SOME TOPICS ON PERMUTABLE SUBGROUPS IN INFINITE GROUPS

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INTRODUCTION

A subgroup H of a group G is said to be *permutable* (or *quasinormal*) if $HK = KH$ for every subgroup K of G. This concept has been introduced by Ore [54] and the condition $HK = KH$ is equivalent to the requirement that the set HK is a subgroup. As a consequence, if H is a permutable subgroup of a group G, then for any subgroup K, $\langle H, K\rangle$ is just the set of all elements hk, where h is in H and k is in K. With an easy argument based on set equality, it can be proved that if H is permutable, then $HK \cap L = (H \cap L)K$ for any subgroups K and L such that $K \leq L$. This property is known as *Dedekind identity* (or *modular law*) and note that, since $K \cap L = K$, the equality $HK \cap L = (H \cap L)(K \cap L)$ is a form of associative law.

The most important examples of permutable subgroups are the normal subgroups, but not every permutable subgroup is normal. In [54] Ore proved that a permutable subgroup of a finite group is always subnormal and, many years later, Stonehewer [61] generalized Ore's result to infinite groups, showing that a permutable subgroup of an arbitrary group is ascendant.

The aim of this thesis is to study permutability in different aspects of the theory of infinite groups. In particular, it will be studied the structure of groups in which all the members of a relevant system of subgroups satisfy a suitable generalized condition of permutability.

Chapter 1 is an introduction to the theory of permutable subgroups and it contains some reminds that are needed in the following chapters. In particular, a proof of the quoted results of Ore and Stonehewer is exhibited and it will be shown that there exists a bound on the ascending length of a permutable subgroup in an arbitrary group. Moreover, several known generalizations of the concept of permutability are presented, where the most natural way, of generalizing it, is to require that, for a fixed subgroup H of a group G, the condition $HK = KH$ is satisfied for subgroups K belonging to a certain system of subgroups of G.

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements and r is the least positive integer with such property; is such an r does not exist, G is said to have *infinite rank*. A recent research topic in group theory is the study of the effect that the imposition of a certain property, to the subgroups of infinite rank of a group, has on the structure of the whole group. Many authors have proved that, in many cases, the behaviour of the subgroups of finite rank can be neglected, at least in a suitable universe of (generalized) soluble groups. For instance, Dixon and Karatas [31] proved that if every subgroup of infinite rank of a locally soluble group G of infinite rank is permutable, then the same property holds also for the subgroups of finite rank. In Chapter 2, some new contributions to this topic are presented, with the investigation of generalized radical groups of infinite rank whose subgroups of infinite rank satisfy a generalized permutability condition (recall that a group G is called *generalized radical* if it has an ascending series of normal subgroups such that every factor is either locally nilpotent or locally finite).

Chapter 3 is devoted to some other aspects of the theory of permutable subgroups in infinite groups. In particular, some results proved for finite groups have been extended to infinite groups. In the first section, polycyclic groups are considered. It is known that the behaviour of the finite homomorphic images of a polycyclic group has an influence on the structure of the group itself. In this context, the first result, and maybe the most important, is by Hirsch [43] who proved that a polycyclic group G is nilpotent whenever its finite homomorphic images are nilpotent. Starting from a structure theorem obtained by Robinson [59], we describe the structure of a polycyclic group such that the subgroups of its finite quotients satisfy different conditions of generalized permutability.

The last section of Chapter 3 deals with groups which coincide with the product HK, for some subgroups H and K, such that H permutes with every subgroup of K and K permutes with every subgroup of H. In this situation, H and K are said to be mutually permutable and clearly any pair of permutable subgroups is mutually permutable. In [14], Beidleman and Heineken proved that if H and K are mutually permutable subgroups of the finite group $G = HK$, then the commutator subgroups H' and K' are subnormal in G and, in the last section, we prove a correspondent result for Černikov groups: it is shown that if H and K are finite-by-abelian mutually permutable subgroups of a Černikov group $G = HK$, then H' and K' are subnormal. As a corollary of the quoted Beidleman and Heineken's theorem, one can observe that under the same hypothesis, if H' and K' are nilpotent, then they are both contained in the Fitting subgroup of G. A generalization of this result to soluble-by-finite group of finite rank is obtained in the conclusion of this work.

Most of our notation is standard and can be found in [56].

CHAPTER 1

PRELIMINARIES

1.1 Permutable subgroups

A subgroup H of a group G is called *permutable* (or *quasinormal*) if $HK = KH$ for every subgroup K of G. This concept has been introduced by Ore [54]. Of course every normal subgroup is permutable, but the converse is not true. However, Ore proved that in a finite group G a permutable subgroup is always subnormal and this is an easy consequence of the following fact.

Lemma 1.1.1 (Ore [54])**.** *If H is a maximal permutable subgroup of a group G, then H is normal in G.*

PROOF. By a contradiction, suppose that H is not normal in G, so that there exists an element x of G such that $H \neq H^x$. Put $K = H^x$, then HK is a permutable subgroup of G containing properly H and, by the maximality of H, we have that $G = HK$. Therefore $x = hk$, for some $h \in H$ and $k \in K$, so $K = H^x = H^k$ and this implies that $H = K$, a contradiction. ■

Corollary 1.1.1 (Ore [54])**.** *If H is a permutable subgroup of a finite group G, then H is subnormal in G.*

In contrast to what happens for finite groups, if G is an infinite group, a permutable subgroup need not to be subnormal. For instance, consider the group $G = \langle x \rangle A$, where A is a group of type p^∞, $p > 2$, and $a^x = a^{1+p}$, for every $a \in A$. By Theorem 2.4.11 of [60], every subgroup of G is permutable in G. Since the automorphism induced by x in the subgroup of order p^n of T has order p^{n-1}, we have $\langle x \rangle_G = \{1\}$; moreover, if H is a normal subgroup of G containing $\langle x \rangle$, then $G' = T$ is contained in H and hence $\langle x \rangle^G = G$. It follows that $\langle x \rangle$ is not subnormal in G.

However, it has been proved by Stonehewer [61] that a permutable subgroup of an arbitrary group G is ascendant. Recall that a subgroup H of a group of G is called *ascendant (of length β)* if there exists a set $\{H_\alpha \mid \alpha < \beta\}$ of subgroups of G, indexed by ordinals less than an ordinal β such that $H_{\alpha_1} \leq H_{\alpha_2}$ if $\alpha_1 \leq \alpha_2$, $H_0 = H$ and $H_\beta = G$, $H_\alpha \triangleleft H_{\alpha+1}$ and $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \beta$.

For the proof of Stonehewer's theorem, the following lemma is needed, which shows that a permutable subgroup is always normalized by an infinite cyclic subgroup disjoint from it.

Lemma 1.1.2 (Stonehewer [61]). *Let H be a permutable subgroup of a group G. If x is an element of infinite order of G such that $H \cap \langle x \rangle = \{1\}$, then $H \triangleleft H\langle x \rangle$.*

Let G be a group and let H be a subgroup of G. The *series of normal closures* of H in G is the sequence of subgroups $(H^{G,n})_{n\in\mathbb{N}}$, defined inductively by the rules $H^{G,0} = G$ and $H^{G,n+1} = H^{H^{G,n}}$, for each non-negative integer n. Note that $H^{G,1} = H^G$ is just the normal closure of H in G and H is subnormal in G of defect d if and only if $H^{G,d} = H$.

Theorem 1.1.1 (Stonehewer [61]). *If H is a permutable subgroup of a group G, then H is ascendant in G.*

PROOF. By Zorn's Lemma, there exists an ascending chain of permutable subgroups of G

$$H = H_0 \triangleleft H_1 \triangleleft \ldots H_\alpha \triangleleft H_{\alpha+1} \triangleleft \cdots \triangleleft H_\rho \leq H^G$$

such that $H_{\alpha+1}/H_\alpha$ is cyclic and finite, for any ordinal $\alpha < \rho$ and there is no permutable subgroup of G $H_{\rho+1}$ in H^G, containing properly H_ρ, such that $H_\rho \triangleleft H_{\rho+1}$ and $H_{\rho+1}/H_\rho$ is cyclic and finite. By contradiction, suppose that $K = H_\rho$ is a proper subgroup of H^G, so that there exists an element x of G such that K is properly contained in KK^x. For some positive integer n, we have that $KK^x = K\langle x^n \rangle$ and, by Lemma 1.1.2, K has finite index in KK^x. Then, it follows by corollary 1.1.1 that K is subnormal in $K\langle x^n \rangle$ and denote with d its defect of subnormality. Let $M = K^{G,d-1}$ be the penultimate term of the normal closure series of K in $K\langle x^n \rangle$. Then M, being generated by conjugates of K, is a permutable subgroup of G contained in H^G and $M = K(K \cap \langle x^n \rangle)$, so M/K is cyclic and finite, a contradiction, since K is properly contained in M. Thus, $H_\rho = H^G$ and H is an ascendant subgroup of G. ■

In all known examples, a permutable subgroup is ascendant of length ω and there is a conjecture that the minimal length of an ascending series of a permutable subgroup is always ω. Some years ago, Napolitani proved that the minimal length of an ascending series of a permutable subgroup is $\omega + 1$ but he did not publish his result. We propose here our version of the proof of this statement.

Proposition 1.1.1. *If H is a permutable subgroup of a group G, then there exists an ascending series of H in G of length $\omega + 1$.*

PROOF. We define inductively an ascending series of H in G by choosing $H_1 = H$ and, for any positive integer n, H_{n+1} is the subgroup generated by all the conjugates H_n^x of H_n in G such that H_n^x normalizes H_n. Clearly, H_n is a permutable subgroup of G contained in H^G, for any $n \geq 1$, so that the subgroup $K = \bigcup_n H_n$ is still a permutable subgroup of G contained in H^G. By a contradiction, suppose that K is properly contained in H^G and let g be an element of G such that K is not normal in $K\langle g \rangle$. By Lemma 1.1.2, K has finite index in $K\langle g \rangle$, so that K is subnormal in $K\langle g \rangle$. Let t be a positive integer and let $L = KK^{x_1} \dots K^{x_t}$ be a subgroup of $K\langle g \rangle$ such that $K \triangleleft L$. For any $i \leq t$, $K^{\langle x_i \rangle} = KK^{x_i}$ ([60], Lemma 6.3.4), so that $K \triangleleft K^{\langle x_i \rangle} \triangleleft K\langle x_i \rangle$. Let n be a positive integer such that $K \cap \langle x_i \rangle = H_j \cap \langle x_i \rangle$ for any $j \geq n$. Then

$$K \cap H_j\langle x_i \rangle \triangleleft K^{\langle x_i \rangle} \cap H_j\langle x_i \rangle \triangleleft K\langle x_i \rangle \cap H_j\langle x_i \rangle$$

and, since $K \cap H_j\langle x_i \rangle = H_j(K \cap \langle x_i \rangle) = H_j$, it follows that

$$H_j \triangleleft K^{\langle x_i \rangle} \cap H_j\langle x_i \rangle \triangleleft H_j\langle x_i \rangle.$$

In particular, $H_j \triangleleft H_j^{\langle x_i \rangle} = H_j H_j^{x_i}$ and $H_j^{x_i} \leq H_{j+1}$. Thus, $H_j^{x_i} \leq K$ for any $j \geq n$. As a consequence, K^{x_i} is contained in K, for any $i \leq t$, and $K = L$.
Since K is a subnormal subgroup of finite index of $K\langle g \rangle$, every term of its normal closure series in $K\langle g \rangle$ is the product of finitely many conjugates of

K in $K\langle g\rangle$ and, by the previous argument, K is normal in $K\langle g\rangle$. This contradiction proves that $K = H^G$ and H is ascendant in G of length $\omega + 1$. ■

A group G is called a *Dedekind group* if every subgroup of G is normal. Of course, any abelian group is a Dedekind group and a non-abelian Dedekind group is called *Hamiltonian*. The dihedral group of order 8 is the least example of a Hamiltonian group. The structure of an arbitrary Dedekind group is well-known. The following theorem describes it and a proof of it can be found in [58].

Theorem 1.1.2 (Dedekind, Baer). *All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.*

A group G is called *quasihamiltonian* if every subgroup of G is permutable. The structure of a quasihamiltonian group has been completely described by Iwasawa [44]. It follows by Stonehewer's theorem that a quasihamiltonian group has all its subgroups ascendant and so it is locally nilpotent. As a consequence, a periodic quasihamiltonian group G is the direct product of its primary components and it is locally finite. Thus, the description of its structure reduces to the study of the structure of a quasihamiltonian p-group, for some prime p.

Theorem 1.1.3 (Iwasawa [44]). *Let p be a prime. The group G is a quasihamiltonian p-group if and only if G is one of the following types:*

(a) *G is abelian,*

(b) *G is the direct product of a quaternion group of order 8 and an elementary abelian 2-group,*

(c) *G contains an abelian normal subgroup A of exponent p^k with cyclic factor group G/A of order p^m and there exists an element $b \in G$ with $G = A\langle b\rangle$ and an integer s, which is at least 2 in case $p = 2$, such that $s < k \leq s + m$ and $b^{-1}ab = a^{1+p^s}$ for all $a \in A$.*

In particular, it turns out that an arbitrary quasihamiltonian p-group G is abelian-by-finite and, if G is not abelian, it has finite exponent.

Theorem 1.1.4 (Iwasawa [44]). *Let G be a non-periodic quasihamiltonian group. Then:*

(a) The set T of all elements of finite order of G is a characteristic subgroup and both T and G/T are abelian.

(b) Every subgroup of T is normal in G.

(c) Either G is abelian or G/T is locally cyclic.

It could be said more about the structure of non-periodic quasihamiltonian groups but it is way beyond the purpose of this thesis, for a detailed account on this and on other topics related to permutability we refer to [60].

1.2 Generalized permutable subgroups

The most natural way to generalize the concept of permutability is to require that a subgroup H of a group G permutes only with the members of a certain system of subgroups of G.
A subgroup H of a periodic group G is said to be *S-permutable* if $HP = PH$ for every Sylow p-subgroup P of G. The consideration of the dihedral group of order 8 shows that there exists S-permutable subgroups which are not permutable. In [45], Kegel proved that in a finite group G an S-permutable subgroup is subnormal, so, in particular, finite groups in which every subgroup is S-permutable are exactly the finite nilpotent groups.
In contrast to what happens for permutable subgroups of infinite groups, an S-permutable subgroup of an arbitrary group need not be ascendant. In fact, in a periodic locally nilpotent group every subgroup is trivially S-permutable and there are many examples of periodic locally nilpotent groups with non-subnormal (or even non-ascendant) subgroups (one can be found in [58], Example 18.2.2).

In general, normality is not a transitive relation in an arbitrary group, i.e. if H and K are subgroups of a group G such that H is normal in K and K is normal in G, then H need not to be normal in G. A group G in which normality is a transitive relation is called a *T-group*. Thus, T-groups are exactly

the groups in which every subnormal subgroup is normal. The structure of a finite soluble T-group has been determined by Gaschütz [37], who showed that if G is a finite soluble T-group and L is the last term of its lower central series, then L is abelian of odd order, G/L is a Dedekind group and the order of L is coprime to $|G : L|$. Later, infinite soluble T-groups have been studied by Robinson [55].
As normality, neither permutability nor S-permutability are transitive relations. A group G in which permutability is a transitive relation is called a *PT-group*, while if S-permutability is a transitive relation G is called a *PST-group*. Thus, an arbitrary group G is a PT-group (resp. PST-group) if and only if its subnormal subgroups are permutable (resp. S-permutable). Zacher and Agrawal described the structure of finite soluble PT-groups and PST-groups respectively.

Theorem 1.2.1 (Zacher [65]). *A finite group G is a soluble PT-group if and only if the last term of the lower central series L of G is an abelian subgroup, the order of L is odd and it is relatively prime to $|G : L|$, every subgroup of L is normal in G and G/L is a quasihamiltonian group.*

Theorem 1.2.2 (Agrawal [1]). *A finite group G is a soluble PST-group if and only if the last term of the lower central series L of G is an abelian subgroup, the order of L is odd and it is relatively prime to $|G : L|$ and every subgroup of L is normal in G.*

We refer to [3] for a detailed account on this topic in the universe of finite groups.

A subgroup H of a periodic group G is said to be *semipermutable* if $HK = KH$ for every subgroup K of G such that $\pi(H) \cap \pi(K) = \emptyset$ (recall that $\pi(G)$ denotes the set of all prime numbers p such that G contains an element of order p). In the symmetric group on 3 letters S_3 every subgroup of order 2 is semipermutable, but clearly none of them is permutable. Semipermutable subgroups behave very differently from permutable subgroups: for instance, in S_3 none of the subgroups of order 2 is subnormal.
In a similar way, we say that a subgroup H of a periodic group G is *S-semipermutable* if $HP = PH$ for every Sylow p-subgroup P of G such that

$p \notin \pi(H)$. A group G in which semipermutability is a transitive relation is called a *BT-group*. A structure theorem for finite soluble BT-groups has been obtained by Wang, Li and Wang.

Theorem 1.2.3 (Wang-Li-Wang [64])**.** *For a finite group G, the following sentences are equivalent:*

(a) G is a soluble BT-group.

(b) Every subgroup of G is semipermutable.

(c) Every subgroup of G is S-semipermutable.

(d) G is a soluble PST-group with nilpotent residual L and if p and q are distinct primes not dividing the order of L then $[P, Q] = \{1\}$*, for any Sylow p-subgroup P and any Sylow q-subgroup Q of G.*

It has been noticed above that in general semipermutable subgroups are not subnormal, so the class of BT-groups and the class of groups in which every subnormal subgroup is semipermutable are distinct. A group G is called an *SP-group* if its subnormal subgroups are semipermutable and G is called an *SPS-group* if its subnormal subgroups are S-semipermutable.

A subgroup H of a periodic group G is called *seminormal* if H is normalized by every subgroup K of G such that $\pi(H) \cap \pi(K) = \emptyset$. A group G is called an *SN-group* if every subnormal subgroup of G is seminormal.

Theorem 1.2.4 (Beidleman-Ragland [15])**.** *For a finite group G, the following sentences are equivalent:*

(a) G is a PST-group.

(b) G is an SP-group.

(c) G is an SPS-group.

(d) G is an SN-group.

A group G is called an *SNT-group* if seminormality is a transitive relation. Ballester-Bolinches et al. [8] proved that any finite SNT-group is a PST-group, but the converse does not hold in general and an example is exhibited.

A subgroup H of a group G is called *SS-permutable* if H has a supplement K in H such that $HP = PH$ for every Sylow p-subgroup of K. A group G in which SS-permutability is a transitive relation is called an *SST-group*. In [19] the authors proved that any finite SST-group is a BT-group, but the converse is not true in general. The following theorem gives a criterion for a BT-group to be an SST-group.

Theorem 1.2.5 (Chen-Guo [19])**.** *Let G be a finite soluble BT-group and let L be the last term of the lower central series of G. Then G is a BT-group if and only if for every p-subgroup P of G with $p \in \pi(G) \setminus \pi(L)$, G has a subgroup K_p such that PK_p is a Sylow p-subgroup of G and $[P, K_p] \leq O_p(G)$.*

A group G is called an *MS-group* if the maximal subgroups of the Sylow subgroups of G are S-semipermutable in G. Ballester-Bolinches et al. [7] studied the structure of finite MS-groups, showing also that the class of MS-groups and the class of BT-groups are not comparable.

Finally, the relation between all the classes of groups introduced in this section can be pictured by the following diagram, with the exception of MS-groups which are not comparable with all the other ones.

$$\text{SST} \Rightarrow \text{BT} \Rightarrow \text{SNT} \Rightarrow \text{SN} \Leftrightarrow \text{SP} \Leftrightarrow \text{SPS} \Leftrightarrow \text{PST}$$

CHAPTER 2

PERMUTABILITY CONDITIONS ON SUBGROUPS OF INFINITE RANK

A property θ pertaining to subgroups of a group G is called an *embedding property* if all the images under automorphisms of G of a θ-subgroup have still the property θ. Normality, subnormality and permutability are some examples of embedding properties. Let $\mathfrak{X}$ and $\mathfrak{U}$ be classes of groups (here, a non-empty collection $\mathfrak{X}$ of groups is called a *group class* if every group isomorphic to a group in $\mathfrak{X}$ belongs to $\mathfrak{X}$) and let θ be an embedding property, we say that $\mathfrak{X}$ controls θ in the universe $\mathfrak{U}$ if the following condition is satisfied: if G is any $\mathfrak{U}$-group, containing some $\mathfrak{X}$-subgroup, and all the $\mathfrak{X}$-subgroups of G satisfy θ, then every subgroup of G has the property θ.
It is immediate to see that, for instance, the class of cyclic groups and the class of finitely generated groups control both normality and permutability in any universe, while neither of them control subnormality. Groups in which every cyclic (resp. finitely generated) subgroup is subnormal are called *Baer groups* (resp. *Gruenberg groups*), and there exist Baer and Gruenberg groups with some non-subnormal subgroup ([56], part 2, Cap. 6).
The reason of this failure may be seen in the fact that cyclic groups and finitely generated groups groups are too small and in a group G the behaviour of its finitely generated subgroups may not have an influence on the structure of the whole group G. So, it is natural to consider classes $\mathfrak{X}$ of groups which are larger in some sense, in order to obtain some results about the control of $\mathfrak{X}$ of many properties pertaining the subgroups of a group.
The most natural class of large groups that could be considered is the class $\mathfrak{I}$ of infinite groups, but the locally dihedral 2-group is an example of a group in which normality is not controlled by the class $\mathfrak{I}$.
A group G is said to have *finite (Prüfer) rank r* if every finitely generated sub-

group of G can be generated by at most r elements and r is the least positive integer with such property; if such an r does not exist then we say that G has *infinite rank*. It is easy to see that any subgroup and any epimorphic image of a group of finite rank has finite rank and, moreover this class of groups is closed by extension. Clearly, every finite group has finite rank and its rank is bounded by its order, so the class of groups of infinite rank may be seen as a class of large groups.
In contrast to the class of infinite groups, in recent years many authors have obtained positive results in terms of properties θ controlled by the class of groups of infinite rank, for different choices of the property θ, at least in a suitable universe of (generalized) soluble groups. We refer to [22] for a detailed survey on this topic.

The existence of groups of infinite rank in which every proper subgroup has finite rank (see [30]) shows that the imposition of an embedding property to the subgroups of infinite rank of an arbitrary group G does not affect the structure of G. So, when dealing with the problem of the control, it is natural to restrict the attention to a suitable universe of groups $\mathfrak{U}$ such that every $\mathfrak{U}$-group of infinite rank contains some proper subgroup of infinite rank. Locally soluble are a good universe in this sense, as the following theorem shows.

Theorem (Dixon-Evans-Smith [30])**.** *Let G be a locally soluble group of infinite rank. Then G contains a proper subgroup of infinite rank.*

For locally nilpotent groups we can say more, in fact Mal'cev proved the following fact.

Theorem (Mal'cev [49])**.** *Let G be a locally nilpotent group of infinite rank. Then G contains an abelian subgroup of infinite rank.*

Later, Šunkov proved an analogous result for locally finite group.

Theorem (Šunkov [62])**.** *Let G be a locally finite group of infinite rank. Then G contains an abelian subgroup of infinite rank.*

Mal'cev's theorem has been generalized by Baer and Heineken [6] to radical groups of infinite rank and, eventually in [22] the authors proved the existence of abelian subgroups of infinite rank in generalized radical groups, where a group is said to be *generalized radical* if it contains an ascending normal series in which every factor is either locally finite or locally nilpotent. So, this last result can be seen as a generalization of both Baer and Heineken's theorem and Šunkov's theorem.

In the end, we list here some results which are related to the subject of this thesis.
The first one has been obtained by Evans and Kim and it relates to normality.

Theorem (Evans-Kim [34])**.** *Let G be a locally soluble group of infinite rank in which every subgroup of infinite rank is normal. Then G is a Dedekind group.*

Later, Dixon and Karatas studied groups in which every subgroup of infinite rank is permutable.

Theorem (Dixon-Karatas [31])**.** *Let G be a locally (soluble-by-finite) group of infinite rank in which every subgroup of infinite rank is permutable. Then G is quasihamiltonian.*

In particular the latter result has been the motivation to consider groups of infinite rank whose subgroups of infinite rank satisfy a certain generalized permutability condition.

2.1 S-permutable subgroups of infinite rank

Since in a finite group any S-permutable subgroup is subnormal, a locally finite group whose subgroups are S-permutable is locally nilpotent. In [11], the authors proved that a periodic hyper-(abelian or finite) group of infinite rank in which every subgroup of infinite rank is S-permutable is locally nilpotent.
The aim of this section is to prove a corresponding result for locally finite groups, obtaining in this way a generalization of the quoted result in [11].

Theorem 2.1.1 (Ballester-Bolinches, Camp-Mora, Dixon, Ialenti, Spagnuolo [10])**.** *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are S-permutable. Then G is locally nilpotent.*

It is easy to see that if H is an S-permutable subgroup of a group G and K is a subgroup of G containing H, then H is S-permutable in K. On the other hand, if N is a normal subgroup of G, then H may not be S-permutable in the factor group G/N. So, in general if N is a normal subgroup of infinite rank of a group G and every subgroup of infinite rank of G is S-permutable, we can not say that G/N has all its subgroups S-permutable.
The situation is different for countable groups, as the following lemma shows.

Lemma 2.1.1. *Let G be a countable locally finite group of infinite rank whose subgroups of infinite rank are S-permutable. If N is a normal subgroup of infinite rank of G, then the factor group G/N is locally nilpotent.*

PROOF. Let K/N be any subgroup of G/N. Then K has infinite rank and so it is S-permutable in G. Application of Corollary 2.3 of [12] yields that K/N is S-permutable in G/N. In particular, every subgroup of the locally finite group G/N is S-permutable and G/N is locally nilpotent. ∎

Proof of Theorem 2.1.1. By contradiction, assume that the theorem is false and let G be a counterexample. Let X be a finitely generated non-nilpotent subgroup of G and let A be any abelian subgroup of infinite rank of G. Then the socle S of A is the direct product of cyclic groups of prime power-order, so that S is countable. It follows that the countable subgroup of infinite rank $\langle X, S\rangle$ of G is still a counterexample and so, replacing G with $\langle X, S\rangle$, we may assume that G is countable. The result will be proved in a series of steps.

Step 1. If G has a p-subgroup X of infinite rank, then $X \cap P$ has infinite rank for every Sylow p-subgroup P of G. In particular, P has infinite rank.
Let P be a Sylow p-subgroup of G. By contradiction, suppose that $X \cap P$ has finite rank and let A be an abelian subgroup of infinite rank of X such that $A \cap (X \cap P) = A \cap P = \{1\}$. Let x be any non-trivial element of A, then A contains a direct product $B = B_1 \times B_2$ such that B_1 and B_2 have infinite rank and $B \cap \langle x\rangle = \{1\}$. Then $B_i\langle x\rangle P$ is a subgroup of G, for $i = 1, 2$, and, since $P \cap B\langle x\rangle = \{1\}$, we have that

$$\langle x\rangle P = (B_1\langle x\rangle \cap B_2\langle x\rangle)\, P = B_1\langle x\rangle P \cap B_2\langle x\rangle P$$

and $\langle x\rangle P$ is a p-subgroup of G. Thus, x belongs to P, a contradiction. Hence,

$X \cap P$ has infinite rank and, in particular, P has infinite rank.

Step 2. If G has a Sylow p-subgroup of infinite rank, then every element of order q of G belongs to $O_q(G)$, for any $q \neq p$.

Let P be a Sylow p-subgroup of infinite rank of G and let x be an element of order q of G, with $q \neq p$. By contradiction, suppose that there exists a Sylow q-subgroup Q of G such that $x \notin Q$. Put $\bar{P} = \bigcap_{i=1}^{q} P^{x^i}$. Clearly, $\bar{P}$ is an $\langle x \rangle$-invariant p-subgroup of G and, by Step 1, it has infinite rank, so it contains an abelian subgroup $B = B_1 \times B_2$, where B_1 and B_2 are both $\langle x \rangle$-invariant and of infinite rank ([41], Theorem 1). Thus, $\langle x \rangle = B_1\langle x \rangle \cap B_2\langle x \rangle$ and $B_i\langle x \rangle Q$ is a subgroup of G, for $i = 1, 2$. If $Q \cap B\langle x \rangle = \{1\}$, then

$$\langle x \rangle Q = (B_1\langle x \rangle \cap B_2\langle x \rangle)\, Q = B_1\langle x \rangle Q \cap B_2\langle x \rangle Q$$

and $\langle x \rangle Q$ is a q-subgroup of G. Thus $x \in Q$, a contradiction by the choice of Q. Hence, $Q \cap B\langle x \rangle$ is not trivial. As $\langle x \rangle$ is a Sylow q-subgroup of $B\langle x \rangle$, $Q \cap B\langle x \rangle = \langle y \rangle$, for some element y of order q ([29], Proposition 2.2.3). Since y belongs to Q, $y \neq x$ and so we may assume that $\langle y \rangle$ is not contained in $B_1\langle x \rangle$. In particular, $B_1\langle x \rangle \cap \langle y \rangle = \{1\}$ and this implies that also $Q \cap B_1\langle x \rangle = \{1\}$. As B_1 is normal in $B_1\langle x \rangle$, it contains a direct product $C_1 \times C_2$, with C_1 and C_2 of infinite rank, such that $\langle x \rangle = C_1\langle x \rangle \cap C_2\langle x \rangle$ ([41], Theorem 1). Thus,

$$\langle x \rangle Q = (C_1\langle x \rangle \cap C_2\langle x \rangle)\, Q = C_1\langle x \rangle Q \cap C_2\langle x \rangle Q$$

and $\langle x \rangle Q$ is a q-subgroup of G, a contradiction again. It follows that x belongs to any Sylow q-subgroup of G and the second step is proved.

Step 3. If G has a p-subgroup X of infinite rank, then every Sylow q-subgroup of G is normal, for any $q \neq p$.

Fix a prime $q \neq p$ and put $\bar{G} = G/O_q(G)$. If $O_q(G)$ has infinite rank, then $\bar{G}$ is locally nilpotent by Lemma 2.1.1, so that, in particular, it has no non-trivial q-subgroups. On the other hand, if $O_q(G)$ has finite rank, then $\bar{G}$ has infinite rank and $\bar{X}$ is a p-subgroup of infinite rank of $\bar{G}$, so that, by Step 2, $\bar{G}$ has no non-trivial elements of order q. Hence, in both cases, the factor group $G/O_q(G)$ is a q'-group and $O_q(G)$ is the unique Sylow q-subgroup of G.

Step 4. If G has a p-subgroup of infinite rank, then every q-subgroup of G has finite rank for any $q \neq p$.
If P and Q are respectively a p-subgroup and a q-subgroup of infinite rank, then it follows by Step 3 that every Sylow subgroup of G is normal. Hence, G is locally nilpotent, a contradiction.

Step 5. G has a Sylow p-subgroup of infinite rank, for some prime p.
By contradiction, suppose that every Sylow subgroup of G has finite rank, so that G satisfies the minimal condition on p-subgroups for every prime p ([56], p.98, part 1). Let F be any finite subgroup of G and put $\pi = \pi(F)$. Then π is a finite set and $G/O_{\pi'}(G)$ is a Černikov group ([29], Theorem 3.5.15 and Corollary 2.5.13). It follows that $O_{\pi'}(G)$ has infinite rank and the factor group $G/O_{\pi'}(G)$ is locally nilpotent by Lemma 2.1.1. In particular, $F \simeq FO_{\pi'}(G)/O_{\pi'}(G)$ is nilpotent and G is locally nilpotent, a contradiction.

Step 6. Final step.
By Step 5, G has a Sylow p-subgroup of infinite rank, for some prime p. Then, by Step 3, G has a unique Sylow q-subgroup G_q, for any $q \neq p$ and, by Step 4, G_q has finite rank. Thus $R = \mathrm{Dr}_{q \neq p} G_q$ is a normal p'-subgroup of G and, since G is countable, it follows by Theorem 2.4.5 of [29] that there exists a Sylow p-subgroup P of G such that $G = RP$. In particular, P has infinite rank by Step 1.
Fix a prime $q \neq p$ and put $Q = G_q$. Since Q has finite rank, Q is a Černikov group and so $P/C_P(Q)$ has finite rank, as it is isomorphic to a periodic group of automorphisms of Q ([56], Theorem 3.29, part 1). It follows that $C_P(Q)$ has infinite rank and $PQ/C_P(Q)$ is locally nilpotent by Lemma 2.1.1. On the other hand, PQ/Q is trivially locally nilpotent. Since $C_P(Q) \cap Q = \{1\}$, PQ embeds in the direct product of $PQ/C_P(Q)$ and PQ/Q, so that PQ is locally nilpotent. In particular, $[P, Q] = \{1\}$ and this holds for every Sylow q-subgroup G_q of G, with $q \neq p$. It follows that $[P, R] = \{1\}$ and $G = P \times R$ is locally nilpotent, a contradiction. This last contradiction completes the proof of the theorem. ■

2.2 Semipermutable subgroups of infinite rank

If p is a prime number, a group G is said to have *finite section p-rank* if every elementary abelian p-section of G has finite order at most p^r and r is the least positive integer with such property; if such an r does not exist, we say that G has *infinite section p-rank*. We say that G has *finite section rank* if G has finite section p-rank for every prime number p.
Note that if G is a group of finite rank, then it has finite section rank, but the converse is not true in general. On the other hand, for a primary group the two concepts coincide, that is a p-group G has finite rank if and only if it has finite section p-rank.
In this section, our aim is to show that both the class of groups of infinite rank and the class of groups of infinite section p-rank control semipermutability in the universe of locally finite groups, obtaining the following two results.

Theorem 2.2.1 (Ballester-Bolinches, Camp-Mora, Dixon, Ialenti, Spagnuolo [10])**.** *Let G be a locally finite group of infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. Then every subgroup of G is semipermutable.*

Theorem 2.2.2 (Ballester-Bolinches, Camp-Mora, Dixon, Ialenti, Spagnuolo [10])**.** *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are semipermutable. Then every subgroup of G is semipermutable.*

In contrast to S-permutability, we will show that, in general, S-semipermutability is not controlled by the class of groups of infinite rank and at the end of this section it will be exhibited an example of a periodic soluble group of infinite rank in which every subgroup of infinite rank is S-semipermutable but not every subgroup of G satisfies the same property.
On the other hand, the situation is different if we restrict the attention to locally finite groups of infinite rank with the minimal condition on p-subgroups.

Theorem 2.2.3 (Ballester-Bolinches, Camp-Mora, Dixon, Ialenti, Spagnuolo [10])**.** *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are S-semipermutable. If G satisfies the minimal condition on p-subgroups for every prime p, then every subgroup of G is S-semipermutable.*

For the convenience of the reader, we put here the following lemma which will be very useful in our proofs.

Lemma 2.2.1 (Wang-Li-Wang [64]). *Let G be a locally finite group. Then all subgroups of G are semipermutable if and only if for every p-element x and q-element y of G, with $q \neq p$, the set $\langle x \rangle \langle y \rangle$ is a subgroup of G.*

We begin with the study of locally finite groups of infinite section p-rank, for a fixed prime number p.
Notice that Lemma 9 of [11] guarantees that any locally finite group of infinite section p-rank contains some p-subgroup of infinite rank. So, in what follows, the cited lemma will be always implicitly used.

Lemma 2.2.2. *Let G be a locally finite group of infinite section p-rank whose subgroups of infinite section p-rank are semipermutable and let S be any p-subgroup of infinite rank of G. Then every subgroup of S permutes with every q-element of G, for any $q \neq p$. In particular, $S\langle x \rangle$ is a $\{p, q\}$-group for any q-element x of G.*

PROOF. Let y be any element of S and let x be an element of G of order q^n, for some positive integer n. By Theorem 1 of [41], S contains an abelian subgroup $A = A_1 \times A_2$, where A_1 and A_2 are $\langle y \rangle$-invariant subgroups of infinite rank and $A \cap \langle y \rangle = \{1\}$. Since $A_i\langle y \rangle$ is a p-subgroup of infinite rank, $\langle x \rangle A_i \langle y \rangle$ is a subgroup of G, for $i = 1, 2$ and, as $A\langle y \rangle \cap \langle x \rangle = \{1\}$, the following equalities hold:

$$\langle x \rangle \langle y \rangle = \langle x \rangle \left(A_1 \langle y \rangle \cap A_2 \langle y \rangle \right) = \langle x \rangle A_1 \langle y \rangle \cap \langle x \rangle A_2 \langle y \rangle$$

and $\langle x \rangle \langle y \rangle$ is a subgroup of G. ■

Lemma 2.2.3. *Let G be a locally finite group of infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. Then every q-element permutes with every r-element of G, for any prime numbers q and r, where $q \neq r$ and q and r are both different from p.*

PROOF. Fix two prime numbers q and r, with $q \neq r$ and both q and r different from p and let x and y be respectively a q-element and an r-element of G. Let $A = A_1 \times A_2$ be an abelian p-subgroup of G, with A_1 and A_2 of infinite rank. Then, by Lemma 2.2.2, for every $i = 1, 2$ $A_i\langle x\rangle$ is a $\{p, q\}$-subgroup, so that it permutes with $\langle y\rangle$, for $i = 1, 2$ and, as $A_i\langle x\rangle \cap \langle y\rangle = \{1\}$, the following equalities hold:

$$\langle x\rangle\langle y\rangle = (A_1\langle x\rangle \cap A_2\langle x\rangle)\,\langle y\rangle = A_1\langle x\rangle\langle y\rangle \cap A_2\langle x\rangle\langle y\rangle$$

and $\langle x\rangle\langle y\rangle$ is a subgroup of G. ■

Next lemma will allow us to reduce the proof of Theorem 2.2.1 to the countable case, as in the proof of Theorem 2.1.1.

Lemma 2.2.4. *Let G be a locally finite group of infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. If every countable subgroup of infinite section p-rank of G has all its subgroups semipermutable, then every subgroup of G is semipermutable.*

PROOF. Let x and y be elements of G with relatively prime orders and let A be an abelian p-subgroup of infinite rank of G. Then, the socle S of A is elementary abelian of infinite rank and, in particular, it is countable. Thus $\langle x, y, S\rangle$ is a countable subgroup of infinite section p-rank and, by hypothesis, $\langle x\rangle\langle y\rangle$ is a subgroup of G. The statement now follows by Lemma 2.2.1. ■

Proof of Theorem 2.2.1. By Lemma 2.2.4, we may assume that G is a countable group. Let q and r be prime numbers with $q \neq r$ and let x and y be a q-element and an r-element of G, respectively. If both q and r are different from p, then $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ by Lemma 2.2.3. So, we only have to consider the case in which one between q and r coincides with p. Without loss of generality, we can assume that $q = p$ and let P be a Sylow p-subgroup of infinite rank of G.
As G is countable, we can consider an ascending chain of finite subgroups of G $(F_n)_{n\in\mathbb{N}}$ such that $G = \bigcup_n F_n$. Clearly, we may assume that $x \in F_1$. Let $(P_n)_{n\in\mathbb{N}}$ be an ascending chain of finite subgroups of P such that $P = \bigcup_n P_n$ and the rank of P_n is at least n for any n. Put $G_n = \langle F_n, P_n\rangle$. For a fixed n, assume that G_n contains a Sylow p-subgroup S_n such that $x \in S_n$ and S_n has rank at least n. As $G_n \leq G_{n+1}$, S_n is contained in some Sylow p-subgroup S_{n+1} of G_{n+1}. Moreover, since S_{n+1} contains a conjugate of P_{n+1},

the rank of S_{n+1} is at least $n+1$. So, by induction, we have that $S = \bigcup_n S_n$ is a p-subgroup of infinite rank of G, containing x. Then, by Lemma 2.2.2, $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ and the theorem is completely proved. ■

Proof of Theorem 2.2.2. If G contains a p-subgroup of infinite rank, for some prime p, then G has also infinite section p-rank and if H is any subgroup of infinite section p-rank of G, then H has infinite rank and, by hypothesis, H is semipermutable. So, in this case, the theorem is a direct consequence of Theorem 2.2.1. Thus, we can assume that every p-subgroup of G has finite rank, so that G satisfies the minimal condition on p-subgroups, for every prime p. By Theorem 3.5.15 of [29], G contains a locally soluble normal subgroup S of finite index in G. Let x and y be respectively a p-element and a q-element of G, with $p \neq q$, and put $\pi = \{p, q\}$. Then $G/O_{\pi'}(S)$ is a Černikov group ([29], Corollary 2.5.13) and $O_{\pi'}(S)$ has infinite rank. By Theorem 1 of [41], $O_{\pi'}(S)$ contains an abelian subgroup $B = B_1 \times B_2$, where B_1 and B_2 are $\langle x\rangle$-invariant subgroups of infinite rank and $\langle x\rangle \cap B = \{1\}$. Therefore, for every $i = 1, 2$, $B_i\langle x\rangle$ is a q'-subgroup of infinite rank of G and $B_i\langle x\rangle\langle y\rangle = \langle y\rangle B_i\langle x\rangle$. As $B\langle x\rangle \cap \langle y\rangle = \{1\}$, the following equalities hold:

$$\langle x\rangle\langle y\rangle = \left(B_1\langle x\rangle \cap B_2\langle x\rangle\right)\langle y\rangle = B_1\langle x\rangle\langle y\rangle \cap B_2\langle x\rangle\langle y\rangle$$

and $\langle x\rangle\langle y\rangle$ is a subgroup of G. The theorem now follows from Lemma 2.2.1. ■

We now turn our attention to locally finite groups of infinite rank whose subgroups of infinite rank are S-semipermutable. The following lemma is correspondent to Lemma 2.2.1 and it is easy to prove.

Lemma 2.2.5. *Let G be a locally finite group. Then every subgroup of G is S-semipermutable if and only if every p-element of G permutes with every Sylow q-subgroup of G, for any prime number p and q with $p \neq q$.*

Proof of Theorem 2.2.3. Let x be a p-element of G and let Q be a Sylow q-subgroup of G, where p and q are different prime numbers. Application of Theorem 3.5.15 of [29] yields that G contains a locally soluble normal subgroup S such that the index of S in G is finite. Since S has infinite rank, the set $\pi(S)$ is infinite ([29], Lemma 3.1.2) and so there exists a finite subset π of $\pi(S)$ such that p, q do not belong to $\pi' \cap \pi(S)$. Moreover, $O_{\pi'}(S)$ has infinite rank ([29], Lemma 2.5.13). Clearly, $O_{\pi'}(S)$ is normal in G and so, by Theorem 1 of [41], it contains an abelian subgroup $B = B_1 \times B_2$, where B_1 and B_2 are $\langle x \rangle$-invariant and have infinite rank. Thus, the q'-subgroups $B_i\langle x \rangle$ permute with Q, for $i = 1, 2$ and from the equality

$$\langle x \rangle Q = B_1\langle x \rangle Q \cap B_2\langle x \rangle Q$$

follows that $\langle x \rangle Q$ is a subgroup of G. In particular, $\langle x \rangle$ is S-semipermutable in G and the theorem now follows from Lemma 2.2.5. ■

The following proposition shows that in Theorem 2.2.3 the hypothesis that G satisfies the minimal condition on p-subgroups cannot be removed.

Proposition 2.2.1. *There exists a periodic metabelian group G of infinite rank whose subgroups of infinite rank are S-semipermutable but not every subgroup is S-semipermutable.*

PROOF. For every positive integer i, let $T_i = \langle a_i, b_i \mid a_i^3 = b_i^2 = 1, b_i^{-1}a_ib_i = a_i^{-1} \rangle$ be an isomorphic copy of the symmetric group on 3 letters and let $T = \mathrm{Dr}_i T_i$.
Let $P = \mathrm{Dr}_i\langle b_i \rangle$ and $Q = \langle a_1 \rangle \times \langle a_2 \rangle$ and put $G = PQ$. Clearly, P is an elementary abelian 2-group of infinite rank and G is a countable metabelian $\{2, 3\}$-group of infinite rank.
Let A be any subgroup of infinite rank of G. Since the unique Sylow 3-subgroup Q of G is finite, either $\pi(A) = \{2\}$ or $\pi(A) = \{2, 3\}$. In both cases, A is trivially S-semipermutable.
Now, suppose that every subgroup of G is S-semipermutable and take $X = \langle a_1a_2 \rangle$. Then, PX is a subgroup of G and $X = PX \cap Q$ is normal in PX, a contradiction, since the element $b_1^{-1}a_1a_2b_1 = a_1^2a_2$ does not belong to X. ■

2.3 Nearly and almost permutable subgroups of infinite rank

A subgroup H of a group G is said to be *almost normal* if H is normal in a subgroup of finite index of G or, equivalently, if H has only finitely many conjugates in G. A famous theorem by B.H. Neumann [53] states that every subgroup of a group G is almost normal if and only if the centre $Z(G)$ of G has finite index in G. Later, this result was generalized by I.I. Eremin [33], who proved that for a group G, the factor group $G/Z(G)$ is finite if and only if its abelian subgroups are almost normal.

A subgroup H of a group G is said to be *nearly normal* if H has finite index in its normal closure H^G. In [53], B.H. Neumann proved also that every subgroup of a group G is nearly normal if and only if the commutator subgroup G' of G is finite. So, in particular, a group G in which every subgroup is nearly normal is a BFC-group, that is a group in which every element has finitely many conjugates and the number of these conjugates is bounded by the order of G'. Moreover, it is worth to notice that, combining Neumann's theorem with a famous result by I. Schur, if every subgroup of a group G is almost normal, then every subgroup of G is nearly normal.

Corresponding properties, where normality is replaced by permutability, have been introduced in [40] and in [23]. A subgroup H of a group G is said to be *almost permutable* if H is permutable in a subgroup of finite index of G and H is said to be *nearly permutable* if it has finite index in a permutable subgroup of G. The structure of a group in which either every subgroup is almost permutable or every subgroup is nearly permutable has been studied.

More precisely, in [40] the authors have proved that a periodic group G has all its subgroups almost permutable if and only if G is simultaneously finite-by-quasihamiltonian and quasihamiltonian-by-finite; while, if G is a non-periodic group in which every subgroup is almost permutable then the set of the elements of finite order T of G is a subgroup and the factor group G/T is abelian, every subgroup of T is almost normal in G and either G is an FC-group or G/T is locally cyclic. Notice that the latter result for non-periodic groups is correspondent to Iwasawa's theorem for non-periodic

quasihamiltonian groups.
In [23] it has been proved that every subgroup of a periodic group G is nearly permutable if and only if G is finite-by-quasihamiltonian. In analogy to Neumann's results, it follows at once that if every subgroup of a group G is almost permutable, then every subgroup of G is nearly permutable. In [39], non-periodic groups with nearly permutable subgroups have been considered and their structure is similar to non-periodic groups with almost permutable subgroups. In fact, the authors proved that if G is a non-periodic group in which every subgroup is nearly permutable, then the set of the elements of finite order T of G is a subgroup and the factor group G/T is abelian, every subgroup of T is nearly normal in G and either G is an FC-group or G/T is locally cyclic.

De Falco, de Giovanni and Musella [21] proved that the class of groups of infinite rank controls almost normality and nearly normality in the universe of generalized radical groups.
The aim of this section is to investigate the structure of a generalized radical group of infinite rank in which either every subgroup of infinite rank is almost permutable or every subgroup of infinite rank is nearly permutable. The non-periodic case and the periodic case will be treated separately.

In section A. structure theorems for non-periodic groups will be obtained. In order to give a common approach to both properties, we have introduced the following property which generalizes almost permutability and nearly permutability: a subgroup H of a group G is said to be *finite-permutable-finite* if there exist subgroups K and L of G such that the indeces $|K : H|$ and $|G : L|$ are finite and K is permutable in L. In case K is normal in L, then H is called a *finite-normal-finite* subgroup of G. The following theorem will be proved.

Theorem 2.3.1 (De Luca, Ialenti [26]). *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are finite-permutable-finite. Then:*

- *The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.*

- *Every subgroup of T is finite-normal-finite in G and either G is an FC-group or G/T is locally cyclic.*

Since any almost permutable and nearly permutable subgroup is finite-permutable-finite, the theorem 2.3.1 guarantees that in a non-periodic generalized radical group G whose subgroups of infinite rank are almost permutable (resp. nearly permutable) the set of the elements of finite order T is a subgroup containing G', and either G is an FC-group or G/T is locally cyclic. Then we will be able to show, as a corollary, that every subgroup of T is almost normal in G (resp. nearly permutable).

Section B. and section C. are devoted to the periodic case. In particular, it will be shown that the class of groups of infinite rank controls nearly permutability (section B.) and almost permutability (section C.) in the universe of locally finite groups.

Theorem 2.3.2 (De Luca, Ialenti [25])**.** *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then every subgroup of G is nearly permutable.*

Theorem 2.3.3 (De Luca, Ialenti [24])**.** *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are almost permutable. Then every subgroup of G is almost permutable.*

A. The non-periodic case

In order to prove the main theorem of this section, first we have to investigate the structure of a group in which every subgroup is finite-permutable-finite and, since every finite-normal-finite subgroup is obviously finite-permutable-finite, we begin with the study of groups in which every subgroup is finite-normal-finite.
Recall that an element x of a group G is said to be an *FC-element* of G if x has finitely many conjugates in G or, equivalently, if the centralizer $C_G(x)$ of x has finite index in G. The *FC-centre* of G is the subgroup of all FC-elements of G and G is called an *FC-group* if it coincides with its FC-centre.

Lemma 2.3.1. *Let G be a group and let x be any element of G. If the subgroup $\langle x\rangle$ is finite-normal-finite in G, then x is an FC-element.*

PROOF. Let H and K be subgroups of G such that $|H : \langle x \rangle|$ and $|G : K|$ are finite and H is normal in K. Then $\langle x \rangle$ is nearly normal in K and so there exists a positive integer n such that the normal subgroup $(\langle x \rangle^K)^n$ of K is contained in $\langle x \rangle$. As $\langle x \rangle^K / (\langle x \rangle^K)^n$ is finite, we have that

$$K/C_K\left(\langle x \rangle^K / (\langle x \rangle^K)^n\right)$$

is finite and, in particular $N_K(x)$ has finite index in K. As $N_K(x)/C_K(x)$ is finite, it follows that the index $|K : C_K(x)|$ is finite. Thus, since K has finite index in G, the index $|G : C_G(x)|$ is finite and x is an FC-element of G. ■

Corollary 2.3.1. *Let G be a group whose subgroups are finite-normal-finite. Then every subgroup of G is nearly normal in G.*

PROOF. By Lemma 2.3.1, G is an FC-group. Let X be any subgroup of G and let H and K be subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is normal in K. Then H is almost normal in the FC-group G and so $|H^G : H|$ is finite ([63], Lemma 7.13). It follows that X has finite index in H^G and so X is nearly normal in G. ■

Now we consider groups in which every subgroup is finite-permutable-finite and, before proving the structure theorem for this class of groups, we introduce some preliminary results.

Proposition 2.3.1. *Let G be a group whose cyclic subgroups are finite-permutable-finite. Then the set of all elements of finite order of G is a subgroup.*

PROOF. Let T be the largest periodic normal subgroup of G. Clearly, every cyclic subgroup of G/T is finite-permutable-finite, so that replacing G with G/T it can be assumed without loss of generality that G has no periodic non-trivial normal subgroups. Let x be any element of finite order of G. As $\langle x \rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x \rangle|$ and $|G : K|$ are finite and H is permutable in K. Then H is a finite permutable subgroup of K and, by Lemma 6.2.15 of [60], H^K is periodic. Let N be the core of K in G, then H^N is periodic and so $[H, N]$ is a subnormal periodic subgroup of G. Thus $[N, H] = 1$ and, in particular, $N \leq C_G(x)$. Then x is a periodic FC-element of G and, by Dietzmann's Lemma ([56], part 1, p.45), $\langle x\rangle^G$ is finite and, therefore, $x = 1$. Thus G is torsion-free and the proposition is proved. ■

Proposition 2.3.2. *Let G be a group whose cyclic subgroups are finite-permutable-finite. If G contains two elements a and b of infinite order such that* $\langle a\rangle \cap \langle b\rangle = \{1\}$*, then G is an FC-group.*

PROOF. Let x be any element of G. As $\langle x\rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is permutable in K. Let y be any element of K of infinite order and suppose, first, that $\langle x\rangle \cap \langle y\rangle = \{1\}$. Then $H \cap \langle y\rangle = \{1\}$ and hence $H^y = H$ ([60], Lemma 6.2.3). Suppose now that $\langle x\rangle \cap \langle y\rangle \neq \{1\}$, so that in particular x has infinite order and there exists an element z of K of infinite order such that

$$\langle x\rangle \cap \langle z\rangle = \langle y\rangle \cap \langle z\rangle = \{1\}.$$

Thus $H^z = H$. Now, as $\langle y\rangle$ is finite-permutable-finite, there exist M and L subgroups of G such that $|M : \langle y\rangle|$ and $|G : L|$ are finite and M is permutable in L. Let k be a positive integer such that z^k is in L. Then $M^{z^k} = M$ and M is normal in $M\langle yz^k\rangle = M\langle z^k\rangle$, so that yz^k must have infinite order and $M \cap \langle yz^k\rangle = \{1\}$. Since $\langle y\rangle \cap \langle yz^k\rangle = \{1\}$, we have also $\langle x\rangle \cap \langle yz^k\rangle = \{1\}$, so that yz^k normalizes H and $H^y = H$. Therefore H is normalized by any element of infinite order of K and, as K is generated by its elements of infinite order, H is normal in K. Thus $\langle x\rangle$ is finite-normal-finite in G and, hence x is an FC-element of G by Lemma 2.3.1. Therefore, G is an FC-group. ■

Theorem 2.3.4. *Let G be a non-periodic group whose subgroups are finite-permutable-finite. Then:*

(a) The set T of all elements of finite order of G is a normal subgroup and the factor group G/T is abelian.

(b) Every subgroup of T is finite-normal-finite in G.

(c) Either G is an FC-group or the group G/T is locally cyclic.

PROOF. (a) By Proposition 2.3.1, T is a subgroup of G. In order to prove that G/T is abelian, we may assume that G is torsion-free. Let x be any element of G. As $\langle x\rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is permutable in K. Since H is a cyclic-by-finite torsion-free group, H is cyclic. Put $H = \langle h\rangle$, so that $\langle x\rangle = \langle h^n\rangle$ for some positive integer n. By a contradiction, assume that H is not normal in K, so that there exists an element y of K such that $H^y \neq H$. Let $L = \langle h\rangle\langle y\rangle$. By Lemma 6.2.3 of [60], $M = \langle h\rangle \cap \langle y\rangle$ is a non-trivial subgroup of L contained in $Z(L)$. Therefore L/M is finite so that also the commutator subgroup L' of L is finite, and so L is abelian. This contradiction proves that $\langle h\rangle$ is normal in K and so also $\langle x\rangle$ is normal in K. Therefore all cyclic subgroups of G are almost normal and G is an FC-group. As G is torsion-free, it follows that G is abelian.

(b) Let X be any subgroup of T. As X is finite-permutable-finite, there exist H and K subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is permutable in K. In particular, H is periodic. Let a be any element of infinite order of K, so that $\langle a\rangle \cap T = \{1\}$. Then $H = H\langle a\rangle \cap T$ is a normal subgroup of $H\langle a\rangle$. It follows that H is normal in K.

(c) This part follows directly from Proposition 2.3.2. ■

Corollary 2.3.2. *Let G be a non-periodic group whose subgroups are finite-permutable-finite and let T be the torsion subgroup of G. Then every subgroup of T is nearly normal in G.*

PROOF. By Theorem 2.3.4, every subgroup of T is finite-normal-finite in G and so by Corollary 2.3.1, T' is finite. Without loss of generality, we may replace G with G/T', so that we may assume that T is abelian. Let X be a finite subgroup of T and let H and K be subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is normal in K. Then H and $|G : N_G(H)|$ are finite, so that H^G is finite ([56], part 1, p. 45). In particular, X^G is finite.
Let H be a periodic subgroup such that $N = N_G(H)$ has finite index in G. If G is an FC-group, then H is nearly normal in G ([63], Lemma 7.13). Suppose that G/T is locally cyclic. Put $G = EN$, where E is finitely generated. If $E \leq T$, $G = N$ and H is normal in G. Suppose that ET is a non-periodic group and, without loss of generality, we may assume that $G = ET$. Then $G = \langle a\rangle \ltimes T$, where a is an element of infinite order of G. Let A and B be subgroups of G such that $|A : \langle a\rangle|$ and $|G : B|$ are finite and A is permutable

in B. As $A \cap T$ is finite, $(A \cap T)^G$ is also finite and, replacing G with $G/(A \cap T)^G$, we can assume that $\langle a \rangle$ is permutable in B. Since $B \cap T$ is a subgroup of finite index of T, $T = (T \cap B)X$, where X is a normal finite subgroup of G. Therefore $G = BT = BX$ and $\langle a \rangle X/X$ is permutable in G/X. Since $\langle a \rangle X \cap T = X$, every subgroup of T/X is G-invariant ([23], Lemma 2.5). Thus, HX is normal in G and the index $|HX : H|$ is finite.
Since every subgroup X of T has finite index in a subgroup H which is almost normal in G, it follows that X is nearly normal in G. ∎

Now, we are in a position to focus on generalized radical groups of infinite rank in which only the subgroups of infinite rank are finite-permutable-finite. In the proof of the main theorem, we will need the following lemmas.

Lemma 2.3.2. *Let G be a periodic group whose subgroups are finite-permutable-finite. Then G is locally finite.*

PROOF. Let E be any finitely generated subgroup of G. Since any subgroup of E is finite-permutable-finite, we may assume without loss of generality that G is finitely generated. Let x be any element of G, then there exist H and K subgroups of G such that $|H : \langle x \rangle|$ and $|G : K|$ are finite and H is permutable in K. It follows that H is finite and K is finitely generated and, hence, by Theorem 6.2.18 of [60], H^K is finite. In particular, x has finitely many conjugates in K and so also in G. Therefore, G is an FC-group and hence it is finite.

Recall that an element x of a group G has finite order modulo a permutable subgroup H of G if the index $|H\langle x \rangle : H|$ is finite; otherwise x is said to have infinite order modulo H.

Lemma 2.3.3. *Let G be a group and let X be a permutable subgroup of G such that any subgroup of G containing X is finite-permutable-finite. If there exists an element of G having infinite order modulo X, then X is normal in G.*

PROOF. Let x and y be elements of G of finite order modulo X and let $L = X\langle x, y \rangle$. The factor group L/X^L has all its subgroups finite-permutable-finite and it is generated by two periodic elements. As a consequence, L/X^L is periodic and so it is also finite by Lemma 2.3.2. It follows that, since the index $|X^L : X|$ is finite ([60], Theorem 6.2.18), X has finite index in L, so that also the product xy has finite order modulo X. Thus the set T of all elements of G having finite order modulo X is a proper subgroup of G. Since G is generated by $G \setminus T$ and any element of $G \setminus T$ normalizes X ([60], Lemma 6.2.3), X is normal in G and the lemma is proved. ■

Here, we highlight the following argument, as it will be frequently used in the next proofs.
Let G be a group of infinite rank whose subgroups of infinite rank are finite-permutable-finite and let $A = A_1 \times A_2$ be an abelian subgroup of G, where both factors A_1 and A_2 have infinite rank. Then, A_1 and A_2 are finite-permutable-finite, so there exist subgroups H_i and K_i of G such that the indeces $|H_i : A_i|$, $|G : K_i|$ are finite and H_i is permutable in K_i, for $i = 1, 2$. It follows that $N = (K_1 \cap K_2)_G$ is a normal subgroup of finite index of G, $H_i \cap N$ is a permutable subgroup of N and $|H_i \cap N : A_i \cap N|$ is finite, for $i = 1, 2$. Then, replacing A with $\bar{A} = (A_1 \cap N) \times (A_2 \cap N)$, we will always assume that H_1 and H_2 are both permutable in a same normal subgroup N of finite index of G. If G is a group of infinite rank whose subgroups of infinite rank are finite-normal-finite, then the same argument can be used, just replacing permutability with normality.
Next proposition shows that, restricting the hypothesis to the subgroups of infinite rank, it is possible to obtain a result similar to Corollary 2.3.1.

Proposition 2.3.3. *Let G be a generalized radical group of infinite rank in which all subgroups of infinite rank are finite-normal-finite. Then every subgroup of G is nearly normal.*

PROOF. Let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a N-invariant subgroup H_i of N, for $i = 1, 2$. Every subgroup of N/H_i is finite-normal-finite and so, by Corollary 2.3.1, $N'H_i/H_i$ is finite, for $i = 1, 2$. Since $H_1 \cap H_2$ is finite, N' is finite and, replacing G with G/N', we may assume that N is abelian. Therefore, N contains a direct product $Y_1 \times Y_2$ of G-invariant subgroups of infinite rank Y_1 and Y_2 ([21], Lemma 6) and it follows that $G'Y_i/Y_i$ is finite, for $i = 1, 2$. Hence, also G' is finite and every subgroup of G is nearly normal. ■

For the convenience of the reader, we state as a Lemma the results of [40] and [39] concerning non-periodic groups in which either every subgroup is almost permutable or every subgroup is nearly permutable.

Lemma 2.3.4 ([40],[39]). *Let G be a non-periodic group in which every subgroup is almost permutable (resp. nearly permutable). Then the set of all elements of finite order T of G is a normal subgroup of G and the factor group G/T is abelian; moreover, every subgroup of T is almost normal (resp. nearly normal) in G and either G is an FC-group or G/T is locally cyclic.*

Proof of Theorem 2.3.1. (a) Suppose firstly that G has no non-trivial periodic normal subgroups. Let x be an element of infinite order of G and let $A = A_1 \times A_2$ be an abelian subgroup of G such that A_1 and A_2 have both infinite rank and $A \cap \langle x \rangle = \{1\}$. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a subgroup H_i which is a permutable subgroup of N, for $i = 1, 2$. Put $\langle y \rangle = \langle x \rangle \cap N$, then $\langle y \rangle \cap H_1 = \langle y \rangle \cap H_2 = \{1\}$ and so, by Lemma 2.3.3, H_1 and H_2 are normal subgroups of N. Therefore every subgroup of N/H_i is finite-permutable-finite and hence, by Theorem 2.3.4, $N'H_i/H_i$ is periodic, for $i = 1, 2$. Since $H_1 \cap H_2$ is finite, N' is periodic and $N' = \{1\}$. Thus N contains two G-invariant subgroups Y_1 and Y_2 of infinite rank such that $Y_1 \cap Y_2 = \{1\}$ ([21], Lemma 6). Since G embed in the direct product $G/Y_1 \times G/Y_2$, it follows from Theorem 2.3.4 that G' is periodic. Thus $G' = \{1\}$ and G is a torsion-free abelian group.

Now, in the general case, let T be the largest normal periodic subgroup of G. If T has finite rank, then G/T has infinite rank and, by the previous argument, G/T is a torsion-free abelian group. On the other hand, if T has infinite rank, then G/T is a non-periodic group whose subgroups are finite-permutable-finite and, by Theorem 2.3.4, G/T is a torsion-free abelian group. In both cases, T coincides with the set of all periodic elements of G and G/T is abelian.

(b) Assume that T has finite rank, so that G has infinite torsion-free rank. Let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank, such that $A \cap T = \{1\}$. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a subgroup H_i which is a permutable subgroup of N, for $i = 1, 2$. Let x be an element of A_2; as $|H_1 : A_1|$ is finite, $H_1 \cap \langle x \rangle = \{1\}$. Therefore, by Lemma 2.3.3, H_1 is normal in N. Similarly, H_2 is a normal subgroup of N. Thus, N/H_i is a non-periodic group with infinite torsion-free rank in which every subgroup is finite-permutable-finite and, by Theorem 2.3.4, N/H_i is an FC-group, for $i = 1, 2$. It follows that N is finite-by-FC and so it is an FC-group. Since $G/Z(N)$ is a periodic group, $Z(N)$ is a G-invariant subgroup with infinite torsion-free rank. Let X be a subgroup of T, so $Z(N)$ contains a torsion-free subgroup $Y = Y_1 \times Y_2$, with Y_1 and Y_2 G-invariant of infinite rank, such that $X \cap Y = \{1\}$ ([21], Lemma 6). Thus, G/Y_i is an FC-group and XY_i is finite-normal-finite in G, for $i = 1, 2$. It follows that G is an FC-group and $X = XY_1 \cap XY_2$ is finite-normal-finite in G.

Assume now that T has infinite rank and let X be any subgroup of T of infinite rank. Then there exist subgroups H and K such that $|H : X|$ and $|G : K|$ are finite and H is permutable in K. In particular, H is periodic. Let a be an element of infinite order of K, then $H = H\langle a \rangle \cap T$ is a normal subgroup of $H\langle a \rangle$. It follows that H is normal in K and X is finite-normal-finite in G. By Proposition 2.3.3, T' is finite. Without loss of generality, we may replace G with G/T' and assume that T is abelian. Let X be a subgroup of T of finite rank and let $A = A_1 \times A_2$ be an abelian subgroup of T, with A_1 and A_2 of infinite rank, such that $A \cap X = \{1\}$. Then XA_i is finite-normal-finite in G, for $i = 1, 2$, and $X = XA_1 \cap XA_2$ is finite-normal-finite in G. Let N be a normal subgroup of finite index of G such that A_i has finite index in a normal subgroup H_i of N, for $i = 1, 2$.

Suppose that G/T is not locally cyclic. Then, by Theorem 2.3.4, N/H_i is an FC-group, for $i = 1, 2$, and so N is an FC-group. It follows that G is FC-by-finite and, in particular, G satisfies locally the maximal condition on subgroups.

In order to prove that G is an FC-group, we first show that any subgroup of T is nearly normal in G. It is enough to show that if a periodic subgroup is almost normal in G, then it is also nearly normal in G. So, let $H \leq T$ be an almost normal subgroup of G. Then, $N = N_G(H)$ has finite index in G and $G = EN$, where E is finitely generated. Put $L = ET$, so $H^G = H^L$. Since G is FC-by-finite, the FC-centre F of L has finite index in

L. Moreover, as every subgroup of T is finite-normal-finite in G, $T \leq F$ and $F = (F \cap E)T$, where $F \cap E$ is finitely generated. As $F/Z(F)$ is locally finite, it follows that the G-invariant abelian subgroup $A = TZ(F)$ has finite index in F and hence also in L. Put $L = AM$, where M is finitely generated, and $A \cap M$ is a normal, finitely generated abelian subgroup of L. Then, $L/(A \cap M)$ is a group of infinite rank whose subgroups of infinite rank are finite-permutable-finite and consequently its periodic subgroups are finite-normal-finite. Hence $M^L/(A \cap M)$ is finite and M^L is finitely generated. Replacing M with M^L, we may assume that M is normal in L. Therefore, $[H, M]$ is a finitely generated subgroup of L' and, by (a), L' is contained in T. Therefore $[H, M]$ is finite and H has finite index in $H[H, M] = H^M = H^L$. Let $A = A_1 \times A_2$ be an abelian subgroup of T, with A_1 and A_2 of infinite rank. Then G/A_i^G is an FC-group and, since $|A_i^G : A_i|$ is finite for $i = 1, 2$, G is finite-by-FC and G is an FC-group. ■

Now we will apply Theorem 2.3.1 to the study of groups of infinite rank in which either every subgroup of infinite rank is almost permutable or every subgroup of infinite rank is nearly permutable.

Corollary 2.3.3. *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are almost permutable. Then*

(a) The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.

(b) Every subgroup of T is almost normal in G.

(c) Either G is an FC-group or G/T is locally cyclic.

PROOF. (a) and (c) follow directly from Theorem 2.3.1.

(b) If T has finite rank and $X \leq T$, take Y_1 and Y_2 as in the proof of (b) in Theorem 2.3.1. Then, by Lemma 2.3.4, XY_i is almost normal in G and also $X = XY_1 \cap XY_2$ is almost normal in G.
If T has infinite rank, let X be a subgroup of T of infinite rank and let K be a subgroup of finite index of G such that X is permutable in K. Then the same argument used in the proof of Theorem 2.3.1 shows that X is normal in K. Therefore, every subgroup of infinite rank of T is almost normal in G and

$T/Z(T)$ is finite by Theorem A of [21]. Now, let X be any subgroup of finite rank of T and let $A = A_1 \times A_2$ be a subgroup of $Z(T)$ with A_1 and A_2 of infinite rank and $X \cap A = \{1\}$. Then $X = XA_1 \cap XA_2$ is almost normal in G. ∎

Corollary 2.3.4. *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are nearly permutable. Then*

(a) The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.

(b) Every subgroup of T is nearly normal in G.

(c) Either G is an FC-group or G/T is locally cyclic.

PROOF. (a) and (c) follow directly from Theorem 2.3.1.

(b) If T has finite rank and $X \leq T$, take Y_1 and Y_2 as in the proof of (b) in Theorem 2.3.1. Then, by Lemma 2.3.4, XY_i is nearly normal in G and also $X = XY_1 \cap XY_2$ is nearly normal in G.
If T has infinite rank, let X be a subgroup of T of infinite rank and let K be a permutable subgroup of G such that X has finite index in K. Then the same argument used in the proof of Theorem 2.3.1 shows that K is normal in G. Therefore, every subgroup of infinite rank of T is nearly normal in G and T' is finite by Theorem B of [21]. We can now replace G with G/T' and assume that T is abelian. Thus, if X is any subgroup of finite rank of T and $A = A_1 \times A_2$ is a subgroup of T, with A_1 and A_2 of infinite rank and $X \cap A = \{1\}$, then $X = XA_1 \cap XA_2$ is nearly normal in G. ∎

B. Nearly permutable subgroups of infinite rank

The study of the periodic case is firstly restricted to primary groups and our first purpose is to show that a locally finite p-group of infinite rank whose subgroups of infinite rank are nearly permutable is finite-by-quasihamiltonian. The following lemma shows that, at least in the universe of locally finite groups, under certain conditions a subgroup of finite rank is the intersection of two subgroups of infinite rank.

Lemma 2.3.5. *Let G be a group and let A be a periodic normal subgroup of infinite rank of G. If X is a subnormal Černikov subgroup of G, then A contains a subgroup of infinite rank B such that $[X,B] = \{1\}$.*

PROOF. Let

$$X = L_0 \lhd L_1 \lhd \cdots \lhd L_{k-1} \lhd L_k = XA$$

be a subnormal series of X in XA and argue by induction on k. If $k = 1$, then X is normal in XA, so the factor group $A/C_A(X)$ is a Černikov group ([56], Theorem 3.29) and we can choose $B = C_A(X)$. Now, let $k > 1$ and put $L = L_{k-1}$. If L has finite rank, then L is a Černikov group and we can choose $B = C_A(L)$. So we can suppose that L has infinite rank. Since $L = X(A \cap L)$, then $A \cap L$ has infinite rank and, by induction, there exists a subgroup B of $A \cap L$ such that $[X,B] = \{1\}$ and the statement is true for any k. ■

Since a locally finite quasihamiltonian p-group is abelian-by-finite, a primary group in which every subgroup is nearly permutable is finite-by-abelian-by-finite. Next proposition shows that this holds only requiring that the subgroups of infinite rank are nearly permutable.

Proposition 2.3.4. *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-abelian-by-finite.*

PROOF. Assume by contradiction that G is not finite-by-abelian-by-finite and put $A = \Omega_1(G)$. Every subgroup of infinite rank of A is nearly normal in A and so A' is finite ([21], Theorem B). Moreover, G/A is finite-by-quasihamiltonian ([23], Theorem). In particular, G/A is finite-by-abelian-by-finite. Let H be a normal subgroup of finite index of G such that H/A is finite-by-abelian. Thus H/A' is still a counterexample to the proposition and we may assume that A is abelian and G/A is finite-by-abelian. Let N/A be a finite normal subgroup of G/A such that G/N is abelian, and let K be a permutable subgroup of G. Then K is normal in KA, the index $|KN : KA|$ is finite and KN is normal in G; it follows that there exists a positive integer n such that every permutable subgroup of G is subnormal of defect at most n. Hence, every subgroup of infinite rank of G has finite index in a subnormal subgroup of defect at most n. In particular, every subgroup of

infinite rank of G is subnormal. Therefore, every subgroup of G is subnormal in G ([46], Theorem 5). Let X be any subgroup of finite rank of G, then X is a Černikov group ([56], Corollary 1, p.38, part 2) and X is subnormal in XA, so by Lemma 2.3.5 A contains a subgroup $C = C_1 \times C_2$, with C_1 and C_2 of infinite rank and $X \cap C = \{1\}$, such that $X = XC_1 \cap XC_2$. As XC_i has infinite rank, for $i = 1, 2$, X has finite index in a subnormal subgroup of G of defect at most n. Thus there exists a finite normal subgroup K of G such that G/K is nilpotent ([28], Theorem 1) and G is nilpotent. Among all counterexamples to the proposition obtained in this way, choose a nilpotent group G with minimal nilpotency class $c > 1$.
If the centre $Z(G)$ of G has infinite rank, then $Z(G)$ contains a subgroup $Z_1 \times Z_2$, with Z_1 and Z_2 of infinite rank. Then G/Z_i is finite-by-abelian-by-finite, for $i = 1, 2$, and so the same holds for G, a contradiction. It follows that $Z(G)$ has finite rank and, by the minimality of c, $G/Z(G)$ is finite-by-abelian-by-finite. Thus, $Z(G) \cap \Omega_1(G)$ is finite and $G/(Z(G) \cap \Omega_1(G))$ is finite-by-abelian-by-finite, so that G is finite-by-abelian-by-finite and this last contradiction completes the proof of the proposition. ■

Lemma 2.3.6. *Let G be a group of infinite rank whose subgroups of infinite rank are nearly permutable. If G contains an elementary abelian normal p-subgroup A of finite index, then the commutator subgroup G' of G is finite.*

PROOF. Let H be any subgroup of infinite rank of G and let K be a permutable subgroup of G such that H has finite index in K. Then, K is normal in KA, KA has finite index in G and it follows from Proposition 3.3 of [26] that G' is finite. ■

Theorem 2.3.5. *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-quasihamiltonian.*

PROOF. By Proposition 2.3.4, G contains a finite normal subgroup N such that G/N is abelian-by-finite. Without loss of generality it can be assumed that $N = \{1\}$, so that G is abelian-by-finite. Let A be an abelian normal subgroup of finite index of G. First, suppose that G has infinite exponent. By Lemma 6 of [21], $\Omega_1(A)$ contains a direct product $Y_1 \times Y_2$ of G-invariant subgroups of infinite rank Y_1 and Y_2 and G/Y_i is finite-by-quasihamiltonian, for

$i = 1, 2$. Since Y_1 and Y_2 have finite exponent, it follows that G/Y_i is finite-by-abelian, for $i = 1, 2$. Hence, G is finite-by-abelian. So, we can suppose that G has finite exponent. Put $G = AE$, where E is a finite subgroup of G and let H be any subgroup of infinite rank of G. Then there exists a permutable subgroup K_1 of G such that $|K_1 : H|$ is finite. Let K be a permutable subgroup of G such that K_1E has finite index in K. It follows that $|K : H|$ is finite and $G = AK$. As a consequence, $|K : H_K|$ is finite and $K \cap A$ is a G-invariant subgroup of finite index of K. Hence, $H_K \cap A$ has finite index in K and, being a normal subgroup of H_KA, it is also normal in G. In particular, every subgroup of infinite rank of G is normal-by-finite, so that every subgroup of G is normal-by-finite ([21], Theorem C). Since A is a bounded abelian group, it is the direct product of cyclic subgroups and so it is clearly residually finite. Application of Lemma 2.1 of [17] yields that A contains a subgroup B of finite index such that every subgroup of B is G-invariant. Then B has finite index in G and, replacing A by B, we may assume that every subgroup of A is G-invariant.
Let $Y = Y_1 \times Y_2$ be a subgroup of A with Y_1 and Y_2 of infinite rank such that $E \cap Y = \{1\}$ and let K_i be a permutable subgroup of G such that $|K_i : EY_i|$ is finite, for $i = 1, 2$. Then $E = EY_1 \cap EY_2$ has finite index in $F = K_1 \cap K_2$, so that F is finite and $G = AF$. Without loss of generality we may assume that $E = K_1 \cap K_2$. Moreover, we can replace G with G/E_G, so that E is a core-free subgroup of G. In particular, $A \cap E = C_E(A) = \{1\}$, and E acts on A as a group of power automorphisms. If $p > 2$, then E is cyclic and by Lemma 2.3.4 of [60] G is locally quasihamiltonian and hence it is also quasihamiltonian. So, we can assume $p = 2$. If A has exponent 2, then G is finite-by-abelian by Lemma 2.3.6. So, we can suppose that the exponent of A is at least 4. Let U be a cyclic subgroup of order 4 of A, then $UK_i/C_{K_i}(U)$ has order at most 8 and, as K_i is permutable, it follows that $[U, K_i] \leq K_i$. Thus, $[U, E] = \{1\}$ and $U \leq Z(G)$. Hence, E is cyclic and, applying Lemma 2.3.4 of [60] again, we obtain that G is quasihamiltonian and so the theorem is completely proved. ■

The investigation of primary groups is completed and we can focus now to the general case of an arbitrary locally finite group.

Since a quasihamiltonian group is locally nilpotent, a periodic group G in which every subgroup is nearly permutable is finite-by-(locally nilpotent) and hence G is also (locally nilpotent)-by-finite. In order to prove the main theorem of this section, the first step is to show that a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable is (locally nilpotent)-by-finite.
Recall that the *Hirsch-Plotkin radical* of a group G is the largest locally nilpotent normal subgroup of G and it contains every locally nilpotent ascendant subgroup of G.

Lemma 2.3.7. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G contains a nilpotent normal subgroup A of infinite rank such that the commutator subgroup A' of A is finite and for every prime p the p-component of A is generated by elements of order p.*

PROOF. Let B be an abelian subgroup of infinite rank of G and let K be a permutable subgroup of G such that $|K : B|$ is finite. Then $|K : B_K|$ is finite and B_K is an abelian ascendant subgroup of infinite rank of G. Thus, B_K is contained in the Hirsch-Plotkin radical R of G and, in particular, R has infinite rank. Let $A = \mathrm{Dr}_p\Omega_1(R_p)$, where R_p is the p-component of R. Then A has infinite rank and every permutable subgroup of A is normal in A. It follows that every subgroup of infinite rank of A is nearly normal in A and A' is finite ([21], Theorem B) and the lemma is proved. ■

Lemma 2.3.8. *Let G be a locally finite group, and let S be a Sylow p-subgroup of G. If S is finite-permutable-finite in G, then $S/O_p(G)$ is finite.*

PROOF. Let H and K be subgroups of G such that the indeces $|H : S|$ and $|G : K|$ are finite and H is permutable in K. The core S_H of S in H is an ascendant p-subgroup of K and so it is contained in $O_p(K)$. Since S/S_H is finite, it follows that $S/O_p(K)$ is finite. Clearly, $O_p(G) = O_p(K) \cap K_G$ and, since G/K_G is finite, we have that also the factor group $S/O_p(G)$ is finite. ■

We put here a technical lemma that will be needed in the following.

Lemma 2.3.9. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are finite-permutable-finite such that every section H/K of G is finite-by-quasihamiltonian, when K has infinite rank. If G contains an abelian normal subgroup of infinite rank A such that for every prime p the p-component of A is elementary abelian, then one of the following conditions holds:*

1. *G is (locally nilpotent)-by-finite,*
2. *G contains a non-(locally nilpotent)-by-finite subgroup $M = QB$, where B is a normal elementary abelian p-subgroup of infinite rank of M and Q is a locally nilpotent p'-group of finite rank, for some prime p.*

PROOF. Assume that G is not (locally nilpotent)-by-finite. If for every prime p the p-component A_p of A has finite rank, then A contains a direct product $B_1 \times B_2$ of G-invariant subgroups of infinite rank B_1 and B_2. Then G/B_i is (locally nilpotent)-by-finite, for $i = 1, 2$, and also G is (locally nilpotent)-by-finite, a contradiction. It follows that for some prime p the rank of $B = A_p$ is infinite. Then there exists a normal subgroup H of finite index of G such that H/B is locally nilpotent. Therefore, H is not (locally nilpotent)-by-finite and we can replace G by H, so that we can assume that G/B is locally nilpotent. Moreover, as G/B is finite-by-quasihamiltonian, its primary components are nilpotent and so every finite subgroup of G/B is subnormal in G/B. Let x be any element of G of order p^n, for some positive integer n, then the p-subgroup $\langle x\rangle B$ is subnormal in G and so it is contained in $O_p(G)$. In particular, x belongs to $O_p(G)$ and, as a consequence, $P = O_p(G)$ is the unique Sylow p-subgroup of G. Clearly, $B \leq P$ and so G/P is a locally nilpotent p'-group. By contradiction, suppose that there exists a Sylow q-subgroup Q of infinite rank of G, with $q \neq p$. Then Q is finite-permutable-finite and $Q/O_q(G)$ is finite by Lemma 2.3.8. Thus, $O_q(G)$ has infinite rank and, as $P \cap O_q(G) = \{1\}$, G is (locally nilpotent)-by-finite, a contradiction. It follows that for every $q \neq p$, every Sylow q-subgroup of G has finite rank and G satisfies the minimal condition on q-subgroups. Therefore, by Lemma 2.5.10 of [29] every q-component of G/P is a Černikov group and so, in particular, G/P is countable. Hence, there exists a locally nilpotent p'-subgroup Q of G such that $G = QP$ ([29], Theorem 2.4.5). Since G/B is locally nilpotent, QB is normal in G, so that QB is not (locally nilpotent)-by-finite. If Q has infinite rank, then there exist subgroups H and K of QB such that the indeces $|H : Q|$ and $|QB : K|$ are finite and H is permutable in K. In particular, $K = H(K \cap B)$ and H is normal in K. It follows that $H \cap B$ is a finite normal subgroup of K and $K/(H \cap B)$ is the product of two (locally nilpotent)-by-finite normal subgroups. Hence, K is finite-by-(locally nilpotent)-by-finite and this implies that K is also (locally nilpotent)-by-finite. Since K has finite index in G, also G is (locally nilpotent)-by-finite, a contradiction. Thus, Q has finite rank and $M = QB$ is the required subgroup. ■

Proposition 2.3.5. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is (locally nilpotent)-by-finite.*

PROOF. By contradiction, assume that G is not (locally nilpotent)-by-finite. By Lemma 2.3.7, G contains a normal subgroup of infinite rank A such that A' is finite and for every prime p the p-component of A is generated by elements of order p. Thus, G/A' is still a counterexample and so, replacing G by G/A', we can suppose that A is abelian. Then it follows from Lemma 2.3.9 that G contains a non-(locally nilpotent)-by-finite subgroup $M = QB$, where B is a normal elementary abelian p-subgroup of infinite rank of M and Q is a locally nilpotent p'-group of finite rank, for some prime p. Without loss of generality we can replace G by M. Put $\pi = \pi(Q)$ and first suppose that π is a finite set. Then Q is a Černikov group. Let J be any quasicyclic subgroup of Q and let x be any element of J. It follows from Lemma 2.9 of [23] that JB and $X = \langle x \rangle B$ are normal subgroups of G. Moreover, by Lemma 2.3.6, $X' = [x, B]$ is finite. As a consequence, J is normal in $J[x, B]$ for every $x \in J$ and, hence, J is normal also in $J[J, B] = J^B$. Therefore the finite residual of Q is subnormal in G and $Q/O_\pi(G)$ is finite. As $O_\pi(G)B$ is contained in the Hirsch-Plotkin radical R of G, we have that G/R is finite. By this contradiction, the set π is infinite. Let $C = C_1 \times C_2$ be a subgroup of B, with C_1 and C_2 of infinite rank, and let K_i be a permutable subgroup of G such that $|K_i : C_i|$ is finite, for $i = 1, 2$. Then $K_1 \cap K_2$ is finite and, by Lemma 1.2.5 of [2], C has finite index in K_1K_2 and, it follows that the set $\sigma = \pi(K_1K_2)$ is finite. Put $Q = Q_\sigma \times Q_{\sigma'}$, then $Q_{\sigma'} \cap K_1K_2 = \{1\}$, so that $Q_{\sigma'}K_1 \cap Q_{\sigma'}K_2 = Q_{\sigma'}(K_1 \cap K_2)$. Let L_i be a permutable subgroup of G such that $Q_{\sigma'}K_i$ has finite index in L_i, for $i = 1, 2$. Then $Q_{\sigma'}$ has finite index in

$L = L_1 \cap L_2$. As $L \lhd LB$, there exists a normal subgroup N of $Q_{\sigma'}B$ such that $|N : Q_{\sigma'}|$ is finite. Hence, $N \cap B$ is finite and $Q_{\sigma'}B/(N \cap B)$ is locally nilpotent, so that $Q_{\sigma'}B$ is (locally nilpotent)-by-finite. On the other hand, as σ is a finite set, the previous argument shows that also $Q_\sigma B$ is (locally nilpotent)-by-finite. Thus, G is the product of its (locally nilpotent)-by-finite normal subgroups $Q_\sigma B$ and $Q_{\sigma'}B$ and this last contradiction completes the proof. ■

Lemma 2.3.10. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. If X is a subgroup of finite rank of G, then X is finite-by-quasihamiltonian.*

PROOF. Let A be an abelian subgroup of infinite rank of G such that $A \cap X = \{1\}$ and let L be any subgroup of X. As A has finite index in a permutable subgroup H of G, $H \cap X$ is finite and L has finite index $HL \cap X = L(H \cap X)$. Let K be a permutable subgroup of G such that $|K : HL|$ is finite, then $K \cap X$ is permutable in X and L has finite index in $K \cap X$. It follows that every subgroup of X is nearly permutable and X is finite-by-quasihamiltonian ([23], Theorem). ■

Next lemma is a generalization of Lemma 3.3 of [23]. We omit the proof since it is analogous to the proof contained in [23].

Lemma 2.3.11. *Let G be a periodic group and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of subgroups of G such that $\pi(E_n)$ is finite for every n, $\pi(E_n) \cap \pi(E_m) = \emptyset$ for $n \neq m$ and all subgroups of E_{n+1} are normalized by $\langle E_1, \dots, E_n\rangle$ for each positive integer n. If every E_n contains a non-permutable subgroup H_n, then the subgroup $H = \langle H_n \mid n \in \mathbb{N}\rangle$ is not nearly-permutable in G.*

We are now in a position to prove the main theorem of this section. First, we consider the locally nilpotent case.

Proposition 2.3.6. *Let G be a periodic locally nilpotent group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-quasihamiltonian.*

PROOF. Assume by contradiction that G is not finite-by-quasihamiltonian. Let n be a positive integer for which there exist n subgroups $E_1, \dots, E_n$ of G such that $\pi(E_i)$ is finite for every $i \leq n$, $\pi(E_i) \cap \pi(E_j) = \emptyset$ for $i \neq j$, every

E_i contains a non-permutable subgroup H_i of rank r_i and $r_i < r_{i+1}$ for every $i < n$. By Theorem 2.3.5 and Lemma 2.3.10, every primary component G_p of G is finite-by-quasihamiltonian. As the set $\pi = \pi(E_1) \cup \cdots \cup \pi(E_n)$ is finite, it follows that G_π is finite-by-quasihamiltonian and, hence, $G_{\pi'}$ contains a finite subgroup $\bar{E}_{n+1}$ and a subgroup $\bar{H}_{n+1}$ of $\bar{E}_{n+1}$ such that $\bar{H}_{n+1}$ is not permutable in $\bar{E}_{n+1}$. Let r_{n+1} be the rank of $\bar{H}_{n+1}$. If $r_n < r_{n+1}$, put $E_{n+1} = \bar{E}_{n+1}$ and $H_{n+1} = \bar{H}_{n+1}$. So, suppose that $r_{n+1} \leq r_n$ and put $\pi_{n+1} = \pi \cup \pi(\bar{E}_{n+1})$. As π_{n+1} is finite, $G_{\pi'_{n+1}}$ is not finite-by-quasihamiltonian and hence it has infinite rank, by Lemma 2.3.10. It follows that there exists a prime $p \notin \pi_{n+1}$ such that r_n is strictly less than the rank of G_p. In this case, put $E_{n+1} = \bar{E}_{n+1} \times G_p$ and $H_{n+1} = \bar{H}_{n+1} \times G_p$.
In both cases, we have that $\pi(E_{n+1})$ is finite, $\pi(E_i) \cap \pi(E_{n+1}) = \emptyset$ for $i \leq n$, H_{n+1} is not permutable in E_{n+1} and $r_n < r_{n+1}$. It follows from Lemma 2.3.11 that $H = \langle H_n \mid n \in \mathbb{N} \rangle$ is not nearly permutable in G and this is a contradiction, since H has infinite rank. ■

Proof of Theorem 2.3.2. By Lemma 2.3.5, G contains a locally nilpotent normal subgroup Q such that the index $|G : Q|$ is finite, so there exists a finite subgroup E of G such that $G = QE$. It follows from Theorem 1 of [41] that Q contains an abelian subgroup $A = A_1 \times A_2$ such that A_1 and A_2 are E-invariant subgroups of infinite rank and $A \cap E = \{1\}$. Let K_i be a permutable subgroup of G such that EA_i has finite index in K_i, for $i = 1, 2$. Then E has finite index in $K_1 \cap K_2$ and $K = K_1 \cap K_2$ is a finite subgroup of G such that $G = QK$. Replacing G with G/K_G, it can be assumed without loss of generality that K is core-free. In particular, $(K_1)_G \cap (K_2)_G = \{1\}$. Since $(K_i)^G/(K_i)_G$ is locally nilpotent, for $i = 1, 2$ ([60], Theorem 6.3.1), K^G is locally nilpotent. Then $G = QK^G$ is locally nilpotent and, by Proposition 2.3.6, G is finite-by-quasihamiltonian. ■

C. Almost permutable subgroups of infinite rank

The study of the periodic case is firstly restricted to primary groups and our first purpose is to show that a locally finite p-group of infinite rank in which every subgroup of infinite rank is almost permutable is abelian-by-finite and finite-by-quasihamiltonian.

Lemma 2.3.12. *Let G be a periodic locally nilpotent group. If there exists a positive integer n such that every subgroup of G is subnormal of defect at most n in a subgroup of finite index of G, then G is nilpotent.*

PROOF. Every subgroup of G is subnormal and by Lemma 1 of [18] there exist a subgroup K of finite index of G, a finite subgroup F of K and a positive integer m such that every subgroup of finite index of K containing F is subnormal in K of defect at most m. Then every subgroup of K containing F is subnormal in K of defect at most $m + n$ and K is nilpotent ([27], p.386). It follows that G is nilpotent-by-finite and, replacing K with K_G, we can assume that K is normal in G, so that there exists a finite subgroup E of G such that $G = KE$. In particular, E is a nilpotent subnormal subgroup of G and G is nilpotent. ■

In [51] and [52], Möhres proved that a group G in which every subgroup is subnormal is soluble and, in addition, if G is the extension of a periodic nilpotent group by a soluble group of finite exponent, then G is nilpotent. An easy argument by induction on the derived length shows that if G is a group generated by elements of finite bounded order and every subgroup of G is subnormal, then G is nilpotent. This remark will be used in the proof of the next theorem.

Theorem 2.3.6. *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are almost permutable. Then G is abelian-by-finite.*

PROOF. Assume by contradiction that G is not abelian-by-finite and put $A = \Omega_1(G)$. Then A has infinite rank and G/A is abelian-by-finite ([40], Theorem 4.13). Let B/A be an abelian normal subgroup of G/A such that B has finite index in G, then B is not abelian-by-finite and, replacing G with B, we may assume that G/A is abelian. Let H be a subgroup of infinite rank of G and let X be a subgroup of finite index of G such that H is permutable in X. Then $N = X_G$ has finite index in G and $\Omega_1(N)$ is a G-invariant subgroup of infinite rank, so that $G/\Omega_1(N)$ is nilpotent and, as $H \triangleleft H\Omega_1(N)$, H is subnormal in G. It follows that every subgroup of infinite rank of G is subnormal and, by Theorem 5 of [46], G has all its subgroups subnormal. In particular, A is a nilpotent group of finite exponent. Put $\bar{G} = G/A'$. If

A' has finite rank, then $\bar{G}$ has infinite rank and $\bar{A}$ is an elementary abelian p-group of infinite rank, containing $\bar{G}'$. It is easy to see that every subgroup of infinite rank of $\bar{G}$ is subnormal of defect at most 2 in a subgroup of finite index of $\bar{G}$. By Lemma 2.3.5 every subgroup of finite rank of $\bar{G}$ is the intersection of two subgroups of infinite rank, so that every subgroup of $\bar{G}$ is subnormal of defect at most 2 in a subgroup of finite index of $\bar{G}$ and, by Lemma 2.3.12, $\bar{G}$ is nilpotent. On the other hand, if A' has infinite rank, then $\bar{G}$ is nilpotent. Therefore G is a nilpotent group by a well-known result of P. Hall. Among all counterexamples to the theorem obtained in this way, choose a nilpotent group G with minimal nilpotency class $c > 1$.

If the centre $Z(G)$ of G has infinite rank, then $Z(G)$ contains a subgroup $Z_1 \times Z_2$, with Z_1 and Z_2 of infinite rank. Then G/Z_i is abelian-by-finite, for $i = 1, 2$, and so the same holds for G, a contradiction. It follows that $Z(G)$ has finite rank and, by the minimality of c, $G/Z(G)$ is abelian-by-finite. Thus, the subgroup $M = Z(G) \cap \Omega_1(G)$ is finite and G/M is abelian-by-finite. If B is a subgroup of finite index of G such that B/M is abelian, then B is still a counterexample and, replacing G with B, we may assume that G is a nilpotent group of nilpotency class 2 and G' has exponent p.

Let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank, and let X_i be a subgroup of finite index of G such that A_i is permutable in X_i, for $i = 1, 2$. Then $N = (X_1 \cap X_2)_G$ has finite index in G and $B_i = A_i \cap N$ is a permutable subgroup of infinite rank of N, for $i = 1, 2$. Application of Lemma 2.10 of [23] yields that $B_i/(B_i)_N$ is finite, so that $(B_i)_N$ has infinite rank and $N/(B_i)_N$ is abelian-by-finite, for $i = 1, 2$. Therefore, N is abelian-by-finite and so the same holds for G, a contradiction. This last contradiction completes the proof of the theorem. ■

We are now in a position to prove that if G is a locally finite p-group of infinite rank whose subgroups of infinite rank are almost permutable, then every subgroup of infinite rank of G is nearly permutable, so that G will be finite-by-quasihamiltonian as an application of Theorem 2.3.5.

Lemma 2.3.13. *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are almost permutable such that the centre $Z(G)$ of G has infinite rank. If H is a core-free subgroup of infinite rank of G, then H is nearly permutable.*

PROOF. Let $A = A_1 \times A_2$ be a subgroup of $Z(G)$ such that A_1 and A_2 have infinite rank. As $H \cap Z(G) = \{1\}$, $H = HA_1 \cap HA_2$. By Theorem 2.3.6, G is abelian-by-finite so that there exist an abelian normal subgroup B and a finite subgroup E of G such that $G = BE$. For any $i = 1, 2$, G/A_i is finite-by-quasihamiltonian ([40], Corollary 4.14), so EA_i/A_i is contained in a finite permutable subgroup F_i/A_i of G/A_i. It follows that $L_i = HA_iF_i$ is a subgroup of G and the index $|L_i : HA_i|$ is finite. Then, the index $|L_1 \cap L_2 : H|$ is finite and, as E is contained in $L_1 \cap L_2$, H has finite index in $K = \langle H, E\rangle$. Clearly, K has infinite rank and there exists a subgroup X of finite index of G such that K is permutable in X. Let C be a finite normal subgroup of G such that $B = C(B \cap X)$, then $G = XC$ and KC is permutable in G. It follows that H is nearly permutable in G. ∎

Theorem 2.3.7. *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are almost permutable. Then G is finite-by-quasihamiltonian.*

PROOF. Assume by contradiction that G is not finite-by-quasihamiltonian. By Theorem 2.3.6, G contains an abelian normal subgroup A of finite index. By contradiction, suppose that G has infinite exponent. By Lemma 6 of [21], $\Omega_1(A)$ contains a direct product $Y_1 \times Y_2$ of G-invariant subgroups of infinite rank Y_1 and Y_2 and G/Y_i is finite-by-quasihamiltonian, for $i = 1, 2$. Since Y_1 and Y_2 have finite exponent, it follows that G/Y_i is finite-by-abelian, for $i = 1, 2$. Hence, G is finite-by-abelian, a contradiction. Therefore G has finite exponent and G is nilpotent ([56], Lemma 6.34, part 2). Among all counterexamples to the theorem obtained in this way, choose a nilpotent group G with minimal nilpotency class $c > 1$.
If the centre $Z(G)$ has finite rank, then $Z(G)$ is finite and, by the minimality of c, $G/Z(G)$ is finite-by-quasihamiltonian, so the same holds for G, a contradiction. From this observation follows that in any counterexample of nilpotency class c the centre has infinite rank. Let H be any subgroup of infinite rank of G. If the core H_G of H in G has infinite rank, then G/H_G is finite-by-quasihamiltonian and H is nearly permutable in G. On the other hand, if H_G has finite rank, then H_G is finite and G/H_G is still a counterexample. By Lemma 2.3.13, H is nearly permutable in G. As a consequence, every subgroup of infinite rank of G is nearly permutable and G is finite-by-

quasihamiltonian by Theorem 2.3.5, a contradiction. This last contradiction completes the proof of the Theorem. ■

By Corollary 5.9 of [40], in a locally finite p-group of infinite rank whose subgroups of infinite rank are almost permutable, every subgroup is almost permutable.

Recall that the *Hirsch-Plotkin radical* of a group G is the largest locally nilpotent normal subgroup of G and it contains every locally nilpotent ascendant subgroup of G. It is easy to see that in a locally finite group G of infinite rank whose subgroups of infinite rank are almost permutable, the Hirsch-Plotkin radical R has infinite rank. In fact, let A be an abelian subgroup of infinite rank of G (see [62]) and let X be a subgroup of finite index of G such that A is permutable in X. Then, $A \cap X_G$ is an abelian ascendant subgroup of infinite rank of G and it is contained in R.

Lemma 2.3.14. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are almost permutable. If X is a p-subgroup of finite rank of G, then the factor group $X/Z(X)$ is finite.*

PROOF. Since X is a Černikov group ([56], Corollary 1, p.38, part 2), if J is its finite residual there exists a finite subgroup E of X such that $X = EJ$. By Theorem 1 of [41], the Hirsch-Plotkin radical of G contains an abelian subgroup $A = A_1 \times A_2$ such that A_1 and A_2 are E-invariant subgroups of infinite rank and $A \cap E = \{1\}$. Let X_i be a subgroup of finite index of G such that EA_i is permutable in X_i, for $i = 1, 2$ and put $N = (X_1 \cap X_2)_G$. Let $R = \mathrm{Dr}_p R_p$ be the Hirsch-Plotkin radical of N, where R_p is the unique Sylow p-subgroup of R. Since N has finite index in G, $B = \mathrm{Dr}_p \Omega_1(R_p)$ is a G-invariant subgroup of infinite rank such that EA_i is normalized by B, for $i = 1, 2$. In particular, $E = EA_1 \cap EA_2$ is normal in EB. On the other hand, the p-subgroup XB/B of G/B is nilpotent, so that EB is subnormal in XB and E is subnormal in X. It follows that X is a Černikov nilpotent group and $X/Z(X)$ is finite. ■

Lemma 2.3.15. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are almost permutable. Then G contains an abelian normal subgroup*

A of infinite rank such that for every prime p the p-component of A is elementary abelian.

PROOF. Let $R = \mathrm{Dr}_p R_p$ be the Hirsch-Plotkin radical of G, where R_p is the unique Sylow p-subgroup of R. Since R has infinite rank, the subgroup $A = \mathrm{Dr}_p \Omega_1(R_p)$ is a G-invariant subgroup of infinite rank. Let H be any subgroup of infinite rank of A and let X be a subgroup of finite index of A such that H is permutable in X. Then there exists a finite subset π of $\pi(R)$ such that $A_{\pi'}$ is contained in X and so $H \triangleleft HA_{\pi'}$. Let $p \in \pi$. If A_p has infinite rank, then A_p is abelian-by-finite and finite-by-quasihamiltonian by Theorem 2.3.6 and Theorem 2.3.7 and hence A_p is also finite-by-abelian. In particular, the centre of A_p has finite index in A_p. On the other hand, the same conclusion holds if A_p has finite rank, by Lemma 2.3.14. It follows that the centre of A_π has finite index in A_π and, as a consequence, $H \cap A_\pi$ is almost normal in A. Thus, $H = (H \cap A_\pi) \times (H \cap A_{\pi'})$ is almost normal in A and by Theorem A of [21] the factor group $A/Z(A)$ is finite. Then, the socle S of $Z(A)$ is a G-invariant subgroup of infinite rank whose primary components are elementary abelian and the lemma is proved. ■

Since a quasihamiltonian group is locally nilpotent, a periodic group in which every subgroup is almost permutable is (locally nilpotent)-by-finite and we will show that this holds requiring that only the subgroups of infinite rank are almost permutable.

Proposition 2.3.7. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are almost permutable. Then G is (locally nilpotent)-by-finite.*

PROOF. By contradiction, assume that G is not (locally nilpotent)-by-finite. By Lemma 2.3.15, G satisfies the hypothesis of Lemma 2.3.9, so that G contains a non-(locally nilpotent)-by-finite subgroup $M = QB$, where B is a normal elementary abelian p-subgroup of infinite rank of M and Q is a locally nilpotent p'-group of finite rank, for some prime p. Without loss of generality we can replace G with M and put $\pi = \pi(Q)$. First, suppose that π is a finite set. Let P be a quasicyclic subgroup of Q and let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank, such that $A \cap P = \{1\}$. Then $P = PA_1 \cap PA_2$ is ascendant in a subgroup of finite index of G and hence P is also ascendant in G. Since π is finite, Q is a Černikov group ([56], Corollary 1, p.38, part 2) and, by the previous argument, its finite residual is ascendant in G. In particular, $Q/O_\pi(G)$ is finite. As $O_\pi(G)B$ is contained in the Hirsch-Plotkin radical R of G, we have that G/R is finite, a contradiction. It follows that π is infinite. Let $C = C_1 \times C_2$ be a subgroup of B, with C_1 and C_2 of infinite rank and let K_i be a subgroup of finite index of G such that C_i is permutable in K_i, for $i = 1, 2$. If $N = (K_1 \cap K_2)_G$, then $C_1 \cap N$ and $C_2 \cap N$ are permutable subgroups of infinite rank of N, so that, without loss of generality, we can assume that C_1 and C_2 are both permutable in a same normal subgroup of finite index N of G. Put $\sigma = \pi(G/N)$, then $Q_{\sigma'}$ is contained in N. Let Y_i be a subgroup of finite index of G such that $Q_{\sigma'}C_i$ is permutable in Y_i, for $i = 1, 2$. It follows that $L = B \cap (Y_1 \cap Y_2)$ is a subgroup of finite index of B such that $Q_{\sigma'} = Q_{\sigma'}C_1 \cap Q_{\sigma'}C_2$ is normal in $Q_{\sigma'}L$. In particular, $Q_{\sigma'}L$ is locally nilpotent and it has finite index in $Q_{\sigma'}B$ ([2], Lemma 1.2.5). On the other hand, as σ is a finite set, $Q_\sigma B$ is (locally nilpotent)-by-finite from the previous argument. Thus, G is the product of its (locally nilpotent)-by-finite normal subgroups $Q_\sigma B$ and $Q_{\sigma'}B$ and this last contradiction completes the proof of the proposition. ■

We are now in a position to prove the main theorem. First, we consider the locally nilpotent case.

Proposition 2.3.8. *Let G be a periodic locally nilpotent group of infinite rank whose subgroups of infinite rank are almost permutable. Then G is finite-by-quasihamiltonian.*

PROOF. Let H be any subgroup of infinite rank of G and let X be a subgroup of finite index of G such that H is permutable in X. Put $\pi = \pi(G/X_G)$, then $G_{\pi'}$ is contained in X and $H \cap G_{\pi'}$ is a permutable subgroup of G. On the other hand, as π is finite, G_π is finite-by-quasihamiltonian by Theorem 2.3.7 and Lemma 2.3.14, so that $H \cap G_\pi$ is nearly permutable in G. It follows that $H = (H \cap G_\pi) \times (H \cap G_\pi)$ is nearly permutable in G and the assertion follows from Theorem 2.3.2. ■

Corollary 2.3.5. *Let G be a periodic locally nilpotent group of infinite rank whose* *subgroups of infinite rank are almost permutable. Then G is quasihamiltonian-by-finite*

PROOF. By Proposition 2.3.8, G contains a finite normal subgroup N such that G/N is quasihamiltonian. Put $\pi = \pi(N)$, then $G_{\pi'}$ is quasihamiltonian. On the other hand, as π is finite, G_π is abelian-by-finite by Theorem 2.3.6 and Lemma 2.3.14. Hence, $G = G_\pi \times G_{\pi'}$ is quasihamiltonian-by-finite. ■

Proof of Theorem 2.3.3. By Proposition 2.3.7, G is (locally nilpotent)-by-finite and by Corollary 2.3.5, G is also quasihamiltonian-by-finite. Let Q be a quasihamiltonian normal subgroup of finite index of G and put $G = QE$, where E is a finite subgroup of G. By Theorem 1 of [41], Q contains an abelian subgroup $A = A_1 \times A_2$ such that A_1 and A_2 are E-invariant subgroups of infinite rank and $A \cap E = \{1\}$. It follows that $E = EA_1 \cap EA_2$ is ascendant in a subgroup of finite index X of G. Put $\pi = \pi(G/X_G) \cup \pi(E)$, then $Q_{\pi'}$ is contained in X and $[Q_{\pi'}, E] = \{1\}$. Hence, $G = Q_{\pi'} \times (Q_\pi E)$, where $Q_{\pi'}$ is quasihamiltonian and $Q_\pi E$ is abelian-by-finite by Theorem 2.3.6 and Lemma 2.3.14. By Theorem B of [40], it is enough to prove that $Q_\pi E$ is finite-by-quasihamiltonian. If $Q_\pi E$ has finite rank, then $Q_{\pi'}$ has infinite rank and every subgroup of $Q_\pi E \simeq G/Q_{\pi'}$ is almost permutable. In particular, $Q_\pi E$ is finite-by-quasihamiltonian by Corollary 5.9 of [40]. Thus, we can assume that $Q_\pi E$ has infinite rank and, without loss of generality, we can replace G with $Q_\pi E$, and assume that G contains an abelian subgroup B of finite index. Let H be any subgroup of finite rank of G, then by Lemma 6 of [21], B contains a direct product $B_1 \times B_2$ of G-invariant subgroups of infinite rank B_1 and B_2 such that $H \cap B_1B_2 = \{1\}$. Then $H = HB_1 \cap HB_2$ is ascendant in a subgroup of finite index of G. Thus, by Theorem A of [47], G is finite-by-(locally nilpotent) and application of Proposition 2.3.8 yields that G is also finite-by-quasihamiltonian. ■

CHAPTER 3

SOME FURTHER PROBLEMS ON PERMUTABILITY

3.1 Polycyclic groups with permutability conditions on finite homomorphic images

If G is a polycyclic group, it is known that its finite homomorphic images have a strong influence on the structure of the whole group. The first result on this topic was obtained by Hirsch [43], who proved that if all the finite homomorphic images of a polycyclic group are nilpotent, then the group itself is nilpotent. Later, Baer [5] proved an analogous theorem, where nilpotency is replaced by supersolubility.

A famous theorem of Mal'cev [50] states that in a polycyclic group any subgroup is closed in the profinite topology, so it is easy to see that a polycyclic group is a T-group if and only if its finite quotients are T-groups. A corresponding result for PT-groups can be obtained using a theorem of Lennox and Wilson [48], which states that a subgroup H of a polycyclic group G is permutable if H^σ is permutable in G^σ for every finite homomorphic image G^σ of G.

As any torsion-free polycyclic group is trivially a PST-group, the requirement that the finite quotients of a polycyclic group are PST-groups can not just have as a consequence that the group itself is a PST-group. Robinson [59] studied this problem in a more general universe, obtaining the following classification.

Theorem 3.1.1 (Robinson [59])**.** *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is a PST-group if and only if G is one of the following:*

(a) A finite soluble PST-group.

(b) A nilpotent group.

(c) A group of infinite dihedral type.

According to [59], a group G is said to be of *infinite dihedral type* if the hypercenter $Z_\infty(G)$ of G is a finite 2-group and the factor group $G/Z_\infty(G)$ is isomorphic with the dihedral group on a finitely generated, infinite abelian group containing no involutions. For the convenience of the reader, we put here the following useful characterization of this kind of groups.

Lemma 3.1.1 (Robinson [59])**.** *A group G is of infinite dihedral type if and only if it contains an abelian normal subgroup A such that the following conditions are satisfied:*

(a) A is finitely generated, infinite and it contains no involutions.

(b) G/A is a finite 2-group and the centralizer $C_G(A)$ has index 2 in G.

(c) every element of $G \setminus C_G(A)$ induce inversion in A.

The aim of this section is to obtain similar results for polycyclic groups whose finite quotients belong to one of the classes of generalized quasi-hamiltonian groups described in Chapter 1. All the following results have been proved in [9] by Ballester-Bolinches, Beidleman and Ialenti.

Let $\mathfrak{X}$ be one of the classes SST, BT, SNT, SN, SP or SPS. In order to extend Robinson's Theorem to finitely generated hyperabelian groups with all finite quotients belonging to $\mathfrak{X}$, we first need to study the finite quotients of groups of infinite dihedral type. The next lemma shows that their structure is quite transparent.

Lemma 3.1.2 (see [59])**.** *Let G be a group of infinite dihedral type. If G/N is a finite quotient of G and L/N is the nilpotent residual of G/N, then L/N is a Hall $2'$-subgroup of G/N.*

PROOF. By Lemma 3.1.1, G/N has a normal abelian subgroup T/N such that G/T is a 2-group, $|G/N : C_{G/N}(T/N)| = 2$ and the elements in $G/N \setminus$

$C_{G/N}(T/N)$ induce inversion in T/N. Let $B/N = O_{2'}(T/N)$. Then G/B is a 2-group so that the nilpotent residual L/N of G/N is contained in B/N. On the other hand, since B/N has odd order and the elements of $G/N \setminus C_{G/N}(T/N)$ induce inversion in B/N, we have that $B/N \leq L/N$. Then $B/N = L/N$ is the nilpotent residual of G/N. ∎

Next, we observe that all the finite quotients of a group of infinite dihedral type are BT-groups.

Lemma 3.1.3. *If G is a group of infinite dihedral type, then all its finite quotients are BT-groups.*

PROOF. By Lemma 3 of [59], every finite quotient of G is a soluble PST-group. Furthermore, if G/N is a finite quotient and L/N is the nilpotent residual of G/N, then G/L is a 2-group by Lemma 3.1.2. Therefore G/N is a BT-group by Theorem 1.2.3. ∎

In [59], it is showed that a group of infinite dihedral type is not a PST-group. However, the class of all infinite dihedral type is a subclass of the class of all BT-groups. In fact, we have:

Lemma 3.1.4. *If G is a group of infinite dihedral type, then every subgroup of G is semipermutable. In particular, G is a BT-group.*

PROOF. Applying Lemma 3.1.1, G has a normal finitely generated, infinite abelian subgroup A containing no involutions, such that G/A is a finite 2-group, $|G : C_G(A)| = 2$ and the elements in $G \setminus C_G(A)$ induce inversion in A. In particular, every subgroup of A is normal in G and, if D is the torsion subgroup of A then $\pi(G) = \{2\} \cup \pi(D)$. Clearly, if A is torsion-free, every subgroup of G is semipermutable. Hence, we may assume that D is not trivial. Let x, y be elements of G of order p^α and q^β respectively, with p and q different prime numbers. Since p and q are different, one of them must belong to $\pi(D)$. Assume $p \in \pi(D)$, so that $x \in A$. In this case, $\langle x \rangle$ is a normal subgroup of G and therefore $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. ∎

Since in a group of infinite dihedral type all finite quotients are BT-groups, we can prove that actually its finite quotients are SST-groups using Theorem 1.2.5.

Lemma 3.1.5. *Let G be a finite group with a normal abelian subgroup A such that:*

(a) *G/A is a 2-group and $|G : C_G(A)| = 2$,*

(b) *elements in $G \setminus C_G(A)$ induce inversion in A.*

Then G is an SST-group.

PROOF. By Theorem 1.2.3, G is a BT-group. Therefore $G = L \rtimes M$, where $M \in \mathrm{Syl}_2(G)$ and L the nilpotent residual of G. In particular, $\pi(G) \setminus \pi(L) = \{2\}$. Clearly, $C_G(A) \leq C_G(L)$. If $G = C_G(L)$ then G is nilpotent and so G is an SST-group. Hence we may assume that G is not nilpotent and $C_G(A) = C_G(L)$.
First, suppose that $O_2(G) = 1$. Since $M \cap C_G(L)$ is a normal 2-subgroup of M of G, it follows that $M \cap C_G(L) = \{1\}$ and M has order 2. If P is a 2-subgroup of G, then $P \in \mathrm{Syl}_2(G)$ and choosing $K_2 = 1$ we have that $[P, K_2^L] \leq O_2(G)$.
Now suppose that $O_2(G)$ is not trivial. Since the factor group $G/O_2(G)$ satisfies the same hypothesis of G, it follows from the previous argument that $|M : O_2(G)| = 2$. Let P a 2-subgroup of G. Without loss of generality, we may assume that $P \leq M$. If $M = PO_2(G)$ then we may choose $K_2 = O_2(G)$ and clearly $[P, K_2^L] \leq O_2(G)$. Otherwise, if $P \leq O_2(G)$, let $K_2 = M$, then $[P, K_2^L] \leq [O_2(G), M^L] \leq O_2(G)$. ∎

Corollary 3.1.1. *Let G be a group of infinite dihedral type. Then every finite homomorphic image of G is an SST-group.*

Theorem 3.1.2. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is an SST-group if and only if G is one of the following:*

(a) *A finite soluble SST-group;*

(b) *A nilpotent group;*

(c) *A group of infinite dihedral type.*

PROOF. Since any finite soluble SST-group is a PST-group, if every finite quotient of G is an SST-group, then the result follows by Robinson's theorem.

Conversely, if G is finite or nilpotent, then trivially G is an SST-group. If G is a group of infinite dihedral type, the assertion follows from Corollary 3.1.1. ■

Bearing in mind the relation between the classes SST, BT, SNT, SN, SP and SPS-group in the finite universe, the following theorem is a direct consequence of Robinson's result and Theorem 3.1.2.

Theorem 3.1.3. *Let $\mathfrak{X}$ be one of the classes BT, SNT, SN, SP or SPS and let G be a finitely generated hyperabelian group. Then every finite quotient of G is an $\mathfrak{X}$-group if and only if G is one of the following:*

1. *a finite soluble $\mathfrak{X}$-group;*
2. *a nilpotent group;*
3. *a group of infinite dihedral type.*

We bring the section to a close by studying the finitely generated hyperabelian MS-groups.

Lemma 3.1.6. *If G is a group of infinite dihedral type, then all its finite quotients are MS-groups.*

PROOF. Let G/N be a finite quotient of G and let L/N be the nilpotent residual of G/N. By Lemma 3 of [59], G/N is a soluble PST-group and, by Lemma 3.1.2, L/N is a normal abelian Hall $2'$-subgroup of G/N. Then conditions (iv) and (v) of Theorem 3.1 of [7] are trivially satisfied and then G/N is an MS-group ([7], Theorem 3.2). ■

In the proof of the main theorem for MS-groups, we used some results showed in [42] about polycyclic groups whose finite quotients are T_0-groups. Here, a finite group G is called a T_0-group if the factor group $G/\Phi(G)$ over the Frattini subgroup is a T-group.

Theorem 3.1.4. *Let G be a polycyclic group. Then every finite quotient of G is an MS-group if and only if G is one of the following:*

- *a finite soluble MS-group;*
- *a nilpotent group;*

- *a group of infinite dihedral type.*

PROOF. Assume that every finite quotient of G is an MS-group. If $G/\Phi(G)$ is abelian, then any maximal subgroup of G is normal, so that any finite quotient of G is a nilpotent group and G itself if nilpotent ([43]).
If $G/\Phi(G)$ is finite, then only finitely many primes are possible for the indices of maximal subgroups and so G has no infinite abelian factors. Then G is finite. In particular, G is a finite MS-group.
Hence we may assume that G is an infinite polycyclic group, such that the Frattini quotient group $G/\Phi(G)$ is an infinite non-abelian group. By Theorem C of [16], any finite MS-group is a T_0-group. Hence, G is the semidirect product of an abelian group A by a cyclic group $\langle t \rangle$ of order 2 ([42], Theorem C). Let N be a normal subgroup of G of finite index. Since A is a maximal subgroup of G, we have that $G = AN$ or $N \leq A$. In the first case, the quotient G/N is abelian. So assume that $N \leq A$. In this case, $G/N = A/N \rtimes \langle t \rangle N/N$ is a finite T_0-group whose nilpotent residual is abelian. By Lemma 4 of [16], G/N is a PST-group. Thus, all finite quotients of G are PST-groups and therefore, by Theorem of [59], G is either nilpotent or a group of infinite dihedral type. ■

Theorem 3.1.5. *Let G be a finitely generated hyperabelian group whose finite quotients are MS-groups. Then G is polycyclic.*

PROOF. We follow the proof of the theorem of [59] and use the same notation. We may assume, arguing by contradiction, that G is just non- polycyclic. Note that the group $\bar{G}$ obtained there is, in our case, a finite MS-group whose nilpotent residual is abelian. Since $\bar{G}$ is a T_0-group, it follows that $\bar{G}$ is a PST-group by Lemma 4 of [16]. Then the contradiction follows as in Theorem [59]. ■

Corollary 3.1.2. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is an SST-group if and only if G is one of the following:*

(a) A finite soluble MS-group.

(b) A nilpotent group.

(c) A group of infinite dihedral type.

A proof of Robinson's theorem

We propose here an alternative proof for Robinson's Theorem. Our proof depends on Theorem D of [42] and the following lemma.

Lemma 3.1.7. *Let G be an infinite non-nilpotent supersoluble group whose finite quotients are PST-groups. Then the hypercentre $Z_\infty(G)$ of G does not have finite index in G.*

PROOF. Applying 5.4.10 of [58], G/F is a finite abelian group. Assume that $G/Z_\infty(G)$ is finite. Then, by Theorem 4.21 of [56], there exists a positive integer k such that $\gamma_k(G) = \gamma_\infty(G)$ is finite. Since G is not nilpotent, there exists a prime p and a positive integer i such that G/F^{p^i} is not nilpotent (otherwise, $\gamma_\infty(G)$ would be contained in every p'-component of F by Lemma 1 of[59] and G would be nilpotent). Then G/F^{p^j} is not nilpotent for all $j \geq i$. Let N_j be a normal subgroup of finite index in G which is maximal with respect to $F \cap N_j = F^{p^j}$. Then G/N_j is not nilpotent. By Theorem 2.1.8 of [3], the nilpotent residual of G/N_j is an abelian Hall subgroup of G/N_j contained in FN_j/N_j with non-central chief factors. Hence $Z_\infty(G) \leq N_j$ and so $Z_\infty(G) \leq F \cap N_j = F^{p^j}$. Thus $Z_\infty(G) \leq \bigcap_{j>i} F^{p^j}$ which is a finite subgroup of F by Lemma 1 of [59]. This contradiction proves the lemma. ∎

Theorem 3.1.6. *Let G be an infinite polycyclic group and let F be its Fitting subgroup. If every finite quotient of G is a PST-group, then either G is nilpotent or the following conditions are satisfied:*

(a) $Z_\infty(G)$ *is a* 2*-group.*

(b) $F/Z_\infty(G)$ *is an abelian group containing no involutions.*

(c) $|G : F| = 2$ *and every element of* $G \setminus F$ *induces inversion in* $F/Z_\infty(G)$.

PROOF. Assume that G is not nilpotent. Since any finite PST-group is supersoluble, G is supersoluble by a result of Baer [5]. By 5.4.10 of [58], G/F is a finite abelian group.
Let N be any normal subgroup of G of finite index and let $\bar{G} = G/N$. Then $\bar{G}$ is a PST-group. By Theorem 2.1.8 of [3], the nilpotent residual $\bar{A}$ of $\bar{G}$ is an abelian Hall subgroup of G. Note that $\bar{A} \leq \bar{G}' \leq \bar{F}$. We have that $\bar{A}$ is

complemented by a Carter subgroup $\bar{D}$ of $\bar{G}$ ([32], IV, Theorem 5.18). Put $\bar{C} = \bar{F} \cap \bar{D}$, then $\bar{F} = \bar{A} \times \bar{C}$ and $\bar{C} \leq C_{\bar{D}}(\bar{A})$. Therefore $\bar{C}$ is contained in the hypercentre $Z_\infty(\bar{G})$ of $\bar{G}$ ([32], IV, Theorem 6.14) and $\bar{G}/\bar{C}$ is a T-group. Hence $\bar{G}/Z_\infty(\bar{G})$ is a T-group and $\bar{G}$ is a T_1-group.
Applying Lemma 3.1.7, $G/Z_\infty(G)$ is infinite. By Theorem D of [42], $G/Z_\infty(G)$ is an extension of an abelian group $A/Z_\infty(G)$ containing no involutions by a cyclic subgroup of order 2 such that the elements of $G \setminus A$ inverts all the elements in $A/Z_\infty(G)$. Since $A/Z_\infty(G)$ is abelian, A is nilpotent and $A = F$. In particular, the 2-component F_2 of F is contained in $Z_\infty(G)$ and $|G : F| = 2$. Applying 5.2.10 of [58], G/F' is not nilpotent and F/F' is the Fitting subgroup of G/F'. By 5.2.6 of [58], F/F' is infinite. Let p be an odd prime and assume that G/F^{p^i} is nilpotent for some positive integer i. Let $j > i$. Since the nilpotent residual of G/F^{p^j} is a Hall subgroup of G/F^{p^j} contained in F^{p^i}/F^{p^j}, it follows that G/F^{p^j} is nilpotent. Let N_j be a normal subgroup of finite index of G which is maximal with respect to $F \cap N_j = F'F^{p^j}$. Then G/N_j is nilpotent. Since FN_j/N_J is a p-group, it follows that G/N_j is a p-group. Moreover, $G = FN_j$. Since F/F' is infinite, F/F' contains a non-trivial torsion-free subgroup M/F' such that M is a normal subgroup of G and G/M is finite. If $j \geq i$, $[M, G] = [M, N_j] \leq M \cap N_j = M^{p^j}(\mathrm{mod}\ F')$. Then $[M/F', G/F']$ is finite by Lemm 1 of [59]. Hence $[M/F', G/F'] = 1$ and $M/F' \leq Z(G/F')$. Then $G/F'/Z(G/F')$ is finite, contrary to Lemma 3.1.7.
Therefore G/F^{p^i} is a non-nilpotent PST-group and F/F^{p^i} is the nilpotent residual of G/F^{p^i} for each odd prime p and positive integer i. Since G/F^{p^i} acts on F/F^{p^i} as a power automorphisms by conjugation ([3], Theorem 2.1.8) and the only power automorphism of order 2 is the inversion, the elements of $G \setminus F$ invert all the elements of F/F^{p^i}. Since $\bigcap_{r>2}\left(\bigcap_{i>0} F^{r^i}\right) = \bigcap_{r>2} F_{r'} = F_2$ by Lemma 2 of [59], it follows that every element of $G \setminus F$ induces inversion on F/F_2 so that $Z(G/F_2) = 1$ and $F_2 = Z_\infty(G)$. ∎

3.2 Groups with finite abelian section rank factorized by mutually permutable subgroups

If a group $G = AB$ is the product of two abelian subgroups A and B, a famous theorem of N. Itô shows that G is metabelian (see [2], Theorem 2.1.1).

This result has been proved by means of a surprisingly short and elementary commutator computation. However, there are only few statements on factorized groups which can be proved without further assumptions on the factors. The situation is much easier to control when the two factors are normal subgroups, as Fitting's theorem and more generally the consideration of Fitting classes show. Although finite supersoluble groups do not form a Fitting class, it was proved by R. Baer [5] that if a finite group G is the product of two supersoluble normal subgroups and its commutator subgroup G' is nilpotent, then G is supersoluble. This result was later improved by M. Asaad and A. Shaalan [4], who were able to show that if a finite group $G = AB$ is factorized by two supersoluble mutually permutable subgroups A and B, then G itself is supersoluble, provided that G' is nilpotent. Here two subgroups A and B of a group G are said to be *mutually permutable* if $AY = Y$A and $XB = BX$ for all subgroups X of A and Y of B; of course any two normal subgroups are mutually permutable. The structure of a product of two mutually permutable subgroups has been recently investigated by several authors, especially in the finite case (we refer to chapters 4 and 5 of the monograph [3] for problems and results on this subject; for the case of infinite groups see also [13] and [20]). In particular, J.C. Beidleman and H. Heineken [14] proved that if a finite group $G = AB$ is the product of its mutually permutable subgroups A and B, then A' and B' are subnormal in G. The arguments used in the proof of this result cannot be adapted to the infinite case, and the aim of this section is to obtain information of a similar type for infinite groups factorized by mutually permutable subgroups. Our first main theorem is an extension of this result to the case of Černikov groups.

Theorem 3.2.1 (de Giovanni, Ialenti [38])**.** *Let $G = AB$ be a Černikov group which is factorized by two mutually permutable subgroups A and B. If A' and B' are finite, then they are subnormal in G.*

It follows clearly from the theorem of Beidleman and Heineken that if the finite group G is factorized by two mutually permutable subgroups A and B such that A' and B' are nilpotent, then the normal closure $\langle A', B'\rangle^G$ is likewise nilpotent. We will prove a corresponding statement for soluble-by-finite groups with finite abelian section rank.

Recall that a group has *finite abelian section rank* if it has no infinite abelian sections of prime exponent. Thus, every primary locally finite group with finite abelian section rank satisfies the minimal condition on abelian subgroups and hence is a Černikov group (see [56] Part 1, Theorem 3.32); it follows that any locally finite group with finite abelian section rank satisfies the condition min-p for all prime numbers p.

Theorem 3.2.2 (de Giovanni, Ialenti [38])**.** *Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable finite-by-nilpotent subgroups A and B. If A' and B' are locally nilpotent, then also the normal closure $\langle A', B'\rangle^G$ is locally nilpotent.*

Since every locally nilpotent group with finite abelian section rank is hypercentral (see [56] Part 2, p.38), and in particular all its subgroups are ascendant, Theorem 3.2.2 has the following consequence.

Corollary 3.2.1. *Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable finite-by-nilpotent subgroups A and B. If A' and B' are locally nilpotent, then they are ascendant in G.*

The monograph [2] can be used as a general reference on products of groups.

Recall that two subgroups A and B of a group G are said to be *totally permutable* if $XY = YX$ for all subgroups X of A and Y of B. It can be proved that if A and B are mutually permutable and $A \cap B = \{1\}$, then A and B are also totally permutable (see [3], Proposition 4.1.16). This result can be improved in the following way.

Lemma 3.2.1. *Let the group $G = AB$ be the product of its mutually permutable subgroups A and B. If N is a normal subgroup of G containing $A \cap B$, then the subgroups AN/N and BN/N of G/N are totally permutable.*

PROOF. Let X/N and Y/N be subgroups of AN/N and BN/N, respectively. Then $X = N(A \cap X)$ and $Y = N(B \cap Y)$. Moreover, $A \cap B$ is contained in $A \cap X$ and $B \cap Y$, and hence

$$(A \cap X)(B \cap Y) = (B \cap Y)(A \cap X)$$

(see [3], Proposition 4.1.16). It follows that $XY = YX$, and in particular AN/N and BN/N are totally permutable. ■

Proof of Theorem 3.2.1 Assume for a contradiction that the statement is false, and choose a counterexample G such that the finite residual J of G has smallest total rank r (recall that the total rank of J is the sum of the ranks of the primary component of J). Since A and B are Černikov groups with finite commutator subgroup, the indeces $|A : Z(A)|$ and $|B : Z(B)|$ are finite, so that the central subgroup $C = Z(A) \cap Z(B)$ of G has finite index in $A \cap B$. Clearly, the factor group G/C is also a counterexample, and hence without loss of generality we may suppose that $A \cap B$ is finite. Moreover, as the subgroup $\langle Z(A), Z(B) \rangle$ has finite index in G, also the index $|G : C_G(A \cap B)|$ is finite, and so the normal closure $N = (A \cap B)^G$ is finite. Let $J(A)$ and $J(B)$ the finite residuals of A and B, respectively. Then $J(A)N/N$ is the finite residual of AN/N and $J(B)N/N$ is the finite residual of BN/N. Application of Lemma 3.2.1 yields that the factor group G/N is the product of its totally permutable subgroups AN/N and BN/N, and hence it follows that $J(A)N$ and $J(B)N$ are normal subgroups of G. Clearly, $J(A)$ is the finite residual of $J(A)N$, and so it is normal in G. Similarly, $J(B)$ is a normal subgroup of G. Assume first that $J(A) \neq \{1\}$. Then the finite residual $J/J(A)$ of $G/J(A)$ has total rank less than r, so that $A'J(A)$ is subnormal in G by the minimal assumption on r, and hence A' is subnormal in G. Suppose now that $J(A) = \{1\}$, i.e. A is finite and $J = J(B)$. If P is any subgroup of type p^∞ of B, then P coincides with the finite residual of $AP = PA$ and hence it is normal in G. Thus the factor group $G/C_G(J)$ is abelian, so that G' centralizes J and in particular A' is normal in $A'J$. As $A'J/J$ is a subnormal subgroup of the finite factorized group G/J, it follows that A' is subnormal in G. A similar argument shows that also B' is a subnormal subgroup of G, and this contradiction completes the proof. ■

If G is any group, the intersection of all normal subgroups N of G such that the factor group is locally nilpotent is called the *locally nilpotent residual* of G. Of course, if G is finite-by-(locally nilpotent), then its locally nilpotent residual M is finite and the group G/M is locally nilpotent.

Lemma 3.2.2. *Let G be a group, and let K be a normal subgroup of G such that K^σ is nilpotent for every homomorphism σ of G onto a finite group G^σ. If K is finite-by-(locally nilpotent), then K is locally nilpotent.*

PROOF. Assume for a contradiction that the statement is false, and choose a counterexample G such that the locally nilpotent residual M of K has smallest order. Then M is a minimal normal subgroup of G. As M is finite, the group $G/C_G(M)$ is also finite, so that $K/C_K(M)$ is nilpotent and hence M is abelian of prime exponent p. Moreover, $[M, K]$ is a non-trivial normal subgroup of G, and hence $[M, K] = M$. It follows that G contains a subgroup L such that $G = ML$ and $M \cap L = \{1\}$ (see [2], Theorem 5.3.11). Clearly, the centralizer $C_L(M)$ is a normal subgroup of G and the factor group $G/C_L(M)$ is finite. Therefore $KC_L(M)/C_L(M)$ is nilpotent, and so K is locally nilpotent. This contradiction proves the lemma. ■

Corollary 3.2.2. *Let the group $G = AB$ be the product of its mutually permutable subgroups A and B. If the subgroups A' and B' are locally nilpotent and the normal closure $\langle A', B'\rangle^G$ is finite-by-(locally nilpotent), then $\langle A', B'\rangle^G$ is locally nilpotent.*

PROOF. Put $W = \langle A', B'\rangle^G$, and let G^σ be any finite homomorphic image of G. Then the nilpotent subgroups $(A')^\sigma$ and $(B')^\sigma$ are subnormal in G^σ, and hence

$$W^\sigma = \langle (A')^\sigma, (B')^\sigma \rangle^{G^\sigma}$$

is nilpotent. Therefore W is locally nilpotent by Lemma 3.2.2. ■

Our next lemma is a slight extension of a result by D.J.S. Robinson [57]. Recall here that if Q is a group and A is any Q-module, then $H_0(Q, A)$ is isomorphic to the additive group of $A/[A, Q]$, the largest homomorphic image of A which is a trivial Q-module, and $H^0(Q, A)$ is isomorphic to the additive group of the Q-submodule of A consisting of all elements fixed by Q.

Lemma 3.2.3. *Let Q be a finite-by-nilpotent group, and let A be a Q-module whose additive group is a divisible p-group of finite rank (where p is a prime number). If $H_0(Q, A) = \{0\}$, then $H^0(Q, A)$ is finite.*

PROOF. Assume for a contradiction that the statement is false, and choose a counterexample such that the divisible part U of the infinite Q-submodule $B = C_A(Q)$ has smallest rank. Suppose first that $Q/C_Q(A)$ is finite. Then there exists a proper Q-submodule V of A such that $U + V = A$ and $U \cap V$ is finite (see [36]). It follows that

$$[A, Q] = [V, Q] \leq V,$$

contradicting the assumption $H_0(A, Q) = A$. Therefore $Q/C_Q(A)$ is infinite, and so its centre contains a non-trivial element $xC_Q(A)$ (see [56] Part 1, Theorem 4.25). The mapping

$$\theta : a \mapsto -a + a^x$$

is a non-zero Q-endomorphism of A, and so

$$A^\theta \simeq A/ker\theta$$

is a non-zero Q-submodule of A whose additive group is divisible. Suppose that $A^\theta \neq A$. Then A^θ and A/A^θ have smaller rank than A, and so our minimal assumption yields that $H^0(Q, A^\theta)$ and $H^0(Q, A/A^\theta)$ are finite. This means that $C_{A^\theta}(Q)$ and $C_{A/A^\theta}(Q)$ a re finite, so that $C_A(Q)$ has finite exponent, and hence it is finite. This contradiction shows that $A^\theta = A$, so that the kernel of θ must be finite. As $C_A(Q)$ is contained in $ker\theta$, it follows that also $C_A(Q)$ is finite, and this last contradiction completes the proof. ■

We are now in a position to prove second main result.

Proof of Theorem 3.2.2 Assume that the statement is false, and choose a counterexample $G = AB$ with minimal torsion-free rank r such that $W = \langle A', B' \rangle^G$ is not locally nilpotent.
Suppose first that G has no periodic non-trivial normal subgroups, and let M be the Hirsch-Plotkin radical of G. Then M is torsion-free and the factor group G/M has torsion-free rank less than r, so that the statement holds for G/M and hence WM/M is locally nilpotent. Moreover, M is nilpotent and $Q = G/M$ is polycyclic-by-finite (see [56] Part 2, Lemma 9.34 and Theorem

10.33). Let x be a non-trivial element of $Z(M)$, and consider the cyclic Q-module $U = \langle x \rangle^G$. Then U contains a free abelian subgroup V such that U/V is periodic and the set $\pi = \pi(U/V)$, consisting of all prime numbers which are orders of elements of U/V, is finite (see [56] Part 2, Corollary 1 to Lemma 9.53). Since U is torsion-free and

$$\bigcap_{p \notin \pi} V^p = \{1\},$$

we have also

$$\bigcap_{p \notin \pi} U^p = \{1\}.$$

For each prime number $p \notin \pi$, the factor group G/U^p has torsion-free rank less than r, so that WU^p/U^p is locally nilpotent and hence it is contained in the Hirsch-Plotkin radical L_p/U^p of G/U^p. Then W is contained in the intersection

$$L_0 = \bigcap_{p \notin \pi} L_p.$$

On the other hand, for each p the group U/U^p has order at most p^r, and so it is contained in $Z_r(L_p/U^p)$. Therefore

$$[U, \underbrace{L_0, \ldots, L_0}_{r}] \leq \bigcap_{p \notin \pi} U^p = \{1\},$$

and hence U lies in $Z_r(L_0)$. As WU/U is locally nilpotent, it follows that W itself is locally nilpotent. This contradiction shows that the largest periodic normal subgroup T of G cannot be trivial, and that WT/T is locally nilpotent.

Assume now that T is residually finite, so that its Sylow subgroups are finite. Clearly, the group $W/\bar{Z}(W)$ is not locally nilpotent, so that $\bar{Z}(W)$ must be periodic and $G/\bar{Z}(W)$ is a counterexample with the same properties of G. Therefore we may suppose without loss of generality that $Z(W) = \{1\}$. For each prime number p, the group $T/O_{p'}(T)$ is finite, so that $WO_{p'}(T)/O_{p'}(T)$ is finite-by-(locally nilpotent) and hence even locally nilpotent by Corollary 3.2.2. It follows that the normal subgroup $O_p(T) \cap W$ is hypercentrally embedded in W, and so $O_p(T) \cap W = \{1\}$ for all p. As the group G is

soluble-by-finite, it follows that $T \cap W$ must be finite, and hence W is locally nilpotent by Corollary 3.2.2, a contradiction. Therefore the largest periodic normal subgroup T of the counterexample G cannot be residually finite, i.e. its finite residual J cannot be trivial and WJ/J is locally nilpotent.
The subgroup J is divisible abelian (see [56] Part 2, Theorem 9.31 and Corollary 2 to Theorem 9.23), and in particular it is the direct product of its primary components. The argument introduced in the previous paragraph shows that there is a prime number p such that $WJ_{p'}/J_{p'}$ is not locally nilpotent. Therefore it can be assumed without loss of generality that J is a p-group. Among all counterexamples with minimal torsion-free rank choose one $G = AB$ such that the finite residual J of the largest periodic normal subgroup T is a p-group of minimal rank. Also in this case, it can be assumed that $Z(W) = \{1\}$. Let P be a minimal infinite G-invariant subgroup of J; clearly, P coincides with its finite residual and so it is divisible. Moreover, WP/P is locally nilpotent by the minimal choice of G, and hence $[P, W] = P$. Since all proper G-invariant subgroups of P are finite, $G/C_G(P)$ is isomorphic to an irreducible soluble-by-finite linear group, and so it is abelian-by-finite (see [2], Lemma 6.6.4). Then there exists a subgroup K of G such that $G = PK$ and $P \cap K = \{1\}$ (see [2], Theorem 5.3.7). The centralizer $C_K(P)$ is a normal subgroup of G, and $WC_K(P)/C_K(P)$ cannot be locally nilpotent, so that $G/C_K(P)$ is a counterexample and hence $C_K(P)$ is periodic. Therefore replacing G by $G/C_K(P)$ we may suppose that $C_K(P) = \{1\}$, so that $C_G(P) = P$. It follows that $J = P$ has no infinite proper G-invariant subgroups and G/J is an abelian-by-finite group whose periodic subgroups are finite. In particular, the set of primes $\pi(G)$ is finite.
If N is any nilpotent normal subgroup of G, then the product NJ is nilpotent and so $[J, N]$ is a proper G-invariant subgroup of J. As $[J, N]$ is divisible, it follows that $[J, N] = \{1\}$, and hence $N \leq C_G(J) = J$. Therefore J is the Fitting subgroup of G. Consider the factorizer

$$X = X(J) = AJ \cap BJ$$

of J in $G = AB$. Then

$$X = (A \cap X)(B \cap X).$$

Let m be a positive integer such that both indices $|A : Z_m(A)|$ and $|B : Z_m(B)|$ are finite. As

$$Z_m(A)J \cap Z_m(B)J = J$$

(see [35]), it follows that J has finite index in X. Assume that the subgroup $A \cap J$ is finite. Then also $A \cap X$ is finite, so that $B \cap X$ has finite index in X and hence $B \cap J$ has finite index in J. This means that $J = B \cap J \leq B$. Since B is finite-by-nilpotent, the intersection $Z(B) \cap J$ is infinite, and so it contains a subgroup Q of type p^∞. But A and B are mutually permutable, so that $AQ = QA$ and $Q(A \cap J) = AQ \cap J$ is a normal subgroup of AQ. Clearly Q is the finite residual of $Q(A \cap J)$, so that it is normalized by A and hence it is normal in G. In this case $J = Q$ is a group of type p^∞, so that G/J is abelian and $G' \leq J$, a contradiction. It follows that $A \cap J$ must be infinite, so that also $Z(A) \cap J$ is infinite. A similar argument proves that $Z(B) \cap J$ is infinite. Clearly, the group G is not metabelian, and so the famous theorem of Itô yields that the subgroups A and B cannot be both abelian. Suppose that A is not abelian, so that in particular $A \neq J$ and $Z(A) \cap J$ is a proper subgroup of J. Since J has no infinite proper G-invariant subgroups, we obtain that

$$J = \big(Z(A) \cap J\big)^G = \big(Z(A) \cap J\big)\big[Z(A) \cap J, G\big],$$

and so $\big[Z(A) \cap J, G\big]$ is infinite. Therefore

$$J = \big[Z(A) \cap J, G\big] = \big[Z(A) \cap J, B\big] \leq [J, B]$$

and hence $[J, B] = J$. It follows that $H_0(BJ/J, J) = \{0\}$, and so $H^0(BJ/J, J)$ is finite by Lemma 3.2.3. Therefore $Z(B) \cap J$ is finite, and this last contradiction completes the proof of the theorem. ∎

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