solution to the problem

$$\begin{cases} -\Delta v = 0 & \text{in } (0,T)^N \times (0,\infty) \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) \\ \frac{\partial v}{\partial \nu} = f(x,v) & \text{on } (0,T)^N \times \{0\} \end{cases},$$

and, as a consequence, the existence of a nontrivial solution to

$$\begin{cases} \sqrt{-\Delta_x} u = f(x,u) & \text{in } (0,T)^N \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, \quad i = 1,\dots,N \end{cases}.$$

Contents

Chapter 1

Introduction and Summary of results

1.1 Introduction

Recently, the study of fractional and non-local operators of elliptic type has attracted the attention of many mathematicians. One of the reasons for this comes from the fact that these operators frequently appear in many different areas of research and find applications in optimization, finance, the thin obstacle problem, phase transitions, anomalous diffusion, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows and water waves. For more details and applications see [10], [11], [17], [22], [26], [32], [33], [52], [57], [71], [76], [80] and references therein.

In this work we focus our attention on nonlocal fractional equations involving the pseudo-differential operator

$$\sqrt{-\Delta + m^2} - m \tag{1.1.1}$$

where m is a real nonnegative number. We note that, when $m = 0$, (1.1.1) reduces to the square root of the Laplacian which has been widely studied: see for instance [17], [18],[21], [72], [79], [80].

One of the main reasons for the study of the operator (1.1.1) is due to its importance in quantum mechanics. It is indeed the kinetic energy of a relativistic particle of mass m. Such operator has been extensively studied in literature: Herbst [44] and Daubechies [28] studied the spectral properties of the operator $\sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} - \sum_{k=1}^{N} Z_k e_k^2 |x - R_k|^{-1}$, Fefferman [35] considered the N-body problem in quantum mechanics, Lieb and Yau in [51]and [52] investigated on the stability of relativistic matter, Fröhlich and Lenzmann [38] proved the existence of solitary waves for a pseudo-relativistic Hartree-Fock equation with a Newton potential.

On the other hand, there is a deep connection between $\sqrt{-\Delta+m^2}-m$ and the theory of stochastic processes: the operator in question is an infinitesimal generator of a Lévy process called the 1-stable relativistic process. For a probabilistic approach to this operator and more general α-stable processes we refer to [7], [24] and [68].

In this work we focus our attention on two problems.

First, in Chapter 2 we study the bounded monotone heteroclinic (or layer) solutions to the problem

$$\begin{cases} \left(\sqrt{-\frac{d^2}{dx^2}+m^2}-m\right)u=-G'(u) \text{ in } \mathbb{R} \\ u(\pm\infty)=\pm 1 \end{cases} \tag{1.1.2}$$

where $m>0$ and the nonlinearity $G\in C^{2,\alpha}(\mathbb{R})$ is even and it has two and only two absolute minima localized in ± 1.

The same problem for $m=0$ has been discussed in [17] by Cabré and Solà-Morales. Our purpose is to generalize their results to the operator (1.1.1).

One of the main difficulty of the analysis of (1.1.2), is the nonlocal character of the involved fractional operator. To circumvent this problem, we follow the approach in [21] (see also [17]), which consists of realizing the operator $\sqrt{-\frac{d^2}{dx^2}+m^2}$ as the Dirichlet to Neumann operator associated to the panharmonic extension (i.e. functions satisfying the equation $-\Delta z+m^2z=0$) in the half-space. This leads us to study the solutions to the following boundary reaction problem in $\mathbb{R}^2_+$

$$\begin{cases} -\Delta v+m^2v=0 & \text{in } \mathbb{R}^2_+=\{(x,y)\in\mathbb{R}^2 : y>0\} \\ \frac{\partial v}{\partial\nu}=mv-G'(v) & \text{on } \partial\mathbb{R}^2_+=\{(x,0): x\in\mathbb{R}\} \end{cases}. \tag{1.1.3}$$

By using variational techniques, we prove that (1.1.3) admits a bounded solution, odd and strictly monotone increasing in x, with limits $\pm e^{-my}$ at $x=\pm\infty$. Moreover, under the additional hypothesis $G''(\pm 1)>0$, we show that such solution is unique. Taking the limit as $m\to 0$ in (1.1.3), we are able to prove the existence of a monotone heteroclinic solution to

$$\begin{cases} \sqrt{-\frac{d^2}{dx^2}}u=-G'(u) \text{ in } \mathbb{R} \\ u(\pm\infty)=\pm 1 \end{cases}. \tag{1.1.4}$$

Second, we deal with the existence of periodic solutions to

$$\begin{cases} (\sqrt{-\Delta_x+m^2}-m)u=f(x,u) & \text{in } (0,T)^N \\ u(x+Te_i)=u(x) & \text{for } x\in\mathbb{R}^N, \quad i=1,\dots,N \end{cases}, \tag{1.1.5}$$

when $f(x,u)$ is a T-periodic function in x, $f(x,u)$ is a locally Lipschitz function satisfying the Ambrosetti-Rabinowitz condition and a polynomial growth at rate p for some subcritical exponent $1<p<2^\sharp-1=\frac{N+1}{N-1}$.

This problem has been particularly motivated by papers [18] and [70], in which has been considered a fractional analogue of the classical boundary value problem

$$\begin{cases} -\Delta_x u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f(x,u)$ behaves like $|u|^{p-2}u$ for some $2 < p < \frac{2N}{N-2}$. We remind that (1.1.6) has been also investigated with periodic boundary conditions by using variational and topological methods; see for instance [8], [46], [56], [67], [77] and references therein. Our aim has been to obtain an analogue to (1.1.6) with periodic boundary conditions, when we replace $-\Delta_x$ by $\sqrt{-\Delta_x + m^2}$.

Analogously to the first problem (1.1.2), to overcome the non-local character of operator $\sqrt{-\Delta_x + m^2}$ in $(0,T)^N$, the study of the equation (1.1.5) is made via a careful analysis of the following elliptic problem in the half-cylinder $(0,T)^N \times (0,\infty)$ with a nonlinear Neumann boundary condition

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } (0,T)^N \times (0,\infty) \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) \\ \frac{\partial v}{\partial \nu} = mv + f(x,v) & \text{on } (0,T)^N \times \{0\} \end{cases}. \tag{1.1.7}$$

Using the Linking Theorem due to Rabinowitz, we prove the existence of solutions to (1.1.7), T-periodic in x and we also study their regularity. Taking the limit as $m \to 0$ in (1.1.7), we prove the existence of a non trivial T-periodic solution to

$$\begin{cases} (\sqrt{-\Delta_x + m^2} - m)u = f(x,u) & \text{in } (0,T)^N \\ u(x + Te_i) = u(x) & \text{for } x \in \mathbb{R}^N, \quad i = 1, \dots, N \end{cases}. \tag{1.1.8}$$

In the next section we give some applications where (1.1.1) appears. Then we introduce the pseudo-relativistic Schrödinger operator and we collect some results about its main properties. Later we motivate the study of heteroclinic solutions and periodic solutions, and we recall some known results about these problems. Finally, we give a summary of the results contained in this thesis.

1.2 Applications

In this section we briefly describe some of the problems where the pseudo-differential operator $\sqrt{-\Delta + m^2} - m$ arises.

1.2.1 Theory of Lévy process

Let $m > 0$ and $\alpha \in (0,2)$. Let $(X_t^{\alpha,m}, P^x)$ be an $\mathbb{R}^N$-valued symmetric Lévy's process (that is a process with independent and stationary increments) having the following characteristic function:

$$\mathbb{E}^0(e^{i\xi\cdot X_t^{\alpha,m}}) = e^{-t[(m^{\frac{2}{\alpha}}+|\xi|^2)^{\frac{\alpha}{2}}-m]}, \quad \xi \in \mathbb{R}^N \tag{1.2.1}$$

where $\mathbb{E}^x$ denotes the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. We assume that sample paths of $X_t^{\alpha,m}$ are right-continuous and have left-hand limits almost surely. The process is Markov with transition probabilities given by $P_t(x, A) = P^x(X_t^{\alpha,m} \in A) = \mu_t(A - x)$, where μ_t is the one-dimensional distribution of $X_t^{\alpha,m}$ with respect to P^0. It is well known that $(X_t^{\alpha,m}, P^x)$ is strong Markov with respect to the so-called standard filtration, see, e.g., [12]. We call $X_t^{\alpha,m}$ the relativistic α-stable process.

The limiting case corresponding to $m = 0$, is a rotationally symmetric α-stable Lévy process with the characteristic function $e^{-t|\xi|^\alpha}$, $\xi \in \mathbb{R}^N$. When $\alpha = 1$ it turns out that $-L = m - \sqrt{-\Delta + m^2}$ is the infinitesimal generator of the contraction semigroup corresponding to $X_t^{1,m}$, that is, denoting with $T_t = e^{-tL}$ with $t \geq 0$, we have

a) $T_{s+t}f = T_s(T_tf) = T_t(T_sf)$, for all $f \in L^2(\mathbb{R}^N)$ and $s, t \geq 0$;

b) The function $t \mapsto T_tf$ is continuous on $L^2(\mathbb{R}^N)$, that is for every $f \in L^2(\mathbb{R}^N)$

$$\lim_{t\to s} ||T_tf - T_sf||_{L^2(\mathbb{R}^N)} = 0;$$

c) $||T_tf||_{L^2(\mathbb{R}^N)} \leq ||f||_{L^2(\mathbb{R}^N)}$, for all $f \in L^2(\mathbb{R}^N)$;

d) $T_0f = f$, for all $f \in L^2(\mathbb{R}^N)$;

e) If $f \in H^1(\mathbb{R}^N)$, then

$$\frac{d}{dt}T_tf = \lim_{\varepsilon\to 0} \frac{T_{t+\varepsilon}f - T_tf}{\varepsilon} = -LT_tf$$

where the limit is taken in L^2-sense.

Various fine properties of relativistic α-stable processes have been studied in the last twenty years. Carmona, Masters and Simon [24] studied the decay of eigenfunctions of relativistic Schrödinger operators of the type

$$\sqrt{-\Delta + m^2} - m + V$$

where V is the operator of multiplication by a function V. Kulczycki and Siudeja [47] studied intrinsic ultracontractivity for the Feynman-Kac semigroup for relativistic α-stable process with generator

$$-((-\Delta + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m) - V$$

where $V \geq 0$ and V is locally bounded. Ryznar [68] derived sharp estimates for the Green function of the relativistic α-stable process on $\mathcal{C}^{1,1}$ domains, and Grzywny and Ryznar [43] found optimal estimates for the Green function of a half space of the relativistic α-stable process. Chen, Kim and Song [25] established sharp two-sided estimates on the transition density of the subprocess of X^m killed upon exiting any $\mathcal{C}^{1,1}$ open sets. Park and Song [61] studied the asymptotic behavior, as $t \to 0$, of the trace of the semigroup of a killed relativistic α-stable process in bounded $\mathcal{C}^{1,1}$ open sets and bounded Lipschitz open sets.

1.2.2 Quantum mechanics

The non-local operator (2.1.3) plays an important role in relativistic quantum mechanics. From the relativity theory, we know that the Hamiltonian for the motion of a free relativistic particle of mass m and momentum p is given by

$$\mathcal{H} = \sqrt{p^2c^2 + m^2c^4}$$

where c denotes the speed of light. Using the usual quantization rule $p \to -i\hbar\nabla$, where $\hbar$ is the Planck's constant, we get the so called pseudo-relativistic Hamiltonian operator and the associated Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi = \sqrt{-\hbar^2c^2\Delta + m^2c^4}\psi.$$

Roughly speaking, we can note that, when $m \to 0$ this equation reduces to

$$i\hbar\frac{\partial\psi}{\partial t} = (-\hbar^2c^2\Delta)^{\frac{1}{2}}\psi$$

the Riesz fractional Schrödinger equation as proposed by Laskin [48]. On the other hand, in the nonrelativistic limit, $c \to \infty$, we recover the classical Schrödinger equation as

$$\begin{aligned}\lim_{c\to\infty}[\sqrt{-\hbar^2c^2\Delta + m^2c^4} - mc^2] &= \lim_{c\to\infty}\Big[\sqrt{-\frac{\hbar^2}{m^2c^2}\Delta + 1} - 1\Big]mc^2 \\ &= -\frac{\hbar^2}{2m}\Delta.\end{aligned}$$

There is a wide literature concerned the study of relativistic Schrödinger operators for physical models. Lieb and Yau [52] considered the quantum mechanical many-body problem of electrons and fixed nuclei interacting via Coulomb forces with relativistic kinetic energy $(p^2c^2 + m^2c^4)^{1/2} - mc^2$. In [51] the authors continued their work on stability of quantum mechanical relativistic matter, considering the

Schrödinger operator for electrically neutral gravitating particles, either fermions or bosons, in the limit of the particle number N tending to infinity, and the gravitational constant G going to zero. Herbst [44] and Daubechies [28] discussed the spectral properties of the operator

$$\sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} - \sum_{k=1}^{N} Z_k e_k^2 |x - R_k|^{-1}.$$

More recently Fröhlich, Jonsson, Lars and Lenzmann [38] studied the existence of travelling solitary wave solutions $\psi(t,x) = e^{i\mu t}\varphi_v(x - vt)$, for some $\mu \in \mathbb{R}$ and $|v| < 1 = c$, of the pseudo-relativistic Hartree equation

$$i\partial_t \psi = (\sqrt{-\Delta + m^2} - m)\psi - (|x|^{-1} * |\psi|^2)\psi \text{ on } \mathbb{R}^3.$$

Coti Zelati and Nolasco [27] proved the existence of positive stationary solutions for a class of nonlinear pseudo-relativistic Schrödinger equations of the type

$$\sqrt{-\Delta + m^2} u = \mu u + \nu |u|^{p-2} u + \sigma (W * u^2) u \text{ on } \mathbb{R}^N,$$

where $N \geq 2$, $p \in (2, \frac{2N}{N-1})$, $\mu < m$, $\nu, \sigma \geq 0$ (but not both zero), $W \geq 0$, $W(x) = W(|x|) \to 0$ as $|x| \to 0$, $W \in L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $r > \frac{N}{2}$.

1.3 Preliminaries

1.3.1 $\sqrt{-\Delta + m^2}$ in $\mathbb{R}^N$

In this chapter we define the pseudo-relativistic operator $\sqrt{-\Delta + m^2} - m$ using Fourier transform. For more details we refer to [49] and [74]. We begin by introducing some notations that will be used throughout this work.

Let $f \in L^1(\mathbb{R}^N)$. The Fourier transform of f, denoted by $\mathcal{F}f$, is the function on $\mathbb{R}^N$ defined by letting

$$\mathcal{F}f(k) = \int_{\mathbb{R}^N} e^{-2\pi i k \cdot x} f(x)dx \tag{1.3.1}$$

and the inverse Fourier transform of f is defined by

$$\mathcal{F}^{-1}f(k) = \int_{\mathbb{R}^N} e^{2\pi i k \cdot x} f(x)dx \tag{1.3.2}$$

where

$$k \cdot x = \sum_{i=1}^{N} k_i x_i.$$

We remark that the above definitions can be extended to $L^2(\mathbb{R}^N)$ functions.
More precisely it holds the following result

Theorem 1 (Plancherel's Theorem)**.** *There exists a map T from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$, called Fourier-Plancherel transform, such that:*

(i) $Tf = \mathcal{F}f$ *for all* $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$*;*

(ii) $||Tf||_{L^2(\mathbb{R}^N)} = ||f||_{L^2(\mathbb{R}^N)}$ *for all* $f \in L^2(\mathbb{R}^N)$*;*

(iii) $(Tf, Tg)_{L^2(\mathbb{R}^N)} = (f, g)_{L^2(\mathbb{R}^N)}$ *for all* $f, g \in L^2(\mathbb{R}^N)$*;*

(iv) The mapping T is an isometric isomorphism from $L^2(\mathbb{R}^N)$ onto $L^2(\mathbb{R}^N)$;

(v) For $f \in L^2(\mathbb{R}^N)$, setting for each $n \in \mathbb{N}$

$$w_n(k) = \int_{|x|\le n} e^{-2\pi i k\cdot x} f(x)dx,$$

we have $w_n \to Tf$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$;

(vi) Conversely, for $f \in L^2(\mathbb{R}^N)$, setting for each $n \in \mathbb{N}$

$$f_n(k) = \int_{|k|\le n} e^{2\pi i k\cdot x} Tf(k)dk,$$

we have $f_n \to f$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$.

With abuse of notation, we will denote again by $\mathcal{F}f$ the Fourier transform of a function $f \in L^2(\mathbb{R}^N)$ and with $\mathcal{F}^{-1}f$ its inverse Fourier transform.

Now we introduce the Sobolev spaces $H^1(\mathbb{R}^N)$ and $H^{\frac{1}{2}}(\mathbb{R}^N)$. A function f is said to be in $H^1(\mathbb{R}^N)$ if $f \in L^2(\mathbb{R}^N)$ and if there exist N functions $g_1, \dots, g_N$ in $L^2(\mathbb{R}^N)$, collectively denoted by ∇f, such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and $i = 1, \dots, N$

$$\int_{\mathbb{R}^N} f(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^N} g_i(x)\varphi(x)dx. \tag{1.3.3}$$

The space $H^1(\mathbb{R}^N)$ can be equipped with the norm

$$||f||_{H^1(\mathbb{R}^N)} = \sqrt{\int_{\mathbb{R}^N} |f(x)|^2 + |\nabla f(x)|^2 dx}. \tag{1.3.4}$$

Using the Fourier transform, we can characterize $H^1(\mathbb{R}^N)$ as follows

Theorem 2. *Let $f \in L^2(\mathbb{R}^N)$ with Fourier transform $\mathcal{F}f$. Then $f \in H^1(\mathbb{R}^N)$ if and only if the function $k \mapsto |k|\mathcal{F}f(k)$ is in $L^2(\mathbb{R}^N)$. If it is in $L^2(\mathbb{R}^N)$, then*

$$\mathcal{F}(\nabla f)(k) = 2\pi i k \mathcal{F}f(k) \tag{1.3.5}$$

and therefore

$$||f||^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1 + 4\pi^2|k|^2)|\mathcal{F}f(k)|^2 dk. \tag{1.3.6}$$

As explained below, this last result allows us to obtain a useful characterization of $||\nabla f||_{L^2(\mathbb{R}^N)}$ in terms of the heat kernel. We remark that the heat kernel is defined on $\mathbb{R}^N \times \mathbb{R}^N$ as

$$e^{t\Delta}(x,y) = (4\pi t)^{-N/2} e^{-\frac{|x-y|^2}{4t}} \quad (t \in \mathbb{R}_+)$$

and its action on a function f is

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^N} e^{t\Delta}(x,y) f(y) dy.$$

If $f \in L^2(\mathbb{R}^N)$ we can see that

$$\mathcal{F}(e^{t\Delta}f)(k) = e^{-4\pi^2|k|^2 t}\mathcal{F}f(k).$$

This last equation expresses the fact that $e^{t\Delta}$ acts on Fourier transform as multiplication by $e^{-t|2k\pi|^2}$. In particular $e^{t\Delta}$ is a fundamental solution to the heat equation $u_t - \Delta_x u = 0$ in $\mathbb{R}_+ \times \mathbb{R}^N$.

Then we can state the following result which explains why $-\Delta$ is the infinitesimal generator of the heat kernel

Theorem 3. *A function f belongs to $H^1(\mathbb{R}^N)$ if and only if $f \in L^2(\mathbb{R}^N)$ and*

$$I^t(f) := \frac{1}{t}[(f,f)_{L^2(\mathbb{R}^N)} - (f, e^{t\Delta}f)_{L^2(\mathbb{R}^N)}]$$

is uniformly bounded in t.

In that case

$$\sup_{t>0} I^t(f) = \lim_{t\to 0} I^t(f) = ||\nabla f||^2_{L^2(\mathbb{R}^N)}.$$

Now we introduce the fractional Slobodeckij-Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^N)$.

An $L^2(\mathbb{R}^N)$ function f is said to be in $H^{\frac{1}{2}}(\mathbb{R}^N)$ if and only if

$$\frac{|f(x) - f(y)|}{|x-y|^{\frac{N+1}{2}}} \in L^2(\mathbb{R}^N). \tag{1.3.7}$$

We endow $H^{\frac{1}{2}}(\mathbb{R}^N)$ with the natural norm

$$||f||_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |f(x)|^2 dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+1}} dxdy\Big)^{\frac{1}{2}} \tag{1.3.8}$$

where

$$[f]_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+1}} dxdy\Big)^{\frac{1}{2}} \tag{1.3.9}$$

is the so called Gagliardo semi-norm of f. $H^{\frac{1}{2}}(\mathbb{R}^N)$ can be also seen as interpolation space between $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$; see for instance [54]. Using the Plancherel's Theorem we can deduce the following result

Theorem 4. *Let $f \in L^2(\mathbb{R}^N)$ and $\mathcal{F}f$ its Fourier transform. Then $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$ if and only if $\int_{\mathbb{R}^N} |k||\mathcal{F}f(k)|^2 dk < \infty$. In particular, for every $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$, $||f||_{H^{\frac{1}{2}}(\mathbb{R}^N)}$ and*

$$\Big(\int_{\mathbb{R}^N} (1+2\pi|k|)|\mathcal{F}f(k)|^2 dk\Big)^{\frac{1}{2}} \tag{1.3.10}$$

are equivalent norms.

From now on, we use this last characterization of $H^{\frac{1}{2}}(\mathbb{R}^N)$ as definition.

We note that $H^{\frac{1}{2}}(\mathbb{R}^N)$ endowed with the inner product

$$(f,g)_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1+2\pi|k|)\overline{\mathcal{F}f(k)}\mathcal{F}g(k)dk$$

is a Hilbert space. Using (1.3.10) and Plancherel's Theorem we can see that

$$\frac{3}{2}||f||^2_{H^1(\mathbb{R}^N)} \geq ||f||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \geq ||f||_{L^2(\mathbb{R}^N)}$$

since $2\pi|k| \leq \frac{1}{2}((2|k|\pi)^2+1)$. This leads to the following chain of inclusions

$$H^1(\mathbb{R}^N) \subset H^{\frac{1}{2}}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \tag{1.3.11}$$

which implies

$$L^2(\mathbb{R}^N) \subset H^{-\frac{1}{2}}(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),$$

where $H^{-\frac{1}{2}}(\mathbb{R}^N)$ and $H^{-1}(\mathbb{R}^N)$ are respectively the dual spaces to $H^{\frac{1}{2}}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. To be more precise, one can define the spaces $H^s(\mathbb{R}^N)$ for all $s \in \mathbb{R}$, and than show that there exists an isometric isomorphism between the topological dual to $H^s(\mathbb{R}^N)$ and $H^{-s}(\mathbb{R}^N)$. For details one can see [31] and [78].

The space $H^{\frac{1}{2}}(\mathbb{R}^N)$ is important in quantum mechanics, since, as explained in Applications 1.2.2, we can consider the Hamiltonian operators of the form

$$\sqrt{-\Delta + m^2}. \tag{1.3.12}$$

for a free relativistic particle of mass m.

Firstly we give the definition of (1.3.12) with $m = 0$ in terms of Fourier transform. The square root of the Laplacian is defined by setting

$$\mathcal{F}(\sqrt{-\Delta} f)(k) = 2\pi|k|\mathcal{F}f(k). \tag{1.3.13}$$

for every $f \in H^1(\mathbb{R}^N)$.

Similarly, the operator (1.3.12) with $m > 0$, is defined in Fourier space as multiplication by $\sqrt{|2\pi k|^2 + m^2}$, that is

$$\mathcal{F}(\sqrt{-\Delta + m^2} f)(k) = \sqrt{|2\pi k|^2 + m^2}\mathcal{F}f(k), \tag{1.3.14}$$

provided that $f \in H^1(\mathbb{R}^N)$. To (1.3.12) we can also associate the following sesquilinear form

$$(g, \sqrt{-\Delta + m^2} f)_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \overline{\mathcal{F}g(k)}\mathcal{F}f(k)\sqrt{|2k\pi|^2 + m^2}dk$$

which makes sense for functions $f, g \in H^{\frac{1}{2}}(\mathbb{R}^N)$.

An alternative way to introduce the operators (1.3.13) and (1.3.14) is that to exploit the theory of the pseudo-differential operators.

We remind that a pseudo-differential operator is of the type

$$(\sigma(x, D)f)(x) = \int_{\mathbb{R}^N} \sigma(x, k)\mathcal{F}f(k)e^{2\pi i k\cdot x}dk \quad f \in \mathcal{S}(\mathbb{R}^N),$$

where $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions and $\sigma \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is a function, called the symbol of order $m \in \mathbb{N}$, with the property that, for every $\alpha, \beta \in \mathbb{N}^N$, there exists a positive constant $C(\alpha, \beta)$ such that

$$|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leq C(\alpha, \beta)(1 + |\xi|)^{m-|\beta|} \quad \forall x, \xi \in \mathbb{R}^N.$$

By this definition we can deduce that $\sqrt{-\Delta}$ and $\sqrt{-\Delta + m^2}$ are pseudo-differential operators of the first order, with symbols $|\xi|$ and $(|\xi|^2 + m^2)^{\frac{1}{2}}$ respectively. For a more careful discussion about the theory of pseudo-differential operators we refer to [45].

By (1.3.14) follows easily that $\sqrt{-\Delta + m^2}$ maps $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$. In particular, it is an isometry from $H^{\frac{1}{2}}(\mathbb{R}^N)$ to $H^{-\frac{1}{2}}(\mathbb{R}^N)$. Indeed, using

$$\int_{\mathbb{R}^N} \sqrt{|2\pi k|^2 + m^2}|\mathcal{F}f(k)|^2 dk$$

as $H^{\frac{1}{2}}(\mathbb{R}^N)$-norm, and recalling that $H^{-\frac{1}{2}}(\mathbb{R}^N)$ is the dual space to $H^{\frac{1}{2}}(\mathbb{R}^N)$ endowed with the norm

$$\int_{\mathbb{R}^N} \frac{1}{\sqrt{|2\pi k|^2 + m^2}}|\mathcal{F}f(k)|^2 dk,$$

by (1.3.14) we deduce

$$\begin{aligned}||\sqrt{-\Delta+m^2}f||^2_{H^{-\frac{1}{2}}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \frac{1}{\sqrt{|2\pi k|^2+m^2}}(|2\pi k|^2+m^2)|\mathcal{F}f(k)|^2 dk \\ &= ||f||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}\end{aligned}$$

that is $\sqrt{-\Delta+m^2}$ is an isometry from $H^{\frac{1}{2}}(\mathbb{R}^N)$ to $H^{-\frac{1}{2}}(\mathbb{R}^N)$.

In order to give an analogue of Theorem 3 for (1.3.12), we need to introduce $\exp\{-t\sqrt{-\Delta+m^2}\}$ for $m \geq 0$.

Firstly we define the Poisson kernel for $m=0$

$$\begin{aligned}e^{-t\sqrt{-\Delta}}(x,y) &:= \mathcal{F}^{-1}(e^{-2\pi t|k|})(x-y) \\ &= \int_{\mathbb{R}^N} e^{-2\pi t|k|} e^{2\pi i k\cdot(x-y)} dk.\end{aligned}$$

This integral has been calculated explicitly, and its result is

$$e^{-t\sqrt{-\Delta}}(x,y) = \Gamma\Big(\frac{N+1}{2}\Big)\pi^{-\frac{N+1}{2}} \frac{t}{(t^2+|x-y|^2)^{\frac{N+1}{2}}}; \tag{1.3.15}$$

see for instance [73] for a proof of the formula (1.3.15). Similarly, when $m>0$, we can introduce the kernel $\exp\{-t\sqrt{-\Delta+m^2}\}$ defined as follows

$$\begin{aligned}e^{-t\sqrt{-\Delta+m^2}}(x,y) &:= \mathcal{F}^{-1}(e^{-t\sqrt{|2\pi k|^2+m^2}})(x-y) \\ &= \int_{\mathbb{R}^N} e^{-t\sqrt{|2\pi k|^2+m^2}} e^{2\pi i k\cdot(x-y)} dk. \end{aligned}\tag{1.3.16}$$

Also for this kernel we have an explicit expression, and it is given by

$$e^{-t\sqrt{-\Delta+m^2}}(x,y) = 2\Big(\frac{m}{2\pi}\Big)^{\frac{N+1}{2}} \frac{t}{[t^2+|x-y|^2]^{\frac{N+1}{4}}} K_{\frac{N+1}{2}}(m\sqrt{t^2+|x-y|^2}) \tag{1.3.17}$$

for $x,y \in \mathbb{R}^N$ and $t>0$. Here K_ν stands for the modified Bessel function of the third kind of order $\nu \in \mathbb{R}_+$, and its integral representation is given by

$$K_\nu(x) = \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty e^{-t}\Big(t+\frac{t^2}{2x}\Big)^{\nu-\frac{1}{2}} dt \quad (x>0).$$

Let us note that K_ν satisfies (see [34])

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2}\Big(\frac{r}{2}\Big)^{-\nu} \text{ as } r\to 0 \tag{1.3.18}$$

and

$$K_\nu(r) \sim \sqrt{\frac{\pi}{2}} r^{-\frac{1}{2}} e^{-r} \text{ as } r \to +\infty. \tag{1.3.19}$$

As explained in [49], the formula (1.3.17) follows from

$$\int_{\mathbb{S}^{N-1}} e^{i\omega\cdot x} d\omega = (2\pi)^{\frac{N}{2}} |x|^{1-\frac{N}{2}} J_{\frac{N}{2}-1}(|x|)$$

and from

$$\int_0^\infty x^{\nu+1} J_\nu(xy) e^{-\alpha\sqrt{x^2+\beta^2}} dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \alpha\beta^{\nu+\frac{3}{2}} (y^2+\alpha^2)^{-\frac{\nu}{2}-\frac{3}{4}} y^\nu K_{\nu+\frac{3}{2}}(\beta\sqrt{y^2+\alpha^2}).$$

Here J_ν is the Bessel function of ν-th order. The above kernel is a positive, $L^1(\mathbb{R}^N)$-function of $(x-y)$, and so, by Young's inequality, it maps $L^p(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ for all $p \geq 1$. Moreover, in any dimension $N \geq 1$

$$\int_{\mathbb{R}^N} e^{-t\sqrt{-\Delta+m^2}}(x,y) dy = e^{-tm} \tag{1.3.20}$$

since the left side of (1.3.20) is the inverse Fourier transform of (1.3.17) evaluated at $k = 0$.

Now, following the proof of Theorem 7.12 in [49], we can prove that

Theorem 5. *Let $m \geq 0$. A function f is in $H^{\frac{1}{2}}(\mathbb{R}^N)$ if and only if $f \in L^2(\mathbb{R}^N)$ and*

$$I^t(f) := \frac{1}{t}[(f,f)_{L^2(\mathbb{R}^N)} - (f, e^{-t\sqrt{-\Delta+m^2}} f)_{L^2(\mathbb{R}^N)}]$$

is uniformly bounded in t, in which case

$$\sup_{t>0} I^t(f) = \lim_{t\to 0} I^t(f) = (f, \sqrt{-\Delta+m^2} f)_{L^2(\mathbb{R}^N)}. \tag{1.3.21}$$

Moreover holds the following formulas for $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$:

(i) If $m = 0$ then

$$\begin{aligned}(f, \sqrt{-\Delta} f)_{L^2(\mathbb{R}^N)} &= \frac{\Gamma(\frac{N+1}{2})}{\pi^{\frac{N+1}{2}}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+1}} dxdy \\ &= \Gamma\left(\frac{N+1}{2}\right) \pi^{-\frac{N+1}{2}} [f]^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}.\end{aligned} \tag{1.3.22}$$

(ii) If $m > 0$ then

$$\begin{aligned}&(f, (\sqrt{-\Delta+m^2} - m) f)_{L^2(\mathbb{R}^N)} \\ &= \left(\frac{m}{2\pi}\right)^{\frac{N+1}{2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{\frac{N+1}{2}}} K_{\frac{N+1}{2}}(m|x-y|)\, dxdy.\end{aligned} \tag{1.3.23}$$

Proof. By Theorem 4 it is sufficient to show that $f \in L^2(\mathbb{R}^N)$ and $I^t(f)$ is uniformly bounded in t if and only if

$$\int_{\mathbb{R}^N} (1+|2k\pi|)|\mathcal{F}f(k)|^2 dk < \infty.$$

Note that by Plancherel's Theorem we know that for every $t > 0$

$$\frac{1}{t}[(f,f)_{L^2(\mathbb{R}^N)} - (f, e^{-t\sqrt{-\Delta+m^2}} f)_{L^2(\mathbb{R}^N)}] = \int_{\mathbb{R}^N} \frac{1-e^{-t\sqrt{|2\pi k|^2+m^2}}}{t} |\mathcal{F}f(k)|^2 dk. \tag{1.3.24}$$

Since the function $r^{-1}(1-e^{-r})$ is decreasing and positive for $r > 0$ we can see that $\frac{1-e^{-t\sqrt{|2\pi k|^2+m^2}}}{t}$ converges monotonically to $\sqrt{|2\pi k|^2+m^2}$ as $t \to 0$.

Therefore if $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$, using the fact $\sqrt{|2\pi k|^2+m^2} \le C(m)(1+|2k\pi|)$ when $m > 0$, we can deduce that $I^t(f)$ is uniformly bounded in t by Dominated Convergence Theorem. Conversely, assume that $I^t(f)$ is uniformly bounded in t and $f \in L^2(\mathbb{R}^N)$. By Monotone Convergence Theorem follows that

$$\infty > \sup_{t>0} I^t(f) = \lim_{t\to 0} I^t(f) = \int_{\mathbb{R}^N} \sqrt{|2k\pi|^2+m^2}|\mathcal{F}f(k)|^2 dk$$

that is $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$.

Now we only prove (ii) because the proof of (i) can be found in [53].

Firstly, we observe that

$$\begin{aligned}
I^t(f) &= \frac{1}{t}[(f,f)_{L^2(\mathbb{R}^N)} - (f, e^{-t(\sqrt{-\Delta+m^2}-m)} e^{-mt} f)_{L^2(\mathbb{R}^N)}] \\
&= \frac{1-e^{-mt}}{t}(f,f)_{L^2(\mathbb{R}^N)} + \frac{e^{-mt}}{t}[(f,(1-e^{-t(\sqrt{-\Delta+m^2}-m)})f)_{L^2(\mathbb{R}^N)}] \\
&=: I_1^t(f) + I_2^t(f).
\end{aligned} \tag{1.3.25}$$

It's clear that

$$I_1^t(f) \to m(f,f)_{L^2(\mathbb{R}^N)} \text{ as } t \to 0. \tag{1.3.26}$$

By applying (1.3.17) and (1.3.20), we can see that

$$\begin{aligned}
I_2^t(f) &= \frac{e^{-mt}}{t}\Big\{\int_{\mathbb{R}^N} |f(x)|^2 \Big[\int_{\mathbb{R}^N} e^{-t(\sqrt{-\Delta+m^2}-m)}(x,y)dy\Big]dx \\
&\quad - \int_{\mathbb{R}^N} f(x)\Big[\int_{\mathbb{R}^N} e^{-t(\sqrt{-\Delta+m^2}-m)}(x,y)f(y)dy\Big]dx\Big\} \\
&= \frac{e^{-mt}}{2t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x)-f(y)|^2 e^{-t(\sqrt{-\Delta+m^2}-m)}(x,y)dxdy \\
&= \left(\frac{m}{2\pi}\right)^{\frac{N+1}{2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{(t^2+|x-y|^2)^{\frac{N+1}{4}}} K_{\frac{N+1}{4}}(m\sqrt{t^2+|x-y|^2})dxdy.
\end{aligned} \tag{1.3.27}$$

Using (1.3.18), (1.3.19) and the hypothesis $f \in H^{\frac{1}{2}}(\mathbb{R}^N)$, we have

$$\lim_{t\to 0} I_2^t(f) = \left(\frac{m}{2\pi}\right)^{\frac{N+1}{2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{\frac{N+1}{2}}} K_{\frac{N+1}{2}}(m|x-y|). \tag{1.3.28}$$

Taking into account (1.3.21), (1.3.25), (1.3.26) and (1.3.28) we can infer that

$$\begin{aligned}(f, \sqrt{-\Delta+m^2} f)_{L^2(\mathbb{R}^N)} =& m(f,f)_{L^2(\mathbb{R}^N)} + \\ &+ \left(\frac{m}{2\pi}\right)^{\frac{N+1}{2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{\frac{N+1}{2}}} K_{\frac{N+1}{2}}(m|x-y|)dxdy\end{aligned}$$

that is (1.3.23). □

1.3.2 $\sqrt{-\Delta+m^2}$ in $(0,T)^N$

The operator $\sqrt{-\Delta+m^2}$ in $\mathbb{R}^N$ has been defined using the Fourier transform as

$$\sqrt{-\Delta+m^2} f(x) = \int_{\mathbb{R}^N} \sqrt{|2\pi k|^2+m^2}\,\mathcal{F}f(k) e^{i2\pi k\cdot x} dk. \tag{1.3.29}$$

Similarly, we can define the operator $\sqrt{-\Delta+m^2}$ on $(0,T)^N$ via the Fourier series. To do this, we have to introduce the Sobolev spaces $\mathbb{H}^1_T$ and $\mathbb{H}^{\frac{1}{2}}_T$ of periodic functions. Let $\mathcal{C}^\infty_T(\mathbb{R}^N)$ be the space of functions $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that u is T-periodic in each x_i, that is

$$u(x+Te_i) = u(x) \text{ for all } x \in \mathbb{R}^N \quad i = 1, \dots, N$$

where (e_i) is the canonical basis in $\mathbb{R}^N$.

Let $\omega = \frac{2\pi}{T}$ and we denote by

$$\varphi_k(x) = \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} \text{ and } \lambda_k = \omega^2|k|^2 + m^2$$

for every $x \in \mathbb{R}^N$ and $k \in \mathbb{Z}^N$, the eigenfunctions and the eigenvalues of $-\Delta+m^2$ on $(0,T)^N$ with periodic conditions. Thus $\{\varphi_k\}_{k\in\mathbb{Z}^N}$ is an orthonormal basis in $L^2(0,T)^N$, that is for all $h, k \in \mathbb{Z}^N$

$$\int_{(0,T)^N} \varphi_h(x)\overline{\varphi_k}(x)dx = \delta_{hk}$$

where $\delta_{hk} = 1$ if $h = k$ and zero otherwise, and

$$||\varphi_k||_{L^2(0,T)^N} = 1.$$

By Fourier expansion we know that for any function $u \in \mathcal{C}_T^\infty(\mathbb{R}^N)$

$$u(x) = \sum_{k\in\mathbb{Z}^N} c_k \varphi_k(x) \quad (x \in \mathbb{R}^N) \tag{1.3.30}$$

where

$$c_k = \int_{(0,T)^N} u(x)\overline{\varphi_k(x)}dx \quad (k \in \mathbb{Z}^N)$$

are the Fourier coefficients of u. We note that the series in (1.3.30) is uniformly convergent, and the derivative of u are given by

$$\frac{\partial u}{\partial x_j} = i\omega \sum_{k\in\mathbb{Z}^N} k_j c_k \varphi_k(x) \quad j = 1, \dots, N.$$

It follows from Parseval's identity that

$$\int_{(0,T)^N} |\nabla u(x)|^2 dx = \omega^2 \sum_{k\in\mathbb{Z}^N} |c_k|^2 |k|^2.$$

This suggests that it is possible to define Sobolev spaces of periodic functions. We denote by $\mathbb{H}_T^1$ the closure of $\mathcal{C}_T^\infty(\mathbb{R}^N)$ under the norm

$$|u|_{\mathbb{H}_T^1}^2 := \sum_{k\in\mathbb{Z}^N} \lambda_k\, |c_k|^2, \tag{1.3.31}$$

and the fractional periodic Sobolev space $\mathbb{H}_T^{\frac{1}{2}}$ as the closure of $\mathcal{C}_T^\infty(\mathbb{R}^N)$ under the norm

$$|u|_{\mathbb{H}_T^{\frac{1}{2}}}^2 := \sum_{k\in\mathbb{Z}^N} \sqrt{\lambda_k}\, |c_k|^2. \tag{1.3.32}$$

Obviously $\mathbb{H}_T^{\frac{1}{2}}$ is a Hilbert space with respect to the inner product

$$(u, v)_{\mathbb{H}_T^{\frac{1}{2}}} = \sum_{k\in\mathbb{Z}^N} \sqrt{\lambda_k} c_k \overline{d}_k$$

for $u = \sum_{k\in\mathbb{Z}^N} c_k\varphi_k(x)$ and $v = \sum_{k\in\mathbb{Z}^N} d_k\varphi_k(x)$ belong to $\mathbb{H}_T^{\frac{1}{2}}$.

The operator $\sqrt{-\Delta + m^2}$ is defined as follows: For any $u \in \mathcal{C}_T^\infty(\mathbb{R}^N)$

$$\sqrt{-\Delta + m^2}\, u = \sum_{k\in\mathbb{Z}^N} c_k \sqrt{\lambda_k} \varphi_k(x). \tag{1.3.33}$$

where

$$u = \sum_{k\in\mathbb{Z}^N} c_k\varphi_k \text{ and } c_k = \int_{(0,T)^N} u\overline{\varphi}_k dx.$$

Then it is clear that $\sqrt{-\Delta+m^2}$ maps $\mathbb{H}^1_T$ in $L^2(0,T)^N$. We want to show that $\sqrt{-\Delta+m^2}$ is an isometric isomorphism from $\mathbb{H}^{\frac{1}{2}}_T$ to its topological dual $(\mathbb{H}^{\frac{1}{2}}_T)^*$. Firstly, we prove the following result

Theorem 6. *Assume $m>0$ and let $(\mathbb{H}^{\frac{1}{2}}_T)^*$ be the topological dual of $\mathbb{H}^{\frac{1}{2}}_T$. Then $(\mathbb{H}^{\frac{1}{2}}_T)^*$ can be identified with the space*

$$\mathbb{H}^{-\frac{1}{2}}_T=\Big\{h=\sum_{k\in\mathbb{Z}^N}h_k\varphi_k:\sum_{k\in\mathbb{Z}^N}\frac{|h_k|^2}{\sqrt{\lambda_k}}<+\infty\Big\}.$$

Proof. Let $l\in(\mathbb{H}^{\frac{1}{2}}_T)^*$ and set $c_k=l(\varphi_k)$ for all $k\in\mathbb{Z}^N$.

Therefore, if $u=\sum_{k\in\mathbb{Z}^N}b_k\varphi_k\in\mathbb{H}^{\frac{1}{2}}_T$ then $l(u)=\sum_{k\in\mathbb{Z}^N}b_kc_k$. Now, we define for every $N\in\mathbb{N}$

$$\psi_N(x)=\sum_{|k|\leq N}\lambda_k^{-\frac{1}{2}}\overline{c_k}\varphi_k(x)\in\mathbb{H}^{\frac{1}{2}}_T$$

So we deduce that

$$l(\psi_N)=\sum_{|k|\leq N}\lambda_k^{-\frac{1}{2}}|c_k|^2=\sum_{|k|\leq N}\lambda_k^{\frac{1}{2}}\lambda_k^{-1}|c_k|^2=||\psi_N||^2_{\mathbb{H}^{\frac{1}{2}}_T}.$$

Hence

$$||l||_{(\mathbb{H}^{\frac{1}{2}}_T)^*}\geq\frac{|l(\psi_N)|}{||\psi_N||_{\mathbb{H}^{\frac{1}{2}}_T}}=\Big(\sum_{|k|\leq N}\lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}\quad\forall N\in\mathbb{N}$$

and letting $N\to\infty$ we obtain

$$\infty>||l||_{(\mathbb{H}^{\frac{1}{2}}_T)^*}\geq\Big(\sum_{k\in\mathbb{Z}^N}\lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}\tag{1.3.34}$$

that is $c\equiv\{c_k\}\in\mathbb{H}^{-\frac{1}{2}}_T$. By Hölder inequality we can see that

$$\begin{aligned}|l(u)|&\leq\Big(\sum_{k\in\mathbb{Z}^N}\lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\in\mathbb{Z}^N}\lambda_k^{\frac{1}{2}}|b_k|^2\Big)^{\frac{1}{2}}\\&=\Big(\sum_{k\in\mathbb{Z}^N}\lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}||u||_{\mathbb{H}^{\frac{1}{2}}_T}\end{aligned}$$

that is

$$||l||_{(\mathbb{H}^{\frac{1}{2}}_T)^*}\leq\Big(\sum_{k\in\mathbb{Z}^N}\lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}.\tag{1.3.35}$$

Taking into account (1.3.34) and (1.3.35) we have, for every $l \in (\mathbb{H}_T^{\frac{1}{2}})^*$

$$||l||_{(\mathbb{H}_T^{\frac{1}{2}})^*} = \Big(\sum_{k\in\mathbb{Z}^N} \lambda_k^{-\frac{1}{2}} |c_k|^2\Big)^{\frac{1}{2}}. \tag{1.3.36}$$

Conversely, for any $c \equiv \{c_k\} \in \mathbb{H}_T^{-\frac{1}{2}}$ such that $\sum_{k\in\mathbb{Z}^N} \lambda_k^{-\frac{1}{2}}|c_k|^2 < \infty$, we can get a corresponding bounded linear functional h on $\mathbb{H}_T^{\frac{1}{2}}$. In fact, we may define h on $\mathbb{H}_T^{\frac{1}{2}}$ by setting

$$h(u) = \sum_{k\in\mathbb{Z}^N} b_k c_k$$

if $u = \sum_{k\in\mathbb{Z}^N} b_k\varphi_k \in \mathbb{H}_T^{\frac{1}{2}}$. Using the Cauchy-Schwartz inequality, we get

$$|h(u)| \leq \Big(\sum_{k\in\mathbb{Z}^N} \lambda_k^{\frac{1}{2}}|b_k|^2\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}^N} \lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}$$

that is

$$||h||_{(\mathbb{H}_T^{\frac{1}{2}})^*} \leq \Big(\sum_{k\in\mathbb{Z}^N} \lambda_k^{-\frac{1}{2}}|c_k|^2\Big)^{\frac{1}{2}}.$$

Hence $h \in (\mathbb{H}_T^{\frac{1}{2}})^*$. Then the mapping of $(\mathbb{H}_T^{\frac{1}{2}})^*$ onto $\mathbb{H}_T^{-\frac{1}{2}}$ defined by

$$l \in (\mathbb{H}_T^{\frac{1}{2}})^* \mapsto c \equiv \{l(\varphi_k)\} \in \mathbb{H}_T^{-\frac{1}{2}}$$

is linear and bijective, and from (1.3.36) we see that $||l||_{(\mathbb{H}_T^{\frac{1}{2}})^*} = ||c||_{\mathbb{H}_T^{-\frac{1}{2}}}$. □

Then we can deduce that

Theorem 7. *The operator $\sqrt{-\Delta+m^2}$ is an isometric isomorphism from $\mathbb{H}_T^{\frac{1}{2}}$ to its topological dual $(\mathbb{H}_T^{\frac{1}{2}})^*$.*

Proof. Let $u = \sum_{k\in\mathbb{Z}^N} c_k\varphi_k \in \mathcal{C}_T^\infty(\mathbb{R}^N)$. Taking into account Theorem 6 and $\sqrt{-\Delta+m^2}u = \sum_{k\in\mathbb{Z}^N} c_k\sqrt{\lambda_k}\varphi_k$, we deduce that

$$||\sqrt{-\Delta+m^2}u||^2_{\mathbb{H}_T^{-\frac{1}{2}}} = \sum_{k\in\mathbb{Z}^N} \frac{|\sqrt{\lambda_k}c_k|^2}{\sqrt{\lambda_k}} = ||u||^2_{\mathbb{H}_T^{\frac{1}{2}}}. \tag{1.3.37}$$

Now, let $u \in \mathbb{H}_T^{\frac{1}{2}}$. By density there exists a sequence $(u_j) \subset \mathcal{C}_T^\infty(\mathbb{R}^N)$ such that $u_j \to u$ in $\mathbb{H}_T^{\frac{1}{2}}$ as $j \to \infty$. Using (1.3.37) we deduce that $\sqrt{-\Delta+m^2}u_j$ converges in $\mathbb{H}_T^{-\frac{1}{2}}$, and we denote by $\sqrt{-\Delta+m^2}u$ such limit. Therefore, $\sqrt{-\Delta+m^2}$ is an isometry from $\mathbb{H}_T^{\frac{1}{2}}$ to $\mathbb{H}_T^{-\frac{1}{2}}$. Finally we prove the surjectivity of the operator in question. Let $h = \sum_{k\in\mathbb{Z}^N} h_k\varphi_k \in \mathbb{H}_T^{-\frac{1}{2}}$ and consider $u = \sum_{k\in\mathbb{Z}^N} \frac{h_k}{\sqrt{\lambda_k}}\varphi_k$. Then we have $u \in \mathbb{H}_T^{\frac{1}{2}}$ and $\sqrt{-\Delta+m^2}u = h$. □

1.4 Extension problem for $\sqrt{-\Delta+m^2}$

In this section we describe $\sqrt{-\Delta+m^2}$ as an operator that maps a Dirichlet condition to Neumann-type condition via an extension problem. This approach is very useful when we deal with nonlinear problems involving fractional powers of second order partial differential operators, since it allows us to write nonlocal problems in a local way and to apply the variational techniques to these kind of problems (see for instance [15],[16], [17], [18], [20], [21], [60], [72]).

To explain this, let us start with the square root of Laplacian $(-\Delta)^{\frac{1}{2}}$ in $\mathbb{R}^N$. We denote by

$$\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : y > 0\}$$

the upper half space. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a smooth bounded function. Then there exists a unique bounded smooth solution v to the extension problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v = u & \text{on } \partial\mathbb{R}^{N+1}_+ \end{cases}. \tag{1.4.1}$$

Consider the map $T : u \to -\partial_y v(x,0)$. Since $-\partial_y v(x,y)$ is still a harmonic function, we have that $w(x,y) = -\partial_y v(x,y)$ is a solution to

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ w = Tu & \text{on } \partial\mathbb{R}^{N+1}_+ \end{cases}. \tag{1.4.2}$$

Then, by applying the operator T twice

$$T(Tu)(y) = -\frac{\partial w}{\partial y}(x,0) = \frac{\partial^2 v}{\partial y^2}(x,0) = -\Delta_x v(x,0)$$

that is $T^2 = -\Delta_x$.

In [21] Caffarelli and Silvestre generalized this situation, by proving that any fractional power of the Laplacian (and other integro-differential operators) in $\mathbb{R}^N$, can be realized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space $\mathbb{R}^{N+1}_+$. Cabré and Tan in [18] and Capella, Davila, Dupaigne and Sire in [23], proved a similar result for $(-\Delta_\Omega)^s$ with zero Dirichlet boundary conditions on $\partial\Omega$, when $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $s \in (0,1)$. Stinga and Torrea in [75] showed that any fractional operator L^σ, where $\sigma \in (0,1)$ and L is a nonnegative self-adjoint linear second order partial differential operator, can be seen as Dirichlet to Neumann operator associated to certain degenerate boundary value problem in $\Omega \times (0,\infty)$. The precise result is the following

Theorem 8. *Let $u \in Dom(L^\sigma)$. A solution of the extension problem*

$$\begin{cases} -L_x v + \frac{1-2\sigma}{y} v_y + v_{yy} = 0 & \text{in } \Omega \times (0,\infty) \\ v(x,0) = u(x) & \text{on } \Omega \times \{0\} \end{cases}$$

is given by

$$v(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma u)(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1-\sigma}}$$

and

$$\lim_{y\to 0} \frac{v(x,y) - v(x,0)}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} L^\sigma u(x) = \frac{1}{2\sigma} \lim_{y\to 0} y^{1-2\sigma} v_y(x,y).$$

Moreover, the following Poisson formula for u holds

$$v(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} u(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4r}L} u(x) e^{-r} \frac{dr}{r^{1-\sigma}}.$$

All identities in Theorem 8 are understood in L^2 sense.

Now, we use the approach due to Caffarelli and Silvestre to give an alternative definition of $\sqrt{-\Delta_x+m^2}$, when $\Omega = \mathbb{R}^N$ and $\Omega = (0,T)^N$. For this reason, we distinguish two cases.

Firstly, we suppose that $\Omega = \mathbb{R}^N$. This case is very similar to that of $(-\Delta)^{\frac{1}{2}}$. For the sake of completeness, we give the details of proof.

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz space of rapidly decaying functions in $\mathbb{R}^N$, that is $u \in \mathcal{S}(\mathbb{R}^N)$ if and only if $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ and

$$\sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta u(x)| < \infty$$

for all multiindex $\alpha, \beta \in \mathbb{N}^N$ and

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_N^{\beta_N}}.$$

For any given function $u \in \mathcal{S}(\mathbb{R}^N)$ there exists a unique $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ such that

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(x,0) = u(x) & \text{on } \partial\mathbb{R}^{N+1}_+ \end{cases}. \tag{1.4.3}$$

The solution of (1.4.3) can be obtained taking the Fourier transform with respect to x in the first equation in (1.4.3) and taking into account of the Dirichlet condition, that is

$$v(x,y) = \mathcal{F}^{-1}(e^{-\sqrt{|2\pi\cdot|^2+m^2}y} \mathcal{F}u(\cdot))(x).$$

We can observe that

$$\begin{aligned}||v||^2_{H^1(\mathbb{R}^{N+1}_+)} &= \iint_{\mathbb{R}^{N+1}_+} |2\pi k|^2|\mathcal{F}v(k,y)|^2 + |\mathcal{F}v_y(k,y)|^2 + m^2|\mathcal{F}v(k,y)|^2 dkdy \\ &= \iint_{\mathbb{R}^{N+1}_+} 2(|2\pi k|^2 + m^2)|\mathcal{F}v(k,y)|^2 dkdy \\ &= \iint_{\mathbb{R}^{N+1}_+} 2(|2\pi k|^2 + m^2)e^{-2\sqrt{|2\pi k|^2+m^2}y}|\mathcal{F}u(k)|^2 dkdy \\ &= \int_{\mathbb{R}^N} \sqrt{|2\pi k|^2 + m^2}|\mathcal{F}u(k)|^2 dk.\end{aligned}$$

Taking the partial derivative of v with respect to y we can see

$$-\frac{\partial v}{\partial y}(x,y) = \int_{\mathbb{R}^N} \sqrt{|2\pi\xi|^2+m^2}e^{2\pi i\xi\cdot x}e^{-\sqrt{|2\pi\xi|^2+m^2}y}\mathcal{F}u(\xi)d\xi$$

and evaluating it at $y=0$ we deduce

$$-\frac{\partial v}{\partial y}(x,0) = \int_{\mathbb{R}^N} \sqrt{|2\pi\xi|^2+m^2}e^{2\pi i\xi\cdot x}\mathcal{F}u(\xi)d\xi = \sqrt{-\Delta+m^2}u(x).$$

By the density of $\mathcal{S}(\mathbb{R}^N)$ in $H^{\frac{1}{2}}(\mathbb{R}^N)$ we can deduce the result for $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$.

Now, we assume that $\Omega = (0,T)^N$. Set $\mathcal{S}_T = (0,T)^N \times (0,\infty)$. We denote by $H^1_{per}(\mathbb{R}^{N+1}_+)$ the completion of the space $\mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+})$ of restrictions to $\mathcal{S}_T$ of infinitely differentiable functions in $\overline{\mathbb{R}^{N+1}_+}$ that are T-periodic in each x_i, with respect to the $H^1(\mathcal{S}_T)$ norm

$$||f||_{H^1(\mathcal{S}_T)} = \iint_{\mathcal{S}_T} |\nabla f(x)|^2 + m^2|f(x)|^2 dx.$$

Let $u = \sum_{k\in\mathbb{Z}^N} c_k\varphi_k(x) \in \mathcal{C}^\infty_T(\mathbb{R}^N)$. Then there exists a unique function $v \in H^1_{per}(\mathbb{R}^{N+1}_+)$ solving

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } (0,T)^N \times (0,\infty) = \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) \\ v(x,0) = u(x) & \text{on } (0,T)^N \times \{0\} \end{cases} . \tag{1.4.4}$$

By standard elliptic theory we deduce that v is smooth for $y \geq 0$.

We may write

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k(y)\varphi_k(x),$$

where $c_k(y) = \int_{(0,T)^N} v(x,y)\varphi_k(x)dx$. Let us note that $c_k(\cdot) \in H^1(0,\infty)$ (since $v \in H^1(\mathcal{S}_T)$) and $c_k(0) = c_k$ for all $k \in \mathbb{Z}^N$.

Then v satisfies (1.4.4) if and only if $c_k(y) = c_k e^{-y\sqrt{\lambda_k}}$ for $y \geq 0$ and $k \in \mathbb{Z}^N$. Hence we find

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k e^{-y\sqrt{\lambda_k}} \varphi_k(x). \tag{1.4.5}$$

Now we can see that

$$\begin{aligned} ||v||^2_{H^1(\mathcal{S}_T)} &= \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 dxdy \\ &= 2\sum_{k\in\mathbb{Z}^N} |c_k|^2 \int_0^\infty \lambda_k e^{-2\sqrt{\lambda_k}} dy \\ &= \sum_{k\in\mathbb{Z}^N} |c_k|^2 \sqrt{\lambda_k} \\ &= |u|^2_{\mathbb{H}^{\frac{1}{2}}_T} < \infty. \end{aligned}$$

Using (1.4.5) and evaluating at $y = 0$ the partial derivative of v with respect to y, we deduce that

$$-\frac{\partial v}{\partial y}(x,0) = \sum_{k\in\mathbb{Z}^N} c_k \sqrt{\lambda_k} \varphi_k(x) = \sqrt{-\Delta + m^2} u(x).$$

By density we obtain the result for functions $u \in \mathbb{H}^{\frac{1}{2}}_T$.

1.5 Heteroclinic solutions

In Chapter 2 (which corresponds to [5]) we consider a fractional Allen-Cahn type equation

$$\Big(\sqrt{-\frac{d^2}{dx^2} + m^2} - m\Big) u = -G'(u) \text{ in } \mathbb{R} \tag{1.5.1}$$

where the potential $G \in C^{2,\alpha}(\mathbb{R})$ is even and

$$G'(\pm 1) = 0 \text{ and } G > G(-1) = G(1) \text{ in } (-1,1). \tag{1.5.2}$$

We are looking for heteroclinic solutions which are increasing from -1 to 1.

In last decades a considerable effort has been devoted to the study of heteroclinic (or layer) solutions to fractional problems of elliptic type.

When $m = 0$ and the nonlinearity $G(u) = \frac{1}{\pi^2}(\cos(\pi u) + 1)$, the above problem is called Pierls-Nabarro problem, and it appears as a model of dislocations in crystals; see [59], [64], [80]. Taking $m = 0$ and $G(u) = \frac{1}{4}(1-u^2)^2$, the problem (1.5.1) reduces to the fractional counterpart of more famous Allen-Cahn equation which was originally introduced by Allen and Cahn in [1] to describe the motion of anti-phase boundaries in crystalline solids.

On the other hand, the study of the problem (1.5.1) is also due to the attempt to generalize the following conjecture due to De Giorgi [29]:

Conjecture 1 (De Giorgi's conjecture). *Let $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^N \tag{1.5.3}$$

such that

$$|u| \leq 1 \text{ and } \partial_{x_N} u > 0$$

in the whole $\mathbb{R}^N$. Is it true that all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $N \leq 8$.

If u satisfies this property, we will say that u is one dimensional. Semilinear Equation (1.5.3) is called Allen-Cahn equation, and it models phase transition. The conjecture has been proven for $N = 2$ by Ghoussoub and Gui in [42], and then by Ambrosio and Cabre in [4] for $N = 3$, even for more general double well potentials. For $4 \leq N \leq 8$, under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \text{ for all } x' \in \mathbb{R}^{N-1}$$

it has been established by Savin in [69]. A counterexample to the conjecture for $N \geq 9$ has been proven by Del Pino, Kowalczyk and Wei in [30].

In fractional setting, one consider the problem

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^N \tag{1.5.4}$$

where $s \in (0, 1)$ and $f(u)$ is a locally Lipschitz function satisfying the assumptions (1.5.2). In an interesting article, Toland [80] has found that the unique bounded heteroclinic solution (up to translations) to

$$\sqrt{-\frac{d^2}{dx^2}} u = \frac{1}{\pi} \sin(\pi u) \text{ in } \mathbb{R}$$

is given by

$$u(x) = \frac{2}{\pi} \arctan x.$$

Cabré and Solà Morales [17] established existence, uniqueness, symmetry, variational properties, and asymptotic behavior of heteroclinic solutions to (1.5.4) when $N = 1$ and $s = \frac{1}{2}$. They proved that the assumption (1.5.2) on G is a necessary and sufficient condition to the existence of heteroclinic solutions in dimension $N = 1$. Under the additional hypothesis $f'(\pm 1) < 0$, they showed that the solution is unique up to translations. More precisely their result can be stated as follows

Theorem 9. *([17]) Let $f \in C^{1,\alpha}(\mathbb{R})$ and $G' = -f$. Then there exists a layer solution to*

$$\sqrt{-\frac{d^2}{dx^2}}u = f(u) \ \text{in } \mathbb{R} \tag{1.5.5}$$

if and only if G satisfies the assumptions (1.5.2). If $f'(\pm 1) < 0$, then the layer solution is unique up to translations.

The proof of the existence of layer solutions is of variational type. The authors construct a sequence of local minimizers u^R, each one increasing in x, defined in a half-ball B_R^+, which vanishes in the origin and satisfying certain elliptic problems with mixed boundary conditions. When R goes to infinity, u^R converges to some solution u of (1.5.5). Using the local minimality and the assumptions (1.5.2), they deduce that the limits of u at infinity are ± 1, and so that u is a heteroclinic solution to (1.5.5).

When $N = 1$ and $s \in (0,1)$, Cabré and Sire [15, 16] proved the existence of heteroclinic solutions to (1.5.4) under the assumptions (1.5.2) on G. They used methods similar to those introduced by Cabré and Solà Morales in [17]. The same result has been proven by Sire and Valdinoci [72], using a different approach based on a geometric inequality of Poincaré type.

In chapter 2 we follow the approach introduced in [17] to study heteroclinic solutions to the fractional equation (1.5.1). In particular, by a limit procedure as $m \to 0$, we rediscover the Theorem 9.

1.6 Periodic solutions

In Chapter 3 (which corresponds to [6]) we study the periodic solutions for a pseudo-relativistic Schrödinger equation

$$\left(\sqrt{-\Delta + m^2} - m\right)u = f(x,u) \text{ in } (0,T)^N \tag{1.6.1}$$

where the nonlinearity $f(x,u)$ satisfies the following hypotheses:

(f1) $f(x,t)$ is locally lipschitz-continuous in $\mathbb{R}^{N+1}$;

(f2) There exist $a_1, a_2 > 0$ and $p \in (1, 2^\sharp - 1)$:

$$|f(x,t)| \le a_1 + a_2|t|^p \quad \forall t \in \mathbb{R} \quad \forall x \in \mathbb{R}^N;$$

(f3) $\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0$ uniformly in $x \in \mathbb{R}^N$;

(f4) There exist $\mu > 2$ and $r > 0$ such that $0 < \mu F(x,t) \leq tf(x,t)$ for all $|t| \geq r$ and for all $x \in \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$;

(f5) f is T-periodic in x: that is $f(x+Te_i,t) = f(x,t)$ for all $x \in \mathbb{R}^N, t \in \mathbb{R}$ and $i = 1,\ldots,N$;

(f6) $tf(x,t) \geq 0$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

By $(f3)$ we deduce that $u = 0$ is a trivial solution to (1.6.1). The hypothesis $(f4)$ gives information about the behavior of $f(x,u)$ and $F(x,u)$ at $u = \infty$. Indeed, a straightforward computation shows that, by $(f4)$, there exist two constants $a, b > 0$ such that

$$F(x,u) \geq a|u|^{\mu} - b \text{ for } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}. \tag{1.6.2}$$

Since $\mu > 2$, (3.1.3) and $(f4)$ imply that $F(x,u)$ grows superquadratically and $f(x,u)$ grows superlinearly as $|u| \to \infty$. As a model for f we can take $f(x,u) = g(x)|u|^{p-1}u$, where g is a smooth positive T-periodic function. We observe that hypotheses $(f1) - (f4)$ are standard when we deal with superlinear second order elliptic partial differential equations. For instance Ambrosetti and Rabinowitz [3] proved, via Mountain Pass Theorem, that there exists a nontrivial solution to the following elliptic boundary value problem

$$\begin{cases} -\sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} + c(x)u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, the a_{ij} are continuously differentiable in Ω with Hölder continuous first derivatives, and $c(x)$ is Hölder continuous in Ω with $c \geq 0$. For a more wide discussion about these types of problem see [67], [77] and [81] and references therein.

In the classical literature, the existence of periodic solutions to ODE and PDE, has been investigated by many authors in different way, by using variational and topological techniques and geometrical approaches. In particular, motivated by celestial mechanics, an enormous interest has been addressed to the study of periodic solutions to the Hamiltonian systems. The variational treatment of these systems goes back to Poincaré [62], who used the least action principle to study the closed orbits of a conservative system with two degree of freedom. Related to the problem of the existence of geodesics for dynamical systems, Lusternik-Schnirelman [55] and Morse [58] introduced the Critical Point Theory on manifold.

Nevertheless, for a long time, it was considered hopeless to approach the existence problem for periodic solutions of

$$\begin{cases} \dot{p} = -H_q(p,q,t) \\ \dot{q} = H_p(p,q,t) \end{cases} \tag{1.6.3}$$

oe equivalently

$$\dot{z} = \mathcal{J} H_z z \tag{1.6.4}$$

where $\mathcal{J} = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ is the symplectic matrix on $\mathbb{R}^{2N}$, as critical points of the associated functional

$$\mathcal{I}(z) = \int_0^{2\pi} p \cdot \dot{q}\, dt - \int_0^{2\pi} H(z) dt$$

defined for $z = (p, q) \in H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$. The reason is that this functional is unbounded from below or from above. In fact, the quadratic form

$$(p, q) \to \int_0^{2\pi} p \cdot \dot{q}\, dt$$

has infinitely many positive and negative eigenvalues. Surprisingly, Rabinowitz [65] was able to overcome these difficulties by using minimax and approximate techniques. Later, Rabinowitz [66] introduced a new variational method, the Linking Theorem, to extend the results obtained in [3] to study semilinear elliptic boundary value problems, and together with Benci [8] developed some critical point theorems for indefinite functionals. These and other variational methods were subsequently introduced also to study periodic solutions of Hamiltonian systems and second order differential equations; see for instance [56] and [77].

To our knowledge, there are not existence results of periodic solutions to nonlocal equations involving the operator $\sqrt{-\Delta + m^2} - m$ with $m \geq 0$. Our main purpose will be, then, to prove the existence of periodic solutions for a fractional Schrödinger equation, by applying the Linking Theorem due to Rabinowitz to the corresponding extension problem.

1.7 Summary of results

In this section we present the main results of this thesis.
The first result concerns the study of the existence of bounded monotone heteroclinic solutions to the problem

$$\begin{cases} \left(\sqrt{-\frac{d^2}{dx^2} + m^2} - m\right) u = -G'(u) & \text{in } \mathbb{R} \\ u(\pm\infty) = \pm 1 \end{cases} \tag{1.7.1}$$

or equivalently the study of bounded solutions to the problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial \nu} = mv - G'(v) & \text{on } \partial\mathbb{R}^2_+ \end{cases}, \tag{1.7.2}$$

monotone increasing in x and with limits $\pm e^{-my}$ as $x \to \pm\infty$. Here the potential $G \in C^{2,\alpha}(\mathbb{R})$ is even and satisfies the assumptions (1.5.2).

The proof of the existence of a solution to (1.7.2), with the above mentioned properties, follows the approach introduced in [17]. We find such a solution as the limit, as $R \to +\infty$, of the solutions of the following elliptic problem with mixed boundary conditions

$$\begin{cases} -\Delta v^R + m^2 v^R = 0 & \text{in } \Omega_R := (-R,R)\times(0,+\infty) \\ \frac{\partial v^R}{\partial \nu} = m v^R - G'(v^R) & \text{on } \partial^0\Omega_R := (-R,R)\times\{0\} \\ v^R(\pm R, y) = \pm e^{-my} & \text{on } \partial^+\Omega_R := \{-R,R\}\times[0,+\infty) \end{cases}. \tag{1.7.3}$$

The solutions of (1.7.3) are obtained as minimizers of the following functional

$$E_{\Omega_R}(v) = \frac{1}{2}\iint_{\Omega_R} |\nabla v|^2 + m^2 v^2 dxdy - \frac{m}{2}\int_{-R}^{R} v^2(x,0)dx - \int_{-R}^{R} G(v(x,0)) - G(1)dx \tag{1.7.4}$$

defined on the space

$$C_{u^R}(\Omega_R) = \{v \in H^1(\Omega_R) : |v| \le e^{-my}, v(-x,y) = -v(x,y) \text{ and } v^R = u^R \text{ on } \partial^+\Omega_R\} \tag{1.7.5}$$

where

$$u^R(x,y) = \frac{\arctan x}{\arctan R} e^{-my}.$$

The minimizers are shown to be monotone increasing with respect to x. The fact that G is even and the minimizers odd, allows us to easily show that the limit as $R \to +\infty$ is a solution to (1.7.2), strictly increasing in x and different from the trivial solutions $v(x,y) = \pm e^{-my}$. We suspect that, as in [17], one can find a nontrivial solution to (1.1.2) also without assuming G to be even, but we have not been able to prove it. Finally, under the additional assumption $G''(\pm 1) > 0$, we obtain the uniqueness of solution by using the sliding method.

The first main result can be stated as follows

Theorem 10. *Let $m > 0$. Let G be an even $C^{2,\alpha}(\mathbb{R})$ function with $\alpha \in (0,1)$, and suppose that G satisfies (1.5.2).*

(I) Then there exists a heteroclinic solution of (1.7.1) which is odd and strictly increasing.

(II) If, in addition, G satisfies

$$G''(\pm 1) > 0, \tag{1.7.6}$$

then this solution is unique.

In order to show the existence of a heteroclinic solution to (1.7.1) with $m = 0$, we prove the following identity similar to the one obtained in [17]

$$G(v(x,0)) - G(1) = \int_0^{+\infty} \frac{v_x^2 - v_y^2 - m^2v^2}{2}(x,t)dt + \frac{m}{2}v^2(x,0) \tag{1.7.7}$$

which holds for every bounded solution $v(x,y)$ to (1.7.2) such that $v(x,0) \to \pm 1$ as $x \to \pm\infty$. When $m < 1$, the $C^{1,\alpha}_{loc}$-norms of solutions v_m of (1.7.2) are controlled by a constant independent from m. Using the fact that $|v_m(x,y)| < e^{-my}$ and the interior gradient estimates for the harmonic functions, we deduce that

$$|\nabla v_m(x,y)| \le \frac{C}{y+1} \text{ for } (x,y) \in \overline{\mathbb{R}}^2_+$$

for some constant C independent from m. As a consequence, we can pass to the limit as $m \to 0$ in (1.7.2) and (1.7.7), so we find a weak bounded solution v to the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial \nu} = -G'(v) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{1.7.8}$$

which satisfies the identity (1.7.7) with $m = 0$. Using this informations and $||\nabla v||_{L^\infty(B^+_R(x,0))} \to 0$ as $x \to \pm\infty$ (see Lemma 2.3 in [17]), by assumptions on G follows that

Theorem 11. *Under the same assumptions of Theorem 10, there exists a strictly increasing heteroclinic solution u to*

$$\begin{cases} \sqrt{-\Delta} u = -G'(u) & \textit{in } \mathbb{R} \\ u(\pm\infty) = \pm 1 \end{cases}. \tag{1.7.9}$$

The second main result concerns the study of solutions to

$$\begin{cases} (\sqrt{-\Delta_x + m^2} - m)u = f(x,u) & \text{in } (0,T)^N \\ u(x + Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, \quad i = 1, \dots, N \end{cases}, \tag{1.7.10}$$

or equivalently the solutions to the problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } (0,T)^N \times (0,\infty) = \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) = \partial_L \mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = mv + f(x,v) & \text{on } (0,T)^N \times \{0\} = \partial^0 \mathcal{S}_T \end{cases}. \tag{1.7.11}$$

which are T-periodic in x. Here the nonlinearity $f(x,t)$ satisfies the hypotheses $(f1) - (f6)$. The problem (1.7.11) has a variational nature, and its solutions can be found as critical points of the functional

$$\mathcal{J}_m(v) = \frac{1}{2}\iint_{\mathcal{S}_T} |\nabla v|^2 + m^2v^2 dxdy - \frac{m}{2}\int_{(0,T)^N} v^2(x,0)dx - \int_{(0,T)^N} F(x,v(x,0))dx$$

defined on the space $\mathbb{X}_T^m$, which is the closure of the set of $\mathcal{C}^\infty(\overline{\mathbb{R}_+^{N+1}})$ functions $v(x,y)$, T-periodic in x, under the $H^1(\mathcal{S}_T)$-norm

$$||v||^2_{\mathbb{X}_T^m} = \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 dxdy.$$

Let us observe that $\mathbb{X}_T^m$ can be splits into two subspaces

$$\mathbb{Y}_T^m =< e^{-my} > \text{ and } \mathbb{Z}_T^m = \{v \in \mathbb{X}_T^m : \int_{(0,T)^N} v(x,0)dx = 0\}$$

where $\dim \mathbb{Y}_T^m < \infty$ and $\mathbb{Z}_T^m$ is the orthogonal complement of $\mathbb{Y}_T^m$ with respect to the inner product in $\mathbb{X}_T^m$. Then, we show that $\mathcal{J}_m$ satisfies the hypotheses of the Linking Theorem, that is

1. $\mathcal{J}_m \leq 0$ on $\mathbb{Y}_T^m$;

2. There exist $\rho > 0$ and $\lambda > 0$ such that
$$\inf\{\mathcal{J}_m(v) : v \in \mathbb{Z}_T^m \text{ and } ||v||_{\mathbb{X}_T^m} = \rho\} \geq \lambda > 0; \tag{1.7.12}$$

3. There exists $z \in \mathbb{X}_T^m$ with $||z||_{\mathbb{X}_T^m} = 1$, and there exist $R > \rho, \delta > 0$ and $R' > 0$, such that
$$\max_{\partial\mathbb{A}_T^m} \mathcal{J}_m = 0 \text{ and } \sup_{\mathbb{A}_T^m} \mathcal{J}_m(v) \leq \delta, \tag{1.7.13}$$
where
$$\mathbb{A}_T^m := \{v = y + sz : y \in \mathbb{Y}_T^m, ||y||_{\mathbb{X}_T^m} \leq R', 0 \leq s \leq R\}$$
and its boundary
$$\partial\mathbb{A}_T^m := \{v = y+sz \in \mathbb{A}_T^m : ||y||_{\mathbb{X}_T^m} = R', 0 \leq s \leq R \text{ or } ||y||_{\mathbb{X}_T^m} \leq R', s \in \{0, R\}\};$$

4. $\mathcal{J}_m$ satisfies the Palais-Smale condition.

Hence, for every $m > 0$, we can find a function $v_m \in \mathbb{X}_T^m$ such that

$$\mathcal{J}_m(v_m) = \alpha_m \text{ and } \mathcal{J}'_m(v_m) = 0$$

where the critical value α_m is defined as

$$\alpha_m := \inf_{\gamma \in \mathcal{P}_T^m} \max_{v \in \mathbb{A}_T^m} \mathcal{J}_m(\gamma(v))$$

and

$$\mathcal{P}_T^m = \{\gamma \in \mathcal{C}^0(\mathbb{A}_T^m, \mathbb{X}_T^m) : \gamma = Id \text{ on } \partial\mathbb{A}_T^m\}.$$

In particular we also study the regularity of these critical points of $\mathcal{J}_m$. Therefore we are able to state the following main result

Theorem 12. *Let $m > 0$ and let $f : \mathbb{R}^{N+1} \to \mathbb{R}$ be a function satisfying $(f1)$–$(f6)$. Then there exists at least a function $u_m \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, T-periodic in x which satisfies the problem (1.7.10).*

If we assume $m \in (0, m_0)$, then we can estimate the critical levels α_m independently from m. Hence there exist two constants $\lambda, \delta > 0$, independent from m, such that

$$\lambda \leq \mathcal{J}_m(v_m) \leq \delta. \tag{1.7.14}$$

Taking the limit as $m \to 0$ in the problem (1.7.11) and exploiting (1.7.14) we can infer that there exists a non trivial solution $v \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}}_+^{N+1}) \cap \mathcal{C}^{\infty}(\mathbb{R}_+^{N+1})$ to

$$\begin{cases} -\Delta v = 0 & \text{in } (0,T)^N \times (0,\infty) = \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial(0,T)^N \times [0,\infty) = \partial_L \mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = f(x,v) & \text{on } (0,T)^N \times \{0\} = \partial^0 \mathcal{S}_T \end{cases}. \tag{1.7.15}$$

This last result can be summarized as follows

Theorem 13. *Under the same assumptions of Theorem 12, there exists a non trivial T-periodic solution $u \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N)$ to*

$$\begin{cases} \sqrt{-\Delta} u = f(x,u) \ \text{in } (0,T)^N & \\ u(x + Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, \quad i = 1, \ldots, N \end{cases}. \tag{1.7.16}$$

Chapter 2

Existence of heteroclinic solutions for a pseudo-relativistic Allen-Cahn type equation

We study the existence and uniqueness of heteroclinic solutions to non-linear Allen-Cahn equation

$$\Big(\sqrt{-\frac{d^2}{dx^2}+m^2}-m\Big)u=-G'(u) \text{ in } \mathbb{R}$$

where G is a double-well potential. We investigate such a problem using variational methods after transforming the problem to an elliptic equation with a nonlinear Neumann boundary conditions. By passing to the limit as $m \to 0$ in the above equation, we show the existence of a monotone heteroclinic solution to the corresponding equation involving the square root of the Laplacian.

2.1 Introduction

This section is devoted to the study of the nonlinear problem

$$\begin{cases} \Big(\sqrt{-\frac{d^2}{dx^2}+m^2}-m\Big)u=-G'(u) \text{ in } \mathbb{R} \\ u(-\infty)=-1, u(+\infty)=1 \end{cases} \tag{2.1.1}$$

where the potential $G \in C^2(\mathbb{R})$ is even and it has two, and only two, absolute minima in the interval $[-1,1]$ located in ± 1.

This problem involves the operator $\sqrt{-\Delta+m^2}$ in one dimension. Let us remark here that in recent years the study of nonlinear equations involving a fractional Laplacian $(-\Delta)^{\alpha}$ has attracted the attention of many mathematicians. Caffarelli, Roquejoffre, Sire [19] and Caffarelli, Salsa, Silvestre [20] have investigated free boundary problems, Silvestre [71] has obtained some regularity results for the obstacle problem, Felmer, Quaas and Tan [37] have studied the existence, regularity and symmetry of positive solutions for some nonlinear problem in the whole space. Cabré and Sola-Morales [17] and then Sire and Valdinoci in [72] studied an analogue of the De Giorgi conjecture for the equation

$$(-\Delta)^{\alpha} u = f(u) \text{ in } \mathbb{R}^N \tag{2.1.2}$$

when $\alpha \in (0,1)$ and f is a locally Lipschitz function. Cabré and Sire in [15, 16] established necessary and sufficient conditions on the nonlinearity f so that the equation (2.1.2) admits solutions monotone increasing in one of its variables (they also study existence of radial solutions having a limit as $|x| \to \infty$).

The fractional operator which appears in (2.1.1) can be defined using Fourier transform $\mathcal{F}$ by the formula

$$\Big(\sqrt{-\frac{d^2}{dx^2} + m^2}\Big)u(x) = \mathcal{F}^{-1}((\sqrt{4\pi^2|\cdot|^2 + m^2})\mathcal{F}u(\cdot))(x), \tag{2.1.3}$$

which is well defined for all functions $u \in H^1(\mathbb{R})$.

As explained in more details in the subsection 1.1 below, problem (2.1.1) can be studied via the nonlinear boundary value problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial \nu} = mv - G'(v) & \text{on } \partial\mathbb{R}^2_+ , \end{cases} \tag{2.1.4}$$

where $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ is the half plane, $\partial\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2_+ : y = 0\}$, $v(x,y)$ is a real valued function. Indeed, using the approach due to Caffarelli and Silvestre in [21], the operator (2.1.3) is equivalent by the Dirichlet to Neumann operator for the operator $-\Delta + m^2$ on $\mathbb{R}^2_+$.

The purpose of this paper is to study bounded solutions of (2.1.1) which are monotone increasing from -1 to 1. These solutions, called heteroclinic o layer solutions, can describe phenomena of phase transition, and in this connection their existence and behavior have been investigated by many authors. Let us recall in particular that when $m = 0$ and $G(u) = \frac{1}{\pi}(1 + \cos(\pi u))$, the problem (2.1.1) is called the Peierls-Nabarro problem and it appears as a model of dislocations in crystals (see [59], [64], [80] and references therein).

Taking $m = 0$ and $G(u) = \frac{1}{4}(1 - u^2)^2$, the problem (2.1.1) reduces to the fractional counterpart of more famous Allen-Cahn equation which was originally introduced by Allen and Cahn in [1] to describe the motion of anti-phase boundaries in crystalline solids.

Our work has been motivated by the paper of Cabré and Solá Morales [17], in which they study the problem

$$\sqrt{-\Delta} u = -G'(u) \text{ in } \mathbb{R}^N, \tag{2.1.5}$$

where the potential G is as above. They proved existence and uniqueness of layer solutions when $N = 1$ and obtain important results about symmetry, variational properties and asymptotic behavior of heteroclinic solutions in any dimension.

In order to state our main result, we introduce some notation. We say that a function $u \in C^2(\mathbb{R})$ is a heteroclinic solution of (2.1.1) if :

(H1) $u(x)$ solves (2.1.1);

(H2) $u(-\infty) = -1$ and $u(+\infty) = 1$;

(H3) $u'(-\infty) = u'(+\infty) = 0$.

Our main result is the following:

Theorem 14. *Let G be an even $C^{2,\alpha}(\mathbb{R})$ function with $\alpha \in (0,1)$.*
We assume that:

$$G'(\pm 1) = 0 \tag{G1}$$

$$G(s) > G(-1) = G(1) \text{ for all } s \in (-1,1). \tag{G2}$$

(I) Then there exists a heteroclinic solution of (2.1.1) which is odd and strictly increasing.

(II) If, in addition, G satisfies

$$G''(\pm 1) > 0, \tag{G3}$$

then this solution is unique.

The proof of the existence of a heteroclinic solution is variational. Following the approach used in [17] we look for a solution of (2.1.4). We find such a solution as the limit, as $R \to +\infty$ of the solutions of the following elliptic problem with mixed boundary conditions

$$\begin{cases} -\Delta v + m^2 v = 0 & (x,y) \in (-R,R) \times (0,+\infty) \\ \frac{\partial v}{\partial \nu} = mv - G'(v) & \text{for } y = 0 \\ v(\pm R, y) = \pm e^{-my} & \text{for } y \in (0,+\infty) \end{cases}. \tag{2.1.6}$$

Let us remark here that this approximate problem is quite different from the one used in [17]. Solutions of (2.1.6) are found as minimizers of a suitable functional on a space of odd functions. The minimizers are shown to be odd and monotone increasing from -1 to 1. The fact that G is even and the minimizers odd allows us to easily show that limit as $R \to +\infty$ is a odd solution and in particular that is not one of the trivial solutions $u(x) = \pm 1$. We suspect that, as in [17], one can find a nontrivial solution also without assuming G to be even, but we have not been able to prove it. Finally the uniqueness of solution is obtained under the additional assumption $(G3)$, and its proof follows the lines of the proof of Lemma 5.2 in [17].

2.1.1 Local realization

Proceeding as in [21], we can characterize the operator (2.1.3) as the Dirichlet to Neumann map:

For any given function $u \in \mathcal{S}(\mathbb{R}^N)$ there is $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ (here $\mathcal{S}$ is the Schwartz space of rapidly decaying functions and $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : y > 0\}$) such that

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(x,0) = u(x) & \text{on } \partial\mathbb{R}^{N+1}_+. \end{cases} \tag{2.1.7}$$

Solving the equation in (2.1.7) via Fourier transform we get

$$v(x,y) = \int_{\mathbb{R}^N} e^{2\pi i k\cdot x} \mathcal{F}u(k) e^{-\sqrt{|2\pi k|^2+m^2}y} dk, \quad (x,y) \in \mathbb{R}^{N+1}_+ \tag{2.1.8}$$

and

$$\begin{aligned} -\frac{\partial v}{\partial y}(x,0) &= \int_{\mathbb{R}^N} e^{2\pi i k\cdot x} \sqrt{|2\pi k|^2 + m^2 \mathcal{F}u(k)} dk \\ &= \sqrt{-\Delta + m^2} u(x). \end{aligned}$$

Then, the non-local pseudo-differential equation

$$\sqrt{-\Delta + m^2} u = mu - G'(u) \quad x \in \mathbb{R}^N$$

is equivalent to the following elliptic Neumann boundary value problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \frac{\partial v}{\partial \nu} = mv - G'(v) & \text{on } \partial\mathbb{R}^{N+1}_+. \end{cases} \tag{2.1.9}$$

2.1.2 Outline of the article

Our paper is organized as follows: In the first section we give some results on the regularity of solutions of (2.1.4), we prove that their gradient tends to zero at infinity and provide the proof (ii) of the Theorem 14; In the second section we study the existence of weak solutions of certain Neumann-Dirichlet problems in the strips; this analysis will be useful to construct a sequence of functions which tends to an increasing layer solution; Finally, the last section is devoted to the proof (i) of the Theorem 14.

Notation 1. *Let $S = (\alpha, \beta) \times (0, +\infty)$ be a strip in $\mathbb{R}^2$. We define*

$$\begin{aligned} \partial^0 S &= \{\alpha < x < \beta, y = 0\} \\ \partial^+ S &= \{x = \alpha, y \geq 0\} \cup \{x = \beta, y \geq 0\}. \end{aligned}$$

We also denote

$$\begin{aligned} B_R^+ &= \{(x,y) \in \mathbb{R}^2 : y > 0, |(x,y)| < R\} \\ \Gamma_R^0 &= \{(x,0) \in \partial\mathbb{R}_+^2 : |x| < R\} \\ \Gamma_R^+ &= \{(x,y) \in \mathbb{R}^2 : y \geq 0, |(x,y)| = R\}. \end{aligned}$$

Finally, we denote by $H^1(S)$ the usual Sobolev space equipped with the norm

$$||u||_{H^1(S)} = \sqrt{||\nabla u||^2_{L^2(S)} + m^2||u||^2_{L^2(S)}}$$

equivalent to the standard one, and

$$H^1_{0,\partial^+}(S) = \{v \in H^1(S) : v \equiv 0 \ \text{on} \ \partial^+ S\}$$

that is the set of $v \in H^1(S)$ such that v belongs to the closure in $H^1(S)$ of $C^1(\overline{S})$ functions with compact support in $S \cup \partial^0 S$.

2.2 Properties of solutions

We begin to provide some results on the properties of solutions of the problem (2.1.4). First we observe that the weak and bounded solutions of (2.1.4) are $C^{2,\alpha}$ up to the boundary $\partial\mathbb{R}^2_+$, that is:

Lemma 1. *Let $\alpha \in (0,1)$ and $R > 0$. Let $u \in L^\infty(B^+_{4R}) \cap H^1(B^+_{4R})$ be a weak solution of*

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } B_R^+ \\ \frac{\partial u}{\partial \nu} = mu - G'(u) & \text{su } \Gamma_R^0 \end{cases} \tag{2.2.1}$$

with $G \in C^{2,\alpha}(\mathbb{R})$. Then $u \in C^{2,\alpha}(\overline{B_R^+})$ and $\|u\|_{C^{2,\alpha}(\overline{B_R^+})} \leq C_R$ where C_R is a constant depending only on α, R and on upper bounds for $||u||_{L^\infty(B_R^+)}$ and $\|G'\|_{C^{1,\alpha}}$. Finally, if u is a bounded weak solution of (2.1.4), we have that $\nabla u \in L^\infty(\mathbb{R}^2_+)$ and $D^2 u \in L^\infty(\mathbb{R}^2_+)$.

The proof of the above Lemma can be obtained along the lines of the Lemma 2.3 in [17]. For the sake of completeness we give a sketch of the proof.

Proof. Let $v(x,y) = \int_0^y u(x,t)dt$ for $(x,y) \in \overline{B}^+_{4R}$. Then v is a weak solution of

$$\begin{cases} -\Delta v + m^2 v = g(x) & \text{in } B^+_{4R} \\ v(x,0) = 0 & \text{su } \Gamma^0_{4R} \end{cases}, \tag{2.2.2}$$

where $g(x) = mu(x,0) - G'(u(x,0))$. Considering the odd reflections of v and g across $\{y = 0\}$ we have that v_{odd} satisfies

$$-\Delta v_{odd} + m^2 v_{odd} = g_{odd} \text{ in } B_{4R}.$$

Since v_{odd} and g_{odd} belong to $L^p(B_{4R})$ for any $p < \infty$, by $W^{2,p}$ regularity theory we deduce that $v \in W^{2,p}(B_{3R}^+) \subset C^{1,\alpha}(\overline{B}_{3R}^+)$ for p sufficiently large. Using the fact that $v_y = u$, we get $u \in C^{0,\alpha}(\overline{B}_{3R}^+)$ and $||u||_{C^{0,\alpha}(\overline{B}_{3R}^+)} \leq C_R$, for some constant C_R depending only on $\alpha, R, ||u||_\infty$ and $||G'||_\infty$. Now, since $G' \in C^{1,\alpha}$ and $u \in C^{0,\alpha}$ in $\overline{B}_{3R}^+$, the boundary $C^{2,\alpha}$ regularity for (2.2.2) and the above $C^{0,\alpha}$ estimate for u leads to $u \in C^{1,\alpha}(\overline{B}_{2R}^+)$ and the corresponding estimate $||u||_{C^{1,\alpha}(\overline{B}_{2R}^+)} \leq C'_R$. Finally considering the Dirichlet problem satisfied by the tangential derivative v_y and the boundary $C^{2,\alpha}$ regularity for such problem, we obtain that $u_x = v_{yx} \in C^{1,\alpha}$. Hence using the fact that $u_{yy} = m^2 u - (v_x)_{yx}$, we can conclude that $u \in C^{2,\alpha}(\overline{B}_R^+)$ and $||u||_{C^{2,\alpha}(\overline{B}_R^+)} \leq C''_R$ for some constant C''_R depending only on α, R and on upper bounds for $\|G'\|_{C^{1,\alpha}}$ and $||u||_{L^\infty(B_R^+)}$. Finally, let u be a weak bounded solution of (2.1.4). By previous estimates in every half-ball $B_4^+(0,y)$, we deduce that ∇u and $D^2 u$ are bounded in $\overline{\mathbb{R}^2_+} \cap \{0 \leq y \leq 1\}$. Now, let $U(x,y,z) = u(x,y)\cos(mz)$. Since U is harmonic in $\mathbb{R}^2_+ \times \mathbb{R}$, using the interior estimates for harmonic functions in the ball $B_y(x,y,0) \subset \mathbb{R}^2_+ \times \mathbb{R}$ we can see that

$$|\nabla u(x,y)| = |\nabla_{(x,y)} U(x,y,0)| \leq \frac{C}{y} ||u||_\infty \text{ in } \mathbb{R}^2_+$$

and

$$|D^2 u(x,y)| = |D^2_{(x,y)} U(x,y,0)| \leq \frac{C'}{y^2} ||u||_\infty \text{ in } \mathbb{R}^2_+$$

for some positive constants C and C'. Then we can conclude that ∇u and $D^2 u$ are bounded in $\overline{\mathbb{R}^2_+}$. □

Now, we prove a maximum principle of the type Phragmen-Lindelöf :

Lemma 2. *Let $v \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ be a solution of the equation*

$$-\Delta v + 2m\partial_y v = 0 \textit{ in } \mathbb{R}^2_+$$

such that $|v| \leq Ce^{my}$ in $\mathbb{R}^2_+$. Then

$$\inf_{\mathbb{R}^2_+} v = \inf_{\partial \mathbb{R}^2_+} v \textit{ and } \sup_{\mathbb{R}^2_+} v = \sup_{\partial \mathbb{R}^2_+} v. \tag{2.2.3}$$

Proof. Subtracting a constant from v, we can suppose that v is non negative on the boundary $\partial\mathbb{R}^2_+$, and we need to show that $v \geq 0$ in $\mathbb{R}^2_+$.

Consider the function $w = \frac{v}{\psi}$, where

$$\psi(x, y) = e^{my} \log \sqrt{x^2 + (y + 2)^2}$$

and we remark that ψ is positive and solves the equation $-\Delta\psi + 2m\partial_y\psi = m^2\psi$ in $\mathbb{R}^2_+$. We note that w and v have the same sign, $w(x, y) \to 0$ as $|(x, y)| \to \infty$ and w satisfies the equation

$$-\Delta w + 2m\partial_y w + (-\Delta\psi + 2m\partial_y\psi)\frac{w}{\psi} - 2m\nabla w \cdot \frac{\nabla\psi}{\psi} = 0 \text{ in } \mathbb{R}^2_+. \tag{2.2.4}$$

If w were negative at some point in $\mathbb{R}^2_+$, then it achieves its negative minimum at some point $(x_0, y_0) \in \mathbb{R}^2_+$.

Consequently we obtain

$$\Big(-\Delta w + 2m\partial_y w + (-\Delta\psi + 2m\partial_y\psi)\frac{w}{\psi} - 2m\nabla w \cdot \frac{\nabla\psi}{\psi}\Big)(x_0, y_0) < 0$$

and this contradicts the (2.2.4). Therefore we have proved that $\inf_{\mathbb{R}^2_+} v = \inf_{\partial\mathbb{R}^2_+} v$. Similarly we obtain the relation for sup considering a possible internal maximum point of w.

□

Next result shows that the solutions of the problem (2.1.4) with limits in x-direction on $\partial\mathbb{R}^2_+$, are such that their gradient converges towards zero when x tends to infinity, and that the limits of solutions are zeros for G'.

Taking into account the previous lemma, we are able to prove the forthcoming

Theorem 15. *Let $u \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ be a bounded solution of the problem*

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial \nu} = mu - G'(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{2.2.5}$$

with $G \in C^{2,\alpha}(\mathbb{R})$. Assume that $\lim_{y\to\pm\infty} u(0, y) = L_\pm$.
Then we have

(i) $e^{my}|u(x, y)| \leq ||u(0, \cdot)||_{L^\infty(\partial\mathbb{R}^2_+)}\ \forall (x, y) \in \overline{\mathbb{R}^2_+}$;

(ii) $G'(L_-) = G'(L_+) = 0$;

(iii) For every fixed $R > 0$,

$$\|e^{my}u - L_\pm\|_{L^\infty(B^+_R(x,0))} \to 0 \text{ as } x \to \pm\infty$$

and

$$\|e^{my}u_y + mL_\pm\|_{L^\infty(B^+_R(x,0))} + \|u_x\|_{L^\infty(B^+_R(x,0))} \to 0 \text{ as } x \to \pm\infty.$$

Proof. Let $z(x,y) = e^{my}u(x,y)$. Then z is a solution of the problem

$$\begin{cases} -\Delta z + 2m\partial_x z = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial z}{\partial \nu} = -G'(z) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{2.2.6}$$

and it verifies $|z| \le Ce^{my}$ and $\lim_{y\to\pm\infty} z(x,0) = L_\pm$. By Lemma 2, we have

$$\inf_{\partial\mathbb{R}^2_+} z \le z \le \sup_{\partial\mathbb{R}^2_+} z \text{ in } \mathbb{R}^2_+$$

and hence we obtain (i).

Now for $t \in \mathbb{R}$, consider the function

$$u^t(x,y) = u(x+t,y),$$

which is a solution of the problem (2.2.5).

We prove that $G'(L_+) = 0$.

If by contradiction $G'(L_+) \neq 0$ then using Lemma 1 we obtain an estimate $C^{2,\alpha}(\overline{B_S^+})$ for u^t (and so for z^t) uniform in t, for every $S > 0$. Hence there exists a sequence (z^{t_n}) which converges in $C^2_{loc}(\overline{\mathbb{R}^2_+})$ to some function $\bar{z} \in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^2_+})$ and it solves (2.2.6). By hypothesis on the limits, we have $\bar{z} \equiv L_+$ on $\partial\mathbb{R}^2_+$ and using the Lemma 2 we deduce that $\bar{z} \equiv L_+$ in $\mathbb{R}^2_+$.

Therefore $0 = \frac{\partial \bar{z}}{\partial \nu}(x,0) = -G'(\bar{z}(x,0)) = -G'(L_+)$ and this contradicts the assumption $G'(L_+) \neq 0$. Similarly, one can show that $G'(L_-) = 0$.

Now we prove that

$$\|e^{my}u - L_\pm\|_{L^\infty(B_R^+(x,0))} + \|e^{my}u_y + mL_\pm\|_{L^\infty(B_R^+(x,0))} + \|u^t_x\|_{L^\infty(B_R^+(x,0))} \to 0$$

as $t \to \pm\infty$ and for every fixed $R > 0$, which is equivalent to prove that

$$\|z^t - L_\pm\|_{L^\infty(B_R^+(x,0))} + \|z^t_y - m(z^t - L_\pm)\|_{L^\infty(B_R^+(x,0))} + \|z^t_x\|_{L^\infty(B_R^+(x,0))} \to 0$$

as $t \to \pm\infty$ and for every fixed $R > 0$. Arguing by contradiction, assume that there exist $R > 0, x \in \mathbb{R}, \varepsilon > 0$ and a sequence $t_n \to +\infty$ such that

$$\|z^{t_n} - L_+\|_{L^\infty(B_R^+(x,0))} + \|z^{t_n}_y - m(z^{t_n} - L_+)\|_{L^\infty(B_R^+(x,0))} + \|z^{t_n}_x\|_{L^\infty(B_R^+(x,0))} \ge \varepsilon \tag{2.2.7}$$

for every n. Since u^{t_n} are solutions of (2.1.4) uniformly bounded in all the half-space $\mathbb{R}^2_+$ indipendently of t_n, Lemma 1 gives $C^{2,\alpha}(\overline{B_S^+})$ estimates for u^{t_n} (and so for z^{t_n}) uniform in n and for every $S > 0$. Hence, for a subsequence that we still denote by z^{t_n}, we have that z^{t_n} converges in $C^2_{loc}(\overline{\mathbb{R}^2_+})$ to some function $z \in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^2_+})$. Then as before $z \equiv L_+$ on $\partial\mathbb{R}^2_+$ and by Lemma 2 we have $z \equiv L_+$ in $\mathbb{R}^2_+$. This contradicts (2.2.7), and so we have the assertion.

□

Finally, we conclude this section proving the uniqueness ((ii) of the Theorem 14) of the solutions of the Neumann problem (2.1.4) in all of $\mathbb{R}^2_+$.

We need to show the following Lemma :

Lemma 3. *Let $z \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ be a bounded function in $\mathbb{R}^2_+$ satisfying:*

$$\begin{cases} -\Delta z + 2m\partial_y z = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial z}{\partial \nu} = -d(x)z & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

where $d(x)$ is some bounded function, and also

$$v(x,0) \to 0 \text{ as } |x| \to \infty. \tag{2.2.8}$$

Assume that there exists a nonempty set $H \subset \mathbb{R}$ such that

$$z(x,0) > 0 \text{ for } y \in H \text{ and } d(x) \geq 0 \text{ for } x \notin H. \tag{2.2.9}$$

Then $z > 0$ in $\overline{\mathbb{R}^2_+}$.

Proof. By Lemma 2 we know that $\inf_{\mathbb{R}^2_+} z = \inf_{\partial\mathbb{R}^2_+} z$. Arguing by contradiction, assume that there exists a point (x_0, y_0) in $\overline{\mathbb{R}^2_+}$ such that $z(x_0, y_0) \leq 0$.

If $\inf_{\mathbb{R}^2_+} z = 0$, then the minimum of z is a achieved at (x_0, y_0). If $\inf_{\mathbb{R}^2_+} z = \inf_{\partial\mathbb{R}^2_+} z < 0$, using (2.2.8), there exists a point $(x_1, 0)$ at which the minimum of z is achieved. In both cases we conclude that z admits a nonpositive minimum at a some point (x_2, y_2). By the strong maximum principle, we cannot have $y_2 > 0$, since z is not identically constant (recall that $z(x,0) > 0$ for $x \in H$). Then $y_2 = 0$. Since $z(x_2, 0) \leq 0$, we have that $x_2 \notin H$ and so $d(x_2) \geq 0$. Using Hopf's Lemma we deduce that $0 > \frac{\partial v}{\partial \nu}(x_2, 0) = -d(x_2)z(x_2, 0)$, and this gives a contradiction . □

Proof of (II) Theorem 14. Under the assumptions $(G1)$, $(G2)$ and $(G3)$, we want to prove that if u_1 and u_2 are two solutions of

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial \nu} = mu - G'(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{2.2.10}$$

satisfying $|u_i| < e^{-my}$, $\lim_{x\to\pm\infty} u_i(x,0) = \pm 1$, $\partial_x u_i > 0$ in $\overline{\mathbb{R}^2_+}$ $(i = 1, 2)$, then they agree. To do this, we prove that $u_1 \leq u_2^t$ in $\overline{\mathbb{R}^2_+}$ for every $t > 0$. This will be enough to conclude the proof because letting $t \to 0$ we deduce $u_1 \leq u_2$ in $\overline{\mathbb{R}^2_+}$ and interchanging u_1 and u_2 we conclude that $u_1 \equiv u_2$.

Since $G''(\pm 1) > 0$, there exists $0 < \tau < 1$ such that

$$G'' > 0 \text{ in } (-1, -\tau) \cup (\tau, 1).$$

By the hypothesis on the limits, there exists a compact interval $[a, b]$ in $\mathbb{R}$ such that

$$\begin{cases} u_i(x,0) \in (-1,-\tau) & \text{if } x \le a \\ u_i(x,0) \in (\tau,1) & \text{if } x \ge b \end{cases}.$$

Since u_2^t solves (2.2.10), $u_2^t - u_1$ is a solution of the problem

$$\begin{cases} -\Delta(u_2^t - u_1) + m^2(u_2^t - u_1) = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial(u_2^t - u_1)}{\partial\nu} = m(u_2^t - u_1) - d^t(x)(u_2^t - u_1) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{2.2.11}$$

where

$$d^t(x) := \begin{cases} \frac{G'(u_2^t) - G'(u_1)}{u_2^t - u_1}(x,0) & \text{if } (u_2^t - u_1)(x,0) \ne 0 \\ 0 & \text{if } (u_2^t - u_1)(x,0) = 0 \end{cases}.$$

Note that d^t is a bounded function (G' is Lipschitz) and $(u_2^t - u_1)(x,0) \to 0$ as $|x| \to \infty$.

We claim $u_1 < u_2^t$ for $t > 0$ large enough. We take $t > 0$ sufficiently large such that $u_1(x,0) < u_2^t(x,0)$ for $x \in [a,b]$. This is allowed since $u_1(x,0) < 1$ and $u_2^t(x,0) \to 1$ as $t \to +\infty$. We aim to apply Lemma 3 to the function $v(x,y) = e^{my}(u_2^t - u_1)(x,y)$ with $H = (a,b) \cup \{y \in \mathbb{R} : (u_2^t - u_1)(x,0) > 0\}$.

Clearly $v(x,0) > 0$ in H. We prove that $d^t \ge 0$ in $\mathbb{R} - H$.

Let $x \notin H$. We distinguish two cases: If $x \ge b$ then $x + t \ge b$ and so $u_1(x,0), u_2^t(x,0) \ge \tau$. If $x \le a$ then $u_1(x,0) \le -\tau$, and since $y \notin H$ we deduce $u_2^t(x,0) \le u_1(x,0) \le -\tau$. Hence $d^t(x) \ge 0$ in both cases, and by Lemma 3 we have $u_2^t - u_1 > 0$ in $\overline{\mathbb{R}^2_+}$.

We show that

$$\text{if } t > 0 \text{ and } u_1 \le u_2^t \text{ then } u_1 \not\equiv u_2^t. \tag{2.2.12}$$

We recall that by hypothesis $\partial_x u_1 > 0$. Suppose $t > 0$ and $u_1 \equiv u_2^t$. Then $u_1(-t,0) = u_2^t(-t,0) = u_2(0,0) = 0$. Since $u_1(0,0) = 0$, it results that $(-t,0)$ and $(0,0)$ are zeros of u_1. But this is impossibile since u_1 is strictly increasing in x.

Now we prove that

$$\text{if } u_1 \le u_2^t \text{ for some } t > 0, \text{ then } u_1 \le u_2^{t+\mu} \text{ for every } \mu \text{ small enough}. \tag{2.2.13}$$

Using (2.2.12) and the strong maximum principle we have $u_1 < u_2^t$ in $\overline{\mathbb{R}^2_+}$.

Let K_t be a compact such that for $x \notin K_t$, $|u_1(x,0)| > 1 - \frac{\tau}{2}$ and $|u_2^t(x,0)| > 1 - \frac{\tau}{2}$. Recall that $(u_2^t - u_1)(x,0) > 0$ in K_t. By continuity and the existence of limits at infinity, we have that $(u_2^{t+\mu} - u_1)(x,0) > 0$ for $x \in K_t$ and $|u_2^{t+\mu}(x,0)| > 1 - \tau$ for $x \notin K_t$, for every $|\mu|$ small enough.

Now we can apply Lemma 3 to $v(x,y) = e^{my}(u_2^{t+\mu} - u_1)(x,y)$ with $H = K_t$, since $d^{t+\mu} \ge 0$ outside K_t. We obtain $u_2^{t+\mu} - u_1 > 0$.

By (2.2.13) we deduce that $\{t > 0 : u_1 \leq u_2^t\}$ is a nonempty, closed and open set in $(0, +\infty)$ and hence coincides with it.

Therefore we can conclude $u_1 \leq u_2^t$ for all $t > 0$.

$\square$

2.3 Minimizers of the Dirichlet - Neumann Problem in the strips

In this section we provide a result on existence of weak solutions for certain Neumann problems in the strips Ω with given Dirichlet boundary conditions on $\partial^+\Omega$. Let $\Omega = \Omega_R = (-R, R) \times (0, \infty) \subset \mathbb{R}^2_+$ be a strip and $R > 1$. We remark some notations:

$$\partial^0\Omega = \{-R < x < R, y = 0\}$$

and

$$\partial^+\Omega = \{x = -R, y \geq 0\} \cup \{x = R, y \geq 0\}.$$

Let u be a $C^1(\overline{\Omega})$ function such that $\nabla u \in L^2(\Omega)$ and $|u| \leq e^{-my}$ and $u(-x, y) = -u(x, y)$ in Ω, and let $G \in C^{2,\alpha}(\mathbb{R})$ be an even function which satisfies $(G1)$ and $(G2)$. We introduce the energy functional

$$E_\Omega(v) = \frac{1}{2}\iint_\Omega |\nabla v|^2 + m^2v^2 dxdy - \frac{m}{2}\int_{\partial^0\Omega} v^2(x, 0)dx + \int_{\partial^0\Omega} [G(v(x, 0)) - G(1)]dx$$

in the set

$$\begin{aligned} C_u(\Omega) = \{v \in H^1(\Omega) : &|v| \leq e^{-my} \text{ a.e. in } \Omega, v \equiv u \text{ on } \partial^+\Omega, \\ &v(-x, y) = -v(x, y) \text{ in } \Omega\}. \end{aligned}$$

This set is a closed convex subset of the affine space

$$H_u(\Omega) = \{v \in H^1(\Omega) : v \equiv u \text{ on } \partial^+\Omega, v(-x, y) = -v(x, y) \text{ in } \Omega\}.$$

Hence $H_u(\Omega)$ is the set of functions $v \in H^1(\Omega)$ odd in x such that $v - u$ belongs to $H^1_{0,\partial^+}(\Omega)$. We can observe that $\|v\|^2_{H^1(S)} - m|v|^2_{L^2(\partial^0 S)} \geq 0$ for every $v \in H^1(S)$, where $S = (\alpha, \beta) \times (0, +\infty)$ is a strip.

In fact given $v \in C_c^1(\mathbb{R}^2)$ we have

$$m\int_\alpha^\beta |v(x,0)|^2 dx = m\int_\alpha^\beta dx \int_{+\infty}^0 \partial_y |v(x,y)|^2 dy$$

$$\leq 2m\Big(\int_\alpha^\beta \int_0^{+\infty} |v|^2 dxdy\Big)^{\frac{1}{2}}\Big(\int_\alpha^\beta \int_0^{+\infty} |\partial_y v|^2 dxdy\Big)^{\frac{1}{2}} \tag{2.3.1}$$

$$\leq m^2 \int_\alpha^\beta \int_0^{+\infty} |v|^2 dxdy + \int_\alpha^\beta \int_0^{+\infty} |\partial_y v|^2 dxdy \tag{2.3.2}$$

$$\leq m^2 \int_\alpha^\beta \int_0^{+\infty} |v|^2 dxdy + \int_\alpha^\beta \int_0^{+\infty} |\nabla v|^2 dxdy \tag{2.3.3}$$

and by density we obtain $\|v\|^2_{H^1(S)} - m|v|^2_{L^2(\partial^0 S)} \geq 0$ for every $v \in H^1(S)$. Hence the quadratic form of the functional, that is

$$\mathcal{Q}(v) = \iint_\Omega |\nabla v|^2 + m^2 v^2 dxdy - m\int_{\partial^0\Omega} v^2(x,0)dx\,,$$

is positive and continuous with respect to H^1-norm.

Finally we set

$$\|v\|_e := \sqrt{\|v\|^2_{H^1(S)} - m|v|^2_{L^2(\partial^0 S)}}.$$

We observe that, in general, $\|\cdot\|_e$ is not a norm in $H^1(\Omega)$, since $\|v\|_e = 0$ does not imply $v \equiv 0$.

In fact taking the equalities in (2.3.1), (2.3.2), (2.3.3) we have that $v(x,y)$ is a function of y alone, $||v_y||^2_{L^2(\Omega)} = m^2||v||^2_{L^2(\Omega)}$ and $v_y^2 = Cv^2$. These conditions force to be $v(x,y) = ce^{-my}$ for some $c \in \mathbb{R}$.

Theorem 16. *Let $\Omega = (-R,R)\times(0,+\infty)$ be a strip in $\mathbb{R}^2_+$ and $u \in C^1(\overline{\Omega})\cap H^1(\Omega)$ such that $|u| \leq e^{-my}$ and $u(-x,y) = -u(x,y)$ in Ω. Let $G \in C^{2,\alpha}(\mathbb{R})$ be an even function, with $\alpha \in (0,1)$. Suppose that the potential G verifies $(G1)$ and $(G2)$. Then the functional E_Ω admits a minimizer w in $C_u(\Omega)$. In particular w is a weak solution of the problem*

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = mw - G'(w) & \text{on } \partial^0\Omega \\ w = u & \text{on } \partial^+\Omega \end{cases} . \tag{2.3.4}$$

The assumptions $G'(\pm 1) = 0$ implies that $-e^{-my}$ and e^{-my} are solutions of the Neumann- Dirichlet problem in (2.3.4).

Proof. We define the following extension of $G(t)$:

$$\overline{G}(t) := \begin{cases} G(t) - G(1) & \text{if } -1 \leq t \leq 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

and we consider, for every $v \in H_u(\Omega)$, the energy functional

$$\overline{E}_\Omega(v) = \frac{1}{2}\iint_\Omega |\nabla v|^2 + m^2 v^2 dxdy - \frac{m}{2}\int_{\partial^0\Omega} v^2(x,0)dx + \int_{\partial^0\Omega} \overline{G}(v(x,0))dx.$$

Note that any minimizer w of $\overline{E}_\Omega$ in $H_u(\Omega)$ such that $|w| \leq e^{-my}$ is also a minimizer of E_Ω in $C_u(\Omega)$. For this reason we minimize the functional $\overline{E}_\Omega$ in $H_u(\Omega)$. First of all we prove the boundedness of minimizer sequences:

Lemma 4. *Let $(w_n) \subset H_u(\Omega)$ be a minimizing sequence for $\overline{E}_\Omega$. Then it is bounded in $H^1(\Omega)$.*

Proof. By (2.3.2) it follows that

$$\overline{E}_\Omega(v) \geq \frac{||\partial_x v||^2_{L^2(\Omega)}}{2}$$

for every $v \in H^1(\Omega)$, and so for every $n \in \mathbb{N}$

$$||\partial_x w_n||_{L^2(\Omega)} \leq C.$$

We show that (w_n) is bounded in $L^2(\Omega)$. Take $w \in C^1(\overline{\Omega})$ odd in x such that $supp(w-u) \subset \Omega \cup \partial^0\Omega$ is compact.

Then

$$\begin{aligned} w^2(x,y) &= w^2(-R,y) + \int_{-R}^x \partial_x w^2(t,y)dt \\ &\leq e^{-2my} + 2\int_{-R}^x \partial_x w(t,y)w(t,y)dt \end{aligned}$$

and integrating this equality over $(-R,R)$ we have

$$\int_{-R}^R w^2(x,y)dx \leq 2R\Big[2\Big(\int_{-R}^R \partial_x w^2(x,y)dx\Big)^{\frac{1}{2}}\Big(\int_{-R}^R w^2(x,y)dx\Big)^{\frac{1}{2}} + e^{-2my}\Big].$$

Using the Young's inequality we obtain

$$\int_{-R}^R w^2(x,y)dx \leq 2Re^{-2my} + \frac{R}{\varepsilon}\Big(\int_{-R}^R \partial_x w^2(x,y)dx\Big) + 4R\varepsilon\Big(\int_{-R}^R w^2(x,y)dx\Big)$$

and integrating over $(0,+\infty)$ we deduce

$$(1-4R\varepsilon)\int_0^{+\infty} dy \int_{-R}^R w^2(x,y)dx \leq \frac{R}{m} + \frac{R}{\varepsilon}\int_0^{+\infty} dy\int_{-R}^R \partial_x w^2(x,y)dx.$$

Choosing $\varepsilon = \frac{1}{8R}$, the above relation becomes

$$\int_0^{+\infty} dy \int_{-R}^{R} w^2(x,y)dx \le \frac{2R}{m} + 16R^2 \int_0^{+\infty} dy \int_{-R}^{R} \partial_x w^2(x,y)dx,$$

and substituting w_n in the place of w we obtain

$$||w_n||_{L^2(\Omega)} \le CR \text{ for all } n \in \mathbb{N}. \tag{2.3.5}$$

Now we prove that $(\partial_y w_n)$ is bounded in $L^2(\Omega)$.

For this reason we consider for every $y \ge 0$

$$\phi(y) = \int_{-R}^{R} w^2(x,y)dx.$$

where $w \in C^1(\overline{\Omega})$ odd in x such that $supp(w-u) \subset \Omega \cup \partial^0\Omega$ is compact. Since $w \in H^1(\Omega)$ we deduce that $\phi \in W^{1,1}(0,\infty) \subset L^\infty(0,\infty)$ and

$$\begin{aligned}
\int_{-R}^{R} w^2(x,0)dx = \phi(0) &\le C||\phi||_{W^{1,1}(0,\infty)} \\
&= C\Big[\int_0^\infty dy \Big|\frac{\partial}{\partial y}\Big(\int_{-R}^{R} w^2(x,y)dx\Big)\Big| + \Big|\int_{-R}^{R} w^2(x,y)dx\Big|\Big] \\
&\le C\Big[\int_0^{+\infty}\int_{-R}^{R} |w||\partial_y w||dxdy + \int_0^{+\infty}\int_{-R}^{R} w^2 dxdy\Big] \\
&\le C[||w||_{L^2(\Omega)}||\partial_y w||_{L^2(\Omega)} + ||w||^2_{L^2(\Omega)}] \\
&\le C\Big[\frac{1}{2\varepsilon}||w||^2_{L^2(\Omega)} + \frac{\varepsilon}{2}||\partial_y w||^2_{L^2(\Omega)} + ||w||^2_{L^2(\Omega)}\Big]
\end{aligned}$$

where in the last inequality we exploited the Young's inequality. Therefore, by density, we obtain

$$|w_n|_{L^2(\partial^0\Omega)} \le C\Big[\frac{1}{2\varepsilon}||w_n||^2_{L^2(\Omega)} + \frac{\varepsilon}{2}||\partial_y w_n||^2_{L^2(\Omega)} + ||w_n||^2_{L^2(\Omega)}\Big] \tag{2.3.6}$$

Taking into account that

$$||\partial_y w_n||^2_{L^2(\Omega)} - m|w_n|^2_{L^2(\partial^0\Omega)} \le \overline{E}_\Omega(w_n) \le C$$

and by estimate (2.3.6) we have

$$\Big(1 - \frac{\varepsilon}{2}C\Big)||\partial_y w_n||^2_{L^2(\Omega)} \le C'\Big(\frac{1}{2\varepsilon} + 1\Big)||w_n||^2_{L^2(\Omega)} + C.$$

Choosing ε in appropriate way and using the estimate (2.3.5) we find

$$||\partial_y w_n||_{L^2(\Omega)} \le C''R.$$

□

Now, we can observe that $\overline{E}_\Omega$ is weakly lower semicontinuous with respect to H_u-norm, since $H_u(\Omega) \Subset L^2(\partial^0\Omega)$ and $\overline{G}(s)$ is a nonnegative continuous function, so using Lemma 4, we deduce that $\overline{E}_\Omega$ admits a minimum w in $H_u(\Omega)$.

Since $\overline{G}'$ is a continuous function ($G \in C^{2,\alpha}$), $\overline{E}_\Omega$ is a C^1 functional in $H_u(\Omega)$. Making first-order variation of $\overline{E}_\Omega$ at the minimum w we obtain that w is a weak solution of (2.3.4) with G' replaced by $\overline{G}'$.

To finish the proof we need to prove that $-e^{-my} \leq w \leq e^{-my}$ a.e. in Ω.

Let, for every $t \in \mathbb{R}$

$$\varphi_t := w + t(\psi_+ - \psi_-)$$

where

$$\psi_+ = \frac{(w - e^{-my})^+}{2} \text{ and } \psi_- = \frac{(w + e^{-my})^-}{2}.$$

Consider the real-valued function

$$\mathcal{E}(t) = \overline{E}_\Omega(\varphi_t) \text{ for } t \in \mathbb{R}.$$

Since w is a minimizer of $\overline{E}_\Omega$ and $\varphi_t \in H_u(\Omega)$, we observe that $\mathcal{E}(t)$ has a minimum at $t = 0$. Then, recalling that $\overline{G}'(s) = 0$ for $|s| \geq 1$, we obtain

$$\begin{aligned}
0 &= \mathcal{E}'(0) \\
&= \iint_\Omega \nabla w \nabla(\psi_+ - \psi_-) + m^2 w(\psi_+ - \psi_-)\, dxdy - m\int_{\partial^0\Omega} w(\psi_+ - \psi_-)\, dx \\
&+ \int_{\partial^0\Omega} \overline{G}'(w)(\psi_+ - \psi_-)\, dx \\
&= \iint_\Omega 2|\nabla\psi_+|^2 - me^{-my}\partial_y\psi_+ + m^2 w\psi_+\, dxdy - m\int_{\partial^0\Omega} w\psi_+\, dx \\
&+ \iint_\Omega 2|\nabla\psi_-|^2 - me^{-my}\partial_y\psi_- - m^2 w\psi_-\, dxdy + m\int_{\partial^0\Omega} w\psi_-\, dx \\
&= 2[||\psi_+||_e^2 + ||\psi_-||_e^2] \\
&= 2||\psi_+ - \psi_-||_e^2.
\end{aligned} \tag{2.3.7}$$

Using the facts $\|v\|_e = 0$ implies $v(x, y) = Ce^{-my}$ and (2.3.7) we have

$$\frac{(w - e^{-my})^+}{2} - \frac{(w + e^{-my})^-}{2} = Ce^{-my}.$$

But $(w - e^{-my})^+ - (w + e^{-my})^-$ is odd in x, hence $(w - e^{-my})^+ = (w + e^{-my})^-$, that is $|w| \leq e^{-my}$.

□

2.4 Existence of heteroclinic solutions

This last section is dedicated to the proof of (i) Theorem 14. We will use Theorem 16 through which we will construct a sequence of functions which converges to a bounded solution $u(x, y)$ of (2.1.4), increasing with respect to x and $u(x, 0) \to \pm 1$ as $x \to \pm\infty$. The proof of forthcoming Theorem 1 is a modification of the one of Lemma 6.2 in [17].

Proof of (I) Theorem 14. We remember that $G \in C^{2,\alpha}(\mathbb{R})$ satisfies the following properties:

$$G \text{ is even },G'(-1) = G'(1) = 0 \text{ and } G > G(-1) = G(1) \text{ in } (-1, 1).$$

Our aim is to prove that there exists a solution $u(x, y)$ of the problem

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial \nu} = mu - G'(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

such that

$$-e^{-my} < u < e^{-my} \text{ in } \overline{\mathbb{R}}^2_+,$$

$$\lim_{x\to\pm\infty} u(x, 0) = \pm 1,$$

$$\partial_x u > 0 \text{ on } \partial\mathbb{R}^2_+.$$

To reach this, we construct a sequence of functions (u^R) which are solutions of particular Neumann-Dirichlet problems in the strips of the form $(-R, R)\times(0, +\infty)$, and every u^R is such that u^R is $C^{2,\alpha}$ in the strip up to the boundary, $|u^R| \leq e^{-my}$, $u^R(0, 0) = 0$ and $\partial_x u^R \geq 0$.

For $R > 1$, let $\Omega_R = (-R, R) \times (0, +\infty)$. Consider the function

$$v^R(x, y) = \frac{\arctan x}{\arctan R} e^{-my} \text{ for } (x, y) \in \overline{\Omega}_R.$$

Since $G'(-1) = 0 = G'(1)$, by the Theorem 16 we know that exists a minimum u^R of E_{Ω_R} in the set $C_{v^R}(\Omega_R)$.

We remark that u^R is a weak solution of

$$\begin{cases} -\Delta u^R + m^2 u^R = 0 & \text{in } \Omega_R \\ \frac{\partial u^R}{\partial \nu} = mu^R - G'(u^R) & \text{on } \partial^0\Omega_R \\ u^R = v^R & \text{on } \partial^+\Omega_R \end{cases} \tag{2.4.1}$$

First of all we show that u^R is $C^{2,\alpha}(\overline{\Omega_R})$; to do this we proceed as in the proof of Lemma 2.10 in [17]. It will be sufficient to study the regularity of u^R in a neighborhood of each of the two corners of the strip Ω_R; for simplicity we only look the corner on $\{x = R\}$, because the other one is treated in the same way.

Consider the odd reflection of u^R across the segment $\{x = R\}$ and say

$$u^R_{odd}(x,y) := \begin{cases} u^R(x,y) & \text{if } -R < x < R \\ 2e^{-my} - u^R(2R-x,y) & \text{if } R < x < 3R \end{cases}.$$

We prove that u^R_{odd} is a weak solution of the problem:

$$\begin{cases} -\Delta u^R_{odd} + m^2 u^R_{odd} = 0 & \text{in } \{-R < x < 3R, y > 0\} \\ \frac{\partial u^R_{odd}}{\partial \nu} = m u^R_{odd} + \tilde{h}(x) & \text{on } \{-R < x < 3R, y = 0\} \\ u^R_{odd} = v^R_{odd} & \text{on } \{y \geq 0, x = -R \text{ or } x = 3R\} \end{cases}$$

where

$$\tilde{h}(x) := \begin{cases} -G'(u^R(x,0)) & \text{if } -R < x < R \\ G'(u^R(2R-x,0)) & \text{if } R < x < 3R \end{cases}.$$

Let $S = (-R, 3R) \times (0, +\infty)$. Take $\varphi \in C^1(\overline{S})$ such that $\varphi \equiv 0$ on $\partial^+ S$. Then the function $\eta(x,y) = \varphi(x,y) - \varphi(2R-x,y)$ defined for every $(x,y) \in \overline{\Omega_R}$ is such that $\eta \in C^1(\overline{\Omega_R})$ and $\eta \equiv 0$ on $\partial^+\Omega_R$. Hence, since u^R is weak solution of (2.4.1), we have

$$\iint_S \nabla u^R_{odd} \nabla \varphi + m^2 u^R_{odd} \varphi dxdy - m \int_{\partial^0 S} u^R_{odd} \varphi dx - \int_{\partial^0 S} \tilde{h} \varphi dx =$$

$$\begin{aligned} =& \iint_{\Omega_R} \nabla u^R \nabla \eta + m^2 u^R \eta dxdy - m \int_{\partial^0 \Omega_R} u^R \eta dx + \int_{\partial^0 \Omega_R} G'(u^R) \eta dx \\ &+ \int_R^{3R} \int_0^{+\infty} \nabla(2e^{-my}) \nabla \varphi + m^2 (2e^{-my}) \varphi dxdy - 2m \int_R^{3R} \varphi dx \\ =& \int_R^{3R} \int_0^{+\infty} (-2me^{-my} \varphi_y + 2m^2 e^{-my} \varphi) dxdy - 2m \int_R^{3R} \varphi dx \\ =& 2m \int_R^{3R} \varphi dx - 2m \int_R^{3R} \varphi dx = 0 \end{aligned}$$

as required.

We study the regularity in $(R, 0)$ considering the function

$$U^R(x,y) = \int_0^y u^R_{odd}(x,t) dt$$

and the Dirichlet problem that it solves:

$$\begin{cases} -\Delta U^R + m^2 U^R = m u^R_{odd}(x,0) + \tilde{h}(x) & \text{in } \{-R < x < 3R, y > 0\} \\ U^R(x,0) = 0 & \text{on } \{-R < x < 3R, y = 0\} \end{cases}.$$

Since $\tilde{h}$ is bounded, we obtain that u^R_{odd} is $C^{0,\alpha}$ up to $\{-R < x < 3R, y = 0\}$. But this leads to $u^R_{odd}(R,0) = 1$ in the classical sense, and using the hypotesis that $G'(1) = 0$, we deduce that $\tilde{h}$ is a $C^{0,\alpha}$ function.

Finally using Schauder estimates we obtain that u^R_{odd} is $C^{1,\alpha}$ up to $\{-R < x < 3R, y = 0\}$. As a consequence $G'(u^R(x,0))$ is $C^{1,\alpha}((-R,R])$. Hence, its extension $\tilde{h}$ is a $C^{1,\alpha}(-R,3R)$ function. Considering the Dirichlet problem satisfied by U^R_y we attain that $U^R_y \in C^{2,\alpha}$ and so $-u^R_{yy} = (U^R_x)_{yx} - m^2 u^R \in C^{0,\alpha}$. Therefore we can conclude that $u^R \in C^{2,\alpha}$ in a neighborhood of $(R,0)$ in $\{y \geq 0\}$.

Now we prove that

$$|u^R| < e^{-my} \text{ in } \Omega_R \cup \partial^0 \Omega_R.$$

Consider the nonnegative function $w(x,y) = e^{-my} - u^R(x,y)$. Observe that w satisfies the equation $-\Delta w + m^2 w = 0$ in Ω_R. Hence, if there exists a point $(x_0,y_0) \in \Omega_R$ such that $w(x_0,y_0) = 0$ then, by the maximum principle, $w \equiv 0$ but this is a contradiction, because of $w(0,0) = 1 - u^R(0,0) = 1$.

If $(x_0,0) \in \partial^0\Omega_R$ is such that $w(x_0,0) = 0$, by Hopf's Lemma we deduce that $\frac{\partial w}{\partial \nu}(x_0,0) < 0$ while $\frac{\partial w}{\partial \nu}(x_0,0) = m - m u^R(x_0,0) + G'(u^R(x_0,0)) = 0$, because of $G'(1) = 0$. Therefore it's true that $u^R < e^{-my}$ in $\Omega_R \cup \partial^0\Omega_R$.

Similarly we can prove that $u^R > -e^{-my}$ in $\Omega_R \cup \partial^0\Omega_R$ considering the function $w(x,y) = e^{-my} + u^R(x,y) \geq 0$.

Finally we show that $\partial_x u^R \geq 0$ in Ω_R. For this purpose we extend u^R to be identically e^{-my} on $[R,+\infty) \times (0,+\infty)$, and for $t > 0$ consider the function

$$u^{R,t}(x,y) = u^R(x+t,y) \text{ for } (x,y) \in \overline{\Omega}_R.$$

For $0 < \varepsilon < 1$, let

$$\Omega_{R,\varepsilon} = (-R, R-\varepsilon) \times (0,+\infty).$$

Fixed $\varepsilon > 0$, we are going to prove

$$u^R \leq u^{R,t} \text{ in } \overline{\Omega}_{R,\varepsilon}\,, \text{ for every } t \geq \varepsilon. \tag{2.4.2}$$

Since u^R is continuous in $\overline{\Omega}_R$, we obtain that $u^{R,t}$ is continuous in $\overline{\Omega}_{R,\varepsilon}$ for every $t \geq \varepsilon$. Besides exploiting $-e^{-my} < u^R < e^{-my}$ in $[0,+\infty) \times (-R,R)$ and $u^R = v^R$ on $\partial^+\Omega_R$ we deduce

$$u^R < u^{R,t} \text{ su } \partial^+\Omega_{R,\varepsilon}\,, \text{ for every } t \geq \varepsilon. \tag{2.4.3}$$

But $u^{R,t} \equiv e^{-my}$ in $\Omega_{R,\varepsilon}$ for t large, and so (2.4.2) holds for t large enough.

Now we consider the set of t's such that $t \geq \varepsilon$ and (2.4.2) holds. By continuity of u^R and $u^{R,t}$ follows easily that such set is closed.

At this point we only need to prove that the above set is also open. For this, we suppose that $u^R \leq u^{R,\tau}$ in $\overline{\Omega}_{R,\varepsilon}$ for some $\tau \geq \varepsilon$. Assume that $u^R(x_0, y_0) = u^{R,\tau}(x_0, y_0)$ at some point $(x_0, y_0) \in \overline{\Omega}_{R,\varepsilon}$.

Then, by (2.4.3), we have $(x_0, y_0) \in \Omega_{R,\varepsilon} \cup \partial^0 \Omega_{R,\varepsilon}$ and, in particular, $u^{R,\tau}(x_0, y_0) = u^R(x_0, y_0) \in (-e^{-my}, e^{my})$. Note that both $u^{R,\tau}$ and u^R are solutions of the same Neumann problem. By the maximum principle and Hopf's Lemma they must agree everywhere and this contradicts (2.4.3).

Hence $u^R < u^{R,\tau}$ in $\overline{\Omega}_{R,\varepsilon}$ and, by continuity, the same inequality will be true for every $t \geq \varepsilon$ in a neighborhood of τ.

We take $(x, y) \in \Omega_R$; then $(x, y) \in \Omega_{R,\varepsilon}$ for every ε small enough. From (2.4.2) applied with $t = \varepsilon$, we obtain $u^R(x, y) \leq u^{R,\varepsilon}(x, y)$ for every small $\varepsilon > 0$. Letting $\varepsilon \to 0$, we deduce that $\partial_x u^R(x, y) \geq 0$.

Now we want to construct a subsequence of (u^R) which converges to a solution u of the problem (2.1.4) such that $|u| < e^{-my}, u(x, 0) \to \pm 1$ as $x \to \pm\infty$ and $\partial_x u > 0$. To obtain this, we show that the functional is uniformly bounded in u^R, that is

Lemma 5. *For every $R > 1$ results*

$$E_{\Omega_R}(u^R) \leq C \tag{2.4.4}$$

for some constant C independent of R.

Proof. Since u^R is a minimum for the functional we have that

$$E_{\Omega_R}(u^R) \leq E_{\Omega_R}(v^R).$$

We estimate the $|| \cdot ||_e$-norm of v^R as follows

$$\begin{aligned}
&\|v^R\|^2_{H^1(\Omega_R)} - m|v^R|_{L^2(\partial^0\Omega_R)} \\
&\quad = \iint_{\Omega_R} |\nabla v^R|^2 + m^2 (v^R)^2 dxdy - m \int_{\partial^0\Omega_R} (v^R(x,0))^2 dx \\
&\quad = \iint_{\Omega_R} 2m^2 \frac{\arctan^2 x}{\arctan^2 R} e^{-2my} + \frac{1}{(1+x^2)^2} \frac{e^{-2my}}{\arctan^2 R} dxdy \\
&\quad - m \int_{\partial^0\Omega_R} \frac{\arctan^2 x}{\arctan^2 R} dx \\
&\quad = \frac{1}{2m} \int_{-R}^{R} \frac{1}{(1+x^2)^2} \frac{1}{\arctan^2 R} dx \leq C
\end{aligned} \tag{2.4.5}$$

where in the last inequality we exploited

$$\begin{aligned}\int_{-R}^{R}\frac{1}{(1+x^2)^2}\frac{1}{\arctan^2 R}dx &\le C'\int_{-R}^{R}\frac{dx}{(1+x^2)^2}\\ &= C''\Big(\arctan R+\frac{R}{1+R^2}\Big)\\ &\le C'''.\end{aligned}$$

Next, by $(G1)$ and $(G2)$, and using Mean Value Theorem we have

$$G(s)-G(1)\le C(1+\cos(\pi s)) \text{ for all } s\in[-1,1]$$

for some constant $C>0$. Hence, remarking that $\frac{\pi}{\arctan R}>2$, we deduce

$$\begin{aligned}G(v^R(x,0))-G(1) &\le C\Big\{1+\cos\Big(\pi\frac{\arctan x}{\arctan R}\Big)\Big\}\\ &\le C\{1+\cos(2\arctan x)\}\\ &= C(2\cos^2(\arctan x))\\ &= \frac{2C}{1+x^2}\end{aligned}$$

from which follows

$$\int_{-R}^{R}\{G(v^R(x,0))-G(1)\}dy\le C\int_{-R}^{R}\frac{dx}{1+x^2}\le C. \tag{2.4.6}$$

Putting together (2.4.5) and (2.4.6) we prove (2.4.4).

□

Hence we can conclude the proof of the Theorem as follows.

Fix $R'>0$. Since $|u^R|<e^{-my}$, Lemma 1 gives $C^{2,\alpha}(\overline{B^+_{R'}})$ estimates for u^R, uniform for $R\ge 4R'$.

Using (2.4.4) we have $||u^R||_{e,S_{R'}}\le C$ where $S_{R'}=(-4R',4R')\times(0,+\infty)$, and since $|u^R(x,0)|<1$, we obtain $||u^R||_{H^1(S_{R'})}\le C+cR'$. Then we can extract a subsequence (u^{R_j}) such that u^{R_j} converges in $C^2_{loc}(\overline{\mathbb{R}^2_+})$ and weakly in $H^1_\#(\mathbb{R}^2_+)$ (where $H^1_\#$ denotes the space of the functions in H^1 of the strips of the type $(0,\infty)\times(\alpha,\beta)$) to some function $u\in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^2_+})\cap H^1_\#(\mathbb{R}^2_+)$ as $j\to\infty$. Consequently u is a solution of the problem (2.1.4) such that

$$|u|\le e^{-my}, u(0,0)=0 \text{ and } \partial_x u\ge 0 \text{ in } \mathbb{R}^2_+.$$

Since $u(0,0)=0$, we have $|u|\not\equiv e^{-my}$ and hence $|u|<e^{-my}$, by the strong maximum principle. Note that ±1 are solutions of the problem since $G'(\pm1)=0$.

By monotonicity of $u(0,\cdot)$ we know that there exist l_-, l_+ such that

$$-1 \le l_- \le l_+ \le 1$$

and

$$\lim_{x\to\pm\infty} u(x,0) = l_\pm.$$

Since u is odd in x we have that $l_- = -l_+$. Moreover $l_+ \ge 0$ since $u(\cdot,0)$ is increasing in x and $u(0,0)=0$. Note that l_+ is positive. Indeed, if $l_+=0$ then $u(\cdot,0)\equiv 0$ and so

$$\int_{-\infty}^{+\infty} [G(u(x,0)) - G(1)]dx = \int_{-\infty}^{+\infty} [G(0)-G(1)]dx = \infty.$$

But this is impossible since using Lemma 5 and Fatou Lemma we can see that

$$\int_{-\infty}^{+\infty} [G(u(x,0)) - G(1)]dx \le C. \tag{2.4.7}$$

Hence $l_+ \in (0,1]$. Let us show that $l_+ = 1$. Arguing by contradiction, assume that

$$l_+ < 1. \tag{2.4.8}$$

By (G2) and continuity of G we deduce that there exist $\delta > 0$ and a neighborhood J of l_+ such that

$$G(s) - G(1) \ge \delta \text{ for all } s \in J. \tag{2.4.9}$$

Recalling that u is non-decreasing in x, we have that there exists $k \in \mathbb{R}$ such that, if $x \ge k$, then $u(x,0)\in J$. Therefore, from (2.4.7) and (2.4.9) we obtain

$$C \ge \int_{-\infty}^{+\infty} [G(u(x,0)) - G(1)]\,dx \tag{2.4.10}$$

$$\ge \int_{k}^{+\infty} [G(u(x,0)) - G(1)]\,dx \tag{2.4.11}$$

$$\ge \int_{k}^{+\infty} \delta\,dx = +\infty, \tag{2.4.12}$$

and this contradiction proves that $l_+ = 1$. Finally, since $\partial_y u$ is a nonnegative solution of

$$\begin{cases} -\Delta \partial_x u + m^2 \partial_x u = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial(\partial_x u)}{\partial \nu} = m\partial_y u - G''(u)\partial_x u & \text{on } \partial\mathbb{R}^2_+ \end{cases},$$

by the strong maximum principle and $u(x,0)\to\pm1$ as $x\to\pm\infty$, we deduce that $\partial_x u > 0$ in $\overline{\mathbb{R}}^2_+$. $\square$

Remark 1. *We can observe that* $\lim_{x\to\pm\infty} u(x,y) = \pm e^{-my}$ *for every* $y \geq 0$. *In fact by monotonicity of* u *with respect to* x, *we know that there exists the limits* $L^{\pm}(y) \in [-e^{-my}, e^{-my}]$ *as* $x \to \pm\infty$. *Since* $u(x,0) \to \pm 1$ *as* $x \to \pm\infty$, *we have that* $L^{\pm}(0) = \pm 1$. *Now, let* $u^k(x,y) = u(x+k,y)$ *for every* $k \in \mathbb{N}$. *Since* u^k *is bounded independently of* k, *by Lemma1 we can extract a subsequence which converges locally to some function* u^∞ *satisfying*

$$\begin{cases} -\Delta u^\infty + m^2 u^\infty = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u^\infty}{\partial \nu} = m u^\infty - G'(u^\infty) & \text{on } \partial\mathbb{R}^2_+. \\ u^\infty(x,0) = 1 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{2.4.13}$$

But $u^\infty(x,y) = L^+(y)$, *so we deduce that* $L^+(y) = e^{-my}$ *for every* $y \geq 0$. *Similarly we can prove that* $L^-(y) = -e^{-my}$ *for every* $y \geq 0$.

2.5 Heteroclinic solutions for $m = 0$

In this section we prove that is possible to take the limit as $m \to 0$ in the problem (2.1.4). We start proving the following

Theorem 17. *Let* u *be a layer solution of the problem (2.1.4). Then for all* $m > 0$

$$G(u(x,0)) - G(1) = \int_0^{+\infty} \frac{u_x^2 - u_y^2 - m^2 u^2}{2}(x,y)dy + \frac{m}{2}u^2(x,0). \tag{2.5.1}$$

Proof. Consider the function

$$v(x) = \int_0^{+\infty} \frac{u_x^2 - u_y^2 - m^2 u^2}{2}(x,y)dy.$$

and we show that v is well defined in $\mathbb{R}$.

By Lemma 1 we know that $|\nabla u|, |D^2 u| \leq C$ in $\overline{\mathbb{R}}^2_+$. Now let $U(x,y,z) = u(x,y)\cos(mz)$ and observe that U is harmonic in $\mathbb{R}^2_+ \times \mathbb{R}$ and $|U| \leq 1$. Using the interior estimates for the first and the second derivatives for harmonic functions in the ball $B_y(x,y,0) \subset \mathbb{R}^2_+ \times \mathbb{R}$, we deduce that

$$|\nabla u(x,y)| = |\nabla_{(x,y)} U(x,y,0)| \leq \frac{C}{y} \text{ and } |D^2 u(x,y)| = |D^2_{(x,y)} U(x,y,0)| \leq \frac{C'}{y^2} \tag{2.5.2}$$

for some constants C and C' independent of m. Hence we can see that

$$|\nabla u| \leq \frac{C}{y+1} \text{ and } |D^2 u| \leq \frac{C'}{y^2+1} \tag{2.5.3}$$

hold in $\mathbb{R}^2_+$. Therefore v is well defined. Integrating by parts and using the fact that u is a solution of (2.1.4) we have

$$v'(x) = \int_0^{+\infty} (u_x u_{xx} - u_y u_{yx} - m^2 u u_x)(x,y)dy \tag{2.5.4}$$
$$= \int_0^{+\infty} (-u_x u_{yy} - u_y u_{yx})(x,y)dy \tag{2.5.5}$$
$$= (u_x u_y)(x,0) = \frac{d}{dx}\Big(-\frac{m}{2}u^2 + G(u)\Big)(x,0). \tag{2.5.6}$$

Therefore $v(x) + \frac{m}{2}u^2(x,0) - G(u(x,0)) + G(1) = const = C$ for all $x \in \mathbb{R}$.

Now we want to compute the value of constant C.

Fix $R > 0$. By Theorem 15, (2.5.3) and $|u| \le e^{-my}$ we can see that

$$\begin{aligned}
\limsup_{|x|\to\infty}\Big|v(x) + \frac{m}{2}\Big| &\le \limsup_{|x|\to\infty}\Big\{\Big|\int_0^R \frac{u_x^2 - u_y^2 - m^2u^2}{2}(x,y)dy + \frac{m}{2}\Big| \\
&\quad + \Big|\int_R^{+\infty} \frac{u_x^2 - u_y^2 - m^2u^2}{2}(x,y)dy\Big|\Big\} \\
&= \Big|\int_0^R -m^2 e^{-2my}dx + \frac{m}{2}\Big| + \limsup_{|x|\to\infty}\Big|\int_R^{+\infty} \frac{u_x^2 - u_y^2 - m^2u^2}{2}(x,y)dy\Big| \\
&= \frac{m}{2}e^{-2mR} + \limsup_{|x|\to\infty}\Big|\int_R^{+\infty} \frac{u_y^2 - u_y^2 - m^2u^2}{2}(x,y)dy\Big| \\
&\le \frac{m}{2}e^{-2mR} + \frac{C}{(R+1)^2} + \frac{m}{4}e^{-2mR} \to 0 \text{ as } R \to +\infty.
\end{aligned}$$

Therefore, using the facts that $u(x,0) \to \pm 1$ as $x \to \pm\infty$ and $G(1) = G(-1)$, we find that $C = 0$. $\square$

Now we suppose that $0 < m \le 1$ and we denote by u_m a heteroclinic solution to (2.1.4). By Lemma 1 and $||u_m||_{L^\infty(\overline{\mathbb{R}}^2_+)} \le 1$, we know that fixed $R > 0$, we have

$$||u_m||_{C^{2,\alpha}(\overline{B}^+_R)} \le C(R, \alpha, ||G'||_{L^\infty([-1,1])}, ||G''||_{L^\infty([-1,1])}) \tag{2.5.7}$$

for some constant $C(R, \alpha, ||G'||_{L^\infty([-1,1])}, ||G''||_{L^\infty([-1,1])})$ independent of m. Hence u_m converges to some function $u \in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^2_+})$ in $C^2_{loc}(\overline{\mathbb{R}^2_+})$ as $m \to 0$ such that $u_x \ge 0$, $u(0,0) = 0$, $|u| \le 1$ and u is a weak solution to the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial \nu} = -G'(u) & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.5.8}$$

By interior gradient estimates for harmonic function we have

$$|\nabla u_m(x,y)| \leq \frac{C}{y} \text{ in } \mathbb{R}^2_+$$

for some constant independent of m (see (2.5.2) of the Theorem 17). Using the estimate (2.5.7) in the ball $B^+_4(x,0)$ we obtain ∇u_m is bounded in $\overline{\mathbb{R}}^2_+ \cap \{0 \leq y \leq 1\}$ by a constant independent of m. Therefore we can deduce that ∇u_m is bounded in $\overline{\mathbb{R}}^2_+$ and in particular we find

$$|\nabla u_m| \leq \frac{C}{y+1} \text{ in } \mathbb{R}^2_+, \tag{2.5.9}$$

for some constant C independent of m. Then using $|u_m| < e^{-my}$ in $\overline{\mathbb{R}^2_+}$, (2.5.9) and the Dominated Convergence Theorem we can pass to the limit as $m \to 0$ in the formula (2.5.1), and we obtain

$$G(u(x,0)) - G(1) = \int_0^{+\infty} \frac{u_x^2 - u_y^2}{2}(x,y)dy \quad \forall x \in \mathbb{R}. \tag{2.5.10}$$

Since u is increasing in x, there exist the limits $u(x,0) \to l_\pm \in [-1,1]$ as $x \to \pm\infty$. Now we argue by contradiction. Assume $l_+ < 1$. By Lemma 2.3 in [17] we know that $||\nabla u||_{L^\infty(B^+_R(x,0))} \to 0$ as $|x| \to \infty$, for every $R > 0$ fixed. Moreover we know that $|\nabla u| \leq \frac{C}{y+1}$ holds in $\mathbb{R}^2_+$. Thus, taking the limit as $x \to +\infty$ in (2.5.10) we deduce that $G(l_+) - G(1) = 0$. But this gives a contradiction since $G(l_+) > G(1)$. Similarly we can prove that $l_- = -1$. Finally, using the fact that u_x is a nonnegative solution to

$$\begin{cases} \Delta u_x = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u_x}{\partial \nu} = -G''(u)u_x & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.5.11}$$

and that $u(x,0) \to \pm 1$ as $x \to \pm\infty$, we can see that u is strictly increasing in x.

Chapter 3

Periodic solution for a pseudo-relativistic Schrödinger equation

We study the existence and the regularity of non trivial T-periodic solutions to the following nonlinear pseudo-relativistic Schrödinger equation

$$\left(\sqrt{-\Delta_x + m^2} - m\right)u(x) = f(x, u(x)) \text{ in } (0,T)^N \tag{3.0.1}$$

where $T > 0$, m is a non negative real number, f is a regular function satisfying the Ambrosetti-Rabinowitz condition and a polynomial growth at rate p for some $1 < p < 2^\sharp - 1$. We investigate such problem using critical point theory after transforming it to elliptic equation in the infinite half-cylinder $(0,T)^N \times (0,\infty)$ with a nonlinear Neumann boundary condition. By passing to the limit as $m \to 0$ in (3.0.1) we also prove the existence of a non trivial T-periodic weak solution to (3.0.1) with $m = 0$.

3.1 Introduction

In this paper we are concerned with periodic solutions for a nonlinear pseudo-relativistic Schrödinger equation. Particularly, we are looking for a function u satisfying the nonlinear problem

$$\begin{cases} \left(\sqrt{-\Delta_x + m^2} - m\right)u(x) = f(x, u(x)) \text{ in } (0,T)^N = \prod_{i=1}^N (0,T) \\ u(x + Te_i) = u(x) \text{ for all } x \in \mathbb{R}^N, i = 1, \dots, N \end{cases} \tag{3.1.1}$$

where $T > 0$ is fixed, $m \geq 0$ and (e_i) is the canonical basis in $\mathbb{R}^N$.

The operator $\sqrt{-\Delta_x + m^2}$ is defined as follows: let $u \in \mathcal{C}_T^\infty(\mathbb{R}^N)$, that is u is infinitely differentiable in $\mathbb{R}^N$ and T-periodic in each variable.

Then u has a Fourier series expansion:

$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} \quad (x \in \mathbb{R}^N)$$

where

$$\omega = \frac{2\pi}{T} \text{ and } c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x) e^{-i\omega k\cdot x} dx \quad (k \in \mathbb{Z}^N)$$

are the Fourier coefficients of u.

The operator $\sqrt{-\Delta_x + m^2}$ is defined by setting

$$\sqrt{-\Delta_x + m^2}\, u = \sum_{k \in \mathbb{Z}^N} c_k \sqrt{\omega^2 |k|^2 + m^2}\, \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}}. \tag{3.1.2}$$

For $u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}}$ and $v = \sum_{k \in \mathbb{Z}^N} d_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}}$, we have that

$$\mathcal{Q}(u,v) = \sum_{k \in \mathbb{Z}^N} \sqrt{\omega^2 |k|^2 + m^2} c_k \overline{d}_k$$

can be extended by density to a quadratic form on the Hilbert space

$$\mathbb{H}_T^m = \Big\{ u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} \in L^2(0,T)^N : \sum_{k \in \mathbb{Z}^N} \sqrt{\omega^2 |k|^2 + m^2}\, |c_k|^2 < \infty \Big\}.$$

We assume that the nonlinear term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ in equation (3.1.1) satisfies the following conditions:

$(f1)$ $f(x,t)$ is locally Lipschitz - continuous in $\mathbb{R}^N \times \mathbb{R}$;

$(f2)$ There exist $a_1, a_2 > 0$ and $p \in (1, 2^\sharp - 1)$such that

$$|f(x,t)| \le a_1 + a_2 |t|^p \quad \forall t \in \mathbb{R} \quad \forall x \in \mathbb{R}^N.$$

Here the critical exponent $2^\sharp$ is infinite if $N = 1$ and $\frac{2N}{N-1}$ if $N \ge 2$;

$(f3)$ $\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0$ uniformly in $x \in \mathbb{R}^N$;

$(f4)$ There exist $\mu > 2$ and $r > 0$ such that $0 < \mu F(x,t) \le t f(x,t)$ for all $|t| \ge r$ and for all $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s) ds$;

$(f5)$ f is T-periodic in each variable x_i, that is $f(x + e_iT, t) = f(x, t)$ for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $i = 1, \ldots, N$;

$(f6)$ $tf(x, t) \geq 0$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

We remark that the hypothesis $(f3)$ guarantees that (3.1.1) possesses the trivial solution $u \equiv 0$. The hypothesis $(f4)$ gives information about the behavior of $f(x, u)$ and $F(x, u)$ at $u = \infty$. Indeed, a straightforward computation shows that, by $(f4)$, there exist two constants $a_3, a_4 > 0$ such that

$$F(x, u) \geq a_3|u|^{\mu} - a_4 \text{ for } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}. \tag{3.1.3}$$

Since $\mu > 2$, (3.1.3) and $(f2)$ imply that $F(x, u)$ grows superquadratically and $f(x, u)$ grows superlinearly as $|u| \to \infty$. As a model for f we can take $f(x, u) = g(x)|u|^{p-1}u$, where g is a smooth positive T-periodic function. We observe that hypotheses $(f1) - (f4)$ are standard when we deal with superlinear second order elliptic partial differential equations: see for example [67], [77] and [81].

The non-local operator $\sqrt{-\Delta_x + m^2}$ in $\mathbb{R}^N$ plays an important role in relativistic quantum mechanics. Indeed the Hamiltonian for a (free) relativistic particle of momentum p and mass m is given by

$$\mathcal{H} = \sqrt{c^2|p|^2 + m^2c^4}$$

and with the usual quantization rule $p \to -i\hbar\nabla$, we get the so called pseudo-relativistic Hamiltonian operator and the associate free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi = \sqrt{-\hbar^2c^2\Delta_x + m^2c^4}\,\psi.$$

Then choosing $\hbar = c = 1$ we obtain the operator above mentioned. For a discussion of the main properties of the operator $\sqrt{-\Delta_x + m^2}$ we refer to [49]. For physical models involving this operator one can see the works of Lieb and Yau [51], [52] where in the first they study boson stars and in the second the stability of relativistic matter. More recently Fröhlich, Jonsson and Lenzmann [38] study the existence of solitary wave solutions of the pseudo-relativistic Hartree equation

$$i\partial_t\psi = (\sqrt{-\Delta_x + m^2} - m)\psi - (|x|^{-1} * |\psi|^2)\psi \text{ on } \mathbb{R}^3;$$

(see also [27, 39, 40] for related models).

From a probabilistic point of view, the operator $-(\sqrt{-\Delta_x + m^2} - m)$ is strictly connected with the potential theory: it is the infinitesimal generator of a Levy process, the so called the 1-stable relativistic process (see [24], [68]).

Recently the study of nonlinear equations involving a fractional Laplacian $(-\Delta_x)^{\alpha}$ has attracted the attention of many mathematicians, since it appears in many different contexts as phase transitions, optimization, finance, minimal surfaces and others. Caffarelli, Roquejoffre, Sire [19] and Caffarelli, Salsa, Silvestre [20] investigated free boundary problems of a fractional Laplacian. Silvestre [71] obtained some regularity results for the obstacle problem of the fractional Laplacian. Cabré and Solá Morales [17] studied an analogue of the De Giorgi conjecture for the equation

$$(-\Delta_x)^\alpha u = -G'(u) \text{ in } \mathbb{R}^N \tag{3.1.4}$$

when $\alpha = \frac{1}{2}$ and $G \in C^2(\mathbb{R})$ has only two absolute minima. The same problem with $\alpha \in (0,1)$ has been studied by Sire and Valdinoci in [72] and Cabré and Sire in [15], [16]. Cabré and Tan [18] proved the existence of positive solutions to the problem

$$\begin{cases} \sqrt{-\Delta_x}\, u = |u|^p \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ u > 0 \text{ in } \Omega \end{cases} \tag{3.1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $1 \leq p < \frac{N+1}{N-1}$ (if $N > 1$), and Servadei and Valdinoci [70] dealt with the existence of non-trivial solutions of the following problem:

$$\begin{cases} (-\Delta_x)^s u - \lambda u = f(x,u) \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded domain, $s \in (0,1)$, λ is a real parameter and $f(x,u)$ is a Carathéodory function which behaves like $u|u|^{p-2}$ for some $2 < p < \frac{2N}{N-2s}$. We can note that the above problems (3.1.5) and (3.1.6) can be seen as the the fractional analogue of the classical problem

$$\begin{cases} -\Delta_x u - \lambda u = f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1.7}$$

This last problem has been also investigated with periodic boundary conditions by using variational and topological methods; see for instance [8], [46], [56], [67], [77] and references therein.

The aim of the present paper is to study an analogue to (3.1.7) with periodic boundary conditions, when we replace $-\Delta_x$ by $\sqrt{-\Delta_x + m^2}$.

The first result is the following:

Theorem 18. *Let $m > 0$. Let $f : \mathbb{R}^{N+1} \to \mathbb{R}$ be a function verifying the conditions $(f1) - (f6)$. Then there exists at least a function $u_m \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, T-periodic which satisfies the problem (3.1.1).*

One of the main difficulty of the analysis of the problem (3.1.1) is the nonlocal character of the involved operator. To circumvent this hitch, we use the approach proposed by Caffarelli and Silvestre [21], which consists to realize the nonlocal problem (3.1.1) into a local problem in one more dimension via the Dirichlet-Neumann map. As explained in detail in section 3 below, for $u \in \mathbb{H}_T^m$ one can find a unique weak solution $v \in \mathbb{X}_T^m$ to the problem

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T = (0,T)^N \times (0,\infty) \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L \mathcal{S}_T = \partial(0,T)^N \times [0,+\infty) \\ v(x,0) = u(x) \text{ on } \partial^0 \mathcal{S}_T = (0,T)^N \times \{0\}, \end{cases} \tag{3.1.8}$$

where the boundary condition on $\partial^0 \mathcal{S}_T$ is in the sense of trace, and $\mathbb{X}_T^m$ is defined as the completion of functions $\mathcal{C}^\infty(\overline{\mathbb{R}_+^{N+1}})$ and T-periodic in each x_i, with respect to the norm

$$||v||^2_{\mathbb{X}_T^m} = \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 \, dxdy.$$

Note that in (3.1.8) the notation $v_{|\{x_i=0\}} = v_{|\{x_i=T\}}$ on $\partial_L \mathcal{S}_T$ means

$$v(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_N, y) = v(x_1, \dots, x_{i-1}, T, x_{i+1}, \dots, x_N, y)$$

for every $i \in \{1, \dots, N\}$ and $y \geq 0$.

Furthermore,

$$-\lim_{y \to 0^+} \frac{\partial v}{\partial y}(x,y) = \sqrt{-\Delta_x + m^2}\, u(x) \text{ in } (\mathbb{H}_T^m)^*,$$

in a weak sense. In order to find solutions of (3.1.1) and to prove their regularity, we will exploit this fact and look for solutions $v \in \mathbb{X}_T^m$ to

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L \mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = mv + f(x,v) \text{ on } \partial^0 \mathcal{S}_T. \end{cases} \tag{3.1.9}$$

The variational structure of the problem (3.1.9) allows us to obtain the existence of T-periodic solutions v_m through known variational methods, namely the Linking Theorem. Such solutions are obtained as critical points in $\mathbb{X}_T^m$ of the functional $\mathcal{J}_m$ associated to (3.1.9), that is

$$\mathcal{J}_m(v) = \frac{1}{2} \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 \, dxdy - \frac{m}{2} \int_{\partial^0 \mathcal{S}_T} |v|^2 \, dx - \int_{\partial^0 \mathcal{S}_T} F(x,v) \, dx.$$

When m is sufficiently small, we are able to prove uniform estimates on critical levels α_m of the functionals $\mathcal{J}_m$. These estimates allows us to deduce uniform estimates on the solutions v_m, and so we can pass to the limit as $m \to 0$ in (3.1.9). As a consequence we can show the existence of a nontrivial T-periodic solution to the problem

$$\begin{cases} \sqrt{-\Delta_x}\, u(x) = f(x, u(x)) \text{ in } (0,T)^N \\ u(x + Te_i) = u(x) \text{ for all } x \in \mathbb{R}^N, i = 1, \dots, N. \end{cases} \tag{3.1.10}$$

This result can be stated as

Theorem 19. *Under the same assumptions of Theorem 18 we can find a non trivial T-periodic weak solution to the problem (3.1.10).*

The paper is organized as follows. In the Section 2 we give some basic results concerning the fractional space $\mathbb{H}^m_T$ and the nonlinear term f. In the Section 3 we show that it's possible to transform the problem (3.1.1) in a Neumann elliptic problem. In the Section 4 we prove the existence of weak solutions of the elliptic problem (3.1.9) through the Theory of Critical Point. In the Section 5 we study the regularity of above critical points and we give the proof of Theorem 18. Finally, in the last section we find a nontrivial periodic solution to the problem (3.1.10).

3.2 Preliminaries

In this section we collect preliminary facts for future reference. Firstly we denote the upper half-space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, y > 0\}.$$

Let $\mathcal{S}_T = (0,T)^N \times (0,\infty)$ be the half-cylinder in $\mathbb{R}^{N+1}_+$ with basis $\partial^0 \mathcal{S}_T = (0,T)^N \times \{0\}$ and we denote by $\partial_L \mathcal{S}_T = \partial(0,T)^N \times [0,+\infty)$ its lateral boundary. With $||v||_{L^r(\mathcal{S}_T)}$ we will always denote the norm of a function $v(x,y)$ in $L^r(\mathcal{S}_T)$ and with $|v|_{L^r(0,T)^N}$ the norm of a function $u(x)$ in $L^r(0,T)^N$.

Let $u \in \mathcal{C}^\infty_T(\mathbb{R}^N)$; that is $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ and u is T-periodic in each x_i, that is

$$u(x + Te_i) = u(x) \text{ for all } x \in \mathbb{R}^N, i = 1, \dots, N.$$

Then we know that for all $x \in \mathbb{R}^N$

$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k \cdot x}}{\sqrt{T^N}},$$

where

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x) e^{-i\omega k \cdot x} dx \quad (k \in \mathbb{Z}^N)$$

are the Fourier coefficients of u. We define the fractional Sobolev space $\mathbb{H}^m_T$ as the closure of $\mathcal{C}^\infty_T(\mathbb{R}^N)$ under the norm

$$|u|^2_{\mathbb{H}^m_T} := \sum_{k \in \mathbb{Z}^N} \sqrt{\omega^2 |k|^2 + m^2}\, |c_k|^2. \tag{3.2.1}$$

When $m = 1$, we set $\mathbb{H}_T = \mathbb{H}^1_T$ and $||\cdot||_{\mathbb{H}_T} = ||\cdot||_{\mathbb{H}^1_T}$. Now we introduce the functional space $\mathbb{X}^m_T$ defined as the completion of

$$\mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+}) = \Big\{ v \in \mathcal{C}^\infty(\overline{\mathbb{R}^{N+1}_+}) : v(x + Te_i, y) = v(x,y) \\ \text{for every } (x,y) \in \overline{\mathbb{R}^{N+1}_+}, i = 1, \dots, N \Big\}$$

under the norm

$$||v||^2_{\mathbb{X}^m_T} = \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 dxdy. \tag{3.2.2}$$

If $m = 1$, we set $\mathbb{X}_T = \mathbb{X}^1_T$ and $||\cdot||_{\mathbb{X}_T} = ||\cdot||_{\mathbb{X}^1_T}$. We begin proving that it's possible to define a trace operator from the space $\mathbb{X}^m_T$ to the fractional space $\mathbb{H}^m_T$:

Theorem 20. *There exists a bounded linear operator* $\mathrm{Tr} : \mathbb{X}^m_T \to \mathbb{H}^m_T$ *such that*

(i) $\mathrm{Tr}(v) = v|_{\partial^0 \mathcal{S}_T}$ *for all* $v \in \mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+}) \cap \mathbb{X}^m_T$*;*

(ii) $|\mathrm{Tr}(v)|_{\mathbb{H}^m_T} \leq ||v||_{\mathbb{X}^m_T}$ *for every* $v \in \mathbb{X}^m_T$*;*

(iii) Tr *is surjective.*

Remark 2. *Sometimes, with abuse of notation, we will denote* $\mathrm{Tr}(v)$ *by* $v(\cdot, 0)$.

Proof. Let $v \in \mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+}) \cap \mathbb{X}^m_T$. Then using the Fourier series we can write

$$v(x, y) = \sum_{k \in \mathbb{Z}^N} c_k(y) \frac{e^{i\omega k \cdot x}}{\sqrt{T^N}} \text{ in } \overline{\mathbb{R}}^{N+1}_+,$$

where $c_k(y) = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} v(x, y) e^{-i\omega k \cdot x} dx$.

Note that $c_k \in \mathcal{C}^\infty([0, \infty)) \cap H^1((0, \infty))$, therefore $c_k(y) \to 0$ as $y \to \infty$, for all $k \in \mathbb{Z}^N$. By the Fundamental Theorem of Calculus and using the Hölder inequality we have

$$\begin{aligned}
|c_k(0)|^2 &= -\int_0^\infty \frac{d}{dy} |c_k(y)|^2 dy \\
&= -2 \int_0^\infty c_k(y) c'_k(y) dy \\
&\leq 2 \Big(\int_0^\infty |c_k(y)|^2 dy \Big)^{\frac{1}{2}} \Big(\int_0^\infty |c'_k(y)|^2 dy \Big)^{\frac{1}{2}}.
\end{aligned}$$

Multiplying both members by $\sqrt{\omega^2 |k|^2 + m^2}$, using the Young inequality and summing over $\mathbb{Z}^N$, we obtain

$$\sum_{k \in \mathbb{Z}^N} \sqrt{\omega^2 |k|^2 + m^2}\, |c_k(0)|^2 \leq \sum_{k \in \mathbb{Z}^N} \int_0^\infty \Big(\omega^2 |k|^2 + m^2 \Big) |c_k(y)|^2 + |c'_k(y)|^2 dy. \tag{3.2.3}$$

Thanks to Parseval Identity, we have

$$||v||^2_{L^2(\mathcal{S}_T)} = \sum_{k\in\mathbb{Z}^N} \int_0^\infty |c_k(y)|^2 dy, \tag{3.2.4}$$

$$||\nabla_x v||^2_{L^2(\mathcal{S}_T)} = \sum_{k\in\mathbb{Z}^N} \int_0^\infty \omega^2 |k|^2 |c_k(y)|^2 dy, \tag{3.2.5}$$

$$||v_y||^2_{L^2(\mathcal{S}_T)} = \sum_{k\in\mathbb{Z}^N} \int_0^\infty |c_k'(y)|^2 dy. \tag{3.2.6}$$

Taking into account (3.2.3),(3.2.4),(3.2.5) and (3.2.6) we obtain

$$\begin{aligned}
||v||^2_{\mathbb{X}^m_T} &= ||\nabla_x v||^2_{L^2(\mathcal{S}_T)} + ||v_y||^2_{L^2(\mathcal{S}_T)} + m^2||v||^2_{L^2(\mathcal{S}_T)} \\
&= \sum_{k\in\mathbb{Z}^N} \int_0^\infty \Big[\Big(\omega^2|k|^2 + m^2\Big)|c_k(y)|^2 + |c_k'(y)|^2\Big] dy \\
&\geq \sum_{k\in\mathbb{Z}^N} \sqrt{\omega^2|k|^2 + m^2}\, |c_k(0)|^2.
\end{aligned} \tag{3.2.7}$$

By density we deduce that (3.2.7) is verified for each $v \in \mathbb{X}^m_T$. This allows us to say that is well defined a bounded linear operator

$$\mathrm{Tr} : \mathbb{X}^m_T \to \mathbb{H}^m_T.$$

Finally we prove that $\mathrm{Tr}(\mathbb{X}^m_T) = \mathbb{H}^m_T$, that is Tr is surjective.

Let $u(x) = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} \in \mathbb{H}^m_T$ where c_k are the Fourier coefficients of u. Consider the function

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}}\, e^{-\sqrt{\omega^2|k|^2+m^2}y} \tag{3.2.8}$$

which is clearly smooth for $y > 0$. We want to show that $v \in \mathbb{X}^m_T$ and $\mathrm{Tr}(v) = u$.

Observe that v is T-periodic in each x_i and v solves $-\Delta v + m^2 v = 0$ in $\mathcal{S}_T$. Moreover $v(x,y) \to u(x)$ as $y \to 0^+$ in $L^2(0,T)^N$. In fact fixed $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ and $k_\varepsilon \in \mathbb{N}$ such that

$$\sum_{|k|>k_\varepsilon} |c_k|^2 < \frac{\varepsilon}{2} \text{ and } \sum_{|k|\leq k_\varepsilon} |c_k|^2 (1 - e^{-\sqrt{\omega^2|k|^2+m^2}y}) < \frac{\varepsilon}{2} \quad \text{for } 0 < y < \delta_\varepsilon.$$

Thus for every $0 < y < \delta_\varepsilon$

$$
\begin{aligned}
|v(\cdot, y) - u(\cdot)|^2_{L^2(0,T)^N} &= \sum_{|k| \le k_\varepsilon} |c_k|^2 (1 - e^{-\sqrt{\omega^2|k|^2+m^2}y})^2 \\
&+ \sum_{|k| > k_\varepsilon} |c_k|^2 (1 - e^{-\sqrt{\omega^2|k|^2+m^2}y})^2 \\
&< \frac{\varepsilon}{2} + \sum_{|k| > k_\varepsilon} |c_k|^2 < \varepsilon.
\end{aligned}
$$

Let us check that $v \in \mathbb{X}^m_T$. By Parseval Identity we can see that

$$
\begin{aligned}
||v||^2_{\mathbb{X}^m_T} &= ||\nabla_x v||^2_{L^2(\mathcal{S}_T)} + ||v_y||^2_{L^2(\mathcal{S}_T)} + m^2 ||v||^2_{L^2(\mathcal{S}_T)} \\
&= 2 \sum_{k \in \mathbb{Z}^N} |c_k|^2 \Big(\omega^2 |k|^2 + m^2\Big) \int_0^\infty e^{-2\sqrt{\omega^2 k^2 + m^2}y} dy \\
&= \sum_{k \in \mathbb{Z}^N} \sqrt{\omega^2 |k|^2 + m^2}\, |c_k|^2 < \infty. \qquad (3.2.9)
\end{aligned}
$$

This proves that $\mathbb{H}^m_T \subseteq \mathrm{Tr}(\mathbb{X}^m_T)$.

$\square$

Then we have the following compact embedding:

Theorem 21. *Let $1 \le q < 2^\sharp$ for $N \ge 2$ and $1 \le q < \infty$ for $N = 1$. Then $\mathrm{Tr}(\mathbb{X}^m_T)$ is compactly embedded in $L^q(0,T)^N$.*

Proof. By Theorem 20 we know that $\mathrm{Tr}(\mathbb{X}^m_T)$ is embedded with continuity in $\mathbb{H}^m_T$. To conclude the proof, it's enough to prove that $\mathbb{H}^m_T$ is compactly embedded in $L^q(0,T)^N$.

Let $u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k \cdot x}}{\sqrt{T^N}} \in \mathbb{H}^m_T$. Fix $\frac{2N}{N+1} < r < 2$ and let r' be its conjugate exponent, that is $\frac{1}{r} + \frac{1}{r'} = 1$. By Hölder inequality we can see that

$$
\begin{aligned}
\Big(\sum_{k \in \mathbb{Z}^N} |c_k|^r\Big)^{\frac{1}{r}} &= \Big(\sum_{k \in \mathbb{Z}^N} |c_k|^r (\sqrt{\omega^2|k|^2 + m^2})^{\frac{r}{2}} (\sqrt{\omega^2|k|^2+m^2})^{-\frac{r}{2}}\Big)^{\frac{1}{r}} \\
&\le |u|_{\mathbb{H}^m_T} \Big(\sum_{k \in \mathbb{Z}^N} (\sqrt{\omega^2|k|^2+m^2})^{-\frac{r}{2-r}}\Big)^{\frac{2-r}{2r}} \qquad (3.2.10)
\end{aligned}
$$

and the last series is finite since $r > \frac{2N}{N+1}$. By Theorem of Hausdorff and Young [82] we deduce that $u \in L^{r'}(0,T)^N$ and

$$
|u|_{L^{r'}(0,T)^N} \le \Big(\frac{1}{\sqrt{T^N}}\Big)^{\frac{2}{r}-1} \Big(\sum_{k \in \mathbb{Z}^N} |c_k|^r\Big)^{\frac{1}{r}}. \qquad (3.2.11)
$$

Hence, by (3.2.10) and (3.2.11), we have

$$|u|_{L^q(0,T)^N} \leq C\Big(\sum_{k\in\mathbb{Z}^N} |c_k|^2\sqrt{\omega^2|k|^2+m^2}\Big)^{\frac{1}{2}} \tag{3.2.12}$$

holds for every $2 \leq q < 2^\sharp$. Using (3.2.12) and interpolation inequality we obtain that for all $q \in [2, 2^\sharp)$

$$|u|_{L^q(0,T)^N} \leq C|u|^\theta_{L^2(0,T)^N}\Big(\sum_{k\in\mathbb{Z}^N} |c_k|^2\sqrt{\omega^2|k|^2+m^2}\Big)^{1-\theta}, \tag{3.2.13}$$

for some real positive number $\theta \in (0,1)$.

At this point we prove that $\mathbb{H}^m_T \subset\subset L^2(0,T)^N$, for any $N \in \mathbb{N}$.

Let $u^j \rightharpoonup 0$ in $\mathbb{H}^m_T$. Then

$$\lim_{j\to\infty} |c^j_k|^2\sqrt{\omega^2|k|^2+m^2} = 0 \quad \forall k \in \mathbb{Z}^N \tag{3.2.14}$$

and

$$\sum_{k\in\mathbb{Z}^N} |c^j_k|^2\sqrt{\omega^2|k|^2+m^2} \leq C \quad \forall j \in \mathbb{N}. \tag{3.2.15}$$

Fix $\varepsilon > 0$. Then there exists $\nu > 0$ such that $(\omega^2|k|^2+m^2)^{-\frac{1}{2}} < \varepsilon$ for $|k| > \nu$. By (3.2.15) we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} |c^j_k|^2 &= \sum_{|k|\leq\nu} |c^j_k|^2 + \sum_{|k|>\nu} |c^j_k|^2\\ &= \sum_{|k|\leq\nu} |c^j_k|^2 + \sum_{|k|>\nu} |c^j_k|^2(\omega^2|k|^2+m^2)^{\frac{1}{2}}(\omega^2|k|^2+m^2)^{-\frac{1}{2}}\\ &\leq \sum_{|k|\leq\nu} |c^j_k|^2 + C\varepsilon.\end{aligned}$$

By (3.2.14) we deduce that $\sum_{|k|\leq\nu} |c^j_k|^2 < \varepsilon$ for j large. So $u^j \to 0$ in $L^2(0,T)^N$.

Then using $\mathbb{H}^m_T \subset\subset L^2(0,T)^N$ and (3.2.13) we can conclude that $\mathbb{H}^m_T$ is compactly embedded in $L^q(0,T)^N$ for every $q \in [2, 2^\sharp)$.

□

Remark 3. *It's possible to prove the existence of a continuous embedding* $\mathbb{H}^m_T \subset L^q(0,T)^N$ *for any* $q \leq 2^\sharp$ *(see for instance [9]).*

Finally, we give some elementary results which will be used in the sequel. We use the growth conditions $(f2), (f3)$ and $(f4)$ to deduce some bounds from above and below for the nonlinear term and its primitive. This part is quite standard and the proofs of the following Lemmas can be found in [67] (see also [70]).

Lemma 6. *Let $f : [0,T]^N \times \mathbb{R} \to \mathbb{R}$ satisfying conditions (f1)-(f3). Then, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(x,t)| \leq 2\varepsilon|t| + (p+1)C_\varepsilon|t|^p \quad \forall t \in \mathbb{R}\, \forall x \in [0,T]^N \tag{3.2.16}$$

and

$$|F(x,t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^{p+1} \quad \forall t \in \mathbb{R}\, \forall x \in [0,T]^N. \tag{3.2.17}$$

Lemma 7. *Assume that $f : [0,T]^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (f1)-(f4). Then, there exist two constants $a_3 > 0$ and $a_4 > 0$ such that*

$$F(x,t) \geq a_3|t|^\mu - a_4 \quad \forall t \in \mathbb{R}\, \forall x \in [0,T]^N. \tag{3.2.18}$$

3.3 Problem in the cylinder

In this section we will show that it is possible to transform the problem (3.1.1) in an elliptic problem with Neumann condition on the boundary $(0,T)^N \times \{0\}$ and periodic conditions on the lateral boundary of the cylinder.

More precisely we prove the following

Theorem 22. *Let $u \in \mathbb{H}^m_T$. Then there exists a unique $v \in \mathbb{X}^m_T$ solution to the problem*

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L \mathcal{S}_T \\ v = u \text{ on } \partial^0 \mathcal{S}_T \end{cases} \tag{3.3.1}$$

where the last boundary condition on $\partial^0 \mathcal{S}_T$ is in the sense of trace.

In addition

$$-\lim_{y \to 0^+} v_y(x,y) = \sqrt{-\Delta_x + m^2}\, u(x) \text{ in } (\mathbb{H}^m_T)^*. \tag{3.3.2}$$

Proof. Let $u(x) = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{i\omega k \cdot x}}{\sqrt{T^N}} \in \mathcal{C}^\infty_T(\mathbb{R}^N)$. Consider the following minimizing problem:

$$\inf\Big\{ \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 dxdy : v \in \mathbb{X}^m_T \text{ and } \mathrm{Tr}(v) = u \Big\}. \tag{3.3.3}$$

By lower weak semi-continuity of the $\mathbb{X}^m_T$-norm and by Theorem 21 we can find a minimizer $v \in \mathbb{X}^m_T$. Moreover from the strict convexity of the functional in (3.3.3), we can see that this minimizer is unique. It follows that v is a weak solution to

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L \mathcal{S}_T \\ v = u \text{ on } \partial^0 \mathcal{S}_T \end{cases}. \tag{3.3.4}$$

By standard elliptic regularity, we deduce that v is smooth for $y \geq 0$. We may write $v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k(y)\frac{e^{i\omega k\cdot x}}{\sqrt{T^N}}$, where $c_k(y) = \int_{(0,T)^N} v(x,y)\frac{e^{-i\omega k\cdot x}}{\sqrt{T^N}}dx$. Since u is the trace of v, $c_k(0)$ are the Fourier coefficients of u. Moreover, for any $k \in \mathbb{Z}^N$, $c_k(y)$ satisfies the equation

$$-c_k''(y) + (\omega^2|k|^2 + m^2)c_k(y) = 0 \text{ for } y > 0. \tag{3.3.5}$$

Therefore we deduce that

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} e^{-\sqrt{\omega^2|k|^2+m^2}\,y}. \tag{3.3.6}$$

Hence

$$\frac{\partial v}{\partial \nu}\Big|_{(0,T)^N\times\{0\}} := -\lim_{y\to 0^+} v_y(x,y) = \sum_{k\in\mathbb{Z}^N} c_k\sqrt{\omega^2|k|^2+m^2}\,\frac{e^{i\omega k\cdot x}}{\sqrt{T^N}} \tag{3.3.7}$$

$$= \sqrt{-\Delta_x + m^2}\,u. \tag{3.3.8}$$

and similarly as in (3.2.9), we obtain

$$||v||^2_{\mathbb{X}^m_T} = \sum_{k\in\mathbb{Z}^N} \sqrt{\omega^2|k|^2+m^2}\,|c_k|^2 = |u|^2_{\mathbb{H}^m_T}. \tag{3.3.9}$$

By density we get the desired result. □

We will call v the extension of u, and we will denote it by $\mathrm{ext}_m(u)$.

Hence to study the problem (3.1.1) it's equivalent to study the solutions $v \in \mathbb{X}^m_T$ to the problem

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L\mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = mv + f(x,v) \text{ on } \partial^0\mathcal{S}_T. \end{cases} \tag{3.3.10}$$

More precisely, we will say that $u \in \mathbb{H}^m_T$ is a weak solution to (3.1.1) if and only if its extension $v = \mathrm{ext}_m(u) \in \mathbb{X}^m_T$ is a weak solution to (3.3.10), that is if

$$\iint_{\mathcal{S}_T} (\nabla v\nabla\eta + m^2 v\eta)dxdy = \int_{(0,T)^N} [mv(x,0) + f(x,v(x,0))]\eta(x,0)dx \tag{3.3.11}$$

holds for all $\eta \in \mathbb{X}^m_T$.

3.4 Linking solutions

In this section we prove the existence of weak solutions to the problem (3.3.10) by using the Linking Theorem due to Rabinowitz [67]:

Theorem 23. *Let $(X, ||\cdot||)$ be a real Banach space with $X = Y \bigoplus Z$, where Y is finite dimensional. Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the following conditions:*

- *J satisfies the Palais-Smale condition,*
- *There exist $\eta, \rho > 0$ such that*

$$\inf\{J(v) : v \in Z \text{ and } ||v|| = \eta\} \geq \rho,$$

- *There exist $z \in \partial B_1 \cap Z$, $R > \rho$ and $R' > 0$ such that*

$$J \leq 0 \text{ on } \partial\mathcal{A}$$

  where

$$\mathcal{A} = \{v = y + sz : y \in Y, ||y|| \leq R' \text{ and } 0 \leq s \leq R\}$$

  and

$$\partial\mathcal{A} = \{v = y + sz : y \in Y, ||y|| = R' \text{ or } s \in \{0, R\}\}.$$

Then J possesses a critical value $c \geq \rho$ which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{v \in \mathcal{A}} J(\gamma(v))$$

where

$$\Gamma = \{\gamma \in \mathcal{C}(\mathcal{A}, X) : \gamma = Id \text{ on } \partial\mathcal{A}\}.$$

Let us consider the following functional

$$\mathcal{J}_m(v) = \frac{1}{2}[||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N}] - \int_{(0,T)^N} F(x, v(x,0))dx$$

for $v \in \mathbb{X}^m_T$. First of all we can observe that $\mathcal{J}_m$ is well defined and $\mathcal{J}_m \in \mathcal{C}^1(\mathbb{X}^m_T)$ by Theorem 20 and by assumptions on f. Moreover, using Theorem 20, we deduce that the quadratic part of $\mathcal{J}_m$ is non-negative, that is for every $v \in \mathbb{X}^m_T$

$$m\int_{(0,T)^N} |v(x,0)|^2 dx \leq \iint_{\mathcal{S}_T} |\nabla v|^2 + m^2 v^2 dxdy. \tag{3.4.1}$$

Indeed this inequality is satisfied by each function in H^1 of the cylinder. In fact for every function $v \in \mathcal{C}^1$ such that $v(x,y) \to 0$ as $y \to +\infty$ and for every $q \in [2, 2^\sharp]$ we have

$$\begin{aligned}\int_{(0,T)^N} |v(x,0)|^q dx &= \int_{(0,T)^N} dx \int_{+\infty}^0 \frac{\partial}{\partial y}|v(x,y)|^q dy \\ &\le q \iint_{\mathcal{S}_T} |v(x,y)|^{q-1}\Big|\frac{\partial}{\partial y}v(x,y)\Big| dxdy \\ &\le q\Big(\iint_{\mathcal{S}_T} |v(x,y)|^{2(q-1)} dxdy\Big)^{\frac{1}{2}}\Big(\iint_{\mathcal{S}_T}\Big|\frac{\partial}{\partial y}v(x,y)\Big|^2 dxdy\Big)^{\frac{1}{2}}\end{aligned}$$

where in the last inequality we have exploited the Cauchy-Schwartz inequality. Then taking $q = 2$ and using $2ab \le a^2 + b^2$ for all $a, b \ge 0$, we deduce that

$$\begin{aligned}m|v|^2_{L^2(0,T)^N} &\le 2m \iint_{\mathcal{S}_T} |v(x,y)||\partial_y v(x,y)|\, dxdy \\ &\le 2m||v||_{L^2(\mathcal{S}_T)}||\partial_y v||_{L^2(\mathcal{S}_T)} \qquad (3.4.2)\\ &\le ||\partial_y v||^2_{L^2(\mathcal{S}_T)} + m^2||v||^2_{L^2(\mathcal{S}_T)} \qquad (3.4.3)\\ &\le ||\nabla v||^2_{L^2(\mathcal{S}_T)} + m^2||v||^2_{L^2(\mathcal{S}_T)}. \qquad (3.4.4)\end{aligned}$$

By density we obtain that (3.4.1) holds for every function in $H^1(\mathcal{S}_T)$. As shown in [5], we can observe that

$$||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N} = 0 \text{ if and only if } v(x,y) = ce^{-my} \qquad (3.4.5)$$

for some constant $c \in \mathbb{R}$. In fact, taking the equality in (3.4.2), (3.4.3) and (3.4.4), we deduce that $m^2v^2 = Cv_y^2$, $m||v||_{L^2(\mathcal{S}_T)} = ||v_y||_{L^2(\mathcal{S}_T)}$ and $v(x,y) = h(y)$ for some $h \in H^1(0,\infty)$. These conditions force to be $v(x,y) = ce^{-my}$ for some $c \in \mathbb{R}$

We note that $\mathbb{X}^m_T$ admits the following decomposition

$$\mathbb{X}^m_T = \mathbb{Y}^m_T \bigoplus \mathbb{Z}^m_T$$

where $\mathbb{Y}^m_T = < e^{-my} >$, $\dim \mathbb{Y}^m_T < \infty$ and

$$\mathbb{Z}^m_T = \Big\{v \in \mathbb{X}^m_T : \int_{(0,T)^N} v(x,0)\, dx = 0\Big\}.$$

Now we give some lemmas to prove Linking hypothesis:

Lemma 8. *The functional $\mathcal{J}_m$ is nonpositive on $\mathbb{Y}^m_T$.*

Proof. If $v \in \mathbb{Y}_T^m$, then $||v||^2_{\mathbb{X}_T^m} - m|v|^2_{L^2(0,T)^N} = 0$. Moreover $F(x,0) = 0$ and by hypothesis (f6) we know that $t\frac{d}{dt}F(x,t) = tf(x,t) \geq 0$ for all $t \in \mathbb{R}$. Therefore $F(x,t)$ is increasing with respect to $t \geq 0$ and decreasing with respect to $t \leq 0$, that is $F(x,t) \geq 0$ for every $x \in [0,T]^N$, $t \in \mathbb{R}$. Hence $\mathcal{J}_m$ is nonpositive on $\mathbb{Y}_T^m$. □

Lemma 9. *There exist $\eta_m, \rho_m > 0$ such that*

$$\inf\{\mathcal{J}_m(v) : v \in \mathbb{Z}_T^m \textit{ and } ||v||_{\mathbb{X}_T^m} = \eta_m\} \geq \rho_m > 0. \tag{3.4.6}$$

Proof. Firstly, we prove that there exists a constant $C > 0$ (depending eventually on m and T) such that

$$||v||_e^2 := ||v||^2_{\mathbb{X}_T^m} - m|v|^2_{L^2(0,T)^N} \geq C||v||^2_{\mathbb{X}_T^m} \tag{3.4.7}$$

for every $v \in \mathbb{Z}_T^m$ (that is $||\cdot||_e$ is equivalent to $||\cdot||_{\mathbb{X}_T^m}$ on $\mathbb{Z}_T^m$).

By contradiction suppose that there exists $(v_n) \subset \mathbb{Z}_T^m$ such that

$$n||v_n||_e^2 < ||v_n||^2_{\mathbb{X}_T^m} \quad \text{for all } n \in \mathbb{N}. \tag{3.4.8}$$

Assuming that $||v_n||_{\mathbb{X}_T^m} = 1$ we deduce that v_n converges weakly in $\mathbb{X}_T^m$ to some function $v \in \mathbb{Z}_T^m$ ($\mathbb{Z}_T^m$ is weakly closed). Then using (3.4.8) we obtain

$$0 \leq ||v_n||_e \leq \frac{1}{n} \to 0 \text{ as } n \to +\infty$$

and so

$$|v_n|_{L^2(0,T)^N} \to \frac{1}{\sqrt{m}} \text{ as } n \to +\infty.$$

Therefore $0 \leq ||v||_e \leq \liminf_{n\to\infty} ||v_n||_e = 0$, and in view of (3.4.5), we have $v = ce^{-my}$. But $v \in \mathbb{Z}_T^m$, hence $c = 0$ and this gives a contradiction because of $|v|_{L^2(0,T)^N} = \frac{1}{\sqrt{m}} > 0$.

Now, we are able to prove (3.4.6). In fact, by Theorem 21, (3.4.1),(3.4.7) and (3.2.17) we can see that for all $v \in \mathbb{Z}_T^m$

$$\begin{aligned}
\mathcal{J}_m(v) &= \frac{1}{2}[||v||^2_{\mathbb{X}_T^m} - m|v|^2_{L^2(0,T)^N}] - \int_{(0,T)^N} F(x, v(x,0))dx \\
&\geq \frac{C}{2}||v||^2_{\mathbb{X}_T^m} - \varepsilon|v|^2_{L^2(0,T)^N} - C_\varepsilon|v|^{p+1}_{L^{p+1}(0,T)^N} \\
&\geq \frac{C}{2}||v||^2_{\mathbb{X}_T^m} - \frac{\varepsilon}{m}||v||^2_{\mathbb{X}_T^m} - c_\varepsilon||v||^{p+1}_{\mathbb{X}_T^m}.
\end{aligned}$$

Then, choosing $\varepsilon \in (0, \frac{mC}{2})$, we can find $\eta_m, \rho_m > 0$ such that

$$\inf\{\mathcal{J}_m(v) : v \in \mathbb{Z}_T^m \text{ and } ||v||_{\mathbb{X}_T^m} = \eta_m\} \geq \rho_m > 0.$$

□

Lemma 10. *There exist* $z \in \partial B_1 \cap \mathbb{Z}_T^m$ *and* $R_m > \eta_m$, $R'_m > 0$ *such that*

$$\max_{\partial \mathbb{A}_T^m} \mathcal{J}_m = 0 \text{ and } \sup_{\mathbb{A}_T^m} \mathcal{J}_m < \infty,$$

where

$$\mathbb{A}_T^m := \{v = y + sz : y \in \mathbb{Y}_T^m, ||y||_{\mathbb{X}_T^m} \leq R'_m \text{ and } 0 \leq s \leq R_m\}$$

and

$$\partial \mathbb{A}_T^m = \{v = y + sz \in \mathbb{A}_T^m : ||y||_{\mathbb{X}_T^m} = R'_m \text{ or } s \in \{0, R_m\}\}.$$

Proof. By Lemma 8 we know tat $\mathcal{J}_m$ is nonpositive on $\mathbb{Y}_T^m$.

Let

$$w(x,y) = \prod_{i=1}^{N} \sin(\omega x_i) e^{-my} \text{ and } z = \frac{w}{||w||_{\mathbb{X}_T^m}}.$$

We note that $z \in \mathbb{Z}_T^m$, $||z||_{\mathbb{X}_T^m} = 1$ and

$$||z||^2_{\mathbb{X}_T^m} - m|z|^2_{L^2(0,T)^N} = \frac{2N\pi^2}{2N\pi^2 + m^2T^2} =: C > 0.$$

Then for every $v \in \mathbb{Y}_T^m \bigoplus \mathbb{R}z$

$$\begin{aligned} |v|^{\mu}_{L^{\mu}(0,T)^N} &\geq C' \Big(\int_{(0,T)^N} |c + sz(x,0)|^2 dx \Big)^{\frac{\mu}{2}} \\ &= C' \Big(\int_{(0,T)^N} c^2 dx + \int_{(0,T)^N} s^2 |z(x,0)|^2 dx \Big)^{\frac{\mu}{2}} \\ &= C'(T^N c^2 + C'' s^2)^{\frac{\mu}{2}} \\ &\geq \overline{C}(c^2 + s^2)^{\frac{\mu}{2}}. \end{aligned} \tag{3.4.9}$$

Therefore , if $v = ce^{-my} + sz \in \mathbb{Y}_T^m \bigoplus \mathbb{R}z$, using (3.2.18) and (3.4.9) we have

$$\begin{aligned} \mathcal{J}_m(v) &= \frac{s^2}{2} \Big[||z||^2_{\mathbb{X}_T^m} - m|z|^2_{L^2(0,T)^N} \Big] - \int_{(0,T)^N} F(x,v) dx \\ &\leq C \frac{s^2}{2} - (a_3 |v|^{\mu}_{L^{\mu}(0,T)^N} - a_4 T^N) \\ &\leq \frac{C}{2} s^2 + a_4 T^N - a_3 \overline{C}(s^2 + c^2)^{\frac{\mu}{2}}. \end{aligned} \tag{3.4.10}$$

Hence

$$\mathcal{J}_m(y + sz) \leq \frac{C}{2} s^2 + a_4 T^N - a_3 \overline{C} s^{\mu},$$

and so we can find $R_m > \eta_m$ such that $\mathcal{J}_m(y+sz) \leq 0$ for every $s \geq R_m$ and $y \in \mathbb{Y}_T^m$. Let $0 \leq s \leq R_m$. By (3.4.10), there exists $R'_m > 0$ such that $\mathcal{J}_m(y + sz) \leq 0$ for every $||y||_{\mathbb{X}_T^m} = \sqrt{mT^N}|c| \geq R'_m$. Finally $\mathcal{J}_m(v) \leq CR_m^2 + a_4 T^N$ for $v \in \mathbb{A}_T^m$. □

To obtain the existence of a critical value of $\mathcal{J}_m$ we must prove the Palais-Smale condition, that is:

Lemma 11. *Let $c \in \mathbb{R}$ and let $(v_n) \subset \mathbb{X}_T^m$ be a sequence such that*

$$\mathcal{J}_m(v_n) \to c \text{ and } \mathcal{J}_m'(v_n) \to 0 \text{ as } n \to \infty. \tag{3.4.11}$$

Then (v_n) has a strongly convergent subsequence in $\mathbb{X}_T^m$.

Proof. We begin proving that (v_n) is bounded in $\mathbb{X}_T^m$.
Fix $\beta \in (\frac{1}{\mu}, \frac{1}{2})$. By Lemma 6 applied with $\varepsilon = 1$ we have

$$\begin{aligned} \Big|\int_{(0,T)^N \cap \{|v_n(x,0)| \le r\}} & \beta f(x, v_n)v_n - F(x, v_n)dx\Big| \le \\ & \le ((2\beta+1)r^2 + C_1(p+2)r^{p+1})T^N =: \kappa_0 \end{aligned} \tag{3.4.12}$$

and

$$\Big|\int_{(0,T)^N \cap \{|v_n(x,0)| \le r\}} F(x, v_n)dx\Big| \le (r^2 + C_1 r^{p+1})T^N = \kappa_0'. \tag{3.4.13}$$

Then, using (3.4.1), (3.4.11), (3.4.12), (3.4.13) and (3.2.18) we have, for n sufficiently large

$$\begin{aligned} c + 1 + ||v_n||_{\mathbb{X}_T^m} & \ge \mathcal{J}_m(v_n) - \beta < \mathcal{J}_m'(v_n), v_n >= \\ & = \Big(\frac{1}{2} - \beta\Big)[||v_n||^2_{\mathbb{X}_T^m} - m|v_n|^2_{L^2(0,T)^N}] + \int_{(0,T)^N} [\beta f(x, v_n)v_n - F(x, v_n)]dx \\ & \ge \int_{(0,T)^N} [\beta f(x, v_n)v_n - F(x, v_n)]dx \\ & = \int_{(0,T)^N \cap \{|v_n(x,0)| \ge r\}} [\beta f(x, v_n)v_n - F(x, v_n)]dx \\ & + \int_{(0,T)^N \cap \{|v_n(x,0)| \le r\}} [\beta f(x, v_n)v_n - F(x, v_n)]dx \\ & \ge (\mu\beta - 1) \int_{(0,T)^N \cap \{|v_n(x,0)| \ge r\}} F(x, v_n)dx - \kappa_0 \\ & \ge (\mu\beta - 1) \int_{(0,T)^N} F(x, v_n)dx - (\mu\beta - 1)\kappa_0' - \kappa_0 \\ & = (\mu\beta - 1) \int_{(0,T)^N} F(x, v_n)dx - \kappa \qquad (3.4.14) \\ & \ge (\mu\beta - 1)[a_3|v_n|^\mu_{L^\mu(0,T)^N} - a_4 T^N] - \kappa \\ & \ge (\mu\beta - 1)[a_3|v_n|^\mu_{L^2(0,T)^N} T^{-N\frac{\mu-2}{2}} - a_4 T^N] - \kappa. \qquad (3.4.15) \end{aligned}$$

where in the last inequality we have used Hölder inequality.

Hence, exploiting (3.4.14) and (3.4.15)

$$\begin{aligned}||v_n||^2_{\mathbb{X}^m_T} &= 2\mathcal{J}_m(v_n) + m|v_n|^2_{L^2(0,T)^N} + 2\int_{(0,T)^N} F(x,v_n)dx \\ &\leq C_1 + C_2(1 + C_3 + ||v_n||_{\mathbb{X}^m_T})^{\frac{2}{\mu}} + \frac{2}{\mu\beta - 1}(1 + c + ||v_n||_{\mathbb{X}^m_T}) \\ &\leq C_4 + C_5||v_n||_{\mathbb{X}^m_T}\end{aligned}$$

and so (v_n) is bounded in $\mathbb{X}^m_T$. Going if necessary to a subsequence, we can assume that v_n converges weakly to some function $v \in \mathbb{X}^m_T$.

Moreover, by Theorem 21, up to a subsequence, we have

$$v_n(\cdot,0) \to v(\cdot,0) \text{ in } L^{p+1}(0,T)^N \tag{3.4.16}$$

$$v_n(\cdot,0) \to v(\cdot,0) \text{ a.e. in } (0,T)^N \tag{3.4.17}$$

$$|v_n(x,0)| \leq h(x) \text{ a.e. in } (0,T)^N \text{ for } n \in \mathbb{N}, \text{ for some } h \in L^{p+1}(0,T)^N. \tag{3.4.18}$$

By hypotheses (f1)and (f2), by (3.4.16)-(3.4.18) and taking into account the Dominated Convergence Theorem we get as $n \to \infty$

$$\int_{(0,T)^N} f(x,v_n(x,0))v_n(x,0)dx \to \int_{(0,T)^N} f(x,v(x,0))v(x,0)dx \tag{3.4.19}$$

and

$$\int_{(0,T)^N} f(x,v_n(x,0))v(x,0)dx \to \int_{(0,T)^N} f(x,v(x,0))v(x,0)dx. \tag{3.4.20}$$

Exploiting the facts $\mathcal{J}'_m(v_n) \to 0$, $v_n \rightharpoonup v$ in $\mathbb{X}^m_T$, $v_n(x,0) \to v(x,0)$ in $L^2(0,T)^N$ and (3.4.19) we can see that as $n \to \infty$

$$\mathcal{J}'_m(v_n)v_n \to 0 \Rightarrow ||v_n||^2_{\mathbb{X}^m_T} - m|v_n|^2_{L^2(0,T)^N} \to \int_{(0,T)^N} f(x,v(x,0))v(x,0)dx$$

and

$$\mathcal{J}'_m(v_n)v \to 0 \Rightarrow ||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N} = \int_{(0,T)^N} f(x,v(x,0))v(x,0)dx.$$

So $||v_n||_{\mathbb{X}^m_T} \to ||v||_{\mathbb{X}^m_T}$ as $n \to \infty$. Finally using $||v_n||_{\mathbb{X}^m_T} \to ||v||_{\mathbb{X}^m_T}$ and $v_n \rightharpoonup v$ in $\mathbb{X}^m_T$ we obtain

$$||v_n - v||^2_{\mathbb{X}^m_T} = ||v_n||^2_{\mathbb{X}^m_T} + ||v||^2_{\mathbb{X}^m_T} - 2 < v_n, v >_{\mathbb{X}^m_T} \to 0.$$

□

Taking into account Lemma 8, Lemma 9, Lemma 10 and Lemma 11 we can state:

Theorem 24. *There exists at least one weak solution $v_m \in \mathbb{X}_T^m$ to the problem (3.3.10).*

3.5 Regularity of solutions

This section is devoted to study the regularity of weak solutions to the problem (3.3.10). We begin proving the following result:

Lemma 12. *Let $v \in \mathbb{X}_T^m$ be a weak solution to*

$$\begin{cases} -\Delta v + m^2 v = 0 \text{ in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \text{ on } \partial_L \mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = g(x, v(x,0)) \text{ on } \partial^0 \mathcal{S}_T \end{cases} \tag{3.5.1}$$

where $g(x, v(x,0)) = mv(x,0) + f(x, v(x,0))$. Then $v(\cdot,0) \in L^q(0,T)^N$ for all $q < \infty$.

Proof. If $N = 1$ the thesis follows by Theorem 21. Let $N \geq 2$. Now we proceed as in the proof of Theorem 3.2. in [27]. We know that

$$\iint_{\mathcal{S}_T} \nabla v(x,y) \nabla \eta(x,y) + m^2 v(x,y)\eta(x,y) dxdy = \int_{(0,T)^N} g(x,v)\eta(x,0)dx \tag{3.5.2}$$

for all $\eta \in \mathbb{X}_T^m$. Let $w = v v_K^{2\beta}$ where $v_K = \min\{|v|, K\}$, $K > 1$ and $\beta \geq 0$. Then $w \in \mathbb{X}_T^m$ and using the fact that v is a critical point for $\mathcal{J}_m$ we deduce that

$$0 = \iint_{\mathcal{S}_T} \nabla v \nabla w + m^2 v w dxdy - m \int_{(0,T)^N} v w dx - \int_{(0,T)^N} f(x,v) w dx. \tag{3.5.3}$$

By direct computation we see

$$\iint_{\mathcal{S}_T} |\nabla (v v_K^{\beta})|^2 dxdy = \iint_{\mathcal{S}_T} v_K^{2\beta} |\nabla v|^2 dxdy + \iint_{D_{K,T}} (2\beta + \beta^2) v_K^{2\beta} |\nabla v|^2 dxdy \tag{3.5.4}$$

where $D_{K,T} = \{(x,y) \in \mathcal{S}_T : |v(x,y)| \leq K\}$. Combining these facts we find that

$$\begin{aligned}
||vv_K^{\beta}||^2_{\mathbb{X}_T^m} &= \iint_{\mathcal{S}_T} |\nabla(vv_K^{\beta})|^2 + m^2 v^2 v_K^{2\beta} dxdy \\
&= \iint_{\mathcal{S}_T} v_K^{2\beta}[|\nabla v|^2 + m^2 v^2] dxdy + \iint_{D_{K,T}} 2\beta\Big(1 + \frac{\beta}{2}\Big) v_K^{2\beta} |\nabla v|^2 dxdy \\
&\leq c_\beta \Big[\iint_{\mathcal{S}_T} v_K^{2\beta}[|\nabla v|^2 + m^2 v^2] dxdy + \iint_{D_{K,T}} 2\beta v_K^{2\beta} |\nabla v|^2 dxdy\Big] \\
&= c_\beta \Big[\iint_{\mathcal{S}_T} \nabla v \nabla(vv_K^{2\beta}) + m^2 v(vv_K^{2\beta}) dxdy\Big] \\
&= c_\beta \int_{(0,T)^N} mv^2 v_K^{2\beta} + f(x,v) vv_K^{2\beta} dx
\end{aligned} \tag{3.5.5}$$

where $c_\beta = 1 + \frac{\beta}{2}$. Using Lemma 6 with $\varepsilon = 1$ we get

$$mv^2 v_K^{2\beta} + f(x,v) vv_K^{2\beta} \leq (m+2) v^2 v_K^{2\beta} + (p+1) C_1 |v|^{p-1} v^2 v_K^{2\beta}. \tag{3.5.6}$$

We also have that

$$|v|^{p-1} = \chi_{\{|v|\leq 1\}} |v|^{p-1} + \chi_{\{|v|>1\}} |v|^{p-1} \leq 1 + h$$

where $h \in L^N(0,T)^N$. In fact if $(p-1)N < 2$ then

$$\int_{(0,T)^N} \chi_{\{|v|>1\}} |v|^{N(p-1)} dx \leq \int_{(0,T)^N} \chi_{\{|v|>1\}} |v|^2 dx < \infty$$

while if $2 \leq (p-1)N$ we have that $(p-1)N \in [2, \frac{2N}{N-1}]$. Therefore we have proved that there exist a constant $c = m + 2 + (p+1)C_1$ and a function $h \in L^N(0,T)^N$, $h \geq 0$ and independent of K and β such that

$$mv^2 v_K^{2\beta} + f(x,v) vv_K^{2\beta} \leq (c+h) v^2 v_K^{2\beta}. \tag{3.5.7}$$

As a consequence

$$||vv_K^{\beta}||^2_{\mathbb{X}_T^m} \leq c_\beta \int_{(0,T)^N} (c+h) v^2 v_K^{2\beta} dx. \tag{3.5.8}$$

Taking the limit as $K \to \infty$ (v_K is increasing with respect to K) we get

$$|||v|^{\beta+1}||^2_{\mathbb{X}_T^m} \leq cc_\beta \int_{(0,T)^N} |v|^{2(\beta+1)} + c_\beta \int_{(0,T)^N} h|v|^{2(\beta+1)} dx. \tag{3.5.9}$$

For any $M > 0$, let $A_1 = \{h \leq M\}$ and $A_2 = \{h > M\}$. Then

$$\int_{(0,T)^N} h|v|^{2(\beta+1)} dx \leq M|||v|^{\beta+1}|^2_{L^2(0,T)^N} + \varepsilon(M)|||v|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N} \tag{3.5.10}$$

where $\varepsilon(M) = \left(\int_{A_2} h^N dx\right)^{\frac{1}{N}} \to 0$ as $M \to \infty$. Taking into account (3.5.9), (3.5.10), we have

$$|||v|^{\beta+1}||^2_{\mathbb{X}^m_T} \leq c_\beta(c+M)||v|^{\beta+1}|^2_{L^2(0,T)^N} + c_\beta\varepsilon(M)||v|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N}. \tag{3.5.11}$$

Choosing M large so that $\varepsilon(M)c_\beta C^2_{2^\sharp} < \frac{1}{2}$, using Theorem 20, Remark 3 and (3.5.11) we obtain

$$||v|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N} \leq C^2_{2^\sharp}|||v|^{\beta+1}||^2_{\mathbb{X}^m_T} \leq 2C^2_{2^\sharp}c_\beta(c+M)||v|^{\beta+1}|^2_{L^2(0,T)^N}. \tag{3.5.12}$$

Then we can start a bootstrap argument: since $v(\cdot,0) \in L^{\frac{2N}{N-1}}$ we can apply (3.5.12) with $\beta_1 + 1 = \frac{N}{N-1}$ to deduce that $v(\cdot,0) \in L^{\frac{(\beta_1+1)2N}{N-1}}(0,T)^N = L^{\frac{2N^2}{(N-1)^2}}(0,T)^N$. Applying (3.5.12) again, after k iterations, we find $v(\cdot,0) \in L^{\frac{2N^k}{(N-1)^k}}(0,T)^N$, and so $v(\cdot,0) \in L^q(0,T)^N$ for all $q \in [2,\infty)$. □

Now we are ready to show that the weak solutions of (3.3.10) are Hölder continuous together with their partial derivatives up to the boundary of the cylinder (hence in the whole of the upper half-space).

Theorem 25. *Let $v \in \mathbb{X}^m_T$ be a weak solution to the problem*

$$\begin{cases} -\Delta v + m^2 v = 0 \ \text{in } \mathcal{S}_T \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} \ \text{on } \partial_L\mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = g(x,v(x,0)) \ \text{on } \partial^0\mathcal{S}_T. \end{cases} \tag{3.5.13}$$

where $g(x,v(x,0)) = mv(x,0) + f(x,v(x,0))$. Assume that v is extended by periodicity to the whole $\mathbb{R}^{N+1}_+$. Then $v \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}}^{N+1}_+) \cap \mathcal{C}^\infty(\mathbb{R}^{N+1}_+)$ for some $\alpha \in (0,1)$.

Proof. We proceed with a useful method introduced by Cabré and Solá Morales in [17], which consists of using the auxiliary function

$$w(x,y) = \int_0^y v(x,s)ds. \tag{3.5.14}$$

Then w weakly solves

$$\begin{cases} -\Delta w + m^2 w = g(x,v) \text{ in } \mathcal{S}_T \\ w_{|\{x_i=0\}} = w_{|\{x_i=T\}} \text{ on } \partial_L\mathcal{S}_T \\ w = 0 \text{ on } \partial^0\mathcal{S}_T. \end{cases} \tag{3.5.15}$$

Denoting again by w the T-periodic extension of w with respect to x to the whole $\mathbb{R}^N$, we can prove that w satisfies

$$\iint_{\mathbb{R}^{N+1}_+} \nabla w \nabla \eta + m^2 w\eta \, dxdy = \iint_{\mathbb{R}^{N+1}_+} g(x,v)\eta \, dxdy \tag{3.5.16}$$

for all $\eta \in \mathcal{C}_c^\infty(\mathbb{R}_+^{N+1})$, that is w is a weak solution to the Dirichlet problem

$$\begin{cases} -\Delta w + m^2 w = g \text{ in } \mathbb{R}_+^{N+1} \\ w = 0 \text{ on } \partial\mathbb{R}_+^{N+1}. \end{cases} \tag{3.5.17}$$

Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}_+^{N+1})$ and $\tau = (T, \ldots, T)$. Then we can see that

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_0^\infty \nabla w(x,y)\nabla\eta(x,y) + m^2 w(x,y)\eta(x,y)dxdy \\ &= \sum_{k\in\mathbb{Z}^N}\int_{T_k}\int_0^\infty \nabla w(x,y)\nabla\eta(x,y) + m^2 w(x,y)\eta(x,y)dxdy \end{aligned} \tag{3.5.18}$$

where $T_k = (0,T)^N + k\tau$ and $k\tau$ is a shortcut for $(k^1 T, \ldots, k^N T)$. Note that in this sum only a finite number of terms are not equal to zero since η is assumed to have a compact support. Making the change of variable $x + k\tau \to x$ in T_k and using the T-periodicity of w and g, we obtain

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^N}\int_{T_k}\int_0^\infty \nabla w(x,y)\nabla\eta(x,y) + m^2 w(x,y)\eta(x,y)\,dxdy \\ &= \sum_{k\in\mathbb{Z}^N}\int_{(0,T)^N}\int_0^\infty \nabla w(x,y)\nabla\eta(x+k\tau,y) + m^2 w(x,y)\eta(x,y)\,dxdy \\ &= \int_{(0,T)^N}\int_0^\infty \nabla w(x,y)\nabla\psi(x,y) + m^2 w(x,y)\psi(x,y)\,dxdy \\ &= \int_{(0,T)^N}\int_0^\infty g(x,v)\psi(x,y)\,dxdy \\ &= \sum_{k\in\mathbb{Z}^N}\int_{(0,T)^N}\int_0^\infty g(x,v)\eta(x+k\tau,y)\,dxdy \\ &= \sum_{k\in\mathbb{Z}^N}\int_{T_k}\int_0^\infty g(x-k\tau,v)\eta(x,y)\,dxdy \\ &= \int_{\mathbb{R}^N}\int_0^\infty g(x,v)\eta(x,y)\,dxdy \end{aligned} \tag{3.5.19}$$

where we have set $\psi(x,y) = \sum_{k\in\mathbb{Z}^N}\eta(x+k\tau,y)$. Clearly this function is admissible since it is indefinitely differentiable, T-periodic in x and it vanishes near $y = 0$. Therefore taking into account (3.5.18) and (3.5.19) we deduce (3.5.16).

Denote by w_{odd} and g_{odd} the extension of w and g to the whole $\mathbb{R}^{N+1}$ by odd reflection with respect to y. Then w_{odd} satisfies the equation

$$-\Delta w_{odd} + m^2 w_{odd} = g_{odd} \text{ in } \mathcal{D}'(\mathbb{R}^{N+1}). \tag{3.5.20}$$

By Lemma 12 we know that $g_{odd} \in L^q_{loc}(\mathbb{R}^{N+1})$ for all $q < \infty$. Using Theorem 3 in [14] we have $w_{odd} \in W^{2,q}_{loc}(\mathbb{R}^{N+1})$ for all $q < \infty$. In particular $w_{odd} \in \mathcal{C}^{1,\alpha}(\mathbb{R}^{N+1})$ for some $\alpha \in (0,1)$. Therefore, $w \in W^{2,q}_{loc}(\mathbb{R}^{N+1}_+) \cap \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}^{N+1}_+})$ and so $v = w_y \in W^{1,q}_{loc}(\mathbb{R}^{N+1}_+) \cap \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}^{N+1}_+})$.

Using $(f1)$ we deduce that $g \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$. By elliptic boundary regularity for the Dirichlet problem (3.5.17) we obtain that $w \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{R}^{N+1}_+})$, and hence $v = w_y \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}^{N+1}_+})$. Finally v satisfies

$$\iint_{\mathbb{R}^{N+1}_+} \nabla v(x,y)\nabla \eta(x,y) + m^2 v(x,y)\eta(x,y)dxdy = 0 \tag{3.5.21}$$

for all $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^{N+1}_+)$, so we can conclude that $v \in \mathcal{C}^\infty(\mathbb{R}^{N+1}_+)$.

□

Proof. (of Theorem 18) This is an immediate consequence of Theorem 24 and Theorem 25. □

3.6 Proof of Theorem 19

Consider the following family of functionals depending on $m \in (0, \frac{\omega}{2})$

$$\mathcal{J}_m(v) = \frac{1}{2}[||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N}] - \int_{(0,T)^N} F(x, v(x,0))dx.$$

In the previous section we proved that the functional $\mathcal{J}_m$ satisfied the hypotheses of Linking Theorem, in this way we obtained, for all fixed $m > 0$, the existence of a function $v_m \in \mathbb{X}^m_T$ such that $\mathcal{J}'_m(v_m) = 0$. In general the estimates obtained on critical levels were dependent on m. Now we want to show that it's possible to take the limit as $m \to 0$ in the problem (3.1.1). For this we need estimates on critical levels independently of m. Therefore we will prove the following properties:

1. There exist $\rho > 0$ and $\lambda > 0$ independent of m such that
$$\inf\{\mathcal{J}_m(v) : v \in \mathbb{Z}^m_T \text{ and } ||v||_{\mathbb{X}^m_T} = \rho\} \geq \lambda > 0; \tag{3.6.1}$$
2. There exists $z \in \mathbb{X}^m_T$ with $||z||_{\mathbb{X}^m_T} = 1$, and there exist $R > \rho, \delta > 0$ independent of m and $R'_m > 0$, such that
$$\max_{\partial \mathbb{A}^m_T} \mathcal{J}_m = 0 \text{ and } \sup_{\mathbb{A}^m_T} \mathcal{J}_m(v) \leq \delta, \tag{3.6.2}$$
where
$$\mathbb{A}^m_T := \{v = y + sz : y \in \mathbb{Y}^m_T, ||y||_{\mathbb{X}^m_T} \leq R'_m, 0 \leq s \leq R\}.$$

We remind that

$$\mathbb{X}_T^m = \mathbb{Y}_T^m \bigoplus \mathbb{Z}_T^m$$

where $\mathbb{Y}_T^m =< e^{-my} >$ and $\mathbb{Z}_T^m = \{v \in \mathbb{X}_T^m : \int_{(0,T)^N} v(x,0)dx = 0\}$.

Then, letting

$$\alpha_m := \inf_{\gamma \in \mathcal{P}_T^m} \max_{v \in \mathbb{A}_T^m} \mathcal{J}_m(\gamma(v)),$$

where

$$\mathcal{P}_T^m = \{\gamma \in \mathcal{C}^0(\mathbb{A}_T^m, \mathbb{X}_T^m) : \gamma = Id \text{ on } \partial\mathbb{A}_T^m\},$$

we deduce by (3.6.1) and (3.6.2) that $\lambda \leq \alpha_m \leq \delta$ for $0 < m < \frac{\omega}{2}$.

We start proving (3.6.1). Firstly we show that it's possible to obtain the uniform estimates in m for the norm $|\cdot|_p$ with $2 \leq r < 2^\sharp$.

Let $v \in \mathbb{Z}_T^m$ and $\varepsilon > 0$. We denote by c_k the Fourier coefficients of the trace of v. Then, taking into account that $c_0 = 0$ and using the trace inequality (ii) of Theorem 20 , we have

$$\begin{aligned}
|\mathcal{T}_m(v)|^2_{L^2(0,T)^N} &= \sum_{|k|\geq 1} |c_k|^2 \\
&\leq \frac{1}{\omega} \sum_{|k|\geq 1} \sqrt{\omega^2|k|^2 + m^2}|c_k|^2 \\
&= \frac{1}{\omega} |\mathcal{T}_m(v)|^2_{\mathbb{H}_T^m} \\
&\leq \frac{1}{\omega} ||v||^2_{\mathbb{X}_T^m}. \qquad (3.6.3)
\end{aligned}$$

Now we want to prove that for every $v \in \mathbb{Z}_T^m$ and $2 < r < 2^\sharp$

$$|\mathcal{T}_m(v)|_{L^r(0,T)^N} \leq C||v||_{\mathbb{X}_T^m}, \qquad (3.6.4)$$

for some constant $C > 0$ independent of m.

Fix $2 < r < 2^\sharp$ and let p' be its conjugate exponent. Taking into account $c_0 = 0$, (3.2.10), (3.2.11) and trace inequality (ii) of Theorem 20 we can see that

$$
\begin{aligned}
|\mathcal{T}_m(v)|_{L^r(0,T)^N} &\leq \Big(\frac{1}{\sqrt{T^N}}\Big)^{\frac{2}{r'}-1}\Big(\sum_{|k|\geq 1}|c_k|^{r'}\Big)^{\frac{1}{r'}} \\
&\leq T^{-\frac{N}{2r'}(2-r')}\Big[|\mathcal{T}_m(v)|_{\mathbb{H}^m_T}\Big(\sum_{|k|\geq 1}(\sqrt{\omega^2|k|^2+m^2})^{-\frac{r'}{2-r'}}\Big)^{\frac{2-r'}{2r'}}\Big] \\
&\leq \frac{T^{-\frac{N}{2r'}(2-r')}}{\sqrt{\omega}}\Big(\sum_{|k|\geq 1}|k|^{-\frac{r'}{2-r'}}\Big)^{\frac{2-r'}{2r'}}|\mathcal{T}_m(v)|_{\mathbb{H}^m_T} \\
&\leq \frac{T^{-\frac{N}{2r'}(2-r')}}{\sqrt{\omega}}\Big(\sum_{|k|\geq 1}|k|^{-\frac{r'}{2-r'}}\Big)^{\frac{2-r'}{2r'}}||v||_{\mathbb{X}^m_T} \qquad (3.6.5)
\end{aligned}
$$

and the last series is finite since $\frac{2N}{N+1} < r' < 2$.

Therefore we are able to estimate the functional $\mathcal{J}_m$ from below. By Lemma 6 and exploiting (3.6.3) and (3.6.4) we can see that for every $0 < m < \frac{\omega}{2}$

$$
\begin{aligned}
\mathcal{J}_m(v) &= \frac{1}{2}[||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N}] - \int_{(0,T)^N} F(x, v(x,0))dx \\
&\geq \frac{1}{2}[||v||^2_{\mathbb{X}^m_T} - m|v|^2_{L^2(0,T)^N}] - \varepsilon|v|^2_{L^2(0,T)^N} - C_\varepsilon|v|^{p+1}_{L^{p+1}(0,T)^N} \\
&\geq \Big(\frac{1}{2} - \frac{m}{2\omega} - \frac{\varepsilon}{\omega}\Big)||v||^2_{\mathbb{X}^m_T} - C'_\varepsilon||v||^{p+1}_{\mathbb{X}^m_T} \\
&\geq \Big(\frac{1}{4} - \frac{\varepsilon}{\omega}\Big)||v||^2_{\mathbb{X}^m_T} - C'_\varepsilon||v||^{p+1}_{\mathbb{X}^m_T}.
\end{aligned}
$$

Choosing $0 < \varepsilon < \frac{\omega}{4}$, we have that $b := \frac{1}{4} - \frac{\varepsilon}{\omega} > 0$.
Let $\rho := \Big(\frac{b}{2C'_b}\Big)^{\frac{1}{p-1}}$. Then, for every $v \in \mathbb{Z}^m_T$ such that $||v||_{\mathbb{X}^m_T} = \rho$

$$
\mathcal{J}_m(v) \geq b\rho^2 - C'_b\rho^{p+1} = \frac{b}{2}\Big(\frac{b}{2C'_b}\Big)^{\frac{2}{p-1}} =: \lambda.
$$

In this way we can deduce that $\alpha_m \geq \lambda$ for every $0 < m < \frac{\omega}{2}$.

Now we prove (3.6.2). Let $w(x,y) = \prod_{i=1}^N \sin(\omega x_i)\frac{1}{y+1}$ and $z = \frac{w}{||w||_{\mathbb{X}^m_T}}$.

We can note that

$$
||w||^2_{\mathbb{X}^m_T} = \Big(\frac{T}{2}\Big)^N\Big\{\int_0^\infty \frac{1}{(y+1)^4}dy + \Big(\omega^2 N + m^2\Big)\int_0^\infty \frac{1}{(y+1)^2}dy\Big\}
$$

and so $||w||^2_{\mathbb{X}^m_T} \in [\frac{1}{3}(\frac{T}{2})^N, \frac{1}{3}(\frac{T}{2})^N + \omega^2 N + \frac{\omega^2}{4}] \equiv [C', C'']$.

Then for every $v = ce^{-my} + sz \in \mathbb{Y}_T^m \bigoplus \mathbb{R}z$, using $0 < m < \frac{\omega}{2}$

$$
\begin{aligned}
|v|^{\mu}_{L^{\mu}(0,T)^N} &\geq T^{-N\frac{\mu-2}{2}}\Big(\int_{(0,T)^N} |c + sz(x,0)|^2 dx\Big)^{\frac{\mu}{2}} \\
&= T^{-N\frac{\mu-2}{2}}\Big(\int_{(0,T)^N} c^2 dx + \int_{(0,T)^N} s^2 |z(x,0)|^2 dx\Big)^{\frac{\mu}{2}} \\
&\geq T^{-N\frac{\mu-2}{2}}(T^N c^2 + \frac{1}{C''}\Big(\frac{T}{2}\Big)^N s^2)^{\frac{\mu}{2}} \\
&\geq T^{-N\frac{\mu-2}{2}} \min\Big\{\frac{2}{\omega}, \frac{1}{C''}\Big(\frac{T}{2}\Big)^N\Big\}^{\frac{\mu}{2}}(||y||^2_{\mathbb{X}_T^m} + s^2)^{\frac{\mu}{2}} \qquad (3.6.6)
\end{aligned}
$$

where in the last inequality we use $||y||^2_{\mathbb{X}_T^m} = mT^N c^2$.

Hence, if $v = ce^{-my} + sz \in \mathbb{Y}_T^m \bigoplus \mathbb{R}z$, using (3.2.18) and (3.6.6)

$$
\begin{aligned}
\mathcal{J}_m(v) &= \frac{s^2}{2}\Big[||z||^2_{\mathbb{X}_T^m} - m|z|^2_{L^2(0,T)^N}\Big] - \int_{(0,T)^N} F(x,v)dx \\
&\leq \frac{s^2}{2} - (a_3|v|^{\mu}_{L^{\mu}(0,T)^N} - a_4 T^N) \\
&= \frac{s^2}{2} + a_4 T^N - a_3|c + sz|^{\mu}_{L^{\mu}(0,T)^N} \\
&\leq s^2 + a_4 T^N - a_3 T^{-N\frac{\mu-2}{2}} \min\Big\{\frac{2}{\omega}, \frac{1}{C''}\Big(\frac{T}{2}\Big)^N\Big\}^{\frac{\mu}{2}}(s^2 + ||y||^2_{\mathbb{X}_T^m})^{\frac{\mu}{2}}. \qquad (3.6.7)
\end{aligned}
$$

As a consequence

$$
\mathcal{J}_m(y + sz) \leq s^2 + a_4 T^N - a_3 \overline{C}(T,\mu) s^{\mu},
$$

and so we can find $R > \rho$ such that $\mathcal{J}_m(y + sz) \leq 0$ for every $s \geq R$ and $y \in \mathbb{Y}_T^m$. Let $0 \leq s \leq R$. By (3.6.7), there exists $R'_m > 0$ such that $\mathcal{J}_m(y+sz) \leq 0$ for every $||y||_{\mathbb{X}_T^m} \geq R'_m$. Thus $\mathcal{J}_m(v) \leq R^2 + a_4 T^N =: \delta$ for $v \in \mathbb{A}_T^m$.

Now we can verify that it's possible to take the limit as $m \to 0$ in (3.1.1). Fix $\beta \in (\frac{1}{\mu}, \frac{1}{2})$ and we proceed as in the first part of Lemma 11. Using the facts that

$\mathcal{J}_m(v_m) = \alpha_m \le \delta$ for all $m \in (0, \frac{\omega}{2})$ and $\mathcal{J}'_m(v_m) = 0$ we have

$$\begin{aligned}
\delta &\ge \mathcal{J}_m(v_m) - \beta < \mathcal{J}'_m(v_m), v_m >= \\
&= \Big(\frac{1}{2} - \beta\Big)[||v_m||^2_{\mathbb{X}^m_T} - m|v_m|^2_{L^2(0,T)^N}] + \int_{(0,T)^N} [\beta f(x, v_m)v_m - F(x, v_m)]dx \\
&\ge \int_{(0,T)^N} [\beta f(x, v_m)v_m - F(x, v_m)]dx \\
&\ge (\mu\beta - 1) \int_{(0,T)^N} F(x, v_m)dx - \kappa \qquad (3.6.8) \\
&\ge (\mu\beta - 1)[a_3|v_m|^{\mu}_{L^{\mu}(0,T)^N} - a_4 T^N] - \kappa \\
&\ge (\mu\beta - 1)[a_3|v_m|^{\mu}_{L^2(0,T)^N} T^{-N\frac{\mu-2}{2}} - a_4 T^N] - \kappa. \qquad (3.6.9)
\end{aligned}$$

By (3.6.9) we deduce the boundedness in L^2 of the trace of v_m, that is

$$|v_m|_{L^2(0,T)^N} \le K(\delta). \qquad (3.6.10)$$

Taking into account $\mathcal{J}_m(v_m) \le \delta$, (3.6.10) and (3.6.8) we obtain

$$\begin{aligned}
||\nabla v_m||^2_{L^2(\mathcal{S}_T)} &\le ||v_m||^2_{\mathbb{X}^m_T} \\
&= 2J_m(v_m) + m|v_m|^2_{L^2(0,T)^N} + 2\int_{(0,T)^N} F(x, v_m(x, 0))dx \\
&\le 2\delta + \frac{\omega}{2}K(\delta) + C(\delta) =: K'(\delta). \qquad (3.6.11)
\end{aligned}$$

Moreover, if c^m_k are the Fourier coefficients of $v_m(\cdot, 0)$, by the trace inequality (ii) of Theorem 20 we can see that

$$K'(\delta) \ge ||v_m||^2_{\mathbb{X}^m_T} \ge |v_m|^2_{\mathbb{H}^m_T} \ge 2T^{N-1}\pi \sum_{k\in\mathbb{Z}^N} |k||c^m_k|^2, \qquad (3.6.12)$$

and so, using (3.6.10) we obtain that $v_m(\cdot, 0)$ is bounded in $\mathbb{H}_T$.

Now we prove that it is possible to estimate v_m in $L^2_{loc}(\mathcal{S}_T)$ by a constant independent of m.

Fix $\alpha > 0$. Since $v_m \in \mathcal{C}^1(\overline{\mathcal{S}_T})$ (see Theorem 25), we have that for any $x \in [0, T]^N$ and $y \in [0, \alpha]$

$$v_m(x, y) = v_m(x, 0) + \int_0^y \partial_y v_m(x, s)ds.$$

By using $(|a| + |b|)^2 \le 2|a|^2 + 2|b|^2$ for all $a, b \in \mathbb{R}$ we obtain

$$|v_m(x, y)|^2 \le 2|v_m(x, 0)|^2 + 2\Big(\int_0^y |\partial_y v_m(x, s)|ds\Big)^2$$

and applying the Hölder inequality we deduce

$$|v_m(x,y)|^2 \leq 2\Big[|v_m(x,0)|^2 + \Big(\int_0^y |\partial_y v_m(x,s)|^2 ds\Big) y\Big]. \tag{3.6.13}$$

Integrating over $(0,T)^N \times (0,\alpha)$ and exploiting (3.6.10) and (3.6.11) we have

$$\begin{aligned} ||v_m||^2_{L^2((0,T)^N \times (0,\alpha))} &\leq 2\alpha |v_m|^2_{L^2(0,T)^N} + \alpha^2 ||\partial_y v_m||^2_{L^2(\mathcal{S}_T)} \\ &\leq 2\alpha K(\delta)^2 + \alpha^2 K'(\delta). \end{aligned}$$

Therefore we can extract a subsequence, that for simplicity we denote again by (v_m), and there exists $v \in L^2_{loc}(\mathcal{S}_T)$ such that $\nabla v \in L^2(\mathcal{S}_T)$, $v_m \rightharpoonup v$ in $L^2_{loc}(\mathcal{S}_T)$, $\nabla v_m \rightharpoonup \nabla v$ in $L^2(\mathcal{S}_T)$ and $v_m(\cdot,0) \to v(\cdot,0)$ in $L^2(0,T)^N$ as $m \to 0$. Now we know that v_m satisfies

$$\iint_{\mathcal{S}_T} \nabla v_m \nabla \eta + m^2 v_m \eta \, dxdy = \int_{(0,T)^N} [m v_m(x,0) + f(x, v_m(x,0))] \eta(x,0) \, dx \tag{3.6.14}$$

for every $\eta \in \mathbb{X}^m_T$. Let $\varphi \in \mathbb{X}_T$ and $\xi \in \mathcal{C}^\infty([0,\infty))$ such that

$$\begin{cases} \xi = 1 & \text{if } 0 \leq y \leq 1 \\ 0 \leq \xi \leq 1 & \text{if } 1 \leq y \leq 2 \\ \xi = 0 & \text{if } y \geq 2. \end{cases} \tag{3.6.15}$$

We set $\xi_R(y) = \xi(\frac{y}{R})$ for $R > 1$. Then choosing $\eta = \varphi \xi_R \in \mathbb{X}^m_T$ in (3.6.14) and taking the limit as $m \to 0$ we have

$$\iint_{\mathcal{S}_T} \nabla v \nabla(\varphi \xi_R) \, dxdy = \int_{(0,T)^N} f(x, v(x,0)) \varphi(x,0) \, dx. \tag{3.6.16}$$

Hence taking the limit as $R \to \infty$ we deduce that v verifies

$$\iint_{\mathcal{S}_T} \nabla v \nabla \varphi \, dxdy - \int_{(0,T)^N} f(x, v(x,0)) \varphi(x,0) \, dx = 0 \quad \forall \varphi \in \mathbb{X}_T.$$

Now we want to prove that $v \not\equiv 0$. Let $\xi(y) \in \mathcal{C}^\infty([0,\infty))$ defined as in (3.6.15); then $\xi v \in \mathbb{X}^m_T$. Hence

$$\begin{aligned} 0 &=< \mathcal{J}'_m(v_m), \xi v > \\ &= \iint_{\mathcal{S}_T} \nabla v_m \nabla(\xi v) + m^2 v_m \xi v \, dxdy - m \int_{(0,T)^N} v_m(x,0) v(x,0) \, dx \\ &- \int_{(0,T)^N} f(x, v_m(x,0)) v(x,0) \, dx \end{aligned}$$

and as $m \to 0$ we find

$$0 = \iint_{\mathcal{S}_T} \nabla v \nabla(\xi v)\, dxdy - \int_{(0,T)^N} f(x, v(x,0))v(x,0)\, dx. \tag{3.6.17}$$

Now, using the facts $\mathcal{J}_m(v_m) \geq \lambda$ and $< \mathcal{J}'_m(v_m), v_m >= 0$, we can see that

$$\begin{aligned} 2\lambda &\leq 2\mathcal{J}_m(v_m) + m|v_m|^2_{L^2(0,T)^N} + 2\int_{(0,T)^N} F(x, v_m(x,0))\, dx \\ &= \|v_m\|^2_{\mathbb{X}^m_T} = m|v_m|^2_{L^2(0,T)^N} + \int_{(0,T)^N} f(x, v_m(x,0))v_m(x,0)\, dx \end{aligned}$$

and taking the limit as $m \to 0$ we obtain

$$2\lambda \leq \int_{(0,T)^N} f(x, v(x,0))v(x,0)\, dx. \tag{3.6.18}$$

Taking into account (3.6.17) and (3.6.18) we deduce that

$$0 < 2\lambda \leq \int_{(0,T)^N} f(x, v(x,0))v(x,0)\, dx = \iint_{\mathcal{S}_T} \nabla v \nabla(\xi v)\, dxdy \tag{3.6.19}$$

and so $v \not\equiv 0$.

3.7 Regularity for $m = 0$

In this section we study the regularity of the solution v to

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{S}_T \\ v|_{\{x_i=0\}} = v|_{\{x_i=T\}} & \text{on } \partial_L \mathcal{S}_T \\ \frac{\partial v}{\partial \nu} = f(x,v) & \text{on } \partial^0 \mathcal{S}_T \end{cases}. \tag{3.7.1}$$

In particular we prove that $v \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}}^{N+1}_+) \cap \mathcal{C}^\infty(\mathbb{R}^{N+1}_+)$. Firstly, we show that $v(\cdot, 0) \in L^q(0,T)^N$ for every $q < \infty$, making the appropriate modifications in the proof of Lemma 12.

Assume $N \geq 2$ and $m \in (0, m_0)$. Let $w_m = v_m v^{2\beta}_{m,K}$ where $v_{m,K} = \min\{|v_m|, K\}$, $K > 1$ and $\beta \geq 0$. Then $w_m \in \mathbb{X}^m_T$ and using the fact that v_m is a critical point for $\mathcal{J}_m$ we deduce that

$$0 = \iint_{\mathcal{S}_T} \nabla v_m \nabla w_m + m^2 v_m w_m dxdy - m\int_{(0,T)^N} v_m w_m dx - \int_{(0,T)^N} f(x, v_m) w_m dx. \tag{3.7.2}$$

By direct computation we see

$$\iint_{\mathcal{S}_T} |\nabla(v_m v_{m,K}^{\beta})|^2 dxdy = \iint_{\mathcal{S}_T} v_{m,K}^{2\beta}|\nabla v_m|^2 dxdy + \iint_{D_{K,m}} (2\beta+\beta^2) v_{m,K}^{2\beta}|\nabla v_m|^2 dxdy \tag{3.7.3}$$

where $D_{K,m} = \{(x,y) \in \mathcal{S}_T : |v_m(x,y)| \leq K\}$. Combining these facts we find that

$$\begin{aligned}
||v_m v_{m,K}^{\beta}||^2_{\mathbb{X}_T^m} &= \iint_{\mathcal{S}_T} |\nabla(v_m v_{m,K}^{\beta})|^2 + m^2 v_m^2 v_{m,K}^{2\beta} dxdy \\
&= \iint_{\mathcal{S}_T} v_{m,K}^{2\beta}[|\nabla v_m|^2 + m^2 v_m^2] dxdy + \iint_{D_{K,m}} 2\beta\Big(1+\frac{\beta}{2}\Big) v_{m,K}^{2\beta}|\nabla v_m|^2 dxdy \\
&\leq c_\beta\Big[\iint_{\mathcal{S}_T} v_{m,K}^{2\beta}[|\nabla v_m|^2 + m^2 v_m^2] dxdy + \iint_{D_{K,m}} 2\beta v_{m,K}^{2\beta}|\nabla v_m|^2 dxdy\Big] \\
&= c_\beta\Big[\iint_{\mathcal{S}_T} \nabla v_m \nabla(v_m v_{m,K}^{2\beta}) + m^2 v_m (v_m v_{m,K}^{2\beta}) dxdy\Big] \\
&= c_\beta \int_{(0,T)^N} m v_m^2 v_{m,K}^{2\beta} + f(x,v_m) v_m v_{m,K}^{2\beta} dx
\end{aligned} \tag{3.7.4}$$

where $c_\beta = 1 + \frac{\beta}{2} \geq 1$. Using Lemma 6 with $\varepsilon = 1$ we get

$$m v_m^2 v_{m,K}^{2\beta} + f(x,v_m) v_m v_{m,K}^{2\beta} \leq (m+2) v_m^2 v_{m,K}^{2\beta} + (p+1) C_1 |v_m|^{p-1} v^2 v_{m,K}^{2\beta}. \tag{3.7.5}$$

Since v_m converges strongly in $L^{N(p-1)}(0,T)^N$ (because of $N(p-1) < 2^\sharp$), we can assume that, up to subsequences, there exists a function $z \in L^{N(p-1)}(0,T)^N$ such that $|v_m| \leq z$ in $(0,T)^N$, for every $m < m_0$.

Therefore, there exist a constant $c = m_0 + 2 + (p+1)C_1$ and a function $h \in L^N(0,T)^N$, $h \geq 0$ and independent of K, m and β such that

$$m v_m^2 v_{m,K}^{2\beta} + f(x,v_m) v_m v_{m,K}^{2\beta} \leq (c+h) v_m^2 v_{m,K}^{2\beta} \text{ on } \partial^0 \mathcal{S}_T. \tag{3.7.6}$$

As a consequence

$$||v_m v_{m,K}^{\beta}||^2_{\mathbb{X}_T^m} \leq c_\beta \int_{(0,T)^N} (c+h) v_m^2 v_{m,K}^{2\beta} dx. \tag{3.7.7}$$

Taking the limit as $K \to \infty$ ($v_{m,K}$ is increasing with respect to K) we get

$$|||v_m|^{\beta+1}||^2_{\mathbb{X}_T^m} \leq c c_\beta \int_{(0,T)^N} |v_m|^{2(\beta+1)} + c_\beta \int_{(0,T)^N} h |v_m|^{2(\beta+1)} dx. \tag{3.7.8}$$

For any $M > 0$, let $A_1 = \{h \leq M\}$ and $A_2 = \{h > M\}$. Then

$$\int_{(0,T)^N} h|v_m|^{2(\beta+1)} dx \leq M||v_m|^{\beta+1}|^2_{L^2(0,T)^N} + \varepsilon(M)||v_m|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N} \tag{3.7.9}$$

where $\varepsilon(M) = \left(\int_{A_2} h^N dx\right)^{\frac{1}{N}} \to 0$ as $M \to \infty$. Taking into account (3.7.8), (3.7.9), we have

$$|||v_m|^{\beta+1}||^2_{\mathbb{X}^m_T} \leq c_\beta(c+M)||v_m|^{\beta+1}|^2_{L^2(0,T)^N} + c_\beta\varepsilon(M)||v_m|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N}. \quad (3.7.10)$$

Now, using the Proposition 2.1 in [9], we know that for every function $w \in \mathcal{C}^\infty_T(\mathbb{R}^N)$ satisfying $\frac{1}{T^N}\int_{(0,T)^N} w(x)dx = 0$, there exists $\mu > 0$ such that

$$|w|^2_{L^{2^\sharp}(0,T)^N} \leq \mu \sum_{|k|\neq 0} |k||b_k|^2,$$

where b_k are the Fourier coefficients of w.

Therefore, for every $w \in \mathcal{C}^\infty_T(\mathbb{R}^N)$, we can see that

$$|w|^2_{L^{2^\sharp}(0,T)^N} \leq \mu_2 \sum_{|k|\neq 0} |k||b_k|^2 + \mu_1|w|^2_{L^2(0,T)^N},$$

where μ_1 and μ_2 are two positive constants.

Using this last fact and taking into account (3.7.10) and Theorem 20, we deduce that

$$\begin{aligned} ||v_m|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N} - \mu_1||v_m|^{\beta+1}|^2_{L^2(0,T)^N} &\leq \mu_3||v_m|^{\beta+1}|^2_{\mathbb{H}^m_T} \\ &\leq \mu_3|||v_m|^{\beta+1}||^2_{\mathbb{X}^m_T} \\ &\leq [c_\beta(c+M)||v_m|^{\beta+1}|^2_{L^2(0,T)^N} \\ &+ c_\beta\varepsilon(M)||v_m|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N}]\mu_3 \end{aligned} \quad (3.7.11)$$

where $\mu_3 > 0$ is independent from m.

Choosing M large enough so that $c_\beta\mu_3\varepsilon(M) < \frac{1}{2}$, by (3.7.11) we obtain

$$||v_m|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N} \leq 2[c_\beta\mu_3(c+M) + \mu_1]||v_m|^{\beta+1}|^2_{L^2(0,T)^N}. \quad (3.7.12)$$

Now we can use a bootstrap argument. Taking $\beta = 0$ in (3.7.12) and using (3.6.10) we infer that ($c_0 = 1$)

$$|v_m|^2_{L^{2^\sharp}(0,T)^N} \leq 2[\mu_3(c+M) + \mu_1]K^2(\delta) =: K_0. \quad (3.7.13)$$

that is $v(\cdot, 0) \in L^{\frac{2N}{N-1}}$.

By applying (3.7.12) with $\beta + 1 = \frac{N}{N-1}$ (that is $\beta = \frac{1}{N-1}$) and using (3.7.13) we have that

$$||v_m|^{\frac{N}{N-1}}|^2_{L^{2^\sharp}(0,T)^N} \leq 2[c_{\frac{1}{N-1}}\mu_3(c+M) + \mu_1]K_0 =: K_1,$$

that is $v(\cdot,0) \in L^{\frac{2N^2}{(N-1)^2}}(0,T)^N$.

After k iterations, we find $v(\cdot,0) \in L^{\frac{2N^k}{(N-1)^k}}(0,T)^N$ for all $k \in \mathbb{N}$, so $v(\cdot,0) \in L^q(0,T)^N$ for all $q \in [2,\infty)$.

Finally, to deduce that $v \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{R}}^{N+1}_+) \cap \mathcal{C}^\infty(\mathbb{R}^{N+1}_+)$ for some $\alpha \in (0,1)$, it's enough to follow the lines of the proof of Theorem 25 with $m = 0$ and $g(x, v(x,0)) = f(x, v(x,0))$.

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