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NEW STABILITY RESULTS FOR REACTION-DIFFUSION MODELS

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Introduction

This thesis deals with linear and nonlinear stability analysis of the equilibria of some reaction-diffusion systems modeling epidemiological models and convection phenomena in porous media.

Reaction-diffusion equations have been deeply studied in recent years thanks to their several applications to ecology, biology, biochemistry, classical theory of heat and mass transfer and so on. At first, reaction-diffusion equations have been especially used to model chemical phenomena, but they are also well suited to model a wide range of other kinds of dynamical processes. In general, they can be used to describe the movement of many individuals in a given environment or media. These individuals could be very small, such as basic particles in physics, bacteria, cells or very large, such as animals, humans or certain kinds of events like epidemics or rumors. In this more general case, the state variable of the reaction-diffusion equations represents the density function of the particles or of the individuals of a given population spreading in a given domain. The dimension of the population density is usually defined as the number of particles or organisms per unit area or unit volume.

According to what happens for a chemical substance, the population size can change either for the movement of the individual particles or for production/killing of individuals due to physical, chemical or biological reasons.

The way in which the particles move is the result of a highly complicated process which can be attributed to a lot of reasons, For example, to the migration processes of humans that can be for looking for a better life, for a job, political, for religious or economical reasons and so on. Although these reasons can be quoted as motivation to move, generally people move from areas where the population density is higher to areas where it is lower. This is similar to what happens in many physical phenomena like heat transfer (from warmer places to colder ones). By making this assumption, the flux of the population is governed by the Fick's law.

On the other hand, the number of individuals may change because of other reasons like birth, death, hunting and so on.

Most of the thesis deals with the analysis of some reaction-diffusion systems modeling the spreading of an infection within a population.

In order to study infectious disease transmission, mathematical models play a central role since they allow to predict the asymptotic behaviour of the infection and, consequently, to take some actions in order to control epidemics. When a population is not infected yet by a disease, all the individuals are regarded as susceptibles. Introducing a few number of infected in the community, in order to know if the epidemic will die out or if it will blow up, it would be useful to study the longtime behaviour and the stability of the so called disease-free equilibrium (DFE). If DFE is stable, then epidemic will decay; then, the problem to determine if endemic equilibria (equilibria with all positive components) exist arises. When endemic equilibria exist, their stability analysis allows to state if epidemic will persist or not.

In the present thesis, first of all, a SEIR epidemic model, under different kinds of boundary conditions, is considered.

In classical epidemic models, the host population is supposed to be divided into three disjoint classes: S, which are individuals susceptible to infection, I, which are infective individuals and R which are the removed ones. However, in several cases, the disease incubation period is not negligible: that means that the disease may require some time for individuals to pass from the infected state to the infective one. In this case, a further class has to be considered, the individuals exposed to infection or individuals in the latent state ([2], [3], [52]).

A key role in epidemic models is played by the so called force of infection or incidence rate, which is the function describing the mechanism of disease transmission. In classical models, the incidence rate is proportional to the product of susceptibles and infectives. However, in order to generalize the dynamics of disease transmission, since 1970's Capasso and his coworkers stressed the importance to consider nonlinear incidence rates, in particular by studying a case of cholera epidemic [8]. Since then, many authors proposed peculiar nonlinear forms for the force of infection [5], [7], [26], [32], [53]. The present thesis will deal with the well known and meaningful incidence rate $g(S, I) = KIS(1 + \alpha I)$, where K and α are positive constants. This functional means an increased rate of infection due to double exposure over a short time period so that the single contacts lead to infection at a rate KIS whereas new infective individuals arise from double exposures at a rate $K\alpha I^2 S$.

In most of epidemic models known in literature, the population is supposed to be homogeneously distributed in the domain at hand so that there is no distinction between individuals in one place and those in another one and the time evolution of the disease is described through a system of ordinary differential equations. However, in order to consider the more general case in which the disease may spread faster in some parts of the domain then in other ones, it is necessary to allow the variables of the model to depend on space as well as on time and hence a reaction-diffusion model.

The same motivation justifies the introduction of a reaction-diffusion system modeling the spreading of a cholera epidemic in a non homogeneously mixed population. Cholera is an acute intestinal infection caused by the bacterium Vibrio Cholerae. The mechanism of transmission occurs, principally, via the ingestion of contaminated food or water and only secondarily via direct human-to-human contacts. A lot of mathematical models are devoted to study cholera outbreaks in different parts of the world. In particular, Capasso and Paveri-Fontana in 1979 studied the cholera epidemic in Bari in 1973 by introducing a system modeling the evolution of infected people in the community and the dynamics of the aquatic population of pathogenic bacteria. In fact, cholera diffusion is strictly linked to the interactions between individuals in the community and bacteria in contaminated water, [8].

In this thesis, the attention is focused on some reaction-diffusion systems modeling the spreading of epidemics within a population or within interacting populations.

Successively, Capasso and Maddalena in 1981, in order to let the model be more realistic, assumed that the bacteria diffuse randomly in the habitat, hence they analyzed a model consisting of two nonlinear parabolic equations [9]. Since many studies found that toxigenic Vibrio Cholerae can survive in some aquatic environments for months to years, many authors began to consider the aquatic environment as a reservoir of Vibrio Cholerae in endemic regions. Codeco in 2001 analyzed the role of aquatic reservoir in promoting cholera outbreaks by introducing an ODE model that includes the dynamics of susceptible population [19].

In this thesis, the above model is generalized taking into account non homo-

geneously mixed toxigenic Vibrio Cholerae reservoir in contaminated water and dividing the total population into three disjoint classes: susceptibles, infected and removed individuals. A central role is played by the study of influence of diffusivity of each population on the model dynamics.
In studying the above reaction-diffusion models, the main aims are to study:

- the longtime behaviour of solutions, in particular their boundedness and the existence of absorbing sets;
- the linear and nonlinear stability of equilibria, especially of the biologically meaningful ones, aiming to obtain the optimal result of coincidence between the linear and nonlinear stability thresholds.

The method applied in order to reach the second aim is the Liapunov direct method which, unlike approximate methods that are often involved in the stability study of partial differential equations, works directly with the system and it is potentially applicable when nonlinearities are involved. However, for the stability analysis, the central problem in using the direct method consists in the construction of a peculiar Liapunov function which allows to find conditions ensuring coincidence between linear and nonlinear stability thresholds as far as the global stability when it is possible. In this thesis a peculiar Liapunov functional, introduced by Rionero [54], [55], [62] is employed; this functional is directly linked, together with its derivative along the perturbations, to the principal invariants of the linear operator of the model at hand.
In this thesis, a similar Liapunov functional is also used in order to study the linear and nonlinear stability of a vertical constant throughflow, which is a stationary solution of a system modeling fluid motions in horizontal porous layers, uniformly heated from below and salted from above by one salt.

The research concerned with fluid motions in porous media, due to their large applications in real world phenomena, is very active nowadays. In fact, porous materials occur everywhere (see for instance geophysical situations, cultural heritage contaminant transport and underground water flow [25], [56] and the references therein). In particular, convection and stability problems in porous layers in the presence of vertical throughflows find relevant applications in cloud physics, in hydrological studies, in subterranean pollution and in many industrial processes where the throughflows can control the onset and evolution of convection (see [12], [13], [29], [30], [43], [44]-[46], [50], [61], [68], [69]). In fact, the effect of vertical throughflow on the onset of convection has been considered in many cases (the effect in a rectangular box in [45]; the effect combined with a magnetic field in [43]; stability analysis in a cubic Forchheimer model in [29] and when the density is quadratic in temperature in [30]; the effect with an inclined temperature gradient in [50]).
In the present thesis the effects of both temperature gradient and salt concentration on the stability of a vertical flow are taken into account. Already in [12] and [18] the authors consider both the effects. Precisely, the effect of variable thermal and solutal diffusivities on the onset of convection for non constant throughflows has been analyzed in [18], while in [12] the stability of a vertical constant throughflow in a porous layer, uniformly heated and salted from below, has been investigated. In particular, sufficient conditions ensuring linear and global nonlinear stability in the L^2-norm have been determined.
In the present thesis, the more destabilizing case of horizontal porous layers uniformly heated from below and salted from above by one salt is analyzed. The thesis is organized as follows.
Chapter 1 is devoted to some general definitions and known results about

reaction-diffusion models, including some existence theorems.
In Chapter 2 some basic properties of dynamical systems and the basic tools for Liapunov direct method are recalled, stressing the differences in its applications to ordinary differential equations and partial differential equations.
Chapter 3 deals with the linear and nonlinear stability analysis of the biologically meaningful equilibria of a SEIR reaction-diffusion model for infections under mixed boundary conditions and then under homogeneous Neumann ones and of a reaction-diffusion system modeling a Cholera epidemic.
Finally, in Chapter 4 the linear and nonlinear stability of a vertical constant throughflow through a porous medium in a horizontal layer, uniformly heated from below and salted from above by one salt, is investigated. By using a new approach concerned with the Routh-Hurwitz conditions and the use of a peculiar Liapunov functional, necessary and sufficient conditions for the linear stability of a vertical constant throughflow are determined. Furthermore, conditions ensuring the global non linear stability for the vertical constant throughflow are obtained.

Chapter 1

Reaction-diffusion systems

Reaction-diffusion equations have been deeply studied in recent years due to their several applications to ecology, biology, biochemistry and the classical theory of heat-mass transfer. At first, they have been especially used to model chemical phenomena. In this sense, they describe how concentration of one or more substances distributed in the space changes under the influence of two processes: local chemical reactions, through which the substances are converted one into each other, and diffusion, which cause the substances' spread out in space.

Let us suppose that the substances, whose diffusion and spreading we are interested in, occupe a domain Ω and let us assume that Ω is an open bounded subset of $\mathbb{R}^n$ with $n \geq 1$ (in particular, for physical reasons, we will be interested in the cases of $n = 1, 2, 3$). Let us denote by $\partial\Omega$ the boundary of Ω and by B an elementary volume at fixed location within the domain. Let us introduce the function $u(x,t) : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ which is the concentration of the chemical substance, being t the time variable and $x \in \Omega$ the location. The change of the amount of a "substance" U within the elementary volume B is given by the flux of matter through the elementary volume boundary

∂B plus the net production rate of a chemical species (the reaction kinetics) in B, and so, in mathematical terms

$$\frac{d}{dt}\int_B u(x,t)\,dx = -\int_{\partial B} \mathbf{J}\cdot\mathbf{n}\,dS + \int_B f\,dx, \tag{1.1}$$

where $\mathbf{J}$ is the flux density, i.e. the scalar product $\mathbf{J}\cdot\mathbf{n}$ is the net rate at which particles cross a unit area in a plane perpendicular to $\mathbf{n}$ (positive in the $\mathbf{n}$ direction, $\mathbf{n}$ being the outward-oriented normal to B on ∂B), and f, the reaction kinetics, is the rate of production and degradation of the reactant. Generally, it consists of a polynomial or rational function of u and of some parameters that represent interaction with other chemicals and external factors.

Using the divergence theorem (assuming the underlying fields are smooth), (1.1) becomes

$$\frac{d}{dt}\int_B u(x,t)\,dx = \int_B \left[-\nabla\cdot\mathbf{J} + f\right]\,dx.$$

If we suppose that the domain is fixed in time, we can differentiate through the integral and, in view of the arbitrary choice of the elementary volume B in Ω, the following local conservation equation holds

$$\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{J} + f,$$

for any flux transport $\mathbf{J}$ and any "supply" f. Obviously, the last term f may depend on u, such as on position x and time t, i.e. $f = f(t,x,u)$.

If we suppose that the instantaneous flux $\mathbf{J}$ is due to isotropic Fickian diffusion, then $\mathbf{J}(x,t) = -D(x)\nabla_x u(x,t)$, where $D(x)$ is called diffusivity and ∇_x is the gradient operator with respect to the x variable. Hence one obtains the following reaction-diffusion equation for species U on a fixed domain

$$\frac{\partial u(x,t)}{\partial t} = \nabla\cdot\left(D(x)\nabla_x u(x,t)\right) + f(t,x,u), \quad (x,t)\in\Omega\times\mathbb{R}^+. \tag{1.2}$$

Now, on considering a changing in time domain $\Omega(t)$ with boundary $\partial\Omega(t)$, let $B(t)$ be an elementary volume which moves with the flow due to domain change. Applying the conservation of matter and the divergence theorem (being instantaneously valid at all times) to any measurable $B(t)$, we obtain

$$\frac{d}{dt}\int_{B(t)} u(x,t)\,dx = \int_{B(t)} \left[-\nabla\cdot\mathbf{J} + f\right]\,dx.$$

In view of the Reynolds transport theorem, one has that

$$\frac{d}{dt}\int_{B(t)} u(x,t)\,dx = \int_{B(t)} \left[\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{v}u)\right]\,dx$$

where $\mathbf{v}$ is the velocity field of the flow. Also in this case, the arbitrary choice of $B(t)$ implies

$$\frac{\partial u}{\partial t} = -u\nabla\cdot\mathbf{v} - \nabla u\cdot\mathbf{v} - \nabla\cdot\mathbf{J} + f,$$

that, for isotropic Fickian flux, becomes

$$\frac{\partial u}{\partial t} = -u\nabla\cdot\mathbf{v} - \nabla u\cdot\mathbf{v} + \nabla\cdot(D(x)\nabla_x u) + f,$$

which is the local form of the diffusion equation with convection. The term $\nabla\cdot\mathbf{v}$ gives the local rate of volume expansion or contraction. In particular, for incompressible flows, $\nabla\cdot\mathbf{v}=0$. The convection or advection term $\nabla u\cdot\mathbf{v}$ represents the transport of chemicals within the domain as it moves and no relative movement of the chemicals with respect to the domain is present.
Since one is generally interested in the interaction of several particles species, for example several chemicals $\{U_1,...,U_n\}$, then equation (1.2) is replaced by a system which describes the evolution of a vector of concentrations $\mathbf{u}=(u_1,...,u_n)$ and now the kinetic term, $\mathbf{f}(t,x,\mathbf{u})=(f_1,...,f_n)$ is a vector describing the interaction of the species.

Reaction-diffusion equations are also well suited to model a wide range of other kinds of dynamical processes. In general, they can be used to describe the movement of many individuals in an environment or media. The individuals can be very small such as basic particles in physics, bacteria, molecules or cells, or very large objects such as animals, plants or certain kind of events like epidemics or rumors.

In this more general case, the state variable $u(x,t)$ represents the density function of the particles or of the individuals of a given population spreading in the domain Ω. The dimension of the population density is usually the number of particles or organisms per unit area (if $n=2$) or unit volume (if $n=3$). We will assume that the function $u(x,t)$ has regularity properties, like continuity and differentiability, which is reasonable when a population with a large number of individuals is considered. Technically, we define the population density function $u(x,t)$ as follows. Let x be a point of the domain Ω and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of spatial regions (which have the same dimension as Ω) surrounding x; here B_n is chosen in such a way that the spatial measurement $|B_n|$ of B_n (length, area, volume, or, mathematically, the Lebesgue measure) tends to zero as $n \longrightarrow \infty$, and $B_n \supset B_{n+1}$; then

$$u(x,t) = \lim_{n\longrightarrow\infty} \frac{\text{number of individuals in } B_n \text{ at time } t}{|B_n|},$$

if the limit exists.

According to what happens for a chemical substance, population can change in two ways: the first one is that the individual particles can move around and the second one is that new individuals may be produced or existing individuals may be killed due to physical, chemical or biological reasons. We shall model these two different phenomena separately.

The way in which the particles move is the result of a highly complicated process which can be attributed to a lot of reasons. For example, the reasons of the emigration of human can be looking for a better life, looking for a better job, political, economical or religious reasons. Although these reasons can be quoted as motivation to move, generally people move from areas where population density is high to areas where it is lower. This is similar to what happens in many physical phenomena, like the heat transfer (from warmer place to colder place) or the dilution of chemical in water. By making this assumption, the vector $\mathbf{J}$, that represents the flux of the population density in this context, is governed by the Fick's law, i.e.

$$\mathbf{J}(x,t) = -D(x)\nabla_x u(x,t),$$

where $D(x)$ is the diffusion coefficient at x and ∇_x is the gradient operator with respect to the x variable.

On the other hand, the number of particles at any point may change because of other reasons like birth, death, hunting and so on. We assume that the rate of change of the density function due to these reasons is $f(t,x,u)$, which we usually call the reaction rate.

For any elementary volume B of the domain, the total population inside B is $\displaystyle\int_B u(t,x)\,dx$ and the rate of change of the total population is

$$\frac{d}{dt}\int_B u(t,x)\,dx.$$

The net growth of the total population inside the region B is

$$\int_B f\,dx$$

and the total out flux is

$$\int_{\partial B} \mathbf{J}(x,t)\cdot\mathbf{n}(x)\,dS,$$

where ∂B is the boundary of B and $\mathbf{n}(x)$ is the outer normal direction to ∂B at x. By using the same procedure as before, one obtains again the reaction-diffusion equation (1.2). The diffusion coefficient $D(x)$ is not a constant in general since the environment is usually heterogeneous. However, when the region of the diffusion is approximately homogeneous, we can assume that $D(x) \equiv D$, then (1.2) can be simplified to

$$\frac{\partial u}{\partial t} = D\Delta u + f(t, x, u),$$

where $\Delta = \nabla \cdot (\nabla) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.

1.1 Boundary conditions

If we consider a reaction-diffusion equation on a bounded domain $\Omega \subset \mathbb{R}^n$, then we need, additional to the initial conditions, well-suited boundary conditions (otherwise, uniqueness cannot be guaranteed). Let us consider the equation

$$\frac{\partial u}{\partial t} = D\Delta u + f(t, x, u), \qquad t > 0, x \in \Omega \tag{1.3}$$

under the initial conditions

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.4}$$

For the existence of solutions of reaction-diffusion equations, the choice of properly posed boundary conditions and reasonable initial data is crucial. In general, suitable boundary conditions for the initial value problem (1.3)-(1.4) can assume one of this form

- Neumann boundary conditions, i.e.

  $$\nabla u \cdot \mathbf{n} = b(x, t), \quad t > 0, x \in \partial\Omega$$

  where $\mathbf{n}$ is the outer normal at $x \in \partial\Omega$ and b is a prescribed function. The homogeneous case, i.e. $b \equiv 0$, corresponds to the no-flux condition:

it means that no particles or individuals can leave or enter the domain Ω via the boundary. In the case of a chemical reaction, it means that there is no additions and subtractions of any chemicals through the boundary and all chemicals are generated inside the reactor and they remain there.

Homogeneous Neumann boundary conditions are also meaningful in many population ecological models. For example, if the spatial domain of the system is an isolated island and if all living species on the island do not attempt to emigrate, then the ecosystem of the island can be considered as a closed one and no-flux boundary conditions can be taken into account.

For the classical heat conduction equation $\dfrac{\partial T}{\partial t} = D\Delta T$, the no-flux boundary conditions mean that the boundary is heat-insulated.

- Dirichlet boundary conditions, i.e.

$$u = b(x,t), \quad t > 0, x \in \partial\Omega$$

where b is a prescribed function. If $b \equiv 0$, these are called homogeneous Dirichlet boundary conditions. Dirichlet boundary conditions assume that the solution takes the value $b(x,t)$ for each point x on the boundary $\partial\Omega$ and for any $t > 0$. In many cases, the homogeneous Dirichlet boundary condition makes the mathematical problem easier, but for most chemical reactions, no-flux condition is more appropriate than Dirichlet boundary condition. When the problem assumes the Dirichlet boundary conditions $u(x,t) \equiv c$, being c a constant, we can think $u(x,t)$ takes the value c for all the points outside of Ω. For example, for the heat equation $\dfrac{\partial T}{\partial t} = D\Delta T$, constant Dirichlet conditions mean that the outside environment has a constant temperature T_0.

For ecological applications, sometimes we can assume homogeneous Dirichlet boundary condition $u(x,t) \equiv 0$ on $\partial\Omega$, which implies that any individual of the species cannot survive outside of or even on the boundary of Ω. Thus, in this case, the boundary is lethal (sometimes called absorbing) and any individual who wanders outside (due to diffusion) is killed by the sterile exterior environment or the deadly boundary.

- Robin boundary conditions (or mixed boundary conditions), i.e.

$$\alpha(x,t)u + \beta(x,t)\nabla u \cdot \mathbf{n} = b(x,t), \quad t > 0, x \in \partial\Omega$$

with $\alpha(x,t), \beta(x,t) \geq 0$ and $b(x,t)$ prescribed functions.

Let us remark that it is also possible to combine different types of boundary conditions on separate parts of the boundary or to consider more complicated cases. In fact, here, the boundary conditions have been introduced as linear conditions in u; however, it is also possible to have nonlinear boundary conditions (but this makes the analysis of the reaction-diffusion system more complicated).

1.2 Existence theorems

Theorems of local (in time) existence and uniqueness of generalized and smooth solutions for reaction-diffusion systems are well known in literature; moreover, when solutions are a-priori bounded, global existence can be also obtained (see, for instance, [28], [34], [67]). Very different techniques may be used to obtain existence results, for example comparison principles or a more topological-functional approach.

In this section we will refer only to a comparison-existence theorem for smooth solutions. Comparison theorem, based on the maximum principles, is a qualitative technique which, in the case of a single nonlinear equation, gives existence and uniqueness theorems for initial-boundary value problem by supplying a-priori bounds on the solution of the equation. It is capable to extension to certain system of parabolic PDEs, but, in general, gives weaker results (for more details we refer to [1], [67]). Among the various existence and comparison theorems that can be established by both functional and classical methods (see [22]), we recall the approach due to Pao [47], since the monotone argument he adopts is constructive and in the mean time it leads to an existence-comparison theorem for the corresponding steady-state problem.

The basic idea of this method is that, by using an upper solution or a lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iterations is monotone and converges to a solution of the problem. For coupled systems of equations, the definition of upper and lower solutions and the construction of monotone sequences depend on the quasi monotone property of the reaction function in the system. To illustrate the method, let us consider the following coupled system of two parabolic equations in the form

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - L_i u_i = f_i(t, x, u_1, u_2) \quad \text{in} \quad D_T \\ B_i u_i = h_i(x, t) \quad \text{on} \quad S_T \qquad\qquad i = 1, 2 \\ u_i(x, 0) = u_{i,0}(x) \quad \text{in} \quad \Omega \end{cases} \tag{1.5}$$

where $D_T = \Omega \times (0, T]$ and $S_T = \partial\Omega \times (0, T]$, being $T > 0$ an arbitrary fixed time, L_i are the following uniformly elliptic operators with smooth coefficients

$$L_i = \sum_{j,l=1}^{n} a_{j,l}^{(i)}(x, t) \frac{\partial^2}{\partial x_j \partial x_l} + \sum_{j=1}^{n} b_j^{(i)}(x, t) \frac{\partial}{\partial x_j}, \qquad i = 1, 2$$

and

$$B_i = \alpha_i(x,t)\frac{\partial}{\partial \nu} + \beta_i(x,t) \qquad\qquad i = 1,2$$

are the boundary operators, with α_i, β_i $(i = 1,2)$ nonnegative smooth functions such that $\alpha_i + \beta_i > 0$, $i = 1,2$. Moreover, let us assume that $\partial\Omega$ has the outside strong sphere property when $\alpha_i = 0$ and is of class $C^{1+\alpha}$ when $\alpha_i > 0$, that the boundary and initial functions $h_i, u_{i,0}(x)$, $i = 1,2$ are nonnegative smooth functions in their respective domains and that the functions f_i are Hölder continuous in $D_T \times J_1 \times J_2$ for some bounded sets $J_1, J_2 \subset \mathbb{R}$.
Let us recall that a function $f_i = f_i(u_1, ..., u_N)$ is said to be a quasimonotone nondecreasing (respectively nonincreasing) function if, for fixed u_i, f_i is nondecreasing (respectively nonincreasing) in u_j for $j \neq i$. Hence, in the case of a vector function $\mathbf{f} = (f_1, f_2)$ of two components, there are three basic types of quasimonotone functions.

Definition 1 *A function* $\mathbf{f} = (f_1, f_2)$ *is called quasimonotone nondecreasing (respectively nonincreasing) in* $J_1 \times J_2$ *if both* f_1 *and* f_2 *are quasimonotone nondecreasing (respectively nonincreasing) for* $(u_1, u_2) \in J_1 \times J_2$*. When* f_1 *is quasimonotone nonincreasing and* f_2 *is quasimonotone nondecreasing (or vice versa), then* $\mathbf{f}$ *is called mixed quasimonotone.*

The function $\mathbf{f}$ is said to be quasimonotone in $J_1 \times J_2$ if it has anyone of the quasimonotone properties in Definition 1.
As usual, we refer to $\mathbf{f}$ as C^1-function in $J_1 \times J_2$ if both f_1 and f_2 are continuosly differentiable in (u_1, u_2) for all $(u_1, u_2) \in J_1 \times J_2$. The function $\mathbf{f}$ is called a quasi C^1-function in $J_1 \times J_2$ if f_1 is continuously differentiable in u_2 and f_2 is continuously differentiable in u_1 for all $(u_1, u_2) \in J_1 \times J_2$. Hence, if $\mathbf{f}$ is a quasi C^1-function, then the three types of quasimonotone functions

in Definition 1 are reduced to the form

$$\begin{aligned}
&\partial f_1/\partial u_2 \geq 0, \quad \partial f_2/\partial u_1 \geq 0 \\
&\partial f_1/\partial u_2 \leq 0, \quad \partial f_2/\partial u_1 \leq 0 \quad \text{for} \quad (u_1, u_2) \in J_1 \times J_2 \\
&\partial f_1/\partial u_2 \leq 0, \quad \partial f_2/\partial u_1 \geq 0
\end{aligned}$$

respectively. These three types of reaction functions appear very often in many physical problems.

For definiteness, when $\mathbf{f}$ is mixed quasimonotone, we always consider f_1 as quasimonotone nonincreasing and f_2 as quasimonotone nondecreasing.

Definition 2 *A pair of functions* $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ *in* $C(\bar{D}_T) \cap C^{1,2}(D_T)$ *are called ordered upper and lower solutions of (1.5) if* $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$, *if, for* $i = 1, 2$,

$$\begin{cases} B_i \tilde{u}_i \geq h_i(x,t) \geq B_i \hat{u}_i & \textit{on} \quad S_T \\ \tilde{u}_i(x,0) \geq u_{i,0}(x) \geq \hat{u}_i(x,0) & \textit{in} \quad \Omega \end{cases}$$

and if

$$\begin{cases} (\tilde{u}_1)_t - L_1 \tilde{u}_1 - f_1(t,x,\tilde{u}_1,\tilde{u}_2) \geq 0 \geq (\hat{u}_1)_t - L_1 \hat{u}_1 - f_1(t,x,\hat{u}_1,\hat{u}_2) \\ (\tilde{u}_2)_t - L_2 \tilde{u}_2 - f_2(t,x,\tilde{u}_1,\tilde{u}_2) \geq 0 \geq (\hat{u}_2)_t - L_2 \hat{u}_2 - f_2(t,x,\hat{u}_1,\hat{u}_2) \end{cases}$$

when (f_1, f_2) *is quasimonotone nondecreasing,*

$$\begin{cases} (\tilde{u}_1)_t - L_1 \tilde{u}_1 - f_1(t,x,\tilde{u}_1,\hat{u}_2) \geq 0 \geq (\hat{u}_1)_t - L_1 \hat{u}_1 - f_1(t,x,\hat{u}_1,\tilde{u}_2) \\ (\tilde{u}_2)_t - L_2 \tilde{u}_2 - f_2(t,x,\hat{u}_1,\tilde{u}_2) \geq 0 \geq (\hat{u}_2)_t - L_2 \hat{u}_2 - f_2(t,x,\tilde{u}_1,\hat{u}_2) \end{cases}$$

when (f_1, f_2) *is quasimonotone nonincreasing and*

$$\begin{cases} (\tilde{u}_1)_t - L_1 \tilde{u}_1 - f_1(t,x,\tilde{u}_1,\hat{u}_2) \geq 0 \geq (\hat{u}_1)_t - L_1 \hat{u}_1 - f_1(t,x,\hat{u}_1,\tilde{u}_2) \\ (\tilde{u}_2)_t - L_2 \tilde{u}_2 - f_2(t,x,\tilde{u}_1,\tilde{u}_2) \geq 0 \geq (\hat{u}_2)_t - L_2 \hat{u}_2 - f_2(t,x,\hat{u}_1,\hat{u}_2) \end{cases} \tag{1.6}$$

when (f_1, f_2) *is mixed quasimonotone.*

In the above definition, $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are required to be in $C(\bar{D}_T)\cup C^{1,2}(D_T)$ in the sense that their components $\tilde{u}_i, \hat{u}_i$, $(i = 1, 2)$, are in $C(\bar{D}_T) \cup C^{1,2}(D_T)$. The ordering relation $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ is meant in the usual componentwise sense, that is $\tilde{u}_i \geq \hat{u}_i$, $(i = 1, 2)$ in $\bar{D}_T$. Moreover, let us remark that upper and lower solutions for quasimonotone nondecreasing functions are independent one of each other and can be constructed separately. The same is true for quasimonotone nonincreasing functions except that the pair $(\tilde{u}_1, \hat{u}_2)$ and $(\hat{u}_1, \tilde{u}_2)$ are independent. However, for mixed quasimonotone functions, upper and lower solutions are coupled and must be determined simultaneously from (1.6). The pair is sometimes referred to as coupled upper and lower solutions. Let us suppose that, for a given type of quasimonotone reaction function, there exists a pair of ordered upper and lower solutions $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$. Let us define the sector

$$\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle = \left\{ (u_1, u_2) \in C(\bar{D}_T) : (\hat{u}_1, \hat{u}_2) \leq (u_1, u_2) \leq (\tilde{u}_1, \tilde{u}_2) \right\}. \tag{1.7}$$

Let us remark that, if $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ is contained in $J_1 \times J_2$, then in the definition of quasimonotone function it suffices to take $J_1 \times J_2 = \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. In the sequel, we will consider each of the three types of reaction functions in the sector $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. In addition, we will assume that there exist bounded functions $\underline{c}_i = \underline{c}_i(x, t), i = 1, 2$, such that, for every (u_1, u_2), (v_1, v_2) in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, (f_1, f_2) satisfies the one-sided Lipschitz condition

$$\begin{cases} f_1(t, x, u_1, u_2) - f_1(t, x, v_1, v_2) \geq -\underline{c}_1(u_1 - v_1) & \text{when} \quad u_1 \geq v_1 \\ f_2(t, x, u_1, u_2) - f_2(t, x, v_1, v_2) \geq -\underline{c}_2(u_2 - v_2) & \text{when} \quad u_2 \geq v_2. \end{cases} \tag{1.8}$$

To ensure the uniqueness of the solution, we also assume that there exist bounded functions $\overline{c}_i = \overline{c}_i(x, t), i = 1, 2$, such that, for every (u_1, u_2), (v_1, v_2)

in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, one has that

$$\begin{cases} f_1(t,x,u_1,u_2) - f_1(t,x,v_1,u_2) \leq \bar{c}_1(u_1 - v_1) \quad \text{when} \quad u_1 \geq v_1 \\ f_2(t,x,u_1,u_2) - f_2(t,x,u_1,v_2) \leq \bar{c}_2(u_2 - v_2) \quad \text{when} \quad u_2 \geq v_2. \end{cases} \tag{1.9}$$

It is clear that, if there exist bounded functions $K_i = K_i(x,t)$ such that (f_1, f_2) satisfies the Lipschitz condition

$$|f_i(t,x,u_1,u_2) - f_i(t,x,v_1,v_2)| \leq K_i\left(|u_1 - v_1| + |u_2 - v_2|\right)$$

for $(u_1,u_2),(v_1,v_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, $(i = 1,2)$, then both conditions (1.8) and (1.9) are satisfied. Let us remark that in the hypotheses (1.8) and (1.9) the functions $\underline{c}_i$ and $\bar{c}_i$, $(i = 1,2)$ are not required to be positive. This weakened condition plays an important role in the study of the qualitative behaviour of the solution. Without any loss of generality, we may assume that the functions $\underline{c}_i$ $(i = 1,2)$ in (1.8) are Hölder continuous in $\bar{D}_T$. This implies that the functions F_1, F_2 given by

$$F_i(t,x,u_1,u_2) = \underline{c}_i(x,t)u_i + f_i(t,x,u_1,u_2), \quad i = 1,2$$

are Hölder continuous in $D_T \times \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and are monotone nondecreasing in u_i, $(i = 1,2)$.

Defining the operators $\mathbb{L}_i$ by

$$\mathbb{L}_i u_i = (u_i)_t - L_i u_i + \underline{c}_i u_i, \quad i = 1,2,$$

one obtains that the differential equations in (1.5) are equivalent to

$$\mathbb{L}_i u_i = F_i(t,x,u_1,u_2) \quad \text{in} \quad D_T \quad i = 1,2.$$

Starting from a suitable initial iteration $(u_1^{(0)}, u_2^{(0)})$, we construct a sequence $\{\mathbf{u}^{(k)}\} = \left\{u_1^{(k)}, u_2^{(k)}\right\}$ from the iteration process

$$\begin{cases} \mathbb{L}_i u_i^{(k)} = F_i\left(u_1^{(k-1)}, u_2^{(k-1)}\right) & \\ B_i u_i^{(k)} = h_i(x,t) & i = 1,2 \\ u_i^{(k)}(x,0) = u_{i,0}(x). & \end{cases} \tag{1.10}$$

It is clear that, for each $k \in \mathbb{N}$, the above system consists of two linear uncoupled initial boundary value problems and therefore the existence of $\left\{u_1^{(k)}, u_2^{(k)}\right\}$ is guaranteed (see [47] for details). To ensure that this sequence is monotone and converges to a solution of (1.5), it is necessary to choose a suitable initial iteration. The choice of this function depends on the type of quasimonotone property of (f_1, f_2). In the following lemmas, we will establish the monotone property of the sequence for each one of the three types of reaction functions.

(i) *Quasimonotone nondecreasing function*: For this type of quasimonotone function, it suffices to take either $(\tilde{u}_1, \tilde{u}_2)$ or $(\hat{u}_1, \hat{u}_2)$ as the initial iteration $(u_1^{(0)}, u_2^{(0)})$. Let us denote the two corresponding sequences by $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ respectively where $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$. The following lemma gives the monotone property of these two sequences.

Lemma 1 *For quasimonotone nondecreasing (f_1, f_2), the two sequences $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ possess the monotone property*

$$\underline{u}_i^{(k)} \leq \underline{u}_i^{(k+1)} \leq \overline{u}_i^{(k+1)} \leq \overline{u}_i^{(k)} \quad \textit{in } \bar{D}_T \quad (i = 1,2) \tag{1.11}$$

where $k \in \mathbb{N}$.

Proof For the proof, see [47].

Remark 1 *In the absence of an upper solution, the monotone nondecreasing property of the sequence* $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ *remains true provided that condition (1.8) holds for every bounded function* $(\tilde{u}_1, \tilde{u}_2)$*. In this case, the sequence* $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ *either converges to some limit as* $k \to \infty$ *or becomes unbounded at some point in* $\bar{D}_T$*. A similar conclusion holds for the sequence* $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$*.*

(ii) *Quasimonotone nonincreasing function*: When the reaction function (f_1, f_2) is quasimonotone nonincreasing, we choose $(\tilde{u}_1, \hat{u}_2)$ or $(\hat{u}_1, \tilde{u}_2)$ as the initial iteration in (1.10). Let us denote the corresponding sequences by $\left\{\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ respectively where $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ and $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$. The following lemma holds.

Lemma 2 *For quasimonotone nonincreasing* (f_1, f_2)*, the two sequences* $\left\{\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ *and* $\left\{\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ *possess the mixed monotone property in the sense that their components* $\overline{u}_i^{(k)}$ *and* $\underline{u}_i^{(k)}$ *satisfy relation (1.11).*

Proof For the proof, see [47].

(iii) *Mixed quasimonotone function*: The construction of monotone sequences for mixed quasimonotone functions requires the use of both upper and lower solutions simultaneously. When f_1 is quasimonotone nonincreasing and f_2 is quasimonotone nondecreasing, the monotone iteration process is given by

$$\begin{cases} \mathbb{L}_1 \overline{u}_1^{(k)} = F_1\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right), \ \ \mathbb{L}_2 \overline{u}_2^{(k)} = F_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}\right), \\ \mathbb{L}_1 \underline{u}_1^{(k)} = F_1\left(\underline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}\right), \ \ \mathbb{L}_2 \underline{u}_2^{(k)} = F_2\left(\underline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right), \end{cases} \tag{1.12}$$

where $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$. The boundary and initial conditions for $\overline{u}_i^{(k)}$ and $\underline{u}_i^{(k)}$ are the same as in (1.10). It follows from this iteration process that the equations in (1.12) are uncoupled but are interrelated in the sense that the k-th iteration $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ or $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ depends on all the four components in the previous iteration. This kind of iteration is fundamental in its extension to coupled system with any finite number of equations. The idea of this construction is to obtain the quasimonotone property of the sequences as shown in the following lemma.

Lemma 3 *For mixed quasimonotone (f_1, f_2), the sequences $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ given by (1.12) possess the monotone property (1.11).*

Proof For the proof, see [47].

The above construction of monotone sequences yelds a sequence of ordered upper and lower solutions for (1.5), which are given in the following

Lemma 4 *Let $(\tilde{u}_1, \tilde{u}_2)$ ad $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (1.5) and let (f_1, f_2) be quasimonotone and satisfy condition (1.8). Then , for each type of quasimonotone (f_1, f_2), the corresponding iterations $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ given by Lemmas 1-3 are ordered upper and lower solutions.*

Proof For the proof, see [47].

The construction of monotone sequences in Lemmas 1-3 is not limited to the process in (1.10) and (1.12). Consider, for instance, the case where (f_1, f_2) is quasimonotone nondecreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Then a different process of iteration is given by

$$\mathbb{L}_1 u_1^{(k)} = F_1\left(t, x, u_1^{(k-1)}, u_2^{(k-1)}\right), \quad \mathbb{L}_2 u_2^{(k)} = F_2\left(t, x, u_1^{(k)}, u_2^{(k-1)}\right). \quad (1.13)$$

The boundary and initial conditions for $u_1^{(k)}$ and $u_2^{(k)}$ are the same as in (1.10). In the above iterative scheme, the component $u_1^{(k)}$ is used in the second equation as soon as it is computed from the first equation. This kind of iteration is similar to the Gauss-Seidel iterative method for algebraic systems which has the advantage to obtain faster convergent sequences. In the following Lemma we show the monotone property of the sequences when the initial iteration is either an upper or a lower solution.

Lemma 5 *Let (f_1, f_2) be quasimonotone nondecreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Then the sequences $\left\{ \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$ and $\left\{ \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$ obtained from (1.13) with $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ and with the boundary and initial conditions in (1.10) possess the monotone property (1.11).*

Proof For the proof, see [47].

It follows by the same argument as in the proof of Lemma 5 that, if (f_1, f_2) is quasimonotone nonincreasing, then the sequences $\left\{ \overline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$ and $\left\{ \underline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$ obtained from (1.13) with $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ and $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$ possess the monotone property (1.11). In the case of mixed quasimonotone (f_1, f_2), an improved iteration process for $\left\{ \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$ and $\left\{ \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$ is given by

$$\begin{cases} \mathbb{L}_1 \overline{u}_1^{(k)} = F_1 \left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)} \right), & \mathbb{L}_2 \overline{u}_2^{(k)} = F_2 \left(\overline{u}_1^{(k)}, \overline{u}_2^{(k-1)} \right), \\ \mathbb{L}_1 \underline{u}_1^{(k)} = F_1 \left(\underline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)} \right), & \mathbb{L}_2 \underline{u}_2^{(k)} = F_2 \left(\underline{u}_1^{(k)}, \underline{u}_2^{(k-1)} \right). \end{cases} \tag{1.14}$$

Following a similar argument as in the proof of Lemma 5, it can be shown that these two sequences also possess the monotone property (1.11).

Lemmas 1-3 imply that, for each one of the three types of quasimonotone functions, the corresponding sequence obtained from (1.10) and (1.12) converges monotonically to some limit function. The same is true for the sequences given by (1.13) and (1.14). Let us define

$$\lim_{k\to\infty} \overline{u}_i^{(k)}(x,t) = \overline{u}_i(x,t), \quad \lim_{k\to\infty} \underline{u}_i^{(k)}(x,t) = \underline{u}_i(x,t), \; i = 1, 2. \tag{1.15}$$

Our aim is now to show that, if conditions (1.8) and (1.9) hold, then $\overline{u}_i \equiv \underline{u}_i \equiv u_i$, $i = 1, 2$, and $\mathbf{u} = (u_1, u_2)$ is the unique solution of (1.5). The proof of this result is based on the integral representation of the linear scalar parabolic boundary value problems and it is contained in the following theorem.

Theorem 1 *Let $(\tilde{u}_1, \tilde{u}_2)$ and $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (1.5) and let (f_1, f_2) be quasimonotone nondecreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and satisfy the conditions (1.8) and (1.9). Then (1.5) has a unique solution $\mathbf{u} = (u_1, u_2)$ in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Moreover, the sequences $\left\{ \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$ and $\left\{ \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$, obtained from (1.10) with $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ converge monotonically to (u_1, u_2) and satisfy the relation*

$$(\hat{u}_1 . \hat{u}_2) \leq \left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right) \leq (u_1, u_2) \leq \left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right) \leq (\tilde{u}_1, \tilde{u}_2) \quad \text{in} \quad \bar{D}_T \tag{1.16}$$

for every $k \in \mathbb{N}$.

Proof For the proof, see [47].

A similar existence-comparison theorem holds also for quasimonotone nonincreasing functions.

Theorem 2 *Let $(\tilde{u}_1, \tilde{u}_2)$ and $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (1.5) and let (f_1, f_2) be quasimonotone nonincreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and satisfy the conditions (1.8) and (1.9). Then (1.5) has a unique solution $\mathbf{u} = (u_1, u_2)$ in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Moreover, the sequences $\left\{ \overline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$ and $\left\{ \underline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$, obtained from (1.10) with $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ and $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$ converge monotonically to (u_1, u_2). The monotone property of the sequences is in the sense of (1.11).*

Finally, one obtains that an analogous existence-comparison theorem holds for mixed quasimonotone functions.

Theorem 3 *Let* $(\tilde{u}_1, \tilde{u}_2)$ *and* $(\hat{u}_1, \hat{u}_2)$ *be coupled upper and lower solutions of (1.5) and let* (f_1, f_2) *be mixed quasimonotone in* $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ *and satisfy the conditions (1.8) and (1.9). Then (1.5) has a unique solution* $\mathbf{u} = (u_1, u_2)$ *in* $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$*. Moreover, the sequences* $\left\{ \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$ *and* $\left\{ \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$*, obtained from (1.10) with* $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ *and* $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ *converge monotonically to* (u_1, u_2) *and satisfy the relation (1.16).*

When the iteration processes in (1.10) and (1.12) are replaced by (1.13) and (1.14) respectively, the result of Lemma 5 and similar results for quasimonotone nonincreasing and mixed quasimonotone functions imply that the corresponding sequences converge to some limit functions in the same way as in (1.15). It is easily seen from the same argument as in the proof of Theorems 1-3 that these limits are solutions of (1.5) in accordance with the quasimonotone property of (f_1, f_2). This observation leads to the following conclusion:

Theorem 4 *Under the hypotheses of Theorems 1-3, except that the iteration processes (1.10) and (1.12) be replaced, respectively, by (1.13) and (1.14), all the conclusions in the corresponding theorems remain true.*

Remark 2 *The monotone method for coupled systems of two parabolic equations can be easily extended to systems with an arbitrary finite number of equations.*

Chapter 2

The Liapunov Direct method

Since a lot of physical systems and real world phenomena can be described through partial differential equations, the problem of studying the properties of the solutions of such equations has been deeply studied. In particular, the analysis of the stability/instability of suitable solutions, for example of the stationary states, with respect to perturbations on the initial data, allows to predict, in such a way, the longtime behaviour of the solutions. To reach this aim, several methods have been developed and used: most of them require linearization, truncation or other kind of approximations of the original equations; Liapunov Direct Method, instead, deals directly with the original system, without using approximation methods.
In this chapter, the outlines of the Liapunov's Direct Method will be described and some peculiar Liapunov functionals, introduced by Rionero in [54], [55], will be used in order to investigate the stability properties of the solutions of evolution equations, aiming to obtain the optimal result of coincidence between the linear and nonlinear stability thresholds.

2.1 Basic properties of dynamical systems

Let $\mathcal{F}$ be a phenomenon taking place into a domain Ω of the physical three dimensional space $\mathbb{R}^3$ and $\mathbf{u} : (\mathbf{x}, t) \in \Omega \times \mathbb{R} \longrightarrow \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$ or $\mathbf{u} : (\mathbf{x}, t) \in \Omega \times [0, T] \longrightarrow \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$, being $n \in \mathbb{N}$ and $T \in (0, +\infty)$, whose components $u_i(\mathbf{x}, t)$, $(i = 1, ..., n)$ are the relevant quantities describing the state of $\mathcal{F}$. The vector $\mathbf{u}$ is called the *state vector*.

If one finds, by experimental data, physical law and so on that there exists a function

$$\mathbf{F}\left(t, \mathbf{x}, \mathbf{u}, \frac{\partial u_i}{\partial x_r}, \frac{\partial^2 u_j}{\partial x_r \partial x_s}, ...\right), \quad i, j = 1, ..., n; \; r, s = 1, 2, 3$$

which governs the behaviour of the time derivative of $\mathbf{u}$, such that, for any positive T

$$\mathbf{u}_t = \mathbf{F} \quad \text{in } \Omega \times (0, T) \tag{2.1}$$

holds, then the phenomenon $\mathcal{F}$ is modeled by the PDE (2.1) to which we associate prescribed initial data

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{2.2}$$

and suitable boundary conditions

$$\mathbf{B}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{u}^* \quad \text{on } \partial\Omega \times [0, T], \tag{2.3}$$

where $\mathbf{B}$ is a given operator and $\mathbf{u}^*(\mathbf{x}, t)$ is an assigned vector.

The initial boundary value problem (I.B.V.P.) (2.1)-(2.3) is the mathematical model describing the evolution of $\mathcal{F}$, also called *evolution equation* of $\mathcal{F}$.

According to the definition due to Hadamard, a given I.B.V.P. is said to be well posed in the state space X, endowed with a suitable topology, if there exists a solution, if this solution is unique and it depends continuously on the initial data. A problem which is not well posed is said to be ill posed.

Remark 3 *Existence, uniqueness and continuous dependence of solutions depend strongly on the choice of the underlying function spaces in which the data are given and in which we are looking for solutions. Depending on the problem, one might use spaces of continuosly differentiable functions* $C^k((0,T)\times\Omega)$ *or spaces of integrable functions* $L^p((0,T)\times\Omega)$*. Thus, the choice of functional topology in the state space is very important and it has to be linked to the physics of the phenomenon.*

In the sequel, we recall some basic concepts of the theory of dynamical systems referring to [25], [70], [72] and assuming that the I.B.V.P. (2.1)-(2.3) is well posed.

Definition 3 *A* ***dynamical system*** *on a metric space* X *is a mapping*

$$v : (v_0, t) \in X \times \mathbb{R} \to v(v_0, t) \in X \tag{2.4}$$

such that

$$v(v_0, 0) = v_0. \tag{2.5}$$

Usually the following additional property is required for a dynamical system (semigroup property):

$$v(v_0, t + \tau) = v(v(v_0, \tau), t), \quad v_0 \in X; t, \tau \in \mathbb{R}^+. \tag{2.6}$$

For example, let $u(u_0, t)$, with $u(u_0, 0) = u_0$, be a global solution of the I.B.V.P. (2.1)-(2.3). Then u is a dynamical system.

The properties (2.4) and (2.6) give to the one parameter family of operators $v(v_0, \cdot)$ the semigroup structure, according to the following definition:

Definition 4 *A* ***semigroup of operators*** *on a metric space* X *is a one parameter family* $\{S(t)\}_{t\geq 0}$ *of operators* $S(t) : X \to X$ *such that*

$$\begin{cases} S(t+s) = S(t)S(s) \\ S(0) = I, \quad (I \textit{ is the identity in } X). \end{cases}$$

The equivalence between the definition of the semigroup operators $\{S(t)\}_{t\geq 0}$ and the dynamical system is immediately obtained by setting

$$v(v_0, t) = S(t)v_0 \quad v_0 \in X; t \in \mathbb{R}^+.$$

Definition 5 *Let v be a dynamical system, then the function*

$$v(v_0, \cdot) : t \in \mathbb{R} \to v(v_0, t) \in X$$

*for a prescribed $v_0 \in X$, is called **motion** associated to the initial conditions v_0 and is denoted by $v(v_0, t)$ or $v(t)$.*

If $v(v_0, t) = v_0$, $\forall t \in \mathbb{R}$, then the motion is **stationary** (or **steady**) and v_0 is an **equilibrium point**.
Let v and w be two motions. If

$$v(0) = w(0) \ \Rightarrow \ v(t) = w(t) \quad \forall t > 0 \text{ (resp.} t < 0)$$

then the motion is **unique forward** (respectively **backward**) in time with respect to the initial data.
The forward uniqueness ensures the semigroup property.
The set $\{t, v(t)\}$ with $t \in \mathbb{R}^+$, is the **positive graph** of the motion v and its projection into X, that is the subset $\gamma^+ = \{v(t) : t \in \mathbb{R}^+\}$ is the **positive orbit** or **trajectory** starting at v_0.

Definition 6 *A dynamical system on a metric space X is a* $\mathbf{C^0}$*-**semigroup** if (2.4)-(2.6) and the following properties hold*

$$v(t, \cdot) : X \ \to \ X \textit{ is continuous } \forall t \geq 0;$$

$$v(\cdot, v_0) : \mathbb{R}^+ \ \to \ X \textit{ is continuous } \forall v_0 \in X.$$

Remark 4 *As we have just seen, a dynamical system may be generated by an evolution equation. In the study of dynamical systems generated by PDEs, the existence of the operators $S(t)$ and their properties is linked to the problem of the existence of solutions for PDEs and so, like for the uniqueness, it must be proved case by case.*

Let us end this section with the important notion of continuous dependence on the initial data, that means wondering if, given a particular (basic) motion $v(v_0, \cdot)$, any other motion $v(v_1, \cdot)$, starting at the same initial time from a position v_1 sufficiently closed to v_0, will remain as closed as desired to $v(v_0, \cdot)$ for every finite time $T > 0$.
This is an important requirement if the mathematical model at stake describes observable natural phenomena. In fact, data in nature cannot be conceived as rigidly fixed and the process of measuring them always involves small errors. Therefore, a mathematical problem cannot be considered as realistically corresponding to physical phenomena unless a variation of the given data in a sufficiently small range leads to an arbitrary small change in the solution.
Let v be a dynamical system on a metric space (X, d) and let $B(x, r)$, with $x \in X$ and $r > 0$, be the open ball centered at x and having radius r.

Definition 7 *A motion $v(v_0, \cdot)$ of a dynamical system **depends continuosly on the initial data** if and only if*

$$\forall T, \epsilon > 0,\ \exists \delta(\epsilon, T) > 0 : v_1 \in B(v_0, \delta) \Rightarrow v(v_1, t) \in B(v(v_0, t), \epsilon),\ \forall t \in [0, T].$$

The following theorems hold.

Theorem 5 *Let v be a dynamical system on a metric space X having the C^0-semigroup properties. Then any motion depends continuosly on the initial data.*

Proof. See [25].

Theorem 6 *A motion which is not unique cannot depend continuosly on the initial data.*

Proof. See [25].

2.2 Liapunov stability

The concept of stability can be interpreted in many different ways. In the sequel, we will refer to the Liapunov stability with respect to perturbations on the initial data, that means, roughly speaking, that for a sufficiently small perturbation on the initial data, the system will remain close to the original solution for all future times.

Definition 8 *A motion $v(v_0, t)$ is Liapunov stable (with respect to perturbations on the initial data) if and only if*

$$\forall \epsilon > 0, \, \exists \, \delta(\epsilon) > 0 : v_1 \in B(v_0, \delta) \, \to \, v(v_1, t) \in B(v(v_0, t), \epsilon), \quad \forall t \in [0, \infty).$$

A motion is unstable if it is not stable.
It results that the Liapunov stability extends the requirements of continuous dependence to the infinite interval of time $(0, \infty)$.

Definition 9 *A motion of a dynamical system $v(v_0, \cdot)$ is said to be an* ***attractor*** *or* ***attractive*** *on a set $Y \subset X$ if*

$$v_1 \in Y \, \Rightarrow \, \lim_{t \to \infty} d \left[v(v_0, t), v(v_1, t) \right] = 0. \tag{2.7}$$

The largest set Y satisfying (2.7) is called the ***basin*** *(or* ***domain****)* ***of attraction*** *of $v(v_0, \cdot)$.*

Definition 10 *A motion $v(v_0, \cdot)$ of a dynamical system is **asymptotically stable** if it is stable and if there exists $\delta_1 > 0$ such that $v(v_0, \cdot)$ is attractive on $B(v_0, \delta_1)$.*
*In particular, $v(v_0, \cdot)$ is **exponentially stable** if there exists $\delta_1 > 0, \lambda(\delta_1) > 0, M(\delta_1) > 0$ such that*

$$v_1 \in B(v_0, \delta_1) \Rightarrow d[v(v_0, t), v(v_1, t)] \leq Me^{-\lambda t} d(v_1, v_0), \; \forall t \geq 0.$$

*If $\delta_1 = \infty$, then $v(v_0, \cdot)$ is **asymptotically** (exponentially) **globally stable**.*

Remark 5 *Let X be a metric linear space. It is always possible to express the stability of a given basic motion $v(v_0, t)$ through the stability of the zero solution of the perturbed dynamical system*

$$u : (u_0, t) \in X \times \mathbb{R}^+ \to v(v_0 + u_0, t) - v(v_0, t)$$

where

$$u(u_0, t) = v(v_1, t) - v(v_0, t) \quad (v_1 = v_0 + u_0)$$

is the perturbation at time t to the basic motion $v(v_0, t)$. Indeed the definition of stability of $v(v_0, \cdot)$ is equivalent to

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 : u_0 \in B(O, \delta) \implies u(u_0, t) \in B(O, \epsilon), \; \forall t \geq 0,$$

where O is the origin of X.

2.3 Liapunov direct method

In 1893 A.M. Liapunov, in order to establish conditions ensuring the stability of solutions of O.D.Es, introduced a method based on studying the sign of the time derivative of an auxiliary function along the solutions of O.D.Es without determining explicitely solutions. Liapunov Direct method or Second Method

[37] has been recognized to be very general and powerful and has been used for over 65 years in the qualitative theory of O.D.Es (cfr. [21], [27], [35], [74]). A first step toward the application of the Liapunov's direct method to P.D.Es was made by Massera [38], who extended this method to denumerably infinite system of O.D.Es. A general stability theory based on the existence of a Liapunov functional for the invariant sets of dynamical systems in general metric spaces was established by Zubov [76]; other pioneering results were due to Movchan [41].
Our aim is to introduce the fundamental ideas and problems of the Liapunov direct method by considering its applications to some phenomena modeled by P.D.Es.

Definition 11 *Let v be a dynamical system on a metric space X. A functional $V : X \to \mathbb{R}$ is a* ***Liapunov function*** *on a subset $I \subset X$ if V is continuous on I and it is a nonincreasing function of time along the solutions having the initial data in I.*

In order to ensure that $V[v(x, \cdot)]$ is a nonincreasing function of time, in the sequel we will assume that V is differentiable with respect to time and that the derivative is nonpositive. However, it is standard in literature to ensure that V is nonincreasing by requiring that the generalized time derivative

$$\dot{V} := \lim_{t \to 0} \inf \frac{1}{t} \left\{ V[v(x,t)] - V(x) \right\}, \quad x \in I,$$

is nonpositive.

Assume that X is a normed linear space. By virtue of Remark 5, the stability of a given motion can be expressed through the stability of the zero solution of the perturbed dynamical system. Therefore, one can introduce the direct method for investigating the stability of an equilibrium position only.

Denoting by $\mathcal{F}_r\,(r > 0)$ the set of functions $\phi : [0, r) \to [0, \infty)$ which are continuous, strictly increasing and such that $\phi(0) = 0$, then the Liapunov method can be summarized by the following theorems.

Theorem 7 *Let u be a dynamical system on a normed space X and let O be an equilibrium point. If V is a Liapunov function on the open ball $B(O, r)$, for some $r > 0$, such that*

i) $V(O) = 0$,

ii) $\exists f \in \mathcal{F}_r : V(u) \geq f(\|u\|)\,,\ u \in B(O, r)$,

then O is stable.

If, in addition,

iii) $\exists g \in \mathcal{F}_r : \dot{V}(u) \leq -g(\|u\|)\,,\ \forall u \in B(O, r)$,

then O is asymptotically stable.

Proof. See [25].

Remark 6 *Let u be a dynamical system on X and let O be an equilibrium point. If V is a Liapunov function on the open ball $B(O, r)$ and is positive definite, i.e.*

$$V(O) = 0, \quad V(u) > 0, \quad u \neq 0,$$

then the stability with respect to the measure V of the perturbation immediately follows. If, moreover, there exists a positive constant c such that, along the solutions, inequality

$$\dot{V} \leq -cV$$

holds, then one has

$$V \leq V(u_0)e^{-ct},$$

i.e. the asymptotic exponential stability with respect to the measure V.

Setting $\Sigma(X,\alpha) = \{x \in X : V(x) < \alpha\}$, the following theorem holds.

Theorem 8 *Let u be a dynamical system on $X \times \mathbb{R}^+$ and let O be an equilibrium point. If V is a Liapunov function on the open set $A_r = B(O,r) \cap \Sigma(X,O)$, for some $r > 0$, and*

i) $V(O) = 0$,

ii) $\exists g \in \mathcal{F}_r : V(u) \leq -g[-V(u)],\ u \in A_r$,

iii) $A_\epsilon \neq \varnothing,\ \forall \epsilon > 0$,

then O is unstable.

Proof. See [25].

All the above theorems are set in a normed linear space X where different kinds of norms can be introduced.
Let us recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if there exist two constants $c_1 \geq c_2 > 0$ such that $c_2 \|x\|_2 \geq \|x\|_1 \geq c_1 \|x\|_2,\ \forall x \in X$.
Therefore, stability/instability properties are invariant under equivalent norms.
If $X = \mathbb{R}^n$, i.e. X is a finite dimensional space, all possible norms are equivalent and so the stability does not depend on the chosen norm. This is the case of phenomena modeled by O.D.Es.
If we consider phenomena with infinite degrees of freedom, modeled by P.D.Es, then it can turn out that a solution is stable with respect to one norm and unstable with respect to another one. In this case, stability depends on the topology of the state space X; this is a relevant difference between O.D.Es and P.D.Es. For a discussion about the importance of the choice of functional topology, see [24], while for examples of topology dependent stability, see [25].

An important application of Liapunov functions is that they can be used in order to determine some invariant sets.

Definition 12 *A set $A \subset X$ is **positively** (respectively **negatively**) **invariant** for the dynamical system v if $v(v_0, t) \in A$ for any $v_0 \in A$ and $t \geq 0$ (respectively $t \leq 0$).*

Positively invariant sets play a fundamental role in studying the longtime behaviour of solutions of a dynamical system; for example, if a bounded set $A \subset X$ can be shown to be positively invariant, then $x \in A \Rightarrow \gamma(x) \in A$ and hence the positive orbit $\gamma(x)$ is bounded.

Definition 13 *A set A is **attractive** on an open set $B \supset A$ if it is positively invariant and*

$$v_0 \in B \Rightarrow \lim_{t \to +\infty} d[v(v_0, t), A] = 0.$$

An important role in the study of the asymptotic behaviuor of solutions is also played by the positive limit sets.

Definition 14 *Let v be a dynamical system on a metric space X and let $x \in X$. A set $\Omega(x) \subset X$ is the **positive limit set** of the motion $v(x, t)$ if $\forall y \in \Omega(x)$ there exists a sequence $(t_n(y))_n$, $t_n \in \mathbb{R}^+$, such that*

$$\begin{cases} \lim_{n \to \infty} t_n = \infty \\ \lim_{n \to \infty} d[v(x, t_n), y] = 0. \end{cases}$$

In particular, $\Omega(x) = x$ if x is an equilibrium point; if $v(x, t)$ is periodic in time, i.e. $\exists \tau : v(v_0, t + \tau) = v(v_0, t)$, then $\Omega(x) = \gamma(x)$, where $\gamma(x)$ is the orbit of $v(x, t)$. In general, $\Omega(x)$ belongs to the closure of $\gamma(x)$.

Information about the asymptotic behaviour of motions by means of Liapunov functions are provided by the following La Salle Invariance Principle.

Theorem 9 *Let v be a dynamical system on a metric space X, with the C^0-semigroup properties and let V be a Liapunov function on a set $A \subset X$. If*

i) $V(x) > -\infty \quad \forall x \in \bar{A}$,

ii) $\gamma(x) \subset A$,

being $\bar{A}$ the closure of A, then $\Omega(x)$ belongs to the largest postive invariant subset $\mathcal{M}^+$ of $\Omega^* = \left\{x \in \bar{A} : \dot{V}(x) = 0\right\}$. Further, if X is complete and $\gamma(x)$ is precompact, then

$$\lim_{t \to \infty} d[v(x,t), \mathcal{M}^+] = 0.$$

Proof. See [35].

As remarked in [25], the La Salle Invariance Principle works very well when $X = \mathbb{R}^n$, but when X is an infinite dimensional space, then, in order to use Theorem 9, one also needs conditions ensuring precompactness of positive orbits; this is another fundamental difference between O.D.Es and P.D.Es.

Chapter 3

On the nonlinear stability of some reaction-diffusion models for infections

This chapter deals with some reaction-diffusion systems modeling the spreading of epidemics within a population or within interacting populations. In order to study infectious disease transmission, mathematical models play a fundamental role, since they allow to predict the asymptotic behaviour of infection and, consequently, they can suggest some actions in order to prevent or control the spreading of epidemic.
Numerous contributions to mathematical theory of epidemics have been given in the last years (cfr.[5]-[7], [26], [31]-[33], [53]).
When a population is not infected by a disease yet, all the individuals are regarded as susceptibles. Introducing a few number of infected in the community, in order to know if the epidemic will die out or if it will blow up, it would be useful to study the stability of the so called disease-free equilibrium. If the disease-free equilibrium is stable, then epidemic will decay. Hence, the

problem to determine if endemic equilibria (equilibria with all positive components) exist arises. When endemic equilibria exist, their stability analysis allows to state if epidemic will persist.
In the first section of this chapter, a SEIR model for infections under mixed boundary conditions is studied; in the second section the same model is analyzed under homogeneous Neumann boundary conditions and, finally, the third section deals with a reaction-diffusion system modeling the spreading of a cholera epidemic in a nonhomogeneously mixed population. For all the previous models, the goal is to study the longtime behaviour of solutions (boundedness, existence of absorbing sets in the phase space,...) and to investigate the linear and nonlinear stability of biologically meaningful equilibria, aiming to obtain the optimal result of coincidence between the linear and nonlinear stability thresholds.

3.1 On the nonlinear stability of an epidemic SEIR reaction-diffusion model under mixed boundary conditions

3.1.1 Introduction

In classical epidemic models (cfr. [5]-[7], [26], [32], [53]), the host population is supposed to be divided into three disjoint classes: $S(t)$, the individuals susceptible to infection, $I(t)$, the infectious individuals and $R(t)$, the removed ones. When the infection requires some time for individuals to pass from the infected state to the infective one, a further class has to be considered: the exposed to the infection, i.e. individuals in the latent state $E(t)$ (cfr. [2], [3], [52]). A key role in epidemic models is played by the so called *force* *of infection*, or *incidence rate*, i.e. a function describing the mechanism of transmission of the disease. In classical models, the incidence rate is proportional to the number of infective individuals. In order to generalize the dynamics of disease transmission, since 1970s, Capasso and his coworkers stressed the importance to consider nonlinear incidence rates. Since then, many authors proposed peculiar nonlinear forms for the force of infection ([5], [7], [26], [32], [53]). In the sequel, the following well known and meaningful force of infection is considered

$$g(S, I) = KIS(1 + \alpha I),$$

K and α being positive constants (cfr. [2], [26], [33] for details), so that an increased rate of infection due to double exposures over a short time period such that the single contacts lead to infection at a rate KIS whereas new infective individuals arise from double exposure at a rate $K\alpha I^2 S$, is taken into account.

Most of epidemic models known in literature are ODE systems, but in order to consder the more general case in which the population is not homogeneously mixed in the domain at hand, it is necessary to allow the variables of the model to depend on space as well as on time and hence a reaction-diffusion model has to be considered (cfr. [10], [42], [51]). In this case, the diffusion of individuals may be connected with searching for food, escaping high infection risks and so on.

For these reasons in the sequel a reaction-diffusion SEIR model is introduced and analyzed: the longtime behaviour of the solutions is studied and, in particular, absorbing sets in the phase space are determined; by using a peculiar Liapunov function, the nonlinear asymptotic stability of endemic equilibrium is investigated. The results are contained in paper [14].

3.1.2 Mathematical model

Let Ω be a bounded domain in which the epidemic is diffusing. We assume that $\Omega \subset \mathbb{R}^3$ is sufficiently smooth and that the reaction-diffusion equations governing the evolution of the infection diffusion model of SEIR type are

$$\begin{cases} \dfrac{\partial S}{\partial t} = \mu(N_0 - S) + \gamma_1 \Delta S - KIS(1+\alpha I), \\ \dfrac{\partial E}{\partial t} = -(\theta+\mu)E + \gamma_2 \Delta E + KIS(1+\alpha I), \\ \dfrac{\partial I}{\partial t} = -(\sigma+\mu)I + \theta E + \gamma_3 \Delta I, \\ \dfrac{\partial R}{\partial t} = \sigma I - \mu R + \gamma_4 \Delta R, \end{cases} \tag{3.1}$$

with

$\gamma_i (> 0)\ (i = 1,2,3,4)$ the diffusion coefficients,

$\mu(> 0)$ the birth/death rate,

$\sigma(> 0)$ the recovery rate,

$\theta(> 0)$ the rate at which exposed individuals become infectious,

$N_0 = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega [S(\mathbf{x},0) + E(\mathbf{x},0) + I(\mathbf{x},0) + R(\mathbf{x},0)]\, d\Omega,$

$|\Omega| =$ the measure of Ω

and (for biological reason)

$$\varphi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \varphi(\mathbf{x}, t) \in \mathbb{R}^+, \quad \forall \varphi \in \{S, E, I, R\}.$$

To (3.1) we associate the following mixed boundary conditions

$$\begin{cases} S = S^*, \quad E = E^*, \quad I = I^*, \quad R = R^* \qquad \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \nabla S \cdot \mathbf{n} = 0,\ \nabla E \cdot \mathbf{n} = 0,\ \nabla I \cdot \mathbf{n} = 0,\ \nabla R \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^* \times \mathbb{R}^+, \end{cases} \tag{3.2}$$

with $\partial\Omega = \Sigma \cup \Sigma^*$, $\Sigma \cap \Sigma^* = \varnothing$, $\Sigma \neq \varnothing$, $\mathbf{n}$ being the unit outward normal on Σ^* and S^*, E^*, I^*, R^* being non negative constants. Further, we suppose that

i) Ω is of class C^p $(p \geq 2)$, with the interior cone property;

ii) $\varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega), \quad \forall\varphi \in \{S, E, I, R\}$;

iii) $\gamma_1 = \gamma_2 = \gamma$.

3.1.3 Absorbing sets

Let us denote by $T > 0$ an arbitrary fixed time and by $\Omega_T = \Omega \times (0, T]$ the parabolic cylinder, Ω_T being the parabolic interior of $\bar{\Omega} \times [0, T]$ (i.e. Ω_T includes the top $\Omega \times \{t = T\}$).
We refer here to the positive smooth solutions of (3.1) under the boundary conditions (3.2) and the smooth positive initial data

$$\begin{cases} S(\mathbf{x}, 0) = S_0(\mathbf{x}), \quad E(\mathbf{x}, 0) = E_0(\mathbf{x}), \\ \\ I(\mathbf{x}, 0) = I_0(\mathbf{x}), \quad R(\mathbf{x}, 0) = R_0(\mathbf{x}), \end{cases} \qquad \mathbf{x} \in \Omega. \tag{3.3}$$

The existence of solutions of (3.1)-(3.3) in $C_1^2(\Omega_T)$ can be proved as done in [40].
The following theorem holds.

Theorem 10 *Let (S, E, I, R) be a positive solution of (3.1)-(3.3) with $\varphi \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$, $\forall\varphi \in \{S, E, I, R\}$. Then (S, E, I, R) is bounded according* *to*

$$
\begin{cases}
S(\mathbf{x},t) < M_1 = \max\left\{N_0,\ \max\limits_{\bar\Omega} S_0(\mathbf{x}), S^*\right\}, \\[2ex]
E(\mathbf{x},t) < M_2 = \max\left\{\dfrac{\mu N_0 + \theta M_1}{\theta+\mu},\ \max\limits_{\bar\Omega}(S_0+E_0)(\mathbf{x}), S^*+E^*\right\}, \\[2ex]
I(\mathbf{x},t) < M_3 = \max\left\{\dfrac{\theta M_2}{(\sigma+\mu)},\ \max\limits_{\bar\Omega} I_0(\mathbf{x}), I^*\right\}, \\[2ex]
R(\mathbf{x},t) < M_4 = \max\left\{\dfrac{\sigma M_3}{\mu},\ \max\limits_{\bar\Omega} R_0(\mathbf{x}), R^*\right\}.
\end{cases}
\tag{3.4}
$$

Proof. By following the procedure used in [11], let us set $\max\limits_{\bar\Omega_T} S = S(\mathbf{x}_1, t_1)$. We have to distinguish the following cases.

1) If $(\mathbf{x}_1, t_1)$ belongs to the interior of Ω_T, then $(3.1)_1$ implies that

$$\left[\frac{\partial S}{\partial t} - \mu(N_0 - S) - \gamma\Delta S\right]_{(\mathbf{x}_1,t_1)} < 0. \tag{3.5}$$

   Since

$$\left[\frac{\partial S}{\partial t}\right]_{(\mathbf{x}_1,t_1)} = 0, \qquad [\Delta S]_{(\mathbf{x}_1,t_1)} < 0,$$

   then (3.5) can hold if and only if

$$\mu(N_0 - S) > 0$$

   and hence if and only if $S(\mathbf{x}_1, t_1) < N_0$.

2) If $(\mathbf{x}_1, t_1) \in \partial\Omega \times [0, T)$, in view of the regularity of the domain Ω, since Ω verifies in any point $\mathbf{x}_0 \in \partial\Omega$ the interior ball condition, there exists an open ball $B^* \subset \Omega$ with $\mathbf{x}_0 \in \partial B^*$. If $S(\mathbf{x}_1, t_1) > N_0$, choosing the radius of B^* sufficiently small, it follows that

$$\gamma\Delta S - \frac{\partial S}{\partial t} > 0 \quad \text{in } B^*$$

and, by virtue of Hopf's Lemma [49], one obtains that

$$\left(\frac{dS}{d\mathbf{n}}\right)_{(\mathbf{x}_1,t_1)} > 0.$$

Since $\dfrac{dS}{d\mathbf{n}} = 0$ on $\Sigma^* \times \mathbb{R}^+$, it follows that $(\mathbf{x}_1, t_1) \in \Sigma \times \mathbb{R}^+$ and hence $S(\mathbf{x}_1, t_1) = S^*$.

3) Finally, if $(\mathbf{x}_1, t_1) \in \Omega \times \{0\}$, then $S(\mathbf{x}_1, t_1) < \max\limits_{\bar{\Omega}} S_0(\mathbf{x})$.

In order to prove $(3.4)_2$, by adding $(3.1)_1$ and $(3.1)_2$, one obtains

$$\frac{\partial(S+E)}{\partial t} = \mu N_0 - \mu(S+E) - \theta E + \gamma\Delta(S+E). \tag{3.6}$$

Let be $\max\limits_{\bar{\Omega}_T}(S+E) = (S+E)(\mathbf{x}_2, t_2)$.

4) If $(\mathbf{x}_2, t_2)$ belongs to the interior of Ω_T, then, by virtue of (3.6) and $(3.4)_1$, it follows

$$\left[\frac{\partial(S+E)}{\partial t} - \mu N_0 - \theta M_1 + (\theta+\mu)(S+E) - \gamma\Delta(S+E)\right]_{(\mathbf{x}_2,t_2)} < 0. \tag{3.7}$$

Since

$$\left[\frac{\partial(S+E)}{\partial t}\right]_{(\mathbf{x}_2,t_2)} = 0, \qquad [\Delta(S+E)]_{(\mathbf{x}_2,t_2)} < 0,$$

then (3.7) can hold if and only if

$$(S+E)(\mathbf{x}_2, t_2) < \frac{\mu N_0 + \theta M_1}{\theta + \mu}$$

and hence $E(\mathbf{x}, t) + < \dfrac{\mu N_0 + \theta M_1}{\theta + \mu}, \quad \forall(\mathbf{x}, t)$.

5) If $(\mathbf{x}_2, t_2) \in \partial\Omega \times [0, T)$ and $(S+E)(\mathbf{x}_2, t_2) > \dfrac{\mu N_0 + \theta M_1}{\theta + \mu}$, then, by following the same procedure of 2) one recovers that

$$\left(\frac{d(S+E)}{d\mathbf{n}}\right)_{(\mathbf{x}_2,t_2)} > 0.$$

Hence, in view of (3.2), since $\dfrac{d(S+E)}{d\mathbf{n}} = 0$ on $\Sigma^* \times \mathbb{R}^+$, it follows that $(\mathbf{x}_2, t_2) \in \Sigma \times \mathbb{R}^+$ and $(S+E)(\mathbf{x}_2, t_2) = S^* + E^*$ and hence $E(\mathbf{x}, t) < S(\mathbf{x}, t) + E(\mathbf{x}, t) < S^* + E^*, \ \forall(\mathbf{x}, t)$.

6) Finally, if $(\mathbf{x}_2, t_2) \in \Omega \times \{0\}$, then $(E+S)(\mathbf{x}_2, t_2) < \max_{\bar{\Omega}}(S_0 + E_0)(\mathbf{x})$ and hence $E(\mathbf{x}, t) < S(\mathbf{x}, t) + E(\mathbf{x}, t) < \max_{\bar{\Omega}}(S_0 + E_0)(\mathbf{x}), \quad \forall(\mathbf{x}, t)$.

7) $(3.4)_3$-$(3.4)_4$ can be obtained by following, step by step, the same procedure of the previous cases accounting for $(3.4)_1$-$(3.4)_2$.

Denoting by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$ respectively, the following uniqueness theorem holds.

Theorem 11 *Model (3.1), under the boundary conditions (3.2) and the (positive) initial data (3.3), admits a unique positive solution.*

Proof. Let (S_1, E_1, I_1, R_1) and (S_2, E_2, I_2, R_2) be two positive solutions of (3.1)-(3.3). Setting

$$\tilde{S} = S_1 - S_2, \quad \tilde{E} = E_1 - E_2, \quad \tilde{I} = I_1 - I_2, \quad \tilde{R} = R_1 - R_2,$$

it follows that

$$\begin{cases} \dfrac{\partial \tilde{S}}{\partial t} = -\mu\tilde{S} + \gamma\Delta\tilde{S} - K(\tilde{I} + I_2)(\tilde{S} + S_2)[1 + \alpha(\tilde{I} + I_2)] + K I_2 S_2(1 + \alpha I_2), \\ \dfrac{\partial \tilde{E}}{\partial t} = -(\theta + \mu)\tilde{E} + \gamma\Delta\tilde{E} + K(\tilde{I} + I_2)(\tilde{S} + S_2)[1 + \alpha(\tilde{I} + I_2)] - K I_2 S_2(1 + \alpha I_2), \\ \dfrac{\partial \tilde{I}}{\partial t} = -(\sigma + \mu)\tilde{I} + \theta\tilde{E} + \gamma_3\Delta\tilde{I}, \\ \dfrac{\partial \tilde{R}}{\partial t} = \sigma\tilde{I} - \mu\tilde{R} + \gamma_4\Delta\tilde{R}, \end{cases} \tag{3.8}$$

under the initial-boundary conditions

$$\begin{cases} \tilde{S}(\mathbf{x}, 0) = 0, \ \tilde{E}(\mathbf{x}, 0) = 0, \ \tilde{I}(\mathbf{x}, 0) = 0, \ \tilde{R}(\mathbf{x}, 0) = 0 \qquad \mathbf{x} \in \Omega \\ \tilde{S} = \tilde{E} = \tilde{I} = \tilde{R} = 0 \qquad \text{on } \Sigma \times \mathbb{R}^+ \\ \nabla\tilde{S} \cdot \mathbf{n} = \nabla\tilde{E} \cdot \mathbf{n} = \nabla\tilde{I} \cdot \mathbf{n} = \nabla\tilde{R} \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^* \times \mathbb{R}^+. \end{cases} \tag{3.9}$$

Setting

$$\tilde{W} = \frac{1}{2}\left(\|\tilde{S}\|^2 + \|\tilde{E}\|^2 + \|\tilde{I}\|^2 + \|\tilde{R}\|^2\right), \tag{3.10}$$

by virtue of the boundedness of solutions and the boundary conditions, it follows that the time derivative of $\tilde{W}$ along the solutions of (3.8)-(3.9) is such that

$$\frac{d\tilde{W}}{dt} \leq c_1\|\tilde{S}\|^2 + c_2\langle|\tilde{I}|, |\tilde{S}| + |\tilde{E}| + |\tilde{R}|\rangle + c_3\langle|\tilde{S}|, |\tilde{E}|\rangle, \tag{3.11}$$

with c_1, c_2 and c_3 positive constants.
By using Hölder and Cauchy inequalities, one has

$$\frac{d\tilde{W}}{dt} \leq c\tilde{W},$$

being c a positive constant. Hence, it turns out that

$$\tilde{W}(t) \leq \tilde{W}(0)e^{ct} \quad \forall t \geq 0.$$

Since $\tilde{W}(0) = 0$ and $\tilde{W}$ is positive definite, one obtains

$$\tilde{W}(t) \equiv 0$$

and hence the thesis follows.

Let us denote by $\bar{\Sigma}$ the subset of the phase space in which solutions of (3.1)-(3.3) are contained, i.e.

$$\begin{aligned} &\bar{\Sigma} = \{(S, E, I, R) \in [\mathbb{R}^+]^4 : \\ &\|S\|^2 + \|E\|^2 + \|I\|^2 + \|R\|^2 \leq (M_1^2 + M_2^2 + M_3^2 + M_4^2)\,|\Omega|\}. \end{aligned} \tag{3.12}$$

Setting

$$E = \|S - S^*\|^2 + \|E - E^*\|^2 + \|I - I^*\|^2 + \|R - R^*\|^2$$

and

$$
\begin{cases}
a_1 = 2\mu\left[1-(N_0+S^*)\varepsilon_1\right], \;\; a_2 = 2(\theta+\mu)(1-E^*\varepsilon_2), \\
a_3 = 2(\sigma+\mu)(1-I^*\varepsilon_3), \;\; a_4 = 2\mu(1-R^*\varepsilon_4), \\
b_1 = 2\left|\Omega\right| \left[\dfrac{\mu(N_0+S^*)}{\varepsilon_1} + KM_1M_3(M_1+S^*)(1+\alpha M_3)\right], \\
b_2 = 2\left|\Omega\right| \left[\dfrac{(\theta+\mu)E^*}{\varepsilon_2} + KM_1M_3(M_2+E^*)(1+\alpha M_3)\right], \\
b_3 = 2\left|\Omega\right| \left[\dfrac{(\sigma+\mu)I^*}{\varepsilon_3} + \theta M_2(M_3+I^*)\right], \\
b_4 = 2\left|\Omega\right| \left[\dfrac{\mu R^*}{\varepsilon_4} + \sigma M_3(M_4+R^*)\right],
\end{cases}
\tag{3.13}
$$

where ε_i are positive constants ($i = 1, 2, 3, 4$), the following theorem holds.

Theorem 12 *Choosing*

$$
\begin{cases}
0 < \varepsilon_1 < \dfrac{1}{N_0+S^*}, \quad 0 < \varepsilon_2 < \dfrac{1}{E^*}, \\
\\
0 < \varepsilon_3 < \dfrac{1}{I^*}, \quad 0 < \varepsilon_4 < \dfrac{1}{R^*},
\end{cases}
\tag{3.14}
$$

then $\forall\varepsilon > 0$ *the manifold*

$$
\begin{aligned}
\Sigma_\varepsilon = \Big\{ &(S,E,I,R) \in [\mathbb{R}^+]^4 : \\
&\|S-S^*\|^2 + \|E-E^*\|^2 + \|I-I^*\|^2 + \|R-R^*\|^2 \le (1+\varepsilon)\frac{\bar{b}}{\bar{a}} \Big\}
\end{aligned}
\tag{3.15}
$$

is an absorbing set for (3.1), with

$$
\bar{a} = \min\left\{a_1, a_2, a_3, a_4\right\}, \quad \bar{b} = b_1 + b_2 + b_3 + b_4.
$$

Proof. Let

$$
\varphi_1 = S, \quad \varphi_2 = E, \quad \varphi_3 = I, \quad \varphi_4 = R.
$$

Multiplying $(3.1)_1$ for $(\varphi_1 - \varphi_1^*)$ and integrating over Ω, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\varphi_1 - \varphi_1^*\|^2 = \mu N_0 \int_\Omega (\varphi_1 - \varphi_1^*)\, d\Omega - \mu \int_\Omega \varphi_1(\varphi_1 - \varphi_1^*)\, d\Omega +$$

$$+\gamma \int_\Omega (\varphi_1 - \varphi_1^*)\Delta(\varphi_1 - \varphi_1^*)\, d\Omega - K \int_\Omega \varphi_3\varphi_1(1+\alpha\varphi_3)(\varphi_1 - \varphi_1^*)\, d\Omega.$$

In view of Hölder and generalized Cauchy inequalities, it turns out that

$$\int_\Omega (\varphi_1 - \varphi_1^*)\, d\Omega \le \|\varphi_1 - \varphi_1^*\||\Omega|^{\frac{1}{2}} \le \varepsilon_1\|\varphi_1 - \varphi_1^*\|^2 + \frac{|\Omega|}{\varepsilon_1},$$

ε_1 being a positive constant. Applying the divergence theorem, taking into account of the boundary conditions (3.2), and by virtue of (3.4), one has

$$\frac{d}{dt}\left\|\varphi_1 - \varphi_1^*\right\|^2 \le -a_1\left\|\varphi_1 - \varphi_1^*\right\|^2 + b_1, \tag{3.16}$$

with a_1 and b_1 such as in $(3.13)_1$,$(3.13)_5$.

Analogously, multiplying $(3.1)_i$ for $(\varphi_i - \varphi_i^*)\,(i = 2,3,4)$ and integrating over Ω, by using Hölder and generalized Cauchy inequalities, applying the divergence theorem, taking into account of the boundary conditions and in view of (3.4), one obtains

$$\frac{d}{dt}\left\|\varphi_i - \varphi_i^*\right\|^2 \le -a_i\left\|\varphi_i - \varphi_i^*\right\|^2 + b_i, \quad i = 2,3,4 \tag{3.17}$$

with a_i and b_i, $(i = 2,3,4)$ such as in (3.13).

Choosing $\varepsilon_i \quad (i = 1,2,3,4)$ such as in (3.14) and adding (3.16) and $(3.17)_i$,$(i = 2,3,4)$, one has

$$\frac{dE}{dt} \le -\bar{a}E + \bar{b}. \tag{3.18}$$

Now, let us prove that (3.18) guarantees that (3.15) is positively invariant and attractive.

Let's start to prove that, $\forall\varepsilon > 0$, Σ_ε is positively invariant.

The trajectories corresponding to the initial data in Σ_ε, i.e.

$$E(0) = \left(1 + \frac{\varepsilon}{2^n}\right)\frac{\bar{b}}{\bar{a}} \in \Sigma_\varepsilon, \quad (n = 1,2,\ldots)$$

could escape from Σ_ε if and only if there exists t^* such that

$$\begin{cases} E(t^*) = (1+\varepsilon)\,\dfrac{\bar{b}}{\bar{a}}, \\ \\ \left(\dfrac{dE}{dt}\right)_{t=t^*} \geq 0. \end{cases}$$

By virtue of (3.18), it follows that

$$\left(\frac{dE}{dt}\right)_{t=t^*} \leq -\bar{a}E(t^*) + \bar{b} = -\varepsilon\bar{b} < 0,$$

so that Σ_ε is positively invariant.

In order to prove that Σ_ε is an attractor, by integrating (3.18), one obtains

$$E(t) \leq E(0)e^{-\bar{a}t} + \frac{\bar{b}}{\bar{a}}\left(1 - e^{-\bar{a}t}\right) \leq E(0)e^{-\bar{a}t} + \frac{\bar{b}}{\bar{a}}.$$

Let B a bounded set of the phase space. Then there exists a positive constant M such that $M = \sup_B E$. Hence, from

$$Me^{-\bar{a}t} + \frac{\bar{b}}{\bar{a}} = (1+\varepsilon)\frac{\bar{b}}{\bar{a}},$$

it turns out that, for

$$t > \bar{t} = \frac{1}{\bar{a}}\ln\frac{\bar{a}M}{\varepsilon\,\bar{b}},$$

every trajectory starting from B reaches Σ_ε and remains there indefinitely, so that Σ_ε is also an attractive set.

We remark that, since by virtue of (3.12) the solutions belong to $\bar{\Sigma}$, obviously, one can choose $B \subset \bar{\Sigma}$ and hence $\Sigma_\varepsilon \subset \bar{\Sigma}$.

Remark 7 *In view of Theorem 12 we can confine ourselves to study the longtime behaviour of system (3.1) taking the initial data belonging to Σ_ε.*

3.1.4 Equilibria and preliminaries to stability

Denoting by R_0 the *basic reproduction number* in absence of diffusion, R_0 is given by

$$R_0 = \frac{N_0 \theta K}{(\theta + \mu)(\sigma + \mu)}$$

and setting

$$R_1^* = \sqrt{\frac{K}{\alpha\mu}}, \qquad R_2^* = 2R_1^* - {R_1^*}^2,$$

the following remark holds.

Remark 8 *It can be easily shown that*

i) $R_1^* < 1$ *if and only if* $R_1^* < R_2^*$,

ii) $R_2^* \leq 1$.

The biologically meaningful equilibria $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ of (3.1) are the non-negative solutions of the system

$$\begin{cases} \mu(N_0 - \bar{S}) - K\bar{I}\bar{S}(1 + \alpha\bar{I}) = 0, \\ -(\theta + \mu)\bar{E} + K\bar{I}\bar{S}(1 + \alpha\bar{I}) = 0, \\ -(\sigma + \mu)\bar{I} + \theta\bar{E} = 0, \\ \sigma\bar{I} - \mu\bar{R} = 0. \end{cases} \tag{3.19}$$

The following two kinds of solutions arise.

i) DISEASE-FREE EQUILIBRIUM:

System (3.1) admits the equilibrium $(S_1, E_1, I_1, R_1) = (N_0, 0, 0, 0)$ which - from biological point of view - means that no infection arises.

ii) ENDEMIC EQUILIBRIA:

The biologically meaningful equilibria (endemic equilibria) of (3.1) are the positive stationary constant solutions $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ of (3.1). It is easily verified that, solving (3.19), one obtains

$$\bar{S} = \frac{(\theta+\mu)(\sigma+\mu)}{K\theta(1+\alpha\bar{I})}, \quad \bar{E} = \frac{\sigma+\mu}{\theta}\,\bar{I}, \quad \bar{R} = \frac{\sigma}{\mu}\bar{I}, \tag{3.20}$$

where $\bar{I}$ has to verify the equation

$$a\bar{I}^2 + b\bar{I} + c = 0, \tag{3.21}$$

with

$$\begin{aligned} &a = K\alpha > 0, \quad b = \mu\alpha\left(R_1^{*^2} - R_0\right), \quad c = \mu\,(1 - R_0)\,, \\ &\Delta = b^2 - 4ac = \mu\alpha(\mu\alpha R_0 + K + 2\sqrt{K\mu\alpha})(R_0 - R_2^*). \end{aligned} \tag{3.22}$$

We have to distinguish the following three cases.

1) If $R_0 < 1$, then $c > 0$. Since

$$\Delta > 0 \;\Leftrightarrow\; R_0 > R_2^*, \tag{3.23}$$

and

$$R_2^* \geq 0 \;\Leftrightarrow\; R_1^* \leq 2,$$

it follows that

- if $R_1^* \geq 2$ or $\{R_2^* < R_0$ with $1 < R_1^* < 2\}$ then $\{b > 0,\ \Delta > 0\}$ and hence (3.21) admits two real negative solutions;
- if $\{R_2^* < R_0$ with $R_1^* < 1\}$ then $\{b < 0,\ \Delta > 0\}$, hence (3.21) admits two real positive solutions;
- if $\{R_0 < R_2^*$ with $R_1^* < 2\}$ then $\Delta < 0$, i.e. (3.21) does not admit real solutions;
- if $\{R_0 = R_2^*$ with $1 < R_1^* < 2\}$ then $\{b > 0,\ \Delta = 0\}$, so there are not any endemic equilibria;

- if $\{R_0 = R_2^* \text{ with } R_1^* < 1\}$ then $\{b < 0,\ \Delta = 0\}$ and hence there exists a unique endemic equilibrium.

2) If $R_0 = 1$, then $c = 0$ and the existence of endemic equilibria depends on R_1^*. In fact, when $R_1^* = 1$, accounting for $(3.22)_2$, $(3.22)_3$, $(3.22)_4$, one has $\Delta = b = c = 0$ and hence do not exist endemic equilibria. When $R_1^* \neq 1$, (3.21) admits the non null solution $\bar{I} = -\dfrac{b}{a}$ that, by virtue of $(3.22)_1$, $(3.22)_2$, is biologically meaningful, i.e. $\bar{I} > 0$, iff $R_1^* < 1$.

3) If $R_0 > 1$, then, by virtue of $(3.22)_3$-$(3.22)_4$, one obtains that $c < 0$ and $\Delta > 0$. Hence in this case, (3.21) admits a unique real positive solution.

The previous results can be summarized in the following theorem.

Theorem 13 *System (3.1)*

i) always admits the disease free equilibrium $(S_1, E_1, I_1, R_1) = (N_0, 0, 0, 0)$*;*

ii) admits a unique endemic equilibrium if

$$R_0 > 1, \tag{3.24}$$

or

$$\begin{cases} R_0 = 1, \\ R_1^* < 1; \end{cases} \tag{3.25}$$

or

$$\begin{cases} R_0 = R_2^* < 1, \\ R_1^* < 1; \end{cases} \tag{3.26}$$

iii) admits two endemic equilibria if

$$\begin{cases} R_2^* < R_0 < 1, \\ R_1^* < 1; \end{cases} \tag{3.27}$$

iv) does not admit any endemic equilibria in the other cases.

In the sequel, let (3.24) or (3.25) or (3.26) holds true. In this case (3.1) admits a unique endemic equilibrium $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$. Setting

$$X_1 = S - \bar{S}, \quad X_2 = E - \bar{E}, \quad X_3 = I - \bar{I}, \quad X_4 = R - \bar{R}$$

(3.1) reduces to

$$\begin{cases} \dfrac{\partial X_1}{\partial t} = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + a_{14}X_4 + \gamma\Delta X_1 - F(X_1, X_3) \\ \dfrac{\partial X_2}{\partial t} = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + \gamma\Delta X_2 + F(X_1, X_3) \\ \dfrac{\partial X_3}{\partial t} = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + a_{34}X_4 + \gamma_3\Delta X_3 \\ \dfrac{\partial X_4}{\partial t} = a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + a_{44}X_4 + \gamma_4\Delta X_4 \end{cases} \tag{3.28}$$

where

$$\begin{cases} a_{11} = -[\mu + K\bar{I}(1 + \alpha\bar{I})],\ a_{12} = 0,\ a_{13} = -K\bar{S}(1 + 2\alpha\bar{I}),\ a_{14} = 0, \\ a_{21} = K\bar{I}(1 + \alpha\bar{I}),\ a_{22} = -(\theta + \mu),\ a_{23} = K\bar{S}(1 + 2\alpha\bar{I}),\ a_{24} = 0, \\ a_{31} = 0,\ \ a_{32} = \theta,\ \ a_{33} = -(\sigma + \mu),\ a_{34} = 0, \\ a_{41} = 0,\ \ a_{42} = 0,\ \ a_{43} = \sigma,\ a_{44} = -\mu, \\ F(X_1, X_3) = KX_3[(1 + 2\alpha\bar{I})X_1 + \alpha X_3(\bar{S} + X_1)]. \end{cases} \tag{3.29}$$

To (3.28)-(3.29) we add the boundary conditions

$$\begin{cases} X_i = 0 \quad \text{on} \quad \Sigma \times \mathbb{R}^+ \\ \nabla X_i \cdot \mathbf{n} = 0 \quad \text{on} \quad \Sigma^* \times \mathbb{R}^+ \end{cases} \qquad i = 1, 2, 3, 4. \tag{3.30}$$

Denoting by $W^*(\Omega)$ the functional space defined by

$$W^*(\Omega)=\left\{\varphi \in W^{1,2}(\Omega)\cap W^{1,2}(\partial\Omega) : \varphi=0 \text{ on } \Sigma\times\mathbb{R}^+, \frac{d\varphi}{d\mathbf{n}}=0 \text{ on } \Sigma^*\times\mathbb{R}^+\right\},$$

our aim is to study the stability of $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ with respect to the perturbations $(X_1, X_2, X_3, X_4) \in [W^*(\Omega)]^4$.

3.1.5 Linear stability of endemic equilibrium

Remark 9 *We remark that the infimum*

$$\bar{\alpha}(\Omega) = \inf_{\varphi\in W^*(\Omega)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2}, \tag{3.31}$$

exists and is a real positive number ([4], [67]);

Adding and subtracting the term $\bar{\alpha}\gamma_i X_i$ to equation $(3.28)_i$, $(i = 1, 2, 3, 4)$, and setting

$$\mathcal{L}^* = \begin{pmatrix} a_{11}-\bar{\alpha}\gamma & 0 & a_{13} & 0 \\ a_{21} & a_{22}-\bar{\alpha}\gamma & a_{23} & 0 \\ 0 & a_{32} & a_{33}-\bar{\alpha}\gamma_3 & 0 \\ 0 & 0 & a_{43} & a_{44}-\bar{\alpha}\gamma_4 \end{pmatrix},$$

in order to analyze the linear stability of the endemic equilibrium, and hence of the null solution of the perturbation system (3.28), we have to find conditions ensuring that all $\mathcal{L}^*$ eigenvalues have negative real parts. The characteristic equation of $\mathcal{L}^*$ is

$$[\lambda - (a_{44} - \bar{\alpha}\gamma_4)](\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3) = 0, \tag{3.32}$$

where $\mathtt{I}_1$, $\mathtt{I}_2$, $\mathtt{I}_3$ are the principal invariants of the matrix

$$\mathcal{J}^* = \begin{pmatrix} a_{11}-\bar{\alpha}\gamma & 0 & a_{13} \\ a_{21} & a_{22}-\bar{\alpha}\gamma & a_{23} \\ 0 & a_{32} & a_{33}-\bar{\alpha}\gamma_3 \end{pmatrix}$$

and are given by

$$\begin{cases} \mathtt{I}_1 = \mathrm{trace}\mathcal{J}^* = a_{11} + a_{22} + a_{33} - \bar{\alpha}(2\gamma + \gamma_3), \\ \\ \mathtt{I}_2 = \begin{vmatrix} a_{11} - \bar{\alpha}\gamma & 0 \\ a_{21} & a_{22} - \bar{\alpha}\gamma \end{vmatrix} + \begin{vmatrix} a_{11} - \bar{\alpha}\gamma & a_{13} \\ 0 & a_{33} - \bar{\alpha}\gamma_3 \end{vmatrix} + \begin{vmatrix} a_{22} - \bar{\alpha}\gamma & a_{23} \\ a_{32} & a_{33} - \bar{\alpha}\gamma_3 \end{vmatrix} \\ \\ \mathtt{I}_3 = \det\mathcal{J}^* = \begin{vmatrix} a_{11} - \bar{\alpha}\gamma & 0 & a_{13} \\ a_{21} & a_{22} - \bar{\alpha}\gamma & a_{23} \\ 0 & a_{32} & a_{33} - \bar{\alpha}\gamma_3 \end{vmatrix}. \end{cases}$$

Accounting for (3.32), the eigenvalues of $\mathcal{L}^*$ are given by the three roots of the characteristic equation of $\mathcal{J}^*$

$$\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3 = 0 \tag{3.33}$$

and by

$$a_{44} - \bar{\alpha}\gamma_4,$$

where, in view of (3.29), $a_{44} - \bar{\alpha}\gamma_4 < 0$. Passing now to the equation (3.33), as it is well known, the necessary and sufficient conditions guaranteeing that all the roots of (3.33) have negative real part are the Routh-Hurwitz conditions [39]

$$\mathtt{I}_1 < 0, \quad \mathtt{I}_3 < 0, \quad \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 < 0. \tag{3.34}$$

Obviously (3.34) require necessarily that $\mathtt{I}_2 > 0$. If one of (3.34) is reversed, then there exists at least one eigenvalue of $\mathcal{J}^*$ with positive real part and hence the null solution of (3.28) is linearly unstable. Denoting by $\mathtt{I}^*$, A^* the principal invariants of the matrix $\begin{pmatrix} a_{22}-\bar{\alpha}\gamma & a_{23} \\ a_{32} & a_{33}-\bar{\alpha}\gamma_3 \end{pmatrix}$, i.e.

$$\begin{aligned} \mathtt{I}^* &= a_{22}+a_{33}-\bar{\alpha}(\gamma+\gamma_3), \\ A^* &= \bar{\alpha}^2\gamma\gamma_3-\bar{\alpha}(\gamma a_{33}+\gamma_3 a_{22})+a_{22}a_{33}-a_{23}a_{32}, \end{aligned} \tag{3.35}$$

it follows that

$$\begin{cases} \mathtt{I}_1 = a_{11}-\bar{\alpha}\gamma+\mathtt{I}^*, \;\; \mathtt{I}_2 = (a_{11}-\bar{\alpha}\gamma)\mathtt{I}^*+A^*, \\ \mathtt{I}_3 = (a_{11}-\bar{\alpha}\gamma)A^*+a_{13}a_{21}a_{32}, \\ \mathtt{I}_1\mathtt{I}_2-\mathtt{I}_3 = (a_{11}-\bar{\alpha}\gamma+\mathtt{I}^*)(a_{11}-\bar{\alpha}\gamma)\mathtt{I}^*+A^*\mathtt{I}^*-a_{13}a_{21}a_{32}. \end{cases} \tag{3.36}$$

The following theorem holds.

Theorem 14 *If*

$$A^* > 0, \tag{3.37}$$

then the endemic equilibrium is linearly stable.

Proof Let us remark that in view of (3.36) and (3.29), if $A^* > 0$, then the Routh Hurwitz conditions (3.34) are verified and the thesis follows.

3.1.6 Nonlinear stability of endemic equilibrium

Introducing the scalings $\mu_i\,(i = 1,2,3,4)$ (μ_i are positive constants to be chosen suitably later) and setting

$$\begin{cases} X_i = \mu_i U_i, \qquad \mathbf{U} = (U_1, U_2, U_3, U_4)^T, \quad \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4)^T, \\ \bar{F} = F(\mu_1 U_1, \mu_3 U_3) = K\mu_3 U_3[(1+2\alpha\bar{I})\mu_1 U_1 + \alpha\mu_3 U_3(\bar{S} + \mu_1 U_1)], \\ \tilde{F}_1 = -\dfrac{1}{\mu_1}\bar{F} + \gamma(\Delta U_1 + \bar{\alpha} U_1), \ \tilde{F}_2 = \dfrac{1}{\mu_2}\bar{F} + \gamma(\Delta U_2 + \bar{\alpha} U_2), \\ \tilde{F}_3 = \gamma_3(\Delta U_3 + \bar{\alpha} U_3), \ \tilde{F}_4 = \gamma_4(\Delta U_4 + \bar{\alpha} U_4), \end{cases} \tag{3.38}$$

(3.28) reduces to

$$\frac{\partial \mathbf{U}}{\partial t} = \tilde{L}\mathbf{U} + \tilde{\mathbf{F}}, \tag{3.39}$$

where $\tilde{L}$ is

$$\tilde{L} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ 0 & b_{32} & b_{33} & 0 \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}$$

with

$$b_{ii} = a_{ii} - \bar{\alpha}\gamma_i, \qquad b_{ij} = \frac{\mu_j}{\mu_i} a_{ij}, \ i \neq j. \tag{3.40}$$

The boundary conditions (3.30) become

$$U_i = 0 \quad \text{on} \quad \Sigma \times \mathbb{R}^+, \quad \nabla U_i \cdot \mathbf{n} = 0, \quad \text{on} \quad \Sigma^* \times \mathbb{R}^+, \quad i = 1,2,3,4. \tag{3.41}$$

Hence the problem to find conditions guaranteeing the stability of ($\bar{S}$, $\bar{E}$, $\bar{I}$, $\bar{R}$) is reduced to determine conditions guaranteeing the stability of the null solution of (3.39)-(3.41). Remarking that, by virtue of (3.35) and (3.40), one obtains that $A^* = b_{22}b_{33} - b_{23}b_{32} = b_{22}b_{33} - a_{23}a_{32}$ and setting

$$\begin{cases} A_1 = A^* + (b_{32})^2 + (b_{33})^2 \\ A_2 = A^* + (b_{22})^2 + (b_{23})^2 \\ A_3 = b_{22}b_{32} + b_{23}b_{33}, \end{cases}$$

in order to study the nonlinear stability of endemic equilibrium, let us introduce the Rionero-Liapunov functional (see [54],[55] for details)

$$W = \frac{1}{2}\|U_1\|^2 + V + \frac{1}{2}\|U_4\|^2,$$

where

$$V = \frac{1}{2}\left[A^*(\|U_2\|^2 + \|U_3\|^2) + \|b_{22}U_3 - b_{32}U_2\|^2 + \|b_{23}U_3 - b_{33}U_2\|^2\right],$$

which is positive definite if $A^* > 0$.

The time derivative of W along the solutions of (3.39) is

$$\begin{aligned} \dot{W} &= b_{11}\|U_1\|^2 + I^*A^*(\|U_2\|^2 + \|U_3\|^2) + b_{44}\|U_4\|^2 + A_1 b_{21}\langle U_1, U_2\rangle + \\ &+ (-A_3 b_{21} + b_{13})\langle U_1, U_3\rangle + b_{43}\langle U_3, U_4\rangle + \Phi_1 + \Phi_2, \end{aligned} \tag{3.42}$$

being

$$\begin{cases} \Phi_1 = \gamma < U_1, \Delta U_1 + \alpha_0 U_1 > + < A_1U_2 - A_3U_3, \gamma(\Delta U_2 + \alpha_0 U_2) > + \\ \quad + < A_2U_3 - A_3U_2, \gamma_3(\Delta U_3 + \alpha_0 U_3) > + \gamma_4 < U_4, \Delta U_4 + \alpha_0 U_4 >, \\ \Phi_2 = \dfrac{1}{\mu_1} < U_1, -\bar{F} > + \dfrac{1}{\mu_2} < A_1U_2, \bar{F} > + \dfrac{1}{\mu_2} < A_3U_3, -\bar{F} >, \end{cases} \tag{3.43}$$

with $\bar{F}$ given by $(3.38)_2$.

The following Lemmas hold.

Lemma 6 *If*

$$A^* > 0 \quad \textit{and} \quad (\gamma + \gamma_3)\,|A_3| < 2\sqrt{A_1 A_2 \gamma\gamma_3}, \tag{3.44}$$

then there exists $\epsilon_1 \in (0,1)$ *such that* $\forall\epsilon \in (0,\epsilon_1)$

$$\Phi_1 \leq -\gamma \left\|\nabla U_1\right\|^2 + \bar{\alpha}\gamma \left\|U_1\right\|^2 - A_2\gamma_3\epsilon \left\|\nabla U_3\right\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \left\|U_3\right\|^2. \qquad (3.45)$$

Proof. By using the divergence theorem, by virtue of the boundary conditions (3.41) and in view of Poincaré inequality (3.31), from $(3.43)_1$ it follows that

$$\Phi_1 \leq -\gamma \left\|\nabla U_1\right\|^2 + \bar{\alpha}\gamma \left\|U_1\right\|^2 - A_2\gamma_3\epsilon \left\|\nabla U_3\right\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \left\|U_3\right\|^2 + \Phi^*,$$

being ϵ a positive constant and

$$\begin{aligned}\Phi^* =& -A_1\gamma \left\|\nabla U_2\right\|^2 - |A_3| (\gamma+\gamma_3) \langle \nabla U_2, \nabla U_3\rangle - A_2\gamma_3(1-\epsilon) \left\|\nabla U_3\right\|^2 + \\ &+ A_1\gamma\bar{\alpha} \left\|U_2\right\|^2 + |A_3|\, \bar{\alpha}(\gamma+\gamma_3) \langle U_2, U_3\rangle + A_2\gamma_3(1-\epsilon)\bar{\alpha} \left\|U_3\right\|^2.\end{aligned} \qquad (3.46)$$

Since $(3.44)_2$ implies that there exists $\epsilon_1 \in (0,1)$ such that $\forall\epsilon \in (0,\epsilon_1)$

$$|A_3| (\gamma+\gamma_3) = 2\sqrt{(1-\epsilon_1)\gamma\gamma_3 A_1 A_2}, \quad |A_3| (\gamma+\gamma_3) \leq 2\sqrt{(1-\epsilon)\gamma\gamma_3 A_1 A_2},$$

by following the same procedure used for the proof of Lemma 3.2 in [54], one obtains that $\Phi^* \leq 0$ and hence the thesis follows.

Lemma 7 *There exists a positive constant* $M(\Omega)$ *such that*

$$\Phi_2 \leq M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).$$

Proof. In view of $(3.43)_2$ and $(3.38)_2$ and since from (3.4) and $(3.38)_1$ one obtains

$$U_i \leq \frac{M_i}{\mu_i}, \quad i = 1,2,3$$

it turns out that

$$\begin{aligned}&\Phi_2 \leq c_1 < U_1^2, |U_3| > + c_2 < |U_1|, U_3^2 > + c_3 < |U_2|, |U_1 U_3| > + \\ &+ c_4 < |U_2|, U_3^2 > + c_5 < |U_3|, U_3^2 >,\end{aligned} \qquad (3.47)$$

where

$$\begin{cases} c_1 = k(1+2\alpha\bar{I})\mu_3, \\ c_2 = k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_1} + A_1 k\alpha M_2 \dfrac{\mu_1\mu_3^2}{\mu_2^2} + |A_3|\, k(1+2\alpha\bar{I})\dfrac{\mu_1\mu_3}{\mu_2} + |A_3|\, k\alpha M_3 \dfrac{\mu_1\mu_3}{\mu_2}, \\ c_3 = A_1 k(1+2\alpha\bar{I})\dfrac{\mu_1\mu_3}{\mu_2},\ c_4 = A_1 k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_2},\ c_5 = |A_3|\, k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_2}. \end{cases}$$

By virtue of the Hölder and embedding inequalities

$$< |f|, g^2 > \leq \|f\|\, \|g\|_4^2, \quad \|g\|_4^2 \leq K_1(\Omega)[\|\nabla g\|^2 + \|g\|^2], \quad K_1(\Omega) > 0,$$

and in view of Cauchy inequality, from (3.47) it follows that

$$\begin{aligned} \Phi_2 \ &\leq\ \eta_1 \|U_3\| \left(\|U_1\|^2 + \|\nabla U_1\|^2\right) + \eta_2 \|U_1\| \left(\|U_3\|^2 + \|\nabla U_1\|^2\right) + \\ &+\ \eta_3 \|U_2\| \left(\|U_1\|^2 + \|\nabla U_1\|^2\right) + \eta_4 \|U_2\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right) + \\ &+\ \eta_5 \|U_3\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right), \end{aligned}$$

being

$$\eta_i = K_1(\Omega)c_i,\ i = 1,2,5 \quad \eta_3 = \frac{1}{2}c_3 K_1(\Omega) \quad \eta_4 = \left(\frac{1}{2}c_3 + c_4\right) K_1(\Omega).$$

Hence the thesis follows with $M(\Omega) = \max\limits_{i=1,\dots,5} \eta_i$.

Remark 10 *We remark that, setting*

$$p = \frac{A^*}{2}, \quad q = \frac{A^*}{2} + \left[(b_{22})^2 + (b_{23})^2 + (b_{32})^2 + (b_{33})^2\right],$$

it follows that

$$p(\|U_2\|^2 + \|U_3\|^2) \leq V \leq q(\|U_2\|^2 + \|U_3\|^2). \tag{3.48}$$

The following lemma holds.

Lemma 8 *Setting*

$$\zeta_1 = \left[1 + \frac{\bar{\alpha}^2\gamma\gamma_3 + \bar{\alpha}\gamma(\sigma+\mu) + \bar{\alpha}\gamma_3(\theta+\mu)}{(\theta+\mu)(\sigma+\mu)}\right] \frac{N_0}{\bar{S}(1+2\alpha\bar{I})}, \tag{3.49}$$

then

$$A^* > 0 \Leftrightarrow R_0 < \zeta_1. \tag{3.50}$$

Proof. By virtue of (3.20), (3.35) and (3.49), one has that

$$A^* = \frac{(\theta+\mu)(\sigma+\mu)}{N_0}\bar{S}(1+2\alpha\bar{I})[\zeta_1 - R_0], \tag{3.51}$$

and hence (3.50) immediately follows.

The following theorem holds.

Theorem 15 *If and only if*

$$R_0 < \zeta_1 \quad \textit{and} \quad (\gamma+\gamma_3)\,|A_3| < 2\sqrt{A_1 A_2 \gamma\gamma_3}, \tag{3.52}$$

then the null solution of (3.28)-(3.30) is (locally) nonlinearly, asymptotically stable with respect to the $L^2(\Omega)$*-norm.*

Proof. By virtue of Lemma 6 and Lemma 7, in view of $(3.40)_1$ and by using the generalized Cauchy inequality, from (3.42) it follows that

$$\begin{aligned}
\dot{W} \leq & -|a_{11}|\,\|U_1\|^2 - |I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - |a_{44}|\|U_4\|^2 + \\
& -\gamma\,\|\nabla U_1\|^2 - A_2\gamma_3\epsilon\,\|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon\,\|U_3\|^2 + \\
& + \frac{A_1^2 a_{21}^2\mu_1^2}{2\mu_2^2|I^*|A^*}\,\|U_1\|^2 + \frac{1}{2}|I^*|A^*\,\|U_2\|^2 + \\
& + (-A_3 b_{21} + b_{13})\,\langle U_1, U_3\rangle + \frac{a_{43}^2\mu_3^2}{2|a_{44}|\mu_4^2}\,\|U_3\|^2 + \frac{1}{2}|a_{44}|\,\|U_4\|^2 + \\
& + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).
\end{aligned} \tag{3.53}$$

Choosing the positive scalings such that

$$\begin{cases}
-A_3 b_{21} + b_{13} = 0 \\
|a_{11}| - \dfrac{A_1^2 a_{21}^2 \mu_1^2}{2\mu_2^2 |I^*|A^*} > \dfrac{1}{2}|a_{11}| \\
|I^*|A^* - \dfrac{a_{43}^2\mu_3^2}{2|a_{44}|\mu_4^2} > \dfrac{1}{2}|I^*|A^*
\end{cases} \tag{3.54}$$

i.e.

$$\begin{cases} \mu_1^2 = \dfrac{a_{13}\mu_2^2\mu_3^2}{a_{21}\left(b_{22}a_{32}\mu_2^2 + b_{33}a_{23}\mu_3^2\right)} \\ \\ \mu_2^2 > \dfrac{\left(A^* + b_{33}^2\right)a_{21}^2\mu_1^2\mu_3^2}{|I^*|A^*|a_{11}|\mu_3^2 + a_{32}^2a_{21}^2\mu_1^2} \\ \\ \mu_4^2 > \dfrac{|a_{43}|^2}{|a_{44}||I^*|A^*}\mu_3^2 \end{cases}$$

and choosing

$$\epsilon < \min\left\{\frac{|I^*|A^*}{4A_2\bar{\alpha}\gamma_3}, \epsilon_1\right\},$$

from (3.53) it turns out that

$$\begin{aligned} \dot{W} \leq & -\frac{1}{2}|a_{11}|\|U_1\|^2 - \frac{1}{4}|I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - \frac{1}{2}|a_{44}|\,\|U_4\|^2 + \\ & -\gamma\,\|\nabla U_1\|^2 - A_2\gamma_3\epsilon\,\|\nabla U_3\|^2 + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2). \end{aligned}$$

Therefore, setting

$$h_1 = \min\left\{\mu, \frac{A^*|I^*|}{2}\right\}, \quad h_2 = \min\{\gamma, \epsilon A_2\gamma_3\},$$

one has that

$$\begin{aligned} \dot{W} \leq & -\frac{h_1}{2}(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2 + \|U_4\|^2) - h_2(\|\nabla U_1\|^2 + \|\nabla U_3\|^2) + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2). \end{aligned}$$

By virtue of (3.48), it turns out that

$$\dot{W} \leq -(\delta_1 - \delta_3 W^{\frac{1}{2}})W - (h_2 - \delta_2 W^{\frac{1}{2}})(\|\nabla U_1\|^2 + \|\nabla U_3\|^2),$$

with

$$\delta_1 = h_1 \min\{1,\ 1/2q\},\ \delta_2 = M(\Omega)\max\{\sqrt{2},\ p^{-1/2}\},\ \delta_3 = M(\Omega)\max\{2\sqrt{2}, p^{-3/2}\}.$$

Hence, if

$$W^{\frac{1}{2}}(0) < \min\{\delta_1/\delta_3,\ h_2/\delta_2\},$$

applying recursive arguments, it follows that

$$\dot{W} \leq -\tilde{K}W, \qquad \tilde{K} = \text{const.} > 0$$

and hence the thesis follows.

3.2 On the stability of a SEIR reaction diffusion model for infections under Neumann boundary conditions

3.2.1 Introduction

In this section, we want to reconsider the model introduced in the prevoius one [17]. In particular, while in the prevoius case it is supposed $\gamma_1 = \gamma_2$, i.e. that both susceptible and infected individuals have the same diffusion coefficient, in the present case we reconsider the problem in the more general case $\gamma_1 \neq \gamma_2$ which, a priori, is biologically the more realistic case. Furthermore, differently from the previous case, we assume that the infection at stake requires to put in quarantine. From mathematical point of view, it means that one has to consider the homogeneous Neumann boundary conditions. Moreover, since the quarantine hospitals normally can have many floors, it seems to prefer to embed the problem in a three-dimensional domain.

3.2.2 Mathematical model

Let Ω be a regular domain of class C^p $(p \geq 2)$, with the interior cone property. Let us consider the reaction-diffusion model of SEIR type (3.1) governing

the evolution of an infection in Ω with all the parameters having the same meaning of the previous section.
To (3.1) we add the following homogeneous Neumann boundary conditions

$$\nabla S \cdot \mathbf{n} = 0,\ \nabla E \cdot \mathbf{n} = 0,\ \nabla I \cdot \mathbf{n} = 0,\ \nabla R \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{3.55}$$

being $\mathbf{n}$ the unit outward normal on $\partial\Omega$.
Let us define

$$N(t) = \frac{1}{|\Omega|} \int_\Omega [S(\mathbf{x},t) + E(\mathbf{x},t) + I(\mathbf{x},t) + R(\mathbf{x},t)]\ d\Omega,$$

i.e. the population size at time t. The following theorem holds.

Theorem 16 *The total population size in Ω is constant for all time, i.e.*

$$N(t) = N_0, \qquad \forall t \geq 0. \tag{3.56}$$

Proof. By adding $(3.1)_1$, $(3.1)_2$, $(3.1)_3$, $(3.1)_4$ and integrating over Ω, one has

$$\begin{aligned} \frac{d}{dt}\int_\Omega (S+E+I+R)\ d\Omega = {} & \mu N_0\,|\Omega| - \mu \int_\Omega (S+E+I+R)\ d\Omega + \\ & + \gamma_1 \int_\Omega \Delta S\, d\Omega + \gamma_2 \int_\Omega \Delta E\, d\Omega + \gamma_3 \int_\Omega \Delta I\, d\Omega + \gamma_4 \int_\Omega \Delta R\, d\Omega. \end{aligned} \tag{3.57}$$

In view of the boundary conditions (3.55), the divergence theorem leads to

$$\int_\Omega \Delta\varphi\, d\Omega = \int_\Omega \nabla \cdot \nabla\varphi\, d\Omega = \int_{\partial\Omega} \nabla\varphi \cdot \mathbf{n}\, d\Sigma = 0, \quad \forall \varphi \in \{S, E, I, R\},$$

hence (3.57) becomes

$$\frac{d}{dt} N(t) + \mu N(t) = \mu N_0. \tag{3.58}$$

Integrating (3.58), one easily obtains (3.56).

3.2.3 Absorbing sets

Let us denote by $T > 0$ an arbitrary fixed time and by $\Omega_T = \Omega \times (0, T]$ the parabolic cylinder, Ω_T being the parabolic interior of $\bar{\Omega} \times [0, T]$ (i.e. Ω_T includes the top $\Omega \times \{t = T\}$).
We refer here to the positive smooth solutions of (3.1) under the boundary conditions (3.55) and the smooth positive initial data

$$\begin{cases} S(\mathbf{x}, 0) = S_0(\mathbf{x}), \quad E(\mathbf{x}, 0) = E_0(\mathbf{x}), \\ \\ I(\mathbf{x}, 0) = I_0(\mathbf{x}), \quad R(\mathbf{x}, 0) = R_0(\mathbf{x}), \end{cases} \qquad \mathbf{x} \in \Omega. \tag{3.59}$$

The following theorems hold.

Theorem 17 *Model (3.1), under the boundary conditions (3.55) and the (positive) initial data (3.59), admits a unique positive solution in* $C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$.

Proof. The existence of solutions of (3.1), (3.55), (3.59) can be proved as done in [40] and the uniqueness as done in [14].
In order to obtain a L^∞-norm estimate of the solution, let us recall a result of [48].

Lemma 9 *Let us consider the parabolic system*

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i = f_i(x, t, u), & x \in \Omega, t > 0, i = 1, ..., l \\ \dfrac{\partial u_i}{\partial \nu} = 0 & x \in \partial\Omega, t > 0 \\ u_i(x, 0) = u_i^0(x) & x \in \Omega \end{cases}$$

where $u = (u_1, ..., u_l)$, $u_i^0 \in C\left(\bar{\Omega}\right)$, $i = 1, ..., l$ *and assume that, for each* $k = 1, ..., l$, *the functions* f_k *satisfy the polynomial growth condition*

$$|f_k(x, t, u)| \leq c_1 \sum_{i=1}^{l} |u_i|^q + c_2 \tag{3.60}$$

for some nonnegative constants c_1 and c_2 and positive constant q. Let p_0 be a positive constant such that $p_0 > \frac{n}{2}\max\{0, q-1\}$ and $\tau(u^0)$ be the maximal existence time of the solution u corresponding to the initial data u^0. Suppose that there exists a positive constant $C_{p_0}(u^0)$ such that $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C_{p_0}(u^0)$, $\forall t \in (0, \tau(u^0))$, then the solution u exists for all time and there is a positive constant C_∞ such that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_\infty(u^0)$, $\forall t \in (0,\infty)$.

Theorem 18 *Any positive solution (S, E, I, R) of (3.1), (3.55), (3.59) is bounded, i.e. there exists a positive constant C, depending on the nonnegative initial data $(S_0(x), E_0(x), I_0(x), R_0(x))$ such that, $\forall t \in (0,\infty)$*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|E(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|R(\cdot,t)\|_{L^\infty(\Omega)} \le C. \quad (3.61)$$

Proof. By following the procedure used in [11], let us set $\max\limits_{\bar{\Omega}_T} S = S(\mathbf{x}_1, t_1)$. We have to distinguish the following cases.

1) If $(\mathbf{x}_1, t_1)$ belongs to the interior of Ω_T, then $(3.1)_1$ implies that

$$\left[\frac{\partial S}{\partial t} - \mu(N_0 - S) - \gamma \Delta S\right]_{(\mathbf{x}_1,t_1)} < 0. \quad (3.62)$$

Since

$$\left[\frac{\partial S}{\partial t}\right]_{(\mathbf{x}_1,t_1)} = 0, \qquad [\Delta S]_{(\mathbf{x}_1,t_1)} < 0,$$

then (3.62) can hold if and only if

$$\mu(N_0 - S) > 0$$

and hence if and only if $S(\mathbf{x}_1, t_1) < N_0$.

2) If $(\mathbf{x}_1, t_1) \in \partial\Omega \times [0, T)$, in view of the regularity of the domain Ω, since Ω verifies in any point $\mathbf{x}_0 \in \partial\Omega$ the interior ball condition, there exists

an open ball $B^* \subset \Omega$ with $\mathbf{x}_0 \in \partial B^*$. If $S(\mathbf{x}_1, t_1) > N_0$, choosing the radius of B^* sufficiently small, it follows that

$$\gamma \Delta S - \frac{\partial S}{\partial t} > 0 \quad \text{in } B^*$$

and, by virtue of Hopf's Lemma [49], one obtains that

$$\left(\frac{dS}{d\mathbf{n}}\right)_{(\mathbf{x}_1, t_1)} > 0.$$

Since $\dfrac{dS}{d\mathbf{n}} = 0$ on $\partial\Omega \times \mathbb{R}^+$, it follows that $S(\mathbf{x}_1, t_1) > N_0$ is not possible.

3) Finally, if $(\mathbf{x}_1, t_1) \in \Omega \times \{0\}$, then $S(\mathbf{x}_1, t_1) < \max\limits_{\bar{\Omega}} S_0(\mathbf{x})$.

Hence,

$$\|S(\cdot, t)\|_{L^\infty} < M_1 = \max\{N_0, \max_{\bar{\Omega}} S_0(\mathbf{x})\}, \, \forall t > 0; \tag{3.63}$$

moreover, in view of (3.63) and of the Young inequality, one obtains that the nonlinear term $g(S, I) = KIS(1 + \alpha I)$ has a polynomial growth such that

$$g(S, I) \leq c_1 I^2 + c_2, \tag{3.64}$$

being $c_1 = KM_1\left(\alpha + \dfrac{1}{2}\right)$ and $c_2 = \dfrac{1}{2}KM_1$. Therefore, (3.64) implies that the hyphotesis (3.60) of Lemma 9 holds with $q = 2$. In order to apply Lemma 9 with $p_0 = 2$ to the model (3.1), (3.55), (3.59), let us first estimate the L^2-norm of each component of the solution (S, E, I, R) with respect to the spatial variable.

In order to do this, multiplying $(3.1)_1$ by $S(\mathbf{x}, t)$ and integrating over Ω, by using the divergence theorem and the boundary conditions (3.55), it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega S^2 \, d\Omega + \gamma_1 \int_\Omega |\nabla S|^2 \, d\Omega + \mu \int_\Omega S^2 \, d\Omega = \\ &= \mu N_0 \int_\Omega S \, d\Omega - \int_\Omega KIS^2(1 + \alpha I) \, d\Omega. \end{aligned} \tag{3.65}$$

In view of (3.63), (3.65) implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega S^2\,d\Omega + \mu\int_\Omega S^2\,d\Omega \le \mu N_0 M_1,\ \forall t \ge 0$$

and hence

$$\frac{1}{2}\frac{d}{dt}\int_\Omega S^2\,d\Omega \le e^{-2\mu t}\int_\Omega S_0^2\,d\Omega + \frac{M_1}{\mu},\ \forall t \ge 0. \tag{3.66}$$

In order to estimate the L^2-norm of $E(\mathbf{x},t)$ with respect to the spatial variable, let us set

$$V(\mathbf{x},t) = S(\mathbf{x},t) + E(\mathbf{x},t)$$

so that, adding $(3.1)_1$ and $(3.1)_2$, one has

$$\frac{\partial V}{\partial t} = \gamma_2\Delta V + (\gamma_1-\gamma_2)\Delta S + \mu N_0 - (\theta+\mu)V + \theta S. \tag{3.67}$$

Multiplying (3.67) by V and integrating over Ω, by virtue of the divergence theorem and of the boundary conditions (3.55) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega V^2\,d\Omega + \gamma_2\int_\Omega |\nabla V|^2\,d\Omega + (\theta+\mu)\int_\Omega V^2\,d\Omega = \\ &(\gamma_2-\gamma_1)\int_\Omega \nabla S\cdot\nabla V\,d\Omega + \mu N_0\int_\Omega V\,d\Omega + \theta\int_\Omega SV\,d\Omega,\ \forall t\ge 0.\end{aligned} \tag{3.68}$$

In view of the Cauchy inequality, for any $t \ge 0$, one obtains that

$$(\gamma_2-\gamma_1)\int_\Omega \nabla S\cdot\nabla V\,d\Omega \le \frac{\gamma_2}{2}\int_\Omega |\nabla V|^2\,d\Omega + \frac{(\gamma_1-\gamma_2)^2}{2\gamma_2}\int_\Omega |\nabla S|^2\,d\Omega, \tag{3.69}$$

$$\mu N_0\int_\Omega V\,d\Omega \le \frac{2\theta+\mu}{4}\int_\Omega V^2\,d\Omega + \frac{\mu^2 N_0^2\,|\Omega|}{2\theta+\mu} \tag{3.70}$$

and

$$\theta\int_\Omega SV\,d\Omega \le \frac{2\theta+\mu}{4}\int_\Omega V^2\,d\Omega + \frac{\theta^2}{2\theta+\mu}\int_\Omega S^2\,d\Omega. \tag{3.71}$$

Taking into account (3.69), (3.70) and (3.71), and by virtue of (3.66), (3.68) implies that

$$\frac{d}{dt}\int_\Omega V^2\,d\Omega + \mu\int_\Omega V^2\,d\Omega \le \frac{(\gamma_1-\gamma_2)^2}{\gamma_2}\int_\Omega |\nabla S|^2\,d\Omega + m_1, \tag{3.72}$$

being $m_1 = \dfrac{2\mu^2 N_0^2 \,|\Omega|}{2\theta+\mu} + \dfrac{2\theta^2 N_0 \,|\Omega|\, M_1}{2\theta+\mu} + \dfrac{2\theta^2}{2\theta+\mu}\displaystyle\int_\Omega S_0^2 \, d\Omega.$
Moreover, (3.72) is equivalent to

$$\frac{d}{dt}\left(e^{\mu t}\int_\Omega V^2 \, d\Omega\right) \le \frac{(\gamma_1-\gamma_2)^2}{\gamma_2} e^{\mu t}\int_\Omega |\nabla S|^2 \; d\Omega + m_1 e^{\mu t}, \tag{3.73}$$

then, by integrating (3.73) from 0 to t, one has

$$\begin{aligned} \int_\Omega V^2 \, d\Omega \;&\le\; e^{-\mu t}\int_\Omega V(\mathbf{x},0)^2 \, d\Omega + \\ &+\; \frac{(\gamma_1-\gamma_2)^2}{\gamma_2} e^{-\mu t}\int_0^t \left[\int_\Omega e^{\mu\tau}|\nabla S|^2 \; d\Omega\right] d\tau + \frac{m_1}{\mu}. \end{aligned} \tag{3.74}$$

On the other hand, multiplying (3.65) by $e^{\mu t}$ then integrating from 0 to t and using (3.63), one obtains that

$$\begin{aligned} &\frac{1}{2}\int_0^t \left[e^{\mu\tau}\frac{d}{d\tau}\int_\Omega S^2 \, d\Omega\right] d\tau + \mu\int_0^t \left[e^{\mu\tau}\int_\Omega S^2 \, d\Omega\right] d\tau + \\ &+\gamma_1\int_0^t \left[e^{\mu\tau}\int_\Omega |\nabla S|^2 \; d\Omega\right] d\tau \le N_0 \,|\Omega|\, M_1 e^{\mu t}. \end{aligned}$$

Integrating by parts, one has that

$$\begin{aligned} \int_0^t \left[e^{\mu\tau}\frac{d}{d\tau}\int_\Omega S^2 \, d\Omega\right] d\tau \;&=\; e^{\mu t}\int_\Omega S^2 \, d\Omega - \int_\Omega S_0^2 \, d\Omega + \\ &-\; \mu\int_0^t \left[e^{\mu\tau}\int_\Omega S^2 \, d\Omega\right] d\tau; \end{aligned} \tag{3.75}$$

therefore, from (3.75) it turns out that

$$2\gamma_1\int_0^t \left[e^{\mu\tau}\int_\Omega |\nabla S|^2 \; d\Omega\right] d\tau \le \int_\Omega S_0^2 \, d\Omega + 2N_0 M_1 \,|\Omega|\, e^{\mu t}, \;\; \forall t \ge 0. \tag{3.76}$$

By virtue of (3.76) and (3.66), (3.74) implies that

$$\begin{aligned} \int_\Omega V^2 \, d\Omega \;&\le\; \int_\Omega \left(S_0^2 + E_0^2\right) \, d\Omega + \frac{(\gamma_1-\gamma_2)^2}{2\gamma_1\gamma_2}\int_\Omega S_0^2 \, d\Omega + \\ &+\; \frac{N_0 \,|\Omega|\, M_1 (\gamma_1-\gamma_2)^2}{2\gamma_1\gamma_2} + \frac{m_1}{\mu}, \;\; \forall t \ge 0. \end{aligned}$$

Setting

$$M_2 = \int_\Omega \left(S_0^2 + E_0^2\right) \, d\Omega + \frac{(\gamma_1-\gamma_2)^2}{2\gamma_1\gamma_2}\int_\Omega S_0^2 \, d\Omega + \frac{N_0 \,|\Omega|\, M_1(\gamma_1-\gamma_2)^2}{2\gamma_1\gamma_2} + \frac{m_1}{\mu},$$

one obtains that

$$\int_\Omega E^2 \, d\Omega \le \int_\Omega (S+E)^2 \, d\Omega = \int_\Omega V^2 \, d\Omega \le M_2, \ \forall t \ge 0. \tag{3.77}$$

Now, multiplying $(3.1)_3$ by I, integrating over Ω and using the divergence theorem and the boundary conditions (3.55), one obtains that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega I^2 \, d\Omega = -(\sigma+\mu)\int_\Omega I^2 \, d\Omega + \theta \int_\Omega EI \, \Omega - \gamma_2 \int_\Omega |\nabla I|^2 \, d\Omega; \tag{3.78}$$

by virtue of the generalized Cauchy inequality and of (3.77), (3.78) implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega I^2 \, d\Omega \le \frac{\theta^2}{2(\sigma+\mu)}\int_\Omega E^2 \, d\Omega - \frac{\sigma+\mu}{2}\int_\Omega I^2 \, d\Omega \le \frac{\theta^2 M_2}{2(\sigma+\mu)} - \frac{\sigma+\mu}{2}\int_\Omega I^2 \, d\Omega.$$

Hence, it follows that

$$\int_\Omega I^2 \, d\Omega \le M_3 \quad \forall t \ge 0 \tag{3.79}$$

being

$$M_3 = \int_\Omega I_0^2 \, d\Omega + \frac{\theta^2 M_2}{(\sigma+\mu)^2}.$$

Following the same procedure for the forth equation $(3.1)_4$, one has that

$$\int_\Omega R^2 \, d\Omega \le M_4 \quad \forall t \ge 0 \tag{3.80}$$

where

$$M_4 = \int_\Omega R_0^2 \, d\Omega + \frac{\theta^2 M_2}{\mu^2}.$$

In view of (3.66), (3.77), (3.79) and (3.80) and applying Lemma 9 with $p_0 = 2$, the thesis follows.

Let us denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively. The following theorem holds.

Theorem 19 *Every ball of radius bigger than* $\dfrac{\bar{b}}{\bar{a}}$ *is an absorbing set, with*

$$\begin{cases} \bar{a} = 2\mu, \\ \bar{b} = 2|\Omega| C^2 \left(\mu + \theta + \sigma + K(1 + \alpha C)\right). \end{cases}$$

Proof. Multiplying $(3.1)_1$ for S, $(3.1)_2$ for E, $(3.1)_3$ for I and $(3.1)_4$ for R, integrating over Ω and adding member by member, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|S\|^2+\|E\|^2+\|I\|^2+\|R\|^2\right) = \mu N_0 \int_\Omega S\, d\Omega - \mu\|S\|^2 + \gamma_1 \int_\Omega S\Delta S\, d\Omega +$$

$$-K\int_\Omega IS^2(1+\alpha I)\, d\Omega - (\theta+\mu)\,\|E\|^2 + \gamma_2 \int_\Omega E\Delta E\, d\Omega +$$

$$+K\int_\Omega ISE(1+\alpha I)\, d\Omega - (\sigma+\mu)\,\|I\|^2 + \theta\int_\Omega EI\, d\Omega +$$

$$+\gamma_3\int_\Omega I\Delta I\, d\Omega + \sigma\int_\Omega IR\, d\Omega - \mu\,\|R\|^2 + \gamma_4\int_\Omega R\Delta R\, d\Omega.$$

Let us define

$$H = \|S\|^2+\|E\|^2+\|I\|^2+\|R\|^2.$$

Applying the divergence theorem, taking into account of the boundary conditions (3.55) and by virtue of (3.61), one obtains

$$\frac{dH}{dt} \leq -\bar{a}H + \bar{b}$$

and hence, following the same procedure used in Theorem 12, Theorem 19 holds.

3.2.4 Equilibria and preliminaries to stability

Let us denote by R_0 the *basic reproduction number* in absence of diffusion,

$$R_0 = \frac{N_0\theta K}{(\theta+\mu)(\sigma+\mu)}$$

and let us set

$$R_1^* = \sqrt{\frac{K}{\alpha\mu}}, \qquad R_2^* = 2R_1^* - {R_1^*}^2.$$

The biologically meaningful equilibria $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ of (3.1) are the non-negative solutions of the system

$$\begin{cases} \mu(N_0 - \bar{S}) - K\bar{I}\bar{S}(1+\alpha\bar{I}) = 0, \\ -(\theta+\mu)\bar{E} + K\bar{I}\bar{S}(1+\alpha\bar{I}) = 0, \\ -(\sigma+\mu)\bar{I} + \theta\bar{E} = 0, \\ \sigma\bar{I} - \mu\bar{R} = 0. \end{cases}$$

The following two kinds of solutions arise.

i) DISEASE-FREE EQUILIBRIUM:
System (3.1) admits the equilibrium $(S_1, E_1, I_1, R_1) = (N_0, 0, 0, 0)$ which - from biological point of view - means that no infection arises.

ii) ENDEMIC EQUILIBRIA:
The biologically meaningful equilibria (endemic equilibria) of (3.1) are the positive stationary constant solutions $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ of (3.1).

The following theorem holds.

Theorem 20 *System (3.1)*

i) always admits the disease free equilibrium $(S_1, E_1, I_1, R_1) = (N_0, 0, 0, 0)$*;*

ii) admits a unique endemic equilibrium if

$$R_0 > 1, \tag{3.81}$$

or

$$\begin{cases} R_0 = 1, \\ R_1^* < 1; \end{cases} \tag{3.82}$$

or

$$\begin{cases} R_0 = R_2^* < 1, \\ R_1^* < 1; \end{cases} \tag{3.83}$$

iii) admits two endemic equilibria if

$$\begin{cases} R_2^* < R_0 < 1, \\ R_1^* < 1; \end{cases} \tag{3.84}$$

iv) does not admit any endemic equilibrium in the other cases.

Proof. For the proof, see that one of Theorem 13.

Let (3.81) or (3.82) or (3.83) holds true. In this case (3.1) admits a unique endemic equilibrium $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$. Setting

$$X_1 = S - \bar{S}, \quad X_2 = E - \bar{E}, \quad X_3 = I - \bar{I}, \quad X_4 = R - \bar{R},$$

(3.1) reduces to

$$\begin{cases} \dfrac{\partial X_1}{\partial t} = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + a_{14}X_4 + \gamma_1 \Delta X_1 - F(X_1, X_3) \\ \dfrac{\partial X_2}{\partial t} = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + \gamma_2 \Delta X_2 + F(X_1, X_3) \\ \dfrac{\partial X_3}{\partial t} = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + a_{34}X_4 + \gamma_3 \Delta X_3 \\ \dfrac{\partial X_4}{\partial t} = a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + a_{44}X_4 + \gamma_4 \Delta X_4 \end{cases} \tag{3.85}$$

where

$$\begin{cases} a_{11} = -[\mu + K\bar{I}(1 + \alpha\bar{I})], \; a_{12} = 0, \; a_{13} = -K\bar{S}(1 + 2\alpha\bar{I}), \; a_{14} = 0, \\ a_{21} = K\bar{I}(1 + \alpha\bar{I}), \; a_{22} = -(\theta + \mu), \; a_{23} = K\bar{S}(1 + 2\alpha\bar{I}), \; a_{24} = 0, \\ a_{31} = 0, \;\; a_{32} = \theta, \;\; a_{33} = -(\sigma + \mu), \; a_{34} = 0, \\ a_{41} = 0, \;\; a_{42} = 0, \;\; a_{43} = \sigma, \; a_{44} = -\mu, \\ F(X_1, X_3) = KX_3[(1 + 2\alpha\bar{I})X_1 + \alpha X_3(\bar{S} + X_1)]. \end{cases} \tag{3.86}$$

To (3.85)-(3.86) we add the boundary conditions

$$\nabla X_i \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ \quad i = 1, 2, 3, 4. \tag{3.87}$$

Denoting by $W^*(\Omega)$ the functional space defined by

$$W^*(\Omega) = \left\{ \varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega) : \frac{d\varphi}{d\mathbf{n}} = 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \ \int_\Omega \varphi \, d\Omega = 0 \right\},$$

our aim is to study the stability of $(\bar{S}, \bar{E}, \bar{I}, \bar{R})$ with respect to the perturbations $(X_1, X_2, X_3, X_4) \in [W^*(\Omega)]^4$.

3.2.5 Linear stability of endemic equilibrium

Remark 11 *We remark that the infimum*

$$\bar{\alpha}(\Omega) = \inf_{\varphi \in W^*(\Omega)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2}, \tag{3.88}$$

exists and is a real positive number ([4], [67]);

Adding and subtracting the term $\bar{\alpha}\gamma_i X_i$ to equation $(3.85)_i$, $(i = 1, 2, 3, 4)$, and setting

$$\mathcal{L}^* = \begin{pmatrix} a_{11} - \bar{\alpha}\gamma_1 & 0 & a_{13} & 0 \\ a_{21} & a_{22} - \bar{\alpha}\gamma_2 & a_{23} & 0 \\ 0 & a_{32} & a_{33} - \bar{\alpha}\gamma_3 & 0 \\ 0 & 0 & a_{43} & a_{44} - \bar{\alpha}\gamma_4 \end{pmatrix},$$

in order to analyze the linear stability of the endemic equilibrium, and hence of the null solution of the perturbation system (3.85), we have to find conditions ensuring that all $\mathcal{L}^*$ eigenvalues have negative real parts. The characteristic equation of $\mathcal{L}^*$ is

$$[\lambda - (a_{44} - \bar{\alpha}\gamma_4)](\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3) = 0, \tag{3.89}$$

where $\mathtt{I}_1$, $\mathtt{I}_2$, $\mathtt{I}_3$ are the principal invariants of the matrix

$$\mathcal{J}^* = \begin{pmatrix} a_{11}-\bar{\alpha}\gamma_1 & 0 & a_{13} \\ a_{21} & a_{22}-\bar{\alpha}\gamma_2 & a_{23} \\ 0 & a_{32} & a_{33}-\bar{\alpha}\gamma_3 \end{pmatrix}$$

and are given by

$$\begin{cases} \mathtt{I}_1 = \mathrm{trace}\mathcal{J}^* = a_{11}+a_{22}+a_{33}-\bar{\alpha}(\gamma_1+\gamma_2+\gamma_3), \\ \mathtt{I}_2 = \begin{vmatrix} a_{11}-\bar{\alpha}\gamma_1 & 0 \\ a_{21} & a_{22}-\bar{\alpha}\gamma_2 \end{vmatrix} + \begin{vmatrix} a_{11}-\bar{\alpha}\gamma_1 & a_{13} \\ 0 & a_{33}-\bar{\alpha}\gamma_3 \end{vmatrix} + \begin{vmatrix} a_{22}-\bar{\alpha}\gamma_2 & a_{23} \\ a_{32} & a_{33}-\bar{\alpha}\gamma_3 \end{vmatrix} \\ \mathtt{I}_3 = \det\mathcal{J}^* = \begin{vmatrix} a_{11}-\bar{\alpha}\gamma_1 & 0 & a_{13} \\ a_{21} & a_{22}-\bar{\alpha}\gamma_2 & a_{23} \\ 0 & a_{32} & a_{33}-\bar{\alpha}\gamma_3 \end{vmatrix}. \end{cases}$$

Accounting for (3.89), the eigenvalues of $\mathcal{L}^*$ are given by the three roots of the characteristic equation of $\mathcal{J}^*$

$$\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3 = 0 \tag{3.90}$$

and by

$$a_{44}-\bar{\alpha}\gamma_4,$$

where, in view of (3.86), $a_{44}-\bar{\alpha}\gamma_4 < 0$. Passing now to the equation (3.90), as it is well known, the necessary and sufficient conditions guaranteeing that all the roots of (3.90) have negative real part are the Routh-Hurwitz conditions [39]

$$\mathtt{I}_1 < 0, \quad \mathtt{I}_3 < 0, \quad \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 < 0. \tag{3.91}$$

Obviously (3.91) require necessarily that $\mathtt{I}_2 > 0$. If one of (3.91) is reversed, then there exists at least one eigenvalue of $\mathcal{J}^*$ with positive real part and hence the null solution of (3.85) is linearly unstable. Denoting by $\mathtt{I}^*$, A^* the principal invariants of the matrix $\begin{pmatrix} a_{22} - \bar{\alpha}\gamma_2 & a_{23} \\ a_{32} & a_{33} - \bar{\alpha}\gamma_3 \end{pmatrix}$, i.e.

$$\begin{aligned} \mathtt{I}^* &= a_{22} + a_{33} - \bar{\alpha}(\gamma_2 + \gamma_3), \\ A^* &= \bar{\alpha}^2\gamma_2\gamma_3 - \bar{\alpha}(\gamma_2 a_{33} + \gamma_3 a_{22}) + a_{22}a_{33} - a_{23}a_{32}, \end{aligned} \tag{3.92}$$

it follows that

$$\begin{cases} \mathtt{I}_1 = a_{11} - \bar{\alpha}\gamma_1 + \mathtt{I}^*, \;\; \mathtt{I}_2 = (a_{11} - \bar{\alpha}\gamma_1)\mathtt{I}^* + A^*, \\ \mathtt{I}_3 = (a_{11} - \bar{\alpha}\gamma_1)A^* + a_{13}a_{21}a_{32}, \\ \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 = (a_{11} - \bar{\alpha}\gamma_1 + \mathtt{I}^*)(a_{11} - \bar{\alpha}\gamma_1)\mathtt{I}^* + A^*\mathtt{I}^* - a_{13}a_{21}a_{32}. \end{cases} \tag{3.93}$$

The following theorem holds.

Theorem 21 *If*

$$A^* > 0, \tag{3.94}$$

then the endemic equilibrium is linearly stable.

Proof Let us remark that in view of (3.93) and (3.86), if $A^* > 0$, then the Routh Hurwitz conditions (3.91) are verified and the thesis follows.

3.2.6 Nonlinear stability of endemic equilibrium

Introducing the scalings μ_i $(i = 1,2,3,4)$ (μ_i are positive constants to be chosen suitably later) and setting

$$\begin{cases} X_i = \mu_i U_i, \qquad \mathbf{U} = (U_1, U_2, U_3, U_4)^T, \quad \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4)^T, \\ \bar{F} = F(\mu_1 U_1, \mu_3 U_3) = K\mu_3 U_3[(1+2\alpha\bar{I})\mu_1 U_1 + \alpha\mu_3 U_3(\bar{S} + \mu_1 U_1)], \\ \tilde{F}_1 = -\dfrac{1}{\mu_1}\bar{F} + \gamma_1(\Delta U_1 + \bar{\alpha} U_1), \ \tilde{F}_2 = \dfrac{1}{\mu_2}\bar{F} + \gamma_2(\Delta U_2 + \bar{\alpha} U_2), \\ \tilde{F}_3 = \gamma_3(\Delta U_3 + \bar{\alpha} U_3), \ \tilde{F}_4 = \gamma_4(\Delta U_4 + \bar{\alpha} U_4), \end{cases} \tag{3.95}$$

(3.85) reduces to

$$\frac{\partial \mathbf{U}}{\partial t} = \tilde{L}\mathbf{U} + \tilde{\mathbf{F}}, \tag{3.96}$$

where $\tilde{L}$ is the Jacobian matrix

$$\tilde{L} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ 0 & b_{32} & b_{33} & 0 \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}$$

with

$$b_{ii} = a_{ii} - \bar{\alpha}\gamma_i, \qquad b_{ij} = \frac{\mu_j}{\mu_i} a_{ij}, \ i \neq j. \tag{3.97}$$

The boundary conditions (3.87) become

$$\nabla U_i \cdot \mathbf{n} = 0, \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \quad i = 1,2,3,4. \tag{3.98}$$

Hence the problem to find conditions guaranteeing the stability of ($\bar{S}$, $\bar{E}$, $\bar{I}$, $\bar{R}$) is reduced to determine conditions guaranteeing the stability of the null solution of (3.96)-(3.98). Remarking that, by virtue of (3.92) and (3.97), one obtains that $A^* = b_{22}b_{33} - b_{23}b_{32} = b_{22}b_{33} - a_{23}a_{32}$ and setting

$$\begin{cases} A_1 = A^* + (b_{32})^2 + (b_{33})^2 \\ A_2 = A^* + (b_{22})^2 + (b_{23})^2 \\ A_3 = b_{22}b_{32} + b_{23}b_{33}, \end{cases}$$

in order to study the nonlinear stability of endemic equilibrium, let us introduce the Rionero-Liapunov functional (see [54], [55] for details)

$$W = \frac{1}{2}\|U_1\|^2 + V + \frac{1}{2}\|U_4\|^2,$$

where

$$V = \frac{1}{2}\left[A^*(\|U_2\|^2 + \|U_3\|^2) + \|b_{22}U_3 - b_{32}U_2\|^2 + \|b_{23}U_3 - b_{33}U_2\|^2\right],$$

which is positive definite if $A^* > 0$.

The time derivative of W along the solutions of (3.96) is

$$\begin{aligned} \dot{W} &= b_{11}\|U_1\|^2 + I^*A^*(\|U_2\|^2 + \|U_3\|^2) + b_{44}\|U_4\|^2 + A_1b_{21}\langle U_1, U_2\rangle + \\ &+ (-A_3b_{21} + b_{13})\langle U_1, U_3\rangle + b_{43}\langle U_3, U_4\rangle + \Phi_1 + \Phi_2, \end{aligned} \tag{3.99}$$

being

$$\begin{cases} \Phi_1 = \gamma_1 < U_1, \Delta U_1 + \bar{\alpha}U_1 > + < A_1U_2 - A_3U_3, \gamma_2(\Delta U_2 + \bar{\alpha}U_2) > + \\ \quad + < A_2U_3 - A_3U_2, \gamma_3(\Delta U_3 + \bar{\alpha}U_3) > + \gamma_4 < U_4, \Delta U_4 + \bar{\alpha}U_4 >, \\ \Phi_2 = \dfrac{1}{\mu_1} < U_1, -\bar{F} > + \dfrac{1}{\mu_2} < A_1U_2, \bar{F} > + \dfrac{1}{\mu_2} < A_3U_3, -\bar{F} >, \end{cases} \tag{3.100}$$

with $\bar{F}$ given by $(3.95)_3$.

The following Lemmas hold.

Lemma 10 *If*

$$A^* > 0 \quad \textit{and} \quad (\gamma_2 + \gamma_3)\,|A_3| < 2\sqrt{A_1A_2\gamma_2\gamma_3}, \tag{3.101}$$

then there exists $\epsilon_1 \in (0,1)$ *such that* $\forall \epsilon \in (0,\epsilon_1)$

$$\Phi_1 \leq -\gamma_1 \|\nabla U_1\|^2 + \bar{\alpha}\gamma_1 \|U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 .$$

Proof. By using the divergence theorem, by virtue of the boundary conditions (3.98) and in view of Poincaré inequality (3.88), from (3.100) it follows that

$$\Phi_1 \leq -\gamma_1 \|\nabla U_1\|^2 + \bar{\alpha}\gamma_1 \|U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 + \Phi^*,$$

being

$$\begin{aligned}\Phi^* =& -A_1\gamma_2 \|\nabla U_2\|^2 - |A_3| (\gamma_2+\gamma_3) \langle \nabla U_2, \nabla U_3\rangle - A_2\gamma_3(1-\epsilon) \|\nabla U_3\|^2 + \\ &+ A_1\gamma_2\bar{\alpha} \|U_2\|^2 + |A_3| \bar{\alpha}(\gamma_2+\gamma_3) \langle U_2, U_3\rangle + A_2\gamma_3(1-\epsilon)\bar{\alpha} \|U_3\|^2 .\end{aligned}$$

Since $(3.101)_2$ implies that there exists $\epsilon_1 \in (0,1)$ such that $\forall \epsilon \in (0,\epsilon_1)$

$$|A_3| (\gamma_2+\gamma_3) = 2\sqrt{(1-\epsilon_1)\gamma_2\gamma_3 A_1 A_2}, \quad |A_3| (\gamma_2+\gamma_3) \leq 2\sqrt{(1-\epsilon)\gamma_2\gamma_3 A_1 A_2},$$

by following the same procedure used for the proof of Lemma 3.2 in [54], one obtains that $\Phi^* \leq 0$ and hence the thesis follows.

Lemma 11 *There exists a positive constant* $M(\Omega)$ *such that*

$$\Phi_2 \leq M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}} (\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).$$

Proof. In view of $(3.100)_2$ and $(3.95)_2$ and since from (3.61) and $(3.95)_1$ one obtains

$$U_i \leq \frac{C}{\mu_i}, \quad i = 1,2,3$$

it turns out that

$$\begin{aligned}\Phi_2 \leq & c_1 < U_1^2, |U_3| > + c_2 < |U_1|, U_3^2 > + c_3 < |U_2|, |U_1 U_3| > + \\ &+ c_4 < |U_2|, U_3^2 > + c_5 < |U_3|, U_3^2 >,\end{aligned} \tag{3.102}$$

where

$$\begin{cases} c_1 = k(1+2\alpha\bar{I})\mu_3, \\ c_2 = k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_1} + A_1 k\alpha C\dfrac{\mu_1\mu_3^2}{\mu_2^2} + |A_3|\, k(1+2\alpha\bar{I})\dfrac{\mu_1\mu_3}{\mu_2} + |A_3|\, k\alpha C\dfrac{\mu_1\mu_3}{\mu_2}, \\ c_3 = A_1 k(1+2\alpha\bar{I})\dfrac{\mu_1\mu_3}{\mu_2},\ c_4 = A_1 k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_2},\ c_5 = |A_3|\, k\alpha\bar{S}\dfrac{\mu_3^2}{\mu_2}. \end{cases}$$

By virtue of the Hölder and embedding inequalities

$$< |f|, g^2 >\leq \|f\|\,\|g\|_4^2, \quad \|g\|_4^2 \leq K_1(\Omega)[\|\nabla g\|^2 + \|g\|^2], \quad K_1(\Omega) > 0,$$

and in view of Cauchy inequality, from (3.102) it follows that

$$\begin{aligned} \Phi_2 \ \leq\ & \eta_1 \|U_3\| \left(\|U_1\|^2 + \|\nabla U_1\|^2\right) + \eta_2 \|U_1\| \left(\|U_3\|^2 + \|\nabla U_1\|^2\right) + \\ +\ & \eta_3 \|U_2\| \left(\|U_1\|^2 + \|\nabla U_1\|^2\right) + \eta_4 \|U_2\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right) + \\ +\ & \eta_5 \|U_3\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right), \end{aligned}$$

being

$$\eta_i = K_1(\Omega)c_i,\ i = 1,2,5, \quad \eta_3 = \frac{1}{2}c_3 K_1(\Omega), \quad \eta_4 = \left(\frac{1}{2}c_3 + c_4\right) K_1(\Omega).$$

Hence the thesis follows with $M(\Omega) = \max\limits_{i=1,\dots,5} \eta_i$.

Remark 12 *We remark that, setting*

$$p = \frac{A^*}{2}, \quad q = \frac{A^*}{2} + \left[(b_{22})^2 + (b_{23})^2 + (b_{32})^2 + (b_{33})^2\right],$$

it follows that

$$p(\|U_2\|^2 + \|U_3\|^2) \leq V \leq q(\|U_2\|^2 + \|U_3\|^2). \tag{3.103}$$

The following lemma holds.

Lemma 12 *Setting*

$$\zeta_1 = \left[1 + \frac{\bar{\alpha}^2\gamma_2\gamma_3 + \bar{\alpha}\gamma_2(\sigma+\mu) + \bar{\alpha}\gamma_3(\theta+\mu)}{(\theta+\mu)(\sigma+\mu)}\right] \frac{N_0}{\bar{S}(1+2\alpha\bar{I})}, \tag{3.104}$$

then

$$A^* > 0 \Leftrightarrow R_0 < \zeta_1. \tag{3.105}$$

Proof. By virtue of (3.92), (3.86) and (3.104), one has that

$$A^* = \frac{(\theta+\mu)(\sigma+\mu)}{N_0}\bar{S}(1+2\alpha\bar{I})[\zeta_1 - R_0],$$

and hence (3.105) immediately follows.

The following theorem holds.

Theorem 22 *If and only if*

$$R_0 < \zeta_1 \quad \textit{and} \quad (\gamma_2+\gamma_3)\,|A_3| < 2\sqrt{A_1 A_2 \gamma_2 \gamma_3}, \tag{3.106}$$

then the null solution of (3.85)-(3.87) is (locally) nonlinearly, asymptotically stable with respect to the $L^2(\Omega)$*-norm.*

Proof. By virtue of Lemma 10 and Lemma 11, in view of (3.97) and by using the generalized Cauchy inequality, from (3.99) it follows that

$$\begin{aligned}\dot{W} \leq\; & -|a_{11}|\,\|U_1\|^2 - |I^*|A^*(\|U_2\|^2+\|U_3\|^2) - |a_{44}|\|U_4\|^2 + \\ & -\gamma_1\,\|\nabla U_1\|^2 - A_2\gamma_3\epsilon\,\|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon\,\|U_3\|^2 + \\ & + \frac{A_1^2 a_{21}^2\mu_1^2}{2\mu_2^2|I^*|A^*}\,\|U_1\|^2 + \frac{1}{2}|I^*|A^*\,\|U_2\|^2 + \\ & + \;(-A_3 b_{21}+b_{13})\,\langle U_1, U_3\rangle + \frac{a_{43}^2\mu_3^2}{2|a_{44}|\mu_4^2}\,\|U_3\|^2 + \frac{1}{2}|a_{44}|\,\|U_4\|^2 + \\ & + M(\Omega)(\|U_1\|^2+\|U_2\|^2+\|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2+\|\nabla U_3\|^2+\|U_1\|^2+\|U_3\|^2).\end{aligned} \tag{3.107}$$

Choosing the positive scalings such that

$$\begin{cases} -A_3 b_{21} + b_{13} = 0 \\ |a_{11}| - \dfrac{A_1^2 a_{21}^2 \mu_1^2}{2\mu_2^2|I^*|A^*} > \dfrac{1}{2}|a_{11}| \\ |I^*|A^* - \dfrac{a_{43}^2\mu_3^2}{2|a_{44}|\mu_4^2} > \dfrac{1}{2}|I^*|A^* \end{cases}$$

i.e.

$$\begin{cases} \mu_1^2 = \dfrac{a_{13}\mu_2^2\mu_3^2}{a_{21}\left(b_{22}a_{32}\mu_2^2 + b_{33}a_{23}\mu_3^2\right)} \\[4ex] \mu_2^2 > \dfrac{\left(A^* + b_{33}^2\right)a_{21}^2\mu_1^2\mu_3^2}{|I^*|A^*|a_{11}|\mu_3^2 + a_{32}^2a_{21}^2\mu_1^2} \\[4ex] \mu_4^2 > \dfrac{|a_{43}|^2}{|a_{44}||I^*|A^*}\mu_3^2 \end{cases}$$

and choosing

$$\epsilon < \min\left\{\frac{|I^*|A^*}{4A_2\bar{\alpha}\gamma_3}, \epsilon_1\right\},$$

from (3.107) it turns out that

$$\begin{aligned} \dot{W} \leq & -\frac{1}{2}|a_{11}|\|U_1\|^2 - \frac{1}{4}|I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - \frac{1}{2}|a_{44}|\,\|U_4\|^2 + \\ & -\gamma_1\,\|\nabla U_1\|^2 - A_2\gamma_3\epsilon\,\|\nabla U_3\|^2 + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2). \end{aligned}$$

Therefore, setting

$$h_1 = \min\left\{\mu, \frac{A^*|I^*|}{2}\right\}, \quad h_2 = \min\{\gamma_1, \epsilon A_2\gamma_3\},$$

one has that

$$\begin{aligned} \dot{W} \leq & -\frac{h_1}{2}(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2 + \|U_4\|^2) - h_2(\|\nabla U_1\|^2 + \|\nabla U_3\|^2) + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2). \end{aligned}$$

By virtue of (3.103), it turns out that

$$\dot{W} \leq -(\delta_1 - \delta_3 W^{\frac{1}{2}})W - (h_2 - \delta_2 W^{\frac{1}{2}})(\|\nabla U_1\|^2 + \|\nabla U_3\|^2),$$

with

$$\delta_1 = h_1 \min\{1,\ 1/2q\},\ \delta_2 = M(\Omega)\max\{\sqrt{2},\ p^{-1/2}\},\ \delta_3 = M(\Omega)\max\{2\sqrt{2}, p^{-3/2}\}.$$

Hence, if

$$W^{\frac{1}{2}}(0) < \min\left\{\delta_1/\delta_3,\, h_2/\delta_2\right\},$$

applying recursive arguments, it follows that

$$\dot{W} \le -\tilde{K}W, \qquad \tilde{K} = \text{const.} > 0$$

and hence the thesis follows.

3.3 A reaction-diffusion system modeling Cholera dynamic under mixed boundary conditions

3.3.1 Introduction

In this section the problem studied in [16], aimed to analyze the spread of Cholera in an heterogeneous environment, is reconsidered.
Cholera is an acute intestinal infection caused by the bacterium *Vibrio cholerae* (*V. cholerae*). The mechanism of transmission occurs, principally, via ingestion of contaminated food or water and, secondarily but more rarely, via direct human-to-human contacts [65]. In the developed world, seafood (in particular consuming contaminated oysters and shellfish) is the usual cause, while in the developing world it is more often water. Generally, the incubation period lasts from less than one day to five days. Symptoms are watery diarrhea and vomiting that can quickly lead to severe dehydration and death if treatment is not promptly given. Without treatment the case-fatality rate for severe cholera is about 50%, [63]. Only 1% to 30% of *V. cholerae* infections develop into severe cholera cases, [64]. People with lowered immunity (for example people with AIDS or malnourished children) are more likely to

experience a severe case if they become infected.
Cholera is endemic in many parts of the world such as Asia, India, Africa and Latin America. It affects 3-5 million people and causes $100.000-130.000$ deaths a year as of 2010, [71]. The primary therapy consists in re-hydrating infected people in order to replace contaminated water in the organism and correct electrolyte imbalance. However, prevention strategy is strongly recommended by the World Health Organization (WHO). It provides water purification, sterilization of all materials that come in contact with cholera patients, improvements in sanitation systems and in personal hygiene. These measurements minimize human contact with contaminated water and consequently spread of the epidemic. Till now, the preventive care consists in active immunization by mean of vaccines. Injectable vaccines are given by two intramuscular or subcutaneous inoculations. Protection lasts not more than six months and it is not complete. Because of the high side effects, this kind of care is actually deterred. Oral vaccines are available by two preparations. The first (*Orochol*) can be given to people being more than two years old. Efficiency is for 60-90%, it starts after seven days and can last up to two years (boosters have to be given every six months). The second preparation (*Cholerix*) is given in two doses far-between two weeks. Efficiency is in 65%.

In order to study infectious diseases transmission, the mathematical models play a central role. In fact, although they represent only an approximation of the problem (they consider only some variables that are involved in the phenomenon), they allow to obtain estimation about the spread of epidemics. In this way it is possible to predict the asymptotic behaviour of infection and, consequentially, to take some actions in order to control epidemics. When a population is not infected by a disease, all the individuals are regarded as susceptibles. Introducing a few number of infected in the community, in order to know if the epidemic will die out or if it will blow up, it would be useful to study the stability of the so called *disease-free* equilibrium. If the disease-free equilibrium is stable, then epidemic will decay. In general, the problem to determine if *endemic equilibria* (i.e. equilibria with positive components) exist, arises. When endemic equilibria exist, their stability analysis allows to state if epidemic will persist. A lot of mathematical models for infectious diseases are devoted to study cholera outbreak in different parts of the world. In particular Capasso and Paveri-Fontana in [8] studied the cholera epidemic in Bari (Italy) in 1973 by introducing a system modeling the evolution of infected people in the community and the dynamics of the aquatic population of pathogenic bacteria. In fact, cholera diffusion is strictly linked to the interactions between individuals in community and bacteria in contaminated water. Successively, Capasso and Maddalena in [9], in order to let the model be more realistic, assumed that the bacteria diffuse randomly in the habitat. Hence they analyzed a model consisting in two nonlinear parabolic equations under boundary conditions of the third type. Many studies (see, for example, [20]) found that toxigenic *V. cholerae* can survive in some aquatic environments for month to years. This suggests to believe that the aquatic environment may be a reservoir of toxigenic *V. cholerae* in endemic regions. Codeco in [19] analyzed the role of aquatic reservoir in promoting cholera outbreak by introducing an ODE model that includes the dynamics of the susceptible population. Three possible scenario, when cholera comes into a new place, have been analyzed: no outbreak (cholera-free); an outbreak followed by few waves (epidemic pattern); an outbreak followed by subsequent outbreaks that can assume a seasonal pattern (endemic pattern). Tian and Wang in [71] introduced a fourth equation in order to study the evolution of

removed individuals.
In [9], the above mentioned models have been generalized taking into account of non-homogeneously mixed toxigenic *V. cholerae* reservoir in contaminated water and dividing the total population in three disjointed and not homogeneously mixed classes (susceptibles - infected - removed) in order to study - among other things - the role of diffusivity of each population on the model dynamics.

3.3.2 Mathematical model

Let $\Omega \subset \mathbb{R}^3$ be a smooth convex domain in which cholera is diffusing. Let us suppose that the population is divided in three disjointed classes: S, the susceptibles; I, the infected; R the removed and let us denote by B the concentration of toxigenic *V. cholerae* in water (cells/ml). The physics of the problem leads to suppose that S, I, R, B are positive, smooth functions. Further, we suppose that these functions depend on time as well as on space. The reaction-diffusion equations which, as far as we know, appear to be new in the existing literature and govern cholera disease, are

$$\begin{cases} \dfrac{\partial S}{\partial t} = \mu(N_0 - S) + \gamma_1 \Delta S - \beta\lambda(B)S, \\ \\ \dfrac{\partial I}{\partial t} = \beta\lambda(B)S - (\sigma + \mu)I + \gamma_2 \Delta I, \\ \\ \dfrac{\partial B}{\partial t} = eI - (\mu_B - \pi_B)B + \gamma_3 \Delta B, \\ \\ \dfrac{\partial R}{\partial t} = \sigma I - \mu R + \gamma_4 \Delta R. \end{cases} \tag{3.108}$$

In comparison with existing models in literature, the additional diffusion terms $\gamma_1 \Delta S$, $\gamma_2 \Delta I$, $\gamma_3 \Delta B$, $\gamma_4 \Delta R$ have been introduced in order to take into account the possibility of each constituent to move in the environment. In (3.108)

$$\lambda(B) = \frac{B}{K_B + B},$$

is the probability to catch cholera, [66], where K_B (cells/ml) is the constant indicating the half saturation rate and it is linked to the concentration of *V. cholerae* in water that yields 50% chance of catching cholera. The constants appearing in (3.108) are positive and have been specified in Table 3.1. Furthermore, according to [19] and [75], it is supposed that $\mu_B > \pi_B$.

Symbols	Description	Units
N_0	total population size at time $t = 0$	person
γ_i	diffusion coefficients (i=1,2,3,4)	t^{-1} m^2
μ	birth/death rate	t^{-1}
σ	recovery rate	t^{-1}
μ_B	loss rate of bacteria	t^{-1}
π_B	growth rate of bacteria	t^{-1}
$e = \dfrac{p}{W}$	contribution of each infected person to the population of V. cholerae	cells/ml t^{-1} person^{-1}
p	rate at which bacterias are produced by an infected individual	cells t^{-1} person^{-1}
W	volume of contaminated water in infected individual	ml
β	contact rate with contaminated water	t^{-1}

Table 3.1: Description of the constants appearing in (3.108)

The diffusion coefficients γ_i, $(i = 1, 2, 3, 4)$ in model (3.108) are strictly linked to the possibility of population to move in the environment. Generally γ_i $(i = 1, 2, 3, 4)$ are such that $\gamma_i \neq \gamma_j$, $(i \neq j)$ and depend on the poor hygiene state, on the country in which the disease is developed.
Let us associate to (3.108) the following mixed boundary conditions

$$\begin{cases} S = S^*, \quad I = I^*, \quad B = B^*, \quad R = R^* \qquad \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = \nabla B \cdot \mathbf{n} = \nabla R \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^* \times \mathbb{R}^+, \end{cases} \tag{3.109}$$

with $\partial\Omega = \Sigma \cup \Sigma^*$, $\Sigma \cap \Sigma^* = \varnothing$, $\Sigma \neq \varnothing$, $\mathbf{n}$ being the unit outward normal on Σ^* and S^*, I^*, B^*, R^* being non negative constants. In the sequel we shall assume that:

(i) $\Omega \subset \mathbb{R}^3$ is a smooth domain having the internal cone property;

(ii) $\varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega)$, $\forall\varphi \in \{S, I, R, B\}$, where $W^{1,2}(A)$ is the Sobolev space $H^1(A) = \{f \in L^2(A)/Df \in L^2(A)\}$.

To (3.108), (3.109) we associate smooth, non-negative initial data

$$\varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}) \qquad \text{with } \mathbf{x} \in \Omega, \qquad \forall\varphi \in \{S, I, B, R\}. \tag{3.110}$$

Let us define

$$N(t) = \frac{1}{|\Omega|} \int_\Omega [S(\mathbf{x}, t) + I(\mathbf{x}, t) + R(\mathbf{x}, t)] \, d\Omega,$$

the population size at time t ($|\Omega|$ is the measure of Ω). Hence

$$N_0 = \frac{1}{|\Omega|} \int_\Omega [S(\mathbf{x}, 0) + I(\mathbf{x}, 0) + R(\mathbf{x}, 0)] \, d\Omega.$$

Let us denote by $T > 0$ an arbitrary fixed time and by $\Omega_T = \Omega \times (0, T]$ the parabolic cylinder, Ω_T being the parabolic interior of $\bar{\Omega} \times [0, T]$ (i.e. Ω_T

includes the top $\Omega \times \{t = T\}$). Let be $\Gamma_T = \partial\Omega \times (0, T)$. Therefore, the parabolic boundary of Ω_T, $\tilde{\Gamma}_T = \Gamma_T \cup \{\partial\Omega \times \{t = 0\}\}$, includes the bottom and vertical sides of $\Omega \times [0, T]$, but not the top. The following theorem holds.

Theorem 23 *Model (3.108)-(3.110) admits a unique positive solution in* $C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$,

Proof. For the proof we refer to [47].

Theorem 24 *Let* $(S, I, B, R) \in [C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)]^4$ *be a non negative solution of (3.108)-(3.110). Then* $\forall \varphi \in \{S, I, B, R\}$, φ *is bounded according to*

$$\begin{cases} S(\mathbf{x}, t) \leq \max\left\{N_0, \max\limits_{\bar{\Omega}} S_0(\mathbf{x}), S^*\right\} := M_1, \\ \\ I(\mathbf{x}, t) \leq \max\left\{\dfrac{\beta}{\sigma + \mu} M_1, \max\limits_{\bar{\Omega}} I_0(\mathbf{x}), I^*\right\} := M_2, \\ \\ B(\mathbf{x}, t) \leq \max\left\{\dfrac{e}{\mu_B - \pi_B} M_2, \max\limits_{\bar{\Omega}} B_0(\mathbf{x}), B^*\right\} := M_3, \\ \\ R(\mathbf{x}, t) \leq \max\left\{\dfrac{\sigma}{\mu} M_2, \max\limits_{\bar{\Omega}} R_0(\mathbf{x}), R^*\right\} := M_4. \end{cases} \tag{3.111}$$

Proof. By following the procedure used in [11], let us set $\max\limits_{\bar{\Omega}_T} S = S(\mathbf{x}_1, t_1)$. We have to distinguish the following cases.

1) If $(\mathbf{x}_1, t_1)$ belongs to the interior of Ω_T, then $(3.108)_1$ implies that

$$\left[\frac{\partial S}{\partial t} - \mu(N_0 - S) - \gamma_1 \Delta S\right]_{(\mathbf{x}_1, t_1)} < 0. \tag{3.112}$$

Since

$$\left[\frac{\partial S}{\partial t}\right]_{(\mathbf{x}_1, t_1)} = 0, \qquad [\Delta S]_{(\mathbf{x}_1, t_1)} < 0,$$

then (3.112) can hold if and only if

$$[\mu(N_0 - S)]_{(\mathbf{x}_1,t_1)} > 0$$

and hence if and only if $S(\mathbf{x}_1, t_1) < N_0$.

2) If $(\mathbf{x}_1, t_1) \in \Gamma_T$ then $S(\mathbf{x}_1, t_1) \leq \max_{\bar{\Omega}} S_0(\mathbf{x})$.

3) If $(\mathbf{x}_1, t_1) \in \partial\Omega \times (0, T)$, then in view of the regularity of the domain Ω, since Ω verifies in any point $\mathbf{x}_0 \in \partial\Omega$ the interior ball condition, there exists an open ball $D^* \subset \Omega$ with $\mathbf{x}_0 \in \partial D^*$. If $S(\mathbf{x}_1, t_1) > N_0$, choosing the radius of D^* sufficiently small, it follows that

$$\gamma_1 \Delta S - \frac{\partial S}{\partial t} > 0, \quad \text{in } D^*,$$

and by virtue of Hopf's Lemma, [49], one obtains that

$$\left(\frac{dS}{d\mathbf{n}}\right)_{(\mathbf{x}_1,t_1)} > 0.$$

Since $\dfrac{dS}{d\mathbf{n}} = 0$ on $\Sigma^* \times \mathbb{R}^+$, it follows that $(\mathbf{x}_1, t_1) \in \Sigma \times \mathbb{R}^+$ and hence $S(\mathbf{x}_1, t_1) = S^*$.

Now let us prove $(3.111)_2$. Let us set $\max_{\bar{\Omega}_T} I = I(\mathbf{x}_2, t_2)$. As in the previous case, we have to distinguish three cases.

1') If $(\mathbf{x}_2, t_2)$ belongs to the interior of Ω_T, then $(3.108)_2$ and $(3.111)_1$ imply that

$$\left[\frac{\partial I}{\partial t} + (\sigma + \mu)I - \beta M_1 - \gamma_2 \Delta I\right]_{(\mathbf{x}_2,t_2)} < 0. \qquad (3.113)$$

Hence, following the same procedure used in 1), one obtains that $I(\mathbf{x}_2, t_2) < \dfrac{\beta}{\sigma + \mu} M_1$.

2') If $(\mathbf{x}_2, t_2) \in \Gamma_T$ then $I(\mathbf{x}_2, t_2) \leq \max_{\bar{\Omega}} I_0(\mathbf{x})$.

3') If $(\mathbf{x}_2, t_2) \in \partial\Omega \times (0, T)$ and $I > \dfrac{\beta}{\sigma + \mu} M_1$, then, by following the same procedure used in 3), one recovers that $(3.111)_2$ holds.

Let us set $\max\limits_{\bar{\Omega}_T} B = B(\mathbf{x}_3, t_3)$ and let us distinguish three cases.

1") If $(\mathbf{x}_3, t_3)$ belongs to the interior of Ω_T, then $(3.108)_3$ and $(3.111)_2$ imply that

$$\left[\frac{\partial B}{\partial t} - eM_2 + (\mu_B - \pi_B)B - \gamma_3 \Delta B \right]_{(\mathbf{x}_3, t_3)} < 0. \tag{3.114}$$

Since

$$\left[\frac{\partial B}{\partial t} \right]_{(\mathbf{x}_3, t_3)} = 0, \qquad [\Delta B]_{(\mathbf{x}_3, t_3)} < 0,$$

then (3.114) can hold if and only if $B(\mathbf{x}_3, t_3) < \dfrac{eM_2}{\mu_B - \pi_B}$.

2") If $(\mathbf{x}_3, t_3) \in \Gamma_T$ then $B(\mathbf{x}_3, t_3) \leq \max\limits_{\bar{\Omega}} B_0(\mathbf{x})$.

3") If $(\mathbf{x}_3, t_3) \in \partial\Omega \times (0, T)$ and $B > \dfrac{eM_2}{\mu_B - \pi_B}$, then, by following the same procedure used in 3), one recovers that $(3.111)_3$ holds.

Let us prove $(3.111)_4$. Setting $\max\limits_{\bar{\Omega}_T} R = R(\mathbf{x}_4, t_4)$, let us distinguish three cases.

1"') If $(\mathbf{x}_4, t_4)$ belongs to the interior of Ω_T, then $(3.108)_4$ and $(3.111)_2$ imply that

$$\left[\frac{\partial R}{\partial t} - \sigma M_2 + \mu R - \gamma_4 \Delta R \right]_{(\mathbf{x}_4, t_4)} < 0. \tag{3.115}$$

Since

$$\left[\frac{\partial R}{\partial t} \right]_{(\mathbf{x}_4, t_4)} = 0, \qquad [\Delta R]_{(\mathbf{x}_4, t_4)} < 0,$$

then (3.115) can hold if and only if $R(\mathbf{x}_4, 3t_4) < \dfrac{\sigma M_2}{\mu}$.

2"') If $(\mathbf{x}_4, t_4) \in \Gamma_T$ then $R(\mathbf{x}_4, t_4) \leq \max\limits_{\bar{\Omega}} R_0(\mathbf{x})$.

3"') If $(\mathbf{x}_4, t_4) \in \partial\Omega \times (0, T)$ and $R > \dfrac{\sigma M_2}{\mu}$, then, by following the same procedure used in 3), one recovers that $(3.111)_4$ holds.

3.3.3 Equilibria and preliminaries to stability

A fundamental role in disease-diffusion is played by the *basic reproduction number*, usually denoted by R_0, which is linked to the ability of disease to invade a population and it is defined as "the expected number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a completely susceptible population", [23]. The basic reproduction number, for model (3.108), has been estimated by Codeco in [19] and Tian and Wang in [71] in the case of S, I, R, B depending only on time. It is given by

$$R_0 = \frac{N_0 \beta e}{K_B(\mu_B - \pi_B)(\sigma + \mu)}. \tag{3.116}$$

Hence, as one is expected, R_0 grows up with β and e, i.e. with the contact rate with contaminated water and contamination of aquatic environment of each infected person. R_0 behaviour with respect to K_B is showed in figure 3.1.

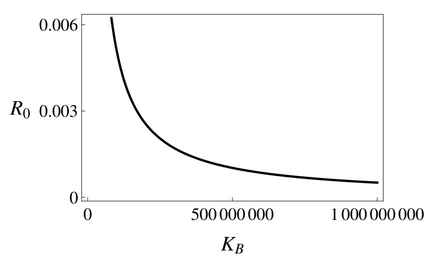

Figure 3.1: Reproduction number in the case $N_0 = 10000$, $\beta = 1$, $e = 10$, $\sigma = 0.2$, $\mu = 0.0001$, $\mu_B - \pi_B = 0.33$

The biologically meaningful equilibria of (3.108) are the non-negative solutions $(\bar{S},\ \bar{I},\ \bar{B},\ \bar{R})$ of the system

$$\begin{cases} \mu(N_0 - S) - \beta \dfrac{BS}{K_B + B} = 0, \\ \beta \dfrac{BS}{K_B + B} - (\sigma + \mu)I = 0, \\ eI - (\mu_B - \pi_B)B = 0, \\ \sigma I - \mu R = 0. \end{cases} \tag{3.117}$$

It is easy to remark that (3.117):

i) always admits the *disease-free equilibrium* $(S_1, I_1, B_1, R_1) = (N_0, 0, 0, 0)$ which - from biological point of view - means that all individuals are susceptibles and no infection arises;

ii) if and only if $R_0 > 1$, admits a unique *endemic equilibrium* (i.e. a solution with positive components)

$$\begin{cases} S_2 = \dfrac{K_B(\sigma + \mu)(\mu_B - \pi_B)(\beta + \mu R_0)}{\beta e(\beta + \mu)}, \\ \\ I_2 = \dfrac{\mu K_B(\mu_B - \pi_B)}{e(\beta + \mu)}(R_0 - 1), \\ \\ B_2 = \dfrac{\mu K_B}{\beta + \mu}(R_0 - 1), \\ \\ R_2 = \dfrac{\sigma K_B(\mu_B - \pi_B)}{e(\beta + \mu)}(R_0 - 1). \end{cases} \tag{3.118}$$

Our aim is to find the best conditions guaranteeing the linear and nonlinear stability of the two constant equilibria when (3.109) holds. Let $(\bar{S},\ \bar{I},\ \bar{B},\ \bar{R})$ be a biologically meaningful equilibrium of (3.108). Setting

$$X_1 = S - \bar{S}, \quad X_2 = I - \bar{I}, \quad X_3 = B - \bar{B}, \quad X_4 = R - \bar{R}, \tag{3.119}$$

model (3.108) becomes

$$\begin{cases} \dfrac{\partial X_1}{\partial t} = \mu(N_0 - X_1 - \bar{S}) + \gamma_1 \Delta X_1 - \beta f(X_1, X_3), \\ \\ \dfrac{\partial X_2}{\partial t} = \beta f(X_1, X_3) - (\sigma + \mu)(X_2 + \bar{I}) + \gamma_2 \Delta X_2, \\ \\ \dfrac{\partial X_3}{\partial t} = e(X_2 + \bar{I}) - (\mu_B - \pi_B)(X_3 + \bar{B}) + \gamma_3 \Delta X_3, \\ \\ \dfrac{\partial X_4}{\partial t} = \sigma(X_2 + \bar{I}) - \mu(X_4 + \bar{R}) + \gamma_4 \Delta X_4, \end{cases} \tag{3.120}$$

where

$$f(X_1, X_3) = \frac{(X_1 + \bar{S})(X_3 + \bar{B})}{K_B + X_3 + \bar{B}}.$$

To (3.120) we append the following initial-boundary conditions

$$\begin{cases} X_i = 0 \qquad \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, 3, 4 \\ \nabla X_i \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^* \times \mathbb{R}^+, \end{cases} \tag{3.121}$$

Denoting by $W^*(\Omega)$ the functional space defined by

$$W^*(\Omega) = \left\{ \varphi \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T) : \varphi = 0 \text{ on } \Sigma \times \mathbb{R}^+, \nabla\varphi \cdot \mathbf{n} = 0 \text{ on } \Sigma^* \times \mathbb{R}^+ \right\},$$

our aim is to study the stability of $(\bar{S}, \bar{I}, \bar{B}, \bar{R})$ with respect to the perturbations $(X_1, X_2, X_3, X_4) \in [W^*(\Omega)]^4$.

Remark 13 *We remark that the infimum*

$$\bar{\alpha}(\Omega) = \inf_{\varphi \in W^*(\Omega)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} \tag{3.122}$$

exists and is a real positive number (cfr. [4], [67]).

In view of the Mac-Laurin expansion

$$f(X_1, X_3) = \frac{\bar{S}\bar{B}}{K_B + \bar{B}} + \frac{\bar{B}}{K_B + \bar{B}} X_1 + \frac{K_B \bar{S}}{(K_B + \bar{B})^2} X_3 - F(X_1, X_3),$$

with

$$F(X_1, X_3) = \frac{K_B X_3}{(K_B + \theta_1 X_3 + \bar{B})^2} \left[\frac{(\theta_1 X_1 + \bar{S}) X_3}{(K_B + \theta_1 X_3 + \bar{B})} - X_1 \right],$$

$(0 < \theta_1 < 1)$.

Hence system (3.108) becomes

$$\begin{cases} \dfrac{\partial X_1}{\partial t} = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + a_{14}X_4 + \gamma_1 \Delta X_1 - \beta F(X_1, X_3) \\ \dfrac{\partial X_2}{\partial t} = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + \gamma_2 \Delta X_2 + \beta F(X_1, X_3) \\ \dfrac{\partial X_3}{\partial t} = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + a_{34}X_4 + \gamma_3 \Delta X_3 \\ \dfrac{\partial X_4}{\partial t} = a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + a_{44}X_4 + \gamma_4 \Delta X_4 \end{cases} \tag{3.123}$$

where

$$\begin{cases} a_{11} = -\left(\mu + \dfrac{\beta \bar{B}}{K_B + \bar{B}}\right), \ a_{12} = 0, \ a_{13} = -\dfrac{\beta K_B \bar{S}}{(K_b + \bar{B})^2}, \ a_{14} = 0, \\ a_{21} = \dfrac{\beta \bar{B}}{K_b + \bar{B}}, \ a_{22} = -(\sigma + \mu), \ a_{23} = \dfrac{\beta K_B \bar{S}}{(K_B + \bar{B})^2}, \ a_{24} = 0, \\ a_{31} = 0, \ \ a_{32} = e, \ \ a_{33} = -(\mu_B - \pi_B), \ a_{34} = 0, \\ a_{41} = 0, \ \ a_{42} = \sigma, \ \ a_{43} = 0, \ a_{44} = -\mu. \end{cases} \tag{3.124}$$

Adding and subtracting the term $\bar{\alpha}\gamma_i X_i$ to equation $(3.120)_i$, $(i = 1, 2, 3, 4)$ introducing the scalings $\mu_i \, (i = 1, 2, 3, 4)$ (μ_i are positive constants to be chosen suitably later) and setting

$$
\begin{cases}
X_i = \mu_i U_i, \qquad \mathbf{U} = (U_1, U_2, U_3, U_4)^T, \\
\\
\bar{F} = \beta F(\mu_1 U_1, \mu_3 U_3) = \dfrac{\beta K_B \mu_3 U_3}{(K_B + \theta_1 \mu_3 U_3 + \bar{B})^2} \left[\dfrac{(\theta_1 \mu_1 U_1 + \bar{S}) \mu_3 U_3}{K_B + \theta_1 \mu_3 U_3 + \bar{B}} - \mu_1 U_1 \right] \\
\\
\tilde{F}_1 = \dfrac{1}{\mu_1} \bar{F} + \gamma_1 (\Delta U_1 + \bar{\alpha} U_1), \;\; \tilde{F}_2 = -\dfrac{1}{\mu_2} \bar{F} + \gamma_2 (\Delta U_2 + \bar{\alpha} U_2), \\
\\
\tilde{F}_3 = \gamma_3 (\Delta U_3 + \bar{\alpha} U_3), \tilde{F}_4 = \gamma_4 (\Delta U_4 + \bar{\alpha} U_4), \; \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4)^T,
\end{cases}
\tag{3.125}
$$

(3.120) reduces to

$$
\frac{\partial \mathbf{U}}{\partial t} = \tilde{L} \mathbf{U} + \tilde{\mathbf{F}}, \tag{3.126}
$$

where $\tilde{L}$ is the Jacobian matrix

$$
\tilde{L} = \begin{pmatrix}
b_{11} & 0 & b_{13} & 0 \\
b_{21} & b_{22} & b_{23} & 0 \\
0 & b_{32} & b_{33} & 0 \\
0 & b_{42} & 0 & b_{44}
\end{pmatrix}
$$

with

$$
b_{ii} = a_{ii} - \bar{\alpha} \gamma_i, \qquad b_{ij} = \frac{\mu_j}{\mu_i} a_{ij}, \; i \neq j \tag{3.127}
$$

The initial-boundary conditions (3.121) become

$$
\begin{cases}
U_i = 0 \qquad \text{on} \quad \Sigma \times \mathbb{R}^+, \\
\\
\nabla U_i \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^* \times \mathbb{R}^+,
\end{cases}
\qquad i = 1, 2, 3, 4 \tag{3.128}
$$

The problem to find conditions guaranteeing the stability of ($\bar{S}$, $\bar{I}$, $\bar{B}$, $\bar{R}$) is reduced to determine conditions guaranteeing the stability of the null solution of (3.126)-(3.128).

3.3.4 Linear stability analysis of biologically meaningful equilibria

The null solution of (3.126) is linearly stable if and only if all the eigenvalues of $\tilde{L}$ have negative real parts. The characteristic equation of $\tilde{L}$ is given by

$$(b_{44} - \lambda)(\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3) = 0, \tag{3.129}$$

where $\mathtt{I}_i$, $(i = 1, 2, 3)$ are the principal invariants of the matrix

$$\tilde{L}_1 = \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}$$

and are given by

$$\begin{cases} \mathtt{I}_1 = \mathrm{trace}\tilde{L}_1 = b_{11} + b_{22} + b_{33} = \lambda_1 + \lambda_2 + \lambda_3, \\ \\ \mathtt{I}_2 = \begin{vmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ 0 & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} = \\ \\ \quad = b_{11}(b_{22} + b_{33}) + b_{22}b_{33} - b_{23}b_{32} = \lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3, \\ \\ \mathtt{I}_3 = \det\tilde{L}_1 = b_{11}(b_{22}b_{33} - b_{23}b_{32}) + b_{13}b_{21}b_{32} = \lambda_1\lambda_2\lambda_3. \end{cases}$$

Accounting for (3.129), the eigenvalues of $\tilde{L}$ are given by λ_i, $(i = 1, 2, 3)$ solutions of

$$\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3 = 0, \tag{3.130}$$

and

$$\lambda_4 = b_{44},$$

where, in view of $(3.127)_9$, $\lambda_4 = b_{44} < 0$. Passing now to the equation (3.130), as it is well known, the necessary and sufficient conditions guaranteeing that all the roots of (3.130) have negative real part, are the Routh-Hurwitz conditions (cfr. [39]):

$$\mathtt{I}_1 < 0, \ \ \mathtt{I}_3 < 0, \ \ \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 < 0. \tag{3.131}$$

Obviously (3.131) require necessarily that $\mathtt{I}_2 > 0$. If one of (3.131) is reversed, then there exists at least one eigenvalue of $\tilde{L}$ with positive real part and hence the null solution of (3.126) is linearly unstable. Denoting by $\mathtt{I}^*$, A^* the principal invariants of the matrix $\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix}$, i.e.

$$\mathtt{I}^* = b_{22} + b_{33}, \quad A^* = b_{22}b_{33} - b_{23}b_{32},$$

it follows that

$$\begin{cases} \mathtt{I}_1 = b_{11} + \mathtt{I}^*, \ \ \mathtt{I}_2 = b_{11}\mathtt{I}^* + A^*, \ \ \mathtt{I}_3 = b_{11}A^* + b_{13}b_{21}b_{32}, \\ \\ \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 = (b_{11} + \mathtt{I}^*)b_{11}\mathtt{I}^* + A^*\mathtt{I}^* - b_{13}b_{21}b_{32}. \end{cases} \tag{3.132}$$

Setting

$$A_1^* = -\frac{b_{13}b_{21}b_{32}}{b_{11}} \ (\leq 0), \qquad A_2^* = \frac{b_{13}b_{21}b_{32} - b_{11}\mathtt{I}^*(b_{11} + \mathtt{I}^*)}{\mathtt{I}^*}, \tag{3.133}$$

the following lemma holds.

Lemma 13 *The Routh-Hurwitz conditions are verified if and only if*

$$A^* > \max\{A_1^*, A_2^*\}. \tag{3.134}$$

Proof. In view of (3.132), (3.131) are equivalent to

$$\begin{cases} b_{11} + \mathtt{I}^* < 0, \quad b_{11}A^* + b_{13}b_{21}b_{32} < 0, \\ \\ (b_{11} + \mathtt{I}^*)b_{11}\mathtt{I}^* + A^*\mathtt{I}^* - b_{13}b_{21}b_{32} < 0. \end{cases} \tag{3.135}$$

From (3.127), since $I^* < 0$, it easily follows that $(3.135)_1$ is always satisfied while $(3.135)_2$-$(3.135)_3$ are verified if and only if (3.134) holds.
Setting

$$R_0^* = 1 + \frac{\bar{\alpha}[\gamma_2(\mu_B - \pi_B) + \gamma_3(\sigma + \mu) + \bar{\alpha}\gamma_2\gamma_3]}{(\sigma + \mu)(\mu_B - \pi_B)}, \tag{3.136}$$

from lemma 13, the following two theorems hold.

Theorem 25 *The disease-free equilibrium is linearly stable if and only if*

$$R_0 < R_0^*. \tag{3.137}$$

Proof. Substituting $(\bar{S}, \bar{I}, \bar{B}, \bar{R}) = (N_0, 0, 0, 0)$ in (3.127), one has that

$$\begin{cases} b_{11} = -(\mu + \bar{\alpha}\gamma_1), \quad b_{13} = -\dfrac{\mu_3 \beta N_0}{\mu_1 K_B}, \quad b_{21} = 0, \\ \\ b_{22} = -(\sigma + \mu + \bar{\alpha}\gamma_2), \quad b_{23} = \dfrac{\mu_3 \beta N_0}{\mu_2 K_B}, \\ \\ b_{32} = \dfrac{\mu_2}{\mu_3} e, \quad b_{33} = -(\mu_B - \pi_B + \bar{\alpha}\gamma_3). \end{cases} \tag{3.138}$$

Hence

$$\begin{aligned} A^* &= (\sigma + \mu + \bar{\alpha}\gamma_2)(\mu_B - \pi_B + \bar{\alpha}\gamma_3) - \frac{\beta N_0 e}{K_B} = \\ &= (\sigma + \mu)(\mu_B - \pi_B)\left[1 - R_0 + \frac{\bar{\alpha}[\gamma_2(\mu_B - \pi_B) + \gamma_3(\sigma + \mu) + \bar{\alpha}\gamma_2\gamma_3]}{(\sigma + \mu)(\mu_B - \pi_B)}\right] = \\ &= (\sigma + \mu)(\mu_B - \pi_B)(R_0^* - R_0) \end{aligned} \tag{3.139}$$

and

$$A_1^* = 0, \quad A_2^* = -b_{11}(b_{11} + \mathtt{I}^*) < 0. \tag{3.140}$$

In view of (3.140), it follows that

$$\max\{A_1^*, A_2^*\} = 0.$$

Hence (3.134) is verified if and only if

$$A^* > 0, \tag{3.141}$$

i.e., by virtue of (3.139), if and only if (3.137) holds.

Theorem 26 *When the endemic equilibrium exists, it is always linearly stable.*

Proof. Substituting $(\bar{S},\ \bar{I},\ \bar{B},\ \bar{R}) = (S_2,\ I_2,\ B_2,\ R_2)$ in (3.127), one has that

$$\begin{cases} b_{11} = -\dfrac{\mu(\beta+\mu)R_0 + \bar{\alpha}\gamma_1(\beta+\mu R_0)}{\beta+\mu R_0}, \\[2ex] b_{13} = -\dfrac{\mu_3(\sigma+\mu)(\mu_B-\pi_B)(\beta+\mu)}{\mu_1 e(\beta+\mu R_0)}, \\[2ex] b_{21} = \dfrac{\mu_1\beta\mu(R_0-1)}{\mu_2(\beta+\mu R_0)}, \quad b_{22} = -(\sigma+\mu+\bar{\alpha}\gamma_2), \\[2ex] b_{23} = -\dfrac{\mu_1}{\mu_2}b_{13}, \quad b_{32} = \dfrac{\mu_2}{\mu_3}e, \quad b_{33} = -(\mu_B-\pi_B+\bar{\alpha}\gamma_3). \end{cases} \tag{3.142}$$

Hence

$$\begin{aligned} A^* &= (\sigma+\mu+\bar{\alpha}\gamma_2)(\mu_B-\pi_B+\bar{\alpha}\gamma_3) - \frac{(\sigma+\mu)(\mu_B-\pi_B)(\beta+\mu)}{\beta+\mu R_0} = \\ &= \frac{\mu(\sigma+\mu)(\mu_B-\pi_B)(R_0-1)}{\beta+\mu R_0} + \bar{\alpha}\left[\gamma_2(\mu_B-\pi_B)+\gamma_3(\sigma+\mu)+\bar{\alpha}\gamma_2\gamma_3\right], \end{aligned} \tag{3.143}$$

and

$$\begin{cases} A_1^* = -\dfrac{\beta\mu(\beta+\mu)(\sigma+\mu)(\mu_B-\pi_B)(R_0-1)}{(\beta+\mu R_0)[\mu R_0(\beta+\mu)+\bar{\alpha}\gamma_1(\beta+\mu R_0)]}, \\[2ex] A_2^* = -\dfrac{K_1R_0^2+K_2R_0+K_3}{(\beta+\mu R_0)^2[\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_2+\gamma_3)]}(<0), \end{cases} \tag{3.144}$$

where K_i, $(i = 1, 2, 3)$ are positive constants given by

$$K_1 \; = \mu^2(\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_2+\gamma_3))\left\{\bar{\alpha}\gamma_1[\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_1+\gamma_2+\gamma_3)]+\right.$$

$$\left. + (\beta+\mu)\left[\beta+\mu+\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_1+\gamma_2+\gamma_3)+\bar{\alpha}\gamma_1\right]\right\},$$

$$K_2 \; = \beta\mu\left\{[\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_2+\gamma_3)]\left[\bar{\alpha}(\beta+\mu)(\gamma_1+\gamma_2+\gamma_3)+\right.\right.$$

$$+ \bar{\alpha}\gamma_1(\beta+\mu)+2\bar{\alpha}\gamma_1(\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_1+\gamma_2+\gamma_3))]+$$

$$\left. + (\beta+\mu)(\mu_B-\pi_B)^2+(\beta+\mu)(\sigma+\mu+\mu_B-\pi_B)[\sigma+\mu+\bar{\alpha}(\gamma_2+\gamma_3)]\right\}$$

and

$$K_3 \; = \beta^2\bar{\alpha}\gamma_1[\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_2+\gamma_3)][\sigma+\mu+\mu_B-\pi_B+\bar{\alpha}(\gamma_1+\gamma_2+\gamma_3)]+$$

$$+\beta\mu(\mu_B-\pi_B)(\sigma+\mu)(\beta+\mu).$$

Since (S_2, I_2, B_2, R_2) exists if and only if $R_0 > 1$, then, from (3.143) and $(3.144)_1$, it turns out that

$$A^* > 0, \qquad A_1^* < 0. \tag{3.145}$$

In view of $(3.144)_2$ and (3.145), it follows that (3.134) is always satisfied.

3.3.5 Nonlinear stability analysis of biologically meaningful equilibria

In epidemic disease models, the nonlinear analysis of the biologically meaningful equilibria has to be investigated in order to take into account the contribution of nonlinear terms. Many papers find that the conditions ensuring the linear stability of equilibria are only sufficient to guarantee the nonlinear stability. Hence, the problem to find if there exists coincidence between linear and nonlinear stability thresholds, arises. In this section we will prove that, for the biologically meaningful equilibria of (3.108), there is coincidence between linear and nonlinear stability thresholds. To this end, let us introduce the Rionero-Liapunov functional (see [54], [55] for more details)

$$W = \frac{1}{2}\|U_1\|^2 + V + \frac{1}{2}\|U_4\|^2,$$

where

$$V = \frac{1}{2}\left[A^*(\|U_2\|^2 + \|U_3\|^2) + \|b_{22}U_3 - b_{32}U_2\|^2 + \|b_{23}U_3 - b_{33}U_2\|^2\right].$$

Remark 14 *Let us remark that if* $(\bar{S}, \bar{I}, \bar{B}, \bar{R}) = (S_2, I_2, B_2, R_2)$ *then* $A^* > 0$ *and* V*,* W *are positive definite. If* $(\bar{S}, \bar{I}, \bar{B}, \bar{R}) = (N_0, 0, 0, 0)$ *then (3.137) is equivalent to require that* $A^* > 0$ *and hence to guarantee that* V *and* W *are positive definite.*

The time derivative of W along the solutions of (3.126) is

$$\begin{aligned} \dot{W} &= b_{11}\|U_1\|^2 + I^*A^*(\|U_2\|^2 + \|U_3\|^2) + b_{44}\|U_4\|^2 + A_1 b_{21}\langle U_1, U_2\rangle + \\ &+ (-A_3 b_{21} + b_{13})\langle U_1, U_3\rangle + b_{42}\langle U_2, U_4\rangle + \Phi_1 + \Phi_2, \end{aligned} \tag{3.146}$$

being

$$\begin{cases} A_1 = A^* + b_{32}^2 + b_{33}^2,\ A_2 = A^* + b_{22}^2 + b_{23}^2,\ A_3 = b_{22}b_{32} + b_{23}b_{33}, \\ \Phi_1 = \gamma_1 < U_1, \Delta U_1 + \bar{\alpha} U_1 > + < A_1 U_2 - A_3 U_3, \gamma_2(\Delta U_2 + \bar{\alpha} U_2) > + \\ \quad + < A_2 U_3 - A_3 U_2, \gamma_3(\Delta U_3 + \bar{\alpha} U_3) > + \gamma_4 < U_4, \Delta U_4 + \bar{\alpha} U_4 >, \\ \Phi_2 = \dfrac{1}{\mu_1} < U_1, \bar{F} > + \dfrac{1}{\mu_2} < A_1 U_2, \bar{F} > - \dfrac{1}{\mu_2} < A_3 U_3, \bar{F} >, \end{cases} \tag{3.147}$$

with $\bar{F}$ given by $(3.125)_2$.

The following Lemmas hold.

Lemma 14 *If*

$$A^* > 0 \quad \text{ and } \quad (\gamma_2 + \gamma_3)\,|A_3| < 2\sqrt{A_1A_2\gamma_2\gamma_3}, \tag{3.148}$$

then there exists $\epsilon_1 \in (0, 1)$ *such that* $\forall \epsilon \in (0, \epsilon 1)$

$$\Phi_1 \leq -\gamma_1 \|\nabla U_1\|^2 + \bar{\alpha}\gamma_1 \|U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 . \tag{3.149}$$

Proof. By using the divergence theorem, by virtue of the boundary conditions (3.128) and in view of Poincaré inequality (3.122), from $(3.147)_2$ it follows that

$$\Phi_1 \leq -\gamma_1 \|\nabla U_1\|^2 + \bar{\alpha}\gamma_1 \|U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 + \Phi^*,$$

being

$$\begin{aligned}\Phi^* =& -A_1\gamma_1 \|\nabla U_2\|^2 - |A_3|\,(\gamma_2+\gamma_3)\,\langle \nabla U_2, \nabla U_3\rangle - A_2\gamma_3(1-\epsilon)\,\|\nabla U_3\|^2 + \\ &+A_1\gamma_2\bar{\alpha}\,\|U_2\|^2 + |A_3|\,\bar{\alpha}(\gamma_2+\gamma_3)\,\langle U_2, U_3\rangle + A_2\gamma_3(1-\epsilon)\bar{\alpha}\,\|U_3\|^2 .\end{aligned} \tag{3.150}$$

Since $(3.148)_2$ implies that there exists $\epsilon_1 \in (0, 1)$ such that $\forall \epsilon \in (0, \epsilon_1)$

$$|A_3|\,(\gamma_2+\gamma_3) = 2\sqrt{(1-\epsilon_1)\gamma_2\gamma_3A_1A_2}, \quad |A_3|\,(\gamma_2+\gamma_3) \leq 2\sqrt{(1-\epsilon)\gamma_2\gamma_3A_1A_2},$$

by following the same procedure used for the proof of Lemma 3.2 in [54], one obtains that $\Phi^* \leq 0$ and hence the thesis follows.

Lemma 15 *There exists a positive constant* $M(\Omega)$ *such that*

$$\Phi_2 \leq M(\Omega)(\|U_1\|^2+\|U_2\|^2+\|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2+\|\nabla U_3\|^2+\|U_1\|^2+\|U_3\|^2).$$

Proof. By virtue of (3.111), (3.119), $(3.125)_1$, the following inequalities hold a.e. in Ω

$$\theta_1\mu_1U_1 + \bar{S} = \theta_1X_1 + \bar{S} = \theta_1(S - \bar{S}) + \bar{S} = \theta_1S + (1-\theta_1)\bar{S} \leq M_1$$

and

$$K_B + \theta_1\mu_3 U_3 + \bar{B} = K_B + \theta_1 B + (1-\theta_1)\bar{B} > K_B.$$

Therefore, it turns out that

$$\begin{aligned} &\Phi_2 \le c_1 < |U_1|, U_3^2 > + c_2 < U_1^2, |U_3| > + c_3 < |U_2|, U_3^2 > + \\ &+ c_4 < |U_2|, |U_1 U_3| > + c_5 < |U_3|, U_3^2 >, \end{aligned} \tag{3.151}$$

where

$$\begin{cases} c_1 = \dfrac{\mu_3^2 \beta M_1}{\mu_1 K_B^2} + \dfrac{\mu_1\mu_3 |A_3| \beta}{\mu_2 K_B}, \; c_2 = \dfrac{\mu_3\beta}{K_B}, \\ \\ c_3 = \dfrac{\mu_3^2 A_1 \beta M_1}{\mu_2 K_B^2}, \; c_4 = \dfrac{\mu_1\mu_3 A_1 \beta}{\mu_2 K_B}, \; c_5 = \dfrac{\mu_3^2 |A_3| \beta M_1}{\mu_2 K_B^2}. \end{cases}$$

By virtue of the Hölder and embedding inequalities

$$< |f|, g^2 > \le \|f\| \, \|g\|_4^2, \quad \|g\|_4^2 \le K_1(\Omega)[\|\nabla g\|^2 + \|g\|^2], \quad K_1(\Omega) > 0,$$

and in view of Cauchy inequality, from (3.151) it follows that

$$\begin{aligned} \Phi_2 \; &\le \; \eta_1 \|U_1\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right) + \eta_2 \|U_3\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right) + \\ &+ \; \eta_3 \|U_2\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right) + \eta_4 \|U_2\| \left(\|U_1\|^2 + \|\nabla U_1\|^2\right) + \\ &+ \; \eta_5 \|U_3\| \left(\|U_3\|^2 + \|\nabla U_3\|^2\right), \end{aligned}$$

being

$$\eta_i = K_1(\Omega) c_i, \; i = 1, 2, 5, \quad \eta_3 = \left(c_3 + \frac{1}{2} c_4\right) K_1(\Omega) \quad \text{and} \quad \eta_4 = \frac{1}{2} c_4 K_1(\Omega).$$

Hence the thesis follows with $M(\Omega) = \max\limits_{i=1,\dots,5} \eta_i$.

Remark 15 *We remark that, setting*

$$p = \frac{A^*}{2}, \quad q = \frac{A^*}{2} + \left[(b_{22})^2 + (b_{23})^2 + (b_{32})^2 + (b_{33})^2\right],$$

it follows that

$$p(\|U_2\|^2 + \|U_3\|^2) \le V \le q(\|U_2\|^2 + \|U_3\|^2). \tag{3.152}$$

The following theorem holds.

Theorem 27 *The disease-free equilibrium and the endemic equilibrium are nonlinearly stable if and only if they are linearly stable, i.e.*

i) the disease-free equilibrium is nonlinearly stable if and only if

$$R_0 < R_0^*. \tag{3.153}$$

ii) the endemic equilibrium is always nonlinearly stable when it exists (i.e. when $R_0 > 1$).

Proof. Necessity follows by remarking that, if one of the Routh-Hurwitz conditions is reversed, then there is linear instability. Passing to prove sufficiency, let us distinguish the two cases.

i) Accounting for the disease-free equilibrium, from (3.127) and (3.124) it follows that $b_{21} = 0$; moreover, the condition (3.153) is equivalent to require that $A^* > 0$. Hence, by virtue of Lemma 14 and Lemma 15 and by using the generalized Cauchy inequality and (3.127), from (3.146) it follows that

$$\begin{aligned}\dot{W} \leq & -|a_{11}|\,\|U_1\|^2 - |I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - |a_{44}|\,\|U_4\|^2 + \\ & - \gamma_1 \|\nabla U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 + \\ & + \frac{a_{13}^2\mu_3^2}{2\mu_1^2|I^*|A^*} \|U_1\|^2 + \frac{1}{2}|I^*|A^* \|U_3\|^2 + \\ & + \frac{a_{42}^2\mu_2^2}{2\,|a_{44}|\,\mu_4^2} \|U_2\|^2 + \frac{1}{2}\,|a_{44}|\,\|U_4\|^2 + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).\end{aligned} \tag{3.154}$$

Choosing the positive scalings such that

$$\frac{\mu_3^2}{\mu_1^2} = \frac{|a_{11}|\,|I^*|\,A^*}{a_{13}^2}, \qquad \frac{\mu_2^2}{\mu_4^2} = \frac{|a_{44}|\,|I^*|\,A^*}{a_{42}^2} \tag{3.155}$$

and choosing

$$\epsilon < \min\left\{\frac{|I^*|A^*}{4A_2\bar{\alpha}\gamma_3}, \epsilon_1\right\},$$

from (3.154) it turns out that

$$\begin{aligned}\dot{W} \leq & -\frac{1}{2}\left|a_{11}\right|\|U_1\|^2 - \frac{1}{4}|I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - \frac{1}{2}\left|a_{44}\right|\|U_4\|^2 + \\ & -\gamma_1\left\|\nabla U_1\right\|^2 - A_2\gamma_3\epsilon\left\|\nabla U_3\right\|^2 + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).\end{aligned} \tag{3.156}$$

Therefore, setting

$$h_1 = \min\left\{\left|a_{11}\right|, \left|a_{44}\right|, \frac{A^*|I^*|}{2}\right\}, \quad h_2 = \min\{\gamma_1, \epsilon A_2\gamma_3\},$$

one has that

$$\begin{aligned}\dot{W} \leq & -\frac{h_1}{2}(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2 + \|U_4\|^2) - h_2(\|\nabla U_1\|^2 + \|\nabla U_3\|^2) + \\ & + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}}(\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).\end{aligned}$$

By virtue of (3.152), it turns out that

$$\dot{W} \leq -(\delta_1 - \delta_3 W^{\frac{1}{2}})W - (h_2 - \delta_2 W^{\frac{1}{2}})(\|\nabla U_1\|^2 + \|\nabla U_3\|^2),$$

with

$$\delta_1 = h_1 \min\{1,\ 1/2q\},\ \delta_2 = M(\Omega)\max\{\sqrt{2},\ p^{-1/2}\},\ \delta_3 = M(\Omega)\max\{2\sqrt{2}, p^{-3/2}\}.$$

Hence, if

$$W^{\frac{1}{2}}(0) < \min\{\delta_1/\delta_3,\ h_2/\delta_2\},$$

applying recursive arguments, it follows that

$$\dot{W} \leq -\tilde{K}W, \qquad \tilde{K} = \text{const.} > 0$$

and hence the thesis follows.

ii) Accounting for the endemic equilibrium, the condition guaranteeing its existence, i.e. $R_0 > 1$, is equivalent to require that $A^* > 0$. Hence, by virtue of Lemma 14 and Lemma 15 and by using the generalized Cauchy inequality, from (3.146) it follows that

$$\begin{aligned}
\dot{W} \leq & -|a_{11}|\,\|U_1\|^2 - |I^*|A^*(\|U_2\|^2 + \|U_3\|^2) - |a_{44}|\,\|U_4\|^2 + \\
& - \gamma_1 \|\nabla U_1\|^2 - A_2\gamma_3\epsilon \|\nabla U_3\|^2 + A_2\gamma_3\bar{\alpha}\epsilon \|U_3\|^2 + \\
& + \frac{A_1 a_{21}^2 \mu_1^2}{2\mu_2^2 |I^*| A^*} \|U_1\|^2 + \frac{1}{2}|I^*|A^* \|U_2\|^2 + (-A_3 b_{21} + b_{13}) \langle U_1, U_3 \rangle + \\
& + \frac{a_{42}^2 \mu_2^2}{2\,|a_{44}|\,\mu_4^2} \|U_2\|^2 + \frac{1}{2}|a_{44}|\,\|U_4\|^2 + \\
& + M(\Omega)(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\frac{1}{2}} (\|\nabla U_1\|^2 + \|\nabla U_3\|^2 + \|U_1\|^2 + \|U_3\|^2).
\end{aligned} \tag{3.157}$$

Choosing the positive scalings such that

$$\begin{cases}
-A_3 b_{21} + b_{13} = 0 \\
|a_{11}| - \dfrac{A_1 a_{21}^2 \mu_1^2}{2\mu_2^2 |I^*| A^*} > \dfrac{1}{2}|a_{11}| \\
|I^*|\,A^* - \dfrac{a_{42}^2 \mu_2^2}{2\,|a_{44}|\,\mu_4^2} > \dfrac{1}{2}|a_{44}|
\end{cases}$$

i.e.

$$\begin{cases}
\mu_1^2 = \dfrac{a_{13}\mu_2^2\mu_3^2}{a_{21}a_{32}b_{22}\mu_2^2 + a_{21}a_{23}b_{33}\mu_2^2} \\
\\
\mu_2^2 > \dfrac{(A^* + b_{33}^2) a_{21}^2 \mu_1^2 \mu_3^2}{|I^*|\,A^*\,|a_{11}|\,\mu_3^2 + a_{32}^2 a_{21}^2 \mu_1^2} \\
\\
\mu_4^2 > \dfrac{a_{42}^2 \mu_3^2}{|a_{44}|\,|I^*|\,A^*}
\end{cases} \tag{3.158}$$

and choosing

$$\epsilon < \min\left\{ \frac{|I^*|A^*}{4A_2\bar{\alpha}\gamma_3}, \epsilon_1 \right\},$$

from (3.157) it turns out that (3.156) holds and the proof follows as in the previous case.

Remark 16 *Summarizing the results contained in Theorem 27, we remark that:*

i) *in the absence of diffusion,* $R_0 = R_0^* = 1$ *is a bifurcation parameter for the disease-free equilibrium. In this case, when* R_0 *is slightly greater than* 1*, then the disease-free equilibrium loses its stability and a globally stable endemic equilibrium (not existing for* $R_0 < 1$*) arises. This is called forward bifurcation;*

ii) *in presence of diffusion,* $R_0 = R_0^*$ *is a bifurcation parameter for the disease-free equilibrium. When* $1 < R_0 < R_0^*$ *there is coexistence of disease-free and endemic equilibrium which are both stable. In a neighborhood of* R_0^* *the following scenario is verified:*

- *for* $R_0 < R_0^*$*, a stable disease-free equilibrium coexists with a stable endemic equilibrium;*
- *for* $R_0 > R_0^*$*, the disease-free equilibrium becomes unstable while the endemic equilibrium remains stable.*

In this case a backward bifurcation is verified.

Chapter 4

On the stability of vertical constant throughflows for binary mixtures in porous layers

The research concerned with fluid motions in porous media, due to their large applications in real world phenomena, is very active in the nowadays. In fact, porous materials occur everywhere (see for instance geophysical situations, cultural heritage, contaminant transport and underground water flow [25], [56] and the references therein). Generally, they are analyzed by mean of reaction-diffusion dynamical systems of P.D.Es, which, as it is well known, play an important role in the modeling and analysis of several phenomena. In particular, convection and stability problems in porous layers in the presence of vertical throughflows find relevant applications in cloud physics, in hydrological studies, in subterranean pollution and in many industrial processes where the throughflows can control the onset and evolution of convection

(see [12], [13], [29], [30], [43], [44]-[46], [50], [68], [69]). In fact, the effect of vertical throughflow on the onset of convection has been considered in many cases (the effect in a rectangular box in [45]; the effect combined with a magnetic field in [43]; stability analysis in a cubic Forchheimer model in [29] and when the density is quadratic in temperature in [30]; the effect with an inclined temperature gradient in [50]). The present section, which deals with the results contained in the paper [15], is devoted to study the effects of both temperature gradient and salt concentration on the stability of a vertical flow. Already in [12] and [18] the authors consider both the effects. Precisely, the effect of variable thermal and solutal diffusivities on the onset of convection for non constant throughflows has been analyzed in [18], while in [12] the stability of a vertical constant throughflow in a porous layer, uniformly heated and salted from below, has been investigated. In particular, sufficient conditions ensuring the linear and the global nonlinear stability in the L^2-norm have been determined.
In the present section, we will analyze the more destabilizing case of horizontal porous layers uniformly heated from below and salted from above by one salt and, by using a new approach concerned with the Routh-Hurwitz conditions, necessary and sufficient conditions for the linear stability of a vertical constant throughflow will be determined. Furthermore, conditions ensuring the global non linear stability for the vertical constant throughflow have been obtained.

4.1 Introduction and mathematical model

Let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). The model describing the fluid motion in a horizontal porous layer of depth d, uniformly heated from below and salted from above by one salt, in the Darcy-Oberbeck-Boussinesq scheme, is given by

$$\begin{cases} \nabla p = -\dfrac{\mu}{k}\mathbf{v} - \rho_f g \mathbf{k}, \\ \\ \nabla \cdot \mathbf{v} = 0, \\ \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = K_T \Delta T, \\ \\ \dfrac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = K_C \Delta C, \end{cases} \tag{4.1}$$

where

$$\rho_f = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)], \tag{4.2}$$

is the fluid mixture density and

p = pressure field, T = temperature field, $\mathbf{v}$ = seepage velocity,
C = solute concentration field, μ = dynamic viscosity, k = permeability,
ρ_0 = reference density, T_0 = reference temperature,
C_0 = reference solute concentration, $-g\mathbf{k}$ = gravitational acceleration,
α_T = thermal expansion coefficient, α_C = solute expansion coefficient,
K_T = thermal diffusivity, K_C = solute diffusivity.

To (4.1) we append the boundary conditions

$$\begin{cases} T(x, y, 0, t) = T_L, \quad T(x, y, d, t) = T_U, \\ \\ C(x, y, 0, t) = C_L, \quad C(x, y, d, t) = C_U, \end{cases} \tag{4.3}$$

where T_L, T_U, C_L, C_U are positive constants such that $T_L > T_U$ and $C_L < C_U$ (i.e. the layer is uniformly heated from below and salted from above).

On considering the following dimensionless variables

$$\begin{cases} \mathbf{x}' = \dfrac{\mathbf{x}}{d},\ t' = \dfrac{K_T}{d^2}t,\ \mathbf{v}' = \dfrac{d}{K_T}\mathbf{v},\ p' = \dfrac{k(p+\rho_0 g z)}{\mu K_T}, \\ \\ T' = (T - T_0)\tilde{T},\ C' = (C - C_0)\tilde{C}, \\ \\ \tilde{T} = \left(\dfrac{\alpha_T \rho_0 g k d}{\mu K_T (T_L - T_U)}\right)^{\frac{1}{2}},\ \tilde{C} = \left(\dfrac{\alpha_C \rho_0 g k d}{\mu K_T L_e (C_U - C_L)}\right)^{\frac{1}{2}}, \end{cases} \tag{4.4}$$

where $L_e = K_T/K_C$ is the Lewis number, system (4.1), omitting all the primes, reduces to

$$\begin{cases} \nabla p = -\mathbf{v} + (\mathcal{R}_T T - \mathcal{R}_S C)\mathbf{k}, , \\ \\ \nabla \cdot \mathbf{v} = 0, \\ \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \Delta T, \\ \\ L_e \left(\dfrac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C\right) = \Delta C, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \mathcal{R}_T^2 &= \frac{k d \rho_0 \alpha_T g |\delta T|}{\mu K_T} \quad \text{is the thermal Rayleigh number,} \\ \mathcal{R}_S^2 &= \frac{L_e k d \rho_0 \alpha_C g |\delta C|}{\mu K_T} \quad \text{is the solute Rayleigh number,} \end{aligned} \tag{4.6}$$

being $\delta T = T_L - T_U$ and $\delta C = C_L - C_U$. The boundary conditions (4.3) become

$$\begin{cases} T(x,y,0,t) = (T_L - T_0)\tilde{T}, \quad T(x,y,1,t) = (T_U - T_0)\tilde{T}, \\ \\ C(x,y,0,t) = (C_L - C_0)\tilde{C}, \quad C(x,y,1,t) = (C_U - C_0)\tilde{C}. \end{cases} \tag{4.7}$$

A throughflow solution of (4.5)-(4.7) is given by

$$\begin{cases} \mathbf{v}^* = Q\mathbf{k}, \quad Q = \text{const}, \\ \\ T^*(z) = \dfrac{\mathcal{R}_T(e^{Qz} - 1)}{1 - e^Q} - \tilde{T}(T_0 - T_L), \\ \\ C^*(z) = -\dfrac{\mathcal{R}_S(e^{L_e Q z} - 1)}{L_e(1 - e^{L_e Q})} - \tilde{C}(C_0 - C_L), \\ \\ p^*(z) = p_0^* - Qz + \mathcal{R}_T \displaystyle\int_0^z T^*(\xi)d\xi - \mathcal{R}_S \int_0^z C^*(\xi)d\xi, \end{cases} \tag{4.8}$$

where p_0^* is a constant.

Setting

$$\mathbf{u} = \mathbf{v} - \mathbf{v}^*, \quad \theta = T - T^*, \quad \Gamma = C - C^*, \quad \pi = p - p^*, \tag{4.9}$$

system (4.5) becomes

$$\begin{cases} \nabla\pi = -\mathbf{u} + (\mathcal{R}_T\theta - \mathcal{R}_S\Gamma)\mathbf{k}, \\ \\ \nabla\cdot\mathbf{u} = 0, \\ \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -f_1(z)w - Q\dfrac{\partial\theta}{\partial z} + \Delta\theta, \\ \\ \dfrac{\partial\Gamma}{\partial t} + \mathbf{u}\cdot\nabla\Gamma = f_2(z)w - Q\dfrac{\partial\Gamma}{\partial z} + \dfrac{1}{L_e}\Delta\Gamma, \end{cases} \tag{4.10}$$

where $\mathbf{u} = (u, v, w)$ and

$$f_1(z) = \frac{\partial T^*}{\partial z} = \frac{Q\mathcal{R}_T e^{Qz}}{1-e^Q}, \quad f_2(z) = -\frac{\partial C^*}{\partial z} = \frac{Q\mathcal{R}_S e^{L_e Qz}}{1-e^{L_e Q}}. \tag{4.11}$$

To (4.10) we append the boundary conditions

$$w = \theta = \Gamma = 0 \quad \text{on} \quad z = 0, 1. \tag{4.12}$$

In the sequel, we will assume that

i) the perturbations $\mathbf{u} = (u, v, w), \theta, \Gamma$ are periodic in the x and y directions of periods $\dfrac{2\pi}{a_x}$ and $\dfrac{2\pi}{a_y}$, respectively;

ii) $\Omega = \left[0, \dfrac{2\pi}{a_x}\right] \times \left[0, \dfrac{2\pi}{a_y}\right] \times [0, 1]$ is the periodicity cell;

iii) u, v, w, θ, Γ belong to $W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$ and can be expanded in Fourier series, uniformly convergent in Ω, together with all their first derivatives and second spatial derivatives.

Remark 17 *Let us observe that, when the throughflow tends to zero* ($Q \longrightarrow 0$)*, (4.8) reverts to the conduction solution*

$$\mathbf{v}_1^* = 0, \quad T_1^*(z) = -\mathcal{R}_T z - \tilde{T}(T_0 - T_L), \quad C_1^*(z) = \frac{\mathcal{R}_S}{L_e} z - \tilde{C}(C_0 - C_L). \tag{4.13}$$

4.2 Absorbing sets

Let us denote by

- $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively;
- $f_+(x) = \max\{0, f(x)\}, \quad f_-(x) = \max\{0, -f(x)\}$ where $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Lemma 16 *Let $(\mathbf{u}, \theta, \Gamma) \in [W^{2,2}(\Omega)]^5$ be a solution of (4.10)-(4.12). Then*

$$\theta = \tilde{\theta} + \bar{\theta}, \quad \Gamma = \tilde{\Gamma} + \bar{\Gamma} \quad \textit{in} \quad \Omega \times \mathbb{R}^+, \tag{4.14}$$

with

$$|\tilde{\theta}| \leq 1, \quad |\tilde{\Gamma}| \leq 1, \tag{4.15}$$

and $\bar{\theta}$, $\bar{\Gamma}$ decreasing functions of t such that

$$\begin{cases} \left\|\bar{\theta}(\cdot, t)\right\| \leq \{\|(\theta - 1)_+\| + \|(\theta + 1)_-\|\}_{t=0}\, e^{-\eta t}, \\ \\ \left\|\bar{\Gamma}(\cdot, t)\right\| \leq \{\|(\Gamma - 1)_+\| + \|(\Gamma + 1)_-\|\}_{t=0}\, e^{-\eta t}, \end{cases} \tag{4.16}$$

where

$$\eta = \pi^2 \min\left\{1, \frac{1}{L_e}\right\}. \tag{4.17}$$

Proof. The proof can be found in [70].

Lemma 17 *Let $(\mathbf{u}, \theta, \Gamma) \in [W^{2,2}(\Omega)]^5$ be a solution of (4.10)-(4.12). Then*

$$\|\mathbf{u}\| \leq \mathcal{R}_T \|\theta\| + \mathcal{R}_S \|\Gamma\|. \tag{4.18}$$

Proof. The proof can be found in [12].

Lemma 18 *Let $(\mathbf{u},\theta,\Gamma)\in[W^{2,2}(\Omega)]^5$ be a solution of (4.10)-(4.12). Then*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\theta\|^2+\|\Gamma\|^2\right)\leq-\pi^2\left(\|\theta\|^2+\frac{1}{L_e}\|\Gamma\|^2\right)+\\&+\left[\frac{|Q|\,\mathcal{R}_Te^Q}{2\,|e^Q-1|}(2\mathcal{R}_T+\mathcal{R}_S)+\frac{|Q|\,\mathcal{R}_T\mathcal{R}_Se^{L_eQ}}{2\,|e^{L_eQ}-1|}\right]\|\theta\|^2+\\&+\left[\frac{|Q|\,\mathcal{R}_Se^{L_eQ}}{2\,|e^{L_eQ}-1|}(\mathcal{R}_T+2\mathcal{R}_S)+\frac{|Q|\,\mathcal{R}_T\mathcal{R}_Se^{Q}}{2\,|e^{Q}-1|}\right]\|\Gamma\|^2.\end{aligned}\tag{4.19}$$

Proof. On multiplying $(4.10)_3$ by θ, $(4.10)_4$ by Γ, integrating over Ω, adding the two resulting equations and on applying the divergence theorem, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\theta\|^2+\|\Gamma\|^2\right)+\int_\Omega\mathbf{u}\cdot\nabla\left(\frac{\theta^2}{2}\right)d\Omega+\int_\Omega\mathbf{u}\cdot\nabla\left(\frac{\Gamma^2}{2}\right)d\Omega=\\&=-\int_\Omega f_1(z)w\theta\,d\Omega+\int_\Omega f_2(z)w\Gamma\,d\Omega-\frac{Q}{2}\int_\Omega\frac{\partial}{\partial z}\left(\theta^2+\Gamma^2\right)d\Omega+\\&-\|\nabla\theta\|^2-\frac{1}{L_e}\|\nabla\Gamma\|^2.\end{aligned}\tag{4.20}$$

By virtue of $(4.10)_2$ and (4.12), the divergence theorem leads to

$$\int_\Omega\mathbf{u}\cdot\nabla\left(\frac{\theta^2}{2}\right)d\Omega\equiv\int_\Omega\mathbf{u}\cdot\nabla\left(\frac{\Gamma^2}{2}\right)d\Omega\equiv\int_\Omega\frac{\partial}{\partial z}\left(\theta^2+\Gamma^2\right)d\Omega\equiv0.\tag{4.21}$$

By using the boundedness of $f_1(z)$ and $f_2(z)$, one obtains

$$\begin{cases}|f_1(z)|<\dfrac{|Q|\mathcal{R}_Te^Q}{|e^Q-1|}\Longrightarrow-\displaystyle\int_\Omega f_1(z)w\theta\,d\Omega\leq\frac{|Q|\mathcal{R}_Te^Q}{|e^Q-1|}\|\mathbf{u}\|\|\theta\|,\\[2ex]|f_2(z)|<\dfrac{|Q|\mathcal{R}_Se^{L_eQ}}{|e^{L_eQ}-1|}\Longrightarrow\displaystyle\int_\Omega f_2(z)w\Gamma\,d\Omega\leq\frac{|Q|\mathcal{R}_Se^{L_eQ}}{|e^{L_eQ}-1|}\|\mathbf{u}\|\|\Gamma\|.\end{cases}\tag{4.22}$$

By virtue of (4.21)-(4.22) and Poincaré inequality, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\theta\|^2+\|\Gamma\|^2\right)\leq\frac{|Q|\mathcal{R}_Te^Q}{|e^Q-1|}\|\mathbf{u}\|\|\theta\|+\frac{|Q|\mathcal{R}_Se^{L_eQ}}{|e^{L_eQ}-1|}\|\mathbf{u}\|\|\Gamma\|+\\&-\pi^2\|\theta\|^2-\frac{\pi^2}{L_e}\|\Gamma\|^2.\end{aligned}\tag{4.23}$$

On applying the Schwartz inequality, the thesis holds in view of Lemma 17.

Theorem 28 *The set*

$$\Sigma = \Big\{(\mathbf{u},\theta,\Gamma) \in [W^{2,2}(\Omega)]^5 : \|(\theta-1)_+\| + \|(\theta+1)_-\| < \eta,$$
$$\|(\Gamma-1)_+\| + \|(\Gamma+1)_-\| < \eta, \|\mathbf{u}\| \le (\mathcal{R}_T + \mathcal{R}_S)(|\Omega|^{\frac{1}{2}} + \eta)\Big\}, \tag{4.24}$$

being $|\Omega|$ *the measure of* Ω *and* η *given by (4.17), is an absorbing set of (4.10)-(4.12) for the solutions* $(\mathbf{u},\theta,\Gamma) \in [W^{2,2}(\Omega)]^5$.

Proof. The proof can be found in [12].

4.3 Independent unknown fields

The main boundary value problem (b.v.p.)

$$\begin{cases} \nabla\pi = -\mathbf{u} + (\mathcal{R}_T\theta - \mathcal{R}_S\Gamma)\mathbf{k}, & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in} \quad \Omega \\ w = \theta = \Gamma = 0, & \text{on} \quad z = 0,1, \end{cases} \tag{4.25}$$

has been studied in details in [12], [57],[58]. For the sake of completeness, we recall here some main results.

Let $L_2^*(\Omega)$ be the set of functions $\psi : (\mathbf{x},t) \in \Omega\times\mathbb{R}^+ \longrightarrow \psi(\mathbf{x},t) \in \mathbb{R}$ such that

i) ψ belongs to $L^2(\Omega)$, $\forall t \in \mathbb{R}^+$, together with its first derivatives and second spatial derivatives;

ii) ψ is periodic in the x and y directions of periods $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$, respectively and

$$[\psi]_{z=0} = [\psi]_{z=1} = 0; \tag{4.26}$$

iii) ψ can be expanded in Fourier series absolutely uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$, together with all its first derivatives and the second spatial derivatives.

Since the set $\{\sin(n\pi z)\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2([0,1])$ under the boundary conditions (4.26), then, $\forall \psi \in L_2^*(\Omega)$, there exists a sequence $\left\{\tilde{\psi}_n(x,y,t)\right\}_{n\in\mathbb{N}}$ such that

$$\psi = \sum_{n=1}^{\infty} \psi_n(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{\psi}_n(x,y,t)\sin(n\pi z), \tag{4.27}$$

being the series appearing in (4.27) absolutely uniformly convergent in Ω.
In view of the periodicity in the x and y directions, one obtains that

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = \displaystyle\sum_{n=1}^{\infty} \dfrac{\partial\tilde{\psi}_n}{\partial t}\sin(n\pi z), \quad \Delta_1\psi_n = -a^2\psi_n, \quad \Delta\psi_n = -\xi_n\psi_n, \\ \\ a^2 = a_x^2 + a_y^2, \quad \xi_n = a^2 + n^2\pi^2, \quad \Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}. \end{cases} \tag{4.28}$$

Lemma 19 *Let* $(\mathbf{u}, \theta, \Gamma)$ *- with* $w, \theta, \Gamma \in L_2^*(\Omega)$ *- be a solution of the b.v.p. (4.25). Then* (w, θ, Γ) *is solution of the b.v.p.*

$$\begin{cases} \Delta w = \mathcal{R}_T\Delta_1\theta - \mathcal{R}_S\Delta_1\Gamma, \quad \textit{in} \quad \Omega, \\ \\ w = \theta = \Gamma = 0, \qquad\qquad \textit{on} \quad z = 0,1. \end{cases} \tag{4.29}$$

Proof. The proof can be found in [57].

Lemma 20 *Let* $(w_n, \theta_n, \Gamma_n) \in [L_2^*(\Omega)]^3$ *verifying the boundary conditions (4.29)*$_2$*,* $\forall n \in \mathbb{N}$*. Then the three components of* $\mathbf{u}$*, solution of (4.25), are* *given by*

$$\begin{cases} u = \displaystyle\sum_{n=1}^{\infty} u_n(x,y,z,t) = \sum_{n=1}^{\infty} \frac{1}{a^2} \frac{\partial \tilde{w}_n(x,y,t)}{\partial x} \frac{d}{dz} \sin(n\pi z), \\ \\ v = \displaystyle\sum_{n=1}^{\infty} v_n(x,y,z,t) = \sum_{n=1}^{\infty} \frac{1}{a^2} \frac{\partial \tilde{w}_n(x,y,t)}{\partial y} \frac{d}{dz} \sin(n\pi z), \\ \\ w = \displaystyle\sum_{n=1}^{\infty} w_n(x,y,z,t) = \sum_{n=1}^{\infty} \left(\frac{\mathcal{R}_T a^2}{\xi_n} \theta_n - \frac{\mathcal{R}_S a^2}{\xi_n} \Gamma_n \right). \end{cases} \tag{4.30}$$

Proof. The proof can be found in [57].

Since $\theta, \Gamma, w \in L_2^*(\Omega)$, on setting

$$\begin{cases} \theta_n = \tilde{\theta}_n(x,y,t) \sin(n\pi z), \\ \\ \Gamma_n = \tilde{\Gamma}_n(x,y,t) \sin(n\pi z), \\ \\ w_n = \tilde{w}_n(x,y,t) \sin(n\pi z), \end{cases} \tag{4.31}$$

the following theorem holds.

Theorem 29 *Let $w_n, \theta_n, \Gamma_n \in L_2^*(\Omega)$, $\forall n \in \mathbb{N}$. Then a complete orthogonal system of solutions of the b.v.p. (4.25) is given by*

$$\begin{cases} w_n = \dfrac{\mathcal{R}_T a^2}{\xi_n} \theta_n - \dfrac{\mathcal{R}_S a^2}{\xi_n} \Gamma_n, \\ \\ \mathbf{u}_n = \dfrac{1}{a^2} \left(\dfrac{\partial^2 w_n}{\partial x \partial z} \mathbf{i} + \dfrac{\partial^2 w_n}{\partial y \partial z} \mathbf{j} \right) + w_n \mathbf{k}. \end{cases} \tag{4.32}$$

Proof. The proof can be found in [57].

Remark 18 *We remark that, in view of (4.32), the independent unknown fields are reduced to θ and Γ.*

4.4 Auxiliary linear stability

Linearizing $(4.10)_3$ and $(4.10)_4$, by virtue of (4.27), $(4.28)_3$ and $(4.30)_3$, one has

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \Gamma_n\end{pmatrix} = \mathcal{L}_n\begin{pmatrix}\theta_n\\ \Gamma_n\end{pmatrix} - \begin{pmatrix}\mathbf{v}^*\cdot\nabla\theta_n\\ \mathbf{v}^*\cdot\nabla\Gamma_n\end{pmatrix} \tag{4.33}$$

with

$$\mathcal{L}_n = \begin{pmatrix} a_n(z) & b_n(z)\\ c_n(z) & d_n(z)\end{pmatrix} \tag{4.34}$$

and

$$\begin{cases} a_n(z) = -\left(\dfrac{f_1(z)\mathcal{R}_T a^2}{\xi_n} + \xi_n\right), \quad b_n(z) = \dfrac{f_1(z)\mathcal{R}_S a^2}{\xi_n}, \\ \\ c_n(z) = \dfrac{f_2(z)\mathcal{R}_T a^2}{\xi_n}, \quad d_n(z) = -\left(\dfrac{f_2(z)\mathcal{R}_S a^2}{\xi_n} + \dfrac{1}{L_e}\xi_n\right). \end{cases} \tag{4.35}$$

To (4.33), we associate the "linear auxiliary system"

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \Gamma_n\end{pmatrix} = \mathcal{L}_n\begin{pmatrix}\theta_n\\ \Gamma_n\end{pmatrix} \tag{4.36}$$

where $\mathcal{L}_n$ is given by (4.34). The equation governing the eigenvalues of $\mathcal{L}_n$ is

$$\lambda_n^2 - \mathtt{I}_n\lambda_n + A_n = 0, \tag{4.37}$$

where

$$\begin{cases} A_n(z) = \det\mathcal{L}_n = a_n d_n - b_n c_n = \lambda_{1n}\lambda_{2n}, \\ \\ \mathtt{I}_n(z) = \mathrm{tr}\mathcal{L}_n = a_n + d_n = \lambda_{1n} + \lambda_{2n}. \end{cases} \tag{4.38}$$

Denoting by $S^* = \mathbb{N}\times\mathbb{R}^+\times[0,1]$, the Routh-Hurwitz conditions, necessary and sufficient to guarantee that all the roots of (4.37) have negative real part $\forall n\in\mathbb{N}$, are

$$A_n(z) > 0, \qquad \mathtt{I}_n(z) < 0, \quad \forall (n, a^2, z)\in S^*, \tag{4.39}$$

i.e.

$$\inf_{S^*} A_n(z) > 0, \qquad \sup_{S^*} \mathtt{I}_n(z) < 0. \tag{4.40}$$

If one of the (4.40) is reversed, then the null solution of system (4.36) is unstable.

The following lemmas hold.

Lemma 21 *The conditions*

$$\begin{cases} \dfrac{Qe^Q}{e^Q-1}\mathcal{R}_T^2 + \dfrac{L_eQe^{L_eQ}}{e^{L_eQ}-1}\mathcal{R}_S^2 < 4\pi^2, \quad \textit{when } Q > 0, \\ \\ \dfrac{Q}{e^Q-1}\mathcal{R}_T^2 + \dfrac{L_eQ}{e^{L_eQ}-1}\mathcal{R}_S^2 < 4\pi^2, \quad \textit{when } Q < 0, \end{cases} \tag{4.41}$$

guarantee that (4.40)$_1$ holds.

Proof. In view of (4.35) and (4.38)$_1$, it follows that, $A_n > 0, \forall(n, a^2, z) \in S^*$, if and only if

$$\frac{1}{L_e}\xi_n^2 + f_2(z)\mathcal{R}_S a^2 + \frac{1}{L_e} f_1(z)\mathcal{R}_T a^2 > 0, \forall(n, a^2, z) \in S^*. \tag{4.42}$$

Let

$$G(z) = |f_1(z)|\mathcal{R}_T + |f_2(z)|\mathcal{R}_s L_e, \tag{4.43}$$

then (4.42) becomes

$$G(z) < \frac{\xi_n^2}{a^2}, \quad \forall(n, a^2, z) \in S^*. \tag{4.44}$$

Since $\displaystyle\min_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+} \frac{\xi_n^2}{a^2} = 4\pi^2$, (4.44) is equivalent to

$$\max_{z\in[0,1]} G(z) < 4\pi^2. \tag{4.45}$$

By virtue of (4.11), from (4.43) one obtains that

$$G'(z) = \frac{Q^2e^{Qz}}{e^Q-1}\mathcal{R}_T^2 + \frac{Q^2L_e^2e^{L_eQz}}{e^{L_eQ}-1}\mathcal{R}_S^2, \tag{4.46}$$

and hence, by remarking that

$$\begin{cases} G'(z) > 0, \, \forall z \in [0,1] \quad \text{when} \quad Q > 0, \\ \\ G'(z) < 0, \, \forall z \in [0,1] \quad \text{when} \quad Q < 0, \end{cases} \tag{4.47}$$

one obtains

$$\begin{cases} \max\limits_{z \in [0,1]} G(z) = G(1) = \dfrac{Qe^Q}{e^Q - 1}\mathcal{R}_T^2 + \dfrac{L_e Q e^{L_e Q}}{e^{L_e Q} - 1}\mathcal{R}_S^2, \text{ when } Q > 0, \\ \\ \max\limits_{z \in [0,1]} G(z) = G(0) = \dfrac{Q}{e^Q - 1}\mathcal{R}_T^2 + \dfrac{L_e Q}{e^{L_e Q} - 1}\mathcal{R}_S^2, \text{ when } Q < 0, \end{cases} \tag{4.48}$$

and the thesis follows.

Lemma 22 *The conditions*

$$\begin{cases} \dfrac{Qe^Q}{e^Q - 1}\mathcal{R}_T^2 + \dfrac{Qe^{L_e Q}}{e^{L_e Q} - 1}\mathcal{R}_S^2 < 4\pi^2 \left(1 + \dfrac{1}{L_e}\right), \ \textit{when } Q > 0, \\ \\ \dfrac{Q}{e^Q - 1}\mathcal{R}_T^2 + \dfrac{Q}{e^{L_e Q} - 1}\mathcal{R}_S^2 < 4\pi^2 \left(1 + \dfrac{1}{L_e}\right), \ \textit{when } Q < 0, \end{cases} \tag{4.49}$$

guarantee that (4.40)$_2$ holds.

Proof. By virtue of (4.35) and (4.38)$_2$, it follows that, $\mathtt{I}_n < 0$, $\forall (n, a^2, z) \in S^*$, if and only if

$$\frac{f_1(z)\mathcal{R}_T a^2}{\xi_n} + \xi_n + \frac{f_2(z)\mathcal{R}_S a^2}{\xi_n} + \frac{1}{L_e}\xi_n > 0, \forall (n, a^2, z) \in S^*. \tag{4.50}$$

Let

$$H(z) = |f_1(z)|\mathcal{R}_T + |f_2(z)|\mathcal{R}_S, \tag{4.51}$$

then (4.50) becomes

$$H(z) < \frac{\xi_n^2}{a^2}\left(1 + \frac{1}{L_e}\right), \tag{4.52}$$

i.e.

$$\max_{z\in[0,1]} H(z) < 4\pi^2\left(1+\frac{1}{L_e}\right). \tag{4.53}$$

In view of (4.11), from (4.51) one obtains that

$$H'(z) = \frac{Q^2 e^{Qz}}{e^Q-1}\mathcal{R}_T^2 + \frac{Q^2 L_e e^{L_e Q z}}{e^{L_e Q}-1}\mathcal{R}_S^2. \tag{4.54}$$

Since

$$\begin{cases} H'(z) > 0,\ \forall z\in[0,1] \quad \text{when} \quad Q>0, \\ \\ H'(z) < 0,\ \forall z\in[0,1] \quad \text{when} \quad Q<0, \\ \end{cases} \tag{4.55}$$

it follows that

$$\begin{cases} \max\limits_{z\in[0,1]} H(z) = H(1) = \dfrac{Qe^Q}{e^Q-1}\mathcal{R}_T^2 + \dfrac{Qe^{L_eQ}}{e^{L_eQ}-1}\mathcal{R}_S^2, \text{ when } Q>0, \\ \\ \max\limits_{z\in[0,1]} H(z) = H(1) = \dfrac{Q}{e^Q-1}\mathcal{R}_T^2 + \dfrac{Q}{e^{L_eQ}-1}\mathcal{R}_S^2, \text{ when } Q<0, \end{cases} \tag{4.56}$$

and the thesis is proved.

In the case $Q>0$, on setting

$$\begin{cases} \mathcal{R}_1 = \dfrac{1-e^Q}{e^{L_eQ}-1}L_e e^{(L_e-1)Q}\mathcal{R}_S^2 + \dfrac{4\pi^2}{Q}\left(\dfrac{e^Q-1}{e^Q}\right), \\ \\ \mathcal{R}_2 = \dfrac{1-e^Q}{e^{L_eQ}-1}e^{(L_e-1)Q}\mathcal{R}_S^2 + \dfrac{4\pi^2}{Q}\left(\dfrac{e^Q-1}{e^Q}\right)\left(1+\dfrac{1}{L_e}\right), \end{cases} \tag{4.57}$$

the following theorem holds.

Theorem 30 *If and only if*

$$\mathcal{R}_T^2 < \min(\mathcal{R}_1, \mathcal{R}_2), \tag{4.58}$$

then the null solution of system (4.36) is stable.

Proof. The proof follows very easily by virtue of Lemmas 21 and 22.

In the case $Q < 0$, on setting

$$\begin{cases} \mathcal{R}_3 = \dfrac{1-e^Q}{e^{L_e Q}-1} L_e \mathcal{R}_S^2 + \dfrac{4\pi^2}{Q}\left(e^Q - 1\right), \\ \\ \mathcal{R}_4 = \dfrac{1-e^Q}{e^{L_e Q}-1} \mathcal{R}_S^2 + \dfrac{4\pi^2}{Q}\left(e^Q - 1\right)\left(1 + \dfrac{1}{L_e}\right), \end{cases} \tag{4.59}$$

the following theorem holds.

Theorem 31 *If and only if*

$$\mathcal{R}_T^2 < \min(\mathcal{R}_3, \mathcal{R}_4), \tag{4.60}$$

hen the null solution of system (4.36) is stable.

Proof. The proof follows very easily by virtue of Lemmas 21 and 22.

Remark 19 *We remark that, in the case $L_e > 1$,*

$$\mathcal{R}_1 - \mathcal{R}_2 = \frac{1-e^Q}{e^{L_e Q}-1} e^{(L_e-1)Q}(L_e - 1)\mathcal{R}_S^2 - \frac{4\pi^2}{L_e Q}\left(\frac{e^Q - 1}{e^Q}\right) < 0, \tag{4.61}$$

and

$$\mathcal{R}_3 - \mathcal{R}_4 = \frac{1-e^Q}{e^{L_e Q}-1}(L_e - 1)\mathcal{R}_S^2 - \frac{4\pi^2}{L_e Q}\left(e^Q - 1\right) < 0. \tag{4.62}$$

Hence the necessary and sufficient conditions guaranteeing the linear stability of the null solution of system (4.36), when $L_e > 1$, become

$$\begin{cases} \mathcal{R}_T^2 < \mathcal{R}_1, \quad \textit{when} \quad Q > 0, \\ \\ \mathcal{R}_T^2 < \mathcal{R}_3, \quad \textit{when} \quad Q < 0. \end{cases} \tag{4.63}$$

4.5 Auxiliary evolution system of the n-th Fourier component of perturbations

We start this Section by proving a uniqueness theorem for (4.10).

Theorem 32 *The problem (4.10) under the initial boundary conditions*

$$\begin{cases} \mathbf{u}(\mathbf{x},0) = \mathbf{u}^{(0)}(\mathbf{x}) = \displaystyle\sum_{n=1}^{\infty} \mathbf{u}_{n0}(\mathbf{x}), \ \pi(\mathbf{x},0) = \pi^{(0)}(\mathbf{x}) = \displaystyle\sum_{n=1}^{\infty} \pi_{n0}(\mathbf{x}), \\ \\ \theta(\mathbf{x},0) = \theta^{(0)}(\mathbf{x}) = \displaystyle\sum_{n=1}^{\infty} \theta_{n0}(\mathbf{x}), \ \Gamma(\mathbf{x},0) = \Gamma^{(0)}(\mathbf{x}) = \displaystyle\sum_{n=1}^{\infty} \Gamma_{n0}(\mathbf{x}), \\ \\ w = \theta = \Gamma = 0 \quad on \quad z = 0,1, \end{cases} \tag{4.64}$$

admits a unique solution $(\mathbf{u}, \pi, \theta, \Gamma) \in [W^{2,2}(\Omega)]^6$.

Proof. Let $(\mathbf{u}_1, \pi_1, \theta_1, \Gamma_1)$ and $(\mathbf{u}_2, \pi_2, \theta_2, \Gamma_2)$ be two solutions of (4.10) under the initial boundary conditions (4.64). Setting

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \quad \theta = \theta_1 - \theta_2, \quad \Gamma = \Gamma_1 - \Gamma_2, \quad \pi = \pi_1 - \pi_2, \tag{4.65}$$

from (4.10) it follows that

$$\begin{cases} \nabla \pi = -\mathbf{u} + (\mathcal{R}_T \theta - \mathcal{R}_S \Gamma)\mathbf{k}, \\ \\ \nabla \cdot \mathbf{u} = 0, \\ \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}_1 \cdot \nabla \theta + \mathbf{u} \cdot \nabla \theta_2 = -f_1(z) w - Q \dfrac{\partial \theta}{\partial z} + \Delta \theta, \\ \\ \dfrac{\partial \Gamma}{\partial t} + \mathbf{u}_1 \cdot \nabla \Gamma + \mathbf{u} \cdot \nabla \Gamma_2 = f_2(z) w - Q \dfrac{\partial \Gamma}{\partial z} + \dfrac{1}{L_e} \Delta \Gamma, \end{cases} \tag{4.66}$$

with

$$w = \theta = \theta_2 = \Gamma = \Gamma_2 = 0 \quad \text{on} \quad z = 0, 1. \tag{4.67}$$

On setting

$$E = \frac{1}{2}\left(\|\theta\|^2 + \|\Gamma\|^2\right), \tag{4.68}$$

in view of Theorem 28 and by virtue of (4.19), one obtains

$$E(t) < E_0 e^{at}, \quad a = \text{ const } > 0 \tag{4.69}$$

and hence

$$E_0 = 0 \quad \Longrightarrow \quad E(t) = 0, \ \forall t \in \mathbb{R}^+ \tag{4.70}$$

and the thesis follows.

Let $(\bar{\mathbf{u}}, \bar{\pi}, \bar{\theta}, \bar{\Gamma})$ be solution of the (nonlinear) initial boundary value problem (i.b.v.p.)

$$\begin{cases} \nabla\bar{\pi} = -\bar{\mathbf{u}} + (\mathcal{R}_T\bar{\theta} - \mathcal{R}_S\bar{\Gamma})\mathbf{k}, \\ \\ \nabla\cdot\bar{\mathbf{u}} = 0, \\ \\ \dfrac{\partial\bar{\theta}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar{\theta} = -f_1(z)\bar{w} - Q\dfrac{\partial\bar{\theta}}{\partial z} + \Delta\bar{\theta}, \\ \\ \dfrac{\partial\bar{\Gamma}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar{\Gamma} = f_2(z)\bar{w} - Q\dfrac{\partial\bar{\Gamma}}{\partial z} + \dfrac{1}{L_e}\Delta\bar{\Gamma}, \end{cases} \tag{4.71}$$

$$\begin{cases} (\bar{w})_{t=0} = \bar{w}^{(0)}, \ (\bar{\theta})_{t=0} = \bar{\theta}^{(0)}, \ (\bar{\Gamma})_{t=0} = \bar{\Gamma}^{(0)}, \\ \\ \bar{w} = \bar{\theta} = \bar{\Gamma} = 0, \quad \text{on } z = 0, 1, \end{cases} \tag{4.72}$$

with $\bar{w}^{(0)}, \bar{\theta}^{(0)}, \bar{\Gamma}^{(0)}$ of type (4.64).

By virtue of the results of the previou sections, it follows that (4.71)-(4.72) is equivalent to

$$\frac{\partial}{\partial t}\begin{pmatrix} \bar{\theta} \\ \bar{\Gamma} \end{pmatrix} = \sum_{n=1}^{\infty} \mathcal{L}_n \begin{pmatrix} \bar{\theta}_n \\ \bar{\Gamma}_n \end{pmatrix} - \begin{pmatrix} \mathbf{v}^* \cdot \nabla\bar{\theta} \\ \mathbf{v}^* \cdot \nabla\bar{\Gamma} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{u}} \cdot \nabla\bar{\theta} \\ \bar{\mathbf{u}} \cdot \nabla\bar{\Gamma} \end{pmatrix}, \tag{4.73}$$

under conditions (4.72).
For any $n \in \mathbb{N}$, let us associate to (4.73), under the initial boundary conditions (4.72), the "*auxiliary system*", linear with respect to φ_{in}

$$\frac{\partial}{\partial t}\begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \end{pmatrix} - \begin{pmatrix} \mathbf{v}^* \cdot \nabla\varphi_{1n} \\ \mathbf{v}^* \cdot \nabla\varphi_{2n} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{u}} \cdot \nabla\varphi_{1n} \\ \bar{\mathbf{u}} \cdot \nabla\varphi_{2n} \end{pmatrix}, \tag{4.74}$$

under the i.b.c.

$$\begin{cases} (\varphi_{1n})_{t=0} = \bar{\theta}_n^{(0)}, \ \ (\varphi_{2n})_{t=0} = \bar{\Gamma}_n^{(0)}, \\ \\ \varphi_{1n} = \varphi_{2n} = 0, \qquad \text{on } z = 0, 1, \end{cases} \tag{4.75}$$

φ_{in} $(i = 1, 2)$, being unknown functions of (x, y, z, t) and $\bar{\mathbf{u}} = \bar{\mathbf{u}}(x, y, z, t)$ divergence free vector determined by solving (4.73) under (4.72) and hence to be considered known. The following theorem holds (cfr. [36], [59], [60] for more details).

Theorem 33 *Let* $(\varphi_{1n}, \varphi_{2n})$ *be a solution of (4.33)-(4.75),* $\forall n \in \mathbb{N}$. *Then the series* $\displaystyle\sum_{n=1}^{\infty} \varphi_{1n}$ *and* $\displaystyle\sum_{n=1}^{\infty} \varphi_{2n}$ *are convergent and it follows that*

$$\sum_{n=1}^{\infty} \varphi_{1n} = \bar{\theta}, \quad \sum_{n=1}^{\infty} \varphi_{2n} = \bar{\Gamma}, \tag{4.76}$$

being $(\bar{\theta}, \bar{\Gamma})$ *solution of (4.71)-(4.72).*

Proof. In the case $\mathbf{v}^* = 0$ the proof is given in [36]. Its generalization in the case $\mathbf{v}^* \neq 0$ can be found in [12].

Remark 20 *In view of Theorem 33, it follows that the asymptotic stability of the null solution of (4.71)-(4.72) is guaranteed by conditions, independent of* n*, ensuring the asymptotic stability of the null solution of (4.33)-(4.75)*$_2$.

4.6 Absence of subcritical instabilities and global nonlinear stability

In order to study the nonlinear stability of the null solution of (4.33), let us consider the standard $L^2(\Omega)$-energy

$$E = \sum_{n=1}^{\infty} E_n, \tag{4.77}$$

where

$$E_n = \frac{1}{2}\left[\mu\|\varphi_{1n}\|^2 + \|\varphi_{2n}\|^2\right], \tag{4.78}$$

being μ a positive scaling to be suitably chosen later. The time derivative of E_n along the solutions of (4.33) is given by

$$\begin{aligned} \frac{dE_n}{dt} &= \int_\Omega \left[\mu a_n \varphi_{1n}^2 + (\mu b_n + c_n)\varphi_{1n}\varphi_{2n} + d_n \varphi_{2n}^2\right] d\Omega + \\ -\frac{1}{2}&\int_\Omega (\mu \bar{\mathbf{u}} \cdot \nabla \varphi_{1n}^2 + \bar{\mathbf{u}} \cdot \nabla \varphi_{2n}^2)\, d\Omega. \end{aligned} \tag{4.79}$$

The following lemma holds.

Lemma 23 *Conditions (4.41) guarantee that*

$$\exists \mu \in \mathbb{R}^+ \;:\; (\mu b_n + c_n)^2 - 4\mu a_n d_n < 0, \quad \forall n \in \mathbb{N},\ \forall z \in [0,1]. \tag{4.80}$$

Proof. Let us consider

$$F(n,z) = (\mu b_n + c_n)^2 - 4\mu a_n d_n, \tag{4.81}$$

i.e.

$$F(n,z) = b_n^2(z)\mu^2 + 2[b_n(z)c_n(z) - 2a_n(z)d_n(z)]\mu + c_n^2(z). \tag{4.82}$$

From (4.35), $b_n(z) < 0$ and $c_n(z) < 0 \quad \forall n \in \mathbb{N},\ \forall z \in [0,1]$, then, since (4.41) guarantees that $A_n(z) > 0, \forall n \in \mathbb{N}, \forall z \in [0,1]$, one obtains

$$a_n(z)d_n(z) - b_n(z)c_n(z) > 0 \Rightarrow a_n(z)d_n(z) > b_n(z)c_n(z) > 0. \tag{4.83}$$

Hence

$$\begin{cases} b_n(z)c_n(z) - 2a_n(z)d_n(z) < 0, \\ \\ \Delta_n = [b_n(z)c_n(z) - 2a_n(z)d_n(z)]^2 - b_n(z)^2 c_n(z)^2 > 0. \end{cases} \tag{4.84}$$

Let us define

$$\begin{cases} \mu_{1_n} = \dfrac{2a_n d_n - b_n c_n - \sqrt{4a_n^2 d_n^2 - 4a_n b_n c_n d_n}}{b_n^2}\ (>0), \\ \\ \mu_{2_n} = \dfrac{2a_n d_n - b_n c_n + \sqrt{4a_n^2 d_n^2 - 4a_n b_n c_n d_n}}{b_n^2}\ (>0), \end{cases} \tag{4.85}$$

and

$$\begin{cases} \bar{\mu} = \sup\limits_{(n,z)\in\mathbb{N}\times[0,1]} \mu_{1_n}, \\ \\ \bar{\bar{\mu}} = \inf\limits_{(n,z)\in\mathbb{N}\times[0,1]} \mu_{2_n}. \end{cases} \tag{4.86}$$

Since$(\bar{\mu}, \bar{\bar{\mu}}) \subset (\mu_{1_n}, \mu_{2_n}),\ \forall n \in \mathbb{N},\ \forall z \in [0,1]$, on choosing

$$\mu \in (\bar{\mu}, \bar{\bar{\mu}}), \tag{4.87}$$

it follows that

$$F(n, z) < 0, \quad \forall n \in \mathbb{N}, \forall z \in [0, 1] \tag{4.88}$$

and the thesis follows.

Theorem 34 *If condition (4.58) (in the case* $Q > 0$*) or (26) (in the case* $Q < 0$*) holds, then the vertical throughflow (4.8) is globally nonlinearly stable in the E-norm.*

Proof. Since E_n is positive definite $\forall n \in \mathbb{N}$ and $\forall z \in [0, 1]$, the stability of (4.8) is guaranteed if (4.79) is negative definite $\forall n \in \mathbb{N}$ and $\forall z \in [0, 1]$. In view of the boundary conditions $(4.75)_2$, the divergence theorem leads to

$$\int_\Omega \bar{\mathbf{u}} \cdot \nabla \varphi_{1n}^2 \, d\Omega \equiv \int_\Omega \bar{\mathbf{u}} \cdot \nabla \varphi_{2n}^2 \, d\Omega \equiv 0. \tag{4.89}$$

Let us remark that, either (4.58) or (4.60) guarantee that

$$\sup_{S^*} a_n(z) < 0, \qquad \sup_{S^*} d_n(z) < 0. \tag{4.90}$$

In fact, since (25) and (26) guarantee that the Routh-Hurwitz conditions (4.40) hold, then

$$a_n(z) d_n(z) > b_n(z) c_n(z), \tag{4.91}$$

hence

$$a_n(z) d_n(z) > 0 \tag{4.92}$$

and, in view of $(4.40)_2$,

$$a_n(z) + d_n(z) < 0. \tag{4.93}$$

Collecting (4.92) and (4.93), one obtains (4.90). Hence, on choosing μ as in (4.87), one obtains that $\dfrac{dE_n}{dt}$ is negative definite and the null solution of (4.71) is globally nonlinearly stable with respect to E-norm.

Remark 21 *Let us underline that the condition ensuring the stability of the null solution of the linear auxiliary system (4.36) also ensures the nonlinear global stability of the null solution of (4.71) under the boundary conditions* $(4.72)_2$*. In fact, by virtue of the boundary conditions*

$$\bar{\theta}_n = \bar{\Gamma}_n = 0, \qquad \textit{on } z = 0, 1, \tag{4.94}$$

the divergence theorem leads to

$$\begin{aligned} &\int_\Omega \bar{\theta}_n \mathbf{v}^* \cdot \nabla \bar{\theta}_n \, d\Omega \equiv \int_\Omega \bar{\Gamma}_n \mathbf{v}^* \cdot \nabla \bar{\Gamma}_n \, d\Omega \equiv 0, \\ &\int_\Omega \bar{\theta}_n \bar{\mathbf{u}} \cdot \nabla \bar{\theta}_n \, d\Omega \equiv \int_\Omega \bar{\Gamma}_n \bar{\mathbf{u}} \cdot \nabla \bar{\Gamma}_n \, d\Omega \equiv 0. \end{aligned} \tag{4.95}$$

Hence, there is no contribution of the terms $\mathbf{v}^* \cdot \nabla \bar{\theta}_n$*,* $\mathbf{v}^* \cdot \nabla \bar{\Gamma}_n$*,* $\bar{\mathbf{u}} \cdot \nabla \bar{\theta}_n$*,* $\bar{\mathbf{u}} \cdot \nabla \bar{\Gamma}_n$ *to the E-norm and, as one is expected, such terms can be avoided at least in the E-energy.*

Remark 22 *We remark that, since*

$$\begin{cases} \lim\limits_{Q \to 0^+} \mathcal{R}_1 \equiv \lim\limits_{Q \to 0^-} \mathcal{R}_3 \equiv -\mathcal{R}_S^2 + 4\pi^2, \\ \\ \lim\limits_{Q \to 0^+} \mathcal{R}_2 \equiv \lim\limits_{Q \to 0^-} \mathcal{R}_4 \equiv -\dfrac{\mathcal{R}_S^2}{L_e} + 4\pi^2 \left(1 + \dfrac{1}{L_e}\right), \end{cases} \tag{4.96}$$

the necessary and sufficient condition guaranteeing the linear stability of the conduction solution (4.13) becomes, as expected,

$$\mathcal{R}_T^2 < \min \left\{ -\mathcal{R}_S^2 + 4\pi^2, -\frac{\mathcal{R}_S^2}{L_e} + 4\pi^2 \left(1 + \frac{1}{L_e}\right) \right\}. \tag{4.97}$$

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