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Regularity results for asymptotic problems

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"Per tre cose vale la pena di vivere:
la matematica, la musica, l'amore"

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This work is dedicated to you.

Abstract

Elliptic and parabolic equations arise in the mathematical description of a wide variety of phenomena, not only in the natural science but also in engineering and economics. To mention few examples, consider problems arising in different contexts: gas dynamics, biological models, the pricing of assets in economics, composite media. The importance of these equations from the applications' point of view is equally interesting from that of analysis, since it requires the design of novel techniques to attack the always valid question of existence, uniqueness and regularity of solutions.
In particular, in recent years parabolic problems came more and more into the focus of mathematicians. Changing from elliptic to the parabolic case means physically to switch from the stationary to the non-stationary case, i.e. the time is introduced as an additional variable. Exactly this natural origin constitutes our interest in parabolic problems: they reflect our perception of space and time. Therefore they often can be used to model physical process, e.g. heat conduction or diffusion process.

In this thesis I will principally concentrate on the regularity properties of solutions of second order systems of partial differential equations in the elliptic and parabolic context. The outline of the thesis is as follows.

After giving some preliminary results, in the 3st Chapter we consider the parabolic analogue of some regularity results already known in the elliptic setting, concerning systems becoming parabolic only in an *asymptotic* sense. In the standard elliptic version, these results prove the *Lipschitz regularity* of solutions to elliptic systems of the type $\operatorname{div} a(Du) = 0$, with $u : \Omega \to \mathbb{R}^N$, under the main assumption that the vector field $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is *asymptotically* close, in C^1-sense, to some regular vector field b. Therefore, one can ask what happens when the vector field a is *asymptotically* close, in a C^0-sense, to the regular vector field $b(\xi) = \xi$. In this direction, in the parabolic framework, the first result obtained shows that the spatial gradient of u belongs to L^∞_{loc}.
The question that naturally arises is what happens in case of power $p \neq 2$, and more in general in case of general growth φ.
Regarding the general growth φ, in Chapter 4, we study variational integrals of the type

$$\mathcal{F}(u) := \int_\Omega f(Du)\,dx \qquad \text{for } u : \Omega \to \mathbb{R}^N$$

where Ω is an open bounded set in $\mathbb{R}^n$, $n \geq 2$, $N \geq 1$. Here $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a quasiconvex

continuous function satisfying a non-standard growth condition:

$$|f(z)| \leq C(1+\varphi(|z|)), \quad \forall z \in \mathbb{R}^{Nn},$$

where C is a positive constant and φ is a given N-function (see Section 2.4 for more details about Orlicz functions). Exhibiting an adequate notion of strict $W^{1,\varphi}$-quasiconvexity at infinity, which we call $W^{1,\varphi}$-asymptotic quasiconvexity, we prove a partial regularity result, namely that minimizers are *Lipschitz* continuous on an open and dense subset of Ω.
In the last Chapter we deal with the study of *local Lipschitz regularity* of weak solutions to non-linear second order parabolic systems of general growth

$$u_t^\beta - \sum_{i=1}^{n} (\mathcal{A}_i^\alpha(Du))_{x_i} = 0, \text{ in } \Omega_T := \Omega \times (-T, 0) \tag{0.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, $u : \Omega_T \to \mathbb{R}^N$, $N > 1$ and $\mathcal{A}$ is a tensor having general growth, that is $\mathcal{A}_i^\alpha(Du) = \dfrac{\varphi'(|Du|)}{|Du|} u_{x_i}^\alpha$, where φ is a given N-function.
Actually, having such result, as observed before, it is possible to prove the analogue of the first problem (studied in Chapter 3) in this case of nonstandard growth, considering an operator $\mathcal{A}$ that is *asymptotically* related to (0.0.1).

Contents

Chapter 1

Introduction

The study of partial differential equations started in the 18th century in the works of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. Partial differential equations play an important role to model natural phenomena; even more, they arise in every field of science. Consequently, the desire to understand the solutions of these equations has always had a prominent place in the efforts of mathematicians. One of the crucial moments was the year 1900 when David Hilbert formulated 23 unsolved mathematical problems of the century in his famous lecture at the International Congress of Mathematicians in Paris, one of them being the 19th:

Are the solutions of regular problems in the Calculus of Variations always necessarily analytic?

Such problem has been solved by Ennio De Giorgi. His result deals with a linear elliptic equation in divergence form:

$$\operatorname{div}(a_{ij}(x)D_j u) = 0 \quad \text{in } \Omega \tag{1.0.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and the coefficients $\{a_{ij}(x)\}$ are assumed to be measurable and such that

$$|a_{ij}(x)| \leq L \quad \text{and} \quad a_{ij}(x)\lambda_i\lambda_j \geq \nu|\lambda|^2 \tag{1.0.2}$$

for almost every $x \in \Omega$ and every $\lambda \in \mathbb{R}^n$, with $0 < \nu \leq L < \infty$. Equation (1.0.1) has to be interpreted in a weak sense: we assume that the integral equation

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\, D_i u\, D_j \phi\, dx = 0 \tag{1.0.3}$$

is satisfied for every $\phi \in C_c^\infty(\Omega)$.

Then we have:

Theorem 1.0.1 (De Giorgi [21])**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution to the equation (1.0.1) under the assumptions (1.0.2). Then there exists a positive number $\alpha = \alpha(n, \frac{L}{\nu}) > 0$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.*

John Nash [78] proved his results also for parabolic equations and few years later a different proof was given by Moser [77]. Each of these three proofs have advantages and drawbacks, but after more than fifty years we can certainly say that De Giorgi's proof is outstanding for its originality, its simplicity and for the many generalizations that were subsequently developed to deal with nonlinear elliptic operators, parabolic operators and of minima of variational integrals.
At the time De Giorgi published his paper, it was known, by Schauder estimates, that for an analytic integrand a solution $u \in C^{1,\alpha}(\Omega)$ to the equation (1.0.1) is necessarily analytic. Therefore, to solve the 19th problem it was sufficient to prove that the solution was $C^{1,\alpha}(\Omega)$. At this point the crucial observation is that by differentiating both sides of the equation one gets that the derivatives $D_i u$ of the solution solve a linear elliptic equation in divergence form with measurable coefficient. Thus to solve the 19th Hilbert problem it was enough to prove the Hölder continuity of weak solutions to (1.0.1).
Let us point out that the linearity of the equation (1.0.3) plays no role in the proof of Theorem 1.0.1, thus the result was extended to a vast class of general nonlinear elliptic equations in divergence form. More precisely, if we consider the following elliptic equation in divergence form

$$\operatorname{div} a(x, u, Du) = 0 \tag{1.0.4}$$

under the assumptions

$$|a(x, v, z)| \leq L(1 + |z|^p), \quad \langle a(x, v, z), z\rangle \geq \nu|z|^p - L \tag{1.0.5}$$

for every $x \in \Omega$, $v \in \mathbb{R}$, $z \in \mathbb{R}^n$ and $p > 1$, then Theorem 1.0.1 holds true.
Subsequently it was observed that for functionals of the type

$$\mathcal{F}(u) = \int_\Omega f(x, u, Du)\, dx,$$

whose associated Euler-Lagrange equation satisfies assumptions of the type (1.0.5), the Hölder regularity of minimizers follows if they are viewed as solutions to elliptic equations. In [50] Frehse, and then Giaquinta & Giusti [53], applied De Giorgi's method to minimizers in a direct way, that is without using the Euler-Lagrange equation, only considering the growth assumptions

$$\nu|z|^p \leq f(x, v, z) \leq L(1 + |z|^p). \tag{1.0.6}$$

In a number of important physical and geometrical situations u is not a scalar but a vector and the corresponding Euler-Lagrange equation is a system. The question arose naturally whether the previous theory extends to systems. In 1968 De Giorgi [22] constructed a surprising counterexample to prove that the regularity theorem does not extend to the vectorial case ($N > 1$): Let $n \geq 3$ and consider the following variational integral

$$\mathcal{F}(u) = \frac{1}{2}\int_{\mathcal{B}_1} |Du|^2 + \Big[\sum_{i,\alpha=1}^{n} \Big((n-2)\delta_{i\alpha} + n\frac{x_i x_\alpha}{|x|^2}\Big) D_\alpha u^i\Big]^2 dx. \tag{1.0.7}$$

Its Euler-Lagrange equation is

$$\int_{\mathcal{B}_1} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i dx = 0, \quad \forall \varphi \in W_0^{1,2}(\mathcal{B}_1, \mathbb{R}^n), \tag{1.0.8}$$

with

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \Big[(n-2)\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2}\Big]\Big[(n-2)\delta_{\beta j} + n\frac{x_j x_\beta}{|x|^2}\Big].$$

Here δ_{ij} denotes the usual Kronecker's symbol. Although these coefficients are bounded and satisfy the Legendre condition, the vector valued map

$$u(x) = \frac{x}{|x|^\gamma}, \quad \gamma := \frac{n}{2}\Big[1 - \frac{1}{\sqrt{(2n-2)^2+1}}\Big], \tag{1.0.9}$$

which belongs to $W^{1,2}(\mathcal{B}_1, \mathbb{R}^n)$ but is not bounded, is an extremal of $\mathcal{F}$, hence it satisfies the elliptic system with bounded coefficients (1.0.8).
The main point in De Giorgi's example is the singularity of the matrix $\{a_{ij}^{\alpha\beta}(x)\}$ at the origin. When the coefficients matrix depends on the solution, Giusti & Miranda [54] showed that the matrix $\{a_{ij}^{\alpha\beta}(u)\}$ can be even analytic.
These counterexamples show that everywhere regularity results for critical points or minimizers of regular variational integrals are in general not possible. So, we can ask:
What kind of regularity we can expect in the vectorial case?
Let us consider the variational integral

$$\mathcal{F}(u) = \int_\Omega f(Du)\, dx \tag{1.0.10}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $n \geq 2$, $u : \Omega \to \mathbb{R}^N$ and $N \geq 1$. It is well known that the convexity of $f(z)$ with respect to z is a sufficient condition for the sequential lower semicontinuity of $\mathcal{F}$, and therefore, when it is combined with the coercivity condition, the existence of a minimizer for $\mathcal{F}$ follows by the Direct Method of the Calculus of Variations. In general the convexity is a necessary condition only in the scalar case $N = 1$. In 1952 Morrey [76] showed that a necessary and sufficient condition for the weak lower semicontinuity of $\mathcal{F}$ is that f has to be quasiconvex. We say that a continuous function $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is quasiconvex if and only if

$$⨍_{\mathcal{B}_1} f(z_0 + D\xi)\, dx \geq f(z_0) \tag{1.0.11}$$

holds for every $z_0 \in \mathbb{R}^{Nn}$ and every smooth function $\xi : \mathcal{B}_1 \to \mathbb{R}^N$ with compact support in the unit ball $\mathcal{B}_1$ in $\mathbb{R}^n$.
Quasiconvexity is weaker than convexity if $N > 1$, while it reduces to convexity if $N = 1$. Note that it is a global condition; if f is of class C^2 in z, it implies the pointwise Legendre-Hadamard condition:

$$f_{z_i^\alpha z_j^\beta}(z)\xi^\alpha\xi^\beta\lambda_i\lambda_j \geq 0 \quad \forall \xi \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}^N.$$

In order to study the regularity of minimizers, it is natural to strengthen (1.0.11), so we introduce the notion of uniformly strictly quasiconvex function. A continuous function $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is said to be uniformly strictly quasiconvex if and only if

$$\fint_{\mathcal{B}_1} f(z_0 + D\xi)\, dx \geq f(z_0) + \nu \fint_{\mathcal{B}_1} (1 + |z_0|^2 + |D\xi|^2)^{\frac{p-2}{2}} |D\xi|^2 dx \tag{1.0.12}$$

holds for every $z_0 \in \mathbb{R}^{Nn}$ and every smooth function $\xi : \mathcal{B}_1 \to \mathbb{R}^N$ with compact support in the unit ball $\mathcal{B}_1$ in $\mathbb{R}^n$.
In 1986 Evans [43], adapting the indirect approach in [51], established the first partial regularity result for minimizers of (1.0.10). More precisely, he considered uniformly strictly quasiconvex integrand f in the quadratic case, and proved that if f is of class C^2 with bounded second derivatives, then there exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{Nn})$ for any $\alpha \in (0, 1)$. This result was generalized by Acerbi & Fusco [2] (see also [15] for the subquadratic case).

On the other hand it could be interesting to identify classes of functionals for which everywhere $C^{1,\alpha}$-regularity of minimizers occurs. A well known result of Uhlenbeck [88] states that the $C^{1,\alpha}$-regularity holds for minimizers if the integrand f is of the type $f(|z|)$, for a convex function f of p-growth, with $p \geq 2$. In [17] Chipot & Evans proved the local Lipschitz regularity for minimizers of (1.0.10) under the main assumption that these functionals become appropriately convex and quadratic at infinity. The heuristic idea is that wherever the gradient of the minimizer is very large, the Euler-Lagrange equations become elliptic and practically linear, so that good estimates are then available. Subsequently Giaquinta & Modica [55] (see also [81]) obtained an analogous result for integrands with superquadratic growth (for the subquadratic case we refer to [67]).
More recent contributions include the works [44] and [49] where the authors use various asymptotic relatedness condition in the context of proving global Lipschitz regularity of minimizers to certain functionals. In addition, in [83, 84] the author have recently produced several results for problems involving asymptotic relatedness conditions; in particular they have shown higher integrability in the case of relatively general functionals, and partial Lipschitz regularity in the case of functionals where the integrand functions depend solely on the gradient of the minimizer. Finally, in [31] the authors established optimal local regularity results for vector-valued extremals and minimizers of variational integrals: the optimality is illustrated by explicit examples showing that, in the non convex case, minimizers need not be locally Lipschitz. This is in contrast to the convex case, where the authors show that extremals are locally Lipschitz continuous.
The regularity of minimizers for the functional (1.0.10) has been intensively studied [44, 28, 29] also when the integrand f behaves asymptotically like a convex, radial Orlicz function φ with growth and coercivity conditions of the type

$$\varphi(|z|) \leq f(z) \leq L(1 + \varphi(|z|)) \tag{1.0.13}$$

(see Section 2.4 for the properties of φ). Let us point out that functionals naturally defined in Orlicz spaces are an important class of functionals of (p, q)-type (see [69, 70, 71, 72]).

The regularity theory for parabolic systems is, to a certain extent, very similar to the elliptic one described above. As already observed, in the general vectorial case only partial regularity results are available, provided that suitable assumptions on growth and regularity of the vector field are satisfied. Partial regularity of solutions has been proved for quasi-linear systems [56, 57, 63, 86], for non-linear systems the regularity theory was developed mainly assuming special structure on the operator (see [5, 6, 74]) or assuming that solutions were a priori more regular, i.e. bounded or even Hölder continuous (see [75]). Let us point out that everywhere regularity is possible only under very special (diagonal type) structures, as for instance in the case of the p-Laplacian system [23, 75], otherwise it fails in general, as shown by counterexamples [87]. The minimal assumptions under which a complete study of regularity properties for non a priori regular weak solutions of parabolic systems with linear growth were considered in [38], where the authors consider a continuous differentiable field with uniformly bounded second derivatives.
As far as the asymptotic framework is concerned, in [66] (see also [8]) the authors considered parabolic problems of the type

$$u_t - \operatorname{div}(\gamma(x,t)a(Du)) = -\operatorname{div} f(x,t) \quad (x,t) \in \Omega \times (-T,0) =: \Omega_T, \tag{1.0.14}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$ and u maps Ω_T into $\mathbb{R}^N$. Under appropriate assumptions on the functions γ, a, f they established the local boundedness of the spatial gradient of solutions to systems which are not everywhere parabolic, but, as before, become parabolic only in an asymptotic sense.
In this context we can insert the first result contained in this thesis. More precisely, we study nonlinear parabolic systems of the type (1.0.14). The main assumptions on the vector field $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ are:
($\mathcal{H}1$) a is a continuous map;
($\mathcal{H}2$) there exist constants L and m such that

$$|a(\xi) - a(\eta)| \leq L|\xi - \eta| \tag{1.0.15}$$

for all $\xi, \eta \in \mathbb{R}^{Nn}$ such that $|\xi| + |\eta| \geq m$;
($\mathcal{H}3$) there exists $\varepsilon > 0$ such that a satisfies the coercivity condition

$$\langle a(\xi) - a(\eta), \xi - \eta \rangle \geq L(1-\varepsilon)|\xi - \eta|^2 \tag{1.0.16}$$

for all $\xi, \eta \in \mathbb{R}^{Nn}$ such that $|\xi| + |\eta| \geq m$.
The key assumption below is that the constant ε is small so that the constant in the coercivity inequality is close to the Lipschitz constant of a. These results can be interpreted as perturbations of classical regularity results for elliptic systems in Campanato spaces (see [11]). The notion of weak solution adopted prescribes that a map

$$u \in C^0(-T,0; L^2(\Omega,\mathbb{R}^N)) \cap L^p(-T,0; W^{1,p}(\Omega,\mathbb{R}^N)), \quad N \geq 1 \tag{1.0.17}$$

is a weak solution to (1.0.14), for $p \geq 2$, if and only if

$$\int_{\Omega_T} -u\varphi_t + \langle \gamma(x,t)a(Du), D\varphi \rangle dx\, dt = \int_{\Omega_T} \langle f, D\varphi \rangle dx\, dt \tag{1.0.18}$$

holds whenever $\varphi \in W_0^{1,2}(\Omega_T, \mathbb{R}^N)$.
We are able to prove two types of regularity results for weak solutions u to (1.0.14). The first one concerns the BMO regularity of Du under suitable assumptions (see Chapter 3). We have:

Theorem 1.0.2. *Let a satisfying the assumptions* $(\mathcal{H}1)-(\mathcal{H}5)$*. Then there exist an* $\varepsilon_0 = \varepsilon_0(n, L) \in (0,1)$ *and a constant* $M = M(a)$ *such that: if* $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in W^{1,2}_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ *is a weak solution of the system (1.0.14) in* Ω_T*, then* $Du \in \text{BMO}_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn})$ *and there exists a constant* $C = C(n, L, \text{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1}))$ *such that*

$$[Du]_{2,n;\Omega_{t_2}} \le C\left(M + [f]_{2,n;\Omega_{t_1}} + \|Du\|_{L^2(\Omega_{t_1})}\right), \tag{1.0.19}$$

where $\Omega_{t_2} \Subset \Omega_{t_1} \Subset \Omega_T$ *are open domains.*

The proof of the BMO bound is based on the fact that a can be written as a perturbation of the identity, $a(\xi) = \xi + b(\xi)$, where $b(\cdot)$ is a bounded function. Moreover, if the function a is a perturbation of the identity with a function of the gradient that has a sufficiently small Lipschitz constant outside of a large ball, $a(\xi) = \xi + e(\xi)$, then this estimate can be improved to an L^∞-bound.

Theorem 1.0.3. *Assume that a satisfies the conditions* $(\mathcal{H}1) - (\mathcal{H}3)$ *and* $(\mathcal{H}4')$*. Then we can find an* $\varepsilon_0 = \varepsilon_0(n, L) \in (0,1)$*, a constant* $M = M(a)$ *and two constants* c_1 *and* c_2 *depending only on* n *and* $\text{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1})$*, such that if* $u \in W^{1,2}_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ *is a weak solution of*

$$u_t - \text{div}(\gamma(x,t)a(Du)) = -\,\text{div}\, f(x,t) \quad \text{in } \Omega_T, \tag{1.0.20}$$

then $u \in W^{1,\infty}_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ *and for all* $\Omega_{t_2} \Subset \Omega_{t_1}$ *holds*

$$\text{esssup}_{\Omega_{t_2}} |Du| \le c_1 \left(M^2 + \int_{\Omega_{t_1}} |Du|^2 dxdt\right)^{\frac{1}{2}} + c_2.$$

As already observed, the question that naturally arise is what happens in case of power $p \neq 2$, and more in general in case of general growth φ.

Concerning the Orlicz setting, in the Chapter 4 we study variational integrals of the type (1.0.10) where $u : \Omega \to \mathbb{R}^N$, Ω is an open bounded set in $\mathbb{R}^n$, $n \ge 2$ and $N \ge 1$. Here f is a continuous function satisfying a φ-growth condition:

$$|f(z)| \le C(1 + \varphi(|z|)), \quad \forall z \in \mathbb{R}^{Nn},$$

where C is a positive constant and φ is a given N-function.
In order to treat the general growth case, we consider the notion of strictly $W^{1,\varphi}$-quasiconvexity introduced in [27].

Definition 1.0.1 (Strict $W^{1,\varphi}$-quasiconvexity)**.** *A continuous function* $f : \mathbb{R}^{Nn} \to \mathbb{R}$ *is said to be strictly* $W^{1,\varphi}$*-quasiconvex if there exists a positive constant* $k > 0$ *such that*

$$\fint_{\mathcal{B}_1} f(z + D\xi)\,dx \ge f(z) + k \fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\,dx$$

for all $\xi \in C^1_0(\mathcal{B}_1)$*, for all* $z \in \mathbb{R}^{Nn}$*, where* $\varphi_a(t) \sim t^2\varphi''(a+t)$ *for* $a, t \ge 0$*.*

We will exploit an adequate notion of strict $W^{1,\varphi}$-quasiconvexity at infinity, which we will call $W^{1,\varphi}$-asymptotic quasiconvexity:

Definition 1.0.2 (Asymptotic $W^{1,\varphi}$-quasiconvexity)**.** *A function $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is asymptotically $W^{1,\varphi}$-quasiconvex if there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that*

$$f(z) = g(z) \ \text{ for } |z| > M.$$

After establishing several characterizations of the notion of asymptotic $W^{1,\varphi}$-quasiconvexity (see Theorem 4.2.1) we will prove the following result.

Theorem 1.0.4. *Let $z_0 \in \mathbb{R}^n$ with $|z_0| > M + 1$, let k be a positive constant so that*

$$\int_{\mathcal{B}_\rho(x_0)} [f(z + D\xi) - f(z)]\, dx \geq \frac{k}{2} \int_{\mathcal{B}_\rho(x_0)} \varphi_{|z_0|}(|D\xi|)\, dx \tag{1.0.21}$$

holds for all $\xi \in C^1_c(\mathcal{B}, \mathbb{R}^N)$, and let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be a minimizer of $\mathcal{F}$. If for some $x_0 \in \Omega$

$$\lim_{r\to 0} \fint_{\mathcal{B}_r(x_0)} |V(Du) - V(z_0)|^2 = 0, \tag{1.0.22}$$

where $V(z) = \sqrt{\frac{\varphi'(|z|)}{|z|}}z$, for all $z \in \mathbb{R}^{Nn} \setminus \{0\}$, then in a neighborhood of x_0 the minimizer u is $C^{1,\overline{\alpha}}$ for some $\overline{\alpha} < 1$.

In order to achieve this regularity result, we have to prove an *excess decay estimate*, where the excess function is defined by

$$\mathbb{E}(\mathcal{B}_R(x_0), u) = \fint_{\mathcal{B}_R(x_0)} |V(Du) - (V(Du))_{\mathcal{B}_R(x_0)}|^2 dx.$$

In the power case the main idea is to use a blow-up argument based strongly on the homogeneity of $\varphi(t) = t^p$. Here we have to face with the lack of the homogeneity since the general growth condition. Thus one makes use of the so-called $\mathcal{A}$-harmonic approximation proved in [27] (see also [85, 34, 35, 37, 39] for the power case). Such tool allows us to compare the solutions of our problem with the solution of the regular one in terms of the closeness of the gradient.
Moreover we will prove that minimizers of $\mathcal{F}$ are Lipschitz continuous on an open and dense subset of Ω.
More precisely we define the set of regular points $\mathcal{R}(u)$ by

$$\mathcal{R}(u) = \{x \in \Omega : u \text{ is Lipschitz near } x\},$$

following that $\mathcal{R}(u) \subset \Omega$ is open.

Corollary 1.0.1. *Assume that f satisfies $(\mathcal{H}1) - (\mathcal{H}5)$. Then, for every minimizer $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$, the regular set $\mathcal{R}(u)$ is dense in Ω.*

We remark that a counterexample [83] shows that it is not possible to establish regularity outside a negligible set (which would be the natural thing in the vectorial regularity theory). So, our regularity result generalizes the ones given in [83] and [16] for integrands with a power growth condition which become strictly convex and strictly quasiconvex near infinity, respectively.

The last Chapter of the thesis deals with a recent problem I am facing with. It concerns the local Lipschitz regularity of weak solutions to non-linear second order parabolic systems of general growth

$$u_t^\beta - \sum_{i=1}^{n} (\mathcal{A}_i^\alpha(Du))_{x_i} = 0, \text{ in } \Omega_T := \Omega \times (-T, 0) \tag{1.0.23}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, $u : \Omega_T \to \mathbb{R}^N$, $N > 1$ and $\mathcal{A}$ is a tensor having certain Orlicz-type growth that generalize p-growth.

In particular we focus on $\mathcal{A}_i^\alpha(Du) = \dfrac{\varphi'(|Du|)}{|Du|} u_{x_i}^\alpha$, where φ is a given N-function.

In the model case $\varphi(s) = s^p$, for some $p > 1$, (1.0.23) gives the evolutionary p-Laplacian. This reveals that (1.0.23) is a natural generalization of the p-Laplacian. Under suitable hypotheses (see Chapter 5), by using a Moser type iteration for systems with general growth conditions, we prove the local Lipschitz regularity of the spatial gradient of solutions to (1.0.23). More precisely:

Theorem 1.0.5. *Let u be a weak solution to (1.0.23). Then $Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn})$. Moreover for every $\mathcal{Q}_{R_0} \Subset \Omega_T$ the following a priori estimate holds with the constant c depending on n and on the characteristic of φ*

$$\sup_{\mathcal{Q}_{\frac{R_0}{2}}} |Du|^2 \leq c \Big(\int_{\mathcal{Q}_{R_0}} \varphi(|Du|)\, dz \Big)^{1+\frac{2}{n}} + c.$$

Finally, let me observe again that having such result, it is possible to prove the analogue of the first problem in the case of nonstandard growth, considering an operator $\mathcal{A}$ that is asymptotically closed to (1.0.23).

The content of the Chapter 3, 4, and 5 corresponds to the papers [58], [59] and [60].

Chapter 2

Preliminaries

This chapter is devoted to a brief exposition of the theory of function spaces that provide the analytic framework for the study of PDEs. There are the Morrey and Campanato's spaces, and the Orlicz's space.

2.1 Notation

We start with some remarks on the notation used throughout the whole work. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; with $x_0 \in \mathbb{R}^n$, we set

$$\mathcal{B}_r(x_0) \equiv \mathcal{B}(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

the open ball of $\mathbb{R}^n$ with radius $r > 0$ and center x_0.
In the following Ω_T will denote the parabolic cylinder $\Omega \times (-T, 0)$, where $T > 0$. If $z \in \Omega_T$, we denote $z = (x, t)$ with $x \in \Omega$ and $t \in (-T, 0)$. When dealing with parabolic regularity, the geometry of cylinders plays an important role. We shall deal with parabolic cylinder with vertex (x_0, t_0) and width $r > 0$ given by

$$\mathcal{Q}_r(x_0, t_0) := \mathcal{B}(x_0, r) \times (t_0 - r^2, t_0).$$

We also consider cylinders with width magnified of a factor $\delta > 0$:

$$\mathcal{Q}_{\delta r}(x_0, t_0) = \mathcal{B}(x_0, \delta r) \times (t_0 - \delta^2 r^2, t_0).$$

given a cylinder $\mathcal{Q} = \mathcal{B} \times (s, t)$, its parabolic boundary is

$$\partial_{\mathcal{P}} \mathcal{Q} := (\mathcal{B} \times \{s\}) \cup (\partial \mathcal{B} \times [s, t]).$$

The parabolic metric is defined as usual by

$$\mathrm{dist}_{\mathcal{P}}(z, z_0) := \sqrt{|x - x_0|^2 + |t - t_0|}$$

whenever $z = (x,t), z_0 = (x_0,t_0) \in \mathbb{R}^{n+1}$.
A function $u : \Omega \to \mathbb{R}^N$ is called Hölder continuous with exponent α on Ω if there exists a constant $c \in (0,+\infty)$ such that for all points $x, y \in \Omega$ the estimate

$$|u(x) - u(y)| \leq c|x-y|^\alpha$$

is satisfied. The Hölder seminorm of u is defined by

$$[u]_{C^{0,\alpha}(\Omega,\mathbb{R}^N)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

The Hölder space $C^{k,\alpha}(\Omega,\mathbb{R}^N)$ consists of all functions $u \in C^k(\Omega,\mathbb{R}^N)$ for which the norm

$$\|u\|_{C^{k,\alpha}(\Omega,\mathbb{R}^N)} := \sum_{|\beta|\leq k} \sup_{x\in\Omega} |D^\beta u(x)| + \sum_{|\beta|=k} [D^\beta u]_{C^{0,\alpha}(\Omega,\mathbb{R}^N)}$$

is finite. Here $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$ denotes a multiindex of lenght $|\beta| = \beta_1 + \cdots + \beta_n$.
The Sobolev space $W^{k,p}(\Omega,\mathbb{R}^N)$ is given by

$$W^{k,p}(\Omega,\mathbb{R}^N) := \{u \in L^p(\Omega,\mathbb{R}^N) : D^\beta u \in L^p(\Omega,\mathbb{R}^N) \quad \forall|\beta| \leq k\},$$

where $D^\beta u$ is the weak derivative of u. Moreover by $W_0^{k,p}(\Omega,\mathbb{R}^N)$ we denote the closure of $C_c^\infty(\Omega,\mathbb{R}^N)$ in the space $W^{k,p}(\Omega,\mathbb{R}^N)$.
The integral average of a function u on $\mathbb{X} \subset \mathbb{R}^n$ measurable subset with positive measure is given by

$$(u)_{\mathbb{X}} = \fint_{\mathbb{X}} u(x)dz := \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} u(x)dz$$

where $|\mathbb{X}|$ is the n-dimensional Lebesgue measure of $\mathbb{X}$.

2.2 Parabolic spaces

We introduce spaces of functions that exhibit different regularity in the space and time variables.
Let $p, q \geq 1$. A function f defined and measurable in Ω_T belongs to $L^{p,q}(\Omega_T) \equiv L^q(-T,0;L^p(\Omega))$ if

$$\|f\|_{L^{p,q}(\Omega_T)} := \left(\int_{-T}^0 \left(\int_\Omega |f|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty.$$

Also $f \in L^{p,q}_{\mathrm{loc}}(\Omega_T)$ if for every compact subset K of Ω and every subinterval $[t_1,t_2] \subset (-T,0)$

$$\left(\int_{t_1}^{t_2} \left(\int_K |f|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty.$$

Whenever $p = q$ we set $L^{p,q}(\Omega_T) \equiv L^p(\Omega_T)$, $L^{p,q}_{\mathrm{loc}}(\Omega_T) \equiv L^p_{\mathrm{loc}}(\Omega_T)$ and $\|f\|_{L^{p,q}(\Omega_T)} \equiv \|f\|_{L^p(\Omega_T)}$.
Let us consider the Banach spaces

$$V^{p,q}(\Omega_T) \equiv L^\infty(-T, 0; L^p(\Omega)) \cap L^q(-T, 0; W^{1,q}(\Omega))$$

and

$$V^{p,q}_0(\Omega_T) \equiv L^\infty(-T, 0; L^p(\Omega)) \cap L^q(-T, 0; W^{1,q}_0(\Omega))$$

both equipped with the norm, $u \in V^{p,q}(\Omega_T)$,

$$\|u\|_{V^{p,q}(\Omega_T)} \equiv \mathrm{esssup}_{-T<t<0} \|v(\cdot, t)\|_{L^p(\Omega)} + \|Du\|_{L^q(\Omega_T)}.$$

When $p = q$ we set $V^{p,p}(\Omega_T) \equiv V^p(\Omega_T)$ and $V^{p,p}_0(\Omega_T) \equiv V^p_0(\Omega_T)$. Both spaces are embedded in $L^r(\Omega_T)$ for some $r > p$.

2.3 Morrey and Campanato spaces

In the sequel we will use the Morrey and Campanato spaces.
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the following property: there exists a constant $A > 0$ such that for all $x_0 \in \Omega$, $\rho < \operatorname{diam}\Omega$ we have

$$|\mathcal{B}_\rho(x_0) \cap \Omega| \geq A\rho^n.$$

Let $p \geq 1$, $\lambda \geq 0$.

Definition 2.3.1. *The Morrey space $L^{p,\lambda}(\Omega, \mathbb{R}^N)$ is the subspace of all functions u in $L^p(\Omega, \mathbb{R}^N)$ satisfying*

$$\|u\|^p_{L^{p,\lambda}(\Omega,\mathbb{R}^N)} := \sup_{\substack{x_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\mathcal{B}_\rho(x_0)\cap\Omega} |u|^p dx < \infty. \tag{2.3.1}$$

It is clear that condition (2.3.1) only depends on the behavior for small radii, i.e. we can fix $\rho_0 > 0$ and replace the definition of $\|u\|^p_{L^{p,\lambda}(\Omega,\mathbb{R}^N)}$ with

$$\sup_{\substack{x_0\in\Omega \\ 0<\rho<\rho_0}} \int_{\mathcal{B}_\rho(x_0)\cap\Omega} |u|^p dx.$$

It is easily seen that $\|u\|^p_{L^{p,\lambda}(\Omega,\mathbb{R}^N)}$ is a norm, and that the space $L^{p,\lambda}(\Omega, \mathbb{R}^N)$ is complete.

Definition 2.3.2. *We denote by $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^N)$ the Campanato space of all functions u in $L^p(\Omega, \mathbb{R}^N)$ such that*

$$[u]^p_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)} := \sup_{\substack{x_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\mathcal{B}_\rho(x_0)\cap\Omega} |u - (u)_{\mathcal{B}_\rho(x_0)\cap\Omega}|^p dx < \infty. \tag{2.3.2}$$

The quantity $[u]_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)}$ is a seminorm in $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)$, equivalent to

$$\sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \inf_{\xi\in\mathbb{R}^N} \int_{\mathcal{B}_\rho(x_0)\cap\Omega} |u-\xi|^p dx.$$

Equipped with the norm $\|\cdot\|_{L^{p,\lambda}(\Omega,\mathbb{R}^N)}$ defined in (2.3.1) the Morrey space $L^{p,\lambda}(\Omega,\mathbb{R}^N)$ is a Banach spaces for all $p\geq 1$ and $\lambda\geq 0$. Furthermore, the Campanato space $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)$ is a Banach spaces endowed with the norm $\|\cdot\|_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)} = \|\cdot\|_{L^p(\Omega,\mathbb{R}^N)} + [\cdot]_{L^{p,\lambda}(\Omega,\mathbb{R}^N)}$.

Proposition 2.3.1. *For $0\leq\lambda<n$ we have $L^{p,\lambda}(\Omega)\cong\mathcal{L}^{p,\lambda}(\Omega)$.*

The Campanato space $\mathcal{L}^{1,n}(\Omega,\mathbb{R}^N)$ has a special role and is usually denoted by $\mathrm{BMO}(\Omega,\mathrm{R}^\mathrm{N})$, the abbreviation for bounded mean oscillation. It is smaller than any Lebesgue space $L^p(\Omega,\mathbb{R}^N)$ with $p<\infty$ but still containing $L^\infty(\Omega,\mathbb{R}^N)$ as a strict consequence. Furthermore, the spaces $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)$ with $n<\lambda\leq n+p$ are known as the integral characterization of Hölder continuity functions (see [11]):

Theorem 2.3.1 (Campanato)**.** *For $n<\lambda\leq n+p$ and $\alpha=\frac{\lambda-n}{p}$ we have $\mathcal{L}^{p,\lambda}(\Omega)\cong C^{0,\alpha}(\overline{\Omega})$. Moreover the seminorm*

$$[u]_{C^{0,\alpha}(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

is equivalent to $[u]_{\mathcal{L}^{p,\lambda}(\Omega)}$.
If $\lambda>n+p$ and $u\in\mathcal{L}^{p,\lambda}(\Omega)$, then u is constant.

2.4 Orlicz spaces

The following definitions and results are standard in the context of N-functions (see [82]).

Definition 2.4.1. *A real function $\varphi:[0,\infty)\to[0,\infty)$ is said to be an N-function if $\varphi(0)=0$ and there exists a right continuous nondecreasing derivative φ' satisfying $\varphi'(0)=0$, $\varphi'(t)>0$ for $t>0$ and $\lim\limits_{t\to\infty}\varphi'(t)=\infty$. Especially φ is convex.*

The concept of N-function generalizes the power function $\varphi(t)=\frac{1}{p}t^p$. Now, let us generalize its Hölder conjugate $\frac{1}{q}t^q$, $q=\frac{p}{p-1}$. To this end, for a non-decreasing real function φ let us denote with φ^{-1} its generalized right-continuous inverse, given by $\varphi^{-1}(t):=\sup\{s\in[0,\infty):\varphi(s)\leq t\}$. Now we introduce

Definition 2.4.2. *A complementary function to an N-function φ is*

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds.$$

Moreover φ^* is again an N-function and for $t > 0$ it results $(\varphi^*)'(t) = (\varphi')^{-1}(t)$. Note that $\varphi^*(t) = \sup_{s\geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$.
The assumption widely used in order to study regularity for systems with Orlicz growths is the following

Definition 2.4.3. *We say that an N-function φ satisfies the Δ_2-condition (we shall write $\varphi \in \Delta_2$) if there exists a positive constant C such that*

$$\varphi(2t) \leq C\,\varphi(t) \quad \text{for all } t \geq 0.$$

We denote the smallest possible constant by $\Delta_2(\varphi)$.

We shall say that two real functions φ_1 and φ_2 are *equivalent* and write $\varphi_1 \sim \varphi_2$ if there exist constants $c_1, c_2 > 0$ such that $c_1\varphi_1(t) \leq \varphi_2(t) \leq c_2\varphi_1(t)$ if $t \geq 0$.
Since $\varphi(t) \leq \varphi(2t)$ the Δ_2-condition implies $\varphi(2t) \sim \varphi(t)$. Moreover if φ is a function satisfying the Δ_2-condition, then $\varphi(t) \sim \varphi(at)$ uniformly in $t \geq 0$ for any fixed $a > 1$. Let us also note that, if φ satisfies the Δ_2-condition, then any N-function which is equivalent to φ satisfies this condition too.
If φ, φ^* satisfy the Δ_2-condition we will write that $\Delta_2(\varphi, \varphi^*) < \infty$. Assume that $\Delta_2(\varphi, \varphi^*) < \infty$. Then for all $\delta > 0$ there exists c_δ depending only on $\Delta_2(\varphi, \varphi^*)$ such that for all $s, t \geq 0$ it holds that

$$t\,s \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(s).$$

This inequality is called Young's inequality. For all $t \geq 0$

$$\begin{aligned} & t \leq \varphi^{-1}(t)(\varphi^*)^{-1}(t) \leq 2t \\ & \frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\varphi'(t) \\ & \varphi\Big(\frac{\varphi^*(t)}{t}\Big) \leq \varphi^*(t) \leq \varphi\Big(\frac{2\varphi^*(t)}{t}\Big). \end{aligned}$$

Therefore, uniformly in $t \geq 0$,

$$\varphi(t) \sim t\varphi'(t), \quad \varphi^*(\varphi'(t)) \sim \varphi(t), \tag{2.4.1}$$

where constants depend only on $\Delta_2(\varphi, \varphi^*)$.

Definition 2.4.4. *We say that an N-function φ is of type (p_0, p_1) with $1 \leq p_0 \leq p_1 < \infty$ if*

$$\varphi(st) \leq C \max\{s^{p_0}, s^{p_1}\}\varphi(t) \quad \forall s, t \geq 0. \tag{2.4.2}$$

The following Lemma can be found in [27] (see Lemma 5).

Lemma 2.4.1. *Let φ be an N-function with $\varphi \in \Delta_2$ together with its conjugate. Then φ is of type (p_0, p_1) with $1 < p_0 < p_1 < \infty$ where p_0 and p_1 and the constant C depend only on $\Delta_2(\varphi, \varphi^*)$.*

If φ is an N-function satisfying the Δ_2-condition, by $L^{\varphi}(\Omega)$ and $W^{1,\varphi}(\Omega)$ we denote the classical Orlicz and Orlicz-Sobolev spaces, i.e. $u \in L^{\varphi}(\Omega)$ if and only if $\int_\Omega \varphi(|u|)\,dx < \infty$ and $u \in W^{1,\varphi}(\Omega)$ if and only if $u, Du \in L^{\varphi}(\Omega)$. The Luxembourg norm is defined as follows:

$$\|u\|_{L^{\varphi}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}.$$

With this norm $L^{\varphi}(\Omega)$ is a Banach space.
Moreover, we denote by $W_0^{1,\varphi}(\Omega)$ the closure of $C_c^{\infty}(\Omega)$ functions with respect to the norm

$$\|u\|_{W^{1,\varphi}(\Omega)} = \|u\|_{L^{\varphi}(\Omega)} + \|Du\|_{L^{\varphi}(\Omega)}$$

and by $W^{-1,\varphi}(\Omega)$ its dual.
Throughout this thesis we will assume that φ satisfies the following assumption.

Assumption 2.4.1. *Let φ be an N-function such that φ is $C^1([0,+\infty))$ and $C^2(0,+\infty)$. Further assume that*

$$\varphi'(t) \sim t\varphi''(t). \tag{2.4.3}$$

We remark that under this assumption $\Delta_2(\varphi, \varphi^*) < \infty$ will be automatically satisfied, where $\Delta_2(\varphi, \varphi^*)$ depends only on the characteristics of φ.
For given φ we define the associated N-function ψ by

$$\psi'(t) = \sqrt{t\varphi'(t)}.$$

Note that

$$\psi''(t) = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\sqrt{\frac{\varphi'(t)}{t}} = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\frac{\psi'(t)}{t}.$$

It is shown in [25] (see Lemma 25) that if φ satisfies Assumption 2.4.1 then also φ^*, ψ and ψ^* satisfy Assumption 2.4.1 and $\psi''(t) \sim \sqrt{\varphi''(t)}$.
We define tensors $\mathcal{A}$ and $\mathcal{V}$ in the following way

$$\begin{aligned} \mathcal{A}(z) &= D\Phi(z) \\ \mathcal{V}(z) &= D\Psi(z), \end{aligned} \tag{2.4.4}$$

where $\Phi(z) := \varphi(|z|)$ and $\Psi(z) := \psi(|z|)$.
Connections between the tensors $\mathcal{A}$ and a N-function φ are given by the the following lemma (see [25] Lemma 21).

Lemma 2.4.2. *Let φ satisfying Assumption 2.4.1, then $\mathcal{A}(z) = \varphi'(|z|)\frac{z}{|z|}$ for $z \neq 0$, $\mathcal{A}(0) = 0$ and $\mathcal{A}$ satisfies*

$$|\,\mathcal{A}(z_1) - \mathcal{A}(z_2)| \leq c\varphi''(|z_1| + |z_2|)|z_1 - z_2| \tag{2.4.5}$$

$$(\mathcal{A}(z_1) - \mathcal{A}(z_2), z_1 - z_2) \geq C\varphi''(|z_1| + |z_2|)|z_1 - z_2|^2, \tag{2.4.6}$$

for $z_1, z_2 \in \mathbb{R}^{Nn}$.

The same conclusions of Lemma 2.4.2 holds with $\mathcal{A}$ and φ replaced by $\mathcal{V}$ and ψ.
Now, let us consider a family of N-functions $\{\varphi_a\}_{a\geq 0}$ setting, for $t \geq 0$,

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\, ds \quad \text{with} \quad \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}.$$

The following lemma can be found in [25] (see Lemma 23 and Lemma 26).

Lemma 2.4.3. *Let φ be an N-function with $\varphi \in \Delta_2$ together with its conjugate. Then for all $a \geq 0$ the function φ_a is an N-function and $\{\varphi_a\}_{a\geq 0}$ and $\{(\varphi_a)^*\}_{a\geq 0} \sim \{\varphi^*_{\varphi'(a)}\}_{a\geq 0}$ satisfy the Δ_2 condition uniformly in $a \geq 0$.*

Let us observe that by the previous lemma $\varphi_a(t) \sim t\varphi_a'(t)$. Moreover, for $t \geq a$ we have $\varphi_a(t) \sim \varphi(t)$ and for $t \leq a$ we have $\varphi_a(t) \sim t^2\varphi''(a)$. This implies that $\varphi_a(st) \leq cs^2\varphi_a(t)$ for all $s \in [0,1]$, $a \geq 0$ and $t \in [0,a]$.
The following lemmas can be found in [25] (see Lemma 24 and Lemma 3).

Lemma 2.4.4. *Let φ satisfy Assumption 2.4.1. Then, uniformly in $z_1, z_2 \in \mathbb{R}^n$, $|z_1| + |z_2| > 0$*

$$\begin{aligned}
\varphi''(|z_1|+|z_2|)|z_1-z_2| &\sim \varphi'_{|z_1|}(|z_1-z_2|),\\
\varphi''(|z_1|+|z_2|)|z_1-z_2|^2 &\sim \varphi_{|z_1|}(|z_1-z_2|).
\end{aligned}$$

The following result show how one can interchangeably use $\mathcal{A}$, $\mathcal{V}$ and φ_a.

Lemma 2.4.5. *Let φ satisfy Assumption 2.4.1 and let $\mathcal{A}$ and $\mathcal{V}$ be defined by (2.4.4). Then, uniformly in $z_1, z_2 \in \mathbb{R}^{Nn}$,*

$$\langle \mathcal{A}(z_1) - \mathcal{A}(z_2), z_1 - z_2\rangle \sim |\mathcal{V}(z_1) - \mathcal{V}(z_2)|^2 \sim \varphi_{|z_1|}(|z_1 - z_2|),$$

and

$$|\,\mathcal{A}(z_1) - \mathcal{A}(z_2)| \sim \varphi'_{|z_1|}(|z_1 - z_2|).$$

Moreover

$$\begin{aligned}
\langle \mathcal{A}(z_1), z_1\rangle &\sim |\mathcal{V}(z_1)|^2 \sim \varphi(|z_1|),\\
|\,\mathcal{A}(z_1)| &\sim \varphi'(|z_1|),
\end{aligned}$$

uniformly in $z_1 \in \mathbb{R}^{Nn}$.

2.5 The method of $\mathcal{A}$-harmonic approximation

In this section we present the $\mathcal{A}$-harmonic approximation technique which is inspired by Simon's proof of the regularity theorem of Allard and which extends the method of harmonic approximation in a natural way to bounded elliptic operators with constant coefficients. In the partial regularity theory this approach was first implemented by Duzaar

& Grotowsky [34] and since then it has been applied to various situations concerning partial regularity of solutions to elliptic and parabolic problems.
Let us consider

$$\operatorname{div}(a(Du)) = 0 \quad \text{in } \Omega \tag{2.5.1}$$

where $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a C^1-vector field such that

$$|Da(z)| \leq L \quad \text{and} \quad \langle Da(z)\lambda, \lambda\rangle \geq \nu|\lambda|^2 \tag{2.5.2}$$

for all $z, \lambda \in \mathbb{R}^{Nn}$ with $0 < \nu \leq L$. In this setting, a weak solution to (2.5.1) is a map $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \langle a(Du), D\phi\rangle\, dx = 0 \tag{2.5.3}$$

for every $\phi \in C_c^\infty(\Omega, \mathbb{R}^N)$.
The regularity statement is that if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to (2.5.1), that is a solution in the usual distributional sense as in (2.5.3), then there is an open subset $\Omega_0 \subset \Omega$ such that

$$u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N) \text{ for every } \alpha < 1 \text{ and } |\Omega \setminus \Omega_0| = 0. \tag{2.5.4}$$

This is actually called partial regularity of solutions.
Let us recall the following definition:

Definition 2.5.1. *Let $\mathcal{A}$ be a bilinear form with constant coefficients satisfying*

$$\nu|\lambda|^2 \leq \mathcal{A}\langle\lambda, \lambda\rangle \quad \textit{and} \quad \mathcal{A}\langle z, \lambda\rangle \leq L|z||\lambda| \tag{2.5.5}$$

for all $z, \lambda \in \mathbb{R}^{Nn}$, $\nu > 0$ and $L > 0$. A map $v \in W^{1,2}(\mathcal{B}_\rho, \mathbb{R}^N)$ is called $\mathcal{A}$-harmonic in the ball $\mathcal{B}_\rho \subset \mathbb{R}^n$ if it satisfies

$$\int_{\mathcal{B}_\rho} \mathcal{A}\langle Dv, D\phi\rangle\, dx = 0 \quad \textit{for all } \phi \in C_c^\infty(\mathcal{B}_\rho, \mathbb{R}^N). \tag{2.5.6}$$

Roughly speaking, an $\mathcal{A}$-harmonic map in $\mathcal{B}_\rho$ is just a weak solution to a constant coefficients elliptic system in the ball $\mathcal{B}_\rho$. Now, the basic idea for proving partial regularity of solutions is to linearize the system (2.5.1) around suitable averages of the gradient, in a small ball $\mathcal{B}_\rho(x_0)$, provided x_0 is a Lebesgue's point for Du, that is

$$\lim_{\rho\to 0} \fint_{\mathcal{B}_\rho(x_0)} |Du - (Du)_{\mathcal{B}_\rho(x_0)}|^2 dx = 0. \tag{2.5.7}$$

In fact, it can be proved that the regular set Ω_0 is exactly the set of Lebesgue's point of the gradient Du, from which the full measure property $|\Omega \setminus \Omega_0| = 0$ immediately follows. In order to achieve this, the idea is to consider the solution v to the system with constant coefficients:

$$\operatorname{div}[Da((Du)_{\mathcal{B}_\rho(x_0)})Dv] = 0 \text{ in } \mathcal{B}_\rho(x_0) \tag{2.5.8}$$

assuming that $\mathcal{B}_\rho(x_0)$ is sufficiently small. Setting $\mathcal{A} := Da((Du)_{\mathcal{B}_\rho(x_0)})$ we have that v is an $\mathcal{A}$-harmonic function, that is smooth in the interior of $\mathcal{B}_\rho(x_0)$ by classical regularity theory [11]. At this point, if we prove that the original solution u to (2.5.1) is close enough to a solution v to (2.5.8), then we may hope that the good regularity estimates available for v are in some sense inherited by u, and we can conclude with the partial regularity. We have the following

Lemma 2.5.1 ($\mathcal{A}$-harmonic approximation lemma)**.** *Consider fixed constants* $0 < \nu \leq L$, *and* $n, N \in \mathbb{N}$ *with* $n \geq 2$. *Then for any given* $\varepsilon > 0$ *there exists* $\delta = \delta(n, N, \nu, L, \varepsilon) \in (0, 1]$ *with the following property: For any bilinear form* $\mathcal{A}$ *satisfying (2.5.5), and for any* $u \in W^{1,2}(\mathcal{B}_\rho(x_0), \mathbb{R}^N)$ *(for some* $\mathcal{B}_\rho(x_0) \subset \mathbb{R}^n$*) satisfying*

$$\rho^{-n} \fint_{\mathcal{B}_\rho(x_0)} |Du|^2 dx \leq 1, \tag{2.5.9}$$

and being approximatively $\mathcal{A}$*-harmonic in the sense that*

$$\left| \rho^{-n} \fint_{\mathcal{B}_\rho(x_0)} \mathcal{A}\langle Du, D\phi \rangle \, dx \right| \leq \delta \sup_{\mathcal{B}_\rho(x_0)} |D\phi| \tag{2.5.10}$$

holds for every $\phi \in C_c^\infty(\mathcal{B}_\rho(x_0), \mathbb{R}^N)$*, there exists an* $\mathcal{A}$*-harmonic map*

$$h \in \mathcal{H} = \left\{ w \in W^{1,2}(\mathcal{B}_\rho(x_0), \mathbb{R}^N) : \rho^{-n} \fint_{\mathcal{B}_\rho(x_0)} |Dw|^2 dx \leq 1 \right\}$$

that is

$$\operatorname{div}(\mathcal{A}\ Dh) = 0 \quad \textit{in} \quad \mathcal{B}_\rho(x_0),$$

satisfying

$$\rho^{-n-2} \fint_{\mathcal{B}_\rho(x_0)} |h - u|^2 \, dx \leq \varepsilon. \tag{2.5.11}$$

By using this Lemma we can conclude there exists an $\mathcal{A}$-harmonic function v that is strongly close to u in the sense of

$$\fint_{\mathcal{B}_\rho(x_0)} |v - u|^2 \, dx \leq \varepsilon^2,$$

and then in turn we would conclude with the regularity of u.
Let us observe that Lemma 2.5.1 still works (see [40]) when considering a bilinear form that satisfies, instead of (2.5.5), the strong Legendre-Hadamard ellipticity condition:

$$\mathcal{A}\langle a \otimes b, a \otimes b \rangle \geq k_{\mathcal{A}} |a|^2 |b|^2$$

for every $a \in \mathbb{R}^N, b \in \mathbb{R}^n$ and for some constant $k_{\mathcal{A}} > 0$.

In [27] the authors give a generalization of the $\mathcal{A}$-harmonic approximation Lemma in Orlicz spaces. In particular, they considered $\mathcal{A} = \dfrac{D^2 f(Q)}{\varphi''(|Q|)}$ where φ is a given N-function and f has a φ-growth, that is

$$|f(Q)| \leq C\varphi(|Q|),$$

and is strictly $W^{1,\varphi}$-quasiconvex. Let us observe that the strictly $W^{1,\varphi}$-quasiconvexity of f implies the following strong Legendre-Hadamard condition:

$$D^2 f(Q)\langle \eta \otimes \xi, \eta \otimes \xi \rangle \geq k_{\mathcal{A}} \varphi''(|Q|)|\eta|^2|\xi|^2$$

for all $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{Nn} \setminus \{0\}$. Given a function $u \in W^{1,2}(\mathcal{B}_R)$, we want to find a function h that is $\mathcal{A}$-harmonic and is close to u. In particular, we are looking for a function $h \in W^{1,2}(\mathcal{B}_R)$ such that

$$\begin{cases} -\operatorname{div}(\mathcal{A}\, Dh) = 0 & \text{in } \mathcal{B}_R \\ h = u & \text{on } \partial \mathcal{B}_R \end{cases}.$$

Let $w := h - u$, then w satisfies

$$\begin{cases} -\operatorname{div}(\mathcal{A}\, Dw) = -\operatorname{div}(\mathcal{A}\, Du) & \text{in } \mathcal{B}_R \\ w = 0 & \text{on } \partial \mathcal{B}_R \end{cases}. \tag{2.5.12}$$

The approximation result is the following:

Theorem 2.5.1. *Let $\mathcal{B}_R \Subset \Omega$ and let $\tilde{\mathcal{B}} \subset \Omega$ denote either $\mathcal{B}_R$ or $\mathcal{B}_{2R}$. Let $\mathcal{A}$ be strongly elliptic in the sense of Legendre-Hadamard. Let ψ be an N-function with $\Delta_2(\psi, \psi^*) < \infty$ and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ depending on $n, N, k_{\mathcal{A}}, |\,\mathcal{A}\,|, \Delta_2(\psi, \psi^*)$ and s such that the following holds: let $u \in W^{1,\psi}(\tilde{\mathcal{B}})$ be almost $\mathcal{A}$-harmonic on $\mathcal{B}_R$ in the sense that*

$$\left| \fint_{\mathcal{B}_R} \langle \mathcal{A}\, Du, D\xi \rangle \, dx \right| \leq \delta \fint_{\tilde{\mathcal{B}}} |Du| \, dx \|D\xi\|_{L^\infty(\mathcal{B}_R)}$$

for all $\xi \in C_0^\infty(\mathcal{B}_R)$. Then the unique solution $w \in W_0^{1,\psi}(\mathcal{B}_R)$ of (2.5.12) satisfies

$$\fint_{\mathcal{B}_R} \psi\Big(\frac{|w|}{R}\Big) \, dx + \fint_{\mathcal{B}_R} \psi(|Dw|) \, dx \leq \varepsilon \left[\left(\fint_{\mathcal{B}_R} \psi^s(|Du|) \, dx \right)^{\frac{1}{s}} + \fint_{\tilde{\mathcal{B}}} \psi(|Du|) \, dx \right].$$

Chapter 3

Bmo regularity for asymptotic parabolic systems with linear growth

In this Chapter we prove local regularity results for the spatial gradient of weak solutions to non-linear problems under the assumption that the involved operator becomes appropriately parabolic at infinity. More precisely, we study nonlinear parabolic systems of the type

$$u_t - \operatorname{div}(\gamma(x,t)a(Du)) = -\operatorname{div} f(x,t) \quad (x,t) \in \Omega \times (-T,0) =: \Omega_T \tag{3.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$ and u maps Ω_T into $\mathbb{R}^N$.
The main assumptions on the vector field $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ are:
$(\mathcal{H}1)$ a is a continuous map;
$(\mathcal{H}2)$ there exist constants L and m such that

$$|a(\xi) - a(\eta)| \leq L|\xi - \eta| \tag{3.0.2}$$

for all $\xi, \eta \in \mathbb{R}^{Nn}$ such that $|\xi| + |\eta| \geq m$;
$(\mathcal{H}3)$ there exists $\varepsilon > 0$ such that a satisfies the coercivity condition

$$\langle a(\xi) - a(\eta), \xi - \eta \rangle \geq L(1-\varepsilon)|\xi - \eta|^2$$

for all $\xi, \eta \in \mathbb{R}^{Nn}$ such that $|\xi| + |\eta| \geq m$.
The notion of weak solution adopted prescribes that a map

$$u \in C^0(-T,0; L^2(\Omega, \mathbb{R}^N)) \cap L^p(-T,0; W^{1,p}(\Omega, \mathbb{R}^N)), \quad N \geq 1 \tag{3.0.3}$$

is a weak solution to (3.0.1), for $p \geq 2$, if and only if

$$\int_{\Omega_T} -u\varphi_t + \langle \gamma(x,t)a(Du), D\varphi \rangle dx\, dt = \int_{\Omega_T} \langle f, D\varphi \rangle dx\, dt \tag{3.0.4}$$

holds whenever $\varphi \in W_0^{1,2}(\Omega_T, \mathbb{R}^N)$.
We are able to prove two types of regularity results for weak solutions u to (3.0.1). The first result concerns the BMO regularity of Du under the further assumptions:
$(\mathcal{H}4)$ $\gamma : \Omega_T \to \mathbb{R}$ is measurable and satisfies the non-degeneracy condition

$$0 < \nu \leq \gamma(\cdot) \leq L \quad \forall (x,t) \in \Omega_T$$

and moreover defining

$$\omega(\rho) := \sup_{\substack{t,s \in (-T,0) \\ x,y \in \mathcal{B}_\rho(x_0) \subset \Omega}} |\gamma(x,t) - \gamma(y,s)|$$

there exists $\alpha > 0$ such that

$$\omega(\rho) \leq c\rho^\alpha. \tag{3.0.5}$$

$(\mathcal{H}5)$ f is $\mathrm{BMO_{loc}}(\Omega_T, \mathbb{R}^{Nn})$.
More precisely, it holds:

Theorem 3.0.2. *Let a satisfying the assumptions* $(\mathcal{H}1) - (\mathcal{H}5)$*. Then there exist an* $\varepsilon_0 = \varepsilon_0(n, L) \in (0,1)$ *and a constant* $M = M(a)$ *such that: if* $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in W^{1,2}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ *is a weak solution of the system (3.0.1) in* Ω_T*, then* $Du \in \mathrm{BMO_{loc}}(\Omega_T, \mathbb{R}^{Nn})$ *and there exists a constant* $C = C(n, L, \mathrm{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1}))$ *such that*

$$[Du]_{2,n;\Omega_{t_2}} \leq C(M + [f]_{2,n;\Omega_{t_1}} + \|Du\|_{L^2(\Omega_{t_1})}),$$

where $\Omega_{t_2} \Subset \Omega_{t_1} \Subset \Omega_T$ *are open domains.*

Next, if we replace hypotheses $(\mathcal{H}4)$ and $(\mathcal{H}5)$ with
$(\mathcal{H}4')$ Functions $\gamma(\cdot)$ and $f(\cdot)$ are measurable, $\gamma(\cdot)$ satisfies the non-degeneracy condition

$$0 < \nu \leq \gamma(\cdot) \leq L \quad \forall (x,t) \in \Omega_T$$

and moreover defining

$$\omega(\rho) := \sup_{\substack{t,s \in (-T,0) \\ x,y \in \mathcal{B}_\rho(x_0) \subset \Omega}} |\gamma(x,t) - \gamma(y,s)| + |f(x,t) - f(y,s)| \tag{3.0.6}$$

there exists $\alpha > 0$ such that

$$\omega(\rho) \leq c\rho^\alpha. \tag{3.0.7}$$

we obtain the following Lipschitz-regularity result:

Theorem 3.0.3. *Assume that a satisfies the conditions* $(\mathcal{H}1) - (\mathcal{H}3)$ *and* $(\mathcal{H}4')$*. Then we can find an* $\varepsilon_0 = \varepsilon_0(n, L) \in (0,1)$*, a constant* $M = M(a)$ *and two constants* c_1 *and* c_2 *depending only on* n *and* $\mathrm{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1})$*, such that if* $u \in W^{1,2}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ *is a weak solution of the system (3.0.1) in* Ω_T *then* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ *and for all* $\Omega_{t_2} \Subset \Omega_{t_1}$ *holds*

$$\mathrm{esssup}_{\Omega_{t_2}} |Du| \leq c_1 \left(M^2 + \int_{\Omega_{t_1}} |Du|^2 dxdt \right)^{\frac{1}{2}} + c_2.$$

3.1 Estimate for a comparison map

Let us now consider, in a fixed cylinder $\mathcal{Q}_r \equiv \mathcal{Q}_r(x_0,t_0) \Subset \Omega_T$, the unique weak solution $v \in W^{1,2}(\Omega_T,\mathbb{R}^N)$ to the Cauchy - Dirichlet problem

$$\begin{cases} v_t - \operatorname{div}[\gamma(x_0,t_0)Dv] = 0 & \text{in } \mathcal{Q}_r \\ v = u & \text{on } \partial_{\mathcal{P}}\mathcal{Q}_r \end{cases} \tag{3.1.1}$$

(see [23]). The central result of this section is the following:

Proposition 3.1.1. *Let a satisfy the assumptions $(\mathcal{H}1)-(\mathcal{H}3)$ and suppose that it holds $(\mathcal{H}4)-(\mathcal{H}5)$. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega_T,\mathbb{R}^N)$ is a weak solution of the system (3.0.1) in Ω_T, fixed $(x_0,t_0)\in\mathbb{R}^{n+1}$, there exists a constant $c \equiv c(n,N,\nu,L)$ such that*

$$\int_{\mathcal{Q}_r} |Du-Dv|^2 dz \leq$$
$$c\left[\int_{\mathcal{Q}_r} (|f-f_{\mathcal{Q}_r}|^2 + M^2)\,dz + \varepsilon\int_{\mathcal{Q}_r} |Du-(Du)_{\mathcal{Q}_r}|^2 dz + \frac{(2\varepsilon+1)}{2}\omega^2(r)\int_{\mathcal{Q}_r}|Du|^2dz\right]. \tag{3.1.2}$$

Proof. We will follow some ideas contained in [31]. For simplicity we assume $L=1$. In view of hypotheses $(\mathcal{H}2)$ and $(\mathcal{H}3)$ we find

$$\begin{aligned}\langle (a(\xi)-\xi)-(a(\eta)-\eta),\xi-\eta\rangle \\ &= \langle a(\xi)-a(\eta),\xi-\eta\rangle - \langle \xi-\eta,\xi-\eta\rangle \\ &\geq (1-\varepsilon)|\xi-\eta|^2 - |\xi-\eta|^2 \\ &= -\varepsilon|\xi-\eta|^2\end{aligned}$$

and

$$\begin{aligned}|[a(\xi)-\xi]-[a(\eta)-\eta]|^2 &= |[a(\xi)-a(\eta)]-[\xi-\eta]|^2 \\ &= |a(\xi)-a(\eta)|^2 + |\xi-\eta|^2 - 2\langle a(\xi)-a(\eta),\xi-\eta\rangle \\ &\leq |\xi-\eta|^2 + |\xi-\eta|^2 - 2(1-\varepsilon)|\xi-\eta|^2 \\ &= 2\varepsilon|\xi-\eta|^2\end{aligned}$$

for all $\xi,\eta\in\mathbb{R}^{Nn}$ such that $|\xi|+|\eta|\geq m$.
Defining $e(\xi)=a(\xi)-\xi$ we have, if $|\xi|+|\eta|\geq m$

$$\begin{aligned}|e(\xi)-e(\eta)|^2 &\leq 2\varepsilon|\xi-\eta|^2 \\ &\Rightarrow |e(\xi)-e(\eta)| \leq \sqrt{2\varepsilon}|\xi-\eta|\end{aligned}$$

that is $e(\xi)$ is a Lipschitz function with constant $\sqrt{2\varepsilon}$.
Let g be the $\sqrt{2\varepsilon}$-Lipschitz extension of the restriction of e to $\mathbb{R}^{Nn}\setminus B(0,m)$ to all $\mathbb{R}^{Nn}$ and let $b(\xi)=e(\xi)-g(\xi)$. Now, $b(\cdot)$ has compact support, is continuous and $b\in L^\infty(\mathbb{R}^{Nn})$. So

we have a reformulation of the equation (3.0.1) as a perturbation of the operator defined in (3.0.1):

$$u_t - \operatorname{div}[\gamma(x,t)Du + \gamma(x,t)b(Du) + \gamma(x,t)g(Du) - f] = 0 \tag{3.1.3}$$

with

$$|g(\xi) - g(\eta)| \le \sqrt{2\varepsilon}|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^{Nn} \tag{3.1.4}$$

$$|b(\xi)| \le M \quad \forall \xi \in \mathbb{R}^{Nn}. \tag{3.1.5}$$

Now, for $t_0 - r^2 < s < t_0$ and $\tilde{\varepsilon} > 0$ small enough, we choose

$$\zeta(t) = \begin{cases} 1 & \text{for } t_0 - r^2 \le t \le s \\ -\frac{1}{\tilde{\varepsilon}}(t - s - \tilde{\varepsilon}) & \text{for } s \le t \le s + \tilde{\varepsilon} \\ 0 & \text{for } s + \tilde{\varepsilon} \le t \le t_0 \end{cases} \tag{3.1.6}$$

and let $\varphi(x,t) = (u - v)\zeta$, where v is the unique weak solution of (3.1.1).
In the weak formulation of (3.1.1) and (3.1.3) respectively we formally use the test function φ obtaining

$$\int_{\mathcal{Q}_r} [-v\varphi_t + \langle \gamma(x_0, t_0)Dv, D\varphi\rangle]\, dz = 0 \tag{3.1.7}$$

and

$$\int_{\mathcal{Q}_r} [-u\varphi_t + \langle \gamma(x,t)Du + \gamma(x,t)(b(Du) + g(Du)) - f, D\varphi\rangle]\, dz = 0. \tag{3.1.8}$$

Subtracting (3.1.7) from (3.1.8) we obtain

$$\int_{\mathcal{Q}_r} [-(u-v)\varphi_t + \langle \gamma(x,t)[b(Du) + g(Du)] - f, D\varphi\rangle]\, dz + \int_{\mathcal{Q}_r} [\langle \gamma(x,t)Du - \gamma(x_0,t_0)Dv, D\varphi\rangle]\, dz = 0,$$

using the definition of φ we deduce

$$\begin{aligned} &\int_{\mathcal{Q}_r} [-(u-v)(u-v)_t\zeta - |u-v|^2\zeta_t\,]\, dz \\ &+ \int_{\mathcal{Q}_r} \langle \gamma(x,t)[b(Du) + g(Du)] - f, \zeta D(u-v)\rangle dz \\ &+ \int_{\mathcal{Q}_r} \langle [\gamma(x,t) - \gamma(x_0,t_0)]Du, \zeta D(u-v)\rangle dz \\ &+ \int_{\mathcal{Q}_r} \langle \gamma(x_0,t_0)D(u-v), \zeta D(u-v)\rangle dz = 0 \end{aligned}$$

that is

$$\begin{aligned}
&\int_{\mathcal{Q}_r} [-(u-v)(u-v)_t\zeta - |u-v|^2\zeta_t + \langle \gamma(x_0,t_0)D(u-v), \zeta D(u-v)\rangle] dz = \\
&\int_{\mathcal{Q}_r} \langle f - \gamma(x,t)(b(Du)+g(Du)), \zeta D(u-v)\rangle dz \\
&- \int_{\mathcal{Q}_r} \langle [\gamma(x,t)-\gamma(x_0,t_0)]Du, \zeta D(u-v)\rangle dz.
\end{aligned}$$

After performing manipulations it follows that

$$\begin{aligned}
\nu \int_{\mathcal{Q}_r} |D(u-v)|^2 \zeta \, dz &= \int_{\mathcal{Q}_r} \langle f - f_{\mathcal{Q}_r} - \gamma(x,t)b(Du), \zeta D(u-v)\rangle dz \\
&- \int_{\mathcal{Q}_r} \langle [\gamma(x,t)-\gamma(x_0,t_0)] g(Du), \zeta D(u-v)\rangle dz \\
&- \int_{\mathcal{Q}_r} \langle \gamma(x_0,t_0)[g(Du) - g((Du)_{\mathcal{Q}_r})], \zeta D(u-v)\rangle dz \\
&- \int_{\mathcal{Q}_r} \langle [\gamma(x,t)-\gamma(x_0,t_0)]Du, \zeta D(u-v)\rangle dz \\
&\leq \int_{\mathcal{Q}_r} |f - f_{\mathcal{Q}_r} - \gamma(x,t)b(Du)||D(u-v)| \, dz \\
&+ \int_{\mathcal{Q}_r} |\gamma(x,t)-\gamma(x_0,t_0)||g(Du)||D(u-v)| dz \\
&+ \int_{\mathcal{Q}_r} \gamma(x_0,t_0)|g(Du) - g((Du)_{\mathcal{Q}_r})||D(u-v)| \, dz \\
&+ \int_{\mathcal{Q}_r} |\gamma(x,t)-\gamma(x_0,t_0)||Du||D(u-v)| \, dz \\
&=: I + II + III + IV.
\end{aligned}$$

We proceed with the estimation of I: by Hölder inequality, (3.1.5) and ($\mathcal{H}4$) it find out that

$$\begin{aligned}
I &\leq \left(\int_{\mathcal{Q}_r} |f - f_{\mathcal{Q}_r} - \gamma(x,t)b(Du)|^2 dz \right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz \right)^{\frac{1}{2}} \\
&\leq \left(2 \int_{\mathcal{Q}_r} |f - f_{\mathcal{Q}_r}|^2 + |b(Du)|^2 dz \right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz \right)^{\frac{1}{2}} \\
&\leq \left(2 \int_{\mathcal{Q}_r} |f - f_{\mathcal{Q}_r}|^2 + M^2 dz \right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz \right)^{\frac{1}{2}}.
\end{aligned}$$

To evaluate the second addendum we take into account that g is a Lipschitz function and hypothesis $(\mathcal{H}4)$:

$$
\begin{aligned}
II &\le \left(\int_{\mathcal{Q}_r} |\gamma(x,t)-\gamma(x_0,t_0)|^2 |g(Du)|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}} \\
&\le \left(\omega^2(r)\int_{\mathcal{Q}_r} |g(Du)|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}} \\
&\le \left(2\varepsilon\omega^2(r)\int_{\mathcal{Q}_r} |Du|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}} + \left(c\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using Hölder inequality and (3.1.4) we find

$$
\begin{aligned}
III &\le \left(\int_{\mathcal{Q}_r} |g(Du)-g((Du)_{\mathcal{Q}_r})|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}} \\
&\le \left(2\varepsilon\int_{\mathcal{Q}_r} |Du-(Du)_{\mathcal{Q}_r}|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}}.
\end{aligned}
$$

Estimate for IV: by hypothesis $(\mathcal{H}4)$

$$
\begin{aligned}
IV &\le \left(\int_{\mathcal{Q}_r} |\gamma(x,t)-\gamma(x_0,t_0)|^2 |Du|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}} \\
&\le \left(\omega^2(r)\int_{\mathcal{Q}_r} |Du|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_r} |D(u-v)|^2 dz\right)^{\frac{1}{2}}.
\end{aligned}
$$

Combining estimates obtained we conclude with

$$
\begin{aligned}
\int_{\mathcal{Q}_r} |D(u-v)|^2 dz \le{}& \frac{6}{\nu^2}\int_{\mathcal{Q}_r} \left(|f-f_{\mathcal{Q}_r}|^2+\tilde{M}^2\right) dz \\
&+\frac{6\varepsilon}{\nu^2}\int_{\mathcal{Q}_r} |Du-(Du)_{\mathcal{Q}_r}|^2 dz + \frac{3(2\varepsilon+1)}{\nu^2}\omega^2(r)\int_{\mathcal{Q}_r} |Du|^2 dz
\end{aligned}
$$

that is (3.1.2).

□

3.2 BMO regularity for spatial gradient

This section is devoted to the proof of the Theorem 3.0.2.

3.2.1 A few lemmas

We start with a preliminary result due to Campanato [12]:

Lemma 3.2.1. *Let v be a solution of the equation*

$$v_t - \operatorname{div}(ADv) = 0 \text{ in } \mathcal{Q}_r(x_0, t_0)$$

with A constant matrix. Then for any $0 < \rho \le r$, we have

$$\int_{\mathcal{Q}_\rho(x_0,t_0)} |Dv|^2 dz \le c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{\mathcal{Q}_r(x_0,t_0)} |Dv|^2 dz \tag{3.2.1}$$

and

$$\int_{\mathcal{Q}_\rho(x_0,t_0)} [Dv - (Dv)_{\mathcal{Q}_\rho(x_0,t_0)}]^2 dz \le c\Big(\frac{\rho}{r}\Big)^{n+4} \int_{\mathcal{Q}_r(x_0,t_0)} [Dv - (Dv)_{\mathcal{Q}_r(x_0,t_0)}]^2 dz \tag{3.2.2}$$

where c is a positive constant depending only on n.

Next Lemma (see for instance Lemma 1 in [20]) plays an important role for the iteration:

Lemma 3.2.2. *Let $\alpha, d > 0$, $A > 0$, $B \ge 0$, $\beta \in [0, \alpha)$. Then there exists $\varepsilon_0, C > 0$ such that for every function Φ nonnegative and nondecreasing defined on $[0, d]$ and satisfying the inequality*

$$\Phi(\sigma) \le \Big(A\Big(\frac{\sigma}{R}\Big)^\alpha + K\Big)\Phi(R) + BR^\beta \quad \forall \sigma, R : 0 < \sigma < R \le d$$

with $K \in (0, \varepsilon_0]$ it holds that

$$\Phi(\sigma) \le C\sigma^\beta (d^{-\beta}\Phi(d) + B) \quad \forall \sigma : 0 < \sigma \le d.$$

3.2.2 Proof of Theorem 3.0.2

Now we are able to prove the BMO regularity of the gradient of solutions to (3.0.1).
In a fixed cylinder $\mathcal{Q}_r \equiv \mathcal{Q}_r(x_0, t_0) \Subset \Omega_T$ we apply (3.2.2) with $\tau \in (0, 1)$

$$\int_{\mathcal{Q}_{\tau r}} |Dv - (Dv)_{\mathcal{Q}_{\tau r}}|^2 dz \le c\tau^{n+4} \int_{\mathcal{Q}_r} |Dv - (Dv)_{\mathcal{Q}_r}|^2 dz,$$

the triangle inequality and (3.1.2) to obtain

$$\begin{aligned}
&\int_{\mathcal{Q}_{\tau r}} |Du - (Du)_{\mathcal{Q}_{\tau r}}|^2 dz \\
&= \int_{\mathcal{Q}_{\tau r}} |Du - Dv + Dv - (Dv)_{\mathcal{Q}_{\tau r}} + (Dv)_{\mathcal{Q}_{\tau r}} - (Du)_{\mathcal{Q}_{\tau r}}|^2 dz \\
&\le 3\Big[\int_{\mathcal{Q}_{\tau r}} |Du - Dv|^2 dz + \int_{\mathcal{Q}_{\tau r}} |Dv - (Dv)_{\mathcal{Q}_{\tau r}}|^2 dz + \int_{\mathcal{Q}_{\tau r}} |(Dv)_{\mathcal{Q}_{\tau r}} - (Du)_{\mathcal{Q}_{\tau r}}|^2 dz\Big] \\
&\le 6 \int_{\mathcal{Q}_r} |Du - Dv|^2 dz + 3 \int_{\mathcal{Q}_{\tau r}} |Dv - (Dv)_{\mathcal{Q}_{\tau r}}|^2 dz \\
&\le c \int_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2)\, dz + c\Big(\varepsilon + \tau^{n+4}\Big) \int_{\mathcal{Q}_r} |Du - (Du)_{\mathcal{Q}_r}|^2 dz \\
&+ c(2\varepsilon + 1)\omega^2(r) \int_{\mathcal{Q}_r} |Du|^2 dz.
\end{aligned} \tag{3.2.3}$$

Moreover using again (3.1.2) and the fact that

$$\int_{\mathcal{Q}_r} |Du - (Du)_{\mathcal{Q}_r}|^2 dz \leq \int_{\mathcal{Q}_r} |Du|^2 dz$$

we have

$$\begin{aligned}
&\int_{\mathcal{Q}_r} |D(u-v)|^2 dz \leq \\
&\leq c\left[\int_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2) dz + \varepsilon \int_{\mathcal{Q}_r} |Du|^2 dz + \frac{(2\varepsilon+1)\omega^2(r)}{2} \int_{\mathcal{Q}_r} |Du|^2 dz\right] \\
&= c\left[\int_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2) dz + \Big(\varepsilon + \frac{(2\varepsilon+1)\omega^2(r)}{2}\Big) \int_{\mathcal{Q}_r} |Du|^2 dz\right] \\
&= c_1 \int_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2) dz + c_2(2\varepsilon + \omega^2(r)) \int_{\mathcal{Q}_r} |Du|^2 dz. \qquad (3.2.4)
\end{aligned}$$

Taking $\rho < r$ in (3.2.4), using triangle inequality and (3.2.1), we deduce

$$\begin{aligned}
&\int_{\mathcal{Q}_\rho} |Du|^2 dz \leq \\
&\leq 2\int_{\mathcal{Q}_\rho} |D(u-v)|^2 dz + 2\int_{\mathcal{Q}_\rho} |Dv|^2 dz \\
&\leq 2\int_{\mathcal{Q}_\rho} |D(u-v)|^2 dz + 2c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{\mathcal{Q}_r} |Dv|^2 dz \\
&\leq 2\int_{\mathcal{Q}_r} |D(u-v)|^2 dz + 2c\Big(\frac{\rho}{r}\Big)^{n+2} \Big[2\int_{\mathcal{Q}_r} |D(u-v)|^2 dz + 2\int_{\mathcal{Q}_r} |Du|^2 dz\Big] \\
&\leq c_3 \int_{\mathcal{Q}_r} |D(u-v)|^2 dz + c_4\Big(\frac{\rho}{r}\Big)^{n+2} \int_{\mathcal{Q}_r} |Du|^2 dz \\
&\leq c_5 \int_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2) dz + \Big(2c_6\,\varepsilon + c_6\,\omega^2(r) + c_4\Big(\frac{\rho}{r}\Big)^{n+2}\Big) \int_{\mathcal{Q}_r} |Du|^2 dz \\
&\leq c_5\, r^{n+2-\delta}\, R_0^\delta \fint_{\mathcal{Q}_r} (|f - f_{\mathcal{Q}_r}|^2 + M^2)\, dz + \Big(c_7\,\varepsilon + c_7\,\omega^2(r) \\
&+ c_4\,\Big(\frac{\rho}{r}\Big)^{n+2}\Big) \int_{\mathcal{Q}_r} |Du|^2 dz.
\end{aligned}$$

Fixed $\varepsilon < \varepsilon_0$ and $r < R_0$ such that $c_7\varepsilon + c_7\omega^2(r) < \varepsilon_0$, by Lemma 3.2.2 we have

$$\int_{\mathcal{Q}_\rho} |Du|^2 dz \leq c\rho^\beta \Big(R_0^{-\beta} \int_{\mathcal{Q}_{R_0}} |Du|^2 dz + B\Big) \leq c\rho^\beta \qquad (3.2.5)$$

for all $\beta < n+2$ and for all $0 < \rho < R_0$ (in particular $\beta = n+2-\delta$, where δ will be choosen in an appropriate way).

At this point we want to prove that there exist a constant $C = C(n, L)$ such that for all $\mathcal{Q}_{R_0} \equiv \mathcal{Q}_{R_0}(x_0, t_0) \Subset \Omega_T$ and $\rho \in (0, R_0)$ the following estimate holds:

$$\begin{aligned} &\fint_{\mathcal{Q}_\rho} |Du - (Du)_{\mathcal{Q}_\rho}|^2 dz \\ &\quad \le C\left(M^2 + [f]^2_{2,n;\Omega_{t_1}} + \fint_{\mathcal{Q}_{R_0}} |Du - (Du)_{\mathcal{Q}_{R_0}}|^2 dz + \tilde{C}\right) \end{aligned} \tag{3.2.6}$$

where $\mathcal{Q}_\rho \equiv \mathcal{Q}_\rho(x_0, t_0)$ and

$$[f]^2_{2,n;\Omega_{t_1}} = \sup_{\substack{(x_0,t_0)\in\Omega_{t_1},\\ \rho\in(0,R_0)}} \fint_{\mathcal{Q}_\rho} |f - f_{\mathcal{Q}_\rho}|^2 dz.$$

Now we define a function Φ:

$$\Phi : (0, R_0) \to [0, \infty)$$

putting

$$\Phi(R) := \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz.$$

Such Φ is non-decreasing and for $0 < \rho < s \le R_0$, by (3.2.3), satisfies the following:

$$\begin{aligned} \Phi(\rho) &= \int_{\mathcal{Q}_\rho} |Du - (Du)_{\mathcal{Q}_\rho}|^2 dz \\ &\le c \int_{\mathcal{Q}_s} (|f - f_{\mathcal{Q}_s}|^2 + M^2)\, dz \\ &+ \left[c\,\varepsilon + \left(\frac{\rho}{s}\right)^{n+4}\right] \int_{\mathcal{Q}_s} |Du - (Du)_{\mathcal{Q}_s}|^2 dz \\ &+ c(2\varepsilon + 1)\omega^2(s)s^{n+2}\omega_n \fint_{\mathcal{Q}_s} |Du|^2 dz \\ &= c\, s^{n+2}\left[\fint_{\mathcal{Q}_s} |f - f_{\mathcal{Q}_s}|^2 dz + M^2\right] + \left[c\,\varepsilon + 3\left(\frac{\rho}{s}\right)^{n+4}\right]\Phi(s) \\ &+ c(2\varepsilon + 1)\omega^2(s)s^{n+2}\omega_n \fint_{\mathcal{Q}_s} |Du|^2 dz. \end{aligned}$$

Taking $0 < s \le R_0$ and $\tau \in (0, 1)$ we have

$$\begin{aligned} \Phi(\tau s) &\le cs^{n+2}\left[\fint_{\mathcal{Q}_s} |f - f_{\mathcal{Q}_s}|^2 dz + M^2\right] + \left[c\varepsilon + 3\tau^{n+4}\right]\Phi(s) \\ &+ c(2\varepsilon + 1)\omega^2(s)s^{n+2}\omega_n \fint_{\mathcal{Q}_s} |Du|^2 dz \\ &= cs^{n+2}\left[\fint_{\mathcal{Q}_s} |f - f_{\mathcal{Q}_s}|^2 dz + M^2\right] + 3\tau^{n+4}\left[\frac{c\varepsilon}{\tau^{n+4}} + 1\right]\Phi(s) \\ &+ c(2\varepsilon + 1)\omega^2(s)s^{n+2}\omega_n \fint_{\mathcal{Q}_s} |Du|^2 dz. \end{aligned} \tag{3.2.7}$$

Let $\tau = \frac{1}{3(c+1)}$ and $\tilde{\varepsilon}_0 = \tau^{n+4}$. Then, for $\varepsilon \in (0, \tilde{\varepsilon}_0)$ we have

$$\frac{c\varepsilon}{\tau^{n+4}} + 1 < c + 1 \Rightarrow 3\tau^{n+4}\Big[\frac{c\varepsilon}{\tau^{n+4}} + 1\Big] < 3\tau^{n+4}(c+1) = \tau^{n+3}$$

and so (3.2.7) becomes

$$\begin{aligned} \Phi(\tau s) \leq c s^{n+2} \left[⨍_{\mathcal{Q}_s} |f - f_{\mathcal{Q}_s}|^2 dz + M^2 \right] + \tau^{n+3}\Phi(s) \\ + c(2\varepsilon + 1)\omega^2(s) s^{n+2} \omega_n ⨍_{\mathcal{Q}_s} |Du|^2 dz. \end{aligned} \tag{3.2.8}$$

By induction, for all $j \in \mathbb{N}$:

$$\begin{aligned} &\Phi(\tau^{j+1}s) \leq (\tau^{j+1}s)^{n+2} \Big\{ \frac{\tau^{j+1}}{s^{n+2}} \Phi(s) \\ &+ c \sup_{0<s<R_0} \Big[⨍_{\mathcal{Q}_s} |f - f_{\mathcal{Q}_s}|^2 dz + M^2 \Big] \sum_{i=0}^{j} \tau^{(j-i)-(n+2)} \\ &+ c(2\varepsilon+1)\omega_n \sum_{i=0}^{j} \tau^{i(n+2)+(j-i)(n+3)-(j+1)(n+2)} \omega^2(\tau^i s) ⨍_{\mathcal{Q}_{\tau^i s}} |Du|^2 dz \Big\}. \end{aligned}$$

Now it holds

$$\begin{aligned} \sum_{i=0}^{j} \tau^{(j-i)-(n+2)} &= \frac{1}{\tau^{n+2}} \sum_{i=0}^{j} \tau^{j-i} \\ &\leq \frac{1}{\tau^{n+2}(1-\tau)}. \end{aligned}$$

Employing hypothesis $(\mathcal{H}4)$ and using (3.2.5)

$$\begin{aligned} &\sum_{i=0}^{j} \tau^{i(n+2)+(j-i)(n+3)-(j+1)(n+2)} \omega^2(\tau^i s) ⨍_{\mathcal{Q}_{\tau^i s}} |Du|^2 dz \\ &\quad \leq \sum_{i=0}^{j} \tau^{i(n+2)+(j-i)(n+3)-(j+1)(n+2)} \frac{c^2 \tau^{2i\alpha} s^{2\alpha}}{\tau^{i(n+2)} s^{n+2} \omega_n} c \tau^{i\beta} s^{\beta} \\ &\quad \leq \frac{C s^{2\alpha+\beta}}{s^{n+2}\omega_n \tau^{n+2}} \sum_{i=0}^{j} \tau^{j-i(n+3-2\alpha-\beta)}. \end{aligned}$$

Taking into account that $\beta = n + 2 - \delta$ and choosing $\delta = 2\alpha$ we deduce

$$\begin{aligned} &\sum_{i=0}^{j} \tau^{i(n+2)+(j-i)(n+3)-(j+1)(n+2)} \omega^2(\tau^i s) ⨍_{\mathcal{Q}_{\tau^i s}} |Du|^2 dz \\ &\quad \leq \frac{C}{\omega_n \tau^{n+2}} \sum_{i=0}^{j} \tau^{j-i} \\ &\quad \leq \frac{C}{\omega_n \tau^{n+2}(1-\tau)}. \end{aligned}$$

Finally

$$\Phi(\tau^{j+1}s) \leq (\tau^{j+1}s)^{n+2}\Big\{\frac{\tau^{j+1}}{s^{n+2}}\Phi(s) + \frac{c}{\tau^{n+2}(1-\tau)}\sup_{0<s<R_0}\Big[\fint_{\mathcal{Q}_s}|f-f_{\mathcal{Q}_s}|^2dz + M^2\Big] + \frac{C(2\varepsilon+1)}{\tau^{n+2}(1-\tau)}\Big\}. \tag{3.2.9}$$

Let $\rho \in (0, R_0)$ and choose $j \in \mathbb{N}$ such that

$$\tau^{j+2}R_0 < \rho \leq \tau^{j+1}R_0. \tag{3.2.10}$$

Then (3.2.9) becomes

$$\begin{aligned}\Phi(\rho) &= \Phi\Big(\tau^{j+1}\frac{\rho}{\tau^{j+1}}\Big)\\ &\leq \Big(\tau^{j+1}\frac{\rho}{\tau^{j+1}}\Big)^{n+2}\Big\{\tau^{j+1}\Big(\frac{\tau^{j+1}}{\rho}\Big)^{n+2}\Phi(\frac{\rho}{\tau^{j+1}})\\ &+ \frac{c}{\tau^{n+2}(1-\tau)}\sup_{0<s<R_0}\Big[\fint_{\mathcal{Q}_s}|f-f_{\mathcal{Q}_s}|^2dz + M^2\Big] + \frac{C(2\varepsilon+1)}{\tau^{n+2}(1-\tau)}\Big\}.\end{aligned}$$

Using the facts

$\Phi\Big(\frac{\rho}{\tau^{j+1}}\Big) \leq \Phi(R_0)$	since Φ is non- decreasing
$\frac{\tau^{j+1}}{\rho} \leq \frac{1}{R_0\tau}$	by (3.2.10)
$\tau^{j+1} < \Big(\frac{\rho}{R_0}\Big)^{\frac{j+1}{j+2}} < 1$	by (3.2.10)

we have

$$\Phi(\rho) \leq \rho^{n+2}\Big\{\frac{1}{(R_0\tau)^{n+2}}\Phi(R_0) + \frac{c}{\tau^{n+2}(1-\tau)}\sup_{0<s<R_0}\Big[\fint_{\mathcal{Q}_s}|f-f_{\mathcal{Q}_s}|^2dz + M^2\Big] + \frac{C(2\varepsilon+1)}{\tau^{n+2}(1-\tau)}\Big\}$$

dividing by $|\,\mathcal{Q}_\rho\,|$ we get

$$\begin{aligned}\frac{1}{\rho^{n+2}\omega_n}\Phi(\rho) &\leq \frac{1}{\tau^{n+2}R_0^{n+2}\omega_n}\Phi(R_0)\\ &+ \frac{c}{\tau^{n+2}(1-\tau)}\sup_{0<s<R_0}\Big[\fint_{\mathcal{Q}_s}|f-f_{\mathcal{Q}_s}|^2dz + M^2\Big] + \frac{C(2\varepsilon+1)}{\omega_n\tau^{n+2}(1-\tau)},\end{aligned}$$

that is (3.2.6).
Let $\delta = \frac{1}{2}\,\mathrm{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1}) > 0$ we may choose, for all $(x_0, t_0) \in \Omega_{t_2}$, a radius $R_0 > \delta$ such that $\mathcal{Q}_{R_0} \equiv \mathcal{Q}_{R_0}(x_0, t_0) \subset \Omega_{t_1}$.

Since

$$\begin{aligned}\frac{1}{\omega_n R_0^{n+2}}\int_{\mathcal{Q}_{R_0}} &|Du-(Du)_{\mathcal{Q}_{R_0}}|^2 dz\\ &\le \frac{1}{R_0^{n+2}\omega_n}\int_{\mathcal{Q}_{R_0}}|Du|^2 dz\\ &\le \frac{1}{\delta^{n+2}\omega_n}\int_{\Omega_{t_1}}|Du|^2 dz,\end{aligned}$$

the inequality (3.2.6) becomes

$$\fint_{\mathcal{Q}_\rho}|Du-(Du)_{\mathcal{Q}_\rho}|^2 dz \le C\Big[M^2+[f]^2_{2,n;\Omega_{t_1}}+\frac{4}{\delta^{n+2}\omega_n}\int_{\Omega_{t_1}}|Du|^2 dz+\tilde{C}\Big]$$

and taking the supremum over all $\mathcal{Q}_r \subset \Omega_{t_2}$ we obtain the assertion.
Let us remark that in (3.2.5) we proved the following result that will be useful later:

Lemma 3.2.3. *Under the same assumptions of Theorem 3.0.2 there exist an ε_0 such that if $\varepsilon < \varepsilon_0$, then for every $0<\rho<R$ we have that*

$$\int_{\mathcal{Q}_\rho}|Du|^2 dz \le c\rho^\beta \quad \forall\beta<n+2.$$

3.3 L^∞ spatial gradient regularity

3.3.1 An intrinsic estimate

From now on:

$$\begin{aligned}\omega_\gamma(\rho) &:= \sup_{\substack{t,s\in(-T,0)\\ x,y\in\mathcal{B}_\rho(x_0)\subset\Omega}}|\gamma(x,t)-\gamma(y,s)|\\ \omega_f(\rho) &:= \sup_{\substack{t,s\in(-T,0)\\ x,y\in\mathcal{B}_\rho(x_0)\subset\Omega}}|f(x,t)-f(y,s)|\end{aligned}$$

and recalling the definition of $\omega(\cdot)$ in (3.0.6) we have

$$[\omega_\gamma(\rho)]^2+[\omega_f(\rho)]^2 \le [\omega(\rho)]^2. \tag{3.3.1}$$

The following Lemma is a key ingredient in the proof of the L^∞ estimate for the system (3.0.1); the proof is due to Dolzmann, Kristensen and Zhang (see Lemma 3.2 in [31]):

Lemma 3.3.1. *Assume that $e:\mathbb{R}^{Nn}\to\mathbb{R}^{Nn}$ is globally Lipschitz continuous with constant L and there exists $\varepsilon, m>0$ such that*

$$|e(\xi)-e(\eta)|\le\varepsilon|\xi-\eta| \quad \forall\xi,\eta\in\mathbb{R}^{Nn}:|\xi|+|\eta|>m.$$

Let $M=\frac{\sqrt{2}mL}{\varepsilon}$. Then, for all $\xi_0\in\mathbb{R}^{Nn}$ and all $\lambda\ge 0$ with $\lambda^2+|\xi_0|^2\ge M^2$ we have

$$|e(\xi)-e(\xi_0)|\le\varepsilon(|\xi-\xi_0|+\lambda) \quad \forall\xi\in\mathbb{R}^{Nn}. \tag{3.3.2}$$

Now we prove a key estimate:

Theorem 3.3.1. *Suppose that $e(\xi) = a(\xi) - \xi$ is globally Lipschitz with constant L and assume that there exist constants $m, \varepsilon > 0$ such that*

$$|e(\xi) - e(\eta)| \leq \varepsilon|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^{Nn} : |\xi| + |\eta| > m.$$

Then we find an $\varepsilon_0 = \varepsilon_0(n) \in (0,1)$ and a constant $M = M(a)$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $u \in W^{1,2}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ is a weak solution of

$$u_t - \operatorname{div}(\gamma(x,t)a(Du)) = -\operatorname{div} f \qquad \text{in } \Omega_T,$$

then there exist a constant $c = c(n, N, \nu, L)$ such that

$$\int_{\mathcal{Q}_R} |D(u-v)|^2 dz \leq$$
$$\leq c\left[\varepsilon^2 \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz + \omega_\gamma^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz + \omega_f^2(R) |\, \mathcal{Q}_R \,|\right]. \tag{3.3.3}$$

Proof. Fix $(x_0, t_0) \in \Omega_T$ and a parabolic cylinder $\mathcal{Q}_R \equiv \mathcal{Q}_R(x_0, t_0) \Subset \Omega_T$, and suppose that

$$\fint_{\mathcal{Q}_R} |Du|^2 dz \geq M^2$$

where $M = \frac{\sqrt{2}mL}{\varepsilon}$ is the same of Lemma 3.3.1.
Define

$$\lambda^2 := \fint_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz$$
$$\xi_0 := \fint_{\mathcal{Q}_R} Du \, dz.$$

Using the fact that

$$\fint_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz = \fint_{\mathcal{Q}_R} |Du|^2 dz - |(Du)_{\mathcal{Q}_R}|^2$$

we have $\lambda^2 + |\xi_0|^2 \geq M^2$, so we can apply Lemma 3.3.1 obtaining

$$|e(\xi) - e(\xi_0)| \leq \varepsilon(|\xi - \xi_0| + \lambda). \tag{3.3.4}$$

Let $v \in W^{1,2}(\Omega_T, \mathbb{R}^N)$ the unique weak solution of (3.1.1). We test the weak form of (3.1.1) and (3.0.1) with $\varphi = (u - v)\zeta$, where ζ is defined in (3.1.6); this is possible modulo a standard use of Steklov averages.
It find out that:

$$\int_{\mathcal{Q}_R} -v\varphi_t + \langle \gamma(x_0, t_0) Dv, D\varphi \rangle \, dz = 0 \tag{3.3.5}$$

and

$$\int_{\mathcal{Q}_R} [-u\varphi_t + \langle \gamma(x,t)a(Du), D\varphi\rangle - \langle f(x,t), D\varphi\rangle]\, dz = 0. \tag{3.3.6}$$

By subtracting (3.3.5) from (3.3.6) we obtain

$$\int_{\mathcal{Q}_R} [-(u-v)\varphi_t + \langle \gamma(x,t)a(Du) - \gamma(x_0,t_0)Dv, D\varphi\rangle \\ - \langle f(x,t), D\varphi\rangle]\, dz = 0,$$

taking into account that $a(Du) = e(Du) + Du$ we have

$$\int_{\mathcal{Q}_R} [-(u-v)(u-v)_t\zeta - |u-v|^2\zeta_t + \langle \gamma(x,t)[e(Du) - e((Du)_{\mathcal{Q}_R})], \zeta D(u-v)\rangle \\ + \langle \gamma(x,t)Du - \gamma(x_0,t_0)Dv, \zeta D(u-v)\rangle \\ - \langle f(x,t) - f(x_0,t_0), \zeta D(u-v)\rangle]\, dz = 0$$

that is

$$\int_{\mathcal{Q}_R} [-(u-v)(u-v)_t\zeta - |u-v|^2\zeta_t + \langle \gamma(x,t)[e(Du) - e((Du)_{\mathcal{Q}_R})], \zeta D(u-v)\rangle \\ + \langle [\gamma(x,t) - \gamma(x_0,t_0)]Du, \zeta D(u-v)\rangle + \langle \gamma(x_0,t_0)D(u-v), \zeta D(u-v)\rangle \\ - \langle f(x,t) - f(x_0,t_0), \zeta D(u-v)\rangle]\, dz = 0.$$

As a consequence we also have

$$\begin{aligned} \nu \int_{\mathcal{Q}_R} |D(u-v)|^2\zeta\, dz &= -\int_{\mathcal{Q}_R} \langle \gamma(x,t)[e(Du) - e((Du)_{\mathcal{Q}_R})], \zeta D(u-v)\rangle\, dz \\ &\quad - \int_{\mathcal{Q}_R} \langle [\gamma(x,t) - \gamma(x_0,t_0)]Du, \zeta D(u-v)\rangle\, dz \\ &\quad + \int_{\mathcal{Q}_R} \langle f(x,t) - f(x_0,t_0), \zeta D(u-v)\rangle\, dz \\ &\leq \int_{\mathcal{Q}_R} |e(Du) - e((Du)_{\mathcal{Q}_R})||D(u-v)|\, dz \\ &\quad + \int_{\mathcal{Q}_R} |\gamma(x,t) - \gamma(x_0,t_0)||Du||D(u-v)|\, dz \\ &\quad + \int_{\mathcal{Q}_R} |f(x,t) - f(x_0,t_0)||D(u-v)|\, dz \\ &=: I + II + III. \end{aligned}$$

We estimate separately the integrals.

Estimate for I: applying (3.3.4), Hölder inequality and triangle inequality we find

$$\begin{aligned} I &\le \varepsilon \int_{\mathcal{Q}_R} (|Du - (Du)_{\mathcal{Q}_R}| + \lambda)|D(u-v)|dz \\ &\le \left(\varepsilon^2 \int_{\mathcal{Q}_R} (|Du - (Du)_{\mathcal{Q}_R}| + \lambda)^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_R} |D(u-v)|^2 dz\right)^{\frac{1}{2}} \\ &\le \left(2\varepsilon^2 \int_{\mathcal{Q}_R} (|Du - (Du)_{\mathcal{Q}_R}|^2 + \lambda^2) dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_R} |D(u-v)|^2 dz\right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand Hölder inequality gives

$$\begin{aligned} II &\le \omega_\gamma(R) \int_{\mathcal{Q}_R} |Du||D(u-v)|dz \\ &\le \left(\omega_\gamma^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_R} |D(u-v)|^2 dz\right)^{\frac{1}{2}}. \end{aligned}$$

To evaluate the last addendum we use hypothesis $(\mathcal{H}4')$

$$III \le \left(\omega_f^2(R)|\,\mathcal{Q}_R\,|\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_R} |D(u-v)|^2 dz\right)^{\frac{1}{2}}.$$

Combining the previous estimates and keeping in mind the definition of λ we conclude with

$$\begin{aligned} &\int_{\mathcal{Q}_R} |D(u-v)|^2 \le \\ &\le \frac{6\varepsilon^2}{\nu^2} \int_{\mathcal{Q}_R} (|Du - (Du)_{\mathcal{Q}_R}|^2 + \lambda^2) dz + \frac{3\omega_\gamma^2(R)}{\nu^2} \int_{\mathcal{Q}_R} |Du|^2 dz + \frac{3\omega_f^2(R)|\,\mathcal{Q}_R\,|}{\nu^2} \\ &= \frac{6\varepsilon^2}{\nu^2} \left[\int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz + \lambda^2 |\,\mathcal{Q}_R\,|\right] + \frac{3\omega_\gamma^2(R)}{\nu^2} \int_{\mathcal{Q}_R} |Du|^2 dz \\ &+ \frac{3\omega_f^2(R)|\,\mathcal{Q}_R\,|}{\nu^2} \\ &= c\left[\varepsilon^2 \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz + \omega_\gamma^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz + \omega_f^2(R)|\,\mathcal{Q}_R\,|\right] \end{aligned}$$

that is (3.3.3). □

3.3.2 Proof of the Theorem 3.0.3

In this section we will prove Theorem 3.0.3 using the inequality proved in last section together with Lemma 3.2.1 and Lemma 3.2.3.
Using (3.2.2) with $\tau \in (0,1)$, the inequality (3.3.3), the fact that

$$\int_{\mathcal{Q}_{\tau R}} |(Dv)_{\mathcal{Q}_{\tau R}} - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz \le \int_{\mathcal{Q}_{\tau R}} |Du - Dv|^2 dz,$$

and the triangle inequality we have

$$\begin{aligned}
&\int_{\mathcal{Q}_{\tau R}} |Du - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz \\
&\quad = \int_{\mathcal{Q}_{\tau R}} |Du - Dv + Dv - (Dv)_{\mathcal{Q}_{\tau R}} + (Dv)_{\mathcal{Q}_{\tau R}} - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz \\
&\quad \leq c \left[\int_{\mathcal{Q}_{\tau R}} |Du - Dv|^2 dz + \int_{\mathcal{Q}_{\tau R}} |Dv - (Dv)_{\mathcal{Q}_{\tau R}}|^2 dz \right] \\
&\quad \leq c \left[\varepsilon^2 \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz + \omega_\gamma^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz + \omega_f^2(R) |\, \mathcal{Q}_R \,| \right] \\
&\quad + c\tau^{n+4} \int_{\mathcal{Q}_R} |Dv - (Dv)_{\mathcal{Q}_R}|^2 dz \\
&\quad \leq c\Big(\varepsilon^2 + \tau^{n+4}\Big) \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz \\
&\quad + c\,\omega_\gamma^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz + c\,\omega_f^2(R) |\, \mathcal{Q}_R \,|.
\end{aligned}$$

Furthermore, using (3.3.1), we infer

$$\begin{aligned}
\int_{\mathcal{Q}_{\tau R}} |Du - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz &\leq \tau^{n+2} c\Big(\frac{\varepsilon^2}{\tau^{n+2}} + \tau^2\Big) \int_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz \\
&\quad + 6c\,\omega^2(R) \int_{\mathcal{Q}_R} |Du|^2 dz + 6c\,\omega^2(R) |\, \mathcal{Q}_R \,|
\end{aligned}$$

and dividing by $|\, \mathcal{Q}_{\tau R} \,|$ we have

$$\begin{aligned}
\fint_{\mathcal{Q}_{\tau R}} |Du - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz &\leq c\Big(\frac{\varepsilon^2}{\tau^{n+2}} + \tau^2\Big) \fint_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz \\
&\quad + \frac{c}{\tau^{n+2}} \omega^2(R) \fint_{\mathcal{Q}_R} |Du|^2 dz + \frac{c}{\tau^{n+2}} \omega^2(R). \qquad (3.3.7)
\end{aligned}$$

For each fixed $\theta < 1$ we can find τ_θ and ε_θ such that

$$\frac{\varepsilon^2}{\tau^{n+2}} + \tau^2 \leq \frac{\theta}{c} \qquad (3.3.8)$$

for

$$\varepsilon_\theta^2 = \tau_\theta^{n+2} \Big(\frac{\theta}{c} - \tau_\theta^2\Big).$$

Choosing $\varepsilon \in (0, \varepsilon_\theta)$ there exists $\tau \in (0,1)$ such that (3.3.7) becomes

$$\begin{aligned}
&\fint_{\mathcal{Q}_{\tau R}} |Du - (Du)_{\mathcal{Q}_{\tau R}}|^2 dz \leq \\
&\quad \leq \theta \fint_{\mathcal{Q}_R} |Du - (Du)_{\mathcal{Q}_R}|^2 dz + \frac{c}{\tau^{n+2}} \omega^2(R) \fint_{\mathcal{Q}_R} |Du|^2 dz + \frac{c}{\tau^{n+2}} \omega^2(R). \qquad (3.3.9)
\end{aligned}$$

At this point we iterate the decay estimate. Fixed $\Omega_{t_2} \Subset \Omega_{t_1} \Subset \Omega_T$, let $d = \text{dist}_{\mathcal{P}}(\Omega_{t_2}, \partial_{\mathcal{P}}\Omega_{t_1})$. Let $(x_0, t_0) \in \Omega_{t_2}$ be an L^2-Lebesgue point of Du:

$$Du(x_0,t_0) = \lim_{\rho\to 0} \fint_{\mathcal{Q}_\rho} Du\, dz$$

$$\lim_{\rho\to 0} \fint_{\mathcal{Q}_\rho} |Du - (Du)_{\mathcal{Q}_\rho}|\, dz = 0$$

where $\mathcal{Q}_\rho \equiv \mathcal{Q}_\rho(x_0,t_0)$.
Now we define an appropriate radius R_0 in the following way:
if $\fint_{\mathcal{Q}_R} |Du|^2 dz > M^2$ for all $R \in (0,d]$, then let $R_0 = d$;
if $\fint_{\mathcal{Q}_R} |Du|^2 dz \le M^2$ for some $R \in (0,d]$, then we set

$$R_0 := \inf\left\{0 < R \le d : \fint_{\mathcal{Q}_R} |Du|^2 dz \le M^2\right\}.$$

We note that for every $\rho \ge R_0$ it happens that $\fint_{\mathcal{Q}_\rho} |Du|^2 dz \le M^2$.
If $R_0 = 0$ we have that

$$\begin{aligned} |Du(x_0,t_0)|^2 &= \left|\lim_{\rho\to 0} \fint_{\mathcal{Q}_\rho} Du\, dz\right|^2 \\ &\le \lim_{\rho\to 0} \fint_{\mathcal{Q}_\rho} |Du|^2 dz \le M^2 \end{aligned}$$

so $|Du(x_0,t_0)| \le M$.
If $0 < R_0 < d$ then $\fint_{\mathcal{Q}_{R_0}} |Du|^2 dz = M^2$ by continuity. Moreover for all $\overline{R} \in \Big\{R \in (0,d] : \fint_{\mathcal{Q}_R} |Du|^2 dz \le M^2\Big\}$ we have that $\fint_{\mathcal{Q}_{\overline{R}}} |Du|^2 dz \le M^2$, so for all $R < R_0$ it follows $\fint_{\mathcal{Q}_R} |Du|^2 dz > M^2$.
Finally, if $R_0 = d$ then

$$\fint_{\mathcal{Q}_{R_0}} |Du|^2 dz = \frac{1}{\omega_n d^{n+2}} \int_{\mathcal{Q}_{R_0}} |Du|^2 dz \le \frac{1}{\omega_n d^{n+2}} \int_{\Omega_{t_1}} |Du|^2 dz$$

and for all $R < d$ we have $\fint_{\mathcal{Q}_R} |Du|^2 dz > M^2$.
This means that for $0 < R_0 \le d$ we have

$$\fint_{\mathcal{Q}_{R_0}} |Du|^2 dz \le \max\left\{M^2, \frac{1}{\omega_n d^{n+2}} \int_{\Omega_{t_1}} |Du|^2 dz\right\} =: \Lambda$$

and for all $R \in (0, R_0)$, $\fint_{\mathcal{Q}_R} |Du|^2 dz > M^2$.
Consequently from (3.3.9) with $\tau \in (0,1)$:

$$\begin{aligned}
&\fint_{\mathcal{Q}_{\tau^j R_0}} |Du - (Du)_{\mathcal{Q}_{\tau^j R_0}}|^2 dz \\
&\qquad \leq \theta \fint_{\mathcal{Q}_{\tau^{j-1} R_0}} |Du - (Du)_{\mathcal{Q}_{\tau^{j-1} R_0}}|^2 dz + \frac{c}{\tau^{n+2}} \omega^2(\tau^{j-1} R_0) \fint_{\mathcal{Q}_{\tau^{j-1} R_0}} |Du|^2 dz \\
&\qquad + \frac{c}{\tau^{n+2}} \omega^2(\tau^{j-1} R_0)
\end{aligned}$$

and by iteration

$$\begin{aligned}
&\fint_{\mathcal{Q}_{\tau^j R_0}} |Du - (Du)_{\mathcal{Q}_{\tau^j R_0}}|^2 dz \leq \theta^j \fint_{\mathcal{Q}_{R_0}} |Du - (Du)_{\mathcal{Q}_{R_0}}|^2 dz \\
&+ \frac{c}{\tau^{n+2}} \sum_{i=0}^{j-1} \theta^i \omega^2(\tau^{j-i-1} R_0) \fint_{\mathcal{Q}_{\tau^{j-i-1} R_0}} |Du|^2 dz + \frac{c}{\tau^{n+2}} \sum_{i=0}^{j-1} \theta^i \omega^2(\tau^{j-1-i} R_0). \qquad (3.3.10)
\end{aligned}$$

We estimate separately the last two terms.
Using hypothesis $(\mathcal{H}4')$, relation (3.3.8) (from which $\tau^2 < \theta$), Lemma 3.2.3 and taking into account that $\beta = n + 2 - \delta$ we deduce

$$\begin{aligned}
&\frac{c}{\tau^{n+2}} \sum_{i=0}^{j-1} \theta^i \omega^2(\tau^{j-i-1} R_0) \fint_{\mathcal{Q}_{\tau^{j-i-1} R_0}} |Du|^2 dz \leq \\
&\qquad \leq \frac{c}{\tau^{n+2}} \sum_{i=0}^{j-1} \theta^i \frac{C \tau^{2\alpha(j-1-i)} R_0^{2\alpha} \tau^{\beta(j-1-i)} R_0^{\beta}}{\tau^{(n+2)(j-1-i)} R_0^{n+2} \omega_n} \\
&\qquad = \frac{\tilde{C}}{\tau^{n+2} \omega_n} R_0^{2\alpha+\beta-(n+2)} \sum_{i=0}^{j-1} \theta^i \tau^{(2\alpha+\beta-(n+2))(j-1-i)} \\
&\qquad \leq \frac{\tilde{C}}{\tau^{n+2} \omega_n R_0^{\delta-2\alpha}} \sum_{i=0}^{j-1} \theta^{i+(j-1-i)\frac{2\alpha-\delta}{2}} \\
&\qquad = \frac{\tilde{C}}{\tau^{n+2} \omega_n R_0^{\delta-2\alpha}} \theta^{(j-1)\frac{2\alpha-\delta}{2}} \sum_{i=0}^{j-1} \theta^{i(1-\frac{2\alpha-\delta}{2})} \\
&\qquad \leq \frac{\tilde{C}}{\tau^{n+2} \omega_n R_0^{\delta-2\alpha}} \frac{\theta^{(j-1)\frac{2\alpha-\delta}{2}}}{1-\theta^{1-\frac{2\alpha-\delta}{2}}}
\end{aligned}$$

and choosing $\delta \le 2\alpha$ we have

$$\begin{aligned}\frac{c}{\tau^{n+2}}\sum_{i=0}^{j-1}&\theta^i\omega^2(\tau^{j-i-1}R_0)\fint_{\mathcal{Q}_{\tau^{j-i-1}R_0}}|Du|^2dz\\ &\le \frac{\tilde{C}}{\tau^{n+2}\omega_n R_0^{\delta-2\alpha}}\frac{\theta^{(j-1)\frac{2\alpha-\delta}{2}}}{1-\theta^{1-\frac{2\alpha-\delta}{2}}}\\ &\le C_1\theta^{(j-1)\frac{2\alpha-\delta}{2}}.\end{aligned}$$

Similarly, keeping in mind (3.0.7), we have

$$\begin{aligned}\frac{c}{\tau^{n+2}}\sum_{i=0}^{j-1}\theta^i\omega^2(\tau^{j-1-i}R_0) &\le \frac{C}{\tau^{n+2}}\sum_{i=0}^{j-1}\theta^i\tau^{2\alpha(j-1-i)}R_0^{2\alpha}\\ &\le \frac{C\,R_0^{2\alpha}}{\tau^{n+2}}\sum_{i=0}^{j-1}\theta^{i+\alpha(j-1-i)}\\ &\le \frac{C\,R_0^{2\alpha}}{\tau^{n+2}}\theta^{\alpha(j-1)}\sum_{i=0}^{j-1}\theta^{i(1-\alpha)}\\ &\le \frac{C\,R_0^{2\alpha}}{\tau^{n+2}}\frac{\theta^{\alpha(j-1)}}{1-\theta^{1-\alpha}} = C_2\theta^{\alpha(j-1)}.\end{aligned}$$

Then (3.3.10) becomes

$$\begin{aligned}&\fint_{\mathcal{Q}_{\tau^j R_0}}|Du-(Du)_{\mathcal{Q}_{\tau^j R_0}}|^2dz \le\\ &\quad\le \theta^j\fint_{\mathcal{Q}_{R_0}}|Du-(Du)_{\mathcal{Q}_{R_0}}|^2dz + C_1\theta^{(j-1)\frac{2\alpha-\delta}{2}} + C_2\theta^{\alpha(j-1)}.\end{aligned}\tag{3.3.11}$$

Now, by triangle inequality

$$\begin{aligned}|(Du)_{\mathcal{Q}_{\tau^j R_0}}| &= |(Du)_{\mathcal{Q}_{\tau^j R_0}} - (Du)_{\mathcal{Q}_{R_0}} + (Du)_{\mathcal{Q}_{R_0}}|\\ &\le |(Du)_{\mathcal{Q}_{R_0}}| + \sum_{i=1}^{j}|(Du)_{\mathcal{Q}_{\tau^i R_0}} - (Du)_{\mathcal{Q}_{\tau^{i-1}R_0}}|\end{aligned}$$

and taking into account that

$$|(Du)_{\mathcal{Q}_{R_0}}| \le (|Du|)_{\mathcal{Q}_{R_0}} \le (|Du|^2)^{\frac{1}{2}}_{\mathcal{Q}_{R_0}} \le \Lambda^{\frac{1}{2}}$$

and

$$\begin{aligned}|(Du)_{\mathcal{Q}_{\tau^i R_0}} - (Du)_{\mathcal{Q}_{\tau^{i-1}R_0}}| &= \left|\fint_{\mathcal{Q}_{\tau^i R_0}} Du\,dz - \fint_{\mathcal{Q}_{\tau^{i-1}R_0}}(Du)_{\mathcal{Q}_{\tau^{i-1}R_0}}\,dz\right|\\ &\le \frac{1}{\tau^{n+2}}\fint_{\mathcal{Q}_{\tau^{i-1}R_0}}|Du-(Du)_{\mathcal{Q}_{\tau^{i-1}R_0}}|\,dz\end{aligned}$$

we conclude

$$
\begin{aligned}
|(Du)_{\mathcal{Q}_{\tau^j R_0}}| &\le \Lambda^{\frac{1}{2}} + \frac{1}{\tau^{n+2}} \sum_{i=1}^{j} \fint_{\mathcal{Q}_{\tau^{i-1} R_0}} |Du - (Du)_{\mathcal{Q}_{\tau^{i-1} R_0}}| \, dz \\
&\le \Lambda^{\frac{1}{2}} + \frac{1}{\tau^{n+2}} \sum_{i=1}^{j} \left(\fint_{\mathcal{Q}_{\tau^{i-1} R_0}} |Du - (Du)_{\mathcal{Q}_{\tau^{i-1} R_0}}|^2 dz \right)^{\frac{1}{2}} \\
&\le \Lambda^{\frac{1}{2}} + \frac{1}{\tau^{n+2}} \sum_{i=1}^{j} \Big[\theta^{i-1} \Lambda + C_1 \theta^{\frac{2\alpha-\delta}{2}(i-2)} + C_2 \theta^{\alpha(i-2)} \Big]^{\frac{1}{2}} \\
&\le \Lambda^{\frac{1}{2}} + \frac{\Lambda^{\frac{1}{2}}}{\sqrt{3}\tau^{n+2}(1-\sqrt{\theta})} + \frac{\tilde{C}_1}{\sqrt{3}\tau^{n+2}} \frac{1}{\theta^{\frac{2\alpha-\delta}{4}}(1-\theta^{\frac{2\alpha-\delta}{4}})} + \frac{\tilde{C}_2}{\sqrt{3}\tau^{n+2}} \frac{1}{\theta^{\frac{\alpha}{2}}(1-\theta^{\frac{\alpha}{2}})} \\
&= \Lambda^{\frac{1}{2}} C + \tilde{C}.
\end{aligned}
$$

Finally it find out that

$$
\begin{aligned}
|Du(x_0, t_0)| &= \Big| \lim_{j \to \infty} (Du)_{\mathcal{Q}_{\tau^j R_0}} \Big| \\
&\le \lim_{j \to \infty} |(Du)_{\mathcal{Q}_{\tau^j R_0}}| \\
&\le \Lambda^{\frac{1}{2}} C + \tilde{C} \\
&\le C \Big(M^2 + \int_{\Omega_{t_1}} |Du|^2 dz \Big)^{\frac{1}{2}} + \tilde{C}
\end{aligned}
$$

and this concludes the proof.

Chapter 4

Partial Regularity Results for Asymptotic Quasiconvex Functionals with General Growth

In this Chapter we study variational integrals of the type

$$\mathcal{F}(u) := \int_\Omega f(Du)\,dx \qquad \text{for } u : \Omega \to \mathbb{R}^N$$

where Ω is an open bounded set in $\mathbb{R}^n$, $n \geq 2$, $N \geq 1$ and f is a continuous function satisfying a φ-growth condition:

$$|f(z)| \leq C(1 + \varphi(|z|)) \quad \forall z \in \mathbb{R}^{Nn},$$

where $C > 0$ and φ is a given N-function.
We will consider the following definition of a minimizer of $\mathcal{F}$.

Definition 4.0.1. *A map $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ is a $W^{1,\varphi}$-minimizer of $\mathcal{F}$ in Ω if*

$$\mathcal{F}(u) \leq \mathcal{F}(u + \xi)$$

for every $\xi \in W_0^{1,\varphi}(\Omega, \mathbb{R}^N)$.

As already observed, the quasiconvexity was originally introduced for proving the lower semicontinuity and the existence of minimizers of variational integrals of the Calculus of Variations. In fact, assuming a power growth condition on f, quasiconvexity is a necessary and sufficient condition for the sequential lower semicontinuity on $W^{1,p}(\Omega, \mathbb{R}^N)$, $p > 1$ (see [1] and [68]). In the regularity theory a stronger definition, the strict quasiconvexity, is needed, a notion which has nowadays become a common condition in the vectorial Calculus of Variations (see [43],[2], [15]).
In order to treat the general growth case, we consider the notion of strictly $W^{1,\varphi}$-quasiconvexity introduced in [27] (see also [9]).

Definition 4.0.2 (Strict $W^{1,\varphi}$-quasiconvexity)**.** *A continuous function $f:\mathbb{R}^{Nn}\to\mathbb{R}$ is said to be strictly $W^{1,\varphi}$-quasiconvex if there exists a positive constant $k>0$ such that*

$$\fint_{\mathcal{B}_1} f(z+D\xi)\,dx \geq f(z)+k\fint_{\mathcal{B}_1}\varphi_{|z|}(|D\xi|)\,dx$$

for all $\xi\in C^1_0(\mathcal{B}_1)$, for all $z\in\mathbb{R}^{Nn}$, where $\varphi_a(t)\sim t^2\varphi''(a+t)$ for $a,t\geq 0$.

We will exploit an adequate notion of $W^{1,\varphi}$-quasiconvexity at infinity, which we will call $W^{1,\varphi}$-asymptotic quasiconvexity:

Definition 4.0.3 (Asymptotic $W^{1,\varphi}$-quasiconvexity)**.** *A function $f:\mathbb{R}^{Nn}\to\mathbb{R}$ is asymptotically $W^{1,\varphi}$-quasiconvex if there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that*

$$f(z)=g(z)\ \text{ for } |z|>M.$$

We note that in recent years a growing literature has considered the subject of asymptotic regular problems: regularity theory for integrands with a particular structure near infinity has been investigated first in [17] and subsequentely in [55], [81], [18], [30], [67], [80], [46], [49], [29], [31].

We deal with the problem wondering if, when you *localize at infinity* the natural assumptions to have regularity, this regularity breaks down or not. It is the same question faced in [7] and [4], where you do not require a global strict convexity or quasiconvexity assumption: all the hypotheses are localized in some point z_0 and you obtain that minimizers are Hölder continuous near points where the integrand function is "close" to the value z_0.
Thus, after establishing several characterizations of the notion of $W^{1,\varphi}$-asymptotic quasiconvexity (see Theorem 4.2.1) we will prove the following result.

Theorem 4.0.2. *Let $z_0\in\mathbb{R}^n$ with $|z_0|>M+1$ so that (4.5.2) holds in $\mathcal{B}_\rho(x_0)$, let $u\in W^{1,\varphi}(\Omega,\mathbb{R}^N)$ be a minimizer of $\mathcal{F}$, and $V(z)=\sqrt{\frac{\varphi'(|z|)}{|z|}}z$. Assume that f satisfies $(\mathcal{H}1)-(\mathcal{H}5)$.*
If for some $x_0\in\Omega$

$$\lim_{r\to 0}\fint_{\mathcal{B}_r(x_0)}|V(Du)-V(z_0)|^2=0 \tag{4.0.1}$$

then in a neighborhood of x_0 the minimizer u is $C^{1,\overline{\alpha}}$ for some $\overline{\alpha}<1$.

In order to achieve this regularity result, we have to prove an *excess decay estimate*, where the excess function is defined by

$$\mathbb{E}(\mathcal{B}_R(x_0),u)=\fint_{\mathcal{B}_R(x_0)}|V(Du)-(V(Du))_{\mathcal{B}_R(x_0)}|^2dx.$$

In the power case the main idea is to use a blow-up argument based strongly on the homogeneity of $\varphi(t)=t^p$. Here we have to face with the lack of the homogeneity since the

general growth condition. Thus one makes use of the so-called $\mathcal{A}$-harmonic approximation proved in [27] (see also [85, 34, 35, 37, 39] for the power case). Such tool allows us to compare the solutions of our problem with the solution of the regular one in terms of the closeness of the gradient.
Moreover we will prove that minimizers of $\mathcal{F}$ are Lipschitz continuous on an open and dense subset of Ω.
More precisely we define the set of regular points $\mathcal{R}(u)$ by

$$\mathcal{R}(u) = \{x \in \Omega : u \text{ is Lipschitz near } x\},$$

following that $\mathcal{R}(u) \subset \Omega$ is open.

Corollary 4.0.1. *Assume that f satisfies $(\mathcal{H}1) - (\mathcal{H}5)$. Then, for every minimizer $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$, the regular set $\mathcal{R}(u)$ is dense in Ω.*

We remark that a counterexample of [83] shows that it is not possible to establish regularity outside a negligible set (which would be the natural thing in the vectorial regularity theory). So, our regularity result generalizes the ones given in [83] and [16] for integrands with a power growth condition which become strictly convex and strictly quasiconvex near infinity, respectively.

4.1 Assumptions and Technical Lemmas

4.1.1 Assumptions

The specific assumptions we are considering are now listed:
Let $f : \mathbb{R}^{Nn} \to \mathbb{R}$ be such that
$(\mathcal{H}1)$ $f \in C^1(\mathbb{R}^{Nn}) \cap C^2(\mathbb{R}^{Nn} \setminus \{0\})$;

$(\mathcal{H}2)$ $\forall z \in \mathbb{R}^{Nn},\ |f(z)| \le C(1 + \varphi(|z|))$;

$(\mathcal{H}3)$ f is asymptotically $W^{1,\varphi}$-quasiconvex;

$(\mathcal{H}4)$ $\forall z \in \mathbb{R}^{Nn} \setminus \{0\},\ |D^2 f(z)| \le C\,\varphi''(|z|)$;

$(\mathcal{H}5)$ $\forall z_1, z_2 \in \mathbb{R}^{Nn}$ such that $|z_1| \le \frac{1}{2}|z_2|$ it holds

$$|D^2 f(z_2) - D^2 f(z_2 + z_1)| \le C\,\varphi''(|z_2|)|z_2|^{-\beta}|z_1|^{\beta}.$$

Remark 4.1.1. *Due to hypothesis $(\mathcal{H}2)$, $\mathcal{F}$ is well defined on the Sobolev-Orlicz space $W^{1,\varphi}(\Omega, \mathbb{R}^N)$.*
Let us also observe that Assumption $(\mathcal{H}5)$, that is a Hölder continuity of $D^2 f$ away from zero, has been used to show everywhere regularity of radial functionals with φ-growth (see [29]). We will use it in Lemma 4.4.2 below.

4.1.2 Technical Lemmas

For $z_1, z_2 \in \mathbb{R}^{Nn}$, $\theta \in [0,1]$ we define $z_\theta = z_1 + \theta(z_2 - z_1)$. The following fact can be found in [3] (see Lemma 2.1).

Lemma 4.1.1. *Let $\beta > -1$, then uniformly in $z_1, z_2 \in \mathbb{R}^{Nn}$ with $|z_1| + |z_2| > 0$, it holds:*

$$\int_0^1 |z_\theta|^\beta \, d\theta \sim (|z_1| + |z_2|)^\beta.$$

Next result is a slight generalization of Lemma 20 in [25].

Lemma 4.1.2. *Let φ be an N-function with $\Delta_2(\{\varphi, \varphi*\}) < \infty$; then, uniformly in $z_1, z_2 \in \mathbb{R}^{Nn}$ with $|z_1| + |z_2| > 0$, and in $\mu \geq 0$, it holds*

$$\frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|} \sim \int_0^1 \frac{\varphi'(\mu + |z_\theta|)}{\mu + |z_\theta|} d\theta.$$

From the previous lemmas we derive the following one.

Lemma 4.1.3. *Let φ be an N-function satisfying Assumption 2.4.1. Then, uniformly in $z_1, z_2 \in \mathbb{R}^{Nn}$ with $|z_1| + |z_2| > 0$, and in $\mu \geq 0$, it holds*

$$\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt \sim \varphi''(\mu + |z_1| + |z_2|).$$

Proof. Using $\varphi'(t) \sim t\varphi''(t)$, applying twice Lemma 4.1.2, and taking into account that

$$\mu + |z_1| + |z_1 + z_2| \sim \mu + |z_1| + |z_2|$$

and $\varphi'(2t) \sim \varphi'(t)$, we obtain

$$\begin{aligned}
\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt &\leq c \int_0^1 \int_0^1 t \frac{\varphi'(\mu + |z_1 + stz_2|)}{\mu + |z_1 + stz_2|} \, dsdt \\
&\leq c \frac{\varphi'(\mu + |z_1| + |z_1| + |z_1 + z_2|)}{\mu + |z_1| + |z_1| + |z_1 + z_2|} \\
&\leq c \frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|} \\
&\leq c\varphi''(\mu + |z_1| + |z_2|).
\end{aligned}$$

Similarly, for the other inequality, we have

$$\begin{aligned}
\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt &\geq c \int_0^1 \int_0^1 t \frac{\varphi'(\mu + |z_1 + stz_2|)}{\mu + |z_1 + stz_2|} \, ds \, dt \\
&\geq c \int_0^1 t \frac{\varphi'(\mu + |z_1| + |z_1 + tz_2|)}{\mu + |z_1| + |z_1 + tz_2|} \, dt \\
&\geq \frac{c}{(\mu + |z_1| + |z_2|)^2} \int_0^1 \varphi(\mu + |z_1| + |z_1 + tz_2|) \, t \, dt,
\end{aligned}$$

where, in the last line, we used that $\varphi(t) \sim t\varphi'(t)$.
Due to the Jensen inequality, we go ahead and we obtain

$$\begin{aligned}\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|)\, ds\, dt &\geq \frac{c}{(\mu + |z_1| + |z_2|)^2}\, \varphi\left(\int_0^1 (\mu + |z_1| + |z_1 + tz_2|)\, t\, dt\right) \\ &\geq \frac{c}{(\mu + |z_1| + |z_2|)^2}\, \varphi(\mu + |z_1| + |z_2|) \\ &\geq c\frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|} \\ &\geq c\, \varphi''(\mu + |z_1| + |z_2|),\end{aligned}$$

thanks also to the equivalence between $\varphi(2t)$ and $\varphi(t)$, $\varphi(t)$ and $t\varphi'(t)$, and $\varphi'(t)$ and $t\varphi''(t)$. □

Remark 4.1.2. *From the previous lemma we easily deduce that*

$$\int_0^1 \int_0^1 t\varphi''(\sqrt{1 + |z_1 + stz_2|^2})\, ds\, dt \sim \varphi''(1 + |z_1| + |z_2|),$$

since $\varphi'(t) \sim t\varphi''(t)$, φ' is increasing and $\varphi'(2t) \sim \varphi'(t)$.

The following version of the Sobolev-Poincaré inequality can be found in [25] (Theorem 7):

Theorem 4.1.1. *Let φ be an N-function with $\Delta_2(\varphi, \varphi^*) < \infty$. Then there exist $\alpha \in (0, 1)$ and $k > 0$ such that, if $\mathcal{B} \subset \mathbb{R}^n$ is a ball of radius R and $u \in W^{1,\varphi}(\mathcal{B}, \mathbb{R}^N)$, then*

$$\fint_{\mathcal{B}} \varphi\left(\frac{|u - (u)_{\mathcal{B}}|}{R}\right) dx \leq k\left(\fint_{\mathcal{B}} \varphi^{\alpha}(|Du|)\, dx\right)^{\frac{1}{\alpha}}.$$

The following two lemmas will be useful later.

Lemma 4.1.4. *Let φ satisfy Assumption 2.4.1 and p_0, p_1 be as in Lemma 2.4.1. Then for each $\eta \in (0, 1]$ it holds*

$$\begin{aligned}\varphi_{|a|}(t) &\leq C\eta^{1-\bar{p}'}\varphi_{|b|}(t) + \eta|V(a) - V(b)|^2, \\ (\varphi_{|a|})^*(t) &\leq C\eta^{1-\bar{q}}(\varphi_{|b|})^*(t) + \eta|V(a) - V(b)|^2\end{aligned}$$

for all $a, b \in \mathbb{R}^n$, $t \geq 0$ and $\bar{p} = \min\{p_0, 2\}, \bar{q} = \max\{p_1, 2\}$. The constants depend only on the characteristics of φ.

For the proof see Lemma 2.5 in [26].

Lemma 4.1.5. *Let φ be an N-function satisfying Assumption 2.4.1 and let us consider the function $z \in \mathbb{R}^{Nn} \mapsto \varphi(\sqrt{1 + |z|^2}\,)$. Then, uniformly in $y, z \in \mathbb{R}^{Nn}$ it holds*

$$\langle D^2\varphi(\sqrt{1 + |z + y|^2}\,)y, y\rangle \sim \varphi''(\sqrt{1 + |z + y|^2}\,)|y|^2.$$

Proof. We can see that

$$D\varphi(\sqrt{1+|z+y|^2}\,) = \varphi'(\sqrt{1+|z+y|^2}\,)\frac{z+y}{\sqrt{1+|z+y|^2}},$$

and

$$\begin{aligned} D^2\varphi(\sqrt{1+|z+y|^2}\,) &= \varphi''(\sqrt{1+|z+y|^2}\,)\frac{z+y}{\sqrt{1+|z+y|^2}} \otimes \frac{z+y}{\sqrt{1+|z+y|^2}} \\ &+ \frac{\varphi'(\sqrt{1+|z+y|^2}\,)}{\sqrt{1+|z+y|^2}}\left[\mathbb{I} - \frac{z+y}{\sqrt{1+|z+y|^2}} \otimes \frac{z+y}{\sqrt{1+|z+y|^2}}\right], \end{aligned}$$

where $\mathbb{I} \in \mathbb{R}^{Nn}$ is the identity matrix. Therefore

$$\begin{aligned} \langle D^2\varphi(\sqrt{1+|z+y|^2}\,)y, y\rangle &= \varphi''(\sqrt{1+|z+y|^2}\,)\frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2} \\ &+ \frac{\varphi'(\sqrt{1+|z+y|^2}\,)}{\sqrt{1+|z+y|^2}}\left[|y|^2 - \frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2}\right]. \end{aligned}$$

Using Assumption 2.4.1 and the fact that $\dfrac{|\langle z+y, y\rangle|^2}{1+|z+y|^2} \leq |y|^2$ we deduce

$$\begin{aligned} \langle D^2\varphi(\sqrt{1+|z+y|^2}\,)y, y\rangle &\leq \varphi''(\sqrt{1+|z+y|^2}\,)\frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2} \\ &+ C\varphi''(\sqrt{1+|z+y|^2}\,)\left[|y|^2 - \frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2}\right] \\ &\leq C\varphi''(\sqrt{1+|z+y|^2}\,)|y|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} &\langle D^2\varphi(\sqrt{1+|z+y|^2}\,)y, y\rangle \geq \\ &\geq \varphi''(\sqrt{1+|z+y|^2}\,)\frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2} + C\varphi''(\sqrt{1+|z+y|^2}\,)\left[|y|^2 - \frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2}\right] \\ &= C\varphi''(\sqrt{1+|z+y|^2}\,)|y|^2 + (1-C)\varphi''(\sqrt{1+|z+y|^2}\,)\frac{|\langle z+y, y\rangle|^2}{1+|z+y|^2} \\ &\geq C\varphi''(\sqrt{1+|z+y|^2}\,)|y|^2. \end{aligned}$$

□

4.2 Characterization of asymptotic $W^{1,\varphi}$-quasiconvexity

In this section we will establish some characterizations of asymptotic $W^{1,\varphi}$-quasiconvexity.

Theorem 4.2.1. *Each of the following assertions is equivalent to the asymptotic $W^{1,\varphi}$-quasiconvexity of a function $f:\mathbb{R}^{Nn}\to\mathbb{R}$:*

(i) If f is C^2 outside a large ball there exists a uniformly strictly $W^{1,\varphi}$-quasiconvex function g which is C^2 outside a large ball with

$$\lim_{|z|\to\infty}\frac{|D^2f(z)-D^2g(z)|}{\varphi''(|z|)}=0. \tag{4.2.1}$$

(ii) If f is locally bounded from below, then there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that

$$f(z)=g(z)\ \text{ for } |z|>M$$

and

$$g\le f\ \text{ on } \mathbb{R}^{Nn}.$$

(iii) If f is locally bounded from above, then there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that

$$f(z)=g(z)\ \text{ for } |z|>M$$

and

$$g\ge f\ \text{ on } \mathbb{R}^{Nn}.$$

(iv) If f satisfies $(\mathcal{H}2)$ there exist positive constants M,k,L such that

$$\fint_{\mathcal{B}_1} f(z+D\xi)\,dx\ge f(z)+k\fint_{\mathcal{B}_1}\varphi_{|z|}(|D\xi|)\,dx \tag{4.2.2}$$

for $|z|>M$ and $\xi\in C_c^\infty(\mathcal{B}_1,\mathbb{R}^N)$, and

$$|f(z_2)-f(z_1)|\le L|z_1-z_2|\varphi'(1+|z_1|+|z_2|) \tag{4.2.3}$$

for all $|z_1|,|z_2|>M$.

Proof. The proof stands on four steps.
Step 1: We want to prove that f asymptotically $W^{1,\varphi}$-quasiconvex is equivalent to (i). Let us show that (i) implies the asymptotic $W^{1,\varphi}$-quasiconvexity of f, the other implication being evidently true.
Let g be as in (i). We may assume that f,g are $C^2(\mathbb{R}^{Nn}\setminus\overline{\mathcal{B}}_{\frac12})$ and taking $h=f-g$ we have that $h\in C^2(\mathbb{R}^{Nn}\setminus\overline{\mathcal{B}}_{\frac12})$. In particular, by (4.2.1) it holds

$$\lim_{|z|\to\infty}\frac{|D^2h(z)|}{\varphi''(|z|)}=0. \tag{4.2.4}$$

Our aim is to prove that

$$\lim_{|z|\to\infty} \frac{|Dh(z)|}{\varphi'(|z|)} = 0, \tag{4.2.5}$$

and

$$\lim_{|z|\to\infty} \frac{|h(z)|}{\varphi(|z|)} = 0. \tag{4.2.6}$$

Let us consider $|z| > 1$. Take $\overline{z} := \frac{z}{|z|}$, then $|\overline{z}| = 1$ and

$$\begin{aligned}
\frac{|Dh(z)|}{\varphi'(|z|)} &\leq \frac{1}{\varphi'(|z|)}\left[\int_0^1 |D^2h(\overline{z}+t(z-\overline{z}))||z-\overline{z}|\,dt + |Dh(\overline{z})|\right] \\
&= \int_0^{\frac{1}{\sqrt{|z|}}} \frac{|D^2h(\overline{z}+t(z-\overline{z}))|}{\varphi''(|\overline{z}+t(z-\overline{z})|)} \frac{\varphi''(|\overline{z}+t(z-\overline{z})|)}{\varphi'(|z|)}|z-\overline{z}|\,dt \\
&+ \int_{\frac{1}{\sqrt{|z|}}}^1 \frac{|D^2h(\overline{z}+t(z-\overline{z}))|}{\varphi''(|\overline{z}+t(z-\overline{z})|)} \frac{\varphi''(|\overline{z}+t(z-\overline{z})|)}{\varphi'(|z|)}|z-\overline{z}|\,dt \\
&+ \frac{|Dh(\overline{z})|}{\varphi'(|z|)} \\
&= \mathcal{I} + \mathcal{II} + \mathcal{III}.
\end{aligned}$$

Estimate for $\mathcal{I}$:

$$\begin{aligned}
\mathcal{I} &\leq \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \int_0^{\frac{1}{\sqrt{|z|}}} \frac{\varphi''(|\overline{z}+t(z-\overline{z})|)}{\varphi'(|z|)}|z-\overline{z}|\,dt \\
&\leq \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \int_0^{\frac{1}{\sqrt{|z|}}} \varphi''(1+t(|z|-1))(|z|-1)\,dt \\
&= \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \left[\varphi'(1+t(|z|-1))\right]_0^{\frac{1}{\sqrt{|z|}}} \\
&\leq \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \varphi'\Big(1+\frac{|z|-1}{\sqrt{|z|}}\Big).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
\frac{\varphi'\Big(1+\frac{|z|-1}{\sqrt{|z|}}\Big)}{\varphi'(|z|)} &\leq \frac{\varphi'(1+\sqrt{|z|})}{\varphi'(|z|)} \\
&\leq c\frac{\varphi(1+\sqrt{|z|})}{1+\sqrt{|z|}} \frac{|z|}{\varphi(|z|)} \\
&\leq c\frac{\varphi(1+\sqrt{|z|})}{\varphi(|z|)}\sqrt{|z|} \\
&\leq c\frac{\varphi(\sqrt{|z|})}{\varphi(|z|)}\sqrt{|z|}
\end{aligned}$$

and using Lemma 2.4.1 we can find $p_0 > 1$ and $C > 0$ such that

$$\varphi(\sqrt{|z|}) = \varphi\left(\frac{|z|}{\sqrt{|z|}}\right) \leq C\left(\frac{1}{\sqrt{|z|}}\right)^{p_0} \varphi(|z|).$$

Then we obtain

$$\frac{\varphi'\left(1+\frac{|z|-1}{\sqrt{|z|}}\right)}{\varphi'(|z|)} \leq C\frac{\varphi(\sqrt{|z|})}{\varphi(|z|)}\sqrt{|z|} \leq C\frac{\sqrt{|z|}}{(\sqrt{|z|})^{p_0}} \to 0 \text{ as } |z| \to \infty. \tag{4.2.7}$$

At this point, using (4.2.4) and (4.2.7), we can conclude that $\mathcal{I} \to 0$ as $|z| \to +\infty$.
Now we estimate $\mathcal{II}$:

$$\begin{aligned}
\mathcal{II} &\leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \int_{\frac{1}{\sqrt{|z|}}}^{1} \frac{\varphi''(|\overline{z}+t(z-\overline{z})|)}{\varphi'(|z|)}|z-\overline{z}|\,dt \\
&\leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \int_{\frac{1}{\sqrt{|z|}}}^{1} \varphi''(1+t(|z|-1))(|z|-1)\,dt \\
&\leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \left[\varphi'(1+t(|z|-1))\right]_{\frac{1}{\sqrt{|z|}}}^{1} \\
&= \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \left[\varphi'(|z|) - \varphi'\left(1+\frac{|z|-1}{\sqrt{|z|}}\right)\right] \\
&= \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \left[1 - \frac{\varphi'\left(1+\frac{|z|-1}{\sqrt{|z|}}\right)}{\varphi'(|z|)}\right] \to 0 \text{ as } |z| \to \infty
\end{aligned}$$

where we used (4.2.4) and (4.2.7) to conclude.
Finally

$$\mathcal{III} \leq \frac{1}{\varphi'(|z|)} \max_{|y|=1} |Dh(y)| \to 0 \text{ as } |z| \to \infty.$$

Analogously we also obtain (4.2.6).
We can see that if $\dfrac{|D^ih(z)|}{\varphi^{(i)}(|z|)} \to 0$ as $|z| \to \infty$ for $i = 0,1,2$, then $\dfrac{|D^ih(z)|}{\varphi^{(i)}(1+|z|)} \to 0$ as $|z| \to \infty$.
Taking into account (4.2.4),(4.2.5) and (4.2.6), fixed $\nu > 0$, that we will choose later, there exists $M >> 1$ such that if $|z| > M$ then

$$\begin{aligned}
&|D^2h(z)| \leq \nu\varphi''(1+|z|) \\
&|Dh(z)| \leq \nu\varphi'(1+|z|) \\
&|h(z)| \leq \nu\varphi(1+|z|).
\end{aligned}$$

Let us consider a cut-off function η defined by

$$\begin{cases} 0 \le \eta \le 1 & \text{if } 1 < |x| \le 2 \\ \eta = 1 & \text{if } |x| > 2 \\ \eta = 0 & \text{if } |x| \le 1. \end{cases}$$

Set

$$\alpha := \max \left\{ \sup_{\mathbb{R}^{Nn}} |D\eta|, \sup_{\mathbb{R}^{Nn}} |D^2\eta| \right\}$$

and let us consider $\eta_M(z) = \eta(\frac{z}{M+1})$. Then we have

$$|D\eta_M| \le \frac{\alpha}{M+1} \quad \text{and} \quad |D^2\eta_M| \le \frac{\alpha}{(M+1)^2}.$$

Let $\Phi := \eta_M h$; then for $M \le |z| \le 2M$ we have

$$|D^2\Phi(z)| \le |D^2\eta_M(z)||h(z)| + 2|D\eta_M(z)||Dh(z)| + |\eta_M(z)||D^2h(z)|.$$

Taking into account the previous estimates, (2.4.1), (2.4.3) and $M \le |z| \le 2M$, we have

$$\begin{aligned} |D^2\Phi(z)| &\le \frac{\nu\alpha}{(M+1)^2}\varphi(1+|z|) + \frac{2\nu\alpha}{(M+1)}\varphi'(1+|z|) + \nu\varphi''(1+|z|) \\ &\le \left[\frac{\nu\alpha}{(M+1)^2}(1+|z|)^2 + \frac{2\nu\alpha}{(M+1)}(1+|z|) + \nu \right] \varphi''(1+|z|) \\ &= \lambda\nu\varphi''(1+|z|). \end{aligned}$$

In particular we can conclude that

$$|D^2\Phi(z)| \le \lambda\nu\varphi''(1+|z|) \quad \forall z \in \mathbb{R}^{Nn}. \tag{4.2.8}$$

Let $\xi \in C_c^\infty(\mathcal{B}_1)$; we can write

$$\Phi(z + D\xi) = \Phi(z) + \langle D\Phi(z), D\xi \rangle + \int_0^1 (1-t)\langle D^2\Phi(z + tD\xi)D\xi, D\xi \rangle \, dt.$$

Integrating over $\mathcal{B}_1$ and by using (4.2.8), (2.4.3), Lemma 4.1.2, the fact

$$1 + |z| + |D\xi + z| \sim 1 + |z| + |D\xi| \tag{4.2.9}$$

and $\varphi_a(t) \sim \varphi''(a+t)t^2$ we get

$$
\begin{aligned}
\fint_{\mathcal{B}_1} \Phi(z+D\xi)\,dx &= \fint_{\mathcal{B}_1} \Phi(z)\,dx + \fint_{\mathcal{B}_1} \langle D\Phi(z), D\xi\rangle\,dx \\
&\quad + \fint_{\mathcal{B}_1} \int_0^1 (1-t)\langle D^2\Phi(z+tD\xi)D\xi, D\xi\rangle\,dtdx \\
&\geq \Phi(z) - \fint_{\mathcal{B}_1} \int_0^1 (1-t)|D^2\Phi(z+tD\xi))||D\xi|^2\,dtdx \\
&\geq \Phi(z) - \lambda\nu \fint_{\mathcal{B}_1} \int_0^1 (1-t)\varphi''(1+|z+tD\xi)|)|D\xi|^2\,dtdx \\
&\geq \Phi(z) - \lambda\nu c \fint_{\mathcal{B}_1} \int_0^1 \frac{\varphi'(1+|z+tD\xi|)}{1+|z+tD\xi|}|D\xi|^2\,dtdx \qquad (4.2.10)\\
&\geq \Phi(z) - \lambda\nu c \fint_{\mathcal{B}_1} \frac{\varphi'(1+|z|+|D\xi+z|)}{1+|z|+|D\xi+z|}|D\xi|^2\,dtdx \\
&\geq \Phi(z) - \lambda\nu c \fint_{\mathcal{B}_1} \frac{\varphi'(1+|z|+|D\xi|)}{1+|z|+|D\xi|}|D\xi|^2\,dtdx \\
&\geq \Phi(z) - \lambda\nu c \fint_{\mathcal{B}_1} \varphi''(1+|z|+|D\xi|)|D\xi|^2\,dx \\
&\geq \Phi(z) - \lambda\nu c \fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx.
\end{aligned}
$$

Let us take $G := g + \Phi$ with g uniformly strictly $W^{1,\varphi}$-quasiconvex with constant $k > 0$ and Φ satisfying (4.2.10). Consequently

$$
\begin{aligned}
\fint_{\mathcal{B}_1} G(z+D\xi)\,dx &\geq G(z) + \Big(k - \lambda\nu c\Big)\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx \\
&= G(z) + \tilde{k}\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx
\end{aligned}
$$

where $\tilde{k} > 0$ if we choose $\nu < \frac{k}{\lambda c}$.
Thus G is uniformly strictly $W^{1,\varphi}$-quasiconvex with constant $\tilde{k} > 0$ and $G(z) = f(z)$ for $|z| > 2(M+1)$. This proves the asymptotic quasiconvexity of f.

Step 2: We want to prove that f asymptotically $W^{1,\varphi}$- quasiconvex is equivalent to (ii), and it suffices to prove that asymptotic $W^{1,\varphi}$-quasiconvexity of f implies (ii). Assume f asymptotic $W^{1,\varphi}$- quasiconvex, i.e. there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that $f(z) = g(z)$ for $|z| > M$.
Now g is locally bounded and f is locally bounded from below, so we have that

$$
\alpha := \sup_{|z|\leq M} [g(z) - f(z)] < \infty.
$$

Let $R > M$ and η be a $C_c^\infty(\mathcal{B}_R)$ function, non-negative on $\mathbb{R}^{Nn}$ and such that

$$
|D^2\eta(z)| \leq \nu\varphi''(1+|z|) \text{ on } \mathbb{R}^{Nn} \text{ and } \eta(z) \geq \alpha \text{ for } |z| \leq M \qquad (4.2.11)
$$

where ν will be chosen later. Let $\xi \in C_c^\infty(\mathcal{B}_1)$; then we can write

$$\eta(z+D\xi) = \eta(z) + \langle D\eta(z), D\xi\rangle + \int_0^1 (1-t)\langle D^2\eta(z+tD\xi)D\xi, D\xi\rangle\, dt.$$

Integrating over $\mathcal{B}_1$ it holds, by (4.2.11),

$$\begin{aligned}
\fint_{\mathcal{B}_1} \eta(z+D\xi)\, dx &= \fint_{\mathcal{B}_1} \eta(z)\, dx + \fint_{\mathcal{B}_1} \langle D\eta(z), D\xi\rangle\, dx \\
&\quad + \fint_{\mathcal{B}_1} \int_0^1 (1-t)\langle D^2\eta(z+tD\xi)D\xi, D\xi\rangle\, dtdx \\
&\le \eta(z) + \fint_{\mathcal{B}_1} \int_0^1 (1-t)|D^2\eta(z+tD\xi)||D\xi|^2\, dtdx \qquad (4.2.12)\\
&\le \eta(z) + \nu\fint_{\mathcal{B}_1} \int_0^1 \varphi''(1+|z+tD\xi|)|D\xi|^2\, dtdx \\
&\le \eta(z) + \nu c\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx.
\end{aligned}$$

where we used, as before, $\varphi'(t) \sim t\varphi''(t)$, Lemma 4.1.2, (4.2.9) and $\varphi_a(t) \sim \varphi''(a+t)t^2$. Now taking $G = g - \eta$, with g and η satisfying (1.0.12) and (4.2.12), we have

$$\fint_{\mathcal{B}_1} G(z+D\xi)dx \ge G(z) + \tilde{k}\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx$$

where $\tilde{k} > 0$ if we choose $\nu = \frac{k}{2c}$. This means that G is uniformly strictly $W^{1,\varphi}$- quasiconvex. But $\eta(z) \ge \alpha \ge g(z) - f(z)$ and $\eta(z) = g(z) - G(z)$, so $G(z) \le f(z)$ for $|z| \le M$.

Step 3: The proof is similar to the previous one.

Step 4: Assume that f is a Borel measurable function, satisfying ($\mathcal{H}2$). Since quasiconvex functions are locally Lipschitz (see [45]), we can see that (ii) implies (iv). So it suffices to show that a function satisfying (iv) is asymptotically $W^{1,\varphi}$- quasiconvex.
Assume that f satisfies (iv) and consider the function

$$F(z) := f(z) - \varepsilon\varphi(\sqrt{1+|z|^2})$$

for $z \in \mathbb{R}^{Nn}$. Here $\varepsilon > 0$ will be chosen later appropriately. Now we prove that F satisfies (4.2.2) and (4.2.3).
Let $\xi \in C_c^\infty(\mathcal{B}_1)$. Since f satisfies (4.2.2), we can write

$$\begin{aligned}
\fint_{\mathcal{B}_1} F(z+D\xi)\, dx &= \fint_{\mathcal{B}_1} f(z+D\xi)\, dx - \varepsilon\fint_{\mathcal{B}_1} \varphi(\sqrt{1+|z+D\xi|^2})\, dx \\
&\ge f(z) + k\fint_{\mathcal{B}_1} \varphi_{|z|}(D\xi)\, dx - \varepsilon\fint_{\mathcal{B}_1} \varphi(\sqrt{1+|z+D\xi|^2})\, dx \qquad (4.2.13)\\
&= F(z) + k\fint_{\mathcal{B}_1} \varphi_{|z|}(D\xi)\, dx - \varepsilon\fint_{\mathcal{B}_1} \left[\varphi(\sqrt{1+|z+D\xi|^2}) - \varphi(\sqrt{1+|z|^2})\right]\, dx.
\end{aligned}$$

Note that

$$\varphi(\sqrt{1+|z+D\xi|^2}) = \varphi(\sqrt{1+|z|^2}) + \langle D\varphi(\sqrt{1+|z|^2}), D\xi\rangle + \int_0^1\int_0^1 t\langle D^2\varphi(\sqrt{1+|z+stD\xi|^2})D\xi, D\xi\rangle\, dsdt.$$

Thus integrating over $\mathcal{B}_1$ and applying Lemma 4.1.5, Remark 4.1.2 and $\varphi''(a+t)t^2 \sim \varphi_a(t)$ for $a, t \geq 0$, it follows that

$$\begin{aligned}
&\fint_{\mathcal{B}_1} \left[\varphi(\sqrt{1+|z+D\xi|^2}) - \varphi(\sqrt{1+|z|^2})\right] dx = \fint_{\mathcal{B}_1} \langle D\varphi(\sqrt{1+|z|^2}), D\xi\rangle\, dx \\
&\quad + \fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\langle D^2\varphi(\sqrt{1+|z+stD\xi|^2})D\xi, D\xi\rangle\, ds\, dt\, dx \\
&\quad \leq C\fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\varphi''(\sqrt{1+|z+stD\xi|^2})|D\xi|^2\, ds\, dt\, dx \\
&\quad \leq C\fint_{\mathcal{B}_1} \varphi''(1+|z|+|D\xi|)|D\xi|^2\, dx \\
&\quad \leq C\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx \\
&\quad \leq C\fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\, dx
\end{aligned} \tag{4.2.14}$$

for $|z|$ sufficiently large. Using together (4.2.13) and (4.2.14), and choosing ε small enough, we have

$$\fint_{\mathcal{B}_1} F(z+D\xi)\, dx \geq F(z) + K\fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\, dx.$$

Moreover, taking into account that f satisfies (4.2.3) we deduce, for $|z_1|, |z_2| > M$,

$$\begin{aligned}
|F(z_2) - F(z_1)| &\leq |f(z_2) - f(z_1)| + \varepsilon|\varphi(\sqrt{1+|z_1|^2}) - \varphi(\sqrt{1+|z_2|^2})| \\
&\leq L|z_2 - z_1|\varphi'(1+|z_1|+|z_2|) + \varepsilon|\varphi(\sqrt{1+|z_1|^2}) - \varphi(\sqrt{1+|z_2|^2})| \\
&\leq (L+c)|z_2 - z_1|\varphi'(1+|z_1|+|z_2|).
\end{aligned}$$

Next we let

$$G(z) := \inf\left\{\fint_{\mathcal{B}_1} F(z+D\xi)\, dx : \xi \in C_c^\infty(\mathcal{B}_1, \mathbb{R}^N)\right\}$$

for $z \in \mathbb{R}^{Nn}$. With this definition we have that $G(z) \leq F(z)$ on $\mathbb{R}^{Nn}$ and $G(z) = F(z)$ for $|z| > M$. Now our aim is to prove that G is locally bounded from below.
Fix $z \in \mathbb{R}^{Nn}$ such that $|z| \leq M+1$ and take $\overline{z} \in \mathbb{R}^{Nn}$such that $|\overline{z}| = 2(M+1)$. We have

$$\begin{aligned}
\fint_{\mathcal{B}_1} F(z+D\xi)\, dx &= \fint_{\mathcal{B}_1} [F(z+D\xi) - F(\overline{z}+D\xi)]\, dx + \fint_{\mathcal{B}_1} F(\overline{z}+D\xi)\, dx \\
&= I + II
\end{aligned}$$

Since F satisfies (4.2.2) we get

$$II = \fint_{\mathcal{B}_1} F(\overline{z} + D\xi)\,dx \geq F(\overline{z}) + k \fint_{\mathcal{B}_1} \varphi_{|\overline{z}|}(|D\xi|)\,dx.$$

Now we estimate I:

$$\begin{aligned} I = \frac{1}{|\,\mathcal{B}_1\,|}\Big[\int_{\{|D\xi|\leq 3(M+1)\}} [F(z+D\xi) - F(\overline{z}+D\xi)]\,dx \\ + \int_{\{|D\xi|>3(M+1)\}} [F(z+D\xi) - F(\overline{z}+D\xi)]\,dx\Big] = \frac{1}{|\,\mathcal{B}_1\,|}[I_1 + I_2]. \end{aligned}$$

To estimate I_1 we use the fact that F is locally bounded: $I_1 \geq \tilde{C}$. Regarding I_2 we take into account that F satisfies (4.2.3), then we apply Young's inequality, $\varphi^*(\varphi'(t)) \sim \varphi(t)$ and the Δ_2 condition to deduce

$$\begin{aligned} I_2 &= \int_{\{|D\xi|>3(M+1)\}} [F(z+D\xi) - F(\overline{z}+D\xi)]\,dx \\ &\geq -L \int_{\{|D\xi|>3(M+1)\}} |z-\overline{z}|\varphi'(1+|z+D\xi|+|\overline{z}+D\xi|)\,dx \\ &\geq -L\delta c \int_{\{|D\xi|>3(M+1)\}} \varphi(1+|z+D\xi|+|\overline{z}+D\xi|)\,dx - LC_\delta \int_{\{|D\xi|>3(M+1)\}} \varphi(|z-\overline{z}|)\,dx \\ &\geq -L\delta c \int_{\{|D\xi|>3(M+1)\}} \varphi(1+|\overline{z}|+|D\xi|)\,dx - C_\delta \\ &\geq -L\delta c \int_{\{|D\xi|>3(M+1)\}} \varphi_{1+|\overline{z}|}(|D\xi|)\,dx - C_\delta \end{aligned}$$

where in the last inequality we used $\varphi_{1+|\overline{z}|}(|D\xi|) \sim \varphi(1+|\overline{z}|+|D\xi|)$ since $|D\xi| > 1+|\overline{z}|$. Putting together estimates on I_1, I_2 and II, taking into account that $\varphi_{|\overline{z}|}(t) \sim \varphi_{1+|\overline{z}|}(t)$ and choosing δ suitably we have

$$\begin{aligned} \fint_{\mathcal{B}_1} F(z+D\xi)\,dx &\geq -C\delta \int_{\{|D\xi|>3(M+1)\}} \varphi_{|\overline{z}|}(|D\xi|)\,dx + F(\overline{z}) + k\fint_{\mathcal{B}_1} \varphi_{|\overline{z}|}(|D\xi|)\,dx - C \\ &\geq -C. \end{aligned}$$

So we get $G(z) \geq -C$ for $|z| \leq M+1$. Moreover for $|z| > M+1$ we gain

$$G(z) = f(z) - \varepsilon\varphi(\sqrt{1+|z|^2}) \geq -C(1+\varphi(|z|))$$

and this proves the local boundedness of G from below.
By Dacorogna's formula [1] we have that G coincides with the quasiconvex envelope QF of F, and thus it is quasiconvex.

[1] In [19] Theorem 5 it is assumed that there exists a quasiconvex function from below F, and the verification of this hypothesis is not immediate in our situation. However, we may still apply the Theorem since the missing hypothesis is only needed to conclude that G is locally bounded from below. Moreover, by (2.4.2) we can say that $\varphi(|z|) \leq c(1+|z|^{p_1})$, $p_1 > 1$.

Finally we can prove that

$$g(z) = G(z) + \varepsilon\varphi(\sqrt{1+|z|^2}) \quad \text{for } z \in \mathbb{R}^{Nn}$$

is a uniformly strictly $W^{1,\varphi}$-quasiconvex function.
By the quasiconvexity of G we get

$$\begin{aligned}
\fint_{\mathcal{B}_1} g(z+D\xi)\,dx &= \fint_{\mathcal{B}_1} G(z+D\xi)\,dx + \varepsilon\fint_{\mathcal{B}_1} \varphi(\sqrt{1+|z+D\xi|^2})\,dx \\
&\geq G(z) + \varepsilon\varphi(\sqrt{1+|z|^2}) + \varepsilon\fint_{\mathcal{B}_1}\left[\varphi(\sqrt{1+|z+D\xi|^2}) - \varphi(\sqrt{1+|z|^2})\right]\,dx \\
&= g(z) + \varepsilon\fint_{\mathcal{B}_1}\left[\varphi(\sqrt{1+|z+D\xi|^2}) - \varphi(\sqrt{1+|z|^2})\right]\,dx.
\end{aligned}$$

Using Lemma 4.1.5, Remark 4.1.2 and $\varphi_a(t) \sim \varphi''(a+t)t^2$ it holds

$$\begin{aligned}
&\fint_{\mathcal{B}_1}\left[\varphi(\sqrt{1+|z+D\xi|^2}) - \varphi(\sqrt{1+|z|^2})\right]\,dx = \\
&= \fint_{\mathcal{B}_1} \langle D\varphi(\sqrt{1+|z|^2}), D\xi\rangle\,dx + \fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\langle D^2\varphi(\sqrt{1+|z+stD\xi|^2})D\xi, D\xi\rangle\,dsdtdx \\
&= \fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\langle D^2\varphi(\sqrt{1+|z+stD\xi|^2})D\xi, D\xi\rangle\,dsdtdx \\
&\geq C\fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\varphi''(\sqrt{1+|z+stD\xi|^2})|D\xi|^2\,dsdtdx \\
&\geq C\fint_{\mathcal{B}_1} \varphi''(1+|z|+|D\xi|)|D\xi|^2\,dx \\
&\geq C\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx.
\end{aligned}$$

We deduce that g is uniformly strictly $W^{1,\varphi}$-quasiconvex, i.e.

$$\fint_{\mathcal{B}_1} g(z+D\xi)\,dx \geq g(z) + \varepsilon c\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx.$$

Moreover we have that $g(z) = f(z)$ for $|z| > M+1$. This proves that f is asymptotically quasiconvex.

□

4.3 Caccioppoli estimate

The starting point for the investigation of the regularity properties of weak solutions is a Caccioppoli-type inequality.
We need the following Lemma (see Lemma 10 [27]):

Lemma 4.3.1. *Let ψ be an N-function with $\psi \in \Delta_2$, let $r > 0$ and let $h \in L^\psi(\mathcal{B}_{2r}(x_0))$. Further, let $f : [\frac{r}{2}, r] \to [0, \infty)$ be a bounded function such that for all $\frac{r}{2} < s < t < r$*

$$f(s) \leq \theta f(t) + A \int_{\mathcal{B}_t(x_0)} \psi\Big(\frac{|h(y)|}{t-s}\Big)\, dy,$$

where $A > 0$ and $\theta \in [0, 1)$. Then

$$f\Big(\frac{r}{2}\Big) \leq C(\theta, \Delta_2(\psi)) A \int_{\mathcal{B}_r(x_0)} \psi\Big(\frac{|h(y)|}{2r}\Big)\, dy.$$

Theorem 4.3.1. *Let $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ be a minimizer of $\mathcal{F}$ and let $\mathcal{B}_R$ be a ball such that $\mathcal{B}_{2R} \Subset \Omega$. Then*

$$\int_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|)\, dx \leq c \int_{\mathcal{B}_{2R}} \varphi_{|z|}\Big(\frac{|u-q|}{R}\Big) dx$$

for all $z \in \mathbb{R}^{Nn}$ with $|z| > M$ and all linear polynomials q on $\mathbb{R}^n$ with values in $\mathbb{R}^N$ such that $Dq = z$.

Proof. Let $0 < s < t$ and consider $\mathcal{B}_s \subset \mathcal{B}_t \subset \Omega$. Let $\eta \in C^\infty_c(\mathcal{B}_t)$ be a standard cut-off function between $\mathcal{B}_s$ and $\mathcal{B}_t$, such that $|D\eta| \leq \frac{c}{t-s}$.
Define $\xi = \eta(u - q)$ and $\zeta = (1 - \eta)(u - q)$; then $D\xi + D\zeta = Du - z$.
Consider

$$\mathcal{I} := \int_{\mathcal{B}_t} [f(z + D\xi) - f(z)]\, dx.$$

By hypothesis f is asymptotically $W^{1,\varphi}$-quasiconvex, and by Theorem 4.2.1 we know that f satisfies (iv), so for $|z| > M$ we have

$$\mathcal{I} \geq k \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|)\, dx. \tag{4.3.1}$$

Moreover

$$\begin{aligned}
\mathcal{I} &= \int_{\mathcal{B}_t} [f(z + D\xi) - f(Du) + f(Du) - f(Du - D\xi) + f(Du - D\xi) - f(z)]\, dx \\
&= \int_{\mathcal{B}_t} [f(z + D\xi) - f(z + D\xi + D\zeta)]\, dt + \int_{\mathcal{B}_t} [f(Du) - f(Du - D\xi)]\, dt \\
&+ \int_{\mathcal{B}_t} [f(z + D\zeta) - f(z)]\, dx = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

Note that $\mathcal{I}_2 \leq 0$ since u is a minimizer. Let us concentrate on $\mathcal{I}_1$:

$$\mathcal{I}_1 = -\int_{\mathcal{B}_t} \int_0^1 Df(z + D\xi + \theta D\zeta) D\zeta\, d\theta dx.$$

Analogously concerning $\mathcal{I}_3$, we have

$$\mathcal{I}_3 = \int_{\mathcal{B}_t} \int_0^1 Df(z + \theta D\zeta) D\zeta\, d\theta\, dx.$$

Thus we obtain that

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_3 &= \int_{\mathcal{B}_t} \int_0^1 \left[Df(z + \theta D\zeta) - Df(z + D\xi + \theta D\zeta) \right] \, D\zeta \, d\theta dx \\
&= \int_{\mathcal{B}_t} \int_0^1 \left[Df(z + \theta D\zeta) - Df(z) + Df(z) - Df(z + D\xi + \theta D\zeta) \right] \, D\zeta \, d\theta dx \\
&= \int_{\mathcal{B}_t} \int_0^1 \left[Df(z + \theta D\zeta) - Df(z) \right] \, D\zeta \, d\theta dx \\
&- \int_{\mathcal{B}_t} \int_0^1 \left[Df(z + D\xi + \theta D\zeta) - Df(z) \right] \, D\zeta \, d\theta dx
\end{aligned}
$$

from which

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_3 \leq & \int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta| \, d\theta dx \\
& + \int_{\mathcal{B}_t} \int_0^1 |Df(z + D\xi + \theta D\zeta) - Df(z)||D\zeta| \, d\theta dx.
\end{aligned}
$$

By using hypothesis ($\mathcal{H}4$) and Lemma 4.1.2 we have

$$
\begin{aligned}
& \int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta| \, d\theta dx \\
& \quad \leq \int_{\mathcal{B}_t} \int_0^1 \int_0^1 |D^2 f(tz + (1-t)(z + \theta D\zeta)| \, |\theta D\zeta| \, |D\zeta| \, dt d\theta dx \\
& \quad \leq c \int_{\mathcal{B}_t} \int_0^1 \int_0^1 \varphi''(|tz + (1-t)(z + \theta D\zeta)|) \, |D\zeta|^2 dt d\theta dx \\
& \quad \leq c \int_{\mathcal{B}_t} \varphi''(2|z| + |z + D\zeta|) \, |D\zeta|^2 dx \\
& \quad \leq c \int_{\mathcal{B}_t} \frac{\varphi'(2|z| + |z + D\zeta|)}{2|z| + |z + D\zeta|} |D\zeta|^2 dx.
\end{aligned}
$$

Taking into account the Δ_2 condition for φ' and $\varphi_a(t) \sim \varphi''(a+t)t^2$, it follows that

$$
\begin{aligned}
\int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta| \, d\theta dx & \leq c \int_{\mathcal{B}_t} \frac{\varphi'(|z| + |D\zeta|)}{|z| + |D\zeta|} |D\zeta|^2 dx \\
& \leq c \int_{\mathcal{B}_t} \varphi''(|z| + |D\zeta|) |D\zeta|^2 dx \\
& \leq c \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) dx.
\end{aligned}
$$

Analogously we can deduce

$$
\begin{aligned}
&\int_{\mathcal{B}_t}\int_0^1 |Df(z+D\xi+\theta D\zeta)-Df(z)||D\zeta|\,d\theta dx \leq \\
&\leq \int_{\mathcal{B}_t}\int_0^1\int_0^1 |D^2f(t(z+D\xi+\theta D\zeta)+(1-t)z)|\,|D\xi+\theta D\zeta|\,|D\zeta|\,dtd\theta dx \\
&\leq c\int_{\mathcal{B}_t}\int_0^1\int_0^1 \varphi''(|t(z+D\xi+\theta D\zeta)+(1-t)z|)\,|D\xi+\theta D\zeta|\,|D\zeta|\,dtd\theta dx \\
&\leq c\int_{\mathcal{B}_t}\varphi''(|z|+|D\xi|+|D\zeta|)\,(|D\xi|+|D\zeta|)\,|D\zeta|\,dx \\
&\leq c\int_{\mathcal{B}_t}\varphi'_{|z|}(|D\xi|+|D\zeta|)\,|D\zeta|\,dx \\
&\leq c\int_{\mathcal{B}_t}\varphi'_{|z|}(|D\xi|)\,|D\zeta|\,dx + c\int_{\mathcal{B}_t}\varphi'_{|z|}(|D\zeta|)\,|D\zeta|\,dx \\
&\leq c\int_{\mathcal{B}_t}\varphi'_{|z|}(|D\xi|)\,|D\zeta|\,dx + c\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx
\end{aligned}
$$

where in the last line we used the equivalence $\varphi'_a(t)\sim t\varphi''(a+t)$ and the fact that

$$\varphi'_{|z|}(|D\xi|+|D\zeta|)\leq c\varphi'_{|z|}(|D\xi|)+c\varphi'_{|z|}(|D\zeta|).$$

Applying Young's inequality for φ_a we have

$$
\begin{aligned}
\mathcal{I}_1+\mathcal{I}_3 &\leq c\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx + c\int_{\mathcal{B}_t}\varphi'_{|z|}(|D\xi|)\,|D\zeta|\,dx \\
&\leq c\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx + c\delta\int_{\mathcal{B}_t}\varphi_{|z|}(|D\xi|)\,dx + C_\delta\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx \\
&\leq C'_\delta\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx + c\delta\int_{\mathcal{B}_t}\varphi_{|z|}(|D\xi|)\,dx.
\end{aligned}
$$

Taking into account (4.3.1) and choosing δ such that $k-c\delta>0$ we conclude

$$\int_{\mathcal{B}_t}\varphi_{|z|}(|D\xi|)\,dx \leq C\int_{\mathcal{B}_t}\varphi_{|z|}(|D\zeta|)\,dx.$$

Now, by the definition of ζ we have $D\zeta=(1-\eta)(Du-z)-D\eta(u-q)$ and we can note that $D\zeta=0$ in $\mathcal{B}_s$. Moreover using the convexity of $\varphi_{|z|}$ and the fact that $|D\eta|\leq\frac{c}{t-s}$, we have

$$\varphi_{|z|}(|D\zeta|)\leq\varphi_{|z|}\left((1-\eta)|Du-z|+\frac{c}{t-s}|u-q|\right)\leq c\varphi_{|z|}(|Du-z|)+c\varphi_{|z|}\left(\frac{|u-q|}{t-s}\right).$$

Hence

$$
\begin{aligned}
\int_{\mathcal{B}_t}\varphi_{|z|}(|D\xi|)\,dx &\leq C\int_{\mathcal{B}_t\setminus\mathcal{B}_s}\varphi_{|z|}(|D\zeta|)\,dx \\
&\leq c\int_{\mathcal{B}_t\setminus\mathcal{B}_s}\varphi_{|z|}(|Du-z|)\,dx + c\int_{\mathcal{B}_t}\varphi_{|z|}\left(\frac{|u-q|}{t-s}\right)\,dx.
\end{aligned}
$$

Thus we have

$$\begin{aligned}\int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|)\, dx &= \int_{\mathcal{B}_s} \varphi_{|z|}(|D\xi|)\, dx \\ &\leq \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|)\, dx \\ &\leq c \int_{\mathcal{B}_t \setminus \mathcal{B}_s} \varphi_{|z|}(|Du - z|)\, dx + c \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u-q|}{t-s}\right) dx.\end{aligned}$$

We fill the hole by adding to both sides the term $c \int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|)\, dx$ and we divide by $c+1$, thus obtaining

$$\begin{aligned}\int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|)\, dx &\leq \frac{c}{c+1} \int_{\mathcal{B}_t} \varphi_{|z|}(|Du - z|)\, dx + C \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u-q|}{t-s}\right) dx \\ &= \lambda \int_{\mathcal{B}_t} \varphi_{|z|}(|Du - z|)\, dx + \alpha \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u-q|}{t-s}\right) dx\end{aligned}$$

where $\lambda := \frac{c}{c+1} < 1$ and $\alpha > 0$. Now we can apply Lemma 4.3.1 to get the desired result. □

An immediate consequence of the previous result is the following:

Corollary 4.3.1. *There exists $\alpha \in (0,1)$ such that for all minimizers $u \in W^{1,\varphi}(\Omega)$ of $\mathcal{F}$, all balls $\mathcal{B}_R$ with $\mathcal{B}_{2R} \Subset \Omega$, and all $z \in \mathbb{R}^{Nn}$ with $|z| > M$*

$$⨍_{\mathcal{B}_R} |V(Du) - V(z)|^2 dx \leq c \left(⨍_{\mathcal{B}_{2R}} |V(Du) - V(z)|^{2\alpha} dx \right)^{\frac{1}{\alpha}}$$

Proof. By using Lemma 2.4.5, applying Theorem 4.3.1 with q such that $(u-q)_{\mathcal{B}_{2R}} = 0$ and Theorem 4.1.1 we have

$$\begin{aligned}⨍_{\mathcal{B}_R} |V(Du) - V(z)|^2 dx &\leq c ⨍_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|)\, dx \\ &\leq c ⨍_{\mathcal{B}_{2R}} \varphi_{|z|}\left(\frac{|u-q|}{R}\right) dx \\ &\leq c \left(⨍_{\mathcal{B}_{2R}} \varphi_{|z|}^{\alpha}(|Du - z|)\, dx \right)^{\frac{1}{\alpha}} \\ &\leq c \left(⨍_{\mathcal{B}_{2R}} |V(Du) - V(z)|^{2\alpha} dx \right)^{\frac{1}{\alpha}}.\end{aligned}$$

□

Using Gehring's Lemma we deduce the following result.

Corollary 4.3.2. *There exists $s > 1$ such that for all minimizers $u \in W^{1,\varphi}(\Omega)$ of $\mathcal{F}$, all balls $\mathcal{B}_R$ with $\mathcal{B}_{2R} \Subset \Omega$, and all $z \in \mathbb{R}^{Nn}$ with $|z| > M$*

$$\left(\fint_{\mathcal{B}_R} |V(Du) - V(z)|^{2s} dx \right)^{\frac{1}{s}} \le c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 dx$$

4.4 Almost $\mathcal{A}$-harmonicity

In this section we recall a generalization of the $\mathcal{A}$-harmonic approximation Lemma in Orlicz space (see [27]).
We say that $\mathcal{A} = (\mathcal{A}_{ij}^{\alpha\beta})_{\substack{i,j=1,\cdots,N \\ \alpha,\beta=1,\cdots,n}}$ is strongly elliptic in the sense of Legendre- Hadamard if

$$\mathcal{A}(a \otimes b, a \otimes b) \ge k_{\mathcal{A}} |a|^2 |b|^2$$

holds for all $a \in \mathbb{R}^N, b \in \mathbb{R}^n$ for some constant $k_{\mathcal{A}} > 0$. We say that a Sobolev function w on $\mathcal{B}_R$ is $\mathcal{A}$-harmonic if

$$-\operatorname{div}(\mathcal{A}Dw) = 0$$

is satisfied in the sense of distributions.
Given a function $u \in W^{1,2}(\mathcal{B}_R)$, we want to find a function h that is $\mathcal{A}$-harmonic and is close to u. In particular, we are looking for a function $h \in W^{1,2}(\mathcal{B}_R)$ such that

$$\begin{cases} -\operatorname{div}(\mathcal{A}Dh) = 0 & \text{in } \mathcal{B}_R \\ h = u & \text{on } \partial \mathcal{B}_R \end{cases}.$$

Let $w := h - u$, then w satisfies

$$\begin{cases} -\operatorname{div}(\mathcal{A}Dw) = -\operatorname{div}(\mathcal{A}Du) & \text{in } \mathcal{B}_R \\ w = 0 & \text{on } \partial \mathcal{B}_R \end{cases}. \tag{4.4.1}$$

We recall Theorem 14 in [27]:

Theorem 4.4.1. *Let $\mathcal{B}_R \Subset \Omega$ and let $\tilde{\mathcal{B}} \subset \Omega$ denote either $\mathcal{B}_R$ or $\mathcal{B}_{2R}$. Let $\mathcal{A}$ be strongly elliptic in the sense of Legendre-Hadamard. Let ψ be an N-function with $\Delta_2(\psi, \psi^*) < \infty$ and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ depending on $n, N, k_{\mathcal{A}}, |\mathcal{A}|, \Delta_2(\psi, \psi^*)$ and s such that the following holds: let $u \in W^{1,\psi}(\tilde{\mathcal{B}})$ be almost $\mathcal{A}$-harmonic on $\mathcal{B}_R$ in the sense that*

$$\left| \fint_{\mathcal{B}_R} (\mathcal{A}\, Du, D\xi)\, dx \right| \le \delta \fint_{\tilde{\mathcal{B}}} |Du|\, dx \|D\xi\|_{L^\infty(\mathcal{B}_R)}$$

for all $\xi \in C_0^\infty(\mathcal{B}_R)$. Then the unique solution $w \in W_0^{1,\psi}(\mathcal{B}_R)$ of (4.4.1) satisfies

$$\fint_{\mathcal{B}_R} \psi\Big(\frac{|w|}{R}\Big)\, dx + \fint_{\mathcal{B}_R} \psi(|Dw|)\, dx \le \varepsilon \left[\left(\fint_{\mathcal{B}_R} \psi^s(|Du|)\, dx \right)^{\frac{1}{s}} + \fint_{\tilde{\mathcal{B}}} \psi(|Du|)\, dx \right].$$

The following results can be found in [27].

Lemma 4.4.1. *Let $\mathcal{B}_R \subset \mathbb{R}^n$ be a ball and let $u \in W^{1,\varphi}(\mathcal{B}_R)$. Then*

$$\fint_{\mathcal{B}_R} |V(Du) - (V(Du))_{\mathcal{B}_R}|^2 dx \sim \fint_{\mathcal{B}_R} |V(Du) - V((Du)_{\mathcal{B}_R})|^2 dx.$$

Lemma 4.4.2. *Let $z := (Du)_{\mathcal{B}_{2R}}$. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $u \in W^{1,\varphi}(\Omega)$ minimizer of $\mathcal{F}$ and every $\mathcal{B}_R$ such that $\mathcal{B}_{2R} \Subset \Omega$, and for*

$$\fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 dx \leq \delta \fint_{\mathcal{B}_{2R}} |V(Du)|^2 dx$$

it holds

$$\left| \fint_{\mathcal{B}_R} D^2 f(z)(Du - z, D\xi)\, dx \right| \leq \varepsilon \varphi''(|z|) \fint_{\mathcal{B}_{2R}} |Du - z|\, dx \|D\xi\|_{L^\infty(\mathcal{B}_R)}, \tag{4.4.2}$$

for every $\xi \in C_c^\infty(\mathcal{B}_R)$.

4.5 Excess decay estimate

Following the ideas in [3] we will prove the following Lemma.

Lemma 4.5.1. *Let $z_0 \in \mathbb{R}^n$ such that $|z_0| > 1$. Let $f \in C^2(B_{2\sigma}(z_0))$ be strictly $W^{1,\varphi}$-quasiconvex at z_0, that is*

$$\int_{\mathcal{B}} [f(z_0 + D\xi) - f(z_0)]\, dx \geq k \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|)\, dx \tag{4.5.1}$$

holds for all $\xi \in C_c^1(\mathcal{B}, \mathbb{R}^N)$. Then, there exists $\rho > 0$ such that for all $z \in \mathcal{B}_\rho(z_0)$

$$\int_{\mathcal{B}} [f(z + D\xi) - f(z)]\, dx \geq \frac{k}{2} \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|)\, dx \tag{4.5.2}$$

holds for all $\xi \in C_c^1(\mathcal{B}, \mathbb{R}^N)$.

Proof. Let

$$\omega_\rho := \sup \left\{ |D^2 f(z_1) - D^2 f(z_2)| : z_1, z_2 \in \mathcal{B}_\sigma(z_0), |z_1 - z_2| < \rho \right\}$$

and fix z such that $|z - z_0| < \rho < \frac{\sigma}{2}$.
For $\eta \in \mathbb{R}^{Nn}$, define

$$G(\eta) = f(z + \eta) - f(z_0 + \eta).$$

By using (4.5.1) we have

$$\begin{aligned}
&\int_{\mathcal{B}} [f(z + D\xi) - f(z)]\, dx = \\
&= \int_{\mathcal{B}} [f(z_0 + D\xi) - f(z_0)]\, dx + \int_{\mathcal{B}} [f(z + D\xi) - f(z_0 + D\xi) + f(z_0) - f(z)]\, dx \\
&\geq k \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|)\, dx + \int_{\mathcal{B}} [G(D\xi) - G(0) - \langle DG(0), D\xi \rangle]\, dx.
\end{aligned}$$

Now we split $\mathcal{B}$ as

$$\mathbb{X} = \left\{ x \in \mathcal{B} : |D\xi| \le \frac{\sigma}{2} \right\} \text{ and } \mathbb{Y} = \left\{ x \in \mathcal{B} : |D\xi| > \frac{\sigma}{2} \right\}.$$

Let us observe that

$$G(D\xi) - G(0) - \langle DG(0), D\xi \rangle = \frac{1}{2} \langle D^2 G(\theta D\xi) D\xi, D\xi \rangle$$

with $\theta \in (0,1)$. Moreover if $x \in \mathbb{X}$ then $|D\xi| \le \frac{\sigma}{2}$, so $z + D\xi \in \mathcal{B}_\sigma(z_0)$. Hence

$$\begin{aligned}
\int_{\mathbb{X}} [G(D\xi) - G(0) - DG(0)D\xi]\, dx &= \frac{1}{2} \int_{\mathbb{X}} \langle D^2 G(\theta D\xi) D\xi, D\xi \rangle \, dx \\
&\ge -\frac{1}{2} \int_{\mathbb{X}} |D^2 f(z + \theta D\xi) - D^2 f(z_0 + \theta D\xi)||D\xi|^2 dx \\
&\ge -\frac{\omega_\rho}{2} \int_{\mathbb{X}} |D\xi|^2 dx \\
&\ge -\frac{c\,\omega_\rho}{2} \int_{\mathbb{X}} \varphi''(|z_0| + |D\xi|)|D\xi|^2 dx \\
&\ge -\frac{c\,\omega_\rho}{2} \int_{\mathbb{X}} \varphi_{|z_0|}(|D\xi|) dx
\end{aligned}$$

where we used the fact that on $\mathbb{X}$ we have

$$\varphi''(|z_0| + |D\xi|) \ge c \frac{\varphi'(|z_0| + |D\xi|)}{|z_0| + |D\xi|} \ge c \frac{\varphi'(1)}{|z_0| + \frac{\sigma}{2}} > 0.$$

Let us define

$$H(z,x) = f(z + D\xi(x)) - f(z) - (Df(z), D\xi(x))$$

so that

$$\int_{\mathbb{Y}} [H(z,x) - H(z_0,x)]\, dx = \int_{\mathbb{Y}} [G(D\xi) - G(0) - (DG(0), D\xi)]\, dx.$$

We can see that

$$\begin{aligned}
\int_{\mathbb{Y}} |H(z,x) - H(z_0,x)|\, dx &\le \int_{\mathbb{Y}} |z - z_0||D_z H(\tau, x)|\, dx \\
&\le \rho \left[\int_{\mathbb{Y}} |Df(\tau + D\xi) - Df(\tau)|\, dx + \int_{\mathbb{Y}} |D^2 f(\tau)||D\xi|\, dx \right] \\
&= \rho[\mathcal{I} + \mathcal{II}].
\end{aligned}$$

Now we estimate $\mathcal{I}$. We use hypothesis $(\mathcal{H}4)$, Lemma 4.1.2 and the fact that

$$|\tau| + |D\xi + \tau| \sim |\tau| + |D\xi|$$

to get

$$\begin{aligned}\mathcal{I} &\leq \int_{\mathbb{Y}} \int_0^1 |D^2 f(\tau + tD\xi)||D\xi|\, dtdx \\ &\leq c \int_{\mathbb{Y}} \int_0^1 \varphi''(|\tau + tD\xi|)|D\xi|\, dtdx \\ &\leq c \int_{\mathbb{Y}} \frac{\varphi'(|\tau| + |D\xi|)}{|\tau| + |D\xi|}|D\xi|\, dx \\ &\leq c_\sigma \int_{\mathbb{Y}} \varphi'(|z_0| + |D\xi|)|D\xi|\, dx\end{aligned}$$

where in the last inequality we used $|\tau| + |D\xi| \leq |z| + |z_0| + |D\xi| \leq \rho + 2|z_0| + |D\xi| < c(|z_0| + |D\xi|)$ as well as $|\tau| + |D\xi| > 1 + \frac{\sigma}{2} =: c_\sigma$ on $\mathbb{Y}$, if ρ is small enough.
Analogously, we estimate $\mathcal{II}$:

$$\begin{aligned}\mathcal{II} &\leq c \int_{\mathbb{Y}} \varphi''(|\tau|)|D\xi|\, dx \\ &\leq c \int_{\mathbb{Y}} \varphi'(|z_0| + |D\xi|)|D\xi|\, dx\end{aligned}$$

since $|\tau| \leq |z| + |z_0| \leq c(|z_0| + |D\xi|)$.
On the other hand, since on $\mathbb{Y}$

$$\begin{aligned}\varphi'(|z_0| + |D\xi|)|D\xi| &\leq c\varphi''(|z_0| + |D\xi|)(|z_0| + |D\xi|)|D\xi| \\ &\leq c(|z_0|, \sigma)\varphi''(|z_0| + |D\xi|)|D\xi|^2 \\ &\leq c(|z_0|, \sigma)\varphi_{|z_0|}(|D\xi|),\end{aligned}$$

we can say that

$$\int_{\mathbb{Y}} [G(D\xi) - G(0) - \langle DG(0), D\xi\rangle]\, dx \geq -\tilde{c}\rho \int_{\mathbb{Y}} \varphi_{|z_0|}(|D\xi|)\, dx$$

where $\tilde{c}$ depends on the characteristics of φ, σ and $|z_0|$. Choosing ρ such that $\frac{c\,\omega_\rho}{2} + \tilde{c}\rho < \frac{k}{2}$ we have the result.

□

In the sequel we assume that $z_0 \in \mathbb{R}^n$, with $|z_0| > M + 1$, so that (4.5.2) holds in $\mathcal{B}_\rho(z_0)$ with $\rho < 1$.
We define the excess function

$$\mathbb{E}(\mathcal{B}_R(x_0), u) = \fint_{\mathcal{B}_R(x_0)} |V(Du) - (V(Du))_{\mathcal{B}_R(x_0)}|^2 dx.$$

The main ingredient to prove our regularity result is the following decay estimate:

Proposition 4.5.1. *For all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \varphi) > 0$ and $\beta \in (0,1)$, such that, if u is a minimizer and if for some ball $\mathcal{B}_R(x_0)$ with $\mathcal{B}_{2R}(x_0) \Subset \Omega$ the following estimates*

$$\mathbb{E}(\mathcal{B}_{2R}(x_0), u) \leq \delta \fint_{\mathcal{B}_{2R}(x_0)} |V(Du)|^2 dx, \qquad |(Du)_{\mathcal{B}_{2R}(x_0)} - z_0| < \rho \tag{4.5.3}$$

hold true, then for every $\tau \in (0, \frac{1}{2}]$

$$\mathbb{E}(\mathcal{B}_{\tau R}(x_0), u) \leq C\tau^{\beta}(\varepsilon\tau^{-n-1} + 1)\, \mathbb{E}(\mathcal{B}_{2R}(x_0), u)$$

where $C = C(\varphi, n)$ and it is independent of ε.

Proof. Let q be a linear function such that $(u-q)_{\mathcal{B}_{2R}} = 0$ and $z := Dq = (Du)_{\mathcal{B}_{2R}}$. Let $w := u - q$. Fix $\varepsilon > 0$ and δ as in Lemma 4.4.2, then w is almost $\mathcal{A}$-harmonic with $\mathcal{A} = \dfrac{D^2 f(z)}{\varphi''(|z|)}$. Let us observe that by Lemma 4.5.1 such $\mathcal{A}$ is strongly elliptic in the sense of Legendre-Hadamard, since for every $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$

$$\frac{D^2 f(z)}{\varphi''(|z|)}(a \otimes b, a \otimes b) \geq \frac{\varphi''(|z_0|)}{\varphi''(|z|)}|a|^2|b|^2 \geq c|a|^2|b|^2$$

for $z_0 \in \mathbb{R}^n$ with $|z_0| > 1$ and z such that $|z - z_0| < \rho$, where c depends on z_0, ρ and φ. Let h be the $\mathcal{A}$-harmonic approximation of w with $h = w$ on $\partial\mathcal{B}_R$. At this point we can apply Theorem 4.4.1 and conclude that, for $|z| > M$, h satisfies

$$\fint_{\mathcal{B}_R} \varphi_{|z|}(|Dw - Dh|)\, dx \leq \varepsilon \left[\left(\fint_{\mathcal{B}_R} \varphi^s_{|z|}(|Du - z|)\, dx \right)^{\frac{1}{s}} + \fint_{\mathcal{B}_{2R}} \varphi_{|z|}(|Du - z|)\, dx \right]$$

where s is the same exponent of Corollary 4.3.2.
Applying Lemma 2.4.5 and Corollary 4.3.2 we have

$$\begin{aligned} \left(\fint_{\mathcal{B}_R} \varphi^s_{|z|}(|Du - z|)\, dx \right)^{\frac{1}{s}} &\leq c \left(\fint_{\mathcal{B}_R} |V(Du) - V(z)|^{2s} dx \right)^{\frac{1}{s}} \\ &\leq c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 dx \end{aligned}$$

from which, taking into account that $z = (Du)_{\mathcal{B}_{2R}}$ and using Lemma 4.4.1 we have

$$\begin{aligned} \fint_{\mathcal{B}_R} \varphi_{|z|}(|Dw - Dh|)\, dx &\leq \varepsilon c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 dx \\ &\leq \varepsilon c \fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 dx \\ &= \varepsilon c\, \mathbb{E}(\mathcal{B}_{2R}, u). \end{aligned} \tag{4.5.4}$$

Now we want to compute $\mathbb{E}(\mathcal{B}_{\tau R}, u)$. Applying Lemma 4.4.1, Lemma 2.4.5 and Lemma 4.1.4 we get

$$\begin{aligned}
\mathbb{E}(\mathcal{B}_{\tau R}, u) &= \fint_{\mathcal{B}_{\tau R}} |V(Du) - (V(Du))_{\mathcal{B}_{\tau R}}|^2 dx \\
&\leq c \fint_{\mathcal{B}_{\tau R}} |V(Du) - V((Dh)_{\mathcal{B}_{\tau R}} + z)|^2 dx \\
&\leq c \fint_{\mathcal{B}_{\tau R}} \varphi_{|(Dh)_{\mathcal{B}_{\tau R}}+z|}(|Du - (Dh)_{\mathcal{B}_{\tau R}} - z|)\, dx \\
&= c \fint_{\mathcal{B}_{\tau R}} \varphi_{|(Dh)_{\mathcal{B}_{\tau R}}+z|}(|Dw - (Dh)_{\mathcal{B}_{\tau R}}|)\, dx \\
&\leq C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dw - (Dh)_{\mathcal{B}_{\tau R}}|) dx + \eta \fint_{\mathcal{B}_{\tau R}} |V((Dh)_{\mathcal{B}_{\tau R}} + z) - V(z)|^2 dx \\
&= \mathcal{I} + \mathcal{II}.
\end{aligned}$$

Using Jensen's inequality, (4.5.4), the fact that

$$\sup_{\mathcal{B}_{\tau R}} |Dh - (Dh)_{\mathcal{B}_{\tau R}}| \leq c\,\tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}|\, dx$$

(see [51]), the convexity of φ, and the Δ_2-condition, we have

$$\begin{aligned}
\mathcal{I} &\leq C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dw - Dh|)\, dx + C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dh - (Dh)_{\mathcal{B}_{\tau R}}|)\, dx \\
&\leq C_\eta \tau^{-n} \varepsilon\, \mathbb{E}(\mathcal{B}_{2R}, u) + C_\eta \varphi_{|z|}\left(\tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}|\, dx\right).
\end{aligned}$$

Taking into account that $\varphi_a(st) \leq cs\varphi_a(t)$ for all $a \geq 0$, $s \in [0,1]$ and $t \geq 0$, using Jensen inequality and (4.5.4) we have

$$\begin{aligned}
&\varphi_{|z|}\Big(\tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}|\, dx\Big) \leq \\
&\quad \leq c\,\tau \varphi_{|z|}\left(\fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}|\, dx\right) \\
&\quad \leq c\,\tau \varphi_{|z|}\left(\fint_{\mathcal{B}_R} |Dh - Dw|\, dx + \fint_{\mathcal{B}_R} |Dw - (Dw)_{\mathcal{B}_R}|\, dx\right) \\
&\quad \leq c\,\tau \varphi_{|z|}\left(\fint_{\mathcal{B}_R} |Dh - Dw|\, dx\right) + c\,\tau \varphi_{|z|}\left(\fint_{\mathcal{B}_R} |Dw - (Dw)_{\mathcal{B}_R}|\, dx\right) \\
&\quad \leq c\,\tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Dh - Dw|)\, dx + c\,\tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - (Du)_{\mathcal{B}_R}|)\, dx \\
&\quad \leq c\,\tau \varepsilon\, \mathbb{E}(\mathcal{B}_{2R}, u) + c\,\tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - (Du)_{\mathcal{B}_R}|)\, dx \\
&\quad \leq c\,\tau \varepsilon\, \mathbb{E}(\mathcal{B}_{2R}, u) + c\,\tau\, \mathbb{E}(\mathcal{B}_{2R}, u)
\end{aligned}$$

where in the last inequality we used

$$\begin{aligned}\fint_{\mathcal{B}_R} \varphi_{|z|}(|Du-(Du)_{\mathcal{B}_R}|)\,dx &\le c\fint_{\mathcal{B}_R}\varphi_{|z|}(|Du-z|)\,dx + c\fint_{\mathcal{B}_R}\varphi_{|z|}(|z-(Du)_{\mathcal{B}_R}|)\,dx\\ &\le c\,\mathbb{E}(\mathcal{B}_{2R},u)+c\,\varphi_{|z|}\left(\left|\fint_{\mathcal{B}_R}[Du-z]\,dx\right|\right)\\ &\le c\,\mathbb{E}(\mathcal{B}_{2R},u)+c\fint_{\mathcal{B}_R}\varphi_{|z|}(|Du-z|)\,dx\\ &\le c\,\mathbb{E}(\mathcal{B}_{2R},u).\end{aligned}$$

So we have

$$\mathcal{I}\le C_\eta\tau^{-n}\varepsilon\,\mathbb{E}(\mathcal{B}_{2R},u)+C_\eta\tau\varepsilon\,\mathbb{E}(\mathcal{B}_{2R},u)+C_\eta\,\tau\,\mathbb{E}(\mathcal{B}_{2R},u).$$

Now we estimate $\mathcal{II}$; taking into account that

$$\sup_{\mathcal{B}_{\tau R}}|Dh|\le\fint_{\mathcal{B}_R}|Dh|\,dx,$$

using Jensen's inequality, and (4.5.4) we obtain

$$\begin{aligned}\mathcal{II}&\le c\,\eta\fint_{\mathcal{B}_{\tau R}}\varphi_{|z|}(|(Dh)_{\mathcal{B}_{\tau R}}|)\,dx\\ &\le c\,\eta\,\varphi_{|z|}\left(\fint_{\mathcal{B}_R}|Dh|\,dx\right)\\ &\le c\,\eta\,\varphi_{|z|}\left(\fint_{\mathcal{B}_R}|Dh-Dw|\,dx+\fint_{\mathcal{B}_R}|Dw|\,dx\right)\\ &\le c\,\eta\,\varphi_{|z|}\left(\fint_{\mathcal{B}_R}|Dh-Dw|\,dx\right)+c\,\eta\,\varphi_{|z|}\left(\fint_{\mathcal{B}_R}|Du-z|\,dx\right)\\ &\le c\,\eta\fint_{\mathcal{B}_R}\varphi_{|z|}(|Dh-Dw|)\,dx+c\,\eta\fint_{\mathcal{B}_R}\varphi_{|z|}(|Du-z|)\,dx\\ &\le c\,\eta\,\varepsilon\ \mathbb{E}(\mathcal{B}_{2R},u)+c\,\eta\ \mathbb{E}(\mathcal{B}_{2R},u).\end{aligned}$$

Putting together estimates for $\mathcal{I}$ and $\mathcal{II}$ we have

$$\mathbb{E}(\mathcal{B}_{\tau R},u)\le C\,\mathbb{E}(\mathcal{B}_{2R},u)\left[C_\eta\,\tau^{-n}\,\varepsilon+C_\eta\,\tau\,\varepsilon+C_\eta\,\tau+\eta\,\varepsilon+\eta\right],$$

choosing $\eta=\tau^\alpha$, and consequently $C_\eta=\dfrac{1}{\tau^{\alpha(\bar p-1)}}$, with $\alpha<\dfrac{1}{\bar p-1}$, we have

$$\mathbb{E}(\mathcal{B}_{\tau R},u)\le C\tau^\beta(\varepsilon\tau^{-n-1}+1)\,\mathbb{E}(\mathcal{B}_{2R},u)$$

where $\beta=\min\{\alpha,1-\alpha(\bar p-1)\}$.

□

Proposition 4.5.2. *Let $\gamma \in (0,1)$. Then there exists δ that depends on γ and on the characteristics of φ such that: if for some ball $\mathcal{B}_R(x_0) \subset \Omega$*

$$\mathbb{E}(\mathcal{B}_{2R}(x_0), u) \le \delta \fint_{\mathcal{B}_{2R}(x_0)} |V(Du)|^2 dx, \qquad |(Du)_{\mathcal{B}_{2R}(x_0)} - z_0| < \frac{\rho}{2} \tag{4.5.5}$$

hold, then for any $\rho \in (0,1]$

$$\mathbb{E}(\mathcal{B}_{\rho R}(x_0), u) \le c\rho^{\gamma\beta}\, \mathbb{E}(\mathcal{B}_{2R}(x_0), u) \tag{4.5.6}$$

where c depends on the characteristics of φ.

Proof. Let $\Lambda(\varepsilon, \tau) = C\tau^{\beta}(\varepsilon\tau^{-n-1} + 1)$ where C depends on the characteristics of φ and on n. Let $\varepsilon = \varepsilon(\tau)$ such that

$$\Lambda(\varepsilon, \tau) \le \min\left\{ \left(\frac{\tau}{2}\right)^{\gamma\beta}, \frac{1}{4} \right\}.$$

Let $\delta = \delta(\tau)$ such that Proposition 4.5.1 holds true and so small that are verified

$$(1 + \tau^{-\frac{n}{2}})\delta^{\frac{1}{2}} < \frac{1}{2} \quad \text{and} \quad c\frac{\delta^{\frac{1}{p}}}{\tau^{\frac{n}{p}}} < \frac{\rho}{2},$$

where c and p will be specified later.
With these choices we can prove that the inequalities in (4.5.3) hold when we replace $\mathcal{B}_{2R}$ with $\mathcal{B}_{\tau R}$, the first one being necessary to obtain the first inequality following exactly the lines of the proof of Proposition 28 in [42].
Concerning the second inequality we first observe that

$$|(Du)_{\mathcal{B}_{\tau R}} - z_0| < |(Du)_{\mathcal{B}_{\tau R}} - (Du)_{\mathcal{B}_{2R}}| + \frac{\rho}{2}.$$

Moreover, taking into account that φ is of type (p_0, p_1) and using Lemma 2.4.5, for some $p > 1$ we get

$$\begin{aligned}
|(Du)_{\mathcal{B}_{\tau R}} - (Du)_{\mathcal{B}_{2R}}| &\le \fint_{\mathcal{B}_{\tau R}} |Du - (Du)_{\mathcal{B}_{2R}}|\, dx \\
&\le \left(\fint_{\mathcal{B}_{\tau R}} |Du - (Du)_{\mathcal{B}_{2R}}|^p dx \right)^{\frac{1}{p}} \\
&\le c \left(\fint_{\mathcal{B}_{\tau R}} \varphi_{|(Du)_{\mathcal{B}_{2R}}|} |Du - (Du)_{\mathcal{B}_{2R}}| dx \right)^{\frac{1}{p}} \\
&\le c \left(\fint_{\mathcal{B}_{\tau R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 dx \right)^{\frac{1}{p}} \\
&\le \frac{c}{\tau^{\frac{n}{p}}} \left(\fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 dx \right)^{\frac{1}{p}} \\
&\le \frac{c}{\tau^{\frac{n}{p}}} \delta^{\frac{1}{p}} \left(\fint_{\mathcal{B}_{2R}} |V(Du)|^2 dx \right)^{\frac{1}{p}} \le c\frac{\delta^{\frac{1}{p}}}{\tau^{\frac{n}{p}}}
\end{aligned}$$

where in the last inequality we use that by Lemma 2.4.5 and Jensen inequality

$$\begin{aligned}\fint_{\mathcal{B}_{2R}(x)} |V(Du)|^2 dy &\sim \fint_{\mathcal{B}_{2R}(x)} \varphi(|Du|)\, dy \\ &\geq \varphi\left(\fint_{\mathcal{B}_{2R}(x)} |Du|\, dy\right) && (4.5.7)\\ &\geq \varphi(|(Du)_{\mathcal{B}_{2R}}|) \geq \varphi(M) > 0. && (4.5.8)\end{aligned}$$

So, the smallness assumptions in (4.5.3) are satisfied for $\mathcal{B}_{\tau R}$. By induction we get

$$\mathbb{E}\Big(\mathcal{B}_{(\frac{\tau}{2})^k 2R}\Big) \leq \min\Big\{\Big(\frac{\tau}{2}\Big)^{\gamma\beta k}, \frac{1}{4^k}\Big\}\,\mathbb{E}(\mathcal{B}_{2R})$$

which is the claim.

□

Proof of Theorem 4.0.2. By Jensen inequality and Lemma 2.4.5 we have

$$\begin{aligned}\varphi_{|z_0|}(|(Du)_{\mathcal{B}_r(x)} - z_0|) &\leq \varphi_{|z_0|}\left(\fint_{\mathcal{B}_r(x)} |Du - z_0|\, dy\right)\\ &\leq \fint_{\mathcal{B}_r(x)} \varphi_{|z_0|}(|Du - z_0|)\, dy\\ &\leq c\fint_{\mathcal{B}_r(x)} |V(Du) - V(z_0)|^2 dy\end{aligned}$$

from which by (4.0.1) we can conclude that

$$|(Du)_{\mathcal{B}_{2R}(x)} - z_0| < \rho$$

for a suitable $R > 0$. Moreover by Lemma 2.4.5, Jensen's inequality, (4.5.7), and (4.0.1) we get

$$\mathbb{E}(\mathcal{B}_{2R}(x), u) \leq \fint_{\mathcal{B}_{2R}(x)} |V(Du) - V(z_0)|^2 dy \leq \delta \fint_{\mathcal{B}_{2R}(x)} |V(Du)|^2 dy.$$

Hence we have that the assumptions of Proposition 4.5.2 are verified in a neighborhood of x_0, say in $\mathcal{B}_s(x_0)$. Then by (4.5.6) we have

$$\mathbb{E}(\mathcal{B}_{\rho R}(x), u) \leq c\rho^{\gamma\beta}\,\mathbb{E}(\mathcal{B}_{2R}(x), u) \quad \forall x \in \mathcal{B}_s(x_0)$$

and by Campanato's characterization of Hölder continuity we deduce that $u \in C^{1,\overline{\alpha}}(\mathcal{B}_s(x_0))$ for some $\overline{\alpha} < 1$.

□

For $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$, we define the set of regular points $\mathcal{R}(u)$ by

$$\mathcal{R}(u) = \{x \in \Omega : u \text{ is Lipschitz near } x\}.$$

It follows that $\mathcal{R}(u) \subset \Omega$ is open. Finally we prove the following partial regularity result.

Proof of Corollary 4.0.1. Using the characterization (iv) of Theorem 4.2.1 we can find $M > 0$ such that the assumptions of Theorem 4.0.2 are satisfied near every $z_0 \in \mathbb{R}^{Nn}$: $|z_0| > M$. By Theorem 4.0.2 we have that $u \in C^{1,\alpha}$ near every $x_0 \in \Omega$ that satisfies

$$\lim_{r \to 0} \fint_{\mathcal{B}_r(x_0)} |V(Du) - V(z_0)|^2 dx = 0$$

and these points x_0 belong to $\mathcal{R}(u)$.
By contradiction assume that some $x \in \Omega$ is not contained in $\overline{\mathcal{R}(u)}$; then in a neighborhood of x we cannot find x_0 as before. Thus, $V(Du)$ is essentially bounded by M on this neighborhood and u is Lipschitz near x. Consequently $x \in \mathcal{R}(u)$ and we have reached the desired contradiction.

□

Chapter 5

Lipschitz regularity for a wide class of parabolic systems with general growth

This last Chapter concerns the local Lipschitz regularity of weak solutions to non-linear second order parabolic systems of general growth

$$u_t^\beta - \sum_{i=1}^{n} (\mathcal{A}_i^\alpha(Du))_{x_i} = 0 \text{ in } \Omega_T := \Omega \times (-T, 0) \tag{5.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, $u : \Omega_T \to \mathbb{R}^N$, $N > 1$ and $\mathcal{A}$ is a tensor having certain Orlicz-type growths that generalize p-growth.
In particular we focus on $\mathcal{A}_i^\alpha(Du) = \dfrac{\varphi'(|Du|)}{|Du|} u_{x_i}^\alpha$, where φ is a given Orlicz function, and we assume that $\dfrac{\varphi'(s)}{s}$ is increasing. In the model case $\varphi(s) = s^p$, for some $p > 1$, (5.0.1) gives the evolutionary p-Laplacian.

Definition 5.0.1. *A function $u \in L^\varphi(-T, 0; W^{1,\varphi}(\Omega, \mathbb{R}^N))$ is a weak solution for (5.0.1) if*

$$\int_{\Omega_T} [u\, \phi_t - \langle \mathcal{A}(Du), D\phi \rangle]\, dz = 0 \tag{5.0.2}$$

is satisfied for all testing function $\phi \in C_c^\infty(\Omega_T, \mathbb{R}^N)$.

For the existence of weak solutions to problem with full space gradient, we refer for instance to Elmahi & Meskine [41], Theorem 2.
By using a Moser type iteration for systems with general growth conditions, we prove the local Lipschitz regularity of the spatial gradient of solutions to (5.0.1). More precisely:

Theorem 5.0.1. *Let u be a weak solution to (5.0.1). Then $Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn})$. Moreover for every $\mathcal{Q}_{R_0}(z_0) \Subset \Omega_T$ the following a priori estimate holds with the constant c*

depending on n and on the characteristic of φ

$$\sup_{\mathcal{Q}_{\frac{R_0}{2}}(z_0)} |Du|^2 \leq c \left(\fint_{\mathcal{Q}_{R_0}(z_0)} \varphi(|Du|)\, dz \right)^{1+\frac{2}{n}} + c. \tag{5.0.3}$$

5.1 Technical lemmas

The following lemmas will be useful for the proof of the main Theorem.

Lemma 5.1.1. *By using the representation $\mathcal{A}_i^\alpha(\xi) = \varphi'(|\xi|)\dfrac{\xi_i^\alpha}{|\xi|}$, for all $\xi, \lambda \in \mathbb{R}^{Nn}$ we have*

$$\sum_{i,j,\alpha,\beta} D_{\xi_j^\beta} \mathcal{A}_i^\alpha(\xi)\lambda_i^\alpha \lambda_j^\beta \sim \varphi''(|\xi|)|\lambda|^2. \tag{5.1.1}$$

Proof. Let us compute $D_{\xi_j^\beta}\mathcal{A}_i^\alpha(\xi)$:

$$D_{\xi_j^\beta}\mathcal{A}_i^\alpha(\xi) = \varphi''(|\xi|)\frac{\xi_j^\beta}{|\xi|}\frac{\xi_i^\alpha}{|\xi|} + \frac{\varphi'(|\xi|)}{|\xi|}\left[\delta_{\xi_i^\alpha \xi_j^\beta} - \frac{\xi_i^\alpha \xi_j^\beta}{|\xi|^2}\right]$$

where δ_{ij} denotes the Kronecker's symbol. Now for all $\xi, \lambda \in \mathbb{R}^{Nn}$

$$\sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(\xi)\lambda_i^\alpha\lambda_j^\beta = \varphi''(|\xi|)\frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2} + \frac{\varphi'(|\xi|)}{|\xi|}\left[|\lambda|^2 - \frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2}\right].$$

By using (2.4.3) and taking into account that $\dfrac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2} \leq |\lambda|^2$ we have

$$\begin{aligned}\sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(\xi)\lambda_i^\alpha\lambda_j^\beta &\leq \varphi''(|\xi|)\frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2} + c\varphi''(|\xi|)\left[|\lambda|^2 - \frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2}\right] \\ &\leq c\varphi''(|\xi|)|\lambda|^2.\end{aligned}$$

Moreover, on the other hand

$$\begin{aligned}\sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(\xi)\lambda_i^\alpha\lambda_j^\beta &\geq \varphi''(|\xi|)\frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2} + c\varphi''(|\xi|)\left[|\lambda|^2 - \frac{|\langle \xi, \lambda\rangle|^2}{|\xi|^2}\right] \\ &\geq c\varphi''(|\xi|)|\lambda|^2.\end{aligned}$$

□

Lemma 5.1.2. *Let φ be an N-function satisfying Assumption (2.4.1) and let $\gamma \geq 0$. Then, for every $\beta > 0$, there exists a positive constant C, independent of γ depending only on the characteristic of φ, such that for all $s > 0$*

$$\left[\int_0^s \sqrt{\zeta^{2\gamma}\frac{\varphi'(\zeta)}{\zeta}}\, d\zeta\right]^\beta \geq \left(\frac{C}{(2\gamma+1)} s^{\gamma+\frac{1}{2}}\sqrt{\varphi'(s)}\right)^\beta.$$

Proof. Integrating by parts and applying (2.4.3) we have

$$
\begin{aligned}
\int_0^s \sqrt{\zeta^{2\gamma}\frac{\varphi'(\zeta)}{\zeta}}\,d\zeta &= \lim_{\delta\to 0}\int_\delta^s \zeta^{\gamma-\frac{1}{2}}\sqrt{\varphi'(\zeta)}\,d\zeta \\
&= \lim_{\delta\to 0}\left\{\left[\frac{2}{2\gamma+1}\zeta^{\gamma+\frac{1}{2}}\sqrt{\varphi'(\zeta)}\right]_\delta^s - \frac{1}{2\gamma+1}\int_\delta^s \zeta^{\gamma+\frac{1}{2}}\frac{\varphi''(\zeta)}{\sqrt{\varphi'(\zeta)}}\,d\zeta\right\} \\
&\geq \frac{2}{2\gamma+1}s^{\gamma+\frac{1}{2}}\sqrt{\varphi'(s)} - \lim_{\delta\to 0}\frac{c}{2\gamma+1}\int_\delta^s \frac{\zeta^{\gamma+\frac{1}{2}}}{\sqrt{\varphi'(\zeta)}}\frac{\varphi'(\zeta)}{\zeta}\,d\zeta \\
&= \frac{2}{2\gamma+1}s^{\gamma+\frac{1}{2}}\sqrt{\varphi'(s)} - \frac{c}{2\gamma+1}\int_0^s \sqrt{\zeta^{2\gamma-1}\varphi'(\zeta)}\,d\zeta.
\end{aligned}
$$

Thus

$$
\frac{2\gamma+1+c}{2\gamma+1}\int_0^s \sqrt{\zeta^{2\gamma}\frac{\varphi'(\zeta)}{\zeta}}\,d\zeta \geq \frac{2}{2\gamma+1}s^{\gamma+\frac{1}{2}}\sqrt{\varphi'(s)}
$$

from which

$$
\left[\int_0^s \sqrt{\zeta^{2\gamma}\frac{\varphi'(\zeta)}{\zeta}}\,d\zeta\right]^\beta \geq \left(\frac{C}{(2\gamma+1)}s^{\gamma+\frac{1}{2}}\sqrt{\varphi'(s)}\right)^\beta.
$$

□

5.2 Proof of the main result

In this section we prove the boundedness of the spatial gradient of solutions to (5.0.1). To achieve this result, we will use a Moser type iteration for systems with general growth conditions. Fundamental to start this procedure is the following result due to [10], which allows us the existence of spatial second derivative of solutions to (5.0.1):

Theorem 5.2.1. *If $\mathcal{A}$ satisfies (2.4.5) and (2.4.6), then a local weak solution u to*

$$
u_t - \operatorname{div}(\mathcal{A}(Du)) = 0,
$$

in Ω_T satisfies the following estimate

$$
\begin{aligned}
\operatorname{esssup}_{t_0\in(-T,0)}\int_{\mathcal{B}_r(x_0)} |Du(x,t)|^2 dx + \int_{\mathcal{Q}_r(z_0)} |DV(Du)|^2 dxdt \\
\leq \frac{C}{(R-r)^2}\int_{\mathcal{Q}_R(z_0)} [\varphi(|Du|)+c]\,dxdt
\end{aligned}
\tag{5.2.1}
$$

for any $r<R$ and concentric parabolic cylinder $\mathcal{Q}_r(z_0), \mathcal{Q}_R(z_0) \Subset \Omega_T$.

Proof of Theorem 5.0.1. Let $0 < \rho < R$ and $z_0 = (x_0, t_0)$. By using Theorem 5.2.1, we can differentiate (5.0.1) with respect to x_k:

$$(u^\alpha_{x_k})_t - \sum_{i=1}^n \Big(D_{\xi^\beta_j} \mathcal{A}^\alpha_i (Du) u^\beta_{x_k x_j} \Big)_{x_i} = 0$$

from which for all $\phi = (\phi^\alpha) \in W^{1,2}_0(\Omega_T, \mathbb{R}^N)$ we have

$$\int_{\Omega_T} \left[-u_{x_k} \phi_t + \sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i (Du) u^\beta_{x_k x_j} \phi^\alpha_{x_i} \right] dxdt = 0. \tag{5.2.2}$$

Let $\chi \in C^1_c(\mathcal{B}_R(x_0))$ be a cut-off function in space such that

$$\begin{cases} 0 \le \chi(x) \le 1 & \\ \chi(x) = 1 & \text{in } \mathcal{B}_\rho(x_0) \\ |D\chi| \le \frac{c}{R-\rho}. & \end{cases} \tag{5.2.3}$$

and let $\eta_\varepsilon \in C^1(\mathbb{R})$ be a cut-off function in time such that, with $\varepsilon > 0$ being arbitrary

$$\begin{cases} \eta_\varepsilon = 1 & \text{on } (t_0 - \rho^2, \tau) \\ \eta_\varepsilon = 0 & \text{on } (-T, t_0 - R^2) \cup (\tau + \varepsilon, 0) \\ 0 \le \eta_\varepsilon(t) \le 1 & \text{on } \mathbb{R} \\ (\eta_\varepsilon)_t = -\frac{1}{\varepsilon} & \text{on } (\tau, \tau + \varepsilon) \\ |(\eta_\varepsilon)_t| \le \frac{C}{(R-\rho)^2} & \text{on } (t_0 - R^2, t_0 - \rho^2) \end{cases} \tag{5.2.4}$$

where $\tau \in (t_0 - \rho^2, t_0)$ such that $\tau + \varepsilon < t_0$.
Taking into account Theorem 5.2.1, it is lawful to take as test function in (5.2.2)

$$\phi^\alpha = |Du|^{2\gamma} u^\alpha_{x_k} \chi^2(x) \eta^2_\varepsilon(t),$$

where $\gamma \ge 0$ and $k = 1, \dots, n$ is fixed. Then it results

$$\begin{aligned} 0 = & \int_{\mathcal{Q}_R(z_0)} - \Big[(|Du|^{2\gamma})_t \, u^\alpha_{x_k} u^\beta_{x_k} \eta^2_\varepsilon + |Du|^{2\gamma} u^\beta_{x_k} (u^\alpha_{x_k})_t \, \eta^2_\varepsilon + |Du|^{2\gamma} u^\alpha_{x_k} u^\beta_{x_k} (\eta^2_\varepsilon)_t \Big] \, \chi^2 dxdt \\ & + \int_{\mathcal{Q}_R(z_0)} 2\gamma |Du|^{2\gamma - 1} \eta^2_\varepsilon \chi^2 \sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i (Du) u^\beta_{x_k x_j} (|Du|)_{x_i} u^\alpha_{x_k} \, dxdt \\ & + \int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma} \eta^2_\varepsilon \chi^2 \sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i (Du) u^\beta_{x_k x_j} u^\alpha_{x_k x_j} \, dxdt \\ & + \int_{\mathcal{Q}_R(z_0)} 2|Du|^{2\gamma} \eta^2_\varepsilon \chi \sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i (Du) u^\beta_{x_k x_j} u^\alpha_{x_k} \chi_{x_i} \, dxdt \\ = & \, I + II + III + IV. \end{aligned} \tag{5.2.5}$$

Now we estimate I. Let us observe that

$$\int_{\mathcal{Q}_R(z_0)} -(|Du|^{2\gamma})_t\, u^\alpha_{x_k} u^\beta_{x_k} \eta^2_\varepsilon \chi^2\, dxdt = \int_{\mathcal{B}_R(x_0)} \chi^2 dx \int_{t_0-R^2}^{t_0} -(|Du|^{2\gamma})_t\, u^\alpha_{x_k} u^\beta_{x_k} \eta^2_\varepsilon\, dt$$
$$= \int_{\mathcal{Q}_R(z_0)} \chi^2 |Du|^{2\gamma} (u^\alpha_{x_k} u^\beta_{x_k} \eta^2_\varepsilon)_t\, dxdt$$

from which

$$\begin{aligned}
I &= \int_{\mathcal{Q}_R(z_0)} \chi^2 |Du|^{2\gamma} u^\alpha_{x_k} (u^\beta_{x_k})_t\, \eta^2_\varepsilon\, dxdt \\
&= \frac{1}{2(\gamma+1)} \int_{\mathcal{Q}_R(z_0)} \frac{d}{dt}(|Du|^{2\gamma+2}\eta^2_\varepsilon \chi^2) dxdt - \frac{1}{2(\gamma+1)} \int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2} (\eta^2_\varepsilon)_t\, \chi^2\, dxdt \\
&= -\frac{1}{2(\gamma+1)} \int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2} (\eta^2_\varepsilon)_t\, \chi^2\, dxdt \\
&= -\frac{1}{2(\gamma+1)} \int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} |Du|^{2\gamma+2} (\eta^2_\varepsilon)_t\, \chi^2\, dxdt - \frac{1}{2(\gamma+1)} \int_{\mathcal{B}_R(x_0)} \int_{\tau}^{\tau+\varepsilon} |Du|^{2\gamma+2} (\eta^2_\varepsilon)_t\, \chi^2\, dxdt.
\end{aligned}$$

Taking into account the definition of η_ε we have that

$$I = -\frac{1}{\gamma+1} \int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} |Du|^{2\gamma+2} \eta_\varepsilon (\eta_\varepsilon)_t\, \chi^2\, dxdt + \frac{1}{\gamma+1} \int_{\mathcal{B}_R(x_0)} ⨍_{\tau}^{\tau+\varepsilon} |Du|^{2\gamma+2} \eta_\varepsilon\, \chi^2\, dxdt.$$

Exploiting the definition of $\mathcal{A}^\alpha_i(Du)$ we compute

$$D_{\xi^\beta_j} \mathcal{A}^\alpha_i(Du) = \left[\frac{\varphi''(|Du|)}{|Du|} - \frac{\varphi'(|Du|)}{|Du|^2} \right] \frac{u^\alpha_{x_i} u^\beta_{x_j}}{|Du|} + \frac{\varphi'(|Du|)}{|Du|} \delta_{\xi^\alpha_i \xi^\beta_j}.$$

By using the fact that $(|Du|)_{x_i} = \sum_{k,\alpha} \frac{u^\alpha_{x_k x_i} u^\alpha_{x_k}}{|Du|}$ we infer

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i} = \\
&= \left[\frac{\varphi''(|Du|)}{|Du|} - \frac{\varphi'(|Du|)}{|Du|^2} \right] \sum_{i,j,\alpha,\beta} \frac{u^\alpha_{x_i} u^\beta_{x_j}}{|Du|} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i} + \frac{\varphi'(|Du|)}{|Du|} \sum_{i,\alpha} u^\alpha_{x_i x_k} u^\alpha_{x_k} (|Du|)_{x_i}
\end{aligned}$$

now we sum over $k = 1, \ldots, n$ and we obtain

$$\begin{aligned}
&\sum_k \sum_{i,j,\alpha,\beta} D_{\xi^\beta_j} \mathcal{A}^\alpha_i(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i} = \\
&= \left[\frac{\varphi''(|Du|)}{|Du|} - \frac{\varphi'(|Du|)}{|Du|^2} \right] \sum_{i,k,\alpha} u^\alpha_{x_k} (|Du|)_{x_k} u^\alpha_{x_i} (|Du|)_{x_i} + \varphi'(|Du|) \sum_i \left[(|Du|)_{x_i} \right]^2 \qquad (5.2.6) \\
&= \left[\frac{\varphi''(|Du|)}{|Du|} - \frac{\varphi'(|Du|)}{|Du|^2} \right] \sum_\alpha \Big[\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \Big]^2 + \varphi'(|Du|) |D(|Du|)|^2.
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\sum_{\alpha}\Big[\sum_{i} u^{\alpha}_{x_i}(|Du|)_{x_i}\Big]^2 \le \sum_{i,\alpha}(u^{\alpha}_{x_i})^2\sum_{i}[(|Du|)_{x_i}]^2 \le |Du|^2|D(|Du|)|^2. \tag{5.2.7}$$

Putting together (5.2.6), and (5.2.7) we obtain

$$\begin{aligned}
II &= \int_{\mathcal{Q}_R(z_0)} 2\gamma|Du|^{2\gamma-1}\eta_\varepsilon^2\chi^2 \sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)u^\beta_{x_kx_j}(|Du|)_{x_i}u^\alpha_{x_k}\,dxdt\\
&= \int_{\mathcal{Q}_R(z_0)} 2\gamma|Du|^{2\gamma-1}\eta_\varepsilon^2\chi^2\Big\{\frac{\varphi''(|Du|)}{|Du|}\sum_{\alpha}\Big[\sum_{i}u^\alpha_{x_i}(|Du|)_{x_i}\Big]^2\\
&- \frac{\varphi'(|Du|)}{|Du|^2}\sum_{\alpha}\Big[\sum_{i}u^\alpha_{x_i}(|Du|)_{x_i}\Big]^2 + \varphi'(|Du|)|D(|Du|)|^2\Big\}dxdt\\
&\ge \int_{\mathcal{Q}_R(z_0)} 2\gamma|Du|^{2\gamma-1}\eta_\varepsilon^2\chi^2\frac{\varphi''(|Du|)}{|Du|}\sum_{\alpha}\Big[\sum_{i}u^\alpha_{x_i}(|Du|)_{x_i}\Big]^2 dxdt \ge 0
\end{aligned}$$

Estimate for IV: applying Cauchy-Schwarz inequality we infer

$$\begin{aligned}
&|IV| \le\\
&\le 2\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\Big[\chi^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)u^\beta_{x_kx_j}u^\alpha_{x_kx_i}\Big]^{\frac12}\Big[\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)\chi_{x_i}u^\alpha_{x_k}\chi_{x_j}u^\beta_{x_k}\Big]^{\frac12}dxdt\\
&\le \frac12\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\chi^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)u^\beta_{x_kx_j}u^\alpha_{x_kx_i}dxdt\\
&+ 2\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)\chi_{x_i}u^\alpha_{x_k}\chi_{x_j}u^\beta_{x_k}dxdt\\
&= \frac12 III + 2\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)\chi_{x_i}u^\alpha_{x_k}\chi_{x_j}u^\beta_{x_k}dxdt.
\end{aligned}$$

So (5.2.5) becomes

$$\begin{aligned}
&\frac{1}{\gamma+1}\int_{\mathcal{B}_R(x_0)}⨍_{\tau}^{\tau+\varepsilon}|Du|^{2\gamma+2}\eta_\varepsilon\,\chi^2\,dxdt\\
&+ \frac12\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\chi^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)u^\beta_{x_kx_j}u^\alpha_{x_kx_i}dxdt\\
&\le \frac{1}{\gamma+1}\frac{C}{|R-\rho|^2}\int_{\mathcal{B}_R(x_0)}\int_{t_0-R^2}^{t_0-\rho^2}|Du|^{2\gamma+2}\chi^2\,dxdt\\
&+ 2\int_{\mathcal{Q}_R(z_0)}|Du|^{2\gamma}\eta_\varepsilon^2\sum_{i,j,\alpha,\beta}D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)\chi_{x_i}u^\alpha_{x_k}\chi_{x_j}u^\beta_{x_k}dxdt.
\end{aligned} \tag{5.2.8}$$

Now observe that by (5.1.1)

$$\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\eta_\varepsilon^2 \sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)\chi_{x_i}u^\alpha_{x_k}\chi_{x_j}u^\beta_{x_k}dxdt \\ \leq c\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\eta_\varepsilon^2\varphi''(|Du|)|D\chi|^2\,dxdt.$$

By Cauchy-Schwarz inequality we have $|(|Du|)_{x_i}|^2 \leq \sum_{k,\alpha}(u^\alpha_{x_ix_k})^2$ for all $i = 1,\dots,n$ from which $|D(|Du|)|^2 \leq |D^2u|^2$, so by using this last fact and (5.1.1) we get

$$\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\eta_\varepsilon^2\chi^2 \sum_{i,j,\alpha,\beta} D_{\xi_j^\beta}\mathcal{A}_i^\alpha(Du)u^\beta_{x_kx_j}u^\alpha_{x_kx_j}\,dxdt \\ \geq c\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\eta_\varepsilon^2\chi^2\varphi''(|Du|)|D(|Du|)|^2\,dxdt.$$

So (5.2.8) becomes

$$\frac{1}{\gamma+1}\int_{\mathcal{B}_R(x_0)}\fint_\tau^{\tau+\varepsilon} |Du|^{2\gamma+2}\eta_\varepsilon\chi^2\,dxdt + c\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\eta_\varepsilon^2\chi^2\varphi''(|Du|)|D(|Du|)|^2\,dxdt \\ \leq \frac{1}{\gamma+1}\frac{C}{|R-\rho|^2}\int_{B_R(x_0)}\int_{t_0-R^2}^{t_0-\rho^2} |Du|^{2\gamma+2}\chi^2\,dxdt + \frac{C}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\varphi''(|Du|)|D\chi|^2\,dxdt$$

and by passing to the limit as $\varepsilon \to 0$ we have

$$\frac{1}{\gamma+1}\int_{\mathcal{B}_R(x_0)} \chi^2|Du|^{2\gamma+2}(\tau)\,dx + c\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\chi^2\varphi''(|Du|)|D(|Du|)|^2\,dxdt \\ \leq \frac{1}{\gamma+1}\frac{C}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\chi^2\,dxdt + \frac{C}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\varphi''(|Du|)\,dxdt,$$

from which

$$\sup_{\tau\in(t_0-R^2,t_0)}\int_{\mathcal{B}_R(x_0)} \chi^2|Du|^{2\gamma+2}\,dx + c\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma}\chi^2\varphi''(|Du|)|D(|Du|)|^2\,dxdt \\ \leq \frac{C}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\chi^2\,dxdt + \frac{C(\gamma+1)}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\varphi''(|Du|)\,dxdt. \tag{5.2.9}$$

Let us define

$$F(s) = \int_0^s \sqrt{\zeta^{2\gamma}\frac{\varphi'(\zeta)}{\zeta}}\,d\zeta \quad \forall s \geq 0.$$

We can observe that by Hölder inequality, and (2.4.3) we get

$$
\begin{aligned}
[F(s)]^2 &\le \left(\int_0^s \zeta^{2\gamma} d\zeta\right)\left(\int_0^s \frac{\varphi'(\zeta)}{\zeta} d\zeta\right) \\
&\le \frac{s^{2\gamma+1}}{2\gamma+1}\int_0^s \frac{\varphi'(\zeta)}{\zeta} d\zeta \\
&\le s^{2\gamma+1}\int_0^s \frac{\varphi'(\zeta)}{\zeta} d\zeta \\
&\le s^{2\gamma+1} c \int_0^s \varphi''(\zeta) d\zeta \\
&\le c\, s^{2\gamma+1}\varphi'(s) \\
&\le c\, s^{2\gamma+2}\varphi''(s).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
D(\chi F(|Du|)) &= D\chi\, F(|Du|) + \chi\, F'(|Du|) D(|Du|) \\
&= D\chi\, F(|Du|) + \chi \sqrt{|Du|^{2\gamma}\frac{\varphi'(|Du|)}{|Du|}} D(|Du|)
\end{aligned}
$$

and by using (2.4.3) we deduce

$$
\begin{aligned}
|D(\chi F(|Du|))|^2 &\le 2\left[|D\chi|^2 F(|Du|)^2 + \chi^2 |Du|^{2\gamma}\frac{\varphi'(|Du|)}{|Du|}|D(|Du|)|^2\right] \\
&\le 2c\left[|D\chi|^2 |Du|^{2\gamma+2}\varphi''(|Du|) + \chi^2|Du|^{2\gamma}\varphi''(|Du|)|D(|Du|)|^2\right].
\end{aligned}
$$

Integrating over $\mathcal{Q}_R(z_0)$ and taking into account (5.2.9), we have

$$
\begin{aligned}
&\int_{\mathcal{Q}_R(z_0)} |D(\chi F(|Du|))|^2 dxdt \le \\
&\le 2c\int_{\mathcal{Q}_R(z_0)} |D\chi|^2|Du|^{2\gamma+2}\varphi''(|Du|)\, dxdt + 2c\int_{\mathcal{Q}_R(z_0)} \chi^2|Du|^{2\gamma}\varphi''(|Du|)|D(|Du|)|^2 dxdt \\
&\le \frac{C(\gamma+1)}{|R-\rho|^2}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\varphi''(|Du|)\, dxdt + \frac{c}{|R-\rho^2|}\int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\chi^2\, dxdt
\end{aligned}
\tag{5.2.10}
$$

Now, applying Hölder and Sobolev inequalities,(5.2.9) and finally (5.2.10) we get

$$
\begin{aligned}
&\int_{\mathcal{Q}_\rho(z_0)} |Du|^{(2\gamma+2)\frac{2}{n}} F^2(|Du|)dxdt \\
&\quad = \int_{t_0-\rho^2}^{t_0} dt \int_{\mathcal{B}_\rho(x_0)} (\chi|Du|)^{(2\gamma+2)\frac{2}{n}} (\chi F(|Du|))^2 dx \\
&\quad \leq \int_{t_0-\rho^2}^{t_0} dt \left(\int_{\mathcal{B}_\rho(x_0)} (\chi|Du|)^{2\gamma+2} dx \right)^{\frac{2}{n}} \left(\int_{\mathcal{B}_\rho(x_0)} (\chi F(|Du|))^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\quad \leq \left(\sup_{t\in(t_0-\rho^2,t_0)} \int_{\mathcal{B}_\rho(x_0)} \chi^2 |Du|^{2\gamma+2} dx \right)^{\frac{2}{n}} \int_{\mathcal{Q}_R(z_0)} |D(\chi F(|Du|))|^2 dxdt \\
&\quad \leq \left(\frac{C}{|R-\rho|^2} \int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2}\, dxdt + \frac{c(\gamma+1)}{|R-\rho|^2} \int_{\mathcal{Q}_R(z_0)} |Du|^{2\gamma+2} \varphi''(|Du|)\, dxdt \right)^{1+\frac{2}{n}} .
\end{aligned} \tag{5.2.11}
$$

By using Lemma 5.1.2 with $\beta = 2$ and $\varphi'(s) \sim s\varphi''(s)$ we have

$$
F^2(|Du|) \geq \frac{c}{(2\gamma+1)^2} |Du|^{2\gamma+1} \varphi'(|Du|) \geq \frac{c}{(2\gamma+1)^2} |Du|^{2\gamma+2} \varphi''(|Du|)
$$

and by the previous estimate we get

$$
\begin{aligned}
&\int_{\mathcal{Q}_\rho(z_0)} |Du|^{2(\gamma+1)\frac{2}{n}+2(\gamma+1)} \varphi''(|Du|)\, dxdt \\
&\quad \leq \left\{ \frac{c(2\gamma+1)^{\frac{2n}{n+2}+1}}{|R-\rho|^2} \left[\int_{\mathcal{Q}_R(z_0)} |Du|^{2(\gamma+1)} dxdt + \int_{\mathcal{Q}_R(z_0)} |Du|^{2(\gamma+1)} \varphi''(|Du|)\, dxdt \right] \right\}^{1+\frac{2}{n}} .
\end{aligned}
$$

Let us observe that, for $|Du| \geq 1$ then $\varphi''(|Du|) \geq \dfrac{\varphi'(|Du|)}{|Du|} \geq \varphi'(1) = c > 0$, so we get

$$
\begin{aligned}
&\int_{\mathcal{Q}_R(z_0)} |Du|^{2(\gamma+1)}\, dxdt \\
&= \int_{\mathcal{Q}_R(z_0)\cap\{|Du|\leq 1\}} |Du|^{2(\gamma+1)}\, dxdt + \int_{\mathcal{Q}_R(z_0)\cap\{|Du|\geq 1\}} |Du|^{2(\gamma+1)}\, dxdt \\
&\leq |\,\mathcal{Q}_R(z_0)| + c \int_{\mathcal{Q}_R(z_0)\cap\{|Du|\geq 1\}} |Du|^{2(\gamma+1)} \varphi''(|Du|)\, dxdt \\
&\leq |\,\mathcal{Q}_R(z_0)| + c \int_{\mathcal{Q}_R(z_0)} |Du|^{2(\gamma+1)} \varphi''(|Du|)\, dxdt
\end{aligned}
$$

Thus we have

$$\begin{aligned}
&\int_{\mathcal{Q}_\rho(z_0)} |Du|^{2(\gamma+1)\frac{2}{n}+2(\gamma+1)}\varphi''(|Du|)\,dxdt \leq \\
&\leq \left\{ \frac{c(2\gamma+1)^{\frac{2n}{n+2}+1}}{|R-\rho|^2} \left[|\,\mathcal{Q}_R(z_0)| + C\int_{\mathcal{Q}_R(z_0)} |Du|^{2(\gamma+1)}\varphi''(|Du|)\,dxdt \right] \right\}^{1+\frac{2}{n}}.
\end{aligned} \tag{5.2.12}$$

Let $\sigma := 1 + \frac{2}{n}$. For some $\gamma_0 > 0$, we set

$$\gamma_{i+1} = (\gamma_i + 1)\frac{2}{n} + \gamma_i = \sigma^{i+1}\gamma_0 + (\sigma^{i+1} - 1).$$

Define $R_i = \frac{R_0}{2}(1 + \frac{1}{2^i})$ and take $\rho = R_{i+1}$ and $R = R_i$ in (5.2.12). We define $\Phi_i = \int_{\mathcal{Q}_{R_i}(z_0)} |Du|^{2(\gamma_i+1)}\varphi''(|Du|)\,dxdt$, and $\beta_i = 2\gamma_i + 1$, thus we have

$$\Phi_{i+1} \leq c^{i+1}\beta_i^{2+\sigma}\Phi_i^{\sigma} + c^{i+1}\beta_i^{2+\sigma}.$$

Iterating we get

$$\begin{aligned}
\Phi_{i+1} \leq\, & c^{\sum_{k=0}^{i}(i-k+1)\sigma^k} \prod_{k=0}^{i} \beta_{i-k}^{(2+\sigma)\sigma^k}\, 2^{\sum_{k=1}^{i}(\sigma^k-1)}\, \Phi_0^{\sigma^{i+1}} \\
& + \sum_{j=1}^{i} 2^{\sum_{k=1}^{j}(\sigma^k-1)}\, c^{\sum_{k=0}^{j}(i-k+1)\sigma^k} \prod_{k=0}^{j} \beta_{i-k}^{(2+\sigma)\sigma^k}.
\end{aligned}$$

Now,

$$\log\left(\prod_{k=0}^{i}\beta_{i-k}^{(2+\sigma)\sigma^k}\right) = \sum_{k=0}^{i}\log\left(\beta_{i-k}^{(2+\sigma)\sigma^k}\right) = (2+\sigma)\sum_{k=0}^{i}\sigma^k\log(\beta_{i-k})$$

by the definition of β we get $\beta_{i-k} \leq 2(\gamma_0+1)\sigma^{i-k+1}$, thus

$$\begin{aligned}
\log\left(\prod_{k=0}^{i}\beta_{i-k}^{(2+\sigma)\sigma^k}\right) &\leq (2+\sigma)\log\sigma\sum_{k=0}^{i}\sigma^k(i-k+1) + (2+\sigma)\log(2(\gamma_0+1))\sum_{k=0}^{i}\sigma^k \\
&\leq c\sigma^{i+1} + c\frac{\sigma^{i+1}-1}{\sigma-1} \leq c\sigma^{i+1}.
\end{aligned}$$

From which

$$\prod_{k=0}^{i}\beta_{i-k}^{(2+\sigma)\sigma^k} \leq e^{c\sigma^{i+1}}.$$

Hence we can infer

$$\Phi_{i+1} \leq M^{\sigma^{i+1}}\Phi_0^{\sigma^{i+1}} + M^{\sigma^{i+1}}(i+1).$$

Now,

$$\Phi_{i+1}^{\frac{1}{2(\gamma_{i+1}+1)}} \leq M^{\frac{\sigma^{i+1}}{2(\gamma_{i+1}+1)}} \Phi_0^{\frac{\sigma^{i+1}}{2(\gamma_{i+1}+1)}} + M^{\frac{\sigma^{i+1}}{2(\gamma_{i+1}+1)}} (i+1)^{\frac{1}{2(\gamma_{i+1}+1)}}.$$

and by the definition of γ_i we have that $\gamma_{i+1}+1 \to +\infty$ as $i \to +\infty$, so we have

$$\Phi_{i+1}^{\frac{1}{2(\gamma_{i+1}+1)}} \leq M^{\frac{1}{2(\gamma_0+1)}} \Phi_0^{\frac{1}{2(\gamma_0+1)}} + M^{\frac{1}{2(\gamma_0+1)}} (i+1)^{\frac{1}{2\sigma^{i+1}(\gamma_0+1)}}.$$

Hence we can infer

$$\sup_{\mathcal{Q}_{\frac{R_0}{2}}(z_0)} |Du|^2 \leq c \left(\int_{\mathcal{Q}_{R_0}(z_0)} |Du|^{2\gamma_0+2} \varphi''(|Du|)\, dz \right)^{\frac{\sigma}{\gamma_0+1}} + c. \tag{5.2.13}$$

Let us consider the estimate (5.2.11) with $\gamma = 0$:

$$\int_{\mathcal{Q}_\rho(z_0)} |Du|^{\frac{4}{n}} F^2(|Du|) dz \leq \left\{ \frac{c}{|R-\rho|^2} \left[\int_{\mathcal{Q}_R(z_0)} |Du|^2 dz + \int_{\mathcal{Q}_R(z_0)} |Du|^2 \varphi''(|Du|)\, dz \right] \right\}^{1+\frac{2}{n}}$$

by Lemma 5.1.2 we infer

$$\int_{\mathcal{Q}_\rho(z_0)} |Du|^{2(1+\frac{2}{n})} \varphi''(|Du|)\, dz \leq \left\{ \frac{c}{|R-\rho|^2} \left[|\,\mathcal{Q}_R(z_0)| + \int_{\mathcal{Q}_R(z_0)} |Du|^2 \varphi''(|Du|)\, dz \right] \right\}^{1+\frac{2}{n}}.$$

Choosing $\gamma_0 = \frac{2}{n}$, an average of the last estimate with (5.2.13) and (2.4.3) implies

$$\begin{aligned} \sup_{\mathcal{Q}_{\frac{R_0}{2}}(z_0)} |Du|^2 &\leq c \left(\int_{\mathcal{Q}_{R_0}(z_0)} |Du|^2 \varphi''(|Du|)\, dz \right)^{\sigma} + c \\ &\leq c \left(\int_{\mathcal{Q}_{R_0}(z_0)} \varphi(|Du|)\, dz \right)^{\sigma} + c. \end{aligned}$$

□

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