Università degli Studi di Napoli
Federico II

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI RENATO CACCIOPPOLI
Corso di Dottorato di Ricerca in Scienze matematiche
XXVIII Ciclo

TESI DI DOTTORATO DI RICERCA

Some properties of large subgroups of groups

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Anno Accademico 2015/2016

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Introduction

Starting from the nineteen twenties until now, a relevant part of investigation in infinite groups was based on the fact that in many case the imposition of finiteness conditions on an infinite group forces the group to be "close" to a finite group.

Recently, a new point of view has been adopted, focusing the attention on groups which are far from finiteness. The subject of this thesis is the investigation of groups which are large in some sense, providing some new contributions to this topic.

A subgroup property θ is an *embedding property* if in any group G all images under automorphism of G of θ-subgroups likewise have the property θ.

For our purposes, the notion of a large group can be formalized in the following way. Let $\mathfrak{X}$ be a classe of groups. Then $\mathfrak{X}$ is said to be a class of *large groups* if it satisfies the following conditions:

- if a group G contains an $\mathfrak{X}$-subgroup, then G belongs to $\mathfrak{X}$;
- if G is any $\mathfrak{X}$-group and N is a normal subgroup of G, then at least one of the groups N and G/N belongs to $\mathfrak{X}$;
- no finite cyclic group belongs to $\mathfrak{X}$.

An obvious example of class of large groups is the class of groups of infinite rank. A group G is said to have finite *(Prüfer) rank* if there exists a positive integer r such that all finitely generated subgroups of G can be generated by at most r elements and r is the least positive integer with such property; otherwise, if such r does not exist, the group is said to have *infinite rank*. It is not difficult to see that in some universe of (generalized) soluble groups, if a group G has infinite rank, then it must be rich in subgroups of infinite rank. Moreover, in recent years, a series of relevant papers has been published by many authors (among which M.R. Dixon, J. Evans, L.A. Kurdachenko, N.N. Semko, H. Smith) which shows that the subgroups of infinite rank of a group of infinite rank have the power to influence the structure of the whole group and to force also the behaviour of the "small" subgroups of G (i.e. the subgroups of finite rank). In fact, it has been proved that, for some choises of group theoretical properties $\mathcal{X}$, if G is a group in which all subgroups of infinite rank satisfy the property $\mathcal{X}$, then the same happens also to the subgroups of finite rank.

The first chapter consists of an overview of the main results concerning the role played by the subgroups of infinite rank in the structure of a large group.

In the second and the third chapter the following embedding properties of normal type are considered: the property of being either normal or self-nomalizing; the property of being either normal or contranormal; the property of being either subnormal or contranormal. It turns out that, at least in a suitable class of generalized soluble groups, if every large subgroup of a group G of infinite rank satisfies one of these conditions, then all subgroups of G are forced to verify it.

It was proved by B.H. Neumann [41] that if every infinite subset of a group G contains a pair of permutable elements, then G is central-by-finite. Many authors have studied similar problems, replacing commutativity by a given group theoretical property (see [5], [18]). In some sense, this topic is connected with the study of large groups, because the above quoted result of Neumann and the other results of the same type can be seen as a signal of the fact that, in many cases, the behaviour of the infinite subsets of an (infinite) group can influence the structure of the whole group. In accordance with this, the fourth chapter is devoted to the study of groups in which every infinite set of cyclic subgroups contains a pair such that at least one of them (or both) is modular in the subgroup generated by them: it is shown that these groups are central-by-finite.

Finally, the last chapter is a work in progress on groups of infinite rank which are isomorphic to their non-abelian subgroups of infinite rank. The structure of groups G containing proper non-abelian subgroup all of which are isomorphic to G has been investigated by Smith and Wiegold in [56], and the corresponding problems for the class of nilpotent groups and for the class of groups with modular subgroup lattice have also been considered (see [57], [58], [17]). Following these results, it is proved that if G is a periodic soluble group of infinite rank which are isomorphic to their non-abelian subgroup of infinite rank, then G is abelian-by-finite and in the torsion-free nilpotent case it is even abelian.

Most of our notation is standard, and it refers to [46].

I would like to thank Professor Maria De Falco that in these years has constantly guided me in the research work and in the writing of this thesis, Professor Carmen Musella for her availability and kindness, Professor Francesco de Giovanni that has been for me a source of ispiration and Anna, colleague and above all friend, with whom I have realized most of my papers.

Chapter 1

The role of large subgroups in groups of infinite rank

Let θ be a property pertaining to subgroups of a group. We shall say that θ is *absolute* if in any group G all subgroups isomorphic to some θ-subgroup are likewise θ-subgroups. Thus θ is absolute if and only if there exists a group class $\mathfrak{X} = \mathfrak{X}(\theta)$ such that in any group G a subgroup X has the property θ if and only if X belongs to $\mathfrak{X}$. Thus among the most natural absolute properties we have those of being an abelian subgroup, a nilpotent subgroup, a finite subgroup.

A subgroup property θ is called an *embedding property* if in any group G all images of θ-subgroups under automorphisms of G likewise have the property θ. Of course, any absolute property is trivially an embedding property, but the most relevant embedding properties, like normality and subnormality, are embedding properties which are not absolute.

If θ is an embedding property for subgroups, a group class $\mathfrak{X}$ is said to *control* θ if it satisfies the following condition:

- *If G is any group containing some $\mathfrak{X}$-subgroup, and all $\mathfrak{X}$-subgroups of G have the property θ, then θ holds for all subgroups of G.*

Observe that if a group class $\mathfrak{X}$ controls the property θ and $\mathfrak{X} \subseteq \mathfrak{X}'$, then θ is controlled also by $\mathfrak{X}'$. Clearly, the class of cyclic groups controls periodicity, and the class of finitely generated groups controls every local property. In particular, the class of finitely generated groups controls commutativity. On the other hand, it is well-known that the class of finitely generated groups neither controls nilpotency nor solubility while the class of countable groups controls both.

Although normality is controlled by the class of finitely generated groups (and even by that of cyclic groups), it is easy to see that most of the significant embedding properties cannot be controlled by the class of finitely generated groups. For instance, it is well-known that there exist unsoluble groups in which all finitely generated subgroups are subnormal, while an important result by W. Möhres [39] shows that every group in which all subgroups are subnormal is soluble. Therefore subnormality cannot be controlled by the class of finitely generated groups. This failure depends on the fact that finitely generated groups are too small. Therefore it is natural to consider the problem of how large should be $\mathfrak{X}$-groups in order to obtain that the group class $\mathfrak{X}$ controls the main embedding properties, at least within an appropriate universe of groups.

Let $\mathfrak{X}$ be a group class. We will say that $\mathfrak{X}$ is a class of *large groups* if it satisfies the following conditions:

- *if a group G contains an $\mathfrak{X}$-subgroup, then G belongs to $\mathfrak{X}$;*
- *if N is a normal subgroup of an $\mathfrak{X}$-group G, then at least one of the groups N and G/N belongs to $\mathfrak{X}$;*
- *$\mathfrak{X}$ contains no finite cyclic groups.*

Let $\mathfrak{X}$ be a class of large groups, and let θ be a subgroup property. Since every group containing an $\mathfrak{X}$-subgroup likewise belongs to $\mathfrak{X}$, it follows that $\mathfrak{X}$

controls θ if and only if whenever in an $\mathfrak{X}$-group G all $\mathfrak{X}$-subgroups have the property θ, then θ holds for all subgroups of G. The easiest non-trivial example of a class of large groups is provided by the class $\mathfrak{I}$ consisting of all infinite groups; however, the consideration of the locally dihedral 2-group shows that normality cannot be controlled by the class $\mathfrak{I}$, even in the universe of periodic metabelian groups.

The idea of these definitions arises from result obtained by several authors who have worked on the class of groups of infinite rank. A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. In particular, a group has rank 1 if and only if it is locally cyclic. It is easy to see that the class of groups of finite rank is closed with respect to subgroups, homomorphic images and extensions, and hence groups of infinite rank form a class of large groups. In a series of recent papers it has been proved that in a (generalized) soluble group of infinite rank the behaviour of subgroups of finite rank with respect to an embedding property can be neglected in many cases, so that the class of groups of infinite rank controls such embedding property in a suitable universe of (generalized) soluble groups in order to avoid that these groups contain Tarski groups (infinite simple groups whose proper subgroups have prime order) as a section.

So, now it will give a survey of results on this subject. The results described in this section will be usually stated for locally (soluble-by-finite) groups. However, many of them can be proved in a larger class of generalized soluble groups. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\acute{\boldsymbol{P}}$, $\grave{\boldsymbol{P}}$, $\boldsymbol{R}$, $\boldsymbol{L}$. It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, and that the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. This class has been introduced by N.S. Černikov [3], who proved in particular that every $\bar{\mathfrak{D}}$-group of finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free non-abelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. For this reason, it is better in some cases to replace $\bar{\mathfrak{D}}$-groups by *strongly locally graded groups*, i.e. groups in which every section belongs to $\bar{\mathfrak{D}}$. The class of strongly locally graded groups has been introduced in [14].

In many problems concerning groups of infinite rank the existence of particular subgroups of infinite rank plays a crucial role.

Lemma 1.1 (M.R. Dixon - M.J. Evans - H. Smith [22]). *Let G be a locally soluble group of infinite rank. Then G contains a proper subgroup of infinite rank.*

Lemma 1.2 (A.I. Malcev [37]). *Let G be a locally nilpotent group of infinite rank. Then G contains an abelian subgroup of infinite rank.*

Lemma 1.3 (V.P. Šunkov [61]). *Let G be a locally finite group of infinite rank. Then G contains an abelian subgroup of infinite rank.*

Lemma 1.4 (V.S. Čarin [2]). *Let G be a locally soluble group of finite rank. Then there exists a positive integer k such that the subgroup $G^{(k)}$ is periodic and hypercentral.*

The first relevant theorem for our purposes was obtained by M.J. Evans and Y. Kim, and deals with the control of normality by the class of groups of infinite rank.

Theorem 1.5. (M.J. Evans - Y. Kim [26]) *If G is a locally soluble group of infinite rank and all subgroups of infinite rank of G are normal, then every subgroups of G is normal.*

In our language, the theorem of Evans and Kim just says that groups of infinite rank form a class of large groups with respect to normality in the universe of locally soluble groups.
It is well-known that normality is not a transitive relation in an arbitrary group (this is for instance the case of the alternating group $Alt(4)$). This remark led H. Wielandt to introduce in 1939 the concept of a subnormal subgroup: a subgroup H of a group G is said to be *subnormal* in G if there exists a finite series of G containing H and G, and the minimal length of a series of this type is called the *defect* of H in G.

Theorem 1.6 (M.J. Evans - Y. Kim [26]). *For every positive integer k there exists a positive integer $f(k)$, depending only on k, such that, if G is a locally soluble group of infinite rank in which all subgroups of infinite rank are subnormal with defect at most k, then G is nilpotent of class at most $f(k)$.*

This result must be seen in relation with a famous theorem of J.E. Roseblade (see [51]) which states that for every positive integer k there exists a positive integer $h(k)$, depending only on k, such that, if G is any group in which all subgroups are subnormal with defect at most k, then G is nilpotent of class at most $h(k)$. It is well-know that in the above quoted theorem of Roseblade, the bound condition on the defect cannot be omitted, but without this assumption, W. Mohres [39] proved that if G is a group in which all subgroups are subnormal, then G is at least soluble. Also this theorem has been generalized looking at subgroups of infinite rank.

Theorem 1.7 (L.A. Kurdachenko - H. Smith [33]). *If G is any locally (soluble-by-finite) group of infinite rank in which all subgroups of infinite rank are subnormal, then G is soluble.*

It is natural to ask if it is possible to obtain analogous results to these of Evans and Kim, replacing normality by other embedding properties of normal type.

A subgroup H of a group G is said to be *permutable* in G if $HX = XH$ for every subgroup X of G. The following result shows that this embedding property is also controlled by the class of groups of infinite rank.

Theorem 1.8 (M.R. Dixon - Z.Y. Karatas [25]). *The class of groups of infinite rank controls permutability in the universe of locally (soluble-by-finite) groups, i.e. if G is a locally (soluble-by-finite) group of infinite rank and all subgroups of infinite rank of G are permutable, then all subgroups of G are permutable.*

Some interesting results in this context are due to M.R. Dixon, M.J. Evans and H. Smith.

Theorem 1.9 (M.R. Dixon - M.J. Evans - H. Smith [24]). *If G is a locally soluble group of infinite rank whose proper subgroups of infinite rank are nilpotent with class at most c, then G itself is nilpotent with class at most c.*

Theorem 1.10 (M.R. Dixon - M.J. Evans - H. Smith [23]). *If G is a soluble group of infinite rank whose proper subgroups of infinite rank have derived length at most k, then G itself has derived length at most k.*

This latest result means, in particular, that if all proper subgroups of infinite rank are abelian, then G is abelian.

Recall that a group G is called *metahamiltonian* if all its non-abelian subgroups are normal. Metahamiltonian groups were introduced and investigated by G.M. Romalis and N.F. Sesekin (see [47],[48],[49]). In particular, they proved that if G is any locally soluble metahamiltonian group, then the commutator subgroup G' of G is finite of prime-power order. Finally, a group G is called *quasihamiltonian* if all its subgroups are permutable.
Maria De Falco, Francesco de Giovanni, Carmela Musella and Nadir Trabelsi have proved, in [14], that if $\mathfrak{X}$ is any class of groups with some natural properties of closure, and G is a locally (soluble-by-finite) group in which all proper

subgroups of infinite rank belong to $\mathfrak{X}$, then some infomations can be obtained on the structure of G.

Theorem 1.11 (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]). *If $\mathfrak{X}$ is the class of metahamiltonian groups, or that of quasihamiltonian groups, then whenever G is a locally (soluble-by-finite) group of infinite rank in which all proper subgroups of infinite rank are $\mathfrak{X}$-group, then G itself is an $\mathfrak{X}$-group.*

Theorem 1.12 (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]). *If G is a locally (soluble-by-finite) group of infinite rank in which all proper subgroups of infinite rank have locally finite commutator subgroup, then also the commutator subgroup G' of G is locally finite.*

As we remarked before normality is not a transitive relation in an arbitrary group. A group G is called a *T-group* if normality in G is a transitive relation, or equivalently if every subnormal subgroup of G is normal. Obviously, every simple group has the T-property, but soluble T-groups have a restricted structure, that was first studied by W. Gaschütz [27] in the finite case and later by D.J.S. Robinson [44] in the general case. In particular, it turns out that every soluble T-group is metabelian and hypercyclic (i.e. it has an ascending normal series with cyclic factors). Moreover, any finitely generated soluble group with the T-property is either finite or abelian.

Theorem 1.13 (M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak [13]). *If G is a periodic soluble group of infinite rank and all subnormal subgroups of infinite rank of G are normal, then G is a T-group.*

Theorem 1.14 (M. De Falco - F. de Giovanni - C. Musella [11]). *If G is a locally soluble group of infinite rank in which all proper subgroups of infinite rank are T-groups, then all subgroups of G are T-group.*

Chapter 2

Groups of infinite rank with a normalizer condition on subgroups

In this Chapter it will show that the class of groups of infinite rank controls the embedding property $\mathcal{E}$ in the universe of strongly locally graded groups.

2.1 $\mathcal{E}$-groups

A group G is an $\mathcal{E}$*-group* if every non-normal subgroups is self-normalizing.
Recall that a group G is said to be a T*-group* (or have the T*-property*) if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. Although the class of T-groups is not subgroups closed (because any simple group is obviously a T-group), it is know that subgroups of finite soluble T-groups have likewise the T-property. A group G is called a $\overline{T}$*-group* if all its subgroups are T-groups.
The following propositions, easy to verify, will be very useful later.

Proposition 2.1.1. *Every finite* $\overline{T}$*-group is soluble.*

Proposition 2.1.2. *The class $\mathcal{E}$ is $\boldsymbol{S}$-closed and $\boldsymbol{H}$-closed.*

Proposition 2.1.3. *Every $\mathcal{E}$-group is a $\overline{T}$-group and so it is a T-group.*

Recall that a group G is said to be a *Dedekind group* if all its subgroups are normal. Obviously, every abelian group has this property, and each Dedekind group is not too far from being abelian. In fact, a classical result of Baer and Dedekind proves that a non-abelian group G has only normal subgroups if and only if $G = Q \times A$, where Q is a quaternion group of order 8 and A is a periodic abelian group with no elements of order 4.

The following theorem characterizes the $\mathcal{E}$-groups with some aperiodic non-trivial element.

Theorem 2.1.4 (G. Giordano [28])**.** *Every non-periodic $\mathcal{E}$-group is abelian.*

Proof. Note that if X is any infinite cyclic subgroup of G and $H < X$, then $H < X \leq N_G(H)$ and so $H \lhd G$ whence $X \lhd G$. Let Y be a finite non-trivial cyclic subgroup of G. Now XY is soluble and also a $\overline{T}$-group, so it is abelian ([44] Theorem 6.1.1) and $Y \lhd G$. Therefore G is a non-periodic Dedekind group so it is abelian. □

The following propositions are useful properties of locally finite $\mathcal{E}$-groups.

Proposition 2.1.5. *A locally finite $\mathcal{E}$-group is soluble.*

Proof. By Proposition 2.1.1 G is a locally soluble $\overline{T}$-group so it is soluble (see [44] p. 36). □

Proposition 2.1.6. *A locally finite p-group in the class $\mathcal{E}$ is a Dedekind group.*

Proof. Let H be a proper cyclic subgroup of G such that $H < K \leq G$, for some finite subgroup K; but K is a finite p-group so it is nilpotent and then $H < N_K(H) \leq N_G(H)$, thus $H \lhd G$. □

The following results characterize the locally finite and finite $\mathcal{E}$-groups.

Theorem 2.1.7 (G. Giordano [28]). *Let G be a locally finite group. The following statements are equivalent:*

(*i*) *G is a non Dedekind $\mathcal{E}$-group.*

(*ii*) *G is a soluble T-group with G/G' cyclic of order p^n (p prime), $|Z(G)| = p^{n-1}$, $\pi(G') \cap \pi(G/G') = \varnothing$ and $2 \notin \pi(G')$.*

Corollary 2.1.8. *For a finite group G the following are equivalent:*

(*i*) *G is a non Dedekind $\mathcal{E}$-group.*

(*ii*) *G is a soluble T-group, the maximal nilpotent factor group of G is cyclic of order p^n (p prime) and $|Z(G)| = p^{n-1}$.*

2.2 $\mathcal{E}(\mathfrak{X})$-groups

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathcal{E}(\mathfrak{X})$-group if every $\mathfrak{X}$-subgroup H of G such that $H < N_G(H)$ is normal in G. It is clear that the class $\mathcal{E}(\mathfrak{X})$ is always subgroup closed and that from $\mathfrak{X} \subseteq \mathfrak{Y}$ it follows $\mathcal{E}(\mathfrak{Y}) \subseteq \mathcal{E}(\mathfrak{X})$. If $\mathfrak{X}$ is the class of all groups, an $\mathcal{E}(\mathfrak{X})$-group is an $\mathcal{E}$-group.

In the sequel the structure of $\mathcal{E}(\mathfrak{X})$-groups will be studied for some relevant classes $\mathfrak{X}$, namely the classes of finite groups, of infinite groups, of abelian groups.

We recall that an *IT-group* is a group in which every infinite subnormal subgroup is normal; moreover a group G is called an *$\overline{IT}$-group* if every subgroup of G is an IT-group.

The first lemma is a general result which shows that in many cases the class $\mathcal{E}(\mathfrak{X})$ is a local class.

Lemma 2.2.1. *Let $\mathfrak{X}$ be a class of finitely generated groups. Then the class $\mathcal{E}(\mathfrak{X})$ is $\boldsymbol{L}$-closed.*

Proof. Let the group G satisfy locally the property $\mathcal{E}(\mathfrak{X})$ and consider an $\mathfrak{X}$-subgroup H of G such that $H < N_G(H)$. Choose $x \in N_G(H) \setminus H$ and let g be any element of G. The subgroup $K = \langle H, x, g\rangle$ is finitely generated, and so it is an $\mathcal{E}(\mathfrak{X})$-group. Since $H < N_K(H)$, it follows that H is normal in K; hence $H^g = H$ and H is normal in G. Therefore G is an $\mathcal{E}(\mathfrak{X})$-group. □

By the Theorem 2.1.4 it follows that every non-periodic $\mathcal{E}$-group is abelian. The situation is less clear for periodic $\mathcal{E}$-groups, since clearly every Tarski group is an $\mathcal{E}$-group. Therefore it is natural to impose some restrictions on the group G, in order to avoid that G contains a Tarski group as a section. We will be concerned with the class of locally graded groups. It will be proved that locally graded groups which satisfy locally the condition $\mathcal{E}$ are soluble $\mathcal{E}$-groups. We note first the following

Lemma 2.2.2. $\mathcal{E}(\mathfrak{F}) = \mathcal{E}(\boldsymbol{L}\mathfrak{F})$. *In particular every locally finite* $\mathcal{E}(\mathfrak{F})$*-group is an* $\mathcal{E}$*-group.*

Proof. Clearly $\mathcal{E}(\boldsymbol{L}\mathfrak{F}) \subseteq \mathcal{E}(\mathfrak{F})$. Conversely, let G be an $\mathcal{E}(\mathfrak{F})$-group, and consider a locally finite non normal subgroup H of G. Then there exists a cyclic subgroup $\langle x\rangle$ of H which is not normal in G and clearly $N_G(\langle x\rangle) = \langle x\rangle$. Hence $\langle x\rangle$ is a Sylow subgroup of H. If $N = N_G(H)$, it follows from the Frattini Argument that $N = HN_N(\langle x\rangle) = H$, so that G is an $\mathcal{E}(\boldsymbol{L}\mathfrak{F})$-group. □

We can prove the following theorem, which is an improvement of the Theorem 2.1.7.

Theorem 2.2.3 (G. Cutolo [6])**.** *Let* G *be a periodic group. The following statements are equivalent:*

(*i*) *G is a locally graded group which satisfies locally the property* $\mathcal{E}(\mathfrak{L})$.

(*ii*) *G is a soluble* $\mathcal{E}$*-group.*

(*iii*) *G is either a Dedekind group or it is a soluble T-group and $G = G' \rtimes K$, where $K = \langle x \rangle$ is a cyclic p-subgroup of G and $Z(G) = \langle x^p \rangle$.*

(If these conditions hold, G' is a Hall subgroup of G and has no elements of order 2).

Proof. (i) implies (ii). It follows from Lemma 2.2.1 that G is an $\mathcal{E}(\mathfrak{L})$-group, so that every finitely generated subgroup of G is either normal or self-normalizing in G. Let H be any finitely generated subgroup of G. If N is a normal subgroup of finite index of H, every subgroup of H containing N is finitely generated, and hence the factor group H/N is an $\mathcal{E}$-group. In particular H/N is a $\overline{T}$-group, so it is metabelian. Let R be the finite residual of H. Then H/R is a finitely generated metabelian periodic group, so it is finite. Therefore R is finitely generated and has no proper subgroup of finite index. Since G is locally graded, this means that $R = \{1\}$, so that H is finite and G is locally finite. It follows from Lemma 2.2.2 that G is an $\mathcal{E}$-group. Moreover G is soluble as a locally finite $\overline{T}$-group (see [44]).

Clearly, (ii) implies (i), while the equivalence between (ii) and (iii) follows from Theorem 2.1.7. □

Now we consider the class $\mathcal{E}(\mathfrak{A})$. This class coincides with the class $\mathcal{E}(\mathfrak{X})$ for some other group classes $\mathfrak{X}$.

Proposition 2.2.4. $\mathcal{E}(\mathfrak{C}) = \mathcal{E}(\mathfrak{A}) = \mathcal{E}(\boldsymbol{L}\mathfrak{N})$.

Proof. Since $\mathcal{E}(\boldsymbol{L}\mathfrak{N}) \subseteq \mathcal{E}(\mathfrak{A}) \subseteq \mathcal{E}(\mathfrak{C})$, it is enough to prove that every $\mathcal{E}(\mathfrak{C})$-group G is also an $\mathcal{E}(\boldsymbol{L}\mathfrak{N})$-group. Let H be a locally nilpotent non-normal subgroup of G; then H contains a cyclic subgroup $\langle x \rangle$ which is not normal in G, so that $N_G(\langle x \rangle) = \langle x \rangle$. Therefore $\langle x \rangle$ is not properly contained in any nilpotent subgroup of H. But H is locally nilpotent and hence $H = \langle x \rangle$. Therefore $N_G(H) = H$ and G is an $\mathcal{E}(\boldsymbol{L}\mathfrak{N})$-group. □

Corollary 2.2.5. *The class* $\mathcal{E}(\mathfrak{A})$ *is* $\boldsymbol{L}$*-closed.*

Proof. Since $\mathcal{E}(\mathfrak{A}) = \mathcal{E}(\mathfrak{C})$, the result follows from Lemma 2.2.1. □

The following lemma is useful in the characterization of $\mathcal{E}(\mathfrak{A})$-groups.

Lemma 2.2.6. *Let* G *be a group, and let* H *be a locally graded subgroup of* G *which is minimal with respect to the condition* $H < N_G(H) < G$*. Then* H *is a cyclic group of prime power order.*

Proposition 2.2.7. *The following hold:*

(*a*) *Every* $\mathcal{E}(\mathfrak{A})$*-group is an* $\mathcal{E}(\mathfrak{F})$*-group.*

(*b*) *Every locally finite* $\mathcal{E}(\mathfrak{A})$*-group is an* $\mathcal{E}$*-group.*

(*c*) *For a periodic group the properties* $\mathcal{E}(\mathfrak{A})$ *and* $\mathcal{E}(\mathfrak{F})$ *are equivalent.*

Proof. (*a*) Let F be a finite subgroup of G. If F is neither normal nor self-nomalizing in G, then F contains a subgroup H which is minimal with respect to this condition. It follows from Lemma 2.2.6 that H is cyclic, which is a contradiction, since G is an $\mathcal{E}(\mathfrak{A})$-group. Therefore G is an $\mathcal{E}(\mathfrak{F})$-group.
(*b*) It follows from (*a*) and Lemma 2.1.2.
(*c*) It follows from (*a*) that $\mathcal{E}(\mathfrak{A}) \subseteq \mathcal{E}(\mathfrak{F})$. On the other hand from Proposition 2.2.4 we obtain

$$\mathcal{E}(\mathfrak{F}) \cap \mathfrak{T} \subseteq \mathcal{E}(\mathfrak{C}) \cap \mathfrak{T} = \mathcal{E}(\mathfrak{A}) \cap \mathfrak{T}.$$

□

Note that $\mathcal{E}(\mathfrak{F})$-groups which are not $\mathcal{E}(\mathfrak{A})$-groups exist, since every torsion-free group is an $\mathcal{E}(\mathfrak{F})$-group.

Our next result gives a complete description of non-periodic $\mathcal{E}(\mathfrak{A})$-groups.

Theorem 2.2.8 (G. Cutolo [6])**.** *For a non-periodic group* G *the following are equivalent:*

(i) *G is a non-abelian* $\mathcal{E}(\mathfrak{A})$*-group.*

(ii) $G = \langle C, z\rangle$*, where C is an abelian subgroup of G,* $|G : C| = 2$*,* $c^z = c^{-1}$ *for every* $c \in C$ *and* $C_2 \leq \langle z\rangle$ *(in particular the 2-component of C has order at most 2).*

In the last part we will characterize locally graded and non periodic $\mathcal{E}(\mathfrak{I})$-groups. It is clear that the class $\mathcal{E}(\mathfrak{I})$ is quotient closed and that every factor of an $\mathcal{E}(\mathfrak{I})$-group respect to an infinite normal subgroup is an $\mathcal{E}$-group.

Proposition 2.2.9. *A non-periodic group G is an* $\mathcal{E}(\mathfrak{I})$*-group if and only if it is abelian.*

Proof. Let G be an $\mathcal{E}(\mathfrak{I})$-group. Thus clearly G is an $\overline{IT}$-group. Let K be an infinite cyclic subgroup of G. Then K is normal in G and, if H is any finite cyclic subgroup of G, the subgroup HK is a non-periodic soluble $\overline{IT}$-group, and so it is abelian (see [29], p. 579). Hence HK is generated by its elements of infinite order, so that HK, and hence its torsion subgroup H, is normal in G. Therefore G is a non-periodic Dedekind group and so it is abelian. □

Theorem 2.2.10 (G. Cutolo [6])**.** *Let G be an infinite periodic locally graded group. Then G is an* $\mathcal{E}(\mathfrak{I})$*-group if and only if it satisfies one of the following conditions:*

(a) *G is an* $\mathcal{E}$*-group.*

(b) *G is an extension of a Prüfer group by a finite* $\mathcal{E}$*-group.*

(c) $G = H \times P$*, where P is a Prüfer p-group and H is an* $\mathcal{E}$*-group whose commutator subgroup* H' *is a Prüfer q-group, with* $p \neq q \neq 2$*.*

2.3 $\mathcal{E}_\infty$-groups

A group G is said to be an *$\mathcal{E}_\infty$-group* if every subgroup H of G of infinite rank such that $H < N_G(H)$ is normal in G. It is clear that the class $\mathcal{E}_\infty$ is quotient closed and that every factor of an $\mathcal{E}_\infty$-group respect to a normal subgroup of infinite rank is an $\mathcal{E}$-group.

As in many problems concerning groups of infinite rank, also in this case, the existence of abelian subgroups of infinite rank plays a crucial role.

Lemma 2.3.1. *Let G be an $\mathcal{E}_\infty$-group. If G contains an abelian subgroup of infinite rank, then G is an $\mathcal{E}$-group.*

Proof. Let A be an abelian subgroup of infinite rank of G. Let H be any subgroup of G of finite rank such that $H < N_G(H)$, and take an element $x \in N_G(H) \setminus H$. Then A contains a direct product $A_1 \times A_2$, such that the subgroups A_1 and A_2 have both infinite rank and $(A_1 \times A_2) \cap H\langle x\rangle = \{1\}$. Clearly the subgroups A_1 and A_2 are normal in G. Moreover, $HA_1 \cap H\langle x\rangle = H$, so that $x \in N_G(HA_1) \setminus HA_1$, and hence HA_1 is normal in G. Similary, HA_2 is normal in G, so that $H = HA_1 \cap HA_2$ is normal in G, and G is an $\mathcal{E}$-group. □

Proposition 2.3.2. *Let G be a periodic locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G is an $\mathcal{E}$-group.*

Proof. Assume that G contains a finitely generated subgroup H of infinite rank. If K is any normal subgroup of finite index of H, then K has infinite rank and hence the factor group H/K is an $\mathcal{E}$-group. Therefore H/K is a finite $\overline{T}$-group, and hence it is metabelian. If R is the finite residual of H, then H/R is a finitely generated metabelian periodic group, and so it is finite. Thus R is a finitely generated subgroup of G which has no proper subgroups of finite index; it follows that $R = \{1\}$, so that H is finite. This contradiction shows that every finitely generated subgroup of G has finite rank and so it is finite by Černikov's theorem (see [3]). Therefore G is a locally finite group, so that it contains an abelian subgroup of infinite rank (Lemma 1.3), and hence G is an $\mathcal{E}$-group by Lemma 2.3.1. □

Lemma 2.3.3. *Let G be a strongly locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G contains a proper normal subgroup of infinite rank.*

Proof. Assume by contradiction that all proper normal subgroups of G have finite rank, so that G is not soluble, and hence it is not an $\mathcal{E}$-group (see Theorem 2.2.3). Moreover, it follows from Lemma 2.3.1 that G has no abelian subgroups of infinite rank, so that G is not locally nilpotent. Therefore G has a simple homomorphic image $\bar{G}$ of infinite rank (see [14], Lemma 2.4). Clearly every subgroup of infinite rank of $\bar{G}$ is self-normalizing in $\bar{G}$. Let $\bar{H}$ be any finitely generated subgroup of infinite rank of $\bar{G}$; since $\bar{G}$ is locally graded, $\bar{H}$ contains a proper normal subgroup $\bar{K}$ of finite index, so that $\bar{K}$ is properly contained in its normalizer, a contradiction. Therefore every finitely generated subgroup of $\bar{G}$ has finite rank, so that G is locally (soluble-by-finite) by Černikov's thoerem (see [3]). As $\bar{G}$ has infinite rank, it must contains a proper locally soluble subgroup $\bar{L}$ of infinite rank (see [22]). Since every subgroup of infinite rank of $\bar{G}$ is self-normalizing, $\bar{L}$ has no proper normal subgroups of infinite rank, and hence $\bar{L}$ has a simple homomorphic image of infinite rank (see [14], Lemma 2.4), a contradiction since simple locally soluble groups have prime order. This contradiction completes the proof of the lemma. □

We are now ready to prove the main result of this section.

Theorem 2.3.4 (A.V. De Luca - G. di Grazia [20])**.** *Let G be a strongly locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G is an $\mathcal{E}$-group.*

Proof. It follows from 2.3.3 that G contains a proper normal subgroup N of infinite rank. The factor group G/N is an $\mathcal{E}$-group, so that $G'' \leq N$ (see

Theorem 2.2.3). If G'' has infinite rank then G'' is strongly locally graded $\mathcal{E}_\infty$-group of infinite rank and so G'' has a proper normal subgroup M of infinite rank. As above, $G^{(4)} \leq M$; if $G^{(4)}$ has infinite rank then $G/G^{(4)}$ has derived length at most 2, and hence $G'' = G^{(4)} \leq M$, a contradiction. Therefore in any case, $K = G^{(4)}$ has finite rank. So K is (locally soluble)-by-finite by Černikov's theorem (see [3]). If S is the locally soluble radical of K then K/S is finite, and therefore $G/C_G(K/S)$ is finite. Hence $C_G(K/S)$ has infinite rank and so $G/C_G(K/S)$ is metabelian; it follows that $K \leq C_G(K/S)$ and hence K is locally soluble (see [25] Lemma 2.7). There exists a positive integer n such that $K^{(n)}$ is a periodic hypercentral group with Černikov primary components (see [45], Lemma 10.39), so that the divisible radical R of $K^{(n)}$ is a divisible normal abelian subgroup of G and $K^{(n)}/R$ has finite primary components. In order to prove that K is soluble we may assume that $R = \{1\}$, so that each primary component of $K^{(n)}$ is finite.

Let P be any primary component of $K^{(n)}$. Since $G/C_G(P)$ is finite, $C_G(P)$ has infinite rank. Again $G/C_G(P)$ is metabelian and we have $P \leq K \leq G'' \leq C_G(P)$, so that P is abelian and $K^{(n)}$ is also abelian. Thus K is soluble and hence G is soluble. By Lemma 2.3.1 G is an $\mathcal{E}$-group. □

Now we will consider groups which are rich in $\mathcal{E}$-subgroups. In particular, it turns out that the class of groups of infinite rank controls the embedding property $\mathcal{E}$ in the universe of strongly locally graded groups.

The proof is accomplished through some lemmas; the first of them shows that the class $\mathcal{E}$ is local, at least within the universe of locally graded groups.

Proposition 2.3.5. *Let G be a locally graded group such that every finitely generated subgroup is an $\mathcal{E}$-group. Then G is an $\mathcal{E}$-group.*

Proof. Assume first that G is not periodic, and let H be any finitely generated subgroup of G. If g is an element of G of infinite order, then the finitely generated subgroup $\langle H, g\rangle$ is a non-periodic $\mathcal{E}$-group, so that it is abelian. Therefore G is abelian. Assume now that G is a periodic group and let H be any finitely generated subgroup of G such that $H < N_G(H)$. Choose an element x in the set $N_G(H) \setminus H$, and let g be any element of G. The subgroup $K = \langle H, x, g\rangle$ is finitely generated and so it is an $\mathcal{E}$-group. Since $H < N_K(H)$, the subgroup H is normal in K. It follows that H is normal in G. Therefore G is an $\mathcal{E}$-group (see Theorem 2.2.3). □

Lemma 2.3.6. *Let G be a locally graded group such that all its proper subgroup are $\mathcal{E}$-groups. Then G is soluble.*

Proof. We can assume that G is not an $\mathcal{E}$-group. It follows from Proposition 2.3.5 that G is finitely generated, so that there exists a proper normal subgroup N of G of finite index. The subgroup N is an $\mathcal{E}$-group and so it is soluble. On the other hand, all proper subgroup of G/N are finite $\mathcal{E}$-groups and hence they are supersoluble. Therefore G/N is soluble (see [46], 10.3.4), so that G is soluble. □

Lemma 2.3.7. *Let F be an infinite locally finite field. Then the simple groups $PSL(2, F)$ and $S_Z(F)$ contain proper subgroups of infinite rank which are not $\mathcal{E}$-groups.*

Proof. Let G be one of the groups $PSL(2, F)$ and $S_Z(F)$. In [43] is proved that G contains a subgroup H such that it is not a Dedekind group and H/H' is not cyclic. It follows from Theorem 2.2.3 that H is not an $\mathcal{E}$-group. □

Lemma 2.3.8. *Let G be a locally soluble group of infinite rank whose proper subgroups of infinite rank are $\mathcal{E}$-groups. Then G is an $\mathcal{E}$-group.*

Proof. The group G is a soluble $\overline{T}$-group (see [11]). We can assume that G is not an abelian group, so that it is a periodic metabelian group.

Assume first that all proper subgroups of infinite rank of G are Dedekind groups, so that the commutator subgroup G' of G is finite (see [14] Proposition 3.1). Let H be any cyclic subgroup of G. Since the factor group G/G' has infinite rank, there exists a subgroup K of G such that $G' \leq K$ and the groups K and G/K have both infinite rank. Now HK is a proper subgroup of infinite rank of G and so it is a Dedekind group; therefore H is subnormal and hence normal in G.

We can now assume that there exists a proper subgroup L of infinite rank of G which is not a Dedekind group. Since L is a locally finite $\mathcal{E}$-group, the factor group L/L' is finite (see Theorem 2.1.7), so that the commutator subgroup L' has infinite rank, and hence G' has likewise infinite rank. Let H be any finitely generated subgroup of G. Since G' is abelian, it contains a subgroup A such that the groups A and G'/A have both infinite rank. The subgroup A is subnormal and hence normal in G; moreover, HA is a proper subgroup of infinite rank of G, and so it is an $\mathcal{E}$-group, so that H is likewise an $\mathcal{E}$-group. Therefore G is an $\mathcal{E}$-group (see Theorem 2.2.3). □

Theorem 2.3.9 (A.V. De Luca - G. di Grazia [20]). *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are $\mathcal{E}$-groups, then G is an $\mathcal{E}$-group.*

Proof. Assume for a contradiction that G is not an $\mathcal{E}$-group, so that, by Lemma 2.3.8, G is not locally soluble. If the commutator subgroup G' of G has finite rank, then all proper subgroups of G are $\mathcal{E}$-groups (see [14], Lemma 2.7), so that by Lemma 2.3.6 G is locally soluble. This contradiction shows that G' has infinite rank, and hence G/G' is finitely generated (see [14] Lemma 2.8). Since the commutator subgroup of any $\mathcal{E}$-group is locally finite (see Theorem 2.2.3), then G' is locally finite (see [14], Theorem A). In particular the set T of all elements of finite order of G form a subgroup and the factor group G/T is a free abelian group of finite rank. Since G is not soluble, the subgroup T cannot be an $\mathcal{E}$-group; it follows that $G = T$ is a periodic group.

Assume now that every proper normal subgroups of G has finite rank, so that G has a simple section G/K of infinite rank (see [14], Lemma 2.4); since all proper subgroups of G are (locally soluble)-by-finite (see [3]), the factor group G/K must be isomorphic either to $PSL(2,F)$ or to $S_Z(F)$ for some infinite locally finite field F (see [43]), and this is impossible by Lemma 2.3.7. Therefore G is locally nilpotent. This contradiction shows that G contains a proper normal subgroup N of infinite rank; in particular N is soluble so that G/N is not soluble. On the other hand, every proper subgroup of G/N is an $\mathcal{E}$-group, so that by Lemma 2.3.6, G/N is soluble. This last contradiction proves the theorem. □

Chapter 3

Groups of infinite rank with normality conditions on subgroup with small normal closure

In this chapter groups of infinite rank in which every subgroup is either normal (subnormal) or contranormal are characterized in terms of their subgroups of infinite rank.

3.1 $\mathcal{AN}$-groups and $\mathcal{SC}$-groups

A subgroup H of a group G is said to be *contranormal* in G if it is not contained in a proper normal subgroup of G, i.e. $H^G = G$ (see for istance [50]). The notion of contranormal subgroup is opposite, in a way, to the notion of normal subgroup.

We shall say that a subgroup H is *alternatively normal* in a group G if H is either normal or contranormal in G. Groups whose subgroups are alternatively normal are called $\mathcal{AN}$*-groups.* Clearly the Dedekind groups and the simple groups are $\mathcal{AN}$-groups.

The following theorem due to Subbotin describes the structure of $\mathcal{AN}$-groups.

Theorem 3.1.1 (I.Ya. Subbotin [60])**.** *A group G is an $\mathcal{AN}$-group if and only if G is a group of following types:*

(i) *G is a Dedekind group.*

(ii) *$G = A\langle b\rangle$, where A is a periodic abelian group whose subgroups are G-invariant, $\langle b\rangle$ is a p-group with p prime, $[A, \langle b\rangle] = A$, $[A, \langle b^p\rangle] = \{1\}$, $\pi(A) \cap \{p\} \subset \{2\}$.*

(iii) *$G = A \rtimes \langle b\rangle$, where A is an abelian group, b is an element of order 2 or 4 and $a^b = a$ for all $a \in A$.*

(iv) *G is an extension of its center by a non-abelian group.*

A natural extension of the class $\mathcal{AN}$ is the class of groups in which every subgroup is either subnormal or contranormal ($\mathcal{SC}$*-groups*). Of course, all homomorphic images of an $\mathcal{SC}$-group are $\mathcal{SC}$-groups. On the other hand, since every simple groups is an $\mathcal{SC}$-group, it is clear that the class of $\mathcal{SC}$-groups is not subgroup closed.

We also note that if H is a contranormal subgroup of a group G, then $HG' = G$. The converse is also true in the soluble case.

Lemma 3.1.2. *Let G be a soluble group and let H be a subgroup of G. Then the following statements are equivalent:*

(1) *H is contranormal in G.*

(2) *$HG' = G$.*

Proof. We have only to prove that (2) implies (1). Let H be a subgroup of G such that $G = HG'$. Then $G = H^G G'$, so that G/H^G is perfect and hence $H^G = G$. Therefore H is contranormal in G. □

Since every group in which all subgroups are subnormal is soluble (see [39]), we have the following:

Lemma 3.1.3. *Let G be an $\mathcal{SC}$-group such that $G' \neq G$. Then G is soluble.*

Proof. Since $G' \neq G$, G' cannot contain contranormal subgroup of G. Thus every subgroup of G' is subnormal, so that G' is soluble, and hence also G is soluble. □

The structure of perfect $\mathcal{SC}$-groups is described by the following result, at least in the case of groups which are not locally nilpotent.

We recall that the *Hirsch-Plotkin radical* is the largest maximal normal locally nilpotent subgroup of a group G.

Proposition 3.1.4. *Let G be a perfect group which is not locally nilpotent. Then G is an $\mathcal{SC}$-group if and only if it contains a largest proper normal subgroup N and all subgroups of N are subnormal.*

Proof. Let G be an $\mathcal{SC}$-group and let N be the Hirsh-Plotkin radical of G. If M is any proper normal subgroup of G, it cannot contain contranormal subgroups of G, so that all its subgroups are subnormal, and it is contained in N. Therefore N is the largest proper normal subgroup of G. Moreover it is clear that all subgroups of N are subnormal.

Conversely, if G has the required structure and H is a subgroup of G which is not contranormal, then H^G is a proper normal subgroup of G, and so it is contained in N. Therefore H is subnormal in G, and G is an $\mathcal{SC}$-group. □

The next result give a description of non perfect $\mathcal{SC}$-group (recall here that a *Baer group* is a group in which all cyclic subgroups are subnormal).

Theorem 3.1.5 (M. De Falco - L.A. Kurdachenko - I.Ya. Subbotin [16]). *Let G be a group such that $G' \neq G$. Then G is an $\mathcal{SC}$-group if and only if one of the following conditions holds:*

(*i*) *All subgroups of G are subnormal.*

(*ii*) *G is a Baer group such that G/G' is a Prüfer group, and all subgroups of G' are subnormal.*

(*iii*) *$G = G'\langle x\rangle$, and there exists a prime number p such that all subgroups of $G'\langle x^p\rangle$ are subnormal and $\langle x^n\rangle$ is contranormal in G for every integer n which is not divisible by p.*

Proof. Suppose first that G is an $\mathcal{SC}$-group which is not of type (i), so that it contains a proper contranormal subgroup B, and in particular $G = G'B$. Assume that $\bar{B} = B/(B \cap G')$ is generated by two proper subgroups $\bar{X}$ and $\bar{Y}$. Then, by Dedekind's modular law, the normal subgroups XG' and YG' are properly contained in G, so that X and Y are not contranormal, and hence they are both subnormal in G. It follows that also $B = XY$ is a subnormal subgroup of G, a contradiction. Therefore $\bar{B}$ cannot be generated by two proper subgroups, and hence it is either a cyclic p-group or a group of type p^∞, where p is a prime.

Suppose first that $\bar{B}$ is a group of type p^∞. Thus also G/G' is a group of type p^∞. Since G' is a proper normal subgroup of G, all its subgroups are subnormal in G. Moreover, if $\langle x\rangle$ is a cyclic subgroup of G which is not contained in G', then $\langle x\rangle G'$ is a proper normal subgroup of G, and hence $\langle x\rangle$ is subnormal in G. Therefore G is a Baer group and so it is a group of type (ii). If $\bar{B}$ is a cyclic p-group, then also G/G' is a cyclic p-group, and G is a group of type (iii).

Suppose now that G satisfies one of the conditions (i)-(iii). Obviously, if (i) holds, then G is an $\mathcal{SC}$-group. Assume that G satisfies (ii), and let H be a subgroup of G. If $G = G'H$, then H is contranormal in G. If $G \neq G'H$, then $G'H/G'$ is cyclic, so that also $H/(H \cap G')$is cyclic. Thus $H = (H \cap G')\langle y \rangle$ for some y. Since G is a Baer group and $H \cap G'$ is subnormal in G, also H is subnormal in G, and G is an $\mathcal{SC}$-group. Suppose finally that G satisfies (iii) and let H be a subgroup of G. If $G = G'H$, then H is contranormal in G. If $G \neq G'H$, then $G'H$ is contained in $G'\langle x^p \rangle$, and hence H is subnormal in G. So G is an $\mathcal{SC}$-group. □

Corollary 3.1.6. *Let G be a torsion non-perfect group which is not a p-group for any prime p. Then G is an $\mathcal{SC}$-group if and only if one of the following conditions holds:*

(i) *All subgroups of G are subnormal.*

(ii) *There exists a prime p such that $G = (Q \times P)\langle x \rangle$, where $Q = O_{p'}(G')$, $P = O_p(G')$, x has p-power order, and all subgroups of $(Q \times P)\langle x^p \rangle$ are subnormal.*

Proof. Suppose that G is an $\mathcal{SC}$-group, and assume first that G is locally nilpotent. Then G is the direct product of its Sylow subgroups. If C is a proper subgroup of G, it follows that C is direct product of subnormal subgroups of G, so that C is not contranormal, and hence it is subnormal in G. Therefore all subgroups of G are subnormal. If G is not locally nilpotent, it follows from Theorem 3.1.5 that G has the structure described in (iii).

The converse statement follows immediately from Theorem 3.1.5. □

Note that the above results generalize the Theorem 3.1.1.

3.2 $\mathcal{AN}_\infty$-groups

A group G is an *$\mathcal{AN}_\infty$-group* if every subgroup of infinite rank is either normal or contranormal in G. It is clear that the class $\mathcal{AN}_\infty$ is quotient closed and that every factor of an $\mathcal{AN}_\infty$-group respect to a normal subgroup of infinite rank is an $\mathcal{AN}$-group.

As in many problems concerning groups of infinite rank, also in our case, the existence of a proper normal subgroup of infinite rank plays a crucial role. Moreover, recall that a group G is said to be a *Dedekind group* if all its subgroups are normal.

Lemma 3.2.1. *Let G be a strongly locally graded $\mathcal{AN}_\infty$-group and let N be a proper normal subgroup of infinite rank of G. Then every subgroup of N is normal in G.*

Proof. Every subgroup of infinite rank of N is normal in G, so that in particular N is a Dedekind group (see Theorem C in [26]).
Let L be a subgroup of finite rank of N. Since N is nilpotent, it contains a direct product $A_1 \times A_2$ such that the subgroups A_1 and A_2 have both infinite rank and $L \cap (A_1 \times A_2) = \{1\}$ (see [38]). Clearly the subgroups A_1 and A_2 are normal in G. Hence the subgroups of infinite rank LA_1 and LA_2 are normal in G, so that $L = LA_1 \cap LA_2$ is normal in G. □

Our next lemma shows in particular that any strongly locally graded group of infinite rank whose proper normal subgroups have finite rank must admit a simple homomorphic image of infinite rank.

Lemma 3.2.2. *Let G be a strongly locally graded group. Then every proper normal subgroup of G has finite rank if and only if the subgroup generated by all proper normal subgroups of G has finite rank.*

Proof. Suppose that G has infinite rank but all its proper normal subgroups have finite rank. Clearly G is perfect and so it is not locally nilpotent by Lemma 2.3 of [7]. Hence G contains a proper normal subgroup N such that G/N is a simple group of infinite rank (see [14], Lemma 2.4). Therefore N

has finite rank. Let H be any proper normal subgroup of G. Since H has finite rank, also HN has finite rank and so it is a proper subgroup of G. Then $HN = N$ and it follows that $H \leq N$ so that N is the subgroup generated by all proper normal subgroups of G. □

The following result will be often used in our proofs.

Lemma 3.2.3. *Let G be a group containing an abelian subgroup A of infinite rank and let H be a subgroup of G such that H^G has finite rank. Then there exists a subgroup B of A such that B has infinite rank and $H^G B$ is a proper subgroup of G.*

Proof. Since H^G is a proper subgroup of G, we can take an element $x \in G \backslash H^G$. Then A contains a direct product $B \times C$ such that the subgroups B and C have both infinite rank and $BC \cap H^G\langle x \rangle = \{1\}$. Now

$$H^G B \cap H^G\langle x \rangle = H^G(B \cap H^G\langle x \rangle) = H^G$$

so $x \notin H^G B$, and hence $H^G B$ is a proper subgroup of G. □

Proposition 3.2.4. *Let G be a strongly locally graded $\mathcal{AN}_\infty$-group. If G contains a proper normal subgroup of infinite rank, then G is an $\mathcal{AN}$-group.*

Proof. Let N be a proper normal subgroup of infinite rank of G. By Lemma 3.2.1 every subgroup of N is normal in G and so N is a Dedekind group. Let H be any subgroup of finite rank of G which is not contranormal, so that H^G is a proper normal subgroup of G. If H^G has infinite rank, then every subgroup of H^G is normal in G by the Lemma 3.2.1 and so H is normal in G.
Suppose now that H^G has finite rank. Since N is a Dedekind group, it contains an abelian subgroup A of infinite rank. By Lemma 3.2.3 there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper normal subgroup of G. Therefore H is normal in G by Lemma 3.2.1 and G is an $\mathcal{AN}$-group. □

It is now easy to prove the main result of this section.

Theorem 3.2.5 (A.V. De Luca - G. di Grazia [21]). *Let G be a locally soluble $\mathcal{AN}_\infty$-group. Then G is an $\mathcal{AN}$-group.*

Proof. Since G is locally soluble, G contains a proper normal subgroup of infinite rank. Therefore G is an $\mathcal{AN}$-group by Proposition 3.2.4. □

In other words, the class of groups of infinite rank controls the embedding property $\mathcal{AN}$ in the universe of locally soluble groups.

3.3 $\mathcal{SC}_\infty$-groups

In this section we will consider groups G in which every subgroup of infinite rank is either subnormal or contranormal. Groups satisfying such property will be called $\mathcal{SC}_\infty$*-groups*. We observe that the class $\mathcal{SC}_\infty$ is closed for homomorphic images and every factor of an $\mathcal{SC}_\infty$-group respect to a normal subgroup of infinite rank is an $\mathcal{SC}$-group.

We need the following elementary property.

Lemma 3.3.1. *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group and let K be a proper subnormal subgroup of infinite rank of G. Then every subgroup of infinite rank of K is subnormal in G.*

In particular we obtain that every proper subnormal subgroup of infinite rank of an $\mathcal{SC}_\infty$-group is soluble (see Theorem 1.7).

Theorem 3.3.2 (A.V. De Luca - G. di Grazia [21]). *Let G be a torsion-free locally (soluble-by-finite) $\mathcal{SC}_\infty$-group. If G contains a proper normal subgroup of infinite rank, then G is an $\mathcal{SC}$-group.*

Proof. Let N be a proper normal subgroup of G of infinite rank; then N is soluble by Lemma 3.3.1. Let H be any subgroup of G of finite rank such that H is not contranormal in G. Then H^G is a proper normal subgroup of G. Clearly, there exists a proper subnormal subgroup K of infinite rank of G which contains H. In fact, if H^G has infinite rank, we can put $K = H^G$; if H^G has finite rank, since N contains an abelian subgroup A of infinite rank, by Lemma 3.2.3 there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper subnormal subgroup of G and in this case we can choose $K = H^G B$. By Lemma 3.3.1 all subgroups of infinite rank of K are subnormal in G and hence K is nilpotent by Theorem 3 (see [33]), so that H is subnormal in G. □

Recall that the *periodic radical* of a group G is the largest periodic normal subgroup of G.

The following lemma will be used to prove the last theorem of the section.

Lemma 3.3.3. *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group containing a proper normal subgroup N of infinite rank. If the periodic radical of G has infinite rank, then every subgroup of N is subnormal in G.*

Proof. By Lemma 3.3.1 every subgroup of infinite rank of N is subnormal in G. So N is soluble and in particular it is a Baer group (see [33], Theorem 2). Let H be any subgroup of finite rank of N. We can suppose that the largest periodic subgroup K of N has finite rank (otherwise H is subnormal in G by Theorem 5 of [33]). Denote by T the periodic radical of G and consider the subgroup NT. If NT is a proper normal subgroup of G then all subgroups of infinite rank of NT are subnormal in G and, since T has infinite rank, H is subnormal in NT by Theorem 5 of [33], and so it is subnormal in G.
Suppose that $G = NT$. Clearly, K is a periodic normal subgroup of G and hence it is contained in T. On the other hand $T \cap N$ is contained in K, so $T \cap N = K$.
Hence, we have that

$$\frac{N}{T \cap N} \simeq \frac{NT}{T} = \frac{G}{T}$$

is a torsion-free group and so T is the set of all elements of finite order of G. Now G/T has infinite rank and all its subgroups of infinite rank are subnormal; so that by Theorem 3 of [33], it is nilpotent. Hence HT is a proper subnormal subgroup of G. By Lemma 3.3.1 every subgroup of infinite rank of HT is subnormal, but T has infinite rank and so H is subnormal in HT by Theorem 5 of [33]. Therefore H is subnormal in G. □

Theorem 3.3.4 (A.V. De Luca - G. di Grazia [21]). *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group containing a proper normal subgroup of infinite rank. If the periodic radical of G has infinite rank, then G is an $\mathcal{SC}$-group.*

Proof. Let H be any subgroup of G of finite rank which is not contranormal in G. Then H^G is a proper normal subgroup of G. If H^G has infinite rank, by Lemma 3.3.3 we have that H is subnormal in G.
Suppose now that H^G has finite rank. If N is a proper normal subgroup of G of infinite rank, then N is soluble by Lemma 3.3.1 and so it contains an abelian subgroup A of infinite rank. By Lemma 3.2.3 there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper subgroup of G. Therefore $H^G B$ is subnormal in G and by Lemma 3.3.1 all its subgroups of infinite rank are subnormal in G, so that H is subnormal in G by Lemma 3.3.3. This completes the proof of the theorem. □

The hypotheses of Theorem 3.3.2 and Theorem 3.3.4 cannot be weakened. Kurdachenko and Smith have proved the existence of a metabelian locally nilpotent group of infinite rank such that the largest periodic subgroup has finite rank, all subgroups of infinite rank are subnormal but there exists a non-subnormal subgroup of finite rank (see [33], Theorem 4). Obviously, this subgroup cannot be even contranormal.

Chapter 4

Groups with a modularity condition on infinite subsets

In this chapter we study the influence on a group G of the condition that every infinite set of cyclic subgroups satisfies a suitable modularity condition.

4.1 Groups with modular subgroup lattice

A lattice $\mathfrak{L}$ is called *modular* if for all $x, y, z \in \mathfrak{L}$ such that $x \leq z$, the modular law holds:

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

We say that an element m of the lattice $\mathfrak{L}$ is *modular* in $\mathfrak{L}$ if, for all $x, y, z \in \mathfrak{L}$

$$x \vee (m \wedge z) = (x \vee m) \wedge z \quad \text{with} \quad x \leq z, \text{ and}$$
$$m \vee (y \wedge z) = (m \vee y) \wedge z \quad \text{with} \quad m \leq z.$$

Of course, a lattice $\mathcal{L}$ is modular if and only if all its elements are modular. A subgroup of a group G is *modular* if it is a modular element of the subgroup lattice $\mathfrak{L}(G)$ of G, and G is called an *M-group* if it has modular subgroup lattice. Abelian groups and Tarski groups are obvious examples of M-groups. Recall also that a subgroup H of a group G is said to be *permutable* (or *quasinormal*) if $HK = KH$ for every subgroup K of G, and G is *quasihamiltonian* if all its subgroups are permutable. Clearly, every normal subgroup is permutable and every permutable subgroup is modular in a group G. It is known that a subgroup is permutable if and only if it is ascendant and modular (see [53] Theorem 6.2.10). Also, we note the following useful properties.

Lemma 4.1.1. *Let G be a soluble group. If M is a maximal subgroup of G that is modular in G, then $|G : M|$ is a prime.*

Proof. We use induction on the derived length of G. If $G' \leq M$, then M is normal in G and $|G : M|$ is a prime; so suppose that $G' \not\leq M$. Since M is a maximal subgroup of G, it follows that $G = G'M$ and, since M is modular in G, $M \cap G'$ in maximal and modular in G'. By induction, $|G' : M \cap G'| = |G : M|$ is a prime. □

Theorem 4.1.2 (S.E. Stonehewer [59])**.** *Let G be generated by two infinite cyclic subgroups M, K intersecting trivially. If M is modular in G, then M is normal in G.*

In this section we determine all groups with modular subgroup lattice.

Let p a prime and $n \geq 2$ be a cardinal number. We say that a group G belongs to the class $P(n, p)$ if G is either elementary abelian of order p^n, or a semidirect product of an elementary abelian normal subgroup A of order p^{n-1} by a group of prime order $q \neq p$ which induces a non-trivial power automorfism on A. Here, if n is infinite, by an elementary abelian group of order p^n or p^{n-1} we shall mean a direct product of n cyclic groups of order p. We call G a *P-group* if $G \in P(n, p)$ for some prime p and some cardinal number $n \geq 2$.

Now we describe first the finite p-group with modular subgroup lattice. We start with an elementary remark.

Lemma 4.1.3. *A finite p-group has modular subgroup lattice if and only if any two of its subgroups permute.*

Theorem 4.1.4 (Iwasawa [53]). *A finite p-group G has modular subgroup lattice if and only if*

(a) *G is direct product of a quaternion group Q_8 of order 8 with an elementary abelian 2-group, or*

(b) *G contains an abelian normal subgroup A with cyclic factor group G/A; further there exists an element $b \in G$ with $G = A\langle b\rangle$ and a positive integer s such that $a^b = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$.*

We say that G is a P^**-group* if G is the semidirect product of an elementary abelian normal subgroup A by a cyclic group $\langle t\rangle$ of prime power order such that t induces a power automorphism of prime order on A. Clearly, every non abelian P-group is a P^*-group.

Theorem 4.1.5 (Iwasawa [53]). *A finite group has modular subgroup lattice if and only if it is a direct product of P^*-groups and modular p-groups with relatively prime orders.*

As an immediate consequence we note the following results.

Theorem 4.1.6. *Every finite group with modular subgroup lattice is metabelian.*

Lemma 4.1.7. *Let G be a finite M-group and $x, y \in G$. If x is a p-element and y a q-element where p and q are primes with $p > q$, then $\langle x\rangle^y = \langle x\rangle$.*

Proof. By Theorem 4.1.5, G is a direct product of groups $G_1, \ldots, G_r$ with relatively prime orders such that every G_i is either a P^*-group or an M-group of prime power order. If x and y are contained in different components of this decomposition, then $xy = yx$ and hence $\langle x\rangle^y = \langle x\rangle$. So suppose that $x, y \in G_i$ for some i. Then G_i is a P^*-group and hence $\langle x\rangle$ is normal in G_i since p is the largest prime dividing the order of G_i. □

Now we give the precise structure of non abelian M-groups with elements of infinite order.

Theorem 4.1.8 (Iwasawa [53]). *Let G be a non-abelian group with elements of infinite order. Then G is an M-group if and only if the following conditions hold:*

(*i*) *The set $T(G)$ of all elements of finite order is a characteristic abelian subgroup of G,*

(*ii*) *$G/T(G)$ is a torsion-free abelian group of rank one,*

(*iii*) *Every subgroup of $T(G)$ is normal in G; all subgroups of prime order and of order 4 are central in G.*

In particular, G is quasihamiltonian.

Consider the periodic case.

Theorem 4.1.9 (Iwasawa [53]). *A group G is a locally finite M-group if and only if it is a direct product of P^*-groups and locally finite p-groups with modular subgroup lattice such that elements of different direct factors have relatively prime orders.*

We say that a group G is a *Tarski group* if it is infinite simple but every proper non trivial subgroup of G has prime order. For a long time it was not known whether Tarski groups exist until Olshanskii (1979) produced the first example of such groups. Cleary the subgroup lattice of a Tarski group is modular. We call a group G an *extended Tarski group* if it contains a normal subgroup N such that

1. G/N is a Tarski group,

2. N is a cyclic p-group, with p prime,

3. for every subgroup H of G, $H \leq N$ or $N \leq H$.

So, we have the following result as our description of periodic M-groups.

Theorem 4.1.10 (Schmidt [53]). *Let G be a torsion group. Then G has modular subgroup lattice if and only if G is a direct product of Tarski groups, extended Tarski groups and a locally finite M-group such that elements of different direct factors have relatively prime orders.*

In 4.1.6 we noted that every finite M-group is metabelian. This is also true for locally finite M-groups and M-groups with elements of infinite order.

Theorem 4.1.11. *Let G be an M-group. If G contains elements of infinite order or if no Tarski group is involved in G, then G is metabelian.*

If any two subgroups of the group G permute, then G is an M-group and, clearly, G is locally finite if it is a torsion group. So the Theorem 4.1.8 and the results on locally finite M-groups give the structure of these groups.

Proposition 4.1.12. *Let G be a torsion group. The following statements are equivalent:*

(a) *Any two subgroups of G permute.*

(b) *G is the direct product of its p-components and these are locally finite p-groups with modular subgroup lattice.*

(c) *G is a locally nilpotent M-group.*

4.2 Groups with many permutable subgroups

In response to a question of Paul Erdös, B.H. Neumann proved in [41] that a group is central-by-finite if and only if the subsets consisting of mutually non commuting elements are finite. It will be discussed here the following rather similar case.

A group G is said *pseudo-Hamiltonian*, or a *PH-group*, if every infinite set of subgroups of G contains a pair that permute. More in general, G is a *PH^*-group* if every infinite set of cyclic subgroups contains a pair that permute. Of course, all central-by-finite groups satisfy PH, though the converse is false. Infact Napolitani and Iwasawa [31] have constructed quasihamiltonian groups as follows:

$$H_{n,q} = \langle a, b : a^{q^n} = 1,\ b^{q^{n-1}} = 1,\ a^b = a^{1+q} \rangle$$

where q is a prime and n a positive integer. The direct product of any number of groups of this type of coprime order with a suitable choise of n and q give rise to PH-groups that are not nilpotent and not central-by-finite.

We need some lemma for prove the main theorems of this section.
Recall that a group G is said *eremitic* if there exists a positive integer e such that whenever an element of G has some positive power in a centralizer C, it has its eth power in C.

Lemma 4.2.1. *Let G be a finitely generated soluble group such that for all x, y in G there exist integers $n, i \geq 1$, such that $[x^i, y^m, y^m] = 1$. Then G is nilpotent-by-finite.*

Proof. By induction on the derived length of G, we may assume that G is (abelian-by-nilpotent)-by-finite, and thus, ignoring the finite factor at the top, that G is abelian-by-nilpotent. Thus G is eremitic (see [35] Theorem B); this means that there is an integer $e \geq 1$ depending only on G such that $[u, v^e] = 1$ whenever $[u, v^m] = 1$ for elements u, v of G and an integer $m > 1$. Let A be an abelian normal subgroup of G such that G/A is nilpotent. For $a \in A$ and $y \in G$, we have $[(ya)^i, y^n, y^n] = 1$ for some $i, n > 1$; so, after a small calculation, we have $[b, y^n, y^n] = 1$, where $b = a^{1+y+y^2+\cdots+y^{i-1}}$. But $\langle a, y \rangle$ is metabelian, so $[b, y^e, y^n] = 1$ and $[b, y^e, y^e] = 1$. Write A additively; then $a(1 - y^i)(1 - y^e)^2 = 0$, and so by eremiticity, $a(1 - y^e)^3 = 0$.

It will be enough if we show that G^e is nilpotent, since it is of finite index. All the elements of the form y^e act nilpotently on $A \cap G^e$, so $\langle A \cap G^e \rangle$ is nilpotent. But this subgroup is subnormal in G^e, so G^e is locally nilpotent and thus nilpotent since it is finitely generated. □

Lemma 4.2.2. *Let G be a finitely generated soluble group such that for every cyclic subgroup H and every element x of G, there exists a positive integer i such that $HH^{x^i} = H^{x^i}H$. Then G is nilpotent-by-finite.*

Proof. For arbitrary x, y in G, there must exists $i > 0$ such that

$$\langle y \rangle \langle y \rangle^{x^i} = \langle y \rangle^{x^i} \langle y \rangle.$$

Thus, by [30], the product $K = \langle y \rangle \langle y \rangle^{x^i}$ has a torsion-free abelian subgroup of finite index. Thus $[y^m, (y^m)^{x^i}] = 1$, for some $m \geq 1$, so $[x^i, y^m, y^m] = 1$ for some $m \geq 1$ and some $i > 0$. Therefore G is nilpotent-by-finite by Lemma 4.2.1. □

Lemma 4.2.3. *Every torsion-free nilpotent PH^*-group is abelian.*

Proof. We may assume that G is 2-generated, say $G = \langle x, y \rangle$. Let Z be the centre of G. Then G/Z is torsion-free and so by an obvious induction, it is abelian and hence G is of class 2. Setting $H_i = \langle xy^i \rangle$, $i \in \mathbb{Z}$, we have $H_n H_m = H_m H_n$ for some $n, m > 0$, $n \neq m$ so $H_n H_m$ is a metabelian group by Itô Theorem and by [30] has an abelian subgroup of finite index. But this group is also torsion-free nilpotent of class 2 and an easy argument shows that it is abelian. Hence xy^n and xy^m commute, and once again, the fact that G is torsion-free nilpotent of class 2 gives that x and y must commute. □

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G with finitely many conjugates, and a group is said to be an

FC*-group* if it coincides with FC-centre. It is well-known that any finitely generated FC-group is central-by-finite.

We are now in a position to prove the main results.

Theorem 4.2.4 (M. Curzio - J. Lennox - A. Rhemtulla - J. Wiegold [5])**.** *Every finitely generated soluble* PH^**-group is central-by-finite.*

Proof. Let G be a finitely generated soluble PH^*-group. By Lemma 4.2.2, G is nilpotent-by-finite, so it has a torsion-free nilpotent subgroup A of finite index; by Lemma 4.2.3 A is abelian. We proceed by induction on $|G/A|$, all being well when $G = A$.

If $\langle A, x\rangle < G$ for all $x \in G$, then $\langle A, x\rangle$ is an FC-group for all x in G, so G is an FC-group. Finitely generated FC-groups are central-by-finite and thus we may assume that $\langle A, x\rangle = G$ for some x. Let n be the order of x modulo A. If n is not a prime power, then $n = rs$ with $(r, s) = 1$. Since $\langle A, x^r\rangle$ and $\langle A, x^s\rangle$ are proper subgroups of G, the centralizers $C_G(x^r)$ and $C_G(x^s)$ are of finite index in G, as above, and $C_G(x) > C_G(x^r) \cap C_G(x^s)$.

We know now that $n = p^m$ for some prime p. By induction, $\langle A, x^p\rangle$ is central-by-finite. Thus $[A, x^p]$ is finite; since A is torsion-free and normal, this means that $[A, x^p] = 1$. Thus the group $B = \langle A, x^p\rangle$ is abelian, and of course x^p is in the centre of G.

We can assume that x has infinitely many conjugates in G, else the centre $Z(G)$ has finite index, since it contains $A \cap C_G(x)$. Thus, there must exist an a such that $\langle x^{a^i}\rangle \neq \langle x^{a^j}\rangle$ if $i \neq j$. Property PH^* now means that $Y = \langle x\rangle\langle x^b\rangle = \langle x^b\rangle\langle x\rangle$ for some $b = a^i$. Modulo the central subgroup $\langle x^p\rangle$, Y has order p or p^2, so $[x, b]^p \in \langle x^p\rangle$; since $[x, b]^p = [x, b^p]$, we have the contradiction that $\langle x^{b^p}\rangle = \langle x\rangle$. Thus x has only finitely many conjugates, and G is central-by-finite. This completes the proof of the theorem. □

Theorem 4.2.5 (M. Curzio - J. Lennox - A. Rhemtulla - J. Wiegold [5])**.** *All torsion-free* PH^**-groups are abelian.*

Proof. It is sufficient to assume that $G = \langle g_1, \ldots, g_k\rangle$ is finitely generated. Suppose that a, b are elements of G such that $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$. Since this product is metabelian, it follows from Theorem 4.2.4 and the fact that G is torsion-free that $\langle a, b\rangle$ is abelian. Now for any pair x, y of elements of G, there exists $i > 0$ such that $\langle y\rangle\langle y^{x^i}\rangle = \langle y^{x^i}\rangle\langle y\rangle$. Thus $\langle y, y^{x^i}\rangle$ is abelian. Similary, $\langle x, x^{y^j}\rangle$ is abelian for some $j > 0$. Hence $\langle x^i, y^j\rangle$ is nilpotent and hence abelian by Theorem 4.2.4. Since $[x^i, y, y] = 1$, we have $1 = [x^i, y^j] = [x^i, y]^j$, so $[x^i, y] = 1$. In particular, by considering the pairs (x, g_j), for $j = 1, \ldots, k$, we get $[x^t, G] = \{1\}$ for some $t > 0$. This shows that $G/Z(G)$ is periodic.

Obtain, if possible, a sequence $(a_1, a_2, \ldots)$ of elements of G as follows. Pick any $a_1 \in G\backslash Z(G)$ and for $i \geq 2$, pick a_i from $G \setminus \bigcup_{j=1}^{i-1} C_G(a_j)$. If $\bigcup_{j=1}^{n} C_G(a_j) = G$ for some $n \in \mathbb{N}$, then $C_G(a_i)$ is of finite index in G for some $i \leq n$. Set $A = \langle a_i^G\rangle$. Then A is in the FC-centre of G and $[A, G]$ is finite. But G is torsion-free, whence $A \leq Z(G)$, contradicting our choice of a_i. We conclude that in this case G is abelian.

The other alternative is the existence of an infinite sequence $(a_1, a_2, \ldots)$ as constructed above. By hypothesis, $\langle a_j\rangle\langle a_i\rangle = \langle a_i\rangle\langle a_j\rangle$ for some $0 < i < j$. In this case $\langle a_i, a_j\rangle$ is abelian, as shown earlier in the proof. But then $a_j \in C_G(a_i)$, a contradiction. This completes the proof of the theorem. □

4.3 Groups with a modularity condition on infinite subsets

It was shown in the previous section that if G is a finitely generated soluble group such that every infinite set of cyclic subgroups of G contains two distinct elements H and K such that $HK = KH$, then G is finite over its centre. It is proved here that a similar result holds when the permutability is replaced by modularity.

Lemma 4.3.1. *Let G be a finitely generated soluble group such that for every cyclic subgroup H and every element x of G, there exists a positive integer i such that H and H^{x^i} are modular in $\langle H, H^{x^i}\rangle$. Then G is nilpotent-by-finite.*

Proof. Let x, y be elements of G, and let i be a positive integer such that $\langle y\rangle$ and $\langle y\rangle^{x^i}$ are modular in $\langle y, y^{x^i}\rangle$. Assume that y has infinite order. If $\langle y\rangle \cap \langle y\rangle^{x^i} = \{1\}$ then $\langle y, y^{x^i}\rangle = \langle y\rangle \times \langle y\rangle^{x^i}$ (see Theorem 4.1.2). On the other hand, if $\langle y\rangle \cap \langle y\rangle^{x^i} \neq \{1\}$, then y^m lies in the centre of $\langle y, y^{x^i}\rangle$, for some $m \geq 1$. Therefore, in any case there exists a positive integer n such that $[x^i, y^n, y^n] = 1$, and hence the group G is nilpotent-by-finite (see Lemma 4.2.1). □

Lemma 4.3.2. *Let G be a torsion-free nilpotent group such that every infinite set of cyclic subgroups contains two distinct subgroups H and K such that at least one of them is modular in the subgroup $\langle H, K\rangle$. Then G is abelian.*

Proof. Since G is nilpotent, all its subgroups are subnormal, and in particular, in any subgroups of G every modular subgroup is permutable. Therefore the statement follows from Lemma 4.2.3. □

We can now prove the main result of the section.

Theorem 4.3.3 (A.V. De Luca - G. di Grazia [19])**.** *Let G be a finitely generated soluble group such that every infinite set of cyclic subgroups contains two distinct subgroups H and K which are modular in the subgroup $\langle H, K\rangle$. Then the centre $Z(G)$ has finite index in G.*

Proof. The group G is nilpotent-by-finite by Lemma 4.3.1, so that it follows from Lemma 4.3.2 that G contains a torsion-free abelian normal subgroup A of finite index. Assume that the statement is false, and choose a counterexample G such that the factor group G/A has smallest possible order.

Let g be any element of G such that $\langle A, g\rangle$ is properly contained in G. It follows from the minimal assumption on $|G/A|$ that $\langle A, g\rangle$ is an FC-group, so that it is contained in the FC-centre of G. As the finitely generated group G cannot be an FC-group, there exists an element x such that $\langle A, x\rangle = G$ and the cyclic group G/A must have prime-power order p^m. Now A is centralized by x^p, since it is a torsion-free normal subgroup of the FC-group $\langle A, x^p\rangle$, and hence x^p lies in the centre of G.

Clearly A is finitely generated, and the element x has infinitely many conjugates in G, so that there exists an element a of A such that $\langle x^{a^i}\rangle \neq \langle x^{a^j}\rangle$ for all distinct integers i and j. It follows that there exists a positive integer k such that the subgroups $\langle x\rangle$ and $\langle x^{a^k}\rangle$ are modular in $Y = \langle x, x^{a^k}\rangle$. Then $\langle x\rangle/\langle x^p\rangle$ and $\langle x^{a^k}\rangle/\langle x^p\rangle$ are modular subgroups of $Y/\langle x^p\rangle$, so that they are maximal in $Y/\langle x^p\rangle$, since they both have order p. It follows that $Y/\langle x^p\rangle$ is finite (see Lemma 4.1.1). Therefore there exists a positive integer t such that $(x^{-1}x^{a^k})^t$ belongs to $\langle x^p\rangle$. On the other hand,

$$(x^{-1}x^{a^k})^t = [x, a^k]^t = [x, a^{kt}],$$

so that $[x, a^{kt}]$ belongs to $\langle x\rangle$. Then $\langle x\rangle = \langle x^{a^{kt}}\rangle$, as also $(x^{-1}x^{a^k})^{-t}$ belongs to $\langle x^p\rangle$. This contradiction completes the proof of the statement. □

It is well-known that a group generated by two cyclic modular subgroups need not have modular subgroup lattice. It can be proved that the hypotheses of the above theorem can be weakened, provided that we require that every infinite set of cyclic subgroups of G contains two distinct elements generating a group with modular subgroup lattice. In fact, we have:

Theorem 4.3.4 (A.V. De Luca - G. di Grazia [19]). *Let G be a finitely generated group such that every infinite set of elements of G contains two distinct elements a and b such that the lattice $\mathfrak{L}(\langle a, b\rangle)$ is modular. Then the factor group $G/Z(G)$ is periodic.*

Proof. Let $G = \langle x_1, \dots, x_n \rangle$, and let g be any element of infinite order of G. For $i \in \{1, \dots, n\}$, consider the set $\Gamma_i = \{g^n x_i \mid n \in \mathbb{Z}\}$. Since Γ_i is infinite, there exist distinct positive integers h and k such that $L_i = \langle g^h x_i, g^k x_i \rangle$ has modular subgroup lattice. Moreover, L_i contains the element of infinite order $g^{h-k} = (g^h x_i)(g^k x_i)^{-1}$, and hence it is quasihamiltonian. It follows that L_i has finite commutator subgroup (see Theorem 4.1.8), and hence the index $|L_i : Z(L_i)|$ is finite. In particular, there exists a positive integer m such that $g^{(h-k)m} \in Z(L_i)$, and so $g^{(h-k)m} \in C_G(x_i)$. Therefore $\langle g \rangle \cap Z(G) \neq \{1\}$, and hence $G/Z(G)$ is periodic. □

The above theorem provides in particular an extension of Theorem 4.3.3 to the case of finitely generated hyper-(abelian or finite) groups.

Chapter 5

Groups of infinite rank which are isomorphic to their non abelian subgroups of infinite rank

In this chapter we give another rank condition. Also in this case the imposition of some conditions on subgroups of infinite rank can influnce the structure of whole group.

5.1 Groups isomorphic to their non-abelian subgroups

A group G is an $\mathfrak{X}$-*group* if it contains proper non-abelian subgroups, all of which are isomorphic to G. Clearly, every $\mathfrak{X}$-group is infinite and 2-generated. There is a satisfactory classification of soluble groups in the class $\mathfrak{X}$ and, although it does not know whether there exist insoluble $\mathfrak{X}$-groups, nevertheless we are able to show that such groups would have to be very restricted in structure.

Let begin with a very easy result.

Lemma 5.1.1. *If G is an abelian-by-finite $\mathfrak{X}$-group, then G is metabelian.*

Proof. If G is central-by-finite then G' is finite and therefore abelian. Otherwise, there exists a non-central normal abelian subgroup A and then, for some $g \in G$, we have $\langle A, g\rangle$ non-abelian and therefore isomorphic to G. Then result follows. □

Theorem 5.1.2 (H. Smith - J. Wiegold [56]). *Let G be an insoluble $\mathfrak{X}$-group, and let Z denote the centre of G. Then G is 2-generated and G/Z is infinite simple. Moreover, Z is contained in every non-abelian subgroup of G.*

Proof. Let G and Z be as stated, and let A denote the Hirsch-Plotkin radical of G. Certainly G is not locally nilpotent, and so A is abelian. Infact $A = Z$, otherwise $G \simeq \langle A, g\rangle$ for some $g \in G$, giving the contradiction that G is soluble. By Lemma 5.1.1, G/Z is infinite. Suppose, for a contradiction, that there exists a normal subgroup N of G such that $Z < N < G$. For some $g \in G \setminus N$ we have $\langle N, g\rangle$ non-abelian and hence isomorphic to G, and so G has a non-trivial finite image. It follows that G is locally graded and hence, by Lemma 1 of [55], that G/Z is locally graded. Now Z is also the Hirsh-Plotkin radical of N and, since $N \simeq G$, we deduce that $N/Z \simeq G/Z$, that is, G/Z is isomorphic to all of its non-trivial normal subgroup. Since G/Z has a non-trivial finite image, we may apply the main result of [36] to obtain the contradiction that G/Z is cyclic. Thus G/Z is simple and $G'Z = G$, and so $G' = G''$. But $G \simeq G'$ and so G is perfect. Thus if H is any non-abelian subgroup of G then we have $HZ = (HZ)' = H' = H$, and the proof is complete. □

Now consider the nilpotent case. It convenient to state the conclusion here in the form of lemmas.

Lemma 5.1.3. *Let G be a nilpotent group. Then G is an $\mathfrak{X}$-group if and only if G has one of the following presentations (where "nil − 2" denotes the pair of relations $[a, b, b] = 1$, $[a, b, a] = 1$, p is an arbitrary prime and k is an arbitrary positive integer).*

(i) $\langle a, b \mid \text{nil} - 2,\ [a, b]^p = 1\rangle$,

(ii) $\langle a, b \mid \text{nil} - 2,\ [a, b]^p = b^{p^k} = 1\rangle$,

(iii) $\langle a, b \mid \text{nil} - 2,\ [a, b]^2 = 1,\ b^{2^k} = [a, b]\rangle$,

(iv) $\langle a, b \mid \text{nil} - 2,\ [a, b]^3 = 1,\ b^{3^k} = [a, b]\rangle$.

Proof. First assume that G is a nilpotent $\mathfrak{X}$-group. Certainly G has class exactly 2 and is generated by two elements a and b, say. Suppose that $[a, b]$ has infinite order; then G is free nil − 2 and $G' = Z(G)$. For each $n > 1$, set $H_n = \langle a^n, b, [a, b]\rangle$. Then $H_n' = \langle [a, b]^n\rangle$ and $Z(H_n) = \langle [a, b]\rangle$, so $Z(H_n)/H_n'$ is cyclic of order n, and we even have that G contains infinitely many pairwise non-isomorphic of non abelian subgroups. By this contradiction, $[a, b]$ has finite order m, say. If $m = rs$ for some $r, s > 1$, then the subgroup $H = \langle a^r, b\rangle$ has its commutator subgroup of order $s \neq |G'|$, another contradiction. So $|G'| = p$, a prime,and there are just the following cases to consider.

Case 1. G/G' is free abelian, so G has the presentation (i).

Case 2. $G/G' = \langle aG'\rangle \times \langle bG'\rangle$, where $|aG'|$ is infinite and $|bG'|$ is finite. Suppose here that bG' has order $p^k l$ where $(p, l) = 1$, and set $K = \langle a, b^l\rangle$. Then $K' = G'$ and b^l has order p^k modulo G', and $K \simeq G$ implies $l = 1$. Thus b has order p^k modulo G', for some positive integer k, and either $b^{p^k} = 1$, in which case we have the presentation (ii), or $b^{p^k} = [a, b]^s$, for some integer s prime to p, and we now assume that this relation holds. Let H be a non-abelian subgroup of G. Then $H' = G'$ and $H/H' = \langle bH'\rangle \times \langle a^\alpha H'\rangle$ for some (arbitrary) integer α prime to p. Thus

$H = \langle a^\alpha, b\rangle$. Suppose that $\theta : G \to H$ is an isomorphism. Then $\theta(a) = a^{\alpha\varepsilon}b^r$, $\theta(b) = b^t$, where $\varepsilon = \pm 1$ and r, t are integers, with $(t, p) = 1$. Now we have $[a, b]^{st} = b^{p^k t} = \theta(b^{p^k}) = [a^{\alpha\varepsilon}b^r, b^t]^s = [a, b]^{\alpha\varepsilon st}$, and so p divides $st(\alpha\varepsilon - 1)$. Thus $\alpha\varepsilon \equiv 1(\text{mod } p)$, that is, $\alpha \equiv \pm 1(\text{mod } p)$. But this must hold for all α prime to p, and so $p = 2$ or 3, and we have the presentation (iii), (iv)- note that replacing a by a^{-1} allows us to assume that $s = 1$ in the case where $p = 3$.

It remains only to show that a group G having one of the presentations (i)-(iv) is an $\mathfrak{X}$-group. We shall retain the appropriate notation at each stage.

In case (i), an arbitrary non-abelian subgroup H of G satisfies $H' = G'$ and H/H' free abelian, and so $H \simeq G$. In case (ii), every non-abelian subgroup H is of the form $\langle a^\alpha, b\rangle$, where $(p, \alpha) = 1$, and the map $a \to a^\alpha$, $b \to b$ extends to an isomorphism from G to H. Each non-abelian subgroup H is also of this type in the remaining two cases. The map $a \to a^{\alpha\delta}$, $b \to b$, with $\delta = 1$ if $\alpha \equiv 1(\text{mod } p)$ and $\delta = -1$ if $\alpha \equiv -1(\text{mod } p)$ (where $p = 2$ or 3) again extends to an isomorphism θ from G to H, as the following calculations shows: $\theta(b)^{p^k} = [\theta(a), \theta(b)]^s$ iff $b^{p^k} = [a^{\alpha\varepsilon}, b]^s$ iff $[a, b]^s = [a, b]^{\alpha\varepsilon s}$, which is true and so all relations are satisfied, and θ extends to a homomorphism onto H. Also $\theta(a^m b^n) = 1$ implies $a^{\alpha\varepsilon m}b^n = 1$, which implies that p divides n and $m = 0$ and so θ is injective. The lemma is thus proved. □

Now we deal with central-by-finite $\mathfrak{X}$-groups. Again it is convenient to isolate this part of the argument.

Lemma 5.1.4. *Let G be a non-nilpotent central-by-finite $\mathfrak{X}$-group. Then $G = \langle A, x\rangle$, where A is a finite elementary abelian p-subgroup of order p^n which is minimal normal in G, x is of infinite order and has order q modulo $Z(G)$, where p, q are distint primes, and for each k in the interval $1 \leq k \leq q - 1$, $\bar{x}$ is conjugate to $\bar{x}^k$ or $\bar{x}^{-k}$ in $GL(n, p)$, where now $\bar{x}$ denotes the image of x under the natural map from $\langle x\rangle$ to $GL(n, p)$.*

Proof. As in the proof of Lemma 5.1.1, G' is finite and therefore abelian. Since G' is not central it has a non-central Sylow p-subgroup, and we may write $G = A \rtimes \langle x \rangle$, where A is finite normal abelian p-subgroup of G and x has infinite order. Now $G' = [A, \langle x \rangle]$ and so $[a, x, x] \neq 1$ for some element a of A, and we have $\langle [a, x], x \rangle \simeq G$. But $[a, x]^p = [a^p, x] = 1$, since $\langle A^p, x \rangle$ is certainly not isomorphic to G. It follows that A has exponent p. Suppose that x has order n modulo $Z(G)$. If $n = rs$, where $r, s > 1$, then $\langle A^r, x \rangle$ is not abelian and is therefore isomorphic to G. But this easily gives a contradiction, and so $n = q$, a prime. Certainly $q \neq p$, since G is not nilpotent. Further, if A contains a proper non trivial G-invariant subgroup B then, by Maschke's Theorem, we have $A = B \times C$, where C is also non-trivial and G-invariant. Now either $\langle B, x \rangle$ or $\langle C, x \rangle$ is isomorphic to G, another contradiction. Finally, if q does not divide k then $\langle A, x^k \rangle$ is isomorphic to G and so x^k acts like $x^{\pm 1}$ on A and the conjugacy condition follows.

Now suppose that G is a group having the structure indicated, and let H be a non-abelian subgroup of G. Then H contains a non-trivial element b of A and an element of the form $g = ux^\lambda$, where $u \in A$ and $\lambda \not\equiv 0 (\text{mod } q)$. Since A is minimal normal we have $\langle b \rangle^{\langle g \rangle} = A$, and so H is normal in G and $H = \langle A, x^\mu \rangle$, for some μ which is not a multiple of q. Clearly then $H \simeq G$, and the result follows. □

The following lemma is fundamental.

Lemma 5.1.5. *Let G be a soluble $\mathfrak{X}$-group. Then G has a normal abelian subgroup of prime index.*

Proof. By Lemma 5.1.3, a nilpotent $\mathfrak{X}$-group has a normal abelian subgroup, namely $\langle a^p, b \rangle G'$ in the notation there employed, of prime index p. Assuming G is not nilpotent, we see that the Hirsh-Plotkin radical A of G is abelian and self-centralizing, and it is clear that if G/A is finite then it is of prime order. Thus we may assume that G is not abelian-by-finite, and hence that $G = A \rtimes \langle x \rangle$ for some element x of infinite order.

Suppose that A contains a non-central element a of finite order. Then $H = \langle a, x \rangle$ is isomorphic to G. Now $H \cap A = \langle a \rangle^{\langle x \rangle}$ is the Hirsch-Plotkin radical of H, else $\langle a, x^n \rangle$ is locally nilpotent and hence abelian for some $n > 0$, giving the contradiction that H is abelian-by-finite. Thus $A \simeq H \cap A$ and A has finite exponent. We may now argue as in the proof of Lemma 5.1.4 to deduce that A has prime exponent n, say. If A is finite then G is central-by-finite, a contradiction. It follows that G is (isomorphic to) the wreath product $\langle a \rangle \wr \langle x \rangle$. But the non-abelian subgroup $A\langle x^n \rangle$ is not even 2-generated. By this contradiction, the torsion group T of A is central in G. If G/T is abelian-by-finite then G is nilpotent-by-finite, another contradiction. Let H/T be a non-abelian subgroup of G/T. Then, of course, H is not abelian and T is the maximal torsion subgroup of H, and so $H/T \simeq G/T$ and G/T belongs to $\mathfrak{X}$.

Factoring by T if necessary, we may assume that G is torsion-free. Let Z denote the centre and C the second centre of G. If $C > Z$ then $\langle C, x \rangle$ is non-abelian and therefore isomorphic to G, giving the contradiction that G is nilpotent. So Z is the hypercentre of G; also A/Z is torsion-free. Let $D = C_A(x^n)$, where n is some positive integer. Then D is normal in G and $\langle D, x \rangle$ is abelian-by-finite and hence abelian, giving $D = Z$. Suppose that H/Z is a non-abelian subgroup of G/Z. Certainly H is non-abelian, and so it contains an element of the form ax^n, for some positive integer n and element a of A. Also, of course, $H \cap A \neq \{1\}$, and it follows that $C_G(H \cap A) = A$ and hence that $Z(H) = Z$. Thus $H/Z \simeq G/Z$, and we have $G/Z \in \mathfrak{X}$. Factoring as before, we may assume that $Z = \{1\}$ and hence that $C_A(x^n) = \{1\}$ for all positive integer n. We claim that G has finite rank; assuming this to be false, we have from [32] that G contains a section L/K which is isomorphic to $C_p \wr C_\infty$ for some prime p. Since L is isomorphic to G, we may as well write

$G = L$. Let B/K denote the base group of G/K. Then B is not isomorphic to G, since it is not finitely generated. Hence B is abelian and, since G/B is infinite cyclic, we see that $B = A$. But then $\langle A, x^p \rangle$ is not 2-generated (modulo K) and we have the contradiction that $\langle A, x^p \rangle$ is abelian. This estabilishes the claim.

Next, suppose that A is finitely generated, of rank r, say, and let $\{a_1, \dots, a_r\}$ be a $\mathbb{Z}$-basis for A. Relative to this basis, the action of x on A may be represented by an invertible $r \times r$ matrix X with integer entries. Set $H = \langle A, x^2 \rangle$, and let θ be an isomorphism from G to H. Since A is the Hirsch-Plotkin radical of both G and H, it is fixed by θ, which is therefore determined by some assignment $a_i \to b_i$ $(i = 1, \dots, r)$, $x \to ax^{\pm 2}$, where $\{b_1, \dots, b_r\}$ is also a basis for A and a is some element of A. By taking the composite with the isomorphism $b_i \to b_i$, $ax^{\pm 2} \to x^{\pm 2}$, we may assume that $a_i \to b_i$, $x \to x^{\pm 2}$. Suppose M represents the change of basis $\{a_1, \dots, a_r\} \to \{b_1, \dots, b_r\}$; then θ restricted to A is rappresented by M and, since θ is an isomorphism, we have $MX^{\pm 2} = XM$, or $M^{-1}XM = X^{\pm 2}$. Consider the subgroup $K = \langle X, M \rangle$ of $GL(r, \mathbb{Z})$; this is a homomorphic image of either $U = \langle a, b \mid a^b = a^2 \rangle$ or $V = \langle a, b \mid a^b = a^{-2} \rangle$ via the assignment $a \to X$, $b \to M$. Now each of U and V is an extension of the dyadic rationals by the infinite cyclic group $\langle b \rangle$ and so K is soluble and hence polycyclic (see Chapter 2 of [52], for example). But, in every polycyclic image of U or V, the image of the subgroup $\langle a \rangle^{\langle b \rangle}$ is finite, and so X has finite order n, say. This gives the contradiction that $[A, x^n] = \{1\}$. Thus A is not finitely generated and, since $C_A(x) = \{1\}$, we see that A contains no non-trivial finitely generated G-invariant subgroups. Since A has rank r, there exists an r-generated subgroup A_0 of A such that A/A_0 is periodic. Further, since $\langle A_0, x \rangle$ is not abelian, we may assume $A_0^{\langle x \rangle} = A$. Write $A_1 = A_0 A_0^x A_0^{x^{-1}}$. Then $|A_1 : A_0| = m$, where m is an integer greater than 1, since A_0 is not normal in G. Setting $A_2 = A_1 A_1^x A_1^{x^{-1}}$, we note that each of the indices $|A_1^x : A_0^x|$ and $|A_1^{x^{-1}} : A_0^{x^{-1}}|$ is also m and hence that $|A_2 : A_0|$ divides m^3. We deduce that A/A_0 is a π-group for some finite set π of primes, namely those dividing m. Since A has finite rank but is not finitely generated, it has a subgroup A^* such that $A/A^* \simeq C_{q^\infty}$ for some prime q. Let $\mathcal{L}$ be the set of all subgroups L of A such that $A/L \simeq C_{q^\infty}$. It easy to see that no member L of $\mathcal{L}$ can be isomorphic to A, and it follows from the $\mathfrak{X}$-property that, for each L in $\mathcal{L}$, $L^{\langle x \rangle} = A$. Choose $B \in \mathcal{L}$ such that the index $|BB^xB^{x^{-1}} : B| = q^\beta$, say, is minimal. Since B is not normal in G we have $\beta > 0$ and thus, for all $L \in \mathcal{L}$, $|LL^{x^2}L^{x^{-2}} : L| > q^\beta$ (here we are using the fact that A/L is locally cyclic). Now let $J = \langle A, x^2 \rangle$.

There exists an isomorphism ϕ from G onto J and, as for our previous isomorphism θ, we may assume that $\phi(x) = x^{\pm 2}$. Also, $\phi(A) = A$ and so the set $\mathcal{L}$ is invariant under ϕ. Thus

$$|BB^xB^{x^{-1}} : B| = |\phi(B)\phi(B)^{x^{\pm 2}}\phi(B)^{x^{\mp 2}} : \phi(B)| > q^\beta,$$

a contradiction which concludes the proof of lemma. □

So we have the main theorem of this section.

Theorem 5.1.6 (H. Smith - J. Wiegold [56]). *Let G be a soluble group.*

(*a*) *If $G \in \mathfrak{X}$ then G contains an abelian normal subgroup of prime index.*

(*b*) *If G is nilpotent then $G \in \mathfrak{X}$ if and only if G has one of the following presentations (where "$\mathrm{nil} - 2$" denotes the pair of relations $[a,b,b] = 1$, $[a,b,a] = 1$, p is an arbitrary prime and k is an arbitrary positive integer).*

(*i*) $\langle a, b \mid \mathrm{nil} - 2,\ [a,b]^p = 1 \rangle$,

(*ii*) $\langle a, b \mid \mathrm{nil} - 2,\ [a,b]^p = b^{p^k} = 1 \rangle$, (*iii*) $\langle a, b \mid \mathrm{nil}-2,\ [a,b]^2 = 1,\ b^{2^k} = [a,b]\rangle$,

(*iv*) $\langle a, b \mid \mathrm{nil}-2,\ [a,b]^3 = 1,\ b^{3^k} = [a,b]\rangle$.

(*c*) *If* G *is not nilpotent then* $G \in \mathfrak{X}$ *if and only if either*

(1) $G = \langle A, x\rangle$, *where* A *is a finite elementary abelian* p*-subgroup of order* p^n *which is minimal normal in* G, x *is of infinite order and has order* q *modulo* $Z(G)$, *where* p, q *are distint primes, and for each* k *in the interval* $1 \leq k \leq q-1$, $\bar{x}$ *is conjugate to* $\bar{x}^k$ *or* $\bar{x}^{-k}$ *in* $GL(n,p)$, *where now* $\bar{x}$ *denotes the image of* x *under the natural map from* $\langle x\rangle$ *to* $GL(n,p)$, *or*

(2) G *contains a normal abelian subgroup* $B = A \times \langle b\rangle$, *where* $A = \langle a_1\rangle \times \cdots \times \langle a_{p-1}\rangle$ *is free abelian of rank* $p-1$ *and normal in* G, b *is of infinite order or of order* p^k *(for some non negative integer* k*) and is central in* G, *and* $G = A \rtimes \langle x\rangle$ *for some* x, *where* $x^p = b$, $a_i^x = a_{i+1}$ *for* $i = 1, \ldots, p-2$ *and* $a_{p-1}^x = (a_1 \cdots a_{p-1})^{-1}$, *where* p *is a prime at most* 19.

5.2 Groups isomorphic to their non-nilpotent subgroups

In this section we examinate a property which represents a natural generalization of the property $\mathfrak{X}$.

A group G is an $\mathfrak{W}$-*group* if it contains proper non-nilpotent subgroups, all of which are isomorphic to G. Of course, $\mathfrak{W}$-groups do not satisfy the minimal condition. Also, we observe that a $\mathfrak{W}$-group fails to be finitely generated if and only if it is locally nilpotent. Again here, there is a quite classification of this group at least in the soluble case.

Now we give some preliminary results.

Lemma 5.2.1. *Let G be a group, N a normal nilpotent subgroup of G and suppose that $G = N\langle x\rangle$ for some element x. If M is a G-invariant subgroup of N such that $M\langle x\rangle$ is nilpotent, then $M \leq Z_n(G)$ for some integer n.*

For the second result, we recall that if G is a locally nilpotent group and H is a subgroup of G, the *isolator* $I_G(H)$ of H in G is $I_G(H) = \{g \in G \mid g^n \in H\}$ for some non-zero integer n, and it is a subgroup of G.

Lemma 5.2.2. *Let $G = A \rtimes \langle g\rangle$ be a countable torsion-free locally nilpotent group with A abelian, and suppose that H is isomorphic to G whenever H is a subgroup of G with $I_G(H) = G$. Then G is free abelian.*

Proof. Let $S = \{b_i \mid i \in \mathbb{N}\}$ or $\{b_1, \dots, b_k\}$ be a maximal $\mathbb{Z}$-indipendent subset of A. Since $A_1 = \langle b_1\rangle^G$ is contained in $\langle b_1, x\rangle$, it is finitely generated by local nilpotency. Write $I_1 = I_A(A_1)$ and $a_1 = b_1$. Now choose i_2 least such that $b_2 \in S \setminus I_1$ and write $B_2 = \langle b_{i_2}\rangle^G$. Then $(B_2 \cap I_1)/A_1$ is finitely generated periodic abelian and is therefore finite, of order n_1 say, so that $(B_2 \cap I_1)^{n_1} \leq A_1$. Set $a_2 = b_{i_2}^{n_1}$; then $\langle a_2\rangle^G \cap I_1 = B_2^{n_1} \cap I_1 = (B_2 \cap I_1)^{n_1} \leq A_1$. Since $\langle a_1, a_2\rangle^G/A_1$ is free abelian, it follows that $\langle a_1, a_2\rangle^G = A_1 \times A_2$ for some finitely generated subgroup A_2 of A. Write $I_2 = I_A(A_1A_2)$ and continue in the obvious manner to obtain in the end a $\langle g\rangle$-invariant free abelian subgroup $A^* = A_1 \times A_2 \times \cdots$ of A whose isolator is A. By hypothesis, $G \simeq A^*\langle g\rangle$, and thus we assume that A is free abelian.

By local nilpotency again, every element of A is in some term $Z_i(G)$ of the upper central series of G, so that A is generated by all the $A \cap Z_i(G)$. We claim that we may write $A = C_1 \times C_2 \times \cdots$ (possibly, with finitely many factors), in such a way that $C_1 \times C_2 \times \cdots \times C_i = A \cap Z_i(G)$ for each i. To see this, recall that $G = \langle A, g\rangle$ and that A is abelian, so the map $d \to [d,_i g]$ is a homomorphism from $A \cap Z_{i+1}(G)$ into A with kernel $A \cap Z_i(G)$. Since A is free abelian, $A \cap Z_i(G)$ is a direct factor of $A \cap Z_{i+1}(G)$ and the claim follows.

Suppose that G is not abelian and choose a prime p such that $[C_1C_2, \langle g\rangle]$ $(= [C_2, \langle g\rangle])$ is not contained in C_1^p. Such exists because C_1 is free abelian. Set $B = C_1A^p = C_1 \times C_2^p \times C_3^p \times \cdots$, so that $G \simeq B\langle g\rangle$ and thus $[Z_2(B\langle g\rangle), \langle g\rangle] \not\leq (Z_1(B\langle g\rangle))^p = C_1^p$. But $Z_2(B\langle g\rangle) = C_1C_2^p$ and thus we have $[C_1C_2^p, \langle g\rangle] \leq C_1^p$, a contradiction that proves the lemma. □

Lemma 5.2.3. *Let G be a soluble locally nilpotent group that is isomorphic to each of its non-nilpotent subgroup. If G is torsion-free it is a Fitting group.*

Proof. Let G be a group that satisfies these hypothesis and suppose for a contradiction that G is not a Fitting group. Since G is not nilpotent it has a countable non-nilpotent subgroup and so itself countable. Choose g in G such that $\langle g\rangle^G$ is not nilpotent, so that $G'\langle g\rangle$ is not nilpotent. But G' is nilpotent since it has smaller derived length than G and so is not isomorphic to G. Since G is isomorphic to $G'\langle g\rangle$, we deduce that G has a normal nilpotent subgroup N with $G = N\langle x\rangle$ for some element x. From [45] Lemma 6.33 we see that $N \cap \langle x\rangle = \{1\}$ and $N = Fitt(G)$. It follows easily that $N'\langle x\rangle$ is not isomorphic to G and is therefore nilpotent, so that $N' \leq Z_a(G)$ for some integer a, by Lemma 5.2.1. Let H be an arbitrary subgroup of G containing $Z_a(G)$ such that $I_G(H) = G$; again, H is not nilpotent. Now in any torsion-free locally nilpotent group centralizers are isolated; thus, if $g \in Z(H)$ then $G \leq C_G(g)$, that is, $g \in Z(G)$ and so $Z(G) = Z(H)$. Furthermore, $G/Z(G)$ is torsion-free and so an easy induction shows that $Z_a(G) = Z_a(H)$, where a is as above. It follows that $Z_a(G)$ is invariant under every isomorphism from G to H; thus, as $N' \leq Z_a(G)$, we see that $G/Z_a(G)$ is abelian-by-cyclic. It is also torsion-free, so that Lemma 5.2.2 applies to give that $G/Z_a(G)$ is abelian and hence that $G = Z_{a+1}(G)$, a contradiction that estabilishes the result. □

Lemma 5.2.4. *Let $G = A\langle x\rangle$ be a locally nilpotent, residually nilpotent group, where A is a normal abelian p-subgroup of G for some prime p. If G is not* *nilpotent, then it has a non-nilpotent subgroup $B\langle x\rangle$ for some subgroup B of A that is the direct product of finite G-invariant subgroups.*

Proof. Firstly, there must exist a finite G-invariant subgroup B_1 of A such that $[B_1, x] \neq \{1\}$, else G would be nilpotent. Thus there is a normal subgroup N_1 of G such that G/N_1 is nilpotent and $N_1 \cap B_1 = \{1\}$. Next, since $G/(N_1 \cap A)$ is nilpotent and G not nilpotent, G does not act nilpotently on $N_1 \cap A$; thus, there is a finite G-invariant subgroup B_2 of $N_1 \cap A$ such that $[B_2, x, x] \neq \{1\}$ (that is, B_2 is not second central) and a G-invariant subgroup N_2 of N_1 such that G/N_2 is nilpotent and $N_2 \cap B_2 = \{1\}$. Continuing in the obvious way, we obtain a subgroup B of A of the form $B = B_1 \times B_2 \times \cdots$, where, for each i, B_i is finite and $[B_{i,i}\, x] \neq \{1\}$. Clearly, $B\langle x\rangle$ is a subgroup of the required kind. □

Now we prove that there are no torsion-free groups in $\mathfrak{W}$ that are soluble and locally nilpotent. Recall that a group G is an *n-Engel group* if every n-commutator $[x,_n y]$ is trivial for all elements x, y of G. Of course, every nilpotent group is an n-Engel group for any integer n.

Theorem 5.2.5 (H. Smith - J. Wiegold [58]). *Let G be a soluble group that is not finitely generated, and suppose that G is isomorphic to each of its non-nilpotent subgroups. If G is torsion-free then G is nilpotent.*

Proof. Suppose for a contradiction that G is a non-nilpotent group satisfying the hypothesis of that theorem. Since G' is nilpotent, so its isolator ([45] Lemma 6.33). There exists a free abelian subgroup K/I of G/I such that $I_G(K) = G$, and K is not nilpotent by the same lemma. Therefore, K is isomorphic to G, which means that G has a normal nilpotent subgroup N with G/N free abelian. Now if G is n-Engel for some integer n then it is nilpotent, by [45] Theorem 7.36 and the fact that G is torsion-free.

Thus G is not n-Engel for any n, whereas it is a Fitting group by Lemma 5.2.3. Let c be the nilpotency class of N. Since G is not $(c+2)$-Engel, we may choose $g_0 \in G$ such that $\langle g_0 \rangle^G$ has nilpotency class $n_0 > c$. Let $I_0 = I_G(N\langle g_0 \rangle)$, so that $G/N = I_0/N \times U_0/N$ for some subgroup U_0, since G/N is free abelian. Since $[G, \langle g_0 \rangle]$ is nilpotent, so is I_0. But $G = I_0 U_0$ and so U_0 is not nilpotent since I_0 and U_0 are normal. Thus, we may choose $g_1 \in U_0$ such that the nilpotency class n_1 of $\langle g_1 \rangle^G$ is strictly greater than $c + n_0$. Clearly, $\{g_0, g_1\}$ is $\mathbb{Z}$-independent modulo N. As so often, we iterate. Suppose that for some $k \geq 1$ we have constructed a subset $\{g_0, g_1, \ldots, g_k\}$ of G that is $\mathbb{Z}$-independent modulo N and such that the nilpotency class n_j of $\langle g_j \rangle^G$ is greater then $c + n_0 + \cdots + n_{j-1}$, for each $j = 1, \ldots, k$. Write $I_k = I_G(N\langle g_0, \ldots, g_k \rangle)$ and $G/N = I_k/N \times U_k/N$. As before, U_k is non-nilpotent and therefore contains an element g_{k+1} whose normal closure in G has nilpotency class n_{k+1} greater than $c + n_0 + \cdots n_k$. We have defined inductively an infinite subset $\{g_0, g_1, \ldots\}$ that is $\mathbb{Z}$-independent modulo N and such that the classes n_i of $\langle g_i \rangle^G$ satisfy $n_{i+1} > c + n_0 + \cdots + n_i$ for all $i \geq 1$.

Set $H = N\langle g_1, g_2, \ldots \rangle$. Since $\langle g_k \rangle^G \leq \langle g_k \rangle G'$ and $G' \leq N$, we see that H is non-nilpotent, and so there is an isomorphism θ from G onto H. Write $y = \theta(g_0)$; then $\langle y \rangle^H$ has nilpotency class n_0. Certainly $y \notin N$ since $n_0 > c$, and so $y = g_{i_1}^{\alpha_1} \cdots g_{i_k}^{\alpha_k} h$ for integer $i_1, \ldots, i_k$ with each α_j non-zero, $0 < i_1 < \cdots < i_k$ and some element h of N. Put $\langle g_{i_1}, \ldots, g_{i_k}, N \rangle$, so that $y \in K$. Since α_k is non-zero, the subgroup $L = \langle y, g_{i_1}, \ldots, g_{i_{k-1}}, N \rangle$ has finite index in K; here we interpret L as $\langle y, N \rangle$ if $k = 1$. It follows that L has exactly the same nilpotency class as K. Now L is the product of the H-invariant subgroups N, $\langle y \rangle^H, \langle g_{i_1} \rangle^H, \ldots, \langle g_{i_{k-1}} \rangle^H$, so that it has class at most $d = c + n_0 + n_{i_1} + \cdots + n_{i_{k-1}}$, by Fitting's Theorem. However, K contains the subgroup $\langle g_{i_k} \rangle^K$ of class n_{i_k}, which is certainly greater than d. This contradiction complete the proof of the theorem. □

Now give a complete description of $\mathfrak{W}$-groups of finite rank. The infinite rank case is much harder, and the best we can say is that they are Fitting groups, of which we omit the proof.

Theorem 5.2.6 (H. Smith - J. Wiegold [58])**.** *Let G be a soluble group in $\mathfrak{W}$ that is not finitely generated. If G is of infinite rank then it is a Fitting group.*

Theorem 5.2.7 (H. Smith - J. Wiegold [58])**.** *Let G be a soluble group in $\mathfrak{W}$ that is not finitely generated. The following three conditions are equivalent:*

(a) *G has finite rank.*

(b) *G is not a Fitting group.*

(c) *$G = P \rtimes \langle x \rangle$ for some divisible abelian p-group P (p a prime) and some element x of infinite order such that*

(i) *$[P, x^p] = \{1\}$, so that $x^p \in Z(G)$:*

(ii) *P has no infinite proper $\langle x \rangle$-invariant subgroups;*

(iii) *$G/\langle x^p \rangle$ has all proper subgroup nilpotent but is not itself nilpotent.*

Proof. It easily seen that (c) implies (b), for if $\langle x \rangle^G$ is nilpotent we have the contradiction that G is nilpotent. Theorem 5.2.6 means that (b) implies (a), and thus it suffices to prove that (a) implies (c).

Suppose then that G is a soluble locally nilpotent $\mathfrak{W}$-group of finite rank. As in the proof of Lemma 5.2.3, we have $G = N\langle x \rangle$ for some normal nilpotent subgroup N and element x; without loss we may choose N to be the Fitting subgroup of G. Each primary component of the torsion subgroup T of G has finite rank and is therefore Černikov. Since a $\mathfrak{W}$-group does not satisfy the minimal condition for subgroups, we deduce that G is not a p-group for any prime p. Furthermore, G is not the direct product of a non-trivial p-group and a non-trivial p'-group (one of them would be non-nilpotent), and it follows that

G is not a torsion group. Thus T is nilpotent and therefore contained in N. We claim that $T\langle x\rangle$ is not nilpotent. If it is, then $T \leq Z_a(G)$ for some integer a, by Lemma 5.2.1. But G/T is a torsion-free locally nilpotent group of finite rank and therefore nilpotent ([45] Theorem 6.36) and we have the contradiction that G is nilpotent, thus establishing our claim. Thus G is isomorphic to $T\langle x\rangle$ and, since $T \leq N$, we may as well assume that $G = T\langle x\rangle = T \rtimes \langle x\rangle$ since x has infinite order. The next claim is that T is a p-group for some prime p. Otherwise $T = P \times Q$, where P is a non-trivial p-group and Q a non-trivial p'-group, and we have that each of $P\langle x\rangle$ and $Q\langle x\rangle$ is nilpotent (neither is isomorphic to G). Two applications of Lemma 5.2.1 yield the contradiction that PQ is in some term of the upper central series.

What we have now is that $G = P \rtimes \langle x\rangle$ for some nilpotent p-group P, in the right direction for establishing (c). Indeed, we show that this P and this x have all the properties required by (c). Let D denote the divisible part of P. We shall show that $D = P$, so that P is divisible abelian. If $D < P$, then $D\langle x\rangle$ is nilpotent; however, G/D is finite-by-cyclic since G has finite rank, and so G/D is nilpotent. Lemma 5.2.1 then supplies the contradiction that G is nilpotent, and so P is a divisible abelian group, as claimed. Since P is certainly not central there is an integer k such that $[\Omega_k(P), x] \neq \{1\}$. Since $\Omega_k(P)$ is finite and $\langle\Omega_k(P), x\rangle$ is nilpotent, there is a positive integer n such that $[\Omega_k(P), x^n] = \{1\}$. For this n, $P\langle x^n\rangle$ is not isomorphic to G and is therefore nilpotent. Let m be the least positive integer such that $P\langle x^m\rangle$ is nilpotent. Write $m = qr$ where q is prime; we see that m is itself prime, indeed that $m = p$. For $P\langle x^r\rangle$ is not nilpotent and hence isomorphic to G, while $P\langle x^{rq}\rangle$ is nilpotent, and so m is prime. There is a finite G-invariant series of P centralized by x^m, and we can choose a factor A of it such that $[A, x] \neq \{1\}$. Then x acts on A as an element of prime order m, and from local nilpotency it follows that the only value possible for m is p. Thus $P\langle x^p\rangle$ is nilpotent.

We claim that $P\langle x^p\rangle$ is in fact abelian, that is, that $[P, x^p] = \{1\}$. Assume not. Certainly $[P, x^p] < P$ since $P\langle x^p\rangle$ is nilpotent, and it follows without difficulty that the homomorphism $a \to [a, x^p]$ from P to itself has infinite kernel K, say. If H is the divisible part of K, then H is normal in G and has smaller rank than P, so that $\langle H, x\rangle$ is nilpotent and therefore abelian, [45] Lemma 3.13. It follows that $C_P(x)$ is infinite, and that the homomorphism $a \to [a, x]$ from P to itself is not onto. Since $[P, x]$ is divisible it has smaller rank than P, so that $[P, x]$ is not isomorphic to P. Thus $\langle [P, x], x\rangle$ is nilpotent and so $P\langle x\rangle$ is nilpotent. This contradiction shoows that $[P, x^p] = \{1\}$ and hence that $x^p \in Z(G)$. Part (i) of (c) is thereby established.

Now let H be a non-nilpotent subgroup of G. Then $H \cap P$ is isomorphic to P, being the periodic part of H. It follows that $H \cap P = P$ by rank considerations, and so H contains P. If now H contains x^p then, since $P\langle x^p\rangle$ is abelian, H must equal G and this confirms part (iii) of (c): all proper subgroup of G containing x^p are nilpotent.

Finally, we prove part (ii). As we saw above, $[P, x]$ is isomorphic to P and therefore is equal to it. It follows that $P = G'$, and hence that the factor group $B = G/\langle x^p\rangle$ is isomorphic to the group $B(p, 1, 0)$ (see [42] 4.5). But then [42] 4.6 applies to give that all proper G-invariant subgroups of P are finite, as required to conclude the proof of theorem. □

5.3 Groups isomorphic to their non-abelian subgroups of infinite rank

A group G is an $\mathfrak{X}_\infty$-*group* if it contains proper non-abelian subgroups of infinite rank and G is isomorphic to all of them. We observe that every $\mathfrak{X}_\infty$-group is countable. Moreover, every factor group G/N of an $\mathfrak{X}_\infty$-group G respect to a normal subgroup N invariant under every isomorphism from G to any subgroup containing N is an $\mathfrak{X}_\infty$-group and if this subgroup has infinite rank, then G/N is an $\mathfrak{X}$-group.

Note this elementary property.

Lemma 5.3.1. *Let G be a periodic soluble $\mathfrak{X}_\infty$-group of infinite rank. If G contains a normal abelian subgroup of infinite rank then G is abelian-by-finite.*

Proof. Let A be a normal abelian subgroup of infinite rank of G. Clearly, there exists a non-abelian finite group E. By the property $\mathfrak{X}_\infty$ the subgroup AE is isomorphic to G and so G is abelian-by-finite. □

Now we prove that every periodic $\mathfrak{X}_\infty$-group contains a normal abelian subgroup of finite index.

Lemma 5.3.2. *Let G be a periodic locally nilpotent soluble $\mathfrak{X}_\infty$-group. Then G is abelian-by-finite.*

Proof. G is the direct product of its primary component. Let G_{π_1} the product of all abelian components and G_{π_2} the product of all non-abelian components. Let Z be the centre of G. It follows from Lemma 5.3.1 that we can assume that Z has finite rank. Therefore G_{π_1} has finite rank, so that G_{π_2} has infinite rank, and hence G is isomorphic to G_{π_2}. Clearly this means that $G = G_{\pi_2}$ so that all primary components are non-abelian. If all primary components of G have finite rank, then G can decomposed in the direct product of two coprime subgroups of infinite rank, a contradiction that shows that there exist a prime p such that G_p has infinite rank. Then G is a p-group. Consider the commutator subgroup of G, G', that it is not isomorphic to G. We can assume that G' has finite rank by Lemma 5.3.1. Then it is a Černikov group and so $C = C_G(G')$ has infinite rank. If C is non-abelian then G' is contained in Z. By Lemma 1.2 G contains an abelian subgroup B of infinite rank and BZ is an abelian normal subgroup of G of infinite rank. Thus G is abelian-by-finite by Lemma 5.3.1. □

For our second result, we recall the definitions of $\mathfrak{X}$-residual and socle. Let $\mathfrak{X}$ be a class of groups and let G be any group. The *$\mathfrak{X}$-residual* of G is the intersection of all normal subgroups of G whose factor groups in G are $\mathfrak{X}$-group and it is denoted by $\varrho^*_{\mathfrak{X}}(G)$. The *socle* of G is the product of all minimal normal subgroup of G, and we denote it by $\mathrm{Soc}(G)$.

Theorem 5.3.3 (G. di Grazia)**.** *Let G be a periodic soluble $\mathfrak{X}_\infty$-group. Then G is abelian-by-finite.*

Proof. We can assume that G is not locally nilpotent, so that every locally nilpotent subgroup of infinite rank is abelian. Let F be the Fitting subgroup of G. If F has infinite rank then is abelian and we have the claim by Lemma 5.3.1. So F has finite rank and every p-component F_p is a Černikov group. By Lemma 1.2 there exists an abelian subgroup of infinite rank B of G such that $[B, F_p] \neq \{1\}$ and $G \simeq BF_p$. If $[B, \varrho^*_{\mathfrak{F}}(F_p)]$ is trivial then BF_p (and hence G) is abelian-by-finite. If not there exist an integer n such that $[B, S] \neq \{1\}$ where $S = \mathrm{Soc}_n(\varrho^*_{\mathfrak{F}}(F_p))$. So G is finite-by-abelian that implies G nilpotent-by-finite and hence G is abelian-by-finite. □

When the group is not periodic, we have the following results at least in the nilpotent case.

Proposition 5.3.4. *Let G be a nilpotent $\mathfrak{X}_\infty$-group. If the subgroup T consisting of all elements of finite order has infinite rank, then G is abelian-by-finite.*

Proof. It follows from Theorem 5.3.3 that we can assume that G is not periodic, so that T is abelian. Moreover G/T is a free abelian group of rank at most 2 (see Theorem 5.1.6). Assume first that $G/T = \langle a_1T\rangle \times \langle a_2T\rangle$; in particular the subgroups $\langle a_1\rangle T$ and $\langle a_2\rangle T$ are abelian, so that $T \leq Z = Z(G)$. In particular G has class 2, so that if n is the order of $\langle [a, b]\rangle$, we have

$$1 = [a, b]^n = [a^n, b] = [a, b^n],$$

and hence $\langle a^n, b^n\rangle T$ is an abelian subgroup of finite index of G.

Assume now that G/T is cyclic so that $G = \langle g\rangle \ltimes T$, where g is an element of infinite order of G. Put $\pi_1 = \{p \in \mathbb{P} \mid T_p \leq Z\}$ and $\pi_2 = \{p \in \mathbb{P} \mid T_p \not\leq Z\}$. For any $p \in \pi_2$, let E_p be a finite subgroup of T_p such that $E_p \not\leq Z$. Assume first that T_{π_1} has infinite rank; then G is isomorphic to $(\langle g\rangle \times T_{\pi_1})E_p$, where p is a prime in π_2, and hence G is abelian-by-finite.

Assume now that T_{π_1} has finite rank, so that T_{π_2} has infinite rank. It is easy to see that there exists a prime $p \in \pi_2$ such that T_p has infinite rank, so that G is isomorphic to $\langle g\rangle \ltimes T_p$, and hence $T = T_p$. Moreover, there exists a positive integer n such that $T[p^n] \not\leq Z$, and hence G is isomorphic to $\langle g\rangle \ltimes T[p^n]$, so that $T = T[p^n]$ is a p-group of finite exponent.

It easy to prove that $Z = (\langle g\rangle \cap Z) \times (T \cap Z)$. Since $Z \not\leq T$ (see [46] Theorem 5.2.22), we have that $\langle g\rangle \cap Z \neq \{1\}$, so there exists a positive integer m such that $g^m \in Z$, and hence $\langle g^m\rangle \times T$ is an abelian subgroup of finite index of G. □

Theorem 5.3.5 (G. di Grazia)**.** *Let G be a torsion-free nilpotent $\mathfrak{X}_\infty$-group. Then G is abelian.*

Proof. Assume by contradiction that G is not abelian, so that it contains two elements a and b such that the subgroup $H = \langle a, b\rangle$ has class two. Then the centralizer $C = C_G(H)$ has infinite rank (see [62]), therefore C contains an abelian torsion-free group of infinite rank A such that $A \cap H = \{1\}$. Consider for each $n \geq 1$, the subgroups $H_n = \langle a^n, b, [a, b]\rangle$ and $K_n = A \times H_n$. Clearly $Z(K_n)/K_n' = Z(K_n)/H_n'$ is the direct product $AH_n'/H_n' \times H_n/H_n'$, where H_n/H_n' is a cyclic group of order n. In particular $(K_n)_{n\in\mathbb{N}}$ is a family of non-abelian subgroups of infinite rank of G pairwise non-isomorphic. This contradiction shows that G is abelian. □

Bibliography

[1] R. Baer: Abzählbar erkennbare gruppentheoretische Eigenshaften, Math. Z. 79, 344-363 (1962).

[2] V.S. Čarin: On groups of finite rank, Ukrain. Math. Z. 16, 212–219 (1964).

[3] N.S. Černikov: A theorem on groups of finite special rank, Ukrain. Math. J. 42, 855–861 (1990).

[4] S.N. Černikov: Groups with given properties of classes of infinite subgroups, Ukrain. Math. J. 19, 715–731 (1967).

[5] M. Curzio, J. Lennox, A. Rhemtulla, J. Wiegold: Groups with many permutable subgroups, J. Austral. Math. Soc (Series A) 48, 397-401 (1990).

[6] G. Cutolo: A normalizer condition on system of subgroups, Boll. Un. Mat. Ital. (7) 3, 215–223 (1989).

[7] M. De Falco - F. de Giovanni - C. Musella: Groups with finitely many conjugacy classes of non-normal subgroups of infinite rank, Colloquium Math. 131, 233–239 (2013).

[8] M. De Falco - F. de Giovanni - C. Musella: Groups with normality conditions for subgroups of infinite rank, Publ. Mat., to appear.

[9] M. De Falco - F. de Giovanni - C. Musella: A note on groups of infinite rank with modular subgroup lattice, Monatsh. Math. 176 (1), 81–86 (2015).

[10] M. De Falco - F. de Giovanni - C. Musella: Groups of infinite rank whose subgroups are closed in the profinite topology, to appear.

[11] M. De Falco - F. de Giovanni - C. Musella: Groups whose proper subgroup of infinite rank have a transitive normality relation, Mediterr. J. Math 10, 1999–2006 (2013).

[12] M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak: On metahamiltonian groups of infinite rank, J. Algebra 407, 135–148 (2014).

[13] M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak: Groups of infinite rank in which normality is a transitive relation, Glasgow Math. J. 56, 387–393 (2014).

[14] M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi: Groups with restrictions on subgroups of infinite rank, Rev. Mat. Iberoamericana 30, 535–548 (2014).

[15] M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi: Groups whose proper subgroups of infinite rank have finite conjugacy classes, Bull. Austral. Math. Soc. 89, 41–48 (2014).

[16] M. De Falco - L.A. Kurdachenko - I.Ya. Subbotin: Groups with only Abnormal and Subnormal Subgroups, Atti Sem. Mat. Fis. Univ. Modena 46, 435–442 (1998).

[17] M. De Falco - C. Musella: Groups which are isomorphic to their subgroups with non-modular subgroup lattice Algebra Discrete Math. 1, 41–50 (2006).

[18] C. Delizia, C. Nicotera: Groups with conditions on infinite subsets, Ischia group theory 2006, World Sci. Publ., Hackensack, NJ, 46–55 (2007).

[19] A.V. De Luca - G. di Grazia: A note on groups with a modularity condition on infinite subsets, Ricerche di Matematica (2) 64, 321–324 (2015).

[20] A.V. De Luca - G. di Grazia: Groups of infinite rank with a normalizer condition on subgroups, Int. J. Group Theory (4) 1, 41–46 (2015).

[21] A.V. De Luca - G. di Grazia: Groups of infinite rank with a normality conditions on subgroups with small normal closure, Bull. Aust. Math. Soc., to appear (DOI:10.1017/S0004972715001355).

[22] M.R. Dixon - M.J. Evans - H. Smith: Locally soluble-by-finite groups of finite rank, J. Algebra 182, 756–769 (1996).

[23] M.R. Dixon - M.J. Evans - H. Smith: Locally soluble-by-finite groups with all proper insoluble subgroups of finite rank, Arch. Math. (Basel) 68, 100–109 (1997).

[24] M.R. Dixon - M.J. Evans - H. Smith: Locally soluble-by-finite groups with all proper non-nilpotent subgroups of finite rank, J. Pure Appl. Algebra 135, 33–44 (1999).

[25] M.R. Dixon - Z.Y. Karatas: Groups with all subgroups permutable or of finite rank, Centr. Eur. J. Math. 10, 950–957 (2012).

[26] M.J. Evans - Y. Kim: On groups in which every subgroup of infinite rank is subnormal of bounded defect, Comm. Algebra 32, 2547–2557 (2004).

[27] W. Gaschütz: Gruppen in denen das Normalteilersein tansitiv ist, J. Reine Angew. Math. 198, 87–92 (1957).

[28] G. Giordano: Gruppi con normalizzatori estremali, Matematiche (Catania) 26, 291–296 (1971).

[29] F. de Giovanni - S. Franciosi: Groups in which every infinite subnormal subgroup is normal, J. Algebra 96, 566–580 (1985).

[30] H. Heineken - J. C. Lennox: A note on products of abelian group, Arch. Math. 41, 498–501 (1983).

[31] K. Iwasawa: Über die endlichen Gruppen und die Verbände ihrer Untergruppen, J. Fac. Sci. Imp. Univ. Tokyo 4, 171–199 (1941).

[32] P.H. Kropholler: On finitely generated soluble groups with no large wreath product sections, Proc. London Math. Soc. (3) 49, 155–169 (1984).

[33] L.A. Kurdachenko - H. Smith: Groups in which all subgroups of infinite rank are subnormal, Glasgow Math. J. 46, 83–89 (2004).

[34] L.A. Kurdachenko - H. Smith: Groups with rank restrictions on non-subnormal subgroups, Turkish J. Math. 28, 165–176 (2004).

[35] J.C. Lennox - J.E. Roseblade: Centrality in finitely generated soluble groups, J. Algebra 16, 399–435 (1970).

[36] J.C. Lennox - H. Smith - J. Wiegold: A problem on normal subgroup, Proc. London Math. Soc. (3) 49, 155–169 (1993).

[37] A.I. Malcev: On groups of finite rank, Mat. Sbornik N.S. (64) 22, 351–352 (1948).

[38] A. I. Malcev: On some Classes of Infinite Soluble Groups, Mat. Sbornik N. S. 28, 567–588 (1951); Amer. Math. Soc. Transl. Ser. 2, 1–21 (1956).

[39] W. Möhres: Auflösbarkeit von Gruppen deren Untergruppen alle subnormal sind, Arch. Math. (Basel) 54, 232–235 (1990).

[40] B.H. Neumann: Groups with finite classes of conjugate subgroups, Math. Z. 63 , 76–96 (1955).

[41] B.H. Neumann: A problem of Paul Erdös on groups, J. Austral. Math. Soc. 21, 467–472 (1976).

[42] M.F. Newman - J. Wiegold: Groups with many nilpotent subgroups, Arch. Math. 15, 241–250 (1964).

[43] J. Otal - J.M. Peña: Infinite locally finite groups of type $PSL(2, F)$ or $S_Z(K)$ are not minimal under certain conditions, Publ. Mat. 32, 43–47 (1988).

[44] D.J.S. Robinson: Groups in which normality is a transtive reltion, Proc. Cambridge Philos. Soc. 68, 21–38 (1964).

[45] D.J.S. Robinson: Finiteness Conditions and Generalized Soluble Groups, Springer, Berlin (1972).

[46] D.J.S. Robinson: A Course in the Theory of Groups, Springer, Berlin (1996).

[47] G.M. Romalis - N.F. Sesekin: Metahamiltonian groups, Ural. Gos. Univ. Mat. Zap. 5, 101–106 (1966).

[48] G.M. Romalis - N.F. Sesekin: Metahamiltonian groups II, Ural. Gos. Univ. Mat. Zap. 6 52–58 (1968).

[49] G.M. Romalis - N.F. Sesekin: Metahamiltonian groups III, Ural. Gos. Univ. Mat. Zap. 7, 195–199 (1969/70).

[50] J. S. Rose: Finite Soluble Groups with Pronormal System Normalizers, Proc. London Math. Soc. (3) 17, 447–469 (1967).

[51] J.E. Roseblade: On groups in which every subgroup is subnormal, J. Algebra 2, 402–412 (1965).

[52] D. Segal: Polycyclic groups, Cambridge Tracts in Mathematics 82, C.U.P. (1983).

[53] R. Schmidt: Subgroup Lattices of Groups, de Gruyter, Berlin (1994).

[54] N.N. Semko - S.N. Kuchmenko: Groups with almost normal subgroups of infinite rank, Ukrain. Math. J. 57, 621–639 (2005).

[55] H. Smith: On homomorphic images of locally graded groups, Rend. Sem. Mat. Uni. Padova, 91, 53–60 (1994).

[56] H. Smith - J. Wiegold: Groups which are isomorphic to their non-abelian subgroups, Glasgow Math. J. 39, 141–151 (1997).

[57] H. Smith - J. Wiegold: Groups isomorphic to their non-nilpotent subgroups, Glasgow Math. J. 40, 257–262 (1998).

[58] H. Smith - J. Wiegold: Soluble groups isomorphic to their non-nilpotent subgroups, J. Austral. Math. Soc. (Series A) 67, 399–411 (1999).

[59] S.E. Stonehewer: Modular subgroup structure in infinite groups, Proc. London Math. Soc. 32, 63–100 (1976).

[60] I.Ya. Subbotin: Groups with Alternatively Normal Subgroups, Russian Math. 36, 87–89 (1992).

[61] V.P. Šunkov: On locally finite groups of infinite rank, Algebra i Logika 10, 199–225 (1971); English transl., Algebra and Logic 10, 127–142 (1971).

[62] D.I. Zaitsev - V.A. Onishchuk: Locally nilpotent groups with the centralizers satisfying a finiteness condition, Ukrain. Mat. Zh. 43 (7-8), 1084–1087 (1991), translation in Ukranian Math. J. 43 (7-8), 1018–1021 (1992).