Berkeley Cardinals and the search for V

Raffaella Cutolo

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University of Naples "Federico II"
Department of Mathematics and Applications

Supervisors:

Professor Peter Koellner

Professor W. Hugh Woodin

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“*This view* [*of the infinite*],

which I consider to be the sole correct one,

is held by only a few.

While possibly I am the very first in history to take this position so explicitly,

with all of its logical consequences,

I know for sure that I shall not be the last!”

[G. Cantor]

Abstract

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This thesis is concerned with *Berkeley cardinals*, very large cardinal axioms inconsistent with the Axiom of Choice. These notions have been recently introduced by J. Bagaria, P. Koellner and W. H. Woodin; our aim is to provide an introductory account of their features and of the motivations for investigating their consequences. As a noteworthy advance in the topic, we establish the independence from ZF of the cofinality of the least Berkeley cardinal, which is indeed the main point to focus on when dealing with the failure of Choice. We then explore the structural properties of the inner model $L(V_{\delta+1})$ under the assumption that δ is a singular limit of Berkeley cardinals each of which is a limit of extendible cardinals, lifting some of the theory of the axiom I_0 to the level of Berkeley cardinals. Finally, we discuss the role of Berkeley cardinals within the ultimate project of attaining a "definitive" description of the universe of set theory.

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Chapter 1

Introduction

Have you ever asked how far the mathematical infinite goes?

Providing the answer to this question is a primary challenge of set theory: it involves, on the one hand, large cardinal axioms (positing the existence of higher and higher infinities), and, on the other, set-theoretic principles limiting the extent of infinity that can (consistently) be demanded to exist; remarkably, on top of these principles we encounter the Axiom of Choice, for it readily implies that Reinhardt cardinals are inconsistent, and this settles the above question in the context of ZFC. But what if we work in just ZF? For all we know, the large cardinal hierarchy could then extend far beyond...

It is indeed an open problem whether it is consistent with ZF that there exists a Reinhardt cardinal, i.e., the statement that there is a non-trivial elementary embedding of the universe V into itself. This assertion arises as the limit case within a general template for the formulation of *very* large cardinal axioms (namely, by the level of measurable cardinals onward), according to which the large cardinal is just the critical point of a non-trivial elementary embedding $j : V \to M$, where M is a transitive class and the critical point of j is the least ordinal moved (upwardly) by j, and the higher the degree of resemblance required between M and V (i.e., the extent of closure satisfied by M), the stronger the large cardinal property holding at $crit(j)$. The axiom demanding full resemblance between M and V (that is, $M = V$) was proposed by Reinhardt; shortly after its introduction, it was ruled out by Kunen, showing that if $j : V \to M$ is a non-trivial elementary embedding and λ is the supremum of the critical sequence of j then $j``\lambda \notin M$, and hence $M \neq V$. In other words, for any non-trivial $j : V \to M$, the pointwise image of λ is not in M where λ is the least ordinal above $crit(j)$ such that $j(\lambda) = \lambda$. However, the proof of Kunen's theorem makes an essential use of the Axiom of Choice (the original version as well as all subsequent proofs known so far), and so, it actually tells us that "there are no Reinhardt cardinals as long as we work in ZFC".

It is worth to mentioning the following corollary of Kunen's proof, which is indeed a stronger version of his theorem: for any ordinal δ there is no non-trivial elementary embedding $j : V_{\delta+2} \to V_{\delta+2}$. Thus, the interest in studying large cardinals lying just below the Kunen inconsistency invites us to consider the following axiom, known as I_0: we say that I_0 holds at λ if there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $crit(j) < \lambda$; it follows immediately that λ equals the supremum of the critical sequence of j, call it λ', as if $\lambda > \lambda'$ then $j \restriction V_{\lambda'+2} : V_{\lambda'+2} \to V_{\lambda'+2}$ is non-trivial, contradicting the local version of Kunen's theorem. I_0 was introduced by Woodin, and to assume that I_0 holds at λ provides the inner model $L(V_{\lambda+1})$ with a rich structure theory, revealing deep analogies with the theory of $L(\mathbb{R})$ (which is just $L(V_{\omega+1})$) under the assumption that the Axiom of Determinacy holds in $L(\mathbb{R})$; as an example of similarity with $L(\mathbb{R})$, we have that under I_0, $L(V_{\lambda+1})$ does not satisfy the Axiom of Choice. Other remarkable similarities between $L(V_{\lambda+1})$ and $L(\mathbb{R})$ (under the respective cited axioms) concern the Coding Lemma and the existence of measurable cardinals: in particular, the measurability of λ^+ (respectively,

ω_1) and the properties of the ordinal Θ,[1] such as the fact that Θ is limit of measurables and for all $\alpha < \Theta$, $\mathcal{P}(\alpha) \in L_\Theta(V_{\lambda+1})$ (respectively, $L_\Theta(\mathbb{R})$).

Returning to our preliminary question: since the limitative result of Kunen doesn't apply to the context of ZF, if we put aside the Axiom of Choice then it makes sense to look at the possibility that Reinhardt cardinals exist; and in fact, we will examine the case that the large cardinal hierarchy encompasses them and proceeds upward, through even stronger hypotheses (actually the strongest formulated so far) that we will call "Berkeley cardinals". Rather surprisingly, these very large cardinal axioms will enable us to lift the main results of the I_0 theory to a higher level, offering a potential new candidate for the comparison with models of determinacy. Does this provide evidence for their consistency? At this stage, the real point turns out to be the following: these strong axioms of infinity are probably inconsistent with ZF, in that they seem to yield "too much power", and establishing such an inconsistency would be a first (and crucial) step toward a sharper indication on where the large cardinal hierarchy definitely breaks down. Nevertheless, at the present time, it still remains possible that Berkeley cardinals may legitimately exist in ZF. Ultimately, as we shall see, the first of the two cases would bring us very close to determining the final picture of the universe V.

1.1 Preliminary remarks

Our base theory is ZF (Zermelo-Fraenkel set theory).

We denote by V the **universe** of set theory, i.e., $\bigcup_\alpha V_\alpha$, where the sets V_α are defined by transfinite recursion, starting by the empty set and iterating the power set operation $\mathcal{P}$:

[1] In the respective cases, Θ is equal to $\sup\{\alpha : \exists\,\pi : V_{\lambda+1} \xrightarrow{onto} \alpha \ (\pi \in L(V_{\lambda+1}))\}$ and $\sup\{\alpha : \exists\,\pi : \mathbb{R} \xrightarrow{onto} \alpha \ (\pi \in L(\mathbb{R}))\}$.

- $V_0 = \emptyset$;
- for any ordinal α, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$;
- for λ limit, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$.

Recall that for every ordinal α, $V_\alpha = \{x : \mathrm{rank}(x) < \alpha\}$, where $\mathrm{rank}(x)$ is the least ordinal β such that $x \in V_{\beta+1}$.

A set a is **ordinal definable** if it is definable from some finite sequence of ordinals, i.e., there exist a formula $\varphi(x, y_1, \ldots, y_n)$ of the language of set theory and ordinals $\alpha_1, \ldots, \alpha_n$ such that $\{a\} = \{x : \varphi(x, \alpha_1, \ldots, \alpha_n)\}$. OD is the class of ordinal definable sets. A set a is **hereditarily ordinal definable** if a is ordinal definable and its transitive closure is contained in OD (i.e., all members of a, members of members of a, and etc., are in OD). HOD is the class of hereditarily ordinal definable sets; it is provable in ZF that HOD satisfies ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice).

The **constructible universe** $L = \bigcup_\alpha L_\alpha$ is the smallest inner model of ZF; the sets L_α are defined by transfinite recursion, starting by the empty set and taking at any step $\alpha + 1$ just the definable subsets of the set of level α:

- $L_0 = \emptyset$;
- for any ordinal α, $L_{\alpha+1} = \mathrm{def}(L_\alpha)$ is the set of all subsets of L_α which are definable with parameters over $(L_\alpha, \in)$;
- for λ limit, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

L was introduced by Gödel in his proof of consistency of the Axiom of Choice. In fact, $L \models \mathrm{ZFC}$.

For any set A, the **constructible closure** $L(A) = \bigcup_\alpha L_\alpha(A)$ is the smallest inner model of ZF that contains A as a member, with $L_\alpha(A)$ defined as follows:

- $L_0(A) = \operatorname{tr\,cl}(\{A\})$ is the transitive closure of $\{A\}$;
- for any ordinal α, $L_{\alpha+1}(A) = \operatorname{def}(L_\alpha(A))$ is the set of all subsets of $L_\alpha(A)$ which are definable with parameters over $(L_\alpha(A), \in)$;
- for λ limit, $L_\lambda(A) = \bigcup_{\alpha<\lambda} L_\alpha(A)$.

Unless $\operatorname{tr\,cl}(\{A\})$ has a well-ordering in $L(A)$, $L(A)$ does not satisfy the Axiom of Choice.

For M and N transitive models of a fragment of ZF, $j : M \to N$ is an **elementary embedding** iff for all formulas $\varphi(x_1, \dots, x_n)$ of the language of set theory and for all $a_1, \dots, a_n \in M$,

$$M \models \varphi(a_1, \dots, a_n) \text{ iff } N \models \varphi(j(a_1), \dots, j(a_n)).$$

If j is the identity map on M, then M is an **elementary substructure** of N, denoted $M \prec N$. We say that j is **non-trivial** if j is not the identity on the ordinals; then, the **critical point** of j (abbreviated $crit(j)$) is the least ordinal α such that $j(\alpha) > \alpha$.

Chapter 2

The Choiceless Hierarchy

In this chapter we describe the hierarchy of large cardinals incompatible with the Axiom of Choice. We define *Berkeley cardinals*, a notion introduced in [1], and establish some relative consistency implications between them and the other "choiceless" large cardinals presented. We conclude by providing the proof that the failure of the Axiom of Choice is essentially related with the cofinality of the least Berkeley cardinal, whose determination turns out to be the key point of interest. The content of this chapter follows [1] very closely and is included in order to make the thesis self-contained.

2.1 From Reinhardt Cardinals upward

We start by defining the hierarchy of Reinhardt cardinals and by mentioning the fundamental result established by Kunen, showing they are inconsistent in ZFC (for the proof of Kunen's theorem see [3] ch. 5).

Definition 2.1.1 (NBG[1])**.** A cardinal κ is **Reinhardt** if there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$.

[1]Von Neumann-Bernays-Gödel set theory (without the Axiom of Choice), a conservative extension of ZF.

Theorem 2.1.2 (ZFC)**.** (**Kunen's Theorem**) There is no non-trivial elementary embedding $j : V \to V$, i.e., there are no Reinhardt cardinals.

Open Question. Is it consistent with ZF that there exists a Reinhardt cardinal? In other words, can there be a non-trivial elementary embedding $j : V \to V$ in the context of ZF?

Throughout this dissertation we work in ZF; in this context, the notion of being a *strongly inaccessible* cardinal is reformulated as follows:

Definition 2.1.3 (ZF)**.** A cardinal κ is **strongly inaccessible** if for all $\alpha < \kappa$ there is no cofinal map $\rho : V_\alpha \to \kappa$.

Remark 2.1.4. Recall that:

1. If κ is strongly inaccessible then $V_\kappa \models \mathrm{ZF}$.
2. $(V_\kappa, V_{\kappa+1}) \models \mathrm{ZF}_2$[2] if and only if κ is strongly inaccessible.
3. If κ is Reinhardt then κ is strongly inaccessible; moreover, κ is a limit of strongly inaccessible cardinals.

Definition 2.1.5 (NBG)**.** A cardinal κ is **super Reinhardt** if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$ and $j(\kappa) > \lambda$.

It is not known whether the assumption that κ is a super Reinhardt cardinal implies that there exists a Reinhardt cardinal below κ; however, a super Reinhardt rank-reflects a Reinhardt cardinal:

Theorem 2.1.6. Suppose κ is a super Reinhardt cardinal. Then there exists $\gamma < \kappa$ such that $(V_\gamma, V_{\gamma+1}) \models \mathrm{ZF}_2 +$ "*There exists a Reinhardt cardinal*".

[2]Second-order version of ZF in the context of full second-order logic, where the second-order quantifiers range over $\mathcal{P}(V_\kappa) = V_{\kappa+1}$.

Proof. Let $j : V \to V$ be a non-trivial elementary embedding with $crit(j) = \kappa$. Let $\langle \kappa_i : i < \omega \rangle$ be the critical sequence of j, defined by $\kappa_0 = \kappa$, $\kappa_{i+1} = j(\kappa_i)$ for all $i < \omega$. Let $\lambda = \sup \{\kappa_i : i < \omega\}$. Notice that λ is fixed by j, as $j(\lambda) = j(\sup \{\kappa_i : i < \omega\}) = \sup \{j(\kappa_i) : i < \omega\} = \sup \{\kappa_{i+1} : i < \omega\} = \sup \{\kappa_i : i < \omega\} = \lambda$. By assumption, there exists a non-trivial elementary embedding $j' : V \to V$ such that $crit(j') = \kappa$ and $j'(\kappa) > \lambda$. Since κ is a limit of strongly inaccessible cardinals, by elementarity, $j'(\kappa)$ is a limit of strongly inaccessible cardinals. Let γ_0 be the least strongly inaccessible cardinal greater than λ. Since γ_0 is definable from λ and $j(\lambda) = \lambda$, we have that $j(\gamma_0) = \gamma_0$; therefore, $j \restriction V_{\gamma_0} : V_{\gamma_0} \to V_{\gamma_0}$ is an elementary embedding with critical point κ, witnessing that $(V_{\gamma_0}, V_{\gamma_0+1}) \models \mathrm{ZF}_2 +$ "*κ is Reinhardt*". Thus, since $\gamma_0 < j'(\kappa)$, by applying j'^{-1} we get that there exists $\gamma = j'^{-1}(\gamma_0) < \kappa$ such that $(V_\gamma, V_{\gamma+1}) \models \mathrm{ZF}_2 +$ "*There exists a Reinhardt cardinal*". $\square$

Definition 2.1.7 (NBG)**.** Suppose A is a proper class. A cardinal κ is ***A*-Reinhardt** if there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$ and $j(A) = A$. A cardinal κ is ***A*-super Reinhardt** if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j(A) = A$ (where $j(A) = \bigcup_\alpha j(A \cap V_\alpha)$).

Definition 2.1.8. Suppose κ is a strongly inaccessible cardinal. Then κ is **totally Reinhardt** if for each $A \in V_{\kappa+1}$, $(V_\kappa, V_{\kappa+1}) \models$ "*There exists an A-super Reinhardt cardinal*".

By definition, if κ is a totally Reinhardt cardinal then κ reflects a super Reinhardt cardinal (i.e., $(V_\kappa, V_{\kappa+1}) \models \mathrm{ZF}_2 +$ "*There exists a super Reinhardt cardinal*"); it is an open question whether a totally Reinhardt cardinal rank-reflects a super Reinhardt.

We now introduce the Berkeley hierarchy.

Definition 2.1.9. For any transitive set M, let $\mathcal{E}(M)$ be the collection of all non-trivial elementary embeddings $j : M \to M$.

Definition 2.1.10. An ordinal δ is a **proto-Berkeley cardinal** if for all transitive sets M such that $\delta \in M$ there exists $j \in \mathcal{E}(M)$ with $crit(j) < \delta$.

Remark 2.1.11. Notice that:

1. If δ is a proto-Berkeley cardinal then, in particular, for any $\lambda > \delta$ there exists a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$ with $crit(j) < \delta$.

2. If δ_0 is the least proto-Berkeley cardinal then every ordinal δ greater than δ_0 is also a proto-Berkeley cardinal.

It turns out that in the context of Berkeley cardinals we have elementary embeddings fixing any given set; this very powerful feature is a consequence of the following lemma.

Lemma 2.1.12. For any set a there exists a transitive set M such that $a \in M$ and a is definable (without parameters) in M.

Proof. Let λ be a limit ordinal such that $a \in V_\lambda$, and let $M = V_\lambda \cup \{\{\langle a, x\rangle : x \in V_\lambda\}\}$. Then, M is a transitive set and a is definable (without parameters) in M as the first element in the pairs belonging to the set of highest rank. $\square$

Corollary 2.1.13. Suppose δ is a proto-Berkeley cardinal. Then for every set a there exists a transitive set M and a $j \in \mathcal{E}(M)$ such that $crit(j) < \delta$ and $j(a) = a$.

Proof. By Lemma 2.1.12, there exists a transitive set M such that $a \in M$ and a is definable (without parameters) in M. We can choose M such that $\delta \in M$ (by considering a limit ordinal $\lambda > \delta$ such that $a \in V_\lambda$ in the definition of

M), and so there exists a $j \in \mathcal{E}(M)$ such that $crit(j) < \delta$; finally, since a is definable in M and $j``M \subseteq M$, $j(a) = a$. □

The following theorem provides the motivation for generalizing the notion of being a proto-Berkeley cardinal.

Theorem 2.1.14. Let δ_0 be the least proto-Berkeley cardinal. Then the critical points of the witnessing embeddings are cofinal in δ_0, i.e., for all transitive sets M such that $\delta_0 \in M$ and for all $\eta < \delta_0$ there exists $j \in \mathcal{E}(M)$ with $\eta < crit(j) < \delta_0$.

Proof. For contradiction, suppose that $\eta_0 < \delta_0$ is the least ordinal for which the claim is false, i.e., there exists a transitive set M_0 such that $\delta_0 \in M_0$ and there does not exist a $j \in \mathcal{E}(M_0)$ with $\eta_0 < crit(j) < \delta_0$. Let M be any transitive set such that $\eta_0 \in M$. By Lemma 2.1.12, there exists a transitive set M' such that $\langle M_0, M, \eta_0 \rangle$ is definable in M'. Since $\delta_0 \in M'$, there exists $j' \in \mathcal{E}(M')$ with $crit(j') < \delta_0$. Since $j'``M' \subseteq M'$, by definability we have that $j'(\langle M_0, M, \eta_0 \rangle) = \langle M_0, M, \eta_0 \rangle$, and so, $j'$ fixes $M_0$, $M$ and $\eta_0$; therefore, $j' \restriction M_0 \in \mathcal{E}(M_0)$ (because notice that $j'``M_0 \subseteq j'(M_0) = M_0$) and, by the definition of η_0, $crit(j' \restriction M_0) \leq \eta_0$. But $j'(\eta_0) = \eta_0$, hence $crit(j' \restriction M_0) < \eta_0$. Similarly, $j' \restriction M \in \mathcal{E}(M)$ and $crit(j' \restriction M) < \eta_0$. It follows that η_0 is a proto-Berkeley cardinal, which is a contradiction. □

Definition 2.1.15. Suppose α is an ordinal. An ordinal δ is an **$\boldsymbol{\alpha}$-proto-Berkeley cardinal** if for all transitive sets M such that $\delta \in M$ there exists $j \in \mathcal{E}(M)$ with $\alpha < crit(j) < \delta$.

The proof of Theorem 2.1.14 adapts to show the following more general result, motivating the definition of *Berkeley* cardinals:

Theorem 2.1.16. Suppose α is an ordinal. Let δ_α be the least α-proto-Berkeley cardinal. Then for all transitive sets M such that $\delta_\alpha \in M$ and for all

$\eta < \delta_\alpha$ there exists $j \in \mathcal{E}(M)$ with $\eta < crit(j) < \delta_\alpha$.

Definition 2.1.17. A cardinal δ is a **Berkeley cardinal** if for every transitive set M such that $\delta \in M$ and for every ordinal $\eta < \delta$ there exists $j \in \mathcal{E}(M)$ with $\eta < crit(j) < \delta$.

Remark 2.1.18. Notice that:

1. For each ordinal α, the least α-proto-Berkeley cardinal is a Berkeley cardinal. So, the least Berkeley cardinal is also characterized as the least α-proto-Berkeley cardinal, for every ordinal α.

2. If δ is a limit of Berkeley cardinals then δ is a Berkeley cardinal, i.e., the class of Berkeley cardinals is closed.

3. The property of being a Berkeley cardinal is a Π_2 property.

4. If δ is a Berkeley cardinal then for all limit ordinals $\lambda > \delta$, V_λ thinks that δ is a Berkeley cardinal.

We focus on the least Berkeley cardinal, call it δ_0. The next lemma will enable us to show some reflection phenomena occurring at δ_0. Also, we prove that δ_0 is not *extendible.*

Lemma 2.1.19. Let δ_0 be the least Berkeley cardinal. Then for a tail of limit ordinals β, if $j \in \mathcal{E}(V_\beta)$ is such that $crit(j) < \delta_0$ then $j(\delta_0) = \delta_0$ and the set $\{\eta < \delta_0 : j(\eta) = \eta\}$ is cofinal in δ_0 (i.e., δ_0 is a limit of j-fixed points).

Proof. By assumption, every $\delta < \delta_0$ is not a proto-Berkeley cardinal, so there exists a transitive set M_δ such that $\delta \in M_\delta$ and there does not exist $j \in \mathcal{E}(M_\delta)$ with $crit(j) < \delta$; for each $\delta < \delta_0$, let β_δ be least such that V_{β_δ} contains such a witness M_δ. If $\beta > \delta_0$ is a limit ordinal such that $\beta > \beta_\delta$ for all $\delta < \delta_0$, then V_β thinks that δ_0 is a Berkeley cardinal and that any $\delta < \delta_0$ is not a proto-Berkeley cardinal (since V_β contains all of the witnesses), so V_β recognizes that δ_0 is the least proto-Berkeley cardinal; therefore, for any such β, δ_0 is definable in V_β, and so, for any $j \in \mathcal{E}(V_\beta)$, we have that $j(\delta_0) = \delta_0$. Let β be in the above tail and suppose that $j \in \mathcal{E}(V_\beta)$ is such that $crit(j) < \delta_0$. Assume, for contradiction, that the set $\{\eta < \delta_0 : j(\eta) = \eta\}$ is not cofinal in δ_0; let $\eta_0 = \sup\{\eta < \delta_0 : j(\eta) = \eta\}$ and, for $i < \omega$, let $\eta_{i+1} = j(\eta_i)$. It follows that $\delta_0 = \sup\{\eta_i : i < \omega\}$, since the latter is a j-fixed point greater than η_0 and less than or equal to δ_0 (as $\eta_i < \delta_0$ for all $i < \omega$). Let M_0 be a witness in V_β to the fact that η_0 is not a proto-Berkeley cardinal, i.e., a transitive set containing η_0 such that there is no $j \in \mathcal{E}(M_0)$ with $crit(j) < \eta_0$. For $i < \omega$, define $M_{i+1} = j(M_i)$; notice that M_{i+1} turns to be a witness that η_{i+1} is not a proto-Berkeley cardinal. For a tail of limit ordinals β, we have j, $M_0 \in V_\beta$, and since the sequence $\langle M_i : i < \omega\rangle$ is definable from these two elements, it is also $\langle M_i : i < \omega\rangle \in V_\beta$; moreover, by Lemma 2.1.12, there exists a transitive set M such that V_β and $\langle M_i : i < \omega\rangle$ are both definable in M, so that they are fixed by any $j \in \mathcal{E}(M)$. Since $\delta_0 \in M$, we can let $j' \in \mathcal{E}(M)$ be such that $crit(j') < \delta_0$. Then, we have that $j'' = j' \restriction V_\beta \in \mathcal{E}(V_\beta)$ is such that $j''(\langle M_i : i < \omega\rangle) = \langle M_i : i < \omega\rangle$ (as the sequence is definable in M and is in the domain of j'') and $crit(j'') < \delta_0$; moreover, notice that $j''(\langle M_i : i < \omega\rangle) = \langle M_i : i < \omega\rangle$ implies $j''(M_i) = M_i$ for all $i < \omega$. Since $\delta_0 = \sup\{\eta_i : i < \omega\}$, $crit(j'') < \eta_i$ for some $i < \omega$; for such an i, $j''(M_i) = M_i$ and $j'' \restriction M_i \in \mathcal{E}(M_i)$ is such that $crit(j'' \restriction M_i) < \eta_i$, contradicting that M_i is a witness to the fact that η_i is not a proto-Berkeley cardinal. $\square$

Definition 2.1.20 (ZF)**.** A cardinal κ is **extendible** if for all ordinals α there exist α' and an elementary embedding $j : V_{\kappa+\alpha} \to V_{j(\kappa)+\alpha'}$ such that $crit(j) = \kappa$ and $j(\kappa) > \alpha$.

Theorem 2.1.21. Suppose that δ_0 is the least Berkeley cardinal. Then, δ_0 is not extendible.

Proof. Let $\lambda > \delta_0$ be such that $V_\lambda \prec_{\Sigma_2} V$. Since the property of being a Berkeley cardinal is a Π_2 property, V_λ correctly recognizes Berkeley cardinals; in particular, V_λ thinks that δ_0 is a Berkeley cardinal. Suppose, for contradiction, that δ_0 is extendible. Let $j : V_\lambda \to V_{\lambda'}$ be an elementary embedding such that $crit(j) = \delta_0$ and $j(\delta_0) > \lambda$. Since $V_{\lambda'}$ recognizes that δ_0 is a Berkeley cardinal, $V_{\lambda'} \models$ "*There exists a Berkeley cardinal below* $j(\delta_0)$"; it follows that $V_\lambda \models$ "δ_0 *is a limit of Berkeley cardinals*": in fact, for any fixed $\alpha < \delta_0$, since $V_{\lambda'} \models$ "*There exists a Berkeley cardinal between* $j(\alpha) = \alpha$ *and* $j(\delta_0)$", by elementarity we have that $V_\lambda \models$ "*There exists a Berkeley cardinal between* α *and* δ_0". But V_λ correctly recognizes Berkeley cardinals, and so, in V, δ_0 is a limit of Berkeley cardinals, a contradiction. □

Although δ_0 is not itself extendible, the hypothesis that there exists a Berkeley cardinal which is also super Reinhardt (and, therefore, extendible) turns to be consistent relative to a stronger version of a Berkeley cardinal, which will be called a *limit club Berkeley* cardinal; before introducing it, we have to define the notion of being a *club Berkeley* cardinal.

Definition 2.1.22. A cardinal δ is a **club Berkeley cardinal** if δ is regular and for all clubs $C \subseteq \delta$ and for all transitive sets M such that $\delta \in M$, there exists $j \in \mathcal{E}(M)$ with $crit(j) \in C$.

Definition 2.1.23. A cardinal δ is a **limit club Berkeley cardinal** if δ is a club Berkeley cardinal which is a limit of Berkeley cardinals.

2.2 Reflection phenomena

We are now able to show the following relative consistency implications:

1. The least Berkeley cardinal rank-reflects an extendible cardinal and a Reinhardt cardinal.

2. A club Berkeley cardinal rank-reflects a super Reinhardt cardinal.

3. A limit club Berkeley cardinal rank-reflects a Berkeley cardinal which is also super Reinhardt.

Theorem 2.2.1. Suppose that δ_0 is the least Berkeley cardinal. Then there exists $\gamma < \delta_0$ such that $(V_\gamma, V_{\gamma+1}) \models \mathrm{ZF}_2 +$ "*There exists an extendible cardinal and there exists a Reinhardt cardinal*".

Proof. By Lemma 2.1.19, for a tail of limit ordinals β, if $j \in \mathcal{E}(V_\beta)$ is such that $crit(j) < \delta_0$ then $j(\delta_0) = \delta_0$ and the set $\{\eta < \delta_0 : j(\eta) = \eta\}$ is cofinal in δ_0. Fix such a β and $j \in \mathcal{E}(V_\beta)$. Let $\lambda = \sup\{\kappa_i : i < \omega\}$, where $\kappa_0 = crit(j)$ and, for $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$; we have that $j(\lambda) = \lambda$ and $\lambda < \delta_0$ (because λ is the least j-fixed point greater than κ_0, while δ_0 is a limit of j-fixed points). Since δ_0 is a Berkeley cardinal, there exists $j' \in \mathcal{E}(V_{\delta_0+1})$ such that $\lambda < crit(j') < \delta_0$. Let $\lambda' = \sup\{\kappa'_i : i < \omega\}$, where $\kappa'_0 = crit(j')$ and, for $i < \omega$, $\kappa'_{i+1} = j'(\kappa'_i)$. The key point is that $V_{\lambda'} \models \mathrm{ZF} +$ "*There exists an extendible cardinal*", and this is witnessed by κ'_0. In fact, suppose for contradiction that $V_{\lambda'} \models$ "*κ'_0 is not extendible*", and let $\alpha < \lambda'$ be least such that in $V_{\lambda'}$ there do not exist α' and an elementary embedding $j : V_{\kappa'_0+\alpha} \to V_{j(\kappa'_0)+\alpha'}$ such that $crit(j) = \kappa'_0$ and $j(\kappa'_0) > \alpha$. Now, if we let $j_0 = j' \restriction V_{\lambda'} \in \mathcal{E}(V_{\lambda'})$ and, for $i < \omega$, $j_{i+1} = j_0(j_i)$, we have that for some $i < \omega$, $j_i(\kappa'_0) > \alpha$; for such an i, the map $j_i \restriction V_{\kappa'_0+\alpha} : V_{\kappa'_0+\alpha} \to V_{j_i(\kappa'_0)+j_i(\alpha)}$ is an elementary embedding such that $crit(j_i \restriction V_{\kappa'_0+\alpha}) = \kappa'_0$ and $(j_i \restriction V_{\kappa'_0+\alpha})(\kappa'_0) > \alpha$, contradicting the choice of α. Thus, κ'_0 is indeed extendible in $V_{\lambda'}$. Since $V_{\kappa'_0} \prec V_{\lambda'}$ (as witnessed by j') and $\kappa'_0 > \lambda$, it follows that $(V_{\kappa'_0}, V_{\kappa'_0+1}) \models \mathrm{ZF}_2 +$ "*There exists an extendible cardinal greater than λ*". Let $\gamma \leq \kappa'_0$ be least such that $(V_\gamma, V_{\gamma+1}) \models \mathrm{ZF}_2 +$ "*There exists an extendible cardinal greater than λ*", and let δ be the least extendible cardinal greater than λ in $V_{\gamma+1}$; since γ and δ are both definable from λ and $j(\lambda) = \lambda$, j fixes both γ and δ, so $j \restriction V_{\gamma+1} : V_{\gamma+1} \to V_{\gamma+1}$ is an elementary embedding with $crit(j \restriction V_{\gamma+1}) = crit(j) = \kappa_0$ (as $\kappa_0 < \lambda < \delta \in V_{\gamma+1}$), witnessing that κ_0 is a Reinhardt cardinal in $V_{\gamma+1}$. In summary, we have shown that there exist $\gamma < \delta_0$ and $\kappa_0 < \delta < \gamma$ such that $(V_\gamma, V_{\gamma+1}) \models \mathrm{ZF}_2 +$ "*δ is extendible and κ_0 is Reinhardt*", so, the proof is complete. □

Theorem 2.2.2. Suppose that δ is a club Berkeley cardinal. Then, $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*There exists a super Reinhardt cardinal*".

Proof. We begin by showing that for all transitive sets M such that $V_{\delta+1} \in M$ there exists $\kappa < \delta$ such that for all $\alpha < \delta$ there exists $j_\alpha \in \mathcal{E}(M)$ with $crit(j_\alpha) = \kappa$ and $j_\alpha(\kappa) > \alpha$. Suppose the claim fails and let M be a counterexample, i.e., a transitive set such that $V_{\delta+1} \in M$ and for all $\kappa < \delta$ there exists $\alpha < \delta$ such that there does not exist $j \in \mathcal{E}(M)$ with $crit(j) = \kappa$ and $j(\kappa) > \alpha$; for each $\kappa < \delta$, let α_κ be the least such α. Let $C = \{\gamma < \delta : \forall\, \kappa < \gamma\ (\alpha_\kappa < \gamma)\}$. Since δ is regular, C is club in δ. Moreover, since $C \subseteq \delta \in V_{\delta+1}$ and $V_{\delta+1} \in M$, $C \in M$. Finally, since δ is a club Berkeley cardinal and $\delta \in M$, there exists $j \in \mathcal{E}(M)$ such that $crit(j) \in C$ and $j(C) = C$: in fact, if we let M' be a transitive set such that both C and M are definable in M', then, since $\delta \in M'$, we get that there exists $j' \in \mathcal{E}(M')$ such that $crit(j') \in C$, $j'(C) = C$ and $j'(M) = M$, and so $j = j' \restriction M \in \mathcal{E}(M)$ is such that $crit(j) = crit(j') \in C$ and $j(C) = C$. Let $crit(j) = \bar{\kappa}$. By elementarity, $j(\bar{\kappa}) \in j(C) = C$; so, by the definition of C, $\bar{\kappa} < j(\bar{\kappa})$ implies $\alpha_{\bar{\kappa}} < j(\bar{\kappa})$, contradicting the definition of $\alpha_{\bar{\kappa}}$. Thus, we have shown that for all transitive sets M such that $V_{\delta+1} \in M$ there exists $\kappa < \delta$ such that for all $\alpha < \delta$ there exists $j_\alpha \in \mathcal{E}(M)$ with $crit(j_\alpha) = \kappa$ and $j_\alpha(\kappa) > \alpha$. Now, let M be any transitive set such that $V_{\delta+1} \in M$, and let $\kappa < \delta$ be as above; then, for each $\alpha < \delta$, $j_\alpha \restriction V_{\delta+1}$ is such that $crit(j_\alpha \restriction V_{\delta+1}) = \kappa$ and $(j_\alpha \restriction V_{\delta+1})(\kappa) > \alpha$, witnessing that $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*κ is super Reinhardt*". □

Theorem 2.2.3. Suppose that δ is a limit club Berkeley cardinal. Then, $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*There exists a Berkeley cardinal that is super Reinhardt*".

Proof. The first claim we are going to prove is that for all transitive sets M such that $V_{\delta+1} \in M$, and for all $D \subseteq \delta$ which are club in δ, there exists $\kappa \in D$ such that for all $\alpha < \delta$ there exists $j_\alpha \in \mathcal{E}(M)$ such that $crit(j_\alpha) = \kappa$ and $j_\alpha(\kappa) > \alpha$. For contradiction, let $\langle M, D\rangle$ be a counterexample, i.e., let M be a transitive set such that $V_{\delta+1} \in M$ and let $D \subseteq \delta$ be a club in δ such that for all $\kappa \in D$ there exists $\alpha < \delta$ such that there does not exist $j \in \mathcal{E}(M)$ with $crit(j) = \kappa$ and $j(\kappa) > \alpha$; for each $\kappa \in D$, let α_κ be the least such α. Let $C = \{\gamma < \delta : \forall\, \kappa \in D \cap \gamma\ (\alpha_\kappa < \gamma)\}$. Since δ is regular, C is club in δ; thus, $C \cap D$ is club in δ. Since δ is a club Berkeley cardinal and $\delta \in M$, there exists $j \in \mathcal{E}(M)$ such that $crit(j) = \bar{\kappa} \in C \cap D$, $j(C) = C$ and $j(D) = D$ (notice that C and D are both in M). By elementarity, $j(\bar{\kappa}) \in j(C \cap D) = j(C) \cap j(D) = C \cap D$, and so, since $\bar{\kappa} \in D \cap j(\bar{\kappa})$, by the definition of C we have that $\alpha_{\bar{\kappa}} < j(\bar{\kappa})$, which contradicts the definition of $\alpha_{\bar{\kappa}}$. Therefore, we have shown that for all transitive sets M such that $V_{\delta+1} \in M$, and for all $D \subseteq \delta$ which are club in δ, there exists $\kappa \in D$ such that for all $\alpha < \delta$ there exists $j_\alpha \in \mathcal{E}(M)$ such that $crit(j_\alpha) = \kappa$ and $j_\alpha(\kappa) > \alpha$. Now, let M be any transitive set such that $V_{\delta+1} \in M$, let $D \subseteq \delta$ be club in δ, and take $\kappa \in D$ as above; again, for each $\alpha < \delta$, $j_\alpha \restriction V_{\delta+1}$ is such that $crit(j_\alpha \restriction V_{\delta+1}) = \kappa$ and $(j_\alpha \restriction V_{\delta+1})(\kappa) > \alpha$, witnessing that $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*There exists a super Reinhardt cardinal* $\kappa \in D$". Since D was *any* club in δ, it follows that $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*There exist stationarily many super Reinhardt cardinals*". Since the class of Berkeley cardinals is closed and δ is a limit of Berkeley cardinals, the Berkeley cardinals below δ are club in δ, and so finally, $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2 +$ "*There exists a Berkeley cardinal that is super Reinhardt*". □

Chart of Choiceless Large Cardinals[3]

$$
\begin{array}{ccccc}
\text{limit club Berkeley} & & \text{totally Reinhardt} & & \text{club Berkeley} \\
\downarrow & \searrow & \downarrow & \swarrow & \\
\text{Berkeley} & & \text{super Reinhardt} & & \\
 & \searrow & \downarrow & & \\
 & & \text{Reinhardt} & &
\end{array}
$$

2.3 The failure of Choice

The large cardinal axioms we are concerned with are in conflict with the Axiom of Choice: we are now going to illustrate the point at which the contradiction arises. We shall need a preliminary lemma.

Definition 2.3.1. Let δ_0 be the least Berkeley cardinal. For any transitive set M such that $\delta_0 \in M$, let $\kappa_M = \min\{crit(j) : j \in \mathcal{E}(M)\}$.

Remark 2.3.2. Notice that:

1. κ_M is well-defined, since $\delta_0 \in M$ and so there exists $j \in \mathcal{E}(M)$.

2. $\kappa_M < \delta_0$.

Lemma 2.3.3. Let δ_0 be the least Berkeley cardinal. Then for all $\eta < \delta_0$ there exists a transitive set M_η such that $\delta_0 \in M_\eta$ and $\kappa_{M_\eta} > \eta$ (i.e., $crit(j) > \eta$ for all $j \in \mathcal{E}(M_\eta)$).

Proof. For contradiction, let $\eta_0 < \delta_0$ be least such that for all transitive sets M such that $\delta_0 \in M$, $\kappa_M \leq \eta_0$. Let M be any transitive set such that $\eta_0 \in M$. Let $\hat{M}$ be a transitive set such that $\delta_0 \in \hat{M}$ and M and η_0 are definable in $\hat{M}$;

[3]The arrows indicate relative consistency implications.

let $j \in \mathcal{E}(\hat{M})$ be such that $crit(j) = \kappa_{\hat{M}}$. By the definition of η_0, $crit(j) = \kappa_{\hat{M}} \leq \eta_0$; but $j(\eta_0) = \eta_0$, so $crit(j) < \eta_0$. Since $j(M) = M$, $j \restriction M \in \mathcal{E}(M)$; moreover, $crit(j \restriction M) < \eta_0$. Since M was an arbitrary set containing η_0, it follows that η_0 is a proto-Berkeley cardinal, which is a contradiction (recall that the least proto-Berkeley cardinal is δ_0 itself). $\square$

Definition 2.3.4. For any cardinal $\gamma \geq \omega$ and for any set $X \neq \emptyset$, γ-DC(X) (**$\boldsymbol{\gamma}$-Dependent Choice on** X) is the statement that for every function $F : {}^{<\gamma}X \to \mathcal{P}(X) \setminus \{\emptyset\}$ there exists a sequence $f : \gamma \to X$ such that for all $\alpha < \gamma$, $f(\alpha) \in F(f \restriction \alpha)$. γ-DC (**$\boldsymbol{\gamma}$-Dependent Choice**) is the statement that γ-DC(X) holds for all non-empty sets X.

Remark 2.3.5. Recall that:

1. The Axiom of Choice (AC) is equivalent to the statement that γ-DC holds for all cardinals $\gamma \geq \omega$.

2. γ-DC implies that every family $\mathfrak{F}$ of non-empty sets such that $|\mathfrak{F}| = \gamma$ has a choice function.

3. If $\omega \leq \kappa < \gamma$, then γ-DC implies κ-DC.

4. If γ is an infinite singular cardinal and κ-DC holds for all cardinals κ such that $\omega \leq \kappa < \gamma$, then γ-DC holds.[4]

Theorem 2.3.6. Suppose δ_0 is the least Berkeley cardinal. Let $\gamma = \mathrm{cof}(\delta_0)$. Then, γ-DC fails.

Proof. Let $\pi : \gamma \to \delta_0$ be cofinal. For each $\xi < \gamma$, let β_ξ be least such that V_{β_ξ} contains a transitive set M such that $\delta_0 \in M$ and $\kappa_M > \pi(\xi)$ (notice that, for every $\xi < \gamma$, such an M does exist by Lemma 2.3.3). For each $\xi < \gamma$, let $\hat{M}_\xi = \{M \in V_{\beta_\xi} : \text{“}M \textit{ is transitive}\text{”} \wedge \delta_0 \in M \wedge \kappa_M > \pi(\xi)\}$; by the

[4] See [2] ch. 8 for the proof.

definition of β_ξ, we have that for every $\xi < \gamma$, $\hat{M}_\xi \neq \emptyset$. For contradiction, assume that γ-DC holds. Then, there exists a sequence $\langle M_\xi : \xi < \gamma\rangle$ such that $M_\xi \in \hat{M}_\xi$ for all $\xi < \gamma$; in fact: if $S = \bigcup_{\xi<\gamma} \hat{M}_\xi$ and $F : {}^{<\gamma}S \to \mathcal{P}(S) \setminus \{\emptyset\}$ is such that $F(s) = \hat{M}_\xi$ whenever $\xi < \gamma$ and s is a ξ-sequence in S, then, by γ-DC, there exists a γ-sequence $f : \gamma \to S$ such that for all $\xi < \gamma$, $f(\xi) \in F(f \restriction \xi) = \hat{M}_\xi$. Let M' be a transitive set such that the sequence $\langle M_\xi : \xi < \gamma\rangle$ is definable in M'. Let $j' \in \mathcal{E}(M')$ be such that $crit(j') < \delta_0$. Since $j'(\langle M_\xi : \xi < \gamma\rangle) = \langle M_\xi : \xi < \gamma\rangle$, we have that $j'(\gamma) = \gamma$ and for all $\xi < \gamma$, if $j'(\xi) = \xi$ then $j'(M_\xi) = M_\xi$; so, for every such ξ, $j' \restriction M_\xi \in \mathcal{E}(M_\xi)$, which implies $\kappa_{M'} \geq \kappa_{M_\xi} > \pi(\xi)$. Therefore, there cannot exist cofinally many $\xi < \gamma$ such that $j'(\xi) = \xi$ (otherwise, we would have $\kappa_{M'} \geq \delta_0 = \sup\{\pi(\xi) : \xi < \gamma\}$). It follows that there exist $\eta_0 = \sup\{\xi < \gamma : j'(\xi) = \xi\} < \gamma$ and $\langle \eta_i : i < \omega\rangle$ such that $\eta_{i+1} = j'(\eta_i)$ and $\sup\{\eta_i : i < \omega\} = \gamma$. So, since $\delta_0 = \sup\{\pi(\xi) : \xi < \gamma\}$, we have that for every $\alpha < \delta_0$ there exists $i < \omega$ such that $\pi(\eta_i) \geq \alpha$, that is, $\delta_0 = \sup\{\pi(\eta_i) : i < \omega\}$. Now, let M'' be a transitive set such that the sequence $\langle M_{\eta_i} : i < \omega\rangle$ is definable in M'' and let $j'' \in \mathcal{E}(M'')$ be such that $crit(j'') < \delta_0$. Then, $j''(\langle M_{\eta_i} : i < \omega\rangle) = \langle M_{\eta_i} : i < \omega\rangle$. But now $j''(M_{\eta_i}) = M_{\eta_i}$ for all $i < \omega$, hence $j'' \restriction M_{\eta_i} \in \mathcal{E}(M_{\eta_i})$ for all $i < \omega$; it follows that $\kappa_{M''} \geq \kappa_{M_{\eta_i}} > \pi(\eta_i)$ for all $i < \omega$, and so, $\kappa_{M''} \geq \delta_0$, a contradiction. $\square$

At this point, the following key question arises:

Key Question. Which is the cofinality of the least Berkeley cardinal?

Chapter 3

Cofinality of the least Berkeley Cardinal

The aim of this chapter is to show that the fundamental question of the cofinality of the least Berkeley cardinal is undecidable. This leaves open a range of possibilities for the exact amount of choice that it makes sense to assume in the context of Berkeley cardinals in order to demand the consistency of the resulting theory. We briefly analyze these possibilities. In the end, we outline the motivation for our further investigation, developed in the next chapter: uncover the mathematical structure revealed by choiceless large cardinals. We assume the reader is familiar with the technique of forcing (a complete account of forcing can be found in [4]).

3.1 Forcing the cofinality to be countable

First, we prove that there exists a forcing extension in which the least Berkeley cardinal has countable cofinality. To begin, recall the following "lifting criterion" for elementary embeddings:

Lemma 3.1.1 (Lifting Criterion). Suppose $j : M \to N$ is an elementary embedding of two transitive models of (a sufficiently large fragment of) ZF. Let $\mathbb{P} \in M$ and suppose $G \subseteq \mathbb{P}$ is M-generic and $H \subseteq j(\mathbb{P})$ is N-generic. Then, j lifts to an elementary embedding $j^* : M[G] \to N[H]$ (with $j^*(G) = H$) iff $j``G \subseteq H$.

Theorem 3.1.2. Assume ZF + BC.[1] Then there exists a forcing extension $V[G]$ of V such that $V[G] \models \text{ZF} + \text{BC} +$ "$\text{cof}(\delta_0) = \omega$, *where* δ_0 *is the least Berkeley cardinal* (*as computed in* $V[G]$)".

Proof. Let $\gamma_0 = (\delta_0)^V$ denote the least Berkeley cardinal in V. Suppose $(\text{cof}(\gamma_0))^V > \omega$. Let $\langle \mathbb{P}_{\gamma_0}, \leq_{\mathbb{P}_{\gamma_0}} \rangle$ be the forcing whose conditions are of the form $\langle \sigma, C \rangle$ where $\sigma \in [S^{\gamma_0}_\omega]^{<\omega} = \{\sigma \subseteq S^{\gamma_0}_\omega : |\sigma| < \omega\}$ is a finite subset of $S^{\gamma_0}_\omega = \{\alpha < \gamma_0 : \text{cof}(\alpha) = \omega\}$ and C is an ω-club in γ_0, i.e., C is an unbounded subset of γ_0 and C contains all its limit points less than γ_0 of cofinality ω. Let $\leq_{\mathbb{P}_{\gamma_0}} = \leq$ be defined as follows: for all $\langle \sigma_1, C_1 \rangle, \langle \sigma_2, C_2 \rangle \in \mathbb{P}_{\gamma_0}$, $\langle \sigma_2, C_2 \rangle \leq \langle \sigma_1, C_1 \rangle$ iff

1. $C_2 \subseteq C_1$,

2. $\sigma_1 \subseteq \sigma_2$,

3. $\sigma_2 \cap \sup(\sigma_1) = \sigma_1$ (i.e., σ_2 end-extends σ_1) and

4. $\sigma_2 \setminus \sigma_1 \subseteq C_1$.

Notice that $\leq$ is transitive. Let $G \subseteq \mathbb{P}_{\gamma_0}$ be V-generic, and let $\sigma_G = \bigcup\{\sigma : \exists C\ (\langle \sigma, C \rangle \in G)\}$. We claim that $(\text{cof}(\gamma_0))^{V[G]} = \omega$. In fact, we have the following:

Claim 1. In $V[G]$:

1. The order type of σ_G is ω.

[1] We use BC as a shorthand for "*There exists a Berkeley Cardinal*".

2. For all $C \in V$ such that C is ω-club in γ_0, $C \setminus \sigma_G$ is bounded.

Proof. Work in $V[G]$.

1. Since each σ is finite, it is clearly $\mathrm{ot}(\sigma_G) \leq \omega$. But for all $n \in \omega$, there exists a condition $\langle \sigma, C \rangle \in G$ with $\mathrm{ot}(\sigma) = n$: in fact, for $n \in \omega$, since any condition in $\mathbb{P}_{\gamma_0}$ can be extended to a condition $\langle \sigma, C \rangle$ such that $\mathrm{ot}(\sigma) = n$, the set $\{\langle \sigma, C \rangle \in \mathbb{P}_{\gamma_0} : \mathrm{ot}(\sigma) = n\}$ is dense, and so, it hits G. Therefore, $\mathrm{ot}(\sigma_G) = \omega$.

2. First, notice that for all $C \in V$ such that C is ω-club in γ_0, there exists a condition $\langle \sigma, D \rangle \in G$ such that $D \subseteq C$: in fact, if $C \in V$ is an ω-club in γ_0 and $\langle \sigma', C' \rangle \in \mathbb{P}_{\gamma_0}$, then $C \cap C'$ is an ω-club in γ_0, and we have that $\langle \sigma', C \cap C' \rangle \leq \langle \sigma', C' \rangle$, so the set $D_C = \{\langle \sigma, D \rangle : D \subseteq C\}$ is dense in $\mathbb{P}_{\gamma_0}$, and, therefore, $G \cap D_C \neq \emptyset$. Now, let $C \in V$ be an ω-club in γ_0. Then, there exists a condition $\langle \sigma, D \rangle \in G$ with $D \subseteq C$; it follows that $\langle \sigma, C \rangle \in G$ (as $\langle \sigma, D \rangle \leq \langle \sigma, C \rangle$), and for all further extensions $\langle \sigma'', C'' \rangle$, $\sigma'' \setminus \sigma \subseteq C$, hence $C \setminus \sigma_G$ is bounded.

□

It follows that in $V[G]$, σ_G is a club in γ_0 of order type ω. So, $(\mathrm{cof}(\gamma_0))^{V[G]} = \omega$. It remains to prove that $V[G] \models$ "γ_0 *is a Berkeley cardinal*". We preliminarily show the following:

Claim 2. Either

1. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*"

   or

2. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*".

Proof. Either there exists a condition $\langle \sigma_0, C_0 \rangle$ such that $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" or there exists a condition $\langle \sigma_1, C_1 \rangle$ such that $\langle \sigma_1, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*". Suppose $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*"; then, $\langle \emptyset, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*": in fact, since σ_0 is in V (and hence in all generic extensions), for every $G \subseteq \mathbb{P}_{\gamma_0}$ generic through $\langle \sigma_0, C_0 \rangle$, $G \setminus \sigma_0$ is generic through $\langle \emptyset, C_0 \rangle$ and $V[G] = V[G \setminus \sigma_0]$, and, conversely, for every $G \subseteq \mathbb{P}_{\gamma_0}$ generic through $\langle \emptyset, C_0 \rangle$, $G \cup \sigma_0$ is generic through $\langle \sigma_0, C_0 \rangle$ and $V[G] = V[G \cup \sigma_0]$ (therefore, we actually have that $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" iff $\langle \emptyset, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*"). Similarly, if $\langle \sigma_1, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*", then $\langle \emptyset, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*". Thus, there cannot exist $\langle \sigma_0, C_0 \rangle$, $\langle \sigma_1, C_1 \rangle \in \mathbb{P}_{\gamma_0}$ such that $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" and $\langle \sigma_1, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*", otherwise we would have that $\langle \emptyset, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" and $\langle \emptyset, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*", which is a contradiction because $\langle \emptyset, C_0 \rangle$ and $\langle \emptyset, C_1 \rangle$ are compatible conditions (in fact, $\langle \emptyset, C_0 \cap C_1 \rangle$ extends both). So, the conditions in $\mathbb{P}_{\gamma_0}$ that decide the statement $\varphi_{\mathrm{BC}} =_{\mathrm{df}}$ "γ_0 *is a Berkeley cardinal*" must decide it in the same way. Since for all formulas φ of the forcing language the set $D_\varphi = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi \vee p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi\}$ is dense, we have that either $D^+_{\varphi_{\mathrm{BC}}} = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}\}$ is dense or $D^-_{\varphi_{\mathrm{BC}}} = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}\}$ is dense, and so, either $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$ or $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}$ (as for all formulas φ, $p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi$ iff $\{q \leq p : q \Vdash_{\mathbb{P}_{\gamma_0}} \varphi\}$ is dense). □

Claim 3. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$.

Proof. For contradiction, assume not. Then, $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}$. So, let $\langle \sigma_0, C_0 \rangle \in \mathbb{P}_{\gamma_0}$ and $\tau \in \mathrm{Name}_{\mathbb{P}_{\gamma_0}}$ be such that $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "τ *is a transitive set such that* $\gamma_0 \in \tau$ *and there exists* $\xi < \gamma_0$ *such that there does not exist a non-trivial elementary embedding* $j : \tau \to \tau$ *with* $\xi < crit(j) < \gamma_0$". Fix such a ξ. Let $\eta >> \gamma_0$ be a limit ordinal such that $\tau \in V_\eta$ and $V_\eta \models \mathrm{ZF}^*$, where ZF^* indicates a big fragment of ZF which suffices to implement the proof that j lifts. Since γ_0 is a Berkeley cardinal in V, there exists an elementary embedding $j : V_\eta \to V_\eta$ such that $\xi < crit(j) < \gamma_0$, $j(\gamma_0) = \gamma_0$, $j(\tau) = \tau$, $j([S^{\gamma_0}_\omega]^{<\omega}) = [S^{\gamma_0}_\omega]^{<\omega}$, $j(\mathbb{P}_{\gamma_0}) = \mathbb{P}_{\gamma_0}$, $j(\langle \sigma_0, C_0 \rangle) = \langle \sigma_0, C_0 \rangle$. Let $C_j = \{\alpha \in S^{\gamma_0}_\omega : j(\alpha) = \alpha\} = \{\alpha < \gamma_0 : \mathrm{cof}(\alpha) = \omega \textit{ and } j(\alpha) = \alpha\}$. By Lemma 2.1.19, C_j is cofinal in γ_0; moreover, C_j is ω-club in γ_0 (in fact, if $\lambda < \gamma_0$ is a limit point of C_j such that $\mathrm{cof}(\lambda) = \omega$, then $j(\lambda) = \lambda \in C_j$), and we have that $\langle \sigma_0, C_0 \cap C_j \rangle \leq \langle \sigma_0, C_0 \rangle$. Now, let $G \subseteq \mathbb{P}_{\gamma_0}$ be a V-generic filter such that $\langle \sigma_0, C_0 \cap C_j \rangle \in G$. Then, $\langle \sigma_0, C_0 \rangle \in G$. Let $\sigma_G = \bigcup \{\sigma : \exists C \; (\langle \sigma, C \rangle \in G)\}$. Look at $j^{-1}``G = \{p \in \mathbb{P}_{\gamma_0} : j(p) \in G\}$. Let us show the following:

Subclaim. $G \subseteq j^{-1}``G$.

Proof. First, notice that for all $\langle \sigma, C \rangle \in \mathbb{P}_{\gamma_0}$ such that $\langle \sigma, C \rangle \leq \langle \sigma_0, C_0 \cap C_j \rangle$, $j(\sigma) = \sigma$: in fact, $j(\sigma_0) = \sigma_0$ (as $j(\langle \sigma_0, C_0 \rangle) = \langle j(\sigma_0), j(C_0) \rangle = \langle \sigma_0, C_0 \rangle$) and $\sigma = \sigma_0 \frown \bar{\sigma}$, where $\sigma \setminus \sigma_0 = \bar{\sigma} \subseteq C_0 \cap C_j \subseteq C_j$, but $j(\bar{\sigma}) = \bar{\sigma}$ (as $\bar{\sigma} \in [C_j]^{<\omega}$ and j is the identity on $[C_j]^{<\omega}$), so $j(\sigma) = j(\sigma_0 \frown \bar{\sigma}) = j(\sigma_0) \frown j(\bar{\sigma}) = \sigma_0 \frown \bar{\sigma} = \sigma$. Moreover, for all $C \subseteq C_j$, $C = j``C \subseteq j(C)$. It follows that for all $\langle \sigma, C \rangle \in \mathbb{P}_{\gamma_0}$ such that $\langle \sigma, C \rangle \leq \langle \sigma_0, C_0 \cap C_j \rangle$, $\langle \sigma, C \rangle \leq j(\langle \sigma, C \rangle)$: in fact, if $\langle \sigma, C \rangle \leq \langle \sigma_0, C_0 \cap C_j \rangle$, then $C \subseteq C_0 \cap C_j \subseteq C_j$, and so $C \subseteq j(C)$, which implies $\langle \sigma, C \rangle \leq \langle \sigma, j(C) \rangle = \langle j(\sigma), j(C) \rangle = j(\langle \sigma, C \rangle)$. Now, take any $\langle \sigma, C \rangle \in G$. Then, there exists a condition $\langle \sigma', C' \rangle \in G$ which extends both $\langle \sigma, C \rangle$ and $\langle \sigma_0, C_0 \cap C_j \rangle$. Since $\langle \sigma', C' \rangle \leq \langle \sigma_0, C_0 \cap C_j \rangle$, we have that $\langle \sigma', C' \rangle \leq j(\langle \sigma', C' \rangle)$, hence $j(\langle \sigma', C' \rangle) \in G$; but by elementarity, $j(\langle \sigma', C' \rangle) \leq j(\langle \sigma, C \rangle)$, so $j(\langle \sigma, C \rangle) \in G$, which implies $\langle \sigma, C \rangle \in j^{-1}``G$. Therefore, $G \subseteq j^{-1}``G$. □

It follows that $j``G = \{j(p) : p \in G\} \subseteq G$. So, by the "lifting criterion" we have that $j : V_\eta \to V_\eta$ lifts to an elementary embedding $j^* : V_\eta[G] \to V_\eta[G]$. Since $j^*(G) = G$ and $j^*(\tau) = j(\tau) = \tau$, we have that $j^*(\tau_G) = \tau_G$;

moreover, $\xi < crit(j^*) = crit(j) < \gamma_0$. Therefore, $j^* \upharpoonright \tau_G : \tau_G \to \tau_G$ is an elementary embedding such that $\xi < crit(j^* \upharpoonright \tau_G) < \gamma_0$, contradicting the choice of $\langle \sigma_0, C_0 \rangle$ and τ as a counterexample to γ_0 being a Berkeley cardinal in $V[G]$. This completes the proof that $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$, i.e., for all V-generic $G \subseteq \mathbb{P}_{\gamma_0}$, $V[G] \models$ "*γ_0 is a Berkeley cardinal*". □

In summary, we have shown that if $G \subseteq \mathbb{P}_{\gamma_0}$ is V-generic, then $V[G] \models$ "*γ_0 is a Berkeley cardinal*" + "$\mathrm{cof}(\gamma_0) = \omega$". It remains to handle a final issue: although γ_0 is a Berkeley cardinal in $V[G]$, it could be that γ_0 is not *the least* Berkeley cardinal in $V[G]$. So, suppose $\gamma_1 < \gamma_0$ is the least Berkeley cardinal in $V[G]$ and $(\mathrm{cof}(\gamma_1))^{V[G]} = \nu_1 > \omega$. Let $p_0 \in G_0 = G$ be a condition that forces this, and consider the product forcing $\mathbb{P}_{\gamma_0} \times \mathbb{P}_{\gamma_1}$; if $G_0 \times G_1$ is V-generic, then $V[G_0 \times G_1] = V[G_0][G_1] \models$ "*γ_1 is a Berkeley cardinal*" + "$\mathrm{cof}(\gamma_1) = \omega$". However, again, it could be that γ_1 is not *the least* Berkeley cardinal in $V[G_0][G_1]$, in which case, we continue. If the theorem continues to fail in the i^{th}-generic extension of V, then we let $p_i \in G_i$, γ_{i+1} and ν_{i+1} be as above and we force with $\mathbb{P}_{\gamma_0} \times \cdots \times \mathbb{P}_{\gamma_{i+1}}$; since $\gamma_0, \ldots, \gamma_{i+1}$ are decreasing, this procedure must terminate at some finite stage $i+1$, at which point $V[G_0] \ldots [G_{i+1}]$ satisfies that *the least* Berkeley cardinal has countable cofinality. □

3.2 Independence from ZF

The following theorem shows that it is consistent for the cofinality of the least Berkeley cardinal to be ω_1, establishing the announced undecidability result.

Theorem 3.2.1. Assume ZF + BC + DC.[2] Then there exists a forcing extension $V[G]$ of V such that $V[G] \models$ ZF + BC + DC + "$\mathrm{cof}(\delta_0) = \omega_1$, *where δ_0 is the least Berkeley cardinal* (*as computed in* $V[G]$)".

[2] DC is ω-DC.

Proof. Let $\gamma_0 = (\delta_0)^V$ denote the least Berkeley cardinal in V. By Theorem 2.3.6, we have $(\mathrm{cof}(\gamma_0))^V \geq \omega_1$. If $(\mathrm{cof}(\gamma_0))^V = \omega_1$ then we are done. So, assume $(\mathrm{cof}(\gamma_0))^V > \omega_1$. Let $\langle \mathbb{P}_{\gamma_0}, \leq_{\mathbb{P}_{\gamma_0}} \rangle$ be the forcing whose conditions are of the form $\langle \sigma, C \rangle$ where $\sigma \in [S^{\gamma_0}_\omega]^\omega = \{\sigma \subseteq S^{\gamma_0}_\omega : |\sigma| = \omega\}$ is a countable subset of $S^{\gamma_0}_\omega = \{\alpha < \gamma_0 : \mathrm{cof}(\alpha) = \omega\}$ and C is an ω-club in γ_0. Let $\leq_{\mathbb{P}_{\gamma_0}} = \leq$ be defined by putting for all $\langle \sigma_1, C_1 \rangle$, $\langle \sigma_2, C_2 \rangle \in \mathbb{P}_{\gamma_0}$, $\langle \sigma_2, C_2 \rangle \leq \langle \sigma_1, C_1 \rangle$ iff

1. $C_2 \subseteq C_1$,
2. $\sigma_1 \subseteq \sigma_2$,
3. $\sigma_2 \cap \sup(\sigma_1) = \sigma_1$ (i.e., σ_2 end-extends σ_1) and
4. $\sigma_2 \setminus \sigma_1 \subseteq C_1$.

It is immediate that $\leq$ is transitive. Moreover, since we are assuming DC, we have the following:

Claim 1. $\langle \mathbb{P}_{\gamma_0}, \leq_{\mathbb{P}_{\gamma_0}} \rangle$ is ω-closed, i.e., for every decreasing ω-sequence $\langle p_n : n \in \omega \rangle$ of elements of $\mathbb{P}_{\gamma_0}$ (i.e., such that $m < n \to p_n \leq_{\mathbb{P}_{\gamma_0}} p_m$) there exists $q \in \mathbb{P}_{\gamma_0}$ such that $q \leq_{\mathbb{P}_{\gamma_0}} p_n$ for all $n \in \omega$.

Proof. Suppose $\langle \langle \sigma_n, C_n \rangle : n \in \omega \rangle$ is a decreasing ω-sequence in $\mathbb{P}_{\gamma_0}$. Then, $\langle \bigcup_{n \in \omega} \sigma_n, \bigcap_{n \in \omega} C_n \rangle \in \mathbb{P}_{\gamma_0}$: in fact, by DC, $\bigcup_{n \in \omega} \sigma_n \in [S^{\gamma_0}_\omega]^\omega$ and $\bigcap_{n \in \omega} C_n$ is an ω-club in γ_0. Moreover, $\langle \bigcup_{n \in \omega} \sigma_n, \bigcap_{n \in \omega} C_n \rangle \leq \langle \sigma_n, C_n \rangle$ for all $n \in \omega$. □

Let $G \subseteq \mathbb{P}_{\gamma_0}$ be V-generic, and let $\sigma_G = \bigcup \{\sigma : \exists C\, (\langle \sigma, C \rangle \in G)\}$. Since $\mathbb{P}_{\gamma_0}$ is ω-closed, we have that $(\omega_1)^{V[G]} = (\omega_1)^V$. We show that $(\mathrm{cof}(\gamma_0))^{V[G]} \leq \omega_1$. In fact, the following holds:

Claim 2. In $V[G]$:

1. The order type of σ_G is ω_1.
2. For all $C \in V$ such that C is ω-club in γ_0, $C \setminus \sigma_G$ is bounded.

Proof. Work in $V[G]$.

1. Since each σ is countable, it is clearly $\mathrm{ot}(\sigma_G) \leq \omega_1$. But for any $\alpha < \omega_1$, since every condition in $\mathbb{P}_{\gamma_0}$ can be extended to a condition $\langle \sigma, C \rangle$ such that $\mathrm{ot}(\sigma) = \alpha$, the set $\{\langle \sigma, C \rangle \in \mathbb{P}_{\gamma_0} : \mathrm{ot}(\sigma) = \alpha\}$ is dense, and so, there exists a condition $\langle \sigma, C \rangle \in G$ with $\mathrm{ot}(\sigma) = \alpha$. Therefore, $\mathrm{ot}(\sigma_G) = \omega_1$.

2. Let $C \in V$ be an ω-club in γ_0. If $\langle \sigma', C' \rangle \in \mathbb{P}_{\gamma_0}$, then $C \cap C'$ is an ω-club in γ_0 and $\langle \sigma', C \cap C' \rangle \leq \langle \sigma', C' \rangle$; so, the set $\{\langle \sigma, D \rangle : D \subseteq C\}$ is dense in $\mathbb{P}_{\gamma_0}$. Therefore, there exists a condition $\langle \sigma, D \rangle \in G$ such that $D \subseteq C$; it follows that $\langle \sigma, C \rangle \in G$, and for all further extensions $\langle \sigma'', C'' \rangle$, $\sigma'' \setminus \sigma \subseteq C$.

□

It follows that in $V[G]$, σ_G is a club in γ_0 of order type ω_1. So, $(\mathrm{cof}(\gamma_0))^{V[G]} \leq \omega_1$. In order to prove that γ_0 is still a Berkeley cardinal in $V[G]$, we preliminarily show the following:

Claim 3. Either

1. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*"

   or

2. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*".

Proof. Either there exists a condition $\langle \sigma_0, C_0 \rangle$ such that $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" or there exists a condition $\langle \sigma_1, C_1 \rangle$ such that $\langle \sigma_1, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*". Notice that if $\langle \sigma_0, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*", then $\langle \emptyset, C_0 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is a Berkeley cardinal*" (in fact, since σ_0 is in V, for every generic through $\langle \sigma_0, C_0 \rangle$ there is an equivalent generic through $\langle \emptyset, C_0 \rangle$, in the sense of yielding the same generic extension, and conversely). Likewise, if $\langle \sigma_1, C_1 \rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*" then

$\langle\emptyset, C_1\rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "γ_0 *is not a Berkeley cardinal*". But $\langle\emptyset, C_0\rangle$ and $\langle\emptyset, C_1\rangle$ are compatible conditions, and so, every condition in $\mathbb{P}_{\gamma_0}$ that decides the statement $\varphi_{\mathrm{BC}} =_{\mathrm{df}}$ "γ_0 *is a Berkeley cardinal*" must decide it in the same way. Since for all formulas φ of the forcing language the set $D_\varphi = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi \vee p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi\}$ is dense, we have that either $D^+_{\varphi_{\mathrm{BC}}} = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}\}$ is dense or $D^-_{\varphi_{\mathrm{BC}}} = \{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}\}$ is dense, namely, either $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$ or $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}$. □

Claim 4. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$.

Proof. For contradiction, assume not. Then, $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi_{\mathrm{BC}}$. So, let $\langle\sigma_0, C_0\rangle \in \mathbb{P}_{\gamma_0}$ and $\tau \in \mathrm{Name}_{\mathbb{P}_{\gamma_0}}$ be such that $\langle\sigma_0, C_0\rangle \Vdash_{\mathbb{P}_{\gamma_0}}$ "τ *is a transitive set such that* $\gamma_0 \in \tau$ *and there exists* $\xi < \gamma_0$ *such that there does not exist a non-trivial elementary embedding* $j : \tau \to \tau$ *with* $\xi < crit(j) < \gamma_0$". Fix such a ξ. Let $\eta >> \gamma_0$ be a limit ordinal such that $\tau \in V_\eta$ and $V_\eta \models \mathrm{ZF}^*$, where ZF^* indicates a big fragment of ZF which suffices to implement the proof that j lifts. Since γ_0 is a Berkeley cardinal in V, there exists an elementary embedding $j : V_\eta \to V_\eta$ such that $\xi < crit(j) < \gamma_0$, $j(\gamma_0) = \gamma_0$, $j(\tau) = \tau$, $j([S^{\gamma_0}_\omega]^\omega) = [S^{\gamma_0}_\omega]^\omega$, $j(\mathbb{P}_{\gamma_0}) = \mathbb{P}_{\gamma_0}$, $j(\langle\sigma_0, C_0\rangle) = \langle\sigma_0, C_0\rangle$. Let $C_j = \{\alpha \in S^{\gamma_0}_\omega : j(\alpha) = \alpha\} = \{\alpha < \gamma_0 : \mathrm{cof}(\alpha) = \omega \textit{ and } j(\alpha) = \alpha\}$. By Lemma 2.1.19, C_j is cofinal in γ_0; moreover, C_j is ω-club in γ_0, and we have that $\langle\sigma_0, C_0 \cap C_j\rangle \leq \langle\sigma_0, C_0\rangle$. Now, let $G \subseteq \mathbb{P}_{\gamma_0}$ be a V-generic filter such that $\langle\sigma_0, C_0 \cap C_j\rangle \in G$, and let $\sigma_G = \bigcup\{\sigma : \exists C\ (\langle\sigma, C\rangle \in G)\}$. It follows that $\langle\sigma_0, C_0\rangle \in G$. Moreover:

Subclaim. $G \subseteq j^{-1}``G$.

Proof. First, notice that for all $\langle\sigma, C\rangle \in \mathbb{P}_{\gamma_0}$ such that $\langle\sigma, C\rangle \leq \langle\sigma_0, C_0 \cap C_j\rangle$, $j(\sigma) = \sigma$: in fact, $j(\sigma_0) = \sigma_0$ (as $j(\langle\sigma_0, C_0\rangle) = \langle j(\sigma_0), j(C_0)\rangle = \langle\sigma_0, C_0\rangle$) and $\sigma = \sigma_0{}^\frown\bar{\sigma}$, where $\sigma \setminus \sigma_0 = \bar{\sigma} \subseteq C_0 \cap C_j \subseteq C_j$, but $j(\bar{\sigma}) = \bar{\sigma}$ (as $\bar{\sigma} \in [C_j]^\omega$ and j is the identity on $[C_j]^\omega$), so $j(\sigma) = j(\sigma_0{}^\frown\bar{\sigma}) = j(\sigma_0)^\frown j(\bar{\sigma}) =$

$\sigma_0^\frown\bar{\sigma} = \sigma$. Moreover, for all $C \subseteq C_j$, $C = j``C \subseteq j(C)$. It follows that for all $\langle\sigma, C\rangle \in \mathbb{P}_{\gamma_0}$ such that $\langle\sigma, C\rangle \leq \langle\sigma_0, C_0 \cap C_j\rangle$, $\langle\sigma, C\rangle \leq j(\langle\sigma, C\rangle)$: in fact, if $\langle\sigma, C\rangle \leq \langle\sigma_0, C_0 \cap C_j\rangle$, then $C \subseteq C_0 \cap C_j \subseteq C_j$, so $C \subseteq j(C)$, which implies $\langle\sigma, C\rangle \leq \langle\sigma, j(C)\rangle = \langle j(\sigma), j(C)\rangle = j(\langle\sigma, C\rangle)$. Now, take any $\langle\sigma, C\rangle \in G$. Then, there exists a condition $\langle\sigma', C'\rangle \in G$ which extends both $\langle\sigma, C\rangle$ and $\langle\sigma_0, C_0 \cap C_j\rangle$. Since $\langle\sigma', C'\rangle \leq \langle\sigma_0, C_0 \cap C_j\rangle$, we have that $\langle\sigma', C'\rangle \leq j(\langle\sigma', C'\rangle)$, hence $j(\langle\sigma', C'\rangle) \in G$; but by elementarity, $j(\langle\sigma', C'\rangle) \leq j(\langle\sigma, C\rangle)$, so $j(\langle\sigma, C\rangle) \in G$, which implies $\langle\sigma, C\rangle \in j^{-1}``G$. Therefore, $G \subseteq j^{-1}``G$. $\square$

It follows that $j``G \subseteq G$. So, by the "lifting criterion" we have that $j : V_\eta \to V_\eta$ lifts to an elementary embedding $j^* : V_\eta[G] \to V_\eta[G]$. Since $j^*(G) = G$ and $j^*(\tau) = j(\tau) = \tau$, we have that $j^*(\tau_G) = \tau_G$; moreover, $\xi < crit(j^*) = crit(j) < \gamma_0$. Therefore, $j^* \upharpoonright \tau_G : \tau_G \to \tau_G$ is an elementary embedding such that $\xi < crit(j^* \upharpoonright \tau_G) < \gamma_0$, contradicting the choice of $\langle\sigma_0, C_0\rangle$ and τ as a counterexample to γ_0 being a Berkeley cardinal in $V[G]$. This completes the proof that $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi_{\mathrm{BC}}$, i.e., for all V-generic $G \subseteq \mathbb{P}_{\gamma_0}$, $V[G] \models$ "*γ_0 is a Berkeley cardinal*". $\square$

To summarize, we have shown that if $G \subseteq \mathbb{P}_{\gamma_0}$ is V-generic, then $V[G] \models$ "*γ_0 is a Berkeley cardinal*" + "$\mathrm{cof}(\gamma_0) \leq \omega_1$". Notice that, since DC is preserved, if γ_0 is still *the least* Berkeley cardinal in $V[G]$, then it is also $\mathrm{cof}(\gamma_0)^{V[G]} \geq \omega_1$, and so, we are done. Therefore, it remains to handle the case that $(\delta_0)^{V[G]} < \gamma_0$. So, suppose $\gamma_1 < \gamma_0$ is the least Berkeley cardinal in $V[G]$. Since $V[G] \models \mathrm{DC}$, $(\mathrm{cof}(\gamma_1))^{V[G]} = \nu_1 \geq \omega_1$. If $\nu_1 = \omega_1$ then we are done. So, assume $\nu_1 > \omega_1$. Let $p_0 \in G_0 = G$ be a condition that forces this, and consider the product forcing $\mathbb{P}_{\gamma_0} \times \mathbb{P}_{\gamma_1}$; if $G_0 \times G_1$ is V-generic, then $V[G_0 \times G_1] = V[G_0][G_1] \models$ "*γ_1 is a Berkeley cardinal*" + "$\mathrm{cof}(\gamma_1) \leq \omega_1$". However, again, it could be that γ_1 is not *the least* Berkeley cardinal in $V[G_0][G_1]$, in which case, we continue. If the theorem continues to fail in the i^{th}-generic extension of V, then we let $p_i \in G_i$, γ_{i+1} and ν_{i+1} be as above and we force with $\mathbb{P}_{\gamma_0} \times \cdots \times \mathbb{P}_{\gamma_{i+1}}$; since $\gamma_0, \ldots, \gamma_{i+1}$ are decreasing, this procedure must terminate at some finite stage $i+1$, at which point $V[G_0]\ldots[G_{i+1}]$ satisfies that *the least* Berkeley cardinal has cofinality less than or equal to ω_1, which implies, by DC, that the cofinality is equal to ω_1. □

3.3 Toward a very deep Inconsistency

From the results achieved so far, we know that:

1. the cofinality of the least Berkeley cardinal is connected with the failure of AC (the lower the cofinality, the greater the failure)

   and

2. the cofinality of the least Berkeley cardinal is independent of ZF.

The natural question arises as to whether the statement "$\mathrm{cof}(\delta_0) = \omega$" is weaker in consistency strength than the statement "$\mathrm{cof}(\delta_0) = \omega_1$" (where δ_0 is the least Berkeley cardinal), and so on. If so, then we would have a hierarchy of increasingly strong hypotheses, enclosing increasingly big fragments of AC:

- ZF + BC + "$\mathrm{cof}(\delta_0) = \omega$"
- ZF + BC + DC + "$\mathrm{cof}(\delta_0) = \omega_1$"
- ZF + BC + ω_1-DC + "$\mathrm{cof}(\delta_0) = \omega_2$"

  ...

- ZF + BC+ $<\delta_0$-DC + "$\mathrm{cof}(\delta_0) = \delta_0$"[3]

[3] $<\delta_0$-DC is γ-DC for all $\gamma < \delta_0$.

Notice that if ZF + BC + "cof(δ_0) $> \omega$" proves that there exists $\gamma < \delta_0$ such that $V_\gamma \models$ ZF + BC + "cof(δ_0) $= \omega$", then it would follow that ZF + BC + DC is inconsistent.

The more appealing motivation to investigate choiceless large cardinals is indeed that of finding an even deeper inconsistency result. This search for "deep inconsistency" points toward a new proof of inconsistency of a large cardinal axiom in just ZF (after Kunen's proof that Reinhardt cardinals are inconsistent with AC). And in fact, it follows from results of Woodin in [6] that

If the HOD Conjecture holds then there are no Berkeley cardinals.

Recall that the HOD Conjecture is the following statement:

HOD Conjecture. There exists a proper class of uncountable regular cardinals κ such that for all $\gamma < \kappa$, if γ is an infinite cardinal in HOD and $(2^\gamma)^{\mathrm{HOD}} < \kappa$ then there exists a partition $\langle S_\alpha : \alpha < \gamma \rangle$ of $\{\xi < \kappa : \mathrm{cof}(\xi) = \omega\}$ into stationary sets such that $\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$.

In particular, a consequence of the assumption that the HOD Conjecture is provable contradicts Theorem 2.2.1. Therefore, a proof of the HOD Conjecture would actually lead to a new very deep inconsistency result.

But what if choiceless large cardinals enable us to developing a remarkable mathematical theory?

The rest of this dissertation is aimed to show that this could indeed be the case. In fact, we will be concerned with the structural properties of $L(V_{\delta+1})$ under the assumption that δ is a singular limit of Berkeley cardinals which are limit of extendibles. Of course, it could be that this pattern leads to inconsistency (and anyhow, this would be very interesting, in the spirit traced above); on the other side, it could also be that this investigation provides us with motivations for arguing in favour of the consistency (with ZF) of choiceless large cardinal axioms (which would be really interesting as well) . . .

Chapter 4

Berkeley Cardinals and the structure of $L(V_{\delta+1})$

In the present chapter we look at the structure theory of $L(V_{\delta+1})$ when δ satisfies a certain large cardinal property which is proved to be consistent by a limit club Berkeley cardinal: it turns out that, in the resulting framework, we are able to get several remarkable I_0-like results. First, we provide a proof of the Coding Lemma, and the subsequent "strong-limitness" of Θ. We are then interested in showing the regularity of δ^+. Finally, we are concerned with the existence of measurable cardinals in $L(V_{\delta+1})$ (recall that I_0 is the statement that there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point below λ; for a sample of the I_0 case see [7]).

4.1 The framework provided by a limit club Berkeley

Before starting our analysis of $L(V_{\delta+1})$, we preliminarily need to justify the large cardinal axiom we will require to hold at δ. Precisely, we will assume that

δ is a singular limit of Berkeley cardinals each of which is a limit of extendibles. And our first task is to show that this hypothesis is in fact consistent by a limit club Berkeley cardinal.

Remark 4.1.1. Notice that if δ^* is a limit club Berkeley cardinal then $(V_{\delta^*}, V_{\delta^*+1}) \models \mathrm{ZF}_2 +$ "*There exists a proper class of Berkeley cardinals*" + "*There exists a proper class of extendible cardinals*".

The next lemma is just a consequence of the fact that extendible cardinals are Σ_3 reflecting.

Lemma 4.1.2. Suppose κ is extendible, and there exists a Berkeley cardinal above κ. Then, κ is a limit of Berkeley cardinals.

Corollary 4.1.3. Suppose κ is extendible, and there exists a Berkeley cardinal above κ. Then, κ is a Berkeley cardinal.

Proof. Immediate, since the class of Berkeley cardinals is closed. □

Corollary 4.1.4. Suppose κ is a limit of extendibles. Then, κ is a Berkeley cardinal iff κ is a limit of Berkeley cardinals.

Corollary 4.1.5. Suppose κ is a limit of extendibles, and there exists a Berkeley cardinal above κ. Then, κ is a Berkeley cardinal (in fact, κ is a limit of Berkeley cardinals).

By Remark 4.1.1 and Corollary 4.1.5, we immediately have the following:

Corollary 4.1.6. Suppose that δ^* is a limit club Berkeley cardinal. Then, $(V_{\delta^*}, V_{\delta^*+1}) \models \mathrm{ZF}_2 +$ "*There exists δ such that δ is a limit of Berkeley cardinals which are limit of extendibles*".

There is an important feature of extendible cardinals we have to point out here: as showed by Woodin in [6], extendibles enable to force choice. Moreover, the forcing in question preserves extendibles and Berkeley cardinals. In particular, in our case, assuming δ is a limit of Berkeley cardinals each of which is a limit of extendible cardinals and $\mathrm{cof}(\delta) < \delta$, we can force DC past the cofinality of δ and preserve that δ is a limit of Berkeley cardinals which are limit of extendibles. And in fact, in carrying out the study of $L(V_{\delta+1})$ we will assume $(\mathrm{cof}(\delta))^+$-DC.

The following theorem synthesizes the results we will establish in the rest of the chapter. We first recall the definition of Θ.

Definition 4.1.7. $\Theta = \Theta^{L(V_{\delta+1})} =_{\mathrm{df}} \sup\{\alpha : \exists\, \pi : V_{\delta+1} \xrightarrow{onto} \alpha \ (\pi \in L(V_{\delta+1}))\}$.

Theorem 4.1.8. Suppose δ is a limit of Berkeley cardinals which are limit of extendibles. Suppose $\mathrm{cof}(\delta) < \delta$. Assume $(\mathrm{cof}(\delta))^+$-DC.[1] Then:

1. $L(V_{\delta+1}) \models$ "*The Coding Lemma*".
2. $L(V_{\delta+1}) \models$ "*For all $\alpha < \Theta$ there exists a surjection $\rho : V_{\delta+1} \xrightarrow{onto} \mathcal{P}(\alpha)$*".
3. $L(V_{\delta+1}) \models$ "*δ^+ is regular*".
4. $L(V_{\delta+1}) \models$ "*δ^+ is measurable*".
5. $L(V_{\delta+1}) \models$ "*Θ is limit of measurable cardinals*".

Throughout this chapter we will refer to the assumption of Theorem 4.1.8 as the theory T^*:

Definition 4.1.9. $\mathrm{T}^* =_{\mathrm{df}}$ "*δ is a limit of Berkeley cardinals which are limit of extendibles*" + "$\mathrm{cof}(\delta) < \delta$" + $(\mathrm{cof}(\delta))^+$-DC.

[1]Notice that the assumption of $(\mathrm{cof}(\delta))^+$-DC implies $L(V_{\delta+1}) \models (\mathrm{cof}(\delta))^+$-DC.

4.2 The Coding Lemma and the size of Θ

As observed after introducing the Berkeley hierarchy, the distinctive feature of the work environment provided by Berkeley cardinals lies in the fact that we can arrange elementary embeddings fixing any given set. This will also be used in showing that the weaker version of the Coding Lemma holds in $L(V_{\delta+1})$; the Coding Lemma will follow (notice that in the I_0 case the proof of the Coding Lemma employs λ-DC, while we don't have δ-DC here).

Lemma 4.2.1 (**Weak Coding Lemma**). Assume T^*. Suppose $\alpha < \Theta$. Then there exists $\Gamma \subseteq \mathcal{P}(V_{\delta+1} \times V_{\delta+1})$ such that:

1. "Γ *is small*", i.e., there exists $\pi : V_{\delta+1} \xrightarrow{onto} \Gamma$;

2. if $Z \subseteq V_{\delta+1}$, $\leq$ is a pre-well-ordering of Z of length α and $W \subseteq Z \times V_{\delta+1}$, then there exists $W^* \subseteq W$ such that:

   (a) $W^* \in \Gamma$;

   (b) for cofinally many $\eta < \alpha$, if Z_η is the η-component and $W \cap Z_\eta \times V_{\delta+1} \neq \emptyset$, then $W^* \cap Z_\eta \times V_{\delta+1} \neq \emptyset$.

Proof. Fix $\alpha < \Theta$. Let $\beta_0 < \Theta$. Start with $\Gamma = L_{\beta_0}(V_{\delta+1})$. If $L_{\beta_0}(V_{\delta+1})$ doesn't witness the Weak Coding Lemma at α, then there exists a counterexample $\langle Z, \rho, W \rangle$ with $Z \subseteq V_{\delta+1}$, $\rho : Z \xrightarrow{onto} \alpha$ (the norm associated to the pre-well-ordering serving as a counterexample) and $W \subseteq Z \times V_{\delta+1}$, and we try $L_{\beta_1}(V_{\delta+1})$; if $L_{\beta_1}(V_{\delta+1})$ doesn't witness the Weak Coding Lemma at α, then we try $L_{\beta_2}(V_{\delta+1})$, and so forth (notice that, although we cannot choose at any step a counterexample, we can certainly choose successively ordinals $\beta_i < \Theta$ for which $L_{\beta_i}(V_{\delta+1})$ doesn't witness the Weak Coding Lemma at α). For i limit, we let $\beta_i =_{\mathrm{df}} \sup\{\beta_j : j < i\}$. We claim that the sequence $\langle \beta_i : i < \delta \rangle$ is not defined, i.e., there exists $i < \delta$ such that $L_{\beta_i}(V_{\delta+1})$ witnesses the Weak Coding Lemma at α. For contradiction, suppose $\langle \beta_i : i < \delta \rangle$ is defined. Since δ is a Berkeley cardinal, there exists a non-trivial elementary embedding $j : L(V_{\delta+1}) \to L(V_{\delta+1})$ such that $\mathrm{cof}(\delta) < crit(j) < \delta$, $j(\delta) = \delta$, $j(\alpha) = \alpha$ and $j(\langle \beta_i : i < \delta \rangle) = \langle \beta_i : i < \delta \rangle$. Let $crit(j) = \kappa$; we have that $j(\beta_\kappa) = \beta_{j(\kappa)}$. Since $L_{\beta_\kappa}(V_{\delta+1})$ doesn't witness the Weak Coding Lemma at α, there exists a counterexample $\langle Z, \rho, W \rangle$; by elementarity, $\langle j(Z), j(\rho), j(W) \rangle$ is a counterexample for $L_{\beta_{j(\kappa)}}(V_{\delta+1})$ to be witness of the Weak Coding Lemma at $j(\alpha) = \alpha$. Now, notice that $W \in L_{\beta_{\kappa+1}}(V_{\delta+1})$ and $\beta_{\kappa+1} << \beta_{j(\kappa)}$. Look at $j``W$. To compute $j``W$, we only need $j \restriction V_\delta$ (as $W \subseteq Z \times V_{\delta+1} \subseteq V_{\delta+1} \times V_{\delta+1}$ and $j \restriction V_{\delta+1}$ is defined by $j \restriction V_\delta$); but $j \restriction V_\delta \in V_{\delta+1}$, so $j``W \in L_{\beta_{\kappa+1}+1}(V_{\delta+1}) \subseteq L_{\beta_{j(\kappa)}}(V_{\delta+1})$ (in fact, $j``W$ is definable from the parameters $j \restriction V_\delta \in V_{\delta+1}$ and $W \in L_{\beta_{\kappa+1}}(V_{\delta+1})$). Moreover, $j``W \subseteq j(W)$ and $j``\alpha$ is cofinal in $j(\alpha)$ (as $j(\alpha) = \alpha$). It follows that for cofinally many $\eta < j(\alpha) = \alpha$, if there exists $\langle a, b \rangle \in j(W)$ such that $j(\rho)(a) = \eta$ then there exists $\langle a, b \rangle \in j``W$ such that $j(\rho)(a) = \eta$. Therefore, $j``W$ satisfies the Weak Coding Lemma at α, contradicting the fact that $\langle j(Z), j(\rho), j(W) \rangle$ was a counterexample for $L_{\beta_{j(\kappa)}}(V_{\delta+1})$ to be witness of the Weak Coding Lemma at α. $\square$

Theorem 4.2.2 (Coding Lemma). Assume T*. Suppose $\alpha < \Theta$. Then there exists $\Gamma \subseteq \mathcal{P}(V_{\delta+1} \times V_{\delta+1})$ such that:

1. "Γ *is small*", i.e., there exists $\pi : V_{\delta+1} \xrightarrow{onto} \Gamma$;

2. if $Z \subseteq V_{\delta+1}$, $\leq$ is a pre-well-ordering of Z of length α and $W \subseteq Z \times V_{\delta+1}$, then there exists $W^* \subseteq W$ such that:

   (a) $W^* \in \Gamma$;

   (b) for all $\eta < \alpha$, if Z_η is the η-component and $W \cap Z_\eta \times V_{\delta+1} \neq \emptyset$, then $W^* \cap Z_\eta \times V_{\delta+1} \neq \emptyset$.

Proof. For contradiction, assume that the Coding Lemma is false. Let α_0 be the least such that the Coding Lemma fails in $L(V_{\delta+1})$. Notice that if the Coding Lemma holds for α, then the Coding Lemma holds for $\alpha+1$ (in fact, if $W \cap Z_\alpha \times V_{\delta+1} \neq \emptyset$, then we just choose an element in this set and add it to the set W^* which satisfies the Coding Lemma at α). So, α_0 must be limit. For every $\alpha < \alpha_0$, let $\beta_\alpha < \Theta$ be such that $L_{\beta_\alpha}(V_{\delta+1})$ witnesses the Coding Lemma at α. Then, choose $\beta < \Theta$ large enough so that $\Gamma = L_\beta(V_{\delta+1})$ witnesses the Coding Lemma for all $\alpha < \alpha_0$ and the Weak Coding Lemma at α_0. Now, let $\beta_0 > \beta$ be such that in $L_{\beta_0}(V_{\delta+1})$ there exists a map $\pi_0 : V_{\delta+1} \xrightarrow{onto} L_\beta(V_{\delta+1})$. We claim that $L_{\beta_0+1}(V_{\delta+1})$ witnesses the Coding Lemma at α_0. Let $Z \subseteq V_{\delta+1}$, $\rho : Z \xrightarrow{onto} \alpha_0$ and $W \subseteq Z \times V_{\delta+1}$; for all $\alpha < \alpha_0$, let $W_\alpha = W \cap Z_\alpha \times V_{\delta+1}$. Let $Z^* = \{x \in V_{\delta+1} : \pi_0(x) \subseteq W\}$, and define $\rho^* : Z^* \to \alpha_0$ such that $\rho^*(x)$ is the least $\alpha < \alpha_0$ such that $\pi_0(x)$ is not good at α, i.e., $W_\alpha \neq \emptyset$ but $\pi_0(x) \cap Z_\alpha \times V_{\delta+1} = \emptyset$. First, notice that ρ^* is well-defined: in fact, for every $x \in Z^*$, either $\pi_0(x)$ satisfies the Coding Lemma at α_0 (in which case, we are done) or there exists $\alpha < \alpha_0$ such that $\pi_0(x)$ is not good at α. Moreover, ρ^* is onto: in fact, for any $\alpha < \alpha_0$, since $L_\beta(V_{\delta+1})$ witnesses the Coding Lemma at α, there exists $W^* \in L_\beta(V_{\delta+1})$ as in the statement of the Coding Lemma, but π_0 is onto, so $W^* = \pi_0(x)$ for some $x \in V_{\delta+1}$, and since $W^* \subseteq W$, we actually have that $W^* = \pi_0(x)$ for some $x \in Z^*$ and $\pi_0(x)$ is good at η for all $\eta < \alpha$. Now, let $W^* = Z^* \times V_{\delta+1}$. By the Weak Coding Lemma, there exists $Y^* \subseteq Z^*$ such that $Y^* \in L_\beta(V_{\delta+1})$ and $\rho^*``Y^*$ is cofinal in $\rho^*``Z^* = \alpha_0$. Finally, let $Y = \bigcup_{x \in Y^*} \pi_0(x)$. Then, $Y \subseteq W$ and $Y \in L_{\beta_0+1}(V_{\delta+1})$. Moreover, let us show that if $W_\alpha \neq \emptyset$, it is also $Y \cap Z_\alpha \times V_{\delta+1} \neq \emptyset$. The point is that for all $\alpha < \alpha_0$ there exists $x \in Y^*$ such that $\pi_0(x)$ is good at α: in fact, for every $\alpha < \alpha_0$ there exists $x \in Y^*$ such that $\rho^*(x) > \alpha$, and since $\rho^*(x)$ is the least such that $\pi_0(x)$ is not good, $\pi_0(x)$ is good at α. It follows that $Y = \bigcup_{x \in Y^*} \pi_0(x)$ is good at α for all $\alpha < \alpha_0$. Therefore, Y satisfies the Coding Lemma at

α_0, and so, $L_{\beta_0+1}(V_{\delta+1})$ witnesses the Coding Lemma at α_0, contradicting our assumption. $\square$

A remarkable consequence of the Coding Lemma is the following result concerning the size of Θ.

Corollary 4.2.3. Assume T^*. Then in $L(V_{\delta+1})$, for all $\alpha < \Theta$ there exists $\rho : V_{\delta+1} \xrightarrow{onto} \mathcal{P}(\alpha)$.

Proof. Let $\alpha < \Theta$ and $\pi : V_{\delta+1} \xrightarrow{onto} \alpha$. Let $\beta < \Theta$ be the least such that $\pi \in L_\beta(V_{\delta+1})$. Let $L_{\beta^*}(V_{\delta+1})$ witness the Coding Lemma at α, and let $\gamma = \max\{\beta, \beta^*\}$. For every $X \subseteq \alpha$, let $X^* =_{\mathrm{df}} \{\langle a, \emptyset\rangle \in V_{\delta+1} \times V_{\delta+1} : \pi(a) \in X\}$. Then, by the Coding Lemma, there exists $W^* \subseteq X^*$ such that $W^* \in L_\gamma(V_{\delta+1})$ and for all $\eta < \alpha$, if there exists $\langle a, b\rangle \in X^*$ such that $\pi(a) = \eta$ then there exists $\langle a, b\rangle \in W^*$ such that $\pi(a) = \eta$. Since π is a surjection, for every $\eta \in X$ there exists $\langle a, b\rangle \in X^*$ such that $\pi(a) = \eta$; so, we have that $\{\pi(a) : \exists\, b\ (\langle a, b\rangle \in W^*)\} = X$. It follows that $X \in L_{\gamma+1}(V_{\delta+1})$. But X was an arbitrary subset of α, hence $\mathcal{P}(\alpha) \subseteq L_{\gamma+1}(V_{\delta+1})$, and since there exists $\rho : V_{\delta+1} \xrightarrow{onto} L_{\gamma+1}(V_{\delta+1})$, the proof is complete. $\square$

4.3 Regularity of δ^+

We are now going to show that δ^+ is a regular cardinal in $L(V_{\delta+1})$. It will follow that V_δ cannot be mapped cofinally into δ^+. In fact, by the following lemma the existence of such a cofinal map would imply that δ can be mapped cofinally into δ^+.

Lemma 4.3.1. Assume T^*. Suppose $\delta < \kappa < \Theta$ and there exists $\pi : V_\delta \to \kappa$ cofinal. Then there exists $\pi^* : \delta \to \kappa$ cofinal.

Proof. By assumption, $\sup \pi``V_\delta = \kappa$. Since $\sup \pi``V_\delta = \sup_{\alpha<\delta}\{\sup \pi``V_\alpha\}$, we can reduce to the case that there exists $\pi : V_\alpha \to \kappa$ cofinal with $\alpha < \delta$. Look at $\mathrm{HOD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}}$, containing all the sets which are hereditarily ordinal definable with parameters π and V_α in $L(V_{\delta+1})$. Notice that $\mathrm{HOD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}} \models \mathrm{ZFC}$. Let $G \subseteq \mathrm{Coll}((\mathrm{cof}(\delta))^+, V_\alpha)$ be V-generic. Then, $\mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}} \models \mathrm{ZFC}$. Since the forcing $\mathrm{Coll}((\mathrm{cof}(\delta))^+, V_\alpha)$ is homogeneous, we have that $\mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}} = \mathrm{HOD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}}$ (in fact, each of the two models can be defined into the other). Let $\mathbb{B} = \mathcal{P}(\mathcal{P}(\alpha)) \cap \mathrm{OD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}} = \{A \subseteq \mathcal{P}(\alpha) : A \in \mathrm{OD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}\}$. Then, $\mathbb{B}$ is ordinal definable from π and V_α, and there exist $\mathbb{B}^* \in \mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$ and an isomorphism $\rho : \mathbb{B}^* \cong \mathbb{B}$ such that ρ is ordinal definable from π and V_α. For every $X \subseteq \alpha$, let $G_X = \{A \in \mathbb{B} : X \in A\}$ and $G^*_X = \rho^{-1}``G_X = \{B \in \mathbb{B}^* : \rho(B) \in G_X\}$. By Vopěnka's basic argument, G^*_X is $\mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$-generic for $\mathbb{B}^*$ and $\mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}[G^*_X] = \mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha,X\}}$. Let us show the following:

Claim. There exists $\pi^* : \mathbb{B}^* \to \kappa$ cofinal such that $\pi^* \in \mathrm{OD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}}$.

Proof. First, notice that in $\mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$, every $a \in V_\alpha$ is coded by some $X \subseteq \alpha$. For every $\xi \in \pi``V_\alpha$, let $A_\xi = \{X \subseteq \alpha : $ "$X$ *codes some* $a \in V_\alpha$" $\wedge\, \pi(a) = \xi\} \in \mathcal{P}(\mathcal{P}(\alpha))$; then, for every $\xi \in \pi``V_\alpha$, A_ξ is ordinal definable with parameters π and V_α, i.e., $A_\xi \in \mathrm{OD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$, and so, $A_\xi \in \mathbb{B}$. Therefore, there exists $e : \mathbb{B} \xrightarrow{onto} \pi``V_\alpha$ such that $e \in \mathrm{OD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$. It follows that $e \circ \rho : \mathbb{B}^* \xrightarrow{onto} \pi``V_\alpha$ and $e \circ \rho \in \mathrm{OD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}}$, which implies $e \circ \rho \in \mathrm{HOD}^{L(V_{\delta+1})[G]}_{\{\pi,V_\alpha\}} = \mathrm{HOD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}}$. Thus, there exists $\pi^* : \mathbb{B}^* \to \kappa$ cofinal such that $\pi^* \in \mathrm{OD}^{L(V_{\delta+1})}_{\{\pi,V_\alpha\}}$. □

Since $|\mathbb{B}^*| < \delta$, we are done. □

Lemma 4.3.2. Assume T*. Then, $\mathrm{cof}(\delta^+) > \delta$, i.e., δ^+ is regular (in V, and so, in $L(V_{\delta+1})$).

Proof. For contradiction, suppose $\mathrm{cof}(\delta^+) < \delta$. Then, there exist $\kappa < \delta$ and $\pi : \kappa \to \delta^+$ cofinal. Since δ is limit of extendibles, we can choose an extendible γ such that $\kappa < \gamma < \delta$. Now, pick $\eta >> \delta$ such that $V_\eta \prec_{\text{suff.}} V$ (sufficiently elementary). Then, there exists $j : V_\eta \to V_{j(\eta)}$ such that $crit(j) = \gamma$ and $j(\gamma) > \eta$. So, we have that $j``V_\eta \prec V_{j(\eta)}$. Moreover, since $\mathrm{cof}(\delta^+) < crit(j) = \gamma$, $\delta^+$ is a continuity point of $j$, i.e., $\sup j``\delta^+ = j(\delta^+) = (j(\delta))^+$; in fact: by elementarity, $j(\pi) : j(\kappa) = \kappa \to j(\delta^+)$ is cofinal in $j(\delta^+)$, and for all $\beta < \kappa$, $j(\pi)(\beta) = j(\pi)(j(\beta)) = j(\pi(\beta))$, so $j(\delta^+) = \sup\{j(\pi)(\beta) : \beta < \kappa\} = \sup\{j(\pi(\beta)) : \beta < \kappa\} \leq \sup j``\delta^+$; but for all $\alpha < \delta^+$ there exists $\beta < \kappa$ such that $\pi(\beta) \geq \alpha$, and so, it is also $\sup j``\delta^+ = \sup\{j(\alpha) : \alpha < \delta^+\} \leq \sup\{j(\pi(\beta)) : \beta < \kappa\} = j(\delta^+)$. It follows that if $G \subseteq \mathrm{Coll}((\mathrm{cof}(\delta))^+, V_\eta)$ is V-generic, then in $V[G]$, $(j(\delta))^+$ is collapsed to $j(\delta)$ (i.e., $|(j(\delta))^+|^{V[G]} = j(\delta)$). So, in $V[G]$, there exists $e : V_\eta \times j(\delta) \xrightarrow{onto} (j(\delta))^+$. By Vopěnka, it follows that in V, there exist $\kappa^* < j(\delta)$ and $e^* : \kappa^* \times j(\delta) \xrightarrow{onto} (j(\delta))^+$, which is a contradiction (as $(j(\delta))^+$ would have cardinality $j(\delta)$). □

Corollary 4.3.3. Assume T*. Then for every map $\pi : V_\delta \to \delta^+$, the range of π is bounded.

4.4 Measurability of δ^+

In showing that δ^+ is measurable (i.e., there exists a δ^+-complete ultrafilter on δ^+), we will repeatedly employ the following result from [6]:

AC-Lemma. Suppose γ is an extendible cardinal. Then for all $\alpha < \gamma$, there exists a partial order $\mathbb{P} \in V_\gamma$ such that $\mathbb{P}$ is $\mathrm{cof}(\delta)$-closed and there exists a strongly inaccessible cardinal ϵ such that $\alpha < \epsilon < \gamma$ and $V^{\mathbb{P}} \models \epsilon$-DC + "*$\epsilon$ is strongly inaccessible*".

Theorem 4.4.1. Assume T*. Then:

1. $V \models$ "*δ^+ is measurable*".

2. $L(V_{\delta+1}) \models$ "*δ^+ is measurable*".

Proof. Fix $\delta_0 < \delta$ such that $\delta_0 > \mathrm{cof}(\delta)$ is a Berkeley cardinal and a limit of extendibles. Let $\lambda < \delta^+$ be an infinite regular cardinal such that λ is regular in $L(V_{\delta+1})$. Let $S^{\delta^+}_{\lambda} = S =_{\mathrm{df}} \{\alpha < \delta^+ : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda\}$. Let $\mathcal{E} = \{j : V_\delta \to V_\delta : \mathrm{cof}(\delta) < crit(j) < \delta_0 \wedge j(\delta_0) = \delta_0 \wedge j(\lambda) = \lambda \wedge$ "*j extends canonically to $\hat{\jmath} : V_{\delta+1} \to V_{\delta+1}$*"$\}$. Notice that, for all $j \in \mathcal{E}$, the canonical extension $\hat{\jmath}$ is unique, as $\hat{\jmath}(A) = \bigcup_{\alpha<\delta} j(A \cap V_\alpha)$ for all $A \subseteq V_\delta$. For any $\mathcal{X} \subseteq \mathcal{E}$ such that there exists $\pi : V_\delta \xrightarrow{onto} \mathcal{X}$ (i.e., such that "*$\mathcal{X}$ is small*"), let $I_{\mathcal{X}} = \{\alpha < \delta^+ : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge (j(\alpha) = \alpha \ \forall j \in \mathcal{X})\}$. Now, let $\mathcal{F}_{\mathcal{E}}$ be the filter generated by $\{I_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{E} \wedge \exists \pi : V_\delta \xrightarrow{onto} \mathcal{X}\}$, that is, let $\mathcal{F}_{\mathcal{E}} = \{X \subseteq \delta^+ : \exists \mathcal{X} \subseteq \mathcal{E}$ ("*$\mathcal{X}$ is small*" $\wedge I_{\mathcal{X}} \subseteq X)\}$. Notice that $S \in \mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}} \restriction S$ is a filter on S; moreover, $\mathcal{F}_{\mathcal{E}}$ is correctly computed by $L(V_{\delta+1})$, i.e., $(\mathcal{F}_{\mathcal{E}})^V \cap L(V_{\delta+1}) = (\mathcal{F}_{\mathcal{E}})^{L(V_{\delta+1})}$ (in fact, $L(V_{\delta+1})$ has all the embeddings $j : V_\delta \to V_\delta$ and $\mathcal{F}_{\mathcal{E}}$ is generated by sets which are in $L(V_{\delta+1})$). Let us show that the filter $\mathcal{F}_{\mathcal{E}}$ is V_δ-complete (in fact, V_{δ^+}-complete) and there exists a partition of S into $<\delta_0$-many $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which the filter $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter.

Claim 1. In V:

1. $\mathcal{F}_{\mathcal{E}}$ is V_δ-complete, i.e., for all $\alpha < \delta$, for all $\rho : V_\alpha \to \mathcal{F}_{\mathcal{E}}$, $\bigcap \rho``V_\alpha \in \mathcal{F}_{\mathcal{E}}$.

2. $\mathcal{F}_{\mathcal{E}}$ is V_{δ^+}-complete.

Proof. Work in V.

1. Fix $\alpha < \delta$ and choose an extendible γ such that $\alpha < \gamma < \delta$. By the AC-Lemma, there exists a pair $\langle \mathbb{P}, \epsilon \rangle \in V_\gamma$ such that $\mathbb{P}$ is $\mathrm{cof}(\delta)$-closed, $\alpha < \epsilon < \gamma$, ϵ is strongly inaccessible and $V^{\mathbb{P}} \models \epsilon$-DC + "*$\epsilon$ is strongly inaccessible*". Fix $\rho : V_\alpha \to \mathcal{F}_{\mathcal{E}}$, and let $G \subseteq \mathbb{P}$ be V-generic. Then, in $V[G]$, there exists a function $F : V_\alpha \to V$ such that for all $a \in V_\alpha$, $F(a) = \langle \mathcal{X}_a, \pi_a \rangle \in V$ where $\mathcal{X}_a \subseteq \mathcal{E}$, $\pi_a : V_\delta \xrightarrow{onto} \mathcal{X}_a$ and $I_{\mathcal{X}_a} \subseteq \rho(a)$

(by ϵ-DC). So, let $\tau \in V$ be a term for F and choose $p \in G$ such that $p \Vdash$ "τ *is a function*, $\mathrm{dom}(\tau) = V_\alpha$ *and for all* $a \in V_\alpha$, $\tau(a) = \langle \mathcal{X}_a, \pi_a \rangle \in V$, $\mathcal{X}_a \subseteq \mathcal{E}$, $\pi_a : V_\delta \xrightarrow{onto} \mathcal{X}_a$ *and* $I_{\mathcal{X}_a} \subseteq \rho(a)$". Now, let $\mathcal{Y} = \bigcup \{\mathcal{X} \subseteq \mathcal{E} : \exists q < p \, \exists a \in V_\alpha \, \exists \pi \in V \, (q \Vdash \tau(a) = \langle \mathcal{X}, \pi \rangle)\}$. Define a map $e : \mathbb{P} \times V_\alpha \times V_\delta \to \mathcal{Y}$ such that $e(\langle q, a, b \rangle) = j$ iff $q \Vdash \tau(a) = \langle \mathcal{X}, \pi \rangle$ and $\pi(b) = j$. Since e is a surjection and since we can map V_δ onto its domain, we get that there exists a surjection from V_δ onto $\mathcal{Y}$; it follows that $I_{\mathcal{Y}} \in \mathcal{F}_{\mathcal{E}}$, but $I_{\mathcal{Y}} = \{\alpha < \delta^+ : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge (j(\alpha) = \alpha \; \forall j \in \mathcal{Y})\} = \bigcap \{I_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{E} \wedge \exists q < p \, \exists a \in V_\alpha \, \exists \pi \in V \, (q \Vdash \tau(a) = \langle \mathcal{X}, \pi \rangle)\} \subseteq \bigcap \rho``V_\alpha$, hence $\bigcap \rho``V_\alpha \in \mathcal{F}_{\mathcal{E}}$. Therefore, $\mathcal{F}_{\mathcal{E}}$ is V_δ-complete.

2. In order to show that $\mathcal{F}_{\mathcal{E}}$ is V_{δ^+}-complete, let $\pi : V_\delta \to \mathcal{F}_{\mathcal{E}}$. We have that for all $\alpha < \delta$, $\bigcap \pi``V_\alpha \in \mathcal{F}_{\mathcal{E}}$; we need to prove that $\bigcap \pi``V_\delta \in \mathcal{F}_{\mathcal{E}}$. Let $e : \mathrm{cof}(\delta) \to \delta$ be cofinal. Since $\bigcap \pi``V_{e(\alpha)} \in \mathcal{F}_{\mathcal{E}}$ for all $\alpha < \mathrm{cof}(\delta)$, $\mathrm{cof}(\delta) < \delta$ and $\mathcal{F}_{\mathcal{E}}$ is $V_\delta$-complete, we have that $\bigcap_{\alpha < \mathrm{cof}(\delta)} (\bigcap \pi``V_{e(\alpha)}) \in \mathcal{F}_{\mathcal{E}}$; but $\bigcap \pi``V_\delta = \bigcap_{\alpha < \mathrm{cof}(\delta)} (\bigcap \pi``V_{e(\alpha)})$ (in fact, by $V_\delta = \bigcup_{\alpha < \mathrm{cof}(\delta)} V_{e(\alpha)}$, $\bigcap \pi``V_\delta = \bigcap \pi``(\bigcup_{\alpha < \mathrm{cof}(\delta)} V_{e(\alpha)}) = \bigcap (\bigcup_{\alpha < \mathrm{cof}(\delta)} \pi``V_{e(\alpha)}) = \bigcap_{\alpha < \mathrm{cof}(\delta)} (\bigcap \pi``V_{e(\alpha)})$), and so, we are done.

□

Claim 2. There exists a partition of S, $\langle T_\alpha : \alpha < \Omega \rangle$, into $\mathcal{F}_{\mathcal{E}}$-positive sets such that $\Omega < \delta_0$ and $\mathcal{F}_{\mathcal{E}} \restriction T_\alpha$ is an ultrafilter for all $\alpha < \Omega$.[2]

Proof. Pick $\eta >> \delta$ such that $V_\eta \prec_{\text{suff.}} V$. Let γ be an extendible cardinal such that $\delta_0 < \gamma < \delta$. Then, there exists $\pi : V_{\eta+1} \to V_{\pi(\eta)+1}$ such that $crit(\pi) = \gamma$ and $\pi(\gamma) > \eta$. Since $\mathcal{F}_{\mathcal{E}}$ is V_δ-complete and $S \in \mathcal{F}_{\mathcal{E}}$, by elementarity, it follows that $\pi(\mathcal{F}_{\mathcal{E}})$ is $\pi(V_\delta) = V_{\pi(\delta)}$-complete and $\pi(S) \in \pi(\mathcal{F}_{\mathcal{E}})$; so, since $\mathcal{F}_{\mathcal{E}} \in V_\eta$ and $\pi(\delta) > \pi(\gamma) > \eta$, we have that $\bigcap \pi``\mathcal{F}_{\mathcal{E}} \in \pi(\mathcal{F}_{\mathcal{E}})$. Therefore, $\bigcap \pi``\mathcal{F}_{\mathcal{E}} \cap \pi(S) \neq \emptyset$. For each $\alpha \in \bigcap \pi``\mathcal{F}_{\mathcal{E}} \cap \pi(S)$, let $\mathcal{U}_\alpha = \{X \subseteq \delta^+ : \alpha \in \pi(X)\}$. It follows

[2]Recall that a set $T \subseteq \delta^+$ is $\mathcal{F}_{\mathcal{E}}$-**positive** iff the complement of T is not in $\mathcal{F}_{\mathcal{E}}$.

that for each $\alpha \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S)$, $\mathcal{U}_\alpha$ is an ultrafilter on δ^+ extending $\mathcal{F}_\mathcal{E}$; in fact:

1. Since $\alpha \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S) \subseteq \pi(\delta^+)$, $\delta^+ \in \mathcal{U}_\alpha$.

2. Since $\alpha \notin \pi(\emptyset) = \emptyset$, $\emptyset \notin \mathcal{U}_\alpha$.

3. If X_1, $X_2 \in \mathcal{U}_\alpha$, then $\alpha \in \pi(X_1) \cap \pi(X_2) = \pi(X_1 \cap X_2)$, so $X_1 \cap X_2 \in \mathcal{U}_\alpha$.

4. If $X_1 \in \mathcal{U}_\alpha$ and $X_2 \subseteq \delta^+$ is such that $X_1 \subseteq X_2$, then $\alpha \in \pi(X_1) \subseteq \pi(X_2)$, so $X_2 \in \mathcal{U}_\alpha$.

5. For any $X \subseteq \delta^+$ such that $\alpha \notin \pi(X)$, $\alpha \in \pi(\delta^+) \setminus \pi(X) = \pi(\delta^+ \setminus X)$, so $\delta^+ \setminus X \in \mathcal{U}_\alpha$.

6. Since $\alpha \in \pi(X)$ for all $X \in \mathcal{F}_\mathcal{E}$, $\mathcal{U}_\alpha \supset \mathcal{F}_\mathcal{E}$.

Moreover, each $\mathcal{U}_\alpha$ is V_γ-complete, i.e., for all $\beta < \gamma$ and $\rho : V_\beta \to \mathcal{U}_\alpha$, $\bigcap \rho``V_\beta \in \mathcal{U}_\alpha$; in fact: if $\beta < \gamma$ and $\rho : V_\beta \to \mathcal{U}_\alpha$, then $\pi(\rho) : V_{\pi(\beta)} = V_\beta \to \pi(\mathcal{U}_\alpha)$ is such that for all $x \in V_\beta$, $\pi(\rho)(x) = \pi(\rho)(\pi(x)) = \pi(\rho(x))$; but for all $x \in V_\beta$, $\rho(x) \in \mathcal{U}_\alpha$, so for all $x \in V_\beta$, $\alpha \in \pi(\rho(x)) = \pi(\rho)(x)$, which means that $\alpha \in \bigcap \pi(\rho)``V_\beta = \pi(\bigcap \rho``V_\beta)$, hence $\bigcap \rho``V_\beta \in \mathcal{U}_\alpha$. Furthermore, if $\mathcal{U} \supset \mathcal{F}_\mathcal{E}$ is a $V_\gamma$-complete ultrafilter on $\delta^+$, then $\pi(\mathcal{U}) \supset \pi(\mathcal{F}_\mathcal{E})$ is $V_{\pi(\gamma)}$-complete, so $\bigcap \pi``\mathcal{U} \in \pi(\mathcal{U})$, and if $\alpha \in \bigcap \pi``\mathcal{U} \neq \emptyset$, then we have that $\mathcal{U} \subseteq \mathcal{U}_\alpha$, which implies $\mathcal{U} = \mathcal{U}_\alpha$: in other words, $\mathcal{W} = \{\mathcal{U}_\alpha : \alpha \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S)\}$ contains all the V_γ-complete ultrafilters on δ^+ extending $\mathcal{F}_\mathcal{E}$. In order to establish an upper bound on the size of $\mathcal{W}$, we preliminarily need to show the following:

Subclaim 1. $\mathcal{W}$ can be well-ordered.

Proof. Clearly, we can assume $\mathcal{W} \in V_\eta$. Let $\mathcal{U} \in \mathcal{W}$. Then, by elementarity, $\pi(\mathcal{U})$ is a $\pi(V_\gamma)$-complete ultrafilter on $\pi(\delta^+)$, i.e., $\pi(\mathcal{U})$ is $V_{\pi(\gamma)}$-complete; so, since $\mathcal{U} \in V_\eta$ and $\pi(\gamma) > \eta$, we have that $\bigcap \pi``\mathcal{U} \in \pi(\mathcal{U})$. Since $\bigcap \pi``\mathcal{U} \subseteq \pi(\delta^+)$, there exists the minimum of $\bigcap \pi``\mathcal{U}$; let $\nu_{\mathcal{U}} = \min(\bigcap \pi``\mathcal{U})$. Let us show that for $\mathcal{U}_\alpha, \mathcal{U}_\beta \in \mathcal{W}$ such that $\mathcal{U}_\alpha \neq \mathcal{U}_\beta$, $\nu_{\mathcal{U}_\alpha} \neq \nu_{\mathcal{U}_\beta}$. Since $\mathcal{U}_\alpha \neq \mathcal{U}_\beta$, there exists $X \subseteq \delta^+$ such that $X \in \mathcal{U}_\alpha$ and $\delta^+ \setminus X \in \mathcal{U}_\beta$, so $\pi(X) \subseteq \pi(\delta^+)$ is such that $\pi(X) \in \pi(\mathcal{U}_\alpha)$ and $\pi(\delta^+ \setminus X) \in \pi(\mathcal{U}_\beta)$. $\bigcap \pi``\mathcal{U}_\alpha \in \pi(\mathcal{U}_\alpha)$ and $\bigcap \pi``\mathcal{U}_\alpha = \bigcap\{\pi(Y) : Y \in \mathcal{U}_\alpha\} \subseteq \pi(Y)$ for all $Y \in \mathcal{U}_\alpha$, hence $\bigcap \pi``\mathcal{U}_\alpha \subseteq \pi(X)$; similarly, $\bigcap \pi``\mathcal{U}_\beta \in \pi(\mathcal{U}_\beta)$ and $\bigcap \pi``\mathcal{U}_\beta \subseteq \pi(\delta^+ \setminus X)$. So, since $\pi(X) \cap \pi(\delta^+ \setminus X) = \emptyset$, we have that $(\bigcap \pi``\mathcal{U}_\alpha) \cap (\bigcap \pi``\mathcal{U}_\beta) = \emptyset$. Therefore, $\nu_{\mathcal{U}_\alpha} \neq \nu_{\mathcal{U}_\beta}$, and so, $\mathcal{W}$ can be well-ordered. □

Since $\mathcal{W}$ is a well-ordered family of ultrafilters, $\mathcal{W}$ has a cardinality. Our next goal is to show the following:

Subclaim 2. $|\mathcal{W}| < \delta_0$.

Proof. For contradiction, suppose that there exists a sequence $\langle \mathcal{W}_\alpha : \alpha < \delta_0 \rangle$ of distinct elements of $\mathcal{W}$. By the AC-Lemma, there exists a pair $\langle \mathbb{P}, \epsilon \rangle \in V_\gamma$ such that $\mathbb{P}$ is cof(δ)-closed, $\delta_0 < \epsilon < \gamma$, ϵ is strongly inaccessible and $V^{\mathbb{P}} \models \epsilon$-DC + "*$\epsilon$ is strongly inaccessible*". Let $G \subseteq \mathbb{P}$ be V-generic. Work in $V[G]$. First, let us show that in $V[G]$, each ultrafilter $\mathcal{W}_\alpha$ generates a V_γ-complete ultrafilter. Let $T \subseteq \delta^+$, $T \in V[G]$, and fix a term $\tau \in \text{Name}_{\mathbb{P}}$ for T; let $p \in G$ be such that $p \Vdash \tau \subseteq \delta^+$. For all $q < p$, let $T_q = \{\xi \in \delta^+ : q \Vdash \xi \in \tau\}$. It follows that $T = \bigcup\{T_q : q < p \wedge q \in G\}$. Now, define $F : \{q \in \mathbb{P} : q < p\} \to \mathcal{W}_\alpha$ such that $F(q) = T_q$ if $T_q \in \mathcal{W}_\alpha$ and $F(q) = \delta^+ \setminus T_q$ otherwise. Since the domain of F is in V_β for some $\beta < \gamma$ and $\mathcal{W}_\alpha$ is V_γ-complete (in V), we have that $T^* = \bigcap F``\text{dom}(F) \in \mathcal{W}_\alpha$; moreover, notice that either $T^* \subseteq T$ or $T^* \subseteq \delta^+ \setminus T$, and so, either $T \in \mathcal{W}_\alpha$ or $\delta^+ \setminus T \in \mathcal{W}_\alpha$. Analogously, we can prove that the forcing doesn't kill completeness. Let $\beta < \gamma$, $\rho : V_\beta \to \mathcal{W}_\alpha$, $\rho \in V[G]$, and fix a term $\tau \in \text{Name}_{\mathbb{P}}$ for ρ; let $p \in G$ be such that $p \Vdash$ "*τ is a function*, $\text{dom}(\tau) = V_\beta$ *and for all* $x \in V_\beta$, $\tau(x) \in \mathcal{W}_\alpha$". For all $q < p$, let $T_q = \{X \in \mathcal{W}_\alpha : \exists x \in V_\beta\ (q \Vdash \tau(x) = X)\}$. It follows that $\rho``V_\beta = \bigcup\{T_q : q < p \wedge q \in G\} = \{X \in \mathcal{W}_\alpha : \exists q < p\ (q \in G \wedge X \in T_q)\} = \{X \in \mathcal{W}_\alpha : \exists q < p\ (q \in G \wedge \exists x \in V_\beta\ (q \Vdash \tau(x) = X))\}$. Now, define $F : \{q \in \mathbb{P} : q < p\} \times V_\beta \to \mathcal{W}_\alpha$ such that $F(\langle q, x\rangle) = X$ iff $q \Vdash \tau(x) = X$. Since the domain of F is in V_μ for some $\mu < \gamma$ and $\mathcal{W}_\alpha$ is V_γ-complete (in V), we have that $T^* = \bigcap F``\mathrm{dom}(F) \in \mathcal{W}_\alpha$; but $T^* = \bigcap\{X \in \mathcal{W}_\alpha : \exists q < p\ \exists x \in V_\beta\ (q \Vdash \tau(x) = X)\} \subseteq \bigcap \rho``V_\beta$, hence $\bigcap \rho``V_\beta \in \mathcal{W}_\alpha$. Therefore, in $V[G]$, for all $\alpha < \delta_0$, $\mathcal{W}_\alpha$ generates a V_γ-complete ultrafilter. By ϵ-DC, in $V[G]$, there exists a sequence $\langle T_\alpha : \alpha < \delta_0\rangle$ such that $T_\alpha \in \mathcal{W}_\alpha$ for all $\alpha < \delta_0$ and $T_\alpha \cap T_\beta = \emptyset$ for all $\alpha, \beta < \delta_0$ such that $\alpha \neq \beta$ (i.e., there exists a separating family for $\langle \mathcal{W}_\alpha : \alpha < \delta_0\rangle$). It follows that there exists a sequence $\langle \hat{T}_\alpha : \alpha < \delta_0\rangle \in V$ such that $\hat{T}_\alpha \in \mathcal{W}_\alpha$ for all $\alpha < \delta_0$ and $\hat{T}_\alpha \cap \hat{T}_\beta = \emptyset$ for all $\alpha, \beta < \delta_0$ such that $\alpha \neq \beta$. Let us see why. In $V[G]$, let $F : \delta_0 \to \bigcup_{\alpha<\delta_0} \mathcal{W}_\alpha$ be such that $F(\alpha) = T_\alpha \in \mathcal{W}_\alpha$ for all $\alpha < \delta_0$ and $F(\alpha) \cap F(\beta) = \emptyset$ whenever $\alpha \neq \beta$. Let $\tau \in \mathrm{Name}_{\mathbb{P}}$ be a term for F and choose $p \in G$ such that $p \Vdash$ "*τ is a function from δ_0 into $\bigcup_{\alpha<\delta_0} \mathcal{W}_\alpha$, $\tau(\alpha) \in \mathcal{W}_\alpha$ for all $\alpha < \delta_0$ and $\tau(\alpha) \cap \tau(\beta) = \emptyset$ for all $\alpha, \beta < \delta_0$ such that $\alpha \neq \beta$*". Now, for every $\alpha < \delta_0$, let $\hat{T}_\alpha = \bigcap\{X \in \mathcal{W}_\alpha : \exists q < p\ (q \Vdash \tau(\alpha) = X)\}$. Then, for each $\alpha < \delta_0$, by the completeness of $\mathcal{W}_\alpha$ we have that $\hat{T}_\alpha \in \mathcal{W}_\alpha$; moreover, since $\hat{T}_\alpha \subseteq T_\alpha$ for all $\alpha < \delta_0$, it is $\hat{T}_\alpha \cap \hat{T}_\beta = \emptyset$ for all $\alpha, \beta < \delta_0$ such that $\alpha \neq \beta$. Therefore, in V, there exists a separating sequence $\langle \hat{T}_\alpha : \alpha < \delta_0\rangle$ for the sequence of ultrafilters $\langle \mathcal{W}_\alpha : \alpha < \delta_0\rangle$, where the separating sets $\hat{T}_\alpha$ are $\mathcal{F}_\mathcal{E}$-positive. Since δ_0 is a Berkeley cardinal, there exists $j : V_\eta \to V_\eta$ such that $\mathrm{cof}(\delta) < crit(j) < \delta_0$, $j(\delta) = \delta$, $j(\delta_0) = \delta_0$, $j(\lambda) = \lambda$ and $j(\langle \hat{T}_\alpha : \alpha < \delta_0\rangle) = \langle \hat{T}_\alpha : \alpha < \delta_0\rangle$. It follows that $j \restriction V_\delta : V_\delta \to V_{j(\delta)} = V_\delta$ is in $\mathcal{E}$, so $T = I_{\{j \restriction V_\delta\}} = \{\alpha < \delta^+ : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge j(\alpha) = \alpha\} \in \mathcal{F}_\mathcal{E}$. Now, let $\kappa_0 = crit(j) < \delta_0$. Since $T \in \mathcal{F}_\mathcal{E}$ and $\hat{T}_{\kappa_0}$ is $\mathcal{F}_\mathcal{E}$-positive, we have that $T \cap \hat{T}_{\kappa_0} \neq \emptyset$ (otherwise, $T \subseteq \delta^+ \setminus \hat{T}_{\kappa_0}$, hence $\delta^+ \setminus \hat{T}_{\kappa_0}$ would be in $\mathcal{F}_\mathcal{E}$); so, choose $\alpha_0 \in T \cap \hat{T}_{\kappa_0}$. Then, by elementarity, $j(\alpha_0) \in \hat{T}_{j(\kappa_0)}$; but $j(\alpha_0) = \alpha_0$, so

$\hat{T}_{\kappa_0} \cap \hat{T}_{j(\kappa_0)} \neq \emptyset$, a contradiction. $\square$

Now we know that the size of $\mathcal{W}$ is less than δ_0; so, let $\mathcal{W} = \langle \mathcal{W}_\alpha : \alpha < \kappa_\mathcal{W} \rangle$ where $\kappa_\mathcal{W} = |\mathcal{W}| < \delta_0$. Let $\langle T_\alpha : \alpha < \kappa_\mathcal{W} \rangle$ be a separating family of $\mathcal{F}_\mathcal{E}$-positive sets for $\mathcal{W}$ (we proved that there exists such a separating family). Notice that, for all $\alpha < \kappa_\mathcal{W}$, T_α cannot be split into $\mathcal{F}_\mathcal{E}$-positive sets. In fact: fix $\alpha_0 < \kappa_\mathcal{W}$ and suppose for contradiction that T_{α_0} can be split into $\mathcal{F}_\mathcal{E}$-positive sets, A, B (i.e., $A \cup B = T_{\alpha_0}$ and $A \cap B = \emptyset$), and let $\alpha_A \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S) \cap \pi(A)$, $\alpha_B \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S) \cap \pi(B)$ (these sets are non-empty because $\bigcap \pi``\mathcal{F}_\mathcal{E}$, $\pi(S) \in \pi(\mathcal{F}_\mathcal{E})$ and $\pi(A)$, $\pi(B)$ are $\pi(\mathcal{F}_\mathcal{E})$-positive); then, $\mathcal{U}_{\alpha_A} = \{X \subseteq \delta^+ : \alpha_A \in \pi(X)\} \in \mathcal{W}$, $\mathcal{U}_{\alpha_B} = \{X \subseteq \delta^+ : \alpha_B \in \pi(X)\} \in \mathcal{W}$, $A \in \mathcal{U}_{\alpha_A}$ and $B \in \mathcal{U}_{\alpha_B}$, but A, $B \subseteq T_{\alpha_0}$, so $T_{\alpha_0} \in \mathcal{U}_{\alpha_A} \cap \mathcal{U}_{\alpha_B}$, whence $\mathcal{U}_{\alpha_A} = \mathcal{U}_{\alpha_B} = \mathcal{W}_{\alpha_0}$, a contradiction. It follows that for all $\alpha < \kappa_\mathcal{W}$, $\mathcal{F}_\mathcal{E} \upharpoonright T_\alpha$ is an ultrafilter: in fact, if $Y \subseteq T_\alpha$ is such that $Y \notin \mathcal{F}_\mathcal{E} \upharpoonright T_\alpha$ and $T_\alpha \setminus Y \notin \mathcal{F}_\mathcal{E} \upharpoonright T_\alpha$, then Y and $T_\alpha \setminus Y$ are both $\mathcal{F}_\mathcal{E}$-positive, and so, $T_\alpha = Y \cup (T_\alpha \setminus Y)$ can be split into $\mathcal{F}_\mathcal{E}$-positive sets, a contradiction. But $\mathcal{F}_\mathcal{E} \upharpoonright T_\alpha \subseteq \mathcal{W}_\alpha \upharpoonright T_\alpha$, and so, for all $\alpha < \kappa_\mathcal{W}$, $\mathcal{F}_\mathcal{E} \upharpoonright T_\alpha = \mathcal{W}_\alpha \upharpoonright T_\alpha$. Moreover, $\bigcup_{\alpha < \kappa_\mathcal{W}} T_\alpha \in \mathcal{F}_\mathcal{E}$; in fact: every $\mathcal{F}_\mathcal{E}$-positive set is in some $\mathcal{W}_\alpha$ (because for any $\mathcal{F}_\mathcal{E}$-positive set T, since $\bigcap \pi``\mathcal{F}_\mathcal{E}$, $\pi(S) \in \pi(\mathcal{F}_\mathcal{E})$ and $\pi(T)$ is $\pi(\mathcal{F}_\mathcal{E})$-positive, there exists $\alpha \in \bigcap \pi``\mathcal{F}_\mathcal{E} \cap \pi(S) \cap \pi(T)$, and so, $T \in \mathcal{W}_\alpha$), so, $\bigcap_{\alpha < \kappa_\mathcal{W}} \mathcal{W}_\alpha = \mathcal{F}_\mathcal{E}$. Finally, we have shown that there exists a partition of S, $\langle T_\alpha : \alpha < \kappa_\mathcal{W} \rangle$, into $<\delta_0$-many $\mathcal{F}_\mathcal{E}$-positive sets such that for all $\alpha < \kappa_\mathcal{W}$, $\mathcal{F}_\mathcal{E} \upharpoonright T_\alpha$ is an ultrafilter. $\square$

We now turn to $L(V_{\delta+1})$. Recall that the filter $\mathcal{F}_\mathcal{E}$ is correctly computed by $L(V_{\delta+1})$. Since δ_0 is a limit of extendibles, we can choose an extendible γ such that $\kappa_\mathcal{W} < \gamma < \delta_0$. By the AC-Lemma, there exists a pair $\langle \mathbb{P}, \epsilon \rangle \in V_\gamma$ such that $\mathbb{P}$ is cof(δ)-closed, $\kappa_\mathcal{W} < \epsilon < \gamma$, ϵ is strongly inaccessible and $V^\mathbb{P} \models \epsilon$-DC + "$\epsilon$ *is strongly inaccessible*". Let $G \subseteq \mathbb{P}$ be V-generic. Since $V[G] \models \epsilon$-DC, we have that $L(V_{\delta+1}[G]) = L(V_{\delta+1})[G] \models \epsilon$-DC (notice that

$L(V_{\delta+1}[G]) = L(V_{\delta+1})[G]$ follows from the fact that $\mathbb{P}$ is cof(δ)-closed). Let us show the following:

Claim 3. 1. If $T \subseteq S$ is $\mathcal{F}_{\mathcal{E}}$-positive in $L(V_{\delta+1})[G]$, then T contains an $\mathcal{F}_{\mathcal{E}}$-positive subset in $L(V_{\delta+1})$.

2. In $L(V_{\delta+1})[G]$, there is no partition of S into ϵ-many $\mathcal{F}_{\mathcal{E}}$-positive sets.

Proof. 1. Let $\tau \in \mathrm{Name}_{\mathbb{P}}$ be a term for T, $\tau \in L(V_{\delta+1})$. Choose $p \in G$ such that $p \Vdash$ "*τ is an $\mathcal{F}_{\mathcal{E}}$-positive subset of S*". For all $q < p$, let $T_q = \{\xi \in S : q \Vdash \xi \in \tau\}$. Then, $T = \bigcup\{T_q : q < p \wedge q \in G\}$. So, in $L(V_{\delta+1})[G]$, $T \subseteq S$ is $\mathcal{F}_{\mathcal{E}}$-positive and T is the union of sets of the ground model; it follows that some set T_q has to be $\mathcal{F}_{\mathcal{E}}$-positive: in fact, if not, $S \backslash T_q \in \mathcal{F}_{\mathcal{E}}$ for each q, but then, since $\mathcal{F}_{\mathcal{E}}$ is $V_\delta[G]$-complete in $L(V_{\delta+1})[G]$, $\bigcap\{S \setminus T_q : q < p \wedge q \in G\} = S \setminus \bigcup\{T_q : q < p \wedge q \in G\} = S \setminus T \in \mathcal{F}_{\mathcal{E}}$, a contradiction.

2. Suppose for contradiction that $\langle S_\beta : \beta < \epsilon\rangle \in L(V_{\delta+1})[G]$ is a partition of S into ϵ-many $\mathcal{F}_{\mathcal{E}}$-positive sets. We have that each set S_β contains an $\mathcal{F}_{\mathcal{E}}$-positive subset $\hat{S}_\beta$ in the ground model. Then, each set $\hat{S}_\beta$ has to be in some $\mathcal{W}_\alpha$ (in fact, every $\mathcal{F}_{\mathcal{E}}$-positive set of V is in some $\mathcal{W}_\alpha$). Thus, in $V[G]$, each set S_β is in some $\mathcal{W}_\alpha$ (recall that each $\mathcal{W}_\alpha$ generates a V_γ-complete ultrafilter in $V[G]$); but each $\mathcal{W}_\alpha$ can contain a unique S_β (as the sets S_β are pairwise disjoint), and so, it follows that $|\kappa_{\mathcal{W}}|^{V[G]} \geq |\epsilon|^{V[G]}$ (i.e., in $V[G]$, there exists $\rho : \kappa_{\mathcal{W}} \xrightarrow{onto} \epsilon$), contradicting $|\kappa_{\mathcal{W}}|^{V[G]} \leq \kappa_{\mathcal{W}} < \epsilon = |\epsilon|^{V[G]}$.

□

Since δ^+ is regular, we have that the filter $\mathcal{F}_{\mathcal{E}}$ is δ^+-complete. It follows that in $L(V_{\delta+1})[G]$, there exist a cardinal $\sigma < \epsilon$ and a partition $\langle S_\alpha : \alpha < \sigma\rangle$ of

S into $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter; in fact: by ϵ-DC and since $\mathcal{F}_{\mathcal{E}}$ is δ^+-complete, we have that in $L(V_{\delta+1})[G]$, either

1. there exists a partition of S into $<\epsilon$-many $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter

   or

2. there exists a partition of S into ϵ-many $\mathcal{F}_{\mathcal{E}}$-positive sets;

but we have shown that the second case cannot happen. Since for each $\alpha < \sigma$, S_α contains an $\mathcal{F}_{\mathcal{E}}$-positive subset $\hat{S}_\alpha$ in the ground model and S_α cannot be split into $\mathcal{F}_{\mathcal{E}}$-positive sets, we have that for each $\alpha < \sigma$, $\hat{S}_\alpha$ cannot be split into $\mathcal{F}_{\mathcal{E}}$-positive sets in the ground model. Therefore, we can assume that $S_\alpha \in L(V_{\delta+1})$ for all $\alpha < \sigma$. So, in $L(V_{\delta+1})$, $\mathcal{F}_{\mathcal{E}} \restriction S_\alpha$ is an ultrafilter for all $\alpha < \sigma$. Now we need a well-ordering of these ultrafilters. Let us define, in V, an equivalence relation on $\mathcal{W} = \{\mathcal{U}_\alpha : \alpha \in \bigcap \pi``\mathcal{F}_{\mathcal{E}} \cap \pi(S)\} = \langle \mathcal{W}_\alpha : \alpha < \kappa_{\mathcal{W}} \rangle$ (where $\kappa_{\mathcal{W}} = |\mathcal{W}|^V < \delta_0$) as follows: for $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{W}$, $\mathcal{U}_1 \sim \mathcal{U}_2$ iff $\mathcal{U}_1 \cap L(V_{\delta+1}) = \mathcal{U}_2 \cap L(V_{\delta+1})$. First, we show that for all $\mathcal{U} \in \mathcal{W}$, $\mathcal{U} \cap L(V_{\delta+1}) \in L(V_{\delta+1})$. Let $\langle T_\alpha : \alpha < \kappa_{\mathcal{W}} \rangle$ be a separating family of $\mathcal{F}_{\mathcal{E}}$-positive sets for $\mathcal{W}$ (so, $T_\alpha \in \mathcal{W}_\alpha$ for all $\alpha < \kappa_{\mathcal{W}}$, $T_\alpha \cap T_\beta = \emptyset$ for all $\alpha \neq \beta$, $\bigcup_{\alpha < \kappa_{\mathcal{W}}} T_\alpha = S$ and each T_α cannot be split into $\mathcal{F}_{\mathcal{E}}$-positive sets), and let $\bar{\alpha} < \sigma$; since $S_{\bar{\alpha}} = \bigcup \{S_{\bar{\alpha}} \cap T_\alpha : \alpha < \kappa_{\mathcal{W}}\}$ is $\mathcal{F}_{\mathcal{E}}$-positive, there must exist $\alpha^* < \kappa_{\mathcal{W}}$ such that $S_{\bar{\alpha}} \cap T_{\alpha^*}$ is $\mathcal{F}_{\mathcal{E}}$-positive, but T_{α^*} cannot be split into $\mathcal{F}_{\mathcal{E}}$-positive sets, so $\mathcal{W}_{\alpha^*} \cap L(V_{\delta+1}) = (\mathcal{F}_{\mathcal{E}} \restriction S_{\bar{\alpha}})^{L(V_{\delta+1})} \in L(V_{\delta+1})$. Now, let $\mathcal{W}^* = \{\mathcal{U} \cap L(V_{\delta+1}) : \mathcal{U} \in \mathcal{W}\}$. Then, $\mathcal{W}^* \in L(V_{\delta+1})$, as $\mathcal{W}^* = \{(\mathcal{F}_{\mathcal{E}} \restriction T)^{L(V_{\delta+1})} : T \in L(V_{\delta+1}) \wedge$ "*T is $\mathcal{F}_{\mathcal{E}}$-positive and cannot be split*"$\} = \{(\mathcal{F}_{\mathcal{E}} \restriction S_\alpha)^{L(V_{\delta+1})} : \alpha < \sigma\}$. Moreover:

Claim 4. In $L(V_{\delta+1})$, $\mathcal{W}^*$ can be well-ordered.

Proof. First, notice that $\mathcal{W}^*$ can be well-ordered in V, as there exists $\rho^* : \mathcal{W} \xrightarrow{onto} \mathcal{W}^*$ and $\mathcal{W}$ can be well-ordered. Thus, $|\mathcal{W}^*|^V \leq |\mathcal{W}|^V = \kappa_{\mathcal{W}}$. Fix a bijection $\rho : |\mathcal{W}^*|^V \leftrightarrow \mathcal{W}^*$. Let us show that $\rho \in L(V_{\delta+1})$. Since $|\mathcal{W}^*|^V \leq \kappa_{\mathcal{W}}$, $|\mathcal{W}^*|^{V[G]} \leq \kappa_{\mathcal{W}} < \epsilon$; since $\mathcal{W}^* \in L(V_{\delta+1})[G]$ and $L(V_{\delta+1})[G] \models \epsilon$-DC, it follows that $\mathcal{W}^*$ can be well-ordered in $L(V_{\delta+1})[G]$ and $|\mathcal{W}^*|^{L(V_{\delta+1})[G]} < \epsilon$. So, $(\mathcal{P}(\mathcal{W}^* \times \mathcal{W}^*))^{V[G]} = (\mathcal{P}(\mathcal{W}^* \times \mathcal{W}^*))^{L(V_{\delta+1})[G]}$ and, thus, $\rho \in L(V_{\delta+1})[G]$; but G is V-generic, hence $\rho \in L(V_{\delta+1})$. Therefore, $\mathcal{W}^*$ is well-ordered in $L(V_{\delta+1})$ and $|\mathcal{W}^*|^{L(V_{\delta+1})} = |\mathcal{W}^*|^V$. □

Finally, since $\mathcal{W}^*$ is well-ordered in $L(V_{\delta+1})$ and $|\mathcal{W}^*|^{L(V_{\delta+1})} = |\mathcal{W}^*|^V \leq \kappa_{\mathcal{W}} < \delta_0$, we can get a separating family for $\mathcal{W}^*$ and conclude that in $L(V_{\delta+1})$ there exists a partition of S into $<\delta_0$-many $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which the filter $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter. Since $\mathcal{F}_{\mathcal{E}}$ is δ^+-complete, we are done. □

The following theorem generalizes the splitting result achieved in the proof of Theorem 4.4.1 by considering in place of δ^+ any cardinal $\kappa > \delta$ such that $\kappa < \Theta$ and $\mathrm{cof}(\kappa) > \delta$: it turns out that, unlike the I_0 case, here we get a uniform bound (δ_0, that can be the least Berkeley cardinal which is limit of extendibles) independent of the cofinality.

Theorem 4.4.2. Assume T^*. Let δ_0 be a Berkeley cardinal and a limit of extendibles such that $\mathrm{cof}(\delta) < \delta_0 < \delta$. Then in $L(V_{\delta+1})$, for any cardinal $\kappa < \Theta$ such that $\mathrm{cof}(\kappa) > \delta$ and for any infinite regular cardinal $\lambda < \kappa$, there exists a partition of $S^\kappa_\lambda =_{\mathrm{df}} \{\alpha < \kappa : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda\}$ into $<\delta_0$-many $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which the filter $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter, where $\mathcal{F}_{\mathcal{E}}$ is the "fixed points filter".

Proof. We just need to define the general setting in which performing the proof of Theorem 4.4.1. So, let $\kappa < \Theta$ be such that $\mathrm{cof}(\kappa) > \delta$ and let $\lambda < \kappa$ be such that λ is regular in $L(V_{\delta+1})$. Let $\beta < \Theta$ be such that $(\mathcal{P}(\kappa))^{L(V_{\delta+1})} \subseteq L_\beta(V_{\delta+1})$. Then, let $\mathcal{E}^{\delta_0}_{\kappa,\lambda,\beta} = \mathcal{E} = \{j : L_\beta(V_{\delta+1}) \to L_\beta(V_{\delta+1}) : \mathrm{cof}(\delta) < crit(j) < \delta_0 \wedge j(\delta) = \delta \wedge j(\delta_0) = \delta_0 \wedge j(\kappa) = \kappa \wedge j(\lambda) = \lambda \wedge j(\beta) = \beta\}$. Notice that, for all $j \in \mathcal{E}$, j is uniquely determined by $j \restriction V_\delta$. For any $\mathcal{X} \subseteq \mathcal{E}$ such that there exists

$\pi : V_\delta \xrightarrow{onto} \mathcal{X}$, let $I_{\mathcal{X}} = \{\alpha < \kappa : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge (j(\alpha) = \alpha \ \forall j \in \mathcal{X})\}$. Finally, let $\mathcal{F}_{\mathcal{E}}$ be the filter generated by $\{I_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{E} \wedge \exists \pi : V_\delta \xrightarrow{onto} \mathcal{X}\}$, that is, let $\mathcal{F}_{\mathcal{E}} = \{X \subseteq \kappa : \exists \mathcal{X} \subseteq \mathcal{E} \ (\exists \pi : V_\delta \xrightarrow{onto} \mathcal{X} \wedge I_{\mathcal{X}} \subseteq X)\}$. Now the proof proceeds exactly as the proof of Theorem 4.4.1. □

4.5 Measurable Cardinals up to Θ

We finally prove that in $L(V_{\delta+1})$, Θ is a limit of measurable cardinals. We preliminarily note that the proof of the Coding Lemma works locally in any $L_\gamma(V_{\delta+1})$ with $\gamma < \Theta$ such that $\mathrm{cof}(\gamma) > \delta$. Therefore, we have the following local version of the Coding Lemma.

Lemma 4.5.1 (Local version of the Coding Lemma)**.** Assume T^*. Let $\gamma < \Theta$ be such that $\mathrm{cof}(\gamma) > \delta$. Then, the Coding Lemma holds in $L_\gamma(V_{\delta+1})$.

Theorem 4.5.2. Assume T^*. Then in $L(V_{\delta+1})$, Θ is limit of measurable cardinals.

Proof. Work in $L(V_{\delta+1})$. Let us show the following:

Claim. Let $\gamma < \Theta$ be the least such that $L_\gamma(V_{\delta+1}) \prec_{\Sigma_1} L_\Theta(V_{\delta+1})$. Then in $L(V_{\delta+1})$, γ is measurable.

Proof. Let $\delta_0 < \delta$ be such that $\delta_0 > \mathrm{cof}(\delta)$ is a Berkeley cardinal and a limit of extendibles. Let $\lambda < \gamma$ be an infinite regular cardinal such that λ is regular in $L(V_{\delta+1})$. Let $S^\gamma_\lambda = S =_{\mathrm{df}} \{\alpha < \gamma : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda\}$. Let $\mathcal{E} = \{j : L_\gamma(V_{\delta+1}) \to L_\gamma(V_{\delta+1}) : \mathrm{cof}(\delta) < crit(j) < \delta_0 \wedge j(\delta) = \delta \wedge j(\delta_0) = \delta_0 \wedge j(\gamma) = \gamma \wedge j(\lambda) = \lambda\}$. For any $\mathcal{X} \subseteq \mathcal{E}$ such that there exists $\pi : V_\delta \xrightarrow{onto} \mathcal{X}$, let $I_{\mathcal{X}} = \{\alpha < \gamma : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge (j(\alpha) = \alpha \ \forall j \in \mathcal{X})\}$. Let $\mathcal{F}_{\mathcal{E}}$ be the filter generated by $\{I_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{E} \wedge \exists \pi : V_\delta \xrightarrow{onto} \mathcal{X}\}$, that is, let $\mathcal{F}_{\mathcal{E}} = \{X \subseteq \gamma : \exists \mathcal{X} \subseteq \mathcal{E} \ (\exists \pi : V_\delta \xrightarrow{onto} \mathcal{X} \wedge I_{\mathcal{X}} \subseteq X)\}$. We claim that $\mathcal{F}_{\mathcal{E}}$ is γ-complete. Fix a surjection $\pi : \mathrm{dom}(\pi) \xrightarrow{onto} L_\gamma(V_{\delta+1})$ such that $\mathrm{dom}(\pi) \subseteq V_{\delta+1}$, π is

Σ_1-definable from $\{V_{\delta+1}\}$ in $L_\gamma(V_{\delta+1})$ and for all $Z \subseteq \mathrm{dom}(\pi)$, if $Z \in L_\gamma(V_{\delta+1})$ then $\pi \restriction Z \in L_\gamma(V_{\delta+1})$ (such a surjection does exist: in fact, there exists $\pi^* : V_{\delta+1} \xrightarrow{onto} L_\gamma(V_{\delta+1})$ such that π^* is Σ_1-definable from $\{V_{\delta+1}\}$ in $L_\Theta(V_{\delta+1})$). Then, for all $j \in \mathcal{E}$, $j(\pi) = \pi$. Let $\mathcal{E}^* = \{j \restriction V_\delta : j \in \mathcal{E}\}$. So, $\mathcal{E}^* \subseteq V_{\delta+1}$. Notice that, for all $j \in \mathcal{E}$, j is uniquely determined by its action on V_δ (i.e., for all $j_1, j_2 \in \mathcal{E}$, if $j_1 \restriction V_\delta = j_2 \restriction V_\delta$ then $j_1 = j_2$), and so there exists a canonical bijection between $\mathcal{E}$ and $\mathcal{E}^*$. Let $\alpha < \gamma$ and $f : \alpha \to \mathcal{F}_\mathcal{E}$; we have to show that $\bigcap f``\alpha \in \mathcal{F}_\mathcal{E}$. Since $\mathrm{cof}(\gamma) > \delta$, the Coding Lemma holds in $L_\gamma(V_{\delta+1})$; so, we can let $\beta < \gamma$ be such that $L_\beta(V_{\delta+1})$ witnesses the Coding Lemma at α in $L_\gamma(V_{\delta+1})$ (notice that, since $L_\gamma(V_{\delta+1}) \prec_{\Sigma_1} L_\Theta(V_{\delta+1})$, $L_\beta(V_{\delta+1})$ turns to be witness of the Coding Lemma at α in $L_\Theta(V_{\delta+1})$). Choose $\rho : V_{\delta+1} \xrightarrow{onto} \alpha$, $\rho \in L_\gamma(V_{\delta+1})$. Let $W = \{\langle x, y\rangle : y : V_\delta \to \mathcal{E}^* \wedge$ "*y gives a set $\hat{\mathcal{X}} \subseteq \mathcal{E}$ such that $I_{\hat{\mathcal{X}}} \subseteq f(\rho(x))$*"$\} \subseteq V_{\delta+1} \times V_{\delta+1}$, where we mean that $\hat{\mathcal{X}}$ contains the extension to $L_\gamma(V_{\delta+1})$ of every element of $y``V_\delta$ (recall that every element of $\mathcal{E}^*$ extends uniquely to an element of $\mathcal{E}$). By the Coding Lemma, there exists $W^* \in L_\beta(V_{\delta+1})$ such that $W^* \subseteq W$ and for all $\xi < \alpha$, if there exists $\langle x, y\rangle \in W$ such that $\rho(x) = \xi$ then there exists $\langle x, y\rangle \in W^*$ such that $\rho(x) = \xi$. Now, let $\mathcal{E}_0^* = \bigcup\{y``V_\delta : \exists x \in V_{\delta+1}\ (\langle x, y\rangle \in W^*)\}$. Since for all $\langle x, y\rangle \in W^*$, $y``V_\delta \subseteq \mathcal{E}^*$, we have that $\mathcal{E}_0^* \subseteq \mathcal{E}^*$; moreover, since $\mathcal{E}_0^*$ is definable from W^* and $W^* \in L_\beta(V_{\delta+1})$, $\mathcal{E}_0^* \in L_{\beta+1}(V_{\delta+1}) \subseteq L_\gamma(V_{\delta+1})$. In order to prove that $\mathcal{F}_\mathcal{E}$ is γ-complete, it suffices to show that $\bigcap f``\alpha \neq \emptyset$. We preliminarily show the following:

Subclaim. Suppose $\mathcal{Z} \subseteq \mathcal{E}^*$, $\mathcal{Z} \in L_\gamma(V_{\delta+1})$. Then, $\{\eta < \gamma : \forall j \in \mathcal{E}\ (j \restriction V_\delta \in \mathcal{Z} \to j(\eta) = \eta)\}$ is cofinal in γ.

Proof. Let $\eta < \gamma$ and let $a \in \mathrm{dom}(\pi)$ be such that $\pi(a) = \eta$. Let $Y = \{j(a) : j \in \mathcal{E} \wedge j \restriction V_\delta \in \mathcal{Z}\}$. Then, $Y \subseteq \mathrm{dom}(\pi)$ (as for all $j \in \mathcal{E}$, $j(\pi) = \pi$, so $j(a) \in j(\mathrm{dom}(\pi)) = \mathrm{dom}(j(\pi)) = \mathrm{dom}(\pi)$) and $Y \in L_\gamma(V_{\delta+1})$. It follows that $\pi \upharpoonright Y \in L_\gamma(V_{\delta+1})$; thus, $\sup\{j(\eta) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} < \gamma$: in fact, $\sup\{j(\eta) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} = \sup\{j(\pi(a)) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} = \sup\{j(\pi)(j(a)) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} = \sup\{\pi(j(a)) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} = \sup \pi``Y$. So, since $\sup\{j(\eta) : j \in \mathcal{E} \wedge j \upharpoonright V_\delta \in \mathcal{Z}\} < \gamma$ for any $\eta < \gamma$, the set $\{\eta < \gamma : \forall j \in \mathcal{E}\ (j \upharpoonright V_\delta \in \mathcal{Z} \to j(\eta) = \eta)\}$ must be cofinal in γ. □

So, we have that the set $\{\eta < \gamma : \forall j \in \mathcal{E}\ (j \upharpoonright V_\delta \in \mathcal{E}_0^* \to j(\eta) = \eta)\}$ is cofinal in γ. It follows that there exists $\eta < \gamma$ such that $\eta \in S$ and $j(\eta) = \eta$ for all $j \in \mathcal{E}$ such that $j \upharpoonright V_\delta \in \mathcal{E}_0^*$ (since S is stationary and $\{\eta < \gamma : \forall j \in \mathcal{E}\ (j \upharpoonright V_\delta \in \mathcal{E}_0^* \to j(\eta) = \eta)\}$ is club in γ). Then, for all $\langle x, y\rangle \in W^*$, $\eta \in f(\rho(x))$: in fact, η is fixed by any embedding in $y``V_\delta$, each of which extends to an embedding in $\hat{\mathcal{X}} \subseteq \mathcal{E}$ such that $I_{\hat{\mathcal{X}}} = \{\alpha < \gamma : (\mathrm{cof}(\alpha))^{L(V_{\delta+1})} = \lambda \wedge (j(\alpha) = \alpha\ \forall j \in \hat{\mathcal{X}})\} \subseteq f(\rho(x))$. Now, notice that for all $\xi < \alpha$ there exists $\langle x, y\rangle \in W$ such that $\rho(x) = \xi$; in fact: for every $\xi < \alpha$ there exists $x \in V_{\delta+1}$ such that $\rho(x) = \xi$, so $f(\rho(x)) = f(\xi) \in \mathcal{F}_{\mathcal{E}}$, that is, there exists a "small" $\hat{\mathcal{X}} \subseteq \mathcal{E}$ such that $I_{\hat{\mathcal{X}}} \subseteq f(\rho(x))$, and so, if we let $y : V_\delta \to \mathcal{E}^*$ be such that $y``V_\delta = \{j \upharpoonright V_\delta : j \in \hat{\mathcal{X}}\} \subseteq \mathcal{E}^*$, then $\langle x, y\rangle \in W$. Therefore, for all $\xi < \alpha$ there exists $\langle x, y\rangle \in W^*$ such that $\rho(x) = \xi$. It follows that for all $\xi < \alpha$, $\eta \in f(\xi)$, i.e., $\eta \in \bigcap f``\alpha$, and so, $\bigcap f``\alpha \neq \emptyset$ and $\mathcal{F}_{\mathcal{E}}$ is γ-complete. Finally, by Theorem 4.4.2, we have that in $L(V_{\delta+1})$ there exists a partition of S into $<\delta_0$-many $\mathcal{F}_{\mathcal{E}}$-positive sets on each of which $\mathcal{F}_{\mathcal{E}}$ is an ultrafilter. Since $\mathcal{F}_{\mathcal{E}}$ is γ-complete, we are done. □

The same argument applies to show that if we fix $\beta < \Theta$ and let γ_β be the least such that $\gamma_\beta > \beta$ and $L_{\gamma_\beta}(V_{\delta+1}) \prec_{\Sigma_1} L_\Theta(V_{\delta+1})$, then in $L(V_{\delta+1})$, γ_β is measurable. So, the proof is complete. □

Chapter 5

Conclusion

The large cardinal axioms investigated in this dissertation have been introduced in [1] with two major purposes:

1. that of finding a deep inconsistency

   and

2. that of showing they can provide a rich mathematical structure.

In the first aim we noted that the inconsistency of Berkeley cardinals with ZF would actually follow by the HOD Conjecture; in the second aim we explored how choiceless large cardinals affect the structure theory of a remarkable model of ZF. Moreover, we proved that the cofinality of the least Berkeley cardinal is undecidable, and this is concerned with the main feature of Berkeley cardinals: they contradict AC. In this regard, one may observe that the evidence for AC is so compelling, that it would seem somewhat unlikely to drop AC (or even just part of AC) by the motivation that the large cardinal hierarchy can be extended beyond the "upper bound" established by Kunen's theorem in the context of ZFC. But the question here is more subtle...

There is a really basic issue involved in the conflict between very large cardinals and AC: it is *the search for* "V", i.e., the ultimate purpose of discovering the "right" axiom for a definitive description of the universe of set theory. Of course, this would also settle the fundamental question of which strong axioms of infinity must be regarded as true. Our final comments are devoted to outline the recent progress made in this direction, as choiceless large cardinals could actually play a crucial role in the resulting scenario.

5.1 Two futures

The justification of the consistency of a large cardinal axiom in set theory is arguably provided through the determination of the smallest inner model satisfying it; this investigation is the object of inner model theory and results in the construction of enlargements of the constructible universe L. The real question in seeking the proper axiom for V turns in fact to be the following:

Could there be an ultimate version of Gödel's L?

Woodin's *Ultimate-L Conjecture* surmises a positive answer to this question; even a candidate for the final axiom for V, "$V =$ Ultimate-L", has been formulated by Woodin. A suitable understanding of the axiom "$V =$ Ultimate-L" requires much additional material from [6] and [7]; however, some preliminary remarks will allow us to give the statement of the Ultimate-L Conjecture and display its implications.

The following results established by Woodin in [6] are exposed here as revisited in [5].

Theorem 5.1.1 (HOD Dichotomy). Suppose δ is an extendible cardinal. Then exactly one of the following holds.

1. For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$ (i.e., HOD is close to V).

2. Every regular cardinal greater than δ is measurable in HOD (i.e., HOD is far from V).

Theorem 5.1.2. The following statement is absolute between V and its generic extensions by partial orders in V_δ: "*δ is an extendible cardinal and for every singular cardinal $\gamma > \delta$, γ is singular in* HOD *and* $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$".

Definition 5.1.3. A transitive class model N of ZFC is a **weak extender model for δ supercompact** iff for every $\gamma > \delta$ there exists a normal fine measure $\mathcal{U}$ on $\mathcal{P}_\delta(\gamma) = \{X \subseteq \gamma : |X| < \delta\}$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}$ (i.e., $\mathcal{U}$ concentrates on N) and

2. $\mathcal{U} \cap N \in N$ (i.e., $\mathcal{U}$ is amenable to N).

Theorem 5.1.4. Suppose δ is an extendible cardinal. Then the following are equivalent.

1. The HOD Conjecture.

2. HOD is a weak extender model for δ supercompact.

3. For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.

Theorem 5.1.5 (Universality). Suppose δ is an extendible cardinal, N is a weak extender model for δ supercompact and $\gamma > \delta$. Let $j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$ be a non-trivial elementary embedding with $crit(j) \geq \delta$. Then, $j \in N$.

Theorem 5.1.5 says that if δ is an extendible cardinal and N is a weak extender model for δ supercompact, then N sees all elementary embeddings between its levels; this implies a sort of analogue of Kunen's theorem for N:

Theorem 5.1.6. Suppose δ is an extendible cardinal and N is a weak extender model for δ supercompact. Then there is no non-trivial elementary embedding $j : N \to N$ with $crit(j) \geq \delta$.

Proof. Suppose for contradiction that there exists a non-trivial elementary embedding $j : N \to N$ with $crit(j) \geq \delta$. Let $\kappa > crit(j) \geq \delta$ be a fixed point of j. Then, $i = j \restriction (V_{\kappa+2})^N : (V_{\kappa+2})^N \to (V_{\kappa+2})^N$ is a non-trivial elementary embedding with $crit(i) = crit(j) \geq \delta$; by Theorem 5.1.5, $i \in N$, contradicting the local version of Kunen's Theorem within N.[1] □

Corollary 5.1.7. Assume the HOD Conjecture. If δ is an extendible cardinal, then there is no non-trivial elementary embedding j : HOD $\to$ HOD with $crit(j) \geq \delta$.

Notice that by "universality", a weak extender model is capable to recognize all large cardinals (below the Kunen inconsistency), accomplishing this way the idea of being "close to V"; under the HOD Conjecture, this property is satisfied by HOD itself. We now state the Ultimate-L Conjecture, which will complete the picture.

Ultimate-*L* Conjecture. Suppose δ is an extendible cardinal. Then there exists a weak extender model for δ supercompact N such that:

1. $N \subseteq$ HOD.

2. $N \models$ "$V =$ Ultimate-L".

Theorem 5.1.8. Assume ZF + "$V =$ Ultimate-L". Then the following hold.

1. $V =$ HOD.

2. AC.

[1]Recall that the local version of Kunen's theorem states that for any κ, there is no non-trivial elementary embedding $j : V_{\kappa+2} \to V_{\kappa+2}$.

3. CH.[2]

At this point, the perspective emerges clearly: there are two plausible futures for the development of set theory, each of which leading to a real turning point in the subject.

The case for Ultimate-L	*The Choiceless Hierarchy*
Ultimate-L Conjecture	Ultimate-L Conjecture fails
$V = \mathrm{HOD}$	HOD is far from V
HOD Conjecture	HOD Conjecture fails
AC	$\neg$AC
Choiceless large cardinals are inconsistent	Choiceless large cardinals are consistent

The prospect displayed above is significative in bringing together the search for deep inconsistency and the search for V: so finally, the study of choiceless large cardinal axioms is possibly the key for a further, and maybe decisive, understanding of V.

[2]The **Continuum Hypothesis** (CH) is the statement that $2^{\aleph_0} = \aleph_1$.

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