DOTTORATO DI RICERCA
in
SCIENZE COMPUTAZIONALI E INFORMATICHE

Ciclo XVIII

Consorzio tra Università di Catania, Università di Napoli Federico II,
Seconda Università di Napoli, Università di Palermo, Università di Salerno

SEDE AMMINISTRATIVA: UNIVERSITÀ DI NAPOLI FEDERICO II

---

TERESA RADICE

# ON SOME REGULARITY RESULTS OF JACOBIAN DETERMINANTS AND APPLICATIONS

---

*TESI DI DOTTORATO DI RICERCA*

# Contents

# Introduction

In recent years Jacobians of mappings of finite distortion have been studied intensively and have proven to be fundamental tools in the calculus of variations, PDEs and nonlinear elasticity. Some of their rather special properties were already visible in the “div-curl ”lemma of Murat [Mu] and Tartar [Ta]. This lemma initiated the theory of “compensated compactness ”. Because of its wide application and theoretical significance the subject has been greatly expanded.

Before presenting our results, we would like to make a few general comments. One way to look at the compensated theory is to consider it as one consequence of the study of oscillations in nonlinear partial differential equations, arising from Continuum Mechanics, Physics or Differential Geometry. It is far beyond the scope of this work to discuss the reasons for such a study but we would like to mention at least that it is natural for the issue of the existence of global (generalized) solutions for many nonlinear systems of interest. It is quite obvious that, in such a study, a fundamental role should be played by weakly continuous nonlinear quantities. The compensated compactness theory has identified classes of such nonlinear quantities. The terminology stems from the fact that compensations arise in nonlinear quantities as $J(u) = \det(\nabla u)$, compensations which in turn allow the weak continuity or the compactness. Indeed, we shall show that these nonlinear quantities have an improved regularity. This improved regularity has many applications and recently there have been considerable advances in this field. In particular, it is a useful tool to apply to divergence and nondivergence elliptic equations.

It is worth pointing out that the starting point in the theory of the regularity of the Jacobian is the celebrated result due to S. Müller [M] that for an orientation preserving mapping $f \in \mathscr{W}^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, $J$ belongs to the Zygmund space $\mathscr{L}\log\mathscr{L}(K)$ for any compact $K \subset \Omega$.

The following estimate, proved by T. Iwaniec and C. Sbordone in [IS]

$$\int_K J(x,f)\,dx \quad \leqslant \quad c(n,K)\,\int_\Omega \frac{|Df(x)|^n}{\log\left(e+\frac{|Df(x)|}{|Df(x)|_\Omega}\right)}\,dx$$

when $K$ is any compact subset of $\Omega$ and $|Df|_\Omega$ denotes the integral mean of $|Df|$ over $\Omega$, can be viewed as dual to Müller's result.
At this point it became clear that the improved integrability property of the Jacobian could be observed in Orlicz-Sobolev spaces near $\mathscr{W}^{1,n}_{loc}(\Omega,\mathbb{R}^n)$. To illustrate, we mention the inequality by H. Brézis, N. Fusco and C. Sbordone [BFS] :

$$\int_K J(x,f)\log^{1-\alpha}\left(e+\frac{|J(x,f)|}{|J|_K}\right)\,dx \quad \leqslant \quad c(n,K)\,\int_\Omega \frac{|Df(x)|^n}{\log^\alpha\left(e+\frac{|Df(x)|}{|Df(x)|_\Omega}\right)}\,dx$$

for $\alpha\in[0,1]$, which was then extended to all $\alpha\in\mathbb{R}$ by L.Greco [1].
As a matter of fact, under the above hypothesis of Iwaniec and Sbordone, the Jacobian is even slightly higher integrable.
Indeed G. Moscariello in [MO] proved that $J(x,f)$ lies in $\mathscr{L}\log\log\mathscr{L}_{loc}(\Omega)$.
Subsequently, L.Greco,T.Iwaniec and G.Moscariello proved that if $|Df(x)|^n\in\mathscr{L}^{\mathcal{P}}(\Omega)$ with $\mathcal{P}$ a log-convex function and if $J\geqslant 0$ then $J\in\mathscr{L}^{\Psi}_{loc}(\Omega)$ where $\Psi(t)=\mathcal{P}(t)+t\int_0^t\frac{\mathcal{P}(s)}{s^2}\,ds$.
However these results are only a part of more general spectrum about estimates of Jacobians.
In this spirit following the results above, we can realize that it is quite natural to study the Jacobians in more general spaces (see [GIOV] , [G2]).
As suggested in [BFS] it is interesting to study the regularity of the Jacobian when $Df$ belongs to Lorentz spaces. It appears that to get positive results one cannot rely only on these spaces but it is forced to encode the theory in the Lorentz-Zygmund spaces (see Chapter 3).
It is also possible to extend this study to the couple $(B,E)$, $B:\Omega\to\mathbb{R}^n$, $E\to\mathbb{R}^n$, of vector fields on $\Omega$, such that div$B=0$ and curl$E=0$, having the scalar product $\langle B,E\rangle$ nonnegative. In this case we obtain results of higher integrability for the scalar product $\langle B,E\rangle$. Let us remark that if $A(x)$ is a symmetric matrix and we consider the equation div $A(x)\nabla u=0$ we obtain a couple taking $B=A(x)\nabla u$ and $E=\nabla u$ (see Chapter 3). Another example of operator in divergence form is given in Chapter 4.
R. Coifman, P.L. Lions, Y. Meyer and S. Semmes in a famous paper "Compensated compactness and Hardy spaces"studied the regularity of the mappings with Jacobian of arbitrary sign and as a consequence of couple $(B,E)$,

belonging to Lebesgue spaces, where $\operatorname{div} B = 0$ and $\operatorname{curl} E = 0$ whose scalar product is of arbitrary sign. Following this idea, we study analogous regularity properties of couple in the framework of Lorentz spaces (see Chapter 5).

Finally, the last chapter (see Chapter 6) is devoted to study nondivergence elliptic equations.

In 1963 Miranda [M] proved that if the coefficients lie in $\mathscr{W}^{1,n}$ then the Dirichlet problem

$$\begin{cases} Lu = h \\ u \in \mathscr{W}^{2,2}(\Omega) \cap \mathscr{W}_0^{1,2}(\Omega) \end{cases}$$

is well posed. Here $\Omega$ is bounded open set in $\mathbb{R}^n$ and $h \in \mathscr{L}^2(\Omega)$. This result is optimal in the category of $\mathscr{L}^p$-spaces. Indeed, for $a_{ij} \in \mathscr{W}^{1,n-\varepsilon}$, $\varepsilon > 0$, the uniqueness fails.

Somewhat later an improvement of Miranda's result was given by Alvino and Trombetti [AT]. They assume that $\frac{\partial a_{ij}}{\partial x_s}$ lay in the Marcinkiewicz space $\mathscr{L}^n_{\text{weak}}$ and, the constants in the weak type inequality for $\frac{\partial a_{ij}}{\partial x_s}$ are sufficiently small.

Here, we develop a theory for elliptic equations with bounded coefficients having sufficiently small $BMO$-norm and we find a higher integrability of the solution. More delicate is the case of unbounded coefficients and the main result is the following $\mathscr{L}^2 \log \mathscr{L}$ estimate

$$\|\nabla^2 u\|_{\mathscr{L}^2 \log \mathscr{L}(\mathbb{R}^n)} \leqslant c(n) \|h\|_{\mathscr{L}^2 \log \mathscr{L}(\mathbb{R}^n)}.$$

We notice that our assumption, the $BMO$-norm of the coefficients $a_{ij}$ to be sufficiently small, is weaker than the smallness condition for the $\mathscr{L}^n_{\text{weak}}$ norm of their derivatives $\frac{\partial a_{ij}}{\partial x_s}$ which allows the authors in [AT] to obtain their existence and uniqueness theorem in $\mathscr{W}^{2,2} \cap \mathscr{W}_0^{1,2}$ of the solution to the Dirichlet problem

$$Lu = h \in \mathscr{L}^2.$$

# Chapter 1

# Functional spaces

## 1.1 Decreasing rearrangement of a function

Although the Lebesgue spaces $\mathscr{L}^p$, $(1 \leqslant p \leqslant \infty)$ play a primary role in many areas of mathematical analysis, there are other classes of Banach spaces of measurable functions that are also of interest. The larger classes of Orlicz spaces and Lorentz spaces are of intrinsic importance.
From now on, let us consider $\mathbb{R}^n$ or its subsets, endowed with the structure of measurable space of Lebesgue. Let $\Omega$ be a measurable subset of $\mathbb{R}^n$ and denote its measure by $|\Omega|$ and with $\mathcal{M}(\Omega)$, or simply $\mathcal{M}$ the set of the real measurable functions on $\Omega$ a.e. finite. We will identify two functions equal a.e. Let $|\Omega| > 0$.

**Definition 1.1.1.** *The distribution function $\mu_f$ of a function $f \in \mathcal{M}(\Omega)$ is given by*

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

Let us remark that $\mu_f$ depends only on the absolute value $|f|$ and so it may assume the value $+\infty$.

**Definition 1.1.2.** *Two functions $f, g \in \mathcal{M}(\Omega)$ are said to be equimeasurable if they have the same distribution function, that is $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$.*

In the following proposition we list some properties of the distribution function.

**Proposition 1.1.3.** *Suppose $f, g, f_n$, $(n = 1, 2...)$, belong to $\mathcal{M}(\Omega)$ and let $a \in \mathbb{R} \setminus \{0\}$. The distribution function $\mu_f$ is nonnegative, decreasing and* *right-continuous on* $[0, \infty)$. *Furthermore,*

$$|g| \leqslant |f| \ \ a.e \quad \Rightarrow \quad \mu_g \leqslant \mu_f;$$

$$\mu_{af}(\lambda) = \mu_f(\lambda/|a|), \ \ (\lambda \geqslant 0);$$

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leqslant \mu_f(\lambda_1) + \mu_g(\lambda_2), \ \ (\lambda_1, \lambda_2 \geqslant 0);$$

$$|f| \leqslant \liminf_n |f_n| \ \ a.e. \ \Rightarrow \ \mu_f \leqslant \liminf_n \mu_{f_n};$$

*in particular,*

$$|f_n| \uparrow |f| \ \ a.e. \ \Rightarrow \ \mu_{f_n} \uparrow \mu_f.$$

Many integral expressions for a function $f \in \mathcal{M}(\Omega)$ may be written throughout a distribution function $\mu_f$.

**Lemma 1.1.4.** *Let us consider an absolutely continuous function* $\Phi : [0, \infty) \to [0, \infty)$ *and* $f \in \mathcal{M}(\Omega)$. *If* $\Phi$ *is monotone, i.e.* $\Phi' \mu_f \in \mathscr{L}^1(0, \infty)$, *for all* $t \geqslant 0$

$$\int_{|f|>t} \Phi(|f(x)|)\, dx = \Phi(t)\mu_f(t) + \int_t^\infty \Phi'(\rho)\mu_f(\rho)\, d\rho$$

*we will use the convention* $0 \cdot \infty = 0$. *Furthermore if* $\Phi(0) = 0$ *we have*

$$\int_\Omega \Phi(|f(x)|)\, dx \ = \ \int_0^\infty \Phi'(\rho)\mu_f(\rho)\, d\rho.$$

For example, if $\Phi(t) = t^p$ with $p > 0$,

$$\int_\Omega |f(x)|^p\, dx \ = \ p \int_0^\infty \rho^{p-1}\mu_f(\rho)\, d\rho.$$

**Definition 1.1.5.** *Suppose* $f \in \mathcal{M}(\Omega)$. *The decreasing rearrangement of a function* $f$ *is the function* $f^* : [0, \infty) \mapsto [0, \infty]$ *defined by*

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leqslant t\}$$

Here we use the convention that $\inf \emptyset = \infty$. Thus, if $\mu_f(\lambda) > t$ for all $\lambda \geqslant 0$, then $f^*(t) = \infty$. If $|\Omega| < \infty$, the distribution function $\mu_f$ is bounded by $|\Omega|$ and so $f^*(t) = 0$ for all $t \geqslant |\Omega|$. In this case we may regard $f^*$ as a function on the interval $[0, |\Omega|]$. Notice also that if $\mu_f$ happens to be continuous and strictly decreasing then $f^*$ is the inverse of $\mu_f$.
Furthermore we can remark that $f^*(t)$ can be expressed as follows:

$$f^*(t) = \sup\{\lambda \geq 0 : \mu_f(\lambda) > t\}.$$

**Example.** In dimension $n = 1$, let us consider $\Omega = ]0, \infty[$ and $f(x) = 1 - e^{-x}$. The distribution function is infinite in $[0, 1[$ and equal to zero in $[1, \infty[$. Hence $f^*(t) = 1$, $\forall t \geqslant 0$.

In the following proposition we list some properties of the decreasing rearrangement

**Proposition 1.1.6.** *Suppose $f, g$ and $f_n$, $(n = 1, 2...)$, belong to $\mathcal{M}(\Omega)$ and let $a \in \mathbb{R} \setminus \{0\}$. The decreasing rearrangement $f^*$ is a nonnegative, decreasing, right-continuous function on $[0, \infty)$. Furthermore*

$$|g| \leqslant |f| \ \ a.e. \quad \Rightarrow \quad g^* \leqslant f^*;$$

$$(af^*) = |a| f^*;$$

$$(f+g)^* (t_1 + t_2) \ \leqslant \ f^*(t_1) + g^*(t_2), \ \ (t_1, t_2 \geqslant 0);$$

$$|f| \leqslant \liminf_n |f_n| \ \ a.e. \ \Rightarrow \ f^* \leqslant \liminf_n f_n^*$$

*in particular,*

$$|f_n| \uparrow |f| \ \ a.e. \ \Rightarrow \ f_n^* \uparrow f^*;$$

$$(|f|^p)^* = (f^*)^p, \quad 0 < p < \infty; \tag{1.1}$$

*if $\mu_f(\lambda) < \infty$ then $f^*(\mu_f(\lambda)) \leq \lambda$, if $f^*(t) < \infty$ then $\mu_f(f^*(t)) \leqslant t$.*

Next theorem shows that a function $f$ and its decreasing rearrangement have the same norm in $\mathscr{L}^p$.

**Theorem 1.1.7.** *Let $f \in \mathcal{M}(\Omega)$. If $0 < p < \infty$, then*

$$\int_\Omega |f(x)|^p \, dx \ = \ p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) \, d\lambda \ = \ \int_0^{|\Omega|} f^*(t)^p \, dt$$

*Furthermore, in the case $p = \infty$*

$$\operatorname*{ess\,sup}_{x \in \Omega} |f(x)| \ = \inf\{\lambda \geqslant 0 : \mu_f(\lambda) = 0\} \ = \ f^*(0)$$

In particular if $f$ is defined on $[0, \infty[$, $f^*$ is nonnegative, decreasing, right-continuous and equimeasurable with $f$.

It is possible to generalize (1.1) taking a function $\Phi : [0, \infty[ \to [0, \infty[$, increasing and left-continuous on $]0, \infty[$, we have:

$$[\Phi(|f|)]^* \ = \ \Phi(f^*).$$

Trivially $\Phi(f^*)$ is decreasing and right-continuous and equimeasurable with $\Phi(|f|)$.

**Proposition 1.1.8.** *Let $f \in \mathcal{M}(\Omega)$. For all $t \in [0, |\Omega|]$, we have*

$$\int_0^t f^*(\tau)\, d\tau \;=\; \sup\left\{\int_F |f| dx \;:\; F \subset \Omega \textit{ measurable}, |F| \leqslant t\right\} \tag{1.2}$$

*Furthermore if $\Omega$ has finite measure, the supremum of (1.2) is the maximum.*

It is worth to point out that the decreasing rearrangement does not necessarily preserves sums or products of functions, there are nevertheless some basic inequalities that govern these processes.
This is due to an elementary inequality of Hardy and Littlewood. The inequality involves finite sequences $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ of non-negative real numbers, and asserts that

$$\sum_{j=1}^n a_j b_j \leqslant \sum_{j=1}^n a_j^* b_j^*, \tag{1.3}$$

where $(a_j^*)_{j=1}^n$ and $(b_j^*)_{j=1}^n$ denote respectively the sequence of elements $a_j$ and $b_j$ arranged in decreasing order.
It will be convenient to regard such sequence $(a_j)_{j=1}^n$ as a simple function $f = \sum_{j=1}^n a_j \chi_{[j-1,j)}$ defined on the interval $[0, \infty)$. In this case the rearrangement $f^*$ of $f$ is just the simple function $f^* = \sum_{j=1}^n a_j^* \chi_{[j-1,j)}$ corresponding to the rearranged sequence $(a_j^*)_{j=1}^n$. The inequality (1.3) will be seen as a special case of the more general inequality established in the theorem below

**Theorem 1.1.9.** *If $f, g \in \mathcal{M}(\Omega)$ then*

$$\int_\Omega |fg|\; dx \;\leqslant\; \int_0^\infty f^*(s)\, g^*(s)\; ds \tag{1.4}$$

An immediate consequence of the Hardy-Littlewood inequality (1.4) is that

$$\int_\Omega |f\tilde{g}|\; dx \;\leqslant\; \int_0^\infty f^*(s)\, g^*(s)\; ds$$

for every function $\tilde{g}$ on $\Omega$ equimeasurable with $g$. If $g$ is a characteristic function of a set $\Omega$ of positive measure $t$ the Hardy-Littlewood inequality (1.4) becomes:

$$\frac{1}{|\Omega|}\int_\Omega |f|\, dx \;\leqslant\; \frac{1}{t}\int_0^t f^*(s) ds, \;\; f \in \mathcal{M}(\Omega)$$

So the average of $|f|$ over any set of measure $t$ is dominated by the corresponding average of $f^*$ over the interval $(0,t)$. Notice that the latter average is also maximal among all averages of $f^*$ taken over sets of measure $t$, this is an immediate consequence of the fact that $f^*$ is decreasing. For this reason the function on the right-hand of (1.4) is called maximal function.

**Definition 1.1.10.** *We will denote by $f^{**}$ the maximal function of $f^*$ defined by*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \quad t > 0$$

Let us remark that there is a certain subadditivity of the maximal operator.

$$(f+g)^{**} \leqslant f^{**} + g^{**}$$

Furthermore we have the following properties:

**Proposition 1.1.11.** *Suppose $f, g$ and $f_n$, $(n = 1, 2, ...)$, belong to $\mathcal{M}(\Omega)$ and let $a \in \mathbb{R}$.*
*Then $f^{**}$ is nonnegative, decreasing and continuous on $(0,\infty)$. Furthermore*

$$f^{**} \equiv 0 \Leftrightarrow f = 0 \ \ a.e.;$$

$$f^* \leqslant f^{**};$$

$$|g| \leqslant |f| \ \ a.e. \ \Rightarrow g^{**} \leqslant \ f^{**};$$

$$(af)^{**} = |a| f^{**};$$

$$|f_n| \uparrow |f| \ \ a.e. \Rightarrow f_n^{**} \uparrow f^{**}.$$

## 1.2 Lorentz spaces

We collect here some definitions and results related to Lorentz spaces whose proofs are contained in [BR] and[BS].

**Definition 1.2.1.** *Let $\Omega \subset \mathbb{R}^n$ a measurable set and suppose $0 < p, q \leqslant \infty$. The Lorentz space $\mathscr{L}^{p,q} = \mathscr{L}^{p,q}(|\Omega|)$ consists of all $f \in \mathcal{M}(\Omega)$ for which*

$$[f]_{\mathscr{L}^{p,q}} = [f]_{p,q} = \begin{cases} \{\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\}^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & q = \infty. \end{cases} \tag{1.5}$$

*is finite.*

Let us remark that in general $[\,]_{p,q}$ is not a norm,in fact the triangular inequality is not verified. Instead, if we replace $f^*$ with $f^{**}$ in the definition above we obtain a norm that we denote by $\|\,\|_{\mathscr{L}^{p,q}}$ or $\|\,\|_{p,q}$.

**Definition 1.2.2.** *Let us suppose* $1 < p \leqslant \infty$ *and* $0 < q \leqslant \infty$*; for* $f \in \mathcal{M}(\Omega)$ *we set*

$$\|f\|_{\mathscr{L}^{p,q}} = \|f\|_{p,q} = \begin{cases} \{\int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q \frac{dt}{t}\}^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^{**}(t), & q = \infty. \end{cases} \tag{1.6}$$

*is finite.*

**Lemma 1.2.3.** *For* $1 < p \leqslant \infty$ *and* $1 \leqslant q \leqslant \infty$*,* $[\,]_{p,q}$ *and* $\|\,\|_{p,q}$ *are equivalent.*

It is easy to prove this assertion by the Hardy inequality.

**Lemma 1.2.4.** ***(G.H. Hardy)****. Let* $\Psi$ *be a measurable nonnegative function on* $]0,\infty[$ *and consider* $-\infty < \nu < 1$*. If* $1 \leqslant q < \infty$*, we have*

$$\left\{\int_0^\infty \left(t^\nu ⨍_0^t \Psi(s)\,ds\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \leqslant \frac{1}{1-\nu}\left\{\int_0^\infty (t^\nu \Psi(t))^q \frac{dt}{t}\right\}^{\frac{1}{q}} \tag{1.7}$$

*and*

$$\left\{\int_0^\infty \left(t^{1-\nu} \int_t^\infty \Psi(s)\,\frac{ds}{s}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \leqslant \frac{1}{1-\nu}\left\{\int_0^\infty (t^{1-\nu} \Psi(t))^q \frac{dt}{t}\right\}^{\frac{1}{q}} \tag{1.8}$$

*with evident modification if* $q = \infty$*.*

*Proof.* Let us suppose $1 < q < \infty$. Writing $\Psi(s) = s^{\frac{\nu}{q'}} s^{-\frac{\nu}{q'}} \Psi(s)$ and applying the Hölder inequality we have

$$⨍_0^t \Psi(s)\,ds \leqslant \left(⨍_0^t s^{-\nu}\,ds\right)^{\frac{1}{q'}} \left(⨍_0^t s^{\frac{\nu q}{q'}} \Psi(s)^q\,ds\right)^{\frac{1}{q}} =$$

$$= (1-\nu)^{-\frac{1}{q'}} t^{\frac{-\nu}{q'} - \frac{1}{q}} \left(\int_0^t s^{\nu(q-1)} \Psi(s)^q\,ds\right)^{\frac{1}{q}}$$

Hence, by an interchange in the order of integration,

$$\int_0^\infty \left(t^\nu ⨍_0^t \Psi(s)\,ds\right)^q \frac{dt}{t} \leqslant (1-\nu)^{1-q} \int_0^\infty t^{\nu-2} dt \int_0^t s^{\nu(q-1)} \Psi(s)^q\,ds =$$

$$= (1-\nu)^{1-q} \int_0^\infty s^{\nu(q-1)} \Psi(s)^q \, ds \int_s^\infty t^{\nu-2} \, dt.$$

For the conclusion of the proof, it is necessary to perform the integration over $t$ and considering the $\frac{1}{q}$-th roots, we obtain (1.7). If $q = 1$, it is sufficient to change the order of integration. If $q = \infty$ it is sufficient to apply Hölder inequality. With the same arguments it is possible to prove inequality (1.8).

□

To prove the lemma below it is sufficient to apply relation (1.7) with $\nu = \frac{1}{p}$.

**Lemma 1.2.5.** *If $1 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, we have*

$$[f]_{p,q} \leqslant \|f\|_{p,q} \leqslant p'[f]_{p,q}$$

*for all $f \in \mathcal{M}(\Omega)$, where $p' = \frac{p}{(p-1)}$.*

The triangular inequality is a consequence of the subadditivity of $f \to f^{**}$.

**Theorem 1.2.6.** *If $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ or if $p = q = \infty$, $\mathscr{L}^{p,q}$ with the norm $\| \, \|_{\mathscr{L}^{p,q}}$ is a Banach space* (see [BR]).

**Definition 1.2.7.** *If $1 \leqslant p < \infty$ the Lorentz space $\mathscr{L}^{p,\infty}$ is also called Marcinkiewicz space or weak-$\mathscr{L}^p$ space.*

**Definition 1.2.8.** *The Marcinkiewicz space $\mathscr{L}^{p,\infty}(\Omega)$ weak-$\mathscr{L}^p(\Omega)$,$(p > 1)$, where $\Omega \subset \mathbb{R}^n$ is an open set, may be defined according the norm*

$$\|f\|_{\mathscr{L}^{p,\infty}} = \sup_{E \subseteq \Omega} |E|^{\frac{1}{p}} \fint_E |f| \, dx.$$

The following result shows that for a fixed $p$, the Lorentz space $\mathscr{L}^{p,q}$ increases when $q$ increases.

**Proposition 1.2.9.** *Let us suppose $0 < p \leqslant \infty$ and $0 < q \leqslant r \leqslant \infty$. There exists a constant $c$, depending only on $p$, $q$ and $r$, such that*

$$[f]_{p,r} \leqslant c \, [f]_{p,q}$$

*for all $f \in \mathcal{M}(\Omega)$ and therefore $\mathscr{L}^{p,q} \subset \mathscr{L}^{p,r}$.*

The inclusion relations between the Lorentz spaces $\mathscr{L}^{p,q}$ when is the first exponent to change are the same of these of the Lebesgue space and are dependent from the measure of $\Omega$. The second exponent doesn't matter. For example, if $|\Omega| < \infty$, $0 < p \leqslant r \leqslant \infty$ and $0 < q, s \leqslant \infty$, it happens that $\mathscr{L}^{r,s} \subset \mathscr{L}^{p,q}$.

**Theorem 1.2.10.** *Suppose* $1 < p < \infty$, $1 \leqslant q < \infty$. *The dual of the Banach space* $\mathscr{L}^{p,q}$ *can be identified (up to equivalence of norms) with* $\mathscr{L}^{p',q'}$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$.

As in $\mathscr{L}^p$ spaces we can obtain a Sobolev inequality also for Lorentz spaces (for more details see [A]).

**Theorem 1.2.11.** *Let be* $u(x)$ *a function sufficiently regular, with compact support; for all* $p \in [1, n[$ *and for all* $r \in [1, p]$ *we have:*

$$\|u\|_{q,r} \leqslant C' \|Du\|_{p^*,r}$$

*with* $p^* = \frac{np}{(n-p)}$ *and*

$$C' = \frac{\{\Gamma(1 + n/2)\}^{1/n}}{\sqrt{\pi}} \frac{p}{n-p} \tag{1.9}$$

*and (1.9) is the best constant possible.*

## 1.3 Lorentz-Zygmund spaces

Throughout this section $\Omega$ will denote a bounded domain in $\mathbb{R}^n$ and for simplicity, let us assume $|\Omega| = 1$. By a measurable function on $\Omega$ we shall mean an equivalence class of measurable functions on $\Omega$ which differ only on a subset of measure zero.

**Definition 1.3.1.** *When* $1 \leqslant q, p \leqslant \infty$ *and* $-\infty < \alpha \leqslant \infty$, *the Lorentz-Zygmund space* $\mathscr{L}^{p,q}(\log \mathscr{L})^\alpha$ *on* $\Omega$ *consists of all (classes of) measurable functions* $f$ *on* $\Omega$ *for which the functional*

$$\|f\|_{\mathscr{L}^{p,q}(\log \mathscr{L})^\alpha} = \|f\|_{p,q;\alpha} = \begin{cases} \left(\int_0^1 [t^{\frac{1}{p}}(1 - \log t)^\alpha f^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & 0 < q < \infty, \\ \sup\limits_{0<t<1} [t^{\frac{1}{p}}(1 - \log t)^\alpha f^*(t)] & q = \infty. \end{cases} \tag{1.10}$$

*is finite.*

Moreover if $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and $-\infty < \alpha < \infty$, then the functional

$$f \to \left(\int_0^1 [t^{\frac{1}{p}}(1 - \log t)^\alpha f^{**}(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

(with evident modification if $q = \infty$) defines a norm in $\mathscr{L}^{p,q}(\log \mathscr{L})^\alpha$ which is equivalent to the quasinorm (1.10). In particular $\mathscr{L}^{p,q}(\log \mathscr{L})^\alpha$ is equivalent

to a Banach space, see [BR] for more details.
It will be useful to observe that since $\log \frac{1}{t}$ and $(1-\log t)$ are asymptotically the same at $t=0$, we have

$$\|f\|_{p,q;\alpha} \sim \left( \int_0^1 \left[ t^{\frac{1}{p}} \left( \log \frac{1}{t} \right)^{\alpha} f^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{1.11}$$

provided only that the integral converges at $t=0$, i.e., provided $\alpha + \frac{1}{q} > 0$.

It is clear that for $\alpha = 0$ we return to the Lorentz space $\mathscr{L}^{p,q}$.

Moreover in the special case $p = q$ and $-\infty < \alpha < \infty$ it is easy to verify that $f \in \mathscr{L}^{p,p}(\log \mathscr{L})^{\alpha}$ if and only if it belongs to the Orlicz-Zygmund space $\mathscr{L}^{p}(\log \mathscr{L})^{\alpha}$, i.e.

$$\int_{\Omega} |f|^p \log^{\alpha}(e + |f|)\, dx < \infty.$$

Various inclusion relations among the Lorentz Zygmund spaces hold.

**Theorem 1.3.2.** *Suppose* $1 \leqslant p \leqslant \infty$, $1 \leqslant a, b \leqslant \infty$ *and* $-\infty < \alpha, \beta < \infty$ *then*

$$\mathscr{L}^{p,a}(\log \mathscr{L})^{\alpha} \subseteq \mathscr{L}^{p,b}(\log \mathscr{L})^{\beta}$$

*whenever*

1. $a \leqslant b$ *and* $\alpha \geq \beta$;
2. $a > b$ *and* $\alpha + \frac{1}{a} > \beta + \frac{1}{b}$.

see ([BR]) for more details.

It is trivial to prove by the simple definition of Lorentz-Zygmund space the following lemma

**Lemma 1.3.3.** *Let us assume* $f \geqslant 0$,

$$f \in \mathscr{L}^{p,q}(\log \mathscr{L})^{\beta} \Rightarrow f^r \in \mathscr{L}^{\frac{p}{r},\frac{q}{r}}(\log \mathscr{L})^{\beta r}$$

*Proof.* Let us simply apply Definition 1.3.1:

$$\|f^r\|_{\frac{p}{r}\frac{q}{r},\beta r} = \left( \int_0^1 [t^{\frac{r}{p}} (f^*(t))^r (1-\log t)^{\beta r}]^{\frac{q}{r}} \frac{dt}{t} \right)^{\frac{r}{q}} = \|f\|^r_{pq,\beta}.$$

□

**Theorem 1.3.4.** *Suppose* $1 < p < \infty$, $1 \leqslant q < \infty$, $-\infty < \alpha < \infty$ *. Then the dual of the Banach space* $\mathscr{L}^{p,q}(\log \mathscr{L})^{\alpha}$ *can be identified (up to equivalence of norms) with* $\mathscr{L}^{p',q'}(\log \mathscr{L})^{-\alpha}$*, where* $\frac{1}{p}+\frac{1}{p'} = 1$ *and* $\frac{1}{q}+\frac{1}{q'} = 1$.

Let us consider the case $p = q = 1$.

**Lemma 1.3.5.** *The dual of the Zygmund space* $\mathscr{L}(\log \mathscr{L})^{\alpha}$ *is given by the Zygmund* $\mathscr{L}^{\infty}(\log \mathscr{L})^{-\alpha}$.

**Definition 1.3.6.** *When* $0 < p < \infty$ *and* $-\infty < \alpha < \infty$*, the Zygmund space* $\mathscr{L}^{p}(\log \mathscr{L})^{\alpha}$ *consists of all measurable functions* $f$ *on* $\Omega$ *for which*

$$\int_{\Omega} [|f(x)| \log^{\alpha}(2 + |f(x)|)]^{p}\, dx < \infty$$

*If* $\alpha \geqslant 0$*, the Zygmund space* $Exp_{\frac{1}{\alpha}}$ *consists of all measurable functions* $f$ *on* $\Omega$ *for which there is a constant* $\lambda = \lambda(f) > 0$ *such that*

$$\int_{\Omega} exp[\lambda|f(x)|]^{\frac{1}{\alpha}}\, dx < \infty$$

**Definition 1.3.7.** *If* $0 < p < \infty$ *and* $-\infty < \alpha < \infty$ *then a measurable function* $f$ *on* $\Omega$ *belongs to the Zygmund space* $\mathscr{L}^{p}(\log \mathscr{L})^{\alpha}$ *if and only if*

$$\left(\int_{0}^{1} [(1 - \log t)^{\alpha} f^{*}(t)]^{p}\, dt\right)^{\frac{1}{p}} < \infty$$

*If* $\alpha \geqslant 0$ *then a measurable function* $f$ *on* $\Omega$ *belongs to the Zygmund space* $Exp^{\frac{1}{\alpha}}$ *if and only if*

$$\sup_{0<t<1} (1 - \log t)^{-\alpha} f^{*}(t) < \infty$$

Let us observe that a comparison with Definition 1.3.1 shows that

$$\mathscr{L}^{\infty,\infty}(\log \mathscr{L})^{-\alpha} = Exp^{\frac{1}{\alpha}}, \quad \alpha \geqslant 0$$

In view of the forthcoming chapters we now state the following propositions

**Proposition 1.3.8.** *Suppose* $1 < p, q < \infty$, $1 \leqslant a, b < \infty$, $-\infty < \alpha, \beta < \infty$. *If* $f \in \mathscr{L}^{p,a} \log^{\alpha} \mathscr{L}$, $g \in \mathscr{L}^{q,b} \log^{\beta} \mathscr{L}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, $\gamma = \alpha + \beta$ *then* $fg \in \mathscr{L}^{1,c} \log^{\gamma} \mathscr{L}$.
*Furthermore, we obtain the following estimate*

$$\|fg\|_{\mathscr{L}^{1,c} \log^{\gamma} \mathscr{L}} \leqslant \|f\|_{\mathscr{L}^{p,a} \log^{\alpha} \mathscr{L}} \|g\|_{\mathscr{L}^{q,b} \log^{\beta} \mathscr{L}} \tag{1.12}$$

*Proof.* By Definition 1.3.1, we have

$$\|fg\|_{\mathscr{L}^{1,c}\log^\gamma \mathscr{L}} = \left(\int_0^1 [tf^*(t)g^*(t)(1-\log t)^\gamma]^c \frac{dt}{t}\right)^{\frac{1}{c}}$$
$$= \left(\int_0^1 \left(t^{\frac{1}{p}} f^*(t)\right)^c \left(t^{\frac{1}{q}} g^*(t)\right)^c (1-\log t)^{\gamma c} \frac{dt}{t}\right)^{\frac{1}{c}}$$
$$= \left(\int_0^1 \left[f^*(t) t^{\frac{1}{p}} (1-\log t)^\alpha \frac{1}{t^{1/a}}\right]^c \left[g^*(t) t^{\frac{1}{q}} (1-\log t)^\beta \frac{1}{t^{1/b}}\right]^c dt\right)^{\frac{1}{c}} \quad (1.13)$$

Applying Hölder inequality to (1.13), with conjugate exponents $\frac{a+b}{b}$ and $\frac{a+b}{a}$ and setting $c = \frac{ab}{a+b}$, the above term can be estimate by the following espression

$$\left\{\left[\int_0^1 \left(f^*(t) t^{\frac{1}{p}} (1-\log t)^\alpha \frac{1}{t^{1/a}}\right)^a dt\right]^{\frac{b}{a+b}} \left(\int_0^1 \left(g^*(t)\, t^{\frac{1}{q}}\, (1-\log t)^\beta \frac{1}{t^{1/b}}\right)^b dt\right)^{\frac{a}{a+b}}\right\}^{\frac{a+b}{ab}}$$
$$= \|f\|_{\mathscr{L}^{p,a}\log \mathscr{L}^\alpha} \; \|g\|_{\mathscr{L}^{q,b}\log \mathscr{L}^\beta}$$

□

Let us consider, now, the case $p = a = \infty$

**Proposition 1.3.9.** *If $p = a = \infty$, $1 < b < \infty$, $-\infty < \alpha, \beta < \infty$, $\alpha + \beta = \gamma$. Let us suppose $f \in \mathscr{L}^\infty(\log \mathscr{L})^\alpha$ and $g \in \mathscr{L}^{1,b}(\log \mathscr{L})^\beta$, then $fg \in \mathscr{L}^{1,b}(\log \mathscr{L})^\gamma$.*

*Proof.* Using Definition 1.3.1 and by elementary calculation

$$\|fg\|_{\mathscr{L}^{1,b}(\log \mathscr{L})^\gamma} = \left(\int_0^1 [t\, f^*(t)g^*(t)(1-\log t)^\gamma]^b \; \frac{dt}{t}\right)^{\frac{1}{b}}$$
$$= \left(\int_0^1 \left[t\, g^*(t)\, (1-\log t)^\beta\right]^b [f^*(t)\, (1-\log t)^\alpha]^b \, \frac{dt}{t}\right)^{\frac{1}{b}}$$
$$\leqslant \left(\int_0^1 \left[t\, g^*(t)\, (1-\log t)^\beta\right]^b \frac{dt}{t}\right)^{\frac{1}{b}} \left(\sup_{0<t<1} f^*(t)\, (1-\log t)^\alpha\right)$$
$$\leqslant \|g\|_{\mathscr{L}^{1,c}(\log \mathscr{L})^\beta} \; \|f\|_{\mathscr{L}^\infty(\log \mathscr{L})^\alpha}$$

□

**Corollary 1.3.10.** *Suppose $\alpha \geqslant 0$,$-\infty < \beta < \infty$ and $\gamma = \alpha + \beta$. If $f \in \mathscr{L}^\infty(\log \mathscr{L})^\alpha$ and $g \in \mathscr{L}(\log \mathscr{L})^\beta$ then $fg \in \mathscr{L}\log^\gamma \mathscr{L}$.*

## 1.4 Orlicz Spaces

An Orlicz function is a continuously increasing function

$$\mathcal{P} : [0, \infty) \to [0, \infty),$$

$$\mathcal{P}(0) = 0, \qquad \lim_{t\to\infty} \mathcal{P}(t) = \infty,$$

though in most of our applications $\mathcal{P}$ will be convex, in this case we call it a Young function. The Orlicz space, denoted by $\mathscr{L}^{\mathcal{P}}(\Omega)$, consists of all measurable functions $f$ on $\Omega$ such that

$$\int_\Omega \mathcal{P}(k^{-1}|f|) \ < \ \infty, \text{ for some } k = k(f) > 0 \tag{1.14}$$

$\mathscr{L}^{\mathcal{P}}(\Omega)$ is a complete linear metric space with respect to the following distance function :

$$\text{dist}_{\mathcal{P}(f,g)} = \inf \left\{ k > 0 \ \int_\Omega \mathcal{P}\left(k^{-1}|f-g|\right) \leqslant k \right\}.$$

There is also a homogeneous nonlinear functional on $\mathscr{L}^{\mathcal{P}}(\Omega)$ called the Luxemburg functional:

$$\|f\|_{\mathscr{L}^{\mathcal{P}}} = \inf \left\{ k > 0; \int_\Omega \mathcal{P}(k^{-1}|f|) \leqslant \mathcal{P}(1) \right\} \tag{1.15}$$

in the case when $\mathcal{P}$ is a Young function, the expression $\| \ \|_{\mathcal{P}}$ is a norm and $\mathscr{L}^{\mathcal{P}}(\Omega)$ becomes a Banach space.
As a first example, if we put $\mathcal{P}(t) = t^p$, $0 < p < \infty$ then the space $\mathscr{L}^{\mathcal{P}}(\Omega)$ coincide with the usual Lebesgue space $\mathscr{L}^p(\Omega)$. Note that $\mathscr{L}^p(\Omega)$ is a Banach space only when $p \geqslant 1$.
The Zygmund spaces, denoted by $\mathscr{L}^p \log^\alpha \mathscr{L}(\Omega)$, correspond to the Orlicz function $\mathcal{P}(t) = t^p \log^\alpha(a+t)$ with $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$ and suitable large constant $a$.
The defining function $\mathcal{P}(t) = t^p \log^\alpha(e+t)$, $1 \leqslant p < \infty$ is a Young function when $\alpha \geqslant 1 - p$ and there we have the following estimate

$$\|f\|_{\mathscr{L}^p \log^{-1} \mathscr{L}} \ \leqslant \ \|f\|_p \ \leqslant \ \|f\|_{\mathscr{L}^p \log \mathscr{L}}$$

and

$$\|f\|_{\mathscr{L}^p \log \mathscr{L}} \ \leqslant \ \left[ \int |f|^p log \left( e + \frac{|f|}{\|f\|_p} \right) \right]^{\frac{1}{p}} \ \leqslant \ 2 \|f\|_{\mathscr{L}^p \log \mathscr{L}}$$

For $p \geqslant 1$ and $\alpha \geqslant 0$ the non-linear functional

$$[[f]]_{p,\alpha} = \left[ \int_{\mathbb{R}^n} |f|^p \log^\alpha \left( e + \frac{|f|}{\|f\|_p} \right) \right]^{\frac{1}{p}}$$

is comparable with the Luxemburg norm, given at (1.15).
The following estimates are straightforward

$$\|f\|_{\mathscr{L}^p \log^{-1} \mathscr{L}} \leqslant \|f\|_{\mathscr{L}^p} \leqslant \|f\|_{\mathscr{L}^p \log^\alpha \mathscr{L}} \leqslant [[f]]_{p,\alpha} \leqslant 2\|f\|_{\mathscr{L}^p \log^\alpha \mathscr{L}} \tag{1.16}$$

We have the Hölder-type inequalities

$$\|fg\|_{\mathscr{L}^c \log^\gamma \mathscr{L}} \leqslant C_{\alpha\beta}(a,b)\|f\|_{\mathscr{L}^a \log^\alpha \mathscr{L}} \cdot \|g\|_{\mathscr{L}^b \log^\beta \mathscr{L}}$$

whenever $a, b > 1$ and $\alpha, \beta \in \mathbb{R}$ are coupled by the relationships

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \quad \frac{\gamma}{c} = \frac{\alpha}{a} + \frac{\beta}{b}$$

Another important example is the exponential class defined with the Orlicz function $e^t - 1$.

Hölder's inequality for Zygmund spaces will be quite important. It takes the form

$$\|\varphi_1 ... \varphi_k\|_{\mathscr{L}^p \log^\alpha \mathscr{L}} \leqslant c\,\|\varphi_1\|_{\mathscr{L}^{p_1} \log^{\alpha_1} \mathscr{L}} \; ... \; \|\varphi_k\|_{\mathscr{L}^{p_k} \log^{\alpha_k} \mathscr{L}}$$

where $p_1, p_2, ..., p_k > 1$ ; $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k}$, $\frac{\alpha}{p} = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + ... + \frac{\alpha_k}{p_k}$.
The constant here does not depend on the functions $\varphi_i \in \mathscr{L}^{p_i} \log^{\alpha_i} \mathscr{L}$.
Another important case arises when we take any Young function $\Phi$ and set

$$\mathcal{P}(t) = \frac{1}{p}\Phi(t^p), \qquad \mathcal{P}_i = \frac{1}{p_i}\Phi(t^{p_i}) \quad i = 1, 2, ..., k$$

$\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_k}$. Then we have

$$\begin{aligned} \mathcal{P}(t_1, t_2, \; ... \; t_n) &= \frac{1}{p} \; \Phi(t_1^{p_1} ... t_k^{p_k}) \\ &\leqslant \frac{1}{p} \; \Phi \left( \frac{p t_1^{p_1}}{p_1} + \; ... \; + \frac{p t_k^{p_k}}{p_k} \right) \\ &\leqslant \frac{1}{p_1} \Phi(t_1^{p_1}) + \; ... \; + \frac{1}{p_k} \Phi(t_k^{p_k}) \end{aligned}$$

$$= \mathcal{P}_1(t_1) + \ \ \dots \ \ \mathcal{P}_k(t_k)$$

A pair of Orlicz function $(\mathcal{P}, \mathcal{Q})$ are called Hölder conjugate couple, or Young complementary functions, if we have Hölder inequality

$$\left| \int_\Omega \langle f, g\rangle \right| \ \leqslant \ C_{\mathcal{P},\mathcal{Q}} \, \|f\|_{\mathcal{P}} \, \|g\|_{\mathcal{Q}}$$

for $f \in \mathscr{L}^{\mathcal{P}}(\Omega)$ and $g \in \mathscr{L}^{\mathcal{Q}}(\Omega)$.
If we take as Hölder conjugate couple $\mathcal{P}(t) = t\log(e+t)$ and $\mathcal{Q}(t) = e^t - 1$ defining the Zygmund and exponential classes, respectively. We have the following estimate

$$\left| \int_\Omega \langle f, g\rangle \right| \ \leqslant \ 4 \, \|f\|_{\mathscr{L}\log\mathscr{L}} \, \|g\|_{Exp}.$$

In view of the same homogeneities on each side we can assume Luxemburg norm equal 1. From the definition of these norms we find

$$\int_\Omega |f|\log(e+|f|) \ = \ \log(e+1)$$

and

$$\int_\Omega (e^{|g|} - 1) = e - 1$$

Then we have the elementary inequality

$$|f||g| \ \leqslant \ |f|\log(1+|f|) + e^{|g|} - 1 \tag{1.17}$$

to conclude that $\int_\Omega |f||g| \leqslant 4$ as desired.
To define the dual space, we must assume a doubling condition on $\mathcal{P}$.

$$\mathcal{P}(2t) \leqslant 2^\beta \mathcal{P}(t)$$

for some constant $\beta \geqslant 1$ and all $t > 0$.

**Theorem 1.4.1.** *Let $(\mathcal{P}, \mathcal{Q})$ be a Hölder conjugate couple of Young functions with $\mathcal{P}$ satisfying a doubling condition. Then every bounded linear functional defined on $\mathscr{L}^{\mathcal{P}}(\Omega)$ is uniquely represented by a function $g \in \mathscr{L}^{\mathcal{Q}}(\Omega)$ as*

$$f \to \langle f, g\rangle$$

Without a doubling condition the dual of $\mathscr{L}^{\mathcal{P}}(\Omega)$ does not have a nice description.
If we consider the Hölder conjugate couple

$$\mathcal{P}(t) = t\log^{\frac{1}{\alpha}}(e+t) \quad \mathcal{Q}(t) = e^{t^\alpha} - 1$$

with $\alpha > 0$, we find that the dual to $\mathscr{L}\log^{\frac{1}{\alpha}}\mathscr{L}(\Omega)$ is the exponential class $Exp_\alpha(\Omega) = \mathscr{L}^{\mathcal{Q}}(\Omega)$, but not conversely.

**Theorem 1.4.2.** *Let $\mathcal{P}$ be an Orlicz function (not necessarily a Young function) satisfying a doubling condition. Then the space $C_0^\infty(\Omega)$ is dense in the metric space $\mathscr{L}^{\mathcal{P}}(\Omega)$.*

For simplicity, we write $Exp(\Omega)$ for $Exp_1(\Omega)$ as this space will be frequently used.
Thus $Exp(\Omega)$ is the dual space to the Zygmund space $\mathscr{L}\log\mathscr{L}(\Omega)$.
We can identify the space $Exp_\gamma$ with $\mathscr{L}^\infty\log\mathscr{L}^{-\frac{1}{\gamma}}$. In this way Proposition 1.3.9 becomes:

**Theorem 1.4.3.** *If $1 \leqslant b < \infty$, $-\infty < \gamma < \infty$. Let us suppose $f \in Exp_\gamma$ and $g \in \mathscr{L}^{1,b}(\log\mathscr{L})^{\frac{1}{\gamma}}$, then $fg \in \mathscr{L}^{1,b}$.*

$\mathscr{L}\log\mathscr{L}$ and $Exp$ have traditionally be regarded as more general Orlicz spaces, it is neverthless the case that can also be regarded as more general types of Lorentz spaces.

## 1.5 BMO-spaces

In this section we will examine the space $BMO$ of functions of bounded mean oscillation.
Our discussion begins in a local context of a fixed cube $Q_0$ in euclidean space $\mathbb{R}^n$. If $\Omega$ is any measurable set of finite positive measure $|\Omega|$ and $f$ is an integrable function, let us recall that $f_\Omega = \fint_\Omega f(x)dx$ indicates the integral mean of $f$ over $\Omega$.

**Definition 1.5.1.** *If $f$ is integrable over $Q_0$, the sharp function $f^\sharp_{Q_0}$ of $f$ relative to $Q_0$ is defined by*

$$f^\sharp_{Q_0}(x) = \sup_{x\in Q\subset Q_0} \fint_Q |f(y) - f_Q|\, dy,$$

*where the supremum extends over all cubes $Q$ that contain $x$ and are contained in $Q_0$.*

The sharp function $f^\sharp_{Q_0}$ measures locally, at the point $x$, the average oscillation of $f$ from its mean value over cubes containing $x$.

**Definition 1.5.2.** *For a fixed cube $Q_0 \in \mathbb{R}^n$ we will denote by $BMO(Q_0)$ the class of functions $f$, such that*

$$\|f\|_{*,Q_0} = \sup_{Q \subset Q_0} \fint_Q |f(x) - f_Q|\, dx \tag{1.18}$$

*is finite, where the supremum taken over all cubes $Q \subset Q_0$.*

It is clear that (1.18) does not define a norm since it vanishes on constant functions. However it is easy to verify that $BMO(Q_0)$ is a Banach space under the norm

$$\|f\|_{BMO(Q_0)} = \|f\|_{*,Q_0} + \|f\|_{\mathscr{L}^1(Q_0)}$$

**Lemma 1.5.3.** ***(John-Nirenberg lemma)***. *Let $Q_0$ be a fixed cube in $\mathbb{R}^n$. Then there is a constant $c$ such that*

$$[(f - f_Q)\chi_Q]^*(t) \;\leqslant\; c\,\|f\chi_Q\|_{*,Q}\, \log^+\left(\frac{6|Q|}{t}\right), \quad (t > 0),$$

*for all $f \in BMO(Q_0)$ and all subcubes $Q$ of $Q_0$; equivalently,*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \;\leqslant\; 6|Q| exp\left\{\frac{-\lambda}{c\|f\chi_Q\|_{*,Q}}\right\}$$

*holds for all $\lambda > 0$.*

**Corollary 1.5.4.** *Suppose $1 \leqslant p < \infty$.Then an integrable function $f$ on $Q_0$ belongs to $BMO(Q_0)$ if and only if*

$$\|f\|_{BMO^p(Q_0)} \;=\; \|f\|_{\mathscr{L}^1} \;+\; \sup_{Q \subset Q_0}\left\{\frac{1}{|Q|}\int_Q |f(y) - f_Q|^p\, dy\right\}^{\frac{1}{p}}$$

*is finite. In fact, there is a constant $c$ such that*

$$\|f\|_{BMO(Q_0)} \;\leqslant\; \|f\|_{BMO^p(Q_0)} \;\leqslant\; c\|f\|_{BMO(Q_0)}$$

The fact that $BMO$-functions have singularities whose rate of growth is at most logarithmic identifies them as members of the Zygmund class $Exp(Q_0)$. Thus we have the following continuous embeddings, $(0 < p < \infty)$,

$$\mathscr{L}^\infty(Q_0) \hookrightarrow BMO(Q_0) \hookrightarrow Exp(Q_O) \hookrightarrow \mathscr{L}^p(Q_0)$$

There is an observation we wish to make. Only $BMO(Q_0)$ in the above embeddings fails to be rearrangement invariant. Hence, the inclusions persists if $BMO(Q_0)$ is replaced by the rearrangement-invariant hull, that is, the space of all $BMO$-functions and all equimeasurable rearrangement of $BMO$-functions.

Let us introduce $W(Q_0)$ the rearrangement-invariant hull of $BMO(Q_0)$.

**Definition 1.5.5.** *Denote by $W = W(\Omega)$ the set of all measurable function $f$ on $\Omega$ for which $f^*$ is everywhere finite and for which the functional*

$$\|f\|_W = \sup_{t>0}[f^{**}(t) - f^*(t)]$$

*is finite*

The following theorem holds

**Theorem 1.5.6.** *Let $Q_0$ be a cube in $\mathbb{R}^n$. Then $W(Q_0)$ is the rearrangement-invariant hull of $BMO(Q_0)$, that is, a function $f$ belongs to $W(Q_0)$ if and only if $f$ is equimeasurable with some function $g$ in $BMO(Q_0)$.*

**Corollary 1.5.7.** *The following inclusions hold for $(0 < p < \infty)$,*

$$\mathscr{L}^\infty(Q_0) \hookrightarrow BMO(Q_0) \hookrightarrow W(Q_0) \hookrightarrow Exp(Q_O) \hookrightarrow \mathscr{L}^p(Q_0)$$

**Definition 1.5.8.** *A locally integrable function $f$ is said to be of bounded mean oscillation on $\mathbb{R}^n$, in symbols $f \in \mathrm{BMO}(\mathbb{R}^n)$, iff*

$$\|f\|_* = \sup_Q ⨍_Q |f(y) - f_Q|\, dy \tag{1.19}$$

*is finite, where the supremum extends over all cubes $Q$ in $\mathbb{R}^n$.*

It is clear that (1.19) does not define a norm since $\|f\|_* = 0$ whenever $f$ is constant. However, by factoring out the constant functions, that is considering the quotient space $BMO/C$, we have the following result

**Proposition 1.5.9.** *$BMO/C$ is a Banach space under $\| \, \|_*$*

As on a cube $Q_0$, also in $\mathbb{R}^n$ John-Nirenberg lemma holds.

**Corollary 1.5.10.** ***(John-Nirenberg lemma)*** *There is a constant $c = c(n)$ such that the estimate*

$$[(f - f_Q)\chi_Q]^* (t) \leqslant c \, \|f\|_* \log^+ \left( \frac{6|Q|}{t} \right) \quad (t > 0),$$

*holds for all $f \in BMO(\mathbb{R}^n)$ and all cubes $Q$ of $Q_0$; equivalently,*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leqslant 6|Q| exp \left\{ \frac{-\lambda}{c\|f\|_*} \right\}$$

*holds for all $\lambda > 0$.*

Before providing a more detailed background, let us observe that clearly $\mathscr{L}^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, and for $f \in \mathscr{L}^\infty(\mathbb{R}^n)$ we have

$$\|f\|_* \;\leqslant\; 2 \, \|f\|_{\mathscr{L}^\infty}$$

and it is interesting to remark the following equivalences.
Denote

$$\|f\|'_* = \sup_Q \inf_{c \in \mathbb{R}} ⨍ |f(x) - c| \, dx, \qquad \|f\|''_* = ⨍_Q ⨍_Q |f(x) - f(y)| \, dx \, dy.$$

Then

$$\|f\|'_* \leqslant \|f\|_* \leqslant \|f\|'_*; \quad \|f\|_* \leqslant \|f\|''_* \leqslant 2\|f\|_*.$$

This means, that in the definition of the $BMO$-class one can use the mean oscillation $\inf_{c \in \mathbb{R}} ⨍ |f(x) - c| \, dx$ or $⨍_Q ⨍_Q |f(x) - f(y)| \, dx \, dy$.

One reason for the importance of $BMO$ is that it arises as the range of certain singular operators, acting in $\mathscr{L}^\infty$. While $BMO$ contains $L^\infty$, the fundamental John Nirenberg lemma shows that it is “slightly ”larger than $\mathscr{L}^\infty$.
A consequence of Definition (1.5.8) is the stronger condition

$$\sup_Q ⨍_Q |f(x) - f_Q|^2 \, dx \;\leqslant\; A \, \|f\|_*^2 \;<\; \infty$$

which is a corollary of an inequality of John-Nirenberg about functions of bounded mean oscillation. Their inequality is as follows

$$|\{x \in Q : |f(x) - f_Q| > \alpha\}| \leqslant \exp^{-c\alpha/\|f\|_*}|Q|, \quad \text{for every } \alpha > 0.$$

We observe that if $f \in BMO$ then $\int_{\mathbb{R}^n} \frac{|f(x)}{1+|x|^{n+1}}\, dx < \infty$, and more precisely

$$\int_{\mathbb{R}^n} \frac{|f(x) - f_Q}{1+|x|^{n+1}}\, dx \leqslant A\,\|f\|_*$$

where $Q$ is the cube whose sides have length 1, and is centered at the origin. There is another delicate observation to make. Bounded functions lie in $BMO(\mathbb{R}^n)$, but they are not dense. For example $g(x) = \log|x|$ lies in $BMO(\mathbb{R}^n)$ but cannot be locally approximated by bounded functions near the origin. Also the $C_0^\infty$ functions on $\mathbb{R}^n$ lie in $BMO$ and they too are not dense. However, their closure in $BMO(\mathbb{R}^n)$ is the space $VMO(\mathbb{R}^n)$, those functions of vanishing mean oscillation.

Let us define the $VMO(\Omega)$ space as follows

**Definition 1.5.11.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$. We shall say that a locally integrable function $f$ has vanishing mean oscillation on $\Omega$ and write $f \in VMO(\Omega)$ if*

$$\lim_{|Q|\to 0} \frac{1}{|Q|}\int_Q |f - f_Q| = 0$$

*uniformly for cubes $Q$ contained in $\Omega$.*

There is an observation we wish to make. It is possible to replace balls by cubes in the above definition yield the same spaces with comparable norms.

# Chapter 2

# Maximal function

In this chapter we will introduce the Hardy-Littlewood maximal operator and we will discuss some techniques related to it.

## 2.1 Definition and some properties of the maximal function

**Definition 2.1.1.** *Let $f$ be a locally integrable function on $\mathbb{R}^n$. The Hardy-Littlewood maximal function $Mf$ of $f$ is defined by*

$$(Mf)(x) = \sup_{x \in Q} ⨍_Q |f(y)|\, dy, \quad (x \in \mathbb{R}^n)$$

*where the supremum extends over all cubes $Q$ containing $x$ (here, as throughout, cubes will be assumed to have their sides parallel to the coordinate axes). The operator $M : f \to Mf$ is called the Hardy-Littlewood operator.*

The maximal function takes into account the local, opposed to the pointwise, behavior of $f$. It provides the magnitude of $f$ amenable to differentiation and integration theory. Quantitative measurement of magnitude is most naturally made by expressing the functions as members of $\mathscr{L}^p$ and $\mathscr{L}^{p,q} \log^\alpha \mathscr{L}$.

The following properties hold:

1. $Mf(x)$ is measurable in fact $\{x \text{ such that } Mf(x) > t\}$ is open.
2. $M$ is sublinear. In fact

$$M(f+g) \leqslant Mf + Mg, \qquad M(\lambda f) = |\lambda| Mf$$

3. $f \in \mathscr{L}^1 \Rightarrow Mf \in \mathscr{L}^1$

We have e.g. that for $f = \chi_{[0,1]}$, $Mf$ behaves as $\frac{1}{x} \notin \mathscr{L}^1$.
It is worth to underline in this chapter the relevance of Theorem 2.1.9 (see [BR] for more details). To give a complete proof of it it is necessary to show some preliminary results. We would like to estimate the size of $Mf$ for integrable $f$.

**Lemma 2.1.2.** *If $f \in \mathscr{L}^\infty$ then $\|Mf\|_{\mathscr{L}^\infty} \leqslant \|f\|_{\mathscr{L}^\infty}$.*

It will be useful for the sequel to recall the two following theorems

**Theorem 2.1.3.** *Suppose $f$ belongs to $\mathcal{M}(\Omega)$. Then*

$$\inf_{f=g+h} \{\|g\|_{\mathscr{L}^1} + t\|h\|_{\mathscr{L}^\infty}\} = \int_0^t f^*(s)\,ds \;=\; tf^{**}(t),$$

*for all $t > 0$.*

**Theorem 2.1.4.** *If $f$ belongs to $\mathscr{L}^1(\mathbb{R}^n)$, then*

$$t\,(Mf)^*(t) \leqslant 4^n \|f\|_{\mathscr{L}^1}, \qquad (t > 0). \tag{2.1}$$

If $g = Mf$ the estimate (2.1) shows that the areas of the rectangles lying below the graph of $g^*$ are uniformly bounded.
This condition that $\sup_t \, t\,g^*(t)$ be finite, is clearly weaker than integrability. The collection of all such functions is referred to as weak-$\mathscr{L}^1$, being satisfy for every $\mathscr{L}^1$-function and by non-integrable functions such that $\frac{1}{|x|}$ .
In other words, we can say that the maximal function $Mf$ is in weak-$\mathscr{L}^1$ whenever $f$ belongs to $\mathscr{L}^1$.

**Theorem 2.1.5.** *(Lebesgue's differentiation theorem). If $f$ is a locally integrable function on $\mathbb{R}^n$, then*

$$\lim_{|Q|\to 0} \fint_Q |f(y) - f(x)|\,dy \;=\; 0, \quad x \in Q,$$

*for almost every $x$ in $\mathbb{R}^n$.*

**Corollary 2.1.6.** *If $f$ is locally integrable in $\mathbb{R}^n$, then*

$$\lim_{|Q|\to 0} \fint_Q f(y)\,dy \;=\; f(x), \quad x \in Q,$$

**Corollary 2.1.7.** *If $f$ is locally integrable in $\mathbb{R}^n$, then*

$$|f(x)| \leqslant (Mf)(x) \tag{2.2}$$

*for almost every $x \in \mathbb{R}^n$.*

Before proceeding further with our study of the maximal function we shall need a covering lemma. The cover is made up from the collection of dyadic cubes, i.e., the cubes formed by means of dilatations and contractions by a factor of two of the basic partition of $\mathbb{R}^n$ into unit cubes with vertices at the lattice points.

**Lemma 2.1.8.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ with finite measure. Then there is a sequence of dyadic cubes $Q_1, Q_2, ...$, with pairwise disjoint interiors, that covers $\Omega$ and satisfies*

1. $Q_k \cap \Omega^c \neq \emptyset, \quad (k = 1, 2, ...)$;
2. $|\Omega| \leqslant \displaystyle\sum_{k=1}^{\infty} |Q_k| \leqslant 2^n |\Omega|$.

Let us recall that the function $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ is the average of $f^*$ over the interval $(0, t)$. This is maximal among all averages of $f^*$ over intervals containing $t$ because $f^*$ is decreasing. Hence, $f^{**}$ is the Hardy maximal function of $f^*$. It is possible to prove the equivalence of the maximal function of the decreasing rearrangement and the decreasing rearrangement of the maximal function.

**Theorem 2.1.9.** *There are constants $c$ and $c'$, depending only on $n$ , such that*

$$c(Mf)^*(t) \leqslant f^{**}(t) \leqslant c'(Mf)^*(t), \ (t > 0) \tag{2.3}$$

*for every locally integrable function $f$ on $Q$* (see [H],[BS]).

*Proof.* Fix $t > 0$. For the left-hand inequality, we may suppose $f^{**}(t) < \infty$, otherwise there is nothing to prove. Given $\varepsilon > 0$, by Theorem 2.1.3 there are functions $g_t \in \mathscr{L}^1$ and $h_t \in \mathscr{L}^\infty$ such that $f = g_t + h_t$ and

$$\|g_t\|_{\mathscr{L}^1} + t\|h_t\|_{\mathscr{L}^\infty} \leqslant tf^{**}(t) + \varepsilon \tag{2.4}$$

Then, by Theorem 2.1.4 and Lemma 2.1.2, for any $s > 0$,

$$(Mf)^*(s) \leqslant (Mg_t)^*\left(\frac{s}{2}\right) + (Mh_t)^*\left(\frac{s}{2}\right) \leqslant \frac{c}{s}\|g_t\|_{\mathscr{L}^1} + \|h_t\|_{\mathscr{L}^\infty}$$

$$\leqslant \frac{c}{s}(\|g_t\|_{\mathscr{L}^1} + s\|h_t\|_{\mathscr{L}^\infty}).$$

Putting $s = t$, using (2.4) and letting $\varepsilon \to 0$, we obtain the first of the inequality of (2.3).
For the right-hand inequality in (2.3), we may suppose $(Mf)^*(t) < \infty$, otherwise there is nothing to prove.
The lower semicontinuity of $Mf$ ensures that the set

$$\Omega = \{x \in \mathbb{R}^n : (Mf)(x) > (Mf)^*(t)\}$$

is open, and we have $|\Omega| \leqslant t$ because $Mf$ and $(Mf)^*$ are equimeasurable. Applying Lemma 2.1.8 we obtain a sequences of cubes $Q_1, Q_2, ...$, with pairwise disjoint interiors, that cover $\Omega$ and satisfy

$$Q_k \cap \Omega^c \neq \emptyset, \quad (k = 1, 2, ...) \tag{2.5}$$

and

$$\sum |Q_k| \leqslant 2^n |\Omega| \leqslant 2^n t \tag{2.6}$$

With $F = (\bigcup_k Q_k)^c$, we set

$$g = \sum_k f\chi_{Q_k}, \quad h = f\chi_F$$

so $f = g + h$. Then the subadditivity of $f \to f^{**}$ gives

$$f^{**}(t) \leqslant g^{**}(t) + h^{**}(t) \leqslant \frac{1}{t}\|g\|_{\mathscr{L}^1} + \|h\|_{\mathscr{L}^\infty}. \tag{2.7}$$

Now, by (2.5), each $Q_k$ contains a point of $\Omega^c$, and at such a point the maximal function has value at most $(Mf)^*(t)$ because of the way in which $\Omega$ is defined. Thus,

$$\frac{1}{|Q_k|}\int_{Q_k} |f(y)|dy \leqslant (Mf)^*(t), \ \ (k = 1, 2, ...).$$

The $\mathscr{L}^1$-norm of $g$ may be estimate by

$$\|g\|_{\mathscr{L}^1} = \sum_k \int_{Q_k} |f(y)|dy \leqslant \sum_k |Q_k|(Mf)^*(t).$$

Hence, using (2.6), we have

$$\|g\|_{\mathscr{L}^1} \leqslant 2^n t(Mf)^*(t) \tag{2.8}$$

On the other hand, the set $F$ is contained in $\Omega^c$ and so the maximal function is bounded by $(Mf)^*(t)$ on $F$. Hence, using (2.2), we have

$$\|h\|_{\mathscr{L}^\infty} = \|f\chi_F\|_{\mathscr{L}^\infty} \leqslant \|(Mf)\chi_F\|_{\mathscr{L}^\infty} \leqslant (Mf)^*(t).$$

Combining the last estimate with (2.7) and (2.8), we obtain the right-hand inequality in (2.3). □

This equivalence is useful to establish the boundedness of the maximal operator on rearrangement-invariant spaces.
To establish the $\mathscr{L}^p$-boundedness of the Hardy-Littlewood maximal operator inequalities (1.7) and (1.8) are crucial.

**Theorem 2.1.10.** *Let $1 < p \leqslant \infty$ and suppose $f \in \mathscr{L}^p(\mathbb{R}^n)$ then $Mf \in \mathscr{L}^p(\mathbb{R}^n)$ and*

$$\|Mf\|_{\mathscr{L}^p(\mathbb{R}^n)} \leqslant c_p \|f\|_{\mathscr{L}^p(\mathbb{R}^n)}$$

*where $c$ is a constant depending only on $p$ and $n$.*

*Proof.* Since $p > 1$, we may use (1.7) with $\lambda = \frac{1}{p}$ and $q = p$. Since $f^{**}(t) \sim (Mf)^*(t)$, with $c$ depending only on $n$,

$$\|Mf\|_{\mathscr{L}^p(\mathbb{R}^n)} = \left(\int_0^\infty (Mf)^*(t)^p dt\right)^{\frac{1}{p}}$$

$$\leqslant c\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)ds\right)^p dt\right)^{\frac{1}{p}}$$

$$\leqslant cp'\left(\int_0^\infty f^*(t)^p dt\right)^{\frac{1}{p}}$$

$$= cp'\|f\|_{\mathscr{L}^p(\mathbb{R}^n)}.$$

□

We will study the Hardy-Littlewood maximal function in the several spaces introduced in the first chapter.

## 2.2 Maximal function in Lorentz-Zygmund spaces

Let us introduce some notions related to the interpolation of operators.
We have seen in the above section that the fundamental inequality

$$(Mf)^*(t) \leqslant c\,\frac{1}{t}\int_0^t f^*(s)\,ds \tag{2.9}$$

for the Hardy-Littlewood maximal operator $M$ provide the $\mathscr{L}^p$-boundedness of this operator for $p > 1$ (Theorem 2.1.10).
We want to develop an interpolation theory such that the boundedness from $\mathscr{L}^p$ to $\mathscr{L}^q$, for intermediate values of $p$ and $q$ will follow by applying the appropriate Hardy inequalities.

**Definition 2.2.1.** *Suppose $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$ and $q_0 \neq q_1$. Let $\sigma$ denote the interpolation segment*

$$\sigma = \left[\left(\frac{1}{p_0},\frac{1}{q_0}\right),\left\{\frac{1}{p_1},\frac{1}{q_1}\right\}\right],$$

*that is, the line segment in the unit square $\{(x,y) : 0 \leqslant x, y \leqslant 1\}$ with endpoints $(1/p_i, 1/q_i)$,$(i = 0, 1)$. Let $m$ denote the slope*

$$m = \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1}$$

*of the line segment $\sigma$.*
*Consider for each measurable function $f$ on $(0,\infty)$ and each $t > 0$*

$$(S_\sigma f)(t) = t^{-1/q_0}\int_0^{t^m} s^{1/p_0} f(s)\frac{ds}{s}$$
$$+t^{-1/q_1}\int_{t^m}^{\infty} s^{\frac{1}{p_1}} f(s)\frac{ds}{s}.$$

*The operator $S_\sigma : f \to S_\sigma f$ is the Caldéron operator associated with the interpolation segment $\sigma$*

Let us give the definition of operator of joint weak type $(p_0, p_1, q_0, q_1)$

**Definition 2.2.2.** *Suppose $1 \leqslant p_0 < p_1 \leqslant \infty$ and $1 \leqslant q_0, q_1 \leqslant \infty$ with $q_0 \neq q_1$. Let $T$ be a quasilinear operator and suppose $Tf$ is defined for all functions $f$ for which*

$$S_\sigma(f^*)(1) = \int_0^1 s^{1/p_0} f^*(s)\,\frac{ds}{s} + \int_1^\infty s^{1/p_1} f^*(s)\,\frac{ds}{s} \tag{2.10}$$

*Then $T$ is said to be of joint weak type $(p_0, q_0; p_1, q_1)$ if there is a constant $c$ such that*

$$(Tf)^*(t) \leqslant c\, S_\sigma(f^*)(t), \quad (0 < t < \infty)$$

*for all $f$ satisfying (2.10)*

Recalling inequality (2.9) it is easy to verify that the Hardy-Littlewood maximal operator $M$ is an operator of joint weak type $(1, 1; \infty, \infty)$.

We can apply the following theorem (for more details see [BR] and [BS]):

**Theorem 2.2.3.** *Suppose $0 < p < r \leqslant \infty$ and $0 < q, s \leqslant \infty$ with $q \neq s$. Let $T$ be a quasilinear operator of weak type $(p, q; r, s)$. Suppose $0 < \theta < 1$ and let*

$$\frac{1}{u} = \frac{1-\theta}{p} + \frac{\theta}{r}, \quad \frac{1}{v} = \frac{1-\theta}{q} + \frac{\theta}{s}$$

*Suppose $0 < a \leqslant \infty$, $-\infty < \alpha < \infty$. Then*

$$T : \mathscr{L}^{u,a}(\log \mathscr{L})^\alpha \to \mathscr{L}^{v,a}(\log \mathscr{L})^\alpha.$$

Since $M$ the Hardy-Littlewood maximal operator is an operator of joint weak type $(1, 1; \infty, \infty)$

$$M \,:\, \mathscr{L}^{u,a}(\log \mathscr{L})^\alpha \to \mathscr{L}^{u,a}(\log \mathscr{L})^\alpha, \quad u > 1$$

Moreover it is possible to establish the $\mathscr{L}^{p,q}(\log \mathscr{L})^\alpha$-boundedness of the Hardy-Littlewood maximal operator.

**Lemma 2.2.4.** *If $p > 1$, and $1 \leqslant q \leqslant \infty$, $-\infty < \alpha < \infty$ then*

$$M : \mathscr{L}^{p,q}(\log \mathscr{L})^\alpha \to \mathscr{L}^{p,q}(\log \mathscr{L})^\alpha$$

*is a bounded operator.*

Now, it is interesting to study the case $p = 1$.
Let us introduce the definition of maximal function relative to a cube.

**Definition 2.2.5.** *Let $f$ be locally integrable and let $Q$ be a cube on $\mathbb{R}^n$. The Hardy-Littlewood maximal function $M_Q f$ of $f$ relative to $Q$, or simply, the $Q$-maximal function on $f$, is defined by*

$$(Mf)_Q(x) = \sup_{x \in Q' \subset Q} \int_{Q'} |f(y)|\, dy$$

where the supremum extends over all cubes $Q'$ that contain $x$ and are contained in $Q$. The following lemma holds for the Zygmund spaces:

**Lemma 2.2.6.** *For $0 < \alpha \leqslant 1$, $f \in \mathscr{L}\log\mathscr{L}^{\alpha}(Q)$ if and only if $M_Q f \in \mathscr{L}(\log\mathscr{L})^{\alpha-1}(Q)$. In particular there exists $c = c(n,\alpha)$ such that*

$$\int_Q |f| \log^{\alpha}\left(e + \frac{|f|}{|f|_Q}\right) dx \leqslant c \int_Q M_Q f \log^{\alpha-1}\left(e + \frac{M_Q f}{|f|_Q}\right) dx$$

To prove our main result (Proposition 2.2.8) let us state the following lemma, contained in [BR].

**Lemma 2.2.7.** *Suppose $0 < \nu < \infty$, let $\psi$ be a nonnegative decreasing function on $(0,1)$ and fix $t$ with $0 < t < 1$.*
*If $1 \leqslant r \leqslant \infty$, then*

$$\left(\int_0^t [s^{\nu}\psi(s)]^r \frac{ds}{s}\right)^{\frac{1}{r}} \leqslant c \int_0^t s^{\nu}\psi(s)\frac{ds}{s} \qquad (2.11)$$

**Remark**. Let us observe that if $-s + \frac{1}{r} > 0$ we have for any $t$ satisfying $0 < t < 1$

$$\left(\log\frac{1}{t}\right)^{-sr+1} = (-sr+1)\int_t^1 \left(\log\frac{1}{u}\right)^{-sr} \frac{du}{u}$$

**Proposition 2.2.8.** *If $r \geqslant 1$, $0 \leqslant s < \frac{1}{r}$ and $M_Q f \in \mathscr{L}^{1,r}(\log\mathscr{L})^{-s}$ then $f \in \mathscr{L}^{1,r}(\log\mathscr{L})^{-s+\frac{1}{r}}$*

*Proof.* Let us assume, for simplicity, $|Q| = 1$. By Definition 1.2.1 and relation (1.11), we have

$$\|f\|_{\mathscr{L}^{1,r}(\log\mathscr{L})^{-s+\frac{1}{r}}} = \left\{\int_0^1 \left[t(1-\log t)^{-s+\frac{1}{r}} f^*(t)\right]^r \frac{dt}{t}\right\}^{\frac{1}{r}} \leqslant$$

$$\leqslant c\left\{\int_0^1 [tf^*(t)]^r \left(\log\frac{1}{t}\right)^{-sr+1} \frac{dt}{t}\right\}^{\frac{1}{r}} =$$

$$= c\left\{\int_0^1 [tf^*(t)]^r \left[\int_t^1 \left(\log\frac{1}{u}\right)^{-sr} \frac{du}{u}\right] \frac{dt}{t}\right\}^{\frac{1}{r}} =$$

$$= c\left\{\int_0^1 \left(\log\frac{1}{u}\right)^{-sr} \left(\int_0^u [tf^*(t)]^r \frac{dt}{t}\right) \frac{du}{u}\right\}^{\frac{1}{r}}.$$

Using Lemma 2.2.7, the last term is dominated by

$$c\left\{\int_0^1 \left(\log\frac{1}{u}\right)^{-sr} \left(\int_0^u tf^*(t)\frac{dt}{t}\right)^r \frac{du}{u}\right\}^{\frac{1}{r}} =$$

$$= c\left\{\int_0^1 \left(\log\frac{1}{u}\right)^{-sr} u^r \left(\frac{1}{u}\int_0^u f^*(t)\, dt\right)^r \frac{du}{u}\right\}^{\frac{1}{r}} =$$

$$= c\left\{\int_0^1 \left(\log\frac{1}{u}\right)^{-sr} u^r \left(f^{**}(u)\right)^r \frac{du}{u}\right\}^{\frac{1}{r}}.$$

By Theorem 2.1.9, we can conclude

$$\|f\|_{L^{1,r}(\log L)^{-s+\frac{1}{r}}} \leqslant c\left\{\int_0^1 \left[\left(\log\frac{1}{u}\right)^{-s} u(M_Q f)^*(u)\right]^r \frac{du}{u}\right\}^{\frac{1}{r}} =$$

$$= c\|M_Q f\|_{\mathscr{L}^{1,r}(\log\mathscr{L})^{-s}} < \infty.$$

□

If we are dealing with Lorentz spaces, Proposition 2.2.8 becomes

**Corollary 2.2.9.** *If $r \geqslant 1$ and $M_Q f \in \mathscr{L}^{1,r}$ then $f \in \mathscr{L}^{1,r}(\log\mathscr{L})^{\frac{1}{r}}$.*

We can also introduce a more general definition of maximal function.

**Definition 2.2.10.** *Let $Q_0$ be a cube in $R^n$ and $s \geqslant 1$. The Hardy-Littlewood maximal function of $f : Q_0 \to R$ is defined by the rule*

$$(M_s f)(x) \;=\; \sup_{x\in Q\subset Q_0} \left(⨍_Q |f|^s\right)^{\frac{1}{s}} \quad , \quad x \in Q_0$$

we shall conveniently discard the subscript $s = 1$. Now consider a continuous function $\mathscr{A} : [0,+\infty) \to [0,+\infty)$ satisfying:

1. doubling condition: there exists $k > 0$ such that $\mathscr{A}(2t) \leqslant k\mathscr{A}(t)$ for any $t > 0$.
2. growth condition: for some $p > 1$ the function $t \longrightarrow \frac{\mathscr{A}(t)}{t^p}$ is increasing.

**Lemma 2.2.11.** *If the above conditions hold and $\mathscr{A}(|f|) \in \mathscr{L}^1(Q_0)$, then*

$$\int_{Q_0} \mathscr{A}(M_s f) \;\leqslant\; c_0 \int_{Q_0} \mathscr{A}(|f|) \tag{2.12}$$

*for all $1 \leqslant s < p$, where $c_0 = c_0(n,p,s,k)$.*

## 2.3 Maximal function in Orlicz spaces

We shall give a brief review of maximal inequalities in Orlicz spaces. This approach leads to the theory of higher integrability of functions.

**Proposition 2.3.1.** *(Maximal inequality) Given an Orlicz function* $\Phi$ *we define*

$$\Psi(t) = \Phi(t) + t\int_0^t s^{-2}\Phi(s)ds \tag{2.13}$$

*where* $\int_0^1 s^{-2}\Phi(s)ds < \infty$. *Then for each measurable function* $h$ *we have*

$$\|h\|_\Psi \leqslant 3^n \|Mh\|_\Phi \tag{2.14}$$

*and*

$$\int_\Omega \Phi(Mh)\, dx \leqslant 2\cdot 3^n \int_\Omega \Psi(2h)\, dx \tag{2.15}$$

*Moreover, we do not gain any higher integrability if the function* $t \to t^{-p}\Phi(t)$ *is increasing for some* $p > 1$. *In this case*

$$\Phi(t) \leqslant \Psi(t) \leqslant \frac{p}{p-1}\Phi(t)$$

*and we have the following asymptotically sharp bound for* $p$ *near* $1$*:*

$$\|Mh\|_\Phi \leqslant \frac{4\cdot 3^n p}{p-1}\|h\|_\Phi$$

This proposition is a generalization of the result of Stein [S] which assert that $Mh \in \mathscr{L}^1(\Omega)$ if and only if $h \in \mathscr{L}\log\mathscr{L}(\Omega)$. In fact it is sufficient to set $\Phi(t) = t\log(e+t)$, we find the following precise inequalities:

$$\frac{1}{3^{n+2}}\int_\Omega Mh \leqslant \int_\Omega |h|\log\left(e + \frac{|h|}{|h|_\Omega}\right) \leqslant 2^n\int_\Omega Mh$$

To observe a regularity phenomenon of a Borel measure as well as the higher integrability of $\mathscr{L}^1$ functions, one really must assume that $Mh(x)$ lies in a Orlicz space $\mathscr{L}^\Phi$, with $\Phi$ not too far from the identity function. The unexpected twist is that such measures have no singular part with respect to Lebesgue measure and, therefore, are presented by locally integrable functions.

## 2.4 Maximal function in BMO spaces

By Definition 2.2.5 certain estimates for the sharp function are easy to come by. It is clear that

$$f^{\sharp}_{Q_0}(x) \leqslant 2\, M_{Q_0} f(x), \quad (x \in Q_0), \tag{2.16}$$

where $M_{Q_0}$ is the $Q_0$-maximal operator.
In particular $M_{Q_0} f \leqslant Mf$, it follows from Theorem 2.1.9 that

$$(f^{\sharp}_{Q_0})^*(t) \leqslant c f^{**}(t), \quad (0 < t < \infty), \tag{2.17}$$

for every integral function $f$ supported on $Q_0$.
The inequalities (2.16) and (2.17) cannot be reverse since there are unbounded functions $f$ whose sharp functions are bounded (for example $f(x) = |\log |x||$, for $-1 \leqslant x \leqslant 1$) . Nevertheless there is an inequality in the opposite direction to (2.17) when the quantity $f^{**}$ is replaced by $f^{**} - f^*$.
The following theorem holds

**Theorem 2.4.1.** *Let $f$ be an integrable function supported on a cube $Q_0$. Then*

$$f^{**}(t) - f^*(t) \leqslant c(f^{\sharp}_{Q_0})^*(t), \quad \left(0 < t < \frac{|Q_0|}{6}\right).$$

(for the proof see [BS]).

**Lemma 2.4.2.** *Let $Q_0$ be a cube in $\mathbb{R}^n$ and suppose $f \in BMO(Q_0)$.If $Q$ is any subcube of $Q_0$, then*

$$(M_{Q_0} f)_Q \leqslant c\|f\|_{*,Q_0} + \inf_Q M_{Q_0} f,$$

*where $c$ is a constant independent of $f$.*

This lemma implies that $M_{Q_0}$ is a bounded operator on $BMO(Q_0)$. This lemma implies a stronger result.

Let us define the $BLO(\mathbb{R}^n)$ class of function $f$ with bounded lower oscillation. This class consists of all locally summable functions $f$ which are locally essentially bounded from below and such that

$$\|f\|_{BLO} = \sup_Q L(f;Q) < \infty$$

Here the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and

$$L(f;Q) = \frac{1}{|Q|}\int_Q f(x)dx - \operatorname*{ess\,inf}_{x\in Q} f(x)$$

Actually, $Mf$ acts from $BMO$ to $BLO$.

**Theorem 2.4.3.** *Let $f \in BMO(Q_0)$, where $Q_0 \subset \mathbb{R}^n$ is a cube. Then $Mf \in BLO$ and*

$$\|Mf\|_{BLO} \leqslant C\,\|f\|_*$$

where the constant $C$ depends only on the dimension $n$.
The following inclusions hold

$$\mathscr{L}^\infty(Q_0) \subset BLO(Q_0) \subset BMO(Q_0)$$

**Theorem 2.4.4.** *The Hardy-Littlewood maximal operator $M_{Q_0}$ is a bounded operator on $BMO(Q_0)$. Furthermore, $M_{Q_0}$ maps $BMO(Q_0)$ into $BLO(Q_0)$.*

**Remark.** The space $BLO(Q_0)$ is exactly the range of the maximal operator $M_{Q_0}$ on $BMO(Q_0)$.

## 2.5 Maximal operator on distributions

We shall rely on one particular approximation of the identity. That is, we fix a radially symmetric function $\Phi \in C_0^\infty(\mathbb{R}^n)$ supported in the unit ball and having integral 1.
For example

$$\Phi = C(n)\begin{cases} \exp\frac{1}{|x|^2-1} & \text{if } |x| < 1; \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{2.18}$$

where the constant $C(n)$ is chosen so that $\int \phi(x)dx = 1$. For each $t > 0$, we consider a parameter approximation to the Dirac mass $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$.
Given $h \in \mathscr{L}^1_{\text{loc}}(\Omega)$, we recall the mollifiers

$$(h * \Phi_t) \;=\; \int_\Omega \Phi_t(x-y)h(y)dy$$

whenever $0 < t < \text{dist}(x, \partial\Omega)$. So we can extend the mollification to Schwartz distribution $h \in \mathscr{D}'(\Omega)$ as follows

$$(h * \Phi_t) \;=\; h[\Phi_t(x - \cdot)]$$

where we remark that the function $y \to \Phi_t(x-y)$ belongs to $C_0^\infty(\Omega)$ for $0 < t < \mathrm{dist}(x, \partial\Omega)$.
Then the associated maximal function of $h$ can be defined as

$$\mathcal{M}_\Omega h(x) = \sup\{|h * \Phi_t(x)| : 0 < t < \mathrm{dist}(x, \partial\Omega)\}$$

and for $\Omega = \mathbb{R}^n$

$$\mathcal{M}h(x) = \sup\{|h * \Phi_t(x)| : t > 0\}$$

This maximal function works well in connection with $\mathscr{H}^1$-spaces defined in chapter (5).
Let us compare the maximal function $\mathcal{M}h$ with the classical Hardy-Littlewood operator $Mh$.
If $\Omega$ is a cube

$$\mathcal{M}h(x) \leqslant C(n)Mh(x) \quad \text{for all x} \in \Omega$$

Indeed, for $0 < t < \mathrm{dist}(x, \partial\Omega)$ the function $y \to \Phi_t(x-y)$ is supported in the ball $B(x,t) \subset Q(x,t) \subset \Omega$. Thus

$$(h * \Phi_t)(x) \leqslant C_n M_\Omega h$$

A reverse is also true locally. In fact if we consider $\sigma = \sigma(n) = (1/\sqrt{64n}) < 1$ and the cube $\sigma\Omega$ which has the same center as $\Omega$ but is reduced $\sigma$ times in size. Then we have

$$M_{\sigma\Omega} h \leqslant c(n)\mathcal{M}_\Omega h$$

for all $x \in \sigma\Omega$. Indeed for $t = 4\sqrt{n}|Q|^{1/n}$ and all $y \in Q$ we obtain

$$\left|\frac{x-y}{t}\right| \leqslant \frac{\sqrt{n}|Q|^{1/n}}{t} \leqslant \frac{1}{4}.$$

Hence

$$\Phi_t(x-y) = \frac{1}{t^n}\Phi\left(\frac{|x-y|}{t}\right) \geq \frac{\exp(-4/3)}{4^n n^{n/2}|Q|} \geq \frac{C(n)}{|Q|}$$

But $t < \mathrm{dist}(x, \partial\Omega)$. We apply the definition of maximal function $\mathcal{M}_\Omega h$ to obtain the reverse inequality.

# Chapter 3

# Jacobian of Orientation Preserving Mappings

## 3.1 Introduction

In higher dimensions there are many example of nonlinear differential expressions. Most familiar is the Jacobian determinant. Consider a smooth mapping $f = (f^1, f^2, ..., f^n) : \Omega \to \mathbb{R}^n$ defined on an open region $\Omega \subset \mathbb{R}^n$, briefly $f \in \mathscr{C}^\infty(\Omega, \mathbb{R}^n)$. Its differential, sometimes called the gradient matrix, consists of the first order partial derivatives of the coordinates functions.

$$Df = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\ \\ \frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n} \end{bmatrix}$$

The determinant of this matrix is called Jacobian of $f$. We reserve several different symbols to denote the Jacobian, most commons are

$$J(x, f) \;=\; \det[Df(x)] \;=\; \frac{\partial(f^1, f^2, ..., f^n)}{\partial(x_1, x_2, ..., x_n)}$$

The Jacobian function occurs in many different contests such as geometric theory of measure and integration, the mapping degree theory, quasiconformal analysis, nonlinear elasticity, etc...
Most often the expression $J(x, f)$ serves us as a volume element on $\Omega$, which

in conjunction with the formula

$$J(x,f)\,dx \;=\; df^1 \wedge \;\; ... \;\; \wedge df^n \;=\; d(f^1\,df^2 \wedge \;\; ... \;\; \wedge df^n)$$

leads, via integration by parts, to important estimates.
In order to make use of these properties it was necessary to integrate the Jacobian, thus the usual hypothesis was $f \in \mathscr{W}^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$.There arise a natural question: under what condition on $f$ is the Jacobian function locally integrable? There is no reason to expect that the degree of integrability of $J(x,f)$ is different from that of $|Df|^n$. This idea followed from the inequality of Hadamard $|J| \leqslant |Df^1|\,|Df^2|...|Df^n|$, this implies that if $f \in \mathscr{W}^{1,n}$ certainly $J \in \mathscr{L}^1$ .
Surprisingly, just one condition that $J(x,f)$ does not change the sign in $\Omega$, implies the higher integrability of the Jacobian.
It is worth pointing out that the starting point in the theory of the regularity of the Jacobian is the celebrated result due to S. Müller [M] that for an orientation preserving mapping

1. $|Df| \in \mathscr{L}^n(\Omega)$, then $J \in \mathscr{L}\log\mathscr{L}(K)$ for any $K$ compact subset of $\Omega$ (see also [CLMS]).

   In [IS] T.Iwaniec and C.Sbordone proved that

2. if $|Df| \in \mathscr{L}^n(\log\mathscr{L})^{-1}(\Omega)$, then $J \in \mathscr{L}^1_{\text{loc}}(\Omega)$.
   In [BFS], H.Brézis, N.Fusco and C.Sbordone interpolated between 1) and 2) by proving that

3. if $|Df| \in \mathscr{L}^n(\log\mathscr{L})^{-\alpha}(\Omega)$, $0 \leqslant \alpha \leqslant 1$, then $J \in \mathscr{L}(\log\mathscr{L})^{1-\alpha}(K)$, for any $K$ compact subset of $\Omega$.

The spaces $\mathscr{L}^n(\log\mathscr{L})^{-\alpha}(\Omega)$, $\mathscr{L}(\log\mathscr{L})^{1-\alpha}(\Omega)$ for $0 \leqslant \alpha < 1$ are Orlicz spaces generated respectively by the Young functions

$$\chi(t) = t^n \log^{-\alpha}(e+t)$$

$$\Theta(t) = t\log^{1-\alpha}(e+t)$$

It is possible to generalize these results as follows (see for more details ([MO])).
If $|Df|$ belongs to the Orlicz space $\mathscr{L}^\chi(\Omega)$, where $\chi(t)$ is a Young function such that

$$at^n\log^{-1}(e+t) \leqslant \chi(t) \quad \forall t \geqslant t_0$$

with $a > 0$, $t_0 > 0$, and if $\Theta(t)$ is defined by

$$\frac{\Theta(t)}{t} \sim \int_1^t \frac{\chi'(s^{1/n})}{s^{2-1/n}}\, ds$$

where $\sim$ denotes the usual equivalence notation between convex real functions,then $J \in \mathscr{L}^{\Theta}(K)$ for any compact subset of $\Omega$.

An important tool to prove these results, is the following estimate, proved by T. Iwaniec and C. Sbordone in [IS]

$$\int_K J(x,f)\, dx \ \leqslant\ c(n,K) \int_\Omega \frac{|Df|^n}{\log(e + \frac{|Df(x)|}{|Df(x)|_\Omega})}\, dx$$

when $K$ is any compact subset of $\Omega$ and $|Df|_\Omega$ denotes the integral mean of $|Df|$ over $\Omega$. It can be viewed as dual to Müller's result.
At this point it became clear that the improved integrability property of the Jacobian could be observed in Orlicz-Sobolev spaces near $\mathscr{W}^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$. To illustrate, we mention the inequality by H.Brezis, N.Fusco and C.Sbordone [BFS]

$$\int_K J(x,f) \log^{1-\alpha}\left(e + \frac{|J(x,f)|}{|J|_K}\right)\ dx \leqslant c(n,K) \int_\Omega \frac{|Df(x)|^n}{\log^\alpha(e + \frac{|Df(x)|}{|Df(x)|_\Omega})}\ dx$$

for all $\alpha \in [0,1]$, which was extended to all $\alpha \in \mathbb{R}$ by L. Greco [G].
As a matter of fact, under the above hypothesis of Iwaniec and Sbordone, the Jacobian is even slightly higher integrable.
Indeed, G. Moscariello, in [MO], proved that $J(x,f)$ lies in $\mathscr{L} \log\log \mathscr{L}_{loc}(\Omega)$.
Subsequently L.Greco, T.Iwaniec and G. Moscariello proved that il $|Df(x)|^n \in \mathscr{L}^{\mathcal{P}}$ with $\mathcal{P}$ a log-convex function and if $J \geqslant 0$ then $J \in \mathscr{L}^{\Upsilon}$ where $\Upsilon(t) = \mathcal{P}(t) + t\int_0^t \frac{\mathcal{P}(s)}{s^2}\, ds$.
However these results are only a part of a more general spectrum about estimates of Jacobians.
In this spirit following the results above, we can realize is quite natural to study the Jacobians in more special spaces. As suggested in [BFS], it is interesting to study the regularity of the Jacobian when $Df$ belongs to Lorentz spaces . It appears that to get positive results one cannot rely only on these spaces but it is forced to encode the theory in the Lorentz-Zygmund spaces.

In [BFS] it is proved that if the Jacobian $J$ of a mapping $f = (f^1, ..., f^n) \in \mathscr{W}^{1,1}(\Omega, \mathbb{R}^n)$ is nonnegative, and if $|Df|^n \in \mathscr{L}^{1,\frac{q}{n}}$ with $1 < \frac{q}{n} < \infty$, then

we have $M_Q J \in \mathscr{L}^{1,\frac{q}{n}}_{loc}(Q)$ where $M_Q$ is the maximal operator defined in Chapter 2.
Many properties of the Jacobian may be emphasized writing $J$ as divergence of a vectorial field. Let us consider $f \in \mathscr{C}^2(\Omega, \mathbb{R}^n)$; developing $\det Df$ according to the last row, we have

$$\det Df \; = \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} \, \mathrm{adj}_{n_j} Df$$

where we denote by $\mathrm{adj}_{ij}$ the algebraic complement of the element $(i, j)$ in the gradient matrix $Df$. With this notation it is possible to write

$$\det Df \; = \; Df_n \cdot B.$$

With a direct calculation it is easy to prove that $\mathrm{div}\, B \; = \; 0$. These techniques will be examined carefully in Section (3.3).

## 3.2 Regularity result

Let us state an important lemma in the framework of Lorentz-Zygmund spaces.

**Lemma 3.2.1.** *If $g \in \mathscr{L}^{p,q}(\log \mathscr{L})^{\alpha}$, $1 \leqslant p < q$ and $\alpha > -\frac{1}{q}$ then*

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{p}} \fint_{\Omega} |g|^{p-\varepsilon}(x) dx = 0. \tag{3.1}$$

*Proof.* By applying Theorem 1.3.2, we have that $\mathscr{L}^{p,q}(\log \mathscr{L})^{\alpha} \subseteq \mathscr{L}^{p}(\log \mathscr{L})^{-\frac{1}{p}}$ then the lemma follows from the result in [BFS] which implies that for $g \in \mathscr{L}^{p}(\log \mathscr{L})^{-s}(\Omega)$, $0 < s \leqslant 1$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^{s} \fint_{\Omega} |g|^{p-\varepsilon}(x) dx = 0.$$

(see for more details [G1]) □

**Theorem 3.2.2.** *If $|Df|^n \in \mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s}(\Omega)$, $J \geqslant 0$, and $0 \leqslant s < \frac{n}{q} \leqslant 1$ then $J \in \mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s+\frac{n}{q}}(K)$ for any $K$ compact subset of $\Omega$.*

*Proof.* For $f \in \mathscr{W}^{1,n-\varepsilon}(\Omega,\mathbb{R}^n)$, $-\infty < \varepsilon < 1$ and $Q \subset Q_0/2$, $Q_0$ a cube contained in $\Omega$ we have

$$\fint_Q |Df_1|^{-\varepsilon} J(x,f) \leqslant c(n) \mid \varepsilon \mid \fint_{2Q} |Df|^{n-\varepsilon} + c(n) \left[ \fint_{2Q} |Df|^{\frac{n(n-\varepsilon)}{(n+1)}} \right]^{\frac{n+1}{n}} .$$

This inequality firstly proved in [IS], was then extended to more general cases in [GIOV] and [G2].
Since, by Theorem 1.3.2, $\mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s} \subset \mathscr{L}(\log \mathscr{L})^{-1}$ we can pass to the limit as $\varepsilon \to 0$ in the above inequality and use Lemma 3.2.1 to obtain, for any cube $Q \subset \frac{Q_0}{2}$

$$\fint_Q J \leqslant c(n) \left[ \fint_{2Q} |Df|^{\frac{n^2}{n+1}} \right]^{\frac{(n+1)}{n}}$$

and therefore for almost every $x \in Q_0/2$ we have

$$M_{Q_0/2} J(x) \leqslant c(n) \left[ M_{Q_0}(|Df|^{\frac{n^2}{n+1}})(x) \right]^{\frac{(n+1)}{n}} \tag{3.2}$$

Since $|Df|^n \in \mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s}$, then $|Df|^{n^2/(n+1)}$ belongs to the Lorentz-Zygmund space $\mathscr{L}^{(n+1)/n, q(n+1)/n^2}(\log \mathscr{L})^{-s\frac{n}{n+1}}$ and then by Lemma 2.2.4 $M_{Q_0}(|Df|^{n^2/(n+1)})$ belongs to the same space.
From this it follows that $\left[ M_{Q_0}(|Df|^{n^2/(n+1)}) \right]^{n+1/n}$ and also $MJ_{Q_0/2}$ belongs to $\mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s}(Q_0/2)$.
Finally from Proposition 2.2.8 we deduce that $J \in \mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{-s+\frac{n}{q}}(K)$ for any compact $K \subset \Omega$. □

**Remark**. Following the same ideas of [BFS] if we consider the function

$$f(x) = \frac{x}{|x|} |\log |x||^{-\frac{1}{q}} (\log |\log |x||)^{-1/n}$$

where $|x| < a < 1$, $n < q$ then it is easy to check that

$$|Df| \in \mathscr{L}^{n,q} \tag{3.3}$$

and $J \notin \mathscr{L}(\log \mathscr{L})^{\frac{n}{q}}$ as it was reasonable to expect.
Actually, by elementary calculation, we can prove that locally

$$J \in \mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{\frac{n}{q}}. \tag{3.4}$$

*Verification of* (3.3). It is easy to check that $|Df|$ is equivalent to

$$(1/|x|)|\log|x||^{-\frac{1}{q}}(\log|\log|x||)^{-\frac{1}{n}}$$

then the claim follows since

$$\int_0^a \left[\frac{1}{|\log r|^{\frac{1}{q}}(\log|\log r|)^{\frac{1}{n}}}\right]^q \frac{dr}{r} < \infty.$$

*Verification of* (3.4). We can remark that $J$ is equivalent to

$$(1/|x|^n)|\log|x||^{-1-n/q}(\log|\log|x||)^{-1}$$

then, since by our assumption $1-\frac{q}{n}<0$, the claim follows thanks to the fact that

$$\| J \|^{\frac{q}{n}}_{\mathscr{L}^{1,\frac{q}{n}}(\log \mathscr{L})^{\frac{n}{q}}} \simeq \int_0^a \left[\frac{1}{|\log r|\ \log|\log r|}\right]^{\frac{q}{n}} \frac{dr}{r} < \infty.$$

## 3.3 Div-curl fields

Let us assume that $\Omega$ is a cube in $\mathbb{R}^n$ and $\sigma\Omega$ a cube of the same center as $\Omega$ but $\sigma$ times smaller than $\Omega$, for $0<\sigma\leqslant 1$.
If $B:\Omega\to\mathbb{R}^n$, $E:\Omega\to\mathbb{R}^n$ are integrable vector fields on $\Omega$ such that

$$\text{div } B = \sum_{i=1}^n \frac{\partial B_i}{\partial x_i} = 0$$

$$\text{curl } E = \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i}\right)_{i,j=1\ldots n} = 0$$

in the sense of distribution, the scalar product $\langle E,B\rangle$ is referred to as a div-curl product.
Let us state this useful lemma

**Lemma 3.3.1.** *Let $\langle B,E\rangle$ be a nonnegative div-curl product such that $B\in\mathscr{L}^p\log^{-1}\mathscr{L}(\Omega,\mathbb{R}^n)$ and $E\in\mathscr{L}^q\log^{-1}\mathscr{L}(\Omega,\mathbb{R}^n)$ with $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then for any $0<\sigma<1$*

$$⨍_{\sigma\Omega}\langle B,E\rangle\ dx \ \leqslant\ c\left(⨍_\Omega |B|^r\right)^{\frac{1}{r}}\left(⨍_\Omega |E|^s\right)^{\frac{1}{s}}$$

*where $\frac{1}{r}+\frac{1}{s}=1+\frac{1}{n}$, $1\leqslant r\leqslant p$ , $1\leqslant s\leqslant q$ and $c=c(n,p,q)$.*

**Theorem 3.3.2.** *Let* $B \in \mathscr{L}^{p,a}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega, \mathbb{R}^n)$, $E \in \mathscr{L}^{q,b}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega, \mathbb{R}^n)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, $a > p$ *and* $b > q$ *if* $\frac{1}{\theta} = \frac{1}{a} + \frac{1}{b}$ *and* $0 \leqslant \frac{2}{\gamma} < \frac{1}{\theta} < 1$, *then* $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}+\frac{1}{\theta}}(\sigma\Omega, \mathbb{R}^n)$.

*Furthermore*

$$\| \langle B, E\rangle \|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-2\frac{1}{\gamma}+\frac{1}{\theta}}(\sigma\Omega)} \leqslant c\| \, |B| \, \|_{\mathscr{L}^{p,a}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)} \| \, |E| \, \|_{\mathscr{L}^{q,b}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)}$$

If $a > p$, $b > q$ and $\frac{1}{\gamma} < 1$, then

$$|B| \in \mathscr{L}^p(\log \mathscr{L})^{-1}(\Omega)$$

and

$$|E| \in \mathscr{L}^q(\log \mathscr{L})^{-1}(\Omega)$$

We are under the hypothesis of Lemma 3.3.1 and there is a constant $c = c(n, p, q)$ such that for any cube $\sigma\Omega \subset \Omega$

$$\fint_{\sigma\Omega} \langle B, E\rangle \; dx \;\leqslant\; c \left(\fint_\Omega |B|^r\right)^{\frac{1}{r}} \left(\fint_\Omega |E|^s\right)^{\frac{1}{s}} \tag{3.5}$$

where $r$,$s$ are any numbers such that $1 \leqslant r \leqslant p$, $1 \leqslant s \leqslant q$, $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}$. If we denote by M the local maximal function associated to the cube $\sigma\Omega \subset \Omega$ and by $\mathbf{M}$ the maximal function in $\Omega$ this inequality yields:

$$\mathrm{M}(\langle B, E\rangle) \;\leqslant\; c\,[\mathbf{M}|B|^r]^{\frac{1}{r}}[\mathbf{M}|E|^s]^{\frac{1}{s}}$$

pointwise in $\sigma\Omega$. Here the constant depends on $\sigma$. Applying Lemma 2.2.4 $[\mathbf{M}|B|^r]^{\frac{1}{r}} \in \mathscr{L}^{p,a}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)$ and $[\mathbf{M}|E|^s]^{\frac{1}{s}} \in \mathscr{L}^{q,b}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)$. Then applying Proposition 1.3.8 we have that $\mathrm{M}(\langle B, E\rangle) \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}}(\sigma\Omega)$. Then applying Proposition 2.2.8 with $r = \theta$ and $s = \frac{2}{\gamma}$ we have that $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}+\frac{1}{\theta}}(\sigma\Omega)$. Furthermore the inequality yields

$$\|\langle B, E\rangle\|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}+\frac{1}{\theta}}(\sigma\Omega)} \leqslant c\|\mathrm{M}\langle B, E\rangle\|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}}(\sigma\Omega)} \tag{3.6}$$

By (1.12) and Lemma 2.2.4

$$\begin{aligned}
\| \langle B, E\rangle \|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}+\frac{1}{\theta}}(\sigma\Omega)} &\leqslant c\|\mathrm{M}\, \langle B, E\rangle \|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}}(\sigma\Omega)} \\
&\leqslant c\| \, [\mathbf{M}|B|^r]^{\frac{1}{r}} \, [\mathbf{M}|E|^s]^{\frac{1}{s}} \, \|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{2}{\gamma}}(\Omega)} \\
&\leqslant c\| \, [\mathbf{M}|B|^r]^{\frac{1}{r}} \, \|_{\mathscr{L}^{p,a}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)} \| \, [\mathbf{M}|E|^s]^{\frac{1}{s}} \, \|_{\mathscr{L}^{q,b}(\log L)^{-\frac{1}{\gamma}}(\Omega)} \\
&\leqslant c\||B|\|_{\mathscr{L}^{p,a}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)} \||E|\|_{\mathscr{L}^{q,b}(\log \mathscr{L})^{-\frac{1}{\gamma}}(\Omega)}
\end{aligned}$$

## 3.4 Div-curl fields coupled by the distortion inequality

Let us introduce another definition which will be generalized in the following section for div-curl couple

**Definition 3.4.1.** *A mapping $f : \Omega \to \mathbb{R}^n$ is said to have finite distortion if*

1. $f \in \mathscr{W}^{1,1}_{loc}(\Omega, \mathbb{R}^n)$
2. *The Jacobian determinant of $f$ is locally integrable and does not change sign in $\Omega$*
3. *there is a measurable function $K_0 = K_0(x) \geqslant 1$ finite almost everywhere, such that $f$ satisfies the distortion inequality*

$$|Df(x)|^n \leqslant K_0(x)|J(x,f)| \text{ a.e. } \Omega \tag{3.7}$$

Assumptions 1. 2. and 3. are not enough to imply $f \in \mathscr{W}^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ unless of course $K_0$ is a bounded function.
We shall investigate the degree of integrability of a class of div-curl fields $\langle B, E\rangle$ which are coupled by the distortion inequality:

$$\frac{|B|^p}{p} + \frac{|E|^q}{q} \leqslant k(x)\,\langle B, E\rangle \quad \text{a.e.} \quad \text{in } \Omega \tag{3.8}$$

where $1 \leqslant k(x) < \infty$ is a measurable function in $\Omega$ and $1 < p, q < \infty$ are conjugate Hölder exponents, $p + q = pq$.
We shall assume

$$\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\Omega) \tag{3.9}$$

**Theorem 3.4.2.** *Let $\mathcal{F} = \langle B, E\rangle$ be a div-curl field verifying (3.8) and (3.9).*
*If $k(x) \in Exp_\gamma(\Omega)$ for some $\gamma > \theta$ then $B \in \mathscr{L}^p(\log \mathscr{L})^\lambda(\Omega, \mathbb{R}^n)$ and $E \in \mathscr{L}^q(\log \mathscr{L})^\lambda(\Omega, \mathbb{R}^n)$ for any $\lambda > 0$, locally.*
*Furthermore*

$$\|\, |B|^p + |E|^q \,\|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{\lambda - \frac{1}{\gamma}}(\sigma\Omega)} \leqslant c\|\langle B, E\rangle\|_{\mathscr{L}^{1,\theta}(\log \mathscr{L})^{\lambda - \frac{1}{\theta}}(\Omega)} \tag{3.10}$$

*Proof.* Since $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\Omega)$, by (3.8) and Lemma 1.3.3 we deduce that $|B| \in \mathscr{L}^{p,\theta p}(\log \mathscr{L})^{-\frac{1}{\gamma p}}(\Omega)$ and $|E| \in \mathscr{L}^{q,\theta q}(\log \mathscr{L})^{-\frac{1}{\gamma q}}(\Omega)$.

Since $\gamma > \theta$,certainly $|B| \in \mathscr{L}^p(\log \mathscr{L})^{-1}(\Omega)$ and $|E| \in \mathscr{L}^q(\log \mathscr{L})^{-1}(\Omega)$. Therefore, we may apply inequality (3.5) and following the same argument of Theorem 3.3.2 we deduce that $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{-\frac{1}{\gamma}+\frac{1}{\theta}}(\sigma\Omega)$.
Again from distortion inequality using the fact that $K(x) \in Exp_\gamma$ we obtain that $|B| \in \mathscr{L}^{p,\theta p}(\log \mathscr{L})^{\frac{1}{\theta p}-\frac{2}{\theta p}}(\sigma\Omega)$ and $|E| \in \mathscr{L}^{q,\theta q}(\log \mathscr{L})^{\frac{1}{\theta q}-\frac{2}{\theta q}}(\sigma\Omega)$.Let use mathematical induction to deduce that $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{m(\frac{1}{\theta}-\frac{1}{\gamma})}(\sigma\Omega)$ for any integer $m$ and then $\langle B, E\rangle \in \mathscr{L}^{1,\theta}(\log \mathscr{L})^{\lambda}(\sigma\Omega)$ for any $\lambda > 0$. The estimate (3.10) can be deduced by distortion inequality arguing as in Theorem 3.3.2. □

A motivation for this type of inequality arises from the study of the $p$-harmonic equation

$$\operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \tag{3.11}$$

In fact, setting $E = \nabla u$ and $B = |\nabla u|^{p-2}\nabla u$ we obtain

$$\frac{|E|^p}{p} + \frac{|B|^q}{q} = \langle B, E\rangle$$

# Chapter 4

# A Divergence free vector: the $\infty$-Laplacian

## 4.1 Introduction

This chapter is concerned with various linear and nonlinear PDEs whose prototype is the $p$-harmonic equation

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \qquad 1 < p < \infty \tag{4.1}$$

The focus is on the limiting case as $p$ approaches $\infty$, referred to as the $\infty$-Laplacian

$$\Delta_\infty u = 2\sum_{i\,j=1}^{n} \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Upon multiplication by a suitable function $\lambda = \lambda(\nabla u)$ it is possible to express this operator in divergence form. There may be several such functions $\lambda = \lambda(\nabla u)$, called $divergence factors$. Writing the $\infty$-Laplacian in divergence form allows to speak of weak $\infty$-harmonic functions in the Sobolev class $\mathscr{W}^{1,2}_{\text{loc}}(\Omega)$.

## 4.2 Divergence factors and integrating fields

In this chapter it is interesting to underline that dealing with nonlinear partial differential equations, it is often convenient to write them in divergence form. Consider, for example, the question of the domain of definition of a given nonlinear differential operator. Expressing this operator in a divergence form, makes one derivative dispensable in the definition of its domain.

Naturally, there may exist many divergence forms of an operator, leading to different domains of definition. A typical example is furnished by the Hessian determinant in two variables:

$$
\begin{aligned}
\det \mathcal{H}u &= \det \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \\
&= u_{xx}u_{yy} - u_{xy}u_{xy} && \text{for} \quad u \in \mathscr{W}^{2,2}_{\text{loc}}(\Omega) \\
&= (u_x u_{yy})_x - (u_x u_{xy})_y && \text{for} \quad u \in \mathscr{W}^{2,4/3}_{\text{loc}}(\Omega) \\
&= \tfrac{1}{2}(u\, u_{xx})_{yy} + \tfrac{1}{2}(u\, u_{yy})_{xx} - (u\, u_{xy})_{xy} && \text{for} \quad u \in \mathscr{W}^{2,1}_{\text{loc}}(\Omega) \\
&= (u_x u_y)_{xy} - \tfrac{1}{2}(u_x u_x)_{yy} - \tfrac{1}{2}(u_y u_y)_{xx} && \text{for} \quad u \in \mathscr{W}^{1,2}_{\text{loc}}(\Omega)
\end{aligned}
\tag{4.2}
$$

In another example, the reader may try to verify the following identity for the Gaussian curvature of a surface $z = u(x, y)$ in $\mathbb{R}^3$,

$$
\mathcal{K}u = \frac{u_{xx}u_{yy} - u_{xy}u_{xy}}{(1 + u_x^2 + u_y^2)^2} = \frac{\det \mathcal{H}u}{(1 + |\nabla u|^2)^2} \tag{4.3}
$$

First notice that $\mathcal{K}u$ is none other than the Jacobian determinant of the mapping

$$
(A, B) = \left( \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}, \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right), \tag{4.4}
$$

$$
\mathcal{K} = A_x B_y - A_y B_x. \tag{4.5}
$$

We can express $\mathcal{K}u$ in divergence form using two different formulas

$$
\mathcal{K} = (AB_y)_x - (AB_x)_y = (A_x B)_y - (A_y B)_x \tag{4.6}
$$

This leads us to two different divergence forms of the curvature

$$
\mathcal{K} = \operatorname{div} \mathbf{F}. \tag{4.7}
$$

The so-called integrating field $\mathbf{F} = (F^1, F^2)$ can be expressed as

$$
\begin{cases}
F^1 = \dfrac{u_x}{(1 + u_x^2 + u_y^2)^2} \left[ (1 + u_x^2) u_{yy} - u_x u_y u_{xy} \right] \\
F^2 = -\dfrac{u_x}{(1 + u_x^2 + u_y^2)^2} \left[ (1 + u_x^2) u_{xy} - u_x u_y u_{xx} \right]
\end{cases}
\tag{4.8}
$$

or

$$\begin{cases} F^1 = -\dfrac{u_y}{(1+u_x^2+u_y^2)^2}\left[(1+u_y^2)u_{xy} - u_x u_y u_{yy}\right] \\ F^2 = \dfrac{u_y}{(1+u_x^2+u_y^2)^2}\left[(1+u_y^2)u_{xx} - u_x u_y u_{xy}\right] \end{cases} \tag{4.9}$$

Adding up these two solutions we gain a symmetry with respect to $x$ and $y$. Namely,

$$2\mathcal{K} = \left(AB_y - A_yB\right)_x + \left(A_xB - AB_x\right)_y \tag{4.10}$$

$$2\mathbf{F} = \left(\frac{u_x u_{yy} - u_y u_{xy}}{1+u_x^2+u_y^2}\,,\ \frac{u_y u_{xx} - u_x u_{xy}}{1+u_x^2+u_y^2}\right) \tag{4.11}$$

One interesting outcome of this calculation is that the Gaussian curvature can be defined for surfaces parameterized by functions in $\mathscr{W}^{2,1}_{\mathrm{loc}}(\Omega)$. Such parametrizations have integrating factors $\mathbf{F} \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

## 4.3 The $p$-Laplacian

The $p$-harmonic equation

$$\operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \tag{4.12}$$

is the Euler-Lagrange equation of the variational integral

$$\mathcal{E}_p[u] = \frac{1}{p}\int_\Omega |\nabla u(x)|^p \mathrm{d}x \qquad 1 < p < \infty. \tag{4.13}$$

That is why the Sobolev space $\mathscr{W}^{1,p}(\Omega)$ is viewed as the natural domain of definition of this equation. However, this equation can also be expressed as a fully non-linear equation, in nondivergence form, by carrying out the differentiation

$$|\nabla u|^2 \Delta u + (p-2)\sum_{i=1}^n \sum_{j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = 0. \tag{4.14}$$

We are looking for a solution in $\mathscr{W}^{1,r}$ with $r < p$ (very weak solutions).

The border line exponents $p = 1$ and $p = \infty$ can also be considered. We set

$$\Delta_\infty u \overset{\text{def}}{=\!=} 2\sum_{i,j=1}^n \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\frac{\partial^2 u}{\partial x_i \partial x_j} \ = \ \frac{2}{|\nabla u|^2}\mathrm{Tr}\,(\nabla u \otimes \nabla u)\mathcal{H}u$$

and

$$\Delta_1 u \overset{\text{def}}{=\!=} 2\sum_{i,j=1}^{n}\left(\delta_i^j - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)\frac{\partial^2 u}{\partial x_i \partial x_j} \;=\; \frac{2}{|\nabla u|^2}\,\mathrm{Tr}\,(|\nabla u|^2 I - \nabla u \otimes \nabla u)\mathcal{H}u$$

The $p$-Laplacian is then a linear combination of $\Delta_1$ and $\Delta_\infty$,

$$\Delta_p = \tfrac{1}{p}\Delta_1 + \tfrac{p-1}{p}\Delta_\infty .$$

More explicitly we have

$$\begin{aligned}\Delta_p u &= \frac{2}{p}\sum_{i,j=1}^{n}\left(\delta_i^j + (p-2)\frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)\frac{\partial^2 u}{\partial x_i \partial x_j}\\ &= \frac{2}{p|\nabla u|^2}\,\mathrm{Tr}\,\big(|\nabla u|^2 I + (p-2)\nabla u \otimes \nabla u\big)\,\mathcal{H}u.\end{aligned}$$

Thus, the scalar function $\lambda = |\nabla u|^{p-2}$ is a divergence factor of $\Delta_p u$. Precisely we have

$$|\nabla u|^{p-2}\Delta_p u \;=\; \frac{2}{p}\,\mathrm{div}|\nabla u|^{p-2}\nabla u \tag{4.15}$$

The corresponding integrating field equals $\mathbf{F}(V) = |V|^{p-2}V$. Indeed,

$$\mathrm{D}\mathbf{F} = |V|^{p-2}\left(I + (p-2)\frac{V\otimes V}{|V|^2}\right) \tag{4.16}$$

## 4.4 The $p$-harmonic equation in the plane

The class of divergence factors is particularly rich in two dimensions due to the complex structure in $\mathbb{R}^2 \cong \mathbb{C} = \{z = x + \imath y,\ x, y \in \mathbb{R}\}$. Let $\Omega$ be an open subset of the complex plane. A function $u \in \mathscr{C}^2(\Omega)$ is:

- $\infty$-harmonic if

$$\frac{1}{2}\,|\nabla u|^2\Delta_\infty u = u_{xx}\,u_x^2 \;+\; 2u_{xy}\,u_x u_y \;+\; u_{yy}\,u_y^2 = 0, \tag{4.17}$$

- 1-harmonic if

$$\frac{1}{2}\,|\nabla u|^2\Delta_1 u = u_{xx}\,u_y^2 \;-\; 2u_{xy}\,u_x u_y \;+\; u_{yy}\,u_x^2 = 0, \tag{4.18}$$

  and

- $p$-harmonic if

$$\tfrac{1}{2}\,|\nabla u|^2\Delta_p u = \tfrac{1}{p}\,|\nabla u|^2\Delta u + \left(1-\tfrac{2}{p}\right)\,(u_{xx}\,u_x^2 + 2u_{xy}\,u_x u_y + u_{yy}\,u_y^2) = 0. \tag{4.19}$$

We shall make use of the Cauchy-Riemann derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right) \tag{4.20}$$

and the complex gradient of $u$, which is defined by

$$f(z) = u_z = \tfrac{1}{2}\,(u_x - \imath\, u_y)\,.$$

Since

$$\frac{\partial f}{\partial z} = \frac{1}{4}\,[u_{xx} - iu_{xy} - i(u_{xy} - iu_{yy})] = \frac{1}{4}(u_{xx} - 2iu_{xy} - u_{yy});$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{4}\,(u_{xx} + u_{yy})$$

and

$$\overline{\frac{\partial f}{\partial z}} = \frac{1}{4}\,[u_{xx} - u_{yy} + i(u_{xy} + u_{yy})]$$

The $p$-Laplacian of $u$ can be expressed in terms of $f$ as

$$\tfrac{1}{4}\Delta_p u \;= \frac{\partial f}{\partial \overline{z}} + \left(\frac{1}{2} - \frac{1}{p}\right)\left[\frac{\overline{f}}{f}\frac{\partial f}{\partial z} + \frac{f}{\overline{f}}\overline{\frac{\partial f}{\partial z}}\right] \tag{4.21}$$

This is an elliptic operator for all $1 < p < \infty$. However, the borderline cases lead to formally parabolic operators

$$\tfrac{1}{4}\Delta_1 u \;= \frac{\partial f}{\partial \overline{z}} - \frac{1}{2}\left[\frac{\overline{f}}{f}\frac{\partial f}{\partial z} + \frac{f}{\overline{f}}\overline{\frac{\partial f}{\partial z}}\right] \tag{4.22}$$

and

$$\tfrac{1}{4}\Delta_\infty u \;= \frac{\partial f}{\partial \overline{z}} + \frac{1}{2}\left[\frac{\overline{f}}{f}\frac{\partial f}{\partial z} + \frac{f}{\overline{f}}\overline{\frac{\partial f}{\partial z}}\right] \tag{4.23}$$

We can view the complex gradient of the $p$-harmonic function as a solution of the Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z, \qquad \mu(z) = \left(\frac{1}{p} - \frac{1}{2}\right)\left[\frac{\overline{f}}{f} + \frac{f}{\overline{f}}\frac{\overline{f_z}}{f_z}\right] \tag{4.24}$$

which is always elliptic if $1 < p < \infty$. For $p = 1$ and $p = +\infty$ we observe that the distortion function $K(z) = \dfrac{1 + |\mu(z)|}{1 - |\mu(z)|}$ is still finite at the points where

$$\frac{\overline{f}}{f}\frac{\partial f}{\partial z} \notin \mathbb{R}.$$

At these points the equation (4.24) remains elliptic.

Of particular interest to us will be the complex gradients of ∞-harmonic functions.
In fact if we consider the equation $\Delta_\infty u = 0$ we obtain

$$\frac{\partial f}{\partial \overline{z}} = -\frac{1}{2}\left( \frac{\overline{f}}{f} \cdot \frac{\partial f}{\partial z} \; + \; \frac{f}{\overline{f}} \cdot \overline{\frac{\partial f}{\partial z}} \right).$$

Recalling that $\Re e(z) = \frac{z+\overline{z}}{2}$ and $\Im m(z) = \frac{z-\overline{z}}{2}$. These are solutions of the quasilinear first order system

$$\frac{\partial f}{\partial \overline{z}} = -\Re e\left( \frac{\overline{f}}{f} \cdot \frac{\partial f}{\partial z} \right) = -\frac{1}{2}\left( \frac{\overline{f}}{f} \cdot \frac{\partial f}{\partial z} \; + \; \frac{f}{\overline{f}} \cdot \overline{\frac{\partial f}{\partial z}} \right). \tag{4.25}$$

The Jacobian determinant of $f$ is computed as:

$$\mathcal{J}(z, f) = |\, f_z \,|^2 - |\, f_{\overline{z}} \,|^2 = \left| \Im m \; \frac{\overline{f}}{f} \cdot \frac{\partial f}{\partial z} \right|^2. \tag{4.26}$$

Thus $\mathcal{J}(z, f)$ is positive at the points where (4.25) is elliptic.

## 4.5 Divergence factors for ∞-Laplacian

To define ∞-Laplacian in the weak sense for functions having only first order derivatives we need to express $\Delta_\infty$ in a divergence form. In particular, in the plane the ∞-laplacian equation becomes

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$$

multiplying by a suitable divergence factor $\lambda$ we would like to write the ∞-laplacian in a divergence form. Let us find all divergence factors $\lambda = \lambda(u_x, u_y)$ of the equation (4.17). That is, we are looking for solutions to

$$\lambda u_x^2 u_{xx} + 2\lambda u_x u_y u_{xy} + \lambda u_y^2 u_{yy} = \frac{\partial}{\partial x}\mathscr{A}(u_x, u_y) + \frac{\partial}{\partial y}\mathscr{B}(u_x, u_y). \tag{4.27}$$

This identity holds if and only if

$$\frac{\partial \mathscr{A}}{\partial u_x} = \lambda u_x^2, \qquad \frac{\partial \mathscr{B}}{\partial u_y} = \lambda u_y^2 \tag{4.28}$$

and

$$\frac{\partial \mathscr{A}}{\partial u_y} + \frac{\partial \mathscr{B}}{\partial u_x} = 2\lambda\, u_x\, u_y. \tag{4.29}$$

It will be advantageous to work with the complex function

$$\mathscr{F} = \mathscr{F}(w) \stackrel{\text{def}}{=\!=} \mathscr{A} + \imath \mathscr{B},$$

of the complex variable $w = u_x + \imath\, u_y$. In this notation the system takes the form

$$\frac{\partial \mathscr{F}}{\partial \overline{w}} = \frac{1}{2}\frac{w}{\overline{w}}\left(\frac{\partial \mathscr{F}}{\partial w} + \overline{\frac{\partial \mathscr{F}}{\partial w}}\right) = \frac{w}{\overline{w}}\,\Re e \frac{\partial \mathscr{F}}{\partial w} \tag{4.30}$$

and

$$\lambda = \frac{2\,\mathscr{F}_{\overline{w}}}{w^2} = \frac{2\,\Re e \mathscr{F}_w}{|w|^2}. \tag{4.31}$$

Observe that $\mathscr{F}$ is orientation preserving in the sense that $|\mathscr{F}_w|^2 - |\mathscr{F}_{\overline{w}}|^2 = |\mathscr{F}_w|^2 - |\Re e\, \mathscr{F}_w|^2 = |\Im m\, \mathscr{F}_w|^2 \geqslant 0$.

## 4.6 Basic examples

A family of basic solutions of (4.30)

$$\mathscr{F}_k(w) = \frac{(k-1)\gamma_k \left(\frac{w}{|w|}\right)^{k+1} + (k+1)\overline{\gamma_k}\left(\frac{\overline{w}}{|w|}\right)^{k-1}}{|w|^{k^2-1}} \tag{4.32}$$

for $k = 0, 1, 2, \ldots$. The corresponding divergence factors of (4.32) are

$$\begin{aligned} \lambda_k(w) &= \frac{2\,\Re e \mathscr{F}_k(w)}{|w|^2} = -\frac{(k-1)k(k+1)}{|w|^{k^2+2}}\left[\gamma_k \left(\frac{w}{|w|}\right)^k + \overline{\gamma_k}\left(\frac{\overline{w}}{|w|}\right)^k\right] \\ &= \frac{a_k \cos k\theta + b_k \sin k\theta}{r^{k^2+2}}, \qquad a_k,\, b_k \in \mathbb{R}. \end{aligned} \tag{4.33}$$

where $w = re^{\imath\theta}$. In particular, the real valued functions

$$\lambda(w) = \frac{\cos k\theta}{r^{k^2+2}} \quad \text{and} \quad \lambda(w) = \frac{\sin k\theta}{r^{k^2+2}} \tag{4.34}$$

are divergence factors of the operator (4.17). We may, therefore, introduce the complex divergence factors

$$\Lambda_k(w) = \frac{\cos k\theta + \imath \sin k\theta}{r^{k^2+2}} = \frac{e^{\imath k\theta}}{r^{k^2+2}} = \frac{r^k e^{\imath k\theta}}{r^{k^2+k+2}} = \frac{w^k}{|w|^{k^2+k+2}} \tag{4.35}$$

The case $k = 0$ gives $\mathscr{F}_0(w) = \gamma\, w$, where $\gamma \in \imath\mathbb{R}$. Hence the divergence factor is trivial, $\lambda = 0$, being the real part equal to 0. For $k = 1$ we obtain $\mathscr{F}_1 \equiv \gamma \in \mathbb{C}$, so again $\lambda = 0$.

The first nontrivial case occurs when $k = 2$

$$\mathscr{F}_2(w) = \frac{\gamma\, w^3 + 3\,\overline{\gamma}\,\overline{w}|w|^2}{|w|^6} \tag{4.36}$$

Next we look for one solution of particular interest to us by studying the limiting case of (4.32) in which $k$ is considered as real parameter approaching zero. Let $\gamma_k = 1$, so that $\mathscr{F}_0(w) = 0$. Then we have

$$\frac{\mathscr{F}_k(w)}{2k} = \frac{(k-1)e^{\imath(k+1)\theta} + (k+1)e^{-\imath(k-1)\theta}}{2kr^{k^2-1}}. \tag{4.37}$$

Therefore, we can compute the limit

$$\lim_{k \longrightarrow 0} \frac{\mathscr{F}_k(w)}{2k} = w(1 - \imath \mathrm{Arg}\, w). \tag{4.38}$$

Thus

$$\mathscr{F}(w) = w(1 - \imath \mathrm{Arg}\, w) \tag{4.39}$$

might be a solution to (4.30), in any simply connected subset of $\mathbb{C} - \{0\}$. Note that choosing a different branch of $\mathrm{Arg}\, w$ will not affect the equation (4.30) since $\imath\, w$ is also a solution. Direct computations reveal that indeed (4.39) is an integrating field:

$$\frac{\partial \mathrm{Arg}\, w}{\partial w} = \frac{-\imath}{2w} \tag{4.40}$$

and

$$\frac{\partial \mathrm{Arg}\, w}{\partial \overline{w}} = \frac{\imath}{2\overline{w}}. \tag{4.41}$$

Hence, we obtain

$$\frac{\partial \mathscr{F}}{\partial w} = \frac{1}{2} - \imath \mathrm{Arg}\, w. \tag{4.42}$$

We find that

$$2\frac{\partial \mathscr{F}}{\partial \overline{w}} = \frac{w}{\overline{w}} \quad \text{and} \quad \frac{\partial \mathscr{F}}{\partial w} + \overline{\frac{\partial \mathscr{F}}{\partial w}} = 1 \tag{4.43}$$

as desired. The corresponding divergence factor is

$$\lambda = \frac{2\mathscr{F}_{\overline{w}}}{w^2} = \frac{1}{|w|^2}. \tag{4.44}$$

**Proposition 4.6.1.** *The $\infty$-Laplacian has a divergence form in which the integrating field $\mathscr{F}(w) = w(1 - \imath \mathrm{Arg} w)$ is multivalued.*

In other words

$$\frac{u_x^2}{u_x^2+u_y^2}\, u_{xx} + 2\frac{u_x u_y}{u_x^2+u_y^2}\, u_{xy} + \frac{u_y^2}{u_x^2+u_y^2}\, u_{yy} = \frac{\partial}{\partial x}\mathscr{A}(u_x, u_y) + \frac{\partial}{\partial y}\mathscr{B}(u_x, u_y). \tag{4.45}$$

where

$$\mathscr{A}(u_x, u_y) = u_x + u_y \tan^{-1}\frac{u_y}{u_x} \tag{4.46}$$

$$\mathscr{B}(u_x, u_y) = u_y - u_x \tan^{-1}\frac{u_y}{u_x} \tag{4.47}$$

The Jacobian determinant of $\mathscr{F} = \mathscr{A} + \imath\mathscr{B}$ is positive as long as $\mathrm{Arg}\, w \neq 0$. Indeed, we note that $|\mathscr{F}_w|^2 - |\mathscr{F}_{\overline{w}}|^2 = |\frac{1}{2} - \imath \mathrm{Arg}\, w|^2 - |\frac{1}{2}|^2 = (\mathrm{Arg}\, w)^2 > 0$. Another example of the divergence form of the equation (4.17) is obtained by taking into consideration the solution

$$\mathscr{F}_2(w) = \frac{3}{w^2\overline{w}} + \frac{1}{\overline{w}^3}. \tag{4.48}$$

Hence, we can write, remembering that $w \cdot \overline{w} = |w|^2$

$$\begin{aligned} \mathscr{A} + \imath\mathscr{B} &= \frac{1}{(u_x - \imath u_y)^3} + \frac{3}{(u_x + \imath u_y)^2(u_x - \imath u_y)} \\ &= \frac{4u_x^3}{(u_x^2+u_y^2)^3} - \frac{4\,\imath u_y^3}{(u_x^2+u_y^2)^3} \end{aligned} . \tag{4.49}$$

Thus, we have

$$\mathscr{A}(u_x, u_y) = \frac{4\,u_x^3}{(u_x^2+u_y^2)^3}, \tag{4.50}$$

$$\mathscr{B}(u_x, u_y) = \frac{-4\,u_y^3}{(u_x^2+u_y^2)^3}, \tag{4.51}$$

and

$$\lambda = \frac{12(u_y^2 - u_x^2)}{(u_x^2+u_y^2)^4}. \tag{4.52}$$

## 4.7 The conjugate functions

To every integrating field there corresponds a conjugate function. Having written the ∞-Laplace equation in the divergence form

$$[\mathscr{A}(u_x,u_y)]_x + [\mathscr{B}(u_x,u_y)]_y = 0 \tag{4.53}$$

the conjugate function $v$ is defined by the rule

$$\begin{cases} \mathscr{A}(u_x,u_y) = & v_y \\ \mathscr{B}(u_x,u_y) = & -v_x \end{cases}. \tag{4.54}$$

Set

$$\Delta = \left(\frac{\partial \mathscr{A}}{\partial u_y} + \frac{\partial \mathscr{B}}{\partial u_x}\right)^2 - 4\frac{\partial \mathscr{A}}{\partial u_x}\frac{\partial \mathscr{B}}{\partial u_y}.$$

According to the general classification of the first order nonlinear PDEs this system is:

- elliptic at the points where $\Delta < 0$
- hyperbolic at the points where $\Delta > 0$
- parabolic at the points where $\Delta = 0$

For the two examples discussed above we obtain

$$\begin{cases} \dfrac{4u_x^3}{(u_x^2+u_y^2)^3} = & v_y \\ \dfrac{4u_y^3}{(u_x^2+u_y^2)^3} = & v_x \end{cases} \tag{4.55}$$

and

$$\begin{cases} u_x + u_y \tan^{-1}\dfrac{u_y}{u_x} = & v_y \\ u_y - u_x \tan^{-1}\dfrac{u_y}{u_x} = & -v_x \end{cases}. \tag{4.56}$$

In the first example, the system is well defined outside the zeros of $\nabla u$. Both systems (4.55) and (4.56) are parabolic at every point. However, a given pair $(u\,,\,v)$ can also be consider as the solution to an elliptic system. Let us analyze this point of view in a general setting

$$\begin{cases} \mathscr{A}(u_x, u_y) = & v_y \\ \\ \mathscr{B}(u_x, u_y) = & -v_x, \end{cases} \tag{4.57}$$

where we recall that $\mathscr{A} + \imath\mathscr{B} = \mathscr{F}$ and $\mathscr{F}_{\overline{w}} = \frac{1}{2}\frac{w}{\overline{w}}(\mathscr{F}_w + \overline{\mathscr{F}_w})$. In analogy to the Cauchy-Riemann equations we introduce the complex function

$$h(z) = u(z) + \imath\, v(z) \tag{4.58}$$

We want to express the system (4.57) as a nonlinear Beltrami type equation for $h$. Our computation is as follows

$$\mathscr{F}(u_x + \imath u_y) = \mathscr{A}(u_x, u_y) + \imath\mathscr{B}(u_x, u_y) = v_y - \imath v_x. \tag{4.59}$$

In terms of $h$ this reads as

$$\mathscr{F}(\overline{h_z} + h_{\overline{z}}) = \overline{h_z} - h_{\overline{z}} \tag{4.60}$$

$$\mathscr{F}(\overline{h_z} + h_{\overline{z}}) + \overline{h_z} + h_{\overline{z}} = 2\overline{h_z}. \tag{4.61}$$

Next we consider the function

$$\Psi(w) \stackrel{\text{def}}{=\!=} \mathscr{F}(w) + w = w(2 - \imath \mathrm{Arg} w) \tag{4.62}$$

that we need to invert. First compute its complex derivatives

$$\Psi_w = 1 + \mathscr{F}_w = \frac{3}{2} - \imath \mathrm{Arg}\, w \tag{4.63}$$

and

$$\Psi_{\overline{w}} = \mathscr{F}_{\overline{w}} = \frac{1}{2}\,\frac{w}{\overline{w}}. \tag{4.64}$$

Hence the Jacobian determinant of $\Psi$ is positive

$$|\Psi_w|^2 - |\Psi_{\overline{w}}|^2 = \frac{9}{4} + (\mathrm{Arg}\, w)^2 - \frac{1}{4} = 2 + (\mathrm{Arg}\, w)^2 \geqslant 2. \tag{4.65}$$

Therefore, the equation $\Psi$ can be locally inverted. We proceed as follows

$$\overline{h_z} + h_{\overline{z}} = \Psi^{-1}(2\overline{h_z}) \tag{4.66}$$

or, equivalently

$$h_{\overline{z}} = \Psi^{-1}(2\overline{h_z}) - \overline{h_z} \tag{4.67}$$

It takes a form of a nonlinear Beltrami equation

$$h_{\overline{z}} = \mathscr{H}(h_z) \tag{4.68}$$

## 4.8 Analysis of $\mathscr{W}^{1,2}$-solutions

We consider here ∞-harmonic functions in the Sobolev class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$. To make use of the integrating field $\mathscr{F}(w) = w(1 - \imath \operatorname{Arg} w)$ we must specify a brunch of the argument of $w = u_x + \imath u_y$. There are many ways to choose a measurable branch of $\operatorname{Arg} w \overset{\mathrm{def}}{=\!=} \operatorname{Arg}\nabla u$. The divergence equation at (4.53) has a meaning in the distributional sense only if both $\mathscr{A}(u_x, u_y)$ and $\mathscr{B}(u_x, u_y)$ are locally integrable. This will be easily assured by assuming that the branch of $\operatorname{Arg}\nabla u$ lies in $\mathscr{L}^2_{\mathrm{loc}}(\Omega)$.

**Definition 4.8.1.** *A function $u \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$ for which we can choose an $\mathscr{L}^2$–branch of $\operatorname{Arg}\nabla u$, is called a weak solution to the ∞-Laplace equation if*

$$\int_\Omega [\eta_x \,\mathscr{A}(u_x, u_y) + \eta_y \,\mathscr{B}(u_x, u_y)] \,\mathrm{d}x\mathrm{d}y = 0 \tag{4.69}$$

*for every $\eta \in \mathscr{C}^\infty_0(\Omega)$.*

Since ∞-harmonic functions have continuous derivatives by Savin's theorem [S], every ∞-harmonic function is a weak solution in the sense of definition 4.8.1 in a neighborhood of points where the gradient does not vanish.

From now on we assume that $\Omega$ is a simply connected domain in $\mathbb{C}$. Thus the system (4.56) admits a unique (up to a constant) conjugate function $v \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega)$.

**Theorem 4.8.2.** *The mapping $h(z) = u + \imath v \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega)$ solves the elliptic Beltrami type equation*

$$h_{\overline{z}} = \mu(z)\overline{h_z}, \qquad \mu(z) = \frac{\imath\theta(z)}{2 - \imath\theta(z)} \tag{4.70}$$

*where $\theta(z) = \operatorname{Arg}\nabla u$. Moreover, the distortion function of $h$ is locally integrable*

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = \frac{1}{4}\left(|\theta| + \sqrt{4 + \theta^2}\right)^2 \leqslant (1 + |\theta|)^2 \in \mathscr{L}^1_{\mathrm{loc}}(\Omega) \tag{4.71}$$

The Jacobian determinant of $h$ actually does not depend on the choice of $\operatorname{Arg}\nabla u$. Indeed, the first order system takes the form

$$\begin{cases} u_x + \theta u_y = & v_y \\ \\ u_y - \theta u_x = & -v_x, \end{cases} \tag{4.72}$$

or equivalently

$$\operatorname{div}\begin{bmatrix} 1 & \mathrm{Arg}\nabla u \\ -\mathrm{Arg}\nabla u & 1 \end{bmatrix}\nabla u = 0. \tag{4.73}$$

Hence

$$J(z,h) = u_x v_y - u_y v_x = u_x^2 + \theta u_x u_y + u_y^2 - \theta u_x u_y = |\nabla u|^2 \in \mathscr{L}^1_{\mathrm{loc}}(\Omega). \tag{4.74}$$

Next, let us assume that $\mathrm{Arg}\nabla u \in \mathscr{L}^\infty(\Omega)$, say $|\theta| < M$. For example, this is the case if $u_y \geqslant 0$ a.e. in $\Omega$. In this case the distortion function is bounded and $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$.

**Corollary 4.8.3.** *If* $\mathrm{Arg}\nabla u \in \mathscr{L}^\infty(\Omega)$ *a.e. in* $\Omega$ *then* $h$ *is a* $K$*-quasiregular mapping, with* $K = (1 + \|\mathrm{Arg}\,\nabla u\|_\infty)^2$*. In particular,* $\nabla u$ *may vanish only on a set of measure zero.*

In fact, by Astala's area distortion theorem [As] we see that $h \in \mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega)$ with every $p < \frac{2K}{K-1}$. Also, $h$ is Hölder continuous of exponent $\alpha = \frac{1}{K}$. Its Jacobian is positive a.e. and hence $\nabla u$ may vanish only on a set of zero measure.

Whether $\nabla u$ may vanish is not clear. For example, Aronsson [Ar] proved that non-constant $\infty$-harmonic functions of class $\mathscr{C}^2(\Omega)$ have nonvanishing gradient. We believe that $\nabla u \neq 0$ if $u \in \mathscr{C}^{1,\alpha}(\Omega)$, with $\alpha > 1/3$.

**Corollary 4.8.4.** *Suppose that* $\mathrm{Arg}\nabla u \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$*, then* $u$ *has locally integrable second derivatives; that is* $u \in \mathscr{W}^{2,1}_{\mathrm{loc}}(\Omega)$*.*

*Proof.* It suffices to observe that the Laplacian of $u$ lies in the Hardy space $\mathscr{H}^1_{\mathrm{loc}}(\Omega)$. Indeed,

$$u_{xx} + u_{yy} = u_x\theta_y - u_y\theta_x = \det\begin{bmatrix} u_x & u_y \\ \theta_x & \theta_y \end{bmatrix} \in \mathscr{H}^1_{\mathrm{loc}}(\Omega). \tag{4.75}$$

□

The Laplace equation with the Jacobian determinant in the right hand side has been investigated by Wente in 1969 [W]. His work originated intensive study of the Jacobian determinants in Hardy spaces [CLMS], [IV].

Finally we note that if $\theta \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$ a theorem of Hempel, Morris and Trudinger [HMT] implies that there exists $\lambda > 0$ so that $\int_\Omega \exp \lambda\theta^2 < \infty$. Then $h = u + \imath v$ becomes a mapping of exponentially integrable distortion

$$|\mathrm{D}h(z)|^2 \leqslant K(z)\, J(z,h), \qquad K \in \mathrm{Exp}(\Omega). \tag{4.76}$$

Some properties of such mapping have been investigated in chapter (3.1) in the case $K \in Exp_\gamma(\Omega)$, $\gamma > 1$. For other properties of such mappings see [IM], [MM], [IKMS], [IKO].

# Chapter 5

# Div-curl couple of arbitrary sign

This chapter is devoted to the illustration of intrinsic links between the theory of compensated compactness and classical tools of Harmonic and Real Analysis. Before providing a more detailed background we will establish some preliminary results and we will show some examples. Our main result is Theorem 5.1.5, which is a generalization of Theorem 5.1.2.

## 5.1 Some definitions and examples.

Let $u \in \mathscr{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ then its Jacobian $J(u) = \det(\nabla u)$ belongs to the multidimensional Hardy space. This space can be characterized as follows

$$\mathscr{H}^1(\mathbb{R}^n) = \left\{ f \in \mathscr{L}^1(\mathbb{R}^n) : \sup_{t>0} |h_t * f| \ \in \mathscr{L}^1(\mathbb{R}^n) \right\}$$

where $h_t = \frac{1}{t^n} h(./t)$, $h \in C_0^\infty(\mathbb{R}^n)$, $h \geqslant 0$, Supp $h \in B(0,1)$. Let us recall that

$$(h_t * f) = \int_{B(x,t)} h_t(x-y) f(y)\, dy = \int_{B(x,t)} \frac{1}{t^n}\, h\left(\frac{x-y}{t}\right)\ f(y)\, dy$$

One of the most prominent result in harmonic analysis is that by Fefferman and Stein [FS]. They proved that $BMO(\mathbb{R}^n)$ is the dual of the Hardy space. The $BMO$-$\mathscr{H}^1$ pairing denoted by

$$\langle\, f\,,\, g \,\rangle = \int_{\mathbb{R}^n}^{*} f(y)\; g(y)\; dy$$

For $f \in BMO(\mathbb{R}^n)$ and $g \in \mathscr{H}^1(\mathbb{R}^n)$. The latter symbol coincides with converging integral if the product $f \cdot g$ happens to be integrable.

**Lemma 5.1.1.** *If $f \in \mathscr{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$ then* *

$$\left| \int^*_{\mathbb{R}^n} f(y)\, g(y)\, dy \right| \preccurlyeq \|f\|_{\mathscr{H}^1} \|g\|_{BMO} \tag{5.1}$$

Another important result of Sarason is that $\mathscr{H}^1(\mathbb{R}^n)$ is the dual space of $VMO(\mathbb{R}^n)$.

Let us come back to our first example. Let $u \in \mathscr{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n$, trivially $J(u)$ belongs to $\mathscr{L}^1(\mathbb{R}^n)$ for the Hadamard inequality ($|J(u)| \leqslant |D\,u|^n$) but the structure of $J(u)$ allows to find a proper subspace of $\mathscr{L}^1$, namely $\mathscr{H}^1$, which contains the range of the mapping $u \to J(u)$ from $\mathscr{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ into $\mathscr{L}^1(\mathbb{R}^n)$. Furthermore $\mathscr{H}^1$ is the minimal linear vector space containing this range.

The above example indicates an improvement of the $\mathscr{L}^1$ regularity. This improvement can be appreciated recalling Stein's lemma about the $\mathscr{L}^1_{\mathrm{loc}}$ nonnegative functions $f \in \mathscr{H}^1_{\mathrm{loc}}$

$$f \in \mathscr{H}^1_{\mathrm{loc}} \iff f \log f \in \mathscr{L}^1_{loc}.$$

Therefore it is covered the result of Müller : let $u \in \mathscr{W}^{1,n}(\mathbb{R}^n)$ assume $J(u) \geqslant 0$ then $J(u) \log J(u) \in \mathscr{L}^1_{loc}$.

This result inspired many succeeding works. It is worth to show three examples, just like in [CLMS], to succeed in proving Theorem 5.1.5.

The first example, as mentioned above, is the Jacobian under the hypothesis

$$u \in \mathscr{L}^q_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \text{ for all } q < \infty, \quad \nabla u \in \mathscr{L}^n(\mathbb{R}^n, \mathbb{R}^{n \times n}). \tag{5.2}$$

The second example deals with vectors $E$,$B$ on $\mathbb{R}^n$ satisfying

$$E \in \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n), \quad B \in \mathscr{L}^q(\mathbb{R}^n, \mathbb{R}^n), \text{ with } 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \tag{5.3}$$

* Hereafter we propose the following abbreviation $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ for inequalities of the form $|\boldsymbol{A}| \leqslant C \cdot \boldsymbol{B}$, where the constant $C \geqslant 0$ (called implied constant) depends on parameters insignificant to us, such as the dimension $n$ and so forth. One shall easily recognize those parameters from the context.The implied constant will vary from line to line.

$$\operatorname{div} E = 0, \quad \operatorname{curl} B = 0 \ \text{ in } \mathscr{D}'(\mathbb{R}^n) \tag{5.4}$$

Then, we form the scalar product $E \cdot B$ which clearly belongs to $\mathscr{L}^1(\mathbb{R}^n)$.

For the third example we consider a scalar function $u$ and a vector field $v$ on $\mathbb{R}^n$ for $n \geqslant 2$ satisfying

$$\begin{cases} \nabla u \in \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^n), \quad u \in \mathscr{L}^{2n/(n-2)} & \text{if } n \geqslant 3; \\ u \in \mathscr{L}^q_{\text{loc}}(\mathbb{R}^n) \quad \text{for all } q < \infty, & \text{if } n = 2. \end{cases} \tag{5.5}$$

$$\begin{cases} \nabla v \in \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n}), \ \operatorname{div} v = 0 & \\ v \in \mathscr{L}^{2n/(n-2)}(\mathbb{R}^n, \mathbb{R}^n), & \text{if } n \geqslant 3; \\ v \in \mathscr{L}^q_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n) \text{for all } q < \infty, & \text{if } n = 2. \end{cases} \tag{5.6}$$

then

$$\nabla u \cdot \left( \frac{\partial v}{\partial x_i} \right) \quad \text{for some fixed } i \in \{1, ..., n\} \in \mathscr{L}^1(\mathbb{R}^n).$$

In [CLMS], the main result is

**Theorem 5.1.2.** *1. Let $u$ satisfy (5.2) then $J(u) \in \mathscr{H}^1(\mathbb{R}^n)$.*

*2. Let $E$ and $B$ satisfy (5.3)-(5.4), then $E \cdot B \in \mathscr{H}^1(\mathbb{R}^n)$.*

*3. Let $u$ satisfy (5.5)-(5.6) then $\nabla u \cdot \frac{\partial v}{\partial x_i} \in \mathscr{H}^1(\mathbb{R}^n)$.*

**Remark**. The cases 1) and 3) are included in case 2). Indeed in the case 3) of Theorem 5.1.2 we observe that $\nabla u \in \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial v}{\partial x_i} \in \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ while $\operatorname{curl}(\nabla u) = 0$ and $\operatorname{div}\left(\frac{\partial v}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(\operatorname{div} v) = 0$ in $\mathcal{D}'$.
This means that case 3) is a reduction of case 2) with $E = \frac{\partial v}{\partial x_i}$ and $B = \nabla u$. The case 1) is a reduction of 2), since we may write

$$J(u) = \det(\nabla u) = \nabla u^1 \cdot \sigma$$

with

$$\operatorname{div} \sigma = 0 \ \text{ in } \mathscr{D}'(\mathbb{R}^n), \ \ |\sigma| \leqslant \Pi_{j=2}^n |\nabla u^j| \text{ a.e.}$$

This means that is possible to take $E = \sigma \in \mathscr{L}^{n/(n-1)}(\mathbb{R}^n, \mathbb{R}^n)$ and $B = \nabla u^1 \in \mathscr{L}^n(\mathbb{R}^n, \mathbb{R}^n)$.

Therefore we have only to prove the second assertion of Theorem 5.1.2. The proof follows from this lemma

**Lemma 5.1.3.** *Let E,B satisfy div E = 0 and curl B = 0. For all* $\alpha, \beta$ *satisfying*

$$\frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{n}, \quad 1\leqslant \alpha\leqslant p, \ \ 1<\beta\leqslant p'.$$

*There exists a constant C (depending only on h,* $\alpha$*,* $\beta$*) such that*

$$|\{h_t*(E\cdot B)\}(x)| \ \leqslant\ C \left(\fint_{B(x,t)}|E|^\alpha\right)^{\frac{1}{\alpha}}\left(\fint_{B(x,t)}|B|^\beta\right)^{\frac{1}{\beta}} \tag{5.7}$$

*for all* $x\in\mathbb{R}^n, t>0$*. Here we denote by* $B(x,t)$ *the open ball centered at* $x$ *of radius* $t$*.*

Admitting Lemma 5.1.3 we conclude with the proof of Theorem 5.1.2, since $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, one can find $\alpha$ and $\beta$ satisfying (5.7) and also $\alpha<p$, $\beta<p'$

$$\sup_{t>0}\left\{\left(\fint_{B(x,t)}|E|^\alpha\right)^{\frac{1}{\alpha}}\left(\fint_{B(x,t)}|B|^\beta\right)^{\frac{1}{\beta}}\right\}$$

$$\leqslant \left(\sup_{t>0}\fint_{B(x,t)}|E|^\alpha\right)^{\frac{1}{\alpha}}\left(\sup_{t>0}\fint_{B(x,t)}|B|^\beta\right)^{\frac{1}{\beta}}.$$

We deduce from the maximal theorem that $\sup_{t>0}|h_t*(E\cdot B)|\in\mathscr{L}^1(\mathbb{R}^n)$ and that

$$\sup_{t>0}|h_t*(E\cdot B)| \ \leqslant\ C\ M(|E|^\alpha)^{\frac{1}{\alpha}}\ M(|B|^\beta)^{\frac{1}{\beta}}$$

Here, we omit the details of proof of Theorem 5.1.3, since we will argue Theorem 5.1.5 in the same way.

This result was generalized by Dolcini. In [DO], it is proved a regularity result in the framework of Orlicz spaces.
Let us define the $N$-functions.
Let $a$ be a real valued function defined on $[0,\infty)$ and having the following properties:

1. $a(0)=0$, $a(t)>0$ if $t>0$, $\lim_{t\to\infty}a(t)=\infty$;
2. $a$ is nondecreasing, that is, $s>t\geqslant 0$ implies $a(s)\geqslant a(t)$;

3. $a$ is right continuous, that is , if $t \geqslant 0$, then $\lim\limits_{s\to t^+} a(s) = a(t)$.

Then the real valued $A$ defined on $[0,\infty)$ by

$$A(t) = \int_0^t a(\tau)d\tau$$

is called an $N$-function. (see for more details [Ad])
An example of $N$-functions are

$$A(t) = t^p, \quad 1 < p < \infty,$$

$$A(t) = e^t - t - 1,$$

$$A(t) = (1+t)\log(1+t) - t.$$

Given a function $a$ we define $\tilde{a}$ as follows

$$\tilde{a}(s) \ = \ \sup_{a(t)\leqslant t} s.$$

The $N$-functions $A$ and $\tilde{A}$ given by

$$A(t) = \int_0^t a(\tau)\,d\tau, \quad \tilde{A} = \int_0^s \tilde{a}(\sigma)\,d\sigma$$

are said to be conjugate. Examples of such conjugate pairs are:

$$A(t) = \frac{t^p}{p}, \quad \tilde{A}(s) = \frac{s^q}{q}, \ 1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1;$$

$$A(t) = e^t - t - 1, \quad \tilde{A}(s) = (1+s)\log(1+s) - s.$$

In the framework of Orlicz spaces we have the following theorem

**Theorem 5.1.4.** *Let $A(t)$ be an $N$-function, and $\tilde{A}(t)$ its conjugate. Suppose that there exists $1 < p \leqslant q < \infty$ for which $\frac{A(t)}{t^p}$ is increasing, $\frac{A(t)}{t^q}$ is decreasing, with $q < p^*$.*†
*If $B \in \mathscr{L}^A(\mathbb{R}^n,\mathbb{R}^n)$, $E \in \mathscr{L}^{\tilde{A}}(\mathbb{R}^n,\mathbb{R}^n)$, $div B = 0$, $curl E = 0$ then $E \cdot B \in \mathscr{H}^1(\mathbb{R}^n)$. Moreover*

$$\|E\cdot B\|_{\mathscr{H}^1} \ \leqslant \ c\,\|E\|_{\mathscr{L}^{\tilde{A}}}\,\|B\|_{\mathscr{L}^A}. \tag{5.8}$$

† Hereafter $p^* = np/(n-p)$ if $p < n$, any exponent bigger than $p$ if $p \geqslant n$.

The space to which this result implies include as particular case the Zygmund space $\mathscr{L}^p \log^\alpha \mathscr{L}$, with $1 < p < \infty$, $\alpha \in \mathbb{R}$.

Our aim is to encode this theory in the Lorentz spaces. The following theorem holds

**Theorem 5.1.5.** *Let us assume* $1 < p, q \leqslant \infty$. *If* $B \in \mathscr{L}^{p,q}(\mathbb{R}^n)$ *and* $E \in \mathscr{L}^{p',q'}(\mathbb{R}^n)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $div B = 0$, $curl E = 0$ *then* $E \cdot B \in \mathscr{H}^1(\mathbb{R}^n)$.
*Furthermore, the following estimate holds*

$$\|E \cdot B\|_{\mathscr{H}^1(\mathbb{R}^n)} \leqslant c \parallel E \parallel_{\mathscr{L}^{p',q'}(\mathbb{R}^n)} \| \, B \parallel_{\mathscr{L}^{p,q}(\mathbb{R}^n)}$$

*Proof.* Let us take $B \in \mathscr{L}^{p,q}(\mathbb{R}^n)$ with $\operatorname{div} B = 0$ and $E \in \mathscr{L}^{p',q'}(\mathbb{R}^n)$ with $\operatorname{curl} E = 0$.
Assume first of all, $q > p$

$$B \in \mathscr{L}^{p-\varepsilon,r}_{\text{loc}}$$

$$r > 0, \quad 0 < \varepsilon < p - 1$$

since $q > p$, it follows that $q' < p'$ for the duality between respectively the exponents $p$, $p'$ and $q$,$q'$. By Theorem 1.3.2

$$E \in \mathscr{L}^{p'}_{\text{loc}}$$

We can find $\varepsilon > 0$ such that

$$(p - \varepsilon)' < (p')^*$$

simply assuming that $p_* = ([p']^*)' < p - \varepsilon$.
Then there exists $\pi \in \mathscr{L}^{(p')^*,q'}_{\text{loc}} : \nabla\pi = E$.
The assumption that $\operatorname{div} B = 0$ implies that $\operatorname{div}(\pi B) = E \cdot B$ in the distributional sense, then if $h$ is a function in $\mathscr{C}^\infty_0(\mathbb{R}^n)$, $h \geqslant 0$ and $\operatorname{Supp} h \subset B(0,1)$

$$\int_\Omega \pi B \, \nabla\varphi \, = \, - \int_\Omega E \cdot B \, \varphi$$

For every $\varphi \in C^\infty_0(\Omega)$.
If $h$ is the function defined above, for any $x \in \mathbb{R}^n$ and for all $t > 0$, we have:

$$h_t * (E \cdot B)\,(x) \, = \int_{B(x,t)} h_t(x - y)\,[(E \cdot B)(y)]\; dy$$

$$= \int_{B(x,t)} \frac{1}{t^n} h\left(\frac{x-y}{t}\right) ((E \cdot B)(y))\, dy$$

$$= \frac{1}{t^n} \int_{B(x,t)} h\left(\frac{x-y}{t}\right) \operatorname{div}[B(y)\pi(y)]\, dy$$

$$= \frac{1}{t^n} \int_{B(x,t)} \nabla h\left(\frac{x-y}{t}\right) \frac{B(y)\pi(y)}{t}\, dy$$

$$= \frac{1}{t^n} \int_{B(x,t)} \nabla h\left(\frac{x-y}{t}\right) B(y) \left\{\frac{\pi(y)-(\pi)_{x,t}}{t}\right\} dy,$$

since $\operatorname{supp}(h) \subset B(0,1)$ the equality becomes:

$$| h_t{*}(E{\cdot}B)(x) | \leqslant c(n,h) \left(\fint_{B(x,t)} | B |^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\fint_{B(x,t)} \left|\frac{\pi(y)-(\pi)_{x,t}}{t}\right|^{(p-\varepsilon)'}\right)^{\frac{1}{(p-\varepsilon)'}}$$

$$\leqslant c(n,h) \left(\fint_{B(x,t)} | B |^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\fint_{B(x,t)} | \nabla\pi |^{\alpha}\right)^{\frac{1}{\alpha}}$$

with $\alpha = ((p-\varepsilon)')_*$.

Considering the supremum in both sides of the above inequality

$$\sup | h_t{*}(E{\cdot}B)(x) | \leqslant c(n,h) \sup \left\{ \left(\fint_{B(x,t)} |B|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\fint_{B(x,t)} | \nabla\pi |^{\alpha}\right)^{\frac{1}{\alpha}} \right\}$$

$$\leqslant c(n,h) \left(\sup \fint_{B(x,t)} |B|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\sup \fint_{B(x,t)} | E |^{\alpha}\right)^{\frac{1}{\alpha}}$$

$$= c(n,h)\, [M_{p-\varepsilon}(B)]^{\frac{1}{p-\varepsilon}}\, [M_\alpha(E)]^{\frac{1}{\alpha}}$$

Applying Holder's inequality and noting that $(M_{p-\varepsilon}(B))^{\frac{1}{p-\varepsilon}} \in \mathscr{L}^{p,q}_{\text{loc}}$ and $(M_\alpha(E))^{\frac{1}{\alpha}} \in \mathscr{L}^{p',q'}_{\text{loc}}$ we have by Lemma 2.2.4

$$\| \sup |h_t * (E \cdot B)(x)| \|_{\mathscr{L}^1} \leqslant c \| \,(M_{p-\varepsilon}(B))^{\frac{1}{p-\varepsilon}}\, \|_{\mathscr{L}^{p,q}}\, \| \,(M_\alpha(E))^{\frac{1}{\alpha}}\, \|_{\mathscr{L}^{p',q'}} \leqslant$$

$$\leqslant C \|B\|_{\mathscr{L}^{p,q}} \|E\,\|_{\mathscr{L}^{p',q'}}$$

and this proves the result for $p < q$.
If $p = q$ we have the proof in Theorem 5.1.2 in [CLMS].

We have to consider a third case with $E \in \mathscr{L}^{p',q'}(\mathbb{R}^n)$ with $\operatorname{curl} E = 0$ and $B \in \mathscr{L}^{p,q}(\mathbb{R}^n)$ with $\operatorname{div} B = 0$ and $q < p$. Let us remark that

$$E \in \mathscr{L}^{p',q'}(\mathbb{R}^n) \text{ implies } \mathscr{L}^{(p')_*}_{\text{loc}}$$

$$B \in \mathscr{L}^{p,q}(\mathbb{R}^n) \text{ implies } \mathscr{L}^{p}_{\text{loc}}$$

For definition $(p')_* = \frac{np'}{n+p'} < p'$, $\forall p'$.
So there exists $\pi \in \mathscr{L}^{p'}_{\text{loc}}(\Omega)$:$\nabla \pi = E \subset \mathscr{L}^{(p')_*}_{\text{loc}}$. As in the first case

$$|h_t * (E \cdot B)(x)| \leqslant c(n,h) \left( \fint_{B(x,t)} |B|^p \right)^{\frac{1}{p}} \left( \fint_{B(x,t)} \left| \frac{\pi(y) - (\pi)_{x,t}}{t} \right|^{p'} \right)^{\frac{1}{p'}}$$

$$\leqslant c(n,h) \left( \fint_{B(x,t)} |B|^p \right)^{\frac{1}{p}} \left( \fint_{B(x,t)} |E|^{(p')_*} \right)^{\frac{1}{(p')_*}}$$

Considering the supremum in both sides of the above inequality we have

$$\sup |h_t * (E \cdot B)(x)| \leqslant c(n,h) \sup \left\{ \left( \fint_{B(x,t)} |B|^p \right)^{\frac{1}{p}} \left( \fint_{B(x,t)} |E\ |^{(p')_*} \right)^{\frac{1}{(p')_*}} \right\}$$

$$\leqslant c(n,h) \left\{ \sup \left( \fint_{B(x,t)} |B|^p \right)^{\frac{1}{p}} \sup \left( \fint_{B(x,t)} |E\ |^{(p')_*} \right)^{\frac{1}{(p')_*}} \right\}$$

$$= c(n,h) \left[ M_p(B) \right]^{\frac{1}{p}} \left[ M_{(p')_*}(E) \right]^{\frac{1}{(p')_*}}$$

Applying Holder's inequality and the fact that $[M_p(B)]^{\frac{1}{p}} \in \mathscr{L}^p_{\text{loc}}$, $[M(E)_{(p')_*}]^{\frac{1}{(p')_*}} \in \mathscr{L}^{(p')_*}_{\text{loc}}$ by Lemma 2.2.4 we have

$$\| \sup |h_t * (E \cdot B)(x)| \|_{\mathscr{L}^1} \leqslant c \| [M_p(B)]^{\frac{1}{p}} \|_{\mathscr{L}^{p,q}} \| [M_{(p')_*}(E)]^{\frac{1}{(p')_*}} \|_{\mathscr{L}^{p',q'}}$$

$$\leqslant C \|B\|_{\mathscr{L}^{p,q}} \|E\ \|_{\mathscr{L}^{p',q'}}$$

so Theorem 5.1.5 is proved. □

## 5.2 Another example

We want to show in this section a few examples from PDE's theory that this can be pushed further if more cancellations are present.
Let $u$, $v$ satisfy

$$\begin{cases} \nabla u \in \mathscr{L}^2(\mathbb{R}^n), \quad \nabla v \in \mathscr{L}^2(\mathbb{R}^n), \\ \operatorname{div} u = \operatorname{div} v = 0 \text{ in } \mathscr{D}'(\mathbb{R}^n) \end{cases} \tag{5.9}$$

We wish to consider the quantity

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j)$$

**Theorem 5.2.1.** *Assume (5.9), then*

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \in \mathscr{H}^1(\mathbb{R}^n)$$

Before proving Theorem 5.2.1, let us claim the following lemma

**Lemma 5.2.2.** *For all $f$ satisfying $\nabla f \in \mathscr{L}^2(\mathbb{R}^n)$ there exists a constant $C \geqslant 0$ such that*

$$\left\{ \int_{\mathbb{R}^n} \left[ \sup_{t>0} \fint_{B(x,t)} \left\{ \frac{1}{t} \left| f - \fint_{B(x,t)} f \right| \right\}^2 dy \right] dx \right\}^{\frac{1}{2}} \leqslant C \, \|\nabla f\|_{\mathscr{L}^2}$$

*Proof of Theorem 5.2.1.* We have to estimate

$$h_t * \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \right)(x)$$

$$= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} h_t(x-y) \frac{\partial^2}{\partial y_i \partial y_j} \left[ \left( u_i - \fint_{B(x,t)} u_i \right) \left( v_i - \fint_{B(x,t)} v_i \right) \right] dy$$

$$= \sum_{i,j=1}^{n} \int_{B(x,t)} \frac{1}{t^n} \left( \frac{\partial^2 h}{\partial x_i \partial x_j} \right) \left( \frac{x-y}{t} \right) \left\{ \frac{1}{t} \left( u_i - \fint_{B(x,t)} u_i \right) \right\} \left\{ \frac{1}{t} \left( v_j - \fint_{B(x,t)} v_j \right) \right\} dy.$$

Therefore

$$\left| h_t * \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \right) \right|$$

$$\leqslant C \sum_{i,j=1}^{n} \int_{B(x,t)} \left| \frac{1}{t} \left( u_i - \fint_{B(x,t)} u_i \right) \right| \left| \frac{1}{t} \left( v_j - \fint_{B(x,t)} v_j \right) \right| dy.$$

We deduce from Hölder inequality

$$\sup_{t>0} \left| h_t * \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \right) \right|$$

$$\leqslant C \left[ \sup_{t>0} \fint_{B(x,t)} \left| \frac{1}{t} \left( u - \fint_{B(x,t)} u \right) \right|^2 \right]^{\frac{1}{2}} \cdot \left[ \sup_{t>0} \fint_{B(x,t)} \left| \frac{1}{t} \left( v - \fint_{B(x,t)} v \right) \right|^2 \right]^{\frac{1}{2}}$$

Using Lemma 5.2.2 we deduce that

$$\left[ \sup_{t>0} \fint_{B(x,t)} \left| \frac{1}{t} \left( u - \fint_{B(x,t)} u \right) \right|^2 \right]^{\frac{1}{2}} \quad ; \quad \left[ \sup_{t>0} \fint_{B(x,t)} \left| \frac{1}{t} \left( v - \fint_{B(x,t)} v \right) \right|^2 \right]^{\frac{1}{2}}$$

belong both to $\mathscr{L}^2(\mathbb{R}^n)$.

Therefore, $\sup_{t>0} \left| h_t * \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \right) \right|$ belongs to $\mathscr{L}^1(\mathbb{R}^n)$.

□

Many other examples may be illustrated

**Example 5.1** Let us consider the Hessian of a mapping $u \in \mathscr{W}^{2,p}(\mathbb{R}^n)$, where $p > \frac{n^2}{n+2}$. For simplicity, let us consider $n = 2$. We observe that we have

$$\det(D^2 u) = \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} - \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2$$

$$= \frac{\partial^2}{\partial x_1 \partial x_2} \left( \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \right) - \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \left( \left( \frac{\partial u}{\partial x_2} \right)^2 \right) - \frac{1}{2} \frac{\partial^2}{\partial x_2^2} \left( \left( \frac{\partial u}{\partial x_1} \right)^2 \right)$$

This last expression make sense when $\nabla u \in \mathscr{L}^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which is the case $D^2 u \in \mathscr{L}^1_{\text{loc}}$.

**Theorem 5.2.3.** *Let $n = 2$, $u \in \mathscr{W}^{2,2}(\mathbb{R}^2)$. Then the expression*

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right) - \frac{1}{2}\frac{\partial^2}{\partial x_1^2}\left(\left(\frac{\partial u}{\partial x_2}\right)^2\right) - \frac{1}{2}\frac{\partial^2}{\partial x_2^2}\left(\left(\frac{\partial u}{\partial x_1}\right)^2\right)$$

*belongs to $\mathscr{H}^1(\mathbb{R}^2)$.*

**Example 5.2** Let $n = 2$ and $u \in \mathscr{W}^{2,2}(\mathbb{R}^n)$. The quantity $|\Delta u|^2 - \sum_{i,j=1}^n \frac{\partial 2}{\partial x_i \partial x_j}$ that we define to be

$$\sum_{i \neq j} \frac{\partial}{\partial x_i \partial x_j}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right) - \frac{1}{2}\frac{\partial}{\partial x_i^2}\left(\left(\frac{\partial u}{\partial x_j}\right)^2\right) - \frac{1}{2}\frac{\partial}{\partial x_j^2}\left(\left(\frac{\partial u}{\partial x_i}\right)^2\right)$$

belong to $\mathscr{H}^1(\mathbb{R}^2)$.
It is possible to combine the Examples 5.1 and 5.2 by considering all the minors of the Hessian matrix $D^2u$.

The following lemma is due to [CLMS]

**Lemma 5.2.4.** *If $u \in \mathscr{W}^{2,2}(\mathbb{R}^n)$, then all the minors of the Hessian matrix $\nabla^2 u$, which have order 2, belong to $\mathscr{H}^1(\mathbb{R}^n)$.*

# Chapter 6

# Nondivergence elliptic equations with BMO coefficients

## 6.1 Introduction

In dimension $n > 2$ little is known about nondivergence elliptic operators with measurable coefficients. The uniqueness problem has been studied in a number of papers mainly, by Aleksandrov, Bakel'man and Pucci (see [A],[B],[P] and the references therein).
For the existence, the uniform ellipticity condition is generally not enough, additional conditions being needed.
In 1956 a "cone "condition was introduced by Cordes. It deals with the scattering of the eigenvalues of the coefficient matrix $A(x) = [a_{ij}(x)]$, see also Talenti [T]. In the two-dimensional case, the "cone "condition is a consequence of the ellipticity condition, so it is redundant.
Under the cone condition Campanato [C] established higher integrability properties of the second derivatives of the solutions, in the same spirit of the results of Meyers' [ME] for divergence elliptic equations.
In 1963 Miranda [M] proved that, if the coefficients lie in $\mathscr{W}^{1,n}$, then the Dirichlet problem

$$\begin{cases} Lu = h \\ u \in \mathscr{W}^{2,2}(\Omega) \cap \mathscr{W}_0^{1,2}(\Omega) \end{cases}$$

is well posed. Here $\Omega$ is bounded open set in $\mathbb{R}^n$ and $h \in \mathscr{L}^2(\Omega)$. This result is optimal in the category of $\mathscr{L}^p$-spaces. Indeed, for $a_{ij} \in \mathscr{W}^{1,n-\varepsilon}$, $\varepsilon > 0$, the uniqueness fails.

An improvement of Miranda's result was given by Alvino and Trombetti [AT]. They assume that $\frac{\partial a_{ij}}{\partial x_s}$ lay in the Marcinkiewicz space $\mathscr{L}^n_{\text{weak}}$ and the constants in the weak type inequality for $\frac{\partial a_{ij}}{\partial x_s}$ are sufficiently small.
New ingredients to this theory came in 1981 from the celebrated results by Krylov-Safonov [KS] concerning Hölder continuity.
In [CFL] Chiarenza-Frasca-Longo originated a study of the equations with $VMO$-coefficients. They showed that if $f$ belongs to $\mathscr{L}^p_{\text{loc}}(\Omega)$,$1 < p < \infty$, then $u$ belongs to $\mathscr{W}^{2,p}_{\text{loc}}(\Omega)$.
Very recently D'Onofrio and Greco [DG] have studied the degree of regularity of solutions of an elliptic equation of nondivergence form, which does not fall under any of the preceding assumptions.
The aim of this chapter is to develop this theory for elliptic equations with coefficients having sufficiently small $BMO$-norm. To formulate the results we must first set up some notation and terminology (see 6.3). The last part of this chapter is divided into two sections. The first one deals with elliptic equations with bounded coefficients and the main result is a higher integrability of $|\nabla^2 u|$. The second one deals with unbounded coefficients and we obtain the following $\mathscr{L}^2 \log \mathscr{L}$ estimate

$$\|\nabla^2 u\|_{\mathscr{L}^2 \log \mathscr{L}(\mathbb{R}^n)} \preccurlyeq \|h\|_{\mathscr{L}^2 \log \mathscr{L}(\mathbb{R}^n)} \qquad {}^*$$

We notice (see Proposition 6.5.3) that our assumption the $BMO$-norm of the coefficients $a_{ij}$ to be sufficiently small is weaker than the smallness condition for the $\mathscr{L}^n_{\text{weak}}$ norm of their derivatives $\frac{\partial a_{ij}}{\partial x_s}$ which allows the authors in [AT] to obtain their existence and uniqueness theorem in $\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0$ of the solution to the Dirichlet problem

$$Lu = h \in \mathscr{L}^2$$

## 6.2 Hodge decomposition

Given a vector field $F = (f^1, f^2, ..., f^n) \in \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ one can solve uniquely the Poisson equation

$$F = \Delta u = (\Delta u^1, \Delta u^2, ..., \Delta u^n),$$

for $U = (u^1, u^2, ..., u^n) \in \mathscr{D}'(\mathbb{R}^n, \mathbb{R}^n)$. This yields the div-curl decomposition of $F$, also known as Hodge decomposition:

$$F = E + B, \tag{6.1}$$

$^*$ We have already introduced this symbol $\preccurlyeq$ in Lemma 5.1.1

where

$$B \;=\; \Delta U - \nabla\,\mathrm{div}\, U \quad\text{and}\quad E \;=\; \nabla\,\mathrm{div}\, U$$

where $B$ and $E$ are easily seen to be divergence and curl free, respectively. More explicitly, with the aid of the Riesz transforms $\mathbf{R} = (\mathbf{R}_1, ..., \mathbf{R}_n)$, we find that

$$B = (\mathbf{I} + \mathbf{R}\otimes\mathbf{R})F \quad\text{and}\quad E = -(\mathbf{R}\otimes\mathbf{R})F.$$

Hereafter, we use the notation $\mathbf{R}\otimes\mathbf{R} = [\mathbf{R}_{ij}]$ for the matrix of the second order Riesz transforms. We consider the projections of $\mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ onto the spaces $\mathcal{B}^p(\mathbb{R}^n)$ and $\mathcal{E}^p(\mathbb{R}^n)$, where we denote by $\mathcal{B}^p(\mathbb{R}^n)$ and $\mathcal{E}^p(\mathbb{R}^n)$ the spaces of divergence free and curl free vector fields in $\mathscr{L}^p(\Omega, \mathbb{R}^n)$ respectively. These projections are easily expressed in terms of the Riesz transforms $\mathbf{B} = \mathbf{I} + \mathbf{R}\otimes\mathbf{R}$ and $\mathbf{E} = -\mathbf{R}\otimes\mathbf{R}$. The uniqueness of the decomposition at 6.1 gives

$$\ker\mathbf{B} = \mathcal{E}^p(\mathbb{R}^n) \quad\text{and}\quad \ker\mathbf{E} = \mathcal{B}^p(\mathbb{R}^n)$$

There is another way of expressing $F$ in terms of the potential field $U$, namely

$$F = \nabla(\mathrm{div}\, U) + \mathrm{div}\,(\mathrm{curl}\, U) \tag{6.2}$$

Let us note that the divergence of a matrix function is being used here, which is a vector field whose coordinates are obtained by computing the divergence of the row vectors.
Now, if $\mathrm{curl}\, F \in \mathscr{L}^s(\mathbb{R}^n, \mathbb{R}^{n\times n})$, consider the Poisson' equation for $\mathrm{curl}\, U$

$$\Delta(\mathrm{curl}\, U) = \mathrm{curl}(\Delta U) = \mathrm{curl}\, F \in \mathscr{L}^s(\mathbb{R}^n, \mathbb{R}^n)$$

By ellipticity of the Laplace operator we gain some regularity of $\mathrm{curl}\, U$. The second term in the right hand side of (6.2) belongs to the Sobolev class $\mathscr{W}^{1,s}(\mathbb{R}^n, \mathbb{R}^n)$, while the first term, denoted by $F_0$, is a curl free distribution. It brings us to the following Poincaré type inequality

**Lemma 6.2.1.** *For each distribution $F \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ with $curl\, F \in \mathscr{L}^s(\mathbb{R}^n, \mathbb{R}^n)$, $1 < s < \infty$, there exists $F_0 \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^n)$ such that $F - F_0 \in \mathscr{W}^{1,s}(\mathbb{R}^n, \mathbb{R}^n)$ and we have the uniform bound*

$$\|F - F_0\|_{\mathscr{W}^{1,s}} \;=\; \|DF - DF_0\|_{\mathscr{L}^s} \leqslant C_s(n)\, \|curl F\|_{\mathscr{L}^s}$$

In much the same way, we obtain the following dual estimate:

**Lemma 6.2.2.** *For each distribution $F \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ with $div\, F \in \mathscr{L}^s(\mathbb{R}^n)$, $1 < s < \infty$, there exists $F_0 \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$ such that $F - F_0 \in \mathscr{W}^{1,s}(\mathbb{R}^n, \mathbb{R}^n)$ and*

$$\|F - F_0\|_{\mathscr{W}^{1,s}} \leqslant \|DF - DF_0\|_{\mathscr{L}^s} \leqslant C_s(n)\, \|div\, F\|_{\mathscr{L}^s}$$

It is worth mention that in both lemmas $F_0$ is obtained via a singular integral operator acting on $F$. Consequently, if $F$ has compact support, then $F_0$ decays as $C|x|^{-n}$ at infinity.
Some Orlicz-Sobolev variants of these lemmas are also available, but we shall pursue this matter later on.
Let $F \in \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n)$, $1 < p < \infty$, be a given vector field. We decompose it as $F = B + E$, with $B \in \mathcal{B}^p(\mathbb{R}^n)$ and $E \in \mathcal{E}^p(\mathbb{R}^n)$. Then we introduce the operator $\mathbf{S} : \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ by the rule $\mathbf{S}F = E - B$. Precisely, we have

$$-\mathbf{S} = \mathbf{B} - \mathbf{E} = \mathbf{I} + 2\mathbf{R} \otimes \mathbf{R}$$

Here are the basic properties of this operator, showing great resemblance to the Hilbert transform in the real line

1. $\mathbf{S}$ is an involution; $\mathbf{S} \circ \mathbf{S} = \mathbf{I}$

2. $\mathbf{S}$ is self-adjoint;

$$\int_{\mathbb{R}^n} \langle \mathbf{S}F, G \rangle \;=\; \int_{\mathbb{R}^n} \langle F, \mathbf{S}G \rangle$$

   for any $F \in \mathscr{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ and $G \in \mathscr{L}^q(\mathbb{R}^n, \mathbb{R}^n)$, with $1 < p, q < \infty$, satisfying $p + q = pq$. Thus, in particular,

3. $\mathbf{S} : \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ is an isometry.
   These are the legitimate reason for calling $\mathbf{S}$ the Hilbert transform in $\mathbb{R}^n$.

## 6.3 Preliminary results

In this section we introduce the necessary background for the succeeding proofs. The following lemma (see [IMMP]) establishes boundedness of singular integral operators and related commutators in the Orlicz classes.

**Lemma 6.3.1.** *Let $\Phi(t) = t^p \log^\alpha(e + t)$, $1 < p < \infty$, $\alpha \in \mathbb{R}$ and let $T$ be a singular integral operator in $\mathbb{R}^n$. Then*

$$\|T(f)\|_{\mathscr{L}^p \log^\alpha \mathscr{L}} \;\leqslant\; C_p(\alpha)\, \|f\|_{\mathscr{L}^p \log^\alpha \mathscr{L}}$$

*Moreover if $k \in BMO(\mathbb{R}^n)$, then the commutator $kT - Tk : \mathscr{L}^p \log^\alpha \mathscr{L}(\mathbb{R}^n) \to \mathscr{L}^p \log^\alpha \mathscr{L}(\mathbb{R}^n)$ is a bounded operator satisfying*

$$\|(kT - Tk)f\|_{\mathscr{L}^p \log^\alpha \mathscr{L}} \;\leqslant\; C_p(n, \alpha)\, \|k\|_{BMO} \|f\|_{\mathscr{L}^p \log^\alpha \mathscr{L}}.$$

**Lemma 6.3.2.** *Let us assume the following inequality holds*

$$\int_{\mathbb{R}^n} \frac{u^2}{\log(e+u)} \leqslant A \quad \textit{with } A \geqslant 1$$

*then*

$$\int_{\mathbb{R}^n} \frac{\left(\frac{u}{A}\right)^2}{\log\left(e+\frac{u}{A}\right)} \leqslant 1$$

*Proof.* It is sufficient to observe that for $A \geqslant 1$

$$\int_{\mathbb{R}^n} \frac{\left(\frac{u}{A}\right)^2}{\log\left(e+\frac{u}{A}\right)} \leqslant \int_{\mathbb{R}^n} \frac{1}{A} \frac{u^2}{\log(e+u)} \leqslant 1$$

□

Let us recall a useful version of Gehring Lemma, for more details see [Gi].

**Proposition 6.3.3.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and $g \in \mathscr{L}^q(\Omega)$, $q > 1$. If for any cube $Q \subset 2Q \subset\subset \Omega$*

$$⨍_Q g^q\,dx \leqslant c\left(⨍_{2Q} g\,dx\right)^q + ⨍_{2Q} f^q\,dx + \theta ⨍_{2Q} g^q\,dx$$

*where $f \in \mathscr{L}^r_{loc}(\Omega)$, $r > q$ and $0 \leqslant \theta < 1$, then there exist $C = C(n, \theta, c, q)$ and $\varepsilon = \varepsilon(n, \theta, c, q)$ such that $g \in \mathscr{L}^p_{loc}(\Omega)$, $p \in [q, q+\varepsilon)$ and*

$$\left(⨍_Q g^p\,dx\right)^{\frac{1}{p}} \leqslant C\left\{\left(⨍_{2Q} g^q\,dx\right)^{\frac{1}{q}} + \left(⨍_{2Q} f^p\,dx\right)^{\frac{1}{p}}\right\}$$

In view of the forthcoming Theorem 6.4.1 we now state the following lemma.

**Lemma 6.3.4.** *Let $u \in \mathscr{W}^{2,2}(\mathbb{R}^n)$. Then for every $\eta \in C_0^\infty(\mathbb{R}^n)$, the following inequality holds:*

$$\int_{\mathbb{R}^n} \eta^2|\nabla u|^2 \leqslant 4\int_{\mathbb{R}^n} u^2|\nabla\eta|^2 + 2\int_{\mathbb{R}^n} |u\nabla^2 u||u|\eta^2$$

*Proof.* Integrating by parts, we have

$$\int_{\mathbb{R}^n} \eta\,|\nabla u|^2 = \int_{\mathbb{R}^n} \eta\,\langle\nabla u \cdot \nabla u\rangle$$

$$= -\int_{\mathbb{R}^n} u\,\langle \nabla u \cdot \nabla \eta \rangle - \int_{\mathbb{R}^n} \eta \cdot u \Delta u$$

$$\leqslant \int_{\mathbb{R}^n} |u|\,|\nabla u|\,|\nabla \eta| + \int_{\mathbb{R}^n} |\nabla^2 u|\,|u|\,|\eta|$$

Replace $\eta$ by $\eta^2 \in C_0^\infty(\mathbb{R}^n)$ in previous inequality to obtain

$$\int_{\mathbb{R}^n} \eta^2\,|\nabla u|^2 \;\leqslant\; 2\int_{\mathbb{R}^n} |u|\,|\nabla u|\,|\eta|\,|\nabla \eta| + \int_{\mathbb{R}^n} |\nabla^2 u|\,|u|\,\eta^2$$

This implies, in view of the elementary inequality $2xy \leqslant \delta^2 x^2 + \frac{y^2}{\delta^2}$, that

$$\int_{\mathbb{R}^n} \eta^2\,|\nabla u|^2 \;\leqslant\; \delta^2 \int_{\mathbb{R}^n} \eta^2\,|\nabla u|^2 + \frac{1}{\delta^2}\int_{\mathbb{R}^n} u^2\,|\nabla \eta|^2 + \int_{\mathbb{R}^n} |\nabla^2 u|\,|u|\,\eta^2$$

Hence:

$$\int_{\mathbb{R}^n} \eta^2\,|\nabla u|^2 \;\leqslant\; \frac{1}{\delta^2(1-\delta^2)}\int_{\mathbb{R}^n} u^2\,|\nabla \eta|^2 + \frac{1}{1-\delta^2}\int_{\mathbb{R}^n} |\nabla^2 u|\,|u|\,\eta^2$$

Taking $\delta^2 = \frac{1}{2}$ we obtain the desired inequality. □

## 6.4 Bounded coefficients, a higher integrability result

Let $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a matrix valued function on $\mathbb{R}^n$ such that

$$|\xi|^2 \;\leqslant\; \langle A(x)\xi\,,\,\xi\rangle \;\leqslant\; K|\xi|^2 \tag{6.3}$$

for all $\xi \in \mathbb{R}^n$ and for a.e. $x \in \mathbb{R}^n$. Let

$$\|A\|_{BMO(\mathbb{R}^n,\mathbb{R}^{n\times n})} \;\leqslant\; \varepsilon \tag{6.4}$$

where $\varepsilon > 0$ is sufficiently small.

Consider the operator

$$Lu \;=\; \langle A(x)\,,\,\nabla^2 u\rangle \;=\; \sum_{i,j=1}^{n} A^{ij}(x)\,\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{6.5}$$

for $u \in \mathscr{W}^{2,1}(\mathbb{R}^n)$.

Our aim is to give a higher integrability result for second derivatives of a solution $u$ of equation

$$Lu = h \tag{6.6}$$

**Theorem 6.4.1.** *Let* $\Omega$ *be a bounded open set of* $\mathbb{R}^n$ *and* $u \in \mathscr{W}^{2,2}_{loc}(\Omega)$ *a solution to equation (6.6) with* $h \in \mathscr{L}^r(\Omega)$, $r > 2$.
*There exists* $p = p(\varepsilon) > 2$, *such that* $\nabla^2 u$ *is locally* $p$*-integrable and*

$$\left( \fint_Q |\nabla^2 u|^p \, dx \right)^{\frac{1}{p}} \preccurlyeq \left( \fint_{2Q} |\nabla^2 u|^2 \, dx \right)^{\frac{1}{2}}$$

*for any cube* $Q \subset 2Q \subset\subset \Omega$.

The following $\mathscr{L}^2$ estimate is useful to prove Theorem 6.4.1

**Theorem 6.4.2.** *Let* $A \in BMO(\mathbb{R}^n, R^{n\times n})$ *satisfy conditions (6.3), (6.4). If* $u$ *solves equation (6.6) then*

$$\|\nabla^2 u\|_{\mathscr{L}^2(\mathbb{R}^n)} \preccurlyeq \|h\|_{\mathscr{L}^2(\mathbb{R}^n)} \tag{6.7}$$

*provided* $|\nabla^2 u| \in \mathscr{L}^2(\mathbb{R}^n)$ *and* $h \in \mathscr{L}^2(\mathbb{R}^n)$.

*Proof.* We denote by $f$ the gradient of $u$, $f = (f^1, f^2, ..., f^n) = \nabla u$, where the coordinate functions satisfy $f^i_j = u_{x_i x_j} = f^j_i$. Hereafter the subscripts $j$ and $i$ stand for the partial derivatives with respect to $x_j$ and $x_i$, respectively.

We have by (6.3)

$$|\nabla^2 u|^2 = |Df|^2 = \sum_{\alpha=1}^{n} \sum_{i=1}^{n} f^\alpha_i f^\alpha_i \leqslant \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x) f^\alpha_i f^\alpha_j$$

$$= \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x)(f^\alpha_i f^j_\alpha - f^\alpha_\alpha f^j_i) + \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x)(f^\alpha_\alpha f^j_i)$$

the second term of the right hand side is equal to $\Delta u \cdot h$, so we obtain

$$\sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x)(f^\alpha_i f^j_\alpha - f^\alpha_\alpha f^j_i) + (\Delta u)\, h$$

$$\leqslant \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x)(f^\alpha_i f^j_\alpha - f^\alpha_\alpha f^j_i) + \sqrt{n}\, |h|\, |Df|$$

by the elementary inequality $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$ the second term can be estimated by the following expression

$$\sum_{\alpha=1}^{n} \sum_{i,j=1}^{n} A^{ij}(x)(f^\alpha_i f^j_\alpha - f^\alpha_\alpha f^j_i) + \frac{1}{2}|Df|^2 + \frac{n}{2}|h|^2$$

Finally, the following estimate holds true

$$|Df|^2 \leqslant 2\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)(f_i^\alpha f_\alpha^j - f_\alpha^\alpha f_i^j) + n|h|^2 \tag{6.8}$$

Integrating on $\mathbb{R}^n$, recalling by Lemma 5.2.4 that $\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} f_i^\alpha f_\alpha^j - f_\alpha^\alpha f_i^j \in \mathscr{H}^1$ it is possible to use Lemma 5.1.1, Hölder inequality and assumption (6.4):

$$\int_{\mathbb{R}^n} |Df|^2 \leqslant 2\|A\|_{BMO}\|\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} f_i^\alpha f_\alpha^j - f_\alpha^\alpha f_i^j\|_{\mathscr{H}^1} + n\int_{\mathbb{R}^n}|h|^2$$

$$\leqslant 2\varepsilon\int_{\mathbb{R}^n}|Df|^2 + n\int_{\mathbb{R}^n}|h|^2$$

The first term in the right hand side is absorbed by the left hand side, provided $\varepsilon = \varepsilon(n)$ is sufficiently small.

Then we conclude with the desired estimate:

$$\int_{\mathbb{R}^n}|\nabla^2 u|^2 = \int_{\mathbb{R}^n}|Df|^2 \leqslant c(n)\int_{\mathbb{R}^n}|h|^2$$

□

Let us state the following Caccioppoli type estimate

**Lemma 6.4.3.** *Let $u \in \mathscr{W}^{2,2}_{loc}(\mathbb{R}^n)$ a solution to equation (6.6) with $h \in \mathscr{L}^2_{loc}(\mathbb{R}^n)$.*
*Then*

$$\int_{\mathbb{R}^n}\psi^4|\nabla^2 u|^2 \preccurlyeq K^4\int_{\mathbb{R}^n}|u|^2\left(|\psi|^2|\nabla^2\psi|^2 + |\nabla\psi|^4\right) \tag{6.9}$$

$$+\int_{\mathbb{R}^n}\psi^4|Lu|^2$$

*for any $\psi \in C_0^\infty(\mathbb{R}^n)$.*

*Proof.* Consider a test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $v = \varphi u$. The partial derivatives are: $v_i = \varphi u_i + \varphi_i u$ and $v_{ij} = \varphi u_{ij} + \varphi_j u_i + \varphi_{ij} u + \varphi_i u_j$.

By Definition (6.6)

$$Lv = \sum_{i,j=1}^{n} A^{ij} v_{ij} = \sum_{i,j=1}^{n} A^{ij}(\varphi u_{ij} + \varphi_j u_i + \varphi_{ij} u + \varphi_i u_j)$$

$$= \varphi h + \sum_{i,j=1}^{n} A^{ij}[\varphi_j u_i + \varphi_{ij} u + \varphi_i u_j]$$

Using Theorem 6.4.2 and hypothesis (6.3), the following inequality holds

$$\| \nabla^2 v \|_2 \preccurlyeq \| \varphi h \|_2 + \| K \nabla\varphi\nabla u \|_2 + \| K u\nabla^2\varphi \|_2. \tag{6.10}$$

Since $v = \varphi u$ then

$$\varphi\nabla^2 u = \nabla^2 v - 2\nabla\varphi\nabla u - u\nabla^2\varphi,$$

hence

$$\| \varphi\nabla^2 u \|_2 \preccurlyeq \| \nabla^2 v \|_2 + \| \nabla\varphi\nabla u \|_2 + \| u\nabla^2\varphi \|_2$$

Finally, in view of (6.10) we conclude with the inequality

$$\|\varphi\nabla^2 u\|_2 \preccurlyeq \|\varphi h\|_2 + \| K\nabla\varphi\nabla u \|_2 + \| Ku\nabla^2\varphi \|_2$$
$$+\| \nabla\varphi\nabla u \|_2 + \| u\nabla^2\varphi \|_2$$

This means that

$$\int_{\mathbb{R}^n} \varphi^2|\nabla^2 u|^2 \preccurlyeq \int_{\mathbb{R}^n} K^2|\nabla\varphi|^2|\nabla u|^2 + \int_{\mathbb{R}^n} K^2|\nabla^2\varphi|^2 u^2$$
$$+ \int_{\mathbb{R}^n} \varphi^2|Lu|^2$$

Applying Lemma 6.3.4 to the first hand right side with $\eta = |\nabla\varphi|$ and noticing that $|\nabla\eta| \leqslant |\nabla^2\varphi|$, we obtain

$$\int_{\mathbb{R}^n} \varphi^2|\nabla^2 u|^2 \preccurlyeq K^2 \int_{\mathbb{R}^n} u^2 |\nabla^2\varphi|^2 + K^2 \int_{\mathbb{R}^n} |u| |\nabla^2 u| |\nabla\varphi|^2$$
$$+ K^2 \int_{\mathbb{R}^n} u^2 |\nabla^2\varphi|^2 + \int_{\mathbb{R}^n} \varphi^2 |L u|^2$$

If we set $\varphi = \psi^2$, then $|\nabla\varphi| \preccurlyeq |\psi||\nabla\psi|$ and $|\nabla^2\varphi| \preccurlyeq |\psi\nabla^2\psi| + |\nabla\psi|^2$. This yields:

$$\int_{\mathbb{R}^n} \psi^4\,|\nabla^2 u|^2 \preccurlyeq K^2 \int_{\mathbb{R}^n} u^2 \left(|\psi||\nabla^2\psi| + |\nabla\psi|^2\right)^2$$

$$+ K^2 \int_{\mathbb{R}^n} |u|\,|\nabla\psi|^2|\psi|^2\,|\nabla^2 u| + \int_{\mathbb{R}^n} \psi^4\,|L\,u|^2$$

Finally

$$\int_{\mathbb{R}^n} \psi^4\,|\nabla^2 u|^2 \preccurlyeq K^2 \int_{\mathbb{R}^n} u^2 \left(|\psi||\nabla^2\psi| + |\nabla\psi|^2\right)^2 \tag{6.11}$$

$$+ K^4 \int_{\mathbb{R}^n} |u|^2|\,\nabla\psi|^4 + \int_{\mathbb{R}^n} \psi^4\,|L\,u|^2$$

Let us consider the first term of the right hand side in (6.11) by the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ we obtain the desired inequality.

$$\int_{\mathbb{R}^n} \psi^4|\nabla^2 u|^2 \preccurlyeq K^4 \int_{\mathbb{R}^n} u^2 \left(|\psi||\nabla^2\psi| + |\nabla\psi|^2\right) + \int_{\mathbb{R}^n} \psi^4|Lu|^2$$

□

Now we are ready to prove our main result

*Proof of Theorem 6.4.1.* Let us consider a bump function

$$0 \leqslant \psi \leqslant 1, \quad \psi \in C_0^\infty(2Q)$$

$$\psi = 1 \quad \text{on } Q$$

such that

$$|\nabla\psi| \leqslant \frac{1}{\operatorname{diam} Q}$$

and

$$|\nabla^2\psi| \leqslant \frac{1}{(\operatorname{diam} Q)^2}$$

Replacing the function $\psi$ in the inequality (6.9) we obtain

$$\int_Q |\nabla^2 u|^2 \leqslant \frac{K^4}{(\mathrm{diam} Q)^4} \int_{2Q} u^2 + \int_{2Q} |L\,u|^2$$

This inequality is not affected if we subtract any linear function from $u$, say $u_0 = \mathscr{A}x + \mathscr{B}$, so that we can write

$$⨍_Q |\nabla^2 u|^2 \preccurlyeq K^4 ⨍_{2Q} \left[\frac{u-u_0}{(\mathrm{diam} Q)^2}\right]^2 + ⨍_{2Q} |Lu|^2$$

Next we recall Poincarè inequality:

$$\left[⨍_{2Q} \left(\left|\frac{u-u_0}{\mathrm{diam}^2 Q}\right|\right)^2\right]^{\frac{1}{2}} \preccurlyeq \left(⨍_{2Q} |\nabla^2 u|^s\right)^{\frac{1}{s}}$$

which holds for every $s \geq \max\{1, \frac{2n}{n+4}\}$.

Take $s = \frac{2n}{n+2}$, to obtain weak Reverse Hölder inequality

$$\left(⨍_Q |\nabla^2 u|^2\right)^{\frac{1}{2}} \preccurlyeq K^2 \left(⨍_{2Q} |\nabla^2 u|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} + \left(⨍_{2Q} |Lu|^2\right)^{\frac{1}{2}}$$

By Proposition 6.3.3 (with $q = 2$) there is $p = p\ (n, K)\ > 2$ such that $\nabla^2 u$ is locally $p$-integrable, provided $h = Lu \in \mathscr{L}^p(\mathbb{R}^n)$ and so Theorem 6.4.1 is proved. □

## 6.5 Unbounded coefficients, a priori estimate

In this section, we wish to deal with unbounded coefficients.
Consider the equation

$$Lu = h \tag{6.12}$$

where

$$Lu = \langle A(x)\,,\,\nabla^2 u\rangle = \sum_{i,j=1}^{n} A^{ij}(x)\,\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{6.13}$$

for $u \in \mathscr{W}^{2,1}(\mathbb{R}^n)$ and

$$|\xi|^2 \ \leqslant\ \langle A(x)\xi\,,\,\xi\rangle \tag{6.14}$$

for all $\xi \in \mathbb{R}^n$ and for a.e. $x \in \mathbb{R}^n$. Let

$$\|A\|_{BMO(\mathbb{R}^n,\mathbb{R}^{n\times n})} \ \leqslant\ \varepsilon \tag{6.15}$$

where $\varepsilon > 0$ is sufficiently small. Then the following a priori estimate holds.

**Theorem 6.5.1.** *Let $A \in BMO(\mathbb{R}^n, R^{n\times n})$ satisfying conditions (6.14), (6.15). Assume that $A(x) = A_0$, $A_0$ constant, outside a compact set $B \subset \mathbb{R}^n$. If $u$ solves equation (6.12), then*

$$\|\nabla^2 u\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)} \preccurlyeq \|h\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

provided $\nabla^2 u$ and $h$ lie in the space $\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)$.

*Proof.* Let us consider a test function

$$\varphi \ =\ \log\left(e + \frac{M|Df|}{\|Df\|_2}\right)$$

Furthermore a well known result of Coifman and Rochberg [CR] tells us that the function $\varphi \ =\ \log\left(e + \frac{M|Df|}{\|Df\|_2}\right)$ is in $BMO(\mathbb{R}^n)$ and

$$\|\varphi\|_{BMO(\mathbb{R}^n)} \leqslant c(n)$$

Let us multiply $Df$ by the test function $\varphi$. By Hodge decomposition

$$\varphi Df \ =\ Dg \,+\, H. \tag{6.16}$$

where both components are given explicitly by means of Riesz transforms.

Indeed

$$Dg \ =\ \mathbf{S}(\varphi\, Df),\ \ \mathbf{S} : \mathscr{L}^p(\mathbb{R}^n,\mathbb{R}^{n\times n}) \ \rightarrow\ \mathscr{L}^p(\mathbb{R}^n,\mathbb{R}^{n\times n}),\ \ 1 < p < \infty$$

and the divergence free vector $H$ can be expressed as

$$H = T(\varphi Df) = (T\varphi - \varphi T)Df$$

where $T$ is a singular operator in $\mathbb{R}^n$ (for more details see [IMa]).

Since $\varphi \in BMO(\mathbb{R}^n)$, by Lemma 6.3.1

$$\|H\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)} \preccurlyeq \|\varphi\|_{BMO}\ \|Df\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

$$\preccurlyeq \ \|Df\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

Let us recall the pointwise inequality (6.8)

$$|Df|^2 \ \preccurlyeq \ \sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)(f_i^\alpha f_\alpha^j - f_\alpha^\alpha f_i^j) \ + \ |h|^2 \tag{6.17}$$

It is obvious that

$$\log\left(e+\frac{|Df|}{\|Df\|_2}\right) \ \leqslant \ \log\left(e+\frac{M|Df|}{\|Df\|_2}\right) \tag{6.18}$$

Multiplying inequality (6.17) by $\varphi$ , we obtain

$$\varphi|Df|^2 \leqslant \varphi\left[\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)(f_i^\alpha f_\alpha^j - f_\alpha^\alpha f_i^j)\right] + \varphi|h|^2$$

$$= \sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)\left[(\varphi f_i^\alpha)f_\alpha^j - (\varphi f_\alpha^\alpha)f_i^j\right] + \varphi|h|^2 \tag{6.19}$$

Recalling (6.18) and (6.19), we have the following estimate

$$|Df|^2\log\left(e+\frac{|Df|}{\|Df\|_2}\right) \leqslant \sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)\left[(\varphi f_i^\alpha)f_\alpha^j - (\varphi f_\alpha^\alpha)f_i^j\right] + \varphi|h|^2 \tag{6.20}$$

Since, using (6.16)

$$\varphi f_i^\alpha = g_i^\alpha + H^{\alpha i}, \quad \varphi f_\alpha^\alpha = g_\alpha^\alpha + H^{\alpha\alpha}$$

we get by (6.20)

$$|Df|^2\log\left(e+\frac{|Df|}{\|Df\|_2}\right) \preccurlyeq \sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)\left[H^{\alpha i}f_\alpha^j - H^{\alpha\alpha}f_i^j\right]$$

$$+\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n} A^{ij}(x)(g_i^\alpha f_\alpha^j - g_\alpha^\alpha f_i^j) + \varphi|h|^2$$

Integrating the previous inequality, by (1.16)

$$\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)} \preccurlyeq \int_{\mathbb{R}^n} |A|\,|H|\,|Df| \ + \ \|A\|_{BMO}\ \|\sum_{\alpha}^{n}\sum_{i,j}^{n} g_i^\alpha f_\alpha^j - g_\alpha^\alpha f_i^j\|_{\mathscr{H}^1} \tag{6.21}$$

$$+\int_{\mathbb{R}^n} \varphi |h|^2$$

where in the right hand side we have employed Lemma 5.1.1.
The estimate we are going to prove is technical but also fundamental to higher integrability properties of (6.21). Let us go step by steps estimating the right hand side of inequality (6.21).

**Estimate for $\int_{\mathbb{R}^n} |A|\,|H|\,|Df|$.**
Using the elementary inequality $2ab \leqslant a^2 + b^2$ with $a = |H|$ and $b = |Df|$

$$2\int_{\mathbb{R}^n} |A|\,|H|\,|Df| \;\leqslant \int_{\mathbb{R}^n} |A|\,|H|^2 \,+ \int_{\mathbb{R}^n} |A|\,|Df|^2 \qquad (6.22)$$

Adding and subtracting $A_0$, we obtain

$$2\int_{\mathbb{R}^n} |A|\,|H|\,|Df| \;\leqslant \int_{\mathbb{R}^n} |A - A_0|\,|H|^2 \,+ \int_{\mathbb{R}^n} |A - A_0|\,|Df|^2$$

$$+|A_0|\int_{\mathbb{R}^n} (|H|^2 \,+\, |Df|^2) \;=\; I_1 \,+\, I_2 \,+\, I_3$$

Let us denote by $I_1$, $I_2$, $I_3$ respectively the first, second and third term in the right hand side of the above inequality.
Let us estimate $I_1$, using inequality (1.17) and Definition 1.3.1

$$\begin{aligned} I_1 \;&=\; \int_{\mathbb{R}^n} |A - A_0||H|^2 = \int_{\mathbb{R}^n} \left(\frac{|H|^2}{\|H\|_2^2}\; N|A - A_0|\right) \frac{\|H\|_2^2}{N} \qquad (6.23)\\ &\leqslant \int_{\mathbb{R}^n} \left[\frac{|H|^2}{\|H\|_2^2} \log\left(1 + \frac{|H|^2}{\|H\|_2^2}\right) + (e^{N|A-A_0|} - 1)\right] \cdot \frac{\|H\|_2^2}{N}\\ &\leqslant \frac{2}{N}\|H\|^2_{\mathscr{L}^2\log\mathscr{L}} \;+\; \frac{\|H\|_2^2}{N}\int_{\mathbb{R}^n} \left(e^{N|A_0-A|} - 1\right) \end{aligned}$$

By assumption $A(x) = A_0$, for $x \in \mathbb{R}^n - B$ a.e., hence

$$\int_{\mathbb{R}^n} \left(e^{N|A_0-A|} - 1\right) = \int_B \left(e^{N|A_0-A|} - 1\right)$$

Since $A$ belongs to $BMO$, we know by the John-Nirenberg lemma (see Lemma 1.5.3) that for $N$ large and $A \in BMO$

$$\int_B \left(e^{N|A_0-A|} - 1\right) < \infty$$

Recalling that $\varphi \in BMO$ and the $BMO$-norm of $\varphi$ may be bounded by a constant depending only on the dimension and applying Lemma 6.3.1 the following inequality holds

$$\|H\|^2_{\mathscr{L}^2\log\mathscr{L}} \preccurlyeq \|\varphi\|_{BMO}\ \|Df\|^2_{\mathscr{L}^2\log\mathscr{L}} \preccurlyeq \|Df\|^2_{\mathscr{L}^2\log\mathscr{L}} \tag{6.24}$$

Furthermore by Theorem 6.4.2, recalling relation (1.16)

$$\|H\|^2_2 \leqslant \|\varphi\|_{BMO}\ \|Df\|^2_2 \preccurlyeq \|h\|^2_2 \preccurlyeq \|h\|^2_{\mathscr{L}^2\log\mathscr{L}} \tag{6.25}$$

It follows from (6.24) and (6.25) that

$$I_1 = \int_{\mathbb{R}^n} |A - A_0||H|^2 \preccurlyeq \frac{2}{N}\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}} + \|h\|^2_{\mathscr{L}^2\log\mathscr{L}} \tag{6.26}$$

Analogously

$$I_2 = \int_{\mathbb{R}^n} |A - A_0||Df|^2 \preccurlyeq \frac{2}{N}\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}} + \|h\|^2_{\mathscr{L}^2\log\mathscr{L}} \tag{6.27}$$

for the last term $I_3$ using again (6.24), and (6.25) we have

$$\begin{aligned} I_3 &= 2|A_0| \int_{\mathbb{R}^n} (|H|^2 + |Df|^2) = |A_0| \left[\|H\|^2_2 + \|Df\|^2_2\right] \\ &\preccurlyeq |A_0| \left[\|h\|^2_{\mathscr{L}^2\log\mathscr{L}} + \|h\|^2_2\right] \\ &\preccurlyeq \|h\|^2_{\mathscr{L}^2\log\mathscr{L}} \end{aligned} \tag{6.28}$$

According the estimates (6.26), (6.27) and (6.28)

$$2\int_{\mathbb{R}^n} |A|\,|H|\,|Df| \preccurlyeq \frac{4}{N}\,\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}} + \|h\|^2_{\mathscr{L}^2\log\mathscr{L}} \tag{6.29}$$

**Estimate for** $\|g^\alpha_i f^j_\alpha - g^\alpha_\alpha f^j_i\|_{\mathscr{H}^1}$.

According to a generalization of the results in [CLMS]

$$\|g^\alpha_i f^j_\alpha - g^\alpha_\alpha f^j_i\|_{\mathscr{H}^1} \leqslant \|Dg\|_{\mathscr{L}^2\log^{-1}\mathscr{L}}\ \|Df\|_{\mathscr{L}^2\log\mathscr{L}} \tag{6.30}$$

Next we will estimate $\|Dg\|_{\mathscr{L}^2\log^{-1}\mathscr{L}}$

$$\|Dg\|_{\mathscr{L}^2\log^{-1}\mathscr{L}} = \|S(\varphi\, Df)\|_{\mathscr{L}^2\log^{-1}\mathscr{L}} \leqslant \|\varphi\, Df\|_{\mathscr{L}^2\log^{-1}\mathscr{L}}$$

$$= \|Df\|_{\mathscr{L}^2 \log \mathscr{L}} \cdot \|\varphi F\|_{\mathscr{L}^2 \log^{-1} \mathscr{L}} \tag{6.31}$$

where

$$F = \frac{|Df|}{\|Df\|_{\mathscr{L}^2 \log \mathscr{L}}}$$

With the aid of the elementary inequality (1.17) we arrive at the estimate

$$\varphi F = (\varphi - 1)F + F \preccurlyeq F \log(e + F) + e^{\varphi - 1} - 1 + F \tag{6.32}$$

Since

$$e^{\varphi-1} - 1 = e^{\log\left(e + \frac{M|Df|}{\|Df\|_2}\right) - 1} - 1 = \frac{1}{e}\left(e + \frac{M|Df|}{\|Df\|_2}\right) - 1 = \frac{M|Df|}{e\|Df\|_2}$$

then by (6.32)

$$\varphi F \leqslant 2F \log(e + F) + \frac{M|F|}{e\|F\|_2}$$

Squaring

$$\varphi^2 F^2 \leqslant 8F^2 \log^2(e + F) + \left|\frac{MF}{\|F\|_2}\right|^2$$

dividing by $\log(e + \varphi F)$ and applying Theorem 2.1.10 and (1.16)

$$\int_{\mathbb{R}^n} \frac{\varphi^2 F^2}{\log(e + \varphi F)} \leqslant \int_{\mathbb{R}^n} \frac{\varphi^2 F^2}{\log(e + F)} \leqslant 8 + \frac{\|MF\|_2^2}{\|F\|_2^2} \leqslant k$$

where $k = k(n) \geqslant 1$.
Hence, by Lemma 6.3.2 we get

$$\int_{\mathbb{R}^n} \frac{\left(\frac{\varphi}{k} F\right)^2}{\log\left(e + \frac{\varphi F}{K}\right)} = \int_{\mathbb{R}^n} \frac{\varphi^2 F^2}{k^2 \log\left(e + \frac{\varphi F}{k}\right)} \leqslant \int_{\mathbb{R}^n} \frac{\varphi^2 F^2}{k \log(e + \varphi F)} \leqslant 1$$

Then by definition of Luxemburg norm

$$\|\varphi F\|_{\mathscr{L}^2 \log^{-1} \mathscr{L}} \leqslant k(n) \tag{6.33}$$

Hence, replacing (6.33) in (6.31)

$$\|Dg\|_{\mathscr{L}^2 \log^{-1} \mathscr{L}} \leqslant k(n)\, \|Df\|_{\mathscr{L}^2 \log \mathscr{L}} \tag{6.34}$$

It follows from (6.30) and (6.34) that

$$\|g_i^\alpha f_\alpha^j - g_\alpha^\alpha f_i^j\|_{\mathscr{H}^1} \preccurlyeq \|Dg\|_{\mathscr{L}^2 \log^{-1} \mathscr{L}} \cdot \|Df\|_{\mathscr{L}^2 \log \mathscr{L}} \tag{6.35}$$

$$\leqslant k(n)\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}}.$$

**Estimate for $\int_{\mathbb{R}^n}\varphi|h|^2$.**
To estimate the last term in (6.21), we need the following inequality

$$\log(e+x)\leqslant 2+\log(1+x^2) \tag{6.36}$$

Applying (6.36) to $\varphi$, we obtain the following estimate

$$\varphi=\log\left(e+\frac{M|Df|}{\|Df\|_2}\right)\leqslant 2+\log\left[1+\left(\frac{M|Df|}{\|Df\|_2}\right)^2\right] \tag{6.37}$$

Assume $\Psi=\log\left[1+\left(\frac{M|Df|}{\|Df\|_2}\right)^2\right]$.
Multiplying relation (6.37) by $|h|^2$ and using inequality (1.17)

$$\varphi|h|^2\leqslant 2|h|^2+\Psi|h|^2$$

Next using inequality (1.17)

$$\varphi|h|^2\leqslant 2|h|^2+\left[\frac{|h|^2}{\|h\|_2^2}\log\left(1+\frac{|h|^2}{\|h\|_2}\right)+e^{\Psi}-1\right]\|h\|_2^2 \tag{6.38}$$

by definition of $\Psi$ and Lemma 2.2.11

$$\int_{\mathbb{R}^n}e^{\Psi-1}-1 = \int_{\mathbb{R}^n}\frac{(M|Df|)^2}{\|Df\|_2^2}\leqslant C \tag{6.39}$$

According to Definition 1.3.1, by (6.38) and (6.39)

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\,|h|^2 &\preccurlyeq \|h\|^2_{\mathscr{L}^2(\mathbb{R}^n)}+\|h\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}\\ &\preccurlyeq \|h\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}\end{aligned} \tag{6.40}$$

Finally, using the previous Estimates and (6.21), we obtain

$$\int_{\mathbb{R}^n}|Df|^2\log\left(e+\frac{|Df|}{\|Df\|_2}\right)$$

$$\leqslant \frac{4}{N}\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}\;+\;\|h\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}\;+\;\varepsilon k(n)\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

$$+\|h\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

If $N$ is large and $\varepsilon$ is small so it is possible to absorb the first and the third term to the right hand side then

$$\|Df\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)} \preccurlyeq \|h\|^2_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

i.e. :

$$\|\nabla^2 u\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)} \preccurlyeq \|h\|_{\mathscr{L}^2\log\mathscr{L}(\mathbb{R}^n)}$$

□

By using the following theorem, (for more details see [G]) we can deduce a relation between the spaces weak-$\mathscr{L}^n$(1.2.8) and $BMO$ (1.5.8).

**Theorem 6.5.2.** *Let $a \in \mathscr{W}^{1,1}(\Omega)$ where $\Omega$ is convex, and suppose there exists a constant $K$ such that*

$$\int_{\Omega\cap B_R} |\nabla a|\,dx \leqslant KR^{n-1} \quad \textit{for all balls } B_R.$$

*Then there exist positive constants $\sigma_0$ and $C$ depending only on n such that*

$$\int_\Omega \exp\left(\frac{\sigma}{K}|a - a_\Omega|\right)\,dx \leqslant C\,(diam\,\Omega)^n$$

*where $\sigma = \sigma_0|\Omega|(diam\Omega)^{-n}$.*

**Proposition 6.5.3.** *Let $a \in \mathscr{W}^{1,1}(\mathbb{R}^n)$ be such that $|\nabla a|$ belongs to $\mathscr{L}^{n,\infty}(\mathbb{R}^n)$ and $\|\nabla a\|_{\mathscr{L}^{n,\infty}} < \varepsilon$.*
*Then $a \in BMO(\mathbb{R}^n)$ and*

$$\|a\|_{BMO(\mathbb{R}^n)} \leqslant C\varepsilon$$

*where C=C(n).*

*Proof.* Fix a ball $B \subset \mathbb{R}^n$. By Theorem 6.5.2, we know that if for $B_R \subset B$

$$R⨍_{B_R} |\nabla a|\,dx < \varepsilon$$

then there exist constants $\sigma = \sigma(n)$ and $C = C(n)$ such that

$$⨍_B \exp\left(\frac{\sigma}{\varepsilon}|a - a_B|\right)\,dx \leqslant C$$

and this implies that

$$\frac{\sigma}{\varepsilon}\fint_B |a - a_B|\, dx \leqslant C$$

hence, taking the supremum with respect to $B \subset \mathbb{R}^n$

$$\|a\|_{BMO} \leqslant \frac{C}{\sigma}\varepsilon.$$

□

# Bibliography


[Ad] R. Adams, **Sobolev spaces**, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, (1975).

[A] L.V. Ahlfors, *Conditions for quasiconformal deformations in several variables*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, (1974), 19-25.

[A1] L.V. Ahlfors, *Quasiconformal deformations and mappings in* $R^n$, J. Analyse Math. 30 (1976), 74-97.

[A] A.D. Aleksandrov, *Uniqueness conditions and bounds for the solution of the Dirichlet problem*, Vestnik Leningrad Univ. Ser. Mat. Meh. Astronom., 18 (1963), no 3,5-29.

[Alv] A. Alvino, *Sulla disuguaglianza di Sobolev in spazi di Lorentz*, Bollettino U.M.I.((5)14-A)(1997),148-156.

[AT] A. Alvino and G. Trombetti, *Second order elliptic equations whose coefficients have their first derivates weakly-*$\mathscr{L}^n$, Ann. Mat. Pura Appl. (4) 138 (1984) 331-340.

[Ar] G. Aronsson, *On the partial differential equation* ${u_x}^2 u_{xx} + 2u_x u_y u_{xy} + {u_y}^2 u_{yy} = 0$, Ark. Mat. 7 (1968), 1968 395-425 .

[As] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., 173 (1994), no. 1, 37-60.

[B] C. Bennett, *Intermediate spaces and the class* $L\log^+ L$, Ark. Mat. 11(1973),215-228.

[BR] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. 175(1980),1-72.

[BS] C. Bennett and R. Sharpley, *Interpolation of operators* , Pure and applied Mathematics 129, Academic Press , Inc., Boston, Ma,(1988).

[BDM] T.Bhattacharya, E.Di Benedetto and J. Manfredi, *Limits as* $p \to \infty$ *of* $\Delta_p u_p = f$ *and related extremal problems*. Some topics in nonlinear PDEs (Turin, 1989). Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, (1991), 15-68 .

[BI] B. Bojarski and T. Iwaniec, *p-harmonic equation and quasiregular mappings*. Partial differential equations (Warsaw, 1984), 25-38, Banach Center Publ., 19, PWN, Warsaw, 1987, see also Universität Bonn, Sonderforschungsbereich 72, Preprint 617, (1983), 1-20.

[Bou] J. Bourgain, *Estimations de certaines fonctions maximales*, C.R.Acad.Sci.Paris Sr.I Math 301(10)(1985), 499-502.

[BFS] H. Brezis, N. Fusco and C. Sbordone, *Integrability for the Jacobian of orientation preserving mappings* , J. Funct.Anal. 115(1993),no.2, 425-431.

[C] S. Campanato, *Un risultato relativo ad equazioni ellittiche del secondo ordine di tipo non variazionale*, Ann. Scuola Norm. Sup. Pisa 21 (1967) 701-707.

[CFL] F. Chiarenza,M. Frasca and P. Longo, *Interior* $\mathscr{W}^{2,p}$ *estimates for nondivergence elliptic equations with discontinuous coefficients* Ricerche Mat. 40 (1991) no 1 149-168.

[CF] F. Chiarenza,M. Frasca and P. Longo, $\mathscr{W}^{2,p}$*-solvability of the Dirichlet problem for nondivergence elliptic equations with* $VMO$ *coefficients* Trans.Amer.Math.Soc. 336 (1993) no 2 841-853.

[CLMS] R. Coifman, P.L. Lions , Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces* J. Math. Pures Appl. 72 , (1993),247-286.

[CR] R. Coifman and R. Rochberg, *Another characterization of* $BMO$ Proc.A.M.S., 79(1980), 249-254

[C1] R. Coifman,R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann.Math.,13 (1976)

[DO] A. Dolcini, *A div-curl result in Orlicz spaces*, Rend. Accad. Sci. Fis. Mat. Napoli (4) 60 (1993), (1994),113-120.

[DG] L. D'Onofrio and L. Greco, *On the regularity of solutions to a nonvariational elliptic equation*, Ann. Fac. Sci. Toulouse Math. (6) 11 (2002), no. 1, 47-56.

[FS1] R. Fefferman and E.M. Stein, *$H^p$ spaces of several variables*, Acta Math. 129 (1972), no. 3-4, 137-193.

[FS] N. Fusco and C. Sbordone, *On the integrability of the gradient of minimizers of functionals with nonstandard growth conditions*, Comm. Pure Appl. Math. 43(1990), 673-683.

[GIOV] F. Giannetti, T. Iwaniec, J. Onninen and A. Verde, *Estimates of Jacobians by subdeterminants*, J.Geom.Anal. 12(2002), 223-254.

[Gi] M. Giaquinta, ***Multiple integrals in the calculus of variations and nonlinear elliptic systems***, Princeton University Press (1983).

[G] D. Gilbarg and N. S. Trudinger, ***Elliptic Partial Differential Equations of Second Order***, Springer-Verlag (1983).

[1] L. Greco, *Su alcune proprietà dei Determinanti Jacobiani*, Ph.D. thesis.

[G1] L. Greco, *A remark on the equality* $\det \mathrm{Df} = \mathrm{Det}\, \mathrm{Df}$, Differential Integral Equations 6 (1993),no.5, 1089-1100.

[G2] L.Greco, *Sharp integrability of nonnegative Jacobians*, Rend.Mat.Appl. (7) 18 (1998),no.3, 585-600.

[GIM] L. Greco, T. Iwaniec and G. Moscariello, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J. 44 (1995), no. 2, 305-339.

[HMT] J.A. Hempel, G.R. Morris and N.S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Austral. Math. Soc. 3 (1970), 369-373.

[H] H. Herz, *The Hardy-Littlewood maximal theorem*, Symposium on Harmonic Analysis,Univ. of Warwick, (1968)

[I] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Annals of Mathematics (2) 136 (1992), no. 3, 589-624.

[I1] T. Iwaniec, *On the concept of the weak Jacobian and Hessian*, Report. Univ. Jyväskylä 83 (2001) 181-205.

[IKMS] T. Iwaniec, P. Koskela ,G. Martin and C. Sbordone, *Mappings of finite distortion: $L^n \log^\alpha L$-integrability*, J. London Math. Soc. (2) 67 (2003), no. 1, 123-136.

[IKO] T. Iwaniec, P. Koskela and J. Onninen, *Mappings of finite distortion: monotonicity and continuity*, Invent. Math. 144 (2001), no. 3, 507-531.

[IMa] T. Iwaniec and J. Manfredi, *Regularity of p-harmonic functions on the plane*, Revista Matemática Iberoamericana 5 (1989), no. 1-2, 1-19.

[IM] T. Iwaniec and G. Martin, ***Geometric function theory and non-linear analysis***, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[IO] T. Iwaniec and J. Onninen, *$\mathscr{H}^1$-estimates of Jacobians by subdeterminants*, 324 ,(2002), (2002).

[IMMP] T. Iwaniec, L. Migliaccio, G.Moscariello and A.Passarelli di Napoli, *A priori estimates for nonlinear elliptic complexes*, Adv. Differential Equations 8 (2003), no. 5, 513-546.

[IS] T. Iwaniec, C. Sbordone, *On the integrability of the jacobian under minimal hypothesis*, Arch. Rational Mech. Anal. 119(1992), no.2, 129-143.

[IS1] T. Iwaniec and C. Sbordone, *Quasiharmonic fields*, Annales de l'Institut Henri Poincaré Analyse Non Linéaire 18 (2001), no. 5, 519-572.

[IV] T. Iwaniec and A. Verde, *A study of Jacobians in Hardy-Orlicz spaces*, Proc. Roy. Soc. Edinburgh Sect. A 129, (1999), 539-570.

[JLM] P. Juutinen, P. Lindqvist and J. Manfredi, *The $\infty$-eigenvalue problem*, Arch. Ration. Mech. Anal. 148 (1999), no. 2, 89-105

[JLM1] P. Juutinen, P. Lindqvist and J. Manfredi, *The infinity Laplacian: examples and observations*, Papers on analysis, 207-217, Rep. Univ. Jyvskyl Dep. Math. Stat., 83, Univ. Jyvskyl, Jyvskyl, 2001.

[JLM2] P. Juutinen, P. Lindqvist and J. Manfredi, *On the Equivalence of Viscosity Solutions and Weak Solutions for a Quasilinear Equation*, SIAM Journal of Mathematical Analysis Vol. 33 (2001), No. 3, 699-717.

[KS] V. Krylov and M. V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izvestija 16 (1981) 151-164.

[LM] P. Lindqvist and J. Manfredi, *The Harnack inequality for $\infty$-harmonic functions*. Electron. J. Differential Equations 1995.

[LM1] P. Lindqvist and J. Manfredi, *Note on $\infty$-superharmonic functions.* Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 471-480.

[ME] N.G. Meyers, *An $\mathscr{L}^p$-estimate for the gradient of solutions of second order elliptic divergence equations* Ann. Sc. Norm. Sup. Pisa (3) 17 (1963), 189-206.

[MeE] N.G. Meyers and A. Elcrat, *Some results on regularity for solutions of nonlinear elliptic systems and quasiregular functions*, Duke Math.J. 42 (1975), 121-136.

[MM] L. Migliaccio and G. Moscariello *Higher integrability of* $\operatorname{div} - \operatorname{curl}$ *products*. Ricerche Mat. 49 (2000), no. 1, 151-161.

[M] C. Miranda, *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale a coefficienti non discontinui*, Ann. Mat. Pura Appl 63 (1963) 353-386.

[Mont] D. Montgomery and L. Zippin, *Topological Transformation groups*, Interscience, New York, (1995).

[MO] G. Moscariello, *On the integrability of the Jacobian in Orlicz Spaces*,Math.Japon. 40(1994), no.2, 323-329.

[M] S. Müller, *Higher integrability of determinants and weak convergence in $L^1$*, J. Reine Angew. Math 412(1990), 20-34.

[Mu] F. Murat,*Compacité par compensation*, Ann. Sc. Pisa, 5(1978), 489-507.

[PO] C. Pucci, *Un problema varizionale per i coefficienti di equazioni differenziali di tipo ellittico*, Annali Scuola Norm. Sup. Pisa 16 (1962) 159-172.

[P] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann.Mat.Pura Appl. 74 (1966) 15-30.

[P2] C. Pucci, *Equazioni ellittiche con soluzioni in $W^{2,p}$, $p < 2$*, Univ. Genova Pubbl. Ist.Mat. 173 (1967/1968) 35-45.

[PT] C. Pucci and G. Talenti, *Elliptic (second order) partial differential equations with measurable coefficients and approximating integral equations*, Advances in Math. 19,(1976),no 1,45-105.

[S] O. Savin, *$C^1$ regularity for infinity harmonic functions in two dimensions*, Preprint 2004.

[ST] E.M. Stein, *Note on the class $L\log L$*, Studia Math. 32(1969), 305-310.

[T] G. Talenti, *Sopra una classe di equazioni ellittiche a coefficienti misurabili*, Ann. Mat. Pura Appl. 69 (1965) 285-304.

[Ta] L. Tartar, *Compensated compactness and applications to partial differential equations*, Monlinear analysis and mechanics:Heriot-Watt Symposium, Vol.IV, Re.Notes in Math., 39, Pitman,Boston,London, (1979), 136-212.

[T] A. Torchinsky, ***Real variable methods in harmonic analysis***, Pure and Applied Mathematics, Vol. 123, Academic Press, Inc., (1986).

[W] H. Wente, *An existence theorem for surfaces of constant mean curvature*, J. Math. Analysis and Appl. 26,(1969), 318-344.

[Z] W.P. Ziemer, ***Weakly Differentiable Functions, Sobolev spaces and functions of bounded variation***, Springer-Verlag, (New York),(1989).