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Homogenization Strategies and Computational Analyses for Masonry Structures via Micro-mechanical Approach

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INTRODUCTION

Masonries have been largely utilized in the history of architecture, in the past.

Despite their present uncommon use in new buildings, they still represent an important research topic due to several applications in the framework of structural engineering, with particular reference to maintaining and restoring historical and monumental buildings.

Hence, even if new materials (for example the reinforced concrete) are, today, wider spread than masonry ones, the unquestionable importance of a lot of real masonry estate requires researcher’s particular attention for this kind of structures. Therefore, in order to design an efficient response for repairing existing masonry structures, a large number of theoretical studies, experimental laboratory activities and computational procedures have been proposed in scientific literature.

Masonry is a heterogeneous medium which shows an anisotropic and inhomogeneous nature. In particular, the inhomogeneity is due to its biphasic composition and, consequently, to the different mechanical properties of its constituents, mortar and natural or artificial blocks. The anisotropy is, instead, due to the different masonry patterns since the mechanical response is affected by the geometrical arrangement of the constituents. Basically, in literature, two approaches are usually taken into account for materials which have a heterogeneous micro-structure: the heuristic approach and the thermodynamical one. In the former, aprioristic hypotheses on the dependence of the constitutive response on a certain number of parameters are considered and the material's mechanical behaviour is obtained by such hypotheses and by experimental tests. This approach is particularly used in non-linear field, where structural analyses are employed (Heyman, 1966). Our attention was focused on the latter approach. It extends the use of the homogeneous classical elasticity to heterogeneous materials by replacing the elastic constants of the classical homogeneous theory with the effective elastic ones, which average the actual inhomogeneous properties of the medium. Hence, such approach yields the overall compliance tensor and the overall stiffness tensor in a mathematical framework by means of mathematical operations of volume averaging and thermodynamical consistency. In this way, starting from the concepts of the average strain for prescribed macrostress and of the average stress for prescribed macrostrain, the global behaviour is provided from the masonry micro-structure geometry and from the known properties of the individual constituents.

In this framework, advanced analytical and numerical strategies - based on the finite element method - have been recently developed.

The main object of the present work is, in a first moment, to furnish an overall description of the different homogenization approaches utilized in literature for modelling masonry structures in linear-elastic field. Then, it will be given a number of new possible proposals for theoretical models which yield global constitutive relationships. Later, computational analyses are employed in order to compare the analytical results obtained from the proposed homogenization techniques with the literature data.

The work is articulated in seven chapters. Briefly, it is given a description of each one, here:

- Chapter I – it deals with the continuum mechanics for solids whose micro-structure is characterized by some heterogeneities. This means that they can appear to be constituted by various components, inclusions with different properties and shapes, or yet, they can show some defects such as cracks or cavities. A lot of advanced materials have this heterogeneous micro-structure, like ceramics, some metals, reinforcing fibres, polymeric composites and so on, for example. For such materials, a micro-mechanical analysis must be involved. Hence, the two approaches which are usually employed in literature for the micro-mechanical analysis of such media (the *thermodynamical* and *heuristic* approaches) are here described.
- Chapter II – it provides a short introduction to the notion of homogenization and of the essential concepts connected to it. Since most of the composite materials shows a brittle, rather than ductile, behaviour and, so, the elastic behaviour prevails, there is often no need to consider the homogenization in an elasto-plastic range. On the contrary, such an approach cannot be ignored when the plastic behaviour comes into play, like in the composites which have a metallic matrix, for example. This leads to some difficulty since the solution of the elasto-plastic homogenization problem in an exact form is available only for very simple cases. However, we will be interested in the elastic response of the homogenized material.

- Chapter III – it deals with the mechanic characterization of masonries, whose heterogeneity makes it quite a difficult task. Masonry is, indeed, constituted by blocks of artificial or natural origin jointed by dry or mortar joints. Hence, such a biphasic composition implies masonry is an inhomogeneous material. Moreover, since the joints are inherent plane of weakness, notably the mechanical masonry response is affected by behaviour preferred directions, which the joints determine. This fact implies masonry is also an anisotropic material. So, the chapter III describes the fundamental mechanical approaches (*Discrete* and *Continuous* models) which have been developed in literature, in order to formulate an appropriate constitutive description of masonry structures in linear-elastic field. In particular, our attention will be focused on the different homogenization proposals for modelling masonry structures, which are given in literature by some authors (Pietruszczack & Niu, Anthoine, Zucchini & Lourenco, et al…), in order to obtain a general account on the existent homogenization procedures and, contemporaneously, to underline the advantages and disadvantages for each one of them.
- Chapter IV – It furnishes some possible proposals for modelling masonry structures, in linear-elastic field, starting from the results of literature approaches. The main object has been to obtain new homogenization techniques able to overcome the limits of the literature homogenization procedures.
- Chapter V – It provides a short introduction to the formulation of Finite Element Method, propaedeutic knowledge in order to employ numerical analyses with F.E.M. calculation codes.

- Chapter VI – It provides some computational analyses (stress and strain-prescribed), carried out by means of the calculation code Ansys, in its version 6.0. This software offers a large number of appliances in a lot of engineering fields and it is just based on the mathematical F.E.M. model. Such finite element analyses have been employed in order to compare the analytical results obtained by our proposed homogenization techniques with the literature data.
- Chapter VII – It deals with a review of the international codes referred to the design of masonry structures. In this framework, the object of this chapter is to furnish a short summary and a comparison between the examined codes different from a number of countries.

CHAPTER I

Micro-mechanics theory

1.1 Introduction

Continuum mechanics deal with idealized solids consisting of material points and material neighbourhoods, by assuming that the material distribution, the stresses and the strains within an infinitesimal material neighbourhood of a typical point were essentially uniform [47], [48]. On the contrary, at a micro-scale, the infinitesimal material neighbourhood turns out to be characterized by some micro-heterogeneities, in the sense that it can appear to be constituted by various components, inclusions with differing properties and shapes, or yet, it can show some defects such as cracks or cavities. Hence, the actual stress and strain fields are not likely uniform, at this level.

A lot of advanced materials have this heterogeneous micro-structure. For example, the ceramics, some metals, ceramic, reinforcing fibres, polymeric composites and so on. For such materials, a micro-mechanical analysis must be involved.

Basically, two approaches are usually employed for the micro-mechanical analysis of such media [25].

The first one extends the use of homogeneous classical elasticity to heterogeneous materials by replacing the elastic constants of the classical homogeneous theory by effective elastic constants which average the actual inhomogeneous properties of the medium [47], [11]. According to this procedure, the definition of the *overall compliance tensor* and the *overall elasticity tensor* can be attained in a rigorous mathematical framework from the concept of average strain for prescribed macrostresses and from the concept of average stress for prescribed macrostrain, respectively. This kind of approach provides the materials' overall behaviour from the micro-structure geometry and from the known properties of the individual constituents, so that, at a macro-scale, the heterogeneous medium can be replaced by a homogeneous one having the mechanical anisotropic properties previously determined. In other words, it is possible to express in a systematic and rigorous manner the *continuum quantities* of an infinitesimal material neighbourhood in terms of the parameters that characterize the *microstructure* and the *micro-constituents* properties of the examined material neighbourhood.

However, in order to obtain the effective estimates of the overall material properties, the recourse to quite restrictive hypotheses and special averaging procedures is subsequently required [33], [10].

The second approach is somewhat more heuristic and is based on the hypothesis that the overall mechanical properties of the heterogeneous medium must be dependent on a certain number of parameters. Later, general relationships between these parameters and the *overall elasticity tensor* are obtained by means of fundamental theorems of the theory of elasticity. Moreover, certain of the effective material properties must be determined experimentally and cannot be predicted from the properties of the constituent materials [16], [54]. The limitation of the approach is counterbalanced by the fact that the resulting theories are able to make predictions in situations where the equivalent homogeneity approach cannot. A remarkable case is that one in which the deformation of the void volume is significant [15].

In the present chapter, a general treatment of micro-mechanics theory is employed in the spirit of the first of the above mentioned approaches.

According to it, there are several micro-mechanical models which are used for predicting the global mechanical behaviour of the heterogeneous materials, as the dilute approximation, the self-consistent scheme, the spherical model, the Mori-Tanaka and the differential scheme. All these models involve some approximations useful for carrying out the analysis and, therefore, they provide approximate effective global properties. The validity of the prediction depends on the chosen model.

1.2 Definition of the Representative Volume Element: geometrical and stress-condition considerations

A common procedure for developing the analysis of heterogeneous solids in micromechanics consists in making reference to a *Representative Volume Element* (RVE), which is an heterogeneous material volume, statistically representative of the neighbourhood of a certain point of a continuum mass [47], [11]. The continuum material point is called a *macro-element*. The corresponding micro-constituents of the RVE are called the *micro-elements*. Therefore, the concept of an RVE is used to estimate the continuum properties at the examined continuum material point, in terms of its microstructure and its microconstituents. In other words, the goal is to obtain the overall average constitutive properties of the RVE in terms of the properties and structure of the microelements, included in it, in order to calculate the global response of the continuum mass to applied loads and prescribed boundary data. These ones correspond to uniform fields applied on the continuum infinitesimal material neighbourhood which the RVE is aimed to represent.

The Figure 1.1 shows a continuum and identifies a typical material point *P* of it, surrounded by an infinitesimal material element. When the macro-element *P* is magnified, it shows the own complex micro-structure consisting in voids, cracks, inclusions, whiskers and other similar defects. To be representative, the RVE has to contain a very large number of such micro-heterogeneities.

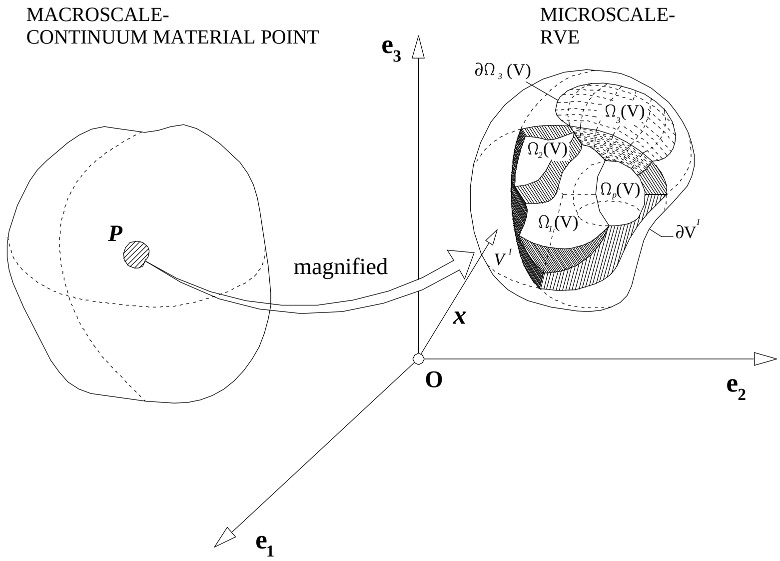


Figure 1.1 Possible microstructure of an RVE for the material neighbourhood of the continuum material point P

In literature, [47], [1], it is often found an RVE definition according to the following geometrical considerations:

1. the RVE has to be structurally typical of the whole medium on average
2. the RVE must include a very large number of micro-elements

Hence, according to the above given concept of the RVE, two length-scales are necessary: one is the continuum or *macro-length* scale, by which the infinitesimal material neighbourhood is measured; the other one is the *micro-length* scale which corresponds to the smallest micro-constituent whose properties and shape are judged to have direct and first-order effects on the continuum overall response and properties. Therefore, it provides a valuable dividing boundary between continuum theories and microscopic ones, being large if compared to the micro-constituents and small if compared to the entire body. So, for scales larger than the representative volume element, continuum mechanics are used and properties of the material as whole are determined, while for scales smaller than the representative volume element, the microstructure of the material has to be considered.

In general, if the typical dimension of the material being modelled is named with *L*, if the typical dimension of the macro-element is named with *D* and if the typical dimension of the micro-element is named with *d*, they have to be in the following relation:

$$
\begin{gathered}
L \;?\; D \;?\; d \\
\frac{L}{D} \;?\; 1; \quad \frac{D}{d} \;?\; 1;
\end{gathered}
\qquad (1.2\text{-}1)
$$

This means that the typical dimension *D* of the RVE should be much larger than the typical size *d* of the micro-element and much smaller than the typical size *L* of the entire body, as it is shown in the following figure, where the RVE is used in the shape of cube.

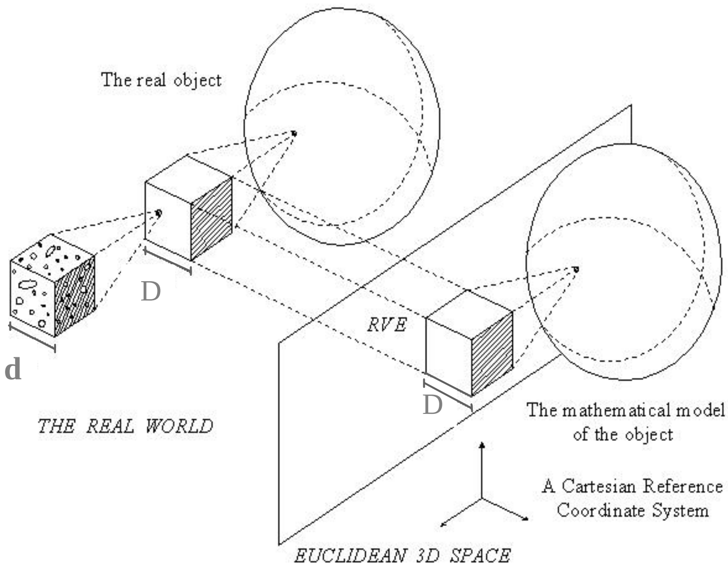


Figure 1.2 The typical dimension a Representative Volume Element.

The relation **Errore. L'origine riferimento non è stata trovata.** has to be valid independently from the fact that the micro-elements have or not have a random, periodic or other distribution within the continuum material, although, of course, the corresponding overall RVE properties are directly affected by this distribution.

It is useful to underline two important concepts in the previous geometrical RVE definition:

- the absolute dimensions of the micro-constituents may be very large or very small, depending on the size of the continuum mass and the objectives of the analysis - it is only the relative dimensions that are of concern.

- the evaluation of the *essential* micro-constituent is another relative concept, depending on the particular problem and on the particular objective. It must be addressed through systematic micro-structural observations at the level of interest and must be guided by experimental results.

Hence, the definition of the RVE is one of the most important decisions that the analyst makes for employing an accurate micro-mechanical analysis. An optimum choice would be one that includes the most dominant features that have first-order influence on the overall properties of interest and, at the same time, yields the simplest model. This can only be done through a coordinate sequence of microscopic and macroscopic observations, experimentations and analyses.

The until now mentioned definition of the RVE, based only on geometrical ratio between the different scales of the whole body, RVE's size and characteristic dimensions of the micro-inclusions or defects, is – in general – not sufficient to ensure the optimality of the choice related to the accuracy and consistency of the micro-mechanical approach, as well as of the homogenization procedure. Indeed, as shown in some works with reference to *configurational body force*, possible significant gradients of stress and strain fields can play a crucial rule for establishing the RVE size, provided that they strongly vary within the RVE characteristic length, [39]. In other words, a consistent criteria to select an RVE has to be also based on the preliminary requirement of a smooth distribution of the physical quantities involved in the analysis. This mathematical property finds its mechanical interpretation in the fact that all micro-mechanics and homogenization theories are based on averaged stress and strain values over the RVE domain, as well as on the overall elastic and inelastic responses. Therefore, no strong field gradients have to be attempted and, in order to ensure this, functional analysis and theory of elasticity theorems have to be invoked and utilized.

On the other hand, some works reported in literature also consider approaches where the weight of geometrical parameters gradients are taken into account. In particular, in this framework, a paper by Fraldi & Guarracino [25] deals with a straightforward homogenization technique for porous media characterized by locally variable values of the volume fraction. From a theoretical point of view, the employed technique corresponds to averaging a continuum model in order to end up with a higher continuum model. According to the opinion of the writers, such procedure offers several advantages. First, making the effective elastic moduli of the homogenized porous medium dependent not only on the value of the matrix volume fraction, g , but on its gradient, ∇g , as well, allows a simple characterization of the micromechanical inhomogeneity of the RVE in a closed mathematical form. Second, the number of parameters necessary to an adequate identification of the mechanical properties of the material is extremely reduced and essentially coincides with the properties of the constituent matrix and with the knowledge of the local values of the density of the medium under analysis. Finally, it seems that, by means of this approach, several problems involving porous media characterized by a non-periodic distribution of voids, such as cancellous bone tissues or radioactively damaged materials, can be effectively tackled from a computational standpoint.

However, once an RVE has been chosen, the micromechanical analysis has to be employed in order to calculate, as said before, its overall response parameters. Since the microstructure of the material, in general, changes in the course of deformation, the overall properties of the RVE also, in general, change. Hence, as anticipated before, an incremental formulation is sometimes necessary, but, for certain problems in elasticity, a formulation in terms of the total stresses and strains may suffice.

In this framework, consider an RVE occupying a volume V^I and bounded by a regular surface ∂V^I, where the superscript *I* stays for inhomogeneous. A typical point in V^I is identified by its position vector $\boldsymbol{x}$, with components $x_i \left(i = 1, 2, 3 \right)$, relative to a fixed rectangular Cartesian coordinate system (see Figure 1.1). The unit base vectors of this coordinate system are denoted by $\boldsymbol{e}_i \left(i = 1, 2, 3 \right)$, so the position vector is given by:

$$\boldsymbol{x} = x_i \, \boldsymbol{e}_i \qquad (1.2\text{-}2)$$

where repeated subscripts are summed.

For the purpose of micromechanical approach, the RVE is regarded as a *heterogeneous continuum with spatially variable, but known, constitutive properties* [47], whose it needs to estimate the average ones.

In general, the displacement field, $\boldsymbol{u} = \boldsymbol{u}\left(\boldsymbol{x}\right)$, the strain field, $\mathsf{E} = \mathsf{E}\left(\boldsymbol{x}\right)$, and the stress one, $\mathsf{T} = \mathsf{T}\left(\boldsymbol{x}\right)$, within the RVE volume, vary in fact from point to point, even if the boundary tractions are uniform or the boundary displacements are linear. Under both the prescribed surface data, the RVE must be in equilibrium and its overall deformation compatible. The governing field equilibrium equations at a typical point $\boldsymbol{x}$ in the volume *V* of the RVE (for simplicity, the superscript *I* will be not repeated) are:

$$\nabla \cdot \boldsymbol{T}\left(\boldsymbol{x}\right) = \boldsymbol{0}; \qquad \boldsymbol{T}\left(\boldsymbol{x}\right) = \boldsymbol{T}^T\left(\boldsymbol{x}\right) \ \textit{in } V \qquad (1.2\text{-}3)$$

where body forces are assumed absent and where the superscript *T* stands for transpose.

In a rectangular Cartesian component form, the (1.2-3) becomes:

$$\sigma_{ij,i} = 0; \qquad \sigma_{ij} = \sigma_{ji} \quad in\ V \tag{1.2.4}$$

where:

$i = j = 1, 2, 3$

and where a comma followed by an index denotes the partial differentiation with respect to the corresponding coordinate variable.

Moreover, the strain-displacement relation has to be verified:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{2}\left\{\nabla \otimes \boldsymbol{u} + (\nabla \otimes \boldsymbol{u})^{T}\right\} \ in\ V \tag{1.2-5}$$

where ∇ is the del operator defined by:

$$\nabla = \partial_i \boldsymbol{e}_i = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \tag{1.2-6}$$

and the superscript T denotes transpose.

The(1.2-5), in a rectangular Cartesian component form, becomes:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad in\ V \tag{1.2-7}$$

When the self-equilibrating tractions, $\boldsymbol{t}^0$, are prescribed on the RVE boundary ∂V, as shown in Figure 1.3 the following boundary equilibrium conditions have to be verified:

$$\boldsymbol{T}(\boldsymbol{x})\boldsymbol{n} = \boldsymbol{t}^0 \quad on\ \partial V_t \tag{1.2-8}$$

or, in Cartesian components:

$$\sigma_{ij} n_i = t_j^0 \ on\ \partial V_t \tag{1.2-9}$$

where:

$\boldsymbol{n} =$ the outer unit normal vector of the RVE boundary ∂V.

$\partial V_t =$ the partition of the RVE boundary where the self-equilibrating tractions, $\boldsymbol{t}^0$, are prescribed.

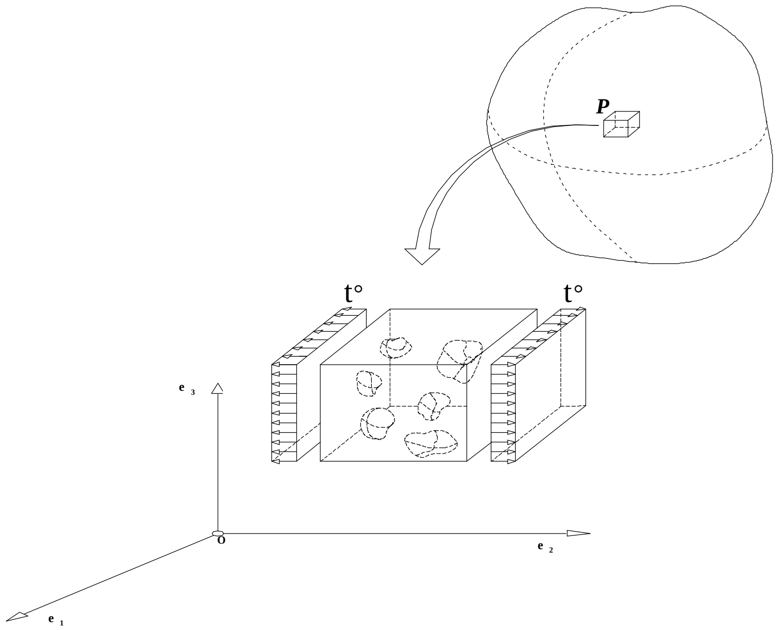


Figure 1.3 Traction boundary conditions

On the other hand, when the self-compatible displacements, $\boldsymbol{u}^0$, (self-compatible in the sense that they don't include rigid-body translations or rotations) are assumed prescribed on the boundary of the RVE, as shown in Figure 1.4, it follows that:

$$\boldsymbol{u} = \boldsymbol{u}^0 \ \ on \ \partial V_u \tag{1.2-10}$$

or, in Cartesian components:

$$u_i = u_i^o \ \ on \ \partial V_u \tag{1.2-11}$$

where:

$\partial V_u =$ the partition of the RVE boundary where the displacements, $\boldsymbol{u}^0$, are prescribed.

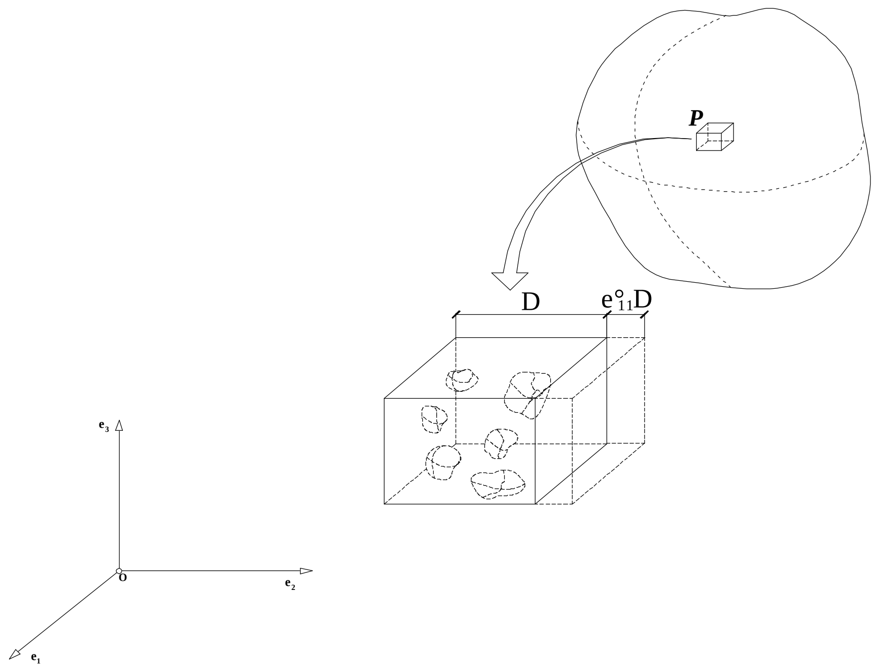


Figure 1.4 Displacement boundary conditions

If the modified microstructure of the material during the deformation has to be taken into account, the incremental formulation is necessary to consider a rate problem, where traction rates $\dot{\boldsymbol{t}}^0$ or velocity $\dot{\boldsymbol{u}}^0$ may be regarded as prescribed on the boundary of the RVE. Here, the rates can be measured in terms of monotone increasing parameter, since no inertia effects are included. Therefore, the basic field equations are obtained by substituting in the above written equations the corresponding rate quantities, i.e. $\dot{\mathrm{T}}(\boldsymbol{x})$ for $\mathrm{T}(\boldsymbol{x})$, $\dot{\mathrm{E}}(\boldsymbol{x})$ for $\mathrm{E}(\boldsymbol{x})$ and $\dot{\boldsymbol{u}}$ for $\boldsymbol{u}$, obtaining that:

$$\nabla \cdot \dot{\boldsymbol{T}}(\boldsymbol{x}) = \boldsymbol{0}; \qquad \dot{\boldsymbol{T}}(\boldsymbol{x}) = \dot{\boldsymbol{T}}^{T}(\boldsymbol{x}) \ \ in\ V \qquad (1.2\text{-}12)$$

and

$$\dot{\boldsymbol{E}}(\boldsymbol{x}) = \frac{1}{2}\left\{\nabla \otimes \dot{\boldsymbol{u}} + \left(\nabla \otimes \dot{\boldsymbol{u}}\right)^{T}\right\} \ \ in\ V \qquad (1.2\text{-}13)$$

When the self-equilibrating tractions, $\dot{\boldsymbol{t}}^{0}$, are prescribed on the RVE boundary ∂V^{I}, the following boundary equilibrium conditions have to be verified:

$$\dot{\boldsymbol{T}}(\boldsymbol{x})\boldsymbol{n} = \dot{\boldsymbol{t}}^{0} \ \ on\ \partial V_{t} \qquad (1.2\text{-}14)$$

On the other hand, when the velocities, $\dot{\boldsymbol{u}}^{0}$, are assumed prescribed on the boundary of the RVE, it follows that:

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}^{0} \ \ on\ \partial V_{u} \qquad (1.2\text{-}15)$$

Once the boundary-value problems associated with an RVE is formulated, the aim is, then, to calculate its overall response parameters and to use these ones in order to describe the local properties of the continuum material element, when returning to the starting macro-scale. In this scale, in fact, the RVE represents a point of the continuum material in which mechanical properties have to be found. Hence, it is necessary to obtain uniform macrofields on the RVE boundary; thus, prescribed surface tractions, $\boldsymbol{t}^{\boldsymbol{0}}$, may be applied as spatially uniform, or prescribed surface displacements, $\boldsymbol{u}^{\boldsymbol{0}}$, may be assumed as spatially linear. In the first case, the goal is to found the average strain field as a function of the corresponding prescribed nominal stress one.

Consequently, the components of the overall compliance tensor are obtained as:

$$\begin{aligned} \bar{S}_{ijhk} &= \delta_{ij}\delta_{hk}\frac{\bar{\varepsilon}_{ij}}{\sigma^{0}_{hk}} \\ \bar{S}_{ijhk} &= \left(1-\delta_{ij}\right)\cdot\left(1-\delta_{hk}\right)\cdot\frac{\bar{\varepsilon}_{ij}}{\sigma^{0}_{hk}} \end{aligned} \qquad i,j,h,k = x,y,z \qquad (1.2\text{-}16)$$

where consistent considerations are done for defining the opportune average strain field to use in the calculation. In the second case, the goal is to found the average stress field as a function of the corresponding prescribed nominal strain one. Consequently, the components of the overall stiffness tensor are obtained as:

$$\begin{aligned} \bar{C}_{ijhk} &= \delta_{ij}\delta_{hk}\frac{\bar{\sigma}_{ij}}{\varepsilon^{0}_{hk}} \\ \bar{C}_{ijhk} &= \left(1-\delta_{ij}\right)\cdot\left(1-\delta_{hk}\right)\cdot\frac{\bar{\sigma}_{ij}}{\varepsilon^{0}_{hk}} \end{aligned} \qquad i,j,h,k = x,y,z \qquad (1.2\text{-}17)$$

where consistent considerations are done for defining the opportune average stress field to use in the calculation.

In order to reach this objectives, fundamental averaging methods are necessary for evaluating average quantities and they will be shown in the following section.

It is worth to notice, here, that an elastic solution obtained via micro-mechanical approach satisfies, at a micro-scale, both the equilibrium and the compatibility in each internal point of the RVE, either in the case of prescribed stress problem either in the other one of prescribed strain. On the contrary, by considering the RVE inside the continuum solid, the first problem satisfies the equilibrium conditions in the points belonging to the interface between two adjacent RVE but doesn't satisfy the compatibility conditions at the same interface. The second problem, instead, satisfies the compatibility conditions in such points but doesn't satisfy, in general, the equilibrium ones. Nevertheless, at a macro-scale the found solution is always an exact one because both field equilibrium and field compatibility conditions are verified. Thanks to the uniformity of the obtained macro-fields, in fact, it will be:

$$\left.\begin{aligned} \nabla \cdot \bar{\boldsymbol{T}} &= \mathbf{0} \\ \nabla \wedge \left(\nabla \wedge \bar{\boldsymbol{E}}\right) &= \mathbf{0} \end{aligned}\right. \quad \forall\, \boldsymbol{x} \in B \qquad (1.2\text{-}18)$$

where:

$\bar{\boldsymbol{T}} =$ average stress field

$\bar{\boldsymbol{E}} =$ average strain field

$\boldsymbol{x} =$ position vector of the points within the volume B.

$B =$ the volume of the continuum solid from which the RVE has been extracted.

It will be seen, in the follows, that the homogenization approach, differently by the micro-mechanical one, starts by assuming constant stress (stress prescribed problem) or constant strain (stress prescribed problem) fields everywhere within the RVE volume, V. Such a procedure whose goal is the evaluation of the overall response parameters, yet, implies that at a micro-scale, in the case of prescribed stress problem, field equilibrium equations are verified in each internal point of the RVE while the compatibility ones are not satisfied for the internal points of the RVE belonging to the interface between two adjacent micro-constituents. In the case of prescribed strain problem, instead, it happens the opposite: compatibility equations are always verified in each internal point of the RVE while the equilibrium ones at the interface between two adjacent micro-constituents are not satisfied. Moreover, by considering the RVE inside the continuum solid, the satisfaction of the equilibrium and the compatibility conditions at the interface between two adjacent RVE is dependent by the shape of the RVE boundary: in the homogenization approach, in fact, either in a prescribed stress problem either in the converse prescribed strain one, the surface tractions and displacements are obtained as a consequence and so they are, in general not uniform or linear respectively. For this reason, such an approach is useful if there is a periodicity of the RVE in the continuum medium and if there is not the presence of voids on the RVE boundary. Nevertheless, at a macro-scale the found solution is always, also for the homogenization approach, an exact one because both field equilibrium and field compatibility equations are verified. Since the homogenization approach, as it will be shown in detail in the Chapter 2, calculates the overall response parameters of the RVE by taking into account the average stress and strain field produced within its volume (and not, like the micro-mechanical approach, the nominal quantities) and thanks to the uniformity of such fields, the (1.2-18) are verified yet.

In particular, for prescribed constant stress field in each point of *V*, the object is to found the average value of the piecewise obtained constant strain field in *V*, (for homogeneous micro-constituents, the strain field is constant in each phase and it assumes different values from phase to phase), as a function of the corresponding prescribed stress field. Consequently, the components of the overall compliance tensor are obtained as given by the(1.2-16), where consistent considerations are done for defining the opportune average strain field to use in the calculation. Analogously, for prescribed constant strain field in each point of *V*, the object is to found the average value of the piecewise obtained constant stress field in *V* (for homogeneous micro-constituents, the stress field is constant in each phase and it assumes different values from phase to phase) as a function of the corresponding prescribed strain field. Consequently, the components of the overall stiffness tensor are obtained as given by the (1.2-17), where consistent considerations are done for defining the opportune average stress field to use in the calculation.

1.3 General theory for evaluating average quantities

In order to obtain further insight into the relation between the microstructure and the overall properties, averaging theorems have to be considered.

In particular, in the case of prescribed self-equilibrating tractions on the boundary ∂V of the RVE, either spatially uniform or not, the *unweighted volume average* of the variable stress field $\boldsymbol{T}(\boldsymbol{x})$, taken on the volume V of the RVE, is completely defined in terms of the prescribed boundary tractions. To show this, denote the volume average of the spatially variable and integrable quantity $\boldsymbol{T}(\boldsymbol{x})$ by:

$$\overline{\boldsymbol{T}} = <\boldsymbol{T}(\boldsymbol{x})> = \frac{1}{V}\int_V \boldsymbol{T}(\boldsymbol{x})\,dV \qquad (1.3\text{-}1)$$

where $\boldsymbol{x}$ is the position vector, that identifies each point in the volume V of the RVE, with components $x_i\ (i = 1, 2, 3)$, relative to a fixed rectangular Cartesian coordinate system (see Figure 1.1).

The gradient of $\boldsymbol{x}$ satisfies:

$$(\nabla \otimes \boldsymbol{x})^T = \partial_j x_i \boldsymbol{e}_i \otimes \boldsymbol{e}_j = x_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \mathbf{1}^{(2)} \qquad (1.3\text{-}2)$$

where:

$\delta_{ij} =$ Kronecker delta

$\mathbf{1}^{(2)} =$ the second-order unit tensor

Hence, in according to the equilibrium equations (1.2-3) and since the stress tensor $\boldsymbol{T}(\boldsymbol{x})$ is divergence-free, the stress field $\boldsymbol{T}(\boldsymbol{x})$ can be written in the following form:

$$\boldsymbol{T}(\boldsymbol{x}) = \mathbf{1}^{(2)} \cdot \boldsymbol{T}(\boldsymbol{x}) = (\nabla \otimes \boldsymbol{x})^T \cdot \boldsymbol{T}(\boldsymbol{x}) = \left\{\nabla \cdot (\boldsymbol{T}(\boldsymbol{x}) \otimes \boldsymbol{x})\right\}^T \qquad (1.3\text{-}3)$$

By means of the Gauss theorem, and by remembering the **Errore. L'origine riferimento non è stata trovata.** and the(1.3-3), the average stress field $\overline{\boldsymbol{T}}$ is expressed as:

$$\overline{\boldsymbol{T}} = <\boldsymbol{T}(\boldsymbol{x})> = \frac{1}{V}\int_V \left\{\nabla \cdot (\boldsymbol{T}(\boldsymbol{x}) \otimes \boldsymbol{x})\right\}^T dV = \frac{1}{V}\int_{\partial V} \left\{\boldsymbol{n} \cdot (\boldsymbol{T}(\boldsymbol{x}) \otimes \boldsymbol{x})\right\}^T ds \quad (1.3\text{-}4)$$

and, for the boundary equilibrium condition (1.2-8), it can be written:

$$\overline{\boldsymbol{T}} = \frac{1}{V}\int_{\partial V} \boldsymbol{x} \otimes \boldsymbol{t}^0 \, ds \qquad (1.3\text{-}5)$$

or, in Cartesian components:

$$\overline{\sigma}_{ij} = \frac{1}{V}\int_{\partial V} x_i t_j^0 \, ds \qquad (1.3\text{-}6)$$

It should be noted that since the prescribed surface tractions, $\boldsymbol{t}^0$, are self-equilibrating, their resultant total force and total moment about a fixed point vanish, i.e.:

$$\int_{\partial V} \boldsymbol{t}^0 ds = \mathbf{0} \qquad \int_{\partial V} \boldsymbol{x} \wedge \boldsymbol{t}^0 ds = \mathbf{0} \qquad (1.3\text{-}7)$$

or, in components:

$$\int_{\partial V} t_j^0 ds = 0 \qquad \int_{\partial V} e_{ijk} x_j \wedge t_k^0 ds = 0 \qquad (1.3\text{-}8)$$

where:

$e_{ijk} =$ the permutation symbol of the third order; $e_{ijk} = (+1, -1, 0)$ when i, j, k form (even, odd, no) permutation of 1, 2, 3.

Hence, the average stress $\bar{\boldsymbol{T}}$ defined by the (1.3-5) is symmetric and independent of the origin of the coordinate system. Indeed, from the (1.3-8), it is:

$$\int_{dV} \boldsymbol{x} \otimes \boldsymbol{t}^0 \, ds = \int_{dV} \boldsymbol{t}^0 \otimes \boldsymbol{x} \, ds \tag{1.3-9}$$

and, so:

$$\bar{\boldsymbol{T}}^{\boldsymbol{T}} = \bar{\boldsymbol{T}} \tag{1.3-10}$$

Let us to assume that the following boundary equilibrium equations were satisfied:

$$\boldsymbol{T}(\boldsymbol{x})\boldsymbol{n} = \boldsymbol{t}^0 \ \ on\ \partial V_t \tag{1.3-11}$$

and:

$$\boldsymbol{T^0 n} = \boldsymbol{t^0} \ \ on\ \partial V_t \tag{1.3-12}$$

where:

$\boldsymbol{n} =$ the outer unit normal vector of the RVE boundary dV.

$\partial V_t =$ the partition of the RVE boundary where the self-equilibrating surface tractions, $\boldsymbol{t}^0$, are prescribed.

$\boldsymbol{t^0} =$ the prescribed surface self-equilibrating tractions, assumed as spatially uniform on the boundary ∂V_t of the RVE.

$\boldsymbol{T}(\boldsymbol{x}) =$ the spatially variable stress tensor obtained from the stress prescribed problem within the volume V of the heterogeneous RVE.

$\boldsymbol{T^0} =$ the spatially constant stress tensor obtained from the stress prescribed problem within the volume V of the RVE, regarded as homogeneous.

From the equation (1.3-5), and by using again the Gauss theorem, it can be written:

$$\bar{\boldsymbol{T}} = \frac{1}{V}\int_{\partial V} \boldsymbol{x}\otimes\boldsymbol{t}^0\, ds = \frac{1}{V}\int_{\partial V}\left(\boldsymbol{T}^0\otimes\boldsymbol{x}\right)\cdot\boldsymbol{n}\, ds = \frac{1}{V}\int_V \nabla\cdot\left(\boldsymbol{T}^0\otimes\boldsymbol{x}\right) dV \quad (1.3\text{-}13)$$

Thus, by taking into account the position (1.3-3), the volume average of the spatially variable and integrable quantity $\boldsymbol{T}\left(\boldsymbol{x}\right)$ can be expressed in the following form:

$$\bar{\boldsymbol{T}} = \frac{1}{V}\int_V \boldsymbol{T}^0\, dV = \boldsymbol{T}^0 \quad (1.3\text{-}14)$$

For the rate problem, the average stress rate is obtained in an analogous manner to what has been done previously, by obtaining that:

$$\bar{\dot{\boldsymbol{T}}} = <\dot{\boldsymbol{T}}\left(\boldsymbol{x}\right)> = \frac{1}{V}\int_{\partial V} \boldsymbol{x}\otimes\dot{\boldsymbol{t}}^0\, ds \quad (1.3\text{-}15)$$

or, in Cartesian components:

$$\bar{\dot{\sigma}}_{ij} = <\dot{\sigma}_{ij}\left(x\right)> = \frac{1}{V}\int_{\partial V} x_i \dot{t}_j^0\, ds \quad (1.3\text{-}16)$$

Hence, it is seen that for the small deformations the average stress rate equals the rate of change of the average stress:

$$\bar{\dot{\boldsymbol{T}}} = <\dot{\boldsymbol{T}}\left(\boldsymbol{x}\right)> = \frac{d}{dt}<\boldsymbol{T}\left(\boldsymbol{x}\right)> = \dot{\bar{\boldsymbol{T}}} \quad (1.3\text{-}17)$$

In particular, in the case of prescribed displacements on the boundary ∂V of the RVE, either spatially linear or not, the *unweighted volume average* of the variable displacement gradient $\nabla\otimes\boldsymbol{u}$ (and so of the variable strain field $\boldsymbol{E}\left(\boldsymbol{x}\right)$), taken over the volume V of the RVE, is completely defined in terms of the prescribed boundary displacements. To show this, denote the volume average of the spatially variable and integrable quantity $\nabla\otimes\boldsymbol{u}$ by:

$$\overline{\nabla\otimes\boldsymbol{u}} = <\nabla\otimes\boldsymbol{u}\left(\boldsymbol{x}\right)> = \frac{1}{V}\int_V \nabla\otimes\boldsymbol{u}\left(\boldsymbol{x}\right) dV \quad (1.3\text{-}18)$$

where $\boldsymbol{x}$ is the position vector, that identifies each point in the volume V of the RVE, with components $x_i\,(i=1,2,3)$, relative to a fixed rectangular Cartesian coordinate system (see Figure 1.1).

From the Gauss theorem, and in view of the boundary conditions (1.2-10), it is:

$$\int_V \text{Ñ}\otimes\boldsymbol{u}(\boldsymbol{x})\,dV=\int_{\partial V}\boldsymbol{n}(\boldsymbol{x})\otimes\boldsymbol{u}(\boldsymbol{x})\,ds=\int_{\partial V}\boldsymbol{n}(\boldsymbol{x})\otimes\boldsymbol{u^0}(\boldsymbol{x})\,ds \qquad (1.3\text{-}19)$$

where:

$\boldsymbol{n}=$ the outer unit normal vector of the RVE boundary dV.

Thus, the average displacement gradient within the RVE volume V can be expressed in the following form:

$$\overline{\text{Ñ}\otimes\boldsymbol{u}}=<\text{Ñ}\otimes\boldsymbol{u}(\boldsymbol{x})>=\frac{1}{V}\int_{\partial V}\boldsymbol{n}(\boldsymbol{x})\otimes\boldsymbol{u^0}(\boldsymbol{x})\,ds \qquad (1.3\text{-}20)$$

or, in Cartesian components:

$$\overline{\boldsymbol{u_{j,i}}}=<\boldsymbol{u_{j,i}}(x)>=\frac{1}{V}\int_{\partial V}n_i u_j^0(x)\,ds \qquad (1.3\text{-}21)$$

Let us to remember that the rotation field $\boldsymbol{R}(\boldsymbol{x})$ is the anti-symmetric part of the displacement gradient, that is:

$$\boldsymbol{R}(\boldsymbol{x})=\frac{1}{2}\left[\text{Ñ}\otimes\boldsymbol{u}(\boldsymbol{x})-\left(\text{Ñ}\otimes\boldsymbol{u}(\boldsymbol{x})\right)^T\right] \qquad (1.3\text{-}22)$$

or, in components:

$$R_{ij}(x)=\frac{1}{2}\left[u_{j,i}-u_{i,j}\right] \qquad (1.3\text{-}23)$$

while the strain field $\boldsymbol{E}(\boldsymbol{x})$ is the corresponding symmetric part of the displacement gradient, hence, the stress field $\boldsymbol{E}(\boldsymbol{x})$ within the RVE volume can be written in the following form:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{2}\left[\nabla \otimes \boldsymbol{u}(\boldsymbol{x}) + \left(\nabla \otimes \boldsymbol{u}(\boldsymbol{x})\right)^T\right] \tag{1.3-24}$$

or, in components:

$$E_{ij}(x) = \frac{1}{2}\left[u_{j,i} + u_{i,j}\right] \tag{1.3-25}$$

Let us to denote the volume average of the spatially variable and integrable quantity $\boldsymbol{E}(\boldsymbol{x})$ by:

$$\overline{\boldsymbol{E}} = < \boldsymbol{E}(\boldsymbol{x}) > = \frac{1}{V}\int_V \boldsymbol{E}(\boldsymbol{x})\, dV \tag{1.3-26}$$

Hence, by means of the (1.3-24) and of the (1.3-19), the average strain field $\overline{\boldsymbol{E}}$ is expressed as:

$$\overline{\boldsymbol{E}} = < \boldsymbol{E}(\boldsymbol{x}) > = \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(\boldsymbol{n} \otimes \boldsymbol{u^0} + \boldsymbol{u^0} \otimes \boldsymbol{n}\right) ds \tag{1.3-27}$$

or, in components:

$$\overline{\varepsilon_{ij}} = < \varepsilon(x)_{ij} > = \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(n_i u_j^0 + u_i^0 n_j\right) ds \tag{1.3-28}$$

while the average rotation field $\overline{\boldsymbol{R}}$ is expressed as:

$$\overline{\boldsymbol{R}} = < \boldsymbol{R}(\boldsymbol{x}) > = \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(\boldsymbol{n} \otimes \boldsymbol{u^0} - \boldsymbol{u^0} \otimes \boldsymbol{n}\right) ds \tag{1.3-29}$$

or, in components:

$$\overline{r_{ij}} = < r(x)_{ij} > = \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(n_i u_j^0 - u_i^0 n_j\right) ds \tag{1.3-30}$$

It is worth to underline that the average strain $\overline{\boldsymbol{E}}$ defined by the (1.3-27) is symmetric and independent of the origin of the coordinate system, and, so, it is:

$$\overline{\boldsymbol{E}}^{\boldsymbol{T}} = \overline{\boldsymbol{E}} \tag{1.3-31}$$

It should be remembered that the prescribed surface displacements, $\boldsymbol{u}^0$, are self-compatible and, so, don't include rigid displacements of the RVE. However, it should be noted that the found average value $\bar{\boldsymbol{E}}$ of the strain field $\boldsymbol{E}(\boldsymbol{x})$ is unchanged even if rigid displacements are added to the surface data. In fact, at a generic point $\boldsymbol{x}$ in the RVE, a rigid translation, $\boldsymbol{u}^r$, and a rigid-body rotation associated with the anti-symmetric, constant, infinitesimal rotation tensor, $\boldsymbol{x} \cdot \boldsymbol{R}^r$, produce an additional displacement given by $\boldsymbol{u}^r + \boldsymbol{x} \cdot \boldsymbol{R}^r$. The corresponding additional average displacement gradient is, then:

$$\begin{aligned} &\overline{\text{Ñ} \otimes \left(\boldsymbol{u}^r + \boldsymbol{x} \cdot \boldsymbol{R}^r \right)} = \\ &= < \text{Ñ} \otimes \left(\boldsymbol{u}^r + \boldsymbol{x} \cdot \boldsymbol{R}^r \right) > = \left\{ \frac{1}{V} \int_{\partial V} \boldsymbol{n}\, ds \right\} \otimes \boldsymbol{u}^r + \left\{ \frac{1}{V} \int_{\partial V} \boldsymbol{n} \otimes \boldsymbol{x}\, ds \right\} \cdot \boldsymbol{R}^r \end{aligned} \tag{1.3-32}$$

By using the Gauss theorem, it follows that:

$$\begin{aligned} &\frac{1}{V} \int_{\partial V} \boldsymbol{n}\, ds = \frac{1}{V} \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{1}^{(2)}\, ds = \frac{1}{V} \int_{V} \text{Ñ} \cdot \boldsymbol{1}^{(2)}\, dV = \mathbf{0} \\ &\frac{1}{V} \int_{\partial V} \boldsymbol{n} \otimes \boldsymbol{x}\, ds = \frac{1}{V} \int_{V} \text{Ñ} \otimes \boldsymbol{x}\, dV = \frac{1}{V} \int_{V} \boldsymbol{1}^{(2)}\, dV = \boldsymbol{1}^{(2)} \end{aligned} \tag{1.3-33}$$

where:

$\boldsymbol{1}^{(2)} =$ the second-order unit tensor

Hence, it is:

$$\overline{\text{Ñ} \otimes \left(\boldsymbol{u}^r + \boldsymbol{x} \cdot \boldsymbol{R}^r \right)} = < \text{Ñ} \otimes \left(\boldsymbol{u}^r + \boldsymbol{x} \cdot \boldsymbol{R}^r \right) > = \boldsymbol{R}^r \tag{1.3-34}$$

which doesn't affect $\bar{\boldsymbol{E}}$. Therefore, whether or not the prescribed surface displacements $\boldsymbol{u}^0$ include rigid-body translation or rotation, is of no significance in estimating the relations between the average stresses and strains or their increments. For simplicity, however, it will be assumed that the prescribed boundary displacements are self-compatible.

Moreover, it can be also found the average value of the displacements field in terms of the surface data. So, by denoting the volume average of the spatially variable and integrable quantity $\boldsymbol{u}(\boldsymbol{x})$ by:

$$\bar{\boldsymbol{u}} = <\boldsymbol{u}(\boldsymbol{x})> = \frac{1}{V}\int_V \boldsymbol{u}(\boldsymbol{x})\, dV \qquad (1.3\text{-}35)$$

and by considering that the displacements field may be written as:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})\cdot \boldsymbol{1}^{(2)} = \boldsymbol{u}(\boldsymbol{x})\cdot(\nabla\otimes\boldsymbol{x}) = \nabla\cdot(\boldsymbol{u}(\boldsymbol{x})\otimes\boldsymbol{x}) - (\nabla\cdot\boldsymbol{u}(\boldsymbol{x}))\,\boldsymbol{x} \qquad (1.3\text{-}36)$$

the average displacements field $\bar{\boldsymbol{u}}$ may be expressed in the following form:

$$\bar{\boldsymbol{u}} = <\boldsymbol{u}(\boldsymbol{x})> = \frac{1}{V}\int_V \nabla\cdot(\boldsymbol{u}(\boldsymbol{x})\otimes\boldsymbol{x}) - (\nabla\cdot\boldsymbol{u}(\boldsymbol{x}))\,\boldsymbol{x}\, dV \qquad (1.3\text{-}37)$$

By making use of the Gauss theorem, it is obtained that:

$$\bar{\boldsymbol{u}} = <\boldsymbol{u}(\boldsymbol{x})> = \frac{1}{V}\int_{\partial V} \boldsymbol{n}\cdot\left(\boldsymbol{u}^{0}(\boldsymbol{x})\otimes\boldsymbol{x}\right) ds - \frac{1}{V}\int_V (\nabla\cdot\boldsymbol{u}(\boldsymbol{x}))\,\boldsymbol{x}\, dV \qquad (1.3\text{-}38)$$

or, in components:

$$\overline{u_i} = <u_i(x)> = \frac{1}{V}\int_{\partial V} n_j u_j^{0} x_i\, ds - \frac{1}{V}\int_V u_{j,j} x_i\, dV \qquad (1.3\text{-}39)$$

which includes the volumetric strain coefficient, $\nabla\cdot\boldsymbol{u}$. So, for incompressible materials whose displacements field is divergence-free, the average displacement, $\bar{\boldsymbol{u}}$ assumes the following expression in terms of the prescribed linear surface displacements, $\boldsymbol{u}^{0}$:

$$\bar{\boldsymbol{u}} = <\boldsymbol{u}(\boldsymbol{x})> = \frac{1}{V}\int_{\partial V} \boldsymbol{n}\cdot\left(\boldsymbol{u}^{0}(\boldsymbol{x})\otimes\boldsymbol{x}\right) ds \qquad (1.3\text{-}40)$$

or, in components:

$$\overline{u_i} = < u_i\left(x\right) > = \frac{1}{V}\int_{\partial V} n_j u_j^0 x_i \, ds \tag{1.3-41}$$

Let us to assume that the following boundary equilibrium equations were satisfied:

$$\boldsymbol{E}\left(\boldsymbol{x}\right)\boldsymbol{x} = \boldsymbol{u}^0 \; on \; \partial V_u \tag{1.3-42}$$

and:

$$\boldsymbol{E^0}\,\boldsymbol{x} = \boldsymbol{u^0} \quad on \; \partial V_u \tag{1.3-43}$$

where:

$\boldsymbol{E}\left(\boldsymbol{x}\right) =$ the spatially variable strain tensor obtained from the strain prescribed problem within the volume *V* of the heterogeneous RVE.

$\boldsymbol{E^0} =$ the spatially constant strain tensor obtained from the strain prescribed problem within the volume *V* of the RVE, regarded as homogeneous.

$\boldsymbol{x} =$ the position vector of the RVE boundary points.

$\partial V_u =$ the partition of the RVE boundary where the self-compatible displacements, $\boldsymbol{u}^0$, are prescribed.

$\boldsymbol{u^0} =$ the prescribed self-compatible displacements, assumed as spatially linear on the boundary ∂V_u of the RVE.

From the equation(1.3-27), the average value of the strain tensor can be written:

$$\begin{aligned} \overline{\boldsymbol{E}} &= < \boldsymbol{E}\left(\boldsymbol{x}\right) > = \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(\boldsymbol{n} \otimes \boldsymbol{u^0} + \boldsymbol{u^0} \otimes \boldsymbol{n}\right) ds = \\ &= \frac{1}{V}\int_{\partial V} \frac{1}{2}\left(\boldsymbol{n} \otimes \left(\boldsymbol{E^0} \cdot \boldsymbol{x}\right) + \left(\boldsymbol{E^0} \cdot \boldsymbol{x}\right) \otimes \boldsymbol{n}\right) ds \end{aligned} \tag{1.3-44}$$

By using the Gauss theorem, it will be:

$$\overline{\boldsymbol{E}} = <\boldsymbol{E}(\boldsymbol{x})> = \frac{1}{V}\int_V \frac{1}{2}\left(\nabla\otimes\left(\boldsymbol{E}^{\boldsymbol{0}}\cdot\boldsymbol{x}\right)+\left(\nabla\otimes\left(\boldsymbol{E}^{\boldsymbol{0}}\cdot\boldsymbol{x}\right)\right)^T\right)ds \quad (1.3\text{-}45)$$

Hence, by operating some calculations and by considering that $\boldsymbol{E}^{\boldsymbol{0}} = \left(\boldsymbol{E}^{\boldsymbol{0}}\right)^T$, the volume average of the spatially variable and integrable quantity $\boldsymbol{E}(\boldsymbol{x})$ can be expressed in the following form:

$$\overline{\boldsymbol{E}} = \frac{1}{V}\int_V \boldsymbol{E}^0\, dV = \boldsymbol{E}^0 \quad (1.3\text{-}46)$$

For the rate problem, the average strain rate is obtained in an analogous manner to what has been done previously, by obtaining that:

$$\begin{aligned}
&\overline{\nabla\otimes\dot{\boldsymbol{u}}} = <\nabla\otimes\dot{\boldsymbol{u}}(\boldsymbol{x})> = \frac{1}{V}\int_{\partial V}\boldsymbol{n}(\boldsymbol{x})\otimes\dot{\boldsymbol{u}}^{\boldsymbol{0}}(\boldsymbol{x})\,ds \\
&\overline{\dot{\boldsymbol{E}}} = <\dot{\boldsymbol{E}}(\boldsymbol{x})> = \frac{1}{V}\int_{\partial V}\frac{1}{2}\left(\boldsymbol{n}(\boldsymbol{x})\otimes\dot{\boldsymbol{u}}^0(\boldsymbol{x})+\dot{\boldsymbol{u}}^0(\boldsymbol{x})\otimes\boldsymbol{n}(\boldsymbol{x})\right)ds \quad (1.3\text{-}47)\\
&\overline{\dot{\boldsymbol{R}}} = <\dot{\boldsymbol{R}}(\boldsymbol{x})> = \frac{1}{V}\int_{\partial V}\frac{1}{2}\left(\boldsymbol{n}(\boldsymbol{x})\otimes\dot{\boldsymbol{u}}^0(\boldsymbol{x})-\dot{\boldsymbol{u}}^0(\boldsymbol{x})\otimes\boldsymbol{n}(\boldsymbol{x})\right)ds
\end{aligned}$$

Hence, it is seen that for the small deformations the average strain rate equals the rate of change of the average strain:

$$\overline{\dot{\boldsymbol{E}}} = <\dot{\boldsymbol{E}}(\boldsymbol{x})> = \frac{d}{dt}<\boldsymbol{E}(\boldsymbol{x})> = \dot{\overline{\boldsymbol{E}}} \quad (1.3\text{-}48)$$

and similarly:

$$\begin{aligned}
&\overline{\dot{\boldsymbol{R}}} = <\dot{\boldsymbol{R}}(\boldsymbol{x})> = \frac{d}{dt}<\boldsymbol{R}(\boldsymbol{x})> = \dot{\overline{\boldsymbol{R}}} \\
&\overline{\nabla\otimes\dot{\boldsymbol{u}}} = <\nabla\otimes\dot{\boldsymbol{u}}> = \frac{d}{dt}<\nabla\otimes\boldsymbol{u}> = \overline{\nabla\otimes\boldsymbol{u}}
\end{aligned} \quad (1.3\text{-}49)$$

Moreover, another useful relation to be considered, valid for either uniform boundary tractions or linear boundary displacements [47], is:

$$<\boldsymbol{T}:\boldsymbol{E}> = <\boldsymbol{T}>:<\boldsymbol{E}> \quad (1.3\text{-}50)$$

It is important to note that this identity, according to which the average value of the product between the stress and the strain tensor is equal to the product of the average values of both mentioned tensors, is valid for materials of any constitutive properties.

As pointed out before, an RVE represents the microstructure of a macro-element (typical continuum material neighbourhood) in a continuum mass, so the stress and strain fields until now considered, that are spatially variable within the volume *V* of the RVE, can be defined as *microstress* or *microstrain* fields. In an analogous manner, the continuum stress and the strain fields, that are spatially variable in function of the position of the macro-elements within the volume *B* of the continuum solid, can be defined as *macrostress* or *macrostrain* fields, to distinguish them from the previous ones. In particular, according to what done before, denote the microstress and microstrain fields by $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x})$ and $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x})$ and the macrostress and macrostrain fields by $\Sigma = \Sigma(\boldsymbol{X})$ and $\mathrm{E} = \mathrm{E}(\boldsymbol{X})$, respectively. In particular, it has been considered the variability of such fields in function of the position vector $\boldsymbol{x}$ that describes the points position within the RVE volume and of the position vector $\boldsymbol{X}$ that describes the points position within the continuum volume. In general, these mechanical quantities are functions of the time t, too.

At a macro-scale, the macro-fields must satisfy the following continuum balance equations:

$$\nabla \cdot \Sigma(\boldsymbol{X}) = \boldsymbol{0}; \qquad \Sigma(\boldsymbol{X}) = \Sigma^{T}(\boldsymbol{X}) \ \textit{in } B \tag{1.3-51}$$

where body forces are assumed absent.

In rectangular Cartesian component form, the (1.3-51) becomes:

$$\sigma^{M}_{ij,i} = 0; \qquad \sigma^{M}_{ij} = \sigma^{M}_{ji} \ \textit{in } B \tag{1.3-52}$$

where:

$i = j = 1, 2, 3$

and where a comma followed by an index denotes partial differentiation with respect to the corresponding coordinate variable. Note that the superscript M stands for *macro*.

Moreover, the strain-displacement relation has to be verified:

$$\mathrm{E}\left(\boldsymbol{X}\right) = \frac{1}{2}\left\{\nabla \otimes \boldsymbol{U}\left(\boldsymbol{X}\right) + \left(\nabla \otimes \boldsymbol{U}\left(\boldsymbol{X}\right)\right)^{T}\right\} \; in\ B \qquad (1.3\text{-}53)$$

where ∇ is the del operator defined by:

$$\nabla = \partial_i \boldsymbol{e}_i = \frac{\partial}{\partial X_i}\boldsymbol{e}_i \qquad (1.3\text{-}54)$$

$\boldsymbol{U}\left(\boldsymbol{X}\right) =$ the macro-displacement field

and the superscript T denotes transpose.

The(1.3-53), in rectangular Cartesian component form, becomes:

$$\varepsilon_{ij}^{M} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right) \; in\ B \qquad (1.3\text{-}55)$$

In general, at a typical point $\boldsymbol{X}$ in the continuum, at a fixed time t, the values of the macrostress and macrostrain tensor, Σ and E, can be determined by the average microstress and microstrain, $\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{E}}$, over the RVE which represents the corresponding macro-element. So, in micromechanics it is assumed that:

$$\Sigma = \bar{\boldsymbol{T}}; \qquad \mathrm{E} = \bar{\boldsymbol{E}} \qquad (1.3\text{-}56)$$

Conversely, the macrostress and the macrostrain tensors, Σ and E, provide the uniform traction or the linear displacement boundary data for the RVE. Hence, when the traction boundary data are prescribed, it is:

$$\Sigma \cdot \boldsymbol{n} = \boldsymbol{t^0} \; on\ \partial V \qquad (1.3\text{-}57)$$

Analogously, when the displacements are assumed to be prescribed on the RVE boundary, it is:

$$\mathsf{E} \cdot \boldsymbol{x} = \boldsymbol{u}^{0} \;\; on \;\; \partial V \tag{1.3-58}$$

In general, the response of the macro-element characterized by relations among macrostress S and macrostrain E will be inelastic and history-dependent, even if the micro-constituents of the corresponding RVE are elastic. This is because, in the course of deformation, flaws, microcracks, cavities and other microdefects develop within the RVE and the microstructure of the RVE changes with changes of the overall applied loads. Therefore, the stress-strain relations for the macro-elements must, in general, include additional parameters which describe the current microstructure of the corresponding RVE.

So, for a typical macro-element, denote the current state of its microstructure by S, which may stand for a set of parameters, scalar or possibly tensorial, that completely defines the microstructure, for example it may stand for the sizes, orientations and distribution of its microdefects.

However, by considering a class of materials whose microconstituents are elastic (linear or non-linear) and by assuming that no change in the microstructure happens under the applied loads, the response of the macro-element will be also elastic. Hence, a Helmholtz free energy, i.e. a macrostress potential, exists and it can be written as:

$$\Phi = \Phi\left(\mathsf{E}, S\right) \tag{1.3-59}$$

which, at a constant state S, yields:

$$\mathsf{S} = \frac{\partial \Phi\left(\mathsf{E}\right)}{\partial \mathsf{E}} \tag{1.3-60}$$

Then, through the Legendre transformation:

$$\Phi\left(\mathsf{E}, S\right)+\Psi\left(\mathsf{S}, S\right)=\mathsf{S}:\mathsf{E} \tag{1.3-61}$$

a macrostrain potential can be analogously defined, as:

$$\Psi=\Psi\left(\mathsf{S}, S\right) \tag{1.3-62}$$

which, at a constant state S, yields:

$$\mathsf{E}=\frac{\partial \Psi\left(\mathsf{S}\right)}{\partial \mathsf{S}} \tag{1.3-63}$$

It is worth to underline that no thermal effects are here considered.

Once these macropotential functions are defined, it is possible to express them in terms of the volume averages of the microstress and microstrain potentials of the microconstituents. Since no thermal effects are considered, yet, and since the material within the RVE is assumed to be elastic, it admits a stress potential, $\mathsf{f}=\mathsf{f}\left(\boldsymbol{x}, \boldsymbol{E}\right)$, and a strain potential, $\mathsf{y}=\mathsf{y}\left(\boldsymbol{x}, \boldsymbol{T}\right)$, such that at a typical point it is:

$$\mathsf{f}\left(\boldsymbol{E}\right)+\mathsf{y}\left(\boldsymbol{T}\right)=\bullet:\bullet \tag{1.3-64}$$

and also:

$$\bullet=\frac{\partial \mathsf{f}\left(\boldsymbol{E}\right)}{\partial \boldsymbol{E}}; \qquad \boldsymbol{E}=\frac{\partial \mathsf{y}\left(\bullet\right)}{\partial \bullet} \tag{1.3-65}$$

As it follows, the cases of the prescribed boundary tractions and of the prescribed boundary displacements for the RVE will be considered separately, for a fixed RVE microstructure so that the dependence on S will not be displayed explicitly.

- Case of prescribed constant macrostrain

Let the RVE be subjected to linear displacements defined through a constant macrostrain E. For such a boundary-value problem, a variable microstrain field and a variable microstress one are obtained within the RVE:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right) \\ \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x},\mathrm{E}\right) \end{aligned} \qquad (1.3\text{-}66)$$

where the argument E emphasizes that a displacement boundary data with constant macrostrain, $\mathrm{E} = \bar{\boldsymbol{E}} = <\boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right)>$, is being considered. Then, the corresponding microstress potential is:

$$\mathrm{f} = \mathrm{f}\left(\boldsymbol{x},\boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right)\right) = \mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right) \qquad (1.3\text{-}67)$$

where the superscript E on f emphasizes the fact that the microstress potential is associated with the prescribed macrostrain E.

Consider now an infinitesimally small variation dE in the macrostrain which produces, consequently, a variation in the microstrain field given by:

$$\mathrm{d}\boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right) = \frac{\partial \boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right)}{\partial \mathrm{E}}\mathrm{dE} \qquad (1.3\text{-}68)$$

Then:

$$\begin{aligned} &<\boldsymbol{T}:\mathrm{d}\boldsymbol{E}> = <\frac{\partial \mathrm{f}\left(\boldsymbol{x},\boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right)\right)}{\partial \boldsymbol{E}}:\frac{\partial \boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right)}{\partial \mathrm{E}}\mathrm{dE}> = \\ &= <\frac{\partial \mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right)}{\partial \mathrm{E}}:\mathrm{dE}> = \frac{\partial}{\partial \mathrm{E}}<\mathrm{f}^{\mathrm{E}}>:\mathrm{dE} \end{aligned} \qquad (1.3\text{-}69)$$

So, by remembering the (1.3-50), it follows that:

$$<\boldsymbol{T}\left(\boldsymbol{x},\mathrm{E}\right)> = \frac{\partial}{\partial \mathrm{E}}<\mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right)> \qquad (1.3\text{-}70)$$

Therefore, by defining the macrostress potential as:

$$\Phi^{\mathrm{E}} = \Phi^{\mathrm{E}}\left(\mathrm{E}\right) = <\mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right)> = \frac{1}{V}\int_V \mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right)dV \qquad (1.3\text{-}71)$$

and the corresponding macrostress field (as before) by:

$$\mathrm{S}^{\mathrm{E}} = \mathrm{S}^{\mathrm{E}}\left(\mathrm{E}\right) = \bar{\boldsymbol{T}} = <\boldsymbol{T}\left(\boldsymbol{x},\mathrm{E}\right)> \qquad (1.3\text{-}72)$$

it is obtained that:

$$\mathsf{S}^{\mathsf{E}}\left(\mathsf{E}\right)=\frac{\partial\Phi^{\mathsf{E}}}{\partial\mathsf{E}}\left(\mathsf{E}\right) \tag{1.3-73}$$

where the superscript E on S emphasizes that Φ^{E} and S^{E} are, respectively the macrostress potential, i.e. the volume average of the microstress potential, and the macrostress field, i.e. the volume average of the microstress field, obtained at a macroscale by the constant prescribed macrostrain E.

- Case of prescribed constant macrostress

Let the RVE be subjected to uniform tractions defined through a constant macrostress S. For such a boundary-value problem, a variable microstrain field and a variable microstress one are obtained within the RVE:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x},\mathsf{S}\right) \\ \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x},\mathsf{S}\right) \end{aligned} \tag{1.3-74}$$

where the argument S emphasizes that a traction boundary data with constant macrostress, $\mathsf{S}=\overline{\boldsymbol{T}}=<\boldsymbol{T}\left(\boldsymbol{x},\mathsf{S}\right)>$, is being considered. Then, the corresponding microstrain potential is:

$$\mathsf{y}=\mathsf{y}\left(\boldsymbol{x},\boldsymbol{T}\left(\boldsymbol{x},\mathsf{S}\right)\right)=\mathsf{y}^{\mathsf{S}}\left(\boldsymbol{x},\mathsf{S}\right) \tag{1.3-75}$$

where the superscript S on y emphasizes the fact that the microstrain potential is associated with the prescribed macrostress S.

Consider now an arbitrary change dS in the macrostress which produces, consequently, a change in the microstress field given by:

$$\mathsf{d}\boldsymbol{T}\left(\boldsymbol{x},\mathsf{S}\right)=\frac{\partial\boldsymbol{T}\left(\boldsymbol{x},\mathsf{S}\right)}{\partial\mathsf{S}}\mathsf{dS} \tag{1.3-76}$$

Then:

$$
\begin{aligned}
&< \mathrm{d}\boldsymbol{T} : \boldsymbol{E} > = < \frac{\partial \boldsymbol{T}(\boldsymbol{x}, \mathrm{S})}{\partial \mathrm{S}} \mathrm{dS} : \frac{\partial \mathrm{y}\left(\boldsymbol{x}, \boldsymbol{T}(\boldsymbol{x}, \mathrm{S})\right)}{\partial \boldsymbol{T}} > = \\
&= < \mathrm{dS} : \frac{\partial \mathrm{y}^{\mathrm{S}}(\boldsymbol{x}, \mathrm{S})}{\partial \mathrm{S}} > = \mathrm{dS} : \frac{\partial}{\partial \mathrm{S}} < \mathrm{y}^{\mathrm{S}} >
\end{aligned}
\tag{1.3-77}
$$

So, by remembering the (1.3-50), it follows that:

$$
< \boldsymbol{E}(\boldsymbol{x}, \mathrm{S}) > = \frac{\partial}{\partial \mathrm{S}} < \mathrm{y}^{\mathrm{S}}(\boldsymbol{x}, \mathrm{S}) > \tag{1.3-78}
$$

Therefore, by defining the macrostrain potential as:

$$
\Psi^{\mathrm{S}} = \Psi^{\mathrm{S}}(\mathrm{S}) = < \mathrm{y}^{\mathrm{S}}(\boldsymbol{x}, \mathrm{S}) > = \frac{1}{V} \int_V \mathrm{y}^{\mathrm{S}}(\boldsymbol{x}, \mathrm{S})\, dV \tag{1.3-79}
$$

and the corresponding macrostrain field (as before) by:

$$
\mathrm{E}^{\mathrm{S}} = \mathrm{E}^{\mathrm{S}}(\mathrm{S}) = \bar{\boldsymbol{E}} = < \boldsymbol{E}(\boldsymbol{x}, \mathrm{S}) > \tag{1.3-80}
$$

it is obtained that:

$$
\mathrm{E}^{\mathrm{S}}(\mathrm{S}) = \frac{\partial \Psi^{\mathrm{S}}}{\partial \mathrm{S}}(\mathrm{S}) \tag{1.3-81}
$$

where the superscript S on E emphasizes that Ψ^{S} and E^{S} are, respectively the macrostrain potential, i.e. the volume average of the microstrain potential, and the macrostrain field, i.e. the volume average of the microstrain field, obtained at a macroscale by the constant prescribed macrostress S.

Define, now, a new macrostress potential function:

$$
\Phi^{\mathrm{S}} = \Phi^{\mathrm{S}}\left(\mathrm{E}^{\mathrm{S}}\right) = \mathrm{S} : \mathrm{E}^{\mathrm{S}} - \Psi^{\mathrm{S}}(\mathrm{S}) \tag{1.3-82}
$$

where the superscript S emphasizes that the corresponding quantity is obtained for prescribed macrostress S.

On the other hand, at the local level, the microstress and the microstrain potential can be expressed, respectively, in the following form:

$$
\begin{aligned}
\varphi^{\Sigma} &= \varphi\left(\boldsymbol{x}, \boldsymbol{E}\left(\boldsymbol{x}, \Sigma\right)\right) = \varphi^{\Sigma}\left(\boldsymbol{x}, \Sigma\right) \\
\psi^{\Sigma} &= \psi\left(\boldsymbol{x}, \boldsymbol{T}\left(\boldsymbol{x}, \Sigma\right)\right) = \psi^{\Sigma}\left(\boldsymbol{x}, \Sigma\right)
\end{aligned}
\tag{1.3-83}
$$

and hence:

$$
\varphi^{\Sigma} + \psi^{\Sigma} = \boldsymbol{T}\left(\boldsymbol{x}, \Sigma\right) : \boldsymbol{E}\left(\boldsymbol{x}, \Sigma\right) \tag{1.3-84}
$$

The volume average over the volume V of the RVE yields:

$$
< \varphi^{\Sigma} > + < \psi^{\Sigma} > = \Sigma : \mathrm{E}^{\Sigma} \tag{1.3-85}
$$

The comparison of the (1.3-85) with the (1.3-82), by taking into account th e(1.3-79), shows that:

$$
\Phi^{\Sigma}\left(\mathrm{E}^{\Sigma}\right) = < \varphi^{\Sigma}\left(\boldsymbol{x}, \Sigma\right) > \tag{1.3-86}
$$

Moreover, by remembering the relation (1.3-81), it is also deduced that:

$$
\Sigma = \frac{\partial \Phi^{\Sigma}}{\partial \mathrm{E}^{\Sigma}}\left(\mathrm{E}^{\Sigma}\right) \tag{1.3-87}
$$

In a similar manner, when the macrostrain E is prescribed through linear boundary displacements, a new macrostrain potential function may be defined:

$$
\Psi^{\mathrm{E}} = \Psi^{\mathrm{E}}\left(\Sigma^{\mathrm{E}}\right) = \Sigma^{\mathrm{E}} : \mathrm{E} - \Phi^{\mathrm{E}}\left(\mathrm{E}\right) \tag{1.3-88}
$$

where the superscript E emphasizes that the corresponding quantity is obtained for prescribed macrostrain E.

On the other hand, at the local level, the microstress and the microstrain potential can be expressed, respectively, in the following form:

$$
\begin{aligned}
\varphi^{\mathrm{E}} &= \varphi\left(\boldsymbol{x}, \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{E}\right)\right) = \varphi^{\mathrm{E}}\left(\boldsymbol{x}, \mathrm{E}\right) \\
\psi^{\mathrm{E}} &= \psi\left(\boldsymbol{x}, \boldsymbol{T}\left(\boldsymbol{x}, \mathrm{E}\right)\right) = \psi^{\mathrm{E}}\left(\boldsymbol{x}, \mathrm{E}\right)
\end{aligned}
\tag{1.3-89}
$$

and hence:

$$
\varphi^{\mathrm{E}} + \psi^{\mathrm{E}} = \boldsymbol{T}\left(\boldsymbol{x}, \mathrm{E}\right) : \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{E}\right) \tag{1.3-90}
$$

The volume average over the volume V of the RVE yields:

$$< \mathrm{f}^{\mathrm{E}} > + < \mathrm{y}^{\mathrm{E}} > = \mathrm{S}^{\mathrm{E}} : \mathrm{E} \tag{1.3-91}$$

The comparison of the (1.3-91) with the (1.3-88), by taking into account the **Errore. L'origine riferimento non è stata trovata.**, shows that:

$$\Psi^{\mathrm{E}}\left(\mathrm{S}^{\mathrm{E}}\right) = < \mathrm{y}^{\mathrm{E}}\left(\boldsymbol{x}, \mathrm{E}\right) > \tag{1.3-92}$$

Moreover, by remembering the relation (1.3-73), it is also deduced that:

$$\mathrm{E} = \frac{\partial \Psi^{\mathrm{E}}}{\partial \mathrm{S}^{\mathrm{E}}}\left(\mathrm{S}^{\mathrm{E}}\right) \tag{1.3-93}$$

At this point, it is useful to make an important consideration.

When the boundary tractions are given by:

$$\boldsymbol{t}^{0} = \boldsymbol{n} \cdot \mathrm{S} \quad on\ \partial V \tag{1.3-94}$$

the microstress and the microstrain fields, as considered before, are:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x}, \mathrm{S}\right) \\ \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{S}\right) \end{aligned} \tag{1.3-95}$$

hence, the overall macrostrain is:

$$\mathrm{E}^{\mathrm{S}} = < \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{S}\right) > \tag{1.3-96}$$

Now, suppose that boundary displacements are defined for this obtained macrostrain by:

$$\boldsymbol{u^0} = \boldsymbol{x} \cdot \mathrm{E}^{\mathrm{S}} \ on\ \partial V \tag{1.3-97}$$

the resulting microstress and microstrain fields are:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x}, \mathrm{E}^{\mathrm{S}}\right) \\ \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{E}^{\mathrm{S}}\right) \end{aligned} \tag{1.3-98}$$

In general, these fields are not identical with those ones shown in the equation (1.3-95). Furthermore, while it is:

$$\mathrm{E}^{\mathrm{S}} = < \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{E}^{\mathrm{S}}\right) > \tag{1.3-99}$$

there is no a priori reason that $< \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}^{\mathsf{S}}\right) >$ should be equal to S for an arbitrary heterogeneous elastic solid.

So, the RVE is regarded as *statistically representative* of the macroresponse of the continuum material neighbourhood if and only if any arbitrary constant macrostress S produces, through the (1.3-94), a macrostrain $\mathsf{E}^{\mathsf{S}} = < \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{S}\right) >$ such that when the displacement boundary conditions (1.3-97) are imposed, then the obtained macrostress must verify the following relation [47]:

$$< \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}^{\mathsf{S}}\right) > \; ; \; \mathsf{S} \tag{1.3-100}$$

where the equality is to hold to a given degree of accuracy.

Conversely, when the boundary displacements are given by:

$$\boldsymbol{u}^{0} = \mathsf{E}\boldsymbol{x} \quad on \ \partial V \tag{1.3-101}$$

the microstress and the microstrain fields, now, are:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}\right) \\ \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{E}\right) \end{aligned} \tag{1.3-102}$$

hence, the overall macrostress is:

$$\mathsf{S}^{\mathsf{E}} = < \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}\right) > \tag{1.3-103}$$

Now, let us suppose that boundary tractions are defined for this obtained macrostress by:

$$\boldsymbol{t}^{\boldsymbol{0}} = \boldsymbol{n} \cdot \mathsf{S}^{\mathsf{E}} \quad on \ \partial V \tag{1.3-104}$$

the resulting microstress and microstrain fields are:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{S}^{\mathsf{E}}\right) \\ \boldsymbol{E} &= \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{S}^{\mathsf{E}}\right) \end{aligned} \tag{1.3-105}$$

In general, these fields are not identical with those ones shown in the equation (1.3-102). Furthermore, while it is:

$$\mathsf{S}^{\mathsf{E}} = < \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{S}^{\mathsf{E}}\right) > \tag{1.3-106}$$

there is no a priori reason that $< \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{S}^{\mathsf{E}}\right) >$ should be equal to E for an arbitrary heterogeneous elastic solid.

So, analogously, the RVE is regarded as *statistically representative* of the macroresponse of the continuum material neighbourhood if and only if any arbitrary constant macrostrain E produces, through the (1.3-101), a macrostress $\mathsf{S}^{\mathsf{E}} = < \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}\right) >$ so that when the traction boundary conditions (1.3-104) are imposed, then the obtained macrostrain must verify the following relation [47]:

$$< \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{S}^{\mathsf{E}}\right) > \; ; \; \mathsf{E} \tag{1.3-107}$$

where the equality is to hold to a given degree of accuracy.

Based on the above given definitions for an RVE, then, the macrostrain potential $\Psi^{\Sigma}\left(\mathsf{S}\right)$ and the macrostress potential $\Phi^{\mathsf{E}}\left(\mathsf{E}\right)$ correspond to each other in the sense that:

$$\frac{\partial \Psi^{\Sigma}}{\partial \mathsf{S}}\left(\mathsf{S}\right) \; ; \; \mathsf{E} \Longleftrightarrow \frac{\partial \Phi^{\mathsf{E}}}{\partial \mathsf{E}}\left(\mathsf{E}\right) \; ; \; \mathsf{S} \tag{1.3-108}$$

and in accordance with the Legendre transformation, it is:

$$\Psi^{\Sigma}\left(\mathsf{S}\right) + \Phi^{\mathsf{E}}\left(\mathsf{E}\right) \; ; \; \mathsf{S} : \mathsf{E} \tag{1.3-109}$$

It should be noted, however, that even for S and E which satisfies the (1.3-108), it will be:

$$\begin{aligned} &\boldsymbol{T}\left(\boldsymbol{x}, \mathsf{S}\right) \neq \boldsymbol{T}\left(\boldsymbol{x}, \mathsf{E}\right) \\ &\boldsymbol{E}\left(\boldsymbol{x}, \mathsf{S}\right) \neq \boldsymbol{E}\left(\boldsymbol{x}, \mathsf{E}\right) \end{aligned} \tag{1.3-110}$$

Moreover, in general, it is:

$$\mathrm{y}^{\Sigma}\left(\boldsymbol{x},\mathrm{S}\right)+\mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{E}\right)\neq\bullet\ \left(\boldsymbol{x},\mathrm{S}\right):\boldsymbol{E}\left(\boldsymbol{x},\mathrm{E}\right) \tag{1.3-111}$$

Similarly, the macropotentials $\Psi^{\mathrm{E}}\left(\mathrm{S}^{\mathrm{E}}\right)$ and $\Phi^{\mathrm{S}}\left(\mathrm{E}^{\mathrm{S}}\right)$ correspond to each other in the sense that:

$$\frac{\partial\Psi^{\mathrm{E}}}{\partial\mathrm{S}^{\mathrm{E}}}\left(\mathrm{S}^{\mathrm{E}}\right)\ ;\ \mathrm{E}^{\mathrm{S}}\Longleftrightarrow\frac{\partial\Phi^{\mathrm{S}}}{\partial\mathrm{E}^{\mathrm{S}}}\left(\mathrm{E}^{\mathrm{S}}\right)\ ;\ \mathrm{S}^{\mathrm{E}} \tag{1.3-112}$$

and in accordance with the Legendre transformation, it is:

$$\Psi^{\mathrm{E}}\left(\mathrm{S}^{\mathrm{E}}\right)+\Phi^{\mathrm{S}}\left(\mathrm{E}^{\mathrm{S}}\right)\ ;\ \mathrm{S}^{\mathrm{E}}:\mathrm{E}^{\mathrm{S}} \tag{1.3-113}$$

whereas, the corresponding micropotentials don’t satisfy a similar relation, i.e.:

$$\mathrm{y}^{\mathrm{E}}\left(\boldsymbol{x},\mathrm{S}^{\mathrm{E}}\right)+\mathrm{f}^{\mathrm{S}}\left(\boldsymbol{x},\mathrm{E}^{\mathrm{S}}\right)\neq\bullet\ \left(\boldsymbol{x},\mathrm{E}\right):\boldsymbol{E}\left(\boldsymbol{x},\mathrm{S}\right) \tag{1.3-114}$$

The following table 1.1 provides a summary of the results presented in this section:

	PRESCRIBED Σ	**PRESCRIBED** E
microstress	$\boldsymbol{T}(\boldsymbol{x},\Sigma)$	$\boldsymbol{T}(\boldsymbol{x},E)$
microstrain	$\boldsymbol{E}(\boldsymbol{x},\Sigma)$	$\boldsymbol{E}(\boldsymbol{x},E)$
macrostress	$\Sigma = \langle \boldsymbol{T}(\boldsymbol{x},\Sigma) \rangle$	$\Sigma^{E} = \langle \boldsymbol{T}(\boldsymbol{x},E) \rangle$
macrostrain	$E^{\Sigma} = \langle \boldsymbol{E}(\boldsymbol{x},\Sigma) \rangle$	$E = \langle \boldsymbol{E}(\boldsymbol{x},E) \rangle$
microstress potential	$\varphi(\boldsymbol{x},\boldsymbol{E}(\boldsymbol{x},\Sigma)) = \varphi^{\Sigma}(\boldsymbol{x},\Sigma)$ $\boldsymbol{T}(\boldsymbol{x},\Sigma) = \dfrac{\partial \varphi(\boldsymbol{x},\boldsymbol{E}(\boldsymbol{x},\Sigma))}{\partial \boldsymbol{E}}$	$\varphi(\boldsymbol{x},\boldsymbol{E}(\boldsymbol{x},E)) = \varphi^{E}(\boldsymbol{x},E)$ $\boldsymbol{T}(\boldsymbol{x},E) = \dfrac{\partial \varphi(\boldsymbol{x},\boldsymbol{E}(\boldsymbol{x},E))}{\partial \boldsymbol{E}}$
microstrain potential	$\psi(\boldsymbol{x},\boldsymbol{T}(\boldsymbol{x},\Sigma)) = \psi^{\Sigma}(\boldsymbol{x},\Sigma)$ $\boldsymbol{E}(\boldsymbol{x},\Sigma) = \dfrac{\partial \psi(\boldsymbol{x},\boldsymbol{T}(\boldsymbol{x},\Sigma))}{\partial \boldsymbol{T}}$	$\psi(\boldsymbol{x},\boldsymbol{T}(\boldsymbol{x},E)) = \psi^{E}(\boldsymbol{x},E)$ $\boldsymbol{E}(\boldsymbol{x},E) = \dfrac{\partial \psi(\boldsymbol{x},\boldsymbol{T}(\boldsymbol{x},E))}{\partial \boldsymbol{T}}$
macrostress potential	$\Phi^{\Sigma} = \Phi^{\Sigma}(E^{\Sigma}) = \langle \varphi^{\Sigma} \rangle$ $\Sigma = \dfrac{\partial \Phi^{\Sigma}}{\partial E^{\Sigma}}(E^{\Sigma})$	$\Phi^{E} = \Phi^{E}(E) = \langle \varphi^{E} \rangle$ $\Sigma^{E} = \dfrac{\partial \Phi^{E}}{\partial E}(E)$
macrostrain potential	$\Psi^{\Sigma} = \Psi^{\Sigma}(\Sigma) = \langle \psi^{\Sigma} \rangle$ $E^{\Sigma} = \dfrac{\partial \Psi^{\Sigma}}{\partial \Sigma}(\Sigma)$	$\Psi^{E} = \Psi^{E}(\Sigma^{E}) = \langle \psi^{E} \rangle$ $E^{\Sigma} = \dfrac{\partial \Psi^{E}}{\partial \Sigma^{E}}(\Sigma^{E})$
microlegendre transformation	$\varphi^{\Sigma} + \psi^{\Sigma} = \boldsymbol{T}(\boldsymbol{x},\Sigma) : \boldsymbol{E}(\boldsymbol{x},\Sigma)$	$\varphi^{E} + \psi^{E} = \boldsymbol{T}(\boldsymbol{x},E) : \boldsymbol{E}(\boldsymbol{x},E)$
macrolegendre transformation	$\Phi^{\Sigma} + \Psi^{\Sigma} = \Sigma : E^{\Sigma}$	$\Phi^{E} + \Psi^{E} = \Sigma^{E} : E$
approximated macrolegendre transformation	$\Phi^{E}(E) + \Psi^{\Sigma}(\Sigma) \approx \Sigma : E$	$\Phi^{\Sigma}(E^{\Sigma}) + \Psi^{E}(\Sigma^{E}) \approx \Sigma^{E} : E^{\Sigma}$
corresponding microlegendre transformation	$\varphi^{E} + \psi^{\Sigma} \neq \boldsymbol{T}(\boldsymbol{x},\Sigma) : \boldsymbol{E}(\boldsymbol{x},E)$	$\varphi^{\Sigma} + \psi^{E} \neq \boldsymbol{T}(\boldsymbol{x},E) : \boldsymbol{E}(\boldsymbol{x},\Sigma)$

Table 1.1 Relation between macro and micro quantities for prescribed macrostress and macrostrain.

1.4 Elasticity, groups of symmetry, anisotropic solids with fourth rank tensors

The heterogeneous materials can be characterized by both inhomogeneity and anisotropy, since the first aspect is due to the multi-phase composition of the medium, while the second one is due to the geometrical arrangement of the different constituents within the examined heterogeneous volume.

In the previous sections, it has been analyzed the first aspect.

In this section, the constitutive relations for anisotropic materials, in linear-elasticity, are presented [64].

A linear anisotropic elastic material, as known, can have as many as 21 elastic constants. However, this number can be opportunely reduced when the examined material possesses certain material symmetry. Moreover, it is also reduced, in most cases, when a two-dimensional deformation is considered. It is worth to remember that the matrices of the elastic constants must be positive definite, because the strain energy must be positive.

Hence, referring to a fixed rectangular coordinate system $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, let $\boldsymbol{T}$ and $\boldsymbol{E}$ be the stress and the strain fields, respectively, in an anisotropic pier-elastic material. The stress-strain relation can be written in the following form:

$$\boldsymbol{T} = \mathbb{C} : \boldsymbol{E} \qquad (1.4\text{-}1)$$

or, in components:

$$\sigma_{ij} = C_{ijhk} \varepsilon_{hk} \qquad (1.4\text{-}2)$$

where:

$\mathbb{C} =$ fourth rank elastic stiffness tensor

and where, for the hypothesis of iper-elasticity, the components C_{ijhk} satisfy the following conditions of full symmetry:

$$C_{ijhk} = C_{jihk} = C_{hkij} \qquad (1.4\text{-}3)$$

The above written equation (1.4-3) groups in it the following equalities:

$$C_{ijhk} = C_{jihk} = C_{ijkh} = C_{jikh} \tag{1.4-4}$$

and

$$C_{ijhk} = C_{hkij} \tag{1.4-5}$$

where the (1.4-4) follows directly from the symmetry of the stress and the strain tensors, while the (1.4-5) is due to the assuming hypothesis of existence of the elastic potential φ, [64]. In other word, the strain energy φ per unit volume of the material, given by:

$$\varphi = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} \tag{1.4-6}$$

is independent of the loading path, i.e. the path that ε_{ij} takes from 0 to ε while it depends on the final value of ε, only.

In linear elasticity, the (1.4-6) may be written as:

$$\varphi = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijhk}\varepsilon_{ij}\varepsilon_{hk} \tag{1.4-7}$$

and since the strain energy must be positive, it has to be:

$$C_{ijhk}\varepsilon_{ij}\varepsilon_{hk} > 0 \tag{1.4-8}$$

for any real, non zero, symmetric tensor ε_{ij}.

Hence, as said before, the stiffness tensor $\mathbb{C}$ is defined positive.

Analogously, the stress-strain relation can be written in the following form, inverse of (1.4-1):

$$\boldsymbol{E} = \mathbb{S} : \boldsymbol{T} \tag{1.4-9}$$

or, in components:

$$\varepsilon_{ij} = S_{ijhk}\sigma_{hk} \tag{1.4-10}$$

where:

$\mathbf{S} =$ fourth rank elastic compliance tensor

and where, for the hypothesis of iper-elasticity, the components S_{ijhk} satisfy the following conditions of full symmetry:

$$S_{ijhk} = S_{jihk} = S_{hkij} \tag{1.4-11}$$

The above written equation (1.4-11) groups in it the following equalities:

$$S_{ijhk} = S_{jihk} = S_{ijkh} = S_{jikh} \tag{1.4-12}$$

and

$$S_{ijhk} = S_{hkij} \tag{1.4-13}$$

where the (1.4-12) follows directly from the symmetry of the stress and the strain tensors, while the (1.4-13) is due to the assuming hypothesis of existence of the elastic complementary potential ψ, [64]. In other word, the stress energy ψ per unit volume of the material, given by:

$$\psi = \int_0^{\sigma} \varepsilon_{ij} d\sigma_{ij} \tag{1.4-14}$$

is independent of the loading path, i.e. the path that σ_{ij} takes from 0 to σ while it depends on the final value of σ, only.

In linear elasticity, the (1.4-14) may be written as:

$$\psi = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}S_{ijhk}\sigma_{ij}\sigma_{hk} \tag{1.4-15}$$

and since the stress energy must be positive, it has to be:

$$S_{ijhk}\sigma_{ij}\sigma_{hk} > 0 \tag{1.4-16}$$

for any real, non zero, symmetric tensor σ_{ij}.

Hence, as said before, the compliance tensor $\mathbf{S}$ is defined positive.

Introducing, now, the contract notation, [36], such that:

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3,$$
$$\sigma_{32} = \sigma_4, \quad \sigma_{31} = \sigma_5, \quad \sigma_{12} = \sigma_6,$$
$$\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3,$$
$$2\varepsilon_{32} = \varepsilon_4, \quad 2\varepsilon_{31} = \varepsilon_5, \quad 2\varepsilon_{12} = \varepsilon_6, \tag{1.4-17}$$

the stress-strain laws (1.4-2) and (1.4-10) may be written, respectively, as:

$$\sigma_\alpha = C_{\alpha\beta}\varepsilon_\beta, \quad C_{\alpha\beta} = C_{\beta\alpha} \tag{1.4-18}$$

and

$$\varepsilon_\alpha = S_{\alpha\beta}\sigma_\beta, \quad S_{\alpha\beta} = S_{\beta\alpha} \tag{1.4-19}$$

With reference, in particular, to the equation (1.4-18), it may be expressed in a matrix form, as it follows:

$$\boldsymbol{T} = \mathbb{C} : \boldsymbol{E}, \quad \mathbb{C} = \mathbb{C}^T \tag{1.4-20}$$

The stress and the strain tensors, $\boldsymbol{T}$ and $\boldsymbol{E}$, are expressed in form of 6x1 column matrices, while the stiffness tensor $\mathbb{C}$ is expressed in form of 6x6 symmetric matrix, as given in the following equation:

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & Sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \tag{1.4-21}$$

where the transformation between C_{ijhk} and $C_{\alpha\beta}$ is accomplished by replacing the subscripts *ij* (or *hk*) by α or β, by using the following rules:

$$
\begin{array}{ccc}
ij\,(or\,hk) & \leftrightarrow & \mathsf{a}\,(or\,\mathsf{b}) \\
11 & \leftrightarrow & 1 \\
22 & \leftrightarrow & 2 \\
33 & \leftrightarrow & 3 \\
32\ or\ 23 & \leftrightarrow & 4 \\
31\ or\ 13 & \leftrightarrow & 5 \\
12\ or\ 21 & \leftrightarrow & 6
\end{array}
\qquad (1.4\text{-}22)
$$

We may write the transformation (1.4-22) as:

$$
\begin{aligned}
\mathsf{a} &= \begin{cases} i & if\ i = j \\ 9 - i - j & if\ i \neq j \end{cases} \\
\mathsf{b} &= \begin{cases} h & if\ h = k \\ 9 - h - k & if\ h \neq k \end{cases}
\end{aligned}
\qquad (1.4\text{-}23)
$$

Analogously, with reference to the equation (1.4-17), the stress-strain law (1.4-19) may be expressed in a matrix form, as it follows:

$$
\boldsymbol{E} = \mathsf{S} : \boldsymbol{T}, \qquad \mathsf{S} = \mathsf{S}^T \qquad (1.4\text{-}24)
$$

where also the compliance tensor S is expressed in form of 6x6 symmetric matrix, as given by:

$$
\mathsf{S} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
 & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
 & & S_{33} & S_{34} & S_{35} & S_{36} \\
 & & & S_{44} & S_{45} & S_{46} \\
 & Sym & & & S_{55} & S_{56} \\
 & & & & & S_{66}
\end{bmatrix} \qquad (1.4\text{-}25)
$$

Here, the transformation between S_{ijhk} and S_{ab} is similar to that one between C_{ijhk} and C_{ab} except the following:

$$
\begin{aligned}
S_{ijhk} &= S_{\alpha\beta} && \textit{if both } \alpha,\beta \le 3 \\
2S_{ijhk} &= S_{\alpha\beta} && \textit{if either } \alpha \textit{ or } \beta \le 3 \\
4S_{ijhk} &= S_{\alpha\beta} && \textit{if both } \alpha,\beta > 3
\end{aligned}
\tag{1.4-26}
$$

From (1.4-20) and (1.4-24), it is obtained the expression of the strain energy, as:

$$\varphi = \frac{1}{2}\boldsymbol{E}^T \mathrm{C} \boldsymbol{E} = \frac{1}{2}\boldsymbol{T}^T \boldsymbol{E} = \frac{1}{2}\boldsymbol{T}^T \mathrm{S} \boldsymbol{T} \tag{1.4-27}$$

and, by considering that φ has to be positive, it must be:

$$
\begin{aligned}
&\boldsymbol{E}^T \mathrm{C} \boldsymbol{E} > 0 \\
&\boldsymbol{T}^T \mathrm{S} \boldsymbol{T} > 0
\end{aligned}
\tag{1.4-28}
$$

This implies that the matrices C and S are both positive definite. Moreover, the substitution of the (1.4-24) in the (1.4-20) yields:

$$\mathrm{CS} = \mathrm{I} = \mathrm{SC} \tag{1.4-29}$$

where the second equality follows from the first one which says that C and S are the inverses of each other and, hence, their product commute.

For a linear anisotropic elastic material, like it has been anticipated before, the matrices C and S have 21 elastic independent constants. However, this number can be reduced when a two-dimensional deformation is considered.

Assume, therefore, the deformation of the examined anisotropic elastic bodies to be a two-dimensional one for which $\varepsilon_3 = 0$. When $\varepsilon_3 = 0$, the stress-strain law given by the first equation of (1.4-18) becomes:

$$\sigma_\alpha = \sum_{\beta \ne 3} C_{\alpha\beta}\varepsilon_\beta \qquad \alpha = 1,2,3,....,6 \qquad \beta = 1,2,....,6 \tag{1.4-30}$$

Ignoring the equation for σ_3, the (1.4-30) may be written as:

$$\hat{\boldsymbol{T}} = \hat{\mathrm{C}}\hat{\boldsymbol{E}} \qquad \hat{\mathrm{C}} = \hat{\mathrm{C}}^T \tag{1.4-31}$$

where:

$$\hat{\boldsymbol{T}}^T = [\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_4, \mathrm{s}_5, \mathrm{s}_6] \tag{1.4-32}$$

$$\hat{\boldsymbol{E}}^T = [\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_4, \mathrm{e}_5, \mathrm{e}_6] \tag{1.4-33}$$

and:

$$\hat{\mathrm{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{24} & C_{25} & C_{26} \\ & & C_{44} & C_{45} & C_{46} \\ Sym & & & C_{55} & C_{56} \\ & & & & C_{66} \end{bmatrix} \tag{1.4-34}$$

Since $\hat{\mathrm{C}}$ is obtained from C by deleting the third row and the third column of it, $\hat{\mathrm{C}}$ is a principal submatrix of C and it also is positive definite. It contains 15 independent elastic constants.

The stress-strain law (1.4-19) for $\mathrm{e}_3 = 0$ is:

$$\mathrm{e}_3 = 0 = S_{3\mathrm{b}} \mathrm{s}_{\mathrm{b}} \tag{1.4-35}$$

Solving for s_3, it is:

$$\mathrm{s}_3 = -\frac{1}{S_{33}} \sum_{\mathrm{b} \neq 3} S_{3\mathrm{b}} \mathrm{s}_{\mathrm{b}} \tag{1.4-36}$$

and by sobstituting the (1.4-36) within the first equation of the (1.4-19), it is obtained:

$$\mathrm{e}_{\mathrm{a}} = \sum_{\mathrm{b} \neq 3} S'_{\mathrm{ab}} \mathrm{s}_{\mathrm{b}} \tag{1.4-37}$$

with:

$$S'_{\mathrm{ab}} = S_{\mathrm{ab}} - \frac{S_{\mathrm{a}3} S_{3\mathrm{b}}}{S_{33}} = S'_{\mathrm{ba}} \tag{1.4-38}$$

where:

$S'_{\alpha\beta}$ = *reduced elastic compliances*.

It is clear, moreover, that:

$$S'_{\alpha 3} = 0, \quad S'_{3\beta} = 0 \qquad \alpha, \beta = 1, 2, ..., 6 \tag{1.4-39}$$

For this reason, there is no need to exclude $\beta = 3$ in the (1.4-37).

By using the notation of the (1.4-32) and (1.4-33), the (1.4-37) can be written in the following form:

$$\hat{\boldsymbol{E}} = \mathsf{S}'\hat{\boldsymbol{T}} \qquad \mathsf{S}' = \mathsf{S}'^{T} \tag{1.4-40}$$

where S' can be defined as *reduced elastic compliance tensor* and it has a symmetric matrix form, given by:

$$\mathsf{S}' = \begin{bmatrix} S'_{11} & S'_{12} & S'_{14} & S'_{15} & S'_{16} \\ & S'_{22} & S'_{24} & S'_{25} & S'_{26} \\ & & S'_{44} & S'_{45} & S'_{46} \\ Sym & & & S'_{55} & S'_{56} \\ & & & & S'_{66} \end{bmatrix} \tag{1.4-41}$$

Like $\hat{\mathsf{C}}$, S' contains 15 independent elastic constants. Moreover, the substitution of the (1.4-40) in the (1.4-31) yields:

$$\hat{\mathsf{C}}\mathsf{S}' = \mathsf{I} = \mathsf{S}'\hat{\mathsf{C}} \tag{1.4-42}$$

where the second equality follows from the first one which says that $\hat{\mathsf{C}}$ and S' are the inverses of each other and, hence, their product commute. This result is independent of whether $\varepsilon_3 = 0$ or not, because it represents a property of elastic constants, [64]. It has to be underlined that the positive definite of $\hat{\mathsf{C}}$ implies that S' is also positive definite.

An alternate proof that $\hat{\mathsf{C}}$ and S' are positive definite is to write the strain energy as:

$$\phi = \frac{1}{2}\hat{\boldsymbol{E}}^T \hat{\mathrm{C}} \hat{\boldsymbol{E}} = \frac{1}{2}\hat{\boldsymbol{T}}^T \hat{\boldsymbol{E}} = \frac{1}{2}\hat{\boldsymbol{T}}^T \mathrm{S}' \hat{\boldsymbol{T}} \tag{1.4-43}$$

and to consider that ϕ has to be positive for any nonzero $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{E}}$, so it must be:

$$\begin{aligned} &\hat{\boldsymbol{E}}^T \hat{\mathrm{C}} \hat{\boldsymbol{E}} > 0 \\ &\hat{\boldsymbol{T}}^T \mathrm{S}' \hat{\boldsymbol{T}} > 0 \end{aligned} \tag{1.4-44}$$

As anticipated at the beginning of this section, the number of the independent elastic constants of the 6x6 matrices C and S can be opportunely reduced, yet, when the examined anisotropic material possesses certain material symmetry.

Hence, with reference to a new rectangular coordinate system $\boldsymbol{e}^*{}_1, \boldsymbol{e}^*{}_2, \boldsymbol{e}^*{}_3$, obtained from the initial fixed one $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ under an orthogonal transformation:

$$\boldsymbol{e}^* = \Omega \boldsymbol{e} \tag{1.4-45}$$

or, in components:

$$e^*{}_i = \Omega_{ij} e_j \tag{1.4-46}$$

in which Ω is an orthogonal matrix that satisfies the following relations:

$$\Omega\Omega^T = \boldsymbol{I} = \Omega^T\Omega \tag{1.4-47}$$

or:

$$\Omega_{ij}\Omega_{kj} = \delta_{ik} = \Omega_{ji}\Omega_{jk} \tag{1.4-48}$$

a material is said to possess a *symmetry* with respect to Ω if the elastic fourth rank stiffness tensor C^* referred to the $e^*{}_i$ coordinate system is equal to that one C referred to the e_i coordinate system, i.e.:

$$\mathsf{C}^* = \mathsf{C} \qquad (1.4\text{-}49)$$

or in components:

$$C^*_{ijhk} = C_{ijhk} \qquad (1.4\text{-}50)$$

where:

$$C^*_{ijhk} = \Omega_{ip}\Omega_{jq}\Omega_{hr}\Omega_{ks}C_{pqrs} \qquad (1.4\text{-}51)$$

An identical equation can be written for S_{ijhk}.

In other words, when:

$$C_{ijhk} = \Omega_{ip}\Omega_{jq}\Omega_{hr}\Omega_{ks}C_{pqrs} \qquad (1.4\text{-}52)$$

the material possesses a symmetry with respect to Ω.

The transformation law (1.4-51) is referred for the C_{ijhk}, but, for simplicity of the calculations, some authors adopt the transformation law for C_{ab}, [64]:

$$C^*_{\mathrm{ab}} = K_{\mathrm{a}r}K_{\mathrm{b}t}C_{rt} \qquad (1.4\text{-}53)$$

where:

$K =$ a 6x6 matrix, whose elements are obtained by means of suitable assembly of the components Ω_{ij}, according to proposals by Mehrabadi, Cowin et al (1995), [43] and Mehrabadi and Cowin (1990), [42].

Then, an anisotropic material possesses the symmetry of *central inversion* (C) if the (1.4-52) is satisfied for:

$$\mathsf{W} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\boldsymbol{I} \qquad (1.4\text{-}54)$$

It is obvious that the (1.4-52) is satisfied by the W given in the (1.4-54) for any C_{ijhk}. Therefore, all the anisotropic materials have the symmetry of central inversion.

If Ω is a proper orthogonal matrix, the transformation (1.4-45) represents a rigid body rotation about an axis. So, an anisotropic material is said to possess a *rotational symmetry* if the (1.4-52) is satisfied for:

$$\Omega^{(r)}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.4-55}$$

which represents, for example, a rotation about the $\boldsymbol{e}_3$-axis an angle θ, as shown in the following figure.

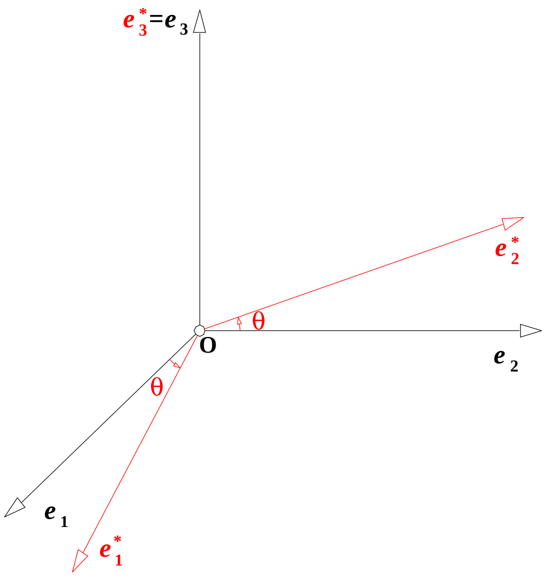


Figure 1.5 Rigid rotation about the e_3-axis.

By extending this property, i.e. if the (1.4-52) is satisfied by the W as given through the (1.4-55) for any q, then the material possesses a rotational symmetry with respect at any rotation in the $\boldsymbol{e}_3 = \mathbf{0}$ plane. In this case, it is said that the $\boldsymbol{e}_3 = \mathbf{0}$ is the *plane of transverse isotropy* or that $\boldsymbol{e}_3$ is *axis of elastic symmetry* with order $p = \infty$ (L^{∞}). More in general, instead, by indicating with:

$$\mathrm{q} = \frac{2\mathrm{p}}{p} \tag{1.4-56}$$

the rotation angle about an axis, this latter is defined as *axis of elastic symmetry* with order p. Since p may assume values equal to $2, 3, 4, 6$ and ∞, the axis of elastic symmetry has indicated, respectively, with L^2, L^3, L^4, L^6 and L^{∞}.

If W is, instead, an orthogonal matrix as defined below:

$$\mathrm{W} = \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\boldsymbol{T}} \tag{1.4-57}$$

where:

$\boldsymbol{n} =$ a unit vector

then, the transformation (1.4-45) represents a reflection about a plane whose normal is $\boldsymbol{n}$, defined as *reflection plane or symmetry plane* (*P*). In particular, if $\boldsymbol{m}$ is any vector on the plane, the following relation is satisfied:

$$\mathrm{W}\boldsymbol{n} = -\boldsymbol{n}, \qquad \mathrm{W}\boldsymbol{m} = \boldsymbol{m} \tag{1.4-58}$$

According to a such orthogonal matrix, therefore, a vector normal to the reflection plane reverses its direction after the transformation while a vector belonging to the reflection plane remains unchanged.

So, an anisotropic material is said to possess a *symmetry plane* if the (1.4-52) is satisfied by the Ω as given by (1.4-57). For example, consider:

$$\boldsymbol{n}^T = [\cos\theta, \sin\theta, 0] \tag{1.4-59}$$

i.e. the symmetry plane contains the $\boldsymbol{e}_3$-axis and its normal vector makes an angle of θ with the $\boldsymbol{e}_1$-axis, as shown in the following figure.

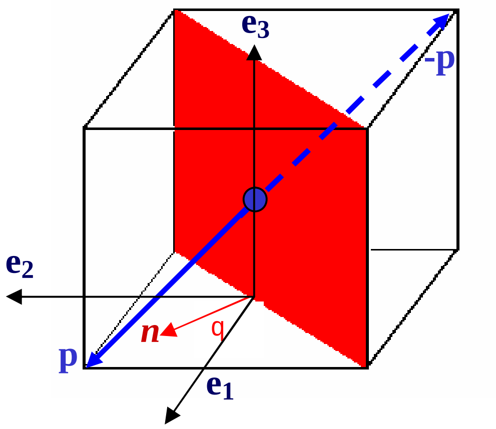


Figure 1.6 Reflection about a plane containing the e_3-axis.

The orthogonal matrix Ω of the (1.4-57), so, has the following expression:

$$\Omega(\theta) = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -\frac{\pi}{2} < \theta \le \frac{\pi}{2} \tag{1.4-60}$$

which is an improper orthogonal matrix and represents a reflection with respect to a plane whose normal is on the $(\boldsymbol{e_1}, \boldsymbol{e_2})$ plane. Since θ and $\theta + \pi$ represent the same plane, θ is limited to the range shown in (1.4-60). In the particular case that $\theta = 0$, Ω becomes:

$$\Omega(\theta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.4-61}$$

which represents a reflection about the plane $\boldsymbol{e}_1 = 0$. Hence, an anisotropic material for which the (1.4-52) is satisfied by the Ω as given through the (1.4-61) is said to possess a *symmetry plane at* $\boldsymbol{e}_1 = 0$. By extending this property, i.e. if the (1.4-52) is satisfied by the Ω as given through the (1.4-60) for any θ, then the material possesses a symmetry plane with respect at any plane that contains the $\boldsymbol{e}_3$-axis. In this case, it is said that the $\boldsymbol{e}_3$-axis is the *axis of symmetry* (*L*).

In analogous manner, it is considered, in the following equation, the expression of an orthogonal matrix which represents a reflection with respect to a plane whose normal is on the $(\boldsymbol{e_2}, \boldsymbol{e_3})$ plane, making an angle φ with the $\boldsymbol{e_2}$-axis:

$$\Omega(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos 2\varphi & -\sin 2\varphi \\ 0 & -\sin 2\varphi & \cos 2\varphi \end{bmatrix} -\frac{\pi}{2} < \varphi \le \frac{\pi}{2} \tag{1.4-62}$$

In particular, the symmetry plane $\boldsymbol{e_2} = \mathbf{0}$ can be represented by either $q = \frac{p}{2}$ or $j = 0$, while the symmetry plane $\boldsymbol{e_3} = \mathbf{0}$ can be represented by $j = \frac{p}{2}$, as shown in the following figure:

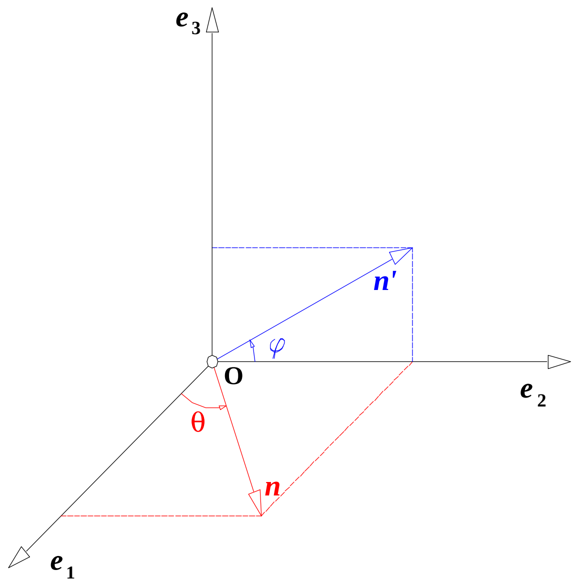


Figure 1.7 The vectors n and n’ are, respectively, the normal vectors to planes of reflection symmetry defined by the (1.4-60) and (1.4-62)

The existence of various combinations of the different symmetry forms implies a corresponding classification of the anisotropy classes of the materials. In particular, two extreme cases of anisotropic elastic materials are the *triclinic* materials and the *isotropic* ones. The first material possesses no rotational symmetry or a plane of reflection symmetry, while the second material possesses infinitely many rotational symmetries and planes of reflection symmetry. For such materials, it can be shown that [64]:

$$C_{ijhk} = \lambda \delta_{ij}\delta_{hk} + G\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right) \tag{1.4-63}$$

where λ and G are the Lamè constants, satisfies the (1.4-52) for any orthogonal Ω.

It is possible to demonstrate that if an anisotropic elastic material possesses a material symmetry with the orthogonal matrix Ω, then it also possesses the material symmetry with $\Omega^{T} = \Omega^{-1}$. This means, for example, that if the material has rotational symmetry with rotation about the x_3-axis an angle θ, it also has the symmetry about the x_3-axis an angle $-\theta$. Moreover, it is possible to demonstrate, yet, that if an anisotropic elastic material possesses a symmetry with Ω' and Ω'', then it also possesses a symmetry with $\Omega = \Omega'\Omega''$, [64]. These statements, valid either for linear or nonlinear material, are useful in determining the structure of the stiffness tensor when the material possesses symmetries.

Depending on the number of rotations and/or reflection symmetry a crystal possesses, Voigt (1910) in fact classified crystals into 32 classes. However, in terms of the 6x6 matrix $\mathbf{C}$, there are only 8 basic groups, since different combinations of symmetry forms may lead to the same structure of the stiffness tensor, [36]. This classification maid for crystals can be extended for non-crystalline materials, so that for them the structure of $\mathbf{C}$ can also be represented by one of the 8 basic groups.

Without loss in generality, in the follows, the list of such groups of materials are presented by choosing the symmetry plane (or planes) to coincide with the coordinate planes whenever possible. If the matrix $\mathbf{C}^*$ referred to a different coordinate system is desired, the (1.4-51) is used to obtain it.

- **Triclinic materials**

They represent the most general case, in which no symmetry form exists. The number of independent constants is, therefore, 21 and the matrix C assumes the following form:

$$\mathsf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & Sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 21 \qquad (1.4\text{-}64)$$

which is equal to that one of the equation (1.4-21).

- **Monoclinic materials**

The symmetry forms are: L^2, P, L^2PC; The number of the independent elastic constants is 13 and the matrix C assumes the following expressions:

a) Symmetry plane coinciding with $\boldsymbol{e_1} = \mathbf{0}$, i.e., $\mathsf{q} = 0$

$$\mathsf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 13 \qquad (1.4\text{-}65)$$

b) Symmetry plane coinciding with $\boldsymbol{e_2} = \mathbf{0}$, i.e., $\mathsf{q} = \frac{\mathsf{p}}{2}$ or $\mathsf{j} = 0$:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{22} & C_{23} & 0 & C_{25} & 0 \\ & & C_{33} & 0 & C_{35} & 0 \\ & & & C_{44} & 0 & C_{46} \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 13 \qquad (1.4\text{-}66)$$

c) Symmetry plane coinciding with $\boldsymbol{e_3} = \mathbf{0}$, i.e., $\varphi = \frac{\pi}{2}$:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 13 \qquad (1.4\text{-}67)$$

- **Orthotropic (or Rhombic) materials**

The symmetry forms are: $3P, 3L^2, L^2 2P, 3L^2 3PC$; With reference to the symmetry form 3*P*, it means that the three coordinate planes, $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{2}$ are the symmetry planes. The number of the independent elastic constants is 9 and the matrix $\mathbf{C}$ assumes the following form:

$$\mathrm{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 9 \qquad (1.4\text{-}68)$$

- **Trigonal materials**

The symmetry forms are: $L^3 3L^2, L^3 3P, L_6^3 3L^2 3PC$; With reference to the symmetry form 3*P*, it is verified that the three coordinate planes, $q = 0$, $q = +\frac{p}{3}$ and $q = -\frac{p}{3}$ are the symmetry planes. The number of the independent elastic constants is 6 and the matrix C assumes the following form:

$$\mathrm{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & C_{14} \\ & & & & & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{bmatrix} \quad n^0 = 6 \qquad (1.4\text{-}69)$$

- **Tetragonal materials**

The symmetry forms are: $L^4, L^4 PC, L_4^2$; It is verified that the tetragonal materials show five symmetry planes at $q = 0$, $q = +\frac{p}{4}$, $q = -\frac{p}{4}$, $q = +\frac{p}{2}$

and $\varphi = +\frac{\pi}{2}$. The number of the independent elastic constants is 6 and the matrix $\mathbf{C}$ assumes the following form:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad n^0 = 6 \qquad (1.4\text{-}70)$$

- **Transversely isotropic (or exagonal) materials**

The symmetry forms are:

$L^3P, L^3 3L^2 4P, L^6, L^6 6L^2, L^6PC, L^6 6P, L^6 6L^2 7PC;$

For the transversely isotropic materials the symmetry planes are $\varphi = \frac{\pi}{2}$, i.e. $\left(\boldsymbol{e_3} = \mathbf{0}\right)$, and any plane that contains the $\boldsymbol{e_3}$-axis. So, the $\boldsymbol{e_3}$-axis is the axis of symmetry. The number of the independent elastic constants is 5 and the matrix $\mathbf{C}$ assumes the following form:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & 0 \\ & & & & & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{bmatrix} \quad n^0 = 5 \qquad (1.4\text{-}71)$$

- **Cubic materials**

The symmetry forms are

$$3L^2 4L^3, 3L^2 4L_6^3 3PC, 3L_4^2 4L^3 6P, 3L^4 4L^3 6L^2, 3L^4 4L_6^3 6L^2 9PC;$$

For the cubic materials there are nine symmetry planes, whose normal vectors are on the three coordinate axes and on the coordinate planes making an angle $\frac{\pi}{4}$ with coordinate axes. The number of the independent elastic constants is 3 and the matrix C assumes the following form:

$$\mathsf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix} n^0 = 3 \qquad (1.4\text{-}72)$$

- **Isotropic materials**

For the isotropic materials any plane is a symmetry plane. The number of the independent elastic constants is 2 and the matrix C assumes the following form:

$$\mathbf{C}=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 & 0 \\ & Sym & & & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 \\ & & & & & \frac{1}{2}\left(C_{11}-C_{12}\right) \end{bmatrix} \quad n^0=2 \qquad (1.4\text{-}73)$$

If λ and G are the Lamè constants, the (1.4-73) assumes the expression given by:

$$\mathbf{C}=\begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2G+\lambda & \lambda & 0 & 0 & 0 \\ & & 2G+\lambda & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & Sym & & & G & 0 \\ & & & & & G \end{bmatrix} \quad n^0=2 \qquad (1.4\text{-}74)$$

It is remarkable that, for isotropic materials, it needs only three planes of symmetry to reduce the number of elastic constants from 21 to 2.

The following figure shows the hierarchical organization of the eight material symmetries of linear elasticity. It is organized so that the lower symmetries are at the upper left and, as one moves down and across the table to the right, one encounters crystal systems with greater and greater symmetry.

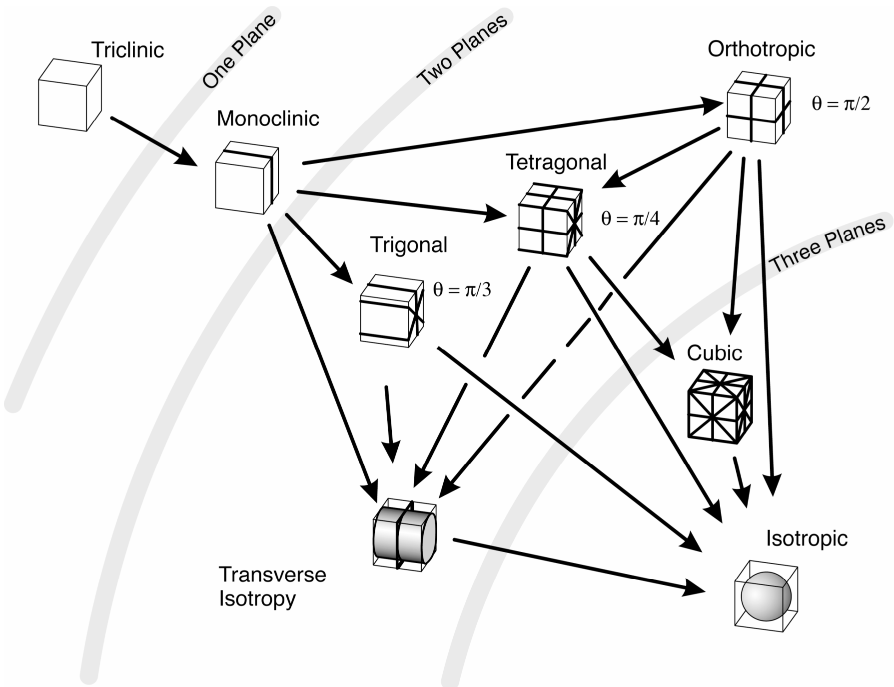


Figure 1.8 Hierarchical organization of the eight material symmetries of linear elasticity

It is worth to underline that the structure of the matrix C above obtained for each class of materials is referred to the specified coordinate system. When different coordinate systems are employed, the transformation law (1.4-51) has to be used for obtaining the structure of the new matrix C, in which, while the number of nonzero elements may increase, the number of independent elastic constants remains constant since it does not depend on the choice of the coordinate systems. In the applications, the choice of the coordinate system is very often dictated by the boundary conditions of the problem and hence it may not coincide with the symmetry planes of the material. In these cases, the transformation of the matrix C to a different coordinates system becomes necessary.

The analysis until here presented for obtaining the structure of the stiffness tensor $\mathbf{C}$ may be applied analogously for obtaining the structure of the compliance tensor $\mathbf{S}$. Like $\mathbf{C}$, the elastic compliance tensor $\mathbf{S}$ is a fourth rank tensor and, under the orthogonal transformation (1.4-45), its components, S^*_{ijhk}, referred to a new coordinate system are related to those ones, S_{ijhk}, referred to the initial coordinate system by:

$$S^*_{ijhk} = \Omega_{ip}\Omega_{jq}\Omega_{hr}\Omega_{ks}S_{pqrs} \tag{1.4-75}$$

which is identical to (1.4-52).

Hence, the structure of the matrix $\mathbf{C}$ appearing in (1.4-64)-(1.4-73) remains valid for the matrix $\mathbf{S}$ with the following modifications required by (1.4-26):

- The relation:

$$C_{56} = -C_{24} = C_{14} \tag{1.4-76}$$

in the (1.4-69) is replaced by:

$$\frac{1}{2}S_{56} = -S_{24} = S_{14} \tag{1.4-77}$$

and the elastic coefficient C_{66} in (1.4-69), (1.4-71) and (1.4-73) is replaced by:

$$S_{66} = 2\left(S_{11} - S_{12}\right) \tag{1.4-78}$$

In engineering applications the matrix $\mathbf{S}$ for isotropic materials is written as:

$$\mathbf{S} = \frac{1}{E}\begin{bmatrix} 1 & -n & -n & 0 & 0 & 0 \\ & 1 & -n & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+n) & 0 & 0 \\ & Sym & & & 2(1+n) & 0 \\ & & & & & 2(1+n) \end{bmatrix} \tag{1.4-79}$$

where:

$$E = \frac{G\left(3\lambda + 2G\right)}{\lambda + G}, \qquad \nu = \frac{\lambda}{2\left(\lambda + G\right)} \tag{1.4-80}$$

are, respectively, the Young's modulus and the Poisson ratio. It can be shown that:

$$\lambda = \frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}, \qquad G = \frac{E}{2\left(1+\nu\right)} \tag{1.4-81}$$

For obtaining the structure of the elastic reduced compliance tensor $\mathbf{S}'$, the same considerations are valid with some modifications required by the (1.4-38) . Hence, for example, the expression of $\mathbf{S}'$ for isotropic materials is the following one:

$$\mathbf{S}' = \frac{1}{2G}\begin{bmatrix} 1-\nu & -\nu & 0 & 0 & 0 \\ & 1-\nu & 0 & 0 & 0 \\ & & 2 & 0 & 0 \\ & & & 2 & 0 \\ & & & & 2 \end{bmatrix} \quad n^0 = 2 \tag{1.4-82}$$

Like stated previously, the strong convexity condition which is equivalent to the positive definiteness of the strain energy, (1.4-8), yields that the stiffness tensor $\mathbf{C}$ is defined positive, as well as, the positive definiteness of the stress energy, (1.4-16), yields that the compliance tensor $\mathbf{S}$ is defined positive. In particular, in the contracted notation, the (1.4-8) implies that the 6x6 matrix $\mathbf{C}$ is also positive definite and, so, all its principal minors are positive, i.e.:

$$C_{ii} > 0 \;\; \left(i \;\; not\, summed\right) \tag{1.4-83}$$

$$\begin{vmatrix} C_{ii} & C_{ij} \\ C_{ij} & C_{jj} \end{vmatrix} > 0 \;\; \left(i, j \;\; not\, summed\right) \tag{1.4-84}$$

$$\begin{vmatrix} C_{ii} & C_{ij} & C_{ih} \\ C_{ij} & C_{jj} & C_{jh} \\ C_{ih} & C_{jh} & C_{hh} \end{vmatrix} > 0 \; \left(i, j, k \;\; not\, summed \right) \tag{1.4-85}$$

$$\vdots \tag{1.4-86}$$

where i, j, h are distinct integers which can have any value from 1 to 6.

In particular, according to the theorem which states that a real symmetric matrix is positive definite if and only if its *leading* principal minors are positive, the *necessary* and *sufficient conditions* for the 6x6 matrix C to be positive definite are the positivity of its 6 *leading* principal minors, i.e.:

$$C_{11} > 0 \; \left(i \;\; not\, summed \right) \tag{1.4-87}$$

$$\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} > 0 \; \left(i, j \;\; not\, summed \right) \tag{1.4-88}$$

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix} > 0 \; \left(i, j, k \;\; not\, summed \right) \tag{1.4-89}$$

$$\vdots \tag{1.4-90}$$

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{vmatrix} > 0 \tag{1.4-91}$$

Analogously, the (1.4-16) implies that the 6x6 matrix S is also positive definite and, so, also its 6 *leading* principal minors are positive. The same consideration can be applied for the matrices $\hat{\mathsf{C}}$ and S'. By imposing these conditions of positivity on the minors of the matrices, the restrictions on the elastic coefficients can be found.

The above done anisotropic classification of the materials according to the number of symmetry planes is based on the assumption that, for each material, the number and the locations of the symmetry planes are known. However, this is not the case when considering an unknown material. So, often, the elastic stiffnesses and the elastic compliances of the material have to be determined to an arbitrarily chosen coordinate system. The result is that, if there exists a symmetry plane, it may not be one of the coordinate planes. Consequently, all elements of the matrices C and S can be nonzero. The problem is to locate the symmetry planes if they exist when C (or S) is given.

When a plane of symmetry exists, as already seen, the (1.4-52) is satisfied by the Ω given in (1.4-57), which has the properties given in (1.4-58) where $\boldsymbol{n}$ is a unit vector normal to the plane symmetry and $\boldsymbol{m}$ is any vector perpendicular to $\boldsymbol{n}$. Cowin and Mehrabadi (1987) have demonstrated that a set of necessary and sufficient conditions for $\boldsymbol{n}$ to be a unit normal vector to a plane of symmetry is, [64]:

$$C_{ijhh} n_j = \left(C_{pqss} n_p n_q \right) n_i \tag{1.4-92}$$

$$C_{ikhk} n_h = \left(C_{pqrq} n_p n_q \right) n_i \tag{1.4-93}$$

$$C_{ijhk} n_j n_k n_h = \left(C_{pqrs} n_p n_q n_r n_s \right) n_i \tag{1.4-94}$$

$$C_{ijhk} m_j m_k m_h = \left(C_{pqrs} n_p m_q n_r m_s \right) n_i \tag{1.4-95}$$

For example, if the plane $\boldsymbol{e_1} = \mathbf{0}$ is considered as plane of symmetry, by substituting in the (1.4-92)-(1.4-95) the vectors $\boldsymbol{n}$ and $\boldsymbol{m}$, defined as:

$$n_i = \delta_{i1}, \qquad m_i = \delta_{i2}\cos\theta + \delta_{i3}\sin\theta \tag{1.4-96}$$

where θ is an arbitrary constant, the independent elastic constants are obtained, according to the (1.4-65) if using the contracted notation.

More in general, the equations (1.4-92)-(1.4-95) tell that $\boldsymbol{n}$ is an eigenvector of the 3x3 symmetric matrices U, V, $Q(\boldsymbol{n})$ and $Q(\boldsymbol{m})$ whose elements are:

$$U_{ij} = C_{ijhh}, \quad V_{ih} = C_{ikhk}, \quad Q_{ih}(\boldsymbol{n}) = C_{ijhk} n_j n_k \tag{1.4-97}$$

and it is stated, here, a modified Cowin-Mehrabadi theorem, as it follows:

- An anisotropic elastic material with given elastic stiffnesses C_{ijhk} has a plane of symmetry if and only if $\boldsymbol{n}$ is an eigenvector of $Q(\boldsymbol{n})$ and $Q(\boldsymbol{m})$, or of U and $Q(\boldsymbol{m})$ or of V and $Q(\boldsymbol{m})$. The vector $\boldsymbol{n}$ is normal to the plane of symmetry, while $\boldsymbol{m}$ is any vector on the plane of symmetry.

Since this theorem is not suitable for determining $\boldsymbol{n}$ because the matrix $Q(\boldsymbol{m})$ depends on $\boldsymbol{m}$ which, in turns, depends on $\boldsymbol{n}$, another theorem is used for computing $\boldsymbol{n}$:

- An anisotropic elastic material with given elastic stiffnesses C_{ijhk} has a plane of symmetry if and only if $\boldsymbol{n}$ (normal vector to the plane of symmetry) is a common eigenvector of U and V and satisfies:

$$C_{ijhk} m_i n_j n_h n_k = 0 \tag{1.4-98}$$

$$C_{ijhk} m_i m_j m_h n_k = 0 \tag{1.4-99}$$

for any two independent vectors $\boldsymbol{m}^{\alpha}$ $(\alpha = 1, 2)$ on the plane of symmetry that do not form an angle a multiple of $\pi/3$.

For the physical interpretation of these matrices and for the demonstration of such theorems, the reader is referred to [64].

1.5 Overall elastic modulus and compliance tensors

In this section, an RVE of volume V bounded by ∂V is considered, which consists of a uniform elastic matrix having elasticity and compliance tensors C^{M} and S^{M}, containing n elastic micro-inclusions with volume Ω^{α}, having elasticity and compliance tensors C^{α} and S^{α} $(\alpha = 1, 2, \ldots, n)$. It is assumed that the micro-inclusions are perfectly bonded to the matrix. All the constituents of the RVE are assumed to be linearly elastic. Hence, the overall response of the RVE is linearly elastic, too. The matrix and each inclusion are assumed to be homogeneous, but neither the matrix nor the inclusions need be isotropic. In general, the overall response of the RVE may be anisotropic, even if its constituents are isotropic. This depends on the geometry and arrangement of the micro-inclusions.

The overall elasticity and compliance tensors of the RVE, denoted by $\bar{\mathsf{C}}$ and $\bar{\mathsf{S}}$, respectively, are, in the follows, estimated in terms of the RVE's micro-structural properties and geometry. As done previously, the cases of a prescribed macrostress and a prescribed macrostrain are considered separately.

- Case of prescribed constant macrostress

For the constant macrostress $\mathsf{S} = \mathsf{S}^{0}$, the boundary tractions are:

$$\boldsymbol{t}^{0} = \boldsymbol{n} \cdot \mathsf{S}^{0} \quad on \ \partial V \tag{1.5-1}$$

Because of the heterogeneity, neither the resulting stress nor the resulting strain fields in the RVE are uniform. *Define* the constant strain field $\boldsymbol{E^0}$ by:

$$\boldsymbol{E^0} \equiv \mathrm{S}^M : \mathrm{S}^0 \tag{1.5-2}$$

and observe that the actual stress field, denoted by $\boldsymbol{T}(\boldsymbol{x})$, and the actual strain field, denoted by $\boldsymbol{E}(\boldsymbol{x})$, can be expressed as:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{x}) &= \mathrm{S}^{\boldsymbol{0}} + \boldsymbol{T^d}(\boldsymbol{x}) \\ \boldsymbol{E}(\boldsymbol{x}) &= \boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x}) \end{aligned} \tag{1.5-3}$$

where the variable stress and strain fields, $\boldsymbol{T^d}(\boldsymbol{x})$ and $\boldsymbol{E^d}(\boldsymbol{x})$, are the disturbances or perturbations in the prescribed uniform stress field $\mathrm{S}^{\boldsymbol{0}}$ and the associated constant strain field $\boldsymbol{E^0}$, due to the presence of the inclusions.

Hence, the total stress and strain tensors, $\boldsymbol{T}$ and $\boldsymbol{E}$, are related by Hooke's law, as it follows:

$$\boldsymbol{T}(\boldsymbol{x}) = \mathrm{S}^{\boldsymbol{0}} + \boldsymbol{T^d}(\boldsymbol{x}) = \begin{cases} \mathrm{C}^M : \boldsymbol{E}(\boldsymbol{x}) = \mathrm{C}^M : \left\{\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})\right\} & in\, M = V - \Omega \\ \mathrm{C}^{\mathrm{a}} : \boldsymbol{E}(\boldsymbol{x}) = \mathrm{C}^{\mathrm{a}} : \left\{\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})\right\} & in\, \Omega^{\mathrm{a}} \end{cases} \tag{1.5-4}$$

and:

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x}) = \begin{cases} \mathrm{S}^M : \boldsymbol{T}(\boldsymbol{x}) = \mathrm{S}^M : \left\{\mathrm{S}^{\boldsymbol{0}} + \boldsymbol{T^d}(\boldsymbol{x})\right\} & in\, M = V - \Omega \\ \mathrm{S}^{\mathrm{a}} : \boldsymbol{T}(\boldsymbol{x}) = \mathrm{S}^{\mathrm{a}} : \left\{\mathrm{S}^{\boldsymbol{0}} + \boldsymbol{T^d}(\boldsymbol{x})\right\} & in\, \Omega^{\mathrm{a}} \end{cases} \tag{1.5-5}$$

where:

$\Omega = \bigcup_{\mathrm{a}=1}^{n} \Omega^{\mathrm{a}}$ = the total volume of all micro-inclusions

$M =$ matrix volume

From the averaging theorems, discussed in Section 1.3, and according to the (1.5-1), it follows that:

$$\bar{\boldsymbol{T}} = <\boldsymbol{T}(\boldsymbol{x})> = \mathsf{S}^0 \tag{1.5-6}$$

On the other hand, the overall average strain field is given by:

$$\bar{\boldsymbol{E}} = <\boldsymbol{E}(\boldsymbol{x})> = <\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})> \tag{1.5-7}$$

i.e., in general, for a prescribed macro-stress, it is:

$$<\boldsymbol{E^d}(x)> \neq 0 \tag{1.5-8}$$

The goal is to calculate the overall compliance tensor, $\bar{\mathsf{S}}$, such that:

$$\bar{\boldsymbol{E}} = \bar{\mathsf{S}} : \bar{\boldsymbol{T}} = \bar{\mathsf{S}} : \mathsf{S}^0 \tag{1.5-9}$$

In order to do it, obtain the average value of the strain field over each micro-inclusion as:

$$\bar{\boldsymbol{E}}^{\mathrm{a}} = <\boldsymbol{E}(\boldsymbol{x})>_{\mathrm{a}} = \frac{1}{\Omega^{\mathrm{a}}} \int_{\Omega^{\mathrm{a}}} \boldsymbol{E}(\boldsymbol{x})\, dV \tag{1.5-10}$$

and the average value of the stress field over each micro-inclusion is:

$$\bar{\boldsymbol{T}}^{\mathrm{a}} = <\boldsymbol{T}(\boldsymbol{x})>_{\mathrm{a}} = \frac{1}{\Omega^{\mathrm{a}}} \int_{\Omega^{\mathrm{a}}} \boldsymbol{T}(\boldsymbol{x})\, dV \tag{1.5-11}$$

In similar manner, the average value of the strain field over the matrix material is obtained as:

$$\bar{\boldsymbol{E}}^{M} = <\boldsymbol{E}(\boldsymbol{x})>_{M} = \frac{1}{M} \int_{M} \boldsymbol{E}(\boldsymbol{x})\, dV \tag{1.5-12}$$

and the average value of the stress field over the matrix material is obtained as:

$$\bar{\boldsymbol{T}}^{M} = <\boldsymbol{T}(\boldsymbol{x})>_{M} = \frac{1}{M} \int_{M} \boldsymbol{T}(\boldsymbol{x})\, dV \tag{1.5-13}$$

The volume average of the (1.5-5) over the matrix and the inclusions yields:

$$\begin{aligned} \bar{\boldsymbol{E}}^{M} &= \mathsf{S}^{M} : \bar{\boldsymbol{T}}^{M} \\ \bar{\boldsymbol{E}}^{\mathrm{a}} &= \mathsf{S}^{\mathrm{a}} : \bar{\boldsymbol{T}}^{\mathrm{a}} \end{aligned} \quad (\mathrm{a}\ \textit{not summed}) \tag{1.5-14}$$

Since:

$$f_M \bar{\boldsymbol{E}}^M = \bar{\boldsymbol{E}} - \sum_{a=1}^{n} f_a \bar{\boldsymbol{E}}^a = \bar{S} : S^0 - \sum_{a=1}^{n} f_a S^a : \bar{\boldsymbol{T}}^a \tag{1.5-15}$$

and:

$$f_M \bar{\boldsymbol{E}}^M = f_M S^M : \bar{\boldsymbol{T}}^M = S^M : \left\{ S^{\boldsymbol{0}} - \sum_{a=1}^{n} f_a \bar{\boldsymbol{T}}^a \right\} \tag{1.5-16}$$

then, it is obtained that:

$$\left(S^M - \bar{S} \right) : S^0 = \sum_{a=1}^{n} f_a \left(S^M - S^a \right) : \bar{\boldsymbol{T}}^a = \sum_{a=1}^{n} f_a \left(S^M - S^a \right) : < S^{\boldsymbol{0}} + \boldsymbol{T}^{\boldsymbol{d}} \left(\boldsymbol{x} \right) >_a \tag{1.5-17}$$

where:

$f_a = \dfrac{\Omega^a}{V} =$ the volume fraction of the $a\,th$ inclusion

$f_M = \dfrac{M}{V} =$ the volume fraction of the matrix

The (1.5-17) leads to an *exact* result. It defines the overall compliance tensor $\bar{S}$ in terms of the average stresses in the inclusions. It is important to note that this result does not require the knowledge of the entire stress field within each inclusion: only the estimate of the average value of it in each inclusion is needed.

Since the overall response is linearly elastic, the disturbances or perturbations in the stress and strain fields due to the presence of the inclusions, $\boldsymbol{T}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$ and $\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$, are linear and homogeneous function of the prescribed constant macro-stress S^0. So, in general:

$$\left(S^a - S^M \right) : < S^{\boldsymbol{0}} + \boldsymbol{T}^{\boldsymbol{d}} \left(\boldsymbol{x} \right) >_a = \left(S^a - S^M \right) : \bar{\boldsymbol{T}}^a = H^a : S^{\boldsymbol{0}} \tag{1.5-18}$$

where the constant fourth-order H^a tensor is defined by:

$$\bar{\boldsymbol{E}}^a - S^M : \bar{\boldsymbol{T}}^a = < \boldsymbol{E}^{\boldsymbol{0}} + \boldsymbol{E}^{\boldsymbol{d}} \left(\boldsymbol{x} \right) >_a - S^M : < S^{\boldsymbol{0}} + \boldsymbol{T}^{\boldsymbol{d}} \left(\boldsymbol{x} \right) >_a = H^a : S^{\boldsymbol{0}} \tag{1.5-19}$$

This is the change in the average strain field of Ω^{a} if S^{a} is replaced by S^{M}.

Since S^{0} is arbitrary, the substitution of the (1.5-18) in the (1.5-17) yields:

$$\overline{S} = S^{M} + \sum_{a=1}^{n} f_{a} H^{a} \tag{1.5-20}$$

which is an *exact* result, yet. It applies to a finite as well as infinitely extended RVE. There is no restriction on the geometry (i.e. shapes) or distribution of the inclusions. The only requirements are that the matrix as well as each inclusion are linearly elastic and homogeneous and that the inclusions are perfectly bonded to the matrix. However, approximations and specializations are generally introduced for obtaining the constant tensors $H^{a}\left(a = 1, 2, ...n\right)$. In fact, in order to estimate such tensor, an usually used approximation is that one to assume the inclusions to be ellipsoidal.

- Case of prescribed constant macrostrain

For the constant macrostrain $E = E^{0}$, the boundary conditions for the RVE are:

$$\boldsymbol{u^{0}} = \boldsymbol{x} \cdot E^{0} \ \ on \ \partial V \tag{1.5-21}$$

Define the constant strain field $\boldsymbol{T^{0}}$ by:

$$\boldsymbol{T^{0}} \equiv C^{M} : E^{0} \tag{1.5-22}$$

and observe that the actual stress field, denoted by $\boldsymbol{T}\left(\boldsymbol{x}\right)$, and the actual strain field, denoted by $\boldsymbol{E}\left(\boldsymbol{x}\right)$, can be expressed as:

$$\begin{aligned} \boldsymbol{T}\left(\boldsymbol{x}\right) &= \boldsymbol{T^{0}} + \boldsymbol{T^{d}}\left(\boldsymbol{x}\right) \\ \boldsymbol{E}\left(\boldsymbol{x}\right) &= E^{0} + \boldsymbol{E^{d}}\left(\boldsymbol{x}\right) \end{aligned} \tag{1.5-23}$$

where the variable stress and strain fields, $\boldsymbol{T}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$ and $\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$, are the disturbances or perturbations in the prescribed uniform strain field $\mathsf{E}^{\boldsymbol{0}}$ and the associated constant stress field $\boldsymbol{T}^{\boldsymbol{0}}$, due to the presence of the inclusions with different elasticity, i.e. of the existence of a material mismatch.

The total stress-strain relations are given by (1.5-4) and (1.5-5).

From the averaging theorems, discusses in the Section 1.3, and according to the (1.5-21), it follows that:

$$\bar{\boldsymbol{E}} = <\boldsymbol{E}\left(\boldsymbol{x}\right)> = \mathsf{E}^{0} \tag{1.5-24}$$

On the other hand, the overall average stress field is given by:

$$\bar{\boldsymbol{T}} = <\boldsymbol{T}\left(\boldsymbol{x}\right)> = <\boldsymbol{T}^{\boldsymbol{0}} + \boldsymbol{T}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)> \tag{1.5-25}$$

i.e., in general, for a prescribed macro-strain, it is:

$$<\boldsymbol{T}^{\boldsymbol{d}}\left(x\right)> \neq 0 \tag{1.5-26}$$

The goal is to calculate the overall stiffness tensor, $\bar{\mathsf{C}}$, such that:

$$\bar{\boldsymbol{T}} = \bar{\mathsf{C}} : \bar{\boldsymbol{E}} = \bar{\mathsf{C}} : \mathsf{E}^{0} \tag{1.5-27}$$

The volume average of the (1.5-4) over the matrix and the inclusions yields:

$$\begin{aligned} \bar{\boldsymbol{T}}^{M} &= \mathsf{C}^{M} : \bar{\boldsymbol{E}}^{M} \\ \bar{\boldsymbol{T}}^{\mathrm{a}} &= \mathsf{C}^{\mathrm{a}} : \bar{\boldsymbol{E}}^{\mathrm{a}} \end{aligned} \quad \left(\mathrm{a}\ \textit{not summed}\right) \tag{1.5-28}$$

Since:

$$f_M \bar{\boldsymbol{T}}^{M} = \bar{\boldsymbol{T}} - \sum_{\mathrm{a}=1}^{n} f_{\mathrm{a}} \bar{\boldsymbol{T}}^{\mathrm{a}} = \bar{\mathsf{C}} : \mathsf{E}^{0} - \sum_{\mathrm{a}=1}^{n} f_{\mathrm{a}} \mathsf{C}^{\mathrm{a}} : \bar{\boldsymbol{E}}^{\mathrm{a}} \tag{1.5-29}$$

and:

$$f_M \bar{\boldsymbol{T}}^{M} = f_M \mathsf{S}^{M} : \bar{\boldsymbol{E}}^{M} = \mathsf{S}^{M} : \left\{ \mathsf{E}^{\boldsymbol{0}} - \sum_{\mathrm{a}=1}^{n} f_{\mathrm{a}} \bar{\boldsymbol{E}}^{\mathrm{a}} \right\} \tag{1.5-30}$$

then, it is obtained that:

$$\left(\mathsf{C}^M-\bar{\mathsf{C}}\right):\mathsf{E}^0=\sum_{\alpha=1}^{n} f_\alpha\left(\mathsf{C}^M-\mathsf{C}^\alpha\right):\bar{\boldsymbol{E}}^\alpha=\sum_{\alpha=1}^{n} f_\alpha\left(\mathsf{C}^M-\mathsf{C}^\alpha\right):<\mathsf{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)>_\alpha \quad (1.5\text{-}31)$$

The (1.5-31) leads to an *exact* result. It defines the overall stiffness tensor $\bar{\mathsf{C}}$ in terms of the average strains in the inclusions. It is important to note that also this result, like the previous one, does not require the knowledge of the entire strain field within each inclusion: only the estimate of the average value of it in each inclusion is needed.

Since the overall response is linearly elastic, the disturbances or perturbations in the stress and strain fields due to the presence of the inclusions, $\boldsymbol{T}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$ and $\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)$, are linear and homogeneous function of the prescribed constant macro-strain E^0. So, again, because of linearity, the change in the average strain field of Ω^α if C^α is replaced by C^M, is expressed as:

$$\bar{\boldsymbol{E}}^\alpha-\mathsf{S}^M:\bar{\boldsymbol{T}}^\alpha=\mathsf{J}^\alpha:\mathsf{E}^{\boldsymbol{0}} \quad (1.5\text{-}32)$$

Since E^0 is arbitrary, from the (1.5-31) it is obtained that:

$$\bar{\mathsf{C}}=\mathsf{C}^M-\sum_{\alpha=1}^{n} f_\alpha\mathsf{C}^M:\mathsf{J}^\alpha=\mathsf{C}^M:\left(\mathbf{1}^4-\sum_{\alpha=1}^{n} f_\alpha\mathsf{J}^\alpha\right) \quad (1.5\text{-}33)$$

which is an *exact* result, yet. At this point, the constant tensors H^α and $\mathsf{J}^\alpha\left(\alpha=1,2,.....n\right)$ has to be estimated for each inclusion.

At this point, in order to introduct the concepts of eigenstrain and eigenstress, a specific elastic problem is considered, where a finite homogeneous linearly elastic (not necessarily isotropic) solid, having elasticity tensor C^M and compliance tensor S^M, contains only one homogeneous linearly elastic (not necessarily isotropic) inclusion Ω, of arbitrary geometry, having elasticity tensor C^α and compliance tensor S^α. The total volume is *V*, bounded by ∂V, and the matrix volume is $M = V - \Omega$, bounded by $\partial V + \partial\Omega^M = \partial V - \partial\Omega$; see the figure below.

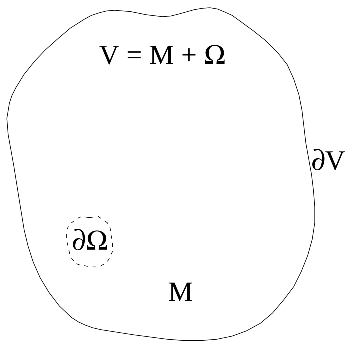


Figure 1.9 Finite homogeneous linearly elastic solid

When self-equilibrating surface tractions, corresponding to the uniform stress field $\mathsf{S}^0 = constant$, are applied on the boundary, because of the RVE heterogeneity, the stress and the strain fields within the volume *V* are spatially variable and they can be expressed by (1.5-3). Analogously, when self-compatible linear surface displacements, corresponding to the uniform strain field $\mathsf{E}^0 = constant$, are applied on the boundary, because of the RVE heterogeneity, the stress and the strain fields within the volume *V* are spatially variable and they can be expressed by (1.5-23).

However, instead of dealing with the above-mentioned heterogeneous solid, it is convenient and effective to consider an *equivalent homogeneous* one which has the uniform elasticity tensor C^M of the matrix material *everywhere*, including in Ω. Then, in order to account for the mismatch of the material properties of the inclusion and of the matrix, a suitable strain field $\boldsymbol{E}^*(\boldsymbol{x})$ is introduced in Ω. Doing so, the *equivalent homogeneous* solid has the same strain and stress fields as the *actual heterogeneous* solid under the applied boundary conditions (tractions or displacements). The introduced strain field $\boldsymbol{E}^*(\boldsymbol{x})$ is called *eigenstrain*. The following figure shows this procedure when boundary tractions corresponding to S^0 are prescribed on ∂V .

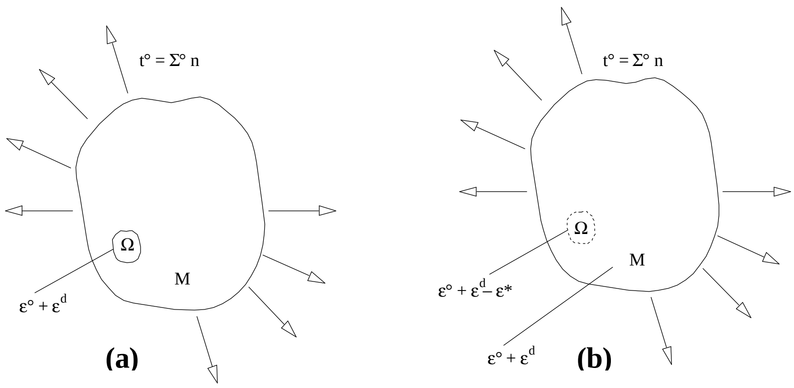


Figure 1.10 (a) heterogeneous solid; (b) equivalent homogeneous solid.

where the assigned eigenstrain field is given by:

$$\boldsymbol{E}^*(\boldsymbol{x}) = \begin{cases} \mathbf{0} & in\, M \\ \boldsymbol{E}^*(\boldsymbol{x}) & in\, \Omega \end{cases} \tag{1.5-34}$$

Since for this equivalent problem the elasticity tensor is, as already mentioned, uniform everywhere and given by C^M , the strain and the stress fields within the solid can be expressed as:

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}^{\boldsymbol{0}} + \boldsymbol{E}^{\boldsymbol{d}}(\boldsymbol{x}) \tag{1.5-35}$$

and:

$$\boldsymbol{T}(\boldsymbol{x}) = \mathsf{C}^M : \left(\boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}^*(\boldsymbol{x})\right) = \begin{cases} \mathsf{C}^M : \left(\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})\right) & in\, M \\ \mathsf{C}^M : \left(\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x}) - \boldsymbol{E}^*(\boldsymbol{x})\right) & in\, \Omega \end{cases} \quad (1.5\text{-}36)$$

which shows that the eigenstrain field disturbs the stress-strain relation.

In order to relate the eigenstrain $\boldsymbol{E}^*(\boldsymbol{x})$ to the corresponding perturbation strain $\boldsymbol{E^d}(\boldsymbol{x})$, consider the equivalent uniform elastic solid of volume V and uniform elasticity C^M and observe that, since by definition:

$$\boldsymbol{T^0} = \mathsf{C}^M : \boldsymbol{E^0} \quad (1.5\text{-}37)$$

or:

$$\boldsymbol{E^0} = \mathsf{S}^M : \boldsymbol{T^0} \quad (1.5\text{-}38)$$

Hence, by considering that:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{x}) &= \boldsymbol{T^0} + \boldsymbol{T^d}(\boldsymbol{x}) \\ \boldsymbol{E}(\boldsymbol{x}) &= \boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x}) \end{aligned} \quad (1.5\text{-}39)$$

and by taking in account the (1.5-35) and (1.5-36), it follows that:

$$\boldsymbol{T^d}(\boldsymbol{x}) = \mathsf{C}^M : \left(\boldsymbol{E^d}(\boldsymbol{x}) - \boldsymbol{E}^*(\boldsymbol{x})\right) \ in\ V \quad (1.5\text{-}40)$$

Since the resulting stress field must be in equilibrium and must produce a compatible strain field, in general the perturbation strain field $\boldsymbol{E^d}(\boldsymbol{x})$ is obtained in terms of an integral operator acting on the corresponding eigenstrain $\boldsymbol{E}^*(\boldsymbol{x})$. In the present context, this integral operator is denoted by $\boldsymbol{\mathsf{S}}$, such that:

$$\boldsymbol{E^d}(\boldsymbol{x}) = \boldsymbol{\mathsf{S}}\left(\boldsymbol{x}; \boldsymbol{E}^*\right) \quad (1.5\text{-}41)$$

or, in components:

$$E_{ij}^d(\boldsymbol{x}) = \mathsf{S}_{ij}\left(\boldsymbol{x}; \boldsymbol{E}^*\right) \quad (1.5\text{-}42)$$

The same procedure of homogenization previously developed can be performed by the introduction of an eigenstress $\boldsymbol{T}^*(\boldsymbol{x})$. To this end, set:

$$\boldsymbol{T}^*(\boldsymbol{x}) = \begin{cases} \mathbf{0} & in\, M \\ \boldsymbol{T}^*(\boldsymbol{x}) & in\, \Omega \end{cases} \tag{1.5-43}$$

Since for this alternative equivalent problem the elasticity tensor is, again, uniform everywhere and given by C^M, the strain and the stress fields within the solid can be expressed as:

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x}) \tag{1.5-44}$$

and:

$$\boldsymbol{T}(\boldsymbol{x}) = \mathsf{C}^M : \boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{T}^*(\boldsymbol{x}) = \begin{cases} \mathsf{C}^M : \left(\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})\right) & in\, M \\ \mathsf{C}^M : \left(\boldsymbol{E^0} + \boldsymbol{E^d}(\boldsymbol{x})\right) + \boldsymbol{T}^*(\boldsymbol{x}) & in\, \Omega \end{cases} \tag{1.5-45}$$

which shows that the eigenstress field disturbs the stress-strain relation.

In order to relate the eigenstress $\boldsymbol{T}^*(\boldsymbol{x})$ to the corresponding perturbation stress $\boldsymbol{T^d}(\boldsymbol{x})$, by considering again the equivalent uniform elastic solid of volume V and uniform elasticity C^M and by taking in account the (1.5-35), (1.5-36) and the (1.5-39), it follows that:

$$\boldsymbol{T^d}(\boldsymbol{x}) = \mathsf{C}^M : \boldsymbol{E^d}(\boldsymbol{x}) + \boldsymbol{T}^*(\boldsymbol{x}) \quad in\ V \tag{1.5-46}$$

In general also the perturbation stress field $\boldsymbol{T^d}(\boldsymbol{x})$ is obtained in terms of an integral operator acting on the corresponding eigenstress $\boldsymbol{T}^*(\boldsymbol{x})$. In the present context, this integral operator is denoted by $\boldsymbol{\mathcal{T}}$, such that:

$$\boldsymbol{T^d}(\boldsymbol{x}) = \boldsymbol{\mathcal{T}}\left(\boldsymbol{x}; \boldsymbol{T}^*\right) \tag{1.5-47}$$

or, in components:

$$T_{ij}^{d}\left(\boldsymbol{x}\right)=T_{ij}\left(\boldsymbol{x};\boldsymbol{T}^{*}\right) \tag{1.5-48}$$

According to this topic, an important result due to Eshelby (1957), whis has played a key role in the micromechanical modelling of elastic and inelastic heterogeneous solids, as well as of nonlinear creeping fluids, is that if:

1. $V-\Omega$ is homogeneous, linearly elastic, and infinitely extended and
2. Ω is an ellipsoid

then:

1. the eigenstrain $\boldsymbol{E}^{*}$ necessary for homogenization is uniform in Ω.
2. the resulting strain $\boldsymbol{E}^{\boldsymbol{d}}$ and, hence, stress $\boldsymbol{T}^{\boldsymbol{d}}$ are also uniform in Ω, the former being given by:

$$\boldsymbol{E}^{\boldsymbol{d}}=S^{\Omega}:\boldsymbol{E}^{*}\ in\ \Omega \tag{1.5-49}$$

where the fourth-order tensor S^{Ω} is called Eshelby's tensor, having the following properties:

a) it is symmetric with respect to the first two indices and the second two indices:

$$S_{ijkl}^{\Omega}=S_{jikl}^{\Omega}=S_{ijlk}^{\Omega} \tag{1.5-50}$$

however, it is not in general symmetric with respect to the exchange of ij and kl, i.e. in general it is:

$$S_{ijkl}^{\Omega}\neq S_{klij}^{\Omega} \tag{1.5-51}$$

b) it is independent of the material properties of the inclusion Ω.

c) it is completely defined in terms of the aspect ratios of the ellipsoidal inclusion Ω and the elastic parameters of the surrounding matrix *M* and

d) when the surrounding matrix *M* is isotropic, it depends only on the Poisson ratio of the matrix and the aspect ratios of Ω.

The components of the Eshelby tensor are listed for several special cases. For them, the reader is referred to the Appendix, at the end of this chapter.

When the eigenstrain $\boldsymbol{E}^*$ and the resulting strain disturbance $\boldsymbol{E}^{\boldsymbol{d}}$ are uniform in Ω, then the corresponding eigenstress $\boldsymbol{T}^*$ and the associated stress disturbance $\boldsymbol{T}^{\boldsymbol{d}}$ are also uniform in Ω. Hence, a fourth-order tensor $\mathcal{T}^{\Omega}$ may be introduced, such that:

$$\boldsymbol{T}^{\boldsymbol{d}} = \mathcal{T}^{\Omega} : \boldsymbol{T}^* \;\; in\,\Omega \tag{1.5-52}$$

The tensor $\mathcal{T}^{\Omega}$ has the same symmetries of Eshelby's tensor.

In order to relate the tensors $\mathcal{T}^{\Omega}$ and $\mathcal{S}^{\Omega}$, it is first noted from the (1.5-40) and the (1.5-46) that the eigenstrain and the eigenstress are related by:

$$\boldsymbol{T}^* + \mathsf{C}^M : \boldsymbol{E}^* = \boldsymbol{0}, \qquad \boldsymbol{E}^* + \mathsf{S}^M : \boldsymbol{T}^* = \boldsymbol{0} \tag{1.5-53}$$

So, from the (1.5-46), the (1.5-49) and the (1.5-52), it follows that:

$$\begin{aligned} \mathcal{S}^{\Omega} : \boldsymbol{E}^* &= \mathsf{S}^M : \left(\mathcal{T}^{\Omega} - \mathbf{1}^{(4)}\right) : \left(-\mathsf{C}^M : \boldsymbol{E}^*\right) \\ \mathcal{T}^{\Omega} : \boldsymbol{T}^* &= \mathsf{C}^M : \left(\mathcal{S}^{\Omega} - \mathbf{1}^{(4)}\right) : \left(-\mathsf{S}^M : \boldsymbol{T}^*\right) \end{aligned} \tag{1.5-54}$$

Therefore, the tensors $\mathcal{T}^{\Omega}$ and $\mathcal{S}^{\Omega}$ must satisfy:

$$\mathcal{S}^{\Omega} + \mathsf{S}^M : \mathcal{T}^{\Omega} : \mathsf{C}^M = \mathbf{1}^{(4)}, \qquad \mathcal{T}^{\Omega} + \mathsf{C}^M : \mathcal{S}^{\Omega} : \mathsf{S}^M = \mathbf{1}^{(4)} \tag{1.5-55}$$

or, in components:

$$\begin{aligned} S^{\Omega}_{ijkl} + S^M_{ijpq} : T^{\Omega}_{pqrs} : C^M_{rskl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \\ T^{\Omega}_{ijkl} + C^M_{ijpq} : S^{\Omega}_{pqrs} : S^M_{rskl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \end{aligned} \tag{1.5-56}$$

However, for the general case, the eigenstrains and the eigenstresses necessary for the homogenization are not uniform in Ω, even if Ω is ellipsoidal, whether V is unbounded or finite. So, the eigenstrains, $\boldsymbol{E}^*(\boldsymbol{x})$ (or the eigenstresses, $\boldsymbol{T}^{*}\left(\boldsymbol{x}\right)$) are defined by the so-called *consistency conditions*, which require the resulting stress field $\boldsymbol{T}\left(\boldsymbol{x}\right)$, (or the strain field $\boldsymbol{E}\left(\boldsymbol{x}\right)$), to be the same under the applied overall loads, whether it is calculated directly from the (1.5-4) or through homogenization from (1.5-40).

Note here that the (1.5-4), when considering an only inclusion, becomes:

$$\boldsymbol{T}(\boldsymbol{x})=\boldsymbol{T}^{\boldsymbol{0}}+\boldsymbol{T}^{\boldsymbol{d}}(\boldsymbol{x})=\begin{cases}\mathsf{C}^{M}:\boldsymbol{E}(\boldsymbol{x})=\mathsf{C}^{M}:\left\{\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}(\boldsymbol{x})\right\} & in\, M=V-\Omega\\ \mathsf{C}^{\Omega}:\boldsymbol{E}(\boldsymbol{x})=\mathsf{C}^{\Omega}:\left\{\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}(\boldsymbol{x})\right\} & in\,\Omega\end{cases} \quad (1.5\text{-}57)$$

Hence, the resulting stress field in Ω, according to the (1.5-57), is given by:

$$\boldsymbol{T}\left(\boldsymbol{x}\right)=\mathsf{C}^{\Omega}:\boldsymbol{E}\left(\boldsymbol{x}\right)=\mathsf{C}^{\Omega}:\left\{\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)\right\} \;\; in\;\Omega \quad (1.5\text{-}58)$$

or else, according to the (1.5-36):

$$\boldsymbol{T}(\boldsymbol{x})=\mathsf{C}^{M}:\left(\boldsymbol{E}(\boldsymbol{x})-\boldsymbol{E}^{*}(\boldsymbol{x})\right)=\mathsf{C}^{M}:\left(\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}(\boldsymbol{x})-\boldsymbol{E}^{*}(\boldsymbol{x})\right)\; in\,\Omega \quad (1.5\text{-}59)$$

By summarizing the equations (1.5-58) and (1.5-59), the stress field in Ω can be expressed in the following form:

$$\boldsymbol{T}\left(\boldsymbol{x}\right)=\mathsf{C}^{\Omega}:\left(\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{d}\left(\boldsymbol{x}\right)\right)=\mathsf{C}^{M}:\left(\boldsymbol{E}^{\boldsymbol{0}}+\boldsymbol{E}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)-\boldsymbol{E}^{*}\left(\boldsymbol{x}\right)\right) \quad (1.5\text{-}60)$$

and analogously for the resulting strain field in Ω:

$$\boldsymbol{E}\left(\boldsymbol{x}\right)=\mathsf{S}^{\Omega}:\left(\boldsymbol{T}^{\boldsymbol{0}}+\boldsymbol{T}^{d}\left(\boldsymbol{x}\right)\right)=\mathsf{S}^{M}:\left(\boldsymbol{T}^{\boldsymbol{0}}+\boldsymbol{T}^{\boldsymbol{d}}\left(\boldsymbol{x}\right)-\boldsymbol{T}^{*}\left(\boldsymbol{x}\right)\right) \quad (1.5\text{-}61)$$

The substitution in the (1.5-60) of $\boldsymbol{E}^{d}\left(\boldsymbol{x}\right)$ as given by (1.5-41) yields an integral equation for $\boldsymbol{E}^{*}\left(\boldsymbol{x}\right)$. Similarly, The substitution in the (1.5-61) of $\boldsymbol{T}^{d}\left(\boldsymbol{x}\right)$ as given by (1.5-47) yields an integral equation for $\boldsymbol{T}^{*}\left(\boldsymbol{x}\right)$.

It is worth to underline that both (1.5-60) and (1.5-61) are valid whether uniform tractions or linear displacements are prescribed on ∂V. In particular, if the overall stress S^{0} is given, then:

$$\begin{aligned} \boldsymbol{T^0} &= \mathsf{S}^0 \\ \boldsymbol{E^0} &= \mathsf{S}^M : \mathsf{S}^0 \end{aligned} \tag{1.5-62}$$

while, if the overall strain E^0 is given, then:

$$\begin{aligned} \boldsymbol{E^0} &= \mathsf{E}^0 \\ \boldsymbol{T^0} &= \mathsf{C}^M : \mathsf{E}^0 \end{aligned} \tag{1.5-63}$$

For any homogeneous linearly elastic inclusion Ω in a homogeneous linearly elastic matrix *M*, consistency conditions (1.5-60) and (1.5-61) yield:

$$\boldsymbol{E^0} + \boldsymbol{E}^d(\boldsymbol{x}) = \mathsf{A}^\Omega : \boldsymbol{E}^*(\boldsymbol{x}), \qquad \boldsymbol{T^0} + \boldsymbol{T}^d(\boldsymbol{x}) = \mathsf{B}^\Omega : \boldsymbol{T}^*(\boldsymbol{x}) \;\; in\ \Omega \tag{1.5-64}$$

where:

$$\mathsf{A}^\Omega = \left(\mathsf{C}^M - \mathsf{C}^\Omega\right)^{-1} : \mathsf{C}^M, \qquad \mathsf{B}^\Omega = \left(\mathsf{S}^M - \mathsf{S}^\Omega\right)^{-1} : \mathsf{S}^M \tag{1.5-65}$$

By definition, the constant tensors A^Ω and B^Ω satisfy:

$$\mathsf{S}^M : \mathsf{C}^\Omega = \mathbf{1}^{(4)} - \left(\mathsf{A}^\Omega\right)^{-1} = \left(\mathbf{1}^{(4)} - \left(\mathsf{B}^\Omega\right)^{-1}\right)^{-T} \tag{1.5-66}$$

or:

$$\mathsf{C}^M : \mathsf{S}^\Omega = \mathbf{1}^{(4)} - \left(\mathsf{B}^\Omega\right)^{-1} = \left(\mathbf{1}^{(4)} - \left(\mathsf{A}^\Omega\right)^{-1}\right)^{-T} \tag{1.5-67}$$

where the superscript $-T$ stands for the inverse of the transpose or the transpose of the inverse.

In the follows, the attention is confined to the case when *V* is unbounded and Ω is ellipsoidal, so that the eigenstrains and the eigenstresses necessary for the homogenization are both uniform in Ω. In particular, when *V* is unbounded there is no distinction between the cases when the strain or the stress is prescribed and, so, it is:

$$\begin{aligned} \mathsf{E}^0 &= \mathsf{S}^M : \mathsf{S}^0 \\ \mathsf{S}^0 &= \mathsf{C}^M : \mathsf{E}^0 \end{aligned} \tag{1.5-68}$$

Moreover, when in addition Ω is ellipsoidal, the substitution, in the first of (1.5-64), for $\boldsymbol{E}^{\boldsymbol{d}}$ given by (1.5-49) and, analogously, the substitution, in the second of (1.5-64), for $\boldsymbol{T}^{\boldsymbol{d}}$ given by (1.5-52) provides explicit expressions for the eigenstrain $\boldsymbol{E}^{*}$ and eigenstress $\boldsymbol{T}^{*}$ which are necessary for homogenization:

$$\boldsymbol{E}^{*} = \left(\mathsf{A}^{\Omega} - \mathcal{S}^{\Omega}\right)^{-1} : \mathsf{E}^{0}, \qquad \boldsymbol{T}^{*} = \left(\mathsf{B}^{\Omega} - \mathcal{T}^{\Omega}\right)^{-1} : \mathsf{S}^{0} \;\; in\ \Omega \tag{1.5-69}$$

Hence, according to the (1.5-64) and the (1.5-69), it can be obtained the strain and stress fields' expressions:

$$\left.\begin{aligned} \boldsymbol{E} &= \mathsf{E}^{0} + \boldsymbol{E}^{\boldsymbol{d}} = \mathsf{A}^{\Omega} : \left(\mathsf{A}^{\Omega} - \mathcal{S}^{\Omega}\right)^{-1} : \mathsf{E}^{0} \\ \boldsymbol{T} &= \mathsf{S}^{0} + \boldsymbol{T}^{\boldsymbol{d}} = \mathsf{B}^{\Omega} : \left(\mathsf{B}^{\Omega} - \mathcal{T}^{\Omega}\right)^{-1} : \mathsf{S}^{0} \end{aligned}\right. \;\; in\ \Omega \tag{1.5-70}$$

The strain field $\boldsymbol{E}$ and the stress field $\boldsymbol{T}$ in Ω given by the (1.5-70) are equivalent. From costitutive relations (1.5-57) and from dual ones:

$$\boldsymbol{E}(\boldsymbol{x}) = \begin{cases} \mathsf{S}^{M} : \boldsymbol{T}(\boldsymbol{x}) = \mathsf{S}^{M} : \left\{\mathsf{S}^{\boldsymbol{0}} + \boldsymbol{T}^{\boldsymbol{d}}(\boldsymbol{x})\right\} & in\, M = V - \Omega \\ \mathsf{S}^{\Omega} : \boldsymbol{T}(\boldsymbol{x}) = \mathsf{S}^{\Omega} : \left\{\mathsf{S}^{0} + \boldsymbol{T}^{\boldsymbol{d}}(\boldsymbol{x})\right\} & in\, \Omega \end{cases} \tag{1.5-71}$$

substitution of the (1.5-65) into (1.5-70), yields, in Ω:

$$\begin{aligned} \boldsymbol{E} &= \mathsf{S}^{\Omega} : \mathsf{B}^{\Omega} : \left(\mathsf{B}^{\Omega} - \mathcal{T}^{\Omega}\right)^{-1} : \mathsf{S}^{0} = \left\{\mathsf{S}^{\Omega} : \left\{\mathbf{1}^{(4)} - \mathcal{T}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{C}^{M} : \mathsf{S}^{\Omega}\right)\right\}^{-1} : \mathsf{C}^{M}\right\} : \mathsf{E}^{0} \\ \boldsymbol{T} &= \mathsf{C}^{\Omega} : \mathsf{A}^{\Omega} : \left(\mathsf{A}^{\Omega} - \mathcal{S}^{\Omega}\right)^{-1} : \mathsf{E}^{0} = \left\{\mathsf{C}^{\Omega} : \left\{\mathbf{1}^{(4)} - \mathcal{S}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{S}^{M} : \mathsf{C}^{\Omega}\right)\right\}^{-1} : \mathsf{S}^{M}\right\} : \mathsf{S}^{0} \end{aligned} \tag{1.5-72}$$

By taking into account the advantage of the identities (1.5-55), the fourth-order tensors in the right-hand sides of (1.5-72) become:

$$\begin{aligned} \mathsf{S}^{\Omega} : \left\{\mathbf{1}^{(4)} - \mathcal{T}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{C}^{M} : \mathsf{S}^{\Omega}\right)\right\}^{-1} : \mathsf{C}^{M} &= \left\{\mathbf{1}^{(4)} - \mathcal{S}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{S}^{M} : \mathsf{C}^{\Omega}\right)\right\}^{-1} \\ \mathsf{C}^{\Omega} : \left\{\mathbf{1}^{(4)} - \mathcal{S}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{S}^{M} : \mathsf{C}^{\Omega}\right)\right\}^{-1} : \mathsf{S}^{M} &= \left\{\mathbf{1}^{(4)} - \mathcal{T}^{\Omega} : \left(\mathbf{1}^{(4)} - \mathsf{C}^{M} : \mathsf{S}^{\Omega}\right)\right\}^{-1} \end{aligned} \tag{1.5-73}$$

Hence, the (1.5-73) compared with the (1.5-70) yields the equivalence relations between $\left(\mathsf{A}^{\Omega}, S^{\Omega}\right)$ and $\left(\mathsf{B}^{\Omega}, T^{\Omega}\right)$, as it follows:

$$\begin{aligned} &\mathsf{S}^{\Omega}:\mathsf{B}^{\Omega}:\left(\mathsf{B}^{\Omega}-T^{\Omega}\right)^{-1}:\mathsf{C}^{M}=\mathsf{A}^{\Omega}:\left(\mathsf{A}^{\Omega}-S^{\Omega}\right)^{-1} \\ &\mathsf{C}^{\Omega}:\mathsf{A}^{\Omega}:\left(\mathsf{A}^{\Omega}-S^{\Omega}\right)^{-1}:\mathsf{S}^{M}=\mathsf{B}^{\Omega}:\left(\mathsf{B}^{\Omega}-T^{\Omega}\right)^{-1} \end{aligned} \tag{1.5-74}$$

Since the total strain in an ellipsoidal inclusion Ω is uniform for the unbounded *V*, the corresponding H^{a} and J^{a}-tensors defined in the (1.5-19) and (1.5-32), respectively, become:

$$<\boldsymbol{E}>_{\Omega}-\mathsf{S}^{M}:<\boldsymbol{T}>_{\Omega}=\bar{\boldsymbol{E}}^{\Omega}-\mathsf{S}^{M}:\bar{\boldsymbol{T}}^{\Omega}=\mathsf{H}^{\Omega}:\mathsf{S}^{0}\ \ in\ \Omega \tag{1.5-75}$$

when the overall stress S^{0} is prescribed, and:

$$<\boldsymbol{E}>_{\Omega}-\mathsf{S}^{M}:<\boldsymbol{T}>_{\Omega}=\bar{\boldsymbol{E}}^{\Omega}-\mathsf{S}^{M}:\bar{\boldsymbol{T}}^{\Omega}=\mathsf{J}^{\Omega}:\mathsf{E}^{0}\ \ in\ \Omega \tag{1.5-76}$$

when the overall strain E^{0} is prescribed. Moreover, since *V* is considered unbounded, H^{a} and J^{a}-tensors satisfy:

$$\mathsf{J}^{\Omega}=\mathsf{H}^{\Omega}:\mathsf{C}^{M}, \qquad \mathsf{H}^{\Omega}=\mathsf{J}^{\Omega}:\mathsf{S}^{M} \tag{1.5-77}$$

By comparing the (1.5-72) with (1.5-75) and (1.5-76), note that H^{a} and J^{a}-tensors may be expressed in terms of Eshelby's tensor S^{Ω} and its conjugate T^{Ω}, as:

$$\begin{aligned} &\mathsf{H}^{\Omega}=\left(\mathsf{S}^{\Omega}-\mathsf{S}^{M}\right):\mathsf{B}^{\Omega}:\left(\mathsf{B}^{\Omega}-\mathsf{T}^{\Omega}\right)^{-1} \\ &\mathsf{J}^{\Omega}=\left(\mathsf{S}^{\Omega}-\mathsf{S}^{M}\right):\mathsf{C}^{\Omega}:\mathsf{A}^{\Omega}:\left(\mathsf{A}^{\Omega}-\mathsf{S}^{\Omega}\right)^{-1} \end{aligned} \tag{1.5-78}$$

or:

$$\begin{aligned} &\mathsf{H}^{\Omega}=\left(\mathsf{S}^{\Omega}-\mathsf{S}^{M}\right):\mathsf{C}^{\Omega}:\mathsf{A}^{\Omega}:\left(\mathsf{A}^{\Omega}-\mathsf{S}^{\Omega}\right)^{-1}:\mathsf{S}^{M} \\ &\mathsf{J}^{\Omega}=\left(\mathsf{S}^{\Omega}-\mathsf{S}^{M}\right):\mathsf{B}^{\Omega}:\left(\mathsf{B}^{\Omega}-\mathsf{T}^{\Omega}\right)^{-1}:\mathsf{C}^{M} \end{aligned} \tag{1.5-79}$$

As pointed out before, the Eshelby tensor S^{Ω} and its conjugate T^{Ω}, in case of uniform ellipsoidal inclusion Ω in an unbounded uniform matrix *M*, depend on the aspect ratios of Ω and on the elastic parameters of the matrix material, but they are independent of the material properties of Ω. On the other hand, H^{Ω} and J^{Ω} depend on the geometry of Ω, as well as on the elasticity of both Ω and the matrix material. For cavities, on the other hand, the (1.5-79) reduce to:

$$\begin{aligned} \mathsf{H}^{\Omega} &= \left(\mathbf{1}^{(\mathbf{4})} - \mathsf{S}^{\Omega}\right)^{-1} : \mathsf{S}^{M} \\ \mathsf{J}^{\Omega} &= \left(\mathbf{1}^{(\mathbf{4})} - \mathsf{S}^{\Omega}\right)^{-1} \end{aligned} \tag{1.5-80}$$

which shows that H^{Ω} and J^{Ω}-tensors are effective tools for homogenization of solids with cavities and cracks, [47].

It is seen, from the above equations, that the equivalence relations between $\left(\mathsf{A}^{\Omega}, S^{\Omega}\right)$ and $\left(\mathsf{B}^{\Omega}, T^{\Omega}\right)$, given by (1.5-74), correspond to the equivalence relations between J^{Ω} and H^{Ω}, given by (1.5-77). It should be kept in mind that:

1. if the solid containing an inclusion is unbounded, these equivalent relations always hold, since the farfield stress $\mathsf{S}^{\text{¥}} = \mathsf{S}^{\mathbf{0}}$ and strain $\mathsf{E}^{\text{¥}} = \mathsf{E}^{\mathbf{0}}$ are related by the (1.5-68) and, hence, the response of the solid is the same whether S^{0} or $\mathsf{E}^{\mathbf{0}}$ is prescribed.
2. if the solid containing an inclusion is bounded, these equivalent relations do not hold, in general, since the response of the solid when uniform boundary tractions are prescribed is different, in general, from that one when linear boundary displacements are prescribed.

1.6 Strategies for obtaining overall elasticity tensors: Voigt and Reuss estimating

In this section, we will introduce some general results about extremely useful bounds for overall elastic stiffness and compliance tensors, in the framework of the micromechanical theory. In particular, Hill (1952) has proved that, independently from the RVE geometry, the actual overall moduli lie somewhere in an interval between the Reuss and Voigt estimates, as shown in the follows.

For semplicity, it is considered the simplest case of an RVE volume V consisting in a linear elastic homogeneous matrix M which contains one only a linear elastic homogeneous inclusion Ω. So, either for stress prescribed and for strain prescribed boundary conditions, the equation (1.5-17) assumes the following form:

$$\left(\mathsf{S}^{M} - \overline{\mathsf{S}}\right) : \overline{\mathsf{T}} = f_{\Omega}\left(\mathsf{S}^{M} - \mathsf{S}^{\Omega}\right) : \overline{\boldsymbol{T}}^{\Omega} \tag{1.6-1}$$

where:

$f_{\Omega} = \dfrac{\Omega}{V} =$ volumetric fraction of the inclusion Ω.

From the (1.6-1), it is possible to obtain a unique dependence of the average value of the stress field in the phase of the inclusion upon the overall stress field in the RVE volume:

$$\overline{\boldsymbol{T}}^{\Omega} = \mathsf{L}^{\Omega} : \overline{\mathsf{T}} \tag{1.6-2}$$

where:

$$\mathsf{L}^{\Omega} = f_{\Omega}^{-1}\left(\mathsf{S}^{M} - \mathsf{S}^{\Omega}\right)^{-1}\left(\mathsf{S}^{M} - \overline{\mathsf{S}}\right) \tag{1.6-3}$$

Analogously, it is possible to express the average value of the stress field in the phase of the matrix in function of the overall stress field in the RVE volume, as it follows:

$$\overline{\boldsymbol{T}}^{M} = \mathsf{L}^{M} : \overline{\mathsf{T}} \tag{1.6-4}$$

where:

L^{Ω} *and* $\mathsf{L}^{M} =$ concentration matrices.

By considering the (1.5-11) and (1.5-13), the overall stress field in the RVE volume can be written in the form:

$$\overline{\mathsf{T}} = f_M \overline{\boldsymbol{T}}^{M} + f_{\Omega} \overline{\boldsymbol{T}}^{\Omega} \tag{1.6-5}$$

From the (1.6-5), by taking in account the relations (1.6-2) and (1.6-4), it has to be verified that:

$$f_M \mathsf{L}^{M} + f_{\Omega} \mathsf{L}^{\Omega} = \boldsymbol{I} \tag{1.6-6}$$

where:

$\boldsymbol{I} =$ the unit matrix

Hence, the concentration matrix L^{M} can be expressed as:

$$\mathsf{L}^{M} = \left(\boldsymbol{I} - f_{\Omega} \mathsf{L}^{\Omega} \right) f_M^{-1} \tag{1.6-7}$$

In similar manner, in the simplest case of an RVE volume V consisting in a linear elastic homogeneous matrix M which contains one only a linear elastic homogeneous inclusion Ω, either for stress prescribed and for strain prescribed boundary conditions, the equation (1.5-31) assumes the following form:

$$\left(\mathsf{C}^{M} - \overline{\mathsf{C}} \right) : \overline{\boldsymbol{E}} = f_{\Omega} \left(\mathsf{C}^{M} - \mathsf{C}^{\Omega} \right) : \overline{\boldsymbol{E}}^{\Omega} \tag{1.6-8}$$

From the (1.6-8), it is possible to obtain a unique dependence of the average value of the strain field in the phase of the inclusion upon the overall strain field in the RVE volume:

$$\bar{\boldsymbol{E}}^{\Omega} = \mathsf{M}^{\Omega} : \bar{\mathsf{E}} \tag{1.6-9}$$

where:

$$\mathsf{M}^{\Omega} = f_{\Omega}^{-1}\left(\mathsf{C}^{M} - \mathsf{C}^{\Omega}\right)^{-1}\left(\mathsf{C}^{M} - \bar{\mathsf{C}}\right) \tag{1.6-10}$$

Analogously, it is possible to express the average value of the strain field in the phase of the matrix in function of the overall strain field in the RVE volume, as it follows:

$$\bar{\boldsymbol{E}}^{M} = \mathsf{M}^{M} : \bar{\mathsf{E}} \tag{1.6-11}$$

where:

M^{Ω} *and* $\mathsf{M}^{M} =$ concentration matrices.

By considering the (1.5-10) and (1.5-12), the overall strain field in the RVE volume can be written in the form:

$$\bar{\mathsf{E}} = f_M \bar{\boldsymbol{E}}^{M} + f_{\Omega} \bar{\boldsymbol{E}}^{\Omega} \tag{1.6-12}$$

From the (1.6-12), by taking in account the relations (1.6-9) and (1.6-11), it has also to be verified that:

$$f_M \mathsf{M}^{M} + f_{\Omega} \mathsf{M}^{\Omega} = \mathsf{I} \tag{1.6-13}$$

where:

$\mathsf{I} =$ the unit tensor

Hence, the concentration matrix M^{M} can be expressed as:

$$\mathsf{M}^{M} = \left(\boldsymbol{I} - f_{\Omega} \mathsf{M}^{\Omega}\right) f_M^{-1} \tag{1.6-14}$$

By taking into account the (1.5-14) and the (1.5-28), the equations (1.6-5) and (1.6-12), respectively, yield:

$$\begin{aligned} \bar{\mathsf{T}} &= f_M \mathsf{C}^{M} : \bar{\boldsymbol{E}}^{M} + f_{\Omega} \mathsf{C}^{\Omega} : \bar{\boldsymbol{E}}^{\Omega} \\ \bar{\boldsymbol{E}} &= f_M \mathsf{S}^{M} : \bar{\mathsf{T}}^{M} + f_{\Omega} \mathsf{S}^{\Omega} : \bar{\mathsf{T}}^{\Omega} \end{aligned} \tag{1.6-15}$$

Because it is:

$$\overline{\mathsf{T}} = \overline{\mathsf{C}} : \overline{\mathsf{E}} \tag{1.6-16}$$

by combining the (1.6-16) with the first equation of the (1.6-15), and according to the (1.6-9) and (1.6-11), the required effective RVE's stiffness tensor is obtained in the following form:

$$\overline{\mathsf{C}} = f_M \mathsf{C}^M : \mathsf{M}^M + f_\Omega \mathsf{C}^\Omega : \mathsf{M}^\Omega \tag{1.6-17}$$

Equivalently, because it is:

$$\overline{\mathsf{E}} = \overline{\mathsf{S}} : \overline{\mathsf{T}} \tag{1.6-18}$$

by combining the (1.6-18) with the second equation of the (1.6-15), and according to the (1.6-2) and (1.6-4), the required effective RVE's compliance tensor is obtained in the following form:

$$\overline{\mathsf{S}} = f_M \mathsf{S}^M : \mathsf{L}^M + f_\Omega \mathsf{S}^\Omega : \mathsf{L}^\Omega \tag{1.6-19}$$

A model for the evaluation of the overall elastic stiffness tensor, probably the simplest one, was introduced by Voigt in 1889 for the estimation of the average constants of polycristals. He assumes that the strain field throughout the RVE is uniform, that yields:

$$\overline{\boldsymbol{E}} = \overline{\boldsymbol{E}}^M = \overline{\boldsymbol{E}}^\Omega = \boldsymbol{E}^{\boldsymbol{0}} \tag{1.6-20}$$

Such a condition can be represented by means of the following simplified model:

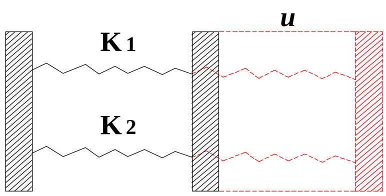


Figure 1.11 Strain-prescribed problem.

It follows that:

$$\mathsf{M}^{M} = \mathsf{M}^{\Omega} = \boldsymbol{I} \tag{1.6-21}$$

and the (1.6-17) becomes:

$$\bar{\mathsf{C}}^{V} = f_M \mathsf{C}^{M} + f_{\Omega} \mathsf{C}^{\Omega} \tag{1.6-22}$$

which the superscript V underlines that the overall stiffness tensor has been obtained in the Voigt approximation.

Moreover, the equation (1.5-32) yields that in Voigt approximation, the J^{Ω} tensor assumes the following form:

$$\mathsf{J}^{\Omega} = \left(\mathbf{1}^{(4)} - \mathsf{S}^{M} : \mathsf{C}^{\Omega} \right) \tag{1.6-23}$$

so that the (1.5-33) leads to (1.6-22).

It is worth to underline that the resulting Voigt stresses are such that the tractions at interface boundaries would not be in equilibrium, so this approximation satisfies the compatibility conditions and do not satisfy the equilibrium ones.

Dually, a model for the evaluation of the overall elastic compliance tensor, probably the simplest one, was introduced by Reuss in 1929. He assumes that the stress field throughout the RVE is uniform, that yields:

$$\bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}^{M} = \bar{\boldsymbol{T}}^{\Omega} = \boldsymbol{T}^{\boldsymbol{0}} \tag{1.6-24}$$

Such a condition can be represented by means of the following simplified model:

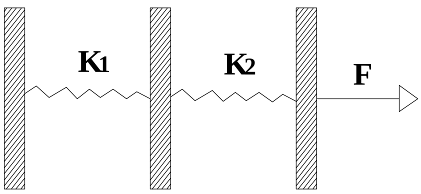


Figure 1.12 Stress-prescribed problem.

It follows that:

$$\mathsf{L}^M = \mathsf{L}^\Omega = \boldsymbol{I} \tag{1.6-25}$$

and the (1.6-19) becomes:

$$\overline{\mathsf{S}}^R = f_M \mathsf{S}^M + f_\Omega \mathsf{S}^\Omega \tag{1.6-26}$$

which the superscript *R* underlines that the overall compliance tensor has been obtained in the Reuss approximation.

Moreover, the equation (1.5-19) yields that in Reuss approximation, the H^Ω tensor assumes the following form:

$$\mathsf{H}^\Omega = \mathsf{S}^\Omega - \mathsf{S}^M \tag{1.6-27}$$

so that the (1.5-20) leads to (1.6-26).

It is worth to underline that the resulting Reuss strains are such that the displacements at interface boundaries would not be compatible, i.e., the inclusion and the matrix could not remain bonded, so this approximation satisfies the equilibrium conditions and do not satisfy the compatibility ones.

As mentioned in the beginning of this section, Hill proved in the 1952 that, independently from the RVE geometry, the actual overall moduli lie somewhere in an interval between the Reuss and Voigt estimates. Thus, the Voigt and Reuss approximations are the upper and lower bounds of the true effective elastic moduli. In order to demonstrate it, let the above cited RVE to be subjected to displacement homogeneous boundary conditions, **Errore. L'origine riferimento non è stata trovata.**:

$$\boldsymbol{u^0} = \mathsf{E}\boldsymbol{x} \;\; on \; \partial V \tag{1.6-28}$$

The external work is given by:

$$W = \frac{1}{2}\int_{\partial V} \boldsymbol{t}\cdot\boldsymbol{u^0} ds = \frac{1}{2}\int_{\partial V} t_i \mathsf{e}_{ij}^M x_j \, ds = \frac{1}{2}\mathsf{e}_{ij}^M \int_{\partial V} t_i x_j \, ds \tag{1.6-29}$$

By remembering that in case of prescribed macrostrain, it is:

$$\bar{\boldsymbol{E}} = < \boldsymbol{E}\left(\boldsymbol{x}, \mathrm{E}\right) > = \mathrm{E} \tag{1.6-30}$$

and by considering the **Errore. L'origine riferimento non è stata trovata.** and the **Errore. L'origine riferimento non è stata trovata.**, it is obtained[1]:

$$W = \frac{1}{2} \mathrm{E}\, \mathrm{S}^{\mathrm{E}} V = \frac{1}{2} \bar{\boldsymbol{E}}\, \bar{\boldsymbol{T}}\, V \tag{1.6-31}$$

Similarly, let the above cited RVE to be subjected to traction homogeneous boundary conditions, **Errore. L'origine riferimento non è stata trovata.**:

$$\boldsymbol{t^0} = \mathrm{S}\boldsymbol{n} \;\; on \; \partial V \tag{1.6-32}$$

The external work is given by:

$$W = \frac{1}{2} \int_{\partial V} \boldsymbol{t^0} \cdot \boldsymbol{u} ds = \frac{1}{2} \int_{\partial V} \mathrm{s}_{ij}^{M} n_j u_i \, ds = \frac{1}{2} \mathrm{s}_{ij}^{M} \int_{\partial V} n_j u_i \, ds \tag{1.6-33}$$

By remembering that in case of prescribed macrostress, it is:

$$\bar{\boldsymbol{T}} = < \boldsymbol{T}\left(\boldsymbol{x}, \mathrm{E}\right) > = \mathrm{S} \tag{1.6-34}$$

and by considering **Errore. L'origine riferimento non è stata trovata.** and the **Errore. L'origine riferimento non è stata trovata.** , it is obtained[2]:

---

[1] It can be noted that, by taking in account the **Errore. L'origine riferimento non è stata trovata.**, the equation (1.6-31) yields the following relation:

$$W = \int_V \mathrm{f}^{\mathrm{E}}\left(\boldsymbol{x}, \mathrm{E}\right) dV = \Phi^{\mathrm{E}}\left(\mathrm{E}\right) V$$

[2] It can be noted that, by taking in account the **Errore. L'origine riferimento non è stata trovata.**, the equation (1.6-35) yields the following relation:

$$W = \int_V \mathrm{y}^{\Sigma}\left(\boldsymbol{x}, \mathrm{S}\right) dV = \Psi^{\Sigma}\left(\mathrm{S}\right) V$$

$$W = \frac{1}{2} \mathrm{E}^{\Sigma} \Sigma V = \frac{1}{2} \bar{\boldsymbol{E}} \bar{\boldsymbol{T}} V \tag{1.6-35}$$

Since, in both cases, it is:

$$W = \frac{1}{2} \int_{\partial V} t_i u_i ds = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} dV \tag{1.6-36}$$

it follows that the following identity is established:

$$\bar{\sigma}_{ij} \bar{\varepsilon}_{ij} = \frac{1}{V} \int_V \sigma_{ij} \varepsilon_{ij} \, dV \tag{1.6-37}$$

Define:

$$\hat{\sigma}_{ij}^{p} = \left(\mathrm{M}_{ijhk}^{p} \right)^{-1} \bar{\sigma}_{hk}^{p}, \qquad \hat{\varepsilon}_{ij}^{p} = \left(\Lambda_{ijhk}^{p} \right)^{-1} \bar{\varepsilon}_{hk}^{p} \tag{1.6-38}$$

therefore, by recalling the equations (1.6-2), (1.6-4), (1.6-9) and (1.6-11), it can be deduced that:

$$\hat{\sigma}_{ij}^{p} = C_{ijhk}^{p} \bar{\varepsilon}_{hk}, \qquad \hat{\varepsilon}_{ij}^{p} = S_{ijhk}^{p} \bar{\sigma}_{hk} \tag{1.6-39}$$

where the superscript p stands for the inclusion phase Ω or for the matrix phase, M, and where C_{ijhk}^{p} and S_{ijhk}^{p} represent the elastic stiffnesses and compliances of the single examined phase.

Furthermore, it is:

$$\sigma_{ij}^{p} = C_{ijhk}^{p} \varepsilon_{hk}^{p}, \qquad \varepsilon_{ij}^{p} = S_{ijhk}^{p} \sigma_{hk}^{p} \tag{1.6-40}$$

Hence, it follows that:

$$\hat{\sigma}_{ij}^{p} \varepsilon_{ij}^{p} = \sigma_{ij}^{p} \bar{\varepsilon}_{ij}, \qquad \sigma_{ij}^{p} \hat{\varepsilon}_{ij}^{p} = \bar{\sigma}_{ij} \varepsilon_{ij}^{p} \tag{1.6-41}$$

and therefore:

$$\begin{aligned} \sigma_{ij}^{p} \varepsilon_{ij}^{p} + \left(\sigma_{ij}^{p} - \hat{\sigma}_{ij}^{p} \right) \left(\varepsilon_{ij}^{p} - \bar{\varepsilon}_{ij} \right) &= \hat{\sigma}_{ij}^{p} \bar{\varepsilon}_{ij} + 2 \left(\varepsilon_{ij}^{p} - \bar{\varepsilon}_{ij} \right) \sigma_{ij}^{p} \\ \sigma_{ij}^{p} \varepsilon_{ij}^{p} + \left(\sigma_{ij}^{p} - \bar{\sigma}_{ij} \right) \left(\varepsilon_{ij}^{p} - \hat{\varepsilon}_{ij}^{p} \right) &= \bar{\sigma}_{ij} \hat{\varepsilon}_{ij}^{p} + 2 \left(\sigma_{ij}^{p} - \bar{\sigma}_{ij} \right) \varepsilon_{ij}^{p} \end{aligned} \tag{1.6-42}$$

The second terms in the left sides in the equations (1.6-42) are positive, since they can be written, respectively, as:

$$\begin{aligned}&\left(\sigma_{ij}^{p}-\hat{\sigma}_{ij}^{p}\right)\left(\varepsilon_{ij}^{p}-\bar{\varepsilon}_{ij}\right)=C_{ijhk}^{p}\left(\varepsilon_{hk}^{p}-\bar{\varepsilon}_{hk}\right)\left(\varepsilon_{ij}^{p}-\bar{\varepsilon}_{ij}\right)\\&\left(\sigma_{ij}^{p}-\bar{\sigma}_{ij}\right)\left(\varepsilon_{ij}^{p}-\hat{\varepsilon}_{ij}^{p}\right)=S_{ijhk}^{p}\left(\sigma_{hk}^{p}-\bar{\sigma}_{hk}\right)\left(\sigma_{ij}^{p}-\bar{\sigma}_{ij}\right)\end{aligned} \qquad (1.6\text{-}43)$$

In addition, by recalling the equation (1.6-37), it is:

$$\int_V \sigma_{ij}^{p}\left(\varepsilon_{ij}^{p}-\bar{\varepsilon}_{ij}\right)dV=0, \qquad \int_V \left(\sigma_{ij}^{p}-\bar{\sigma}_{ij}\right)\varepsilon_{ij}^{p}\,dV=0 \qquad (1.6\text{-}44)$$

With these considerations, the equation (1.6-42) yields:

$$\begin{aligned}V\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}&\le\bar{\varepsilon}_{ij}\int_V\hat{\sigma}_{ij}^{p}\,dV\\V\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}&\le\bar{\sigma}_{ij}\int_V\hat{\varepsilon}_{ij}^{p}\,dV\end{aligned} \qquad (1.6\text{-}45)$$

By considering the (1.6-39), we have finally:

$$\begin{aligned}\bar{C}_{ijhk}&\le\frac{1}{V}\int_V C_{ijhk}^{p}\,dV\\\bar{S}_{ijhk}&\le\frac{1}{V}\int_V S_{ijhk}^{p}\,dV\end{aligned} \qquad (1.6\text{-}46)$$

This result, by taking in account the equations (1.6-22) and (1.6-26), can be expressed in the following form:

$$\begin{aligned}\bar{C}_{ijhk}&\le\bar{C}_{ijhk}^{V}\\\bar{S}_{ijhk}&\le\bar{S}_{ijhk}^{R}\end{aligned} \qquad (1.6\text{-}47)$$

which indicates that the Voigt approximation gives upper bound and the Reuss approximation gives lower bound for the overall stiffness tensor of the homogenized material. Unfortunately, these bounds are of pratical significance only for small volume fractions and slight mismatch of elastic moduli of phases. Better universal bounds are given by Hashin and Shtrikman (1963), as shown in the following section.

1.7 Variational methods- Hashin and Shtrikman's variational principles.

The homogenization problem of an heterogeneous RVE is equivalent to solve one of the following variational problems:

$$\begin{aligned} \frac{1}{2}\bar{\mathsf{C}}\bar{\boldsymbol{E}}\cdot\bar{\boldsymbol{E}} &= \inf_{\boldsymbol{E}^{\boldsymbol{d}}\in\mathsf{E}} \frac{1}{V}\int_V \frac{1}{2}\mathsf{C}\left(\bar{\boldsymbol{E}}+\boldsymbol{E}^{\boldsymbol{d}}\right)\cdot\left(\bar{\boldsymbol{E}}+\boldsymbol{E}^{\boldsymbol{d}}\right)dV \\ \frac{1}{2}\bar{\mathsf{S}}\bar{\boldsymbol{T}}\cdot\bar{\boldsymbol{T}} &= \inf_{\boldsymbol{T}^{\boldsymbol{d}}\in\mathsf{T}} \frac{1}{V}\int_V \frac{1}{2}\mathsf{S}\left(\bar{\boldsymbol{T}}+\boldsymbol{T}^{\boldsymbol{d}}\right)\cdot\left(\bar{\boldsymbol{T}}+\boldsymbol{T}^{\boldsymbol{d}}\right)dV \end{aligned} \qquad (1.7\text{-}1)$$

where:

$\mathsf{E}=$ compatible periodic strain field space, whose average value is equal to zero

$\mathsf{T}=$ equilibrated periodic stress field space, whose average value is equal to zero

$\bar{\mathsf{C}}=$ homogenized stiffness tensor

$\bar{\mathsf{S}}=$ homogenized compliance tensor

$\bar{\boldsymbol{T}}=$ generic stress field belonging to Sym

$\bar{\boldsymbol{E}}=$ generic strain field belonging to Sym

The first members of the (1.7-1) represent, respectively, the elastic energy density and the complementary one of the homogenized material. In particular, solving the first problem of the (1.7-1) is equivalent of determining, between the compatible strain fields, whose prescribed average value is $\bar{\boldsymbol{E}}$, the sole one that is also equilibrated. On the contrary, solving the second problem of the

(1.7-1) is equivalent of determining, between the equilibrated stress fields, whose prescribed average value is $\overline{\boldsymbol{T}}$, the sole one that is also compatible.

It is possible to demonstrate that, if the stiffness tensor C and the compliance one S have, unifomly in *V*, all the eigenvalues lower down bounded by a positive constant, then the equations (1.7-1) admit one and only one solution.

Since the functionals in the first members of the (1.7-1) are conjugate each other, [29], it follows that the homogenized properties of the material are well defined, hence:

$$\overline{\mathsf{S}} = \overline{\mathsf{C}}^{-1} \tag{1.7-2}$$

In this framework, the basic physic idea of the Hashin and Shtrikman's principles is to substitute the heterogeneous medium with a reference homogeneous one, having a stiffness tensor, C^H , and a compliance tensor, S^H . In order to simulate the actual micro-structure, eigenstress and eigenstrain fields are prescribed on the reference homogeneous medium, as already seen in the previous section. So, the Hashin and Shtrikman's variational principles are characterized from two tumbled variational problems:

- The first problem, defined as auxiliary problem, is related to the elastostatic response of the reference homogeneous solid, subjected to a prescribed field of polarization (eigenstress or eigenstrain).
- The second problem, defined as optimization problem, has the objective to found the unknown field of polarization.

In the follows, the four classic Hashin and Shtrikman's variational principles are reported. It is worth to underline that two of these are minimum principles, while the other two are saddle principles. Naturally, the minimum principles are particularly useful, because each numeric approximation of them, for example by using the Finite Element Method, represents an upper estimation of the solution.

In particular, consider a reference homogeneous material which is more deformable than each phase included in the heterogeneous RVE, such that $\mathsf{C}-\mathsf{C}^H$ is positive definite everywhere in *V*. Hence, the following identity is verified:

$$\begin{aligned}&\int_V \left(\frac{1}{2}\mathsf{C}\boldsymbol{E}\cdot\boldsymbol{E} - \frac{1}{2}\mathsf{C}^H\boldsymbol{E}\cdot\boldsymbol{E} \right) dV = \\ &= \sup_{\boldsymbol{T}^*\in\mathsf{H}} \left\{ \int_V \left(\boldsymbol{T}^*\cdot\boldsymbol{E} - \frac{1}{2}\left(\mathsf{C}-\mathsf{C}^H\right)^{-1}\boldsymbol{T}^*\cdot\boldsymbol{T}^* \right) dV \right\}\end{aligned} \tag{1.7-3}$$

where:

$\mathsf{H} =$ the space of symmetric second-order periodic tensors

$\boldsymbol{T}^* =$ polarization field (eigenstress) prescribed on the reference homogeneous medium in order to simulate the actual micro-structure of the heterogeneous RVE.

In particular, by taking:

$$\boldsymbol{E} = \overline{\boldsymbol{E}} + \hat{\boldsymbol{E}} \tag{1.7-4}$$

where $\overline{\boldsymbol{E}} \in Sym$ and $\hat{\boldsymbol{E}} \in \mathsf{E}$, and by remembering that C^H is constant in *V*, the (1.7-3) assumes the following form:

$$\begin{aligned}&\int_V \left(\frac{1}{2}\mathsf{C}\left(\overline{\boldsymbol{E}}+\hat{\boldsymbol{E}}\right)\cdot\left(\overline{\boldsymbol{E}}+\hat{\boldsymbol{E}}\right) - \frac{1}{2}\mathsf{C}^H\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \right) dV = \\ &= \sup_{\boldsymbol{T}^*\in\mathsf{H}} \left\{ \int_V \left(<\boldsymbol{T}^*>\cdot\overline{\boldsymbol{E}} - \frac{1}{2}\left(\mathsf{C}-\mathsf{C}^H\right)^{-1}\boldsymbol{T}^*\cdot\boldsymbol{T}^* \right) dV \right\} + \\ &+\int_V \left(\boldsymbol{T}^*\cdot\hat{\boldsymbol{E}} + \frac{1}{2}\mathsf{C}^H\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}} \right) dV\end{aligned} \tag{1.7-5}$$

where $<\boldsymbol{T}^*>$ denotes the average value of $\boldsymbol{T}^*$ in *V*.

Therefore, by considering the lower bound with respect to $\hat{\boldsymbol{E}}$, changing the minimization with the maximization and by dividing for *V*, it is obtained:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\mathsf{C}^{H}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}=\\&=\frac{1}{V}\sup_{\boldsymbol{T}^{*}\in\mathsf{H}}\left\{\int_{V}\left(<\boldsymbol{T}^{*}>\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\left(\mathsf{C}-\mathsf{C}^{H}\right)^{-1}\boldsymbol{T}^{*}\cdot\boldsymbol{T}^{*}\right)dV+\inf_{\hat{\boldsymbol{E}}\in\mathsf{E}}F_{\mathsf{C}^{H}}^{(\boldsymbol{T}^{*})}\right\}\end{aligned}\tag{1.7-6}$$

where the quadratic functional $F_{\mathsf{C}^{H}}^{(\boldsymbol{T}^{*})}$ is defined by:

$$F_{\mathsf{C}^{H}}^{(\boldsymbol{T}^{*})}:\hat{\boldsymbol{E}}\in\mathsf{E}\rightarrow\int_{V}\left(\boldsymbol{T}^{*}\cdot\hat{\boldsymbol{E}}+\frac{1}{2}\mathsf{C}^{H}\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}}\right)dV\tag{1.7-7}$$

Consider, now, a reference homogeneous material which is stiffer than each phase included in the heterogeneous RVE, such that $\mathsf{C}-\mathsf{C}^{H}$ is negative definite everywhere in *V*. Hence, in analogous manner, it is obtained the following equation:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\mathsf{C}^{H}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}=\\&=\frac{1}{V}\inf_{\boldsymbol{T}^{*}\in\mathsf{H}}\left\{\int_{V}\left(<\boldsymbol{T}^{*}>\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\left(\mathsf{C}-\mathsf{C}^{H}\right)^{-1}\boldsymbol{T}^{*}\cdot\boldsymbol{T}^{*}\right)dV+\inf_{\hat{\boldsymbol{E}}\in\mathsf{E}}F_{\mathsf{C}^{H}}^{(\boldsymbol{T}^{*})}\right\}\end{aligned}\tag{1.7-8}$$

The equations (1.7-6) and (1.7-8) represent the Hashin and Shtrikman's variational principles, based on the eigenstress. In particular, the (1.7-6) is a saddle principle, while the (1.7-8) is a minimum principle. From them, by imposing stationariness principles with respect to $\boldsymbol{T}^{*}$, it is obtained:

$$\left(\mathsf{C}-\mathsf{C}^{H}\right)^{-1}\boldsymbol{T}^{*}=\hat{\boldsymbol{E}}+\overline{\boldsymbol{E}}\tag{1.7-9}$$

that confirms that stress field $\boldsymbol{T}^*$ is the correction which has to be prescribed to the reference homogeneous material stress field $\mathsf{C}^H\left(\hat{\boldsymbol{E}}+\overline{\boldsymbol{E}}\right)$ in order to obtain the stress field in the actual material $\mathsf{C}\left(\hat{\boldsymbol{E}}+\overline{\boldsymbol{E}}\right)$.

It is possible to obtain other two variational principles, having similar expressions to the (1.7-6) and the (1.7-8) and involving the overall compliance tensor $\overline{\mathsf{S}}$. About them, the sole results will be shown, directly, since they are reached with similar considerations to those ones already done.

Therefore, consider a reference homogeneous material which is stiffer than each phase included in the heterogeneous RVE, such that $\mathsf{S}-\mathsf{S}^H$ is positive definite everywhere in *V*. Hence, in analogous manner, it is obtained the following equation:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\mathsf{S}^H\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}=\\&=\frac{1}{V}\sup_{\boldsymbol{E}^*\in\mathsf{H}}\left\{\int_V\left(<\boldsymbol{E}^*>\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\left(\mathsf{S}-\mathsf{S}^H\right)^{-1}\boldsymbol{E}^*\cdot\boldsymbol{E}^*\right)dV+\inf_{\hat{\boldsymbol{T}}\in\mathsf{T}}F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}\right\}\end{aligned}\tag{1.7-10}$$

where the quadratic functional $F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}$ is defined by:

$$F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}:\hat{\boldsymbol{T}}\in\mathsf{T}\rightarrow\int_V\left(\boldsymbol{E}^*\cdot\hat{\boldsymbol{T}}+\frac{1}{2}\mathsf{S}^H\hat{\boldsymbol{T}}\cdot\hat{\boldsymbol{T}}\right)dV\tag{1.7-11}$$

and where:

$\mathsf{H}=$ the space of symmetric second-order periodic tensors

$\boldsymbol{E}^*=$ polarization field (eigenstrain) prescribed on the reference homogeneous medium in order to simulate the actual micro-structure of the heterogeneous RVE.

Consider, on the contrary, a reference homogeneous material which is more deformable than each phase included in the heterogeneous RVE, such that

$\mathsf{S}-\mathsf{S}^H$ is positive definite everywhere in *V*. Hence, in similar form, it is obtained the following equation:

$$\frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\mathsf{S}^H\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}= \\ =\frac{1}{V}\inf_{\boldsymbol{E}^*\in\mathsf{H}}\left\{\int_V\left(<\boldsymbol{E}^*>\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\left(\mathsf{S}-\mathsf{S}^H\right)^{-1}\boldsymbol{E}^*\cdot\boldsymbol{E}^*\right)dV+\inf_{\hat{\boldsymbol{T}}\in\mathsf{T}}F_{\mathsf{S}^H}^{(\boldsymbol{E}^*)}\right\} \tag{1.7-12}$$

The equations (1.7-10) and (1.7-12) represent the Hashin and Shtrikman's variational principles, based on the eigenstrain. In particular, the (1.7-10) is a saddle principle, while the (1.7-12) is a minimum principle. From them, by imposing stationariness principles with respect to $\boldsymbol{E}^*$, it is obtained:

$$\left(\mathsf{S}-\mathsf{S}^H\right)^{-1}\boldsymbol{E}^*=\hat{\boldsymbol{T}}+\overline{\boldsymbol{T}} \tag{1.7-13}$$

that confirms that strain field $\boldsymbol{E}^*$ is the correction which has to be prescribed to the reference homogeneous material strain field $\mathsf{S}^H\left(\hat{\boldsymbol{T}}+\overline{\boldsymbol{T}}\right)$ in order to obtain the strain field in the actual material $\mathsf{S}\left(\hat{\boldsymbol{T}}+\overline{\boldsymbol{T}}\right)$.

It has to be considered that the Hashin and Shtrikman's variational principles involve auxiliary problems, consisting in the minimization of the functionals, $F_{\mathsf{C}^H}^{(\boldsymbol{T}^*)}$ and $F_{\mathsf{S}^H}^{(\boldsymbol{E}^*)}$. The goal is to solve an equilibrium problem and a compatibility problem, respectively, for the reference homogeneous solid, subject to a prescribed eigenstress, $\boldsymbol{T}^*$, and eigenstrain, $\boldsymbol{E}^*$, respectively. For a such problem, however, only few particular cases shows the solution.

In particular, it can be remembered the Eshelby's solution for the case in which the polarization field is constant and different from zero, only in an ellipsoidal region. This solution lets to use the Hashin and Shtrikman's variational principles for determining the homogenized properties of a biphasic composite, with a low concentration of inclusions. In order to do it, the same matrix or the inclusions can be chosen as reference homogeneous material, but the matrix and the inclusions have to be well ordered, that means, $\mathsf{C}^M - \mathsf{C}^\Omega$ has to be defined in sign.

In case of periodic composite, the auxiliary problem is easier to solve, because it is possible to transform the RVE domain into a Fourier domain. It is not our interest to expose this procedure, so the interested reader is referred to [29].

The calculation of the elastic energy density and of the complementary one, according to the two equations of (1.7-1), requires the execution of very difficult minimization with respect of functionals, that are defined on unbounded space. Operating such minimizations is equivalent to solve the elastostatic problem for the RVE, in the cases of displacements approach and tractions approach, respectively. A numeric minimization, obtained, for example, by using the Element Finite Method, can be employed on finite subspaces, E_f and T_f, of the above mentioned spaces, E and T. Consequently, numeric minimization will yield the following expressions of the tensors, C^+ and S^+:

$$\begin{aligned} \frac{1}{2}\mathsf{C}^+\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} &= \inf_{\boldsymbol{E}^d\in\mathsf{E}_f}\frac{1}{V}\int_V \frac{1}{2}\mathsf{C}\left(\overline{\boldsymbol{E}}+\boldsymbol{E}^d\right)\cdot\left(\overline{\boldsymbol{E}}+\boldsymbol{E}^d\right)dV \\ \frac{1}{2}\mathsf{S}^+\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}} &= \inf_{\boldsymbol{T}^d\in\mathsf{T}_f}\frac{1}{V}\int_V \frac{1}{2}\mathsf{S}\left(\overline{\boldsymbol{T}}+\boldsymbol{T}^d\right)\cdot\left(\overline{\boldsymbol{T}}+\boldsymbol{T}^d\right)dV \end{aligned} \tag{1.7-14}$$

which, for constructions, satisfy the following inequalities:

$$\begin{aligned} \frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} &\leq \frac{1}{2}\mathsf{C}^+\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \\ \frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} &\leq \frac{1}{2}\mathsf{S}^+\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \end{aligned} \tag{1.7-15}$$

By naming with C^- and S^-, respectively, the inverse of the tensors S^+ and C^+, upper and lower limitations for the elastic energy, and the complementary one, of the homogenized material are obtained, as given by:

$$\begin{aligned}&\frac{1}{2}\mathsf{C}^-\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \le \frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \le \frac{1}{2}\mathsf{C}^+\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}\\ &\frac{1}{2}\mathsf{S}^-\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \le \frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}} \le \frac{1}{2}\mathsf{S}^+\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}\end{aligned} \tag{1.7-16}$$

Elementary estimations on $\overline{\mathsf{C}}$ and $\overline{\mathsf{S}}$ are obtained by choosing the simplest E_f and T_f, i.e., coinciding with the space constituted by the sole null tensor. In this way, the well known Voigt and Reuss' estimations dall'alto e dal basso are reached; in particular, for a biphasic composite, it is:

$$\begin{aligned}&\left(f_M\mathsf{S}_M + f_\Omega\mathsf{S}_\Omega\right)^{-1} \le \overline{\mathsf{C}} \le f_M\mathsf{C}_M + f_\Omega\mathsf{C}_\Omega\\ &\left(f_M\mathsf{C}_M + f_\Omega\mathsf{C}_\Omega\right)^{-1} \le \overline{\mathsf{S}} \le f_M\mathsf{S}_M + f_\Omega\mathsf{S}_\Omega\end{aligned} \tag{1.7-17}$$

with:

$$\begin{aligned}&f_M\mathsf{C}_M + f_\Omega\mathsf{C}_\Omega = \left(\mathsf{C}^+\right)^V, \qquad \left(f_M\mathsf{C}_M + f_\Omega\mathsf{C}_\Omega\right)^{-1} = \left(\mathsf{S}^-\right)^V\\ &\left(f_M\mathsf{S}_M + f_\Omega\mathsf{S}_\Omega\right)^{-1} = \left(\mathsf{C}^-\right)^R, \qquad f_M\mathsf{S}_M + f_\Omega\mathsf{S}_\Omega = \left(\mathsf{S}^+\right)^R\end{aligned} \tag{1.7-18}$$

where the superscript *V* and *R* stands for Voigt and Reuss.

At the same manner, The Hashin and Shtrikman's variational principles, (1.7-6), (1.7-8), (1.7-10) and (1.7-12) yield estimations dall'alto e dal basso on the stiffness and compliance tensors, if the optimization with regard to the polarization fields is employed above a finite underspace, H_f, of the above unbounded mentioned space H of all possible polarization fields.

In particular:

- if the reference homogeneous material is more deformable than each phase included in the heterogeneous RVE, it is:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\mathsf{C}^H\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}\geq\\&\geq\frac{1}{V}\sup_{\boldsymbol{T}^*\in\mathsf{H}_f}\left\{\int_V\left(<\boldsymbol{T}^*>\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\left(\mathsf{C}-\mathsf{C}^H\right)^{\boldsymbol{-1}}\boldsymbol{T}^*\cdot\boldsymbol{T}^*\right)dV+\inf_{\hat{\boldsymbol{E}}\in\mathsf{E}}F_{\mathsf{C}^H}^{\left(\boldsymbol{T}^*\right)}\right\}\end{aligned}\tag{1.7-19}$$

and:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\mathsf{S}^H\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}\leq\\&\leq\frac{1}{V}\inf_{\boldsymbol{E}^*\in\mathsf{H}_f}\left\{\int_V\left(<\boldsymbol{E}^*>\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\left(\mathsf{S}-\mathsf{S}^H\right)^{\boldsymbol{-1}}\boldsymbol{E}^*\cdot\boldsymbol{E}^*\right)dV+\inf_{\hat{\boldsymbol{T}}\in\mathsf{T}}F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}\right\}\end{aligned}\tag{1.7-20}$$

- if the reference homogeneous material is stiffer than each phase included in the heterogeneous RVE, it is:

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{C}}\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\mathsf{C}^H\overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{E}}\leq\\&\leq\frac{1}{V}\inf_{\boldsymbol{T}^*\in\mathsf{H}}\left\{\int_V\left(<\boldsymbol{T}^*>\cdot\overline{\boldsymbol{E}}-\frac{1}{2}\left(\mathsf{C}-\mathsf{C}^H\right)^{\boldsymbol{-1}}\boldsymbol{T}^*\cdot\boldsymbol{T}^*\right)dV+\inf_{\hat{\boldsymbol{E}}\in\mathsf{E}}F_{\mathsf{C}^H}^{\left(\boldsymbol{T}^*\right)}\right\}\end{aligned}\tag{1.7-21}$$

and

$$\begin{aligned}&\frac{1}{2}\overline{\mathsf{S}}\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\mathsf{S}^H\overline{\boldsymbol{T}}\cdot\overline{\boldsymbol{T}}\geq\\&\geq\frac{1}{V}\sup_{\boldsymbol{E}^*\in\mathsf{H}_f}\left\{\int_V\left(<\boldsymbol{E}^*>\cdot\overline{\boldsymbol{T}}-\frac{1}{2}\left(\mathsf{S}-\mathsf{S}^H\right)^{\boldsymbol{-1}}\boldsymbol{E}^*\cdot\boldsymbol{E}^*\right)dV+\inf_{\hat{\boldsymbol{T}}\in\mathsf{T}}F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}\right\}\end{aligned}\tag{1.7-22}$$

A numeric estimation of the inferior extreme of $F_{\mathsf{C}^H}^{\left(\boldsymbol{T}^*\right)}$ and of $F_{\mathsf{S}^H}^{\left(\boldsymbol{E}^*\right)}$ implies that only the minimum principles (1.7-20) and (1.7-21) yield upper estimations for the density of the elastic complementary energy and for the elastic one, respectively, for the homogenized material. The saddle principles (1.7-19) and (1.7-22), instead, are able to yield an estimation that cannot be read as an upper or lower estimation.

1.8 Inhomogeneous materials: Stress and Displacement Associated Solution Theorems.

As studied in the previous sections, the heterogeneity of the material implies an inhomogeneity of the same medium, so that the elastic properties of the solid are spatially variable in the examined volume.

It is well known the difficulty to find solutions to anisotropic inhomogeneous material problems. A very few restricted classes of these problems, in fact, are solved in a general way. For example, it can be cited the solution for cylinders subjected to pure torsion, possessing cylindrical orthotropy with a variation of the shear moduli with the local normal direction to the family of curves of which lateral boundary is a member, [17]. A second example is the exact solution for the case of an anisotropic half-space with elastic moduli dependent upon the coordinate, the angle q , when the loads on the half-space are represented by a straight line of force, [12]. A third example can be considered, that is the solution for problems in which the variation of the elastic constants is in the radial direction, [4].

In spite of this difficulty, in the last years, it has been a growing interest about the mechanical behaviour of anisotropic and inhomogeneous solids, above all in biomechanics. Moreover, the necessity to build thermodynamically consistent theories for this kind of materials, by means the employment of the mathematical theory of the *homogenization*, has determined the necessity to find exact analytical solutions in the ambit of this more complex section of the theory of elasticity, [37], [41].

In literature, a method has been presented by Fraldi & Cowin, 2004, [24], to overcome the difficulties exposed above: the use of two theorems, *S.A.S. theorem* and *D.A.S. theorem*, introduced by the authors, provides solutions for inhomogeneous, anisotropic elastostatic problems starting from the solution of *associated* anisotropic and homogeneous ones, but they have to be satisfied some conditions, that are exposed in the following.

In particular, the *stress-associated solution* (S.A.S.) theorem lets to find solutions for inhomogeneous anisotropic elastostatic problems, if two conditions are satisfied:

1. The solution of the homogeneous elastic *reference* problem (*the associated* one) is known and it has a stress state with a zero eigenvalue everywhere in the domain of the problem.
2. The inhomogeneous anisotropic elastic tensor is in relation with the homogeneous associated one according to the following equation:

$$\mathbb{C}^I = \varphi(\boldsymbol{x})\mathbb{C}^H, \qquad \varphi(\boldsymbol{x})\big|\forall \boldsymbol{x}\in B,\ \varphi(\boldsymbol{x}) > \alpha > 0,\ \alpha \in R^+ \tag{1.8-1}$$

where:

$\mathbb{C}^H = \mathbb{C}^{H^T} =$ the elasticity tensor of the anisotropic homogeneous elastic *reference problem*.

$\mathbb{C}^I =$ the elasticity tensor of the corresponding anisotropic inhomogeneous elastic problem.

$B =$ the domain occupied by both the homogeneous object (B^H) and the inhomogeneous one (B^I).

$\alpha \in R^+ =$ an arbitrary positive real number.

$\varphi(\boldsymbol{x}) =$ a $C^2(B)$ scalar function.

The second condition implies that the inhomogeneous character of the material is due to the presence of a scalar parameter, $\varphi(\boldsymbol{x})$, producing the inhomogeneity in the elastic constants. It can be also relaxed and, so, written in a weaker form:

$$\hat{C}^I_{ijhk} = \varphi\,\hat{C}^H_{ijhk} \tag{1.8-2}$$

where:

$\hat{C}^H_{ijhk}$ = those only elastic coefficients explicitly involved in the specific anisotropic homogeneous problem used to construct the associated solution.

This means that components of the elasticity tensor not involved in the solution of the homogeneous problem will not be involved in that one of the associated inhomogeneous problem.

If the conditions 1 and 2 are satisfied, starting from the known solution of the homogeneous problem, the *associated solution*, that is the solution to the inhomogeneous problem, is derived.

In particular, the strain-displacement field solution is identical with the strain-displacement field of the homogeneous reference solution, while the stress field of the inhomogeneous problem is equal to $\varphi(\boldsymbol{x})$ times the stress field of the homogeneous problem.

The advantage of this method is in the fact that its use yields both exact solutions for several new inhomogeneous and anisotropic problems and a redefinition of the already known solutions, like those ones for the *shape intrinsic anisotropic materials*, the *angularly inhomogeneous materials* and the *radially inhomogeneous materials.*

More in detail, let us to consider the following anisotropic homogeneous elastic object, that occupies a volume B^H, with mixed boundary-value:

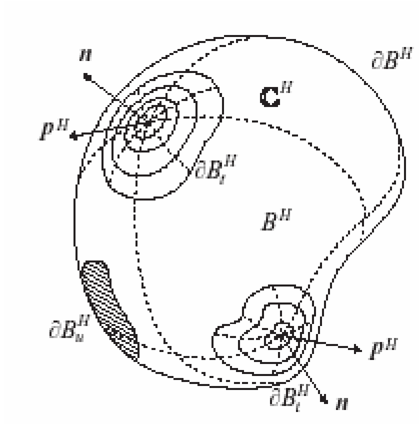


Figure 1.13 Homogeneous solid.

In absence of action-at-a-distance forces and taking into account the compatibility of the solution by writing the equilibrium equations in terms of displacements, the following equilibrium equations can be written:

$$\begin{aligned} &\nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{0} \quad && in\, B^H \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t} \quad && on\, \partial B_t^H \end{aligned} \tag{1.8-3}$$

where:

$\nabla = \partial_i \boldsymbol{e}_i =$ is a vectorial differential operator

$\partial B_t^H =$ the boundary partition of the homogeneous continuum on which the traction field is assigned.

On the boundary partition on which the displacements field is assigned, the following relation has to be satisfied:

$$\boldsymbol{u} = \boldsymbol{u^0} \quad on\, \partial B_u^H \tag{1.8-4}$$

where, in fact:

$\partial B_u^H =$ the boundary partition of the homogeneous continuum on which the displacements field is assigned.

The anisotropic Hooke's law, in a linear elastic stress-strain relation, is written in the form:

$$\boldsymbol{T}(\boldsymbol{u}) = \mathbb{C}^H : \boldsymbol{E}(\boldsymbol{u}) = \mathbb{C}^H : sym(\nabla \otimes \boldsymbol{u}) = \mathbb{C}^H : (\nabla \otimes \boldsymbol{u}) \tag{1.8-5}$$

or, in components:

$$\sigma_{ij} = C_{ijhk}^H \varepsilon_{hk} = C_{ijhk}^H u_{h,k} \tag{1.8-6}$$

Let us consider, now, the following anisotropic inhomogeneous elastic object, that occupies a volume B^I , geometrically the same of B^H , with mixed boundary-value:

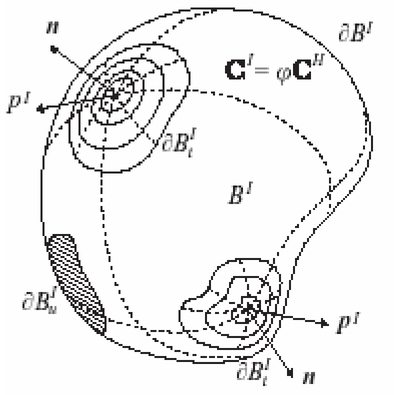


Figure 1.14 Inhomogeneous solid.

In absence of action-at-a-distance forces and taking into account the compatibility of the solution by writing the equilibrium equations in terms of displacements, in an analogous manner to what has been done before, the following equilibrium equations can be written:

$$\begin{aligned} &\nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \mathbf{0} \quad && in\, B^I \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t}^I \quad && on\, \partial B_t^I \end{aligned} \tag{1.8-7}$$

where:

$\partial B_t^I =$ the boundary partition of the inhomogeneous continuum on which the traction field is assigned. It is geometrically the same of that one in the homogeneous problem.

On the boundary partition on which the displacements field is assigned, the following relation has to be satisfied:

$$\boldsymbol{u} = \boldsymbol{u^0} \quad on\,\partial B_u^H \tag{1.8-8}$$

where, in fact:

$\partial B_u^I =$ the boundary partition of the inhomogeneous continuum on which the displacements field is assigned. It is geometrically the same of that one in the homogeneous problem.

The anisotropic Hooke’s law, in a linear elastic stress-strain relation, is written in the form:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{u}) &= \mathbb{C}^I : \boldsymbol{E}(\boldsymbol{u}) = \mathbb{C}^I : sym(\nabla \otimes \boldsymbol{u}) = \\ &= \varphi(\boldsymbol{x})\mathbb{C}^H \boldsymbol{:} sym(\nabla \otimes \boldsymbol{u}) = \varphi(\boldsymbol{x})\mathbb{C}^H \boldsymbol{:} (\nabla \otimes \boldsymbol{u}) \end{aligned} \tag{1.8-9}$$

according to the position (1.8-1).

So, taking into account the equations (1.8-1) and (1.8-9), yet, the first of the equilibrium equations (1.8-7), can be written in the following form:

$$\nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \varphi(\boldsymbol{x})\nabla \cdot \left[\mathbb{C}^H : \boldsymbol{E}(\boldsymbol{u})\right] + \left[\mathbb{C}^H : \boldsymbol{E}(\boldsymbol{u})\right] \cdot \nabla \varphi(\boldsymbol{x}) = \boldsymbol{0} \quad in\,B^I \tag{1.8-10}$$

If it is considered the hypothesis that the displacements field is equal in the homogeneous and inhomogeneous problems, that is:

$$\boldsymbol{u}^H = \boldsymbol{u}^I \tag{1.8-11}$$

where:

$\boldsymbol{u}^H =$ displacements field, solution of the homogeneous problem

$\boldsymbol{u}^I =$ displacements field, solution of the inhomogeneous problem

then, the equation (1.8-10) can be written in the form:

$$\nabla \cdot \boldsymbol{T}(\boldsymbol{u}^H) = \varphi(\boldsymbol{x})\left[\nabla \cdot \boldsymbol{T}^H(\boldsymbol{u}^H)\right] + \left[\boldsymbol{T}^H(\boldsymbol{u}^H)\right] \cdot \nabla \varphi(\boldsymbol{x}) = \boldsymbol{0} \quad in\,B^I \tag{1.8-12}$$

that is obtained by sobstituting:

$$\nabla \cdot \left[\mathbb{C}^H : \boldsymbol{E}(\boldsymbol{u}^H)\right] = \nabla \cdot \left[\boldsymbol{T}^H(\boldsymbol{u}^H)\right] \tag{1.8-13}$$

But, since the equation (1.8-13) is equal to zero, it follows that:

$$\left[\boldsymbol{T}^H\left(\boldsymbol{u}^H\right)\right]\cdot\nabla\varphi\left(\boldsymbol{x}\right)=\boldsymbol{0}\quad\forall x\in B^I \tag{1.8-14}$$

By excluding the trivial case in which $\varphi\left(\boldsymbol{x}\right)$ is constant, this means that:

1. the stress tensor $\boldsymbol{T}^H$ for the reference homogeneous problem has to be plane, at each internal point $\boldsymbol{x}\in B^H$, that is, it has to be a locally variable zero-eigenvalue stress state:

$$\det\boldsymbol{T}^H=0,\quad\forall\boldsymbol{x}\in B^H \tag{1.8-15}$$

2. the vector $\nabla\varphi$, at the corresponding points $\boldsymbol{x}\in B^I$, has to be coaxial with the eigenvector associated to the zero stress eigenvalue in the homogeneous problem.

In the previous statements, it has been implicitly considered the definition about the "*plane stress*": a stress state will be said *plane* if, in a fixed point $\boldsymbol{x}$ of the solid, there is a *plane of the stresses* to which all the stress components σ_{ij} belong. It is easy to demonstrate that this plane exists if the stress tensor $\boldsymbol{T}$ has a zero eigenvalue. So, if $\{\xi_1,\xi_2,\xi_3\}$ is the orthogonal *principal* reference frame of the stress tensor $\boldsymbol{T}$ and if ξ_3 is assumed, for example, as the eigenvector associated to the zero eigenvalue of $\boldsymbol{T}$, the *plane of the stresses* must coincide with $\xi_1-\xi_2$ plane.

It follows that a necessary and sufficient condition for the existence of a *plane stress* is given by:

$$\det\boldsymbol{T}=0 \tag{1.8-16}$$

The geometrical relationship (1.8-14) between the stress tensor $\boldsymbol{T}^H$ and the vector $\nabla\varphi$ may be rewritten in the form:

$$\left\{ \boldsymbol{T}^{H} \cdot \nabla \varphi = \boldsymbol{0} \right\} \Leftrightarrow \left\{ \forall \boldsymbol{v} \in V, \; \boldsymbol{T}^{H} : \left(\nabla \varphi \otimes \boldsymbol{v} \right) = 0 \right\} \qquad (1.8\text{-}17)$$

where:

$\boldsymbol{v} =$ any unit vector defined in the three-dimensional Euclidean space E^3

$V =$ the corresponding vector space

So, it follows that the stress vector on the plane whose normal is $\boldsymbol{v}$ is always orthogonal to the vector $\nabla \varphi$.

More in detail, representing the stress tensor $\boldsymbol{T}^H$ in the principal stress directions space, as:

$$\boldsymbol{T}^{H} = \begin{bmatrix} \sigma_{\xi 1}^{H} & 0 & 0 \\ 0 & \sigma_{\xi 2}^{H} & 0 \\ 0 & 0 & \sigma_{\xi 3}^{H} \end{bmatrix} \qquad (1.8\text{-}18)$$

and representing in the same space the gradient of the scalar function φ, as:

$$\nabla \varphi \left(\xi \right)^{T} = \left[\varphi_{,\xi 1} \;\; \varphi_{,\xi 2} \;\; \varphi_{,\xi 3} \right] \qquad (1.8\text{-}19)$$

the equation (1.8-14) becomes:

$$\sigma_{\xi 1}^{H} \varphi_{,\xi 1} = 0; \quad \sigma_{\xi 2}^{H} \varphi_{,\xi 2} = 0; \quad \sigma_{\xi 3}^{H} \varphi_{,\xi 3} = 0 \qquad (1.8\text{-}20)$$

and it is satisfied only if the two conditions above written are satisfied. The case of three zero eigenvalues of the stress tensor $\boldsymbol{T}^H$ in each point $\boldsymbol{x} \in B^H$ is trivial; The case of only one zero eigenvalue of the stress tensor $\boldsymbol{T}^H$ in each point $\boldsymbol{x} \in B^H$, for example in the ξ_3 direction, the only non zero component of the vector $\nabla \varphi$ at the corresponding points $\boldsymbol{x} \in B^I$ is $\varphi_{,\xi_3}$ (so, too, if there are two zero eigenvalues there can be two non-zero components of $\nabla \varphi$).

In the following figure it is shown the case of stress plane, for each point $\boldsymbol{x} \in B^H$, con eigenvalue σ_{ξ_3} equal to zero.

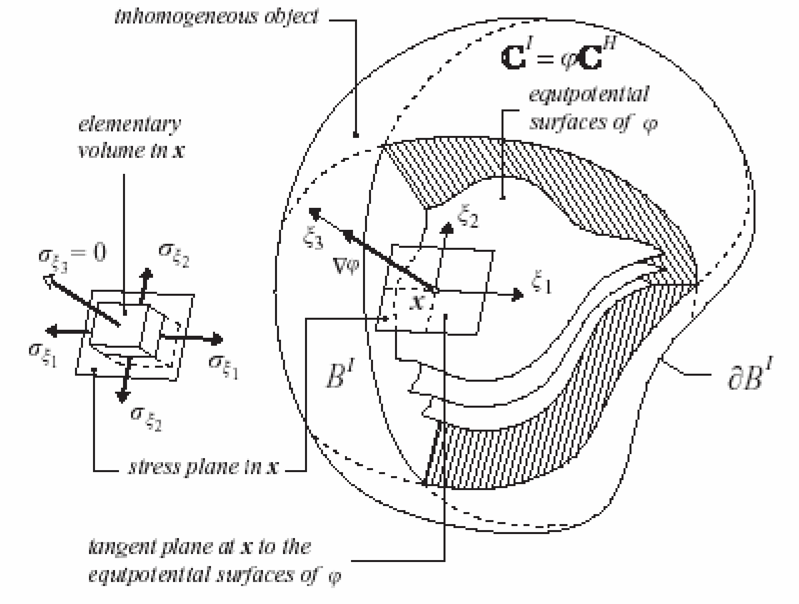


Figure 1.15 Geometrical interpretation of the relationship between the equipotential surfaces of and the distribution of the planes of stresses in the associated anisotropic problem.

It illustrates, in fact, that, at each internal point $\boldsymbol{x} \in B^I$, the equipotential surfaces of φ admit as a tangent plane the plane whose normal (parallel to $\nabla \varphi$) is coaxial with the eigenvector associated with the zero stress eigenvalue.

It can be noted that the assumed position (1.8-1) and the hypothesis (1.8-11) , that is true if the equation (1.8-14) is satisfied, imply:

$$\boldsymbol{T}^I = \varphi \boldsymbol{T}^H \tag{1.8-21}$$

So, the following theorem is established:

Stress associated solution theorem (SAS)

Consider two geometrically identical elastic objects, one homogeneous, B^H, and the other inhomogeneous, B^I, respectively. Let be C^H and $\mathsf{C}^I = \varphi(\boldsymbol{x})\mathsf{C}^H$ the corresponding elasticity tensors. The two elastostatic Cauchy problems associated with the two objects, in presence of the body forces and of mixed boundary-value, are:

$$
\begin{aligned}
&p^H : \left\{ \nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{0} \quad in\, B^H, \; \boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t} \;\; on\; \partial B_t^H, \quad \boldsymbol{u} = \boldsymbol{u^0} \;\; on\; \partial B_u^H \right\} \\
&p^I : \left\{ \nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{0} \quad in\, B^I, \; \boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \varphi\, \boldsymbol{t} \;\; on\; \partial B_t^I, \quad \boldsymbol{u} = \boldsymbol{u^0} \;\; on\; \partial B_u^I \right\}
\end{aligned}
\tag{1.8-22}
$$

where:

$$\varphi(\boldsymbol{x}) \in C^2(B)\Big| \; \forall \boldsymbol{x} \in B, \;\; \varphi(\boldsymbol{x}) > \alpha > 0, \;\; \alpha \in R^+$$

if $\boldsymbol{u}^H$ is the solution of the homogeneous problem p^H, then $\boldsymbol{u}^I = \boldsymbol{u}^H$ if and only if $\left\{ \boldsymbol{T}^H : (\nabla \varphi \otimes \boldsymbol{v}) = 0, \; \forall \boldsymbol{v} \in V \right\}$, i.e.:

$$\left\{ \forall \boldsymbol{x} \in B^I, \forall \boldsymbol{v} \in V, \boldsymbol{T}^H : (\nabla \varphi \otimes \boldsymbol{v}) = 0 \right\} \Leftrightarrow \boldsymbol{u}^I = \boldsymbol{u}^H \tag{1.8-23}$$

In other words, when a solution $\mathsf{B}_s^H = \left\{ \boldsymbol{u}^H, \boldsymbol{E}^H, \boldsymbol{T}^H \right\}$ for an anisotropic homogeneous elastic problem p^H is known , the SAS theorem yields the corresponding solution for an inhomogeneous elastic problem p^I as $\mathsf{B}_s^I = \left\{ \boldsymbol{u}^H, \boldsymbol{E}^H, \varphi\, \boldsymbol{T}^H \right\}$, if and only if $\boldsymbol{T}^H \cdot \nabla \varphi = \boldsymbol{0}$ everywhere in the object and the displacements boundary conditions are the same for both the homogeneous and inhomogeneous objects.

This SAS theorem can be generalized to comprise different types of composite materials, [24].

For example, let us to consider the composite materials for which each phase is characterized by constant elastic moduli within their own phase, but different from phase to phase.

This kind of inhomogeneity can be described by a scalar function j that is *constant* in each phase, but piecewise *discontinuous*.

In this case, in particular, for each phase p of the composite material, the elasticity tensor can be written as:

$$\mathsf{C}_p^H = j_p \mathsf{C}^H \qquad p = \{1, 2, \ldots, n\} \subset N \tag{1.8-24}$$

where:

C^H = the elasticity tensor of a reference isotropic or anisotropic homogeneous material whose geometries are the same of those ones of the composite material object.

C_p^H = the elasticity tensor of the phase p of the anisotropic homogeneous material which is homogeneous in it.

j_p = a positive scalar parameter, different from phase to phase.

Let us to consider a partition of the inhomogeneous body as:

$$\left\{ \Omega_p (B) \middle| B \equiv U_{p=1}^{n} \Omega_p (B) \right\} \tag{1.8-25}$$

and let us to indicate with $\partial\Omega_{(p,q)}$ the interface boundary between two generic sub-domains Ω_p and Ω_q of the partition, with elasticity tensors C_p^H and C_q^H, respectively.

In order to obtain the solution for this kind of composite material, starting from the known solution for the anisotropic homogeneous reference problem, it has to be:

$$\boldsymbol{T}_p^H = j_p \boldsymbol{T}^H \qquad \forall \boldsymbol{x} \in \Omega_p (B) \tag{1.8-26}$$

that is the condition required by the S.A.S theorem.

It is noted that this position satisfies the equilibrium equations in each sub-domain of the partition. In fact, it can be written:

$$\nabla \cdot \boldsymbol{T}_p^H = \varphi_p \nabla \cdot \boldsymbol{T}^H = \mathbf{0} \qquad \forall \boldsymbol{x} \in \Omega_p\left(B\right) \tag{1.8-27}$$

Moreover, according to the constitutive relationship, it can be written:

$$\boldsymbol{E}_p^H = \mathbb{C}_p^{H^{-1}} \boldsymbol{T}_p^H = \mathbb{C}^{H^{-1}} \boldsymbol{T}^H = \boldsymbol{E}^H \qquad \forall p \in N, \forall \boldsymbol{x} \in \Omega_p \tag{1.8-28}$$

that yields that compatibility condition on the discontinuity surfaces between the different phases of the composite is automatically satisfied and the same thing it can be said for the compatibility conditions on the external boundary.

As regards the limit equilibrium equations for the interface surfaces, it follows that:

$$\boldsymbol{T}_p^H \cdot \boldsymbol{n}_{(p,q)} = \boldsymbol{T}_q^H \cdot \boldsymbol{n}_{(q,p)} \qquad \left\{\forall \left\{p, q\right\} \in N, \forall \boldsymbol{x} \in \partial\Omega_{(p,q)}\right\} \tag{1.8-29}$$

where:

$\boldsymbol{n}_{(p,q)} =$ the unit normal vector to the interface between the phases p and q.

According to the equation (1.8-26), the equation (1.8-29) is satisfied if:

$$\boldsymbol{T}^H \cdot \boldsymbol{n}_{(p,q)} = \mathbf{0} \qquad \forall \boldsymbol{x} \in \partial\Omega_{(p,q)} \tag{1.8-30}$$

This means that, for each point belonging to the interface surfaces between two phases, the stress tensor $\boldsymbol{T}^H$ of the reference homogeneous material has to possess at least one zero eigenvalue, that is:

$$det\boldsymbol{T}^H = 0 \qquad \forall \boldsymbol{x} \in \partial\Omega_{(p,q)} \tag{1.8-31}$$

So, the eigenvector associated with the zero eigenvalue of the stress tensor is coaxial with the unit normal vector to the tangent plane to the interface.

Finally, to complete the elastic solution for the composite material, it is necessary that the equilibrium conditions on the external boundary were verified. In particular, indicating with $\partial B_{t(e)}$ the partition of the external boundary on which the tractions $\boldsymbol{t}^H$ are assigned in the homogeneous reference material, it can be written:

$$\boldsymbol{T}_e^H \cdot \boldsymbol{n} = \boldsymbol{t}_e^H = \varphi_e \boldsymbol{t}^H \qquad \forall \boldsymbol{x} \in \partial B_{t(e)} \tag{1.8-32}$$

where the total stress boundary is given by:

$$\partial B_t = \cup_{e=1}^{k} \partial B_{t(e)} \tag{1.8-33}$$

where:

$k =$ the total number of the phases that have a projection of their boundary on the external boundary on which the tractions are prescribed.

At this point, known the stress and strain fields that are elastic solution for the reference homogeneous problem, it is possible to built the elastic associated solution for the composite multi-phase materials with analogous geometry to the homogeneous problem.

It also has to be noted that the case of multi-phase materials, characterized by a scalar parameter φ, constant in each phase, can be seen as a generalization of the S.A.S. theorem where it is sufficient that the condition $det\boldsymbol{T}^H = 0$ were worth only in the points belonging to the internal interfaces between the different phases, and not necessarily in each point of the homogeneous body; in other words, the stress tensor $\boldsymbol{T}^H$ can be a three-dimensional stress field in any point of the domain, except for the points belonging to the interface surfaces.

A further example of materials to which the S.A.S. theorem can be applied is that one of composite multi-phase materials, where, in a more general situation, the following relation can be written for the elasticity tensor in each phase of the heterogeneous solid:

$$\mathsf{C}_p^H = \varphi_p(\boldsymbol{x})\mathsf{C}^H \quad \forall \boldsymbol{x}_p \in \Omega_p(B) \subset B \tag{1.8-34}$$

where:

$$\Omega_p(B)\Big| B \equiv \bigcup_{p=1}^{n} \Omega_p(B) \tag{1.8-35}$$

is again the considered partition of the inhomogeneous object.

The equation (1.8-34) means that, now, φ_p is a positive scalar function, not necessarily constant, but continuous inside each phase.

With analogous procedure to that one used before, it is easy to verify that, in order to extend the S.A.S. theorem to piecewise continuous composite materials, two facts have to be verified:

1. at each internal point of each phase *p*, the stress tensor $\boldsymbol{T}^H$ possesses at least one zero eigenvalue.
2. at every point belonging to the interface surfaces between two adjacent phases, the eigenvector associated with the zero eigenvalue of the stress tensor $\boldsymbol{T}^H$ is coaxial with the normal to the tangent plane.

For further examples of applicability of the S.A.S. theorem and for more details on its formulation, let us to send to the references being in literature, [24]. It is useful to underline, now, that the S.A.S. theorem yields the possibility to find a closed-form solution for inhomogeneous materials and it evidences that this possibility doesn't depend on the relation between geometry of the solid domain and orientation of the planes of the mirror symmetry but on the relation between the geometry of the stress distribution in the homogeneous material and the structural gradient of the inhomogeneous material.

In analogous manner, the *displacement-associated solution* (D.A.S.) theorem lets to find solutions for inhomogeneous anisotropic elastostatic problems, if two conditions are satisfied, [23]:

3. The solution of the homogeneous elastic *reference* problem (*the associated* one) is known and it has a local plane strain state, with a zero eigenvalue everywhere in the domain of the problem.
4. The inhomogeneous anisotropic compliance tensor is in relation with the homogeneous associated one according to the following equation:

$$\mathbf{S}^{I} = \frac{1}{\varphi(\boldsymbol{x})}\mathbf{C}^{H^{-1}} = \lambda(\boldsymbol{x})\mathbf{S}^{H}, \qquad \lambda(\boldsymbol{x})\big|\,\forall \boldsymbol{x}\in B,\ \lambda(\boldsymbol{x}) > \beta > 0,\ \beta \in R^{+} \tag{1.8-36}$$

where:

$\mathbf{S}^{H} = \mathbf{S}^{H^{T}} =$ the compliance tensor of the anisotropic homogeneous elastic *reference problem*.

$\mathbf{S}^{I} =$ the compliance tensor of the corresponding anisotropic inhomogeneous elastic problem.

$B =$ the domain occupied by both the homogeneous object (B^{H}) and the inhomogeneous one (B^{I}).

$\beta \in R^{+} =$ an arbitrary positive real number.

$\lambda(\boldsymbol{x}) =$ a $C^{2}(B)$ scalar function.

The second condition implies that the inhomogeneous character of the material is due to the presence of a scalar parameter, $\lambda(\boldsymbol{x})$, producing the inhomogeneity in the compliance coefficients. It can be also relaxed and, so, written in a weak form:

$$\hat{S}^{I}_{ijhk} = \lambda \hat{S}^{H}_{ijhk} \tag{1.8-37}$$

where:

$\hat{S}^{H}_{ijhk}$ = those only compliance coefficients explicitly involved in the specific anisotropic homogeneous problem used to construct the associated solution.

This means that components of the compliance tensor not involved in the solution of the homogeneous problem will not be involved in that one of the associated inhomogeneous problem.

If the conditions 3 and 4 are satisfied, starting from the known solution of the homogeneous problem, the *associated solution*, that is the solution to the inhomogeneous problem, is derived.

In particular, the stress field solution is identical with the stress field of the homogeneous reference solution, while the strain field of the inhomogeneous problem is equal to $\lambda(\boldsymbol{x})$ times the strain field of the homogeneous problem.

The advantage of this method is in the fact that its use yields exact solutions for several new interesting inhomogeneous and anisotropic problems.

More in detail, let us to consider an anisotropic homogeneous elastic object, that occupies a volume B^H, with mixed boundary-value (see figure 1.13).

In presence of action-at-a-distance forces and taking into account the compatibility of the solution by writing the equilibrium equations in terms of displacements, the following equilibrium equations can be written:

$$\begin{aligned} &\nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{0} && in\, B^H \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t} && on\, \partial B_t^H \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{0} && on\, \partial B_o^H \end{aligned} \qquad (1.8\text{-}38)$$

where:

$\nabla = \partial_i \boldsymbol{e_i} =$ is a vectorial differential operator

$\partial B_t^H =$ the boundary partition of the homogeneous continuum on which the traction field is assigned.

$\partial B_o^H =$ the boundary partition of the homogeneous continuum in absence of both traction and displacements fields.

On the boundary partition on which the displacements field is assigned, the following relation has to be satisfied:

$$\boldsymbol{u} = \boldsymbol{0} \quad on \, \partial B_u^H \tag{1.8-39}$$

where, in fact:

$\partial B_u^H =$ the boundary partition of the homogeneous continuum on which the displacements field is assigned.

The anisotropic Hooke's law, in a linear elastic stress-strain relation, is written in the form:

$$\boldsymbol{T}(\boldsymbol{u}) = \mathbb{C}^H : \boldsymbol{E}(\boldsymbol{u}) = \mathbb{C}^H : sym(\nabla \otimes \boldsymbol{u}) = \mathbb{C}^H : (\nabla \otimes \boldsymbol{u}) \tag{1.8-40}$$

or:

$$sym(\nabla \otimes \boldsymbol{u}) = \boldsymbol{E}(\boldsymbol{u}) = \mathbb{S}^H : \boldsymbol{T}(\boldsymbol{u}) \tag{1.8-41}$$

in components:

$$\sigma_{ij} = C_{ijhk}^H \varepsilon_{hk} = C_{ijhk}^H u_{h,k} \tag{1.8-42}$$

and:

$$\varepsilon_{ij} = S_{ijhk}^H \sigma_{hk} \tag{1.8-43}$$

Let us to consider, now, an anisotropic inhomogeneous elastic object, that occupies a volume B^I, geometrically the same of B^H, with mixed boundary-value (see figure 1.14).

In presence of action-at-a-distance forces and taking into account the compatibility of the solution by writing the equilibrium equations in terms of displacements, in an analogous manner to what has been done before, the following equilibrium equations can be written:

$$\begin{aligned} &\nabla \cdot \boldsymbol{T}(\boldsymbol{u}) = -\boldsymbol{b} \quad && in\, B^I \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t} \quad && on\, \partial B_t^I \\ &\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{0} \quad && on\, \partial B_o^I \end{aligned} \tag{1.8-44}$$

where:

$\partial B_t^I =$ the boundary partition of the inhomogeneous continuum on which the traction field is assigned. It is geometrically the same of that one in the homogeneous problem.

$\partial B_o^I =$ the boundary partition of the inhomogeneous continuum in absence of both traction and displacements fields. It is geometrically the same of that one in the homogeneous problem.

On the boundary partition on which the displacements field is assigned, the following relation has to be satisfied:

$$\boldsymbol{u} = \boldsymbol{0} \quad on\, \partial B_u^H \tag{1.8-45}$$

where, in fact:

$\partial B_u^I =$ the boundary partition of the inhomogeneous continuum on which the displacements field is assigned. It is geometrically the same of that one in the homogeneous problem.

Let us to assume the stress tensor $\boldsymbol{T}^H$ as the solution for the homogeneous problem, and let us to assume, also, the hypothesis that:

$$\boldsymbol{T}^I = \boldsymbol{T}^H \tag{1.8-46}$$

In this way, the equations in the differential system (1.8-38) are automatically satisfied. Moreover, if $\boldsymbol{T}^{H}$ is the solution of the first anisotropic and homogeneous problem, we have that the compatibility condition

$$\nabla \times [\nabla \times (\mathsf{S}^{H} : \boldsymbol{T}^{H})] = \mathbf{O} \tag{1.8-47}$$

have to be also satisfied. As well-known, this ensures that a displacement field $\boldsymbol{u}^{H}$ exists. So, it is possible to write the strain-displacement relationship

$$\boldsymbol{E}^{H} = \mathsf{S}^{H} : \boldsymbol{T}^{H} = sym\, \nabla \otimes \boldsymbol{u}^{H} \tag{1.8-48}$$

where:

$\boldsymbol{u}^{H} =$ displacements field, solution of the homogeneous problem

Then, in order to accept the hypothesis (1.8-46), the following equation:

$$\nabla \times [\nabla \times (\mathsf{S}^{I} : \boldsymbol{T}^{I})] = \nabla \times [\nabla \times (\lambda\, \mathsf{S}^{H} : \boldsymbol{T}^{H})] = \mathbf{O} \tag{1.8-49}$$

becomes necessary and sufficient condition for the existence of a displacement field $\boldsymbol{u}^{I}$, where $\boldsymbol{u}^{I}$ is the displacements field, solution of the inhomogeneous problem, and it is given by:

$$sym\, \nabla \otimes \boldsymbol{u}^{I} = \boldsymbol{E}^{I} = \mathsf{S}^{I} : \boldsymbol{T}^{I} = \lambda\, \mathsf{S}^{H} : \boldsymbol{T}^{H} \tag{1.8-50}$$

The compatibility condition (1.8-49), in general, is not satisfied. Therefore, it is necessary to find the conditions under whose it becomes true, [23].

Without loss of generality, let us consider:

$$\lambda = \lambda(\boldsymbol{x}_3) \tag{1.8-51}$$

This means that $\boldsymbol{x}_3$ is the direction that is locally coaxial with the gradient of λ, i. e.,

$$\nabla \lambda^T = [0, 0, \partial \lambda / \partial x_3] \tag{1.8-52}$$

So, by recalling that $\boldsymbol{u}^H$ is the solution of the homogeneous problem, and by operating some algebraic manipulations, the set of compatibility equations (1.8-49) can be reduced to five differential equations as it is shown:

$$\begin{cases} \lambda_{,33} u^H_{1,1} + \lambda_{,3} (u^H_{1,3} - u^H_{3,1})_{,1} = 0 \\ \lambda_{,33} u^H_{2,2} + \lambda_{,3} (u^H_{2,3} - u^H_{3,2})_{,2} = 0 \\ \lambda_{,3} (u^H_{1,2} - u^H_{2,1})_{,1} = 0 \\ \lambda_{,3} (u^H_{1,2} - u^H_{2,1})_{,2} = 0 \\ \lambda_{,33} (u^H_{1,2} + u^H_{2,1}) + \lambda_{,3} [(u^H_{1,3} - u^H_{3,1})_{,2} + (u^H_{2,3} - u^H_{3,2})_{,1}] = 0 \end{cases} \tag{1.8-53}$$

where, obviously, is absent any prescribed constrain about the relation between the first and the second derivatives of the parameter λ, [23].

It can be noted that the terms in the parentheses represent the skew components of the $\nabla \otimes \boldsymbol{u}^H$, that are local rotations, while the only present strain components are $(1-\delta_{i3})(1-\delta_{j3}) u^H_{i,j}$, having indicated with δ_{hk} the standard Kronecker operator.

It has to be noted that:

1. the displacement field for the reference homogeneous problem has to be related, at each internal point $\boldsymbol{x} \in B^H$, with a local plane

strain field, where any plane with support the axis $\boldsymbol{x}_3$ can be the plane of the strains:

$$\det\boldsymbol{E}^H = 0, \quad \forall \boldsymbol{x} \in B^H \tag{1.8-54}$$

2. the vector Ñl , at the corresponding points $\boldsymbol{x} \in B^I$, has to be coaxial with the support axis $\boldsymbol{x}_3$ of plane of the strains in the homogeneous problem.
3. $curl(\boldsymbol{u}^H)$ must be independent from $\boldsymbol{x}_3$-direction, i.e. the Ñl -direction.

In the previous statements, analogously to what has been done with the stress state, it has been implicitly considered the definition about the "*plane strain*": a strain state will be said *plane* if, in a fixed point $\boldsymbol{x}$ of the solid, there is a *plane of the strains* to which all the strain components e_{ij} belong. It is easy to demonstrate that this plane exists if the strain tensor $\boldsymbol{E}$ has a zero eigenvalue. So, if $\{x_1, x_2, x_3\}$ is the orthogonal *principal* reference frame of the strain tensor $\boldsymbol{E}$ and if x_3 is assumed, for example, as the eigenvector associated to the zero eigenvalue of $\boldsymbol{E}$, the *plane of the strains* must coincide with $x_1 - x_3$ plane.

It follows that a necessary and sufficient condition for the existence of a *plane strain* is given by:

$$det\boldsymbol{E} = 0 \tag{1.8-55}$$

It has to be noted that the satisfaction of the compatibility condition (1.8-49) yields that the displacements field of the homogeneous problem has to satisfy the equations (1.8-53).

This compatibility condition (1.8-49), therefore, may be rewritten in the form:

$$
\begin{aligned}
&\left\{ curl\,\mathbf{[}curl\,\mathbf{(}\lambda\,\mathsf{S}^{H}\mathbf{:}\boldsymbol{T}^{H}\mathbf{)]}=\mathbf{O}\right\} \Leftrightarrow \\
&\Leftrightarrow \left\{ \forall \boldsymbol{h}\in V : \nabla\lambda \cdot \boldsymbol{h}=0,\ (\nabla\otimes curl\,\boldsymbol{u}^{H}\ \boldsymbol{h}=\mathbf{0},\ sym(\nabla\otimes\boldsymbol{u}^{H})\boldsymbol{h}\cdot\boldsymbol{h}=0 \right\}
\end{aligned}
\tag{1.8-56}
$$

where:

$\lambda\left(\boldsymbol{x}\right)\in C^{2}\left(B\right)\Big|\ \forall \boldsymbol{x}\in B,\ \ \lambda\left(\boldsymbol{x}\right)>\alpha>0,\ \ \alpha\in R^{+}$

$\boldsymbol{h} =$ any unit vector defined in the three-dimensional Euclidean space E^{3}

$V =$ the corresponding vector space

Moreover, it is worth to note that the assumed position (1.8-36) and the hypothesis (1.8-46), that is true if the equation (1.8-49) is satisfied, imply:

$$\boldsymbol{E}^{I} = \lambda\,\boldsymbol{E}^{H} \tag{1.8-57}$$

So, at this point, it can be stated that any anisotropic and homogeneous elastic problem that possesses a solution represented by the displacement equations can be considered a *Displacement Auxiliary Solution* for the corresponding dual inhomogeneous elastic problem.

In other words, it can be possible to demonstrate the following theorem:

Displacement associated solution theorem (DAS)

Consider two geometrically identical anisotropic elastic objects, one homogeneous, B^{H}, and the other inhomogeneous, B^{I}, respectively. Let be

S^H and $\mathsf{S}^I = \lambda\left(\boldsymbol{x}\right)\mathsf{S}^H$ the corresponding compliance tensors. The two elastostatic Cauchy problems associated with the two objects, in presence of the body forces and of mixed boundary-value, are:

$$
\begin{aligned}
&p^H:\left\{\nabla\cdot\boldsymbol{T}(\boldsymbol{u})=-\boldsymbol{b}\ in B^H,\ \boldsymbol{T}(\boldsymbol{u})\cdot\boldsymbol{n}=\boldsymbol{t}\ on\,\partial B_t^H,\boldsymbol{T}(\boldsymbol{u})\cdot\boldsymbol{n}=\boldsymbol{0}\ on\,\partial B_o^H,\boldsymbol{u}=\boldsymbol{0}\ on\,\partial B_u^H\right\}\\
&p^I:\left\{\nabla\cdot\boldsymbol{T}(\boldsymbol{u})=-\boldsymbol{b}\ in B^I,\ \boldsymbol{T}(\boldsymbol{u})\cdot\boldsymbol{n}=\boldsymbol{t}\ on\,\partial B_t^I,\boldsymbol{T}(\boldsymbol{u})\cdot\boldsymbol{n}=\boldsymbol{0}\ on\,\partial B_o^I,\boldsymbol{u}=\boldsymbol{0}\ on\,\partial B_u^I\right\}
\end{aligned}
\tag{1.8-58}
$$

If $\boldsymbol{T}^H$ is the solution of the homogeneous problem p^H, then $\boldsymbol{T}^I = \boldsymbol{T}^H$ if and only if the second part of the equation (1.8-56) is verified, i.e.:

if $\boldsymbol{w}^H = curl\,\boldsymbol{u}^H \,\middle|\, \forall\boldsymbol{v}\in V, skew\left(\nabla\otimes\boldsymbol{u}^H\right)\boldsymbol{v} = \boldsymbol{w}^H\wedge\boldsymbol{v}$

we have that:

$$
\begin{aligned}
&\forall\boldsymbol{h}\in V\,\middle|\,\nabla\lambda\cdot\boldsymbol{h}=0,\left\{(\nabla\otimes curl\,\boldsymbol{u}^H\ \boldsymbol{h}=\boldsymbol{0},\ sym(\nabla\otimes\boldsymbol{u}^H)\boldsymbol{h}\cdot\boldsymbol{h}=0\right\}\Leftrightarrow\\
&\qquad\qquad\Leftrightarrow\left\{\boldsymbol{T}^I=\boldsymbol{T}^H\right\}
\end{aligned}
\tag{1.8-59}
$$

In other words, when a solution $\mathsf{B}_e^H = \left\{\boldsymbol{u}^H, \boldsymbol{E}^H, \boldsymbol{T}^H\right\}$ for an anisotropic homogeneous elastic problem p^H is known , the DAS theorem yields the corresponding solution for an inhomogeneous elastic problem p^I as $\mathsf{B}_e^I = \left\{\lambda\boldsymbol{E}^H, \boldsymbol{T}^H\right\}$, if and only if the anisotropic and homogeneous elastic problem possesses, everywhere in the object, a displacement solution satisfying the equations (1.8-53) and if the displacements boundary conditions are the same for both the homogeneous and inhomogeneous objects.

The solution $\boldsymbol{u}^I$, for the inhomogeneous problem, in general, have to be integrated with reference to the specific case.

It is worth to underline that in the case where displacement boundary-value $\boldsymbol{u}$ is not equal to zero, the elastic mixed problem can be rewritten as the corresponding first type one, in which only the traction and reaction fields are considered.

For more details on D.A.S. demonstration, let us to send to the references being in literature, [23].

It is useful to underline, now and again, the geometrical interpretation of the result of the theorem, constituted by the observation that, in order to find an analytical solution for a given elastic inhomogeneous and anisotropic body in the form $B_{\varepsilon}^{I} = \left\{ \lambda \boldsymbol{E}^{H}, \boldsymbol{T}^{H} \right\}$, a necessary and sufficient condition is that the displacement solution for the corresponding anisotropic and homogeneous problem is related with a local plane strain field that has as plane of the strains any plane with support an axis coaxial with the gradient of λ, with rotational part depending on this gradient direction, only.

The D.A.S. theorem can be generalized to comprise different types of composite materials.

For example, it is possible to consider the case of a multi-linear law for λ, i.e.:

$$\lambda = \lambda_0 + \lambda_1 \boldsymbol{x_1} + \lambda_2 \boldsymbol{x_2} + \lambda_3 \boldsymbol{x_3} \tag{1.8-60}$$

with $\lambda_i,\ i = \{0, ..., 3\}$ arbitrary constants.

In this case, it is obtained that the second derivatives of the differential system (1.8-53) go to zero, therefore, the compatibility equation system becomes as it follows:

$$
\begin{cases}
\lambda_1 (u^*_{1,2} - u^*_{2,1})_{,2} = \lambda_2 (u^*_{1,2} - u^*_{2,1})_{,1} \\
\lambda_2 (u^*_{2,3} - u^*_{3,2})_{,3} = \lambda_3 (u^*_{2,3} - u^*_{3,2})_{,2} \\
\lambda_3 (u^*_{1,3} - u^*_{3,1})_{,1} = \lambda_1 (u^*_{1,3} - u^*_{3,1})_{,3} \\
\lambda_1 [(u^*_{1,2} - u^*_{2,1})_{,3} + (u^*_{1,3} - u^*_{3,1})_{,2}] = \lambda_2 (u^*_{1,3} - u^*_{3,1})_{,1} + \lambda_3 (u^*_{1,2} - u^*_{2,1})_{,1} \\
\lambda_2 [(u^*_{2,1} - u^*_{1,2})_{,3} + (u^*_{2,3} - u^*_{3,2})_{,1}] = \lambda_1 (u^*_{2,3} - u^*_{3,2})_{,2} + \lambda_3 (u^*_{2,1} - u^*_{1,2})_{,2} \\
\lambda_3 [(u^*_{3,1} - u^*_{3,1})_{,2} + (u^*_{3,2} - u^*_{2,3})_{,1}] = \lambda_1 (u^*_{3,2} - u^*_{2,3})_{,3} + \lambda_2 (u^*_{3,1} - u^*_{1,3})_{,3}
\end{cases}
\tag{1.8-61}
$$

Because of the arbitrary of the assumption about the constants in the λ law, by setting to zero all skew components of $\nabla \otimes \boldsymbol{u}^H$, a very closed solution of the system can be found in the classical *strain potential* form, [8], that is

$$\boldsymbol{u}^H = \nabla \varphi \tag{1.8-62}$$

where $\varphi = \varphi(\boldsymbol{x})$ is a scalar function. The displacement in the form of the equation (1.8-61) produces, as well-known, an irrotational deformation field and constitutes the irrotational part of the Papkovich-neuber representation in the isotropic elasticity, [8]. The reason for which this particular case could result very useful is related to the fact that many fundamental solutions in isotropic and anisotropic elasticity have a representation as described in (1.8-61), as the axisymmetric, thermoelastic and heat-conduction problems.

It is, also, interesting to observe that, for the case of multi-linear law of λ, not any prescription on the form of the strain tensor $\boldsymbol{E}^H$ is necessary and, so, it is possible to use as *Displacement Associated Solutions* all the three dimensional solutions about anisotropic elasticity, satisfying the equation (1.8-61), that is, all the three dimensional solutions that satisfy the equation:

$$curl\, \boldsymbol{u}^H = \mathbf{0} \tag{1.8-63}$$

For the examples of applicability of the D.A.S. theorem and for more details on its formulation, let us to send to the references being in literature, [23].

It is worth to note that the D.A.S. theorem, like the S.A.S. one, yields the possibility to find a closed-form solution for some inhomogeneous materials and it evidences that this possibility depends, in general, on the relation between the geometry of the strain distribution in the homogeneous material and the structural gradient, $\nabla \lambda$, of the inhomogeneous material.

APPENDIX

The components of the Eshelby tensor S, with respect to a rectangular Cartesian coordinate system, are listed for several special cases, [47]. In particular, it is here considered a matrix *M* to be unbounded and isotropically elastic, and the inclusion Ω to be ellipsoidal with semiprincipal axes, $\boldsymbol{a_i}$, which coincide with the coordinate axes, $\boldsymbol{x_i}\,(i=1,2,3)$, as shown in the following figure:

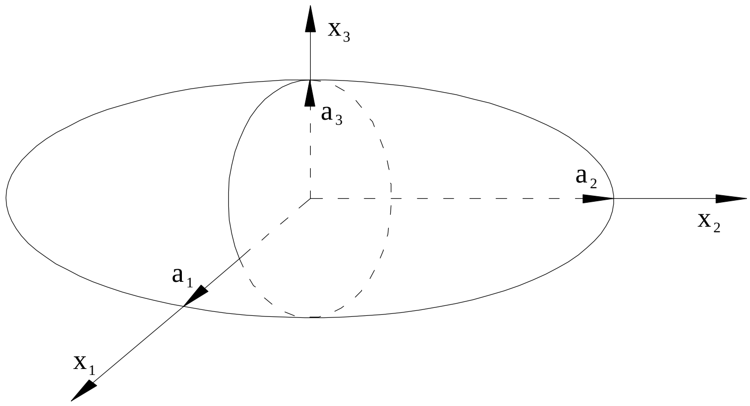


Figure 1.16 An ellipsoidal coaxial with the Cartesian coordinates.

1) General form $(\boldsymbol{a_1} > \boldsymbol{a_2} > \boldsymbol{a_3})$:

$$\mathsf{S}_{1111} = \frac{3}{8\pi(1-\nu)} a_1^2 I_{11} + \frac{(1-2\nu)}{8\pi(1-\nu)} I_1$$

$$\mathsf{S}_{1122} = \frac{1}{8\pi(1-\nu)} a_2^2 I_{12} - \frac{(1-2\nu)}{8\pi(1-\nu)} I_1 \qquad [1\text{-}1]$$

$$\mathsf{S}_{1212} = \frac{1}{16\pi(1-\nu)} \left(a_1^2 + a_2^2\right) I_{12} + \frac{(1-2\nu)}{16\pi(1-\nu)} \left(I_1 + I_2\right)$$

where the I_i and I_{ij} integrals are given by:

$$
\begin{aligned}
I_1 &= \frac{4\pi a_1 a_2 a_3}{\left(a_1^2-a_2^2\right)\left(a_1^2-a_3^2\right)^{1/2}}\left\{F(\theta,k)-E(\theta,k)\right\} \\
I_3 &= \frac{4\pi a_1 a_2 a_3}{\left(a_2^2-a_3^2\right)\left(a_1^2-a_3^2\right)^{1/2}}\left\{\frac{a_2\left(a_1^2-a_3^2\right)^{1/2}}{a_1 a_3}-E(\theta,k)\right\} \\
I_1 &+ I_2 + I_3 = 4\pi
\end{aligned}
\qquad [1\text{-}2]
$$

and:

$$
\begin{aligned}
&3I_{11}+I_{12}+I_{13}=\frac{4\pi}{a_1^2}, \qquad 3a_1^2 I_{11}+a_2^2 I_{12}+a_3^2 I_{13}=3I_1 \\
&I_{12}=\frac{I_2-I_1}{a_1^2-a_2^2}
\end{aligned}
\qquad [1\text{-}3]
$$

where *F* and *E* are the elliptic integrals of the first and the second kind, and:

$$
\theta=\arcsin\left\{\frac{a_1^2-a_3^2}{a_1^2}\right\}^{1/2}, \qquad k=\left\{\frac{a_1^2-a_2^2}{a_1^2-a_3^2}\right\}^{1/2} \qquad [1\text{-}4]
$$

2) Sphere $\left(\boldsymbol{a_1}=\boldsymbol{a_2}=\boldsymbol{a_3}=\boldsymbol{a}\right)$:

$$
S_{ijkl}=\frac{5\nu-1}{15(1-\nu)}\delta_{ij}\delta_{kl}+\frac{4-5\nu}{15(1-\nu)}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right) \qquad [1\text{-}5]
$$

3) Elliptic cylinder $\left(\boldsymbol{a_3}\to\infty\right)$:

$$S_{1111} = \frac{1}{2(1-n)}\left\{\frac{a_2^2 + 2a_1a_2}{(a_1+a_2)^2} + (1-2n)\frac{a_2}{a_1+a_2}\right\}$$

$$S_{2222} = \frac{1}{2(1-n)}\left\{\frac{a_1^2 + 2a_1a_2}{(a_1+a_2)^2} + (1-2n)\frac{a_1}{a_1+a_2}\right\}$$

$$S_{3333} = 0$$

$$S_{1122} = \frac{1}{2(1-n)}\left\{\frac{a_2^2}{(a_1+a_2)^2} - (1-2n)\frac{a_2}{a_1+a_2}\right\}$$

$$S_{2233} = \frac{1}{2(1-n)}\frac{2n\,a_1}{a_1+a_2}$$

$$S_{2211} = \frac{1}{2(1-n)}\left\{\frac{a_1^2}{(a_1+a_2)^2} - (1-2n)\frac{a_1}{a_1+a_2}\right\}$$

$$S_{1133} = \frac{1}{2(1-n)}\frac{2n\,a_2}{a_1+a_2}$$

$$S_{3311} = S_{3322} = 0$$

$$S_{1212} = \frac{1}{2(1-n)}\left\{\frac{a_1^2 + a_2^2}{2(a_1+a_2)^2} + \frac{(1-2n)}{2}\right\}$$

$$S_{2323} = \frac{a_1}{2(a_1+a_2)} \qquad [1\text{-}6]$$

$$S_{3131} = \frac{a_2}{2(a_1+a_2)}$$

4) Penny-shape $(\boldsymbol{a_1} = \boldsymbol{a_2}\ ?\ \boldsymbol{a_3})$:

$$S_{1111} = S_{2222} = \frac{\pi(13-8\nu)}{32(1-\nu)}\frac{a_3}{a_1}$$

$$S_{3333} = 1-\frac{\pi(1-2\nu)}{4(1-\nu)}\frac{a_3}{a_1}$$

$$S_{1122} = S_{2211} = \frac{\pi(-1+8\nu)}{32(1-\nu)}\frac{a_3}{a_1}$$

$$S_{2233} = S_{1133} = \frac{\pi(-1+2\nu)}{8(1-\nu)}\frac{a_3}{a_1}$$

$$S_{3311} = S_{3322} = \frac{\nu}{1-\nu}\left\{1-\frac{\pi(4\nu+1)}{8\nu}\frac{a_3}{a_1}\right\}$$

$$S_{1212} = \frac{\pi(7-8\nu)}{32(1-\nu)}\frac{a_3}{a_1}$$

$$S_{3131} = S_{2323} = \frac{1}{2}\left\{1+\frac{\pi(\nu-2)}{4(1-\nu)}\frac{a_3}{a_1}\right\} \qquad [1\text{-}7]$$

CHAPTER II

Homogenization theory

2.1 Introduction

In this chapter, a short introduction to the notion of the *homogenization* and of the essential concepts connected to it is provided.
In particular, by considering a heterogeneous medium, i.e., a medium whose material properties vary pointwise in a continuous or discontinuous manner, in a periodic or non periodic way, deterministically or randomly, *homogenization* can be defined as the modelling technique of such a heterogeneous medium by means a unique continuous medium, [41]. Furthermore, its goal is to determine the mechanical parameters of the unique fictitious material that "best" represents the real heterogeneous material or composite material. Obviously, homogenization procedure applies itself to all fields of macroscopic physics, but we will focus the attention on the mechanics of elastic bodies, particularly, on composite materials.

Since most of the composite materials present a brittle, rather than ductile, behaviour and, so, the elastic behaviour prevails, often there is no need to consider the homogenization in an elasto-plastic range. Such an approach cannot be ignored when the plastic behaviour comes into play, like in the composites which have a metallic matrix, for example. This leads to some difficulty since the solution of the elasto-plastic homogenization problem in an exact form is available only for very simple cases. However, we will be interested in the elastic response of the homogenized material.

2.2 General theory

In the chapter 1, it has been noticed that, in order to employ a homogenization procedure, two different scales are used in the description of the heterogeneous media. One of these, we remember, is a *macroscopic* scale at which homogeneities are weak, [41]. The other one is the scale of inhomogeneities and it has been defined as the *microscopic* scale. The latter defines the size of the representative volume element.

About the notion of the RVE, it can be said that, from the experimental point of view, there exists a kind of statistical homogeneity, in the sense that any RVE at a specific point looks very much like any other RVE taken at random at another point.

The mechanical problem presents itself in the following manner, [41].

Let $\boldsymbol{T}(\boldsymbol{x})$ and $\boldsymbol{E}(\boldsymbol{x})$ be the stress and the strain field at the microscale in the framework of the examined RVE and denote, analogously to what has been done in the previous chapter, the same mechanic quantities at macroscale by $\mathsf{S}(\boldsymbol{X})$ and $\mathsf{E}(\boldsymbol{X})$, and the averaging operator by $<...>$. Hence, for a volume averaging, we have, as already seen before:

$$\begin{aligned} \mathsf{S}\left(\boldsymbol{X}\right) &= <\boldsymbol{T}\left(\boldsymbol{x}\right)> = \frac{1}{V}\int_V \boldsymbol{T}\left(\boldsymbol{x}\right)dV \\ \mathsf{E}\left(\boldsymbol{X}\right) &= <\boldsymbol{E}\left(\boldsymbol{x}\right)> = \frac{1}{V}\int_V \boldsymbol{E}\left(\boldsymbol{x}\right)dV \end{aligned} \qquad (2.2\text{-}1)$$

where V is the volume of the representative element.

In literature, [41], the following definitions are given:

a) The process that relates the macrofields $\left(\mathsf{S},\mathsf{E}\right)$, by means of the (2.2-1), to the microscopic constitutive equations is called *homogenization.*

b) The “inverse” process which consists in determining the microfields $\left(\boldsymbol{T},\boldsymbol{E}\right)$ from the macrofields $\left(\mathsf{S},\mathsf{E}\right)$ is called *localization*.

Therefore, in the localization process, the data are the prescribed macrostress S, or the prescribed macrostrain E, and such problem corresponds to the following one:

$$\left(L.P.\right)\begin{Bmatrix} <\boldsymbol{T}> = \mathsf{S}, \qquad <\boldsymbol{E}> = \mathsf{E} \\ div\boldsymbol{T} = \boldsymbol{0}, \qquad \text{"micro" equilibrium} \\ \text{the "micro" behaviour is known} \end{Bmatrix} \qquad (2.2\text{-}2)$$

This is a particular ill-posed problem, because of the following two reasons:

1. The prescribed load is not a prescription at points in the bulk or at a limiting surface, but it is the averaged value of a field.
2. There are no boundary conditions.

The missing boundary conditions must, in some way, reproduce the internal state of the RVE in the most satisfactory manner. They therefore depend on the choice of RVE, more specifically on its size. Different choices of RVE, in fact, will provide different macroscopic laws.

Hence, the following relations give some examples of boundary conditions:

$$\boldsymbol{T}\cdot\boldsymbol{n}=\mathsf{S}\cdot\boldsymbol{n}\ \ on\ \partial V \qquad (2.2\text{-}3)$$

$$\boldsymbol{u}=\mathsf{E}\cdot\boldsymbol{x}\ \ on\,\partial V \qquad (2.2\text{-}4)$$

They represent, respectively, the condition of prescribed uniform tractions on ∂V and the condition of prescribed uniform strains on ∂V. These two conditions are so that the (2.2-1) is verified. Indeed, from the(2.2-4), it is obtained:

$$\int_V \frac{1}{2}\left(\tilde{\mathsf{N}}\otimes\boldsymbol{u}+\left(\tilde{\mathsf{N}}\otimes\boldsymbol{u}\right)^T\right)ds=\int_{\partial V}\frac{1}{2}\left(\boldsymbol{n}\otimes\left(\mathsf{E}\cdot\boldsymbol{x}\right)+\left(\boldsymbol{n}\otimes\left(\mathsf{E}\cdot\boldsymbol{x}\right)\right)^T\right)ds \qquad (2.2\text{-}5)$$

This implies:

$$\overline{\boldsymbol{E}}=<\boldsymbol{E}\left(\boldsymbol{u}\right)>=\mathsf{E} \qquad (2.2\text{-}6)$$

and the proof for the (2.2-3) is self-evident.

The above reasoning does not apply for the case of periodic structure. This because, in this case, the stress and strain microfields, $\boldsymbol{T}$ and $\boldsymbol{E}$ are locally periodic and, so, periodicity conditions have to be considered, as it follows:

- The tractions $\boldsymbol{T}\cdot\boldsymbol{n}$ are opposite on opposite faces of ∂V, where the unit normal vector $\boldsymbol{n}$ corresponds to $\boldsymbol{-n}$.
- The local strain field $\boldsymbol{E}\left(\boldsymbol{u}\right)$ is made of two parts, the *mean* E and a fluctuation part $\boldsymbol{E}\left(\boldsymbol{u}^*\right)$ so that:

$$\boldsymbol{E}\left(\boldsymbol{u}\right)=\mathsf{E}+\boldsymbol{E}\left(\boldsymbol{u}^*\right),\qquad <\boldsymbol{E}\left(\boldsymbol{u}^*\right)>=0 \qquad (2.2\text{-}7)$$

where:

$\boldsymbol{u}^*=$ periodic displacement

Therefore, the boundary conditions for this problem are the following ones:

$$\boldsymbol{T}\cdot\boldsymbol{n}\quad \text{is antiperiodic}\ \ on\ \partial V \qquad (2.2\text{-}8)$$

$$\boldsymbol{u}=\mathsf{E}\cdot\boldsymbol{x}+\boldsymbol{u}^*,\qquad \boldsymbol{u}^*\ \text{periodic}\ \ on\ \partial V \qquad (2.2\text{-}9)$$

By taking in account the (2.2-3) and (2.2-4) or the (2.2-8) and (2.2-9), the localization problem (2.2-2) is, now, theoretically well-posed, but this must be verified for each constitutive behaviour.

2.3 Localization and Homogenization problem in pure elasticity

The case of purely elastic components will be examined, in this section. Here, anisotropic linear-elastic components will be considered.

About the localization problem, it is written in the following form:

$$\begin{aligned} &\boldsymbol{T}(\boldsymbol{x}) = \mathsf{C}(\boldsymbol{x}) : \boldsymbol{E}(\boldsymbol{x}) = \mathsf{C}(\boldsymbol{x}) : \left[\mathsf{E} + \boldsymbol{E}\left(\boldsymbol{u}^{*}(\boldsymbol{x})\right)\right] \\ &\boldsymbol{divT}(\boldsymbol{x}) = 0 \\ &+ \text{boundary conditions} \end{aligned} \qquad (2.3\text{-}1)$$

where E or S is prescribed and where:

$\mathsf{C}(\boldsymbol{x}) =$ the tensor of elasticity coefficients at the micro-scale.

Accordingly, the fluctuation displacement $\boldsymbol{u}^{*}$ is the solution of the following problem:

$$\begin{aligned} &\boldsymbol{div}\left(\mathsf{C}(\boldsymbol{x}) : \boldsymbol{E}\left(\boldsymbol{u}^{*}(\boldsymbol{x})\right)\right) = -\boldsymbol{div}\left(\mathsf{C}(\boldsymbol{x}) : \mathsf{E}\right) \\ &+ \text{boundary conditions} \end{aligned} \qquad (2.3\text{-}2)$$

Whenever the stiffness tensor C is constant for each component, it can be shown that:

$$\boldsymbol{div}\left(\mathsf{C} : \mathsf{E}\right) = \left(\S \mathsf{C}^{\cdot\cdot} : \mathsf{E}\right) \boldsymbol{n} \mathsf{d}\,(\mathsf{l}) \qquad (2.3\text{-}3)$$

where:

$$\S \mathsf{C}^{\cdot\cdot} = \mathsf{C}^{+} - \mathsf{C}^{-} \qquad (2.3\text{-}4)$$

and where:

$\mathsf{d}\,(\mathsf{l}) =$ Dirac's distribution.

$\boldsymbol{n} =$ the unit normal oriented from the "-" to the "+" side of the surface I separating components.

Under classical working hypotheses applying to C, the localization problem (2.3-2) admits a unique solution for all the types of boundary conditions, (2.2-3), (2.2-4), (2.2-8) and (2.2-9). In order to prove this, we must distinguish whether it is E or S is prescribed.

- **Case where E is prescribed**

For the demonstration of the existence and uniqueness of the solution, the reader can see Suquet (1981). We are only interested, here, to give the representation of the solution.

Since the problem is linear, the solution $\boldsymbol{E}\left(\boldsymbol{u}^*\right)$ depends linearly by the prescribed macrostrain field, E. Moreover, this latter can be decomposed into six elementary states of macroscopic strains (i.e. stretch in three directions and three shears). So, let $\boldsymbol{E}\left(\mathsf{c}_{hk}\right)$ be the fluctuation strain field at microscopic level, induced by these six elementary states. The solution $\boldsymbol{E}\left(\boldsymbol{u}^*\right)$ for a general macrostrain E is the superposition of the six elementary solutions, as in the following relation:

$$\boldsymbol{E}\left(\boldsymbol{u}^*\right) = \mathsf{E}_{hk}\boldsymbol{E}\left(\mathsf{c}_{hk}\right) \tag{2.3-5}$$

where a summation over h and k is considered.

In all, it is:

$$\boldsymbol{E}\left(\boldsymbol{u}\right) = \mathsf{E} + \boldsymbol{E}\left(\boldsymbol{u}^*\right) = \mathsf{E}\left(\boldsymbol{I} + \boldsymbol{E}\left(\mathsf{c}\right)\right) \tag{2.3-6}$$

This can be also expressed in the form:

$$\boldsymbol{E}\left(\boldsymbol{u}\right) = \mathsf{L} : \mathsf{E} \tag{2.3-7}$$

or, in components:

$$e_{ij}\left(\boldsymbol{u}\right)=L_{ijhk}:\mathrm{E}_{hk} \qquad (2.3\text{-}8)$$

where:

$$L_{ijhk}=I_{ijhk}+e_{ij}\left(\mathrm{c}_{hk}\right) \qquad (2.3\text{-}9)$$

with:

$I_{ijhk}=\frac{1}{2}\left(\mathrm{d}_{ih}\mathrm{d}_{jk}+\mathrm{d}_{ik}\mathrm{d}_{jh}\right)=$ the tensorial representation in R^3 of the unity of R^6.

The tensor L, as already mentioned in the previous section, is called the tensor of *concentrations* (Mandel, 1971) or, depending on the author, the tensor of *strain localization* or also the tensor of *influence* (Hill, 1967).

About the homogenization problem, instead, we can write:

$$\mathrm{S}=<\boldsymbol{T}\left(\boldsymbol{x}\right)>=<\mathrm{C}:\boldsymbol{E}\left(\boldsymbol{u}\right)>=<\mathrm{C}:\mathrm{L}:\mathrm{E}\ >=<\mathrm{C}:\mathrm{L}>:\mathrm{E} \qquad (2.3\text{-}10)$$

so that:

$$\mathrm{S}=\bar{\mathrm{C}}:\mathrm{E} \qquad (2.3\text{-}11)$$

where:

$\bar{\mathrm{C}}=$ homogenized symmetric stiffness tensor, which is given by:

$$\bar{\mathrm{C}}=<\mathrm{C}:\mathrm{L}> \qquad (2.3\text{-}12)$$

It can be noticed that:

$$<\mathrm{L}>=\boldsymbol{I}, \qquad <\mathrm{L}^T>=\boldsymbol{I} \qquad (2.3\text{-}13)$$

where the superscript T denotes transpose.

The obtained equation (2.3-12) shows that the tensor of the macro elasticity coefficients can be determined by taking the average of the micro elasticity coefficients, the latter being weighted by the tensor of strain localization. The symmetry of the homogenized stiffness tensor can be proved in two ways; the interested reader is referred to [41].

- **Case where S is prescribed**

For the demonstration of the existence and uniqueness of the solution, here again, the reader can see Suquet (1981). Here, we are only interested to give the representation of the solution, by starting that a unique solution exists.

In this case, the localization problem becomes:

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{u}) = \mathsf{E} + \boldsymbol{E}(\boldsymbol{u}^*) = \mathsf{S} : \boldsymbol{T}(\boldsymbol{x}) \\
&\boldsymbol{divT}(\boldsymbol{x}) = 0 \\
&< \boldsymbol{T}(\boldsymbol{x}) > = \mathsf{S} \\
&+ \text{boundary conditions}
\end{aligned} \tag{2.3-14}$$

where:

$\mathsf{S} =$ tensor of the micro elastic compliances.

$\mathsf{E} =$ unknown macrostrain field.

Analogously to the previous case, since the problem is linear, the solution $\boldsymbol{T}(\boldsymbol{x})$ depends linearly by the prescribed macrostress field S. Moreover, this latter can be decomposed into six elementary states of macroscopic stresses (i.e. compression in three directions and three shears). So, let $\boldsymbol{M}_{hk}$ be the solution of the problem (2.3-14), induced by these six elementary states. The solution $\boldsymbol{T}(\boldsymbol{x})$ for a general macrostress S is the superposition of the six elementary solutions, as in the following relation:

$$\boldsymbol{T}(\boldsymbol{x}) = \Sigma_{hk} \boldsymbol{M}_{hk} \tag{2.3-15}$$

where a summation over h and k is considered.

In all, it can be written that:

$$\boldsymbol{T}(\boldsymbol{x}) = \mathsf{M} : \mathsf{S} \tag{2.3-16}$$

or, in components:

$$\sigma_{ij}\left(\boldsymbol{x}\right)=M_{ijhk}:\Sigma_{hk} \tag{2.3-17}$$

where:

$$M_{ijhk}=\left(\boldsymbol{M}_{hk}\right)_{ij} \tag{2.3-18}$$

The tensor M, as already mentioned in the previous section, is called the tensor of *concentrations* or the tensor of *stress localization*.

About the homogenization problem, instead, we can write:

$$\mathsf{E}=<\boldsymbol{E}\left(\boldsymbol{u}\right)>=<\mathsf{S}:\boldsymbol{T}\left(\boldsymbol{x}\right)>=<\mathsf{S}:\mathsf{M}:\mathsf{S}>=<\mathsf{S}:\mathsf{M}>:\mathsf{S} \tag{2.3-19}$$

so that:

$$\mathsf{E}=\overline{\mathsf{S}}:\mathsf{S} \tag{2.3-20}$$

where:

$\overline{\mathsf{S}}=$ homogenized symmetric compliance tensor, which is given by:

$$\overline{\mathsf{S}}=<\mathsf{S}:\mathsf{M}> \tag{2.3-21}$$

It can be noticed that:

$$<\mathsf{M}>=\boldsymbol{I},\qquad <\mathsf{M}^{T}>=\boldsymbol{I} \tag{2.3-22}$$

where the superscript T denotes transpose.

The obtained equation (2.3-20) shows that the tensor of the macro compliance coefficients can be determined by taking the average of the micro compliance coefficients, the latter being weighted by the tensor of stress localization. The symmetry of the homogenized compliance tensor can also be proved; the interested reader is referred to [41], again.

2.4 Equivalence between prescribed stress and prescribed strain

It can be highlighted that $\overline{\mathsf{C}}$ and $\overline{\mathsf{S}}$ are inverse tensors of one another if they correspond to the same choice of boundary conditions in the localization problem. By using the symmetry of $\overline{\mathsf{C}}$, it can be written:

$$\overline{\mathsf{C}} : \overline{\mathsf{S}} = \left(\overline{\mathsf{C}}\right)^T : \overline{\mathsf{S}} = < \mathsf{L}^T : \mathsf{C} > : < \mathsf{S} : \mathsf{M} > \tag{2.4-1}$$

in which, for the definition of the tensors L and M, the first factor represents an *admissible stress* field and the second factor is an *admissible strain* field. So, by applying the Hill-Mandel principle, and by considering that $\mathsf{C} : \mathsf{S} = \mathbf{I}$, the (2.4-1) assumes the following form:

$$\overline{\mathsf{C}} : \overline{\mathsf{S}} = < \mathsf{L}^T : \mathsf{C} : \mathsf{S} : \mathsf{M} > = < \mathsf{L}^T : \mathsf{M} > = < \mathsf{L}^T > : < \mathsf{M} > = \mathbf{I} \tag{2.4-2}$$

which, indeed, implies that $\overline{\mathsf{C}}$ and $\overline{\mathsf{S}}$ are inverse tensors of one another.

However, if different boundary conditions are used, according to the estimate of Hill (1967) and Mandel (1971), it is:

$$\overline{\mathsf{C}} : \overline{\mathsf{S}} = \mathbf{I} + \mathrm{O}\left(\left(d / l\right)^3\right) \tag{2.4-3}$$

where $\overline{\mathsf{C}}$ is evaluated by using the condition (2.2-4), while $\overline{\mathsf{S}}$ is computed by using the condition (2.2-3) and where:

$d =$ characteristic size of an inhomogeneity.

$l =$ typical RVE size.

If $l >> d$, then the choice of boundary conditions is hardly important. For periodic media where $d / l = \mathrm{O}(1)$, this choice is very important.

CHAPTER III

Mechanics of masonry structures: experimental, numerical and theoretical approaches proposed in literature

3.1 Introduction

Masonries have been largely used in the history of architecture. Despite their unusual use in new buildings, they still represent an important research topic due to several applications in the framework of structural engineering, with particular reference to maintaining and restoring historical and monumental buildings. Hence, since preservation of existing masonry structures is considered a fundamental issue in the cultural life of modern societies, large investments have been concentrated on this issue, leading to develop a great number of theoretical studies, experimental laboratory activities and computational procedures in the scientific literature. The main interest of many researchers is in finding constitutive models able to simulate the complex response of such structures subjected to static and dynamic loads.

However, the mechanic characterization of a masonry structure shows itself as a very difficult task. This complexity results from its anisotropic composite behaviour. Masonry is, indeed, constituted by blocks of artificial or natural origin jointed by dry or mortar joints. Moreover, since the joints are inherent plane of weakness of such composite material, notably the mechanical masonry response is affected by behaviour preferred directions, which the joints determine, [28]. Two fundamental mechanical approaches have been developed in order to formulate an appropriate constitutive description of masonry structures:

- Discrete Models
- Continuous Models

The main object of this chapter will be to furnish an overall description of the above mentioned kind of approaches. In particular, our attention will be focused on the different homogenization proposals for modelling masonry structures, which are given in literature by some authors.

As it will be seen in the follows, indeed, the analysis of masonry via micro-mechanic and homogenization techniques, have to be included in the approaches which are based on the continuous models. In this framework, advanced analytical and numerical strategies – these latter based on the finite element method - have been recently developed.

3.2 Discrete and “ad-hoc” models

In spite of the considerable solutions which can be derived from continuous approaches, an interesting natural treatment of a masonry structure, which deals more directly with its discontinuous nature, is offered by its numerical modelling.

The numerical modelling of masonry structures shows objective difficulties, due to distinct issues:

- The typological characteristics of such structures don't allow referring to simplified static schemes.
- The material mechanical properties yield to a non-linear behaviour, whose prediction can result to be misleading.
- The incomplete experimental characterization of the masonry makes the calibration of numerical models quite uncertain.

However, they will be exposed, in the follows, three different modelling approaches in which each single structural masonry element is studied and the actual distribution of blocks and joints can be accounted for [30]: the finite element method with micro-modelling, the finite element method with discontinuous elements (FEMDE) and the discrete element method (DEM).

1. Modelling with FEM

Basically, two different approaches have been adopted to model with FEM the masonry behaviour: the 'micro-modelling' or 'two-materials approach' and the macro-modelling or 'equivalent-material approach'. Since this latter regards the masonry structure as a homogeneous equivalent continuum, it is referred to the group of continuous models. So, it will be illustrated in the following section.

As regards the former approach, the discretization follows the actual geometry of both the blocks and mortar joints, adopting different constitutive models for the two components. Particular attention must be paid in the modelling of joints, since the sliding at joint level often starts up the crack propagation. Although this approach may appear very straightforward, its major disadvantage comes from the extremely large number of elements to be generated as the structure increases in size and complexity.

Hence, the use of micro-models becomes unlikely to use for the global analysis of entire buildings, also considering the fact that the actual distribution of blocks and joints might be impossible to detect unless invasive investigations are performed, [30].

2. Modelling with FEMDE

In this approach, the blocks are modelled using conventional continuum elements, linear or non-linear, while mortar joints are simulated by interface elements, the 'joint elements', which are made up of two rows of superimposed nodes with friction constitutive low, (see Fig. 3.1).

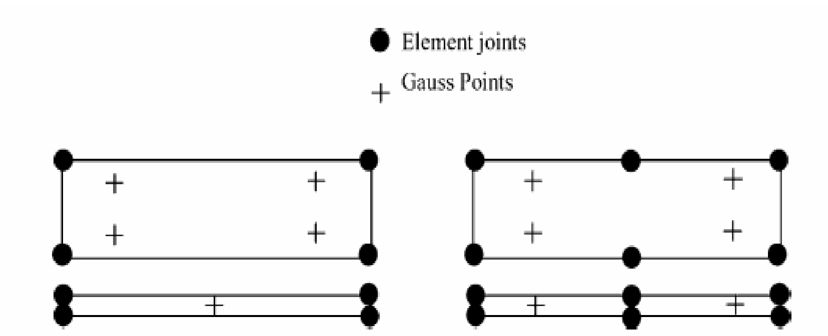


Figure 3.1 Degeneration of the continuum element into "joint element".

The introduction of the joint is easy to implement in a software programme, since the nodal unknowns are the same for continuum and joint elements, though for the latter the stress tensor must be expressed in terms of nodal displacements instead of deformation components.

Two major concerns balance the apparent simplicity of this approach, [30]:

- Block mesh and joint mesh must be connected together, so that they have to be compatible, which is possible only if interface joints are identically located. This compatibility is very difficult to ensure when complex block arrangements are to be handled, like in 3D structures.
- The joint element is intrinsically able to model the contact only in the small displacement field. When large motion is to be dealt, is not possible to provide easy remeshing in order to update existing contacts and/or to create new ones.

3. Modelling with DEM

The above-mentioned limitations are overcome by the DEM (discrete element method).

This methodology originated as distinct element method in geotechnical and granular flow applications (Cundall, 1971) and it is based on the concept that individual material elements are considered to be separate and are (possibly) connected only along their neighbours by frictional/adhesive contact. Here, elements were considered rigid, but later developments (Munjiza et al, 1995) included the addition of element deformations and fracturing, which has permitted a more rigorous treatment of both the contact conditions and fracture requirements. The incorporation of deformation kinematics into the discrete element formulation has also led naturally to a combined finite/discrete element approach in which the problem is analyzed by a combination of the two methods.

With present day computational power, large scale discrete element models can be considered, also for industrial applications in different fields. About 10-50,000 elements are routinely employed, [56].

In this approach, therefore, the structure is considered as an assembly of distinct blocks, rigid or deformable, interacting through unilateral elasto-plastic contact elements which follow a Coulomb slip criterion for simulating contact forces. The method is based on a formulation in large displacement (for the joints) and small deformations (for the blocks), and can correctly simulate collapse mechanisms due to sliding, rotations and impact.

The contacts are not fixed, like in the FEMDE, so that during the analyses blocks can loose existing contacts and make new ones. Once every single block has been modelled both geometrically and mechanically, and the volume and surface forces are known, the time history of the block's displacements is determined by explicitly solving the differential equations of motion. In particular, high viscous damping is used to achieve convergence to static solution or steady failure mechanism, [30].

In other words, two main features of the DEM method lead to its use for the analysis of masonry structures by means of the Cundall's software program UDEC-Universal Distinct Element Code, [18]. One is the allowance for large displacements and rotations between blocks, including their complete detachment. Other is the automatic detection of new contacts as the calculation progresses.

Another advantage of this approach is the possibility of following the displacements and determining the collapse mechanism of structures made up of virtually any number of blocks, [30], [51]. On the contrary, it must be considered that the finite elements used for the internal mesh of the blocks, when deformable, show poor performance, so the method is not accurate for the study of stress states within the blocks.

The discrete element analysis is particularly suitable for problems in which a significant part of the deformation is accounted for by relative motion between elements. Hence, masonry provides just a natural application for this technique since its significant deformation occurs at joints or contact points, i.e. the deformation and failure modes of these structures are strongly dependent on the role of the joints. This approach is, therefore, as already mentioned, well suited for collapse analysis and may thus provide support for studies of safety assessment, for example of historical stone masonry structures under earthquakes. It has been recently applied by Guiffrè *et al* (1994) for the design of masonry walls.

The representation of the interfaces between blocks relies on sets of point contacts. Adjacent blocks can touch along a common edge segment or at discrete points where a corner meets an edge or another corner.

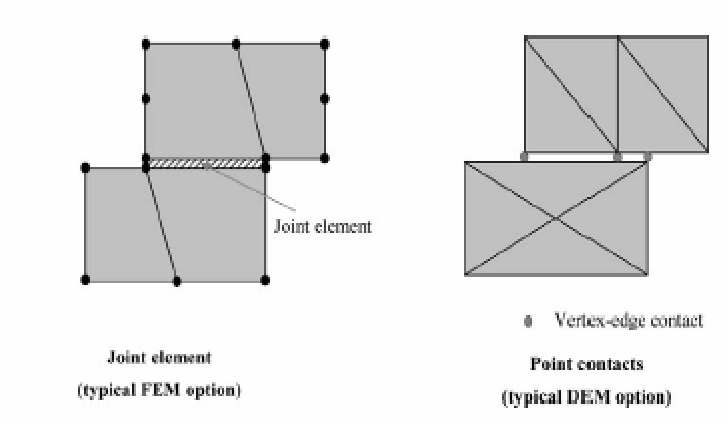


Figure 3.2 Joints elements vs. "point contacts".

Different types of contacts can be handled, depending on the initial geometry and on the displacement history during the analysis, [30]. Typically, the general types of contacts are:

- face-to-face (FF)
- edge-to-face (EF)

- vertex-to-face (VF)
- edge-to-edge (EE)
- vertex-to-edge (VE)
- vertex-to-vertex (VV)

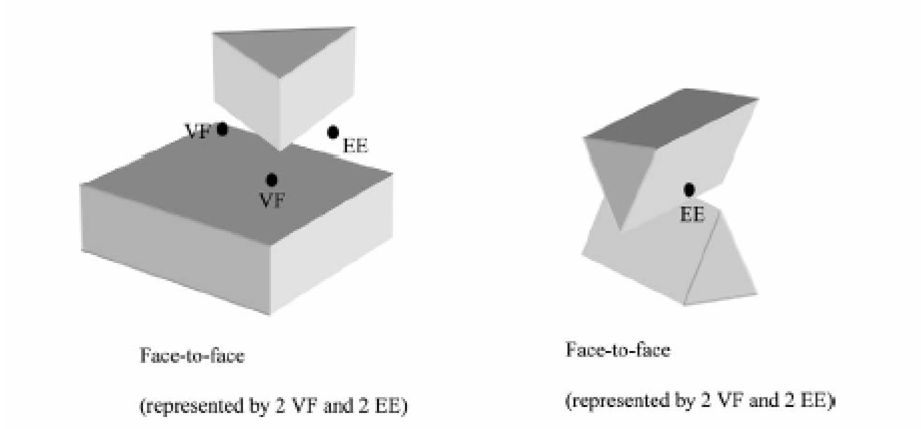


Figure 3.3 Different types of contacts.

All of them can be represented by sets of point contacts of two elementary types: VF and EE (see Fig. 3.3).

At each contact, the mechanical interaction is represented by a force, resolved into a normal and a shear component. Contact displacements are defined as the relative displacements between two blocks at the contact point. In the elastic range, contact force and displacements are related through the contact stiffness parameters (normal and shear).

The discrete element techniques allow describing the masonry constitutive behaviour if an accurate stress-strain relationship is employed for each constituent material, which is, then, discretized individually and by taking in account the necessary parameters to define the contact mechanical behaviour;

since the contact forces are thought to follow a classical Mohr–Coulomb criterion, the following parameters must be assigned:

- kn: normal stiffness
- ks: shear stiffness
- Nt: tensile strength
- f: friction angle
- m: dilatancy angle
- c: cohesion

This numerical approach concurs to investigate, numerically, some distinctive aspects of masonry which are closely related to the behaviour of its micro-constituents and its geometry (bond patterns, thickness of joints), such as anisotropy in the inelastic range and the post peak softening, [44]. In particular, since masonry is analyzed as an assembly of blocks connected each other by interfaces, such numerical technique also yields the investigation of the interactions between the single constituents. In order to do this, frictional properties and appropriate constitutive laws of interfaces are often included in the numerical models. Hence, this approach is able to provide a realistic and rigorous analysis in which the exact joint positions are considered.

Several attempts have been made to categorise, in the framework of the discrete models, the computational approaches for structural masonry, where its inherent discontinuous nature (unit, joints, interface) need to be recognized. Perhaps, the most appropriate categorization comes from the “Delft School”, [9], [38], where the following principal modelling strategies are identified:

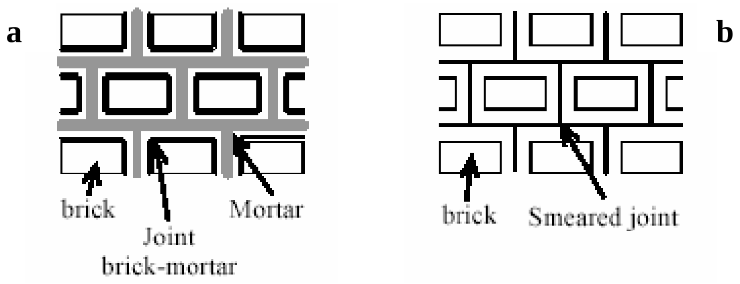


Figure 3.4 Modelling strategies

a) *Detailed micro-modelling*: units and joint are represented by continuous elements, whereas unit/mortar interfaces are modelled by discontinuous elements.

b) *Simplified micro-modelling*: “geometrically expanded” continuum units, with discontinuum elements covering the behaviour of both mortar joints and interfaces.

It has to be underlined that a three-dimensional micro-modelling analysis of a masonry panel involving only a very simple geometry would require a large number of elements in order to enable accurate modelling of each joint and masonry unit.

Hence, the Micro-modelling approaches appears to be too onerous for the analysis of single masonry walls and, in practice, not feasible for the analysis of structures with a large number of masonry panels. However, to overcome this computational difficulty, a different way of tackling the problem is often considered: the Macro-modelling approach, in which masonry is modelled by an equivalent continuous material, as shown in the following section.

3.3 Continuous Models

The accuracy of the discrete approaches, described in the previous section, is much higher than of the continuous ones, because they are able to yield a detailed investigation of the microscopic problem. Even though they show this advantage, according to what it has been already underlined, they also show the disadvantage to be too expensive in terms of computational costs and, moreover, the corresponding high number of degrees limits the applicability, [45]. So, whole buildings are almost impossible to simulate by means of such micro-models.

On the contrary, in the framework of numerical modelling and according to the traditional continuous finite element theory, masonry can also be analyzed like a continuum homogeneous media which, by taking in account the block and the mortar properties in its constitutive law, is able to represent the mechanical behaviour of the discrete and composite starting material.

This approach is called, in literature, Macro-modelling approach. The macro-modelling assumes that the homogeneous equivalent material is discretized with a finite element mesh which does not copy the wall organism, but obeys the method's own criteria.

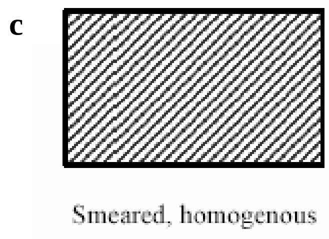


Figure 3.5 Modelling strategies

c) *Macro-modelling*: all the three constituents of structural masonry are represented by an equivalent continuum

The here considered macro-models don't make distinction between blocks and mortar joints but they smear the effect of joint presence through the formulation of the constitutive modelling of the fictitious equivalent material. Such a constitutive model has to reproduce an average mechanical behaviour of masonry. This assumption bypasses the physical characteristics of the problem.

Hence, obviously, these models can not be as accurate as micro-models can be, nevertheless their main advantages are:

- the enormous reduction of the computational cost, that makes possible the numerical analyses of complex structures such as bridges and even buildings, cathedrals, castles and so on, and
- the capability to investigate the global response of the masonry structures without the computing effort needed in the micro-modelling.

The use of a discretization technique based on continuous models implies the previous definition of a carefully tuned constitutive characterization of the equivalent continuous material, which depends on the analysis of masonry micro-structure and averages the macroscopic mechanical behaviour of the structural masonry. To this aim, two different ways have been considered:

- Phenomenological and experimental approaches
- Homogenization Theory based approaches

In the following subsections, major details are given on this topic by starting from the description of first approach. This latter provides valuable information used to establish, via phenomenological considerations or via experimental testing, empirically and semi-empirically based methodologies for the design of masonry structures. However, such an approach finds its limits in the dependence of the results by the conditions in which the data are obtained.

3.3.1 Phenomenological and experimental approaches

Some continuous models which have been proposed for masonry are based on phenomenological laws. In phenomenological analysis, the constitutive response of the masonry is determined by experimental tests, [57]. One of the most adopted phenomenological constitutive law for masonry is the so-called “no-tension material”. According to this model, the masonry is schematized as a homogeneous elastic material which cannot support tensile stresses.

This prevalent feature that distinguishes masonry structures and makes them dissimilar from actual concrete and steel structures was first introduced explicitly by Heyman in 1966, [31], [32]. He proved, after a number of practical studies carried on with special reference to monumental buildings, that proneness to disease or collapse is much more dependent on the activation of cracking mechanisms than on the probability of crushing in compression of masonry. He also proved, on the contrary, that localised fractures do not usually affect the performance of the skeleton, as can be observed in many existing masonry buildings. In other words, fractures should be considered as a physiological feature of the masonry material, unless they are so large as to compromise the local resistance of the material elements, or so well organised that a collapse mechanism may be activated.

The logical conclusion was that the material model should include fracturing as an intrinsic pattern for the stress-strain relationships. Moreover, the structural model should be sensitive to the presence of collapse mechanisms in the neighbourhood of the actual equilibrium configuration.

Some authors have tried to develop a formal theoretical framework for such phenomena, just based on the assumption that the material model, that is intended to be an "analogue" of real masonry, cannot resist tensile stress, but behaves elastically (indefinitely) under pure compression, [7], [6], [19], [14], [66], [20].

It is noted that these conditions give a well-defined specification of the admissible domain for stresses, but allow complete freedom for the path of fracture growth. This means that, in building up the stress-strain relationships for inelastic deformation (fractures, in this case), one is free to include the most appropriate assumptions. Hence, one deals with Standard NRT (Not-Resisting-Tension) material or with Non-standard NRT material, depending on the circumstance that the material is assumed to fracture according to a pattern similar to the Drucker's postulate, or not.

In the NNRT (Non-Standard NRT) case, one can imagine even more patterns.

In the framework of such no-tension structures, with reference to the danger of their collapse, a special extension of Limit Analysis has been developed in literature allowing to formulate basic theorems quite analogous to the kinematical and static theorems of classical Limit Analysis, thus giving the possibility to establish effective procedures to assess structural safety versus the collapse limit state, [6], [14]. Special problems, such as the non-existence in highly depressed arches of collapse mechanisms involving exclusively unilateral hinges, have been identified, and attention has been drawn on the necessity to include the eventuality that sliding mechanisms occur between stones by inadequate friction. More in general, an analysis of the class of structural problems with unilateral constitutive relations has been employed by formulating a general model and by proving its consistency. The authors of this analysis, moreover, have exposed two iterative methods for the numerical solution of this class of problems, [61].

However, as regards the masonry, the basic assumption to study via phenomenological approach this kind of structures, we underline again, is that no-tension stress fields are selected by the masonry through the activation of an additional inelastic strain field, i.e. the fractures. Hence, the stress-strain relationship for NRT materials is of the form:

$$\boldsymbol{E} = \boldsymbol{E}_e + \boldsymbol{E}_f = \mathsf{S}\boldsymbol{T} + \boldsymbol{E}_f \qquad (3.3.1\text{-}1)$$

or in the inverse one:

$$\boldsymbol{T} = \mathsf{C}\left(\boldsymbol{E} - \boldsymbol{E}_f\right) = \mathsf{C}\boldsymbol{E}_e \qquad (3.3.1\text{-}2)$$

where:

$\mathsf{S} =$ compliance tensor

$\mathsf{C} =$ stiffness tensor

Since in a NRT solid, the equilibrium against external loads is required to be satisfied by *admissible* stress fields, which imply pure compression everywhere in the solid, and by assuming stability of the material in the Drucker’s sense, compatibility of the strain field can be ensured, indeed, by superposing to the elastic strain field an additional fracture field, that does not admit contraction in any point and along any direction.

This means that the stress tensor $\boldsymbol{T}$ in equation (3.3.1-1) must be negative semi-definite everywhere in the solid, while the fracture strain field $\boldsymbol{E}_f$ is required to be positive semi-definite.

In other words, it has to be verified:

$$semi-definite\begin{cases}\boldsymbol{E}_f \ \ positive \\ \boldsymbol{T} \ \ negative\end{cases} \rightarrow \begin{cases}\boldsymbol{E}_{fa} \geq 0 \\ \boldsymbol{T}_a \ \ \leq 0\end{cases} \ \forall a \in r_a \qquad (3.3.1\text{-}3)$$

where:

$r_a =$ the set of directions through the generic point in the solid

$a =$ one of such directions

$\boldsymbol{E}_f =$ the tensile fracture inelastic strain that is assumed to superpose to the elastic one $\boldsymbol{E}_e$, in order to anneal tensile stresses, if possible.

Solution stress and strain fields obtained by the authors are proven to satisfy classical variational principles, like the minimum principles of Complementary and Total Energy functionals, respectively on the compatibility and equilibrium side, [6]. Moreover, the solution paths are based on constrained optimisation of the energetic functionals, also enhancing some peculiar features that distinguish structural patterns from each other.

For ulterior details on this kind of approach, the reader is referred to the existing literature in such framework, [6], [7].

3.3.2 Homogenization theory based approaches

In this second subsection, the homogenization theory based approaches are described. They regard the masonry as a heterogeneous biphasic medium, consisting in units (brick or stones) and mortar joints, from which a homogeneous equivalent material is obtained, by using homogenization techniques. With this task, they provide an analytical definition of the average mechanical properties of structural masonry.

However, it has to be underlined that, in this framework, the most of homogenization techniques proposed in literature adopt the hypothesis of “periodic-structure” for masonry. This leads to assume units, head and bed mortar joints of equal dimensions and elastic properties. Moreover, these components must be arranged in a periodic pattern. Nevertheless, this hypothesis can be accepted for new structures only. The periodic approach, indeed, is surely incorrect for a very large number of existing masonry structures, which vice versa have a great cultural and social interest such as in restoring historical buildings. So, in order to apply the homogenization theory to old masonry, which are characterized by chaotic or semi-periodic patterns, a different approach is necessary, [13].

In the chapters 1 and 2, it has been highlighted that the study of composite materials is, in general, referred to the analysis of an RVE that represents, statistically, the microstructure of such materials. Well, in the case of a periodic composite material, like masonry one, the homogenization techniques are based on the identification of a particular RVE, which is able to generate the whole examined structure through opportune translations. This kind of RVE is defined, in literature, the masonry *periodic cell* or the masonry *basic cell*.

However, given the complex geometry of the basic cell, a close-form solution of the homogenization problem seems to be impossible, which leads, basically, to three different lines of action.

The first one is to handle the brickwork structure of masonry by considering the salient features of the discontinuum within the framework of a Cosserat continuum theory, e.g. Muhlhaus (1993), [46]. Particular attention is given to the interface problem. This approach is considered a very elegant solution, but nevertheless very complex both from a mathematical point of view and from the point of view of the development of a systematic methodology for the homogenized properties identification. Hence, the step towards the real application of such an approach is still to be done.

The second one aims at substituting the complex geometry of the basic cell with a simplified microstructure geometry so that a close-form solution of the homogenized problem can be possible. This approach is, in literature, well known as engineering approach. Keeping in mind the objective of performing a non-linear analysis at the structural level, Geymonat [27], Pande et al. [58], Maier et al. [40] and Pietruszczak and Niu, [60], introduced homogenization techniques in such approximate manner. In spite of the fact that these simplified approaches present some limits in the solution accuracy, as it will be illustrated in the follows, they are used by several authors and, nevertheless, perform satisfactorily in the case of linear elastic analysis. In non-linear field, on the contrary, they lead to unacceptable results.

The third one, e.g. Anthoine, [5] and Urbanski et al. [65], is to apply rigorously the homogenization theory for periodic media to the basic cell in a sole step. Because of the complexity of the exact geometry of the basic cell, it becomes necessary to find the solution problem by using an approximate and numerical method such as the finite element method. Since the complete determination of the homogenized constitutive law requires an infinite number of computations, in a nonlinear range, the theory has been used to determine the macro-parameters of masonry and not, actually, to carry out analysis at the structural level.

3.3.2.a A homogenization approach by Pietruszczak & Niu

These authors have proposed a mathematical formulation for describing the average mechanical properties of a periodic structural masonry, [60].

The object of such approach is to present an alternative solution method to the discrete one that becomes quite impractical in the context of large-scale masonry buildings.

The conceptual approach is based on the framework already outlined by Pietruszczak (1991) which regards a representative volume element of structural masonry as a composite medium consisting of the brick matrix intercepted by the sets of head and bed joints. Thus, the presence of discrete sets of mortar joints results in a strong directional dependence of the average mechanical properties. The estimation of them is the main interest of the authors.

Therefore, let us consider a typical element of structural masonry, i.e. a brick panel, as shown schematically in the following figure:

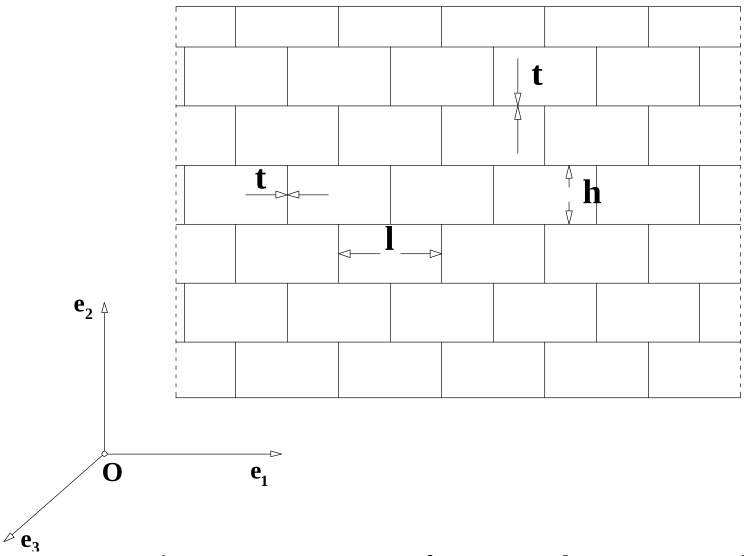


Figure 3.6 Geometry of a structural masonry panel

Let it be subjected to a uniformly distributed load. On the macroscale, the panel is regarded, as already mentioned, as a two-phase composite: brick units interspersed by two orthogonal sets of joints filled with mortar.

In order to describe the average mechanical properties of the system, the authors propose a simplified homogenization procedure, consisting in addressing the influence of head and bed joints separately, i.e. in invoking the concept of a superimposed medium. It is worth to notice that a homogenization process which is performed in several steps (in this case two steps) leads to results depending on the sequence of steps chosen. Moreover, it doesn't take into consideration the geometrical arrangement of the masonry: two different bond patterns yield the same results, [27]. This leads to results that are unacceptable in non-linear range.

With reference to the following figure, it has been considered first a medium (1), consisting in the brick matrix with a family of head joints. The head joints are treated as aligned, uniformly dispersed weak inclusions embodied in the matrix. In particular, they are considered in the form of monotonically aligned rectangular parallelepipeds.

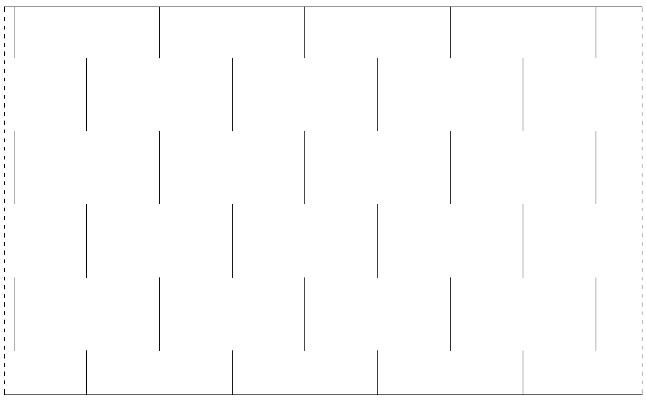


Figure 3.7 Medium (1)

The average properties of the medium (1) can be represented by a constitutive relation, as:

$$\bar{\dot{\boldsymbol{T}}}^{(1)} = \left[\bar{\boldsymbol{C}}^{(1)}\right]\bar{\dot{\boldsymbol{E}}}^{(1)} \tag{3.3.2.a-1}$$

where the volume average of stress rate $\bar{\dot{\boldsymbol{T}}}^{(1)}$ in the medium (1) is considered in vectorial form, as given by:

$$\bar{\dot{\boldsymbol{T}}}^{(1)} = \left\{\dot{\sigma}_{11}^{(1)}, \dot{\sigma}_{22}^{(1)}, \dot{\sigma}_{33}^{(1)}, \dot{\sigma}_{12}^{(1)}, \dot{\sigma}_{13}^{(1)}, \dot{\sigma}_{23}^{(1)}\right\}^T \tag{3.3.2.a-2}$$

and so also for the volume average of strain rate $\dot{\bar{\boldsymbol{E}}}^{(1)}$ in the medium (1):

$$\dot{\bar{\boldsymbol{E}}}^{(1)} = \left\{ \dot{\varepsilon}_{11}^{(1)}, \dot{\varepsilon}_{22}^{(1)}, \dot{\varepsilon}_{33}^{(1)}, \dot{\varepsilon}_{12}^{(1)}, \dot{\varepsilon}_{13}^{(1)}, \dot{\varepsilon}_{23}^{(1)} \right\}^T \qquad (3.3.2.a\text{-}3)$$

while $\left[\bar{\boldsymbol{C}}^{(1)} \right]$ is the volume average of 6x6 stiffness matrix in the medium (1).

In particular, if both bricks and head joints are isotropic, then the homogenized medium (1) can be regarded as an orthotropic elastic-brittle material. In such a case, the components of $\left[\bar{\boldsymbol{C}}^{(1)} \right]$ can be estimated from Eshelby's (1957) solution to an ellipsoidal inclusion problem combined with Mori-Tanaka's (1973) mean-field theory, [60].

The whole masonry panel can now be represented by a homogenized medium (1), stratified by a family of bed joints, considered as a medium (2). It is shown in the following figure:

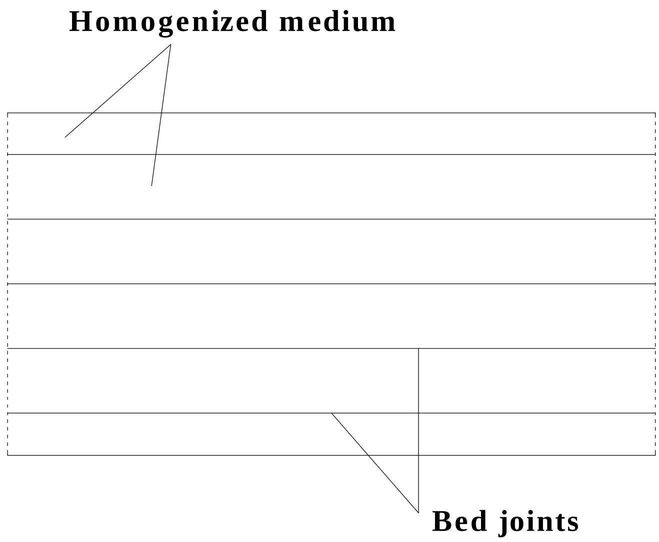


Figure 3.8 Medium (2): medium (1) intercepted by bed joints

The bed joints are regarded as continuous plane of weakness. In other words, they run continuously through the panel and form the weakest link in the microstructure of the system. In particular, the bed joints are considered as an elasto-plastic medium, (2), with mechanical properties defined by:

$$\dot{\bar{\boldsymbol{T}}}^{(2)} = \left[\bar{\boldsymbol{C}}^{(2)}\right]\dot{\bar{\boldsymbol{E}}}^{(2)} \tag{3.3.2.a-4}$$

By assuming that both constituents (1) and (2) exist simultaneously and are perfectly bonded, the overall stress and strain rate averages, $\dot{\bar{\boldsymbol{T}}}$ and $\dot{\bar{\boldsymbol{E}}}$, can be derived from the averaging rule (Hill, 1963):

$$\dot{\bar{\boldsymbol{E}}} = f_1\dot{\bar{\boldsymbol{E}}}^{(1)} + f_2\dot{\bar{\boldsymbol{E}}}^{(2)} \tag{3.3.2.a-5}$$

$$\dot{\bar{\boldsymbol{T}}} = f_1\dot{\bar{\boldsymbol{T}}}^{(1)} + f_2\dot{\bar{\boldsymbol{T}}}^{(2)} \tag{3.3.2.a-6}$$

where f_1 and f_2 are the volume fractions of both constituents, defined as:

$$f_1 = \frac{h}{h+t}; \qquad f_2 = \frac{t}{h+t} \tag{3.3.2.a-7}$$

and where h and t represent the spacing and the thickness of bed joints, respectively.

The assumption of perfect bonding between the constituents and the equilibrium requirements provides additional kinematics and static constraints, given by:

$$\left[\mathrm{d}^*\right]\dot{\bar{\boldsymbol{E}}}^{(1)} = \left[\mathrm{d}^*\right]\dot{\bar{\boldsymbol{E}}}^{(2)} \tag{3.3.2.a-8}$$

$$\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{T}}}^{(1)} = \left[\mathrm{d}\right]\dot{\bar{\boldsymbol{T}}}^{(2)} \tag{3.3.2.a-9}$$

where:

$$\left[\mathrm{d}^*\right]=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix};\quad \left[\mathrm{d}\right]=\begin{bmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix} \tag{3.3.2.a-10}$$

The constraints (3.3.2.a-8) and (3.3.2.a-9), as applied averages, are rigorous provided that $t = h$.

It is evident that the field equations listed above, (3.3.2.a-1), (3.3.2.a-4), (3.3.2.a-5) and (3.3.2.a-6), together with the constraints (3.3.2.a-8) and (3.3.2.a-9), provide a set of 30 equations for 30 unknowns, e.g. $\bar{\dot{\boldsymbol{T}}}$, $\bar{\dot{\boldsymbol{T}}}^{(1)}$, $\bar{\dot{\boldsymbol{T}}}^{(2)}$, $\bar{\dot{\boldsymbol{E}}}^{(1)}$ and $\bar{\dot{\boldsymbol{E}}}^{(2)}$. Thus the problem is mathematically determinate. Moreover, it can be noticed that the number of unknowns can be reduced by introducing certain simplifying assumptions pertaining to the kinematics of bed joints. For example, the formulation can be employed by expressing the local deformation field in bed joints in terms of velocity discontinuities rather than strain rates $\bar{\dot{\boldsymbol{E}}}^{(2)}$, thereby reducing the number of unknowns to 27.

In order to solve a so-posed problem and, so, provide an explicit form of the average constitutive relation, it is convenient to introduce the following identity:

$$[\mathrm{d}]\,\bar{\dot{\boldsymbol{T}}}^{(i)}=[\mathrm{d}]\left[\bar{\boldsymbol{C}}^{(i)}\right]\bar{\dot{\boldsymbol{E}}}^{(i)}=\left[\boldsymbol{F}_1^{(i)}\right]\left[\mathrm{d}^*\right]\bar{\dot{\boldsymbol{E}}}^{(i)}+\left[\boldsymbol{F}_2^{(i)}\right][\mathrm{d}]\,\bar{\dot{\boldsymbol{E}}}^{(i)};\quad i=1,2 \tag{3.3.2.a-11}$$

where:

$$\left[\boldsymbol{F}_1^{(i)}\right]=\begin{bmatrix}\bar{C}_{21}^{(i)}&\bar{C}_{23}^{(i)}&\bar{C}_{25}^{(i)}\\\bar{C}_{41}^{(i)}&\bar{C}_{43}^{(i)}&\bar{C}_{45}^{(i)}\\\bar{C}_{61}^{(i)}&\bar{C}_{63}^{(i)}&\bar{C}_{65}^{(i)}\end{bmatrix};\quad \left[\boldsymbol{F}_2^{(i)}\right]=\begin{bmatrix}\bar{C}_{22}^{(i)}&\bar{C}_{24}^{(i)}&\bar{C}_{26}^{(i)}\\\bar{C}_{42}^{(i)}&\bar{C}_{44}^{(i)}&\bar{C}_{46}^{(i)}\\\bar{C}_{62}^{(i)}&\bar{C}_{64}^{(i)}&\bar{C}_{66}^{(i)}\end{bmatrix} \tag{3.3.2.a-12}$$

By using the equations (3.3.2.a-11) and the (3.3.2.a-8), the static constraint (3.3.2.a-9) can now be expressed in the following form:

$$\left[\boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{1})}\right]\left[\mathrm{d}^{*}\right]\dot{\bar{\boldsymbol{E}}}+\left[\boldsymbol{F}_{\boldsymbol{2}}^{(\boldsymbol{1})}\right]\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{1})}=\left[\boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{2})}\right]\left[\mathrm{d}^{*}\right]\dot{\bar{\boldsymbol{E}}}+\left[\boldsymbol{F}_{\boldsymbol{2}}^{(\boldsymbol{2})}\right]\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{2})} \quad (3.3.2.a\text{-}13)$$

By means of the representation (3.3.2.a-13) and the decomposition (3.3.2.a-5), the strain rates in both constituents can be uniquely related to $\dot{\bar{\boldsymbol{E}}}$. Therefore, in view of the cinematic constraints (3.3.2.a-8), the set of equations (3.3.2.a-5) reduces to:

$$\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{2})}=\frac{1}{f_2}\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}-\frac{f_1}{f_2}\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{1})} \qquad (3.3.2.a\text{-}14)$$

Substitution of the equation (3.3.2.a-14) in the (3.3.2.a-13), after some simple algebra, results in:

$$\left[\mathrm{d}\right]\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{1})}=\left[\mathrm{M}\right]\dot{\bar{\boldsymbol{E}}} \qquad (3.3.2.a\text{-}15)$$

where:

$$\left[\mathrm{M}\right]=\left(\left[\boldsymbol{F}_{\boldsymbol{2}}^{(\boldsymbol{1})}\right]+\frac{f_1}{f_2}\left[\boldsymbol{F}_{\boldsymbol{2}}^{(\boldsymbol{2})}\right]\right)^{-1}\left\{\frac{1}{f_2}\left[\boldsymbol{F}_{\boldsymbol{2}}^{(\boldsymbol{2})}\right]\left[\mathrm{d}\right]+\left(\left[\boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{2})}\right]-\left[\boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{1})}\right]\right)\left[\mathrm{d}^{*}\right]\right\} \quad (3.3.2.a\text{-}16)$$

Thus, in view of equation (3.3.2.a-5), the following relationship is obtained:

$$\dot{\bar{\boldsymbol{E}}}^{(\boldsymbol{1})}=\left[\bar{\mathrm{M}}_1\right]\dot{\bar{\boldsymbol{E}}} \qquad (3.3.2.a\text{-}17)$$

where:

$$\left[\bar{\mathrm{M}}_1\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{M}_{11} & \mathrm{M}_{12} & \mathrm{M}_{13} & \mathrm{M}_{14} & \mathrm{M}_{15} & \mathrm{M}_{16} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathrm{M}_{21} & \mathrm{M}_{22} & \mathrm{M}_{23} & \mathrm{M}_{24} & \mathrm{M}_{25} & \mathrm{M}_{26} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \mathrm{M}_{31} & \mathrm{M}_{32} & \mathrm{M}_{33} & \mathrm{M}_{34} & \mathrm{M}_{35} & \mathrm{M}_{36} \end{bmatrix} \qquad (3.3.2.a\text{-}18)$$

and the components of $\left[\mathrm{M}\right]$ are defined by the equation (3.3.2.a-16).

The strain rates in bed joints can be expressed in a similar functional form to that of equation (3.3.2.a-17). Indeed, after substituting the equation (3.3.2.a-17) in (3.3.2.a-5), one obtains:

$$\bar{\dot{\boldsymbol{E}}}^{(2)} = \left[\bar{\mathrm{M}}_2 \right] \bar{\dot{\boldsymbol{E}}} \qquad (3.3.2.a\text{-}19)$$

where:

$$\left[\bar{\mathrm{M}}_2 \right] = \left(\frac{1}{f_2} [\mathrm{I}] - \frac{f_1}{f_2} \left[\bar{\mathrm{M}}_1 \right] \right) \qquad (3.3.2.a\text{-}20)$$

and $[\mathrm{I}]$ represents the unit matrix (6x6).

Finally, the overall stress rate averages $\bar{\dot{\boldsymbol{T}}}$ remains to be determined. It can be obtained from equation (3.3.2.a-6). Indeed, the substitution of the equations (3.3.2.a-17) and (3.3.2.a-19) in (3.3.2.a-6), results in:

$$\bar{\dot{\boldsymbol{T}}} = \left\{ f_1 \left[\bar{\boldsymbol{C}}^{(1)} \right] \left[\bar{\mathrm{M}}_1 \right] + \left[\bar{\boldsymbol{C}}^{(2)} \right] \left([\mathrm{I}] - f_1 \left[\bar{\mathrm{M}}_1 \right] \right) \right\} \bar{\dot{\boldsymbol{E}}} \qquad (3.3.2.a\text{-}21)$$

Since the following relation:

$$\bar{\dot{\boldsymbol{T}}} = \left[\bar{\boldsymbol{C}} \right] \bar{\dot{\boldsymbol{E}}} \qquad (3.3.2.a\text{-}22)$$

represents the average constitutive relation for the entire composite system, it is obtained that:

$$\left[\bar{\boldsymbol{C}} \right] = \left\{ f_1 \left[\bar{\boldsymbol{C}}^{(1)} \right] \left[\bar{\mathrm{M}}_1 \right] + \left[\bar{\boldsymbol{C}}^{(2)} \right] \left([\mathrm{I}] - f_1 \left[\bar{\mathrm{M}}_1 \right] \right) \right\} \qquad (3.3.2.a\text{-}23)$$

As expected, the macroscopic behaviour depends on the mechanical properties of both constituents and their volume fractions. In the follows, the average elastic properties of the masonry are established in detail.

By remembering, indeed, the assumed hypothesis of orthotropic behaviour of the medium (1), the constitutive matrix of the equation (3.3.2.a-1) assumes the form:

$$\left[\overline{\boldsymbol{C}}^{(1)}\right]=\begin{bmatrix} \overline{C}_{11}^{(1)} & \overline{C}_{12}^{(1)} & \overline{C}_{13}^{(1)} & 0 & 0 & 0 \\ \overline{C}_{12}^{(1)} & \overline{C}_{22}^{(1)} & \overline{C}_{23}^{(1)} & 0 & 0 & 0 \\ \overline{C}_{13}^{(1)} & \overline{C}_{23}^{(1)} & \overline{C}_{33}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{C}_{55}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{C}_{66}^{(1)} \end{bmatrix} \quad (3.3.2.a\text{-}24)$$

The nine independent elastic constants are functions of the properties of both constituents (brick and head joints) as well as the cross-sectional aspect ration and the volume fraction of the inclusions.

The authors use the results reported by Zhao and Weng (1990) for the estimation of the average elastic properties of the medium (1) considered in equation (3.3.2.a-24). These latter authors have identified the average elastic constants of an orthotropic composite reinforced with aligned elliptic cylinders. The estimates, as already mentioned, are based on Eshelby's solution to the ellipsoidal inclusion problem combined with Mori-Tanaka's mean field theory, in order to deal with the finite concentration of inclusions. For the algebraic expressions of such elastic constants the reader is referred to the original publication, [60].

By considering now that the bed joints are isotropic, the constitutive matrix of the equation (3.3.2.a-4) assumes the form:

$$\left[\overline{\boldsymbol{C}}^{(2)}\right]=\begin{bmatrix} \overline{C}_{11}^{(2)} & \overline{C}_{12}^{(2)} & \overline{C}_{12}^{(2)} & 0 & 0 & 0 \\ \overline{C}_{12}^{(2)} & \overline{C}_{11}^{(2)} & \overline{C}_{12}^{(2)} & 0 & 0 & 0 \\ \overline{C}_{12}^{(2)} & \overline{C}_{12}^{(2)} & \overline{C}_{11}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{C}_{44}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{C}_{44}^{(2)} \end{bmatrix} \qquad \overline{C}_{44}^{(2)}=\overline{C}_{11}^{(2)}-\overline{C}_{12}^{(2)} \quad (3.3.2.a\text{-}25)$$

Thus, given the representations (3.3.2.a-24) and (3.3.2.a-25), the matrices $\left[F_1^{(i)}\right]$ and $\left[F_2^{(i)}\right]$ defined in the equation (3.3.2.a-12) reduce to:

$$\begin{gathered} \left[\boldsymbol{F_1}^{(1)}\right]=\begin{bmatrix} \overline{C}_{12}^{(1)} & \overline{C}_{23}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \left[\boldsymbol{F_2}^{(1)}\right]=\begin{bmatrix} \overline{C}_{22}^{(1)} & 0 & 0 \\ 0 & \overline{C}_{44}^{(1)} & 0 \\ 0 & 0 & \overline{C}_{66}^{(1)} \end{bmatrix} \\ \left[\boldsymbol{F_1}^{(2)}\right]=\begin{bmatrix} \overline{C}_{12}^{(2)} & \overline{C}_{12}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \left[\boldsymbol{F_2}^{(2)}\right]=\begin{bmatrix} \overline{C}_{11}^{(2)} & 0 & 0 \\ 0 & \overline{C}_{44}^{(2)} & 0 \\ 0 & 0 & \overline{C}_{44}^{(2)} \end{bmatrix} \end{gathered} \quad (3.3.2.a\text{-}26)$$

The substitution of the (3.3.2.a-26) in the equation (3.3.2.a-16) yields, after some algebraic manipulations:

$$[\mathsf{M}]=\begin{bmatrix} \dfrac{\left(\overline{C}_{12}^{(2)}-\overline{C}_{12}^{(1)}\right)}{a} & \dfrac{1}{f_2}\dfrac{\overline{C}_{11}^{(2)}}{a} & \dfrac{\left(\overline{C}_{12}^{(2)}-\overline{C}_{23}^{(1)}\right)}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{f_2}\dfrac{\overline{C}_{44}^{(2)}}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{f_2}\dfrac{\overline{C}_{44}^{(2)}}{c} \end{bmatrix} \quad (3.3.2.a\text{-}27)$$

where:

$$a = \bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{2}; \qquad b = \bar{C}_{44}^{(1)} + \frac{f_1}{f_2}\bar{C}_{44}^{(2)}; \qquad c = \bar{C}_{66}^{(1)} + \frac{f_1}{f_2}\bar{C}_{44}^{(2)} \tag{3.3.2.a-28}$$

Thus, by means of the definitions (3.3.2.a-24), (3.3.2.a-25) and (3.3.2.a-27), the components of the macroscopic constitutive matrix can be determined by means of the equation (3.3.2.a-23). So, the composite masonry panel is an orthotropic body (on a macro-scale) with a stiffness matrix $\left[\bar{\boldsymbol{C}}\right]$ whose nine components are defined as it follows:

$$\bar{C}_{11} = \left(f_1\bar{C}_{11}^{(1)} + f_2\bar{C}_{11}^{(2)}\right) - \frac{f_1\left(\bar{C}_{12}^{(2)} - \bar{C}_{12}^{(1)}\right)^2}{\bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{(2)}}; \qquad \bar{C}_{22} = \frac{1}{\frac{1}{f_1}\bar{C}_{22}^{(1)} + \frac{1}{f_2}\bar{C}_{11}^{(2)}};$$

$$\bar{C}_{33} = \left(f_1\bar{C}_{33}^{(1)} + f_2\bar{C}_{11}^{(2)}\right) - \frac{f_1\left(\bar{C}_{12}^{(2)} - \bar{C}_{23}^{(1)}\right)^2}{\bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{(2)}}; \qquad \bar{C}_{44} = \frac{1}{\frac{1}{f_1}\bar{C}_{44}^{(1)} + \frac{1}{f_2}\left(\bar{C}_{11}^{(2)} - \bar{C}_{12}^{(2)}\right)};$$

$$\bar{C}_{55} = f_1\bar{C}_{55}^{(1)} + f_2\left(\bar{C}_{11}^{(2)} - \bar{C}_{12}^{(2)}\right); \qquad \bar{C}_{66} = \frac{1}{\frac{1}{f_1}\bar{C}_{66}^{(1)} + \frac{1}{f_2}\left(\bar{C}_{11}^{(2)} - \bar{C}_{12}^{(2)}\right)};$$

$$\bar{C}_{12} = \left(f_1\bar{C}_{12}^{(1)} + f_2\bar{C}_{12}^{(2)}\right) - \frac{f_1\left(\bar{C}_{12}^{(2)} - \bar{C}_{12}^{(1)}\right)\left(\bar{C}_{11}^{(2)} - \bar{C}_{22}^{(1)}\right)}{\bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{(2)}};$$

$$\bar{C}_{13} = \left(f_1\bar{C}_{13}^{(1)} + f_2\bar{C}_{13}^{(2)}\right) - \frac{f_1\left(\bar{C}_{12}^{(2)} - \bar{C}_{12}^{(1)}\right)\left(\bar{C}_{12}^{(2)} - \bar{C}_{23}^{(1)}\right)}{\bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{(2)}};$$

$$\bar{C}_{23} = \left(f_1\bar{C}_{23}^{(1)} + f_2\bar{C}_{12}^{(2)}\right) - \frac{f_1\left(\bar{C}_{12}^{(2)} - \bar{C}_{23}^{(1)}\right)\left(\bar{C}_{11}^{(2)} - \bar{C}_{22}^{(1)}\right)}{\bar{C}_{22}^{(1)} + \frac{f_1}{f_2}\bar{C}_{11}^{(2)}}; \tag{3.3.2.a-29}$$

3.3.2.b Homogenization theory for periodic media by Anthoine

In this paragraph, it will be exposed the homogenization theory for periodic masonry proposed by A. Anthoine, [5].

Hence, the starting step is to choice the basic cell, which, together with the associated frame of reference, depends strongly on the geometry of the considered composite material. Therefore, typical “masonry like” patterns are analyzed and, consequently, appropriate basic cell are chosen.

For example, they can are, here, exposed some basic cells, proposed by the author, for different, simple and complex, masonry patterns, [5].

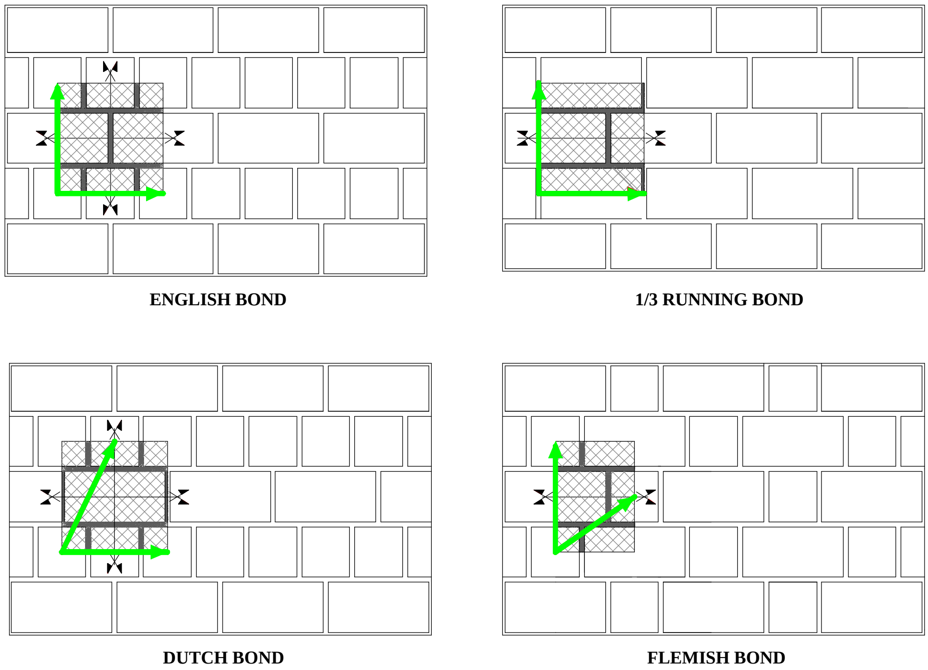


Figure 3.9 Basic cell and frame of reference for more complex bond patterns.

In particular, it can be said that, for two-dimensional periodic media, i.e. for three-dimensional media under the plane stress or plane strain assumption, the periodicity of the arrangement may be characterized by a plane frame of reference $(\boldsymbol{v}_1, \boldsymbol{v}_2)$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two independent vectors having the following property:

- the mechanical characteristics of the media are invariant along any translation $m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$, where m_1 and m_2 are integers, as it is shown in the following figure.

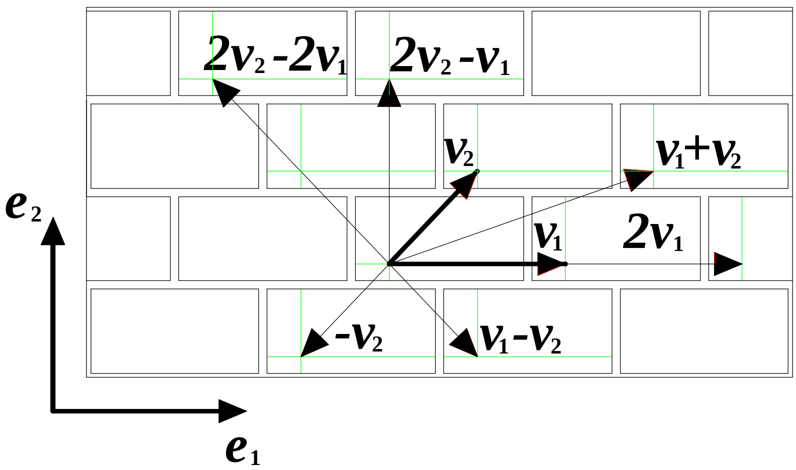


Figure 3.10 Two-dimensional running bond masonry (plane stress) and frame of reference

As a consequence, it is enough to define the mechanical properties of the media in the domain *S* of the basic cell. In particular, the following properties can be considered:

- for a given frame of reference $(\boldsymbol{v}_1, \boldsymbol{v}_2)$, all the possible associated cells have the same area $|S|$, where:

$$|S| = |\boldsymbol{v}_1 \wedge \boldsymbol{v}_2| \qquad (3.3.2.b\text{-}1)$$

- the boundary ∂S of a cell S can always be divided into two or three pairs of identical sides corresponding to each other through a translation along $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ or $\boldsymbol{v}_1 - \boldsymbol{v}_2$ (two such sides will be said *opposite*), as it is shown in the following figure 3.11.

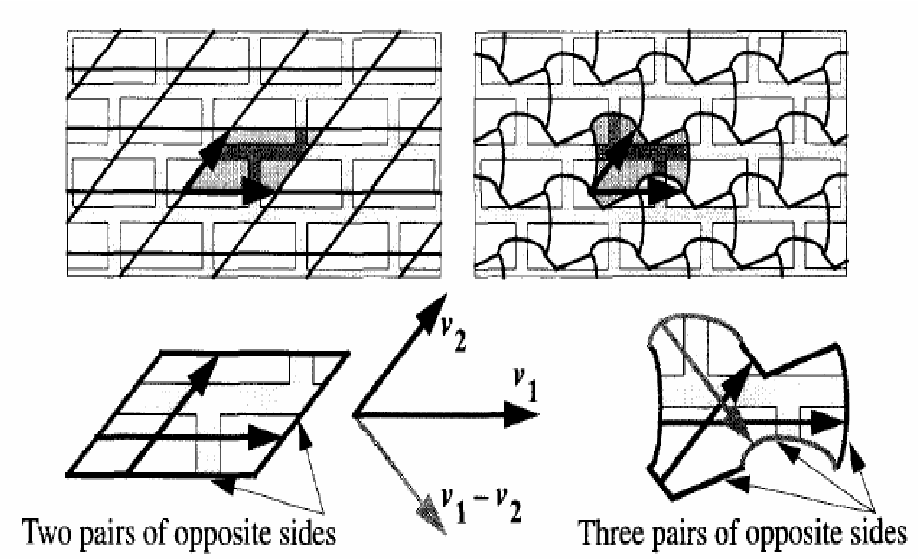


Figure 3.11 Two different cells associated to the same frame of reference and having, respectively, two and three pairs of opposite sides.

However, it is worth to underline that neither the frame of reference, nor the cell, are uniquely defined: the same cell, *S*, leads to different masonry patterns when associated to different frames of reference; and so also, for a given frame of reference more cells can be used, but it is worth choosing that one with the least area and, if possible, with symmetry properties. Such minimum cells and associated frames of reference will be called *basic cell*.

In literature, a distinction is often made between rectangular and hexagonal patterns: the formers admit an orthogonal basic frame, whereas the latter ones do not; so, the first ones can be seen as particular cases of the second ones.

In particular, the more common masonry patterns are analyzed by the author, [5]: stack bond or running bond. In this case, she proposes a cell, made up of one brick surrounded by half mortar joint, as a “good” basic cell, as shown in the following figure 3.12.

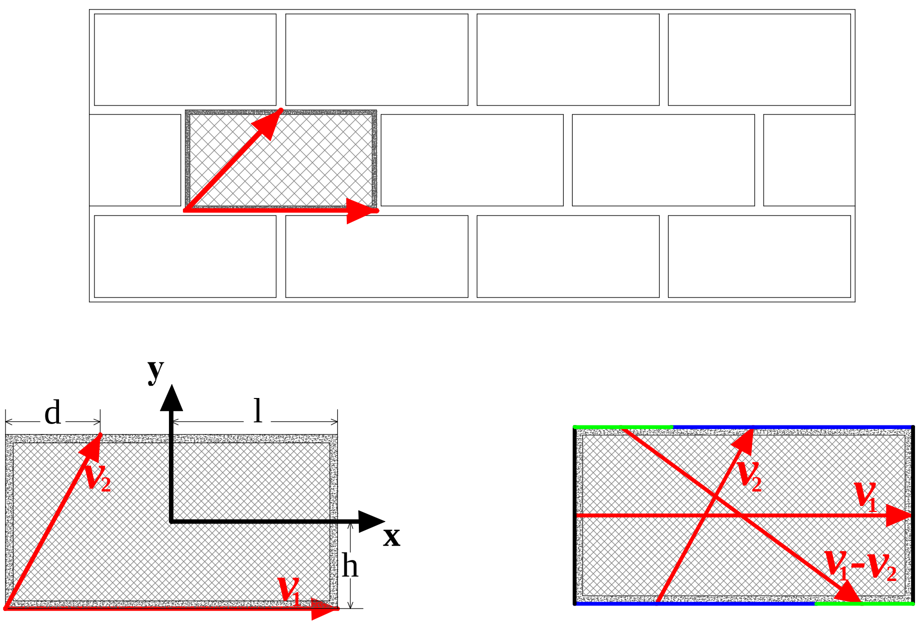


Figure 3.12 Frame of reference, basic cell and opposite sides for common masonry patterns: d=0 for stack bond; d=l for running bond.

The reference frame is then composed by the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that satisfy the following relations:

$$\begin{aligned} \boldsymbol{v}_1 &= 2l\boldsymbol{e}_1 \\ \boldsymbol{v}_2 &= d\boldsymbol{e}_1 + 2h\boldsymbol{e}_2 \end{aligned} \tag{3.3.2.b-2}$$

where:

$2l =$ the length of the brick plus the thickness of the head joint

$2h =$ the height of the brick plus the thickness of the bed joint

$d =$ the overlapping

So, according to the above shown figure, the different frames that can be determined in function of the variation of the parameter d yields to different bond patterns. In fact, it is:

$d = 0 \quad \Rightarrow$ stack bond pattern

$d = l \quad \Rightarrow$ running bond pattern

$d = \frac{2}{3} l \Rightarrow$ another kind of running bond pattern, and so on…

The first two bond patterns are here considered.

The boundary ∂S of the chosen cell is, so, composed of three pairs of opposite sides, if $d \neq 0$ (running bond pattern), or of two pairs of opposite sides, if $d = 0$ (stack bond pattern). In particular:

$d \neq 0 \Rightarrow$ the opposite sides are the vertical ones, the upper left with the lower right, the upper right with the lower left.

$d = 0 \Rightarrow$ the opposite sides are the parallel sides of the rectangle according to the definition of the opposite sides, given before.

Once it is established the appropriate basic cell, the homogenization procedure can be performed.

Here, the author proposes a different approach to the homogenization problem for periodic continuum media that aims to overcome the limits of the simplified homogenization techniques, previously described. Indeed, several limits can be adduced to these approximate procedures:

- a homogenization process which is performed in several steps, by introducing the head joints and the bed ones in different times, leads to results depending on the order of the successive steps. Moreover, it doesn't take into consideration the geometrical arrangement of the masonry: two different bond patterns yield the same results, [27].
- a homogenization procedure which is itself approximated (for example, self-consistent method in [60]) or based on a simplified geometrical arrangement of the media (mortar joints being treated as interfaces or ellipsoidal inclusions,[60]) leads to results that are unacceptable in non-linear range.
- another approximation lies in the fact that the finite thickness of masonry has never been taken into account: the masonry has always been considered or infinitely thin, in the sense of a two-dimensional media under the plane stress assumption, [58], [40], or infinitely thick, in the sense of a three-dimensional bulk [60], [58].

The idea of the homogenization procedure by Anthoine is that one to derive the in-plane characteristics of masonry through a rigorous application of the homogenization theory for periodic media, i.e. by performing a procedure that is in one-step and on the exact geometry, according, therefore, to the actual bond pattern of the masonry and, when considering the three-dimensional media, to its finite thickness. So, first, an appropriate cell and frame of reference are chosen. Here, it will be illustrated the case of two-dimensional media.

So, let us consider a masonry specimen Ω, subjected to a macroscopically homogeneous plane stress state $\boldsymbol{T}_0$.

A stress state is said to be globally (or macroscopically) homogeneous over the domain Ω if all the included basic cells within Ω undergo the same loading conditions. Really, there always is an approximation in the fact that the perturbations near the boundary $\partial\Omega$ imply that the more external cells of the specimen are not subjected to the same loading conditions as those ones lying in the centre. This difficulty is overcome according to the Saint-Venant principle: cells lying far enough from the boundary are subjected to the same loading conditions and so they also show the same deformation. In particular, two joined cells must still fit together in their common deformed state.

In mechanical terms, this yields that:

1. the stress vector has to be continuous when passing from a cell to the next one. Since passing from a cell to the next one that is identical is the same thing that passing from a side to the opposite one in the same cell, this condition can be written in the form:

$$\textit{the vectors } \mathrm{T}\boldsymbol{n} \textit{ are opposite on opposite sides of } \partial S \qquad (3.3.2.b\text{-}3)$$

because the external normal $\boldsymbol{n}$ are opposite. Such a stress field $\boldsymbol{T}$ is said to be periodic on ∂S, while the external normal $\boldsymbol{n}$ and the stress vector $\boldsymbol{Tn}$ are said to be anti-periodic on ∂S.

2. strains are compatible, i.e. neither separation nor overlapping occurs. In order to satisfy the compatibility, the displacement fields on the two opposite sides must be equal up to a rigid displacement. Such a strain field $\boldsymbol{E}$ is said to be periodic on ∂S. So, in the case of the stack bond pattern, shown in the figure 3.13, this condition can be written in the form:

$$\begin{aligned}
&\forall x_2 \in [-h,h],\ \boldsymbol{u}(l,x_2)-\boldsymbol{u}(-l,x_2)=\boldsymbol{U}-Rx_2\boldsymbol{e}_1 \\
&\forall x_1 \in [-l,l],\ \boldsymbol{u}(x_1,h)-\boldsymbol{u}(x_1,-h)=\boldsymbol{V}+Sx_1\boldsymbol{e}_2
\end{aligned} \qquad (3.3.2.b\text{-}4)$$

where:

$\boldsymbol{U}$ *and* $\boldsymbol{V}$ = translation vectors

R *and* S = rotation constants

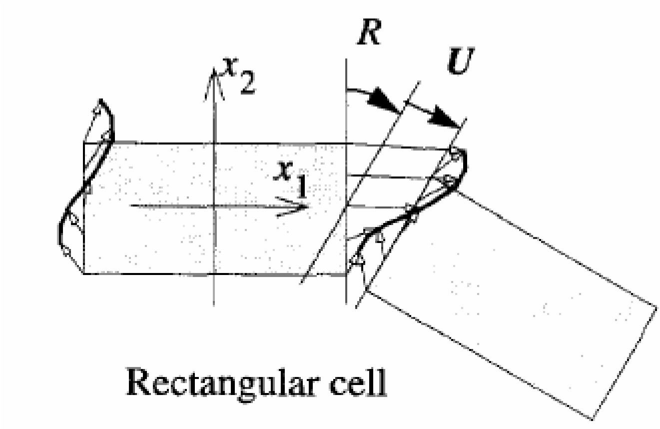


Figure 3.13 Displacement fields for a rectangular cell

Since each corner of the cell belongs both to the vertical and the horizontal side and since it must undergo the same displacement, the equation (3.3.2.b-4) has to be compatible when written for the extreme values of x_1 and x_2:

$$\begin{aligned} x_2 = h &\Rightarrow \boldsymbol{u}(l,h) - \boldsymbol{u}(-l,h) = \boldsymbol{U} - Rh\boldsymbol{e}_1 \\ x_2 = -h &\Rightarrow \boldsymbol{u}(l,-h) - \boldsymbol{u}(-l,-h) = \boldsymbol{U} + Rh\boldsymbol{e}_1 \\ x_1 = l &\Rightarrow \boldsymbol{u}(l,h) - \boldsymbol{u}(l,-h) = \boldsymbol{V} + Sl\boldsymbol{e}_2 \\ x_1 = -l &\Rightarrow \boldsymbol{u}(-l,h) - \boldsymbol{u}(-l,-h) = \boldsymbol{V} - Sl\boldsymbol{e}_2 \end{aligned} \qquad (3.3.2.b\text{-}5)$$

The relations (3.3.2.b-5) can be ensured only if *R* and *S* are zero constants.

What has been written for the stack bond pattern can be reformulated for the running bond pattern, shown in the following figure 3.14:

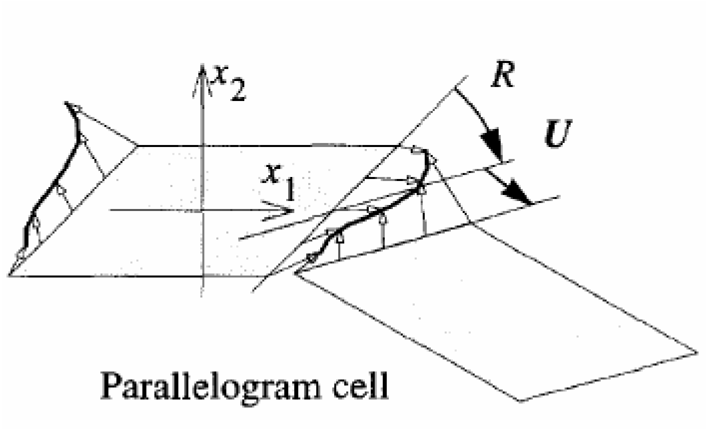


Figure 3.14 Displacement fields for a parallelogram cell

Now, the reference basic cell is a parallelogram cell; so the system (3.3.2.b-4) changes to:

$$\begin{aligned}&\forall x_2 \in [-h,h],\ \boldsymbol{u}\left(l+\frac{lx_2}{2h},x_2\right)-\boldsymbol{u}\left(-l+\frac{lx_2}{2h},x_2\right)=\boldsymbol{U}-R\left(x_2\boldsymbol{e}_1+\frac{lx_2}{2h}\boldsymbol{e}_2\right)\\&\forall x_1 \in [-l,l],\ \boldsymbol{u}\left(\frac{l}{2}+x_1,h\right)-\boldsymbol{u}\left(-\frac{l}{2}+x_1,-h\right)=\boldsymbol{V}+Sx_1\boldsymbol{e}_2\end{aligned} \qquad (3.3.2.b\text{-}6)$$

Analogously, displacements at corners are consistent only if R and S are zero constants.

Taking into consideration the zero values of R and S, the systems (3.3.2.b-4) and (3.3.2.b-6) assume the form:

$$\begin{aligned}&\forall x_2 \in [-h,h],\ \boldsymbol{u}\left(l+\frac{dx_2}{2h},x_2\right)-\boldsymbol{u}\left(-l+\frac{dx_2}{2h},x_2\right)=\boldsymbol{U}\\&\forall x_1 \in [-l,l],\ \boldsymbol{u}\left(\frac{d}{2}+x_1,h\right)-\boldsymbol{u}\left(-\frac{d}{2}+x_1,-h\right)=\boldsymbol{V}\end{aligned} \qquad (3.3.2.b\text{-}7)$$

where:

$d =$ the overlapping

Such a displacement field $\boldsymbol{u}$ is said strain-periodic because it leads to a periodic strain field.

A so done displacement field $\boldsymbol{u}$ may always be written in the following form:

$$\begin{aligned}&\forall \alpha,\beta = 1 \, or \, 2, u_\alpha\left(x_1,x_2\right)=\overline{E}_{\alpha\beta}x_\beta + u_\alpha^p\left(x_1,x_2\right)\Leftrightarrow\\&\Leftrightarrow \boldsymbol{u}\left(x_1,x_2\right)=\overline{\boldsymbol{E}}\boldsymbol{x}+\boldsymbol{u}^p\left(x_1,x_2\right)\end{aligned} \tag{3.3.2.b-8}$$

where:

$\overline{E}_{\alpha\beta} =$ constants

$\boldsymbol{u}^p =$ periodic displacement field, in the sense that it assumes equal values on the opposite sides of ∂S.

By the equation (3.3.2.b-8), it can be obtained that:

$$\boldsymbol{u}^P\left(x_1,x_2\right)=\boldsymbol{u}\left(x_1,x_2\right)-\overline{\boldsymbol{E}}\boldsymbol{x} \tag{3.3.2.b-9}$$

By comparing the equation (3.3.2.b-8) with the equation (3.3.2.b-7), it is deduced that:

$$\begin{aligned}&\overline{E}_{11}=U_1/2l\\&\overline{E}_{21}=U_2/2l\\&\overline{E}_{12}=\left(V_1-U_1\,d/2l\right)/2h\\&\overline{E}_{22}=\left(V_2-U_2\,d/2l\right)/2h\end{aligned} \tag{3.3.2.b-10}$$

From the relations (3.3.2.b-10), it can be noted that, for example, the component $\overline{E}_{11}$ represents the mean elongation of the cell along the x_1 axis and, so, $\overline{\boldsymbol{E}}$ can be considered as the mean strain tensor of the cell. It can be demonstrated by considering the definition of the average of the strain components in the domain of the basic cell. Therefore, it is:

$$\bar{e}_{ab}\left(\boldsymbol{u}\right) = \frac{1}{|S|}\int_{S} e_{ab}\left(\boldsymbol{u}\right) ds \qquad (3.3.2.b\text{-}11)$$

where:

$\bar{e}_{ab}\left(\boldsymbol{u}\right) =$ the average value, marked by the symbol $\bar{\ }$, of the generic strain component e_{ab}.

$a, b =$ 1, 2

$S =$ the area of the basic cell

Since, the generic strain component, obtained as the symmetric part of the gradient of $\boldsymbol{u}$, is given by the following expression:

$$e_{ab}\left(\boldsymbol{u}\right) = \left(\bar{E}_{ab} + u^{p}_{a,b} + \bar{E}_{ba} + u^{p}_{b,a}\right)/2 \qquad (3.3.2.b\text{-}12)$$

and for the assumed symmetry of $\bar{\boldsymbol{E}}$ (only the symmetric part of it is considered, being the anti-symmetric part of it correspondent to a rigid rotation of the cell and being the rigid displacements disregarded), it is:

$$e_{ab}\left(\boldsymbol{u}\right) = \bar{E}_{ab} + \left(u^{p}_{a,b} + u^{p}_{b,a}\right)/2 \qquad (3.3.2.b\text{-}13)$$

the relation (3.3.2.b-11) becomes:

$$\bar{e}_{ab}\left(\boldsymbol{u}\right) = \bar{E}_{ab} + \frac{1}{2|S|}\int_{S}\left(u^{p}_{a,b} + u^{p}_{b,a}\right) ds = \bar{E}_{ab} + \frac{1}{2|S|}\int_{dS}\left(u^{p}_{a} n_{b} + u^{p}_{b} n_{a}\right) dl \qquad (3.3.2.b\text{-}14)$$

Since $\boldsymbol{u}^{p}$ is a periodic vector fields and $\boldsymbol{n}$ is an anti-periodic one on dS, the product $u^{p}_{a} n_{b}$ represents an anti-periodic scalar field on dS. Thus, in the (3.3.2.b-14), the integral in dS is equal to zero because the assumed values on the opposite sides cancel each other. This means that:

$$\bar{\varepsilon}_{\alpha\beta}(\boldsymbol{u}) = \bar{E}_{\alpha\beta} \qquad (3.3.2.b\text{-}15)$$

and, as it was above mentioned, $\bar{\boldsymbol{E}}$ turns out to coincide with the average of $\mathrm{E}(\boldsymbol{u})$ on the cell.

By substituting the (3.3.2.b-15) into the (3.3.2.b-9), it is obtained:

$$\boldsymbol{u}^{P}(x_1, x_2) = \boldsymbol{u}(x_1, x_2) - \bar{\mathrm{E}}(\boldsymbol{u}) \cdot \boldsymbol{x} \qquad (3.3.2.b\text{-}16)$$

At this point, it can be stated that, if $\boldsymbol{T}$ is periodic and $\boldsymbol{u}$ is strain-periodic on the boundary ∂S of the cell, it is possible to study the problem within the single cell rather than on the whole specimen. If the specimen is subjected to the macroscopically homogeneous stress state $\boldsymbol{T}_0$, above defined, such a posed problem to solve is:

$$\begin{aligned}
&\bar{\boldsymbol{T}} = \boldsymbol{T}_0 \\
&div\boldsymbol{T} = 0 \quad on\ S \\
&\boldsymbol{T}\ periodic\ on\ \partial S\ (\boldsymbol{Tn}\ anti-periodic\ on\ \partial S) \\
&\boldsymbol{E} = f^{-1}(\boldsymbol{T}) \\
&\boldsymbol{u} - \bar{\mathrm{E}} \cdot \boldsymbol{x}\ \ periodic\ on\ \partial S
\end{aligned} \qquad (3.3.2.b\text{-}17)$$

where:

$$\bar{\boldsymbol{T}} = \frac{1}{|S|}\int_S \boldsymbol{T}\, ds \qquad (3.3.2.b\text{-}18)$$

with:

$\bar{\boldsymbol{T}} =$ the average value of the stress tensor in the domain of the basic cell

In particular, in this prescribed stress problem (the macroscopically homogeneous stress state $\boldsymbol{T}_0$ is the assigned data of the problem), no body forces are considered and the constitutive law f is a periodic function of the spatial variable $\boldsymbol{x}$ that describes the mechanical behaviour of the different materials in the composite cell.

Naturally, the same problem can be considered as a prescribed strain one; such a similar problem is written in the following form:

$$\begin{aligned}
&\overline{\mathsf{E}} = \boldsymbol{E}_0 \\
&\boldsymbol{u} - \boldsymbol{E}_0 \cdot \boldsymbol{x} \ \ periodic\ on\ \partial S \\
&\boldsymbol{T} = f\left(\boldsymbol{E}\right) \\
&div\boldsymbol{T} = 0 \quad\quad on\ S \\
&\boldsymbol{T}\ \ periodic\ on\ \partial S\ \ (\boldsymbol{Tn}\ \ anti-periodic\ on\ \partial S)
\end{aligned} \tag{3.3.2.b-19}$$

where the assigned data of the problem is the macroscopically homogeneous strain state $\boldsymbol{E}_0$, and analogously to the equation (3.3.2.b-18) it is:

$$\overline{\boldsymbol{E}} = \frac{1}{|S|}\int_S \boldsymbol{E}\, ds \tag{3.3.2.b-20}$$

By passing, first, through a "localization" problem that concurs to determine the local (microscopic) fields $\boldsymbol{T}$, $\boldsymbol{u}$ and $\boldsymbol{E}$ from the global (macroscopic) field $\boldsymbol{T}_0$ or $\boldsymbol{E}_0$, the unknown macroscopic fields $\overline{\boldsymbol{T}}$ and $\overline{\boldsymbol{E}}$ are then evaluated.

In particular, in the case of stress prescribed problem, they will be obtained $\overline{\boldsymbol{T}} = \boldsymbol{T}_0$ and $\overline{\boldsymbol{E}}$ according to the equation (3.3.2.b-20); dually, in the case of strain prescribed problem they will be obtained $\overline{\boldsymbol{T}}$ according to the equation (3.3.2.b-18) and $\overline{\boldsymbol{E}} = \boldsymbol{E}_0$.

At this point, the global (macroscopic) constitutive law of the composite material, that is the $\boldsymbol{T}_0 - \boldsymbol{E}_0$ relationship, can be built by repeating the above described procedure for different values of $\boldsymbol{T}_0$ and $\boldsymbol{E}_0$.

Such a procedure is said homogenization process because the actual composite specimen, subjected to the prescribed macroscopically homogeneous loading, can be, now, substituted with a fictitious homogenized material obeying to the found global constitutive law without changing its mechanical macroscopic answer.

This result has a remarkable importance since, while the discretization of the original composite masonry structure is prohibitive, the discretization of the same structure, subjected to the same loads but replaced by the homogeneous material, is more advantageous.

It has again to be underlined, however, that the homogenization theory is applied only under the assumption that the loading conditions are equal (or enough similar) for adjacent basic cells. This happens if two cases are possible:

1. or the size of the basic cell is quite small when compared with the size of the structure, so that, at a structural scale, two adjacent cells have almost the same position and, therefore, the same loading.
2. or the basic cell is “not so small” when compared with the size of the structure, but the macroscopic stresses induced by the structural loads don’t vary (or vary slowly) within the structure.

It is worth to state that, in presence of concentrated loads and boundary conditions, high gradients or even singularities can be generated in the macroscopic stress field. In these cases, also very small adjacent cells can undergo to different load conditions, so local analyses in such critical regions have to be performed on the original composite material.

If the two constituents of the masonry basic cell are considered linear elastic and perfectly bonded, the two “localization” problems, the prescribed stress one shown in the (3.3.2.b-17) and the prescribed strain one shown in the (3.3.2.b-19), can be rewritten in the respectively following forms:

$$\begin{aligned}
&div\boldsymbol{T} = \boldsymbol{0} \qquad on\ S\\
&\boldsymbol{E}(\boldsymbol{u}) = \mathsf{C}^{-1}:\boldsymbol{T}\\
&\boldsymbol{T}\ periodic\ on\ \partial S\ \ (\boldsymbol{Tn}\ anti-periodic\ on\ \partial S)\\
&\boldsymbol{u}-\left\langle \mathsf{C}^{-1}:\boldsymbol{T}\right\rangle\cdot\boldsymbol{x}\ \ periodic\ on\ \partial S\\
&\overline{\boldsymbol{T}} = \boldsymbol{T}_0
\end{aligned} \tag{3.3.2.b-21}$$

where:

$\left\langle \mathsf{C}^{-1}:\boldsymbol{T}\right\rangle = \overline{\mathsf{E}}(\boldsymbol{u}) =$ the average value of the strain tensor in the basic cell

and

$\mathsf{C}^{-1} = \mathsf{S} =$ the known elastic compliance tensor of the constituents in plane stress.

For the prescribed strain problem:

$$\begin{aligned}
&div\left(\mathsf{C}:\boldsymbol{E}(\boldsymbol{u})\right) = \boldsymbol{0} \qquad on\ S\\
&\boldsymbol{T} = \mathsf{C}:\boldsymbol{E}(\boldsymbol{u})\\
&\mathsf{C}:\boldsymbol{E}(\boldsymbol{u})\ periodic\ on\ \partial S\ \ (\mathsf{C}:\boldsymbol{E}(\boldsymbol{u})\cdot\boldsymbol{n}\ anti-periodic\ on\ \partial S)\\
&\boldsymbol{u}-\boldsymbol{E}_0(\boldsymbol{u})\cdot\boldsymbol{x}\ \ periodic\ on\ \partial S\\
&\overline{\mathsf{E}} = \boldsymbol{E}_0
\end{aligned} \tag{3.3.2.b-22}$$

where:

$\mathsf{C} =$ the known elastic stiffness tensor of the constituents in plane stress.

The problem (3.3.2.b-22) may be rewritten in terms of $\boldsymbol{u}^p$.

In order to make it, let us to consider the equation (3.3.2.b-13), that in a tensorial form becomes:

$$\boldsymbol{E}(\boldsymbol{u}) = \overline{\boldsymbol{E}} + \overline{\boldsymbol{E}}\left(\boldsymbol{u}^p\right) \tag{3.3.2.b-23}$$

Therefore, by substituting the (3.3.2.b-23) in the (3.3.2.b-22), it is obtained:

$$\begin{aligned}
& div\left(\mathsf{C}:\overline{\boldsymbol{E}}\right)+div\left(\mathsf{C}:\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right)=\mathbf{0} \quad on\ S \\
& \boldsymbol{T}=\mathsf{C}:\left(\overline{\boldsymbol{E}}+\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right) \\
& \mathsf{C}:\left(\overline{\boldsymbol{E}}+\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right) periodic\, on\, \partial S \ (\mathsf{C}:\left(\overline{\boldsymbol{E}}+\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right)\cdot \boldsymbol{n}\ anti-periodic\, on\, \partial S) \\
& \boldsymbol{u}^p \ periodic\ on\ \partial S \\
& \overline{\mathsf{E}}=\boldsymbol{E}_0
\end{aligned} \qquad (3.3.2.b\text{-}24)$$

When the boundary of the basic cell is constituted by the same material in each its point and when the two constituents of the cell are homogeneous materials, and so, the stiffness tensor C is constant on each component, the third condition in the (3.3.2.b-24), $(\mathsf{C}:\left(\overline{\boldsymbol{E}}+\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right)\cdot\boldsymbol{n}$ is anti-periodic on $\mathrm{d}S$, reduces to $(\mathsf{C}:\left(\boldsymbol{E}\left(\boldsymbol{u}^p\right)\right)\cdot\boldsymbol{n}$. The solution of such a posed problem is a periodic displacement field that yields a periodic stress field on $\mathrm{d}S$ and is in equilibrium with the concentrated body forces $\boldsymbol{f}$, induced at the interfaces $\boldsymbol{I}$ by the uniform average strain tensor $\overline{\boldsymbol{E}}$:

$$\boldsymbol{f}=div\left(\mathsf{C}:\overline{\boldsymbol{E}}\right)=\left(\mathsf{C}^m-\mathsf{C}^b\right):\overline{\boldsymbol{E}}\cdot\boldsymbol{n}\mathrm{d}_{\mathrm{I}} \qquad (3.3.2.b\text{-}25)$$

where:

$\mathsf{C}^m =$ stiffness tensor, in plane stress, of the mortar

$\mathsf{C}^b =$ stiffness tensor, in plane stress, of the brick

$\boldsymbol{n}$ = the normal oriented from the brick to the mortar (see figure 3.15)

$\mathrm{d}_{\mathrm{I}} =$ the Dirac distribution on the interface $\boldsymbol{I}$ (see figure 3.15)

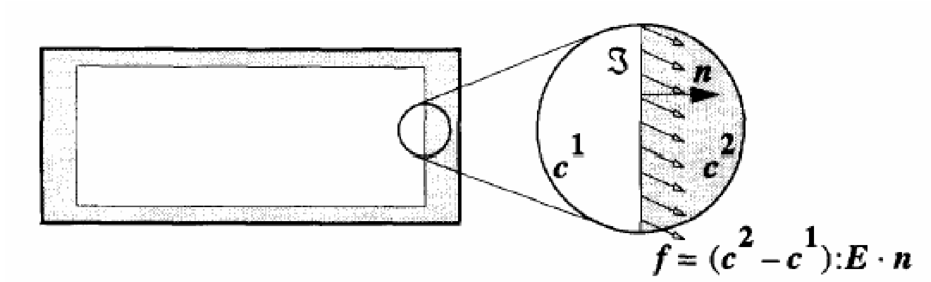


Figure 3.15 Body forces concentrated at the interface of the constituents (brick and mortar).

For the linearity of problems (3.3.2.b-21) and (3.3.2.b-24), the superposition principle (S.P.) can be considered valid.

Therefore, both for the prescribed stress problem and the prescribed strain problem, the given symmetric second-order tensors, respectively $\boldsymbol{T}_0$ and $\boldsymbol{E}_0$ are assigned as linear combination of three elementary tensors $\boldsymbol{I}^{\mathsf{ab}}$, where a and b belong to the range $[1,2]$.

In particular, for the problem (3.3.2.b-21), if $\boldsymbol{T}^{\mathsf{ab}}$ is the elementary solution obtained for $\boldsymbol{T}_0 = \boldsymbol{I}^{\mathsf{ab}}$, then for $\boldsymbol{T}_0 = T_0^{\mathsf{ab}}\boldsymbol{I}^{\mathsf{ab}}$, the solution is:

$$\boldsymbol{T} = \boldsymbol{T}^{\mathsf{ab}}T_0^{\mathsf{ab}} \tag{3.3.2.b-26}$$

This solution can be rewritten in the following form:

$$\boldsymbol{T} = \mathsf{B}\boldsymbol{T}_0 \tag{3.3.2.b-27}$$

where:

$\mathsf{B} =$ a fourth-order tensor, called the tensor of stress concentration, because it gives the local stress field $\boldsymbol{T}$ in terms of the average stress field $\boldsymbol{T}_0$.

Finally, the average value of the strain state $\boldsymbol{E}$ is given by:

$$\overline{\boldsymbol{E}} = \langle \boldsymbol{E} \rangle = \langle \mathsf{C}^{-1} : \boldsymbol{T} \rangle = \langle \mathsf{C}^{-1} : \mathsf{B} : \boldsymbol{T}_0 \rangle = \langle \mathsf{C}^{-1} : \mathsf{B} \rangle : \boldsymbol{T}_0 = \langle \mathsf{C}^{-1} : \mathsf{B} \rangle : \overline{\boldsymbol{T}} \tag{3.3.2.b-28}$$

So, it may be deduced that:

$$\overline{\mathsf{C}}^{-1} = \left\langle \mathsf{C}^{-1} : \mathsf{B} \right\rangle$$
$$or \qquad (3.3.2.b\text{-}29)$$
$$\overline{\mathsf{S}} = \left\langle \mathsf{S} : \mathsf{B} \right\rangle$$

where:

$\overline{\mathsf{C}}^{-1} = \overline{\mathsf{S}} =$ the macroscopic (homogenized) tensor of elastic compliances of the equivalent (homogeneous) two-dimensional plane stress material.

Analogously, for the problem (3.3.2.b-22), if $\boldsymbol{u}^{\mathrm{ab}}$ is the elementary solution obtained for $\boldsymbol{E}_0 = \boldsymbol{I}^{\mathrm{ab}}$, then for $\boldsymbol{E}_0 = E_0^{\mathrm{ab}} \boldsymbol{I}^{\mathrm{ab}}$, the solution is:

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{ab}} E_0^{\mathrm{ab}} \qquad (3.3.2.b\text{-}30)$$

and the local strain field is given by the following relation:

$$\boldsymbol{E}(\boldsymbol{u}) = \boldsymbol{E}\left(\boldsymbol{u}^{\mathrm{ab}} E_0^{\mathrm{ab}}\right) = \boldsymbol{E}\left(\boldsymbol{u}^{\mathrm{ab}}\right) E_0^{\mathrm{ab}} = \boldsymbol{E}^{\mathrm{ab}} E_0^{\mathrm{ab}} \qquad (3.3.2.b\text{-}31)$$

where:

$\boldsymbol{E}^{\mathrm{ab}} = \boldsymbol{E}\left(\boldsymbol{u}^{\mathrm{ab}}\right) =$ the local strain field obtained for the elementary solution when $\boldsymbol{E}_0 = \boldsymbol{I}^{\mathrm{ab}}$

The equation (3.3.2.b-31) can be rewritten in the following form:

$$\boldsymbol{E} = \mathsf{A}\boldsymbol{E}_0 \qquad (3.3.2.b\text{-}32)$$

where:

$\mathsf{A} =$ a fourth-order tensor, called the tensor of strain localization, because it gives the local strain field $\boldsymbol{E}$ in terms of the average strain field $\boldsymbol{E}_0$.

Finally, the average value of the stress state $\boldsymbol{T}$ is given by:

$$\overline{\boldsymbol{T}} = \langle \boldsymbol{T} \rangle = \langle \mathsf{C} : \boldsymbol{E} \rangle = \langle \mathsf{C} : \mathsf{A} : \boldsymbol{E}_0 \rangle = \langle \mathsf{C} : \mathsf{A} \rangle : \boldsymbol{E}_0 = \langle \mathsf{C} : \mathsf{A} \rangle : \overline{\boldsymbol{E}} \qquad (3.3.2.b\text{-}33)$$

So, it may be deduced that:

$$\overline{C} = \langle C : A \rangle \tag{3.3.2.b-34}$$

where:

$\overline{C} =$ the macroscopic (homogenized) tensor of elastic stiffness of the equivalent (homogeneous) two-dimensional plane stress material.

All what has been said for the two-dimensional (plane stress) periodic masonry may be generalized to the case of the three-dimensional periodic masonry, having two or three directions of periodicity, where its actual finite thickness is taken into account. Because this problem is out of our interest, for more details on such topic, the reader is referred to the proposed procedure in literature by A. Anthoine, [5].

It, however, appears quite interesting to illustrate, here, the main results and considerations that the author obtains from her numerical analysis, in linear elasticity, on bi-dimensional and three-dimensional masonry specimens. In both cases, comparisons are done between her proposed formulation and the other simplified approaches, existing in literature. Some of these ones have already mentioned, at the start of this section.

In particular, they may be divided in two groups: the two-dimensional approaches and the three-dimensional ones. The formers, in order of increasing approximation and decreasing complexity, are:

- the method proposed by Maier *et al* (1991); the homogenization approach is performed in three steps for running bond masonry patterns and in two steps for stack bond masonry patterns.
- the method proposed by Pande *et al* (1989); the homogenization approach is performed in two steps, head joints being introduced first.
- a variant of the Pande method; the homogenization approach is yet performed in two steps, but the steps are inverted: bed joints are

introduced first. In the case of stack bond patterns, this method is equivalent to Maiers' one.

- The multi-layer approximation proposed by Maier *et al* (1991); the head joints are disregarded and, therefore, the masonry is considered as composed of alternating layers of mortar (bed joints) and brick. The homogenization approach is so performed in one step.

The latter ones, again in order of increasing approximation and decreasing complexity, are:

- the method proposed by Pande *et al* (1989)
- the inverse of it
- the multi-layer approximation

The methods belonging to both two groups, differently from the formulation proposed by Anthoine, don't need finite element calculations, but, for their simplicity, they can be implemented analytically.

In the follows, in Table 3.1 and in Table 3.2 are summarized the direct comparisons, about the macroscopic properties, between the different methods, respectively for the two-dimensional methods and for the three-dimensional ones:

TWO- DIMENSIONAL HOMOGENIZATION	E_1(MPa)	E_2(MPa)	n_{12}	G_{12}(MPa)
Stack bond	**8530**	**6790**	**0.196**	**2580**
Running bond	**8620**	**6770**	**0.200**	**2620**
Running bond in three steps (Maier et al., 1991)	**9208**	**6680**	**0.2045**	**2569**
Running or stack bond in two steps, head joints first (Pande et al., 1989)	**8464**	**6831**	**0.2182**	**2569**
Running or stack bond in two steps, bed joints first (“Pande inverted” or “Maier” for stack bond)	**8587**	**6768**	**0.1948**	**2569**
Multi-layer (Maier et al., 1991)	**9646**	**6950**	**0.2077**	**2782**

Table 3.1 Elastic constants of the homogenized material; two-dimensional methods.

TWO- DIMENSIONAL HOMOGENIZATION	E_1(MPa)	E_2(MPa)	n_{12}	G_{12}(MPa)
Stack bond	**8600**	**7000**	**0.200**	**2580**
Running bond	**8680**	**6980**	**0.204**	**2620**
Running or stack bond in two steps, head joints first (Pande et al., 1989)	**8566**	**7066**	**0.1974**	**2569**
Running or stack bond in two steps, bed joints first (“Pande inverted”)	**8676**	**7006**	**0.1995**	**2569**
Multi-layer	**9647**	**7198**	**0.2098**	**2782**

Table 3.2 Elastic constants of the homogenized material; three-dimensional methods.

All the results are presented in terms of four material elastic coefficients, E_1, E_2, ν_{12} and G_{12}.

From the Table 3.1, the first two rows obtained with the Anthoine's formulation yield that the different bond pattern, stack bond or running bond, has very little influence (less than 1% difference) on the homogenized elastic properties. This influence is, however, stronger on the local displacements and stress fields, [5]. Moreover, yet from the Table 3.1, it can be noted that the other four simplified methods lead to quite acceptable results. The less accurate is the multi-layer approach (the simplest one) where the homogenized elastic coefficients are overestimated. This was logical, since head joints are substituted by a stiffer material (the brick). The more elaborated approach (homogenization in three steps, Maier *et al* (1991)), instead, doesn't reveal itself the more accurate.

The Table 3.2, substantially, suggests the same considerations as deduced by Table 1, about the global elastic behaviour of masonry, as obtained through the different methods.

It is interesting to underline that, for a given bond pattern and for a given method, the two-dimensional approach always gives lower values of the elastic constants than the three-dimensional one. This fact is quite obvious, since the plane stress assumption neglects the thickness of the wall, by weakening it. In spite of this consideration, the two- and the three-dimensional approaches yield quite similar results on the homogenized elastic coefficients (less than 4% difference).

However, strong differences can be, instead, pointed out in the local stress fields. In particular, this fact is not relative to the in-plane components $(\sigma_{11},\sigma_{12},\sigma_{22})$, but to the out-of-plane ones $(\sigma_{13},\sigma_{23},\sigma_{33})$, which are, by definition, equal to zero in the two-dimensional approach. For this reason, even if the homogenized elastic constants are only slightly modified in the two approaches, it is worth to consider this strong difference in a non-linear analysis: by neglecting the stress component σ_{33} in the two-dimensional method, some failure situations may be not encountered.

It is, therefore, most probable that the conclusions drawn in the elastic range, in the two-dimensional approaches, are wrong in a non-linear range. In fact:

- the plane assumptions may lead to quantitatively wrong results (under-estimation of the ultimate load) and to qualitatively wrong results (erroneous failure mechanism)
- the bond pattern may strongly influence the failure mechanism and consequently the failure load; for example, in the stack bond masonry the cracks may develop easily in the aligned head joints, while in the running bond masonry they need to pass through or around the brick.

Of course, if a non-linear analysis has to be performed, the problems (3.3.2.b-21) and (3.3.2.b-22) have to be solved for a macroscopic loading history and with damage or plasticity constitutive laws. Since the superposition principle doesn't apply anymore, the complete determination of the constitutive law requires an infinite number of computations. The reader is referred to Suquet (1987).

3.3.2.c A homogenization procedure by A. Zucchini - P.B. Lourenco

In the framework of the third homogenization approach shown in the previous paragraph, an interesting analysis has been employed by the authors

A. Zucchini and P.B. Lourenco, [67]. They have proposed a new micro-mechanical model. By taking in account the actual deformations of the basic cell of a periodic masonry arrangement, this micro-mechanical model includes additional internal deformation modes which are neglected in the standard two-step homogenization procedure, which is based on the assumption of continuous perpendicular head joints. The authors show that these mechanisms, which result from the staggered alignment of the bricks in the composite, are important for the medium global response and for reducing the maximum errors in the calculation of the homogenized elastic moduli when large difference in mortar and brick stiffness are expected. Indeed, one of the goals of this approach is constituted just by the overcoming the limitations presented in the standard two-step homogenization technique, often known in literature as simplified homogenization approach that, we remember, are:

- Large errors which occurred if great differences of stiffness between unit and mortar are presented. For the cases in which non-linear analysis is employed, and where the ratio stiffness of unit (brick) on stiffness of mortar becomes larger (>10), this simplified approach leads to non-acceptable errors.
- The standard two-step homogenization technique does not take into account the pattern of units and mortar joints (running bond and stretcher bond lead to the same result).
- The results depend on the order in which the two steps are executed.

In particular, the analysis is employed for a single leaf masonry wall, with typical periodic arrangement in stretcher bond and the hypothesis of linear elastic-brittle behaviour is assumed, so that the S.P. may be used until the collapse. The unit-mortar interface is not considered in the model. On the contrary, the full three-dimensional behaviour is examined and the attention is, finally, given to a comparison between the results from a detailed finite element analysis (FEM) and the proposed micro-mechanical homogenization model, in order to demonstrate the efficiency of the proposed solution, [67].

Hence, such a micro-mechanical model is obtained by extracting a basic periodic cell, which can rebuild the whole structure by making opportune translations of it, as shown in figure below.

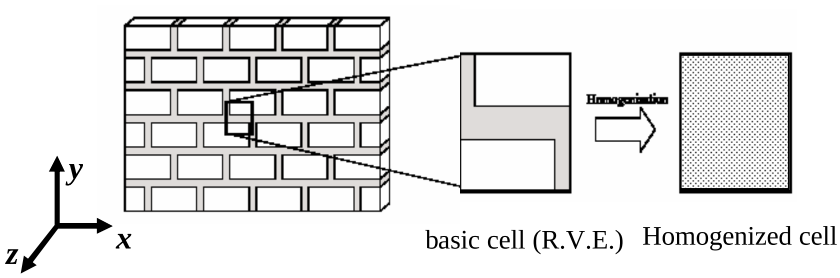


Figure 3.16 Definition of masonry axes and of chosen micro-mechanical model

It has been chosen a right-oriented x-y-z Cartesian coordinate system and the following components for the basic cell are considered:

- Head joint (2)
- Unit (b)
- Cross joint (3)
- Bed joint (1)

The following figure shows, in detail, the geometry of the basic cell, with the definition of its dimensions and of adopted symbols.

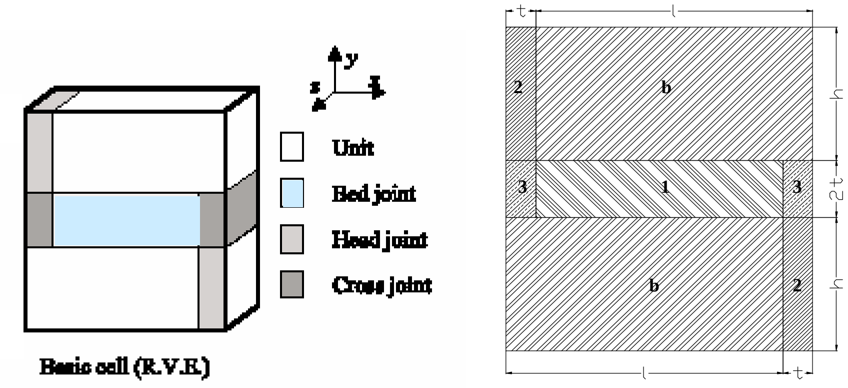


Figure 3.17 Adopted basic cell and geometric parameters

The assumed hypothesis of linear elasticity concurs to the possibility to study the elastic response of the model for a generic loading condition as linear combination of the elastic responses to six elementary loading conditions: three cases of normal stresses and three cases of simple shear (prescribed stress homogenization).

For each of these cases, and – as a consequence – for each constituent of the cell, suitably chosen components of the stress and strain tensors are assumed to be of relevance for the stress-strain state of the basic cell. In particular, such choose derives by observing the basic cell deformations, which are calculated, previously, with a finite element analysis under the same loading conditions. In the figure 3.18 are reported the deformed configurations resulting from the FEM analysis, [67].

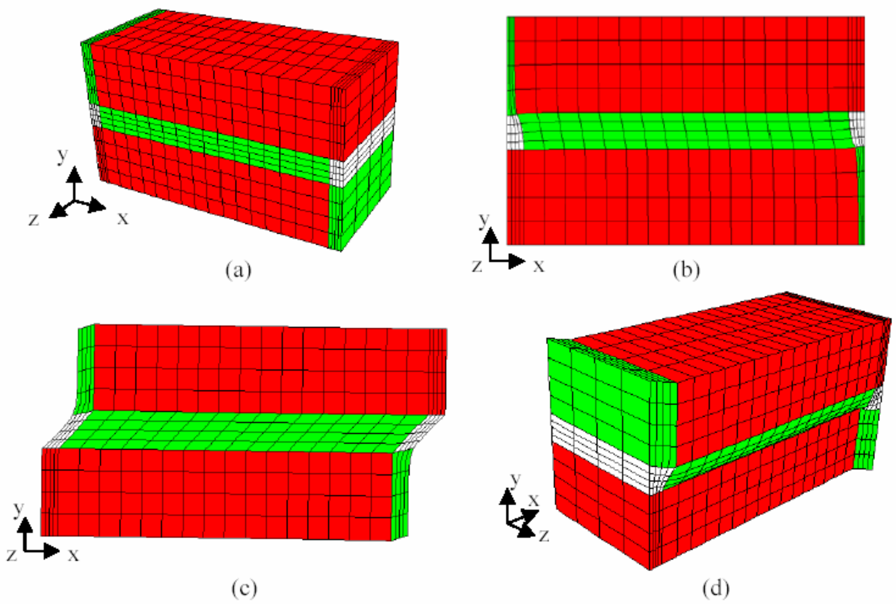


Figure 3.18 (a) finite element mesh, (b) deformation for compression in *x*-direction, (c) deformation for shear *xy*, (d) deformation for shear *xz*

As an example, therefore, in the case of uniform normal stress in *x* direction, the assumed deformation mechanism and, as a consequence, the chosen stress components are shown in the figure below.

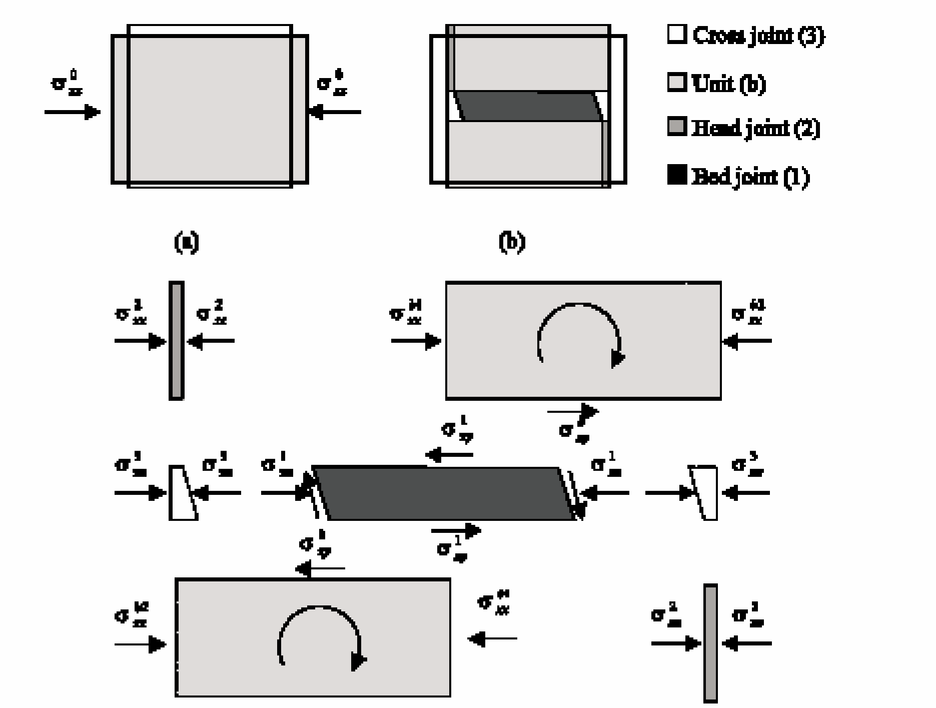


Figure 3.19 Normal stress loading in *x*-direction

Analogously, for the case of uniform normal stress in *y* direction, the assumed deformation mechanism and, as a consequence, the chosen stress components are shown in figure below.

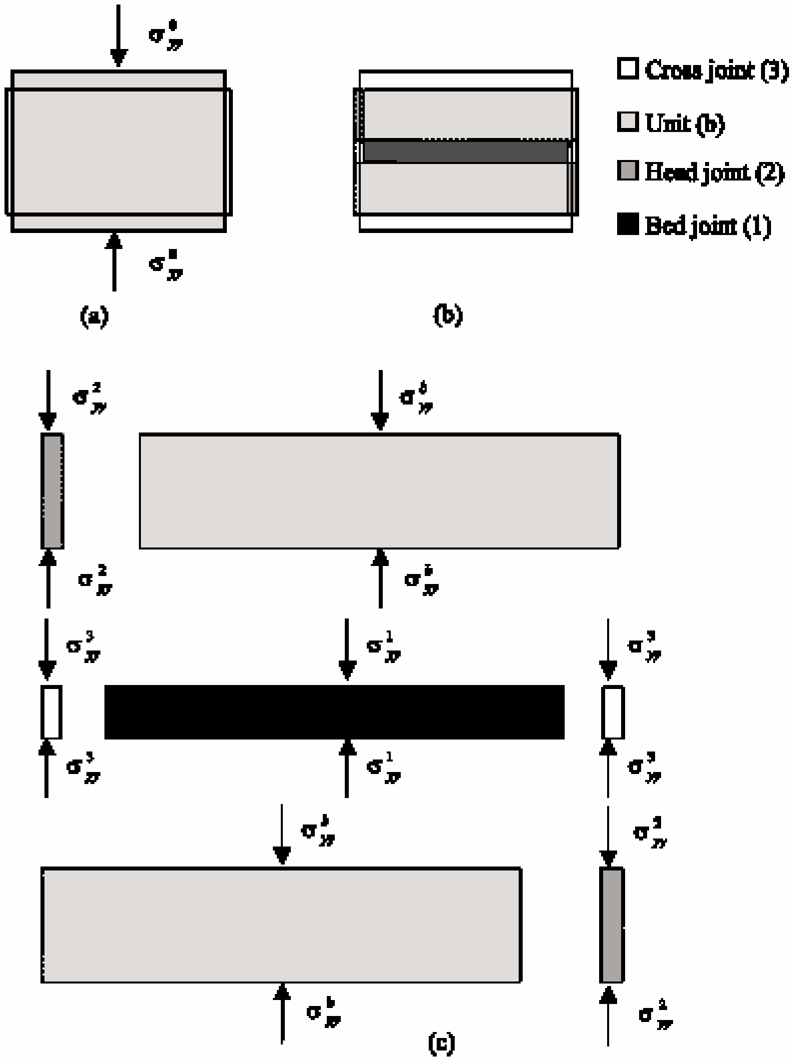


Figure 3.20 Normal stress loading in *y*-direction

In a so-posed elastic problem, the number of unknowns is larger than in a usual homogenization procedure, since the second-order effects are considered by taking in account, as already mentioned, the additional deformation mechanisms, given by:

- Vertical normal stress σ_{yy}^{1}, in the bed joint, when the basic cell is loaded with in-plane shear σ_{xy}^{0}.
- In-plane shear σ_{xy}^{1}, in the bed joint, when the basic cell is loaded with a horizontal in-plane normal stress σ_{xx}^{0}.
- Out-of-plane shear σ_{yz}^{1}, in the bed joint, when the basic cell is loaded with an out-of-plane shear stress σ_{xz}^{0}.

In order to define uniquely the unknown internal stresses and strains of each component, a set of equilibrium, compatibility and constitutive equations, for each loading case, has to be imposed, as it follows. Brick, bed joint, head joint and cross joint variables will be indicated respectively by the superscripts b, 1, 2 and 3. The mean value of the normal stress σ_{xx} and the normal strain ε_{xx} in the unit will be indicated, respectively, by $\bar{\sigma}_{xx}^{b}$ and $\bar{\varepsilon}_{xx}^{b}$. The prescribed uniform normal (macro) stresses on the faces of the homogenized basic cell in the *x*-, *y*- and *z*-direction will be indicated, respectively, by σ_{xx}^{0}, σ_{yy}^{0} and σ_{zz}^{0}.

- **Uniform normal stress loading case in *x*, *y* or *z* direction.**

No other stresses, except σ_{xx}^{0}, σ_{yy}^{0} and σ_{zz}^{0}, are applied on the boundary of the basic cell. In this case, all shear stresses and strains for each component are neglected, except the in-the-plane shear stress and strain $\left(\sigma_{xy}, \varepsilon_{xy}\right)$ in the bed joint and in the brick, as illustrated in the above figure 3.19. We remember that the shear strain component, ε_{xy}^{1}, is one of the deformation mechanisms here considered and, instead, neglected by the standard two step homogenization procedure, since depending on the geometrical arrangement of the bricks in the masonry pattern. Furthermore, the non-zero stresses and strains are assumed to be constant in each basic cell constituent, except the normal stress s^{b}_{xx} in the brick, which must be a linear function of *x* in order to account for the presence of the shear stress component s^{b}_{xy}. The latter, moreover, requires the introduction of a couple for the momentum equilibrium of one-fourth of the brick in the basic cell (see figure 3.19) which derives from the neighbouring cell along *y*-axis. The symmetric brick quarter of the cell above, indeed, reacts at the centre line of the brick with a couple due to a self-equilibrating vertical stress distribution, s^{b}_{yy}, which is neglected in the model. This is shown in the following figure 3.21.

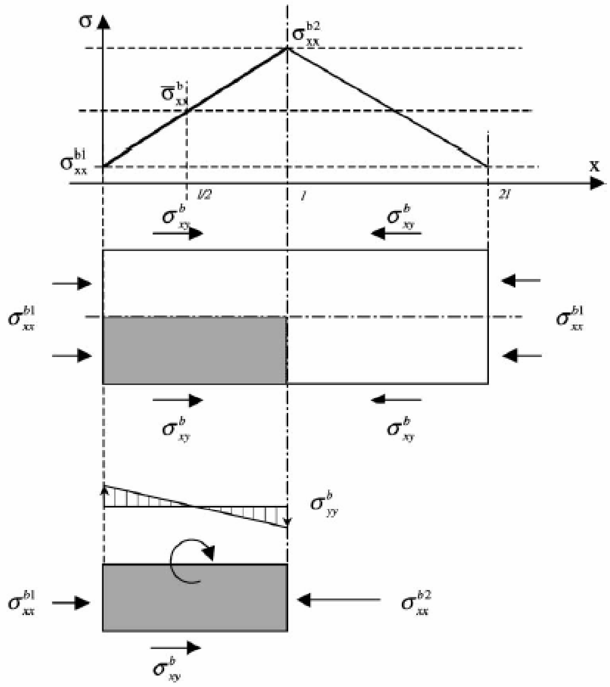


Figure 3.21 Normal stress loading in *x*-direction: unit equilibrium (couple moment equal to self-equilibrating vertical stress distribution).

Is has to be underlined, yet, that, in order to assure the brick equilibrium, the shear stress σ_{xy}^{b} has to be a linear function of y and, in order to assure the bed joint equilibrium, a shear stress σ_{xy}^{1} has to be introduced also for the left and right sides of such joint.

Under these hypotheses, the following equilibrium equations, at internal or boundary interfaces, can be written:

- Limit equilibrium equation at internal interface brick-head joint

$$\sigma_{xx}^{2} = \bar{\sigma}_{xx}^{b} - \sigma_{xy}^{1}\frac{l-t}{2h} \qquad (3.3.2.c-1)$$

where:

$\bar{\sigma}_{xx}^{b}$ = average value of the normal stress in the brick. It is given by:

$$\bar{\sigma}_{xx}^{b} = \frac{\sigma_{xx}^{b1} + \sigma_{xx}^{b2}}{2} \qquad (3.3.2.c-2)$$

with:

$\sigma_{xx}^{b1} =$ the normal stress in the left side of the brick

$\sigma_{xx}^{b2} =$ the normal stress in the right side of the brick

For the equilibrium of the brick, moreover, the following relation has to be verified:

$$h\sigma_{xx}^{b1} + (l-t)\sigma_{xy}^{1} = h\sigma_{xx}^{b2} \qquad (3.3.2.c-3)$$

where the authors assume that the shear acts only on the bed-brick interface $(l-t)$. Hence, some equilibrium conditions at the interfaces are not satisfied.

From the (3.3.2.c-2) and (3.3.2.c-3), it is obtained that:

$$\sigma_{xx}^{b1} = \bar{\sigma}_{xx}^{b} - \sigma_{xy}^{1}\frac{l-t}{2h} \qquad (3.3.2.c-4)$$

and:

$$\sigma_{xx}^{b2} = \bar{\sigma}_{xx}^{b} + \sigma_{xy}^{1}\frac{l-t}{2h} \qquad (3.3.2.c\text{-}5)$$

which have been used in the equation (3.3.2.c-1).

- Limit equilibrium equation at internal interface brick-bed joint

$$\sigma_{yy}^{b} = \sigma_{yy}^{1} \qquad (3.3.2.c\text{-}6)$$

- Limit equilibrium equation at right boundary

$$h\sigma_{xx}^{2} + 2t\sigma_{xx}^{3} + h\left(\bar{\sigma}_{xx}^{b} + \sigma_{xy}^{1}\frac{l-t}{2h}\right) = 2\left(h+t\right)\sigma_{xx}^{0} \qquad (3.3.2.c\text{-}7)$$

- Limit equilibrium equation at upper boundary

$$l\sigma_{yy}^{b} + t\sigma_{yy}^{2} = \left(l+t\right)\sigma_{yy}^{0} \qquad (3.3.2.c\text{-}8)$$

- Limit equilibrium equation at front boundary

$$2th\sigma_{zz}^{2} + 2\left(l-t\right)t\sigma_{zz}^{1} + 2lh\sigma_{zz}^{b} + 4t^{2}\sigma_{zz}^{3} = \left[2th + 2\left(l+t\right)t + 2lh\right]\sigma_{zz}^{0} \qquad (3.3.2.c\text{-}9)$$

Analogously, the following compatibility equations can be written:

- Compatibility equation at upper boundary

$$2t\varepsilon_{yy}^{1} + h\varepsilon_{yy}^{b} = h\varepsilon_{yy}^{2} + 2t\varepsilon_{yy}^{3} \qquad (3.3.2.c\text{-}10)$$

- Compatibility equation at right boundary

$$t\varepsilon_{xx}^{2} + l\bar{\varepsilon}_{xx}^{b} = 2t\varepsilon_{xx}^{3} + \left(l-t\right)\varepsilon_{xx}^{1} \qquad (3.3.2.c\text{-}11)$$

- Compatibility equation at front boundary

$$\varepsilon_{zz}^{b} = \varepsilon_{zz}^{1} \qquad (3.3.2.c\text{-}12)$$

- Compatibility equation at front boundary

$$\varepsilon_{zz}^{b} = \varepsilon_{zz}^{2} \qquad (3.3.2.c\text{-}13)$$

In the above equations, the unknown stresses and strains in the cross joint can be eliminated by means of the following relations:

$$\varepsilon_{yy}^{3}=\frac{E^{2}}{E^{3}}\varepsilon_{yy}^{2}$$
$$\varepsilon_{xx}^{3}=\frac{E^{1}}{E^{3}}\varepsilon_{xx}^{1} \qquad (3.3.2.c\text{-}14)$$
$$\sigma_{zz}^{3}=\frac{E^{3}}{E^{1}}\sigma_{zz}^{1}$$

$$\sigma_{xx}^{3}=\sigma_{xx}^{1} \qquad (3.3.2.c\text{-}15)$$

The equations (3.3.2.c-14) assume that the cross joint behaves as a spring connected in series with the bed joint in the *x*-direction, connected in series with the head joint in the *y*-direction and connected in parallel with the bed joint in the *z*-direction. The equation (3.3.2.c-15) represents the equilibrium at the cross-bed joint interface. It can be noted that the stress-strain state in the cross joint does not play a major role in the problem, because of its usually small volume ratio, [67].

By coupling with the nine linear elastic stress-strain relations in the brick, head joint and bed joint the above considered equilibrium and compatibility equations, a linear system of 18 equations comes out. In particular, the constitutive equations assume the following form:

Ø Constitutive linear elastic equations

$$\varepsilon_{xx}^{k}=\frac{1}{E^{k}}\left[\sigma_{xx}^{k}-\nu_{k}\left(\sigma_{yy}^{k}+\sigma_{zz}^{k}\right)\right]$$
$$\varepsilon_{yy}^{k}=\frac{1}{E^{k}}\left[\sigma_{yy}^{k}-\nu_{k}\left(\sigma_{xx}^{k}+\sigma_{zz}^{k}\right)\right] \qquad k=b,1,2 \qquad (3.3.2.c\text{-}16)$$
$$\varepsilon_{zz}^{k}=\frac{1}{E^{k}}\left[\sigma_{zz}^{k}-\nu_{k}\left(\sigma_{xx}^{k}+\sigma_{yy}^{k}\right)\right]$$

In this linear system, the unknowns are the six normal stresses and strains of each of the three components (brick, head joint and bed joint) and the shear stress and strain in the bed joint, for a total of 20 unknowns.

Therefore, two additional equations are necessary to solve the problem. These ones can be derived introducing the shear deformation of the bed joint: the elastic mismatch between the normal *x* strains in the brick and in the head joint is responsible for the shear in the bed joint because of the staggered alignment of the bricks in the masonry wall. This mechanism, shown in the following figure, leads to the approximated relation:

- Compatibility equation

$$\varepsilon_{xy}^{1} = \frac{1}{2}\frac{\Delta x_2 - \Delta x_b}{2t} = \frac{t\varepsilon_{xx}^{2} - t\varepsilon_{xx}^{b2}}{4t} \; ; \; \frac{\varepsilon_{xx}^{2} - \bar{\varepsilon}_{xx}^{b}}{4} \qquad (3.3.2.c\text{-}17)$$

which is valid in the hypothesis that the bed joint does not slip on the brick and the ε_{xx}^{b} is assumed linear in *x*-direction. In particular, it would be verified that:

$$\varepsilon_{xx}^{b2} = \bar{\varepsilon}_{xx}^{b} + \sigma_{xy}^{1}\frac{(l-t)}{2hE^{b}} \qquad (3.3.2.c\text{-}18)$$

but, usually, the second term in the right-hand side can be neglected.

Another one additional equation is the elastic stress-strain relation:

- Constitutive linear elastic equation

$$\sigma_{xy}^{1} = 2G^{1}\varepsilon_{xy}^{1} \qquad (3.3.2.c\text{-}19)$$

Hence, a linear system of 20 equations and 20 variables is finally obtained. Since a symbolic solution, nevertheless obtained, was too complex for the practical purposes, the linear system has been solved numerically. So, the internal stresses and strains are obtained for uniaxial load in the *i*-direction, given by:

$$\sigma_{ii}^{0} = 1, \quad \sigma_{ij}^{0} = 0 \rightarrow i \neq j \qquad i,\, j = x, y, z \qquad (3.3.2.c\text{-}20)$$

Once the unknowns are found, the shear stress in the brick can be obtained by means of the internal equilibrium equation:

$$\frac{\partial \sigma_{xx}^{b}}{\partial x}+\frac{\partial \sigma_{xy}^{b}}{\partial y}+\frac{\partial \sigma_{xz}^{b}}{\partial z}=0 \qquad (3.3.2.c-21)$$

which leads to:

$$\sigma_{xy}^{b}=\sigma_{xy}^{1}\left(1-\frac{y}{h}\right) \qquad (3.3.2.c-22)$$

At this point, the homogenized Young's moduli and Poisson's coefficient of the basic cell are, finally, obtained by forcing the macro-deformation of the model and of the homogenized material to be the same, meaning that both systems must contain the same strain energy.

By assuming an orthotropic behaviour, the Young's moduli and the Poisson's coefficients are given by:

$$E_i=\frac{\sigma_{ii}^{0}}{\varepsilon_{ii}^{H}},\quad \nu_{ij}=\frac{\varepsilon_{jj}^{H}}{\varepsilon_{ii}^{H}},\qquad i,j=x,y,z \qquad (3.3.2.c-23)$$

where:

ε_{ii}^{H} = homogenized strain, obtained for a prescribed stress case.

In particular, it is obtained that:

$$\begin{aligned}
\varepsilon_{xx}^{H}&=\varepsilon_{xx}^{1}\frac{l-t+2t\,E^{1}/E^{3}}{l+t}\\
\varepsilon_{yy}^{H}&=\frac{\varepsilon_{yy}^{2}\left(h+2t\,E^{2}/E^{3}\right)+h\varepsilon_{yy}^{b}}{l+t}\\
\varepsilon_{zz}^{H}&=\varepsilon_{zz}^{b}
\end{aligned} \qquad (3.3.2.c-24)$$

The procedure for determining the homogenized shear moduli is analogous to the previous one.

- **In-plane shear modulus G_{xy}**

No other stresses, except σ_{xy}^{0}, are applied on the boundary of the basic cell. In this case, all stresses and strains are neglected, except the in-the-plane shear stress and strain $\left(\sigma_{xy}, \varepsilon_{xy}\right)$ in each basic cell component, and the normal stress and strain components σ_{yy}^{1} and ε_{yy}^{1} in the bed joint. We remember that this latter is one of the strain components here considered and, instead, neglected by the standard two step homogenization procedure. Furthermore, the non-zero stresses and strains are assumed to be constant in each basic cell constituent, except the shear stress σ_{xy}^{b} in the brick, which must be a linear function of *x* in order to account for the presence of the normal stress component σ_{yy}^{1} in bed joint. The deformation of the basic cell is approximated in the following figure.

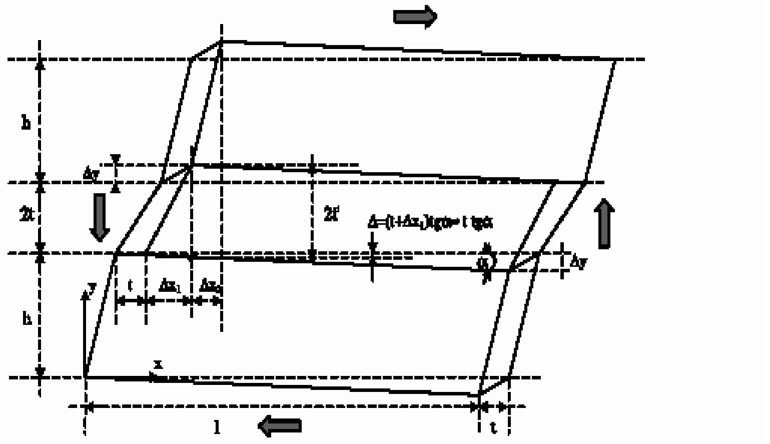


Figure 3.22 Model assumptions for *xy* shear

Under these hypotheses, the following equilibrium equations, at internal or boundary interfaces, can be written:

- Limit equilibrium equation at upper boundary

$$t\sigma_{xy}^{2} + l\bar{\sigma}_{xy}^{b} = \left(t + l\right)\sigma_{xy}^{0} \qquad (3.3.2.c\text{-}25)$$

Ø Limit equilibrium equation at internal interface brick-head joints

$$\sigma_{xy}^{2} = \bar{\sigma}_{xy}^{b} + \frac{l}{2h}\sigma_{yy}^{1} \qquad (3.3.2.c\text{-}26)$$

Ø Limit equilibrium equation at internal interface brick-bed joints

$$\bar{\sigma}_{xy}^{b} = \sigma_{xy}^{1} \qquad (3.3.2.c\text{-}27)$$

where:

$\bar{\sigma}_{xy}^{b} =$ the mean value of σ_{xy}^{b} in the brick

The normal strain ε_{yy}^{1} is derived from geometric considerations on the deformation mechanism illustrated in the figure 3.22 where all the geometric quantities can be defined. So, neglecting the second-order terms, it is straightforward to obtain:

$$\varepsilon_{yy}^{1} \cong \frac{2t' - 2t}{2t} \cong \frac{\Delta y + \left(t/l\right)\Delta y}{2t}, \qquad \varepsilon_{xy}^{2} - \bar{\varepsilon}_{xy}^{b} = \frac{\Delta y}{2t} + \frac{\Delta y}{2l} \qquad (3.3.2.c\text{-}28)$$

This leads to:

Ø Compatibility equation

$$\varepsilon_{yy}^{1} = \varepsilon_{xy}^{2} - \bar{\varepsilon}_{xy}^{b} \qquad (3.3.2.c\text{-}29)$$

By coupling with the four linear elastic stress-strain relations in the brick, head joint and bed joint the above considered equilibrium and compatibility equations, a linear system of 8 equations comes out. In particular, the constitutive equations assume the following form:

Ø Constitutive linear elastic equations

$$\sigma_{yy}^{1} = E^{1}\left(\bar{\varepsilon}_{xy}^{b} - \varepsilon_{xy}^{2}\right) \qquad (3.3.2.c\text{-}30)$$

$$\sigma_{xy}^{k} = 2G^{k}\varepsilon_{xy}^{k} \qquad k = b, 1, 2 \qquad (3.3.2.c\text{-}31)$$

In this linear system, the unknowns are the three shear stresses and strains of the three basic cell components (brick, head joint and bed joint) and the normal stress and strain, σ_{yy}^{1} and ε_{yy}^{1}, in the bed joint, for a total of 8 unknowns.

By solving the obtained linear system, the internal stresses and strains are found.

At this point, the homogenized shear modulus, G_{xy}, of the basic cell is, finally, given by:

$$G_{xy} = \frac{\sigma_{xy}^{0}}{2\varepsilon_{xy}^{H}} = \frac{l(t+l)(t+h)}{k\dfrac{tl(t+h)}{G^{2}} + \dfrac{(t+l-kt)(lh-t^{2})}{G^{b}} + \dfrac{t(t+l)(t+l-kt)}{G^{1}}} \qquad (3.3.2.c\text{-}32)$$

where:

ε_{xy}^{H} = homogenized strain, obtained for a prescribed stress case and given by:

$$\varepsilon_{xy}^{H} = \frac{1}{h+t}\left[\left(l\bar{\varepsilon}_{xy}^{b} + t\varepsilon_{xy}^{2}\right)\frac{h}{l+t} + t\varepsilon_{xy}^{1} + \left(\varepsilon_{xy}^{2} - \bar{\varepsilon}_{xy}^{b}\right)\frac{t^{2}}{l+t}\right] \qquad (3.3.2.c\text{-}33)$$

and k is defined as it follows, [67]:

$$k = \frac{lE^{1} + 4hG^{b}}{lE^{1} + 4hG^{b} + E^{1}\dfrac{l^{2}}{l+t}\left(\dfrac{G^{b}}{G^{2}} - 1\right)} \qquad (3.3.2.c\text{-}34)$$

In analogous manner, the other two homogenized shear moduli are determined.

- **In-plane shear modulus G_{xz}**

No other stresses, except σ_{xz}^{0}, are applied on the boundary of the basic cell. In this case, all stresses and strains are neglected, except the in-the-plane shear stress and strain $(\sigma_{xz}, \varepsilon_{xz})$ in each basic cell component, and the shear stress and strain components σ_{yz}^{1} and ε_{yz}^{1} in the bed joint. We remember that this latter is one of the strain components here considered and, instead, neglected by the standard two step homogenization procedure. Furthermore, the non-zero stresses and strains are assumed to be constant in each basic cell constituent, except the shear stress σ_{xz}^{b}, which must be a linear function of x in order to account for the presence of the shear stress component σ_{yz}^{1} in bed joint. The deformation of the basic cell is approximated in the following figure 3.23.

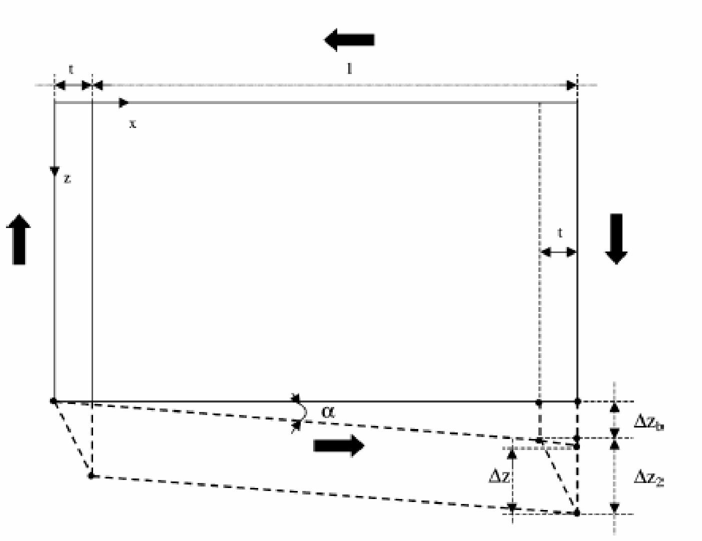


Figure 3.23 Model assumptions for *xz* shear

Under these hypotheses, the following equilibrium equations, at internal or boundary interfaces, can be written:

Ø Limit equilibrium equation at right boundary

$$h\left(\bar{\sigma}_{xz}^{b} + \frac{(l-t)}{2h}\sigma_{yz}^{1}\right) + 2t\sigma_{xz}^{1} + h\sigma_{xz}^{2} = 2(t+h)\sigma_{xz}^{0} \qquad (3.3.2.c\text{-}35)$$

Ø Limit equilibrium equation at internal interface brick-head joints

$$\sigma_{xz}^{2} = \bar{\sigma}_{xz}^{b} - \frac{(l-t)}{2h}\sigma_{yz}^{1} \tag{3.3.2.c-36}$$

Ø Compatibility equation at internal interface brick-bed joints

$$\varepsilon_{xz}^{1} = \bar{\varepsilon}_{xz}^{b} \tag{3.3.2.c-37}$$

Moreover the shear strain ε_{yz}^{1} is derived from geometric considerations on the deformation mechanism illustrated in the figure 3.23 where all the geometric quantities can be defined. So, it can be found to be verified the following compatibility equation, [67]:

Ø Compatibility equation

$$\varepsilon_{yz}^{1} = \frac{1}{2}\left(\varepsilon_{xz}^{2} - \bar{\varepsilon}_{xz}^{b}\right) \tag{3.3.2.c-38}$$

By coupling with the four linear elastic stress-strain relations in the brick, head joint and bed joint the above considered equilibrium and compatibility equations, a linear system of 8 equations comes out. In particular, the constitutive equations assume the following form:

Ø Constitutive linear elastic equations

$$\sigma_{yz}^{1} = 2G^{1}\varepsilon_{yz}^{1} \tag{3.3.2.c-39}$$

$$\sigma_{xz}^{k} = 2G^{k}\varepsilon_{xz}^{k} \qquad k = b,1,2 \tag{3.3.2.c-40}$$

In this linear system, the unknowns are the three shear stresses, σ_{xz}, and strains, ε_{xz}, of the three basic cell components (brick, head joint and bed joint) and the shear stress and strain, σ^1_{yz} and ε^1_{yz}, in the bed joint, for a total of 8 unknowns.

By solving the obtained linear system, the internal stresses and strains are found.

Hence, the homogenized shear modulus G_{xz} of the basic cell is, finally, given by:

$$G_{xz} = \frac{\sigma^0_{xz}}{2\varepsilon^H_{xz}} = \frac{(t+l)\left(tG^1 + hG^b\right)}{(t+h)\left(t\dfrac{4hG^b + (l-t)G^1}{4hG^2 + (l-t)G^1} + l\right)} \qquad (3.3.2.c\text{-}41)$$

where:

ε^H_{xz} = homogenized strain, obtained for a prescribed stress case and given by:

$$\varepsilon^H_{xz} = \frac{t\varepsilon^2_{xz} + l\varepsilon^b_{xz}}{t+l} \qquad (3.3.2.c\text{-}42)$$

- **In-plane shear modulus G_{yz}**

No other stresses, except σ^0_{yz}, are applied on the boundary of the basic cell. In this case, all stresses and strains are neglected, except the in-the-plane shear stress and strain $\left(\sigma_{yz}, \varepsilon_{yz}\right)$ in each basic cell component, assumed to be constant everywhere. The deformation of the basic cell is approximated in the following figure.

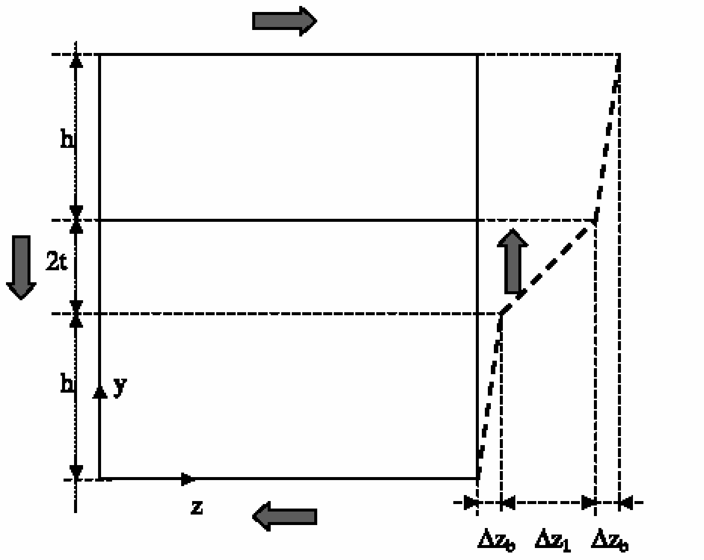


Figure 3.24 Model assumptions for *yz* shear

Under these hypotheses, the following equilibrium and compatibility equations, at internal or boundary interfaces, can be written:

Ø Limit equilibrium equation at upper boundary

$$t\sigma_{yz}^{2}+l\sigma_{yz}^{b}=\left(t+l\right)\sigma_{yz}^{0} \tag{3.3.2.c-43}$$

Ø Limit equilibrium equation at internal interface brick-bed joints

$$\sigma_{yz}^{b}=\sigma_{yz}^{1} \tag{3.3.2.c-44}$$

Ø Compatibility equation at internal interface brick-head joints

$$\varepsilon_{yz}^{b}=\varepsilon_{yz}^{2} \tag{3.3.2.c-45}$$

By coupling with the three linear elastic stress-strain relations in the brick, head joint and bed joint the above considered equilibrium and compatibility equations, a linear system of 6 equations comes out. In particular, the constitutive equations assume the following form:

- Constitutive linear elastic equations

$$\sigma_{yz}^{k} = 2G^{k}\varepsilon_{yz}^{k} \qquad k = b,1,2 \qquad (3.3.2.c\text{-}46)$$

In this linear system, the unknowns are the three shear stresses σ_{yz} and the three shear strains ε_{yz} of the three basic cell components (brick, head joint and bed joint), for a total of 6 unknowns.

By solving the obtained linear system, the internal stresses and strains are found.

Hence, the homogenized shear modulus G_{yz} of the basic cell is, finally, given by:

$$G_{yz} = \frac{\sigma_{yz}^{0}}{2\varepsilon_{yz}^{H}} = \frac{t+h}{t+l}G^{1}\frac{lG^{b}+tG^{2}}{tG^{b}+hG^{1}} \qquad (3.3.2.c\text{-}47)$$

where:

ε_{yz}^{H} = homogenized strain, obtained for a prescribed stress case and given by:

$$\varepsilon_{yz}^{H} = \frac{t\varepsilon_{yz}^{1}+h\varepsilon_{yz}^{b}}{t+h} \qquad (3.3.2.c\text{-}48)$$

For more details, the reader is referred to [67].

However, we want to underline, here, the most results obtained from the authors. Their described model has been applied to a real masonry basic cell and compared with the results of the previous accurate FEM analysis. In the finite element analysis and the analytical model, the properties of the components have been taken absolutely equal.

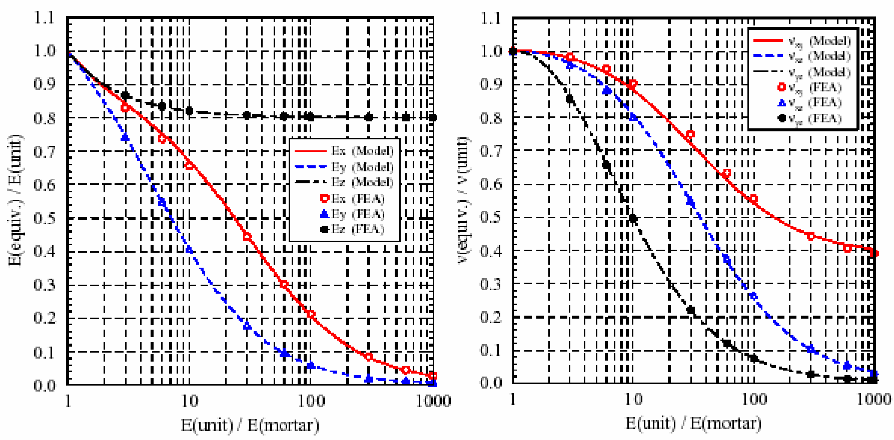
The same elastic properties have been adopted for the bed joint, head joint and cross joint $E_1 = E_2 = E_3 = E_m; \nu_1 = \nu_2 = \nu_3 = \nu_m$. Different stiffness ratios between mortar and unit are considered.

This has allowed assessing the performance of the model for inelastic behaviour. In fact, non-linear behaviour is associated with (tangent) stiffness degradation and homogenisation of non-linear processes will result in large stiffness differences between the components. In the limit, the ratio between the stiffness of the different components is zero or infinity.

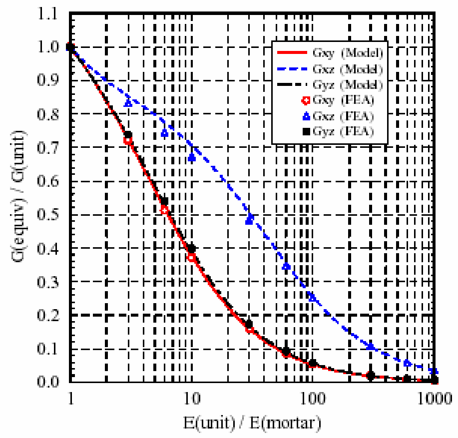
The material properties of the unit are kept constant, whereas the properties of the mortar are varied. In particular, for the unit, the Young's modulus Eb is 20 GPa and the Poisson's ratio is 0.15. For the mortar, the Young's modulus is varied to yield a ratio Eb/Em ranging from 1 to 1000 while the mortar Poisson's ratio is kept constant to 0.15 and equal to that one of the unit.

The adopted range of Eb/Em is very large (up to 1000). Note that the ratio Eb/Em tends to infinity when softening of the mortar is complete and only the unit remains structurally active.

The elastic properties of the homogenised material, calculated by means of the proposed micro-mechanical model, are compared with the values obtained by FEM analysis, in the figures 3.25 and 3.26.



(a) (b)



(c)

Figure 3.25 Comparison between the micro-mechanical model and FEA results for different stiffness ratios: (a) Young's moduli, (b) Poisson's ratio and (c) Shear moduli.

The agreement is very good in the entire considered range Eb/Em. In particular the figure 3.26 yields the relative error of the elastic parameters predicted by the proposed model and show that it is always less than 6%.

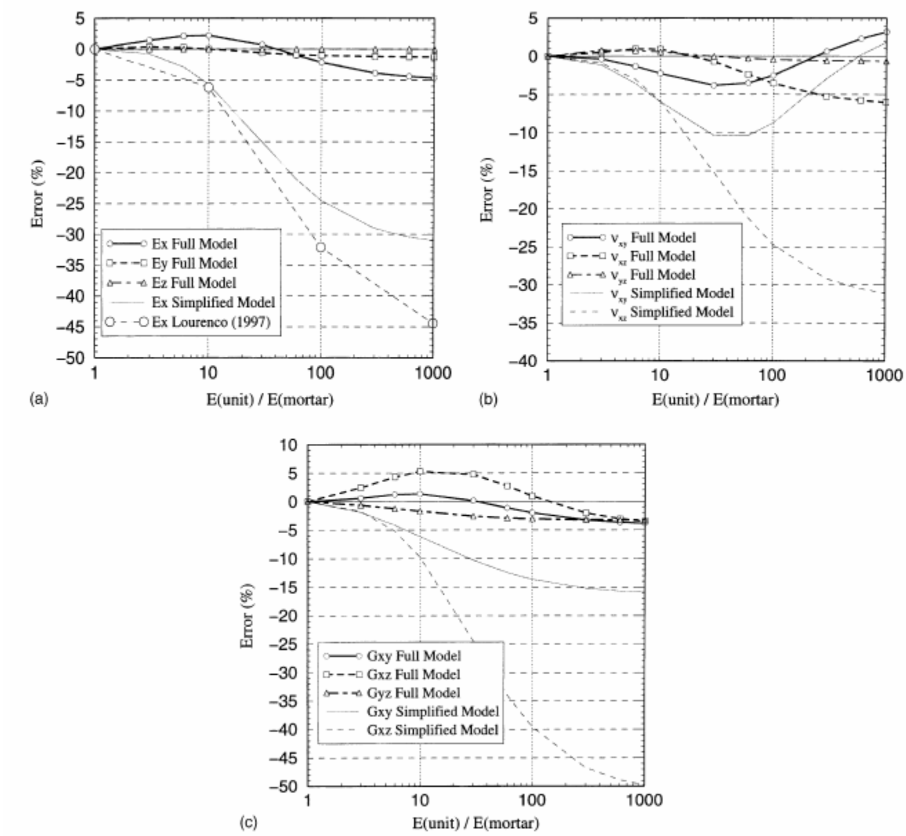


Figure 3.26 Comparison between the proposed micro-mechanical model and the simplified model

The thinner curves in Fig. 3.26 yield the results of a simplified model (E_x only), which is derived from the model presented in the paper, [67], where the additional deformation mechanisms of the bed joint have not been taken into account. The simplified model, therefore, neglects the main effects due to the misalignment of the units in the masonry wall and coincides with the full model when the units are aligned in the wall. For this reason, such a simplified model appears closer to the standard two-step homogenisation techniques.

The figure 3.26a also includes the results of the standard two-step homogenisation of Lourenco (1997), showing that it leads to non-acceptable errors up to 45% for the estimation of Young's modulus E_x in *x*-direction. Less pronounced differences are found in the estimation of young's moduli in *y*- and *z*- directions, but they are not reported in the figure, see Lourenco (1997).

For large ratios Eb/Em the simplified model predicts value of E_x, ν_{xz} and G_{xz} much smaller than the actual values obtained by FEM analysis. The large and increasing errors of such model on these variables (up to 50%) indicate that for much degraded mortar the neglected deformation mechanisms of the bed joint contribute significantly to the overall basic cell behaviour.

In spite of the fact that Lourenco's approach overcomes the limits of the standard two-steps homogenization, it is worth to notice that the proposed homogenized model is obtained on a parametrization-based procedure depending on a specific benchmark FEM model (i.e. selected ratios between elastic coefficients and geometrical dimensions), so it shows a sensitivity to geometrical and mechanical ratios!

CHAPTER IV

Proposal of modified approaches: theoretical models

4.1 Introduction

In this chapter, some possible new procedures for modelling masonry structures, in linear-elastic field, are proposed, starting from the results of literature approaches.

In the previous chapter 3, a general account on such existing homogenization techniques has been shown. In particular, it has been underlined that they can be basically divided in two approaches. The first one employs an approximated homogenization process in different steps by obtaining, on the contrary, a close-form solution (for example, Pietruszczak & Niu, 1992). The second one employs a rigorous homogenization process in one step by obtaining, on the contrary, an approximated numerical solution (for example, Lourenco & al, 2002). Moreover, also the limits for each one of the two approaches have been highlighted in the chapter 3.

Hence, the main object of this chapter is to obtain new homogenization techniques which are able to overcome the limits of the existing ones, in both approaches. More in detail, two procedures have been proposed: a simplified two-step homogenization (S.A.S. approach) and a rigorous one-step homogenization (Statically-consistent Lourenco approach).

4.2 Statically-consistent Lourenco approach

As first approach, a new proposal for the analysis of masonry structures is given, starting from some results already reached by A. Zucchini and P.B. Lourenco and mentioned in the previous chapter 3, [67]. In that work, the authors have employed a numerical strategy to analyze masonry walls, by using a propaedeutic micro-mechanical approach to determine constitutive properties.

Hence, in the present section, by recalling Lourenco's *stress-prescribed* homogenization technique, a new rigorous *one-step* homogenization procedure is proposed. In particular, the overall material properties of the representative volume element (RVE) are determined as functions of both the elastic coefficients of the phases and the geometry of the arrangement, under the hypothesis of orthotropic behaviour. By developing a new modified constitutive model for masonries, it will be seen that the proposed approach leads to a statically-consistent solution for the elastic homogenization problem, but it doesn't take into account for compatibility conditions at the constituent interfaces. However, by means of equilibrium considerations, a different stress distribution in each masonry component, which is more accurate than Lourenco's one, is obtained.

The micro-mechanical model used in the analysis is the same than the one considered by the authors. For clearness of exposition, it is shown again in the following figure:

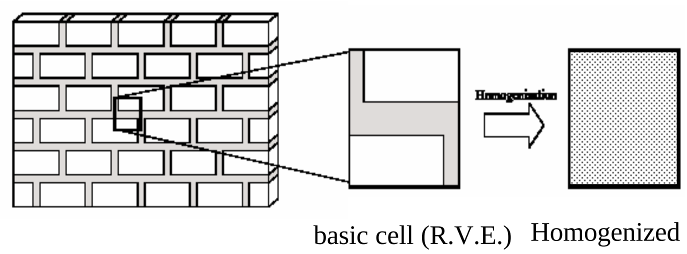


Figure 4.1 Definition of masonry axes and of chosen micro mechanical model

Such micro-mechanical model is obtained by extracting a basic periodic cell from single leaf masonry in stretcher bond.

It has been considered a right-oriented x-y-z Cartesian coordinate system and the following components for the basic cell are considered:

- Head joint (a)
- Unit (b)
- Cross joint (c)
- Bed joint (d)

and for the symmetry of the assembly, we also have:

- Cross joint (e)
- Unit (f)
- Head joint (g)

The following figure shows, in detail, the geometry of the basic cell, with the definition of the dimensions and of adopted symbols.

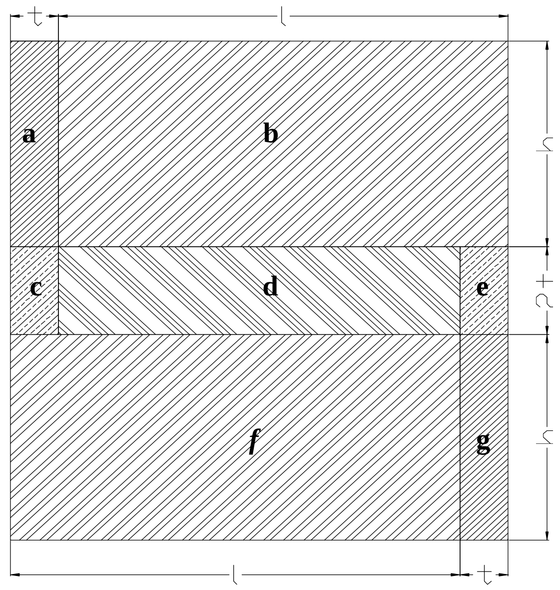


Figure 4.2 Adopted basic cell and geometric parameters

We have maintained the assumed hypothesis of linear elasticity so that it is possible to study, yet, the elastic response of the model for a generic loading condition as linear combination of the responses to six basic loading conditions: three cases of normal stresses and three cases of simple shear (prescribed stress homogenization).

For each of these cases, and – as a consequence – for each constituent of the cell, selected components of stress tensor are involved. In particular, it is done the hypothesis that the stresses vary as bi-linear functions upon the coordinates.

As an example, in the case of uniform normal stress both in x and in y directions, the assumed stress components are taken as shown in the figures below.

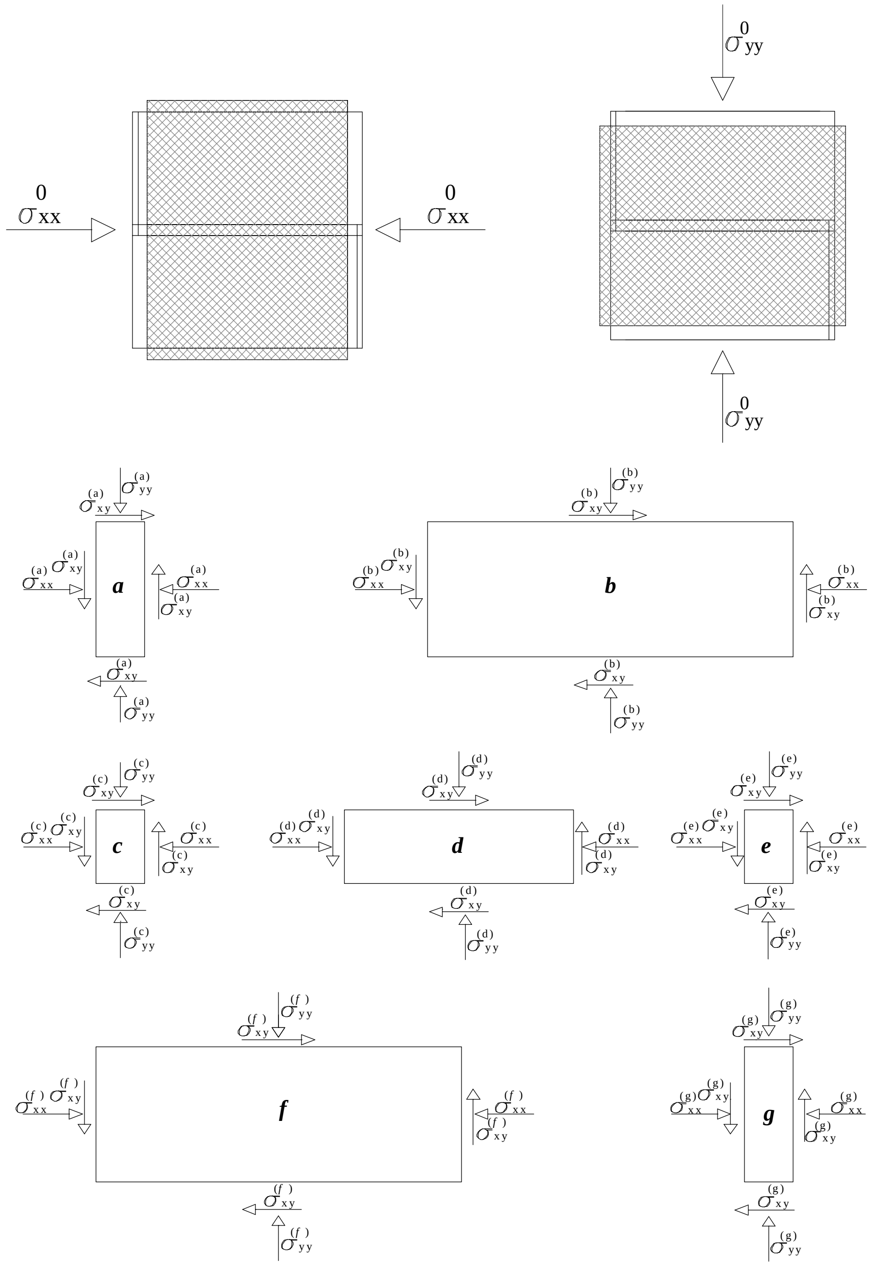


Figure 4.3 Selected components of the stress tensor for prescribed-stress loading conditions

For both these load conditions, a plane stress state is considered: all shear and normal stress components that involve z–direction are neglected and non-zero stress components, s_{xx}, s_{xy}, s_{yy}, are assumed bi-linear functions in x and y in each constituent of the cell. So, it can be written:

$$s_{xx}^{(p)} = A_0^{(p)} + A_1^{(p)} \cdot x + A_2^{(p)} \cdot y + A_3^{(p)} \cdot x \cdot y \qquad (4.2\text{-}1)$$

$$s_{xy}^{(p)} = B_0^{(p)} + B_1^{(p)} \cdot x + B_2^{(p)} \cdot y + B_3^{(p)} \cdot x \cdot y \qquad (4.2\text{-}2)$$

$$s_{yy}^{(p)} = C_0^{(p)} + C_1^{(p)} \cdot x + C_2^{(p)} \cdot y + C_3^{(p)} \cdot x \cdot y \qquad (4.2\text{-}3)$$

where the "*p*" index runs between "*a*" and "*g*" .

With these hypotheses, for these cases, the number of constants to determine is 84, which are 12 unknown constants for each component.

In order to define uniquely the above written functions, a set of equilibrium equations has to be imposed.

In particular, by fixing the origin of the right-oriented local x-y-z coordinate system, each time, in according to our convenience, for the single constituent of the cell, the following relations can be written:

- indefinite equilibrium equations, in absence of volume force:

$$s^{(p)}{}_{ij,i} = 0 \qquad with\ \ i,\ j = \{x,\ y\} \qquad (4.2\text{-}4)$$

- limit equilibrium equations on the boundary of the basic cell, in weak form:

$$\int_{\partial\Omega_e^{(p)}} s_{ij}^{(p)} \cdot a_i \, d\Omega = \int_{\partial\Omega_e^{(p)}} t_j^{(p)} \, d\Omega \quad \forall p \qquad (4.2\text{-}5)$$

with:

$p = a, b, c, e, f, g$

$i, j = x, y$

$\partial\Omega_e^{(p)} =$ the boundary faces of the generic p element.

$a_i =$ the components of the unit vector, normal to the boundary faces.

$t_j^{(p)} =$ the uniform macro-stresses applied on boundary faces of the homogenized basic cell, for the generic p element.

- equilibrium equations at the interfaces between the constituents:

$$\int_{\partial\Omega_i^{(p)}} s_{ij}^{(p)} \cdot a_i \, d\Omega = \int_{\partial\Omega_i^{(p)}} s_{ij}^{(q)} \cdot a_i \, d\Omega \qquad (4.2\text{-}6)$$

where "p" and "q" are two contact elements and where:

$\partial\Omega_i^{(p)} =$ the internal faces of the generic p element.

$a_i =$ components of the unit vector, normal to the internal surfaces.

Moreover, they are written local equilibrium equations for the unloaded boundary faces and global equilibrium equations, in a weak form, for translation and rotation of the whole basic cell:

- local equilibrium equations for the global cell, on the unloaded boundary faces:

$$s_{ij} = 0 \quad \forall x, y \quad with\ i, j = x, y \qquad (4.2\text{-}7)$$

- weak equilibrium equations for translation and rotation of the whole basic cell:

$$\int_{\partial\Omega} s_{ij} \, d\Omega = 0 \qquad (4.2\text{-}8)$$

and:

$$\int_{\partial\Omega_k} s_{ii} x_j \, d\Omega = 0 \quad \forall k \qquad (4.2\text{-}9)$$

where:

$$\partial\Omega = \sum_{k=1}^{4} \partial\Omega_k \tag{4.2-10}$$

with:

$\partial\Omega =$ the boundary surface of the whole basic cell, obtained as the summation of four boundary faces shown in the following figure 4.4.

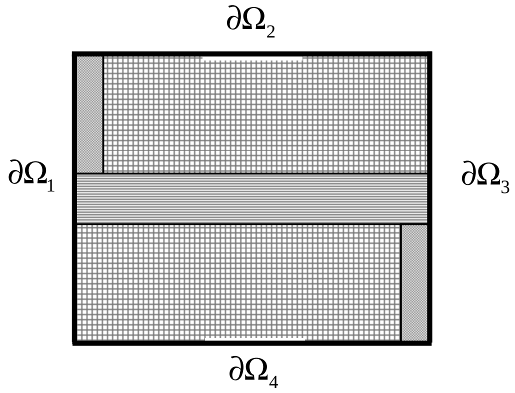


Figure 4.4 Boundary surface of the basic cell

Finally, in order to satisfy the conditions of polar symmetry, the following equations are considered:

- polar-symmetry punctual conditions:

$$\sigma_{ij}(\boldsymbol{P}) = \sigma_{ij}(\boldsymbol{P}^*) \tag{4.2-11}$$

where $\boldsymbol{P}^*$ is the polar-symmetric point of $\boldsymbol{P}$, obtained fixing the origin of the right-oriented local x-y-z coordinate system in the centre of mass of the whole system.

Solving this system of linear equations in the unknown constants, the stress components, for each constituent of the basic cell, are obtained.

In particular, for the compression in x direction, it can be written:

$$\sigma_{xx}^{(p)} = -\sigma_{xx}^{0} \tag{4.2-12}$$

$$\sigma_{xy}^{(p)} = \sigma_{yy}^{(p)} = 0 \tag{4.2-13}$$

where:

$\sigma_{xx}^{0} =$ the uniform normal macro-stress applied on the faces of the homogenized basic cell in x direction.

This result tells us that the sole non-zero stresses are constant in each basic cell component.

In analogous manner, for the compression in *y* direction, it is obtained:

$$\sigma_{yy}^{(p)} = -\sigma_{yy}^{0} \tag{4.2-14}$$

$$\sigma_{xy}^{(p)} = \sigma_{xx}^{(p)} = 0 \tag{4.2-15}$$

where:

$\sigma_{yy}^{0} =$ the uniform normal macro-stress applied on the faces of the homogenized basic cell in y direction.

Also in this case, non-zero stresses are everywhere constant.

In the last loading case, i.e. the basic cell loaded with a uniform normal stress, that one in z direction, the procedure is simplified, because of the *a priori* hypothesis that the sole stresses that can play a significant role are the z-direction normal stresses.

So, in particular, it can be written:

$$\sigma_{zz}^{(p)} = -\sigma_{zz}^{0} \tag{4.2-16}$$

where:

$\sigma_{zz}^{0} =$ the uniform normal macro-stress applied on the faces of the homogenized basic cell in z direction.

All other stress components are neglected.

At this point, known the stress functions, strain ones, in each component, can be derived by considering a linear elastic and isotropic stress-strain relation of all the components.

That means:

$$
\begin{aligned}
\varepsilon_{xx}^{(p)} &= \frac{1}{E^{(p)}}\left[\sigma_{xx}^{(p)} - \nu^{(p)}\left(\sigma_{yy}^{(p)} + \sigma_{zz}^{(p)}\right)\right] \\
\varepsilon_{yy}^{(p)} &= \frac{1}{E^{(p)}}\left[\sigma_{yy}^{(p)} - \nu^{(p)}\left(\sigma_{xx}^{(p)} + \sigma_{zz}^{(p)}\right)\right] \qquad \textit{with } p = a, b, ..., g \qquad (4.2\text{-}17) \\
\varepsilon_{zz}^{(p)} &= \frac{1}{E^{(p)}}\left[\sigma_{zz}^{(p)} - \nu^{(p)}\left(\sigma_{xx}^{(p)} + \sigma_{yy}^{(p)}\right)\right]
\end{aligned}
$$

Here, in the follows, they are presented the values of the strains, induced by the different load conditions. Since in all the three load cases, the stress state is a mono-dimensional one and the non-zero stress component value is equal to the applied load, generalizing the results, it can be written:

$$
\varepsilon_{ij}^{(p)} = \frac{\sigma_{ij}^{0(p)}}{2G^{(p)}} - \delta_{ij}\frac{\nu^{(p)}}{E^{(p)}}\sigma_{hh}^{0} \qquad (4.2\text{-}18)
$$

where:

$G^{(p)}, E^{(p)}, \nu^{(p)} =$ material properties of the single basic cell constituent. In particular, it is:

$G^{(p)} =$ Lamè modulus

$E^{(p)} =$ Young modulus

$\nu^{(p)} =$ Poisson modulus

For the average theorem, when boundary conditions are applied in terms of uniform stresses on the considered RVE (basic cell) and by naming with $\overline{\sigma}_{ij}$ the average value of stress in it, the following relation can be considered:

$$\overline{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ij} dV = \sigma_{ij}^{0} \qquad (4.2\text{-}19)$$

according to what has been studied in the Chapter 1.

In the (4.2-19), V stands for the volume of the basic cell.

The average value of strain, $\overline{\varepsilon}_{ij}$, instead, is defined as:

$$\overline{\varepsilon}_{ij} = \frac{1}{V}\int_V \varepsilon_{ij} dV = \frac{1}{V}\left[\sum_{p=\alpha}^{\gamma} \int_{V^{(p)}} \varepsilon_{ij}^{(p)} dV\right] \qquad (4.2\text{-}20)$$

where:

$V^{(p)} =$ the volume of the single basic cell constituent.

So, the properties of the homogenized cell can be determined through the following relation between the average values of stress and strain, by establishing, at the most, the hypothesis of an orthotropic behaviour. In detail, the components of fourth order homogenized tensor of compliances are found:

$$\overline{S}_{ijhk} = \delta_{ij}\delta_{hk}\frac{\overline{\varepsilon}_{ij}}{\sigma_{hk}^{0}} \quad with\ i, j = x, y, z \qquad (4.2\text{-}21)$$

where:

$\delta_{ij} =$ components of Kronecker delta

This procedure concurs to find the inverse of homogenized Young's moduli and the Poisson's coefficients for the homogenized RVE.

An analogous procedure can be used to determine the homogenized shear moduli, as:

$$\overline{S}_{ijhk} = \left(1-\delta_{ij}\right)\cdot\left(1-\delta_{hk}\right)\cdot\frac{\overline{\varepsilon}_{ij}}{\sigma_{hk}^{0}} \quad with\ i, j, h, k = x, y, z \qquad (4.2\text{-}22)$$

In this case, the analysis carried out in this paper has lead to the same results of those ones reached by Lourenco e Zucchini, [67]. For this reason, the procedure that we have employed for determining the homogenized shear moduli will be not shown here, but we only illustrate the obtained results.

In particular, the fourth order compliance tensor, so determined, assumes the following form:

$$\overline{\mathsf{S}}=\begin{bmatrix} \overline{S}_{1111} & \overline{S}_{1122} & \overline{S}_{1133} & 0 & 0 & 0 \\ \overline{S}_{1122} & \overline{S}_{2222} & \overline{S}_{2233} & 0 & 0 & 0 \\ \overline{S}_{1133} & \overline{S}_{2233} & \overline{S}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{S}_{3131} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{S}_{3232} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{S}_{1212} \end{bmatrix} \quad with\ \ x\equiv 1, y\equiv 2, z\equiv 3 \tag{4.2-23}$$

where

$$\overline{S}_{1111}=\overline{S}_{2222}=\overline{S}_{3333}= =-\frac{-\frac{h\cdot l}{Eb}-\frac{h\cdot l}{Ef}+t\cdot\left(-\frac{h}{Ea}-\frac{h}{Eg}-\frac{2\cdot l}{Ed}-\frac{2\cdot t}{Ec}+\frac{2\cdot t}{Ed}-\frac{2\cdot t}{Ee}\right)}{2\cdot(h+t)\cdot(l+t)} \tag{4.2-24}$$

$$\overline{S}_{1122}=\overline{S}_{1133}=\overline{S}_{2233}= =-\frac{\frac{h\cdot t\cdot \nu a}{Ea}+\frac{h\cdot l\cdot \nu b}{Eb}+\frac{2\cdot t^2\cdot \nu c}{Ec}+\frac{2\cdot(l-t)\cdot t\cdot \nu d}{Ed}+\frac{2\cdot t^2\cdot \nu e}{Ee}+\frac{h\cdot l\cdot \nu f}{Ef}+\frac{h\cdot t\cdot \nu g}{Eg}}{2\cdot(h+t)\cdot(l+t)} \tag{4.2-25}$$

$$\overline{S}_{3131}=\frac{1}{2\cdot G_{xz}}=\frac{(h+t)\cdot\left(t\cdot\frac{4\cdot Gb\cdot h+(l-t)\cdot Gd}{4\cdot Ga\cdot h+(l-t)\cdot Gd}+l\right)}{2\cdot(t+l)\cdot(t\cdot Gd+h\cdot Gb)} \tag{4.2-26}$$

$$\overline{S}_{3232}=\frac{1}{2\cdot G_{yz}}=\frac{1}{2\cdot Gd}\cdot\frac{t+l}{t+h}\cdot\frac{t\cdot Gb+h\cdot Gd}{l\cdot Gb+t\cdot Ga} \tag{4.2-27}$$

$$\bar{S}_{1212}=\frac{1}{2\cdot G_{xy}}=\frac{4\cdot Gb\cdot Gd\cdot h\cdot t\cdot(h+t)+Ed\cdot l\cdot(l+t)\cdot(Gd\cdot h+Gb\cdot t)}{2\cdot Gd\cdot(h+t)\cdot(Ed\cdot Gb\cdot l^2+Ga\cdot(Ed\cdot l\cdot t+4\cdot Gb\cdot h\cdot(l+t)))}+$$
$$+\frac{4\cdot Ga\cdot h\cdot(Gb\cdot t\cdot(l+t)+Gd\cdot(h\cdot l-t^2))}{2\cdot Gd\cdot(h+t)\cdot(Ed\cdot Gb\cdot l^2+Ga\cdot(Ed\cdot l\cdot t+4\cdot Gb\cdot h\cdot(l+t)))} \quad (4.2\text{-}28)$$

These results suggest a hexagonal material symmetry.

By comparing the proposed homogenization technique with Lourenco's one, it can be said that:

***LOURENCO & AL. APPROACH* -** proposes a homogenized model obtained on a parameterization-based procedure depending on a specific benchmark FEM model (i.e. selected ratios between elastic coefficients and geometrical dimensions), so it shows a sensitivity to geometrical and mechanical ratios! Moreover, the numerical estimate of the homogenized coefficients gives some not symmetrical moduli, so a symmetrization becomes necessary!

***LOURENCO MODIFIED APPROACH (Statically-consistent approach)* -** proposes a parametric homogenized model not depending on specific selected ratios between elastic coefficients and geometrical dimensions, so it shows a more generalized applicability. Moreover, since the approach implies a statically-consistent solution, it results extremely useful for its applicability according to the Static Theorem: a statically admissible solution guarantees the structure to be in security as regards the collapse.

4.3 SAS approach: two-step procedure consistency

In the follows, it will be shown an application of the S.A.S. theorem in order to homogenize the masonry material.

Since the description of this theorem has been already highlighted in the Section 8 of the Chapter 1, we will limit here to expose the proposed homogenization procedure.

In particular, it is considered a single leaf masonry wall in stretcher bond. From it, a basic cell (RVE) is considered, as illustrated in the figure, below:

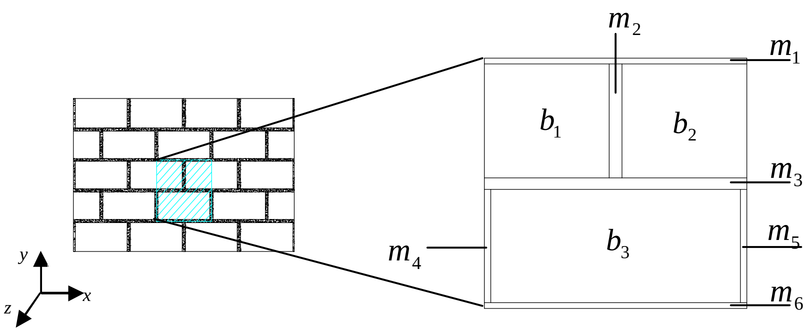


Figure 4.5 Basic cell (RVE)

The different constituents of the RVE are indicated, respectively, with:

$m_1, m_2, m_3, m_4, m_5, m_6 =$ the mortar components

$b_1, b_2, b_3 =$ the unit components

The homogenization approach, which we use, remarks the standard simplified two-step technique. This means that a homogenization process is operated first in one direction, then in the orthogonal one. In such a way, masonry basic cell can be seen as a layered material.

So, by calling with *homogenization* $y \rightarrow x$ the approach that homogenizes first in *x*-direction, then in *y*-direction and with *homogenization* $x \rightarrow y$ the other approach that homogenizes first in *y*-direction, then in *x*-direction, both cases are analyzed.

1. Homogenization *y -> x*

In order to simplify the procedure, it is supposed that the homogenization in *x*-direction was already effected. Therefore, the above illustrated masonry RVE can be considered as the layered material shown below:

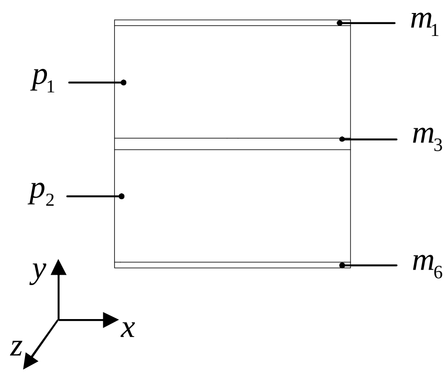


Figure 4.6 Layered material (RVE)

In order to apply the S.A.S. theorem, generally it is required that two conditions have to be satisfied, [24]. These are:

a. $\det \mathbf{T}^H = 0 \qquad \forall \mathbf{x} \in \Omega^H$ (4.3-1)

b. $\left[\mathbf{T}^H\left(\mathbf{u}^H\right)\right] \cdot \nabla \varphi\left(\mathbf{x}\right) = 0 \qquad \forall \mathbf{x} \in \Omega^I$ (4.3-2)

where:

$\mathbf{T}^H =$ the stress tensor of the *reference* homogeneous material.

$\Omega^H =$ domain occupied by the *reference* homogeneous material.

$\Omega^I =$ domain occupied by the inhomogeneous material (RVE).

For this layered material, it is clear that the material inhomogeneity is defined by a function $\varphi\left(\boldsymbol{x}\right)$ that is a constant function, but piecewise discontinuous from a phase to another one, and it is clear that the $\nabla \varphi$ -direction is coincident with the *y*-direction. In this particular case, as already seen, the above written two conditions become:

c. $\det \mathbf{T}^H = 0, \qquad \forall \boldsymbol{x} \in \partial\Omega_{(p,q)}$ (4.3-3)

d. $\mathbf{T}^H \cdot \boldsymbol{n}_{(p,q)} = \mathbf{0} \qquad \forall \boldsymbol{x} \in \partial\Omega_{(p,q)}$ (4.3-4)

where:

$\partial\Omega_{(p,q)} =$ interface surface between two faces *p* and *q*.

This means that it is not necessary to have a stress tensor $\mathbf{T}^H$ everywhere plane in the *reference* homogeneous material, but only in each point belonging to the interface surfaces, and that the eigenvector associated with the zero eigenvalue of the stress tensor $\mathbf{T}^H$ has to be coaxial with the unit normal vector to the tangent plane to the interface.

For simplicity, it will be however considered a plane stress tensor $\mathbf{T}^H$ in each point of the homogeneous *reference* body.

So, by assuming the homogeneous *reference* body coincident with an orthotropic characterization of the mortar, a strain prescribed homogenization in *y*-direction is operated. The sole non-zero components of the stress tensor $\mathbf{T}^H$ have to be:

$$\begin{aligned} &\sigma^H_{xx} \neq 0, \;\; \sigma^H_{zz} \neq 0, \;\; \sigma^H_{xz} \neq 0 \\ &\sigma^H_{yy} = 0, \;\; \sigma^H_{zy} = 0, \;\; \sigma^H_{xy} = 0 \end{aligned} \qquad (4.3\text{-}5)$$

because they are in the respect of the condition (4.3-2).

According to the S.A.S. theorem, for the generic phase "*i*" of the inhomogeneous material, it can be written:

$$\begin{aligned} &\mathbb{C}_i^I = \varphi_i \mathbb{C}^H \\ &\boldsymbol{u}_i^I = \boldsymbol{u}^H \qquad \textit{with} \;\; i = m_1, p_1, m_3, p_2, m_6 \\ &\mathbf{T}_i^I = \varphi_i \mathbf{T}^H \end{aligned} \tag{4.3-6}$$

where:

$\mathbb{C}_i^I, \boldsymbol{u}_i^I, \mathbf{T}_i^I =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of the generic phase of the inhomogeneous material (RVE).

$\mathbb{C}^H, \boldsymbol{u}^H, \mathbf{T}^H =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of *reference* homogeneous material.

Let us assume that the stress tensor is constant everywhere in the homogeneous *reference* domain and let us to consider the stress components separately each from the other, for example:

$$\sigma_{xx}^H \neq 0, \;\; \sigma_{zz}^H = 0, \;\; \sigma_{xz}^H = 0 \tag{4.3-7}$$

So, by using the Voigt notation, the stress tensor $\mathbf{T}^H$ can be written in the form of a vector, [24]. In general, it is:

$$\mathbf{T}^H = \begin{bmatrix} \sigma_{11}^H \\ \sigma_{22}^H \\ \sigma_{33}^H \\ \sigma_{32}^H \\ \sigma_{31}^H \\ \sigma_{12}^H \end{bmatrix} \tag{4.3-8}$$

In the particular case that the sole non-zero stress component is the σ_{xx}^H, it is:

$$\mathbf{T}^H = \begin{bmatrix} \sigma_{xx}^H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-9}$$

where it has been assumed:

$$1 \equiv x; \quad 2 \equiv y; \quad 3 \equiv z; \tag{4.3-10}$$

By calling S^H the fourth order compliance tensor of the homogeneous *reference* orthotropic material, it can be written in the following form:

$$\mathsf{S}^H = \begin{bmatrix} S_{1111}^H & S_{1122}^H & S_{1133}^H & 0 & 0 & 0 \\ & S_{2222}^H & S_{2233}^H & 0 & 0 & 0 \\ & & S_{3333}^H & 0 & 0 & 0 \\ & & & S_{3232}^H & 0 & 0 \\ & Sym & & & S_{3131}^H & 0 \\ & & & & & S_{1212}^H \end{bmatrix} \tag{4.3-11}$$

By remembering that the assumed homogeneous *reference* orthotropic material is the mortar, the compliance tensor S^H can be written as:

$$\mathsf{S}^H = \begin{bmatrix} \frac{1}{E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{E_1^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{21}^{(m)}}{E_2^{(m)}} & \frac{1}{E_2^{(m)}} & -\frac{\nu_{23}^{(m)}}{E_2^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{31}^{(m)}}{E_3^{(m)}} & -\frac{\nu_{32}^{(m)}}{E_3^{(m)}} & \frac{1}{E_3^{(m)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{32}^{(m)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{31}^{(m)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}^{(m)}} \end{bmatrix} \tag{4.3-12}$$

where the symbol (m) indicates the mortar and where, for symmetry, it has to be:

$$\frac{\nu_{12}^{(m)}}{E_1^{(m)}} = \frac{\nu_{21}^{(m)}}{E_2^{(m)}}; \quad \frac{\nu_{13}^{(m)}}{E_1^{(m)}} = \frac{\nu_{31}^{(m)}}{E_3^{(m)}}; \quad \frac{\nu_{23}^{(m)}}{E_2^{(m)}} = \frac{\nu_{32}^{(m)}}{E_3^{(m)}}; \tag{4.3-13}$$

The strain tensor, for the same material, is, therefore, obtained through the following relation:

$$\mathbf{E}^H = \mathsf{S}^H : \mathbf{T}^H \tag{4.3-14}$$

In the case that the sole non-zero stress component is the σ_{xx}^H and by using the Voigt notation, the strain tensor $\mathbf{E}^H$ is:

$$\mathbf{E}^H = \begin{bmatrix} \dfrac{\sigma_{xx}^H}{E_1^{(m)}} \\ -\dfrac{\nu_{21}^{(m)}\sigma_{xx}^H}{E_2^{(m)}} \\ -\dfrac{\nu_{31}^{(m)}\sigma_{xx}^H}{E_3^{(m)}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.3\text{-}15)$$

For the second equation of (4.3-6), it is:

$$\mathbf{E}_i^I = \mathbf{E}^H \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \qquad (4.3\text{-}16)$$

that means that everywhere in each phase $(m_1, p_1, m_3, p_2, m_6)$ of the inhomogeneous material the strain tensor is equal to the strain tensor $\mathbf{E}^H$ of the homogeneous *reference* material, so the compatibility is automatically satisfied in each point of the RVE.

Moreover, being the strain tensor constant in each point of the inhomogeneous material, it is also possible to write:

$$\overline{\mathbf{E}}^I = \mathbf{E}^H \qquad (4.3\text{-}17)$$

where:

$\overline{\mathbf{E}}^I =$ average value of the strain tensor in the homogeneous material.

The equilibrium conditions are, instead, guaranteed by the S.A.S. theorem. According to it, in fact, it is obtained that:

$$\mathbf{T}_i^I = \varphi_i \mathbf{T}^H = \begin{bmatrix} \varphi_i \sigma_{xx}^H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad with \;\; i = m_1, p_1, m_3, p_2, m_6 \qquad (4.3\text{-}18)$$

This means that everywhere in each phase (m_1, p_1, m_3, p_2, m_6) of the inhomogeneous material the stress tensor is equal to φ_i times the stress tensor $\mathbf{T}^H$ of the homogeneous *reference* material.

By indicating the average value of the stress tensor in the inhomogeneous material with $\overline{\mathbf{T}}^I$, it can be calculated as:

$$\overline{\mathbf{T}}^I = \frac{1}{V}\int_V \mathbf{T}^I dV \qquad (4.3\text{-}19)$$

where:

$V =$ the whole volume of the RVE.

The equation (4.3-19) is equivalent to write:

$$\overline{\mathbf{T}}^I = \frac{1}{V}\sum_i \int_{V_i} \mathbf{T}_i^I dV \qquad with \; i = m_1, p_1, m_3, p_2, m_6 \qquad (4.3\text{-}20)$$

By remembering that:

$$\int_{V_i} \mathbf{T}_i^I dV = \overline{\mathbf{T}}_i^I V_i \qquad (4.3\text{-}21)$$

where:

$\overline{\mathbf{T}}_i^I =$ average value of the stress tensor in the generic phase of the RVE

$V_i =$ volume of the generic phase of the RVE

So, the equation (4.3-20) can be rewritten in the form:

$$\overline{\mathbf{T}}^{I} = f_i \, \varphi_i \, \mathbf{T}^{H} \quad with \; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-22}$$

where it has been considered that:

$$f_i = \frac{V_i}{V}; \qquad \overline{\mathbf{T}}_i^{I} = \mathbf{T}_i^{I} = \varphi_i \, \mathbf{T}^{H} \tag{4.3-23}$$

with:

$f_i =$ the volumetric fraction of the generic phase, weighed upon the whole inhomogeneous volume.

The average stress tensor, $\overline{\mathbf{T}}^{I}$, therefore, has the following form:

$$\overline{\mathbf{T}}^{I} = \begin{bmatrix} f_i \varphi_i \sigma_{xx}^{H} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-24}$$

At this point, it is possible to obtain the homogenized compliance tensor for the inhomogeneous layered material, shown in figure 4.6, by means of the relation:

$$\overline{\mathbf{E}}^{I} = \overline{\mathsf{S}}^{I} : \overline{\mathbf{T}}^{I} \tag{4.3-25}$$

with:

$\overline{\mathsf{S}}^{I} := \overset{y \to x}{\mathsf{S}}$ = homogenized compliance tensor of the inhomogeneous layered material where the symbol "$y \to x$" recalls the two-step homogenization process, here considering that we first homogenize in *x*-direction and then in *y*-direction.

By considering in explicit form the equation (4.3-25), it can be written:

$$\begin{bmatrix} \bar{\varepsilon}^I_{11} \\ \bar{\varepsilon}^I_{22} \\ \bar{\varepsilon}^I_{33} \\ \bar{\varepsilon}^I_{32} \\ \bar{\varepsilon}^I_{31} \\ \bar{\varepsilon}^I_{12} \end{bmatrix} = \begin{bmatrix} \bar{S}^I_{1111} & \bar{S}^I_{1122} & \bar{S}^I_{1133} & 0 & 0 & 0 \\ & \bar{S}^I_{2222} & \bar{S}^I_{2233} & 0 & 0 & 0 \\ & & \bar{S}^I_{3333} & 0 & 0 & 0 \\ & & & \bar{S}^I_{3232} & 0 & 0 \\ & Sym & & & \bar{S}^I_{3131} & 0 \\ & & & & & \bar{S}^I_{1212} \end{bmatrix} \cdot \begin{bmatrix} \bar{\sigma}^I_{11} \\ \bar{\sigma}^I_{22} \\ \bar{\sigma}^I_{33} \\ \bar{\sigma}^I_{32} \\ \bar{\sigma}^I_{31} \\ \bar{\sigma}^I_{12} \end{bmatrix} \tag{4.3-26}$$

By taking into account the equations (4.3-17) and (4.3-24), for the assumed hypothesis (4.3-7), the first column of the homogenized compliance tensor $\bar{\mathbf{S}}^I$ is calculated. In particular, it is obtained:

$$\begin{aligned} &\bar{S}^I_{1111} = \frac{1}{f_i \varphi_i E_1^{(m)}}; \\ &\bar{S}^I_{2211} = \bar{S}^I_{1122} = -\frac{\nu_{21}^{(m)}}{f_i \varphi_i E_2^{(m)}}; \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \\ &\bar{S}^I_{3311} = \bar{S}^I_{1133} = -\frac{\nu_{31}^{(m)}}{f_i \varphi_i E_3^{(m)}}; \end{aligned} \tag{4.3-27}$$

By repeating the same procedure for the other two stress conditions, it is possible to determine adding compliance coefficients.

In particular, let us to assume now the following stress condition:

$$\sigma^H_{xx} = 0, \;\; \sigma^H_{zz} \neq 0, \;\; \sigma^H_{xz} = 0 \tag{4.3-28}$$

So, by using the Voigt notation, the stress tensor of the homogeneous *reference* material, $\mathbf{T}^H$, becomes:

$$\mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-29}$$

The strain tensor, for the same material, is, therefore, obtained through the relation (4.3-14), that yields, in Voigt notation:

$$\mathbf{E}^H = \begin{bmatrix} -\dfrac{\nu_{13}^{(m)}\sigma_{zz}^H}{E_1^{(m)}} \\ -\dfrac{\nu_{23}^{(m)}\sigma_{zz}^H}{E_2^{(m)}} \\ \dfrac{\sigma_{zz}^H}{E_3^{(m)}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-30}$$

For the same considerations, already done before, it is still worth to write:

$$\mathbf{E}_i^I = \mathbf{E}^H \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-31}$$

and:

$$\overline{\mathbf{E}}^I = \mathbf{E}^H \tag{4.3-32}$$

Then, according to the S.A.S. theorem, it is now obtained that:

$$\mathbf{T}_i^I = \varphi_i \mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ \varphi_i \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textit{with} \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-33}$$

and by proceeding analogously to what has been already done, it can be written, again:

$$\overline{\mathbf{T}}^I = f_i \, \varphi_i \, \mathbf{T}^H \quad \textit{with } i = m_1, p_1, m_3, p_2, m_6$$

(4.3-34)

The average stress tensor, $\overline{\mathbf{T}}^I$, in this case, has the following form:

$$\overline{\mathbf{T}}^I = \begin{bmatrix} 0 \\ 0 \\ f_i \varphi_i \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textit{with} \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-35}$$

By taking into account the equations (4.3-32) and (4.3-35), for the assumed hypothesis (4.3-28), the third column of the homogenized compliance tensor $\overline{\mathsf{S}}^I$ is calculated. In particular, it is obtained:

$$\begin{aligned} &\overline{S}^I_{1133} = \overline{S}^I_{3311} = -\frac{\nu_{13}^{(m)}}{f_i \varphi_i E_1^{(m)}}; \\ &\overline{S}^I_{2233} = \overline{S}^I_{3322} = -\frac{\nu_{23}^{(m)}}{f_i \varphi_i E_2^{(m)}}; \quad \textit{with} \;\; i = m_1, p_1, m_3, p_2, m_6 \\ &\overline{S}^I_{3333} = \frac{1}{f_i \varphi_i E_3^{(m)}}; \end{aligned} \tag{4.3-36}$$

Finally, let us assume the following stress condition:

$$\sigma_{xx}^{H} = 0,\ \sigma_{zz}^{H} = 0,\ \ \sigma_{xz}^{H} \neq 0 \qquad (4.3\text{-}37)$$

So, the stress tensor of the homogeneous *reference* material, $\mathbf{T}^{H}$, becomes:

$$\mathbf{T}^{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{xz}^{H} \\ 0 \end{bmatrix} \qquad (4.3\text{-}38)$$

About the strain tensor, for the same material, the relation (4.3-14) yields, in Voigt notation:

$$\mathbf{E}^{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\sigma_{xz}^{H}}{2G_{31}^{(m)}} \\ 0 \end{bmatrix} \qquad (4.3\text{-}39)$$

Again, it is still worth to write:

$$\mathbf{E}_{i}^{I} = \mathbf{E}^{H} \quad with\ \ i = m_1, p_1, m_3, p_2, m_6 \qquad (4.3\text{-}40)$$

and:

$$\overline{\mathbf{E}}^{I} = \mathbf{E}^{H} \qquad (4.3\text{-}41)$$

Moreover, according to the S.A.S. theorem, it is now obtained that:

$$\mathbf{T}_i^I = \varphi_i \mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \varphi_i \sigma_{xz}^H \\ 0 \end{bmatrix} \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-42}$$

and by proceeding analogously to what has been already done, it can be written, again:

$$\overline{\mathbf{T}}^I = f_i\, \varphi_i\, \mathbf{T}^H \quad with \; i = m_1, p_1, m_3, p_2, m_6$$

(4.3-43)

The average stress tensor, $\overline{\mathbf{T}}^I$, in this case, has the following form:

$$\overline{\mathbf{T}}^I = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_i \varphi_i \sigma_{xz}^H \\ 0 \end{bmatrix} \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-44}$$

By taking into account the equations (4.3-41) and (4.3-44), for the assumed hypothesis (4.3-37), another coefficient of the homogenized compliance tensor $\overline{\mathsf{S}}^I$ is calculated. In particular, it is obtained:

$$\overline{S}_{3131}^I = \frac{1}{2\, f_i\, \varphi_i\, G_{31}^{(m)}} \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-45}$$

In this way, for the symmetry of the compliance tensor $\overline{\mathsf{S}}^I$, only three coefficients remain undeterminable, and the tensor assumes the form:

$$\bar{\mathrm{S}}^{I}=\begin{bmatrix} \dfrac{1}{f_i \varphi_i E_1^{(m)}} & -\dfrac{\nu_{21}^{(m)}}{f_i \varphi_i E_2^{(m)}} & -\dfrac{\nu_{13}^{(m)}}{f_i \varphi_i E_1^{(m)}} & 0 & 0 & 0 \\ & \bar{S}_{2222}^{I} & -\dfrac{\nu_{23}^{(m)}}{f_i \varphi_i E_2^{(m)}} & 0 & 0 & 0 \\ & & \dfrac{1}{f_i \varphi_i E_3^{(m)}} & 0 & 0 & 0 \\ & & & \bar{S}_{3232}^{I} & 0 & 0 \\ & Sym & & & \dfrac{1}{2 f_i \varphi_i G_{31}^{(m)}} & 0 \\ & & & & & \bar{S}_{1212}^{I} \end{bmatrix} \tag{4.3-46}$$

where:

$$f_i \varphi_i = f_{m1}\varphi_{m1} + f_{p1}\varphi_{p1} + f_{m3}\varphi_{m3} + f_{p2}\varphi_{p2} + f_{m6}\varphi_{m6} \tag{4.3-47}$$

By considering that the mortar is the *reference* orthotropic homogeneous material, it is:

$$\varphi_{m_1} = \varphi_{m_3} = \varphi_{m_6} = 1 \tag{4.3-48}$$

being the phases m_1, m_3, m_6 coincident with the mortar.

Moreover, it can be considered that the two partitions, p_1 and p_2, have the same volumetric fraction weighed upon the RVE volume V :

$$f_{p_1} = f_{p_2} \tag{4.3-49}$$

So, the equation (4.3-47) can be rewritten in the form:

$$f_i \varphi_i = f_{m_{orizz}} + f_{p1}\left(\varphi_{p1} + \varphi_{p2}\right) \tag{4.3-50}$$

with:

$$f_{m_{orizz}} = f_{m1} + f_{m3} + f_{m6} = \frac{V_{m_{orizz}}}{V} \tag{4.3-51}$$

and where:

$f_{m_{orizz}}$ = the volumetric fraction of the horizontal mortar, weighed upon the whole RVE volume, V .

It is worth to underline that the found elastic coefficients represent an exact solution to the homogenization problem, and, therefore, both compatible and equilibrated solution, according to the S.A.S. theorem, [24].

Moreover, it must be said that the S.A.S. theorem also yields the stiffness tensor for the generic phase "*i*" of the inhomogeneous material (RVE), shown in figure 4.6, as:

$$\mathrm{C}_i^I = \varphi_i \mathrm{C}^H \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-52}$$

from whose:

$$\mathrm{S}_i^I = \frac{1}{\varphi_i} \mathrm{S}^H \quad with \;\; i = m_1, p_1, m_3, p_2, m_6 \tag{4.3-53}$$

So, the compliance tensors for the partition p_1 and p_2 can be obtained as it follows:

$$\mathrm{S}_{p_1}^I = \begin{bmatrix} \frac{1}{\varphi_{p_1} E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{\varphi_{p_1} E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{\varphi_{p_1} E_1^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{21}^{(m)}}{\varphi_{p_1} E_2^{(m)}} & \frac{1}{\varphi_{p_1} E_2^{(m)}} & -\frac{\nu_{23}^{(m)}}{\varphi_{p_1} E_2^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{31}^{(m)}}{\varphi_{p_1} E_3^{(m)}} & -\frac{\nu_{32}^{(m)}}{\varphi_{p_1} E_3^{(m)}} & \frac{1}{\varphi_{p_1} E_3^{(m)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\varphi_{p_1} G_{32}^{(m)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{p_1} G_{31}^{(m)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{p_1} G_{12}^{(m)}} \end{bmatrix} \tag{4.3-54}$$

and:

$$\mathbf{S}_{p_2}^{I}=\begin{bmatrix}\frac{1}{\varphi_{p_2}E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{\varphi_{p_2}E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{\varphi_{p_2}E_1^{(m)}} & 0 & 0 & 0\\ -\frac{\nu_{21}^{(m)}}{\varphi_{p_2}E_2^{(m)}} & \frac{1}{\varphi_{p_2}E_2^{(m)}} & -\frac{\nu_{23}^{(m)}}{\varphi_{p_2}E_2^{(m)}} & 0 & 0 & 0\\ -\frac{\nu_{31}^{(m)}}{\varphi_{p_2}E_3^{(m)}} & -\frac{\nu_{32}^{(m)}}{\varphi_{p_2}E_3^{(m)}} & \frac{1}{\varphi_{p_2}E_3^{(m)}} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2\varphi_{p_2}G_{32}^{(m)}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{p_2}G_{31}^{(m)}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{p_2}G_{12}^{(m)}}\end{bmatrix} \tag{4.3-55}$$

where:

$\mathbf{S}_{p_1}^{I} =$ compliance tensor of the partition p_1

$\mathbf{S}_{p_2}^{I} =$ compliance tensor of the partition p_2

At this point, it is possible to explicit the constants φ_{p1} and φ_{p2} related to the partitions p_1 and p_2, obtained by means a homogenization process in *x*-direction of the elements b_1, m_2, b_2 and m_4, b_3, m_5, respectively, as it is shown in the figures below:

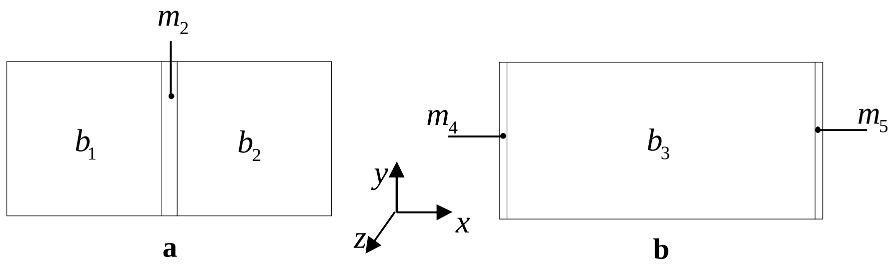


Figure 4.7 a) Partition p_1; b) Partition p_2.

So, let us consider for example the partition p_1.

For this layered material, it is clear that the $\nabla\varphi$ -direction is coincident, now, with the *x*-direction and, analogously to the previous case, the material inhomogeneity is yet defined by a function $\varphi(x)$ that is a constant function, but piecewise discontinuous from a phase to another one.

For analogous considerations to those previous ones, it will be again considered a plane stress tensor $\mathbf{T}^H$ in each point of the homogeneous *reference* body.

So, by remembering that the orthotropic homogeneous *reference* body is coincident with the mortar, a strain prescribed homogenization in *x*-direction is operated. The sole non-zero components of the stress tensor $\mathbf{T}^H$, now, have to be:

$$\begin{array}{l} \sigma_{yy}^H \neq 0,\ \sigma_{zz}^H \neq 0,\ \sigma_{yz}^H \neq 0 \\ \sigma_{xx}^H = 0,\ \sigma_{xy}^H = 0,\ \sigma_{xz}^H = 0 \end{array} \tag{4.3-56}$$

because they are in the respect of the condition (4.3-2).

According to the S.A.S. theorem, for the generic phase "*j*" of the partition p_1, it can be written:

$$\begin{array}{l} \mathsf{C}_j^{p_1} = \varphi_j \mathsf{C}^H \\ \boldsymbol{u}_j^{p_1} = \boldsymbol{u}^H \qquad with \quad j = b_1, m_2, b_2 \\ \mathbf{T}_j^{p_1} = \varphi_j \mathbf{T}^H \end{array} \tag{4.3-57}$$

where:

$\mathsf{C}_j^{p_1}, \boldsymbol{u}_j^{p_1}, \mathbf{T}_j^{p_1} =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of the generic phase of the partition p_1 of the RVE.

$\mathsf{C}^H, \boldsymbol{u}^H, \mathbf{T}^H =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of *reference* homogeneous material.

Let us assume, again, that the stress tensor is constant everywhere in the homogeneous *reference* domain and let us to consider the stress components separately each from the other, for example:

$$\sigma_{yy}^H \neq 0, \ \sigma_{zz}^H = 0, \ \sigma_{yz}^H = 0 \tag{4.3-58}$$

So, by using the Voigt notation, the stress tensor $\mathbf{T}^H$ of the homogeneous material can be written in the form of the following vector:

$$\mathbf{T}^H = \begin{bmatrix} 0 \\ \sigma_{yy}^H \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-59}$$

where it has been assumed:

$$1 \equiv x; \ \ 2 \equiv y; \ \ 3 \equiv z; \tag{4.3-60}$$

The strain tensor, for the same material is obtained by means the following relation:

$$\mathbf{E}^H = \mathsf{S}^H : \mathbf{T}^H \tag{4.3-61}$$

In the case that the sole non-zero stress component is the σ_{yy}^H, by using the Voigt notation and by remembering the (4.3-12), the strain tensor $\mathbf{E}^H$ is:

$$\mathbf{E}^H = \begin{bmatrix} -\frac{\nu_{12}^{(m)}\sigma_{yy}^H}{E_1^{(m)}} \\ \frac{\sigma_{yy}^H}{E_2^{(m)}} \\ -\frac{\nu_{32}^{(m)}\sigma_{yy}^H}{E_3^{(m)}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-62}$$

For the second equation of (4.3-57), it is:

$$\mathbf{E}_j^{p_1} = \mathbf{E}^H \quad with \;\; j = b_1, m_2, b_2 \tag{4.3-63}$$

It means that everywhere in the phases b_1, m_2, b_2 of the inhomogeneous material the strain tensor is equal to the strain tensor $\mathbf{E}^H$ of the homogeneous *reference* material, so the compatibility is automatically satisfied in these phases of the RVE.

Moreover, being the strain tensor constant in each point of the partition p_1, it is also possible to write:

$$\overline{\mathbf{E}}^{p_1} = \mathbf{E}_j^{p_1} = \mathbf{E}^H \tag{4.3-64}$$

where:

$\overline{\mathbf{E}}^{p_1} =$ average value of the strain tensor in the partition p_1.

The equilibrium conditions are, instead, guaranteed by the S.A.S. theorem. According to it, in fact, it is obtained that:

$$\mathbf{T}_j^{p_1} = \varphi_j \mathbf{T}^H = \begin{bmatrix} 0 \\ \varphi_j \sigma_{yy}^H \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textit{with} \quad j = b_1, m_2, b_2 \tag{4.3-65}$$

This means that everywhere in the phases b_1, m_2, b_2 of the partition p_1 the stress tensor is equal to φ_j times the stress tensor $\mathbf{T}^H$ of the homogeneous *reference* material.

By indicating the average value of the stress tensor in the partition p_1 with $\overline{\mathbf{T}}^{p_1}$, it can be calculated as:

$$\overline{\mathbf{T}}^{p_1} = \frac{1}{V_{p_1}} \int_{V_{p_1}} \mathbf{T}^{p_1} dV \tag{4.3-66}$$

where:

$V_{p_1} =$ the volume of partition p_1 of the RVE.

The equation (4.3-66) is equivalent to write:

$$\overline{\mathbf{T}}^{p_1} = \frac{1}{V_{p_1}} \sum_j \int_{V_j} \mathbf{T}_j^{p_1} dV \quad \textit{with} \ j = b_1, m_2, b_2 \tag{4.3-67}$$

By remembering that:

$$\int_{V_j} \mathbf{T}_j^{p_1} dV = \overline{\mathbf{T}}_j^{p_1} V_j \quad \textit{with} \ j = b_1, m_2, b_2 \tag{4.3-68}$$

where:

$\overline{\mathbf{T}}_j^{p_1} =$ average value of the stress tensor in the generic phase j of the RVE, with $j = b_1, m_2, b_2$.

$V_j =$ volume of the generic phase *j* of the RVE, with $j = b_1, m_2, b_2$

So, the equation (4.3-67) can be rewritten in the form:

$$\overline{\mathbf{T}}^{p_1} = f_j^{'} \varphi_j \mathbf{T}^H \quad with \;\; j = b_1, m_2, b_2 \tag{4.3-69}$$

where it has been considered that:

$$f_j^{'} = \frac{V_j}{V_{p_1}}; \qquad \overline{\mathbf{T}}_j^{p_1} = \mathbf{T}_j^{p_1} = \varphi_j \mathbf{T}^H \tag{4.3-70}$$

with:

$f_j^{'} =$ the volumetric fraction of the generic phase "*j*", weighed upon the volume of the partition p_1.

The average stress tensor in the partition p_1, $\overline{\mathbf{T}}^{p_1}$, therefore, has the following form:

$$\overline{\mathbf{T}}^{p_1} = \begin{bmatrix} 0 \\ f_j^{'} \varphi_j \sigma_{yy}^H \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad with \;\; j = b_1, m_2, b_2 \tag{4.3-71}$$

At this point, it is possible to obtain the homogenized compliance tensor for the inhomogeneous layered partition p_1, shown in figure 4.7a, by means of the relation:

$$\overline{\mathbf{E}}^{p_1} = \overline{\mathbb{S}}^{p_1} : \overline{\mathbf{T}}^{p_1} \tag{4.3-72}$$

with:

$\overline{\mathbb{S}}^{p_1} := \overset{x}{\mathbb{S}}$ = homogenized compliance tensor of the inhomogeneous layered partition p_1 where the symbol "x" recalls the homogenization process, which is in x-direction.

By considering in explicit form the equation (4.3-72), it can be written:

$$\begin{bmatrix} \overline{\varepsilon}_{11}^{p_1} \\ \overline{\varepsilon}_{22}^{p_1} \\ \overline{\varepsilon}_{33}^{p_1} \\ \overline{\varepsilon}_{32}^{p_1} \\ \overline{\varepsilon}_{31}^{p_1} \\ \overline{\varepsilon}_{12}^{p_1} \end{bmatrix} = \begin{bmatrix} \overline{S}_{1111}^{p_1} & \overline{S}_{1122}^{p_1} & \overline{S}_{1133}^{p_1} & 0 & 0 & 0 \\ & \overline{S}_{2222}^{p_1} & \overline{S}_{2233}^{p_1} & 0 & 0 & 0 \\ & & \overline{S}_{3333}^{p_1} & 0 & 0 & 0 \\ & & & \overline{S}_{3232}^{p_1} & 0 & 0 \\ & Sym & & & \overline{S}_{3131}^{p_1} & 0 \\ & & & & & \overline{S}_{1212}^{p_1} \end{bmatrix} \cdot \begin{bmatrix} \overline{\sigma}_{11}^{p_1} \\ \overline{\sigma}_{22}^{p_1} \\ \overline{\sigma}_{33}^{p_1} \\ \overline{\sigma}_{32}^{p_1} \\ \overline{\sigma}_{31}^{p_1} \\ \overline{\sigma}_{12}^{p_1} \end{bmatrix} \qquad (4.3\text{-}73)$$

By taking into account the equations (4.3-64) and (4.3-71), for the assumed hypothesis (4.3-58), the second column of the homogenized compliance tensor $\overline{\mathbb{S}}^{p_1}$ is calculated. In particular, it is obtained:

$$\begin{aligned} &\overline{S}_{1122}^{p_1} = \overline{S}_{2211}^{p_1} = -\frac{\nu_{12}^{(m)}}{f_j' \varphi_j E_1^{(m)}}; \\ &\overline{S}_{2222}^{p_1} = \frac{1}{f_j' \varphi_j E_2^{(m)}}; \qquad\qquad with \quad j = b_1, m_2, b_2 \qquad (4.3\text{-}74) \\ &\overline{S}_{3322}^{p_1} = \overline{S}_{2233}^{p_1} = -\frac{\nu_{32}^{(m)}}{f_j' \varphi_j E_3^{(m)}}; \end{aligned}$$

By repeating the same procedure for the other two stress conditions, it is possible to determine adding compliance coefficients.

In particular, let us assume now the following stress condition:

$$\sigma_{yy}^{H} = 0, \ \sigma_{zz}^{H} \neq 0, \ \sigma_{yz}^{H} = 0 \qquad (4.3\text{-}75)$$

So, by using the Voigt notation, the stress tensor of the homogeneous *reference* material, $\mathbf{T}^H$, becomes:

$$\mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-76}$$

The strain tensor, for the same material, is, therefore, obtained by means the relation (4.3-61), that yields, in Voigt notation:

$$\mathbf{E}^H = \begin{bmatrix} -\frac{\nu_{13}^{(m)}\sigma_{zz}^H}{E_1^{(m)}} \\ -\frac{\nu_{23}^{(m)}\sigma_{zz}^H}{E_2^{(m)}} \\ \frac{\sigma_{zz}^H}{E_3^{(m)}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.3-77}$$

For the same considerations, already done before, it is still worth to write:

$$\mathbf{E}_j^{p_1} = \mathbf{E}^H \quad with \quad j = b_1, m_2, b_2 \tag{4.3-78}$$

and:

$$\overline{\mathbf{E}}^{p_1} = \mathbf{E}_j^{p_1} = \mathbf{E}^H \tag{4.3-79}$$

Then, according to the S.A.S. theorem, it is now obtained that:

$$\mathbf{T}_j^{p_1} = \varphi_j \mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ \varphi_j \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-80}$$

and by proceeding analogously to what has been already done, it can be written, again:

$$\overline{\mathbf{T}}^{p_1} = f_j^{'} \varphi_j \mathbf{T}^H \quad with \;\; j = b_1, m_2, b_2 \tag{4.3-81}$$

The average stress tensor, $\overline{\mathbf{T}}^{p_1}$, in this case, has the following form:

$$\overline{\mathbf{T}}^{p_1} = \begin{bmatrix} 0 \\ 0 \\ f_j^{'} \varphi_j \sigma_{zz}^H \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-82}$$

By taking into account the equations (4.3-79) and (4.3-82), for the assumed hypothesis (4.3-75), the third column of the homogenized compliance tensor $\overline{\mathrm{S}}^{p_1}$ is calculated. In particular, it is obtained:

$$\begin{aligned} &\overline{S}_{1133}^{p_1} = \overline{S}_{3311}^{p_1} = -\frac{\nu_{13}^{(m)}}{f_j^{'} \varphi_j E_1^{(m)}}; \\ &\overline{S}_{2233}^{p_1} = \overline{S}_{3322}^{p_1} = -\frac{\nu_{23}^{(m)}}{f_j^{'} \varphi_j E_2^{(m)}}; \quad with \quad j = b_1, m_2, b_2 \\ &\overline{S}_{3333}^{p_1} = \frac{1}{f_j^{'} \varphi_j E_3^{(m)}}; \end{aligned} \tag{4.3-83}$$

Finally, let us assume the following stress condition:

$$\sigma_{yy}^{H} = 0, \ \sigma_{zz}^{H} = 0, \ \ \sigma_{yz}^{H} \neq 0 \tag{4.3-84}$$

So, the stress tensor of the homogeneous *reference* material, $\mathbf{T}^{H}$, becomes:

$$\mathbf{T}^{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_{yz}^{H} \\ 0 \\ 0 \end{bmatrix} \tag{4.3-85}$$

About the strain tensor, for the same material, the relation (4.3-61) yields, in Voigt notation:

$$\mathbf{E}^{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\sigma_{yz}^{H}}{2G_{32}^{(m)}} \\ 0 \\ 0 \end{bmatrix} \tag{4.3-86}$$

Again, it is still worth to write:

$$\mathbf{E}_{j}^{p_1} = \mathbf{E}^{H} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-87}$$

and:

$$\overline{\mathbf{E}}^{p_1} = \mathbf{E}_{j}^{p_1} = \mathbf{E}^{H} \tag{4.3-88}$$

Moreover, according to the S.A.S. theorem, it is now obtained that:

$$\mathbf{T}_j^{p_1} = \varphi_j \mathbf{T}^H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varphi_j \sigma_{yz}^H \\ 0 \\ 0 \end{bmatrix} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-89}$$

and by proceeding analogously to what has been already done, it can be written, again:

$$\overline{\mathbf{T}}^{p_1} = f_j^{'} \varphi_j \mathbf{T}^H \quad with \;\; j = b_1, m_2, b_2 \tag{4.3-90}$$

The average stress tensor, $\overline{\mathbf{T}}^{p_1}$, in this case, has the following form:

$$\overline{\mathbf{T}}^{p_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_j^{'} \varphi_j \sigma_{yz}^H \\ 0 \\ 0 \end{bmatrix} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-91}$$

By taking into account the equations (4.3-88) and (4.3-91), for the assumed hypothesis (4.3-84), another coefficient of the homogenized compliance tensor $\overline{\mathbf{S}}^{p_1}$ is calculated. In particular, it is obtained:

$$\overline{S}_{3232}^{p_1} = \frac{1}{2 f_j^{'} \varphi_j G_{32}^{(m)}} \quad with \quad j = b_1, m_2, b_2 \tag{4.3-92}$$

In this way, for the symmetry of the compliance tensor $\overline{\mathbf{S}}^{p_1}$, only three coefficients remain undeterminable, and the tensor assumes the form:

$$\bar{S}^{p_1}=\begin{bmatrix} \bar{S}^{p_1}_{1111} & -\dfrac{\nu^{(m)}_{12}}{f'_j\varphi_j E^{(m)}_1} & -\dfrac{\nu^{(m)}_{13}}{f'_j\varphi_j E^{(m)}_1} & 0 & 0 & 0 \\ & \dfrac{1}{f'_j\varphi_j E^{(m)}_2} & -\dfrac{\nu^{(m)}_{23}}{f'_j\varphi_j E^{(m)}_2} & 0 & 0 & 0 \\ & & \dfrac{1}{f'_j\varphi_j E^{(m)}_3} & 0 & 0 & 0 \\ & & & \dfrac{1}{2f'_j\varphi_j G^{(m)}_{32}} & 0 & 0 \\ & Sym & & & \bar{S}^{p_1}_{3131} & 0 \\ & & & & & \bar{S}^{p_1}_{1212} \end{bmatrix} \quad (4.3\text{-}93)$$

where:

$$f'_j\varphi_j = f'_{b_1}\varphi_{b_1} + f'_{m_2}\varphi_{m_2} + f'_{b_2}\varphi_{b_2} \quad (4.3\text{-}94)$$

By considering that the mortar is the *reference* orthotropic homogeneous material, it is:

$$\varphi_{m_2} = 1 \quad (4.3\text{-}95)$$

being the phase m_2 coincident with the mortar, while it will be:

$$\varphi_{b_1} = \varphi_{b_2} = \varphi_b \quad (4.3\text{-}96)$$

where:

$\varphi_b =$ the brick elastic ratio

Moreover, it can be considered that the two constituents, b_1 and b_2, of the partition p_1, have the same volumetric fraction weighed upon the volume V_{p_1} of the partition:

$$f'_{b_1} = f'_{b_2} \quad (4.3\text{-}97)$$

So, the equation (4.3-94) can be rewritten in the form:

$$f_j^{'}\varphi_j = 2 f_{b_1}^{'}\varphi_b + f_{m_2}^{'} \tag{4.3-98}$$

By comparing the (4.3-93) with the (4.3-54) and by imposing their equivalence, that is:

$$\mathsf{S}_{p_1}^{I} = \overline{\mathsf{S}}^{p_1} \tag{4.3-99}$$

it is obtained that:

$$\varphi_{p_1} = f_j^{'}\varphi_j = 2 f_{b_1}^{'}\varphi_b + f_{m_2}^{'} \tag{4.3-100}$$

Then, a similar homogenization process in *x*-direction has been executed for the other partition p_2, shown in the figure 4.7b. Because of the analogy of the procedure, it is here shown the result, only, that is the compliance tensor $\overline{\mathsf{S}}^{p_2}$:

$$\overline{\mathsf{S}}^{p_2} = \begin{bmatrix} \overline{S}_{1111}^{p_2} & -\dfrac{\nu_{12}^{(m)}}{f_k^{'}\varphi_k E_1^{(m)}} & -\dfrac{\nu_{13}^{(m)}}{f_k^{'}\varphi_k E_1^{(m)}} & 0 & 0 & 0 \\ & \dfrac{1}{f_k^{'}\varphi_k E_2^{(m)}} & -\dfrac{\nu_{23}^{(m)}}{f_k^{'}\varphi_k E_2^{(m)}} & 0 & 0 & 0 \\ & & \dfrac{1}{f_k^{'}\varphi_k E_3^{(m)}} & 0 & 0 & 0 \\ & & & \dfrac{1}{2 f_k^{'}\varphi_k G_{32}^{(m)}} & 0 & 0 \\ & Sym & & & \overline{S}_{3131}^{p_1} & 0 \\ & & & & & \overline{S}_{1212}^{p_1} \end{bmatrix} \tag{4.3-101}$$

with:

$$f_k^{'} = \frac{V_k}{V_{p_2}} \qquad with\ \ k = m_4, b_3, m_5 \tag{4.3-102}$$

and where:

$f_k^{'} =$ the volumetric fraction of the generic phase "k", weighed upon the volume of the partition p_2.

So, it can be written:

$$f_k^{'}\varphi_k = f_{m_4}^{'}\varphi_{m_4} + f_{b_3}^{'}\varphi_{b_3} + f_{m_5}^{'}\varphi_{m_5} \tag{4.3-103}$$

By considering, again, that the mortar is the *reference* orthotropic homogeneous material, it is:

$$\varphi_{m_4} = \varphi_{m_5} = 1 \tag{4.3-104}$$

being the phases m_4 and m_5 coincident with the mortar, while it will be:

$$\varphi_{b_3} = \varphi_b \tag{4.3-105}$$

Moreover, it can be considered that the two constituents, m_4 and m_5, of the partition p_2, have the same volumetric fraction weighed upon the volume V_{p_2} of the partition:

$$f_{m_4}^{'} = f_{m_5}^{'} \tag{4.3-106}$$

So, the equation (4.3-103) can be rewritten in the form:

$$f_k^{'}\varphi_k = 2 f_{m_4}^{'} + f_{b_3}^{'}\varphi_b \tag{4.3-107}$$

By comparing the (4.3-101) with the (4.3-55) and by imposing their equivalence, that is:

$$\sigma_{p_2}^I = \overline{\sigma}^{p_2} \tag{4.3-108}$$

it is obtained that:

$$\varphi_{p_2} = f_k^{'}\varphi_k = 2 f_{m_4}^{'} + f_{b_3}^{'}\varphi_b \tag{4.3-109}$$

According to the geometry of the RVE, it can be considered that:

$$2\,f'_{m_4} = f'_{m_2}$$
$$2\,f'_{b_1} = f'_{b_3}$$

(4.3-110)

and so, it can be written:

$$\varphi_{p_1} = \varphi_{p_2} = \varphi_p = f'_{m_{vert}} + f'_b \varphi_b \qquad (4.3\text{-}111)$$

with:

$$f'_{m_{vert}} = \frac{V'_{m_{vert}}}{V_{p_1}} \quad and \quad f'_b = \frac{V'_b}{V_{p_1}} \qquad (4.3\text{-}112)$$

where:

$V'_{m_{vert}} =$ the volume of the vertical mortar in a single row.

$V'_b =$ the volume of the brick in a single row.

So, the equation (4.3-50) can be rewritten in the form:

$$f_i \varphi_i = f_{m_{orizz}} + 2\,f_{p1} \varphi_p \qquad (4.3\text{-}113)$$

that is:

$$f_i \varphi_i = f_{m_{orizz}} + 2\,f_{p1} \left(f'_{m_{vert}} + f'_b \varphi_b \right) \qquad (4.3\text{-}114)$$

By considering that:

$$2\,f_{p_1} = \frac{V_{p_1} + V_{p_2}}{V} = \frac{2V_{p_1}}{V} \qquad (4.3\text{-}115)$$

and by remembering the (4.3-51) and the (4.3-112), the equation (4.3-114) can be rewritten in the form:

$$f_i \varphi_i = f_m + f_b \varphi_b \qquad (4.3\text{-}116)$$

where:

$$
\begin{aligned}
f_m &= f_{m_{orizz}} + f_{m_{vert}} \\
f_{m_{vert}} &= \frac{V_{m_{vert}}}{V} \\
f_b &= \frac{V_b}{V}
\end{aligned}
\qquad (4.3\text{-}117)
$$

At this point, it is possible to write the homogenized compliance tensor of the RVE as it follows:

$$
\bar{\Sigma}^I = \begin{bmatrix}
\frac{1}{E_1^{(m)}\phi} & -\frac{\nu_{21}^{(m)}}{E_1^{(m)}\phi} & -\frac{\nu_{13}^{(m)}}{E_1^{(m)}\phi} & 0 & 0 & 0 \\
 & \bar{S}^I_{2222} & -\frac{\nu_{23}^{(m)}}{E_2^{(m)}\phi} & 0 & 0 & 0 \\
 & & \frac{1}{E_3^{(m)}\phi} & 0 & 0 & 0 \\
 & & & \bar{S}^I_{3232} & 0 & 0 \\
 & Sym & & & \frac{1}{2G_{31}^{(m)}\phi} & 0 \\
 & & & & & \bar{S}^I_{1212}
\end{bmatrix}
\qquad (4.3\text{-}118)
$$

where $\phi = \left(f_m + f_b \varphi_b \right)$.

It has to be underlined, again, that the found elastic coefficients represent a solution to the homogenization problem that is both exact and very simple. The exactness is given according to the S.A.S. theorem, [24]. The simplicity is related to the fact that the homogenized compliance tensor is obtained from that one of the reference homogeneous material by multiplying for a scalar factor ϕ^{-1}, depending from the geometry of the micro-constituents and from the elastic ratio φ_b.

2. Homogenization x ® y

Analogously to what has been done for the homogenization y-> x , in order to simplify the procedure, it is supposed that the homogenization in y-direction was already effected. Therefore, the illustrated above masonry RVE can be considered as the layered material shown below:

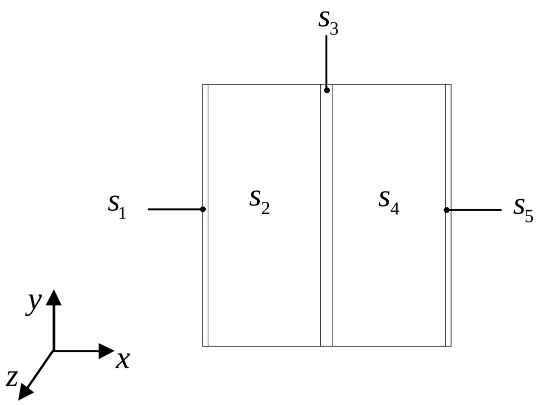


Figure 4.8 Layered material (RVE)

For this layered material, it is clear that the ∇j -direction is coincident with the x-direction and that the material inhomogeneity is defined by a function j (x) that is a constant function, but piecewise discontinuous from a phase to another one.

For analogous considerations to those previous ones, also in this case, it will be considered a plane stress tensor $\mathbf{T}^H$ in each point of the homogeneous *reference* body (the orthotropic mortar), so that $\mathbf{T}^H$ satisfies the condition (4.3-2) .

So, a strain prescribed homogenization in x-direction is operated. The sole non-zero components of the stress tensor $\mathbf{T}^H$, in this case, have to be:

$$\begin{aligned} &\sigma_{yy}^{H} \neq 0, \ \sigma_{zz}^{H} \neq 0, \ \ \sigma_{yz}^{H} \neq 0 \\ &\sigma_{xx}^{H} = 0, \ \sigma_{xy}^{H} = 0, \ \ \sigma_{xz}^{H} = 0 \end{aligned} \tag{4.3-119}$$

just because they are in the respect of the condition (4.3-2).

According to the S.A.S. theorem, for the generic phase "q" of the inhomogeneous material, shown in figure 4.8, it can be written:

$$\begin{aligned} &\mathbb{C}_q^I = \varphi_q \mathbb{C}^H \\ &\boldsymbol{u}_q^I = \boldsymbol{u}^H \qquad\quad with \ \ q = s_1, s_2, s_3, s_4, s_5 \\ &\mathbf{T}_q^I = \varphi_q \mathbf{T}^H \end{aligned} \tag{4.3-120}$$

where:

$\mathbb{C}_q^I, \boldsymbol{u}_q^I, \mathbf{T}_q^I =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of the generic phase of the inhomogeneous material (RVE), shown in figure 4.8.

$\mathbb{C}^H, \boldsymbol{u}^H, \mathbf{T}^H =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of *reference* homogeneous material.

By reiterating the procedure used for the homogenization $y \rightarrow x$, it is possible to obtain the homogenized compliance tensor for such inhomogeneous layered material, by means of the relation:

$$\overline{\overline{\mathbf{E}}}^I = \overline{\overline{\mathbb{S}}}^I : \overline{\overline{\mathbf{T}}}^I \tag{4.3-121}$$

where:

$$\overline{\overline{\mathbf{E}}}^I = \mathbf{E}^H \tag{4.3-122}$$

$$\overline{\overline{\mathbf{T}}}^I = f_q \varphi_q \mathbf{T}^H \quad\ with \ \ q = s_1, s_{2,} s_3, s_4, s_5 \tag{4.3-123}$$

with:

$\overline{\overline{\mathbf{E}}}^{I}$ = the average value of the strain tensor in the inhomogeneous layered material, shown in figure 4.8.

$\overline{\overline{\mathbf{T}}}^{I}$ = the average value of the stress tensor in the inhomogeneous layered material, shown in figure 4.8.

$\overline{\overline{\mathsf{S}}}^{I} := \overset{x \to y}{\mathsf{S}}$ = homogenized compliance tensor of such inhomogeneous layered material, where the symbol "$x \to y$" recalls the two-step homogenization process, here considering that we first homogenize in y-direction and then in x-direction.

By considering in explicit form the equation (4.3-121), it can be written:

$$\begin{bmatrix} \overline{\overline{\mathsf{e}}}^{I}_{11} \\ \overline{\overline{\mathsf{e}}}^{I}_{22} \\ \overline{\overline{\mathsf{e}}}^{I}_{33} \\ \overline{\overline{\mathsf{e}}}^{I}_{32} \\ \overline{\overline{\mathsf{e}}}^{I}_{31} \\ \overline{\overline{\mathsf{e}}}^{I}_{12} \end{bmatrix} = \begin{bmatrix} \overline{\overline{S}}^{I}_{1111} & \overline{\overline{S}}^{I}_{1122} & \overline{\overline{S}}^{I}_{1133} & 0 & 0 & 0 \\ & \overline{\overline{S}}^{I}_{2222} & \overline{\overline{S}}^{I}_{2233} & 0 & 0 & 0 \\ & & \overline{\overline{S}}^{I}_{3333} & 0 & 0 & 0 \\ & & & \overline{\overline{S}}^{I}_{3232} & 0 & 0 \\ & Sym & & & \overline{\overline{S}}^{I}_{3131} & 0 \\ & & & & & \overline{\overline{S}}^{I}_{1212} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathsf{s}}}^{I}_{11} \\ \overline{\overline{\mathsf{s}}}^{I}_{22} \\ \overline{\overline{\mathsf{s}}}^{I}_{33} \\ \overline{\overline{\mathsf{s}}}^{I}_{32} \\ \overline{\overline{\mathsf{s}}}^{I}_{31} \\ \overline{\overline{\mathsf{s}}}^{I}_{12} \end{bmatrix} \qquad (4.3\text{-}124)$$

By considering the non-zero stress components (4.3-119) separately each one from the other, and by taking into account the equations (4.3-122) and (4.3-123), the second and the third column of the homogenized compliance tensor $\overline{\overline{\mathsf{S}}}^{I}$ are obtained, and so, also the coefficient $\overline{\overline{S}}^{I}_{3232}$. In particular, for the second column, it is obtained:

$$\overline{\overline{S}}^{I}_{1122} = \overline{\overline{S}}^{I}_{2211} = -\frac{\nu_{12}^{(m)}}{f_q \varphi_q \cdot E_1^{(m)}}$$
$$\overline{\overline{S}}^{I}_{2222} = \frac{1}{f_q \varphi_q \cdot E_2^{(m)}} \qquad with \;\; q = s_1, s_2, s_3, s_4, s_5 \tag{4.3-125}$$
$$\overline{\overline{S}}^{I}_{3322} = \overline{\overline{S}}^{I}_{2233} = -\frac{\nu_{32}^{(m)}}{f_q \varphi_q \cdot E_3^{(m)}}$$

For the third column, it is obtained:

$$\overline{\overline{S}}^{I}_{1133} = \overline{\overline{S}}^{I}_{3311} = -\frac{\nu_{13}^{(m)}}{f_q \varphi_q \cdot E_1^{(m)}}$$
$$\overline{\overline{S}}^{I}_{2233} = \overline{\overline{S}}^{I}_{3322} = -\frac{\nu_{23}^{(m)}}{f_q \varphi_q \cdot E_2^{(m)}}$$
$$\overline{\overline{S}}^{I}_{3333} = \frac{1}{f_q \varphi_q \cdot E_3^{(m)}}$$

(4.3-126)

and then:

$$\overline{\overline{S}}^{I}_{3232} = \frac{1}{2 f_q \varphi_q \cdot G_{32}^{(m)}}$$

(4.3-127)

In this way, for the symmetry of the compliance tensor $\overline{\overline{S}}^{I}$, only three coefficients remain undeterminable, and the tensor assumes the form:

$$\overline{\overline{\mathrm{S}}}^{I} = \begin{bmatrix} \overline{\overline{S}}^{I}_{1111} & -\dfrac{\nu_{12}^{(m)}}{f_q \varphi_q E_1^{(m)}} & -\dfrac{\nu_{13}^{(m)}}{f_q \varphi_q E_1^{(m)}} & 0 & 0 & 0 \\ & \dfrac{1}{f_q \varphi_q E_2^{(m)}} & -\dfrac{\nu_{23}^{(m)}}{f_q \varphi_q E_2^{(m)}} & 0 & 0 & 0 \\ & & \dfrac{1}{f_q \varphi_q E_3^{(m)}} & 0 & 0 & 0 \\ & & & \dfrac{1}{2 f_q \varphi_q G_{32}^{(m)}} & 0 & 0 \\ & Sym & & & \overline{\overline{S}}^{I}_{3131} & 0 \\ & & & & & \overline{\overline{S}}^{I}_{1212} \end{bmatrix} \quad (4.3\text{-}128)$$

where:

$$f_q \varphi_q = f_{s_1} \varphi_{s_1} + f_{s_2} \varphi_{s_2} + f_{s_3} \varphi_{s_3} + f_{s_4} \varphi_{s_4} + f_{s_5} \varphi_{s_5} \quad (4.3\text{-}129)$$

It can be considered, now, that the three partitions, s_1, s_3 and s_5, have the volumetric fractions, weighed upon the RVE volume, V, that are in the following relation:

$$f_{s_3} = 2 f_{s_1} = 2 f_{s_5} \quad (4.3\text{-}130)$$

while the partitions s_2 and s_4 have the same volumetric fraction, weighed upon the RVE volume, V:

$$f_{s_2} = f_{s_4} \quad (4.3\text{-}131)$$

where:

$$f_{s_t} = \frac{V_{s_t}}{V} \qquad t = 1,\ 2,\ 3,\ 4,\ 5 \quad (4.3\text{-}132)$$

Moreover, it can be considered that, for the geometric symmetry, it is:

$$\begin{aligned} \varphi_{s_1} &= \varphi_{s_5} \\ \varphi_{s_2} &= \varphi_{s_4} \end{aligned} \tag{4.3-133}$$

So, the equation (4.3-129) can be rewritten in the form:

$$f_q \varphi_q = 2 f_{s_1} \left(\varphi_{s_1} + \varphi_{s_3} \right) + 2 f_{s_2} \varphi_{s_2} \tag{4.3-134}$$

It is worth to underline, again, that the found elastic coefficients represent an exact solution to the homogenization problem, and, therefore, both compatible and equilibrated solution, according to the S.A.S. theorem, [24].

Moreover, as already said for the homogenization $y \to x$, the S.A.S. theorem also yields the stiffness tensor for the generic phase "q" of the inhomogeneous layered material, shown in figure 4.8. By recalling the first equation of the (4.3-120), it is:

$$\mathsf{C}_q^I = \varphi_q \mathsf{C}^H \quad with \;\; q = s_1, s_2, s_3, s_4, s_5 \tag{4.3-135}$$

from whose:

$$\mathsf{S}_q^I = \frac{1}{\varphi_q} \mathsf{S}^H \;\; with \;\; q = s_1, s_2, s_3, s_4, s_5 \tag{4.3-136}$$

So, the compliance tensors for the partitions s_1, s_2, s_3, s_4 and s_5 can be obtained as it follows:

$$\mathrm{S}_{s_t}^{I}=\begin{bmatrix} \frac{1}{\varphi_{s_t}E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{\varphi_{s_t}E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{\varphi_{s_t}E_1^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{21}^{(m)}}{\varphi_{s_t}E_2^{(m)}} & \frac{1}{\varphi_{s_t}E_2^{(m)}} & -\frac{\nu_{23}^{(m)}}{\varphi_{s_t}E_2^{(m)}} & 0 & 0 & 0 \\ -\frac{\nu_{31}^{(m)}}{\varphi_{s_t}E_3^{(m)}} & -\frac{\nu_{32}^{(m)}}{\varphi_{s_t}E_3^{(m)}} & \frac{1}{\varphi_{s_t}E_3^{(m)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\varphi_{s_t}G_{32}^{(m)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{s_t}G_{31}^{(m)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\varphi_{s_t}G_{12}^{(m)}} \end{bmatrix} \quad (4.3\text{-}137)$$

$with\ \ t\in[1,5]$

At this point, it is possible to explicit the constants φ_{s_t}, where *t* is within the range [1-5], related to the partitions s_1, s_2, s_3, s_4 and s_5, obtained by means a homogenization process in *y*-direction of the elements $\left\{m_1^{s_1}, b_1^{s_1}, m_3^{s_1}, m_4^{s_1}, m_6^{s_1}\right\}$, $\left\{m_1^{s_2}, b_1^{s_2}, m_3^{s_3}, b_3^{s_4}, m_6^{s_5}\right\}$, $\left\{m_1^{s_3}, m_2^{s_3}, m_3^{s_3}, b_3^{s_3}, m_6^{s_3}\right\}$, $\left\{m_1^{s_4}, b_2^{s_4}, m_3^{s_4}, b_3^{s_4}, m_6^{s_5}\right\}$ and $\left\{m_1^{s_5}, b_2^{s_5}, m_3^{s_5}, m_5^{s_5}, m_6^{s_5}\right\}$, respectively, as it is shown in the figure below.

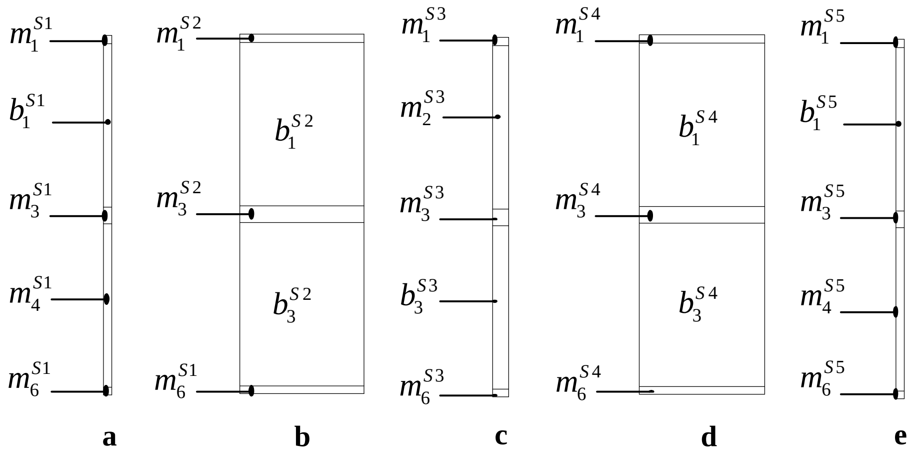


Figure 4.9 a) partition s_1; b) partition s_2; c) partition s_3; d) partition s_4; e) partition s_5.

So, let us consider for example the partition s_1.

For this layered material, it is clear that the $\nabla \varphi$ -direction is coincident, now, with the *y*-direction and, analogously to the previous case, the material inhomogeneity is yet defined by a function $\varphi(x)$ that is a constant function, but piecewise discontinuous from a phase to another one.

For analogous considerations to those previous ones, it will be again considered a plane stress tensor $\mathbf{T}^H$ in each point of the homogeneous *reference* body.

So, by remembering that the orthotropic homogeneous *reference* body is coincident with the mortar, a strain prescribed homogenization in *y*-direction is operated. The sole non-zero components of the stress tensor $\mathbf{T}^H$, now, have to be:

$$\begin{aligned} &\sigma_{xx}^H \neq 0, \ \sigma_{zz}^H \neq 0, \ \sigma_{xz}^H \neq 0 \\ &\sigma_{yy}^H = 0, \ \sigma_{xy}^H = 0, \ \sigma_{yz}^H = 0 \end{aligned} \tag{4.3-138}$$

because they are in the respect of the condition (4.3-2).

According to the S.A.S. theorem, for the generic phase "*r*" of the partition s_1, it can be written:

$$
\begin{aligned}
&\mathsf{C}_r^{s_1} = \mathrm{j}_r \mathsf{C}^H \\
&\boldsymbol{u}_r^{s_1} = \boldsymbol{u}^H \qquad with \;\; r = m_1^{S_1},\, b_1^{S_1},\, m_3^{S_1},\, m_4^{S_1}, m_6^{S_1} \\
&\mathbf{T}_r^{s_1} = \mathrm{j}_r \mathbf{T}^H
\end{aligned}
\tag{4.3-139}
$$

where:

$\mathsf{C}_r^{s_1}, \boldsymbol{u}_r^{s_1}, \mathbf{T}_r^{s_1} =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of the generic phase of the partition s_1 of the RVE.

$\mathsf{C}^H, \boldsymbol{u}^H, \mathbf{T}^H =$ respectively, the stiffness tensor, the displacements solution and the stress tensor of *reference* homogeneous material.

By reiterating the procedure until here used, it is possible to obtain the homogenized compliance tensor for the inhomogeneous layered partition s_1, shown in figure 4.9a, by means of the relation:

$$\overline{\overline{\mathbf{E}}}^{s_1} = \overline{\overline{\mathsf{S}}}^{s_1} : \overline{\overline{\mathbf{T}}}^{s_1} \tag{4.3-140}$$

where:

$$\overline{\overline{\mathbf{E}}}^{s_1} = \mathbf{E}^H \tag{4.3-141}$$

and:

$$\overline{\overline{\mathbf{T}}}^{s_1} = f_r^{'} \mathrm{j}_r \mathbf{T}^H \quad with \;\; r = m_1^{s_1},\, b_1^{s_1},\, m_3^{s_1},\, m_4^{s_1}, m_6^{s_1} \tag{4.3-142}$$

with:

$\overline{\overline{\mathbf{E}}}^{s_1} =$ the average value of the strain tensor in the inhomogeneous layered partition s_1, shown in figure 4.9a.

$\overline{\overline{\mathbf{T}}}^{s_1}$ = the average value of the stress tensor in the inhomogeneous layered partition s_1, shown in figure 4.9a.

$\overline{\overline{\mathsf{S}}}^{s_1} := \overset{y}{\mathsf{S}} =$ homogenized compliance tensor of the same inhomogeneous layered partition s_1 where the symbol "y" recalls the homogenization process, which is in y-direction.

In (4.3-142) it has been considered that:

$$f_r^{'} = \frac{V_r}{V_{s_1}}; \quad \overline{\mathbf{T}}_r^{s_1} = \mathbf{T}_r^{s_1} = \mathrm{j}_r \mathbf{T}^H \quad with\ r = m_1^{s_1},\, b_1^{s_1},\, m_3^{s_1},\, m_4^{s_1},\, m_6^{s_1} \qquad (4.3\text{-}143)$$

with:

$f_r^{'} =$ volumetric fraction of the generic phase "r", weighed upon the volume V_{s_1} of the partition s_1

$V_r =$ volume of the generic phase "r" of the partition s_1

By considering in explicit form the equation (4.3-140), it can be written:

$$\begin{bmatrix} \overline{\overline{\mathrm{e}}}^{s_1}_{11} \\ \overline{\overline{\mathrm{e}}}^{s_1}_{22} \\ \overline{\overline{\mathrm{e}}}^{s_1}_{33} \\ \overline{\overline{\mathrm{e}}}^{s_1}_{32} \\ \overline{\overline{\mathrm{e}}}^{s_1}_{31} \\ \overline{\overline{\mathrm{e}}}^{s_1}_{12} \end{bmatrix} = \begin{bmatrix} \overline{\overline{S}}^{s_1}_{1111} & \overline{\overline{S}}^{s_1}_{1122} & \overline{\overline{S}}^{s_1}_{1133} & 0 & 0 & 0 \\ & \overline{\overline{S}}^{s_1}_{2222} & \overline{\overline{S}}^{s_1}_{2233} & 0 & 0 & 0 \\ & & \overline{\overline{S}}^{s_1}_{3333} & 0 & 0 & 0 \\ & & & \overline{\overline{S}}^{s_1}_{3232} & 0 & 0 \\ & Sym & & & \overline{\overline{S}}^{s_1}_{3131} & 0 \\ & & & & & \overline{\overline{S}}^{s_1}_{1212} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathrm{s}}}^{s_1}_{11} \\ \overline{\overline{\mathrm{s}}}^{s_1}_{22} \\ \overline{\overline{\mathrm{s}}}^{s_1}_{33} \\ \overline{\overline{\mathrm{s}}}^{s_1}_{32} \\ \overline{\overline{\mathrm{s}}}^{s_1}_{31} \\ \overline{\overline{\mathrm{s}}}^{s_1}_{12} \end{bmatrix} \qquad (4.3\text{-}144)$$

By considering the non-zero stress components (4.3-138) separately each one from the other, and by taking into account the equations (4.3-141) and

(4.3-142), the first and the third column of the homogenized compliance tensor $\overline{\overline{\mathrm{S}}}^{s_1}$ are obtained, and so, also the coefficient $\overline{\overline{S}}^{s_1}_{3131}$. In particular, for the first column, it is obtained:

$$
\begin{aligned}
&\overline{\overline{S}}^{I}_{1111} = \frac{1}{f_r^{'} \mathrm{j}_r E_1^{(m)}} \\
&\overline{\overline{S}}^{I}_{2211} = \overline{\overline{S}}^{I}_{1122} = -\frac{\mathrm{n}_{21}^{(m)}}{f_r^{'} \mathrm{j}_r E_2^{(m)}} \quad \textit{with} \;\; r = m_1^{s_1},\, b_1^{s_1},\, m_3^{s_1},\, m_4^{s_1}, m_6^{s_1} \qquad (4.3\text{-}145) \\
&\overline{\overline{S}}^{I}_{3311} = \overline{\overline{S}}^{I}_{1133} = -\frac{\mathrm{n}_{31}^{(m)}}{f_r^{'} \mathrm{j}_r E_3^{(m)}}
\end{aligned}
$$

For the third column, it is obtained:

$$
\begin{aligned}
&\overline{\overline{S}}^{I}_{1133} = \overline{\overline{S}}^{I}_{3311} = -\frac{\mathrm{n}_{13}^{(m)}}{f_r^{'} \mathrm{j}_r E_1^{(m)}} \\
&\overline{\overline{S}}^{I}_{2233} = \overline{\overline{S}}^{I}_{3322} = -\frac{\mathrm{n}_{23}^{(m)}}{f_r^{'} \mathrm{j}_r E_2^{(m)}} \qquad (4.3\text{-}146) \\
&\overline{\overline{S}}^{I}_{3333} = \frac{1}{f_r^{'} \mathrm{j}_r E_3^{(m)}}
\end{aligned}
$$

and then:

$$
\overline{\overline{S}}^{I}_{3131} = \frac{1}{2 f_r^{'} \mathrm{j}_r G_{31}^{(m)}}
$$

(4.3-147)

In this way, for the symmetry of the compliance tensor $\overline{\overline{\mathrm{S}}}^{s_1}$, only three coefficients remain undeterminable, and the tensor assumes the form:

$$\overline{\overline{S}}^{s_1} = \begin{bmatrix} \frac{1}{f_r^{'}\varphi_r E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{f_r^{'}\varphi_r E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{f_r^{'}\varphi_r E_1^{(m)}} & 0 & 0 & 0 \\ & \overline{\overline{S}}_{1111}^{s_1} & -\frac{\nu_{23}^{(m)}}{f_r^{'}\varphi_r E_2^{(m)}} & 0 & 0 & 0 \\ & & \frac{1}{f_r^{'}\varphi_r E_3^{(m)}} & 0 & 0 & 0 \\ & & & \overline{\overline{S}}_{3232}^{s_1} & 0 & 0 \\ & Sym & & & \frac{1}{2 f_r^{'}\varphi_r G_{31}^{(m)}} & 0 \\ & & & & & \overline{\overline{S}}_{1212}^{s_1} \end{bmatrix} \quad (4.3\text{-}148)$$

where:

$$f_r^{'}\varphi_r = f_{m_1^{s_1}}^{'}\varphi_{m_1^{s_1}} + f_{b_1^{s_1}}^{'}\varphi_{b_1^{s_1}} + f_{m_3^{s_1}}^{'}\varphi_{m_3^{s_1}} + f_{m_4^{s_1}}^{'}\varphi_{m_4^{s_1}} + f_{m_6^{s_1}}^{'}\varphi_{m_6^{s_1}} \quad (4.3\text{-}149)$$

By considering that the mortar is the *reference* orthotropic homogeneous material, it is:

$$\varphi_{m_1^{s_1}} = \varphi_{m_3^{s_1}} = \varphi_{m_4^{s_1}} = \varphi_{m_6^{s_1}} = 1 \quad (4.3\text{-}150)$$

being the phases m^{s_1} coincident with the mortar, while it will be:

$$\varphi_{b_1^{s_1}} = \varphi_b \quad (4.3\text{-}151)$$

where:

$\varphi_b =$ the brick elastic ratio

Moreover, it can be considered that the constituents, $m_1^{s_1}, m_3^{s_1}$ and $m_6^{s_1}$, of the partition s_1, have the volumetric fractions, weighed upon the volume V_{s_1} of the partition, that are in the following relation:

$$f_{m_3^{s_1}}^{'} = 2 f_{m_1^{s_1}}^{'} = 2 f_{m_6^{s_1}}^{'} \quad (4.3\text{-}152)$$

while, for the constituents $b_1^{s_1}$ and $m_4^{s_1}$, it can be written:

$$f^{'}_{b_1^{s_1}} = f^{'}_{m_4^{s_1}} \tag{4.3-153}$$

So, the equation (4.3-149) can be rewritten in the form:

$$f_r^{'} \mathrm{j}_r = 4 f^{'}_{m_1^{s_1}} + f^{'}_{b_1^{s_1}} \left(1 + \mathrm{j}_b\right) \tag{4.3-154}$$

By comparing the (4.3-148) with the (4.3-137) and by imposing their equivalence, that is:

$$\mathrm{S}_{s_1}^{I} = \overline{\overline{\mathrm{S}}}^{s_1} \tag{4.3-155}$$

it is obtained that:

$$\mathrm{j}_{s_1} = f_r^{'} \mathrm{j}_r \tag{4.3-156}$$

Moreover, because the partitions s_1 and s_5 have the same micro-structure, as already considered in the first equation of the (4.3-133) , it will be:

$$\mathrm{S}_{s_1}^{I} = \overline{\overline{\mathrm{S}}}^{s_1} = \overline{\overline{\mathrm{S}}}^{s_5} = \mathrm{S}_{s_2}^{I} \Rightarrow \mathrm{j}_{s_1} = \mathrm{j}_{s_5} \tag{4.3-157}$$

This consideration yields that:

$$\mathrm{j}_{s_1} = \mathrm{j}_{s_5} = 4 f^{'}_{m_1^{s_1}} + f^{'}_{b_1^{s_1}} \left(1 + \mathrm{j}_b\right) \tag{4.3-158}$$

Then, a similar homogenization process in *y*-direction has been executed for another partition, s_2, shown in the figure 4.9b. Because of the analogy of the procedure, it is here shown the result, only, that is the compliance tensor $\overline{\overline{\mathrm{S}}}^{s_2}$:

$$\overline{\overline{S}}^{s_2}=\begin{bmatrix} \frac{1}{f_u^{'}\varphi_u E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{f_u^{'}\varphi_u E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{f_u^{'}\varphi_u E_1^{(m)}} & 0 & 0 & 0 \\ & \overline{\overline{S}}_{2222}^{s_2} & -\frac{\nu_{23}^{(m)}}{f_u^{'}\varphi_u E_2^{(m)}} & 0 & 0 & 0 \\ & & \frac{1}{f_u^{'}\varphi_u E_3^{(m)}} & 0 & 0 & 0 \\ & & & \overline{\overline{S}}_{3232}^{s_2} & 0 & 0 \\ & Sym & & & \frac{1}{2 f_u^{'}\varphi_u G_{31}^{(m)}} & 0 \\ & & & & & \overline{\overline{S}}_{1212}^{s_2} \end{bmatrix} \quad (4.3\text{-}159)$$

where:

$$f_u^{'}\varphi_u = f_{m_1^{S_2}}^{'}\varphi_{m_1^{S_2}} + f_{b_1^{S_2}}^{'}\varphi_{b_1^{S_2}} + f_{m_3^{S_2}}^{'}\varphi_{m_3^{S_2}} + f_{b_3^{S_2}}^{'}\varphi_{b_3^{S_2}} + f_{m_6^{S_2}}^{'}\varphi_{m_6^{S_2}} \quad (4.3\text{-}160)$$

and:

$$f_u^{'} = \frac{V_u}{V_{S_2}} \qquad with\ \ u = m_1^{S_2},\, b_1^{S_2},\, m_3^{S_2},\, b_3^{S_2},\, m_6^{S_2} \quad (4.3\text{-}161)$$

with:

$f_u^{'} =$ the volumetric fraction of the generic phase “*u*”, weighed upon the volume of the partition s_2.

By considering, again, that the mortar is the *reference* orthotropic homogeneous material, it is:

$$\varphi_{m_1^{S_2}} = \varphi_{m_3^{S_2}} = \varphi_{m_6^{S_2}} = 1 \quad (4.3\text{-}162)$$

being the phases $m_1^{S_2}$, $m_3^{S_2}$ and $m_6^{S_2}$ coincident with the mortar, while it will be:

$$\varphi_{b_1^{S_2}} = \varphi_{b_3^{S_2}} = \varphi_b \quad (4.3\text{-}163)$$

Moreover, it can be considered that the constituents, $m_1^{S_2}, m_3^{S_2}$ and $m_6^{S_2}$, of the partition s_2, have the volumetric fractions, weighed upon the volume V_{S_2} of the partition, that are in the following relation:

$$f'_{m_3^{S_2}} = 2\,f'_{m_1^{S_2}} = 2\,f'_{m_6^{S_2}} \tag{4.3-164}$$

while, for the constituents $b_1^{S_2}$ and $b_3^{S_2}$, it can be written:

$$f'_{b_1^{S_2}} = f'_{b_3^{S_2}} \tag{4.3-165}$$

So, the equation (4.3-160) can be rewritten in the form:

$$f'_u \mathrm{j}_u = 4\,f'_{m_1^{S_2}} + 2\,f'_{b_1^{S_2}} \mathrm{j}_b \tag{4.3-166}$$

By comparing the (4.3-159) with the (4.3-137) and by imposing their equivalence, that is:

$$\mathrm{S}^I_{s_2} = \overline{\overline{\mathrm{S}}}^{s_2} \tag{4.3-167}$$

it is obtained that:

$$\mathrm{j}_{s_2} = f'_u \mathrm{j}_u \tag{4.3-168}$$

Moreover, because the partitions s_2 and s_4 have the same micro-structure, as already considered in the second equation of the (4.3-133) , it will be:

$$\mathrm{S}^I_{s_2} = \overline{\overline{\mathrm{S}}}^{s_2} = \overline{\overline{\mathrm{S}}}^{s_4} = \mathrm{S}^I_{s_4} \Rightarrow \mathrm{j}_{s_2} = \mathrm{j}_{s_4} \tag{4.3-169}$$

This consideration yields that:

$$\mathrm{j}_{s_2} = \mathrm{j}_{s_4} = 4\,f'_{m_1^{S_2}} + 2\,f'_{b_1^{S_2}} \mathrm{j}_b \tag{4.3-170}$$

Finally, the same homogenization process in *y*-direction has been executed for the partition s_3, shown in the figure 4.9c. The obtained compliance tensor $\overline{\overline{\mathrm{S}}}^{s_3}$ is:

$$\overline{\overline{S}}^{s_3} = \begin{bmatrix} \frac{1}{f_v^{'}\varphi_v E_1^{(m)}} & -\frac{\nu_{12}^{(m)}}{f_v^{'}\varphi_v E_1^{(m)}} & -\frac{\nu_{13}^{(m)}}{f_v^{'}\varphi_v E_1^{(m)}} & 0 & 0 & 0 \\ & \overline{\overline{S}}_{2222}^{s_3} & -\frac{\nu_{23}^{(m)}}{f_v^{'}\varphi_v E_2^{(m)}} & 0 & 0 & 0 \\ & & \frac{1}{f_v^{'}\varphi_v E_3^{(m)}} & 0 & 0 & 0 \\ & & & \overline{\overline{S}}_{3232}^{s_3} & 0 & 0 \\ & Sym & & & \frac{1}{2 f_v^{'}\varphi_v G_{31}^{(m)}} & 0 \\ & & & & & \overline{\overline{S}}_{1212}^{s_3} \end{bmatrix} \quad (4.3\text{-}171)$$

where:

$$f_v^{'}\varphi_v = f_{m_1^{S_3}}^{'}\varphi_{m_1^{S_3}} + f_{m_2^{S_3}}^{'}\varphi_{m_2^{S_3}} + f_{m_3^{S_3}}^{'}\varphi_{m_3^{S_3}} + f_{b_3^{S_3}}^{'}\varphi_{b_3^{S_3}} + f_{m_6^{S_3}}^{'}\varphi_{m_6^{S_3}} \quad (4.3\text{-}172)$$

and:

$$f_v^{'} = \frac{V_v}{V_{S_3}} \qquad with \;\; v = m_1^{S_3},\, m_2^{S_3},\, m_3^{S_3},\, b_3^{S_3},\, m_6^{S_3} \quad (4.3\text{-}173)$$

with:

$f_v^{'} =$ the volumetric fraction of the generic phase "*v*", weighed upon the volume of the partition s_3.

Again, it is possible to write:

$$\varphi_{m_1^{S_3}} = \varphi_{m_2^{S_3}} = \varphi_{m_3^{S_3}} = \varphi_{m_6^{S_3}} = 1 \quad (4.3\text{-}174)$$

being the phases $m_1^{S_3}$, $m_2^{S_3}$, $m_3^{S_3}$ and $m_6^{S_3}$ coincident with the mortar, while it will be:

$$\varphi_{b_3^{S_3}} = \varphi_b \quad (4.3\text{-}175)$$

Analogously to the previous case, it can be considered that the constituents, $m_1^{S_3}, m_3^{S_3}$ and $m_6^{S_3}$, of the partition s_3, have the volumetric fractions, weighed upon the volume V_{S_3} of the partition, that are in the following relation:

$$f'_{m_3^{S_3}} = 2 f'_{m_1^{S_3}} = 2 f'_{m_6^{S_3}} \tag{4.3-176}$$

while, for the constituents $m_2^{S_3}$ and $b_3^{S_3}$, it can be written:

$$f'_{m_2^{S_3}} = f'_{b_3^{S_3}} \tag{4.3-177}$$

So, the equation (4.3-172) can be rewritten in the form:

$$f'_v \mathrm{j}_v = 4 f'_{m_1^{S_3}} + f'_{b_3^{S_3}} \left(1 + \mathrm{j}_b\right) \tag{4.3-178}$$

Because, it is:

$$\begin{aligned} f'_{m_1^{S_3}} &= f'_{m_1^{S1}} \\ f'_{b_3^{S_3}} &= f'_{b_1^{S1}} \end{aligned} \tag{4.3-179}$$

it is obtained that:

$$f'_v \mathrm{j}_v = f'_r \mathrm{j}_r \tag{4.3-180}$$

By comparing the (4.3-171) with the (4.3-137) and by imposing their equivalence, that is:

$$\mathrm{S}_{S_3}^{I} = \overline{\overline{\mathrm{S}}}^{S_3} \tag{4.3-181}$$

it is obtained that:

$$\mathrm{j}_{S_3} = f'_v \mathrm{j}_v \tag{4.3-182}$$

and so:

$$\mathrm{j}_{S_3} = \mathrm{j}_{S_1} = \mathrm{j}_{S_5} = 4 f'_{m_1^{S1}} + f'_{b_3^{S1}} \left(1 + \mathrm{j}_b\right) \tag{4.3-183}$$

By substituting the (4.3-158), the (4.3-170) and the (4.3-183) in the equation (4.3-134), and by operating some manipulation according to the definition of the involved volumetric fractions , it is reached that:

$$f_q \varphi_q = 16 \frac{V_{m_1^{S1}}}{V} + 4 \frac{V_{b_1^{S1}}}{V} \left(1 + \varphi_b\right) + 8 \frac{V_{m_1^{S2}}}{V} + 4 \frac{V_{b_1^{S2}}}{V} \varphi_b \qquad (4.3\text{-}184)$$

So, according to the geometry of the RVE, it can be written that:

$$\begin{aligned} 16 V_{m_1^{S1}} + 8 V_{m_1^{S2}} &= V_{m_{orizz}} \\ 4 V_{b_1^{S1}} = 4 V_{m_4^{S1}} &= V_{m_{vert}} \\ 4 V_{b_1^{S1}} + 4 V_{b_1^{S2}} &= V_b \end{aligned} \qquad (4.3\text{-}185)$$

where:

$V_{m_{orizz}} =$ the volume of the horizontal mortar in the RVE.

$V_{m_{vert}} =$ the volume of the vertical mortar in the RVE.

$V_b =$ the volume of the bricks in the RVE.

$V =$ the volume of the representative element.

So, the equation (4.3-184) can be rewritten in the form:

$$f_q \varphi_q = \frac{V_{m_{orizz}}}{V} + \frac{V_{m_{vert}}}{V} + \frac{V_b}{V} \varphi_b \qquad (4.3\text{-}186)$$

By considering that:

$$
\begin{aligned}
&f_m = f_{m_{orizz}} + f_{m_{vert}} \\
&f_{m_{vert}} = \frac{V_{m_{vert}}}{V} \\
&f_{m_{orizz}} = \frac{V_{m_{orizz}}}{V} \\
&f_b = \frac{V_b}{V}
\end{aligned}
\tag{4.3-187}
$$

where:

$f_m =$ the volumetric fraction of the mortar, weighed upon the volume V of the RVE.

$f_b =$ the volumetric fraction of the brick, weighed upon the volume V of the RVE.

The equation (4.3-186) becomes:

$$
f_q \varphi_q = f_m + f_b \varphi_b \tag{4.3-188}
$$

At this point, it is possible to write the homogenized compliance tensor of the RVE as it follows:

$$
\bar{\bar{S}}^{I} = \begin{bmatrix}
\bar{\bar{S}}^{I}_{1111} & -\dfrac{\nu_{12}^{(m)}}{E_1^{(m)}\varphi} & -\dfrac{\nu_{13}^{(m)}}{E_1^{(m)}\varphi} & 0 & 0 & 0 \\
 & \dfrac{1}{E_2^{(m)}\varphi} & -\dfrac{\nu_{23}^{(m)}}{E_2^{(m)}\varphi} & 0 & 0 & 0 \\
 & & \dfrac{1}{E_3^{(m)}\varphi} & 0 & 0 & 0 \\
 & & & \dfrac{1}{2G_{32}^{(m)}\varphi} & 0 & 0 \\
 & Sym & & & \bar{\bar{S}}^{I}_{3131} & 0 \\
 & & & & & \bar{\bar{S}}^{I}_{1212}
\end{bmatrix} \tag{4.3-189}
$$

where $\mathrm{f} = \left(f_m + f_b \mathrm{j}_b \right)$.

By comparing the equation (4.3-118) with the equation (4.3-118), it is noted that the found elastic coefficients of the homogenized compliance tensor obtained after the homogenization process $y \rightarrow x$ are equal to those ones of the homogenized compliance tensor obtained after the homogenization process $x \rightarrow y$. This means that the proposed two-step homogenization appears to be a consistent procedure, which, differently from the standard two-step homogenization approaches found in literature, doesn't lead to results depending on the order of the step execution.

So, by unifying the two results, a more complete homogenized compliance tensor is obtained:

$$\mathrm{S}^{Hom} = \begin{bmatrix} \frac{1}{E_1^{(m)}\mathrm{f}} & -\frac{\mathrm{n}_{12}^{(m)}}{E_1^{(m)}\mathrm{f}} & -\frac{\mathrm{n}_{13}^{(m)}}{E_1^{(m)}\mathrm{f}} & 0 & 0 & 0 \\ & \frac{1}{E_2^{(m)}\mathrm{f}} & -\frac{\mathrm{n}_{23}^{(m)}}{E_2^{(m)}\mathrm{f}} & 0 & 0 & 0 \\ & & \frac{1}{E_3^{(m)}\mathrm{f}} & 0 & 0 & 0 \\ & & & \frac{1}{2G_{32}^{(m)}\mathrm{f}} & 0 & 0 \\ & Sym & & & \frac{1}{2G_{31}^{(m)}\mathrm{f}} & 0 \\ & & & & & \mathrm{S}_{1212}^{Hom} \end{bmatrix} \quad (4.3\text{-}190)$$

with:

$\mathrm{S}^{Hom} =$ the found complete homogenized compliance tensor.

It has to be underlined that there is a sole elastic coefficient remaining unknown.

By comparing the proposed homogenization technique with Pietruszczak & Niu’s one, it can be said that:

***PIETRUSZCZAK & NIU APPROACH* -** implies an approximated homogenization procedure in two steps, whose results are dependent on the sequence of the steps chosen. It represents the limit of this kind of the existing approaches.

***S.A.S. APPROACH* -** employs a parametric homogenization which results consistent in the two-step process, by implying exact solutions in some direction. Hence, the proposed procedure overcomes the limit of the simplified approaches. Moreover, it has to be underlined, yet, the simplicity of the procedure which yields to obtain the homogenized compliance tensor from that one of the reference homogeneous material, by multiplying for a scalar factor f^{-1} depending on the geometry of the micro-constituents and on the elastic ratio, j_b.

CHAPTER V

Remarks on finite element method (F.E.M.)

5.1 Introduction

The procedure of subdividing a complex system into its components, or elements, whose behaviour is more easily described, represents a natural path followed in every science branch, as well as the engineering one.

Such a treatment, defined as "*discrete problems*", is often used in order to overcome the difficult solution of the "*continuous problems*", where a complex mathematical continuous model is hold by local differential equations. However, both mathematics and engineers have developed general techniques that are directly applicable to the differential equations of the "*continuous problems*", as well as:

- approximations to the finite differences.
- weighted residual techniques

- appropriate techniques on the determination of the stationariness of some functional.

On the other side, engineers study the problem by establishing an analogy between the elements of the discrete model and the portions of the continuous domains. For example, in the solids mechanics area, Mc Henry, Hremikoff e Newmark have shown, in the early fifty years, that suitable solutions to the elastic problem of the continuum can be obtained by means of substitution of little portions of such medium with an assembly of simple elastic beams. Later, Argyris and Turner also demonstrated that the mechanical behaviour of the continuum can be obtained by analyzing the elements in which it is subdivided.

With the use of personal computers, furthermore, the “discrete problems” are easily solved even if the number of elements, necessary to obtain a suitable model, is enough great.

The term “*finite element*” was born for direct analogy.

The goal of this chapter is to show that the finite element method corresponds to a continuum discretization, based on consistent mathematical models.

Standard methodologies are developed, in the last years, in the analysis of discrete problems. The civil engineering, for the structures, first estimates the relations between forces and displacements for each element of the structure and then provides to assembly the whole system by means of a well defined procedure: it requires establishing the local equilibrium for each node or each connection point of the structure. The solution of the unknown displacements becomes, so, feasible.

It is possible to define some systems of standard discretization. The existence of a unified treatment of the *discretization standard problem* allows us to define the finite element procedure as an approximation method for the continuous problems, so that:

- the continuum is divided into a finite number of parts (elements) whose behaviour is individualized by a finite number of parameters
- the solution of the whole system is obtained by assembling its single constituent elements.

5.2 Structural elements and systems

In order to introduce the general concept of the discrete systems, it is initially considered, for example, the structure of figure 5.1 with a linear mechanical behaviour, [59].

The connexions are given by hinges, so that the moments cannot be transferred. It is assumed that, from pulled apart calculations, the characteristics of each element are exactly known. Hence, if a typical element, marked with (1) and associated to the nodes 1, 2, 3, is analyzed, the forces acting on such nodes are univocally defined by the same nodal displacements. Both forces and displacements are defined by appropriate components (*U, V, u, v*) in a global coordinate system. The distributed load is named *p*. Furthermore, it is presumed an initial deformation, for example due to a thermal variation.

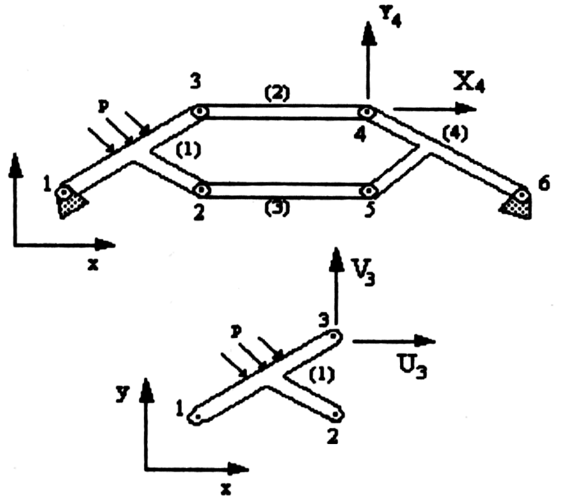


Figure 5.1 Typical structure constituted from interconnected elements.

By listing the forces acting on all nodes (in the examined case, node 3) of the elements (in the examined case, element 1) in matrix form, it is obtained:

$$\boldsymbol{q}^1 = \begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \end{Bmatrix} \qquad \boldsymbol{q}_1^1 = \begin{Bmatrix} U_1 \\ V_1 \end{Bmatrix}, \ etc \tag{5.2-1}$$

and for the corresponding nodal displacements:

$$\boldsymbol{a}^1 = \begin{Bmatrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{Bmatrix} \qquad \boldsymbol{a}_1^1 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}, \ etc \tag{5.2-2}$$

By considering a linear-elastic behaviour of the element, the characteristic relations assume the form:

$$\mathbf{q}^1 = \mathbf{K}^1\mathbf{a}^1 + \mathbf{f}_\mathbf{p}^1 + \mathbf{f}_{\mathrm{e}_\mathbf{o}}^1 \tag{5.2-3}$$

where:

$\boldsymbol{f}_P^1 =$ the nodal forces necessary to balance the distributions of acting loads on the element.

$\boldsymbol{f}_{\mathrm{e}_0}^1 =$ the nodal forces necessary to balance the reactions due, for example, to thermal strains.

$\boldsymbol{K}^1\boldsymbol{a}^1 =$ the nodal forces due to the nodal displacements.

At the same manner, the preliminary analysis lets to define a unique distribution of stresses and internal reactions in a specific point, in terms of nodal displacements.

Hence, the stresses are defined by means of a matrix S^1 and relations having the following form:

$$\mathsf{S}^1 = \mathbf{S}^1\mathbf{a}^1 + \mathsf{S}_\mathbf{p}^1 + \mathsf{S}_\mathbf{eo}^1 \tag{5.2-4}$$

where the last two terms are, respectively, the stresses due to the distribution of load on the element and the stresses due to the initial strains when the displacement results to be constrained.

The matrix $\mathbf{K}^\mathbf{e}$ and the matrix $\mathbf{S}^\mathbf{e}$ are known, respectively, as stiffness matrix and stress matrix of the element.

The relations (5.2-3) and (5.2-4) have been illustrated in an example of three nodes element, with interconnection points that are able to transfer only two force components. However, the same considerations and the same definitions can be applied to a general case.

The element 2 has only two interconnection points, but in general it is possible to have a higher number of such points. Moreover, if the connections are rigid and built-in, the three components of the generalized forces and of the generalized displacements corresponding to moments and rotations, respectively, have to be considered. For rigid connections in three-dimensional structures, the number of components for each node is six.

$$\mathbf{q^e} = \begin{Bmatrix} \mathbf{q}_1^\mathbf{e} \\ . \\ . \\ \mathbf{q}_\mathbf{m}^\mathbf{e} \end{Bmatrix} \qquad \text{and} \qquad \mathbf{a^e} = \begin{Bmatrix} \mathbf{a}_1^\mathbf{e} \\ . \\ . \\ \mathbf{a}_\mathbf{m}^\mathbf{e} \end{Bmatrix} \tag{5.2-5}$$

where $\mathbf{q}_\mathbf{i}^\mathbf{e}$ and $\mathbf{a}_\mathbf{i}^\mathbf{e}$ have the same components number of the freedom grades. These quantities are connected the ones with the others.

The stiffness element matrix is always a square matrix and it assumes the form:

$$\mathbf{K^e} = \begin{bmatrix} \mathbf{K}_\mathbf{ii}^\mathbf{e} & \mathbf{K}_\mathbf{ij}^\mathbf{e} & \dots & \mathbf{K}_\mathbf{im}^\mathbf{e} \\ . & . & & . \\ . & . & & . \\ \mathbf{K}_\mathbf{mi}^\mathbf{e} & \dots & \dots & \mathbf{K}_\mathbf{mm}^\mathbf{e} \end{bmatrix} \tag{5.2-6}$$

where:

$\mathbf{K}_\mathbf{ii}^\mathbf{e}$ = are square submatrices *l x l*, with *1* the number of force components which have to be considered at nodes.

For example, it can be considered a hinged beam, having an uniform section *A* and Young's modulus *E*, in a two-dimensional problem, as it is shown in the following figure.

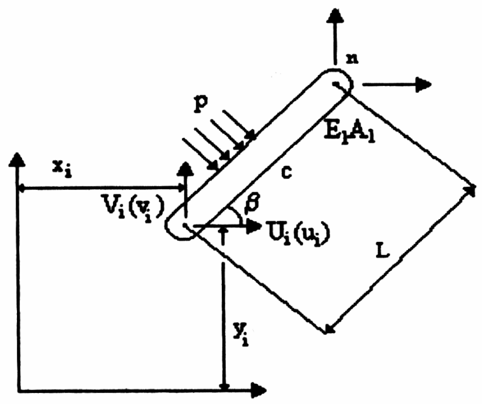


Figure 5.2 Hinged beam.

The examined beam is subjected to an uniform lateral load p and to a uniform thermal deformation:

$$\varepsilon_o=\alpha T \tag{5.2-7}$$

By denoting with x_i , y_i and x_n , y_n the extreme nodes coordinates, the beam length is defined as:

$$L = \sqrt{[(x_n - x_i)^2 + (y_n - y_i)^2]} \tag{5.2-8}$$

and its slope with regards to the horizontal axis as:

$$b = \tan^{-1} \frac{y_n - y_i}{x_n - x_i} \tag{5.2-9}$$

At nodes, only two components of the forces and of displacements have to be considered.

The nodal forces due to lateral loads are:

$$\mathbf{f}_{\mathbf{p}}^{\mathbf{e}} = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix} = -\begin{Bmatrix} -\text{sen}\,\mathrm{b} \\ \cos\mathrm{b} \\ -\text{sen}\,\mathrm{b} \\ \cos\mathrm{b} \end{Bmatrix} \frac{pL}{2} \tag{5.2-10}$$

and represent the components of the reactions on the beam, *pL/2*.

In order to block the thermal strain e_o , an axial force is necessary *(E*a*TA)*, whose components are:

$$\mathbf{f}_{\mathrm{e}_0}^{\mathbf{e}} = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix} = -\begin{Bmatrix} -\cos\mathrm{b} \\ -\text{sen}\,\mathrm{b} \\ \cos\mathrm{b} \\ \text{sen}\,\mathrm{b} \end{Bmatrix} (E\mathrm{a}\,TA) \tag{5.2-11}$$

The element displacements are, finally:

$$\mathbf{a}^{\mathbf{e}} = \begin{Bmatrix} u_i \\ v_i \\ u_n \\ v_n \end{Bmatrix} \tag{5.2-12}$$

They cause an elongation equal to $(u_n - u_i)\cos\mathrm{b} + (v_n - v_i)\text{sen}\,\mathrm{b}$. This one, multiplied for *EA/L*, yields the axial forces whose components can be found again.

By using a matrix notation, it s obtained:

$$\mathbf{K}^{\mathbf{e}}\mathbf{a}^{\mathbf{e}} = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix} =$$

$$= \frac{EA}{L}\begin{bmatrix} \cos^2 b & \text{sen}\, b \cos b & -\cos^2 b & -\text{sen}\, b \cos b \\ \text{sen}\, b \cos b & \text{sen}^2 b & -\text{sen}\, b \cos b & -\text{sen}^2 b \\ -\cos^2 b & -\text{sen}\, b \cos b & \cos^2 b & \text{sen}\, b \cos b \\ -\text{sen}\, b \cos b & -\text{sen}^2 b & \text{sen}\, b \cos b & \text{sen}^2 b \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_n \\ v_n \end{Bmatrix} \qquad (5.2\text{-}13)$$

The general components of the equation (5.2-3) have been established by means of the analysis of an elementary case. It is very simple to obtain the stress in a general element section in the form given by (5.2-4).

For example, if our attention is focused on the middle section C of the beam, the extreme stress in the fibres is determined by the axial forces and bending moments acting on the element. By using the matrix notation, it can be written:

$$\begin{aligned} s_C^e &= \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix}_C \\ &= \frac{E}{L}\begin{bmatrix} -\cos b & -\text{sen}\, b & \cos b & \text{sen}\, b \\ -\cos b & -\text{sen}\, b & \cos b & \text{sen}\, b \end{bmatrix} a^e + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \frac{pL^2}{8}\frac{d}{I} - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} E a T \end{aligned} \qquad (5.2\text{-}14)$$

where:

d = half deepness of the section.

I = inertial moment of the area.

All the terms in the (5.2-4) can be easily known.

For more complex elements, more advanced analytical procedures have been required, but the results are formally identical.

It is worth to notice that the complete stiffness matrix, obtained for the simple examined element, results to be symmetric. This is the consequence of the energy conservation and its corollaries (the well known Maxwell-Betti theorem).

The properties of the elements are assumed by considering simple linear relations. Generally, similar relations could be also established for non-linear materials.

5.3 Assembly and analysis of a structure

Let us consider the whole structure of the figure 5.1. In order to obtain the complete solution, both the following conditions have to be satisfied:

a- compatibility

b- equilibrium

A general system of nodal displacements **a**, having the form:

$$\mathbf{a} = \begin{Bmatrix} \mathbf{a}_1 \\ . \\ . \\ \mathbf{a}_\mathbf{n} \end{Bmatrix} \tag{5.3-1}$$

and built by taking into account all the structure elements, satisfies automatically the first condition.

In this way, the equilibrium condition within the single element is satisfied, while it is necessary to establish the equilibrium condition at structure nodes. The resulting equations will carry the unknown displacements, so that the structural problem is determined after founding such displacements.

The internal elements forces or the stresses can be easily attained from the equation (5.2-4) by using priori-established characteristics for each element.

Let us consider that the structure is loaded with external nodal forces **r**, in addition to distribute loads acting on the single elements.

$$\mathbf{r} = \begin{Bmatrix} \mathbf{r}_1 \\ . \\ . \\ \mathbf{r}_\mathbf{n} \end{Bmatrix} \tag{5.3-2}$$

The generic force $\mathbf{r_i}$, moreover, must have the same number of components than those ones of the reactions of the examined elements.

For example, in this case, since the hypothesis of hinged nodes has been done, it is:

$$\mathbf{r_i} = \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} \tag{5.3-3}$$

However, in order to generalize the problem, an arbitrary number of components are taken into consideration.

If the equilibrium conditions for a general node i are imposed, each $\mathbf{r_i}$ component is equal to the summation of the components of the forces acting on the elements concurred in the node.

Hence, by considering all the components of the forces, it is obtained:

$$\mathbf{r_i} = \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{q_i^e} = \mathbf{q_i^1} + \mathbf{q_i^2} + \ldots \tag{5.3-4}$$

where:

$\mathbf{q_i^1}$ = the force contribute to the node i from the element 1.

$\mathbf{q_i^2}$ = the force contribute to the node i from the element 2.

Only the elements concurred in the node evidently give a non-zero contribute to the forces. For not losing in generality, the summation is here thought to be extended to all elements.

By substituting in the equation (5.2-3) the forces contribute to the nodes *i* and by noting that the nodal variables $\mathbf{a_i}$ are common (so that the index *e* can be omitted), it is obtained:

$$\mathbf{r_i} = \left(\sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{K}_{\mathbf{i}1}^{\mathbf{e}} \right) \mathbf{a}_1 + \left(\sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{K}_{\mathbf{i}2}^{\mathbf{e}} \right) \mathbf{a}_2 + \ldots + \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{f}_{\mathbf{i}}^{\mathbf{e}} \qquad (5.3\text{-}5)$$

where:

$$\mathbf{f}^{\mathbf{e}} = \mathbf{f}_{\mathbf{p}}^{\mathbf{e}} + \mathbf{f}_{\mathbf{eo}}^{\mathbf{e}} \qquad (5.3\text{-}6)$$

and where the summation is again pertained to the sole elements concurred in the node *i*.

By assembling the equations, relative to all nodes, it is simply obtained:

$$\mathbf{Ka} = \mathbf{r} - \mathbf{f} \qquad (5.3\text{-}7)$$

where the submatrices are:

$$\mathbf{K_{ij}} = \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{K}_{\mathbf{ij}}^{\mathbf{e}} \qquad (5.3\text{-}8)$$

$$\mathbf{f_i} = \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{f}_{\mathbf{i}}^{\mathbf{e}} \qquad (5.3\text{-}9)$$

and where the summation includes all elements.

Such a rule for assembling is very suitable because as soon as a coefficient is determined, for a typical element, this one can be immediately introduced in its own location within the global stiffness matrix of the structure.

This general process can be easily extended and generalized to any process which adopts the finite elements methodology.

It is worth to be noted, moreover, that the structure is constituted by different elements and that, in order to carry out the matrix summation, all the matrices must have the same dimensions. Furthermore, the single matrices to sum have to be constructed with the same number of components of forces and displacements. For example, if a force component is able to transfer moments at a node and if another hinged node is coupled with it, it is necessary to complete the stiffness matrix by inserting appropriate null coefficients corresponding to rotations or moments.

5.4 Boundary conditions

The system of equations resulting from the (5.3-7) can be solved afterwards having substituted the pre-determined displacement field. In the example of the figure 5.1, where both the displacement components of nodes 1 and 6 are equal to zero, this means the substitution of:

$$\mathbf{a}_1 = \mathbf{a}_6 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.4-1}$$

This is equivalent to reduce the number of equilibrium equations (12 for this case) by deleting the first and the last couple and, so, by reducing to eight the number of unknown displacements.

It is always suitable, nevertheless, to include all nodes when assembling the equations according to the relations (5.3-7). Obviously, without a number minimum of constrained displacements, (that is a number minimum of constrains which blocks the rigid displacement of the structure), it is not possible to solve the system, since the displacements cannot univocally be determined. Such an obvious physical problem can be mathematically read in the fact that the matrix **K** becomes singular and has not an inverse matrix.

The assignment of suitable displacements, after the phase of the assembly, allows obtaining a unique solution by deleting suitable rows and columns from the various matrices.

If all the equations of a system are assembled, the assumed form is the following one:

$$\begin{aligned} \mathbf{K}_{11}\mathbf{a}_1 + \mathbf{K}_{12}\mathbf{a}_2 + \ldots = \mathbf{r}_1 - \mathbf{f}_1 \\ \mathbf{K}_{21}\mathbf{a}_1 + \mathbf{K}_{22}\mathbf{a}_2 + \ldots = \mathbf{r}_2 - \mathbf{f}_2 \end{aligned} \tag{5.4-2}$$

It is well known that, if some displacements as $\mathbf{a}_1 = \bar{\mathbf{a}}_1$ are set, the external forces $\mathbf{r}_1$ cannot be set and they remain unknown. The first equation can be deleted and the $\bar{\mathbf{a}}_1$ value can be substituted in the remaining equations.

Such a computational process is uncomfortable and the same result can be reached by adding a very large number, $\mathrm{a}\mathbf{I}$, to the coefficient $\mathbf{K}_{11}$ and then by rectifying with $\bar{\mathbf{a}}_1\mathrm{a}$ the right member of the equation $\mathbf{r}_1 - \mathbf{f}_1$.

If a is quite greater than the other stiffness coefficients, such a correction really substitutes the first equation of the (5.4-2) with the following one:

$$\mathrm{a}\,\mathbf{a}_1 = \mathrm{a}\,\bar{\mathbf{a}}_1 \tag{5.4-3}$$

which represents the required condition.

The whole system remains symmetric and only few and little changes are necessary in the computational sequence. Such a procedure is employed for the assigned displacements. It was introduced by Payne and Irons.

When all the boundary conditions have been introduced, the system equations can be solved according to unknown displacements and strains. Hence, the internal forces for each element are obtained.

5.5 General model

In order to straighten the topic discussed in this chapter, we consider an example where five elements are interconnected, as shown in the figure 5.3.

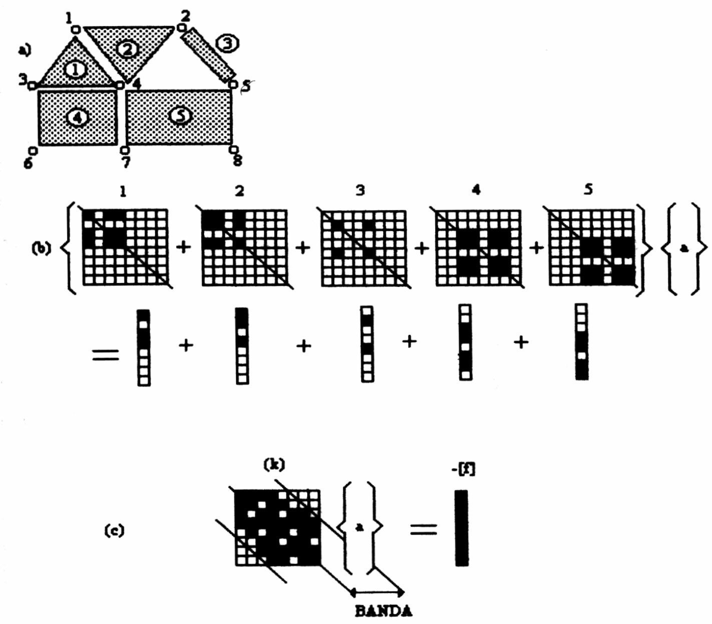


Figure 5.3 An assembly example for the stiffness matrix.

The first step is to determine the element properties from geometry and loads.

For each element, the stiffness matrix, and so the corresponding nodal forces, are found in the form given by the (5.2-3). Each element is identified by its own number and by nodal connections. For example:

element 1	connection	1 3 4
element 2		1 4 2
element 3		5 2
element 4		3 6 7 4
element 5		4 7 8 5

Tabe 5.1 Nodal connections.

By assuming that the properties are evaluated in a global reference system, the stiffness values and the forces can be introduced in their location in the global matrix (see fig. 5.3b). Each shaded square represents a single coefficient or submatrix of a kind $\mathbf{K}_{\mathbf{ij}}$.

The second step is the assembly of the final equation. According to the equation (5.3-8), this is obtained by means of a simple summation of all numbers set in the apposite spaces of the global matrix.

The result is shown in the figure 5.3c, where the coefficients are blackened.

All the non-zero coefficients are edged within the BAND, which can be a priori-calculated from the nodal connections.

In a computer programming, only the elements located to one side of the diagonal have to be memorized.

The third step is the input of the boundary conditions in the assembled matrix.

The fourth step is the solution of the system with any methodology.

5.6 The systems of standard discretization

In the standard discretization systems, either structural ones or other different ones, it is worth:

1) A set of parameters, called $\mathbf{a_i}$, can be detected. These parameters simultaneously describe the behaviour of each element **e** and of the whole system. They are called *system parameters*.
2) For each element, a set of quantities $\mathbf{q_i^e}$ can be calculated, in function of the system paramaters $\mathbf{a_i}$, as shown in the following equation:

$$\mathbf{q_i^e} = \mathbf{q_i^e}(\mathbf{a}) \qquad (5.6\text{-}1)$$

Such relations can generally be non-linear, but in a lot of cases they assume a linear form, of a kind:

$$\mathbf{q}_\mathbf{i}^\mathbf{e} = \mathbf{K}_{i1}\mathbf{a}_1 + \mathbf{K}_{i2}\mathbf{a}_2 + \ldots + \mathbf{f}_\mathbf{i}^\mathbf{e} \tag{5.6-2}$$

3) The system of equations is obtained by a simple sum:

$$\mathbf{r}_\mathbf{i} = \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{q}_\mathbf{i}^\mathbf{e} \tag{5.6-3}$$

In the linear case, it is obtained:

$$\mathbf{Ka} + \mathbf{f} = \mathbf{r} \tag{5.6-4}$$

so that:

$$\begin{aligned} \mathbf{K}_{\mathbf{ij}} &= \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{K}_{\mathbf{ij}}^{\mathbf{e}} \\ \mathbf{f}_\mathbf{i} &= \sum_{\mathbf{e}=1}^{\mathbf{m}} \mathbf{f}_\mathbf{i}^\mathbf{e} \end{aligned} \tag{5.6-5}$$

From such system, after the suitable boundary conditions have been imposed, the solutions in the variables **a** are found.

It is noticed that these statements are very general and they include structural problems, hydraulic or electronic ones, etc. Generally, there is neither linearity nor the symmetry of the matrices, even if both linearity and symmetry naturally come in a lot of problems.

5.7 Coordinate transformations

It is often suitable to establish the characteristics of a single element in a coordinate system different from the one in which the external forces and displacements of the assembled structure will be measured.

A new coordinate system can be used for each element, so it becomes a simple problem of transformation of the force and displacement components contained in equation (5.2-3) in another coordinate system.

Obviously, the passage from the local reference system to the global one has to be carried out before to employ the assembly.

The local coordinate system in which the element properties have been evaluated is marked by the superscript ′.

The displacement components can be transformed through a suitable matrix of the direction cosines **L** with:

$$\mathbf{a}' = \mathbf{L}\mathbf{a} \tag{5.7-1}$$

The corresponding force components have to carry out the same work in the systems:

$$\mathbf{q}^{\mathbf{T}}\mathbf{a} = \mathbf{q}'^{\mathbf{T}}\,\mathbf{a}' \tag{5.7-2}$$

By considering the (5.7-1), it is obtained:

$$\mathbf{q}^{\mathbf{T}}\mathbf{a} = \mathbf{q}'^{\mathbf{T}}\,\mathbf{L}\mathbf{a} \tag{5.7-3}$$

or

$$\mathbf{q} = \mathbf{L}^{\mathbf{T}}\mathbf{q}' \tag{5.7-4}$$

The set of the transformations given by the (5.7-1), (5.7-2), (5.7-3) and (5.7-4) is called *controvariant.*

The stiffness matrix could also be obtained in the local reference system, and so opportunely transformed. It can be, therefore, written:

$$\mathbf{q}' = \mathbf{K}'\mathbf{a}' \tag{5.7-5}$$

From the (5.7-1), (5.7-3), (5.7-4) and (5.7-5), it is obtained:

$$\mathbf{q} = \mathbf{L}^{\mathbf{T}}\mathbf{K}'\mathbf{L}\mathbf{a} \tag{5.7-6}$$

or in the global coordinate system:

$$\mathbf{K} = \mathbf{L}^{\mathbf{T}}\mathbf{K}'\mathbf{L} \tag{5.7-7}$$

The above mentioned transformation results to be very useful. This fact can be verified by calculating the stiffness values of the previous example (structure with hinged nodes) in the local reference system.

Generally, the problem is to substitute a group of parameters **a**, in which the system of equations has been written, with another one **b** by means of a transformation matrix T. Hence, it is obtained:

$$\mathbf{a} = \mathbf{T}\mathbf{b} \tag{5.7-8}$$

In the linear problem, the system of equations assumes the following form:

$$\mathbf{K}\mathbf{a} = \mathbf{r} - \mathbf{f} \tag{5.7-9}$$

and, by substituting, it can be written:

$$\mathbf{K}\mathbf{T}\mathbf{b} = \mathbf{r} - \mathbf{f} \tag{5.7-10}$$

By pre-multiplying for $\mathbf{T}^{\mathbf{T}}$, it is obtained a new system:

$$\left(\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}\right)\mathbf{b} = \mathbf{T}^{\mathrm{T}}\mathbf{r} - \mathbf{T}^{\mathrm{T}}\mathbf{f} \tag{5.7-11}$$

where the equations symmetry is preserved if the matrix **K** is symmetric.

However, sometimes the matrix **T** is not a square matrix and the (5.7-8) represents an approximation in which a large number of parameters *a* is constrained.

Evidently, the system of equation (5.7-10) has more equations than those closely necessary in order to solve the transformed group of parameters *b* and the final expression (5.7-11) shows a reduced system which rounds the original one.

5.8 General concepts

The approximation process of a continuum behaviour by means of “finite elements” (whose behaviour is quite similar to the actual structure make “discrete”) was introduced, in the first place, on mechanical structures.

In a lot of engineering problems, the solution of the stress and strain distributions in an elastic continuum is required and different problems can be encountered: plane stress field, plane strain field, solids with an axial symmetry, plates, shells, three-dimensional solids and so on.

In all cases, the number of interconnections between each finite element and its contiguous ones, through imaginary boundaries, is infinite. For this reason, it is very difficult to understand, in a first approach, how some problems may be discretized.

Such a difficulty, however, can be overcome in the following manner:

1. The continuum is divided, by imaginary lines or areas, in a finite number of elements.
2. The elements are assumed to be interconnected by means of a discrete number of nodal points located on their boundary. The nodal displacements are the unknown quantities of the problem.
3. A set of functions is selected to univocally define the displacement field within each “finite element” in terms of the nodal displacements.
4. The displacement functions univocally define the strain field within the element in terms of nodal displacements. Such strain field, together with some initial strains and with material properties, define the stress state in the element and on its boundary.
5. A force system, acting on nodes, in equilibrium with the boundary tractions and some distribute loads, is obtained by means of the stiffness relation (5.2-3).

Finally, the solving procedures follow the general models described in the previous paragraphs.

Obviously, in a first approach, it is not always easy to select displacement functions which satisfy the requirement of continuous displacements between contiguous elements. This means that the compatibility conditions on boundary can be violated. On the contrary, the compatibility conditions within each element are obviously satisfied, due to the uniqueness of the displacement underlying their continuous representation.

Furthermore, the equilibrium conditions are satisfied only in a global form, due to the equivalent point-wise forces at nodes. Hence, local violations to the equilibrium conditions can rise within the element and its boundary.

The choice of the element form and of the displacement functions form is depending by the engineer's genius and, evidently, the approximation degree in the results is strongly dependent by this choice.

Such a till now described approach is known as *displacement formulation.* It is equivalent to minimize the total potential energy of the system in terms of an assigned displacement field. The right definition of such displacement field provides a convergence in the results.

The acknowledgment of the equivalence between the finite element method and a minimization procedure for the total potential energy has been guessed late.

However, Courant in 1934 and Prager at Synge in 1947 suggested methods essentially identical.

It is worth to notice that the finite element method allows to be extended to various continuum problems where it is possible to have variational formulations.

5.9 Direct formulation of the Finite Element Method

In this paragraph, some indications are given in a more detailed mathematical form, in order to obtain the characteristics of a finite element. It is preferred to obtain the results in a generalized form, which is, so, applicable to various situations.

In order to avoid difficult concepts, the general relations are illustrated by means of very simple examples regarding the analysis of plane stress fields for a thin structure.

A general region is divided in triangular elements, as shown in the figure 5.4.

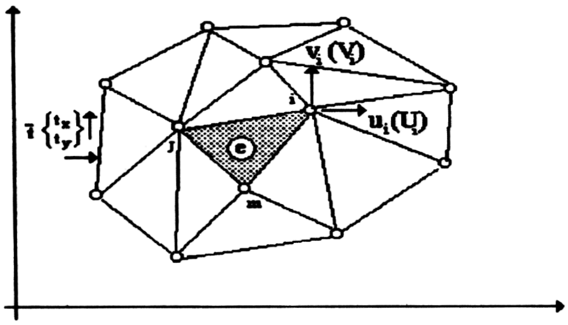


Figure 5.4 Plane stress field for a region divided in triangular elements.

5.9.1 Shape functions

A typical finite element, *e*, is defined by the nodes *i*, *j*, *m*,... etc. and by boundary lines. The displacement vector **u** in a point within the element is approximated by a column vector $\hat{\mathbf{u}}$.

$$\mathbf{u} \approx \hat{\mathbf{u}} = \sum \mathbf{N_i a_i^e} = [\mathbf{N_i}, \mathbf{N_j}, \ldots] \begin{Bmatrix} \mathbf{a_i} \\ \mathbf{a_j} \\ . \\ . \end{Bmatrix}^{\mathbf{e}} = \mathbf{N a^e} \tag{5.9.1-1}$$

where:

$\mathbf{N_i}$ = pre-established functions depending on the nodes coordinates

a_i^e = nodal displacements for a particular element.

In plane stress field, it is obtained, for example:

$$\mathbf{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} \tag{5.9.1-2}$$

This represents the column vector (horizontal and vertical displacements) of a typical point within the element. Moreover, it also is:

$$\mathbf{a_i} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{5.9.1-3}$$

which represents the column vector of the corresponding nodal displacements (node *i*).

The functions $\mathbf{N_i}$, $\mathbf{N_j}$, $\mathbf{N_m}$ are chosen so that they provide the respective nodal displacements when the corresponding node coordinates are introduced in the equation (5.9.1-1).

In general, it can be written:

$$\mathbf{N_i}(x_i, y_i) = \mathbf{I} \text{ (identity matrix)} \tag{5.9.1-4}$$

while:

$$\mathbf{N_i}(x_j, y_j) = \mathbf{N_i}(x_m, y_m) = 0, \quad \text{etc} \tag{5.9.1-5}$$

which is always satisfied by suitable linear functions in *x* and *y*.

If both displacement components are identically interpolated, it can be written:

$$\mathbf{N_i}=N_i\,\boldsymbol{I} \tag{5.9.1-6}$$

where N_i is obtained by the (5.9.1-1), by noticing that $N_i=1$ for the vertex with coordinates x_i, y_i but it is equal to zero for the other ones.

In the case of triangular elements, the more obvious linear interpolation is shown in the following figure.

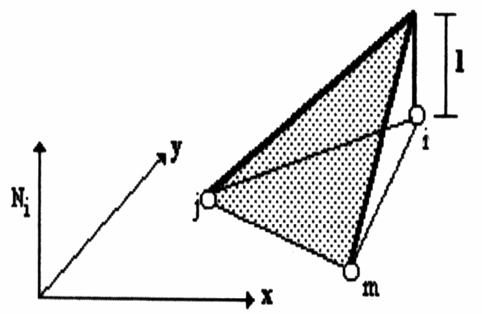


Figure 5.5 Shape function for triangular elements.

The function **N** is named **shape functions** and they are very important, as it will be seen, in the analysis to finite elements.

5.9.2 Strain fields

By the knowledge of all displacements within the element the strain field can be determined everywhere, in each element point, according to the following matrix relation:

$$\varepsilon = \mathbf{Su} \tag{5.9.2-1}$$

where:

S = a suitable linear operator.

By considering the equation (5.9.1-1), the (5.9.2-1) is approximated as:

$$\varepsilon = \mathbf{Ba} \tag{5.9.2-2}$$

with:

$$\mathbf{B} = \mathbf{SN} \tag{5.9.2-3}$$

For plane strain fields, the strain values are obtained in terms of displacement fields by means of the well-known relations which define the operator **S**:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{5.9.2-4}$$

If the shape functions, $\mathbf{N_i}$, $\mathbf{N_j}$, $\mathbf{N_m}$, are already established, the matrix **B** can be easily reached. If a linear form is assumed for such functions, the strain field is constant everywhere in the element.

5.9.3 Stress fields

Generally, an element is subjected to initial strains, on its boundary, due, for example, to temperature gradients. Such strains are denoting with ε_0.

The stresses are caused by the difference between the actual strains and the initial ones. Furthermore, it is suitable to assume that there was also an initial stress state in the element, due to residual stresses σ_0 ,which can be measured but cannot be known if the material stress history is unknown.

Hence, for a linear-elastic behaviour, they are assumed the following stress-strain linear relations:

$$\sigma = \mathbf{D}(\varepsilon - \varepsilon_0) + \sigma_0 \tag{5.9.3-1}$$

where:

D = elastic stiffness matrix which contains the material properties.

For plane stress fields, there are only three stress components, denoted with:

$$\sigma = \begin{Bmatrix} \sigma_{\mathbf{x}} \\ \sigma_{\mathbf{y}} \\ \tau_{\mathbf{xy}} \end{Bmatrix} \tag{5.9.3-2}$$

The matrix **D** is simply obtained from the stress-strain for an isotropic material:

$$\begin{aligned} \varepsilon_x - (\varepsilon_x)_0 &= \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y \\ \varepsilon_y - (\varepsilon_y)_0 &= \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y \\ \gamma_{xy} - (\gamma_{xy})_0 &= \frac{2(1+\nu)}{E}\tau_{xy} \end{aligned} \tag{5.9.3-3}$$

Hence, it is obtained:

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{5.9.3-4}$$

5.9.4 Equivalent nodal forces

Let us assign the following nodal forces:

$$\mathbf{q^e} = \begin{Bmatrix} \mathbf{q_i^e} \\ \mathbf{q_j^e} \\ . \\ . \end{Bmatrix} \qquad (5.9.4\text{-}1)$$

which are statically equivalent to the boundary tractions and to the element distribute loads.

Each force $\mathbf{q_i^e}$ has the same number of components than the corresponding nodal displacements $\mathbf{a_i}$ and also the same directions.

The distribute mass forces **b** are defined as forces on unit volume at an element point and they have directions corresponding to those ones of the displacements **u** in such point.

For example, in a plane stress state, the nodal forces are:

$$\mathbf{q_i^e} = \begin{Bmatrix} U_i \\ V_i \end{Bmatrix} \qquad (5.9.4\text{-}2)$$

where the components *U* and *V* correspond to the displacement directions *u* and *v*, while it is:

$$\mathbf{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \qquad (5.9.4\text{-}3)$$

where b_x, and b_y, are the mass force components.

In order to found the statically equivalent nodal forces, it can be imposed a virtual displacement, $\delta\mathbf{a^e}$, at nodes and then the external work has to be balanced to the internal one.

According to the equations (5.9.1-1) and (5.9.2-2), the assignment of the virtual displacement $\delta\mathbf{a^e}$ implies the following displacements and strains, respectively:

$$\delta\mathbf{u} = \mathbf{N}\delta\mathbf{a}^{\mathbf{e}} \qquad \text{and} \qquad \delta\varepsilon = \mathbf{B}\delta\mathbf{a}^{\mathbf{e}} \tag{5.9.4-4}$$

The external work can be written in the following matrix form as:

$$\delta\mathbf{a}^{\mathbf{e}^{\mathbf{T}}}\mathbf{q}^{\mathbf{e}} \tag{5.9.4-5}$$

while the internal work for unit volume is given by:

$$\delta\varepsilon^{\mathbf{T}}\sigma - \delta\mathbf{u}^{\mathbf{T}}\mathbf{b} \tag{5.9.4-6}$$

or

$$\delta\mathbf{a}^{\mathbf{T}}(\mathbf{B}^{\mathbf{T}}\sigma - \mathbf{N}^{\mathbf{T}}\mathbf{b}) \tag{5.9.4-7}$$

By integrating the internal work in the element volume $V^{\mathbf{e}}$ and, then, by balancing the two works, it is obtained:

$$\delta\mathbf{a}^{\mathbf{e}^{\mathbf{T}}}\mathbf{q}^{\mathbf{e}} = \delta\mathbf{a}^{\mathbf{e}^{\mathbf{T}}}\left(\int_{\mathbf{V^e}}\mathbf{B}^{\mathbf{T}}\sigma\mathbf{dV} - \int_{\mathbf{V^e}}\mathbf{N}^{\mathbf{T}}\mathbf{bdV}\right) \tag{5.9.4-8}$$

The equation (5.9.4-8) is valid for each virtual displacement and, therefore, it has to be verified the following relation:

$$\mathbf{q}^{\mathbf{e}} = \int_{\mathbf{V^e}}\mathbf{B}^{\mathbf{T}}\sigma\mathbf{dV} - \int_{\mathbf{V^e}}\mathbf{N}^{\mathbf{T}}\mathbf{bdV} \tag{5.9.4-9}$$

By taking into consideration the equation (5.9.3-1), the equation (5.9.4-9) can be written in the following form:

$$\mathbf{q}^{\mathbf{e}} = \mathbf{K}^{\mathbf{e}}\mathbf{a}^{\mathbf{e}} + \mathbf{f}^{\mathbf{e}} \tag{5.9.4-10}$$

where:

$$\mathbf{K}^{\mathbf{e}} = \int_{\mathbf{V^e}}\mathbf{B}^{\mathbf{T}}\mathbf{DBdV} \tag{5.9.4-11}$$

and

$$\mathbf{f}^{\mathbf{e}} = -\int_{\mathbf{V^e}}\mathbf{N}^{\mathbf{T}}\mathbf{bdV} - \int_{\mathbf{V^e}}\mathbf{B}^{\mathbf{T}}\mathbf{D}\varepsilon_0\mathbf{dV} + \int_{\mathbf{V^e}}\mathbf{B}^{\mathbf{T}}\sigma_0\mathbf{dV} \tag{5.9.4-12}$$

In the last equation, the three terms in the right hand represent the forces due, respectively, to the mass forces, to the initial strains and to the initial stresses.

If the initial stress field is self-equilibrated, like in the case of residual stresses, the contribute of such forces in the (5.9.4-12) is identically null. For this reason, their estimation is often omitted.

In the particular case of plane stress field for triangular elements, the matrix **B** doesn't depend on the coordinates, so the volume integral becomes particularly simple.

The structure interconnection and the structure solution, given by the elements assembly, follow the simple procedures until now described.

Generally, nodal concentrated forces can be applied at nodes, and so the matrix:

$$\mathbf{r} = \begin{Bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ . \\ . \\ \mathbf{r}_\mathbf{n} \end{Bmatrix} \qquad (5.9.4\text{-}13)$$

is added up to the equivalent nodal forces.

If some boundary displacements are individualized, they can be satisfied by establishing some of the nodal parameters **a**.

Let us consider that the boundary is subjected to distribute loads $\bar{\mathbf{t}}$ for unit area. Thus, a load condition at the nodes of an element having a boundary face $\mathbf{A^e}$ has to be taken into consideration. By means of the virtual work principle, it is read as:

$$-\int_{\mathbf{A^e}} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{t}} \mathbf{dA} \qquad (5.9.4\text{-}14)$$

It is worth to notice that $\bar{\mathbf{t}}$ must have the same number of **u** components in order to satisfy the (5.9.4-14).

Once the nodal displacements are determined from the global structure solution, the stress field in some elements points are found by means of the equations (5.9.2-2) and (5.9.3-1), that is:

$$\sigma = \mathbf{DBa^e} - \mathbf{D}\varepsilon_0 + \sigma_0 \tag{5.9.4-15}$$

where the terms of the equation (5.2-4) are immediately recognized. The stress matrix is given by:

$$\mathbf{S^e} = \mathbf{DB} \tag{5.9.4-16}$$

and the stresses:

$$\sigma_{\varepsilon 0} = -\mathbf{D}\varepsilon_0 + \sigma_0 \tag{5.9.4-17}$$

have to be summed.

In the (5.9.4-15) there are not the stresses due to the distribute load, $\sigma_\mathbf{p}^\mathbf{e}$, because the internal element equilibrium has not been considered, since only global equilibrium condition have been established.

5.10 Generalization to the whole region

In the previous paragraph, the virtual work is applied to the single elements and it is introduced the concept of the equivalent nodal forces. The assembly follows the traditional approach of the direct equilibrium.

The idea of considering the contribute of the nodal forces on the element substitutes the actual interactions in the continuum. If such introduction of nodal forces which are equivalent to nodes is quite obvious from an engineering point of view, it is less obvious from the mathematical one.

The reasoning previously done can be directly applied on the whole continuum. However, each element has to be separately considered, yet. Thus, the equation (5.9.1-1) can be read as applied on the whole structure, and so it assumes the following form:

$$\mathbf{u} = \overline{\mathbf{N}}\mathbf{a} \tag{5.10-1}$$

where **a** comprises all the nodal displacements and where:

$$\overline{\mathbf{N}}_{\mathbf{i}} = \mathbf{N}_{\mathbf{i}}^{\mathbf{e}} \tag{5.10-2}$$

if the point *i* is internal to a particular element **e** while it is:

$$\overline{\mathbf{N}}_{\mathbf{i}} = 0 \tag{5.10-3}$$

if the point *i* is not internal to a particular element **e**.

The matrix $\overline{\mathbf{B}}$ is also defined and it is considered that the shape functions are defined on the whole region **V**. For simplicity, we omit the superscript $\left[\,^{-}\,\right]$. For virtual displacements $\delta\mathbf{a}$, the sum of the internal and external work for the whole region assumes the following form:

$$-\delta\mathbf{a}^{\mathbf{T}}\mathbf{r} = \int_{\mathbf{V}} \delta\mathbf{u}^{\mathbf{T}}\mathbf{b}\mathbf{dV} + \int_{\mathbf{A}} \delta\mathbf{u}^{\mathbf{T}}\bar{\mathbf{t}}\mathbf{dA} - \int_{\mathbf{V}} \delta\varepsilon^{\mathbf{T}}\sigma\mathbf{dV} \tag{5.10-4}$$

In the equation (5.10-4), the quantities $\delta\mathbf{a}$**,** $\delta\mathbf{u}$**,** $\delta\varepsilon$ are arbitrary only if they derive from continuous displacements. Let us assume, for simplicity, that such quantities are simple variations according to (5.9.2-2) and (5.10-1). Hence, by considering the (5.9.3-1), an algebraic system is obtained:

$$\mathbf{Ka} + \mathbf{f} = \mathbf{r} \tag{5.10-5}$$

where:

$$\mathbf{K} = \int_{\mathbf{V}} \mathbf{B}^{\mathbf{T}}\mathbf{DBdV} \tag{5.10-6}$$

and

$$\mathbf{f} = -\int_{\mathbf{V}} \mathbf{N}^{\mathbf{T}}\mathbf{b}\,\mathbf{dV} - \int_{\mathbf{A}} \mathbf{N}^{\mathbf{T}}\bar{\mathbf{t}}\,\mathbf{dA} - \int_{\mathbf{V}} \mathbf{B}^{\mathbf{T}}\mathbf{D}\varepsilon_0\,\mathbf{dV} + \int_{\mathbf{V}} \mathbf{B}^{\mathbf{T}}\sigma_0\,\mathbf{dV} \tag{5.10-7}$$

The integrals are calculated on the whole volume **V** and on the whole surface **A**.

It is obvious that:

$$\mathbf{K}_{\mathbf{ij}} = \sum \mathbf{K}_{\mathbf{ij}}^{\mathbf{e}} \qquad\qquad \mathbf{f}_{\mathbf{i}} = \sum \mathbf{f}_{\mathbf{i}}^{\mathbf{e}} \tag{5.10-8}$$

since, for the property of the definite integrals, it is:

$$\int_V (\ )dV = \sum \int_{V^e} (\ )dV \tag{5.10-9}$$

The same thing is true for the area integrals.

The assembling rules are so obtained without building the interaction forces between the elements.

By considering the equation (5.10-4) and by making it equal to the sum of each element contributes, it is implicitly assumed that there are not discontinuities between adjacent elements. In other words, it is required that the integrated terms in the equation (5.10-9) are continuous functions. Such terms derive from the function $\mathbf{N_i}$ used to define the displacement ***u.*** (eq. (5.10-1)). Hence, for an example, if the strains are obtained by the first derivatives of the function **N**, the latter have to be continuous, that is, it has to be a C_0 class function. In some more general problems, the strain can be defined by means of the second derivatives of the function **N**. In such cases, it is required that the function **N** and its derivatives have to be continuous, that is, they have to be C_1 class functions.

5.11 Displacement method as the minimum of the total potential energy

The virtual work principle used in the previous paragraphs guarantees the satisfaction of the equilibrium conditions, in the pre-established limits of the displacement model. If the number of the parameters **a**, defining the displacement field, increases beyond some limits, then all the equilibrium conditions can be assured since the approximation is very close to the reality.

Hence, the equation (5.10-4) can be rewritten in a different form if the virtual quantities $\delta\mathbf{a}$, $\delta\mathbf{u}$, $\delta\varepsilon$ are considered as variations of the actual ones. For example, it becomes:

$$\delta\left(\mathbf{a}^{\mathbf{T}}\mathbf{r} + \int_{\mathbf{V}}\mathbf{u}^{\mathbf{T}}\mathbf{b}\mathbf{dV} + \int_{\mathbf{A}}\mathbf{u}^{\mathbf{T}}\bar{\mathbf{t}}\mathbf{dA}\right) = -\delta\mathbf{W} \tag{5.11-1}$$

where:

W = external forces potential

This is true if $\mathbf{r}_i$, $\mathbf{b}$, $\bar{\mathbf{t}}$ are conservative.

The last terms of the equation (5.10-4) for elastic materials can be written in the following form:

$$\delta\mathbf{U} = \int_{\mathbf{V}}\delta\varepsilon^{\mathbf{T}}\sigma\mathbf{dV} \tag{5.11-2}$$

where:

U = elastic system energy.

For a linear-elastic material, whose behaviour is described in the equation (5.9.3-1), it is:

$$\mathbf{U} = \frac{1}{2}\int_{\mathbf{V}}\varepsilon^{\mathbf{T}}\mathbf{D}\varepsilon\mathbf{dV} - \int_{\mathbf{V}}\varepsilon^{\mathbf{T}}\mathbf{D}\varepsilon_0\mathbf{dV} + \int_{\mathbf{V}}\varepsilon^{\mathbf{T}}\sigma_0\mathbf{dV} \tag{5.11-3}$$

where:

D = elastic symmetric matrix

By considering the equation (5.10-4), it can be simply written:

$$\delta(\mathbf{U}+\mathbf{W})=\delta(\mathsf{P})=0 \tag{5.11-4}$$

where:

P = total potential energy.

This means that, if the equilibrium is assured, the total potential energy is stationary for admissible displacement variations.

The equations previously obtained (from the (5.10-5) to the (5.10-8)) are simply the result of such variations with respect to constrained displacements of a finite parameters number a . It can be written:

$$\frac{\partial \mathsf{P}}{\partial \mathbf{a}}=\left\{\begin{array}{c}\dfrac{\partial \mathsf{P}}{\partial \mathbf{a}_1}\\ \dfrac{\partial \mathsf{P}}{\partial \mathbf{a}_2}\\ \cdot\\ \cdot\\ \cdot\end{array}\right\}=0 \tag{5.11-5}$$

It is demonstrated that in condition of elastic stability, the total potential energy is not only stationary but it touches a minimum. So, the finite element method looks for a minimum within an assumed displacement model.

It is worth to notice that the actual equilibrium requires an absolute minimum of the total potential energy P .

5.12 Convergence criterions

By assuming accurate shape functions, the endless system grades of freedom are contained and the minimum of the total potential energy can be not found independently from the refinement of the mesh.

In order to assure the convergence to the correct result, some requirements have to be satisfied.

For example, the shape functions must be able to represent the displacements distribution in a form which has to be the closest one to the actual distribution. This means that the shape function must be chosen according to determined criterions:

- CRITERION 1: The shape functions have to be so to describe a null strain state. This occurs in case of rigid displacements.
- CRITERION 2: The shape functions have to be so to describe a constant strain state. It is worth to notice that the second criterion includes the first one, since the rigid displacements yield a particular case of constant strain field, which is null everywhere.

Both the criterions have to be satisfied if the elements dimension, at the limit, tends to zero. By imposing such criterions on finite sizes, a greater solution accuracy is yielded.

- CRITERION 3: The shape functions have to be chosen so that the interface strains result to be finite. This criterion implies a certain continuity of displacements cross the elements.

5.13 Error discretization and convergence classes

Generally, the approximation in the displacement field given by the equation (5.9.1-1) makes the solution to be exact, in the limits in which the h element dimension decreases. In some cases, the exact solution is obtained with a finite number of subdivisions (or with a single element) if the used polynomial growth exactly complies with the solution. So, if the exact solution is, for example, a square polynomial and if the shape function includes all the square terms, the approximations will yield just the exact solution. This latter can always be expanded in series in the around of a point or a node with a polynomial growth, as a kind:

$$\mathbf{u} = \mathbf{u_i} + \left(\frac{\partial \mathbf{u_i}}{\partial \mathbf{x}}\right)_{\mathbf{i}}(\mathbf{x} - \mathbf{x_i}) + \left(\frac{\partial \mathbf{u_i}}{\partial \mathbf{y}}\right)_{\mathbf{i}}(\mathbf{y} - \mathbf{y_i}) + \ldots \tag{5.13-1}$$

If a polynomial expression with *p* grade is used within an *h* dimension element, it can locally comply with the expansion in Taylor's series. Moreover, if *x* and *y* have the same order of magnitude than *h*, the *u* error will be of the order $O(h^{p+1})$. Thus, for example, for a plane stress field where a linear expression of *p*=1 grade is used, an order $O(h^2)$ convergence class is expected and the error in displacement field is reduced to 1/4 for a halved mesh.

With analogous reasoning, the strains (or stresses) which are given by the m-th derivative of the displacement can converge with an error of $O(h^{p+1-m})$ (in the above cited example where m=1 the error is $O(h)$).

The elastic energy given by the square value of the stresses converges with an error of $O(h^{2(p+1-m)})$ or, in case of plane stress field, $O(h^2)$.

In a lot of problems, the simple determination of the convergence order is sufficient to extrapolate the correct result.

Hence, if the displacements converge with an error of $O(h^2)$ and two approximate solutions u^1 and u^2 are obtained with a mesh size of *h* and *h/2*, it can be written:

$$\frac{u^1 - u}{u^2 - u} = \frac{O(h^2)}{O(h/2)^2} = 4 \tag{5.13-2}$$

where u is the exact solution.

The discretization errors are not the only possible in the finite elements computation, since the rounding errors produced by the electronic computer on the decimal digits have to be summed to the discretization ones. Such errors are minimized if computers which use a great number of significant digits are used.

5.14 Analysis of a three-dimensional stress field

In this paragraph, the finite element method is applied to a generic three-dimensional stress state.

The simplest element in three-dimensions is the tetrahedral one, which is a four nodes element, whose characteristics are analyzed in the follows.

5.14.1 Displacement functions

Let us consider the tetrahedral element *i, j, m, p* in the reference system x,y,z, as it is shown in the following figure.

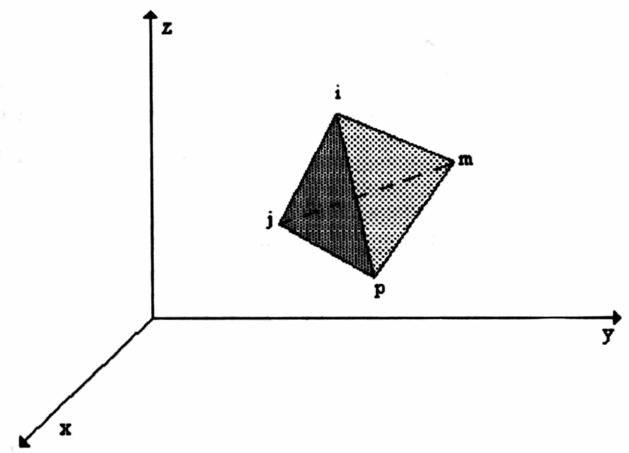


Figure 5.6 A tetrahedral volume.

It is:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{5.14.1-1}$$

where a linear function is assumed like displacement function, as given in the following equation:

$$\mathbf{u} = \alpha_1 + \alpha_2 \mathbf{x} + \alpha_3 \mathbf{y} + \alpha_4 \mathbf{z} \tag{5.14.1-2}$$

By making equal the displacement values at node, four equations are obtained, given by:

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i \tag{5.14.1-3}$$

where the coefficients from α_1 to α_4 can be evaluated.

It is possible to write the solution by using a form as a kind:

$$\mathbf{u} = \frac{1}{6\mathbf{V}} \begin{bmatrix} (a_i + b_i x + c_i y + d_i z)u_i + (a_j + b_j x + c_j y + d_j z)u_j + \\ (a_m + b_m x + c_m y + d_m z)u_m + (a_p + b_p x + c_p y + d_p z)u_p \end{bmatrix} \tag{5.14.1-4}$$

with:

$$6\mathbf{V} = \det \begin{bmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_m & y_m & z_m \\ 1 & x_p & y_p & z_p \end{bmatrix} \tag{5.14.1-5}$$

where the value **V** represents the tetrahedral volume. Furthermore, it is:

$$a_i = \det\begin{bmatrix} x_j & y_j & z_j \\ x_m & y_m & z_m \\ x_p & y_p & z_p \end{bmatrix} \qquad b_i = -\det\begin{bmatrix} 1 & y_j & z_j \\ 1 & y_m & z_m \\ 1 & y_p & z_p \end{bmatrix}$$
$$c_i = -\det\begin{bmatrix} x_j & 1 & z_j \\ x_m & 1 & z_m \\ x_p & 1 & z_p \end{bmatrix} \qquad d_i = -\det\begin{bmatrix} x_j & y_j & 1 \\ x_m & y_m & 1 \\ x_p & y_p & 1 \end{bmatrix} \tag{5.14.1-6}$$

where the constants can be defined by means of an index rotation, in order *p*, *i*, *j*, *m*.

The ordination of the nodal numbers *p*, *i*, *j*, *m* is done in counter clockwise, as shown in the figure 5.6.

The displacement element is defined by twelve nodal displacements components:

$$\mathbf{a^e} = \begin{Bmatrix} \mathbf{a_i} \\ \mathbf{a_j} \\ \mathbf{a_m} \\ \mathbf{a_p} \end{Bmatrix} \tag{5.14.1-7}$$

with

$$\mathbf{a_i} = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \quad \text{etc} \tag{5.14.1-8}$$

The displacement of an arbitrary point can be written in the form:

$$\mathbf{u}=[\mathbf{I}N_i, \mathbf{I}N_j, \mathbf{I}N_m, \mathbf{I}N_p]\mathbf{a^e} \tag{5.14.1-9}$$

where the shape functions are defined as:

$$N_i = \frac{a_i + b_i x + c_i y + d_i z}{6V} \qquad \text{etc} \tag{5.14.1-10}$$

with

I = identical matrix.

The used displacement functions obviously satisfy the required continuity at the interface between two different elements, as natural consequence of their linearity.

5.14.2 Strain matrix

The six strain components are all considerable in the three-dimensional analysis. Hence, the strain matrix is so defined:

$$\varepsilon=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial z}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\\ \dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\end{Bmatrix}=\mathbf{Su} \tag{5.14.2-1}$$

where the standard Timoshenko’s notation is assumed.

By using the equations (5.14.1-3) and (5.14.1-9), it is easily verified that:

$$\varepsilon=\mathbf{Ba^e}=\left[\mathbf{B_i},\mathbf{B_j},\mathbf{B_m},\mathbf{B_p}\right]\mathbf{a^e} \tag{5.14.2-2}$$

where

$$\mathbf{B_i} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{6\mathbf{V}} \begin{bmatrix} b_i & 0 & 0 \\ 0 & c_i & 0 \\ 0 & 0 & d_i \\ c_i & b_i & 0 \\ 0 & d_i & c_i \\ d_i & 0 & b_i \end{bmatrix} \tag{5.14.2-3}$$

The initial strain field is written in the form:

$$\varepsilon_0 = \begin{Bmatrix} \alpha\theta^{\mathbf{e}} \\ \alpha\theta^{\mathbf{e}} \\ \alpha\theta^{\mathbf{e}} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5.14.2-4}$$

with

α = coefficient of thermal dilation

θ^e = mean raise of temperature in the element.

5.14.3 Elasticity matrix

In case of complete anisotropy, the matrix **D** contains 21 independent constants. Generally, it is:

$$\sigma = \begin{Bmatrix} \sigma_{x} \\ \sigma_{y} \\ \sigma_{z} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \mathbf{D}\left(\varepsilon - \varepsilon_0\right) + \sigma_0 \qquad (5.14.3\text{-}1)$$

In case of isotropic material, the matrix **D** is described in function of only two independent elastic constants, E and ν, and it assumes the following form:

$$\boldsymbol{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & Sym & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \qquad (5.14.3\text{-}2)$$

5.14.4 Stiffness, stress and loads matrix

The stiffness matrix is defined by means of the equation (5.9.4-8) and it can be easily expressed since the stresses and the strains within the element are constants. The general stiffness submatrix is given by:

$$\mathbf{K}_{ij}^{e} = \mathbf{B}_{i}^{T}\mathbf{D}\mathbf{B}_{j}\mathbf{V}^{e} \qquad (5.14.4\text{-}1)$$

where:

$\mathbf{V}^{e}$ = tetrahedral volume.

The nodal forces, due to the initial strain field, have the following form:

$$\mathbf{f}_{\mathbf{i}}^{\mathbf{e}} = -\mathbf{B}_{\mathbf{i}}^{\mathbf{T}}\mathbf{D}\varepsilon_0\mathbf{V}^{\mathbf{e}} \tag{5.14.4-2}$$

The forces due to the initial stress field have an analogous form.

The mass distribute forces are expressed in terms of the components b_x, b_y, b_z. It is possible to show that the nodal forces equivalent to the distribute mass ones result equal each others and they are equal to ¼ of the resulting force in case where the mass forces are constant.

CHAPTER VI

Computational Analyses

6.1 Introduction

In the previous chapter, it has been studied the finite element method from a theoretical point of view. Hence, it has been seen that the FEM can be thought as a mathematical model able to include in it the continuum theories. Such method, in fact, overcomes the difficulties of the analysis of a continuum solid structural response by operating a discretization of the same continuum. This means, as already seen, that the solid is divided in a finite number of elements, whose structural behaviours are known. Such elements, when assembled with accurate relation laws among the nodes, are able to yield the global behaviour of the primitive solid, even if approximately. Obviously, the solution is as much close to the actual mechanical response as the mesh is heightened.

After having introduced these fundamental and essential notes on the F.E.M. theory, the goal of this chapter will be to show some computational analyses, carried out by means of the calculation code Ansys, in its version 6.0.

This software offers a large number of appliances in a lot of engineering fields and it is just based on the mathematical F.E.M. model.

In the follows, the used micro-mechanical model and the effected analyses will be described. Since, in linear-elastic field, a numerical analysis can efficaciously replace an experimental test, such finite element analyses have been employed in order to compare the analytical results obtained by our proposed homogenization techniques, shown in the chapter 4, and the literature data.

6.2 Micro-mechanical model

The micro-mechanical model used in the finite element analyses is the same one considered in the S.A.S. homogenization approach and illustrated in the chapter 4. In particular, it is constituted by a periodic basic cell extracted from a single leaf masonry wall in stretcher bond, as shown in the figure below:

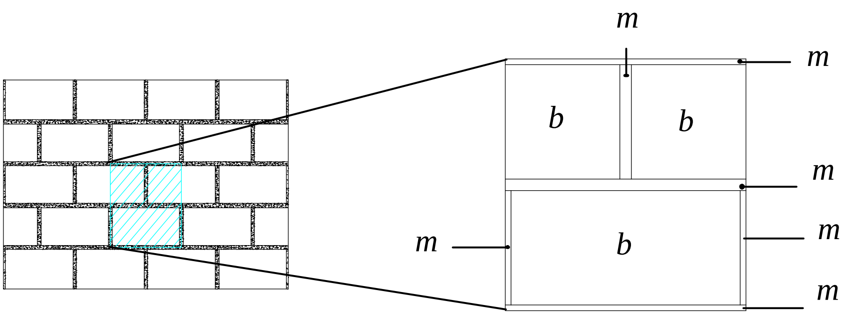


Figure 6.1 Definition of masonry axes and of chosen micro mechanical model.

where m stands for the mortar components and b stands for the brick ones.

The assigned dimensions are so that the equivalence in the volumetric fractions, between the above mentioned model and the one considered in the Lourenco-Zucchini analysis and in the statically-consistent Lourenco approach, is obtained.

In particular, the two models must have the following dimensions:

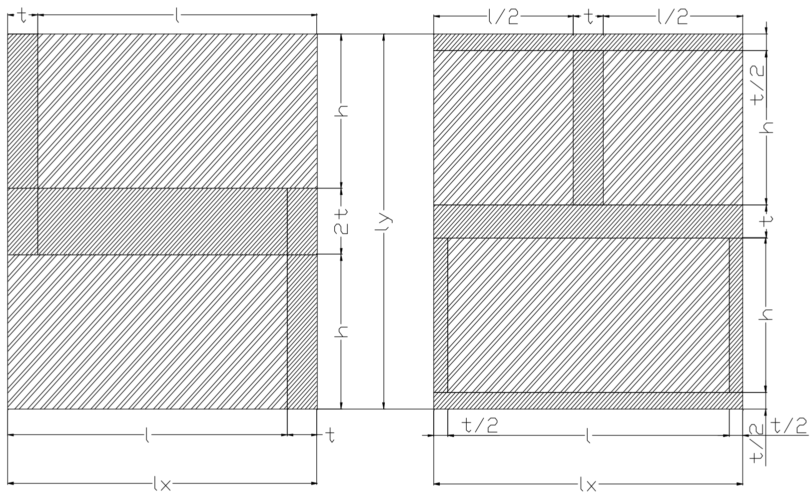


Figure 6.2 Equivalence in the volumetric fractions between the two micro mechanical models.

The input data considered in the analysis are:

$$
\begin{aligned}
&E_b = 2*10^5 \;\; daN/cm^2 && \textit{brick dimensions}: \\
&E_m = 2*10^4 \;\; daN/cm^2 && l = 21cm;\; h = 5cm;\;\; s = 10cm \\
&\nu_b = \nu_m = 0.15;\;\; \frac{E_b}{E_m} = 10 && \textit{mortar tickness}: \;\; t = 1cm \qquad (6.2\text{-}1) \\
&\textit{model dimensions}: \\
&l_x = 22\,cm;\;\; l_y = 12\,cm;\;\; l_z = 10\,cm
\end{aligned}
$$

with:

$E_b =$ Young modulus for the brick, considered isotropic

$E_m =$ Young modulus for the mortar, considered isotropic

$\nu_b =$ Poisson modulus for the brick

$\nu_m =$ Poisson modulus for the mortar

$s =$ Thickness of the brick in *z*-direction

$l_z =$ Dimension of the micro-mechanical model in *z*-direction

The assumed hypothesis of linear elasticity lets to study the elastic response of the model for a generic loading condition as linear combination of the elastic responses for six elementary loading conditions. In particular, both stress-prescribed and strain-prescribed F.E.M. analyses have been carried out.

In the following paragraph, the results obtained with the stress-prescribed analysis will be described.

However, it is first illustrated, in the figure below, the finite element model which has been used in the numerical analysis.

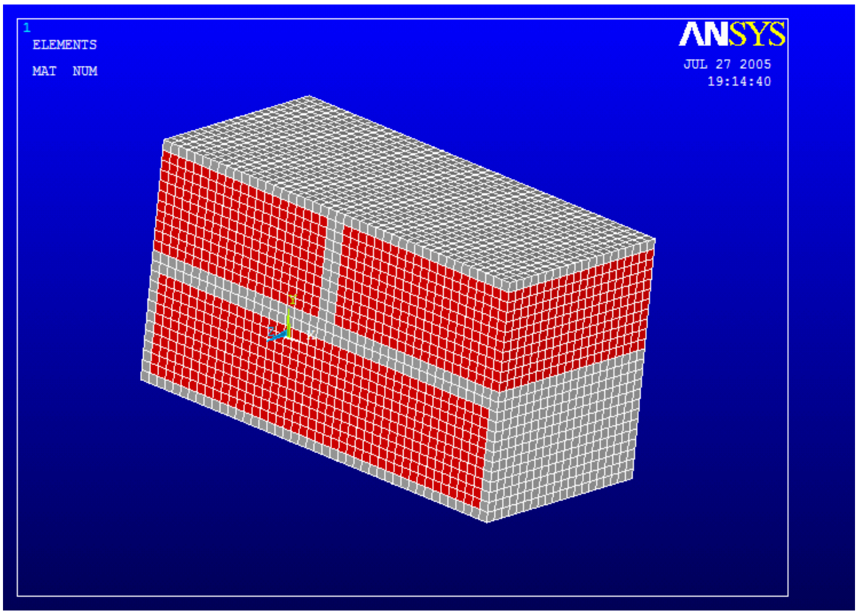


Figure 6.3 Finite element model-mesh.

The element type considered was the structural solid 45, in particular the brick 8 nodes. The mesh was obtained by a process of regular subdivisions of all model lines, by taking into account a mesh size of 0.5 cm. Thus, the model has been discretized in a number of the elements *n* equal to 21120.

6.3 Stress-prescribed analysis

In the stress-prescribed analyses, the goal has been to obtain the overall compliance tensor by means of six numerical analyses. Since an orthotropic mechanical behaviour is considered, only nine elastic coefficients will be independent and different from zero.

By using the Voigt notation, so that:

$$\begin{array}{llll} e_1 \rightarrow e_{xx} & e_4 \rightarrow 2e_{zy} & s_1 \rightarrow s_{xx} & s_4 \rightarrow s_{zy} \\ e_2 \rightarrow e_{yy} & e_5 \rightarrow 2e_{zx} & s_2 \rightarrow s_{yy} & s_5 \rightarrow s_{zx} \\ e_3 \rightarrow e_{zz} & e_6 \rightarrow 2e_{xy} & s_3 \rightarrow s_{zz} & s_6 \rightarrow s_{xy} \end{array} \tag{6.3-1}$$

the stress-strain relation can be written in the following form:

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \\ \bar{e}_4 \\ \bar{e}_5 \\ \bar{e}_6 \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & 0 \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{66} \end{bmatrix} \cdot \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \\ \bar{s}_4 \\ \bar{s}_5 \\ \bar{s}_6 \end{bmatrix} \tag{6.3-2}$$

where the superscript $\left[\bar{\ } \right]$ means that the above written equations refer to the average values of the corresponding quantities within the considered RVE.

By applying the six loading conditions one at a time, it is possible to obtain the single columns of the compliance tensor, one at a time too, according to the following relation:

$$\bar{S}_{ij} = \frac{\bar{e}_i}{\bar{s}_j} \qquad i, j = 1,2,3,4,5,6 \tag{6.3-3}$$

More in detail, both homogenized compliance coefficients and physic ones are determined, as described in the follows.

- **Homogenized elastic compliances**

In the chapter 1, it has been seen that, for the average theorem, when the boundary conditions are applied in terms of uniform stresses on the considered RVE (basic cell), the following relation furnishes the average stress value in the RVE volume:

$$\bar{\sigma}_j = \frac{1}{V}\int_V \sigma_j \, dV = \sigma_j^0 \qquad j = 1, 2, 3, 4, 5, 6 \tag{6.3-4}$$

where V stands for the volume of the basic cell and σ_j^0 is the generic stress-prescribed component.

The same result is attained if the above shown RVE is considered subjected, for an example, to a unit stress component σ_j^0, i.e:

$$p = \sigma_j^0 = -1 \tag{6.3-5}$$

Hence, the resulting force F_j on loaded face is obtained by:

$$F_j = \sum_{r=1}^{m} \bar{\sigma}_j^{(r)} A^{(r)} \tag{6.3-6}$$

where:

m = the number of the elements in which the loaded face is discretized.

$A^{(r)}$ = the area of the generic element

$\bar{\sigma}_j^{(r)}$ = the average value of the *j*-stress component, for the generic element

Since the used mesh size is constant everywhere, all the areas of the elements are equal, too. So, the equation (6.3-6) can be rewritten in the form:

$$F_j = A\sum_{r=1}^{m} \bar{\sigma}_j^{(r)} \Rightarrow \sum_{r=1}^{m} \bar{\sigma}_j^{(r)} = \frac{F_j}{A} \tag{6.3-7}$$

By dividing both members for the elements number *m*, it is obtained:

$$\frac{1}{m}\sum_{r=1}^{m} \bar{\sigma}_j^{(r)} = \frac{F}{mA} = \frac{F}{A_{tot}} \Rightarrow \hat{\sigma}_j = p = \sigma_j^0 \tag{6.3-8}$$

where:

$\hat{\sigma}_j$ = the average value of the *j*-stress component on the examined loaded face

A_{tot} = the area of such loaded face

The equation (6.3-8) remains unaltered if it is multiplied and divided for l, where l is given by:

$$l = \frac{n}{m} \tag{6.3-9}$$

and with:

$n =$ the number of the elements, equal to 21120, in which the whole RVE has been discretized.

Since such operation yields the average value of the *j*-stress component within the whole RVE, it is obtained that:

$$\bar{\sigma}_j = p = \sigma_j^0 = -1 \tag{6.3-10}$$

At this point, it occurs to calculate the volume average value of strain, $\bar{\varepsilon}_i$, obtained as:

$$\bar{\varepsilon}_i = \frac{\sum_{r=1}^{n} \bar{\varepsilon}_i^{(r)}}{n} \tag{6.3-11}$$

where:

$n =$ the number of elements in which the whole RVE is discretized and equal to 21120.

$\bar{\varepsilon}_i^{(r)}$ = the average value of the *i*-strain component, for the generic element.

Hence, the properties of the homogenized cell can be determined by means of equation (6.3-3). In detail, in the follows, the found coefficients of the homogenized tensor of compliances are shown for the six loading conditions:

- case of compression in *x*-direction:

$$\bar{S}_{11} = \frac{\bar{e}_1}{\bar{s}_1} = 7,98 \cdot 10^{-6}$$
$$\bar{S}_{21} = \frac{\bar{e}_2}{\bar{s}_1} = -1,09 \cdot 10^{-6} \qquad (6.3\text{-}12)$$
$$\bar{S}_{31} = \frac{\bar{e}_3}{\bar{s}_1} = -0,95 \cdot 10^{-6}$$
$$\bar{S}_{41} = \bar{S}_{51} = \bar{S}_{61} = 0$$

- case of compression in *y*-direction:

$$\bar{S}_{12} = \frac{\bar{e}_1}{\bar{s}_2} = -1,09 \cdot 10^{-6}$$
$$\bar{S}_{22} = \frac{\bar{e}_2}{\bar{s}_2} = 12,6 \cdot 10^{-6} \qquad (6.3\text{-}13)$$
$$\bar{S}_{32} = \frac{\bar{e}_3}{\bar{s}_2} = -1,04 \cdot 10^{-6}$$
$$\bar{S}_{42} = \bar{S}_{52} = \bar{S}_{62} = 0$$

- case of compression in *z*-direction:

$$\bar{S}_{13} = \frac{\bar{e}_1}{\bar{s}_3} = -0.95 \cdot 10^{-6}$$
$$\bar{S}_{23} = \frac{\bar{e}_2}{\bar{s}_3} = -1,04 \cdot 10^{-6} \qquad (6.3\text{-}14)$$
$$\bar{S}_{33} = \frac{\bar{e}_3}{\bar{s}_3} = 6,6 \cdot 10^{-6}$$
$$\bar{S}_{43} = \bar{S}_{53} = \bar{S}_{63} = 0$$

- case of shear stress in *zy*-plane:

$$\bar{S}_{14} = \bar{S}_{24} = \bar{S}_{34} = 0$$
$$\bar{S}_{44} = \frac{\bar{\varepsilon}_4}{\bar{\sigma}_4} = 14,78 \cdot 10^{-6} \tag{6.3-15}$$
$$\bar{S}_{54} = \bar{S}_{64} = 0$$

- case of shear stress in *zx*-plane:

$$\bar{S}_{15} = \bar{S}_{25} = \bar{S}_{35} = \bar{S}_{45} = 0$$
$$\bar{S}_{55} = \frac{\bar{\varepsilon}_5}{\bar{\sigma}_5} = 9,19 \cdot 10^{-6} \tag{6.3-16}$$
$$\bar{S}_{65} = 0$$

- case of shear stress in *xy*-plane:

$$\bar{S}_{16} = \bar{S}_{26} = \bar{S}_{36} = \bar{S}_{46} = \bar{S}_{56} = 0$$
$$\bar{S}_{66} = \frac{\bar{\varepsilon}_6}{\bar{\sigma}_6} = 15,9 \cdot 10^{-6} \tag{6.3-17}$$

Hence, the homogenized compliance tensor assumes the following form:

$$\mathbf{S}_{F.E.M.}^{Hom} = 10^{-6} \begin{bmatrix} 7.98 & -1.09 & -0.95 & 0 & 0 & 0 \\ -1.09 & 12.6 & -1.04 & 0 & 0 & 0 \\ -0.95 & -1.04 & 6.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.78 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9.19 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15.9 \end{bmatrix} \tag{6.3-18}$$

- **Physic elastic compliances**

The procedure used for determining the physic elastic compliances is analogous to the one used for determining the homogenized compliance tensor.

In order to find the volume average stress value on the boundary faces, the equation (6.3-8) can be again used. Thus, it is, yet:

$$\bar{\sigma}_j = p = \sigma_j^0 = -1 \qquad (6.3\text{-}19)$$

At this point, it occurs to calculate the volume average value of nominal strain, $\bar{\varepsilon}_i$. In case of normal strain components, it is obtained as:

$$\bar{\varepsilon}_i = \frac{\overline{\Delta l_i}}{l_i} \qquad (6.3\text{-}20)$$

where:

l_i = the characteristic RVE lengths

$\overline{\Delta l_i}$ = the average characteristic lengths variation, equal to:

$$\overline{\Delta l_i} = \frac{\sum_{r=1}^{m} \bar{u}_i^{(r)}}{m} \qquad (6.3\text{-}21)$$

where:

m = the number of the elements for the generic loaded surface.

$\bar{u}_i^{(r)}$ = the average value of the *i*-displacement component for the generic element.

Analogously, the average value of the nominal shear strain components can be obtained.

Hence, the properties of the homogenized cell can be determined by means of the equation (6.3-3). In detail, in the follows, the found coefficients of the physic tensor of compliances are shown for the six loading conditions:

- case of compression in *x*-direction:

$$\begin{aligned}
&\bar{S}_{11} = \frac{\bar{e}_1}{\bar{s}_1} = 8,05 \cdot 10^{-6} \\
&\bar{S}_{21} = \frac{\bar{e}_2}{\bar{s}_1} = -1,13 \cdot 10^{-6} \\
&\bar{S}_{31} = \frac{\bar{e}_3}{\bar{s}_1} = -0,97 \cdot 10^{-6} \\
&\bar{S}_{41} = \bar{S}_{51} = \bar{S}_{61} = 0
\end{aligned} \tag{6.3-22}$$

- case of compression in *y*-direction:

$$\begin{aligned}
&\bar{S}_{12} = \frac{\bar{e}_1}{\bar{s}_2} = -1,13 \cdot 10^{-6} \\
&\bar{S}_{22} = \frac{\bar{e}_2}{\bar{s}_2} = 12,6 \cdot 10^{-6} \\
&\bar{S}_{32} = \frac{\bar{e}_3}{\bar{s}_2} = -1,1 \cdot 10^{-6} \\
&\bar{S}_{42} = \bar{S}_{52} = \bar{S}_{62} = 0
\end{aligned} \tag{6.3-23}$$

- case of compression in *z*-direction:

$$\begin{aligned}
&\bar{S}_{13} = \frac{\bar{e}_1}{\bar{s}_3} = -0.97 \cdot 10^{-6} \\
&\bar{S}_{23} = \frac{\bar{e}_2}{\bar{s}_3} = -1,1 \cdot 10^{-6} \\
&\bar{S}_{33} = \frac{\bar{e}_3}{\bar{s}_3} = 6,8 \cdot 10^{-6} \\
&\bar{S}_{43} = \bar{S}_{53} = \bar{S}_{63} = 0
\end{aligned} \tag{6.3-24}$$

- case of shear stress in *zy*-plane:

$$\bar{S}_{14} = \bar{S}_{24} = \bar{S}_{34} = 0$$
$$\bar{S}_{44} = \frac{\bar{\varepsilon}_4}{\bar{\sigma}_4} = 15,07 \cdot 10^{-6} \tag{6.3-25}$$
$$\bar{S}_{54} = \bar{S}_{64} = 0$$

- case of shear stress in *zx*-plane:

$$\bar{S}_{15} = \bar{S}_{25} = \bar{S}_{35} = \bar{S}_{45} = 0$$
$$\bar{S}_{55} = \frac{\bar{\varepsilon}_5}{\bar{\sigma}_5} = 8,61 \cdot 10^{-6} \tag{6.3-26}$$
$$\bar{S}_{65} = 0$$

- case of shear stress in *xy*-plane:

$$\bar{S}_{16} = \bar{S}_{26} = \bar{S}_{36} = \bar{S}_{46} = \bar{S}_{56} = 0$$
$$\bar{S}_{66} = \frac{\bar{\varepsilon}_6}{\bar{\sigma}_6} = 14,54 \cdot 10^{-6} \tag{6.3-27}$$

Hence, the physic compliance tensor assumes the following form:

$$\mathbf{S}^{Phys}_{F.E.M.} = 10^{-6} \begin{bmatrix} 8.05 & -1.13 & -0.97 & 0 & 0 & 0 \\ -1.13 & 12.6 & -1.1 & 0 & 0 & 0 \\ -0.97 & -1.1 & 6.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.61 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14.54 \end{bmatrix} \tag{6.3-28}$$

6.4 Strain-prescribed analysis

In the strain-prescribed analyses, the goal has been to obtain the overall stiffness tensor by means of six numerical analyses. Since an orthotropic mechanical behaviour is considered, only nine elastic coefficients will be independent and different from zero.

By remembering the Voigt notation, the stress-strain relation can be written in the following form:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \cdot \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} \qquad (6.4\text{-}1)$$

where the superscript $\left[\,\overline{\;\;}\,\right]$ means that the above written equations refer to the average values of the corresponding quantities within the considered RVE.

By applying the six loading conditions one at a time, it is possible to obtain the single columns of the stiffness tensor, one at a time too, according to the following relation:

$$\bar{C}_{ij} = \frac{\bar{\sigma}_i}{\bar{\varepsilon}_j} \qquad i, j = 1, 2, 3, 4, 5, 6 \qquad (6.4\text{-}2)$$

More in detail, both homogenized stiffness coefficients and physic ones are determined, as described in the follows.

- **Homogenized elastic stiffness**

In the chapter 1, it has been seen that, for the average theorem, when the boundary conditions are applied in terms of surface displacements on the considered RVE (basic cell), the following relation furnishes the average strain value in the RVE volume:

$$\bar{\varepsilon}_j = \frac{1}{V}\int_V \varepsilon_j \, dV = \varepsilon_j^0 \tag{6.4-3}$$

where V stands for the volume of the basic cell and ε_j^0 is the generic strain component so that:

$$\boldsymbol{\varepsilon^0} \cdot \boldsymbol{x} = \boldsymbol{u^0} \tag{6.4-4}$$

with:

$\boldsymbol{u^0}$ = prescribed surface displacement.

More in detail, in order to found the homogenized stiffness tensor, the average strain value within the RVE volume is obtained as:

$$\bar{\varepsilon}_j = \varepsilon_j^0 = \sum_{r=1}^{n} \frac{\bar{\varepsilon}_j^{(r)}}{n} \tag{6.4-5}$$

where:

n = the number of elements in which the whole RVE is discretized and equal to 21120.

$\bar{\varepsilon}_j^{(r)}$ = the average value of the j-strain component, for the generic element.

At this point, it occurs to calculate the average value of stress, $\bar{\sigma}_i$, obtained as:

$$\bar{\sigma}_i = \frac{\sum_{r=1}^{n} \sigma_i^{(r)}}{n} \tag{6.4-6}$$

where:

$\bar{\sigma}_i^{(r)}$ = the average value of the i-stress component, for the generic element.

Hence, the properties of the homogenized cell can be determined by means of equation (6.4-2). In detail, in the follows, the found coefficients of the homogenized stiffness tensor are shown for the six loading conditions:

- case of normal strain in *x*-direction:

$$\begin{aligned} \bar{C}_{11} &= \frac{\bar{s}_1}{\bar{e}_1} = 0.13 \cdot 10^6 \\ \bar{C}_{21} &= \frac{\bar{s}_2}{\bar{e}_1} = 0.013 \cdot 10^6 \\ \bar{C}_{31} &= \frac{\bar{s}_3}{\bar{e}_1} = 0.02 \cdot 10^6 \\ \bar{C}_{41} &= \bar{C}_{51} = \bar{C}_{61} = 0 \end{aligned} \tag{6.4-7}$$

- case of normal strain in *y*-direction:

$$\begin{aligned} \bar{C}_{12} &= \frac{\bar{s}_1}{\bar{e}_2} = 0.013 \cdot 10^6 \\ \bar{C}_{22} &= \frac{\bar{s}_2}{\bar{e}_2} = 0.08 \cdot 10^6 \\ \bar{C}_{32} &= \frac{\bar{s}_3}{\bar{e}_2} = 0.014 \cdot 10^6 \\ \bar{C}_{42} &= \bar{C}_{52} = \bar{C}_{62} = 0 \end{aligned} \tag{6.4-8}$$

- case of normal strain in *z*-direction:

$$\begin{aligned} \bar{C}_{13} &= \frac{\bar{s}_1}{\bar{e}_3} = 0.02 \cdot 10^6 \\ \bar{C}_{23} &= \frac{\bar{s}_2}{\bar{e}_3} = 0.014 \cdot 10^6 \\ \bar{C}_{33} &= \frac{\bar{s}_3}{\bar{e}_3} = 0.17 \cdot 10^6 \\ \bar{C}_{43} &= \bar{C}_{53} = \bar{C}_{63} = 0 \end{aligned} \tag{6.4-9}$$

- case of shear strain in *zy*-plane:

$$
\begin{aligned}
&\bar{C}_{14} = \bar{C}_{24} = \bar{C}_{34} = 0 \\
&\bar{C}_{44} = \frac{\bar{\sigma}_4}{\bar{\varepsilon}_4} = 0.07 \cdot 10^6 \\
&\bar{C}_{54} = \bar{C}_{64} = 0
\end{aligned}
\tag{6.4-10}
$$

- case of shear strain in *zx*-plane:

$$
\begin{aligned}
&\bar{C}_{15} = \bar{C}_{25} = \bar{C}_{35} = \bar{C}_{45} = 0 \\
&\bar{C}_{55} = \frac{\bar{\sigma}_5}{\bar{\varepsilon}_5} = 0.11 \cdot 10^6 \\
&\bar{C}_{65} = 0
\end{aligned}
\tag{6.4-11}
$$

- case of shear strain in x*y*-plane:

$$
\begin{aligned}
&\bar{C}_{16} = \bar{C}_{26} = \bar{C}_{36} = \bar{C}_{46} = \bar{C}_{56} = 0 \\
&\bar{C}_{66} = \frac{\bar{\sigma}_6}{\bar{\varepsilon}_6} = 0.06 \cdot 10^6
\end{aligned}
\tag{6.4-12}
$$

Hence, the homogenized stiffness tensor assumes the following form:

$$
\mathbf{C}_{F.E.M.}^{Hom} = 10^6 \begin{bmatrix} 0.13 & 0.013 & 0.02 & 0 & 0 & 0 \\ 0.013 & 0.08 & 0.014 & 0 & 0 & 0 \\ 0.02 & 0.014 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.06 \end{bmatrix}
\tag{6.4-13}
$$

- **Physic elastic compliances**

The procedure used for determining the physic elastic stiffness coefficients is analogous to the one used for determining the homogenized stiffness tensor.

It occurs to calculate the volume average value of nominal strain, $\bar{\varepsilon}_j$. In case of normal strain components, it is obtained as:

$$\bar{\varepsilon}_j = \frac{u_j^0}{l_j} \qquad (6.4\text{-}14)$$

where:

l_j = the characteristic RVE lengths

u_j^0 = the unit prescribed displacement

Analogously, the volume average value of the nominal shear strain components can be obtained.

At this point, it occurs to calculate the average value of nominal stress, $\bar{\sigma}_i$, obtained as:

$$\bar{\sigma}_i = \frac{F_i}{A} \qquad (6.4\text{-}15)$$

where:

F_i = the resulting force on the loaded surface

A = the area of the loaded surface

Hence, the properties of the homogenized cell can be determined by means of equation (6.4-2). In detail, in the follows, the found coefficients of the physic stiffness tensor are shown for the six loading conditions:

- case of normal strain in *x*-direction:

$$\begin{aligned} \bar{C}_{11} &= \frac{\bar{\sigma}_1}{\bar{\varepsilon}_1} = 0.13 \cdot 10^6 \\ \bar{C}_{21} &= \frac{\bar{\sigma}_2}{\bar{\varepsilon}_1} = 0.013 \cdot 10^6 \\ \bar{C}_{31} &= \frac{\bar{\sigma}_3}{\bar{\varepsilon}_1} = 0.02 \cdot 10^6 \\ \bar{C}_{41} &= \bar{C}_{51} = \bar{C}_{61} = 0 \end{aligned} \qquad (6.4\text{-}16)$$

- case of normal strain in *y*-direction:

$$\begin{aligned}
&\bar{C}_{12} = \frac{\bar{\sigma}_1}{\bar{\varepsilon}_2} = 0.013 \cdot 10^6 \\
&\bar{C}_{22} = \frac{\bar{\sigma}_2}{\bar{\varepsilon}_2} = 0.08 \cdot 10^6 \\
&\bar{C}_{32} = \frac{\bar{\sigma}_3}{\bar{\varepsilon}_2} = 0.014 \cdot 10^6 \\
&\bar{C}_{42} = \bar{C}_{52} = \bar{C}_{62} = 0
\end{aligned} \tag{6.4-17}$$

- case of normal strain in *z*-direction:

$$\begin{aligned}
&\bar{C}_{13} = \frac{\bar{\sigma}_1}{\bar{\varepsilon}_3} = 0.02 \cdot 10^6 \\
&\bar{C}_{23} = \frac{\bar{\sigma}_2}{\bar{\varepsilon}_3} = 0.014 \cdot 10^6 \\
&\bar{C}_{33} = \frac{\bar{\sigma}_3}{\bar{\varepsilon}_3} = 0.17 \cdot 10^6 \\
&\bar{C}_{43} = \bar{C}_{53} = \bar{C}_{63} = 0
\end{aligned} \tag{6.4-18}$$

- case of shear strain in *zy*-plane:

$$\begin{aligned}
&\bar{C}_{14} = \bar{C}_{24} = \bar{C}_{34} = 0 \\
&\bar{C}_{44} = \frac{\bar{\sigma}_4}{\bar{\varepsilon}_4} = 0.07 \cdot 10^6 \\
&\bar{C}_{54} = \bar{C}_{64} = 0
\end{aligned} \tag{6.4-19}$$

- case of shear strain in *zx*-plane:

$$\begin{aligned}
&\bar{C}_{15} = \bar{C}_{25} = \bar{C}_{35} = \bar{C}_{45} = 0 \\
&\bar{C}_{55} = \frac{\bar{\sigma}_5}{\bar{\varepsilon}_5} = 0.11 \cdot 10^6 \\
&\bar{C}_{65} = 0
\end{aligned}$$

- case of shear strain in *xy*-plane:

$$\bar{C}_{16} = \bar{C}_{26} = \bar{C}_{36} = \bar{C}_{46} = \bar{C}_{56} = 0$$
$$\bar{C}_{66} = \frac{\bar{\sigma}_6}{\bar{\varepsilon}_6} = 0.06 \cdot 10^6 \qquad (6.4\text{-}21)$$

Hence, the homogenized stiffness tensor assumes the following form:

$$\mathbb{C}^{Phys}_{F.E.M.} = 10^6 \begin{bmatrix} 0.13 & 0.013 & 0.02 & 0 & 0 & 0 \\ 0.013 & 0.08 & 0.014 & 0 & 0 & 0 \\ 0.02 & 0.014 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.06 \end{bmatrix} \qquad (6.4\text{-}22)$$

It is worth to notice that the homogenized and the physic stiffness tensor are equal. It is due to the linearity of the problem.

The reader is referred to the appendix to this chapter for the plotting of the obtained results.

6.5 Numerical Voigt and Reuss estimation

In the chapter 1, general concepts were illustrated about the Reuss and Voigt estimations of the overall elastic stiffness and compliance coefficients. Moreover, it was underlined that such estimations are extremely useful bounds since the actual overall moduli of a heterogeneous material lie somewhere in an interval between the Reuss and Voigt estimates.

Thus, in order to obtain the Reuss and Voigt elastic tensors, mortar and brick volumetric fractions are calculated for the simple above illustrated RVE. For clearness of exposition, this latter is shown again, in the following figure 6.4, with its numerical characteristic dimensions. The unit of measurement is the centimetre. Since the examined basic cell referrers to a single leaf masonry wall, the characteristic length in *z*-direction ($l_z = 10\,cm$) doesn't influence the volumetric fractions. Hence, the figure 6.4 shows the RVE in *xy*-plane.

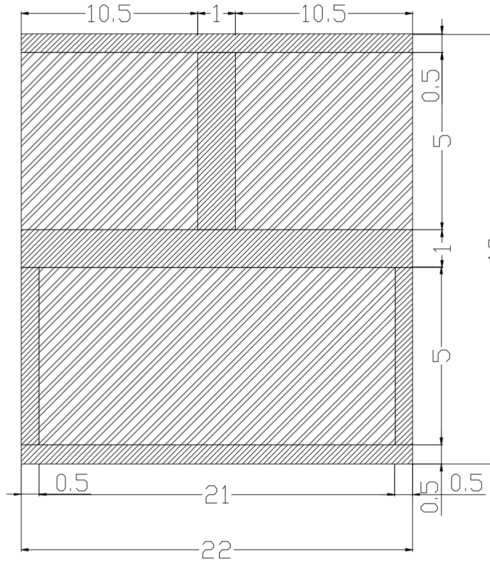


Figure 6.4 Utilized RVE (*xy*-plane) in F.E.M. analysis.

The volumetric fractions can, so, be determined as:

$$f_i = \frac{A_i}{A} \qquad i = mortar, brick \tag{6.5-1}$$

where:

A_i = mortar and brick surface in *xy*-plane

A = RVE surface in *xy*-plane

By operating some calculation, it is obtained that:

$$\begin{aligned} f_m &= 0.2 \\ f_b &= 0.8 \end{aligned} \tag{6.5-2}$$

with:

f_m = mortar volumetric fraction

f_b = brick volumetric fraction

By remembering that the overall compliance tensor obtained in the Reuss approximation is given as:

$$\overline{\mathsf{S}}^R = f_m \mathsf{S}^m + f_b \mathsf{S}^b \qquad (6.5\text{-}3)$$

where the superscript R just stands for Reuss and where:

S^m = mortar compliance tensor

S^b = brick compliance tensor

Since both mortar and brick are considered homogeneous isotropic materials, the Reuss compliance tensor shows, evidently, only three elastic coefficients different from zero, and only two of these are independent. By taking in account the assumed numerical data in (6.2-1), the Reuss compliance tensor assumes the following form:

$$\mathsf{S}^R = 10^{-6} \begin{bmatrix} 14 & -2.1 & -2.1 & 0 & 0 & 0 \\ -2.1 & 14 & -2.1 & 0 & 0 & 0 \\ -2.1 & -2.1 & 14 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16.1 \end{bmatrix} \qquad (6.5\text{-}4)$$

From it, the Reuss stiffness tensor can be determined as the inverse of the Reuss compliance one. Thus, it assumes the following form:

$$\mathsf{C}^R = \left(\mathsf{S}^R\right)^{-1} = 10^6 \begin{bmatrix} 0.08 & 0.01 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.08 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0.08 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.06 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.06 \end{bmatrix} \quad (6.5\text{-}5)$$

Analogous procedure is employed for determining the overall stiffness tensor in the Voigt approximation. It is given as:

$$\overline{\mathsf{C}}^V = f_m \mathsf{C}^m + f_b \mathsf{C}^b \quad (6.5\text{-}6)$$

where the superscript V just stands for Voigt and where:

C^m = mortar stiffness tensor

C^b = brick stiffness tensor

Also the Voigt stiffness tensor shows, evidently, only three elastic coefficients different from zero, and only two of these are independent. By taking in account the assumed numerical data in (6.2-1), yet, the Voigt stiffness tensor assumes the following form:

$$\mathsf{C}^V = 10^6 \begin{bmatrix} 0.17 & 0.03 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.17 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.03 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.14 \end{bmatrix} \quad (6.5\text{-}7)$$

From it, the Voigt compliance tensor can be determined as the inverse of the Voigt stiffness one. Thus, it assumes the following form:

$$\mathsf{S}^{V}=\left(\mathsf{C}^{V}\right)^{-1}=10^{6}\begin{bmatrix} 6.21 & -0.93 & -0.93 & 0 & 0 & 0 \\ -0.93 & 6.21 & -0.93 & 0 & 0 & 0 \\ -0.93 & -0.93 & 6.21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.14 \end{bmatrix} \quad (6.5\text{-}8)$$

6.6 Numerical results for the analyzed homogenization techniques

In the chapter 3, an account of the literature data on masonry homogenization procedures has been analyzed and, in this framework, some techniques have been studied more in detail. Then, in the following chapter IV, starting from those literature approaches, two theoretical homogenization procedures have been proposed: the statically-consistent Lourenco approach and the S.A.S. one.

The goal of this chapter is to obtain the numerical results from these proposed techniques and, then, to compare them with the ones obtained from the most recent literature approach: the one-step Lourenco-Zucchini homogenization.

6.6.1 Numerical results for Lourenco-Zucchini approach

It is shown, in the following figure 6.5, the RVE adopted by the authors Lourenco and Zucchini with its numerical characteristic dimensions. It is worth to remember that the assigned dimensions are so that there is equivalence, in the volumetric fractions, between such RVE and the examined F.E.M. model.

In particular, the model has the following dimensions:

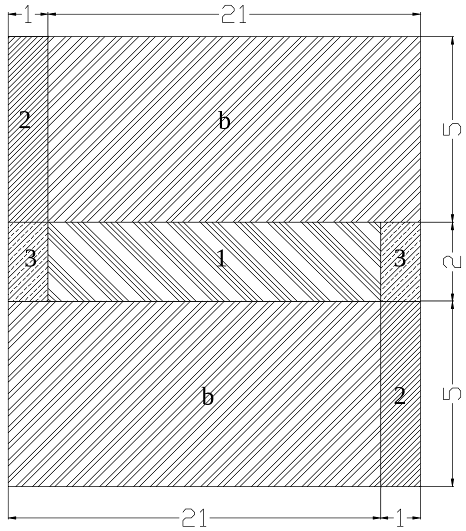


Figure 6.5 Lourenco's RVE in *xy*-plane.

The unit of measurement is the centimetre. Since also this examined basic cell referrers to a single leaf masonry wall, the characteristic length in *z*-direction ($l_z = 10\,cm$) doesn't influence the volumetric fractions. Hence, the figure 6.5 shows the RVE in *xy*-plane, again.

Let us recall the analytical results, which are obtained by the authors. Thus, it is:

$$\bar{S}_{1111} = \frac{\bar{\mathrm{e}}_{xx}^{(\mathrm{s}_{xx}^{0})}}{\mathrm{s}_{xx}^{0}} = \mathrm{e}_{xx}^{(1)^{(\mathrm{s}_{xx}^{0})}} \frac{l - t + 2t\,E^{(1)} / E^{(3)}}{l + t}$$

$$\bar{S}_{2211} = \frac{\bar{\mathrm{e}}_{yy}^{(\mathrm{s}_{xx}^{0})}}{\mathrm{s}_{xx}^{0}} = \frac{\mathrm{e}_{yy}^{(2)^{(\mathrm{s}_{xx}^{0})}} \left(h + 2t\,E^{(2)} / E^{(3)}\right) + h\mathrm{e}_{yy}^{(b)^{(\mathrm{s}_{xx}^{0})}}}{l + t}$$

$$\bar{S}_{3311} = \frac{\bar{\mathrm{e}}_{zz}^{(\mathrm{s}_{xx}^{0})}}{\mathrm{s}_{xx}^{0}} = \mathrm{e}_{zz}^{(b)^{(\mathrm{s}_{xx}^{0})}}$$

$$\bar{S}_{1122} = \frac{\bar{\mathrm{e}}_{xx}^{(\mathrm{s}_{yy}^{0})}}{\mathrm{s}_{yy}^{0}} = \mathrm{e}_{xx}^{(1)^{(\mathrm{s}_{yy}^{0})}} \frac{l - t + 2t\,E^{(1)} / E^{(3)}}{l + t}$$

$$\bar{S}_{2222} = \frac{\bar{\mathrm{e}}_{yy}^{(\mathrm{s}_{yy}^{0})}}{\mathrm{s}_{yy}^{0}} = \frac{\mathrm{e}_{yy}^{(2)^{(\mathrm{s}_{yy}^{0})}} \left(h + 2t\,E^{(2)} / E^{(3)}\right) + h\mathrm{e}_{yy}^{(b)^{(\mathrm{s}_{yy}^{0})}}}{l + t}$$

$$\bar{S}_{3322} = \frac{\bar{\mathrm{e}}_{zz}^{(\mathrm{s}_{yy}^{0})}}{\mathrm{s}_{yy}^{0}} = \mathrm{e}_{zz}^{(b)^{(\mathrm{s}_{yy}^{0})}}$$

$$\bar{S}_{1133} = \frac{\bar{\mathrm{e}}_{xx}^{(\mathrm{s}_{zz}^{0})}}{\mathrm{s}_{zz}^{0}} = \mathrm{e}_{xx}^{(1)^{(\mathrm{s}_{zz}^{0})}} \frac{l - t + 2t\,E^{(1)} / E^{(3)}}{l + t}$$

$$\bar{S}_{2233} = \frac{\bar{\mathrm{e}}_{yy}^{(\mathrm{s}_{zz}^{0})}}{\mathrm{s}_{zz}^{0}} = \frac{\mathrm{e}_{yy}^{(2)^{(\mathrm{s}_{zz}^{0})}} \left(h + 2t\,E^{(2)} / E^{(3)}\right) + h\mathrm{e}_{yy}^{(b)^{(\mathrm{s}_{zz}^{0})}}}{l + t}$$

$$\bar{S}_{3333} = \frac{\bar{\mathrm{e}}_{zz}^{(\mathrm{s}_{zz}^{0})}}{\mathrm{s}_{zz}^{0}} = \mathrm{e}_{zz}^{(b)^{(\mathrm{s}_{zz}^{0})}}$$

$$\bar{S}_{3232} = \frac{\bar{\mathrm{e}}_{yz}}{\mathrm{s}_{yz}^{0}} = \frac{t + l}{(2t + h)G^{(1)}} \frac{tG^{(b)} + hG^{(1)}}{lG^{(b)} + tG^{(2)}}$$

$$\bar{S}_{3131} = \frac{\bar{\mathrm{e}}_{xz}}{\mathrm{s}_{xz}^{0}} = \frac{\left(t + h\right)\left(t \dfrac{4hG^{(b)} + \left(l - t\right)G^{(1)}}{4hG^{(2)} + \left(l - t\right)G^{(1)}} + l\right)}{2\left(t + l\right)\left(tG^{(1)} + hG^{(b)}\right)}$$

$$\bar{S}_{1212} = \frac{\bar{\mathrm{e}}_{xy}}{\mathrm{s}_{xy}^{0}} = \frac{k \dfrac{tl\left(t + h\right)}{G^{(2)}} + \dfrac{\left(t + l - kt\right)\left(lh - t^{(2)}\right)}{G^{(b)}} + \dfrac{t\left(t + l\right)\left(t + l - kt\right)}{G^{(1)}}}{2l\left(t + l\right)\left(t + h\right)} \qquad (6.6.1\text{-}1)$$

By taking into account the numerical data given by the equation (6.2-1) and by substituting the numerical value of the strains which the authors have obtained for each basic cell constituent, the overall compliance tensor is finally determined. It assumes the following form:

$$\mathsf{S}^{L\text{-}Z} = 10^{-6}\begin{bmatrix} 7.47 & -1.02 & -0.92 & 0 & 0 & 0 \\ -0.54 & 6.83 & -0.5 & 0 & 0 & 0 \\ -0.91 & -0.92 & 6.13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.99 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 \end{bmatrix} \qquad (6.6.1\text{-}2)$$

where the superscript $^{L-Z}$ stands for Lourenco-Zucchini.

From it, also the overall stiffness tensor is obtained as the inverse of the overall compliance one. Hence, it assumes the following form:

$$\mathsf{C}^{L\text{-}Z} = \left(\mathsf{S}^{L\text{-}Z}\right)^{-1} = 10^{6}\begin{bmatrix} 0.14 & 0.024 & 0.023 & 0 & 0 & 0 \\ 0.013 & 0.15 & 0.014 & 0 & 0 & 0 \\ 0.022 & 0.026 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.067 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.064 \end{bmatrix} \qquad (6.6.1\text{-}3)$$

6.6.2 Numerical results for the statically-consistent Lourenco-approach

The RVE adopted in this proposed approach is the same one used by Lourenco and Zucchini. For clearness of exposition, it is shown here, again, with its numerical characteristic dimensions:

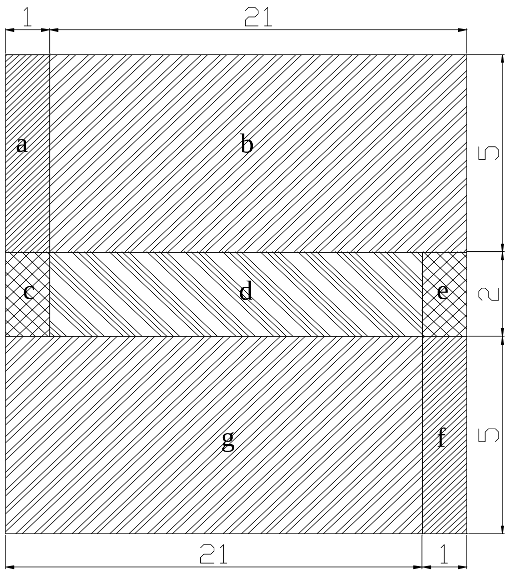


Figure 6.6 Utilized RVE (*xy*-plane) in Lourenco modified approach.

The unit of measurement is the centimetre again.

Let us recall the analytical results which we have obtained with this homogenization procedure. Thus, it is:

$$\bar{S}_{1111} = \bar{S}_{2222} = \bar{S}_{3333} = = -\frac{-\frac{h\cdot l}{Eb} - \frac{h\cdot l}{Ef} + t\cdot\left(-\frac{h}{Ea} - \frac{h}{Eg} - \frac{2\cdot l}{Ed} - \frac{2\cdot t}{Ec} + \frac{2\cdot t}{Ed} - \frac{2\cdot t}{Ee}\right)}{2\cdot(h+t)\cdot(l+t)} \quad (6.6.2\text{-}1)$$

$$\bar{S}_{1122} = \bar{S}_{1133} = \bar{S}_{2233} = = -\frac{\frac{h\cdot t\cdot \mathrm{n}a}{Ea} + \frac{h\cdot l\cdot \mathrm{n}b}{Eb} + \frac{2\cdot t^2\cdot \mathrm{n}c}{Ec} + \frac{2\cdot(l-t)\cdot t\cdot \mathrm{n}d}{Ed} + \frac{2\cdot t^2\cdot \mathrm{n}e}{Ee} + \frac{h\cdot l\cdot \mathrm{n}f}{Ef} + \frac{h\cdot t\cdot \mathrm{n}g}{Eg}}{2\cdot(h+t)\cdot(l+t)} \quad (6.6.2\text{-}2)$$

$$\bar{S}_{3131} = \frac{1}{2 \cdot G_{xz}} = \frac{(h+t) \cdot \left(t \cdot \frac{4 \cdot Gb \cdot h + (l - t) \cdot Gd}{4 \cdot Ga \cdot h + (l - t) \cdot Gd} + l \right)}{2 \cdot (t+l) \cdot (t \cdot Gd + h \cdot Gb)} \tag{6.6.2-3}$$

$$\bar{S}_{3232} = \frac{1}{2 \cdot G_{yz}} = \frac{1}{2 \cdot Gd} \cdot \frac{t+l}{t+h} \cdot \frac{t \cdot Gb + h \cdot Gd}{l \cdot Gb + t \cdot Ga} \tag{6.6.2-4}$$

$$\begin{aligned} \bar{S}_{1212} = \frac{1}{2 \cdot G_{xy}} = & \frac{4 \cdot Gb \cdot Gd \cdot h \cdot t \cdot (h+t)}{2 \cdot Gd \cdot (h+t) \cdot (Ed \cdot Gb \cdot l^2 + Ga \cdot (Ed \cdot l \cdot t + 4 \cdot Gb \cdot h \cdot (l+t)))} + \\ & + \frac{Ed \cdot l \cdot (l+t) \cdot (Gd \cdot h + Gb \cdot t) + 4 \cdot Ga \cdot h \cdot (Gb \cdot t \cdot (l+t) + Gd \cdot (h \cdot l - t^2))}{2 \cdot Gd \cdot (h+t) \cdot (Ed \cdot Gb \cdot l^2 + Ga \cdot (Ed \cdot l \cdot t + 4 \cdot Gb \cdot h \cdot (l+t)))} \end{aligned} \tag{6.6.2-5}$$

By taking into account the numerical data given by the equation (6.2-1), the overall compliance tensor is finally determined. It assumes the following form:

$$\mathsf{S}^{S\text{-}c} = 10^{-6} \begin{bmatrix} 14.2 & -2.1 & -2.1 & 0 & 0 & 0 \\ -2.1 & 14.2 & -2.1 & 0 & 0 & 0 \\ -2.1 & -2.1 & 14.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.99 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 \end{bmatrix} \tag{6.6.2-6}$$

where the superscript $^{S-c}$ stands for statically-consistent approach.

From it, also the overall stiffness tensor is obtained as the inverse of the overall compliance one. Hence, it assumes the following form:

$$\mathsf{C}^{S\text{-}c} = \left(\mathsf{S}^{S\text{-}c}\right)^{-1} = 10^{6} \begin{bmatrix} 0.074 & 0.013 & 0.013 & 0 & 0 & 0 \\ 0.013 & 0.074 & 0.013 & 0 & 0 & 0 \\ 0.013 & 0.013 & 0.074 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.067 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.064 \end{bmatrix} \tag{6.6.2-7}$$

6.6.3 Numerical results for the S.A.S. approach

It is shown, in the following figure 6.7, the RVE adopted in this proposed approach with its numerical characteristic dimensions. It is worth to remember, again, that the assigned dimensions are so that there is equivalence, in the volumetric fractions, between such RVE and the examined F.E.M. model.
In particular, the model has the following dimensions:

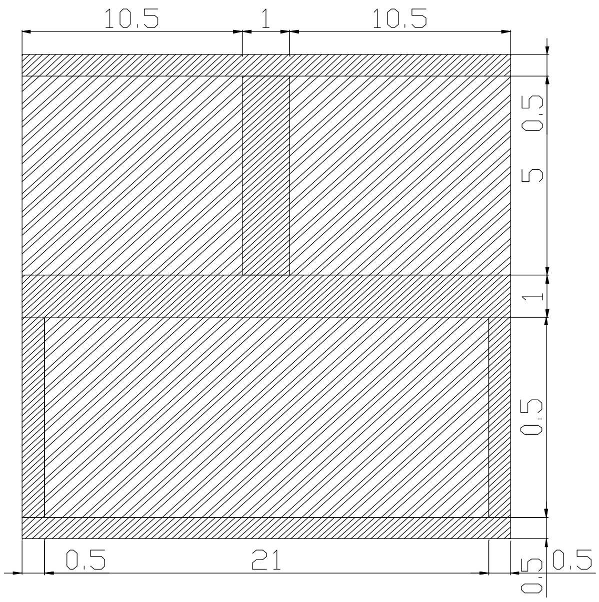


Figure 6.7 Utilized RVE (*xy*-plane) in S.A.S. approach.

The unit of measurement is the centimetre again.

Let us recall the analytical results, which we obtained with this homogenization procedure. Thus, it is:

$$\bar{\bar{\mathsf{S}}}^{I} = \begin{bmatrix} \bar{\bar{S}}^{I}_{1111} & -\dfrac{\nu_{12}^{(m)}}{E_1^{(m)}\phi} & -\dfrac{\nu_{13}^{(m)}}{E_1^{(m)}\phi} & 0 & 0 & 0 \\ & \dfrac{1}{E_2^{(m)}\phi} & -\dfrac{\nu_{23}^{(m)}}{E_2^{(m)}\phi} & 0 & 0 & 0 \\ & & \dfrac{1}{E_3^{(m)}\phi} & 0 & 0 & 0 \\ & & & \dfrac{1}{2G_{32}^{(m)}\phi} & 0 & 0 \\ & Sym & & & \bar{\bar{S}}^{I}_{3131} & 0 \\ & & & & & \bar{\bar{S}}^{I}_{1212} \end{bmatrix} \tag{6.6.3-1}$$

where:

$$\phi = \left(f_m + f_b \varphi_b \right) \tag{6.6.3-2}$$

By taking into account the numerical data given by the equation (6.2-1), the overall compliance tensor is finally determined. It assumes the following form:

$$\mathsf{S}^{S.A.S.} = 10^{-6} \begin{bmatrix} 6.13 & -0.92 & -0.92 & 0 & 0 & 0 \\ -0.92 & 6.13 & -0.92 & 0 & 0 & 0 \\ -0.92 & -0.92 & 6.13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.05 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.05 \end{bmatrix} \tag{6.6.3-3}$$

where the superscript $^{S.A.S.}$ stands for S.A.S. approach.

From it, also the overall stiffness tensor is obtained as the inverse of the overall compliance one. Hence, it assumes the following form:

$$\mathrm{C}^{S.A.S} = (\mathrm{S}^{S.A.S})^{-1} = 10^6 \begin{bmatrix} 0.17 & 0.03 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.17 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.03 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.14 \end{bmatrix} \quad (6.6.3\text{-}4)$$

6.7 Comparisons for numerical results

In this paragraph, a comparison between the numerical results, obtained by the examined homogenization techniques, illustrated in the previous sections, is made. In particular, it is worth to notice that the numerical estimate of the homogenized coefficients proposed by Lourenco et al. furnishes a not symmetrical elasticity tensor, as it is shown in the following table.

	FEM^{HOM} stress-prescr.	FEM^{HOM} strain-prescr.		REUSS		VOIGT		AVERAGE VOIGT/REUSS		LOURENCO		LOURENCO MODIFIED		S.A.S.	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value %	Error %	Value	Error %
$S_{11}*10^6$	7.98	7.94	-0.5	14	75.4	6.21	-22.2	10.105	26.6	7.47	-6.4	14.2	77.9	6.13	-23.2
$S_{12}*10^6$	-1.09	-1.14	4.6	-2.1	92.7	-0.93	-14.7	-1.515	39	~~-1.02~~	-6.4	-2.1	92.7	-0.92	-15.6
$S_{13}*10^6$	-0.95	-0.84	-11.6	-2.1	121	-0.93	-2.1	-1.515	59.5	-0.92	-3.2	-2.1	121	-0.92	-3.2
$S_{21}*10^6$	-1.09	-1.14	4.6	-2.1	92.7	-0.93	-14.7	-1.515	39	~~-0.54~~	-50.5	-2.1	92.7	-0.92	-15.6
$S_{22}*10^6$	12.6	12.8	1.6	14	11.1	6.21	-50.7	10.105	-19.8	6.83	-45.8	14.2	12.7	6.13	-51.3
$S_{23}*10^6$	-1.04	-0.92	-11.5	-2.1	102	-0.93	-10.6	-1.515	45.7	~~-0.5~~	-51.9	-2.1	102	-0.92	-11.5
$S_{31}*10^6$	-0.95	-0.84	-11.6	-2.1	121	-0.93	-2.1	-1.515	59.5	-0.91	-4.2	-2.1	121	-0.92	-3.2
$S_{32}*10^6$	-1.04	-0.92	-11.5	-2.1	102	-0.93	-10.6	-1.515	45.7	~~-0.92~~	-11.5	-2.1	102	-0.92	-11.5
$S_{33}*10^6$	6.6	6.06	-8.2	14	112	6.21	-5.9	10.105	53.1	6.13	-7.1	14.2	115	6.13	-7.1
$S_{44}*10^6$	14.78	14.3	-3.2	16.1	8.9	7.14	-51.7	11.62	-21.4	14.9	0.8	14.9	0.8	7.05	-52.3
$S_{55}*10^6$	9.19	9.09	-1.1	16.1	75.2	7.14	-22.3	11.62	26.4	8.15	-11.3	8.15	-11.3	7.05	-23.3
$S_{66}*10^6$	15.9	16.7	5.03	16.1	1.3	7.14	-55.1	11.62	-26.9	15	-5.7	15	-5.7	7.05	-55.7

Table 6.1 Comparisons.

Hence, in order to ensure consistency to the model a symmetrization of Lourenco's elasticity tensor should be suggested.

Thus, in the following tables, a new comparison is made between the numerical results obtained by the proposed homogenization techniques and those ones obtained by means of the effected Lourenco's tensor symmetrization, with reference to all the employed F.E.M. analyses. In each table, the procedure which furnishes the numerical reference results is marked with a red ring and our proposed techniques are in red fonts.

	FEM^{HOM} stress-prescr.	FEM^{HOM} strain-prescr.		REUSS		VOIGT		AVERAGE VOIGT/REUSS		LOURENCO SYM		LOURENCO MODIFIED		S.A.S.	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value %	Error %	Value	Error %
$S_{11}*10^6$	7.98	7.94	-0.5	14	75.4	6.21	-22.2	10.105	26.6	7.47	-6.4	14.2	77.9	6.13	-23.2
$S_{12}*10^6$	-1.09	-1.14	4.6	-2.1	92.7	-0.93	-14.7	-1.515	39	-0.78	-28.4	-2.1	92.7	-0.92	-15.6
$S_{13}*10^6$	-0.95	-0.84	-11.6	-2.1	**121**	-0.93	-2.1	-1.515	59.5	-0.91	-4.2	-2.1	121	-0.92	-3.2
$S_{21}*10^6$	-1.09	-1.14	4.6	-2.1	92.7	-0.93	-14.7	-1.515	39	-0.78	-28.4	-2.1	92.7	-0.92	-15.6
$S_{22}*10^6$	12.6	12.8	1.6	14	11.1	6.21	-50.7	10.105	-19.8	**6.83**	**-45.8**	**14.2**	**12.7**	**6.13**	**-51.3**
$S_{23}*10^6$	-1.04	-0.92	-11.5	-2.1	102	-0.93	-10.6	-1.515	45.7	**-0.71**	**-31.7**	**-2.1**	**102**	**-0.92**	**-11.5**
$S_{31}*10^6$	-0.95	-0.84	-11.6	-2.1	**121**	-0.93	-2.1	-1.515	59.5	-0.91	-4.2	-2.1	121	-0.92	-3.2
$S_{32}*10^6$	-1.04	-0.92	-11.5	-2.1	102	-0.93	-10.6	-1.515	45.7	-0.71	-31.7	-2.1	102	-0.92	-11.5
$S_{33}*10^6$	6.6	6.06	-8.2	14	112	6.21	-5.9	10.105	53.1	6.13	-7.1	14.2	115	6.13	-7.1
$S_{44}*10^6$	14.78	14.3	-3.2	16.1	8.9	7.14	-51.7	11.62	-21.4	14.9	**0.8**	14.9	**0.8**	7.05	-52.3
$S_{55}*10^6$	9.19	9.09	-1.1	16.1	75.2	7.14	-22.3	11.62	26.4	8.15	-11.3	8.15	-11.3	7.05	-23.3
$S_{66}*10^6$	15.9	16.7	5.03	16.1	1.3	7.14	-55.1	11.62	-26.9	15	-5.7	15	-5.7	7.05	-55.7

Table 6.2 Comparisons.

	FEM^{PHYS} stress-prescr.	FEM^{PHYS} strain-prescr.		REUSS		VOIGT		AVERAGE VOIGT/REUSS		LOURENCO SYM		LOURENCO MODIFIED		S.A.S.	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value %	Error %	Value	Error %
$S_{11}*10^6$	8.05	7.94	-1.37	14	73.9	6.21	-22.9	10.105	25.5	7.47	-7.2	14.2	76.4	6.13	-23.9
$S_{12}*10^6$	-1.13	-1.14	0.88	-2.1	85.8	-0.93	-17.7	-1.515	34.1	-0.78	-31	-2.1	85.8	-0.92	-18.6
$S_{13}*10^6$	-0.97	-0.84	-13.4	-2.1	**116**	-0.93	-4.1	-1.515	56.2	-0.91	-6.2	-2.1	116	-0.92	-5.2
$S_{21}*10^6$	-1.13	-1.14	0.88	-2.1	85.8	-0.93	-17.7	-1.515	34.1	-0.78	-31	-2.1	85.8	-0.92	-18.6
$S_{22}*10^6$	12.6	12.8	1.59	14	11.1	6.21	-50.7	10.105	-19.8	**6.83**	**-45.8**	**14.2**	**12.7**	**6.13**	**-51.3**
$S_{23}*10^6$	-1.1	-0.92	-16.4	-2.1	90.9	-0.93	-15.5	-1.515	37.7	**-0.71**	**-35.5**	**-2.1**	**90.9**	**-0.92**	**-16.4**
$S_{31}*10^6$	-0.97	-0.84	-13.4	-2.1	**116**	-0.93	-4.1	-1.515	56.2	-0.91	-6.2	-2.1	116	-0.92	-5.2
$S_{32}*10^6$	-1.1	-0.92	-16.4	-2.1	90.9	-0.93	-15.5	-1.515	37.7	-0.71	-35.5	-2.1	90.9	-0.92	-16.4
$S_{33}*10^6$	6.8	6.06	-10.9	14	106	6.21	-8.7	10.105	48.6	6.13	-9.9	14.2	109	6.13	-9.9
$S_{44}*10^6$	15.07	14.3	-5.11	16.1	6.84	7.14	-52.6	11.62	-22.9	14.9	**-1.1**	14.9	**-1.1**	7.05	-53.2
$S_{55}*10^6$	8.61	9.09	5.57	16.1	86.9	7.14	-17.1	11.62	35	8.15	-5.3	8.15	-5.3	7.05	-18.1
$S_{66}*10^6$	14.54	16.7	14.9	16.1	10.7	7.14	-50.9	11.62	-20.1	15	3.16	15	3.2	7.05	-51.5

Table 6.3 Comparisons.

	FEM^{HOM} strain-prescr.	FEM^{HOM} stress-prescr.		REUSS		VOIGT		AVERAGE VOIGT/REUSS		LOURENCO SYM		LOURENCO MODIFIED		S.A.S.	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %
$C_{11}*10^{-6}$	0.13	0.13	0	0.08	38.5	0.17	30.8	0.125	-3.8	0.14	7.7	0.074	-43.1	0.17	30.8
$C_{12}*10^{-6}$	0.013	0.013	0	0.01	-23.1	0.03	131	0.02	53.8	0.019	46.2	0.013	**0**	0.03	131
$C_{13}*10^{-6}$	0.02	0.021	5	0.01	-50	0.03	50	0.02	**0**	0.023	15	0.013	-35	0.03	50
$C_{21}*10^{-6}$	0.013	0.013	0	0.01	-23.1	0.03	131	0.02	53.8	0.019	**46.2**	0.013	**0**	0.03	**131**
$C_{22}*10^{-6}$	0.08	0.082	2.5	0.08	**0**	0.17	113	0.125	56.2	0.15	**87.5**	0.074	**-7.5**	0.17	**112**
$C_{23}*10^{-6}$	0.014	0.015	7.14	0.01	-28.6	0.03	114	0.02	42.9	0.02	42.9	0.013	-7.1	0.03	114
$C_{31}*10^{-6}$	0.02	0.021	5	0.01	-50	0.03	50	0.02	**0**	0.023	15	0.013	-35	0.03	50
$C_{32}*10^{-6}$	0.014	0.015	7.14	0.01	-28.6	0.03	114	0.02	42.9	0.02	42.9	0.013	-7.1	0.03	114
$C_{33}*10^{-6}$	0.17	0.16	-5.9	0.08	-52.9	0.17	**0**	0.125	-26.5	0.17	**0**	0.074	-56.5	0.17	**0**
$C_{44}*10^{-6}$	0.07	0.068	-2.9	0.06	-14.3	0.14	100	0.1	42.9	0.067	-4.3	0.067	-4.3	0.14	100
$C_{55}*10^{-6}$	0.11	0.11	0	0.06	-45.5	0.14	27.3	0.1	-9.09	0.12	9.09	0.12	9.09	0.14	27.3
$C_{66}*10^{-6}$	0.06	0.063	5	0.06	**0**	0.14	**133**	0.1	66.7	0.064	6.7	0.064	6.7	0.14	**133**

Table 6.4 Comparisons.

	FEM^{PH} strain-prescr.	FEM^{PHYS} stress-prescr.		REUSS		VOIGT		AVERAGE VOIGT/REUSS		LOURENCO SYM		LOURENCO MODIFIED		S.A.S.	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %
$C_{11}*10^{-6}$	0.13	0.13	0	0.08	-38.5	0.17	30.8	0.125	-3.8	0.14	7.7	0.074	-43.1	0.17	30.8
$C_{12}*10^{-6}$	0.013	0.013	0	0.01	-23.1	0.03	131	0.02	53.8	0.019	46.2	0.013	**0**	0.03	131
$C_{13}*10^{-6}$	0.02	0.02	0	0.01	-50	0.03	50	0.02	**0**	0.023	15	0.013	-35	0.03	50
$C_{21}*10^{-6}$	0.013	0.013	0	0.01	-23.1	0.03	131	0.02	53.8	0.019	**46.2**	0.013	**0**	0.03	**131**
$C_{22}*10^{-6}$	0.08	0.082	2.5	0.08	**0**	0.17	113	0.125	56.3	0.15	**87.5**	0.074	**-7.5**	0.17	**113**
$C_{23}*10^{-6}$	0.014	0.015	7.14	0.01	-28.6	0.03	114	0.02	42.9	0.02	42.9	0.013	-7.1	0.03	114
$C_{31}*10^{-6}$	0.02	0.02	0	0.01	-50	0.03	50	0.02	**0**	0.023	15	0.013	-35	0.03	50
$C_{32}*10^{-6}$	0.014	0.015	7.14	0.01	-28.6	0.03	114	0.02	42.9	0.02	42.9	0.013	-7.1	0.03	114
$C_{33}*10^{-6}$	0.17	0.15	-11.8	0.08	-52.9	0.17	**0**	0.125	-26.5	0.17	**0**	0.074	-56.5	0.17	**0**
$C_{44}*10^{-6}$	0.07	0.066	-5.7	0.06	-14.3	0.14	100	0.1	42.9	0.067	-4.3	0.067	-4.3	0.14	100
$C_{55}*10^{-6}$	0.11	0.12	9.09	0.06	-45.5	0.14	27.3	0.1	-9.09	0.12	9.09	0.12	9.09	0.14	27.3
$C_{66}*10^{-6}$	0.06	0.069	15	0.06	**0**	0.14	**133**	0.1	66.7	0.064	6.7	0.064	6.7	0.14	**133**

Table 6.5 Comparisons.

By observing differences among the elastic coefficients shown in comparison-tables, it is worth to notice that, due to consistency, some elastic moduli appear to be closer than those proposed by Lourenco.

As a result, it is possible to determine an elasticity tensor obtained by means of those parametric terms, yielded by the examined homogenization procedures, which are closer to the reference numerical data. Such elasticity tensor is, so, defined on the knowledge of elastic ratios as well as of geometrical parameters characterizing the RVE.

APPENDIX

- Stress-prescribed analyses: compression in *x*-direction

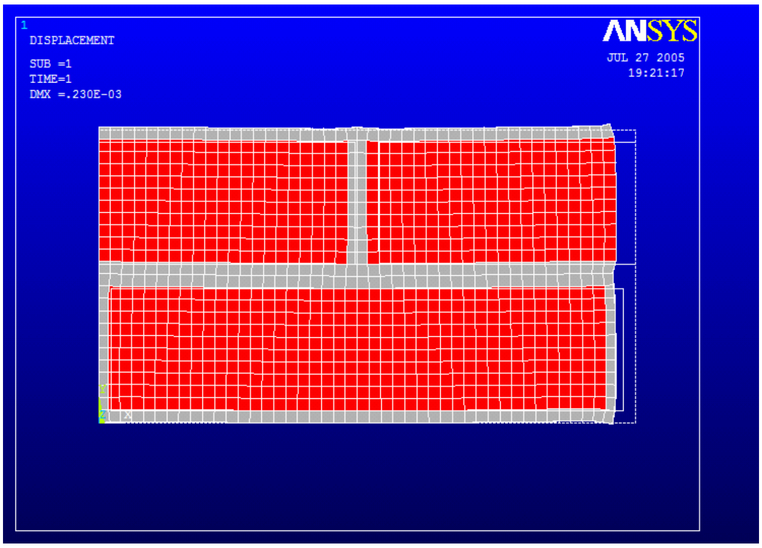


Figure 1 Deformed configuration.

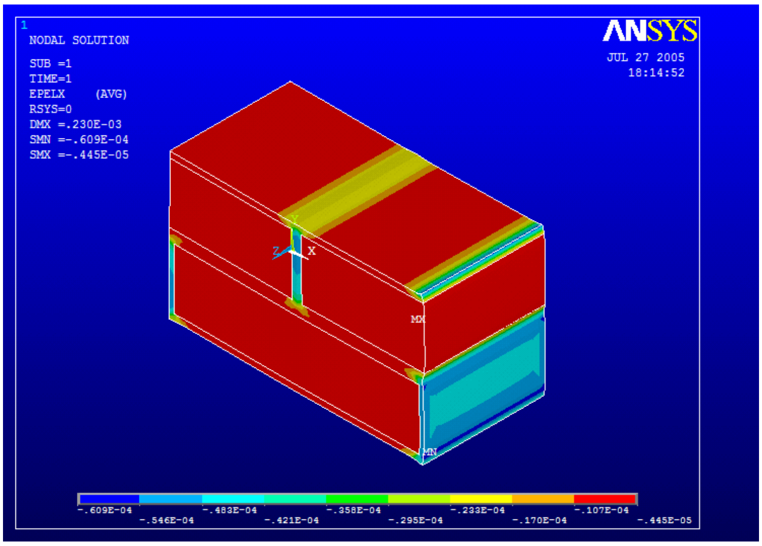


Figure 2 Normal strain in *x*-direction.

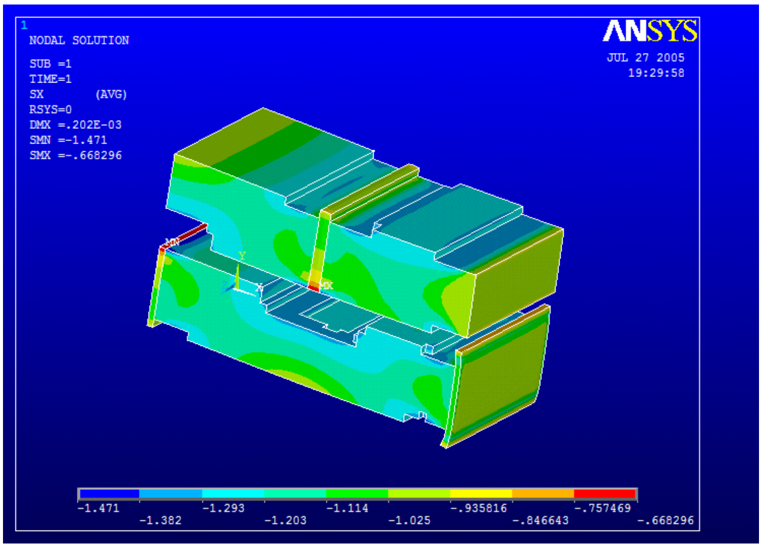


Figure 3 Normal stress in *x*-direction, $|-0.7| < \sigma_x < |-1.3|$.

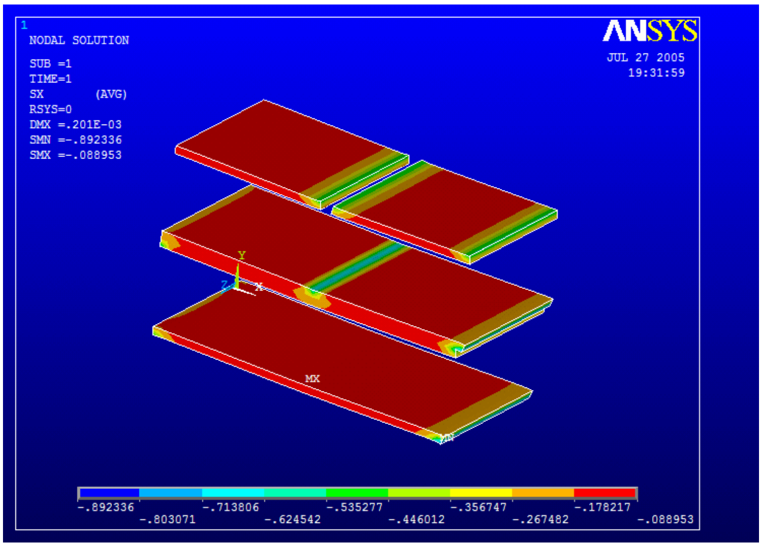


Figure 4 Normal stress in *x*-direction, $0 < \sigma_x < |-0.6|$.

- **Stress-prescribed analyses: compression in *y*-direction**

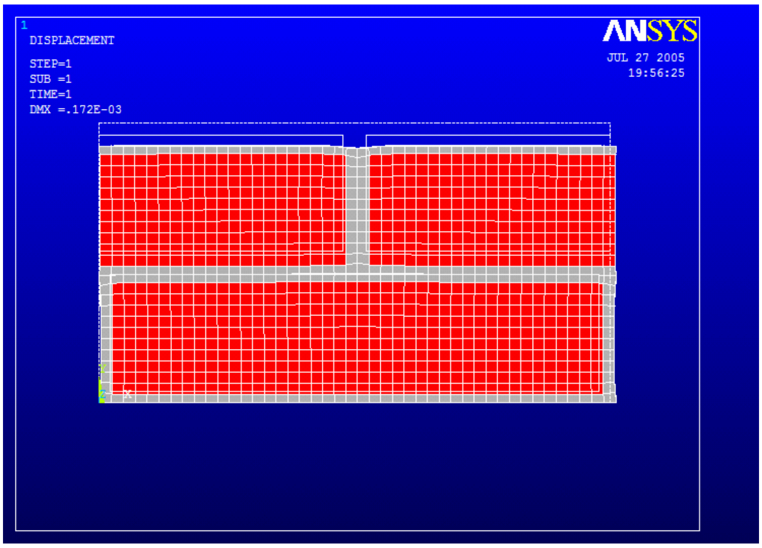


Figure 5 Deformed configuration.

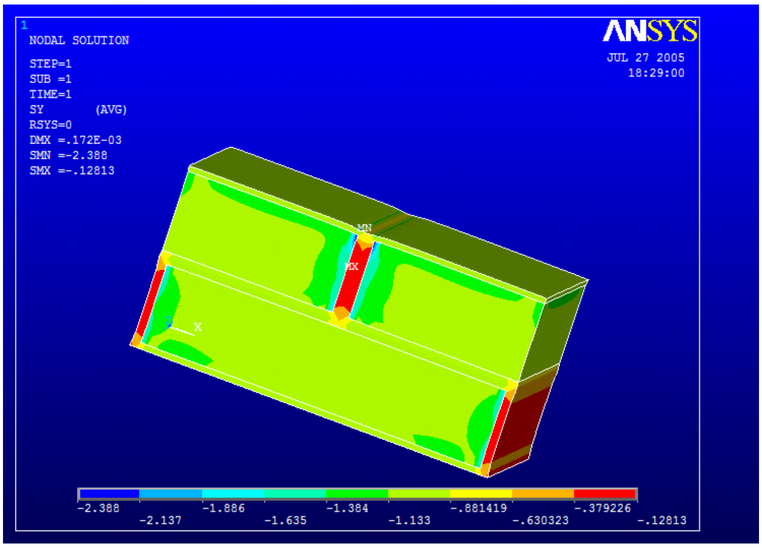


Figure 6 Normal stress in *y*-direction.

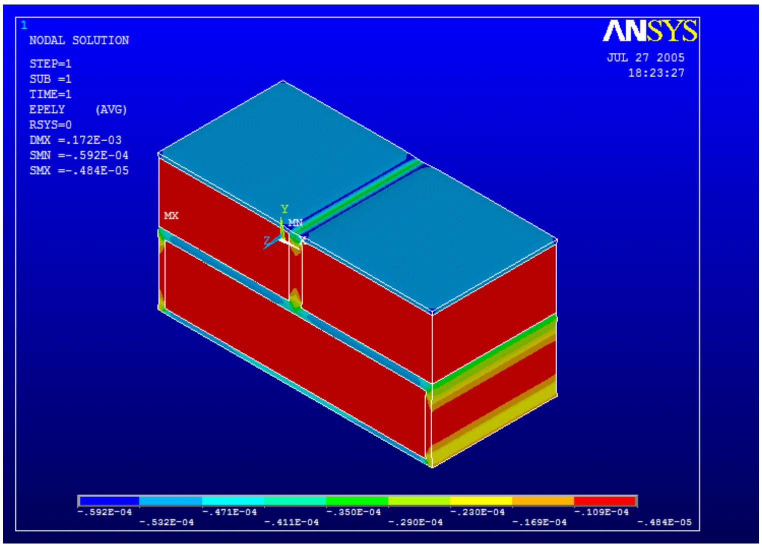


Figure 7 Normal strain in *y*-direction.

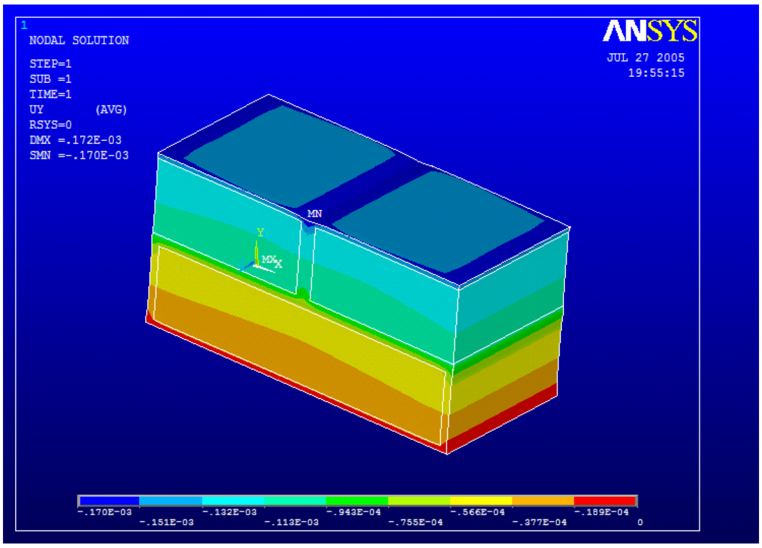


Figure 8 Displacement in *y*-direction.

- **Stress-prescribed analyses: compression in *z*-direction**

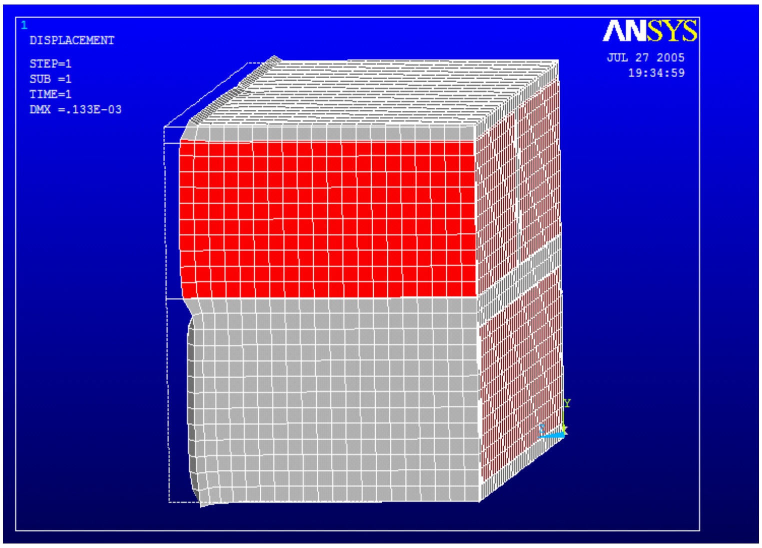


Figure 9 Deformed configuration.

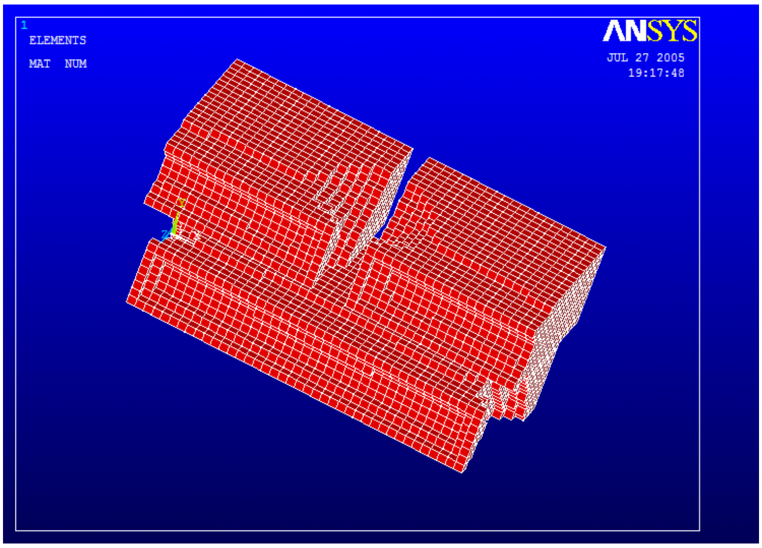


Figure 10 Normal stress in *z*-direction, $|-1| < \sigma_z < |-1.3|$.

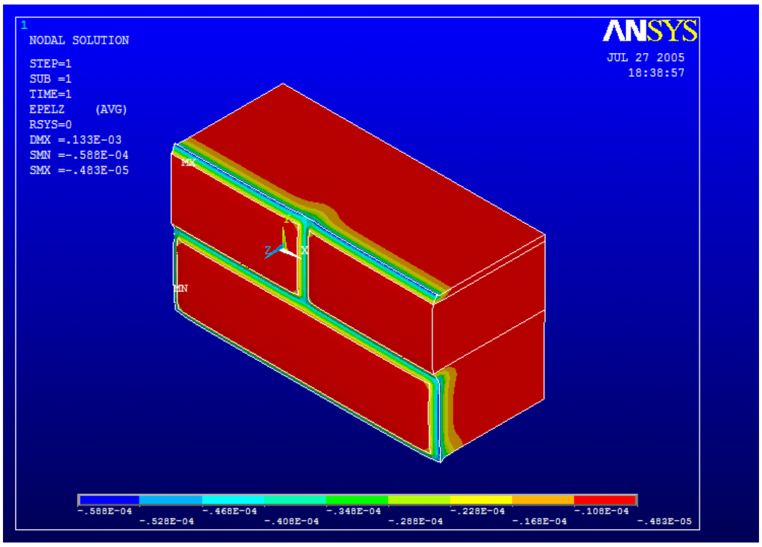


Figure 11 Normal strain in *z*-direction.

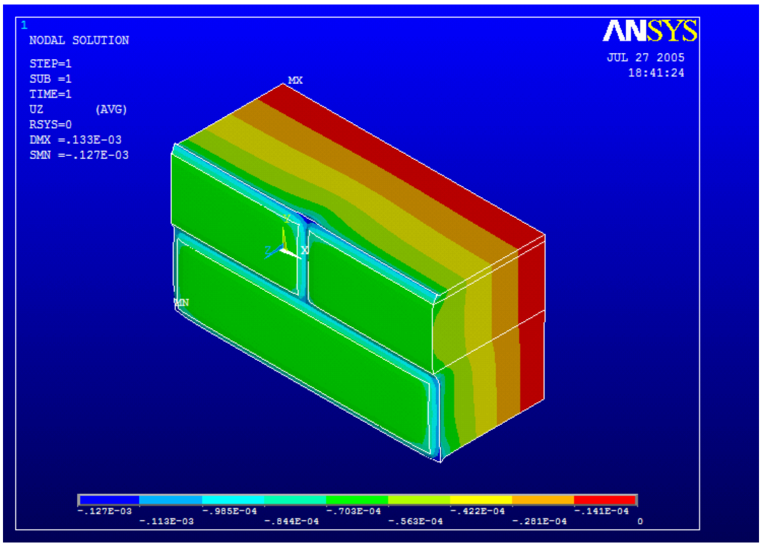


Figure 12 Displacement in *z*-direction.

- **Stress-prescribed analyses: shear in *xy*-plane**

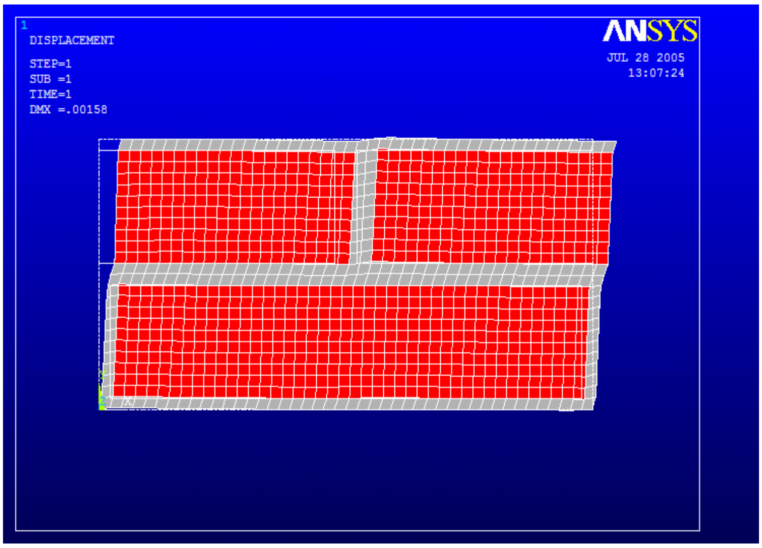


Figure 13 Deformed configuration.

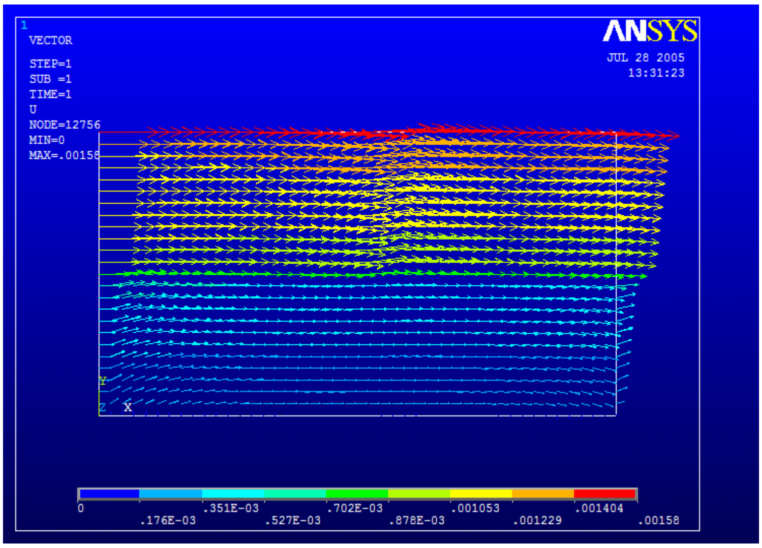


Figure 14 Vector-plot for displacement.

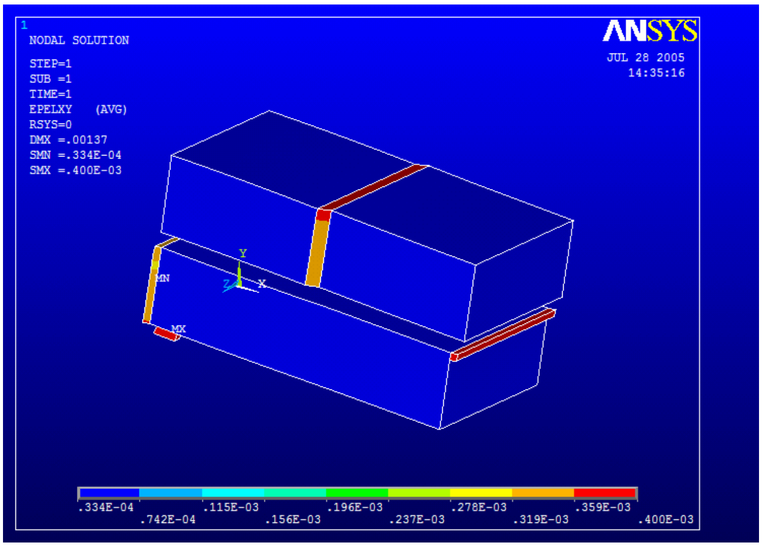


Figure 15 Shear strain in *xy*-plane, $\varepsilon_{xy} < 0.4*10^{-3}$.

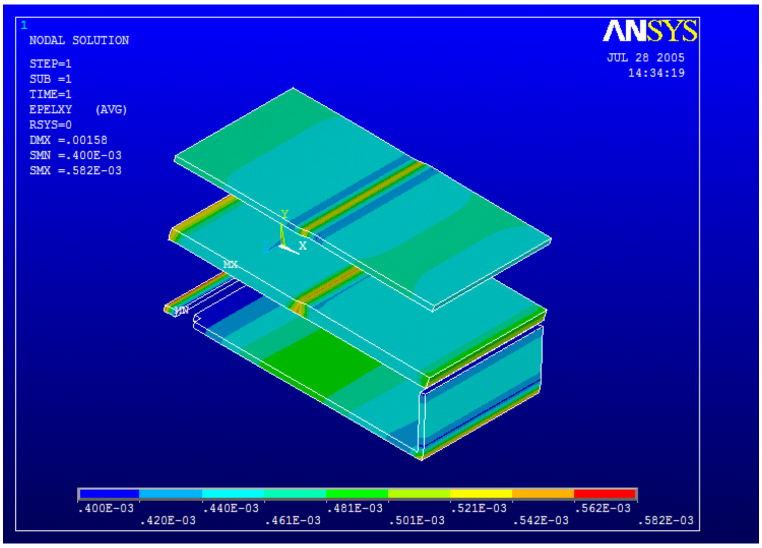


Figure 16 Shear strain in *xy*-plane, $0.4*10^{-3} < \varepsilon_{xy} < 0.6*10^{-3}$.

- **Stress-prescribed analyses: shear in *xz*-plane**

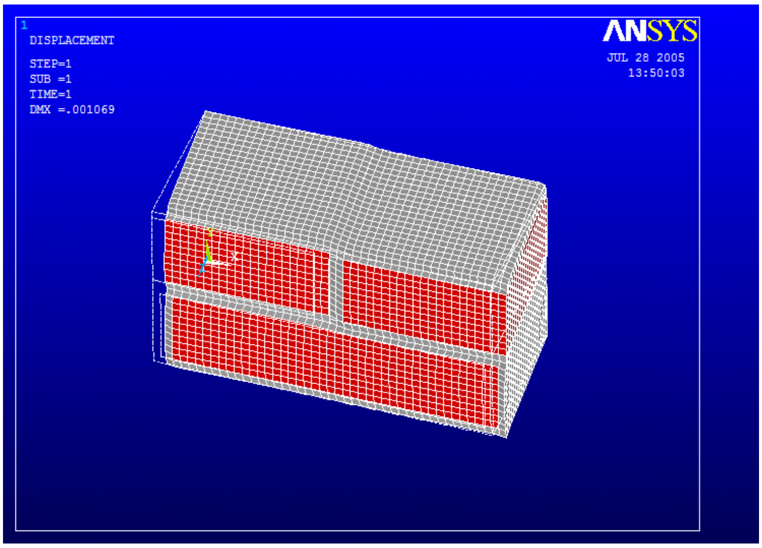


Figure 17 Deformed configuration.

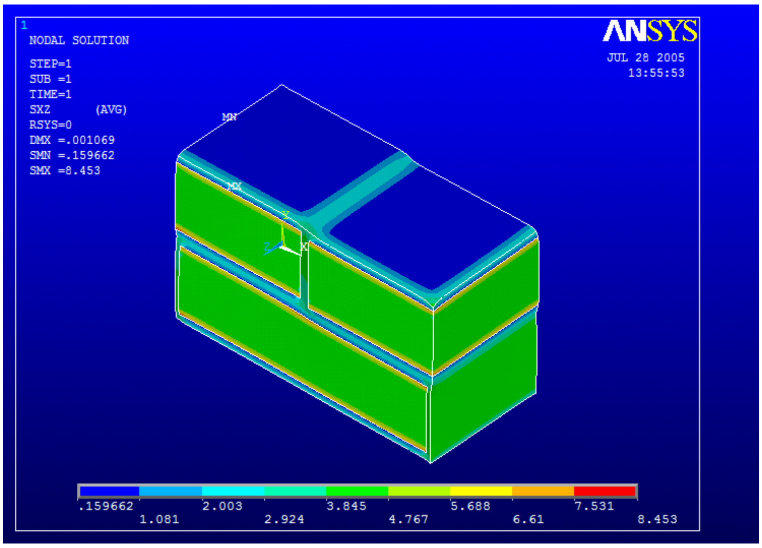


Figure 18 Shear stress in *xz*-plane.

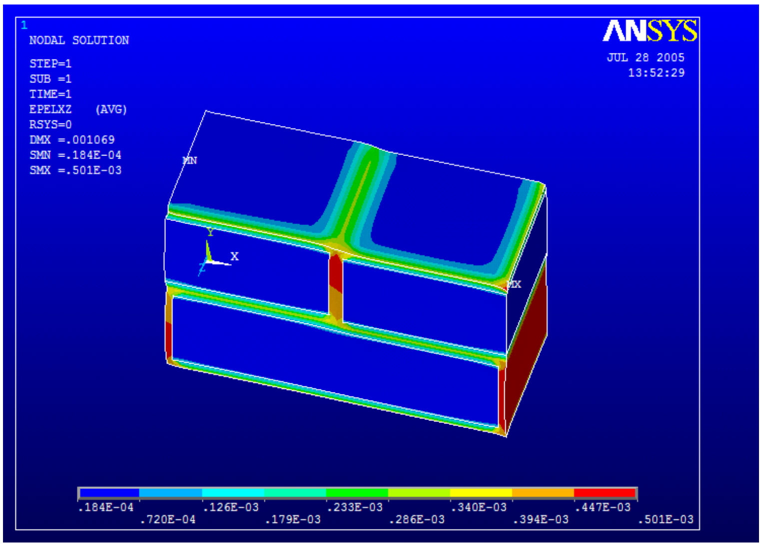


Figure 19 Shear strain in *xz*-plane.

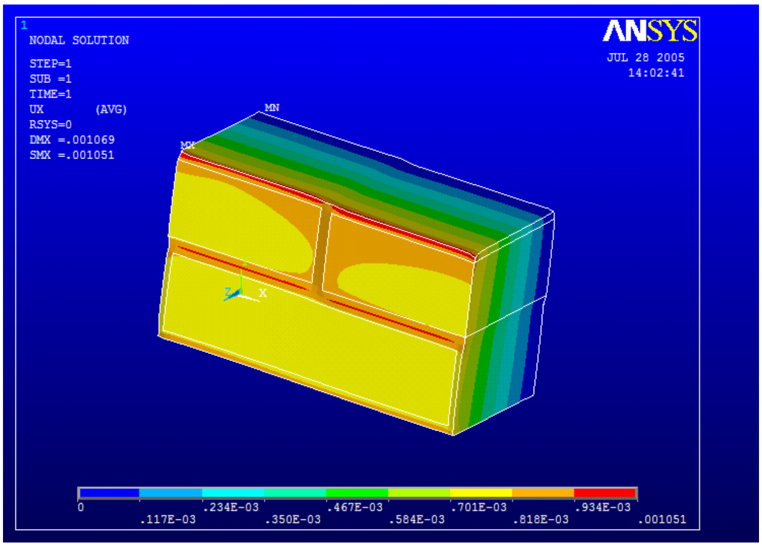


Figure 20 Displacement in *x*-direction.

- **Stress-prescribed analyses: shear in *yz*-plane**

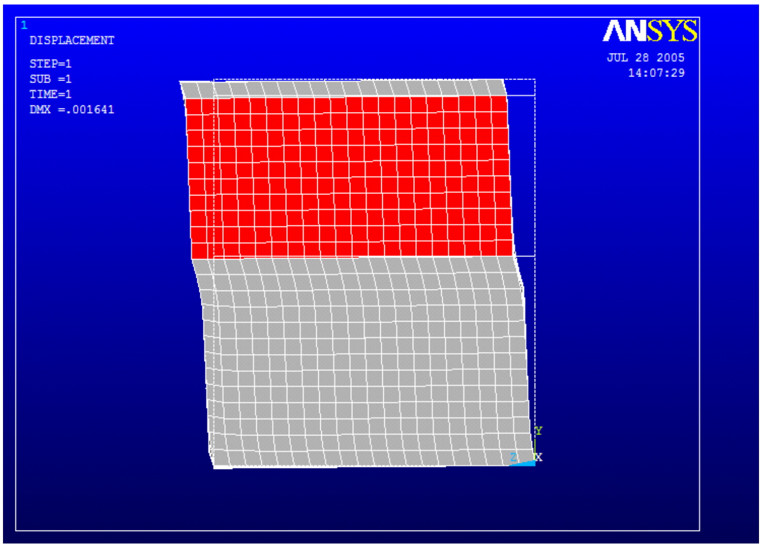


Figure 21 Deformed configuration.

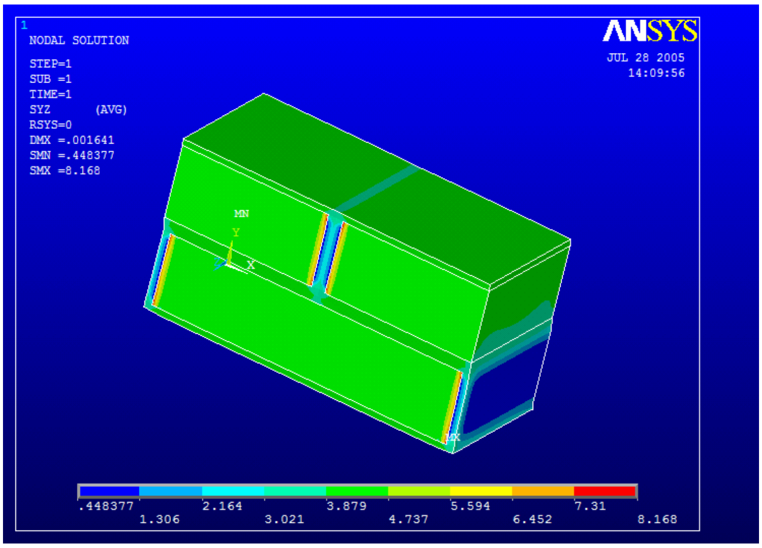


Figure 22 Shear stress in *yz*-plane.

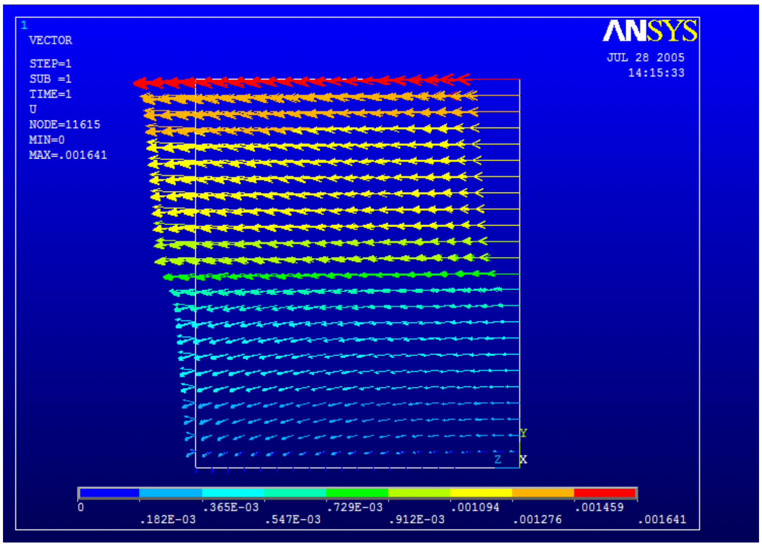


Figure 23 Vector plot for displacement.

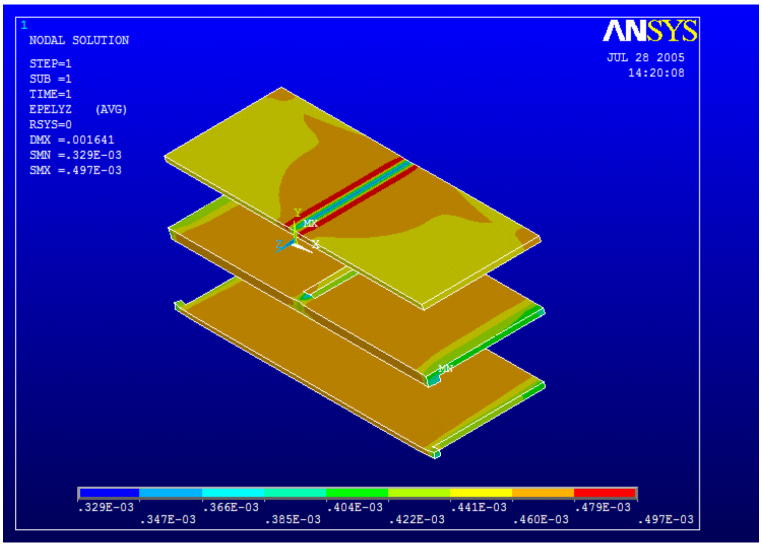


Figure 24 Shear strain in *yz*-plane, $0.4*10^{-3} < \varepsilon_{yz} < 0.5*10^{-3}$.

- **Strain-prescribed analyses: normal strain in *x*-direction**

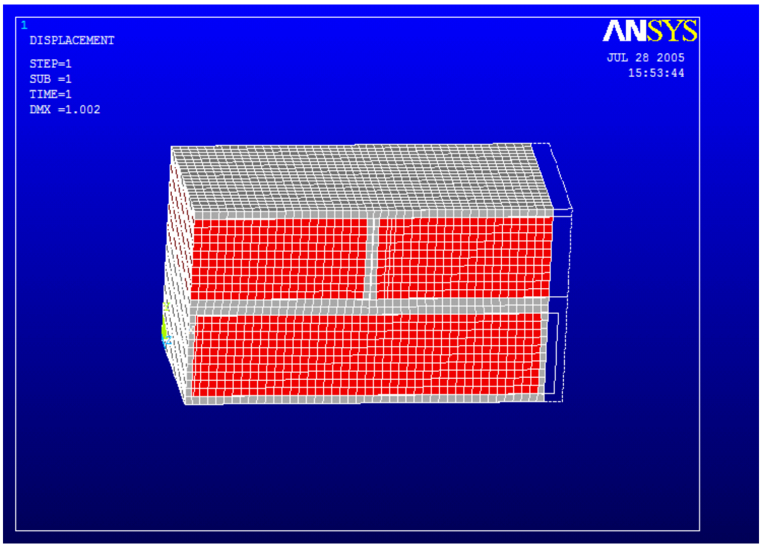


Figure 25 Deformed configuration.

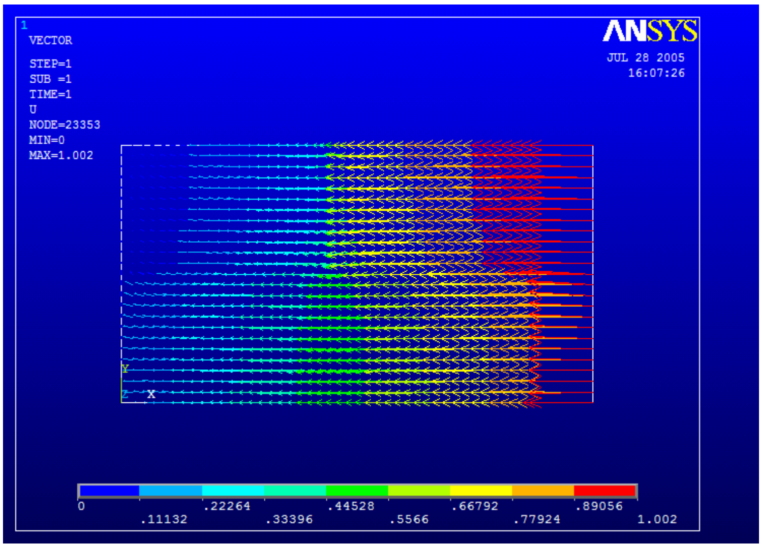


Figure 26 Vector-plot for displacement.

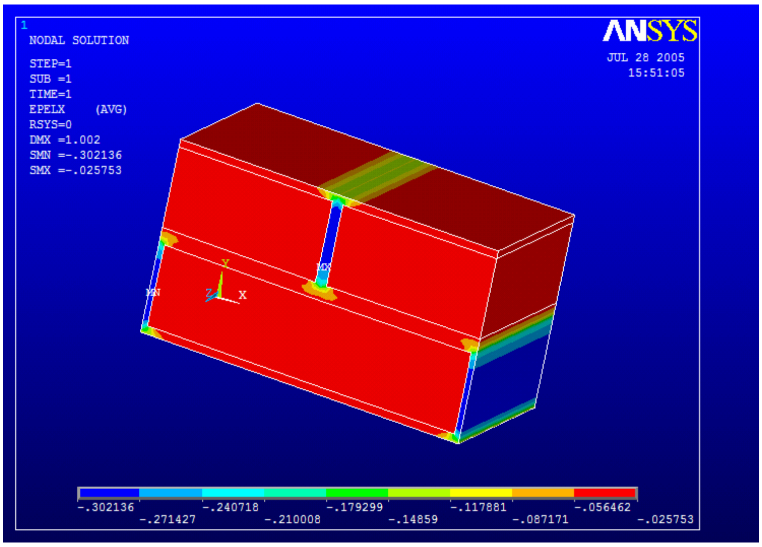


Figure 27 Normal strain in *x*-direction.

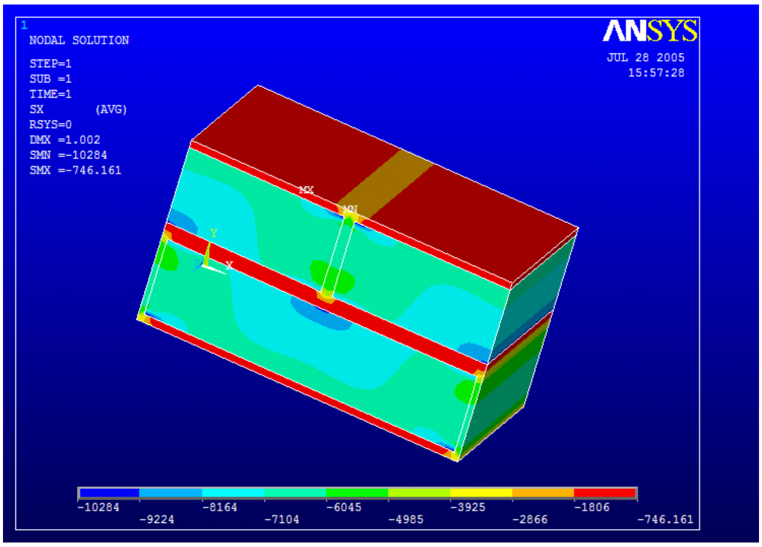


Figure 28 Normal stress in *x*-direction.

- **Strain-prescribed analyses: normal strain in *y*-direction**

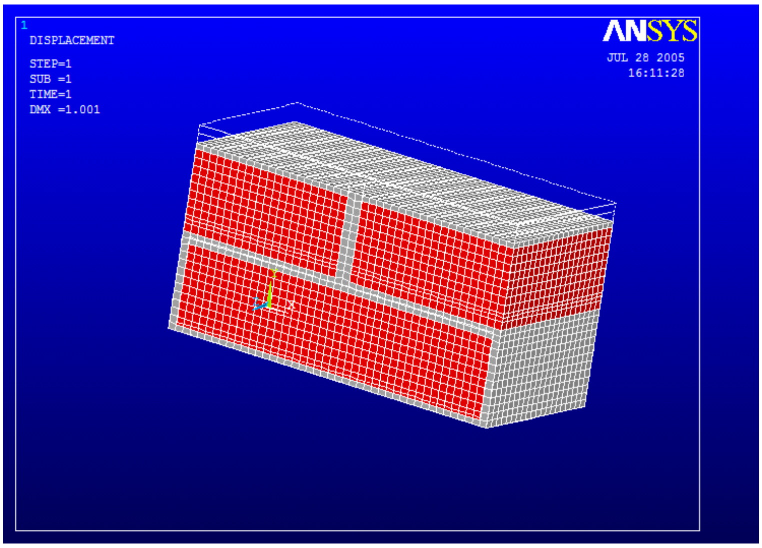


Figure 29 Deformed configuration.

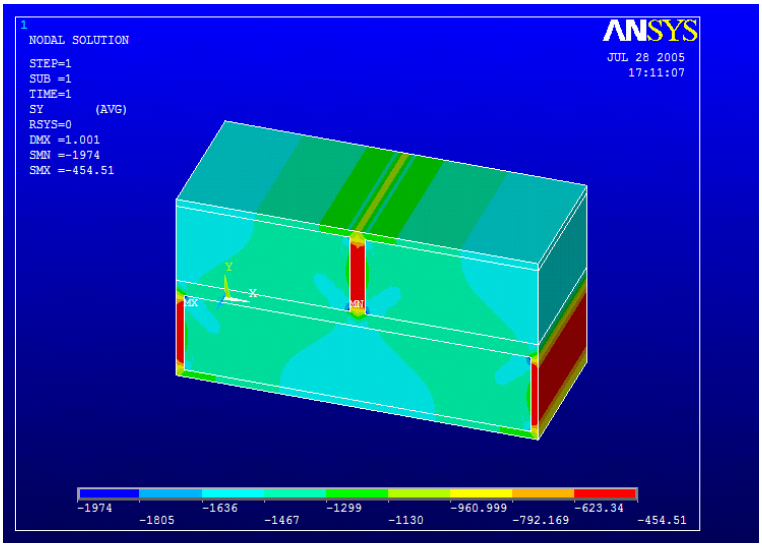


Figure 29 Normal stress in *y*-direction.

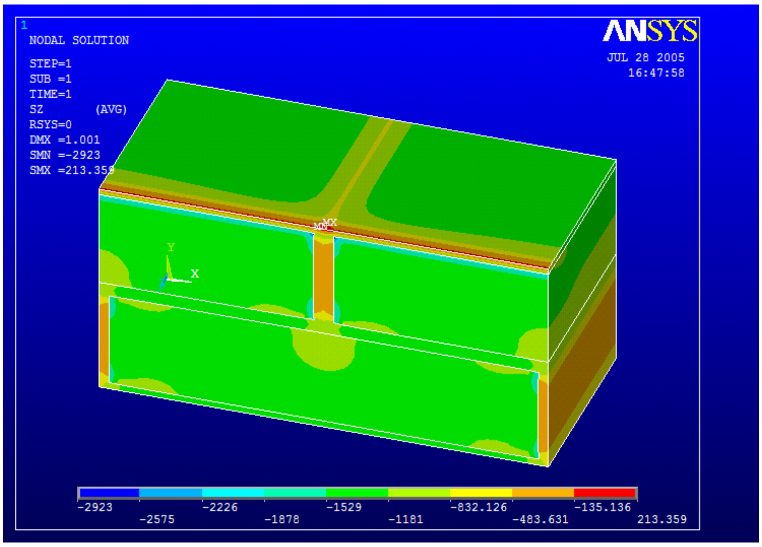


Figure 30 Normal stress in *z*-direction.

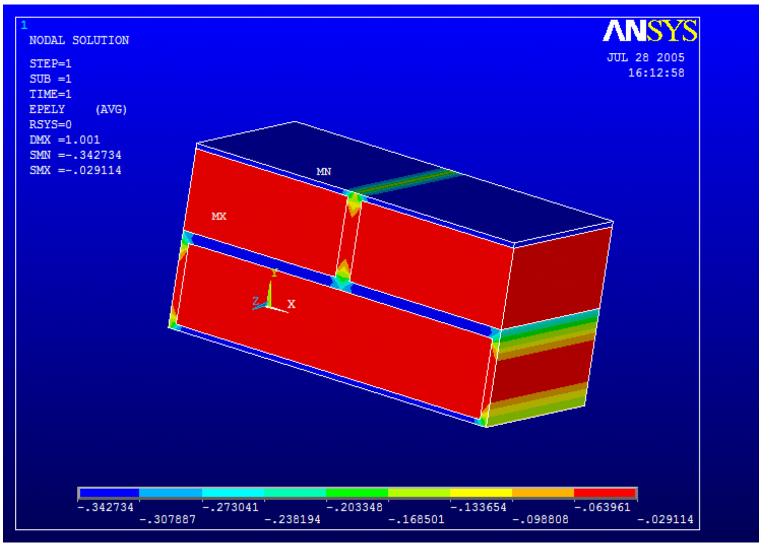


Figure 31 Normal strain in *y*-direction.

- **Strain-prescribed analyses: normal strain in *z*-direction**

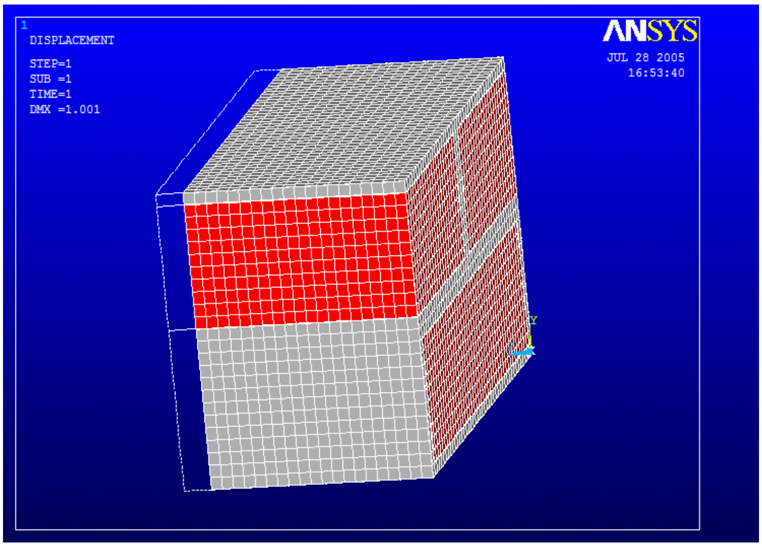


Figure 32 Deformed configuration.

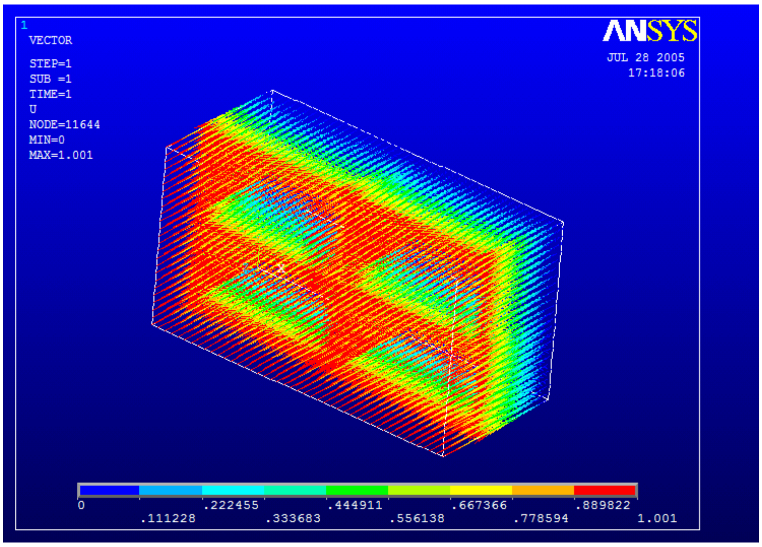


Figure 33 Vector-plot for displacement.

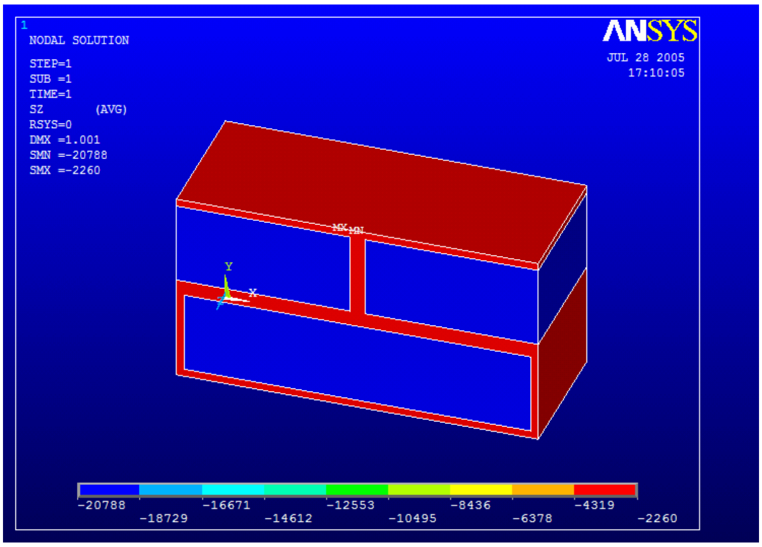


Figure 34 Normal stress in *z*-direction.

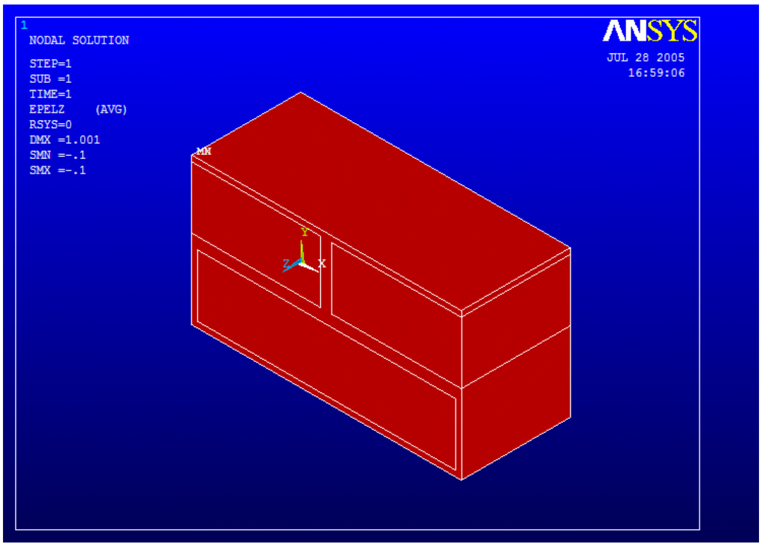


Figure 35 Normal strain in *z*-direction.

- **Strain-prescribed analyses: shear strain in *xy*-plane**

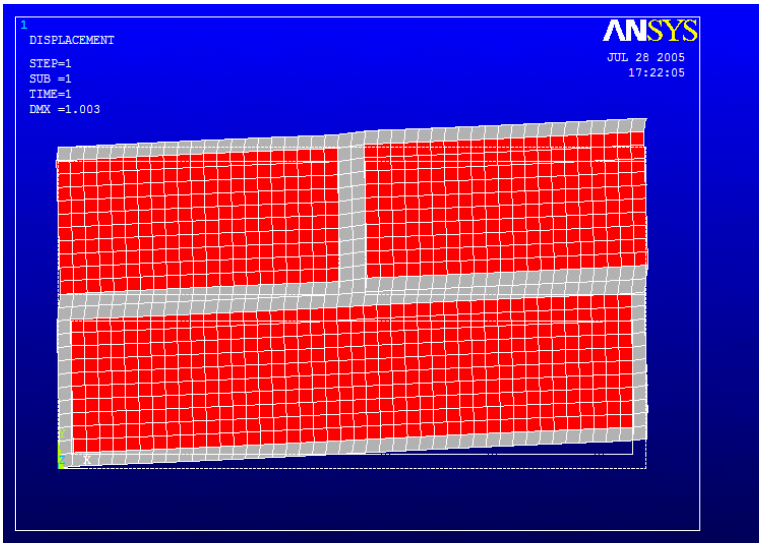


Figure 36 Deformed configuration.

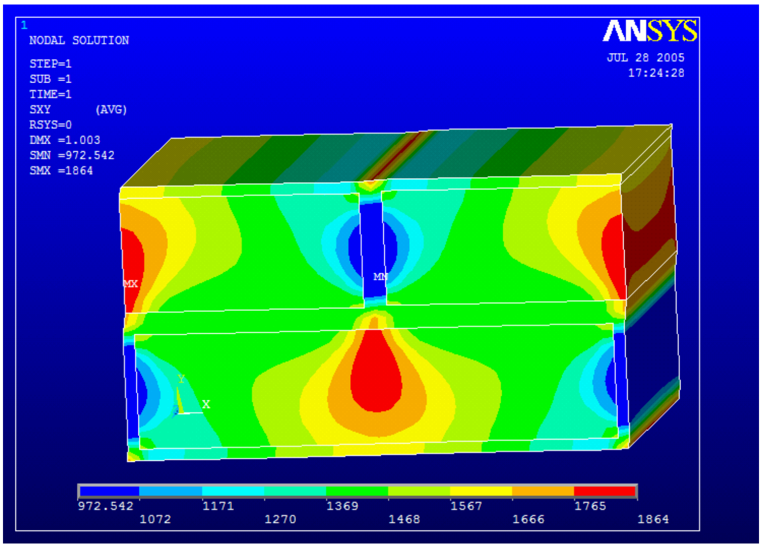


Figure 37 Shear stress in *xy*-plane.

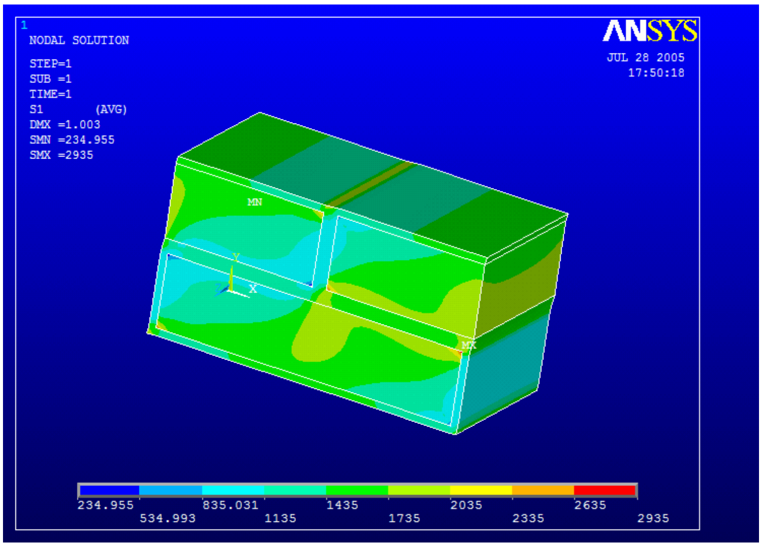


Figure 40 Principal stress s_1.

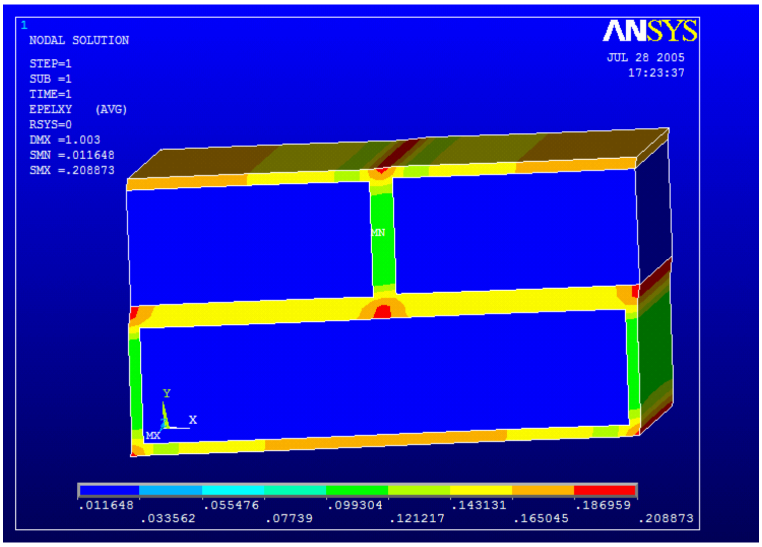


Figure 41 Shear strain in *xy*-plane.

- **Strain-prescribed analyses: shear strain in *xz*-plane**

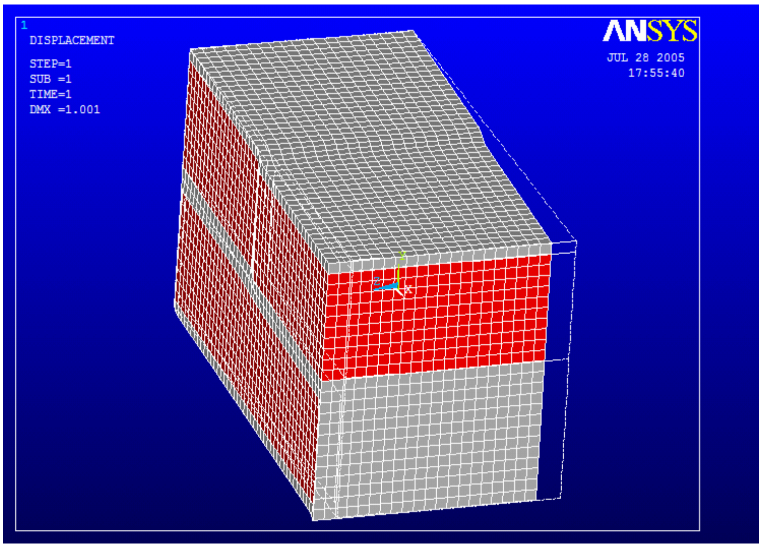


Figure 42 Deformed configuration.

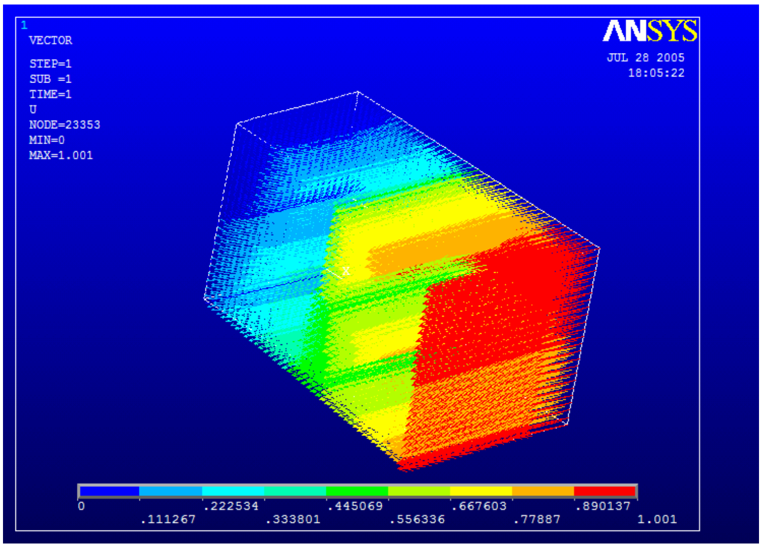


Figure 43 Vector-plot for displacement.

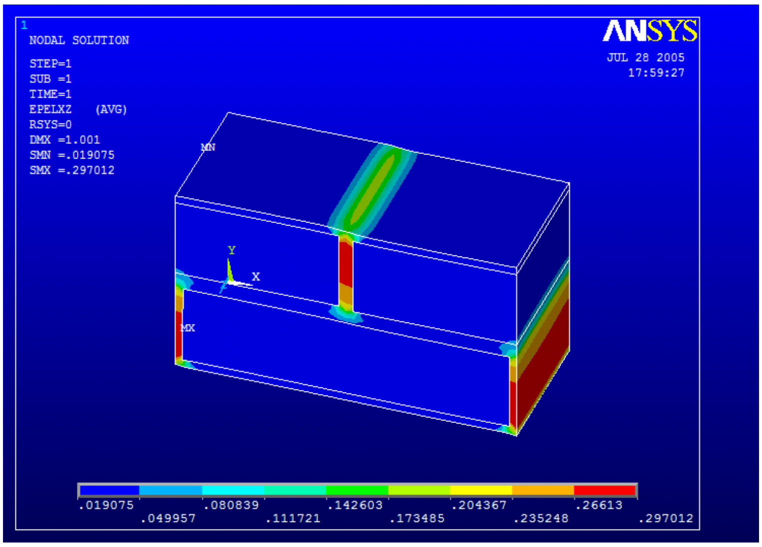


Figure 44 Shear strain in *xz*-plane.

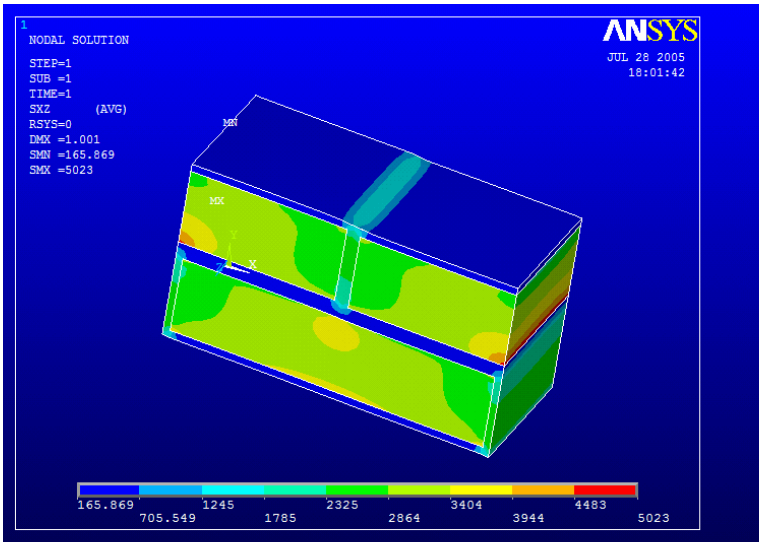


Figure 45 Shear stress in *xz*-plane.

- **Strain-prescribed analyses: shear strain in *yz*-plane**

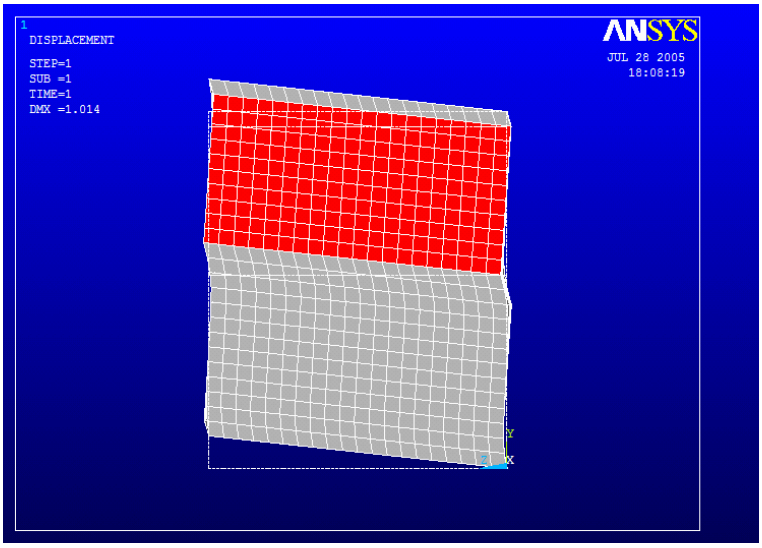


Figure 46 Deformed configuration.

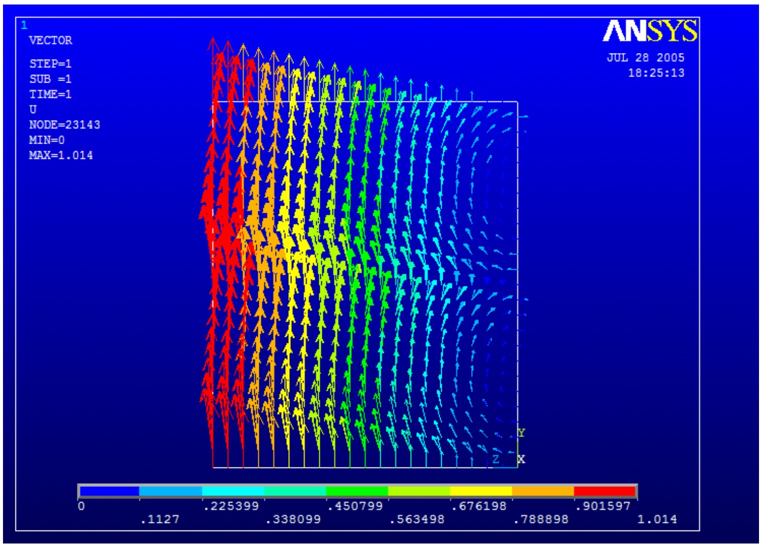


Figure 47 Vector-plot for displacement.

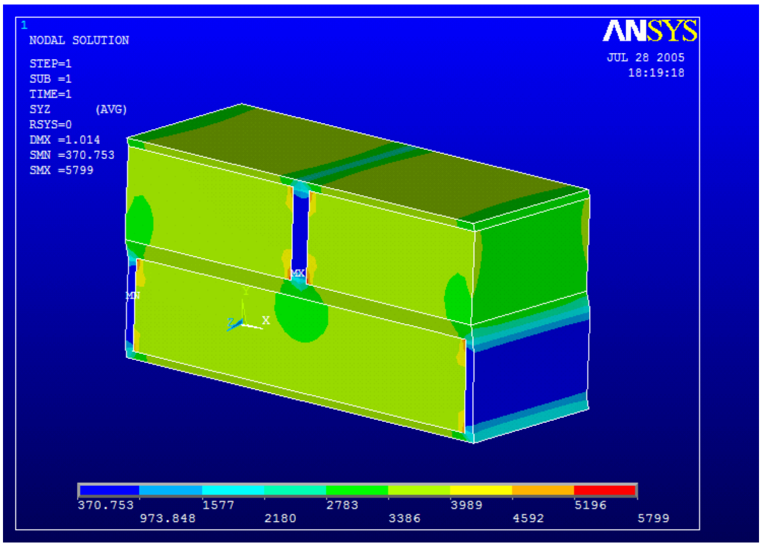


Figure 48 Shear stress in *yz*-plane.

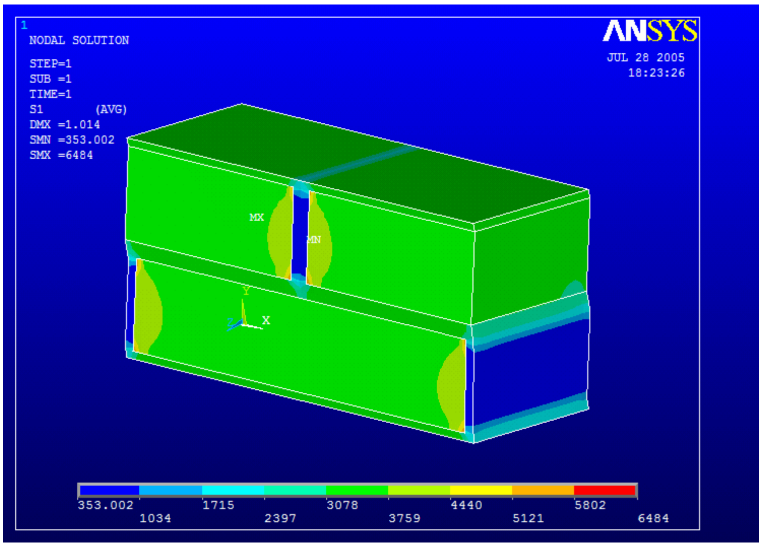


Figure 49 Principal stress s_1.

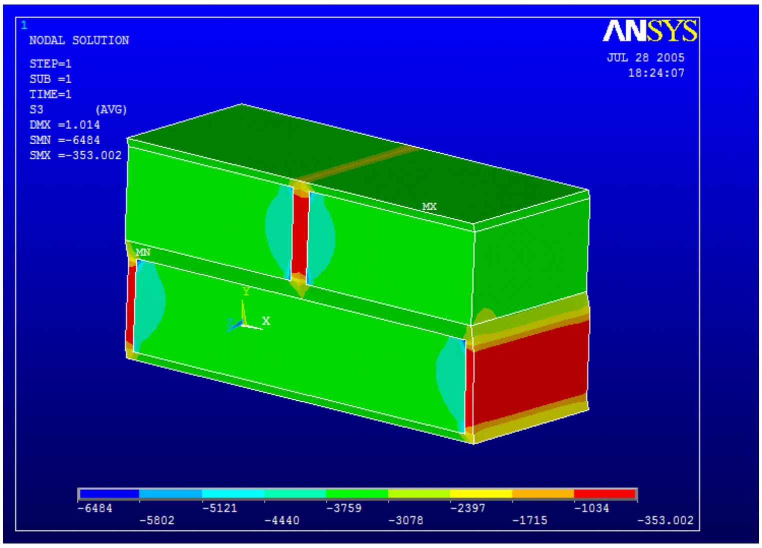


Figure 50 Principal stress s_3.

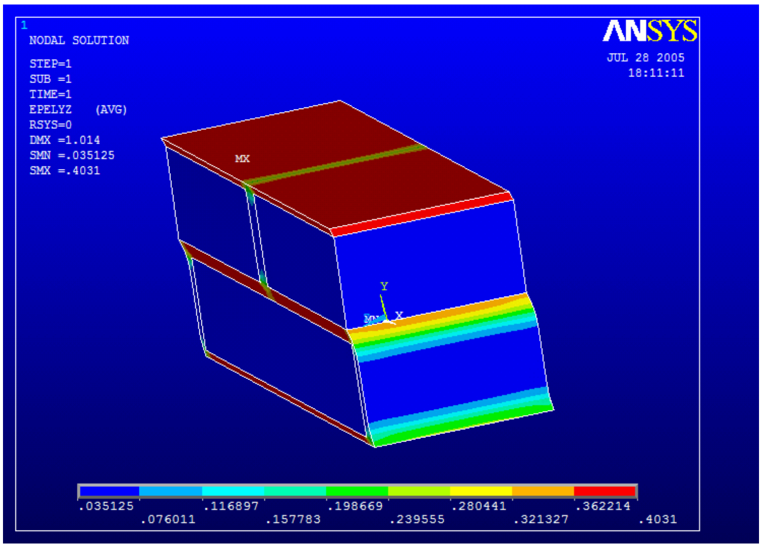


Figure 51 Shear strain in *zy*-plane.

CHAPTER VII

Design codes for masonry buildings

7.1 Introduction

The most effective use of masonry construction is seen in load bearing structures wherein it performs a variety of functions, namely, supporting loads, subdividing space, providing thermal and acoustic insulation and so on.

Until 1950's there were no engineering methods of designing masonry for buildings and thickness of walls was based on *Rules-of-Thumb* tables given in Building Codes and Regulations, [21]. As result, walls used to be very thick and masonry structures were found to be very uneconomical. Hence, since intensive theoretical and experimental research has been conducted in advanced countries, factor affecting strength, stability and performance of masonry structures have been identified, which need to be considered in design.

Recently mechanized brick plants, moreover, are producing brick units having nominal strength which ranges from 17.5 to 25 N/mm^2 and, so, sufficiently greater than the ordinary manufactured ones, with strength of only 0.07 to 0.1 N/ mm^2. Therefore, nowadays, it is possible to construct 5 to 6 storeyed load bearing structures at costs which are less than those of RC framed structures, [21].

The use of reinforcement in masonry can further improve its load carrying capacity and above all its flexure and shear behaviour under earthquake loads. In particular, masonry units are being manufactured in shapes and sizes that make reinforcement embedding in masonry less cumbersome.

With these developments, structural design of load bearing masonry buildings has been undergoing considerable modifications as underlined by the changes which are taking place in the masonry guidelines throughout the world.

In this framework, the object of this chapter is to furnish a short summary and a comparison of the different codes from a number of countries, which are referred to the design of masonry structures.

7.2 Review of masonry codes

A brief description and the major highlights of the various codes are presented below and the comparison between them is summarized in tables, related to design approach, member sizing and details, as given in the follows, [21].

§ BUILDING CODE REQUIREMENTS FOR MASONRY STRUCTURES
(ACI 530-02/ASCE 5-02/TMS 402-02)

This code has been drawn up by the joint efforts of the American Concrete Institute, the Structural Engineering Institute of the American Society of the Civil Engineers and the Masonry Society, [2], [3]. Such a code covers the design and the construction of masonry structures, by providing minimum requirements for the structural analysis and by using both allowable stress design as well as limit state design for unreinforced and reinforced masonries. An empirical design method applicable to buildings meeting specific location and construction criteria is also included.

§ INTERNATIONAL BUILDING CODE 2000

The International Building Code (IBC 2000) has been designed to meet the need for a modern, up-to-date building instrument addressing the design of building systems through requirements emphasizing performance, [34]. This model code encourages the international consistency in the application of provisions and it is available for adoption and use by jurisdictions internationally.

The provisions of this code for the design of masonry members have been heavily borrowed from ACI 530-02, ASCE 5-02, TMS 402-02.

§ EUROCODE 6: DESIGN OF MASONRY STRUCTURES (DD ENV 1-1-1996: 1996)

The Eurocode 6 has been published by the European Committee for Standardization (CEN) and it is to be used with the National Application Document (NAD) of member countries, [22].

This code provides a general basis for the design of buildings and civil engineering works in unreinforced and reinforced masonry made with clay and concrete masonry units imbedded in mortar. It adopts the limit state design method.

However, Eurocode 6 doesn't cover the special requirements of seismic design: provisions related to such requirements are given in Eurocode 8, *Design of Structures in Seismic Regions*.

The designer should consider the relative contribution of concrete infill and masonry in resisting load and, where the concrete infill makes a much greater contribution to the load resistance than the masonry, Eurocode 2 should be used and the strength of masonry should be ignored, [21].

§ TESTO UNICO-NORME TECNICHE PER LE COSTRUZIONI (C.S.LL.PP.30-05-2005)

Such a code covers the design and the construction of masonry structures, in order to guarantee pre-established safety coefficients. It adopts the limit state design for unreinforced and reinforced masonries. In particular, the constructions have to satisfy the following requirements:

- safety towards ultimate limit state – overcoming of an ultimate limit state is not reversible and provides a structural collapse.
- safety towards serviceability limit state – overcoming of a serviceability limit state can be or not be reversible. In the first case, the damage or

the deformations will disappear when the external actions which have caused such an overcoming will stop. In the second case, the damage and deformations will be permanent and unacceptable; this limit state is identified with damage limit state.

- strength towards accidental loads.

§ NEW ZEALAND STANDARD – CODE OF PRACTICE FOR THE DESIGN OF CONCRETE MASONRY STRUCTURES (NZS 4230: Part 1: 1990)

The New Zealand Standard has been prepared under the direction of the Building and Civil Engineering Divisional Committee for the Standards Council, established under the Standards Act 1988. It is set in two parts: Code and Commentary, [55].

Such a code is largely dictated by seismic considerations. In this framework, it is intended to provide a satisfactory structural performance for masonry structures during a major earthquake. Minimum reinforcing requirements for different structural systems and the reinforcing and separation of non-structural elements will limit non-structural damage during moderate earthquakes.

The NZS 4230 adopts a design philosophy based on strength design, using reinforced masonry only. It contains cross-references to NZS 3101, which is the primary code for the seismic design of structure.

§ INDIAN STANDARD: CODE OF PRACTICE FOR STRUCTURAL USE OF UNREINFORCED MASONRY (IS: 1905-1987)

The Indian Standard on masonry design has been first published in 1960 and later on revised in 1969, 1980 and 1987, [35]. This latter has been reaffirmed in 1998. A separate handbook to this code, SP 20 (S&T), 1991, is also available, [21].

Such a code provides recommendations for structural design aspect of load bearing and non-load bearing walls of unreinforced masonry only, by using a design procedure based on the allowable stress design, along with several empirical formulae.

These guidelines are referred to IS 4326 for strengthening unreinforced masonry building for seismic resistance and it doesn't provide any calculation for the design of reinforcement.

7.3 Comparison on design philosophies

In this section, design philosophies of various codes have been compared with regard to their design assumptions and assumed factor of safety.

§ EMPIRICAL DESIGN

Empirical rules for the design of masonry structures were developed by experience and, traditionally, they have been used as a procedure, not as a design analysis for sizing and proportioning masonry elements, [21].

This method predates any engineering analysis and the effect of any steel reinforcement, if used, is neglected. However, this design procedure is applicable to very simple structures with severe limitations on building height proportions and on horizontal loads, due to wind and earthquake.

Empirical design method is still being continued in ACI 530-2002 and, with some changes, in IBC 2000. The Indian Standard also mixes empirical procedure with allowable stress design method.

§ ALLOWABLE STRESS DESIGN

This method states that, under the working loads, the stresses developed in a member must be less than admissible ones.

In case of unreinforced masonry, it is assumed that tensile stresses, not exceeding the allowable limits, are resisted by masonry material, while in the case of reinforced structures masonry tensile strength is neglected.

The ACI code has followed this approach for both reinforced and unreinforced masonry, while the IS code has applied it only to unreinforced masonry. On the contrary, such a design method doesn't find place in Eurocode and in the New Zealand Standard.

§ STRENGTH DESIGN OR LIMIT STATE DESIGN

This method requires masonry members be proportioned so that the design strength equals or exceeds the required strength.

Design strength is the nominal strength multiplied by a strength reduction factor, j. The required strength shall be determined in accordance with the strength design load combinations of a legally adopted building code, [21].

The ACI code has adopted this procedure, as well as the IBC 2000 and the New Zealand code, with more emphasis on the reinforced masonry rather than unreinforced ones. The Eurocode 6 specifies a limit state design for collapse and serviceability, wherein instead of strength reduction factors, partial safety factors for loads and materials are specified separately. In particular, partial safety factor for loads depends on the load combinations and partial safety factor for materials depends on the type of masonry units and the failure mode. Also the Italian code (T.U. 30/03/2005) adopts the ultimate and serviceability limit state design, for reinforced and unreinforced masonry.

In these codes, the strength of reinforced masonry members is calculated by basing on the following hypothesis:

a. There is strain continuity between the reinforcement, grout and masonry.
b. The maximum compressive strain $\left(\varepsilon_{mu}\right)$ at the extreme masonry compression fibre shall be assumed to be 0.0035 for clay masonry and 0.0025 for concrete one. The New Zealand code also specifies that the maximum usable strain will be 0.008 for confined concrete masonry.
c. Reinforcement stress below specified yield strength $\left(f_y\right)$ shall be taken as E_s times steel strain. For strain greater than the ones corresponding to $\left(f_y\right)$, stress in reinforcement shall be taken equal to $\left(f_y\right)$.
d. The tensile strength of masonry shall be neglected in calculating flexural strength but shall be considered in calculating deflection.
e. Masonry stress of 0.80 times the compressive strength of masonry, f_m, (ACI code) or 0.85 f_m (IBC 2000, New Zealand Standards) shall be assumed uniformly distributed over an equivalent

compression zone bounded by the edges of the cross section and a straight line located parallel to the neutral axis at a distance of $a = 0.80c$ or $a = 0.85c$ respectively from the fibre of the maximum compressive strain, as shown in figure 7.1. In particular, a is defined as the depth of equivalent compression zone at nominal strength and c is the distance from extreme compression fibre to neutral axis. The value of uniformly distributed masonry stress for confined masonry, as specified in New Zealand Standards, is $0.9Kf_m$ up to a distance $a = 0.96c$, as shown in figure 7.1 where K is a factor greater than 1, for increase in masonry strength due to confinement provided by confining plates.

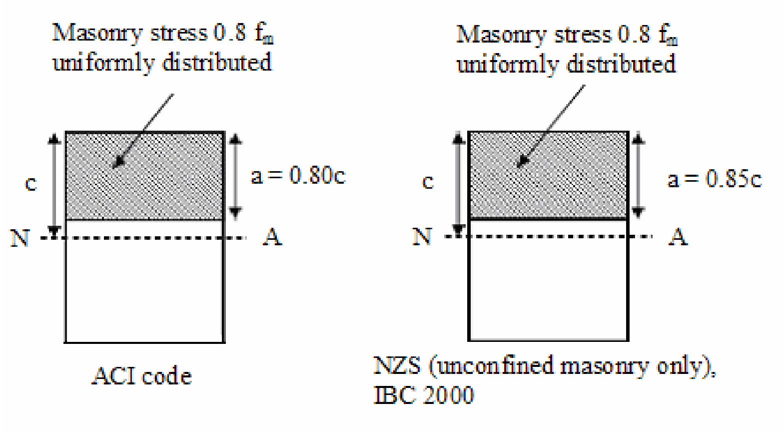


Figure 7.1 Equivalent rectangular masonry stress distribution.

7.4 Comparison of the key concepts for unreinforced masonry

In this section, provisions of both allowable stress and strength (limit state) design, specified in various codes, will be discussed and compared with reference to unreinforced masonry subjected to axial compression, flexure and shear. The Italian code will be exposed separately.

7.4.1 Allowable stress design

7.4.1a Axial compression

Masonry is generally subjected to axial compression due to vertical loads, dead and live ones.

Compression tests of masonry prisms are used for determining specified compressive strength of masonry f_m, which is further modified for slenderness, eccentricity, shapes of cross-section and so on, in order to derive allowable compressive stress values.

In ACI code, calculated compressive stress, f_a, should be less than the allowable compressive stress F_a, which is obtained by multiplying f_m with 0.25 and slenderness ratio R. In particular, the factor 0.25 accounts for material uncertainty and reduces f_m to working stress level. R is the capacity reduction factor for slenderness, as given in the following equations, [21]:

$$R = 1 - \left(\frac{h}{40t}\right)^2 \quad for\ h/t \le 29 \tag{7.4.1a-1}$$

$$R = \left(\frac{20t}{h}\right)^2 \quad for\ h/t > 29 \tag{7.4.1a-2}$$

where:

$h =$ height of masonry structural element

$t =$ thickness of masonry structural element

Slenderness can affect capacity either as a result of inelastic buckling or because of additional bending moments due to the deflection. Applied axial load must be less than 25% of the Euler buckling load, P_e, as given in the following relation:

$$P_e = \frac{\pi^2 E_m I_n}{h^2}\left(1 - \frac{2e}{t}\right)^3 \qquad (7.4.1a\text{-}3)$$

where:

$e =$ the eccentricity of the axial load

$E_m =$ modulus of elasticity of masonry in compression

$I_n =$ moment of inertia of net cross-sectional area of a member

Hence, according to ACI code, the permissible value is function of the slenderness ratio whereas the limiting value of axial load is depending on both slenderness ratio and eccentricity of the axial load.

In IS: 1905 code a stress reduction factor, k_s, is multiplied with the basic compressive stress for slenderness ratio of the element and also the eccentricity of loading. The basic compressive stress is valued both from prism tests and a standard table which is based on compressive strength of unit and mortar type. A limit to the maximum slenderness ratio for a load bearing wall is considered, depending on the number of storeys and the type of mortar.

7.4.1b Axial compression with flexure

Masonry is generally subjected to flexural stresses due to eccentricity of loading or application of horizontal loads, as well as wind or earthquake.

According to ACI code, if a member is subjected to bending only, calculated bending compressive stress f_b should be less than allowable bending stress F_b in masonry, taken as $0.33 f_m$, which is 1.33 times the basic compressive stress allowed for direct loads (0.25 f_m). This increase is due to the restraining effect of less highly strained compressive fibres on those ones of maximum strain and is supported by experiment.

For combined axial and flexural loads, a masonry member is acceptable if the sum of the quotients of the resulting compression stresses to the allowable stresses does not exceed 1, as given in the following relation and figure:

$$\frac{f_a}{F_a} + \frac{f_b}{F_b} \le 1 \qquad (7.4.1b\text{-}1)$$

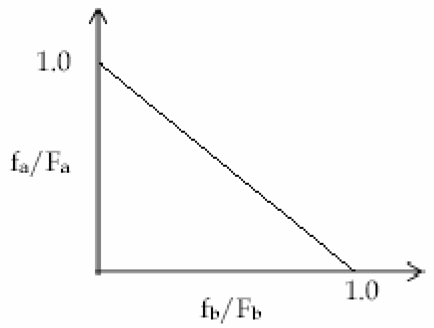


Figure 7.2 Interaction diagram for unreinforced masonry using allowable stress design.

The unity formula (7.4.1b-1) is widely used and very conservative.

IS: 1905 code checks bending compression and tensile stresses independently against permissible values. The permissible values for bending compression are obtained first by increasing the basic compressive stress by 25% and then reducing it for the eccentric loading causing flexure. The code furnishes permissible loads for three eccentricity values, [21]:

(a) $e < t/24$

(b) $t/24 < e < t/6$

(c) $t/6 < e$

An applied moment can be converted into equivalent eccentricity.

7.4.1c Shear

Masonry is generally subjected to shear stresses due to in-plane lateral wind or seismic forces. So, masonry load bearing walls also act as shear walls to resist to such a kind of load.

The lateral load carrying capacity of shear wall structures mainly depends on their in-plane resistances because the in-plane stiffness is far greater than its out-of-plane stiffness. Three modes of shear failure in unreinforced masonry are possible, [21]:

(a) Diagonal tension crack form through the mortar and masonry units.

(b) Sliding occurs along a straight crack at horizontal bed joints.

(c) Stepped cracks form, alternating from head joint to bed joint.

The ACI code recognizes these modes and addresses them while specifying permissible shear stresses. For prevention of diagonal cracks, in-plane shear stress should not exceed $0.125\sqrt{f_m}$. For sliding failure, the allowable shear stress is based on a Mohr-Coulomb type failure criterion and for preventing stepped cracks, different values of permissible shear stress are given for various bond masonry patterns, [21].

The IS: 1905 code, instead, only takes into account the sliding failure by specifying that the allowable shear stress $F_v = 0.1 + \sigma_d/6$, which is a Mohr-Coulomb type failure criterion, where σ_d is average axial stress. However, this linear relation is valid up to axial compression of 2.4 MPa, at which it reaches the maximum limiting value of 0.5 MPa, [21].

7.4.2 Strength design or limit state design

7.4.2a Axial Compression

According to ACI code, the nominal axial strength is based on compressive strength of masonry, modified for unavoidable minimum eccentricity and slenderness ratio, in addition to the strength reduction factor. The expression for effect of the slenderness is the same as in allowable stress design.

Eurocode 6 also considers the effect of slenderness and eccentricity by using capacity reduction factor. However, this capacity reduction factor is based on eccentricity not only at the ends of member but also at middle one–fifth, wherever the moment may be maximum, [21].

7.4.2b Axial Compression with Flexure

According to all codes, the two failure modes of wall considered are parallel and perpendicular to bed joints. The codes require the section to be checked by calculating axial and flexural strength.

7.4.2c Shear

The ACI code considers the previously discussed three modes of failure for evaluating the nominal shear strength of masonry.

Analogously, the IBC 2000 also considers those factors for determining the masonry nominal shear strength and differs only in magnitude from the ACI code.

On the contrary, Eurocode 6 only considers a sliding mode of shear failure and prescribes an equation of Mohr-Coulomb type $\left(F_v = 0.1 + 0.4\sigma_d\right)$.

7.5 Comparison of the key concepts for reinforced masonry

In this section, provisions of both allowable stress and strength (limit state) design, specified in various codes, will be discussed and compared with reference to reinforced masonry subjected to axial compression, flexure and shear.

Reinforced masonry is a construction system where steel reinforcement, in the form of reinforcing bars or mesh, is embedded in the mortar or placed in the holes and filled with concrete or grout, [21].

By reinforcing masonry with steel reinforcement, the resistance to seismic loads and energy dissipation capacity can be improved significantly. In such reinforced structures, tension is developed in masonry but it is not considered to be effective in resisting design loads: reinforcement is assumed to resist all the tensile stresses.

7.5.1 Allowable stress design

Only the ACI code contains provisions on allowable stress design for reinforced masonry.

7.5.1a Axial Compression

In ACI code, the allowable axial compressive load $\left(P_a\right)$ in reinforced masonry shall not exceed $\left(0.25 f_m A_n + 0.65 A_{st} F_s\right) R$, which is obtained by adding the contribution of masonry and reinforcement, and where:

$A_n =$ net cross-sectional area of masonry

$A_{st} =$ total area of longitudinal reinforcing steel

$F_s =$ allowable tensile or compressive stress in reinforcement

The second term in the addition is the contribution of the longitudinal steel. In particular, the coefficient 0.65 was determined from tests of reinforced masonry columns. The coefficient 0.25 provides a factor of safety of about 4 against the crushing of masonry. Strength is further modified for slenderness effects by the factor R, which is the same for unreinforced masonry.

7.5.1b Axial Compression with Flexure

For combined axial compression and flexure, the unity formula for interaction is not used in designing masonry members in case of reinforced masonry, since it becomes very conservative.

In such cases, emphasis has been to compute nonlinear interaction diagram taking the effect of reinforcement and compression behaviour of masonry into account. The equations and the assumptions used for developing the axial load-bending moment interaction diagram are very similar to those ones used in the analysis and design of reinforced concrete members. Interaction diagrams thus produced permit a rapid graphical solution.

7.5.1c Shear

The shear resistance of masonry also increases when reinforcements are added. However, they are effective in providing resistance only if they are designed to carry the full shear load.

According to ACI code, the minimum area of shear reinforcement is given by the following relation:

$$A_V = \frac{V_S}{F_S d} \tag{7.5.1c-1}$$

where:

$A_V =$ cross section area of shear reinforcement.

$V_S =$ shear strength provided by reinforcement.

$d =$ distance from extreme compression fibre to centroid of tension reinforcement.

This can be derived by assuming a 45° shear crack extended from the extreme compression fibre to the centroid of the tension steel, summing the forces in the direction of the shear reinforcement neglecting the doweling resistance of the longitudinal reinforcement, [21]. However, the shear stress shall not exceed the permissible shear stress of masonry, which depends on the M/Vd_V ratio for shear walls, where:

$M =$ maximum moment at the section under consideration.

$V =$ shear force.

$d_V =$ actual depth of masonry in direction of shear considered.

Such a ratio is the product of h/d_V ratio and a factor depending on end restraints, as shown in the following figure.

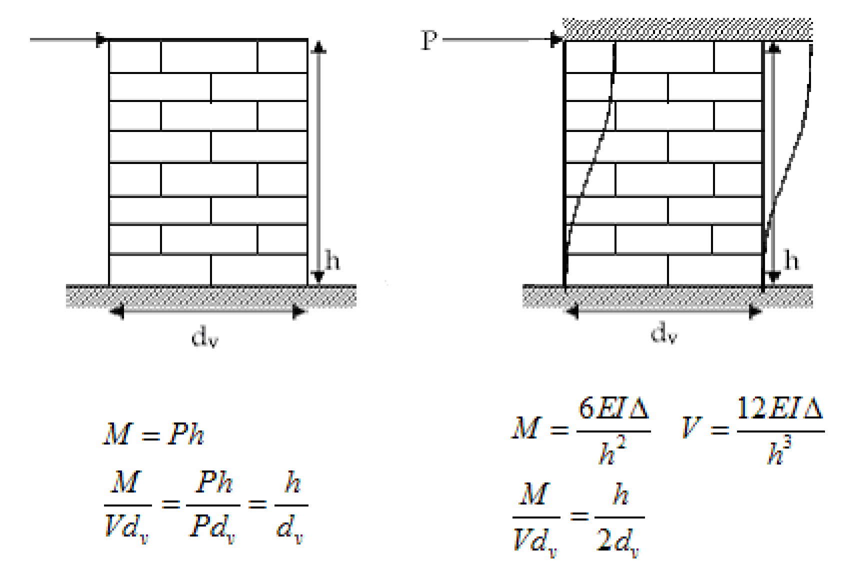


Figure 7.2 Significance of M/Vd_v factor.

7.5.2 Strength design or limit state design

7.5.2a Axial Compression

The nominal strength of a member may be calculated by using the assumptions of an equivalent rectangular stress block. Slenderness effect on axial load carrying capacity is also taken into account, except in IBC 2000.

In New Zealand Standards, nominal axial strength of a load bearing wall is given by $0.5 f_m A_g R'$, where R' is always equal to $\left[1-\left(h/40t\right)^2\right]$ and where A_g is the gross cross-sectional area of masonry.

7.5.2b Axial Compression with Flexure

The nominal axial and flexure strength, for combined axial compression and flexure, are computed similar to RC members with the design assumptions as discussed earlier, which vary from one code to another.

According to ACI code and IBC 2000, the maximum usable strain e_{mu} shall be 0.0035 for clay masonry and 0.002 for concrete masonry. In wall design for out-of-plane loads, according to both the codes, the required moment due to the lateral loads, eccentricity of axial load and lateral deformation are assumed maximum at mid-height of the wall. In certain design conditions, like large eccentricities acting simultaneously with small lateral loads, the design maximum moment may occur elsewhere. When this occurs, the designer should use the maximum moment at the critical section.

In Eurocode 6, the maximum tensile strain in reinforcement should be limited to 0.01. According to this code, no redistribution of the moment is allowed with normal ductility steel. In this case the ratio of depth of neutral axis to the effective depth should not be greater than 0.4. Redistribution of moments in a continuous beam should be limited to 15% when high ductility steel is to be used.

The New Zealand Standards, which deals with only concrete masonry, specifies that e_{mu} shall be 0.0025 for unconfined masonry and 0.008 for confined masonry. Confinement is provided to the masonry walls to impart ductility to them, [21].

7.5.2c Shear

Shear force is assumed to be resisted by both, masonry and reinforcement.

The formulas given in the ACI code and IBC 2000 to derive nominal shear strength of masonry and reinforcement are empirically derived from research. The concept of the minimum shear reinforcement is to help restrain growth of inclined cracking and provide some ductility for members (by confining masonry) subjected to unexpected force or catastrophic loading.

In Eurocode 6, there is a maximum limit to the shear strength provided by masonry and shear reinforcement together, which is given by $0.3 f_m bd / \gamma_m$, [21], where:

$b =$ width of the section.

$d =$ distance from extreme compression fibre to centroid of tension reinforcement.

$\gamma_m =$ partial safety factor for materials.

In the New Zealand Standards, it is mentioned that for masonry members subjected to shear and flexure together with axial load, the shear stress provided by the masonry shall be multiplied by the factor $\left(1+12P_u / A_g f_m\right)$, where the axial load, *P*, is negative for tension and where P_u is the factored axial load, [21].

It is evident that the shear strength provided by masonry, V_m, will decrease because of a reduction of aggregate interlock resulting from axial tension. The code considers instances where shear transfer is required by shear friction along a known or likely crack path. Resistance to sliding along a potential shear failure plane is provided by frictional forces between the sliding surfaces. The frictional forces are proportioned to the coefficient of friction and the total normal force acting across the joint, which may be provided by axial force, P_u, and distributed reinforcement, $A_{vf} f_y$, where A_{vf} is the area of shear friction reinforcement. The effective clamping force across the crack will be $A_{vf} f_y + P_u$. Thus the dependable factored shear force, V_u, which can be transmitted across the crack by shear friction, is $\phi \mu_f \left(A_{vf} f_y + P_u \right)$, where ϕ is the strength reduction factor and μ_f is the coefficient of friction. Thus, the required area of shear friction reinforcement shall be computed from:

$$A_{vf} = \frac{1}{f_y}\left(\frac{V_u}{\mu_f \phi} - P_u \right) \qquad (7.5.2c\text{-}1)$$

During the placing of grout, if the interface has been intentionally roughened, $\mu_f = 1$; else μ_f is taken to be 0.7.

7.6 Discussion

Presently, most design codes prefer Limit State Design approach because of better reliability and economy, which is a major departure from the conventional empirical design method. Moreover, for reinforced masonry, only the ACI code contains provisions based on allowable stress values, whereas all other codes follow only Limit State Design approach. The International Building Code 2000 specifies some minor changes to the ACI code in the form of design assumptions and strength reduction factors.

For allowable strength of masonry shear walls, ACI code emphasizes on the aspect ratio and boundary conditions by a parameter M/Vd_V. Also the strength of masonry is based on prism tests, instead of placing reliance on standard tables, which relate it to the strength of unit and type of mortar. The advantage of prism test is that the prisms are built of similar materials under the same conditions with the same bonding arrangement as in the structure.

The design approach in IS: 1905-1987 is semi-empirical, which combines allowable stress design with rules of thumb for unreinforced masonry only, especially for stresses arising from vertical and moderate lateral loads, such as wind. The permissible stress values are not directly linked to prism test values and do not address the strength and ductility of masonry members under large lateral loads due to earthquakes. Neither limit state methodology has been adopted in this code nor there are any provisions related to reinforced masonry for any design philosophies. So, this code should be expanded to incorporate such provisions.

It is worth to underline that, among such these codes, only the New Zealand Standard contains provisions on ductility of masonry structures and confined masonry. Regarding shear, it contains provisions on shear friction reinforcement and also considers the case when masonry members are subjected to shear and flexure together with axial tension. These salient features are not covered in other documents, [21].

7.7 The Italian code (T.U. 30/03/2005)

They will be exposed, in this paragraph, the main aspects of the Italian code- T.U. 30/03/2005. The interested reader is referred, for major details, to [63].

The first aspect regards the determination of the characteristic resistances for masonry and its constituents. In particular, it is:

- Ø Compressive characteristic strength for masonry element, f_{bk}, in the direction of vertical loads (UNI EN 772-1), in the case of 30 examined specimens, is obtained as it follows:

$$f_{bk} = f_{bm} - 1.64s \tag{7.7-1}$$

where:

$f_{bm} =$ the arithmetic media of the resistances of the elements.

$s =$ the mean square deviation.

When the number *n* of the examined specimens is between 10 and 29, the coefficient s assumes the following *k* values:

n	10	12	16	20	25
k	2.13	2.06	1.98	1.93	1.88

Table 7.1 Mean square deviation.

When the number *n* of the examined specimens is between 6 and 9, the compressive characteristic strength is assumed equal to the minimum value between:

a. $0.7 f_{bm} \left(N/mm^2 \right)$

b. the minimum value of the unit resistance of the single specimen.

When masonry is constituted by natural elements, the compressive characteristic strength of the element is assumed, conventionally, equal to:

$$f_{bk} = 0.75 f_{bm} \tag{7.7-2}$$

- Compressive characteristic in-plane strength for masonry element, $\overline{f}_{bk}$, in the orthogonal direction of vertical loads (UNI EN 772-1) is obtained as it follows:

$$\overline{f}_{bk} = 0.7\,\overline{f}_{bm} \tag{7.7-3}$$

- Compressive characteristic strength for mortar, f_m, (UNI EN 998-2) is given by the following table 2:

Class	**M 2.5**	**M 5**	**M 10**	**M 15**	**M 20**	**M d**
Compressive strength N/mm^2	2.5	5	10	15	20	d

Table 7.2 Compressive characteristic strength for mortar.

where d is a compressive strength $\geq$ 25 N/mm^2, declared by the producer.

- Compressive characteristic strength for masonry, f_k, is given by the following relation:

$$f_k = f_m - ks \tag{7.7-4}$$

where:

$f_m =$ average resistance

$s =$ deviation estimate

$k =$ a coefficient is dependent from the number *n* of the specimens, [63].

Compressive characteristic strength for masonry, f_k, can be also obtained according to the compressive characteristic strength of the masonry elements, f_{bk}, and to the mortar category, as shown in the following tables 3 and 4:

Compressive strength f_{bk} for artificial masonry elements N/mm^2	**Mortar type**			
	M 15	**M 10**	**M 5**	**M 2.5**
2.0	1.2	1.2	1.2	1.2
3.0	2.2	2.2	2.2	2.0
5.0	3.5	3.4	3.3	3.0
7.5	5.0	4.5	4.1	3.5
10.0	6.2	5.3	4.7	4.1
15.0	8.2	6.7	6.0	5.1
20.0	9.7	8.0	7.0	6.1
30.0	12.0	10.0	8.6	7.2
40.0	14.3	12.0	10.4	--

Table 7.3 Compressive characteristic strength for masonry - Artificial elements.

Compressive strength f_{bk} for natural masonry elements N/mm^2	**Mortar type**			
	M 15	**M 10**	**M 5**	**M 2.5**
2.0	1.0	1.0	1.0	1.0
3.0	2.2	2.2	2.2	2.0
5.0	3.5	3.4	3.3	3.0
7.5	5.0	4.5	4.1	3.5
10.0	6.2	5.3	4.7	4.1

15.0	8.2	6.7	6.0	5.1
20.0	9.7	8.0	7.0	6.1
30.0	12.0	10.0	8.6	7.2
$\geq$ 40.0	14.3	12.0	10.4	--

Table 7.4 Compressive characteristic strength for masonry – Natural elements.

- Shear characteristic strength for masonry in absence of normal stresses, f_{vk0}, is given by the following relation:

$$f_{vk0} = 0.7 f_{vm} \tag{7.7-5}$$

where:

$f_{vm} =$ average shear resistance.

Shear characteristic strength for masonry, f_{vk0}, can be also obtained according to the compressive characteristic strength of the masonry element, f_{bk}, and to the mortar category, as shown in the following tables:

Compressive strength f_{bk} for artificial brick elements	**Mortar type**	f_{vk0}
≤ 15	$\leq M15$	0.2
>15	$\leq M15$	0.3

Table 7.5 Shear characteristic strength for masonry – Artificial brick elements.

Compressive strength f_{bk} for artificial concrete elements	**Mortar type**	f_{vk0}
≤ 3	$M15, M10, M5$	0.1
	$M2.5$	0.1
> 3	$M15, M10, M5$	0.2
	$M2.5$	0.1

Table 7.6 Shear characteristic strength for masonry – Artificial concrete elements.

Compressive strength f_{bk} for natural elements	**Mortar type**	f_{vk0}
≤ 3	$M15, M10, M5$	0.1
	$M2.5$	0.1
> 3	$M15, M10, M5$	0.2
	$M2.5$	0.1

Table 7.7 Shear characteristic strength for masonry – Natural elements.

- Shear characteristic strength for masonry in presence of normal stresses, f_{vk}, is given by the following relation:

$$f_{vk} = f_{vk0} + 0.4\sigma_n \tag{7.7-6}$$

where:

$\sigma_n =$ average normal stress, due to the vertical loads acting on the examined section.

- Elastic secant modulus, E, is given by the following relation:

$$E = 1000\, f_k \tag{7.7-7}$$

- Elastic tangential modulus, G, is given by the following relation:

$$G = 0.4E \tag{7.7-8}$$

The second aspect regards the specification of the provisions for the structural organization of a masonry building and for the structural analysis of unreinforced and reinforced masonry with reference to both allowable stress and limit state design.

7.7.1 Structural organization

A bearing masonry building has to be conceived as a three-dimensional box where the bearing walls, the ceilings and the foundations are opportunely connected each other in order to resist to the vertical and horizontal loads, [63].

The thickness of the masonry walls cannot be less than:

- masonry in artificial resistant full elements: 120 mm
- masonry in artificial resistant half-full elements: 200 mm
- masonry in artificial resistant perforated elements: 250 mm
- masonry in squared stone: 240 mm
- lined masonry: 400 mm

The following tables relate a classification for the artificial brick and concrete elements:

Brick elements	**Hole percentage**	***f***
Full	$j \leq 15\%$	$f \leq 900\, mm^2$
Half full	$15\% < j \leq 45\%$	$f \leq 1200\, mm^2$
Perforated	$45\% < j \leq 55\%$	$f \leq 1500\, mm^2$

Table 7.8 Brick elements classification.

Concrete elements	**Hole percentage**	***f***	
		$A \leq 90000\, mm^2$	$A > 90000\, mm^2$
Full	$j \leq 15\%$	$\leq 10A$	$\leq 15A$
Half full	$15\% < j \leq 45\%$	$\leq 10A$	$\leq 15A$
Perforated	$45\% < j \leq 55\%$	$\leq 10A$	$\leq 15A$

Table 7.9 Concrete elements classification.

where:

$j =$ hole percentage.

$f =$ average area of a single hole section.

$A =$ the gross area of the element face, which is delimited by its perimeter.

The *conventional thinness* of the masonry walls has to be defined according to the following equation:

$$l = h_0/t \tag{7.7.1-1}$$

where:

$h_0 =$ free bending length of the wall equal to $r\,h$.

$h =$ internal level height.

$r =$ lateral factor of constraint (see table 10).

$t =$ thickness of the wall.

	r
h/a* ≤ *0.5	**1**
0.5* < *h/a* ≤ *1.0	**3/2 –*h/a***
1.0* < *h/a	**1/[1+(*h/a*)^2]**

Table 7.10 Lateral factor of constraint.

with:

$a =$ wheelbase between two transversal walls.

In any case, the *conventional thinness* of the masonry walls cannot result more than 20, [63].

7.7.2 Structural analyses and resistance controlling

The structural analyses can be non linear analyses or linear ones, these latter being obtained by assuming the secant value for the elastic moduli. For each structural element, they must yield:

- the axial load given by the vertical loads and, for buildings with height more than 10 m, the variation of the axial load given by the horizontal actions.
- the shear force given by the vertical and horizontal loads.
- the eccentricity of the axial loads.
- the bending moments given by the vertical and horizontal loads.

The actions have to be combined so to determine the most disadvantageous load conditions for the single resistance controlling. However, it has to be considered the reduced probability of a simultaneous intervention of all actions with their most unfavourable values, like the in force codes prescribe, [63].

Two hypotheses are considered in a strength controlling: the assumption that the sections remain plane and the masonry tensile strength is neglected.

Each masonry wall has to be verified with reference to both allowable stress and limit state design, under the following load conditions:

(a) axial compression with flexure for lateral loads
(b) axial compression with flexure for in-plane loads
(c) shear for in-plane loads
(d) concentrated loads

The design strength f_d to be used in the cases (a), (b) and (d) is:

$$f_d = \frac{f_k}{\gamma_m}\frac{1}{\gamma_{R,d}} \qquad (7.7.2\text{-}1)$$

where:

$f_k =$ compressive characteristic strength for masonry.

$\gamma_m =$ partial safety coefficient on the masonry compressive strength. It is equal to 2 or 2.5 depending on the kind of resistant elements, artificial or natural, [63].

$\gamma_{R,d} =$ partial safety coefficient (see table 11).

The design strength f_{vd} to be used in the case (c) is:

$$f_{vd} = \frac{f_{vk}}{\gamma_m}\frac{1}{\gamma_{R,d}} \qquad (7.7.2\text{-}2)$$

where:

$f_{vk} =$ shear characteristic strength for masonry, in presence of normal stresses, calculated in function of the f_{vk0}.

$\gamma_m =$ partial safety coefficient on the masonry compressive strength. It is equal to 2 or 2.5 depending on the kind of resistant elements, artificial or natural, [63].

$\gamma_{R,d} =$ partial safety coefficient (see table 11).

Calculation method	$\gamma_{R,d}$
Allowable stress	≥ 2
Limit state design	≥ 1.2

Table 7.11 Partial safety coefficient $\gamma_{R,d}$.

7.7.3 Allowable stress design for unreinforced masonry

7.7.3a Axial Compression with Flexure

The strength controlling is satisfied if:

$$\sigma = \frac{N_d}{\Phi_l \Phi_t A} \leq f_d \qquad (7.7.3a\text{-}1)$$

where:

$N_d =$ design axial force.

$\Phi_l =$ restrictive coefficient of the strength for longitudinal eccentricity.

$\Phi_t =$ restrictive coefficient of the strength for transversal eccentricity.

$f_d =$ design compressive strength for masonry.

$A =$ area of the wall section.

7.7.3b Shear for in-plane loads

The strength controlling is satisfied if:

$$t = \frac{V_d}{b A} \le f_{vd} \tag{7.7.3b-1}$$

where:

$f_{vd} =$ design shear strength for masonry.

$V_d =$ design shear force.

$b =$ choking coefficient of the wall.

$A =$ net area of the wall section.

7.7.3c Concentrated loads

The strength controlling is satisfied if:

$$s = \frac{N_{dc}}{b_c A_c} \le f_d \tag{7.7.3c-1}$$

where:

$N_{dc} =$ design value of the concentrated load.

$b_c =$ amplifying coefficient of the concentrated loads.

$f_d =$ design compressive strength for masonry.

$A_c =$ support area.

7.7.4 Limit state design for unreinforced masonry

7.7.4a Axial Compression with Flexure for out-of-plane loads

The strength controlling is satisfied if:

$$N_d \leq N_{Rd} = \Phi_t f_d A \qquad (7.7.4a\text{-}1)$$

where:

$N_d =$ design axial force.

$N_{Rd} =$ design strength.

$\Phi_t =$ restrictive coefficient of the strength for load transversal eccentricity and for the wall thinness.

$f_d =$ design compressive strength for masonry.

$A =$ area of the wall section.

7.7.4b Axial Compression with Flexure for in-plane loads

The strength controlling is satisfied if:

$$M_d \leq M_{Rd} = \frac{tl^2}{2}\frac{N_d}{A}\left(1 - \frac{N_d}{A\alpha f_d}\right) \qquad (7.7.4b\text{-}1)$$

where:

$M_d =$ design bending moment.

$N_d =$ design axial force.

$M_{Rd} =$ design strength.

$t =$ wall thickness.

$l =$ wall length.

$f_d =$ design compressive strength for masonry.

$A =$ area of the wall section.

a ≤ 0.85; it is a restrictive coefficient of the strength.

7.7.4c Shear for in-plane loads

The strength controlling is satisfied if:

$$V_d \leq V_{Rd} = \mathrm{b}\, A f_{vd} \tag{7.7.4c-1}$$

where:

$V_d =$ design shear force.

$V_{Rd} =$ design strength.

$f_{vd} =$ design shear strength for masonry.

$A =$ area of the wall section.

b $=$ choking coefficient of the wall.

7.7.4d Concentrated loads

The strength controlling is satisfied if:

$$N_{dc} \leq N_{Rdc} = \mathrm{b}_c A_c f_d \tag{7.7.4d-1}$$

where:

$N_{dc} =$ design concentrated force.

$N_{Rdc} =$ design strength.

$f_d =$ design compressive strength for masonry.

$A_c =$ support area.

$b_c =$ amplifying coefficient for the concentrated loads.

It has been illustrated, here, a short review of the Italian code. For more detail on it, the reader is referred to [63].

CONCLUSIONS

The present work deals with the mechanic characterization of masonries (heterogeneous materials) via micro-mechanical approach, in linear-elastic field. In order to provide a definition of the constitutive laws for masonry, both the aspects of inhomogeneity and anisotropy are taken into consideration, since the first one is due to the biphasic composition and the second one is due to the geometrical arrangement of the constituents within the masonry RVE.

In this framework, both *heuristic* and *thermodynamical* approaches, which are used in literature in order to study the heterogeneous materials, are described. In particular, the attention has been focused on the latter one and, more in detail, on the homogenization techniques and micro-mechanical analyses which are furnished by the scientific literature, with reference to masonries.

By applying the homogenization theory to the masonry material, it is possible to obtain a "*homogeneous equivalent material*" whose mechanical properties are able to average the actual and variable ones of the heterogeneous medium. Hence, by means of mathematical operations of volume averaging and consistency, the global mechanical behaviour of masonry can be determined, depending on its micro-structure geometry and on the known elastic properties of its micro-constituents.

The present work has, so, two main objects:

- to furnish a general account on the homogenization procedures for periodic masonries existing in literature in linear-elastic field and, contemporaneously, to underline the advantages and disadvantages for each one of them. More in detail, the existing homogenization procedures can be basically divided in two approaches. The first one employs a simplified homogenization process in different steps for obtaining, on the contrary, a close-form solution (Pietruszczak & Niu, 1992, for example). The second one employs a rigorous homogenization process in one step for obtaining, on the contrary, an approximated numerical solution (Lourenco & al, 2002, for example).
- to furnish some possible proposals for modelling periodic masonry structures, in linear-elastic field, by starting from the results of literature approaches, in order to obtain new homogenization techniques able to overcome the limits of the existing ones. More in detail, two procedures have been proposed: a simplified two-step homogenization (S.A.S. approach) and a rigorous one-step homogenization (Lourenco modified approach-statically consistent).

By comparing the homogenization techniques, it can be said that:

PIETRUSZCZAK & NIU APPROACH - implies an approximated homogenization procedure in two steps, whose results are dependent on the sequence of the steps chosen. It represents the limit of this kind of the existing approaches.

S.A.S. APPROACH **-** employs a parametric homogenization which, on the contrary, results consistent in the two-step process, by implying exact solutions in some directions. Hence, the proposed procedure overcomes the limit of the above mentioned simplified approaches.

LOURENCO & AL. APPROACH **-** proposes a homogenized model which is obtained on a parameterization-based procedure depending on a specific benchmark FEM model (i.e. selected ratios between elastic coefficients and geometrical dimensions); so it shows a sensitivity to geometrical and mechanical ratios! Moreover, the numerical estimate of the homogenized coefficients gives some not symmetrical moduli, so a symmetrization becomes necessary!

LOURENCO MODIFIED APPROACH **-** proposes a parametric homogenized model which, on the contrary, is not dependent on specific selected ratios between elastic coefficients and geometrical dimensions, so it shows a more generalized applicability. Moreover, since the approach implies a statically-consistent solution, it results extremely useful according to the Static Theorem.

Some computational analyses (stress and strain-prescribed) are finally carried out by means of the calculation code Ansys, in its version 6.0, in order to compare the analytical results obtained by our proposed homogenization techniques with the literature theoretical and experimental data. Such comparison has yielded the elaboration of useful tables. They contain both the elastic homogenized moduli, which are obtained by means of the different examined homogenization techniques, and the estimate of the errors from which each procedure is affected. By observing differences among the elastic coefficients which are shown in the comparison-tables, it is worth to highlight that:

- due to consistency, some proposed elastic moduli appear to be closer than those ones yielded by Lourenco.
- as a result, it is possible to determine an elasticity tensor by means of those parametric moduli, yielded by the examined homogenization procedures, which are closer to the reference numerical data. Such elasticity tensor is, so, defined on the knowledge of elastic ratios as well as of geometrical parameters characterizing the RVE.

The last chapter, finally, deals with a review of the international codes referred to the design of masonry structures. In this framework, the goal is to furnish a short summary and a comparison between the examined codes different from a number of countries. This review will be particularly useful in a possible continuation of the research activity, whose perspectives are:

- the extension of the proposed strategies to post-elastic range.
- the introduction of anisotropic failure criteria.
- the comparison of the proposed models with the experimental data, by considering the possibility of applying them to reinforced masonry walls.

Hence, in this framework, a comparison between the theoretical constitutive characterizations, obtained by means of the examined homogenization procedures, and that ones yielded by the examined codes will be made.

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