Università degli Studi di Napoli Federico II
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On some methods and applications of ordered Banach spaces

Edited by
Achille Basile

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Fuori Collana

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Editor's Preface

Achille Basile*

This book collects some recent contributions of Ciro Tarantino to mathematical research in infinite dimensional spaces.

Spaces of infinite dimension have played an increasing role in the last four decades concerning their applications in models both for economics and for finance. The intertemporal allocation of resources, commodity differentiation, uncertainty, dynamics of the fundamental variables of financial markets, are some of the issues that can be properly captured by means of the mathematical techniques that are typical in such spaces.

The contribution devoted to fairness properties of some kind of allocations that emerge in very general models of markets, the so-called mixed markets, mainly focuses on the absence of envy among coalitions of agents. Naturally the object of envy is something that can be subject to trade and not some moral entity (like beauty, say, or some other natural talent). Due to the exchange activities economic agents move from an initial endowment of goods to a new feasible one and may be possible that a group of agents envies the net trade of another coalition. When an allocation represents an efficient redistribution of resources and do not exhibit envy among coalitions, then we say that it is coalitionally fair, definitely a socially desirable property. Are competitive equilibria coalitionally fair? Is it the case that coalition proof allocations, namely the Core allocation, are also robust with respect to this further stability property? These are the themes addressed by Tarantino in *Coalitional fairness with many agents and commodities*. The answers that are provided improve the previous results that are known in the literature. Here markets consider the possibility of the interaction of oligopolies and price takers, under different level of information about the future states of the world and with reference to infinite dimensional random consumption.

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Joint with Canale, in *Embedding and compactness results for multiplication operators in Sobolev spaces*, Tarantino, by means of a multiplicative embedding of Sobolev spaces into Lebesgue spaces, is able to provide an inequality of Fefferman type. Moreover the multiplicative operator is also shown to be compact. Here the multiplicator is a point in a Banach space that generalizes the concept of Morrey space.

Clearly such results are of a more purely mathematical nature. However, it is worthwhile to recall that Fefferman type inequalities have been used in mathematical finance for example to approximate a given contingent claim by means of trading strategies on a multivariate semimartingale describing the price evolution of a certain number of assets. Morrey spaces are as well involved in applications. The interest in this case bases on the fact that we generalize the possibility of controlling the oscillation of functions, what leads to results concerning the regularity of the abstract solutions of the relevant partial differential equations.

Coalitional fairness with many agents and commodities*

Ciro Tarantino**

Abstract: We investigate fairness properties of Rational expectations equilibria (RE equilibria) in economies with uncertainty and asymmetric information. We consider mixed models consisting of a set of atoms and an atomless sector interpreted, respectively, as traders that are large and small in terms of their influence on the market. Moreover, we assume that infinitely many commodities are present on the market. We provide characterizations of RE equilibria as allocations that are ex-post coalitionally fair.
Keywords: Asymmetric information; Ex-post core; c-fair allocations; Rational expectations equilibria.
JEL class. C71, D43, D51, D82.

1. *Introduction*

In this paper we investigate economic equity and fairness properties of Rational expectations equilibria in mode..ls of exchange economies with uncertainty and asymmetric information. The existing literature on fair allocations has established, in the case of complete information economies, several version of the equivalence between the fair and the competitive allocations (see H. R. Varian, *Equity, Envy*; J. J. Gabszewicz, *Coalitional Fairness*; B. Shitovitz, *Coalitional fair* and L. Zhou, *Strictly fair*). The study of equity and fairness notions has been initiated in the seminal papers by D. Foley, *Resource allocation.* and H. R. Varian, *Equity, Envy.* where an allocation of resources is said to be (individually) fair if it is both efficient and envy-free. It is said to be envy-free if no agent prefers or envies another agent's bundle to his own. It is an immediate fact that equal income competitive allocations are fair (see H. R. Varian, *Equity Envy*). But in general the converse is not true, even in large economies. In the attempt of fully characterizing the set of equal-income competitive allocations, various concepts of fairness have been introduced and investigated. In particular, the notion of envy is ex-

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tended in H. R. Varian, *Equity Envy.* from individual to coalitions (see also K. Vind, *Lecture notes*). An allocation is said to be coalitionally envy-free if no coalition envies any other coalition. It is said to be coalitionally fair if it is both efficient and coalitionally envy-free. The comparison between agents is made on a coalitional basis instead of an individual one, since each coalition of agents is required to compare its aggregate bundle to the aggregate bundle of any other coalition with the same size. It turns out that, under this stronger notion of equity, competitive allocations with equal-income coincide with the intersection of the sets of coalitionally fair allocations of all replica economies. Based on the notion of net trade, a different property of coalitional fairness has been investigated in J. J. Gabszewicz, *Coalitional Fairness.* and B. Shitovitz, *Coalitional fair*. For a given allocation of commodities among the traders, the net trade of agent t is the vector $x(t)-e(t)$, where $x(t)$ is the bundle received by t and $e(t)$ the initial endowment of t . An allocation is said to be coalitionally fair if no coalition S_1 can benefit from the aggregate net trade of some other disjoint coalition S_2 . That is, no coalition could redistribute among its members the net trade of a disjoint coalition in such a way that each of its members is better off. As a consequence, the market is endowed with a form of stability deriving from the fact that no coalition envies the net trade of any other. It is an immediate property that competitive allocations are coalitionally fair and that coalitionally fair allocations are in the core. Hence, as a consequence of core equivalence theorems, it is proved that in the case of large nonatomic economies, coalitionally fair allocations coincide with core allocations and, consequently, with the competitive ones. When the set of atoms is non empty, these inclusions may be strict J. J. Gabszewicz, *Coalitional Fairness.* Therefore, the study of coalitionally fair allocations and the investigation of fairness properties of competitive allocations deserve particular interest in the case of mixed models consisting of a set of atoms and of an atomless sector. This problem is investigated in our paper in the general framework of exchange economies with uncertainty and asymmetric information. Moreover, we do not limit consideration to finitely many commodities, but infinitely many commodities are taken into account.

The uncertainty is represented by a measurable space of events; in our model we only consider the case of a finite set of possible states of nature, then only finitely many events may occur. Agents have different information regarding the events, in the sense that not only they are not aware of which state of nature will realize, but they also have different prior beliefs. Uncertainty is reflected on the initial endowments and on the utility functions that depend on the state of nature which will be realized. The information of each agent is described by a partition of the set of states of nature. Each atom of this partition is an event; when a state occurs, the agent only observes the event containing the true state, but he cannot distinguish among states belonging to the same atom.

On the commodity space side, we deal with the general case of ordered Banach spaces. Economic problems involving stochastic and dynamic models, naturally lead to the consideration of infinite dimensional commodity spaces. It is due to G. Debreu, *Valuation Equilibrium*, the first examination of validity of welfare theorems in economies with infinitely many private goods. In a subsequent contribution, C. D. Aliprantis et al., *Equilibria in markets.* placed such economic models in the general framework of Riesz spaces. From a technical point of view, we remark that our results do not require a lattice structure on the commodity space. On the other hand, the general representation adopted in the paper makes us unable to use standard version of Lyapunov's theorem in our main characterizations. Indeed, this convexity result fails to be true not only for the restriction of resource allocations on the atomic part of the market (i.e. on large traders), but also for their restriction on the atomless sector (i.e. on negligible traders).

Dealing with general mixed models, coalitional fairness properties of Rational expectation equilibria (RE equilibria) are derived by means of an artificial family of complete information economies associated to the original economy. It is proved that ex-post coalitionally fair allocations coincide with the selections from the coalitionally fair correspondence of the associated complete information economies. Moreover, assuming that each trader knows his utility function, it is true that RE equilibrium allocations are exactly the selections from the competitive equilibrium correspondence of the associated complete information economies. As a consequence, we show that in general atomless models of economies with asymmetric information RE equlibrium allocations coincide with ex-post coalitionally fair allocations. Similar characterizations are discussed for the case of monopolistic economies or models with several large traders.

The paper is organized as follows. In Section 2 we introduce the model and the main assumptions. In Section 3 we study relations among the main equilibrium notions with reference to a general mixed model of exchange economy with differential information. In Section 4 we study fairness properties of RE equilibrium allocations.

2. *The differential information economy setting*

We shall deal with a pure exchange economy with uncertainty and differential information. The mathematical model describing uncertainty is standard: a measurable space $(\Omega, \mathbf{F})$ where the set Ω denotes the possible states of nature and the algebra $\mathbf{F}$ (of subsets of Ω) represents the family of all possible events. There are finitely many possible states of nature.

Agents are not endowed with the same information regarding the states of the world. Different agents t may have different information algebras $\mathbf{F}_t$ generated by a partition Π_t of Ω made of elements of $\mathbf{F}$. According to the usual interpretation: if the prevailing state is ω , the only information available to t is that the unique event $\Pi_t(\omega)$ of Π_t to whom ω belongs, prevails. Without any possibility to distinguish among states belonging to the same element of the partition.

The space of agents is an arbitrary complete finite measure space $(T, \mathbf{T}, \mu)$, being $\mathbf{T}$ the σ -algebra of all coalitions whose economic weight on the market is given by the measure μ . It follows that in this way we can simultaneously cover the discrete economy case (T is finite and the measure μ is the counting measure), the nonatomic" case of economies with infinitely many agents, each with no influence on the market (this type of economy is usually described by the interval [0,1] with the Lebesgue σ-algebra and the Lebesgue measure on it) or even the case of mixed markets (in the measure space there are atoms together with a nonatomic part), namely the case with both many uninfluential and some influential agents (or oligopolies) acting on the market.

In the case of a mixed market, we assume that $T = T_0 \cup T_1$, where T_1 is the set of atoms and T_0 the atomless part of T .

For what concerns the commodity space, with the aim of taking into account different kinds of models, for example to allow for infinite variations of the goods, or an infinite time horizon, we do not impose any restriction about its dimension. We represent it by a separable ordered Banach space IB whose positive cone IB_+ has non-empty norm interior. Physical commodities to be consumed are points in the positive cone IB_+ . Due to uncertainty, agents consume random commodities $x \in IB_+^{\Omega}$.

A consumption profile or assignment is defined next.

Definition 1. *An assignment is a function x associating to each agent in any state of the world an element of his consumption set*

$$x \ : \ \Omega \times T \to IB_+$$

such that

- *the function* $x(\cdot,t) =: x_t$ *is* F *--measurable for each* $t \in T$ *;*
- *the function* $x(\omega,\cdot) =: x_\omega$ *is* μ *--integrable on* T *for each* $\omega \in \Omega$ *.*

Further characteristics of an agent are:

- a state-dependent utility function u_t representing his preference:

$$u_t \ : \ \Omega \times IB_+ \to IR$$

where $u_t(\cdot, x)$ is F --measurable for all $t \in T$ and for all $x \in IB_+$ [1] and, for each $\omega \in \Omega$, $u_t(\omega,\cdot)$ is continuous, strictly increasing and strictly quasi-concave on IB_+ .

- an initial endowment contingent to the state of nature

$$e_t \ : \ \Omega \to IB_+$$

and F_t --measurable This allows to take into account the private information of the agent t , in such a way that t does not obtain any additional information about states of nature belonging to the same element of Π_t , i.e. those states which are indistinguishable for agent t .
For each $\omega \in \Omega$ the function $e_t(\omega)$ is assumed to be μ --integrable with respect to *t* in the sense of Bochner and such that $\int_T e_t(\omega)\, d\mu$ is a strictly positive vector.
We always write integrability to mean Bochner integrability.

We say that an assignment is an **allocation** if for each $\omega \in \Omega$,

$$\int_T x_\omega \, d\mu \leq \int_T e_\omega \, d\mu.$$

We denote by E our *differential information economy*, i.e. the following collection:

[1]We assume the $(\mathsf{T} \otimes \mathsf{B})$ -measurability of the mapping $(t,x) \mapsto u_t(\omega, x)$ for any $\omega \in \Omega$. B is the Borel σ -algebra of IB_+

$$\mathbf{E} = \{(\Omega, \mathbf{F});\ (T, \mathbf{T}, \mu);\ IB;\ (\Pi_t, u_t, e_t)_{t\in T}\}.$$

To a differential information economy $\mathbf{E}$, we associate a family of complete information economies $\{\mathbf{E}(\omega)\}_{\omega\in\Omega}$ by fixing for each state ω the initial endowment $e(\omega,\cdot)$ and the utility functions $(u_t(\omega,\cdot))_{t\in T}$.

3. *Coalitional fairness*

Let us take an economy $\mathbf{E}$, an allocation *x* and a coalition *S* of agents.

Definition 2. *We say that the allocation x is blocked ex--post by the coalition* S *in the state* ω_o *if there exists an assignment y such that*

$$\int_S y(\omega_0,t)\,d\mu \le \int_S e(\omega_0,t)\,d\mu;$$
$$u_t(\omega_0, y(\omega_0,t)) > u_t(\omega_0, x(\omega_0,t)) \quad \text{for a.a. } t\in S.$$

Definition 3. *An allocation x is an ex--post core allocation if there exists no coalition that blocks it in any state of nature. The set of all ex--post core allocations, the ex--post core of* $\mathbf{E}$ *, is denoted by* $\mathbf{C}(\mathbf{E})$ *(*cf. Einy et al., *Rational expectations*).

We move now our attention to a notion of *coalitional fairness* inspired to J. J. Gabszewicz, *Coalitional Fairness.* and already adapted by C. Donnini et al., *Coalitional fairness.* in differential information economies with reference to the interim case.

Let us take an allocation x . Suppose that, corresponding to a non-negligible coalition S_1 , we find a disjoint coalition S_2 , an assignment y and a state ω_0 such that

2) $u_t(\omega_0, y(\omega_0,t)) > u_t(\omega_0, x(\omega_0,t))$ for a.a. $t\in S_1$;

3) $\int_{S_1}\left[y(\omega_0,t) - e(\omega_0,t)\right]d\mu \le \int_{S_2}\left[x(\omega_0,t) - e(\omega_0,t)\right]d\mu$.

It seems natural to say, under the above circumstances, that agents of S_1 in the state ω_0 discover ex-post that they envy the net trade[2] of coalition S_2 . Indeed, they could redistribute it among themselves becoming better off.

[2]If x is an allocation and S a coalition, the vector $\int_S [x_\omega - e_\omega,]\,d\mu$ is the *net trade* of coalition S at ω .

We may well say, then, that *under x there is an enviuos coalition*. Consequently, the allocation x will be said *ex-post c-fair (in the sense of Gabszewicz)* if there are no envious coalitions.

The formal definitions follow below.

Definition 4. *Given two disjoint coalitions S_1 and S_2 , an allocation x* ***is ex--post coalitionally fair (c-fair) relative to*** S_1 ***and*** S_2 *(or it* ***does not show ex--post envy relative to*** S_1 ***and*** S_2 *), if there exist no state of nature $\omega_0 \in \Omega$ and no assignment y , such that for some i , $i = 1, 2$:*

1) $\mu(S_i) > 0$;
2) $u_t(\omega_0, y(\omega_0, t)) > u_t(\omega_0, x(\omega_0, t))$ for a.a. $t \in S_i$;
3) $\int_{S_i} [y_{\omega_0} - e_{\omega_0}] d\mu \leq \int_{S_j} [x_{\omega_0} - e_{\omega_0}] d\mu \quad j \neq i$.

Definition 5. *Given a class* $\mathbf{S}$ *of coalitions, an allocation x is an* ***ex--post c-fair allocation with respect to*** $\mathbf{S}$ *if it doesn't show ex--post envy relative to S_1 and S_2 , whenever S_1 and S_2 belong to* $\mathbf{S}$.

The set of all ex-post c-fair allocations with respect to $\mathbf{S}$ is denoted by $\mathbf{S}$ - $\mathbf{C}_{fair}(\mathbf{E})$. The set of all ex-post c-fair allocations with respect to $\mathbf{T}$ is shortly denoted by $\mathbf{C}_{fair}(\mathbf{E})$ and we usually omit to say with respect to $\mathbf{T}$. If no ex--post envy emerges relative to any possible pair of disjoint coalitions S_1, S_2 with $S_1 \in \mathbf{S}_1$ and $S_2 \in \mathbf{S}_2$, then we speak of c--fairness with respect to the classes $\mathbf{S}_1$ and $\mathbf{S}_2$ of coalitions.

It follows directly from the definitions that

Proposition 6. *Any ex--post coalitionally fair allocation in* $\mathbf{T}$ *is in the ex--post core:*

$$\mathbf{C}_{fair}(\mathbf{E}) \subseteq \mathbf{C}(\mathbf{E})$$

Proof: We suppose that x is in $\mathbf{C}_{fair}(\mathbf{E})$, but it does not belong to the ex-post core. Then, there exist a coalition A and an assignment y such that in a fixed state of nature A blocks x . Setting $S_1 = A$ and S_2 empty we get a contradiction.

The inclusion above cannot be reversed in general. In the case of complete information economies, this is shown to be true in J. J. Gabszewicz, *Coalitional Fairness* Proposition 2. For atomless economies, however, ex-post core and ex-post c-fair allocations do coincide since one can properly extend both J. J. Gabszewicz, *Coalitional Fairness*, Theorem 1 and E. Einy et al, *Rational expectations*, Theorem 3.1.

In general mixed market with infinitely many commodities, generalizing J. J. Gabszewicz, *Coalitional Fairness*, Theorem 2, one can show that

Theorem 7. *Let x be a strictly positive ex--post core allocation. Then x is ex--post coalitionally fair with respect to* $\mathbf{T}_0 = \{S \in \mathbf{T} \mid S \subseteq T_0\}$ *and* $\mathbf{T}_1 = \{S \in \mathbf{T} \mid S \supseteq T_1\}$.

The result can be interpreted by saying that, in a market economy with many commodities and atoms, under a core allocation coalitions of negligible traders do not envy ex--post coalitions containing all large traders and conversely.

First, observe that analogously to E. Einy et al, *Rational expectations*, Theorem 3.1, we can obtain the following characterization:

Theorem 8. *Let* $\mathbf{E}$ *be an economy. For any class of coalitions* $\mathbf{S}$ *, we get*

$$\mathbf{S}\text{-}\mathbf{C}_{fair}(\mathbf{E}) = \{x \mid x \text{ is an assignment and } x(\omega,\cdot) \in \mathbf{S}\text{-}\mathbf{C}_{fair}(\mathbf{E}(\omega)) \ \ \forall \omega \in \Omega\}.$$

Then we proceed with the proof of Theorem 7.

Proof of Theorem 7: Since both, ex--post core and c--fair allocations are selections (E. Einy et al, *Rational expectations.,* Theorem 3.1 and Theorem 8), it is enough to prove the statement for an economy with complete information or, in other words, to extend J. J. Gabszewicz, *Coalitional Fairness*, Theorem 2 to the case of a general commodity space IB for which int(IB_+) is non-empty.

Let, then, $S_1 \in \mathbf{T}_0$, $S_2 \in \mathbf{T}_1$, y be such that $\mu(S_1) > 0$, $u_t(y(t)) > u_t(x(t))$, a.e. in S_1 and $\int_{S_1} (y-e)\, d\mu \le \int_{S_2} (x-e)\, d\mu$. Consider the assignment $z = \frac{1}{2}(x+y)$. We have that it is strictly positive and $u_t(z(t)) > u_t(x(t))$, a.e. in S_1 . Moreover

$$\int_{S_1} (z-e)\, d\mu = \frac{1}{2}\int_{S_1} (x-e)\, d\mu + \frac{1}{2}\int_{S_1} (y-e)\, d\mu \le \frac{1}{2}\int_{S_1 \cup S_2} (x-e)\, d\mu.$$

Since $\int_{S_1} z\, d\mu >> 0$ and by continuity of preferences, there exist $\varepsilon > 0$ and a subset C of S_1 such that $u_t(\varepsilon z(t)) > u_t(x(t))$, a.e. in C . Define the function $\bar{z}$ to be the same as z on $S_1 \subseteq C$ and equal to εz on C . Then $u_t(\bar{z}(t)) > u_t(x(t))$, a.e. in S_1 and

$$\int_{S_1} \bar{z}\, d\mu = \varepsilon \int_C z\, d\mu + \int_{S_1 \setminus C} z\, d\mu << \int_{S_1} z\, d\mu,$$

so

$$\int_{S_1} (\bar{z} - e)\, d\mu << \int_{S_1} (z - e)\, d\mu \le \frac{1}{2} \int_{S_1 \cup S_2} (x - e)\, d\mu.$$

Since

$$v := \tfrac{1}{2} \int_{S_1 \cup S_2} (x - e)\, d\mu - \int_{S_1} (\bar{z} - e)\, d\mu << 0$$

we can take $\varepsilon > 0$ in such a way that the disk centered in v and radius ε is fully contained into int (IB_+) .

Observe now that since x is a core allocation, the monotonicity gives that the feasibility actually guarantees $\int_T x d\mu = \int_T e d\mu$. Now, if the set $S_1 \cup S_2$ exhausts T , then the integral $\int_{S_1 \cup S_2} (x - e)\, d\mu$ is zero and $\int_{S_1} (\bar{z} - e)\, d\mu < 0$ gives the feasibility of $\bar{z}$, namely we violate that x is in the core. We can therefore assume that the set $T \subseteq (S_1 \cup S_2)$, which does not contain atoms, is of positive μ -measure. By the relative convexity of the range of the integral function over T_0 , we find a set

$$\bar{S} \subseteq T \subseteq (S_1 \cup S_2) \text{ with } \left\| \int_{\bar{S}} (x - e)\, d\mu - \frac{1}{2} \int_{T \setminus (S_1 \cup S_2)} (x - e)\, d\mu \right\| < \varepsilon.$$

Define s to be the same as $\bar{z}$ on the coalition S_1 and the same as x on $\bar{S}$, then $u_t(s(t)) \ge u_t(x(t))$, a.e. in $S_1 \cup \bar{S}$ and $u_t(s(t)) > u_t(x(t))$, a.e. in S_1 . Moreover

$$\left\| \int_{S_1 \cup \bar{S}} (s - e)\, d\mu - \left(\int_{S_1} (\bar{z} - e)\, d\mu - \frac{1}{2} \int_{S_1 \cup S_2} (x - e)\, d\mu \right) \right\| =$$

$$\left\| \int_{\bar{S}} (x - e)\, d\mu - \frac{1}{2} \int_{T \setminus (S_1 \cup S_2)} (x - e)\, d\mu + \frac{1}{2} \int_T (x - e)\, d\mu) \right\| < \varepsilon.$$

It follows that

$$\int_{S_1 \cup \bar{S}} (s - e)\, d\mu << 0$$

so we can take a vector $w >> 0$ such that $w + \int_{S_1 \cup \bar{S}} s\, d\mu = \int_{S_1 \cup \bar{S}} e\, d\mu$. Finally, modifying s by taking $x + \frac{w}{\mu(\bar{S})}$ instead of x on $\bar{S}$, we violate that x is in the core since the coalition $S_1 \cup \bar{S}$ blocks x via the new assignment s . Indeed $u_t(s(t)) > u_t(x(t))$, a.e. in $S_1 \cup \bar{S}$ and $\int_{S_1 \cup \bar{S}} s\, d\mu = \int_{S_1} \bar{z}\, d\mu + \int_{\bar{S}} x\, d\mu + w = \int_{S_1 \cup \bar{S}} e\, d\mu.$

For the case where $S_1 \in \mathbf{T}_1$, $S_2 \in \mathbf{T}_0$ a similar proof applies.

4. *Fairness of Rational Expectations equilibria*

We analyze now coalitional fairness of rational expectations equilibria of the economy $\mathbf{E}$.

In this context agents restrict their consumption choices to budget sets defined as follows:

$$B_t(\omega, p) = \{a \in IB_+ \ : \ p(\omega)\cdot a \le p(\omega)\cdot e(\omega,t)\},$$

for the agent t , in the state ω and with respect to a prevailing system of prices p belonging to $\{p \in (IB_+^{'})^{\Omega} \ : \ p \ is \ \mathrm{F}-\mathbf{measurable} \ \}$ [3]. Moreover, since agents evaluate choices with a state by state comparison of the conditional expectation of their utility, taking into account both private information and the information revealed by prices, *we assume that every agent* t *is given with a strictly positive probability over* $\mathbf{F}$. Consequently we denote, with reference to such probability, by $E_t(f \,|\, \mathbf{G})$ the conditional expectation of an $\mathbf{F}$ -measurable function f given the subalgebra $\mathbf{G}$ of $\mathbf{F}$.

We denote by $\sigma(p)$ the smallest σ -algebra of $\mathbf{F}$ that makes the function p measurable ($\sigma(p)$ represents the information contained in p). $\mathbf{F}$ being generated by a partition Π , the σ --algebra $\sigma(p)$ is generated by the partition of Ω , coarser than Π , in its turn generated by the function p . In this case, moreover, $E_t(f \,|\, \mathbf{G})$ is a $\mathbf{G}$ --measurable random variable having the same mean of f on the elements of the partition that generates $\mathbf{G}$.

Definition 9. *A rational expectations equilibrium (RE equilibrium) is a pair* (p, x) *where* p *is a price system, and* x *is an allocation such that:*

(i) $x(\cdot,t)$ is $(\sigma(p) \vee \mathbf{F}_t)$ -measurable for all $t \in T$;

(ii) $x(\omega,t) \in B_t(\omega, p)$ *for each* $\omega \in \Omega$ *and for each* $t \in T$;

(iii) *almost everywhere in* T *, if y is* $(\sigma(p) \vee \mathbf{F}_t)$ *--measurable and satisfies* $y(\omega) \in B_t(\omega, p)$ *for each* $\omega \in \Omega$ *, then*

$$E_t(u_t(\cdot, x(\cdot,t)) \,|\, \sigma(p) \vee \mathbf{F}_t) \ge E_t(u_t(\cdot, y(\cdot)) \,|\, \sigma(p) \vee \mathbf{F}_t)$$

pointwise on Ω .

[3]Here we use standard notation for the topological dual of IB , its positive cone and the duality mapping.

In the above definition, the private information $\mathbf{F}_t$ of agent t is refined by means of the information generated by p . In case the price reveals all the information, i.e. $\sigma(p) = \mathbf{F}$, the RE equilibrium is said *fully revealing*. The set of allocations that are RE equilibria for a suitable price is denoted by $RE(\mathbf{E})$ [4].

Given the inclusions above, we see that:

$$RE(\mathbf{E}) \not\subset \mathbf{C}_{fair}(\mathbf{E}).$$

therefore, a RE equilibrium needs not be coalitionally fair. This can only happen when agents do not know their utility functions. In fact, we shall show next that assuming the functions $u(\cdot,t)$ to be $\mathbf{F}_t$ --measurable, then any RE equilibrium allocation is ex--post coalitionally fair.

Theorem 10. *Assume that in the economy* $\mathbf{E}$ *the functions* $u_t(\cdot,x)$ *are* $\mathbf{F}_t$ *--measurable. Then*

$$RE(\mathbf{E}) \subseteq \mathbf{C}_{fair}(\mathbf{E}).$$

Proof: Let $W(\mathbf{E}(\omega))$ be the Walrasian allocations of $\mathbf{E}(\omega)$, the using A. De Simone et al, *Some,* Theorem 11 and Theorem 8, we have to prove that

$$\{x \mid x \text{ is an assignment and } x(\omega,\cdot) \in W(\mathbf{E}(\omega)) \ \ \forall \omega \in \Omega\} \subseteq$$

$$\subseteq \{x \mid x \text{ is an assignment and } x(\omega,\cdot) \in \mathbf{C}_{fair}(\mathbf{E}(\omega)) \ \ \forall \omega \in \Omega\}$$

So, it is sufficient to show that in a complete information economy any Walrasian allocation is also coalitionally fair, i.e.

$$W(\mathbf{E}(\omega)) \subseteq \mathbf{C}_{fair}(\mathbf{E}(\omega)) \ \text{ for any fixed } \omega \in \Omega .$$

Let us consider a fixed state of nature ω_0 and the corresponding deterministic information economy $\mathbf{E}(\omega_0)$. Let x be a competitive allocation in $W(\mathbf{E}(\omega_0))$, and assume by way of contradiction that $x \notin \mathbf{C}_{fair}(\mathbf{E}(\omega_0))$. A contradiction easily follows like in J. J. Gabszewicz, *Coalitional Fairness*.

In the case of atomless economies, under suitable hypotheses, RE equilibrium allocations are exactly ex-post c-fair allocations.

[4]In E. Einy et al, *Rational expectations*, Example 4.2, an economy is constructed with a RE equilibrium, which can be blocked by an ex--post core allocation.

Theorem 11. *Let us assume that the set* T_1 *of large traders is empty. Moreover, for any agent and any state, suppose that* $u_t(\cdot, x)$ *are* $\mathbf{F}_t$ *--measurable. Then*

$$RE(\mathbf{E}) = \mathbf{C}_{fair}(\mathbf{E}).$$

Proof: Only the inclusion $RE(\mathbf{E}) \supseteq \mathbf{C}_{fair}(\mathbf{E})$ has to be justified. But if x is an ex--post c-fair allocation, and therefore in the ex--post core, then E. Einy et al, *Rational expectations.*, Theorem 3.1, applies. So fix a state ω . Hence, $x(\omega,\cdot) \in C(\mathbf{E}(\omega))$ and using the core equivalence Theorem in A. Rustichini et al, *Edgeworth's conjecture*, we have $x(\omega,\cdot) \in W(\mathbf{E}(\omega))$ and by A. De Simone et al, *Some new characterization*, Theorem 11, $x \in RE(\mathbf{E})$.

In the case of mixed markets one has to observe that ex--post c-fair allocation need not be RE equilibrium allocations in general. Such a result is too strong for oligopolistic models (T_1 non-empty), even in the case of complete information economies (see J. J. Gabszewicz, *Coalitional Fairness*, Proposition 2). If T_1 consists of a single atom, it is possible to prove that the RE equilibrium allocations are exactly ex-post c-fair allocations with respect to the class of coalitions which do not contain the atom.

Theorem 12. *Let* $\mathbf{E}$ *be a mixed market with only one atom, i.e.* $|T_1| = 1$. *Moreover, for any agent and any state, suppose that* $u_t(\cdot, x)$ *are* $\mathbf{F}_t$ *--measurable. Then*

$$RE(\mathbf{E}) = \mathbf{T}_0 \text{ - } \mathbf{C}_{fair}(\mathbf{E}).$$

Proof: By Theorem 8 we can write

$$\mathbf{T}_0 \text{ - } \mathbf{C}_{fair}(\mathbf{E}) = \{x \mid x \text{ is an assignment and } x(\omega,\cdot) \in \mathbf{T}_0 \text{ - } \mathbf{C}_{fair}(\mathbf{E}(\omega)) \quad \forall \omega \in \Omega\}.$$

Let us fix a state ω . Using the weak form of Lyapunov's convexity theorem, we can reproduce the proof of B. Shitovitz, *Coalitional fair*, Corollary A*, and obtain that

$$\mathbf{T}_0 \text{ - } \mathbf{C}_{fair}(\mathbf{E}) = \{x \mid x \text{ is an assignment and } x(\omega,\cdot) \in W(\mathbf{E}(\omega)) \quad \forall \omega \in \Omega\}.$$

The statement follows by E. Einy et al, *Rational expectations,* Theorem 4.3.

So, in the case of just one large trader, under a RE equilibrium allocation coalitions made by small traders do not envy each other ex--post. Let us discuss now the general case of mixed markets. Moving from the observation that core allocations are not necessarily c-fair, we have proved in Theorem 7 that a strictly positive allocation x which is in the ex--post core is ex-post c-fair with respect to $\mathbf{T}_0$ and $\mathbf{T}_1$. Here we wish to point out that if the (ex--post) core allocation x is also a restricted rational expectation equilibrium, then it comes out to be c-fair with respect to the class $\mathbf{T}_0 \cup \mathbf{T}_1$ of coalitions. In the following, we limit consideration to the case of finitely many commodities.

By a restricted equilibrium we mean what follows.

Definition 13. *A restricted rational expectations equilibrium (RRE equilibrium) is a pair* (p, x) *where* p *is a price system, and* x *is an allocation such that:*

(i) $x(\cdot,t)$ *is* $(\sigma(p) \vee \mathbf{F}_t)$ *--measurable for all* $t \in T$ *;*

(ii) $p(\omega)\cdot x(\omega,t) = p(\omega)\cdot e(\omega,t)$ *for each* $\omega \in \Omega$ *and for each* $t \in T_0$ *;*

(iii) almost everywhere in T *, if y is* $(\sigma(p) \vee \mathbf{F}_t)$ *--measurable and satisfies* $y(\omega) \in B_t^*(\omega, p) := \{a \in IB_+ \ : \ p(\omega)\cdot a \le p(\omega)\cdot x(\omega,t)\}$, *for each* $\omega \in \Omega$ *,*

then

$$E_t(u_t(\cdot, x(\cdot,t)) \mid \sigma(p) \vee \mathbf{F}_t) \ge E_t(u_t(\cdot, y(\cdot)) \mid \sigma(p) \vee \mathbf{F}_t)$$

pointwise on Ω .

The usual interpretation goes as follows: under a RRE equilibrium allocation x , there exists a price system p such that: 1) Each trader maximizes (ex--post) over his/her efficiency set; 2) small traders are in a RE equilibrium with respect to p . As for restricted competitive equilibria of complete information economies, in general, it is not true that the restriction of the allocation x on $\mathbf{T}_0$ is a RE equilibrium, since the feasibility condition in the atomless sub-economy may be violated. However, RRE equilibria which are also in the ex-post core satisfy additional fairness properties.

Proposition 14. *Let IB be finite dimensional. Every strictly positive, ex--post core allocation x which is also a RRE equilibrium is ex--post coalitionally fair with respect to $\mathbf{T}_0 \cup \mathbf{T}_1$.*

Proof: Let p be the equilibrium price. Since $x \in C(\mathbf{E})$, then, for any $\hat{\omega} \in \Omega$, we have that $x_{\hat{\omega}} \in C(\mathbf{E}(\hat{\omega}))$ and therefore $x_{\hat{\omega}}$ is a restricted competitive equilibrium in the economy $\mathbf{E}(\hat{\omega})$ with respect to the price $p(\hat{\omega})$. To check latter statement it is enough to go like in the proof of E. Einy et al, *Rational expectations…*, Theorem 4.3, by taking $a \in B_t^*(\hat{\omega}, p)$ and replacing $e(\omega,t) = e(\hat{\omega},t)$ by $x(\omega,t) = x(\hat{\omega},t)$. Of course the equalities $p(\hat{\omega}) \cdot x(\hat{\omega},t) = p(\hat{\omega}) \cdot e(\hat{\omega},t)$ over $\mathbf{T}_0$ are due to the definition of RRE equilibrium.

Now, $x_{\hat{\omega}}$ is a restricted competitive equilibrium in $\mathbf{E}(\hat{\omega})$ and also belongs to the core. Since we have proved the c--fairness of x with respect to $\mathbf{T}_0$ and $\mathbf{T}_1$ in Theorem 7, then the concluding argument goes like in the proof of J. J. Gabszewicz, *Coalitional Fairness*, Theorem 3.

The previous result implies that under conditions ensuring that ex-post core allocations are RRE equilibria, the ex--post core consists of allocations coalitionally fair with respect to the whole class $\mathbf{T}_0 \cup \mathbf{T}_1$. Such conditions are well known in the case of finitely many commodities (see for example J. J. Gabszewicz, *Coalitional Fairness*, B. Shitovitz, *Coalitional fair*, M. G. Graziano et al, *A note on the private...*). We leave the identification of analogous conditions in the presence of infinitely many commodities as subject of future investigation. A further line of open research is represented by the case in which the commodity space IB does not exhibit interior points in its positive cone. Such case comprises interesting models for applications like investigation of infinite-horizon economies, asset pricing models, differentiated commodity models. To cover this general case, the set of assumptions imposed in section 2 will require a modification inspired by classical properness assumptions on preferences. In particular, we expect that the commodity space enjoys a Riesz space structure. This requirement together with the lattice structure of the price space seems to be indispensable to carry over lattice theoretical arguments connected with properness conditions.

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Embedding and compactness results for multiplication operators in Sobolev spaces

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Abstract: The paper deals with the operator $u \to gu$ defined in the Sobolev space $W^{r,p}(\Omega)$ and which takes values in $L^p(\Omega)$ when Ω is an unbounded open subset in R^n The functions g belong to Morrey type spaces which provide an intermediate space between $L^\infty(\Omega)$ and $L^p_{loc}(\Omega)$. The main result is a embedding result from which we can deduce a Fefferman type inequality. L^p estimates and a compactness result are also stated.
Keywords: multiplication operator, L^p estimates, compactness results.
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1. *Introduction*

Let Ω be an unbounded open subset in R^n .
In literature there are different results about the study of *multiplication operator* for a suitable function $g \ : \ \Omega \to C$

$$u \to gu, \tag{1.1}$$

as an operator defined in a Sobolev space (with or without weight) and which takes values in a $L^p(\Omega)$ space.
In $W_0^{1,p}(\Omega)$ or in $W^{1,p}(\Omega)$ with Ω regular enough, reference results are some well-known inequalities which state the boundedness of (1.1): Hardy type inequalities (see H. Brezis, *Sobolev spaces,* A. Kufner, *Weighted*, J. Nečas, *Les methods*) when $g(x)$ is an appropriate power of the distance of x from a subset of $\partial\Omega$, Fefferman inequality (C. Fefferman, *The uncertainty principle,* see, e.g. F. Chiarenza - M. Franciosi, *A generalization of*, F. Chiarenza - M. Frasca, *A remark on*) obtained when g belongs to suitable Morrey spaces.

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Our aim is to study the *multiplication operator* in the Sobolev space $W^{r,p}(\Omega)$, $r \in N$, $1 \le p < +\infty$ when g belongs to suitable Morrey type spaces $\mathbf{M}^p(\Omega)$ which include $L^\infty(\Omega)$ and which are wider than classical Morrey space $L^{p,n}$ when $\Omega = R^n$. We observe that the space $\mathbf{M}^p(\Omega)$ provides an intermediate space between $L^\infty(\Omega)$ and $L^p_{loc}(\overline{\Omega})$.

The main results we stated concern embedding and compactness results for multiplication operators.

One of the aspects of our interest in these estimates lies in the fact that this type of inequalities are useful tools to prove a priori bounds when studying elliptic equations. Furthermore a priori bounds enable us to state existence and uniqueness results. For some applications in the study of the a priori bounds see A. Canale, *A priori* 2003, *On some results* 2006, *On some results* 2007, *Bounds in spaces* 2008, *Bounds in space* 2009.

In Section 2 and Section 3 we introduce the spaces $\mathbf{M}^p$ and some suitable subspaces analyzing their inclusion properties and the relations between such spaces and the classical Morrey spaces $L^{p,n}$. In Section 4 we state the embedding result and L^p estimates when the functions g belong to the subspaces $\overline{\mathbf{M}}^p$ and $\mathbf{M}^p_0$ defined in Section 2. Moreover we deduce from embedding result a Fefferman type result.

The compactness result for the multiplication operator is stated in Section 5.

2. *Notations and Morrey type spaces*

Let R^n be the n -dimensional real euclidean space, and $B_r(x)$ the set

$$B_r(x) = \{y \in R^n : |y - x| < r\}, \quad B_r = B_r(0) \quad \forall x \in R^n, \ \forall r \in R_+,$$

For any $x \in R^n$, we call *open infinite cone* having vertex at x every set of the type

$$\{x + \lambda(y - x) : \ \lambda \in R_+, \ |y - z| < r\},$$

where $r \in R_+$ and $z \in R^n$ are such that $|z - x| > r$.

For all $\theta \in]0, \pi/2[$ and for all $x \in R^n$ we denote by $C_\theta(x)$ an open infinite cone having vertex at x and opening θ .
For a fixed $C_\theta(x)$, we set

$$C_\theta(x,h) = C_\theta(x) \cap B_h(x), \quad \forall h \in R_+.$$

Let Ω be an open set in R^n . We denote by $\Gamma(\Omega, \theta, h)$ the family of open cones $C \subset\subset \Omega$ of opening θ and height h .
We assume that the following hypothesis is satisfied:

$h_1)$ There exists $\theta \in]0, \pi/2[$ such that

$$\forall x \in \Omega \quad \exists C_\theta(x) \text{ such that } \overline{C_\theta(x,\rho)} \subset \Omega.$$

Let $(\Omega_\rho(x))_{x\in\Omega}$ be the family of open sets in R^n defined as

$$\Omega_\rho(x) = B_\rho(x) \cap \Omega, \quad x \in \Omega, \quad \rho > 0.$$

If $1 \le p < +\infty$ we define by $\mathtt{M}^p(\Omega)$ as the space of functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$\| g \|_{\mathtt{M}^p(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \left(\rho^{-n/p} \| g \|_{L^p(\Omega_\rho(x))} \right) < +\infty, \quad d > 0, \tag{2.1}$$

equipped with the norm defined by (2.1). Furthermore let $\overline{\mathtt{M}}^p(\Omega)$ be the closure of $L^\infty(\Omega)$ in $\mathtt{M}^p(\Omega)$ and $\mathtt{M}^p_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $\mathtt{M}^p(\Omega)$.

Remark 2.1 *We observe that:*

1. *If Ω is a bounded open set and $d = \operatorname{diam}\Omega$, then $\mathtt{M}^p(\Omega)$ coincides with the space $L^{p,n}(\Omega)$, the classical Morrey space.*
2. *If $\Omega = R^n$ then $L^{p,n}(\Omega) \subset \mathtt{M}^p(\Omega)$.*
3. *The norm (2.1) is a sort of average integral on $B_\rho(x)$. To this end we note that if the family $\{\Omega_\rho(x)\}$ shrinks nicely to x , that is if $\Omega_\rho(x) \subset B_\rho(x)$ for any $\rho > 0$ and there is a constant $\alpha > 0$, independent of ρ , such that $|\Omega_\rho(x)| > \alpha |B_\rho(x)|$, the measure of $\Omega_\rho(x)$ is equivalent to the measure of $B_\rho(x)$.*

For functions that belong to the spaces $\overline{\mathsf{M}}^{p}(\Omega)$ we can state the following alternative definition. Let $\Sigma(\Omega)$ the σ -algebra of Lebesgue measurable subsets of Ω .

Lemma 2.2 *A function $g \in \overline{\mathsf{M}}^{p}(\Omega)$ if and only if $g \in \mathsf{M}^{p}(\Omega)$ and the function*

$$\sigma_g^p : \ t \in [0,1] \to \sup_{\substack{E \in \Sigma(\Omega) \\ \sup\limits_{\substack{x \\ \rho \in]0,d]}} \rho^{-n}|E \cap B_\rho(x)| \le t}} \left(\| g\chi_E \|_{M^p(\Omega)} \right) \tag{2.2}$$

is continuous in zero, where χ_E denotes the characteristic function of E .

Proof. Let g be a function belonging to the space $\mathsf{M}^{p}(\Omega)$ and let σ_g^p be continuous in zero. We denote by δ_ε a positive number such that

$$E \in \Sigma(\Omega), \qquad \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \left(\rho^{-n} \mid E \cap B_\rho(x) \mid \right) \le \delta_\varepsilon \ \Rightarrow \| g\chi_E \|_{M^p(\Omega)} \le \varepsilon$$

and by Ω_r the set

$$\Omega_r(g) = \{x \in \Omega \ : \ | g(x) | \ge r\} \quad \forall r \in R_+. \tag{2.3}$$

Then there exists a positive constant $c \in R_+$, independent of r and g , such that

$$\sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \left(\rho^{-n} \mid \Omega_r \ \cap B_\rho(x) \mid \right) \le c \frac{\| g \|^p_{\mathsf{M}^p} \ (\Omega)}{r^p}.$$

If we set

$$r_\varepsilon = c \left(\frac{\| g \|^p_{\mathsf{M}^p(\Omega)}}{\delta_\varepsilon} \right)^{1/p},$$

we get

$$\sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \left(\rho^{-n} \mid \Omega_{r_\varepsilon} \cap B_\rho(x) \mid \right) \le \delta_\varepsilon$$

and, as a consequence,

$$\| g\chi_{\Omega_{k_\varepsilon}} \|_{\mathbf{M}^p(\Omega)} \leq \varepsilon.$$

Let us introduce now the function

$$g_\varepsilon = g - g\chi_{\Omega_{k_\varepsilon}}.$$

Evidently $g_\varepsilon \in L^\infty(\Omega)$ and

$$\| g - g_\varepsilon \|_{\mathbf{M}^p(\Omega)} = \| g\chi_{\Omega_{k_\varepsilon}} \|_{M^p(\Omega)} \leq \varepsilon.$$

Conversely, let us assume $g \in \overline{\mathbf{M}}^p(\Omega)$. Then there exists $g_\varepsilon \in L^\infty(\Omega)$ such that

$$\| g - g_\varepsilon \|_{\mathbf{M}^p(\Omega)} \leq \varepsilon / 2.$$

So we get

$$\| g\chi_E \|_{\mathbf{M}^p(\Omega)} \leq \| (g - g_\varepsilon)\chi_E \|_{M^p(\Omega)} + \| g_\varepsilon \chi_E \|_{\mathbf{M}^p(\Omega)} \leq$$

$$\leq \varepsilon / 2 + \| g_\varepsilon \|_{L^\infty(\Omega)} \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \left(\rho^{-n} \,| E \cap B_\rho(x) |\, \right)^{1/p} \qquad (2.4)$$

from which we deduce that

$$\| g\chi_E \|_{\mathbf{M}^p\Omega)} \leq \varepsilon.$$

for any $E \in \Sigma(\Omega)$ such that

$$\sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \rho^{-n} \,| \Omega_\rho(x) \cap E | \leq \left(\frac{\varepsilon}{2 \| g_\varepsilon \|_{L^\infty(\Omega)}} \right)^p$$

and the lemma is proved.

3. *Inclusion properties*

The following Theorem states inclusion properties of the spaces $\mathbf{M}^p(\Omega)$.

Theorem 3.1 *The following inclusions hold:*

a) $L^\infty(\Omega) \to \mathbf{M}^p(\Omega) \qquad \forall p \in [1,+\infty[$.
b) $\mathbf{M}^q(\Omega) \to \mathbf{M}^p(\Omega), \quad 1 \le p \le q < +\infty$.
c) $\mathbf{M}^q(\Omega) \subset \widetilde{\mathbf{M}}^p(\Omega), \qquad 1 \le p < q < +\infty$.

Proof.

a) If $g \in L^\infty(\Omega)$ we get

$$\begin{aligned} \| g \|_{\mathbf{M}^p(\Omega)} &= \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \left(\rho^{-n/p} \| g \|_{L^p(\Omega_\rho(x))}\right) \le \\ &\le \| g \|_{L^\infty(\Omega)} \sup_{\substack{x\in\Omega \\ \rho+\in]0,d]}} \left(\rho^{-n/p} | \Omega_\rho(x) |^{1/p}\right) \le c \| g \|_{L^\infty(\Omega)} \end{aligned} \tag{3.1}$$

where, if ω_n denotes the volume of the unit ball $B_1(0)$, we get $c = \omega_n^{\frac{1}{p}}$.

b) By Hölder inequality it follows

$$\begin{aligned} \| g \|_{\mathbf{M}^p(\Omega)} &\le \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} (\rho^{-\frac{n}{p}} \| g \|_{L^q(\Omega_\rho(x))} | \Omega_\rho(x) |^{\frac{1}{p}-\frac{1}{q}}) \le \\ &\le c_1 \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \rho^{-\frac{n}{q}} \| g \|_{L^q(\Omega_\rho(x))}, \end{aligned} \tag{3.2}$$

with $c_1 = \omega_n^{\frac{1}{p}-\frac{1}{q}}$.

c) We observe that, if $g \in \mathbf{M}^q(\Omega)$, from $b)$ we have $g \in \mathbf{M}^p(\Omega)$. Furthermore

$$\| g\chi_E \|_{M^p(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \rho^{-n/p} \| g \|_{L^p(\Omega_\rho(x)\cap E)} \leq$$

$$\leq \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \rho^{-n/p} | \Omega_p(x) \cap E |^{\frac{1}{p}-\frac{1}{q}} \| g \|_{L^q(\Omega_\rho(x)\cap E)} \leq \tag{3.3}$$

$$\leq \| g \|_{M^q(\Omega)} \sup_{\substack{x\in\Omega \\ \rho\in]0,d]}} \left(\rho^{-n} | \Omega_\rho(x) \cap E | \right)^{\frac{1}{p}-\frac{1}{q}},$$

and we deduce that the function σ_g^p , defined by (sigma), is continuous in zero.
From Lemma 2.2 it follows that $g \in \overline{M}^p(\Omega)$.

4. *Embedding result*

Let us consider the function

$$\phi \ : \ (x,y) \in \Omega\times\Omega \to \begin{cases} 1 & \text{if } \ y \in \Omega_\rho(x) \\ 0 & \text{if } \ y \notin \Omega_\rho(x). \end{cases} \tag{4.1}$$

and, for any $x \in \Omega$, we set

$$E(x) = \{ y \in \Omega \ : \ x \in \Omega_\rho(y) \}. \tag{4.2}$$

Lemma 4.1 *For any* $x \in \Omega$, $E(x)$ *is a measurable set and there exist* $c_1, c_2 \in R_+$ *such that*

$$c_1\rho^n \leq | E(x) | \leq c_2\rho^n \quad \forall x \in \Omega. \tag{4.3}$$

Proof. Clearly the function ϕ is a measurable function. Then, for any fixed $y \in \Omega$, the function $\phi^y \ : \ x \in \Omega \to \phi(x,y)$ is measurable. Since ϕ^y is the characteristic function of $E(y)$, we have that $E(y)$ is measurable.
Now we prove that (4.3) holds. The inequality on the right is easily proved. We will prove the inequality on the left.

Let us consider $C_\theta(x,\rho)$ such that $\overline{C_\theta(x,\rho)} \subset \Omega$. We get

$$C_\theta(x,\rho) \subset E(x).$$

In fact let $y \in C_\theta(x,\rho)$. Then there exists a cone $C \in \Gamma(\Omega,\theta,h)$ such that $x, y \in C$. So

$$x \in B(y,\rho) \cap \Omega \Rightarrow y \in E(x).$$

Thus the inequality (4.3) is stated.

Now we state a Lemma which we will use in the proof of the embedding result.

Lemma 4.2 *If* $h_1)$ *holds, then we have* $v \in L^1(\Omega)$ *if and only if the map* $x \in \Omega \to \rho^{-n} \| v \|_{L^1(\Omega_\rho(x))}$ *belongs to* $L^1(\Omega)$. *Therefore there exist* $c_1, c_2 \in R_+$ *such that*

$$c_1 \| v \|_{L^1(\Omega)} \leq \int_\Omega \rho^{-n} \| v \|_{L^1(\Omega_\rho(x))} \, dx \leq c_2 \| v \|_{L^1(\Omega)} \quad \forall v \in L^1(\Omega). \tag{4.4}$$

Proof. The result is a consequence of the relation

$$\begin{aligned} \int_\Omega \rho^{-n} \| v \|_{L^1(\Omega_\rho(x))} \, dx &= \int_\Omega \rho^{-n} \int_\Omega | v(y) | \phi(x,y) dx dy = \\ &= \int_\Omega | v(y) | dy \int_{E(y)} \rho^{-n} dx \end{aligned} \tag{4.5}$$

and of the Lemma 4.1.

Let r, p, q be real number with the condition

$$h_2) \;\; r \in N, \quad 1 \leq p \leq q < +\infty, \quad q \geq \frac{n}{r}, \quad q > \frac{n}{r} \text{ if } \frac{n}{r} = p > 1.$$

Let $u \in W^{r,p}(\Omega)$. For any $x \in \Omega$ we set

$$\Psi^x \ : \ y \in \Omega \to x + \frac{y-x}{\rho} \quad , \qquad \Omega^*_\rho(x) = \Psi^x(\Omega_p(x)).$$

$$u^* = (u^x)^* \ : \ z \in \Omega^*_\rho(x) \to u(x + \rho(z-x)).$$

We note that

$$u^* \in W^{r,p}(\Omega^*_\rho(x)).$$

We also note that, in consequence of h_1) $\Omega^*_\rho(x)$ has the cone property, with the characteristic cone having height and opening independent of x .
On the other hand, if $\tau = q/p$, from h_2) we get

$$\tau \geq 1, \quad \tau > 1 \text{ if } \frac{n}{r} = p > 1, \quad \frac{\tau - 1}{p\tau} \geq \frac{1}{p} - \frac{r}{n}.$$

From well-known embedding theorems of Sobolev spaces (see, e.g., R.A. Adams, *Sobolev ……*), we deduce that

$$u^* \in L^{\frac{p\tau}{\tau-1}}(\Omega^*_\rho(x))$$

and the following bound holds

$$\| u^* \|_{L^{\frac{p\tau}{\tau-1}}(\Omega^*_\rho(x))} \leq c_0 \| u^* \|_{W^{r,p}(\Omega^*_\rho(x))}, \tag{4.6}$$

where $c_0 = c_0(p,q,r,n)$ is a constant independent of x and u^* .
From (4.6) easily it follows that

$$\rho^{-n\frac{(\tau-1)}{p\tau}} \| u \|_{L^{\frac{p\tau}{\tau-1}}(\Omega_\rho(x))} \leq c_0 \sum_{|\alpha| \leq r} \rho^{|\alpha| - \frac{n}{p}} \| \partial^\alpha u \|_{L^p(\Omega_\rho(x))} . \tag{4.7}$$

Theorem 4.3 *If* h_1) *and* h_2) *hold, then for any* $g \in \mathsf{M}^q(\Omega)$ *and for any* $u \in W^{r,p}(\Omega)$ *we get* $gu \in L^p(\Omega)$ *and*

$$\| gu \|_{L^p(\Omega)} \leq c \| g \|_{\mathsf{M}^q(\Omega)} \| u \|_{W^{r,p}(\Omega)}, \tag{4.8}$$

where the constant $c = c(p,q,r,n)$ *is independent of* g *and* u .

Proof. Let $u \in W^{r,p}(\Omega)$ and $g \in \mathbf{M}^q(\Omega)$. By (4.4) and by Hölder inequality it follows that

$$\int_\Omega |gu|^p \, dx \le c_1 \int_\Omega \rho^{-n} \int_{\Omega_\rho(x)} |gu|^p \, dy\, dx \le c_1 \int_\Omega \rho^{-n} \|g\|^p_{L^{p\tau}(\Omega_\rho(x))} \|u\|^p_{L^{\frac{p\tau}{\tau-1}}(\Omega_\rho(x))} dx \le$$
$$\le c_1 \|g\|^p_{\mathbf{M}^q(\Omega)} \rho^{-\left(n\frac{\tau-1}{\tau}\right)} \int_\Omega \|u\|^p_{L^{\frac{p\tau}{\tau-1}}(\Omega(x))} dx. \tag{4.9}$$

On the other hand from (4.7) and Lemma (4.2) we obtain

$$\rho^{-n\frac{(\tau-1)}{\tau}} \int_\Omega \|u\|^p_{L^{\frac{p\tau}{\tau-1}}(\Omega_\rho(x))} dx \le c_0 \sum_{|\alpha|\le r} \rho^{|\alpha|p-n} \int_\Omega \|\partial^\alpha u\|^p_{L^p(\Omega_\rho(x))} dx \le$$
$$\le c_1 \sum_{|\alpha|\le r} \rho^{|\alpha|p} \|\partial^\alpha u\|^p_{L^p(\Omega)} . \tag{4.10}$$

From (4.9) and (4.10) the inequality (4.8) follows.

The following theorem is a consequence of the embedding result stated in the Theorem 4.3 (see result of C. Fefferman, *The uncertainty,* and also F. Chiarenza - M. Frasca, *A remark on*, for a simplified proof).

Theorem 4.4 *If* $h_1)$ *and* $h_2)$ *hold then for any* $g \in \mathbf{M}^q(\Omega)$ *, we get*

$$\|gu\|_{L^p(\Omega)} \le c \|g\|_{\mathbf{M}^q(\Omega)} \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

where the constant $c = c(p,q,n)$ *is independent of* g *and* u .

Proof. Taking in mind the Poincaré inequality, the proof is a direct consequence of the Theorem 4.3 when $r = 1$.

The L^p estimates below are obtained when the function g belongs to the closure of $L^\infty(\Omega)$ in $\mathbf{M}^q(\Omega)$ or to the closure of $C_0^\infty(\Omega)$ in $\mathbf{M}^q(\Omega)$. In the applications to the elliptic differential equations we deal with multiplication operators belonging to these spaces.

Lemma 4.5 *If* $h_1)$, $h_2)$ *hold and* $g \in \overline{\mathtt{M}}^q(\Omega)$ *, then for any* $\varepsilon \in R_+$ *there exists* $c(\varepsilon) \in R_+$ *such that*

$$\| gu \|_{L^p(\Omega)} \leq \varepsilon \| u \|_{W^{r,p}(\Omega)} + c(\varepsilon) \| u \|_{L^p(\Omega)} \quad \forall u \in W^{r,p}(\Omega).$$

Proof. Let $\phi_\varepsilon \in L^\infty(\Omega)$ such that

$$\| g - \phi_\varepsilon \|_{\mathtt{M}^q(\Omega)} \leq \varepsilon / c, \tag{4.11}$$

where the constant c is the constant in the bound (4.8). Then by Theorem 4.3

$$\begin{aligned} \| gu \|_{L^p(\Omega)} &\leq \| (g - \phi_\varepsilon) u \|_{L^p(\Omega)} + \| \phi_\varepsilon u \|_{L^p(\Omega)} \leq \\ &\leq c \| (g - \phi_\varepsilon) u \|_{\mathtt{M}^q(\Omega)} \| u \|_{W^{r,p}(\Omega)} + \| \phi_\varepsilon \|_{L^\infty(\Omega)} \| u \|_{L^p(\Omega)} \leq \\ &\leq \varepsilon \| u \|_{W^{r,p}(\Omega)} + c(\varepsilon) \| u \|_{L^p(\Omega)} \end{aligned} \tag{4.12}$$

with $c(\varepsilon) = \| \phi_\varepsilon \|_{L^\infty(\Omega)}$.

Lemma 4.6 *If* $h_1)$, $h_2)$ *hold and* $g \in \mathtt{M}_0^q(\Omega)$ *, then for any* $\varepsilon \in R_+$ *there exist* $c(\varepsilon) \in R_+$ *and an open set* $\Omega_\varepsilon \subset\subset \Omega$ *with cone property such that*

$$\| gu \|_{L^p(\Omega)} \leq \varepsilon \| u \|_{W^{r,p}(\Omega)} + c(\varepsilon) \| u \|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{r,p}(\Omega). \tag{4.13}$$

Proof. Let $\phi_\varepsilon \in C_0^\infty(\Omega)$ be such that (4.11) holds.

Reasoning as in the proof of the Lemma 4.5 we get

$$\| gu \|_{L^p(\Omega)} \leq \varepsilon \| u \|_{W^{r,p}(\Omega)} + c(\varepsilon) \| u \|_{p, \sup p\phi_\varepsilon}, \tag{4.14}$$

with $c(\varepsilon)$ as in the Lemma 4.5.

Then let us fix $\theta \in]0, \pi/2[$ and $h_\varepsilon \in]0, \mathrm{dist}(\partial\Omega, \mathrm{supp}\,\phi_\varepsilon)/2[$. If we denote by Ω_ε the open set of R^n union of the cones $C \in \Gamma(\Omega, \theta, h_\varepsilon)$ such that $C \cap$ supp g_ε is not empty, then (4.13) follows from (4.14).

5. *Compactness result*

Now we state the compactness result.

Theorem 5.1 *If* $h_1)$, $h_2)$ *hold and* $g \in \mathbf{M}_0^q(\Omega)$ *, then the operator*

$$u \in W^{r,p}(\Omega) \to gu \in L^p(\Omega)$$

is compact.

Proof. We remark that for any open set $\Omega' \subset\subset \Omega$ the operator

$$u \in W^{r,p}(\Omega) \to u|_{\Omega'} \in W^{r,p}(\Omega')$$

is linear and bounded.
On the other hand, if Ω' verifies the cone property too, by Rellich-Kondrachov's theorem the operator

$$u \in W^{r,p}(\Omega') \to u \in L^p(\Omega')$$

is compact. Then also the operator

$$u \in W^{r,p}(\Omega) \to u|_{\Omega'} \in L^p(\Omega')$$

is compact. Therefore if $\{u_n\} \in W^{r,p}(\Omega')$ is a bounded sequence there exists a subsequence $\{u_{n_k}\}$ converging to u in $L^p(\Omega')$. By Lemma 4.6 we get

$$\| g(u_{n_k} - u) \|_{L^p(\Omega)} \leq \varepsilon \| u_{n_k} - u \|_{W^{r,p}(\Omega)} + c(\varepsilon) \| u_{n_k} - u \|_{L^p(\Omega')} \quad \forall u \in W^{r,p}(\Omega)$$

from which we can deduce the result.

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Spaces of infinite dimension have played an increasing role in the last four decades concerning their applications in models both for economics and for finance. The intertemporal allocation of resources, commodity differentiation, uncertainty, dynamics of the fundamental variables of financial markets, are some of the issues that can be properly captured by means of the mathematical techniques that are typical in such spaces.
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