UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

Dipartimento di Matematica e Applicazioni
"Renato Caccioppoli"
Dottorato di Ricerca in Scienze Matematiche e Informatiche

Tesi di Dottorato di
Mattia Brescia

Double Chain Conditions in Group Theory

CICLO XXIX, S.S.D. MAT/02

Contents

Introduction

Historical

During the history of Group Theory long has been (and long *is*) the journey to escape from the finite, still bringing along the nice-working and somehow reassuring properties of the finite world. Long has been the path through which academics from all over the world tried to reconcile the fairly impressive and sometimes (or maybe very often) unsettling matter of infinity even with the centuries-old difficulty of talking about it, driven everyone just by their deep love for knowledge itself. As we may clearly rule out, with small exceptions, any mere practical wish from this, we could then even say that the urge to keep on looking for the finite inside the infinite is nothing but that first momentum which once gave birth to Group Theory as the study of groups of permutations and groups of matrices and which endures still today, after more than two centuries (not really much compared with even the countable infinite, but no little time for us). In the early 1880s Walther von Dyck, a student of Felix Klein, laid the first systematic foundations for presentations of groups and after him many were the scholars, for instance in Erlangen under Felix Klein and in Berlin under Issai Schur, who began taking a serious interest in the matter. Such was the florid background which saw the finiteness conditions as a distinct branch of Infinite Group Theory coming to prominence and eventually flourishing. On the other side of the research, in 1921 Emmy Noether, under the aegis of David Hilbert, introduced and deeply studied the maximal condition on ideals of a ring in her paper *Idealtheorie in Ringbereichen*, and around 1928 Emil Artin dealt with the minimal condition on right ideals, both of them bringing with their works great development to modern algebra. In the mutual exchange of awareness and hints typical of the mathematical international community, everything is important and can have almost instantly an influence over something in the same net. These two examples, in fact, are not only useful in a general meaning as finiteness conditions but have here greater importance qua chain conditions, properly speaking, and not only in ring theory, where they were firstly developed, but in Group Theory, too, as one would have been able to experience shortly afterwards.

During the following years such conditions were studied in a proper group theoretical sense by famous algebraists such as Otto Schmidt, Robert Remak and Helmut Wielandt. Later on, Sergei N. Černikov, Reinhold Baer and Anatoly I. Mal'cev extended the study of maximal and minimal chain condition further more. On the one hand, it was discovered that many classes of groups satisfying the maximal condition on subgroups shared the same good behavior, being the classes of finite extensions of a polycyclic group; on the other hand, the inspection on the minimal condition revealed that several relevant classes (and more or less the same as in the maximal case) were well-behaved, being the classes whose elements are finite extensions of an abelian group satisfying the minimal condition. Notice that abelian groups satisfying the minimal condition on subgroups and polycyclic groups are well-studied and completely described. Thus soon arose the question about to what extent these good behaviors could hold and there was a mild hope for almost every class of groups to have it. Unfortunately, in 1979 Alexander Ol'šanskiĭ in his *Infinite groups with cyclic subgroups* proved the existence of infinite simple groups having both the maximal and the minimal condition on subgroups, building a stainless roof over every further development on the subject.

Nonetheless, the research must go on and several different forms of minimal and maximal chain conditions were examined; among the others a significant role was taken by the maximal and minimal conditions on non-θ-subgroups, where θ is a subgroup theoretical property.

For the purpose of our work, we now mention that, together with exploring such conditions on θ or non-θ subgroups, someone came up with the idea of studying chain conditions other than the minimal and the maximal one. Eventually, the result was that the imposition of weaker forms of the classical chain conditions in many cases produces remarkable effects. In particular, the so-called *double chain condition* was taken into account. In 1971 Dmitrij Zaicev in his paper *On the theory of minimax groups* and in 1973 Thomas Shores in *A chain condition for groups* independently proved that if G is a generalized soluble group admitting no chains of subgroups with the same order type of the set of integers, then G is a finite extension of a soluble group and it either satisfies the minimal or the maximal condition on subgroups. These results are a remarkable example of both the aesthetics and the intrinsic force of the subject: even in the more complex hypotheses an intimate truth will appear, which will reduce the question to simpler, more accessible ones. In particular, most of the times it is possible to reduce a suitable condition to its *extremal* cases, namely those cases for which the statement is trivial, as minimal and maximal condition are for the double chain condition on subgroups. Throughout this work many interesting examples of this behavior will be shown, not only with regards to double chain conditions, and still some case will be revealed, for which a strict reduction into extremal cases is not possible. At the occurrence of such (bad, indeed, but *imperfections makes perfection*) situations, an example will be given.

It should also be mentioned that the double chain condition for other algebraic structures, like for instance rings and modules, has also been studied (see for instance [14]).

In conclusion, along this work a triadic structure is going to be exhibited, so that each chapter will concern a subgroup property θ and will show fundamental or useful results about the maximal and the minimal condition on θ-subgroups, then moving on to the related double chain condition and showing the behavior of several reasonable classes of groups satisfying it, in comparison with the results for the other two conditions, which it extends.

Practical

A group class $\mathfrak{X}$ is said to be a finiteness class if

$$\mathfrak{F} \leq \mathfrak{X} \leq \mathfrak{U}$$

with $\mathfrak{F}$ and $\mathfrak{U}$ being the class of finite groups and the class of all groups, respectively. In other words, a class of groups $\mathfrak{X}$ is said to be a finiteness class if the class of finite groups lies in it. The group theoretical property related with such a class is said *finiteness condition* and it is satisfied by every finite group by definition. Among finiteness conditions great importance has been given to the so-called *chain conditions*, which are the fulcrum of this thesis. Since the word *chain* is generally used to define a totally ordered set (i.e. an ordered set in which every pair of elements are comparable under the order relation), it is now clear that totally ordered sets must be involved.

Let θ be a group theoretical property and T a totally ordered set. Then we say that a group G satisfies the *T-chain condition on θ-subgroups* if the lattice of θ-subgroups of G ordered by inclusion does not contain any T-ordered subset. For instance, if we choose $T = (\mathbb{N}, <)$ and θ as the property of being a subgroup, then the related chain condition is the well-known *maximal condition on subgroups*, which in literature can also be found under other names such as *maximum condition (on subgroups)*, *ascending chain condition (on subgroups)* or Max, often referring to the related class. Precisely a group G belongs to Max if and only if for each ascending chain

$$X_0 \leq X_1 \leq \ldots \leq X_n \leq \ldots$$

of subgroups of G there exists an integer k such that $X_n = X_k$ for all $n \geq k$.

On the other hand, if we choose $T = (\mathbb{N}, >)$ and θ as the property of being a subgroup, then the related chain condition is the renowned *minimal condition on subgroups*, which in literature can also be found under other names such as *minimum condition (on subgroups)*, *descending chain condition (on subgroups)* or Min, frequently referring to the related class. Precisely a group G belongs to Min if and only if for each descending chain

$$X_0 \geq X_1 \geq \ldots \geq X_n \geq \ldots$$

of subgroups of G there exists an integer k such that $X_n = X_k$ for all $n \geq k$. From the first historical definition a generalization of these two properties to θ-subgroups for other properties θ is straightforward.

Finally, coming to the core of this work, we have the *double chain condition on θ-subgroups* if we take $T = (\mathbb{Z}, <)$ and θ as a subgroup theoretical property. We shall say that a group G satisfies the double chain condition on θ-subgroups if for each double chain

$$\ldots \leq X_{-n} \leq \ldots \leq X_{-1} \leq X_0 \leq X_1 \leq \ldots \leq X_n \leq \ldots$$

of θ-subgroups of G there exists an integer k such that either $X_n = X_k$ for all $n \leq k$ or $X_n = X_k$ for all $n \geq k$. Obviously, both the minimal and the maximal conditions on θ-subgroups imply the double chain condition on θ-subgroups, forming its first two trivial extremal cases.

We are going to give here some of the basic instruments or concepts which will be of great use in the following chapters. Firstly we are bringing the reader into the mood of dealing with double chain conditions, showing simple cases where double chains appear and some related results. Secondly we are giving some preliminary results which we will use extensively in the following.

The appearance of double chains

Let us show here some cases where double chains immediately are formed, which will make it possible to use the double chain conditions (and maximal and minimal conditions, too, clearly) to obtain well-behaved subgroups to proceed in our study. One of them is the following lemma, which allows us to take into direct account the perhaps most eye-catching results from satisfying double chain conditions.

Lemma 0.0.1. *Let χ be a subgroup theoretical property such that the intersection of two χ-subgroups is still a χ-subgroup and let G be a group satisfying the double chain condition on non-χ-subgroups. Let H/K be a section of G which is a direct product of infinitely many non-trivial subgroups. Then K is a χ-subgroup of G and such is every direct term of H/K.*

Proof. H/K is a direct product of infinitely many non-trivial subgroups if and only if it contains a direct product of countably many non-trivial subgroups so without loss of generality we can let $\{H_n|n \in \mathbb{Z}\}$ be a countably infinite collection of subgroups of H properly containing K and such that $H/K = \mathrm{Dr}_{i\in\mathbb{Z}} H_i/K$. We can then split this collection into two infinite collections, namely $\{U_i|i \in \mathbb{Z}\}$ and $\{V_j|j \in \mathbb{Z}\}$, such that

$$H/K = \mathrm{Dr}_{i\in\mathbb{Z}} U_i/K \times \mathrm{Dr}_{j\in\mathbb{Z}} V_j/K$$

and for each integer n define

$$U_n^* = \langle U_k | k < n \rangle$$

$$V_n^* = \langle V_k | k < n \rangle.$$

It is clear that these two families form two infinite double chains of subgroups of G. Therefore, since the group satisfies the double chain condition on non-χ-subgroups, there exist two integers s and r such that $U_r^* \chi G$ and $V_s^* \chi G$. Obviously we have that $U_r^* \cap V_s^* = K$ and K is a χ-subgroup by the hypothesis on χ.

If now we take a subgroup H_m from the collection $\{H_n|n \in \mathbb{Z}\}$, we have that H/H_m satisfies the same hypotheses of the lemma and then by the first part of the proof and by the generality of H_m we have that every direct term of H/K is a χ-subgroup of G. □

This result underlines, in a rather simple way, how the presence of a direct product of subgroups may have potentially a strong influence on the whole group in case it satisfies a double chain condition, giving us infinitely many well-behaved subgroups to employ. The extent of the effect is obviously greatly influenced by the nature of the property χ, as we will show along this work.

Now we are going to show a couple of usual circumstances in which infinite direct products of subgroups (and hence double chains) show up.

For our purposes we are going to define the concept of independence for a set of indices. Let G be a group and let $\{S_\alpha|\alpha \in I\}$ be a family of subgroups of G. A subset B of I is said to be *independent* if $\langle S_\beta | \beta \in B \rangle = \mathrm{Dr}_{\beta\in B} S_\beta$.

In this case $\{S_\beta|\beta \in B\}$ is said to be an *independent* set of subgroups of G. It is useful to notice that for every non-empty set of indices A the set of all independent subset of A is an inductive set and hence Zorn's Lemma can be applied to it.

For an abelian group G the rank $r_p(G)$ or *p-rank* of G is the cardinality of a maximal independent subset of cyclic p-subgroups of G. Similarly the 0-rank or *torsion-free rank* of G is defined as the cardinality of a maximal independent subset of infinite cyclic subgroups of G.

Now, for an abelian p-group to have finite rank is equivalent to satisfying the minimal condition on subgroups [see for instance [52] p. 107]. With this in mind the following is an easy consequence.

Proposition 0.0.2. *Let G be an abelian periodic group not satisfying the minimal condition on subgroups. Then G contains a subgroup which is direct product of infinitely many subgroups.*

From the above we can see how to find double chains by means of abelian subgroups not satisfying Min and this will be used broadly in the following.

Let us now use the definition of an independent set to establish some more facts about infinite direct products and conditions for their existence in groups.

Lemma 0.0.3. *Let G be a group and let $\{M_\alpha|\alpha \in I\}$ be a family of minimal normal subgroups of G. Then the subgroup generated by the family is the direct product of certain members of that family.*

Proof. Let $J = \langle M_\alpha|\alpha \in I\rangle$. By Zorn's Lemma we can find a maximal independent set $B \subseteq I$. Let $K = \mathrm{Dr}_{\beta\in B} M_\beta$ and suppose $K \neq J$. Then there is an $\alpha \in I$ such that M_α is not contained in K and clearly $M_\alpha \cap K = \{1\}$ by the minimality of M_α. Hence $M_\alpha K = M_\alpha \times K$ and the set $B \cup \{\alpha\}$ contradicts the maximality of B. Therefore $J = K$ and the lemma is proved. □

Recall that, given a group G, we define the *socle* of G to be equal to $\{1\}$ if G does not have minimal normal subgroups and equal to the product of all its minimal normal subgroups if G does.

Corollary 0.0.4. *The socle of a group G is the direct product of a (possibly empty) set of minimal normal subgroups of G.*

Other relevant preliminaries

The elementary consideration that every group has a chief series, together with the knowledge about some classes of groups possessing particular series of this kind, lead many to investigate about possible extensions of those classes for which the properties of their chief series still hold. Important steps in this direction were firstly made by A. I. Mal'cev [35] in 1941. Here we present these theorems as proved, in a more *modern* fashion, by McLain [37] in 1956 .

Theorem 0.0.5 (Mal'cev [35])**.** *The property "every chief factor is abelian" is* $\boldsymbol{L}$*-closed*

Proof. Let us suppose the existence of a group G such that satisfies "every chief factor is abelian" locally and which has a minimal normal subgroup N containing a and b such that $c = [a, b] \neq 1$. By minimality, $N = \langle c\rangle^G$ and clearly there exist $g_1, \dots, g_n$ in G such that a and b belong to $\langle c^{g_1}, \dots, c^{g_n}\rangle$. By hypotesis all chief factors of $H = \langle c, g_1, \dots, g_n\rangle$ are abelian and since $c \in (\langle c\rangle^H)'$ we have that $\langle c\rangle^H = (\langle c\rangle^H)'$. On the other hand by Zorn's Lemma we can find in $\langle c\rangle^H$ a subgroup M which is normal in H, does not contain c and it is maximal with respect to these conditions. Then $\langle c\rangle^H/M$ is a chief factor of H, so $(\langle c\rangle^H)' \leq M < \langle c\rangle^H$, a contradiction. □

Theorem 0.0.6 (Mal'cev [35])**.** *The property "every chief factor is central" is* $\boldsymbol{L}$*-closed*

Proof. Let us suppose the existence of a group G such that satisfies "every chief factor is central" locally and having a minimal normal subgroup N such that there are $a \in N$ and $g \in G$ such that $c = [a, g] \neq 1$. By minimality, $N = \langle c\rangle^G$ and clearly there exist $g_1, \dots, g_n$ in G such that a belongs to $\langle c^{g_1}, \dots, c^{g_n}\rangle$. By hypothesis all chief factors of $H = \langle c, g, g_1, \dots, g_n\rangle$ are central. Let us say $A = \langle a\rangle^H$. Since obviously $[A, H]$ is normal in H, we have that $[A, H]$ contains c and all its conjugates in H, then $A \leq [A, H]$ and clearly $A = [A, H]$. If we now take M as a maximal H-invariant subgroup of A not containing a we have that A/M is a chief factor of H and hence $[A, H] \leq M < A$, which is a contradiction. □

Some significant results follow from these parallel theorems.

Corollary 0.0.7. *A chief factor of a locally soluble group is abelian and a chief factor of a locally nilpotent group is central.*

From this we can straightforwardly see that

Corollary 0.0.8. *A locally soluble simple group must be cyclic of prime order.*

Anyway many consequences of these results will be given in the following.

We now give a conclusion to this practical introduction by presenting the concept of (right) *Engel element*, which is one of the many possible ways of extending properties pertaining nilpotency.
If G is a group and x an element of G, we can define x to be a *right Engel* element of G if for each y in G there exists a natural number n such that

$$[x, \underbrace{y, \ldots, y}_{n}] = 1.$$

We write here $R(G)$ for the set of right Engel elements of G. In general one does not know whether $R(G)$ is a subgroup of G or not, but clearly there are classes of groups, such as the class of nilpotent groups (where obviously every element is a right Engel element), for which $R(G)$ is a rather well-behaved subgroup. One relevant study about the problem of Engel elements was spotlighted in a paper by Baer in 1957. One of the most important results is that in a group G satisfying Max $R(G)$ is the hypercentre of G ([4], p. 257). As concerns our case we have the following statement.

Proposition 0.0.9. *Let G be a locally (finite-by-nilpotent) group, then $R(G)$ is the largest normal subgroup of G respect with the condition of having all G-chief factors being central factors of G.*

Proof. Firstly notice that $R(G)$ is easily seen to be a locally nilpotent characteristic subgroup of G by the quoted result of Baer. Now we want to prove that, for each X and Y such that X/Y is a chief factor of G and $X \leq R(G)$, X/Y is a central factor of G. We can assume without loss of generality that $Y = \{1\}$ and take by a contradiction $x \in X$ and $g \in G$ such that $t = [x, g] \neq 1$. Since clearly $X = \langle y \rangle^G$ there exists a finitely generated subgroup H of G such that $x \in \langle y \rangle^H$. Say $K = \langle t, g, H \rangle$. Then $t \in [\langle t \rangle^K, K]$ and so $\langle t \rangle^K = [\langle t \rangle^K, K]$. On the other hand, we have that $\langle t \rangle^K \leq R(K)$ and that $R(K)$ coincides with the hypercentre of K, since K is Max, so $[\langle t \rangle^K, K] < \langle t \rangle^K$ and this proves our first claim.
Let now S be a normal subgroup of G having a G-central series and fix an element x of G. $K = S\langle x \rangle$ has a central series and so does in K every finitely generated subgroup, which is by hypothesis finite-by-nilpotent and then nilpotent. So K is locally nilpotent and this means, by the arbitrariness of x, that S lies inside $R(G)$. □

On the notation:

Maximal and minimal condition on θ-subgroups will be referred to as Max-θ and Min-θ, respectively. Regarding the double chain condition on θ-subgroups, we will instead write DC_{θ}. In particular, *ab*, *n*, *nn*, *sn*, *snab*, *snn*, *np* will stand for the subgroup property of being "abelian", "normal", "non-normal", "subnormal", "subnormal abelian", "subnormal non-normal" and "non-pronormal", respectively.

Apart from this, most of our notation is standard and can be found in [48].

Chapter 1

Double chain condition on subgroups

Here we are going to introduce the fundamental chain conditions, in which θ is the property of being merely a group. We should remember in advance that even in this seemingly very elementary case serious pathologies can come out. In fact, in 1979 Alexander Olšanskiĭ [41] has shown that there exist infinite simple groups satisfying both minimal and maximal condition on their subgroups, making then the study of those properties being reasonably restricted to a universe of generalized soluble groups in order to exclude such *ill-behaved* groups.

1.1 Maximal condition

As we just noticed, the maximal condition on subgroups, together with the minimal condition, cannot aim to reach the deepest generality and mostly has to deal with the fact of its study being restricted time by time to some class of generalized soluble groups. At the beginning, until the quoted paper of Olšanskiĭ, many approached the study of this property and many achieved remarkable results; among them Hirsch, Mal'cev, Baer, Schmidt, Bowers and Robinson, who came with claiming that the general question might have not a positive (and certainly not an easy) answer. Together with these considerations, many papers were published in which the study was restricted with regards to the type of property (see Zappa [62]) or to the class of groups.

Since the theme is well-studied and can be easily found throughout the literature, here we will present some basic results for future use, only showing the more relevant proofs.

In his paper of 1951 Mal'cev proved that a soluble group of automorphisms of a finitely generated abelian group is polycyclic (Mal'cev [36], Theorem 2).

This result was eventually generalized by Baer ([3], Satz B') and finally by Robinson ([48], Theorem 3.27), whose result we report here.

Theorem 1.1.1. *A radical group of automorphisms of a polycyclic-by-finite group is polycyclic.*

This result will be very useful in several circumstances and it is in fact used in more then one occasion in the proof of Theorem 1.1.2 below.

Moving somehow away from Max, many were the group theorists concerning about the maximal condition on abelian subgroups. Having to face several restrictions in their approach to the pure maximal condition, many asked themselves to what extent the less pure but not less interesting property of not having proper ascending chains of abelian subgroups, namely Max-*ab*, could lay its influence upon certain classes of groups. Clearly these classes should be, in some sense, influenced by the structure of their abelian subgroups and therefore they are expected to have *many* of them. The restriction to generalized soluble groups was just one step forward. In [36] (Theorem 8) Mal'cev showed that for soluble groups Max-*ab* and Max coincide and this, together with the result of Schmidt (see the section below) gave birth to a prolific literature about this kind of finiteness conditions.

Here we give the statement of the generalization of Mal'cev's result.

Theorem 1.1.2 (Robinson [48], Theorem 3.31)**.** *Let G be a radical group and R the Hirsch-Plotkin radical of G. If R satisfies Max-ab then G satisfies Max.*

1.2 Minimal condition

As we noticed for the maximal condition on subgroups, even the study about the minimal condition, which is sometimes a more striking property, cannot aim to reach the deepest generality and mostly has to be restricted time by time to some class of generalized soluble groups. Before the quoted paper of Olšanskiĭ, many approached the study of this property and many achieved remarkable results; among them Černikov, from whom finite extensions of abelian groups satisfying Min took their name, Kuroš, Baer, Schmidt, Polovickiĭ and Robinson. Together with these considerations, many papers were published in which the study was limited with regards to the type of property or to the class of groups. As we will see in this section, there is a strong analogy between results about the minimal and the maximal condition on subgroups especially when looking at our final theorems. Nevertheless, one has to point out that from the very beginning of our brief inspection through the minimal condition a strong asymmetry shows up. In fact, as we are going to see in a moment, in the Min case one cannot avoid the hypothesis of something being periodic, which was not present, even in the analogous case of being finitely generated, in the Max case. The crucial thing here is the following: in the generalized Min case (see Theorem 1.2.1, see Theorem 4.2.5) the group has to satisfy some periodicity hypotheses while in the generalized Max case it has to be a radical group; however, as we have already discussed, the generalized soluble hypothesis is a very common one and it is basically the fundamental premise for working on many finiteness conditions avoiding pathologies, so it is required anyway. That is the main reason the work with the generalized Min condition is most of the times more clumsy.

Since the theme is well-studied and can be easily found throughout the literature, here we will present some basic results for future use, only showing the more relevant proofs. Moreover some results can be deduced from more general ones that can be found below in the dedicated chapter (see, for example, the results in Section 4.2).

In his paper of 1955 Baer proved an important result about the periodic group of automorphisms of Černikov groups; in 1960 Polovickiĭ [44] independently proved the same result. We state it here.

Theorem 1.2.1. *A periodic group of automorphisms of a Černikov group is Černikov.*

This result will be very useful in several circumstances and it is in fact used in more then one occasion in the proof of Theorem 1.2.2 below.

Moving away from Min, many were the group theorists concerning about the minimal condition on abelian subgroups, namely Min-*ab*, which they studied, in the same ideal background described for the maximal condition on subgroups, in a generalized soluble environment. In [53] (Theorem 9) Schmidt showed that for hyperabelian groups Min-*ab* and Min coincide and this, together with the quoted result of Mal'cev about soluble groups satisfying Max-*ab*, gave birth to a wide literature about this kind of finiteness conditions.

Here we give the statement of the generalization of Schmidt's result.

Theorem 1.2.2 (Robinson [48], Theorem 3.32)**.** *Let G be a radical group and R the Hirsch-Plotkin radical of G. If R satisfies Min-ab then G is a soluble Černikov group.*

We conclude this section by proving an important result by Černikov [12], the prove of which can be retrieved in [48] and is based on the following

Lemma 1.2.3. *Let G a locally finite group. If G satisfies the minimal condition on abelian p-subgroups, then so does every section of G. If G has finite Sylow p-subgroups, then so has every section of G.*

Theorem 1.2.4. *Let G be a locally soluble group satisfying Min-ab. Then G is a Černikov group.*

Proof. Let H be a chief factor of G. Since by Corollary 0.0.7 H is abelian, we know that H is an elementary abelian p-group. By Lemma 1.2.3 H is Min and hence it is finite. The class $\mathfrak{F}^{-\mathbf{H}}$, namely the largest quotient-closed class of groups with trivial intersection with $\mathfrak{F}$, is clearly a radical class and the $\mathfrak{F}^{-\mathbf{H}}$-radical of G, say R, belongs to it, so it must induce the trivial automorphism on H. Because of the arbitrary choice of H, we see that R has a central series, so it is also clearly locally nilpotent. By Theorem 1.2.2 R is Černikov. On the other hand G/R has no Prüfer subgroups by the definition of R and it is easily seen (see, for example, the argument used inside Theorem 4.2.5) that G/R satisfies Min-*ab*, too, so its abelian subgroups are all finite. Now by the well-known theorem of Hall-Kulatilaka-Kargapolov G/R is finite and the theorem is proved. □

We conclude this section by stating an interesting result by Zaicev, which will be used in the following.

Theorem 1.2.5 (Zaicev [61])**.** *Let G be a periodic locally solvable group and F be a finite group of automorphisms of G. If each F-invariant abelian subgroup of G satisfies Min, then G satisfies Min-ab.*

1.3 Double chain condition

In this section we are giving account of the quoted paper of Shores about groups with no double chains of subgroups, which we will call here DC-groups. The paper, mostly unaware of the recent paper of Zaicev about minimax groups, deals with two questions: "To which extent do DC-groups satisfy either Max or Min?" and "What are the classes of groups for which DC and DC_{ab}, namely the double chain condition on abelian subgroups, do and do not coincide?". As we will see the answer is somewhat positive for the first question, being that it is positive for a reasonably large class of groups, and somewhat negative for the second one.

Lemma 1.3.1. *Let G be an abelian DC-group. Then G is either Max or Min. In particular, if G is not periodic it is Max.*

Proof. Let G be a periodic abelian group. Then it is clearly Min, otherwise $Soc(G)$ would be the direct sum of infinitely many subgroups which we could re-arrange as a double chain.
Let now G have an element x of infinite order. Then, if we consider

$$\ldots < \langle x^n \rangle < \ldots < \langle x \rangle \leq \langle x, x_1 \rangle \leq \ldots \leq \langle x, x_1, \ldots, x_n \rangle \leq \ldots$$

with $x_1, \ldots, x_n, \ldots$ elements of G it has to stop and so G satisfies Max. □

Theorem 1.3.2. *Let G be a nilpotent group. Then the following are equivalent:*

(a) G is either Max or Min;

(b) G is a DC-group;

(c) G is a DC_{ab}-group.

Proof. Let us show the only non-trivial implication, namely $(c) \implies (a)$. We firstly claim that if G is DC_{ab}-group $G/Z(G)$ is the same. In fact, let $A/Z(G)$ be an abelian subgroup of $G/Z(G)$ and suppose that it is neither Max nor Min. Then $Z(G)$ satisfies Min. Let M a maximal abelian subgroup of the nilpotent group A, we have that M cannot be Max and hence is Min by Lemma 1.3.1. So A is not periodic by Theorem 1.2.1 and $Z(G)$ is finite. From this we get at once that A' is finite, too, and that $A/Z(G)$ is residually finite. Therefore M is finite, a contradiction.
By means of this claim, we can now use induction on the nilpotency length of G. Let G be periodic. By Lemma 1.3.1 $Z(G)$ is Min and by induction $G/Z(G)$ is either Min or Max, that is Min in both cases and so is G. Let G be non-periodic. Then $Z(G)$ has to be finitely generated. If $Z(G)$ is not Min, then $G/Z(G)$ is Max and so is G. So we can assume that $Z(G)$ is finite. Now, by induction, $G/Z(G)$ has to be Max or Min, hence it is Max and so is G. □

Let p be a prime, P be a Prüfer p-group and x an automorphism of P of infinite order. If we consider $G = \langle x \rangle \ltimes P$ we readily see that G satisfies the double chain condition on abelian subgroups while it does not satisfy the one on subgroups. This gives a negative answer to our second question, at least in the case of soluble groups, whereas, as we have just seen, it is positive for nilpotent groups. Nevertheless, from our Lemma 1.3.1 and from Satz A.3 in [1] one can directly deduce the following

Proposition 1.3.3. *Let G be a hyper-(finite or locally nilpotent) DC_{ab}-group. Then G is a soluble-by-finite minimax group.*

Theorem 1.3.4. *Let G be a hyper-(finite or locally nilpotent) group. Then the following are equivalent:*

(a) G is soluble-by-finite and satisfies either Max or Min;

(b) G is a DC-group;

Proof. Clearly $(a) \implies (b)$. Conversely let G be a hyper-(finite or locally nilpotent) DC-group. By Proposition 1.3.3 we know that G is soluble-by-finite. Clearly we may suppose that G is soluble and proceed by induction on the derived length of G. Since G' is either Max or Min, if it is not periodic it is Max and so has to be G/G', together with G, so we may assume G' being periodic and hence Min. Consider $C = C_G(G')$, which is clearly nilpotent of class 2 and then either Max or Min by Theorem 1.3.2. If G' is finite, then G/C is finite and G is either Max or Min so we can suppose that G' is not finitely generated. From this it follows that G itself has to be periodic and so G/G' satisfies Min and the theorem is proved. □

Theorem 1.3.5. *Let G be a locally radical group. Then the following are equivalent:*

(a) G is soluble and satisfies either Max or Min;

(b) G is a DC-group;

Proof. Clearly $(a) \implies (b)$. Conversely let G be a locally radical DC-group. By Theorem 1.3.4 every finitely generated subgroup of G is soluble and satisfies either Min or Max, so we can suppose that G is not finitely generated. From this follows that G has to be periodic. Then every abelian subgroup of G is Min and, by Theorem 1.2.4, G satisfies Min. □

Chapter 2

Double chain condition on normal subgroups

This chapter will talk about chain conditions on normal subgroups. It is clear that every simple group satisfies every chain condition on normal (and subnormal) subgroups, so here we have further reasons to work in a suitable universe of generalized soluble groups, regardless of the results of Olšanskiĭ.

As we will show, these three properties are closed under homomorphic images and, maybe surprisingly, also under subgroups of finite index.

2.1 Maximal condition

The maximal condition on normal subgroups, which we will denote by Max-n, is clearly **P** and **H**-closed, while it is not closed under **S** or $\mathbf{S}_n$ operators. This is easily shown by the following

Example 2.1.1. *Let $G = \langle a, b : b^{-1}ab = a^2 \rangle$. Then G has only one normal subgroup which is isomorphic with the group of rational numbers whose denominators are powers of 2.*

The situation is different if we look at the inheritance for subgroups of finite index.

Theorem 2.1.2 (Wilson [59])**.** *Let G be a group satisfying Max-n and let H be a subgroup of finite index of G. Then H satisfies Max-n.*

Proof. Clearly we may assume H normal in G. Suppose by a contradiction that H does not satisfy the maximal condition on H-invariant subgroups, so we can find in H a normal subgroup K of G maximal among the subgroups belonging to an H-invariant ascending chain. If we fix an ascending chain of normal subgroups of H, for instance $K = K_1 < K_2 < \ldots$, and if we call T a right transversal of H in G we notice that, for every i, $K = \mathrm{core}_T K_i$, so it is not empty the set of all finite non-empty sets such that the previous condition is satisfied for every i. Call X an element of minimal cardinality of this set. If $x \in X$, then Xx^{-1} is a set with the same property, so we may assume that $1 \in X$. Now call $Y = X \setminus \{1\}$ and notice that it cannot be empty since otherwise K would not be included in any ascending chain of H-invariant subgroups of H, which is, indeed. Let $K = K_1 < K_2 < \ldots$ be an ascending chain of normal subgroups of H and let $L_i = K_i \mathrm{core}_Y K_i$, which defines an ascending chain of normal subgroups of H (clearly containing K), otherwise we would have

$$K_i = K_i \mathrm{core}_X K_{i+1} = K_i(K_{i+1} \cap \mathrm{core}_Y K_{i+1}) \geq K_{i+1} \cap L_i = K_{i+1} \cap L_{i+1} = K_{i+1}$$

from a certain integer i on.

So, by the property of X, we have that $\mathrm{core}_X L_i = K$ and this means that

$$K_i = K_i \mathrm{core}_X L_i = K_i(L_i \cap \mathrm{core}_Y L_i) = L_i \cap (K_i \mathrm{core}_Y L_i) = L_i.$$

But now from the definition of L_i it follows that $\mathrm{core}_Y K_i = \mathrm{core}_X K_i = K$ and this contradicts the minimality of X. □

As with the maximal condition on subgroups, we have the following result.

Theorem 2.1.3 (Hall [29], p. 420)**.** *A soluble group satisfying Max-n is finitely generated.*

Proof. Let G be a soluble group in Max-n. Let us work by induction on the derived length of G. Let A be the last non-trivial term of the derived series and let $X = \langle x_1, \ldots, x_m \rangle$ be such that $G = AX$. Since A satisfies Max-G we have that there exist the elements in A, say $y_1, \ldots, y_n$, such that $A = \langle y_1 \rangle^G \cdots \langle y_m \rangle^G$ and since A is abelian it follows easily that G is generated by $x_1, \ldots, x_m, y_1, \ldots, y_n$. □

As shown by Example 2.1.1 there exist metabelian groups satisfying Max-n which does not satisfy Max, so in the universe of soluble groups we have that Max is a proper subclass of Max-n, which is a proper subclass of the class of finitely generated groups. Nevertheless, P. Hall showed, in particular, that for metabelian groups these two letter classes coincide. This result is based on the following lemma on modules which we just state here.

Lemma 2.1.4 (Hall [29], Corollary to Lemma 3)**.** *Let H be a normal subgroup of a group G such that G/H is polycyclic. Let N be an H-module contained in the G-module M and such that M is generated as a G-module by N. Then, if N satisfies Max-H, M satisfies Max-G.*

Theorem 2.1.5 (Hall [29], Theorem 3)**.** *Every finitely generated abelian-by-polycyclic group satisfies Max-n.*

Proof. Let G be a group and let A be a normal abelian subgroup of G such that G/A is polycyclic. It is known that in A there exist $a_1, \ldots, a_n$ such that $A = \langle a_1 \rangle^G, \ldots, \langle a_n \rangle^G$. Let us take an arbitrary a_i and let $C = C_G(\langle a_i \rangle^G)$, so G/C is polycyclic. Clearly $\langle a_i \rangle$ is an A-module which satisfies Max-A since $A \leq C$. Then by Lemma 2.1.4, taking $H = A$ and $M = \langle a_i \rangle^G$, we have that each $\langle a_i \rangle^G$ satisfies Max-G and since Max-G is **P**-closed we have that G satisfies Max-n. □

Different is the situation regarding locally nilpotent groups.

Locally Nilpotent and Locally Soluble groups

Theorem 2.1.6 (McLain [37])**.** *For a locally nilpotent group Max and Max-n coincide.*

Proof. Let G be a locally nilpotent group satisfying Max-n and not Max, hence being not a finitely generated nilpotent group. Since clearly $G = XG'$ where X is a finitely generated nilpotent subgroup of nilpotent class, say c, $G = X\gamma_{c+2}$ for a well-known result of McLain (see [38], Lemma 2). So G/γ_{c+2} is nilpotent and the lower central series of G stops after finitely many steps at, say, L, which is clearly non-trivial by our first assumption. Take now a normal subgroup M of G maximal with respect to being properly contained in L. By Corollary 0.0.7 L/M is a central factor of G and hence $L = [L, G] \leq M < L$, a contradiction. □

On the other hand things do not work as well for locally soluble groups, how it is shown by a construction by McLain [38], who built a periodic locally soluble group satisfying Max-n which is not soluble, still.

Example 2.1.7 (McLain [38])**.** *There exists a locally soluble group satisfying Max-n which is not soluble.*

2.2 Minimal condition

The minimal condition on normal subgroups, which we will denote by Min-n, is clearly **P** and **H**-closed, while it is not closed under **S** or $\mathbf{S}_n$ operators, as shown in an example by Čarin [9].
On the other hand, similarly to the Max-n case, we have that Min-n is inherited by subgroups of finite index.

Theorem 2.2.1 (Wilson [59])**.** *Let G be a group satisfying Min-n and let H be a subgroup of finite index of G. Then H satisfies Min-n.*

Proof. Clearly we may assume H normal in G. Suppose by a contradiction that H does not satisfy the minimal condition on H-invariant subgroups, so there exists in H a subgroup M which is normal in G and minimal with respect to the condition to belong to a descending chain of normal subgroups of H. If we fix a descending chain of normal subgroups of H, namely $M = M_1 > M_2 > \ldots$, and if we call T a right transversal of H in G we notice that, for every i, $M = M_i^T$, so it is not empty the set of all finite non-empty sets such that the previous condition is satisfied for every i. Call then X an element of minimal cardinality of this set. We can assume that $1 \in X$. Now call $Y = X \setminus \{1\}$ and notice that it cannot be empty since otherwise K would satisfy the minimal condition on H-invariant subgroups, which is not. Let $M = M_1 > M_2 > \ldots$ be a descending chain of normal subgroups of H and let $L_i = M_i \cap M_i^Y$, which defines a descending chain of normal subgroups of H (clearly contained in M), otherwise we would have

$$M_i = M_i \cap M_{i+1}^X = M_i \cap (M_{i+1} M_{i+1}^Y) \leq M_{i+1} L_i = M_{i+1}$$

from a certain integer i on.
So, by the property of X, we have that $L_i^X = M$ and this means that

$$M_i = M_i \cap L_i^X = M_i \cap (L_i L_i^Y) \leq L_i (M_i \cap M_i^Y) = L_i.$$

But now from the definition of L_i it follows that $M_i^Y = M_i^X = M$ and this contradicts the minimality of X. □

Corollary 2.2.2 (Baer [2], p. 14, Theorem 4)**.** *Let G be a group satisfying Min-n. Then the centre and the FC-hypercentre of the finite residual of G coincide and satisfy Min.*

Proof. Let R be the finite residual of G, which is, by Min-n, minimal among the normal subgroups of G having finite index. Say F_1 the FC-centre of G and

F_2/F_1 the FC-centre of R/F_1, then $F_1 = \zeta(R)$ and $F_2 = \zeta_2(R)$. If we take $y \in R$ and $x \in \zeta_2(R)$, we have that $x^y = xz$, where $z \in \zeta(R)$. Since the order of z divides the order of x and $\zeta(R)$ clearly satisfies Min by Theorem 2.2.1, we can say that $x \in F_1$ and so $F_2 = \zeta_2(R) = \zeta(R)$, from which the thesis follows immediately. □

Corollary 2.2.3. *Let χ be the following subgroup theoretical property: $H\chi G$ if and only if $H \triangleleft G$ and $Aut_G H$ is residually finite. Then a hyper-χ group which satisfies Min-n is a Černikov group and the properties Min and Min-n coincide for hyper-χ groups.*

Proof. Let G be a hyper-χ group satisfying Min-n and R be the finite residual of G, which is hence minimal among the normal subgroups of G having finite index. Take an ascending normal χ-series of G and a factor F of the series. Clearly $\mathrm{Aut}_R F$ is residually finite and R must centralize F. We have found that R centralizes every term of the ascending χ-series and then R has an ascending central series. So by Corollary 2.2.2 R is abelian and Černikov and so is G. □

By means of these useful corollaries it is revealed, in a pretty straightforward way, how Min and Min-n coincide for some relevant classes.

Proposition 2.2.4 (Duguid and McLain [22])**.** *For the FC-hypercentral groups Min-n and Min coincide.*

Proposition 2.2.5 (Baer [2], p. 16, Theorem 3)**.** *For the hyperfinite groups Min-n and Min coincide.*

Moreover, as we will see in the next paragraph, the same holds for locally nilpotent groups.

Theorem 2.2.6 (Baer [5])**.** *Let G be a soluble group satisfying Min-n. Then G is locally finite.*

Proof. Let us use induction on the derived length of G, say $n > 1$, and let $A = G^{(n-1)}$. Since A does not have descending chains of G-invariant subgroups it has an ascending chief G-series. If we take a chief factor of that series, say F, $G/C_G(F)$ is locally finite by induction and is isomorphic with an irreducible group of automorphisms of F. Then F is an elementary abelian p-group [See [5], Proposition (b)] and so G is locally finite. □

We state here a direct corollary of Lemma 0.0.3.

Corollary 2.2.7. *Let M be a minimal normal subgroup of a group. Then M satisfies Min-n if and only if M is the direct product of finitely many isomorphic simple groups.*

Locally Nilpotent and Locally Soluble groups

Analogously to what proved for Max-n, things work pretty well for locally nilpotent groups, while for locally soluble groups things are more troublesome.

Theorem 2.2.8 (Čarin [9], McLain [37]). *For a locally nilpotent group Min and Min-n coincide.*

Proof. If G is a locally nilpotent group satisfying Min-n, then it has an ascending chief series which is an ascending central series by Corollary 0.0.7. Hence G satisfies Min by Corollary 2.2.3 □

This way, we have shown that a locally nilpotent group satisfying Min-n is a hypercentral Černikov group, and examples such as that of the locally dihedral 2-group show that they need not be nilpotent. Anyway, the consideration that in a Baer group the finite residual lies in the centre allows us to state the following stronger result.

Proposition 2.2.9. *Let G be a Baer group satisfying Min-n. Then G is a nilpotent Černikov group.*

On the other hand things do not work as well for locally soluble groups, how it is shown by a construction by McLain [38], who built a periodic locally soluble group satisfying Min-n which is not soluble.

Example 2.2.10 (McLain [38]). *There exists a locally soluble group satisfying Min-n which is not soluble.*

Finally the following is still apparently unknown.

Open Problem. *Let G be a locally soluble group satisfying Min-n. Is G periodic?*

2.3 Double chain condition

Here we will discuss on the double chain condition on normal subgroups. A group G belonging to this class will be said to satisfy the DC_n-condition. This property is clearly closed under **H**, though it is not closed under **P**, as shown by the direct product between a Prüfer group and an infinite cyclic group. It is less obvious whether it is closed under subgroups of finite index or not, but, similarly to Max-n and Min-n, we can prove this being true.

Theorem 2.3.1. *Let G be a DC_n-group and let H be a subgroup of finite index of G. Then H is a DC_n-group, too.*

Proof. Clearly we may assume H normal in G. Suppose by a contradiction that H does not satisfy the double chain condition on H-invariant subgroups. If we now say S to be the set of all normal subgroups of G which belong to a double chain of normal subgroups of H, if S is non-empty then it contains (at least) either a maximal element K or a minimal element M. If otherwise S is empty we can take $K = \{1\}$.

Case 1 Suppose that there exists in H such a normal subgroup K of G. If we fix an ascending chain of normal subgroups of H, for instance $K = K_1 < K_2 < \ldots$, and if we call T a right transversal of H in G we notice that, for every i, $K = core_T K_i$, so it is not empty the set of all finite non-empty sets such that the previous condition is satisfied for every i. Call X an element of minimal cardinality of this set. If $x \in X$, then Xx^{-1} is a set with the same property, so we may assume that $1 \in X$. Now call $Y = X \setminus \{1\}$ and notice that it cannot be empty since otherwise K would not be included in any ascending chain of H-invariant subgroups of H. Let $K = K_1 < K_2 < \ldots$ be an ascending chain of normal subgroups of H and let $L_i = K_i \mathrm{core}_Y K_i$, which defines an ascending chain of normal subgroups of H (clearly containing K), otherwise we would have

$$K_i = K_i \mathrm{core}_X K_{i+1} = K_i(K_{i+1} \cap \mathrm{core}_Y K_{i+1}) \geq K_{i+1} \cap L_i = K_{i+1} \cap L_{i+1} = K_{i+1}$$

from a certain integer i on.

So, by the property of X, we have that $\mathrm{core}_X L_i = K$ and this means that

$$K_i = K_i \mathrm{core}_X L_i = K_i(L_i \cap \mathrm{core}_Y L_i) = L_i \cap (K_i \mathrm{core}_Y L_i) = L_i.$$

But now from the definition of L_i it follows that $\mathrm{core}_Y K_i = \mathrm{core}_X K_i = K$ and this contradicts the minimality of X.

Case 2 Suppose that there exists in H a minimal subgroup M of S. Since S is not empty, if we fix a descending chain of normal subgroups of H, namely

$M = M_1 > M_2 > \ldots$, and if we call T a right transversal of H in G we notice that, for every i, $M = M_i^T$, so it is not empty the set of all finite non-empty sets such that the previous condition is satisfied for every i. Call then X an element of minimal cardinality of this set. We can assume that $1 \in X$. Now call $Y = X \backslash \{1\}$ and notice that it cannot be empty since otherwise K would satisfy the minimal condition on H-invariant subgroups. Let $M = M_1 > M_2 > \ldots$ be a descending chain of normal subgroups of H and let $L_i = M_i \cap M_i^Y$, which defines a descending chain of normal subgroups of H (clearly contained in M), otherwise we would have

$$M_i = M_i \cap M_{i+1}^X = M_i \cap (M_{i+1} M_{i+1}^Y) \leq M_{i+1} L_i = M_{i+1}$$

from a certain integer i on.
So, by the property of X, we have that $L_i^X = M$ and this means that

$$M_i = M_i \cap L_i^X = M_i \cap (L_i L_i^Y) \leq L_i (M_i \cap M_i^Y) = L_i.$$

But now from the definition of L_i it follows that $M_i^Y = M_i^X = M$ and this contradicts the minimality of X. □

Let us now go into deeper details and show how we can reduce the double chain condition on normal subgroups to the simpler cases of the minimal and maximal conditions on normal subgroups. The following can be found in [19].

Lemma 2.3.2. *The class of groups with trivial socle is closed under* $\boldsymbol{R}$.

Proof. Assume by a contradiction that there exist a group G and a collection of normal subgroups of G $\{K_i | i \in I\}$ such that each G/K_i has trivial socle and G/K has non-trivial socle where

$$K = \bigcap_{i \in I} K_i.$$

Since G/K has non-trivial socle it contains a minimal normal subgroup N/K. Then there exists an $i \in I$ such that $N \cap K_i = K$ and we have that NK_i/K_i is a minimal normal subgroup of G/K_i, a contradiction. □

It follows in particular from Lemma 2.3.2 that any group G contains one smallest normal subgroup $S(G)$ such that $G/S(G)$ has trivial socle.

Theorem 2.3.3. *Let G be a DC_n-group. Then $S(G)$ satisfies the minimal condition on G-invariant subgroups and $G/S(G)$ satisfies the maximal condition on normal subgroups.*

Proof. Clearly, $G/S(G)$ does not contain minimal normal subgroups and satisfies the property DC_n, so it has to satisfy the maximal condition on normal subgroups.

Let now N be a normal subgroup of G properly contained in $S(G)$. G/N has non-trivial socle, which clearly lies inside $S(G)/N$ and this has then some non-trivial minimal G-invariant subgroups, which are finitely many by the double chain condition. This proves (see [2] p.3) that $S(G)$ satisfies the minimal condition on G-invariant subgroups. □

Corollary 2.3.4. *Let G be a residually finite DC_n-group. Then G satisfies the maximal condition on normal subgroups.*

Proof. Since, by Theorem 2.3.3, $S(G)$ is finite and $G/S(G)$ satisfies the maximal condition on normal subgroups, so does G. □

Corollary 2.3.5. *Let G be a periodic hyper-(abelian or finite) DC_n-group. Then G satisfies the minimal condition on normal subgroups.*

Proof. By Theorem 2.3.3 we can say that $G/S(G)$ is soluble-by-finite and also finitely generated by Theorem 2.1.3. Thus $G/S(G)$ is finite. But again by Theorem 2.3.3 $S(G)$ satisfies the minimal condition on G-invariant subgroups and so does G. □

We can now provide a characterization of soluble groups satisfying Min-n and this can be achieved by means of the above results and Theorem 2.2.6.

Corollary 2.3.6. *Let G be a soluble group. Then G satisfies the minimal condition on normal subgroups if and only if it is a periodic DC_n-group.*

We recall that a group G is said to have *Černikov conjugacy classes* or, equivalently, to be a CC-group if $G/C_G(\langle x\rangle^G)$ is a Černikov group for each element x of G.

Proposition 2.3.7. *Let G be a DC_n-group having Černikov conjugacy classes. Then G satisfies either Min or Max.*

Proof. Suppose that there exists an $x \in G$ such that $G/C_G(\langle x\rangle^G)$ is infinite, hence not satisfying Max-n. It follows that $C_G(\langle x\rangle^G)$ satisfies the minimal condition on G-invariant subgroups and in particular $Z(G)$ is a Černikov group. On the other hand, $G/Z(G)$ is clearly residually Černikov, then $C_G(\langle x\rangle^G)$ is Černikov and so is G.

Hence we can suppose that G is an FC-group, so $G/Z(G)$ is locally normal and finite and then by Corollaries 2.3.4 and 2.3.5 it satisfies both the maximal and the minimal condition on normal subgroups. So $G/Z(G)$ is finite and, together with the fact that $Z(G)$ satisfies either the maximal or the minimal condition, the proposition is proved. □

Chapter 3

Double chain condition on non-normal subgroups

We are now entering a classical and broad topic of group theory, the one which deals with answering the question "What happens if a group G has only a few θ-subgroup", with θ a subgroup theoretical property. Clearly requiring a group to have some finiteness conditions on non-θ subgroups is indeed an answer to that question.

In the case of this chapter we are approaching chain conditions on non-normal subgroups. The study about the class of groups satisfying the minimal condition on non-normal subgroups began with Černikov in [13] and then was widely broadened by Phillips and Wilson in [43], while groups satisfying the maximal condition on non-normal subgroups has been investigated by Cutolo [17]. Starting from these results the corresponding double chain condition was studied by De Mari and de Giovanni in [19].

All these classes are clearly **S** and **H**-closed while they are not closed even under extensions by finite groups.

3.1 Maximal condition

The aim of this section is the study of groups having *many* normal subgroups in the interpretation of chain conditions and in particular the study of the maximal condition on non-normal subgroups, which we will call Max-nn. As we will see, in this case the situation is more complicated than in the Min-nn case: in the latter case we are going to see a simple splitting into the two extremal properties of being a Dedekind or a Černikov group, while here non-

Dedekind non-Max groups satisfying Max-*nn* will be shown. This section will follow the steps of [17].

Let us begin with a useful lemma.

Lemma 3.1.1. *Let G be a group satisfying Max-nn. Then:*

(a) Every non-normal subgroup of G is finitely generated;

(b) G' satisfies Max. In particular G is locally polycyclic;

(c) Either G is soluble or it satisfies Max-ab.

Proof. (a) Let H be a non-finitely generated subgroup of G and let x be an element of H. There exists an infinite sequence $h_1, h_2, \ldots, h_n, \ldots$ in H such that $\langle x, h_1 \rangle < \langle x, h_1, h_2 \rangle < \ldots < \langle x, h_1, h_2, \ldots, h_n \rangle < \ldots$, and there is a positive integer m such that $\langle x, h_1, h_2, \ldots, h_m \rangle$ is normal in G so $\langle x \rangle^G \leq H$ and H is normal in G.
(b) If G is Dedekind we are done, so let H be a maximal non-normal subgroup of G. By *(a)* H is finitely generated, so let $H = \langle x_1, \ldots, x_n \rangle$. If we fix x_i with $0 \leq i \leq n$, then $\langle x_i \rangle^G$ satisfies Max, otherwise one would have an ascending chain of subgroups between $\langle x_i \rangle$ and $\langle x_i \rangle^G$ while there is not. Then H^G satisfies Max, but G/H^G is a Dedekind group, so G' satisfies Max.
(c) Suppose that G contains an abelian group A which is not finitely generated. Then by *(a)* A is normal in G and by *(b)* every finitely generated subgroup X/A of G/A is Max, so X is not finitely generated, hence it is normal, G/A is Dedekind and G is soluble. □

Lemma 3.1.2. *Let G be a group satisfying Max-nn. If G is not Dedekind, then $Z(G) = F \times P \times K$, where F is a finite group, P is $\{1\}$ or Z_{p^∞} for a prime p and K is torsion-free of finite rank. Moreover*

(a) if $P \neq \{1\}$, K is finitely generated and every p' subgroup is normal in G;

(b) if G has a finite non-normal subgroup, then K is finitely generated.

Proof. By Lemma 3.1.1 (a) every non-normal subgroup of G is finitely generated. If $\langle c \rangle$ is a non-normal subgroup of G and A is a non-finitely generated subgroup of $Z(G)$, then $\langle c \rangle A$ is clearly normal in G and so in $Z(G)$ there are not two disjoint non-finitely generated subgroups. Then there is at most one Prüfer p-group P for a prime p and K is of finite rank and is finitely generated if $P \neq \{1\}$. Finally, since $\langle c \rangle$ is a characteristic subgroup of $\langle c \rangle K$, it follows that $\langle c \rangle K$ is non-normal in G, hence it is finitely generated and so is K. □

The following lemma is also needed to prove the main theorem of this section. We state it here.

Lemma 3.1.3 (Cutolo [17], Lemma 2.3)**.** *Let G be a group satisfying Max-nn. If G contains a central torsion-free subgroup A such that G/A is not periodic and A is not finitely generated, then G is abelian.*

Proposition 3.1.4. *Let G be torsion-free nilpotent group satisfying Max-nn. Then G is either finitely generated or abelian.*

Proof. Let us suppose G is not abelian. By Lemma 3.1.3 we know that $Z(G)$ is finitely generated. By Lemma 3.1.1 (b) G' is Max, so $G/Z(G)$ is Max [See [25], Theorem 5.9] and G is Max, too. □

Lemma 3.1.5. *Let G be a soluble-by-finite group satisfying Max-nn. If G is neither Max nor Dedekind, it is either a central extension of Z_{p^∞} for a rime p by a finitely generated Dedekind group or the direct product of $\mathbb{Q}_2$ and a finite hamiltonian group.*

Proof. By Lemma 3.1.1 (b) G' is Max, so $G/Z(G)$ is Max [See [25], Theorem 5.9] and $Z(G)$ cannot be finitely generated. Then $G/Z(G)$ is a Dedekind group and G is nilpotent. By Proposition 3.1.4 $G/Tor(G)$ is either abelian or finitely generated, which means abelian in any case, since in the former case $Tor(G)$ would be not finitely generated. So G' is periodic, then finite and so is $G/Z(G)$.

Let $T = Tor(Z(G))$ and firstly suppose G/T finitely generated. By Lemma 3.1.2 there is in T a Prüfer p-subgroup A for a prime p such that G/A is finitely generated, which means that G/A is Dedekind.

So we can suppose G/T being not finitely generated so, by Lemma 3.1.2, T is finite and every finite subgroup of G is normal, but G is not Dedekind so we find an infinite non-normal cyclic subgroup $\langle c\rangle$. Since clearly $\langle c\rangle \cap Z(G) \neq \{1\}$ we have that $\langle c\rangle/\langle c\rangle_G$ is a finite non-normal subgroup of $G/\langle c\rangle_G$, which by the previous argument has a central Prüfer p-subgroup $P/\langle c\rangle_G$ such that G/P is a finitely generated Dedekind group. So $P \leq Z(G)$ and $P = P_0 \times P_1$ where P_0 is finite and P_1 is isomorphic with $\mathbb{Q}_p$, namely the additive group of rational numbers whose denominators are powers of p. Clearly G/P_1 is finitely generated Dedekind, but $P_1 \cap G' = \{1\}$, so G/P_1 is finite. Then $|G'| = 2$ and $G/Z(G)$ has exponent 2. By Lemma 3.1.2 every p' subgroup of $G/\langle c\rangle_G$ is normal, so $\langle c\rangle/\langle c\rangle_G$ is a p group and $p = 2$.

Let us show that G is in fact the direct product of $\mathbb{Q}_2$ by a finite hamiltonian group. Let V/P_1 be the $2'$-component of G/P_1. Then V is abelian and $V = V_0 \times V_1$ with V_0 being a finite group and $V_1 \simeq \mathbb{Q}_2$. As before G/V_1 is a finite hamiltonian group. If we say U/V_1 the Sylow 2-subgroup of G/V_1, then $G' \leq U$ and $V_1G'/G' \simeq \mathbb{Q}_2$, therefore in U/G' there exist a subgroup B/G' such that $U = (V_1G')B$ and $V_1G' \cap B = G'$. Hence $U = V_1 \times B$ and B is finite. By this we have, finally, that $G = UV = V_1 \times (BV_0)$ and $BV_0 \simeq G/V_1$, hence our thesis. □

Theorem 3.1.6. *Let G be a locally graded group. Then G satisfies Max-nn if and only if it either*

(a) satisfies Max or

(b) is a Dedekind group or

(c) is a central extension of Z_{p^∞} for a prime p by a finitely generated Dedekind group or

(d) is the direct product of $\mathbb{Q}_2$ and a finite hamiltonian group.

Proof. Let $N = G''$ and let H be a subgroup of finite index of N. By Lemma 3.1.1 (b), $|N : H_G|$ is finite, too, and by Lemma 3.1.5 G/H_G is metabelian. So $H_G = N$ and N has no proper subgroup of finite index, but G is locally graduated, so $N = \{1\}$ and G itself is metabelian so we can apply again Lemma 3.1.5. Conversely, suppose first that $G = NA$ where A is a Prüfer p-group for a prime p contained in $Z(G)$, G/A is a finitely generated Dedekind group and N polycyclic and normal. Assume by a contradiction that

$$K_1 < K_2 < \ldots < K_n < \ldots$$

is an ascending chain of non-normal subgroups of G and let

$$K = \bigcup_{i \in \mathbb{N}} K_i.$$

Since G/A is finitely generated $K/(A \cap K)$ satisfies Max, so $A \cap K = A$ and K is normal in G. Since N is polycyclic there exists an integer n such that $K \cap N = K_n \cap N$, so $K/K \cap N$ has a proper subgroup, though it is a Prüfer group, a contradiction.

Let finally $G = Q \times F$ such that $Q \simeq \mathbb{Q}_2$ and F is a finite hamiltonian group. Let

$$K_1 \leq K_2 \leq \ldots \leq K_n \leq \ldots$$

be an ascending chain of non-normal subgroups of G, let $K = \cup_i K_i$ and $N = Q \cap K_1$. Clearly N is not trivial and $Q/N = A \times D$ where A is a Prüfer 2-group and D is a finite abelian group of odd order. Therefore $G/N = A \times D \times (FN/N)$ satisfies Max-*nn* by what we proved above and there exists an integer n such that $K = K_n$, so G satisfies Max-*nn*. □

Corollary 3.1.7. *Let G be a locally graded group satisfying Max-nn. If G is not Max, then G is a nilpotent central-by-finite group of class at most 2.*

Proof. By Theorem 3.1.6 we can assume that G is a group of type (c). The only case to inspect is that of G being a 2-group and a central extension of a subgroup $A \simeq Z_{2^\infty}$ by a finite hamiltonian group. Since the Schur multiplier of $G/Z(G)$ has exponent 2, $G' \cap Z(G)$ has exponent 2 and the order of $A \cap G'$ is at most 2. Then clearly $|G'| \leq 4$ and, for each $x \in G$, $|G : C_G(x)| \leq 4$. But $A \leq C_G(x)$ and each subgroup B/A of G/A such that $|G/A : B/A| \leq 4$ contains $G'A/A$, then $G' \leq Z(G)$. □

3.2 Minimal condition

This condition has been studied, together with many others, in [43] by Phillips and Wilson in which the theorem about the minimal condition on non-normal subgroups is called Theorem B(v). It can be seen as a typical example of *splitting* into extremal cases, since it proves that a locally graded group satisfying the minimal condition on non-normal subgroups is either a Černikov or a Dedekind group.

It is useful for the following results to prove here a simple lemma firstly proved in [13] by Černikov and then slightly extended by Phillips and Wilson.

Lemma 3.2.1. *Let H be a finite subgroup of a group G and $\{R_\alpha | \alpha \in I\}$ be a descending chain of H-invariant subgroups of G. Then $\bigcap_{\alpha\in I}(R_\alpha H) = (\bigcap_{\alpha\in I} R_\alpha)H$.*

Proof. Let g be an element of $\bigcap_{\alpha\in I}(R_\alpha H)$, then for each $\alpha \in I$ we can find $r_\alpha \in R_\alpha$ and $h_\alpha \in H$ such that $g = r_\alpha h_\alpha$. Define $I_h = \{\alpha \in I | h_\alpha = h\}$ for each $h \in H$. Clearly $I = \bigcup_{h\in H} I_h$ and the set $\{\bigcap_{\alpha\in I_h}(R_\alpha)H | h \in H\}$ is finite and totally ordered so there exist $h_1 \in H$ such that $\bigcap_{\alpha\in I_{h_1}} R_\alpha = \bigcap_{\alpha\in I}(R_\alpha)$. Since

$$g \in (\bigcap_{h\in H}\bigcap_{\alpha\in I_h} R_\alpha)H,$$

the lemma is proved. □

Lemma 3.2.2. *Let G be a soluble locally finite group satisfying Min-nn. Then it is either a Černikov or a Dedekind group.*

Proof. Suppose G to be not a Černikov group. Let n be the biggest natural number such that G^n is not Černikov and let H be a finite subgroup of G. If we say A/G^{n+1} the socle of G^n/G^{n+1}, then HA/G^{n+1} is residually finite and hence there exists a proper descending chain of H-invariant subgroups of G/G^{n+1} with trivial intersection. By Lemma 3.2.1 HG^{n+1}/G^{n+1} is normal in G/G^{n+1} and $G/C_G(HG^{n+1})$, as a periodic group of automorphisms of a Černikov group, is Černikov. So $C_G(HG^{n+1})$ is not Černikov and it contains a non-Černikov abelian subnormal subgroup whose socle is an infinite abelian residually finite group centralizing H so by Lemma 3.2.1 H is normal in G and G is a Dedekind group. □

Lemma 3.2.3. *Let G be a locally finite group satisfying Min-nn. Then it is either a Černikov or a Dedekind group.*

Proof. By Lemma 3.2.2 it suffices to prove that G is soluble. Let us assume G has an infinite descending chain of normal subgroups and say L the intersection of such a chain and say H a finite subgroup of G. By Lemma 3.2.1 LK is normal in G for every finite subgroup $K \geq H$, so G/HL is a Dedekind group and G/L is finite-by-abelian. By Theorem A in [43] G is locally soluble and G/L is soluble. This way we have shown, in particular, that there exist a natural number k and a perfect subgroup P of G such that $P = G^n$ for each $n \geq k$. Moreover, by the same reasoning, P has to satisfy Min-G and Theorem A in [43] shows that $P \leq R(G)$, so by Proposition 0.0.9 P is hypercentral. But we saw that P is perfect, hence it is trivial and G is soluble. □

In proving the main theorem of this section we define the class of groups, say $\mathfrak{M}$, in which every finitely generated subgroup is either nilpotent or has a finite non-nilpotent homomorphic image. Clearly every $\mathfrak{M}$-group is locally graded and the class itself is *good* enough to avoid many typical pathologies. The class $\mathfrak{M}$ is, indeed, large enough to contain the class of locally finite groups, that of linear groups (Wehrfritz [57]) and that of hyper-(abelian or finite) groups (Robinson [47]).

Theorem 3.2.4. *Let G be a locally graded group. Then if G satisfies Min-nn, it is either a Černikov or a Dedekind group.*

Proof. Firstly take H as a finitely generated subgroup of G. By Min-*nn* there exists $K < H$ such that $|H : K|$ is finite and K is a Dedekind group. So H is in particular an $\mathfrak{M}$-group and so is G.

By Lemma 3.2.3 we can suppose that G is not locally finite, therefore neither a Černikov nor a Dedekind group. So there exists $H < G$ such that there are

$h \in H$ and $g \in G$ and $h^g \notin H$. Since H cannot be locally finite, so there exists a finitely generated infinite subgroup H_1 of H. Hence we can suppose without loss of generality that G is finitely generated and residually finite by replacing it with $\langle h, g, H_1 \rangle$.

Let M be a minimal non-normal subgroup of G, which cannot be finite by Lemma 3.2.1. Clearly M cannot be the product of two proper subgroups so it has to be of Prüfer type, but G is residually finite and this is our final contradiction. □

3.3 Double chain condition

Now we will discuss about groups satisfying the double chain condition on non-normal subgroups, in other words the DC_{nn} condition. As we will see, for reasonably large classes of groups to satisfy the double chain condition on non-normal subgroups simply means to satisfy either the maximal or the minimal condition on non-normal subgroups. The following is based on [19].

We begin with a fundamental lemma which is a transposition of Lemma 0.0.1 to the case of non-normal subgroups. As we noticed while presenting that lemma, it shows a useful tool in working with double chains.

Lemma 3.3.1. *Let G be a DC_{nn}-group and let H/K be a section of G which is the direct product of infinitely many non-trivial subgroups. Then K is normal in G and G/K is a Dedekind group.*

Proof. Let $\{H_n | n \in \mathbb{Z}\}$ be a countably infinite collection of subgroups of H properly containing K. We can then split this collection into two infinite collections, namely $\{U_i | i \in \mathbb{Z}\}$ and $\{V_j | j \in \mathbb{Z}\}$, such that

$$H/K = \mathrm{Dr}_{i\in\mathbb{Z}} U_i/K \times \mathrm{Dr}_{j\in\mathbb{Z}} V_j/K$$

and for each integer n define

$$U_n^* = \langle U_k | k < n \rangle$$

$$V_n^* = \langle V_k | k < n \rangle.$$

Since the group satisfies the double chain condition, there exist two integers s and r such that $U_r^* \triangleleft G$ and $V_s^* \triangleleft G$. Obviously we have that $U_r^* \cap V_s^* = K$ and K is normal. In order to prove that G/K is a Dedekind group we can suppose, without loss of generality, that $K = \{1\}$, which means that $H = \mathrm{Dr}_{i\in\mathbb{Z}} U_i \times \mathrm{Dr}_{j\in\mathbb{Z}} V_j$ where each term is normal in G by the first part of the proof. Let now x be an element of G and notice that $\langle x\rangle \cap H$ can be contained only in a finite number of direct terms of H, so we can assume that $\langle x\rangle \cap H = \{1\}$. This way, again by the double chain condition, we have that there exist two integers such that $\langle x\rangle U_r^* \triangleleft G$ and $\langle x\rangle V_s^* \triangleleft G$, from which follows that $\langle x\rangle \triangleleft G$. Therefore G is a Dedekind group. □

Corollary 3.3.2. *Let G be a DC_{nn}-group. Than either G is a Dedekind group or each abelian subgroup of G is minimax.*

Proof. Suppose G is not a Dedeind group. Let A be an abelian subgroup of G and B a maximal free subgroup B of A, which, by Lemma 3.3.1, is finitely generated. If we assume by a contradiction that A/B is not a Černikov group we have that, for each $n \geq 3$, A/B^{2^n} has infinite socle and so by Lemma 3.3.1 B^{2^n} is a normal subgroup of G and G/B^{2^n} is abelian, since it has elements of order 8. Then G is residually abelian and then abelian, a contradiction. □

A straightforward consequence of the previous corollary together with the well-known result by Šunkov that locally finite groups satisfying the minimal condition on their abelian subgroups are Černikov groups (see [55]), is the following

Corollary 3.3.3. *Let G be a locally finite DC_{nn}-group. Than G is either a Dedekind or a Černikov group.*

As we said, we will not deal with any *weak* chain condition, using some of the results concerning them, nonetheless. An important result of these ones is the following, which we state here.

Theorem 3.3.4 (Zaičev [60], Theorem 1)**.** *In the class of locally soluble-by-finite groups, the Min-∞, Max-∞ and DC-∞ conditions are equivalent and they determine there the class of soluble-by-finite minimax groups.*

Lemma 3.3.5. *Let G be a DC_{nn}-group. Then every non-normal subgroup of G is either periodic or finitely generated.*

Proof. Let H be a subgroup which is neither finitely generated nor periodic. If there is in H a subgroup $\langle a\rangle$ of infinite order which is not normal in G, then it contains an infinite descending chain of non-normal subgroups of G and hence there exists a positive integer n such that $\langle a, x, h_1, \ldots, h_n\rangle$, with $h_1, \ldots, h_n$ in H, is normal in G for any x in H, so H is normal. Suppose now that all infinite cyclic subgroups of H are normal in G. Take in H b and y of infinite and finite order, respectively. If $y^{-1}by = b$, $\langle b, y\rangle = \langle b, by\rangle$ is normal in G and so $\langle y\rangle^G \leq H$. If $y^{-1}by = b^{-1}$, we have that

$$\ldots < \langle b^{2^n}, y\rangle < \ldots < \langle b^2, y\rangle < \langle b, y\rangle$$

is a descending chain of non-normal subgroup so, as in the first part of the proof, $\langle y\rangle$ is normal in G and H is normal as well. □

Theorem 3.3.6. *Let G be a locally radical DC_{nn}-group. Then G is soluble.*

Proof. Clearly we can suppose that G is not Dedekind, so by Corollary 3.3.2 every abelian subgroup of G is minimax. From this all finitely generated subgroups are soluble and minimax [see [48] 10.35] and together with the quoted result of Šunkov follows that each periodic subgroup of G is Černikov. Hence G is locally soluble. Say $L = \gamma_3(G)$. If we show that L satisfies the weak double chain condition on subgroups, and hence that it is soluble by Theorem 3.3.4, we are done, so suppose that there is in L a double chain

$$\ldots < H_{-n} < \ldots < H_{-1} < H_0 < H_1 < \ldots < H_n < \ldots$$

such that $|H_{i+1} : H_i| = \infty$ for each integer i. Since there is an integer m such that H_m is normal in G, by Theorem 3.3.4 follows that both H_m and G/H_m are not minimax and moreover any subgroup X containing H_m is clearly not minimax, too, so it is normal in G by Lemma 3.3.5. So G/H_m is Dedekind and $L \leq H_m$ which is impossible. □

Lemma 3.3.7. *Let G be a DC_{nn}-group and let*

$$H_1 < H_2 < \ldots < H_n < \ldots$$

be an infinite ascending chain of non-normal subgroup of G. Then H_i is periodic for each positive integer i.

Proof. Assume there exists an integer m such that H_m is not periodic. Then, since clearly every infinite cyclic subgroup of H_m is normal in G, H_m cannot be generated by its elements of infinite order. Thus we can find two elements of finite order, say x and y, such that xy is of infinite order. But we have, then, that $\langle x, y\rangle / C_{\langle x,y\rangle}(xy)$ is an infinite dihedral group and so $\langle x, y\rangle$ does not satisfy Min-nn, which is impossible. □

Now we get to the main theorem of this section, which shows the desired result for double chain conditions.

Theorem 3.3.8. *Let G be a locally radical DC_{nn}-group. Then G satisfies either the minimal or the maximal condition on non-normal subgroups.*

Proof. By Theorem 3.3.6 G is soluble and assume that G does not satisfy our thesis. Then by Corollary 3.3.2 G is minimax. If we say R the Hirsch-Plotkin radical of G, it cannot satisfy Min by Theorem 1.2.2 and so it is not periodic. We claim that R' is finite so we can assume the existence in R of an infinite non-normal subgroup $\langle a \rangle$. Clearly between $\langle a \rangle$ and $\langle a \rangle^G$ there are only finitely many subgroups, so $\langle a \rangle^G$ satisfies Max-n, by Theorem 2.1.6 it is a finitely generated nilpotent group and its elements of finite order form a finite subgroup E. Let us assume the existence of an infinite ascending chain

$$H_1 < H_2 < \ldots < H_n < \ldots$$

of non-normal subgroups of R containing E. Clearly an integer m must exist such that $H_m \langle a \rangle^G$ is normal in G, but H_m, being by Lemma 3.3.7 the torsion subgroup of $H_m \langle a \rangle^G$ is also normal in G, which shows that R satisfies Max-nn and so by Theorem 1.1.2 and Corollary 3.1.7 R' is finite. In particular, R is an FC-group and the finite residual J of G is contained in $Z(R)$, so every subgroup of J has to be normal in G. In fact, if every infinite cyclic subgroup of R is normal in G the fact is easy-checked and if there is an infinite subgroup $\langle a \rangle$ of R which is non-normal in G it clearly cannot centralize more than one Prüfer group and then J is trivial or a Prüfer group and so it has all of its subgroups normal in G.

Since G is minimax every periodic subgroup of G/J is finite and so G/J satisfies Max-nn by Lemma 3.3.7 and $J \neq \{1\}$. So each infinite cyclic subgroup of G has finitely many conjugates. Let now X/J be a non-normal subgroup of G. By Lemma 3.1.1 (a), X/J is polycyclic, so X cannot be polycyclic and hence it is periodic by Lemma 3.3.5, which implies that X/J is finite. So G/J satisfies Min-nn and hence is Dedekind by Theorem 3.2.4 and hence abelian. Therefore G' is periodic, G is generated by its aperiodic elements and is an FC-group. Thus $J \leq Z(G)$, G is nilpotent and G' is finite, indeed. Let Y be a non-normal subgroup of G. If Y is finitely generated it has clearly finitely many conjugates and if it is periodic the same holds, since $Y/Y \cap J$ is finite. However, by Lemma 3.3.5 every non-normal subgroup of G is either periodic or finitely generated, so $G/Z(G)$ is finite [see [40], Theorem 13.2]. Write $Z(G) = K \times L$ with K torsion-free and L Černikov. K cannot be finitely generated, otherwise G would satisfy Max-nn by Theorem 3.1.6. Then we can take in K a double chain of finitely generated subgroups, for instance

$$\ldots < K_1^{2^n} < \ldots < K_1^2 < K_1 < K_2 < \ldots < K_n < \ldots.$$

If we now say T a periodic subgroup of G, clearly there exists a subgroup V from that chain such that VT is normal in G, so T, as a characteristic subgroup of VT, is normal in G, which hence satisfies Max-*nn* by Lemma 3.3.7 and this is our final contradiction. □

Chapter 4

Double chain condition on subnormal subgroups

In this chapter we will take into account the results about chain conditions for subnormal subgroups heading towards an inspection on the double chain condition on subnormal subgroups and a characterization of radical groups satisfying such a property.

Notice firstly that clearly the classes of groups satisfying a chain condition on subnormal subgroups are subclasses of those containing groups satisfying the corresponding chain conditions on normal subgroups so all those results can be used here. Hence, in some cases, our treatise will proceed faster.

The study about the class of groups satisfying the minimal and maximal condition on subnormal subgroups has been breefly afforded by Kuroš in [30] and Robinson in [48]. Starting from these results and from those about normal subgroups in [19] the corresponding double chain condition was studied by Brescia and de Giovanni in [8].

All these classes are clearly **Sn** and **H**-closed.

4.1 Maximal condition

Firstly, we can easily notice that Max-*sn* is closed under the following operations: **Sn**, **H**, **P** and $\mathbf{R_0}$. By these closure operations we can steadily deduce that, for instance, for soluble groups Max and Max-*sn* coincide. The same cannot be said about Max-*n* and Max-*sn* because of the existence of groups such as those shown by McLain (see Example 2.1.7). On the other hand it is well-known the existence of simple groups not satisfying Max, so it is proved

that Max-*sn* is a class which comes strictly between Max and Max-*n*.
As we were saying in advance, the exposition about the class of groups satisfying the maximal condition on subnormal subgroups can be heavily helped by looking at the results for the corresponding condition on normal subgroups. For example, by using Theorem 0.0.5 we can deduce that

Theorem 4.1.1. *Let G be a radical group. It satisfies Max-sn, if and only if it satisfies Max.*

On the other hand, it seems still being unknown the following

Open Problem. *Let G be a locally soluble group satisfying Max-sn. Is G polycyclic?*

Finally, we end this section with a rapid incursion in the land of Max-*snab*, for the study of which the reader can refer, among other properties, to Baer [6] and to Robinson [46].

Theorem 4.1.2. *Let G be a soluble group satisfying Max-snab. Then G is a polycyclic group.*

Proof. If we show that Max-*snab* in our case is preserved under taking quotient over abelian normal subgroups, then the thesis follows by induction on the derived length of G so let H be a normal abelian subgroup of G. Clearly H is Max, take an abelian subgroup A/H of G/H and let $C = C_A(H)$. Since $[C', C] \leq [H, C] = \{1\}$, C is nilpotent. Say M a maximal normal abelian subgroup of C, which is Max by hypothesis. Then by Theorem 1.1.1 both C/M and A/C are p̌olycyclic and so A is Max, which completes the proof. □

In the two quoted works of Baer and Robinson this result is indeed proved true for subsoluble groups.

4.2 Minimal condition

Firstly, we can easily notice that Min-*sn* is closed under the following operations: **Sn**, **H**, **P** and $\mathbf{R_0}$. By this closure operations we can steadily deduce that, for instance, for soluble groups Min and Min-*sn* coincide. The same cannot be said about Min-*n* and Min-*sn* because of the existence of groups such as those show by McLain (see Example 2.2.10). On the other hand, it is well-known the existence of simple groups not satisfying Min, so it is proved that Min-sn is a class coming strictly between Min and Min-n.

As we were saying in advance, the exposition about the class of groups satisfying the minimal condition on subnormal subgroups can be heavily helped by looking at the results for the corresponding condition on normal subgroups. Nevertheless, in this case, we cannot apply the analogue of Theorem 0.0.5 in showing that a radical group satisfying Min-sn is a Černikov group. We are going to need a little more work to get there and beyond that, eventually avoiding ending with an open problem for locally soluble groups as in the maximal case.

Let us prove a preliminary result, which will be useful more than once in the following.

Theorem 4.2.1 (Wielandt [58]). *Let H and K be subnormal subgroups of a group G and let $H \cap K = \{1\}$. If H is a non-abelian simple group, then $[H, K] = \{1\}$.*

Proof. Let $J = \langle H, K \rangle$ and proceed by induction on the subnormal defect s of H in J. If $s \leq 1$, H is normal in J, so $[H, K]$ either equals H or is trivial. In the former case any subnormal subgroup of J containing K must also contain H, so $K = J$ and $H = H \cap J = \{1\}$. Then $[H, K] = \{1\}$. Let $s > 1$ and assume the thesis holds for $s-1$. Now there exists a $k \in K$ such that $H^k \neq H$. Clearly $H \cap H^k = \{1\}$ and the subnormal defect of H in H^J is $s - 1$, so by induction $[H, H^k] = \{1\}$. Then, for each h and h_1 in H we have

$$\{1\} = [h, h_1^k] = [h, h_1[h_1, k]] = [h, [h_1, k]][h, h_1]^{[h_1,k]},$$

from which we have that $H = H' \leq [H, K]$. Hence $K = J$ and $H = \{1\}$, a contradiction. □

The proof of the following lemma is totally analogous with that of Lemma 0.0.3, though slightly different in the conclusion. Also here, then, we are going to make use of the concept of an independent set of indices which we have seen together with the quoted lemma.

Lemma 4.2.2. *Let G be a group and let $\{S_\alpha | \alpha \in I\}$ a family of subnormal non-abelian simple subgroups of G. Then the subgroup generated by the family is the direct product of certain members of that family.*

Proof. Let $J = \langle S_\alpha | \alpha \in I \rangle$. By Zorn's Lemma we can find a maximal independent set $B \subseteq I$. Let $K = \mathrm{Dr}_{\beta \in B} S_\beta$ and suppose $K \neq J$. Then there is an $\alpha \in I$

such that S_α is not contained in K and clearly $S_\alpha \cap K = \{1\}$. So, by Theorem 4.2.1, $[S_\alpha, K] = \{1\}$ and $S_\alpha K = S_\alpha \times K$ and the set $B \cup \{\alpha\}$ contradicts the maximality of B. Therefore $J = K$ and the lemma is proved. □

Theorem 4.2.3. *Let G be a group satisfying Min-sn. Then G is a Černikov group if and only if G is a hyper-(finite or locally nilpotent) group.*

Proof. If G is a Černikov group then the result is trivially true, so let us prove the converse. In this case, if we show that G has a finite non-trivial normal subgroup then, since Min-*sn* is **H**-closed, we will have shown that G is hyperfinite and the theorem will follow from Proposition 2.2.5.

Let G be a non-trivial hyper-(finite or locally nilpotent) group satisfying Min-*sn* and let R be the Hirsch-Plotkin radical of G. If R is non-trivial, by Theorem 2.2.8 it contains a finite non-trivial characteristic subgroup so we can suppose $R = \{1\}$. In this case, there is in G no non-trivial locally nilpotent ascendant subgroup and by hypothesis G contains a finite non-trivial ascendant subgroup and then a simple non-abelian ascendant subgroup H. If we consider H^G it satisfies Min-*sn* and by Lemma 4.2.2 it is the direct product of conjugates of H, which is finite, and hence H^G is a finite non-trivial normal subgroup of G. □

We can now finally state the results concerning Min-*sn* for radical and locally soluble groups.

Theorem 4.2.4. *Let G be a group which is radical or locally soluble. Then G satisfies Min-sn if and only if it satisfies Min.*

Proof. Let G be a radical group satisfying Min-*sn*. Then by Theorem 4.2.3 it satisfies Min.

Let G be a locally soluble group satisfying Min-*sn*. Then by Corollary 0.0.7 G is hyperabelian and by Theorem 4.2.3 it satisfies Min. □

Finally, we end this section with a rapid incursion in the land of Min-*snab*, for the study of which the reader can refer, among other properties, to Baer [6] and to Robinson [46].

Theorem 4.2.5. *Let G be a periodic soluble group satisfying Min-snab. Then G is a Černikov group.*

Proof. If we show that Min-*snab* in our case is preserved under taking quotient over abelian normal subgroups, then the thesis follows by induction on the derived length of G so let H be a normal abelian subgroup of G. Clearly H is Min, take an abelian subgroup A/H of G/H and let $C = C_A(H)$. Since $[C', C] \leq [H, C]$, C is nilpotent. Say M a maximal normal abelian subgroup of C, which is Min by hypothesis. Then by Theorem 1.2.1 both C/M and A/C are Černikov and so A is Min, which completes the proof. □

In the two quoted works of Baer and Robinson this result is indeed proved true for subsoluble groups and, moreover, has been stressed how the periodicity hypothesis is unavoidable. In fact, it can be found a metabelian non-periodic group with no descending chains of abelian subnormal subgroups, which is not Černikov.

Example 4.2.6. *Let P be a group of type p^∞ for some prime number p and U_p be the additive group of the ring of p-adic integers. Then $G = U_p \ltimes P$ is a non-Černikov group satisfying DC_{snab}*

4.3 Double chain condition

Here we come to the final section of this chapter, which will involve the study of groups satisfying the double chain condition on subnormal subgroups. The main theorem will show how the desired theorem of the form "if G is a group satisfying the double chain condition on θ-subgroups, then G satisfies either Max-θ or Min-θ" is not always achievable; in this case, effectively, we will show a soluble group satisfying the double chain condition on subnormal subgroups but which satisfies neither the maximal nor the minimal condition on subnormal subgroups. The results in this section can be found in [8].

Our first main theorem shows that for residually finite groups the double chain condition and the maximal condition on subnormal subgroups are equivalent. In particular, any radical residually finite group satisfying the double chain condition on subnormal subgroups is polycyclic.

Theorem 4.3.1. *Let G be a residually finite DC_{sn}-group. Then G satisfies the maximal condition on subnormal subgroups.*

Proof. Assume for a contradiction that the group G does not satisfy the maximal condition on subnormal subgroups. As G satisfies the maximal condition on normal subgroups by Corollary 2.3.4, there exists a maximal finite normal subgroup M of G. Clearly G/M is likewise a counterexample to the statement, and hence a replacement of G by G/M allows us to suppose without loss of generality that G has no finite non-trivial normal subgroups.

Let H be the Hirsch-Plotkin radical of G. Again Corollary 2.3.4 yields that H satisfies the maximal condition on normal subgroups, so that it is a finitely generated nilpotent group by Theorem 2.1.6. Then G/H cannot satisfy the maximal condition on subnormal subgroups, and hence H satisfies also the minimal condition on subgroups. It follows that H is finite, and so even trivial. Assume that G contains a finite non-trivial subnormal subgroup, and so also a finite simple non-abelian subnormal subgroup Y. Let K be a normal subgroup of finite index of G such that $Y \cap K = \{1\}$. Then $[Y, K] = \{1\}$ by Theorem 4.2.1, so that Y has finitely many conjugates in G and hence its normal closure Y^G is finite. This contradiction shows that all subnormal non-trivial subgroups of G are infinite. Let

$$X_1 < X_2 < \ldots < X_n < \ldots$$

be an infinite ascending chain of subnormal subgroups of G. Since X_1 is infinite and residually finite, it admits an infinite descending normal series

$$X_1 > X_0 > X_{-1} > \ldots > X_{-n} > \ldots$$

and so

$$\ldots < X_{-n} < \ldots < X_{-1} < X_0 < X_1 < \ldots < X_n < \ldots$$

is a double chain consisting of subnormal subgroups of G. This last contradiction completes the proof. □

Lemma 4.3.2. *Let G be a locally nilpotent DC_{sn}-group. Then G is hypercentral.*

Proof. Let N any proper normal subgroup of G. In order to prove that G/N has a non-trivial centre, it can obviously be assumed that G/N is not nilpotent, so that it cannot satisfy the maximal condition on normal subgroups. It follows that G/N contains a minimal normal subgroup, and hence $Z(G/N) \neq \{1\}$. Therefore G is hypercentral. □

We will also need the following elementary result on maximal nilpotent normal subgroups of hyperabelian groups.

Lemma 4.3.3 ([18], Lemma 3)**.** *Let G be a hyperabelian group, and let M be a maximal element of the set of all nilpotent normal subgroups of G of class at most 2. Then $C_G(M) = Z(M)$.*

Proof. Assume for a contradiction that $Z(M) < C_G(M)$. As G is hyperabelian, there exists an abelian non-trivial normal subgroup $A/Z(M)$ of $G/Z(M)$ such that $A \leq C_G(M)$. Then A' is contained in $Z(M)$, and hence

$$(AM)' = A'M'[A, M] = A'M' \leq Z(M) \leq Z(AM).$$

Therefore, the normal subgroup AM of G is nilpotent with class at most 2 so that $AM = M$ and A is contained in M, which means that $A \leq Z(M)$, a contradiction. □

We can now prove that radical groups with the double chain condition on subnormal subgroups are soluble. Observe here that it seems still unknown whether a locally soluble group satisfying the maximal condition on subnormal subgroups must be soluble (and so polycyclic).

Lemma 4.3.4. *Let G be a radical DC_{sn}-group. Then G is soluble.*

Proof. Let

$$\{1\} = H_0 < H_1 < \ldots < H_\alpha < H_{\alpha+1} < \ldots < H_\tau = G$$

be an ascending normal series of G whose factors are locally nilpotent. For each ordinal $\alpha < \tau$ the group $H_{\alpha+1}/H_\alpha$ is hypercentral by Lemma 4.3.2, and so G is hyperabelian. Assume for a contradiction that G is insoluble, and let T be the largest periodic normal subgroup of G. If $T < G$, the factor group G/T contains a torsion-free abelian non-trivial normal subgroup A/T. Then G/A must satisfy the maximal condition on subnormal subgroups, and in particular it is soluble, so that G/T is a soluble group. Therefore the subgroup T is not soluble, and hence without loss of generality it can be assumed that G is periodic.

The set of all nilpotent normal subgroups of G of class at most 2 contains a maximal element M by Zorn's Lemma, and $C_G(M) = Z(M)$ by Lemma 4.3.3. As G/M is not soluble, it cannot satisfy the maximal condition on subnormal subgroups, so that M satisfies the minimal condition on subnormal subgroups and hence it is a Černikov group . But $G/Z(M)$ is isomorphic to a periodic group of automorphisms of M, and so it is finite (see [48], p.85). This last contradiction completes the proof. □

We are now ready to prove the main theorem of this section, in which a characterisation of soluble groups is given.

Theorem 4.3.5. *Let G be a radical group. Then G is a DC_{sn}-group if and only if one of the following conditions holds:*

(a) G satisfies Min-sn;

(b) G satisfies Max-sn;

(c) $G = HJ$, where J is the finite residual of G, H is polycyclic, $C_H(J)$ is finite and every subnormal subgroup of G either properly contains J or is a Černikov group.

Proof. Suppose first that the radical group G satisfies the double chain condition on subnormal subgroups, but it neither satisfies the minimal nor the maximal condition on subnormal subgroups. It follows from Lemma 4.3.4 that G is soluble, and hence it is minimax, because each factor of the derived series of G either satisfy the minimal or the maximal condition on subgroups. The finite residual J of G is the direct product of finitely many Prüfer subgroups (see [49], Theorem 10.33), and the infinite factor group G/J is polycyclic by Theorem 4.3.1. Thus the Fitting subgroup F/J of G/J contains an element of infinite order aJ. Assume that $[J, F]$ is properly contained in J, so that $J/[J, F]$ is a non-trivial direct product of Prüfer subgroups. For each positive integer n, let $S_n/[J, F] = Soc_n(J/[J, F])$ be the n-th term of the upper socle series of $J/[J, F]$. Then

$$\ldots < \langle a^{2^n}\rangle[J, F] < \ldots < \langle a^2\rangle[J, F] < \langle a\rangle[J, F] < \langle a\rangle S_1 < \ldots < \langle a\rangle S_n < \ldots$$

is a double chain of subnormal subgroups of G. This contradiction shows that $[J, F] = J$. If p is any prime number and J_p is the p-component of J, it follows that $[J_p, F] = J_p$. Thus the cohomology group $H_n(G/J, J_p)$ has finite exponent for each non-negative integer n and for all primes p (see [50]), and so

$$H^n(G/J, J) = \bigoplus_p H^n(G/J, J_p)$$

is periodic for all n. In particular, the cohomology class of the extension

$$J \rightarrowtail G \twoheadrightarrow G/J$$

has finite order, and so G nearly splits over J (see [51], Lemma 10). As J is divisible, it follows that there exists a subgroup H of G such that $G = HJ$ and $H \cap J$ is finite, and H is obviously polycyclic. The centralizer $C_H(J)$ is a normal subgroup of G, and

$$J = \bigcup_{n\in\mathbb{N}} Soc_n(J)$$

, so that the sequence

$$C_H(J) \leq C_H(J)Soc_1(J) \leq \ldots \leq C_H(J)Soc_n(J) \leq \ldots$$

is an infinite ascending chain of normal subgroups of G. Thus the double chain condition on subnormal subgroups yields that $C_H(J)$ satisfies the minimal condition on subnormal subgroups, and hence it is finite.

Let X be any subnormal subgroup of G which is not a Černikov group. Then X does not satisfy the minimal condition on subnormal subgroups, and so the set of all subnormal subgroups of G containing X must satisfy the maximal condition. In particular, there exists a positive integer m such that $XJ = XSoc_m(J)$, so that X has finite index in XJ. But J has no proper subgroups of finite index, and hence it is properly contained in X.

Conversely, suppose that $G = HJ$ has the structure described in (c), and let

$$\ldots \leq X_{-n} \leq \ldots \leq X_{-1} \leq X_0 \leq X1 \leq \ldots \leq X_n \leq \ldots$$

be a chain of subnormal subgroups of G indexed by the linearly ordered set of integers. By the assumption we have that either X_0 is a Černikov group or it contains J. As the factor group G/J is polycyclic, it follows that there exists a non-negative integer m such that either $X_n = X_m$ for all $n \geq m$ or $X_n = X_m$ for all $n \leq m$. Therefore G satisfies the double chain condition on subnormal subgroups. □

In the last part of the section it will be proved that in the situation described in case (c) of Theorem 4.3.5, the group G need not to split over its finite residual.

We shall denote by $\mathfrak{M}(sn)$ the class of all groups satisfying both the minimal and the maximal condition on subnormal subgroups. It is well-known that $\mathfrak{M}(sn)$ is just the class of groups admitting a subnormal composition series of finite length. Obviously, our next lemma holds in particular for finite normal subgroups.

Lemma 4.3.6. *Let G be a group containing a normal $\mathfrak{M}(sn)$-subgroup N such that G/N is a DC_{sn}-group. Then G itself is a DC_{sn}-group.*

Proof. Assume for a contradiction that

$$\ldots < X_{-n} < \ldots < X_{-1} < X_0 < X_1 < \ldots < X_n < \ldots$$

is a double chain of subnormal subgroups of G. As $X_i \cap N$ is a subnormal subgroup of G for all integers i and N belongs to $\mathfrak{M}(sn)$, there exists a positive integer k such that $X_i \cap N = X_k \cap N$ for each $i \geq k$ and $X_i \cap N = X_{-k} \cap N$ for each $i \leq -k$. Then it follows from the Dedekind modular law that $X_iN < X_{i+1}N$ for each $i \geq k$ and $X_iN > X_{i-1}N$ for each $i \leq -k$, which is impossible because X_iN is subnormal in G for all i and G/N has no double chains of subnormal subgroups. This contradiction proves the statement. □

Consider a non-split central extension

$$C \rightarrowtail U \twoheadrightarrow Q$$

where $C = \langle c \rangle$ is a group of prime order $p > 2$ and Q is a free abelian group of rank 2, and let P be a group of type p^∞, and $\langle x \rangle$ its unique subgroup of order p. Then Q is isomorphic to a group of automorphisms of P, and so we may construct a semidirect product $K = U \ltimes P$, with $C_U(P) = C$. As C is contained in the centre of K, the subgroup $M = \langle c^{-1}x \rangle$ is normal in K, and we may consider the factor group $G = K/M$. Then $J = PM/M$ is the finite residual of G, and $G = HJ$, where $H = UM/M$. Moreover, $H \cap J$ has order p, and G cannot split over J because U does not split over C. Finally, $N = \langle c, x \rangle / M$ is a finite normal subgroup of G and

$$G/N \simeq K/C$$

satisfies the double chain condition on subnormal subgroups, so that G is a DC$_{sn}$-group by Lemma 4.3.6.

We conclude this chapter with a brief note on groups satisfying the double chain condition on subnormal abelian subgroups, shortly said DC$_{snab}$-groups.

Groups satisfying the classical chain conditions on abelian subnormal subgroups have been studied (see [6] and [46]); in particular, referring to Theorem 4.2.5, it has been proved that any periodic soluble group with the minimal condition on abelian subnormal subgroups is a Černikov group, while, referring to Theorem 4.1.2, it has been proved that soluble groups with the maximal condition on abelian subnormal subgroups are polycyclic. Observe that, by looking at Example 4.2.6, the periodicity assumption is crucial in the study of soluble groups with the minimal condition on abelian subnormal subgroups. On the other hand, for periodic soluble groups the double chain condition on (abelian) subnormal subgroups and the minimal condition on (abelian) subnormal subgroups obviously coincide, and so we will not discuss here the double chain condition on abelian subnormal subgroups.

Chapter 5

Double chain condition on subnormal non-normal subgroups

In this chapter we will take into account the results about chain conditions on subnormal non-normal subgroups heading towards an inspection on the double chain condition on subnormal non-normal subgroups.

Notice firstly that clearly the classes of groups satisfying a chain condition on subnormal non-normal subgroups are strictly related with the class of T-groups, namely those groups in which every subnormal subgroup is normal, which is an additional extermal case for our conditions, together with the respective maximal and minimal chain conditions. Finite soluble T-groups has been described by Gaschütz [26] in 1957, who proved, among the other results, that these groups are metabelian and hypercyclic. These results has been extended to the infinite case by Robinson [45] in 1964 and other interesting facts concerning generalized soluble groups satisfying the property T were investigated. After these works many were the paper dealing with the structure of generalized T-groups, always in a generalized soluble environment. Some of these generalizations arose by imposing restrictions on the set of the subnormal non-normal subgroups. For instance, in [23] and in [24] Franciosi and de Giovanni studied (generalized) soluble groups in which, respectively, every infinite subnormal subgroup is normal and in which subnormal non-normal subgroups have finite index.

The study about the class of groups satisfying the maximal and minimal condition on subnormal non-normal subgroups has been afforded by de Giovanni and de Mari in [20] and [21], respectively.

All these classes are clearly **Sn** and **H**-closed.

Notice that hereafter through this chapter we will make more or less implicitly use of the results in the paper of Cooper on power automorphisms [15].

Now we will report some useful lemmas on DC_{snn}-groups which obviously still hold for groups satisfying Max-*snn* and Min-*snn*. In particular notice the analogy between the following lemma and Lemma 3.3.1.

Lemma 5.0.1. *Let G be a DC_{snn}-group and let H/K be a section of G which is a direct product of infinitely many non-trivial cyclic subgroups. Suppose that H is subnormal in G, then K and H are normal in G and every cyclic subgroup of the Baer radical B/K of G/K is normal in G/K.*

Proof. H/K is a direct product of infinitely many cyclic non-trivial subgroups if and only if it contains a direct product of countably many cyclic non-trivial subgroups so, without loss of generality, we can let $\{H_n | n \in \mathbb{Z}\}$ be a countably infinite collection of subgroups of H properly containing K, such that H_i/K is cyclic for all $i \in \mathbb{Z}$ and that $H/K = \mathrm{Dr}_{i\in\mathbb{Z}} H_i/K$. We can then split this collection into two infinite collections, namely $\{U_i | i \in \mathbb{Z}\}$ and $\{V_j | j \in \mathbb{Z}\}$, such that

$$H/K = \mathrm{Dr}_{i\in\mathbb{Z}} U_i/K \times \mathrm{Dr}_{j\in\mathbb{Z}} V_j/K$$

and for each integer n define

$$U_n^* = \langle U_k | k < n \rangle$$

$$V_n^* = \langle V_k | k < n \rangle.$$

Since the group satisfies the double chain condition on subnormal non-normal subgroups, there exist two integers r and s such that $U_r^* \lhd G$ and $V_s^* \lhd G$ both of them being normal in G. Obviously we have $U_r^* \cap V_s^* = K$ and K is normal. On the other hand by the same reasoning we see that each direct term of H/K is normal in G/K, so H/K is normal in G/K, too, and H is normal in G.

In order to prove that every cyclic subgroup of the Baer radical B/K of G/K is normal in G/K, we can suppose, without loss of generality, that $K = \{1\}$. Let now x be an element of B and notice that $\langle x \rangle \cap H$ can be contained only in a finite number of direct terms of H, so we can assume that $\langle x \rangle \cap H = \{1\}$. This way, again by the double chain condition, we have that there exist two integers k and h such that $\langle x \rangle U_k^*$ and $\langle x \rangle V_h^*$ are normal in G. So $\langle x \rangle = \langle x \rangle U_k^* \cap \langle x \rangle V_h^*$ is normal in G. □

Lemma 5.0.2. *Let G be a Baer DC_{snn}-group. Then either G is a Dedekind group or each subnormal abelian subgroup of G is minimax.*

Proof. Suppose G is not a Dedekind group and let A be an abelian subnormal subgroup of G and B a free abelian subgroup of A such that A/B is periodic. By Lemma 5.0.1, B is finitely generated. Suppose by a contradiction that A/B is not Min, then, if we take an integer n, A/B^{2^n} has an infinite socle which is product of infinitely many subgroups. Again by Lemma 5.0.1 B^{2^n} is normal in G and G/B^{2^n} is a Dedekind group, and hence even an abelian group for every $n > 2$. Thus G is abelian and this is a contradiction. □

Lemma 5.0.3. *Let G be a DC_{snn}-group and let X be a subnormal non-normal subgroup of G. If X is a Baer group, then X^G satisfies either Max or Min.*

Proof. X is contained in the Baer radical and so X^G is a Baer group, too. Let us suppose X^G is not Černikov and so it is not periodic, otherwise it would contain an infinite direct product of cyclic subgroups and then, by Lemma 5.0.1, X would be even normal in G. Since it is generated by its elements of infinite order, there exists in X an element x which contains an infinite descending chain of subnormal non-normal subgroups of G. We now claim that X is finitely generated. Suppose it is not, then we may consider a maximal element M in the family of subnormal subgroups of X which are not normal in G and which contain x. But for each element $y \in X \setminus M$ we have that $\langle M, y\rangle$ is normal in G and hence that $X = \langle\langle M, y\rangle | y \in X \setminus M\rangle$ is normal in G, so we proved that X is finitely generated and that is Max. Clearly, thanks to the presence of $\langle x\rangle$, there is no ascending chain of subnormal subgroups between X and X^G, so each factor of the subnormal series of X satisfies Max and hence X^G itself is polycyclic. □

Corollary 5.0.4. *Let G be a DC_{snn}-group and let X be a subnormal non-normal subgroup of G. If X is a Baer group, then it is either periodic or finitely generated.*

Here we report a simple lemma we will use in the following [see [19], Lemma 2.2].

Lemma 5.0.5. *Let G be a locally nilpotent group whose derived subgroup G' is finitely generated. Then G is nilpotent.*

Proposition 5.0.6. *Let G be a DC_{snn}-group and let B be the Baer radical of G. Then B is nilpotent and in particular B coincides with the Fitting subgroup of G.*

Proof. By Proposition 2.2.9, we can suppose that B is not a Černikov group and not a T-group so it is not even periodic, otherwise by Lemma 5.0.2 it would satisfy Min-*snab* hence being Černikov by Theorem 4.2.5. So we can find an aperiodic element x such that $\langle x \rangle$ contains an infinite descending chain of subnormal non-normal subgroups of G. Then we may consider a maximal element M in the family of subnormal non-normal subgroups of B which contain x, so B/M^B is a Dedekind group and $B'M^B/M^B$ is finite. Since $x \in M$, M^B is not Černikov and so by Lemma 5.0.3 it satisfies Max. Hence B' itself satisfies Max and then, by Lemma 5.0.5, B is nilpotent. □

Now remembering that a soluble group is minimax if and only if all of its abelian subgroups are minimax (see [49], Theorem 10.35) and combining Lemma 5.0.2 and Proposition 5.0.6 we have the following

Corollary 5.0.7. *Let G be a DC_{snn}-group and let B be the Baer radical of G. Then B is either a Dedekind or a minimax group.*

Proposition 5.0.8. *Let G be a DC_{snn}-group and let H be a subnormal Baer subgroup of G. Then H satisfies either the minimal or the maximal condition on non-normal subgroups. In particular, if H is not Černikov, then it satisfies the maximal condition on its non-normal subgroups.*

Proof. By Proposition 5.0.6, H satisfies the double chain condition on its non-normal subgroups and so by Theorem 3.3.8 it satisfies either the minimal or the maximal condition on non-normal subgroups. □

We conclude this introduction by showing how finitely generated soluble DC_{snn}-groups behave likewise finitely generated soluble T-groups.

Proposition 5.0.9. *Let G be a finitely generated soluble group with the DC_{snn} condition. Then G is polycyclic.*

Proof. Let A be the least non-trivial term of the derived series of G and assume by induction that G/A is polycyclic. Then there exists F, a finite subgroup of A, such that $F^G = A$. So, if $F = A$, G is polycyclic, otherwise F is a subnormal non-normal subgroup of G and by Lemma 5.0.3 we can assume A is Černikov. Let J be the finite residual of A. Since G/J is polycyclic, J is the normal closure of a finite subgroup. But every finite subgroup of J has finite normal closure in G, hence $J = \{1\}$ and G is polycyclic. □

5.1 Maximal condition

This section is based on the paper of De Mari and de Giovanni [20]. Some results are here extended from the same hypotheses while some others are reported in their double chain (hence still extended) form in the dedicated section. The wanted and chased parallelisms with the work of Robinson about T-groups are evident.

Theorem 5.1.1. *Let G be a subsoluble group satisfying Max-snn. Then G is soluble.*

Proof. Let τ be an ordinal number and suppose the existence of an ascending normal series of G with abelian factors

$$\{1\} = G_0 < G_1 < \ldots < G_\tau = G.$$

Consider by a contradiction the least ordinal, clearly a limit ordinal, $\mu \leq \tau$ such that G_μ is not soluble. Since G satisfies Max-*snn*, the set of ordinal numbers $\alpha < \mu$ such that there exists a subnormal non-normal subgroup X such that $G_\alpha < X < G_\beta$ for some $\alpha < \beta < \mu$, then there is an ordinal $\delta < \mu$ such that G_β/G_δ is a metabelian T-group for every $\delta < \beta < \mu$. Therefore, since

$$G_\mu = \bigcup_{\delta<\beta<\mu} G_\beta,$$

G_μ/G_δ is metabelian and this cannot be.
In general let

$$\{1\} = X_0 < X_1 < \ldots < X_\tau = G$$

an ascending subnormal series of G with abelian factors. Say k the finite number of terms of the series which are not normal in G. By the first part of the proof we may assume that $k > 0$ and let $\rho < \tau$ be the largest ordinal such that X_ρ is not normal in G. Then $X_{\rho+1}$ is normal in G and $G/X_{\rho+1}$ is soluble by the above. Moreover, by induction, $X_{\rho+1}$ itself is soluble and so is G. □

For the periodic case we can clearly partially refer to the general statements in the section about the double chain condition on subnormal non-normal subgroups. Here, instead, we are going to give account of some results about the general case which will differ, *a posteriori*, from the ones in the double chain section.

Lemma 5.1.2. *Let G be a group satisfying Max-snn and let F be the Fitting subgroup of G. If X is a subgroup of F which is not finitely generated, then X is normal in G.*

Proof. Since F is nilpotent by Proposition 5.0.6, X is subnormal in G. For each element x of X we can construct an ascending chain of subnormal subgroups of G, for instance

$$\langle x \rangle < \langle x, x_1 \rangle < \ldots < \langle x, x_1, \ldots, x_n \rangle < \ldots,$$

where $x_1, \ldots, x_n$ are elements of X. By hypothesis there exists a positive integer k such that $\langle x, x_1, \ldots, x_k \rangle$ is normal in G, which means that $\langle x \rangle^G \leq X$ and X is normal in G. □

Recall that the set of all automorphisms of G fixing every infinite subgroup of G, namely $\mathrm{IAut}(G)$, is a subgroup of $\mathrm{Aut}(G)$ and contains $\mathrm{PAut}(G)$, the group of every power automorphisms of G. The behavior of $\mathrm{IAut}(G)$ has been studied in [16] by Curzio, Franciosi and de Giovanni.

Lemma 5.1.3. *Let G be a an infinite soluble group satisfying Max-snn with periodic Fitting subgroup F. If F is not a finite extension of a Prüfer group, then every subgroup of F is normal in G.*

Proof. By Lemma 5.1.2 every infinite subgroup fo F is normal in G, so $G/C_G(F)$ is isomorphic to a subgroup of $\mathrm{IAut}(F)$. Since we can assume that $\mathrm{PAut}(F)$ $\neq \mathrm{IAut}(F)$, we have that F is a finite extension of a Prüfer group [see [16], Proposition 2.5]. □

The following is the main theorem of [20] regarding the non-periodic case and it uses arguments which are more generally developed in the section about the double chain condition in this chapter. We state it here.

Theorem 5.1.4. *Let G be a soluble non-polycyclic group satisfying Max-snn and let F be the Fitting subgroup of G. If either F is torsion-free or the torsion subgroup T of F is infinite and not a finite extension of a Prüfer group, then every subnormal non-normal subgroup of G has finite index in G.*

Corollary 5.1.5. *Let G be a torsion-free soluble group satisfying Max-snn. If G is not polycyclic, then it is abelian.*

Proof. The thesis follows from Theorem 5.1.4 and from Corollary 3.4 in [24] □

5.2 Minimal condition

This section is based on the paper of De Mari and de Giovanni [21]. Like for the previous section, some results are here extended from the same hypotheses while some others are reported in their double chain (hence still extended) form in the dedicated section. The aim of the paper is mostly to highlight the similarities between T-groups and groups satisfying Min-*snn*.

As for the case of the maximal condition on subnormal non-normal subgroups, here we can apply the results proved at the beginning of the current chapter. So we have at once that in a group satisfying Min-*snn* the Baer radical is nilpotent, that it is either a Dedekind or a minimax group, that the class of nilpotent DC_{snn}-groups splits into its extremal classes, that finitely generated groups satisfying Min-*snn* are finitely generated and so on. *A posteriori*, by looking at Proposition 5.3.1, which does not use any fact pertaining Min-*snn*, we can also say that subsoluble groups satisfying the minimal condition on subnormal non-normal subgroups are soluble, indeed.

For the periodic case we can clearly refer to the general statements in the section about the double chain condition on subnormal non-normal subgroups. Here, instead, we are going to give account of some results about the general case.

Below, we present the main results of [21] for the non-periodic case and before them some lemmas, which are *stronger* than the corresponding ones in the double chain case.

Lemma 5.2.1. *Let G be a group satisfying Min-snn and let F be the Fitting subgroup of G. If F is not a Černikov group, then all subgroups of F are normal in G.*

Proof. If $\langle x\rangle$ is an infinite cyclic subgroup of F, it has to be normal in G by Min-*snn*. Then, since F is generated by its aperiodic elements, if we assume that F is not periodic, then every subgroup of F is normal in G. On the other hand, if F is periodic, as it is not Černikov, the thesis follows from Lemma 5.0.1. □

And now for an argument which is not a rare encounter when dealing with subnormal non-normal subgroups.

Lemma 5.2.2. *Let G be a group satisfying Min-snn if $G = \langle z, A\rangle$, where A is an abelian normal subgroup of G, $z^2 \in A$ and $a^z = a^{-1}$ for each $a \in A$, then G has finitely many subnormal non-nomal subgroups.*

Proof. Clearly, $\gamma_{n+1} = A^{2^n}$ for each positive integer n. Say $L = \bigcap_{n\in\mathbb{N}} A^{2^n}$ and assume that $A^{2^n} > A^{2^{n+1}}$ for each n. Since $z^4 = 1$, we have that $\langle z\rangle A^{2^n} > \langle z\rangle A^{2^{n+1}}$ so, since they are all subnormal in G, there exists a positive integer k such that $\langle z\rangle A^{2^k}$ is normal in G; then $A^2 = G' = [G, z]$ lies inside $\langle z\rangle A^{2^k}$ and $\langle z\rangle A^2 = \langle z\rangle A^{2^k}$, which is a contradiction. So we know that there exists a natural number c such that $L = A^{2^c}$, hence L is an abelian 2-divisible group and G/L is a nilpotent 2-group of finite exponent. Since G/L satisfies Min-nn, by Theorem 3.2.4 it is either finite or a Dedekind group. If we now say H to be a subnormal non-normal subgroup of G, it has an element h that acts as the inversion on each element of A, so $[L, H] = L^2 = L$ and $L \leq H$. Therefore, if G is not a T group, every subnormal non-normal subgroup of G contains L and G/L is finite, hence our claim. □

Theorem 5.2.3. *Let G be a soluble group satisfying Min-snn whose Fitting subgroup F is non-periodic. Then G has finitely many subnormal non-normal subgroups.*

Proof. By Lemma 5.2.1 every subgroup of F is normal in G, hence F is abelian and we can assume that $|G/F| = 2$. Then G has finitely many subnormal non-normal subgroups by Lemma 5.2.2. □

Now the main result for non-periodic groups satisfying Min-*snn* with Fitting subgroup being, this time, periodic is stated as follows.

Theorem 5.2.4. *Let G be a soluble non-periodic group satisfying Min-snn whose Fitting subgroup F is periodic and let T be the largest periodic normal subgroup of G. If G is not a Černikov group, then T/F is finite and G/T is a T-group.*

5.3 Double chain condition

As we have already pointed out, the class of DC_{snn}-groups is closed under **Sn** and **H**. Anyway it is not closed under other closure operators, such as $\mathbf{D}_0$, namely it is not closed under direct products. In fact, firstly take $A = C_{p^\infty} \wr C_p$ as the wreath product between the Prüfer group C_{p^∞} and C_p, the cyclic group of order p, for a given prime p; secondly let $B = F/[F, [F, F]]$ where F is a 2-generator free group. Then $A \times B$ does not belong to DC_{snn} while both A and B do, since they satisfy, respectively, the minimal and the maximal condition on subgroups.

Proposition 5.3.1. *Let G be a subsoluble DC_{snn}-group. Then it is soluble.*

Proof. Since in every DC_{snn}-group the Baer radical is nilpotent by Lemma 5.0.6 and G is hyper-Baer, then G is hyperabelian.

Suppose by contradiction that G is not soluble. Let F be the Fitting subgroup of G. Since G is not soluble not all the subgroups of F can be normal in G. Firstly assume that F is not Černikov, then it must be non-periodic, otherwise it would contain a subgroup which is a direct product of infinitely many non-trivial cyclic subgroups and hence, by Lemma 5.0.1, every subgroup of F would be normal in G. On the other hand, if F is not periodic there exists an aperiodic element x in F such that $\langle x\rangle$ is not normal in G. Then there exists a sequence of integer numbers $k_1, \dots, k_n, \dots$ such that

$$\dots < \langle x^{k_n}\rangle < \dots < \langle x^{k_1}\rangle$$

is a descending chain of subnormal non-normal subgroups of G, so that G/F satisfies the maximal condition on subnormal non-normal subgroups and that is a contradiction by Theorem 5.1.1.

Hence we have found that in this case F has to be Černikov. Denote T as the maximal torsion normal subgroup of G. Since F is nilpotent, F is the Fitting subgroup of T, too, so T itself is Černikov. If we now call J the finite residual of T, we have that T/J is finite and so the Fitting subgroup K/J of G/J cannot be periodic, since G/J cannot be finite. But clearly not every subgroup in K/J is normal in G/J, so there exists an aperiodic element y of K such that $\langle y\rangle J$ is not normal in G/J. Then there exists a sequence of integer numbers $h_1, \dots, h_n, \dots$ such that

$$\dots < \langle y^{h_n}\rangle J < \dots < \langle y^{h_1}\rangle J$$

is a descending chain of subnormal non-normal subgroups of G. So G/K satisfies the maximal condition on subnormal non-normal subgroups and it is hence soluble by Theorem 5.1.1 and this is our final contradiction. □

The periodic case

In this section we will inspect some results on the periodic case and see how in the primary case our description of DC_{snn}-groups splits in two extremal cases: that of Černikov groups and that of groups having only a finite number of subnormal non-normal subgroups.

Unfortunately, in 1986 Leinen [33] solved an open problem firstly stated by Robinson in [45] by proving that there exists a locally nilpotent T-group which is not soluble (and hence also not Černikov). This poses a clear limit to our hopes about possible *upwards* extensions.

Lemma 5.3.2. *Let G be a periodic soluble DC_{snn}-group and let $F = Fit(G)$. If G is not Černikov, then every subgroup of F is normal in G. In particular, G is metabelian and hypercyclic.*

Proof. Since G is not Černikov, F is not Černikov, too, so it contains an infinite direct product of subnormal subgroups of G and every subgroup of F is normal in G by Lemma 5.0.1. Moreover $G/C_G(F)$ is a group of power automorphisms of F and hence it is abelian, so $G' \leq C_G(F) \leq F$ is abelian and has all of its subgroups normal in G. Hence G is metabelian and hypercyclic. □

Lemma 5.3.3. *Let G be a periodic soluble DC_{snn}-group. Then, if G is not a T-group, G/G' is a Černikov group and G is abelian-by-finite.*

Proof. Since we can suppose that G is not Černikov, by Lemma 5.3.2 every subgroup of the Fitting subgroup F of G is normal in G and G is metabelian. Let now H be a subnormal non-normal subgroup of G. Since H' is contained in F, it is normal in G. Then H/H' is an abelian subnormal non-normal subgroup of G/H', which is hence Černikov by Lemma 5.3.2. So clearly G/G' is Černikov, too. Moreover we have that G/F is Černikov, but it is residually finite, so it is finite and G is abelian-by-finite, provided that F is a Dedekind group. □

Now we have a proposition which in the periodic case connects our property with one studied by Casolo [10].

Proposition 5.3.4. *Let G be a periodic soluble group with the DC_{snn} condition and let H be a subnormal non-normal subgroup of G. Then G/H_G is Černikov. Moreover, if G is not Černikov, H^G/H_G is finite.*

Proof. We can clearly assume that G is not Černikov, so by Lemma 5.3.2 every subgroup of the Fitting subgroup F of G is normal in G and G is metabelian. Since H' is contained in F, it is normal in G. H/H_G is an abelian subnormal non-normal subgroup of G/H_G, which is hence Černikov by Lemma 5.3.2. By Lemma 5.3.3, F has finite index in G and clearly $F \cap H = H_G$, so H/H_G is finite. Since H/H_G is contained in the Fitting subgroup of G/H_G and so it has finitely many conjugates in G/H_G [See Robinson Part 1, 5.49] we have that H^G/H_G is finite. □

Let L be the last term of the lower central central series of a group G. In the context of periodic T-groups we have that L just coincides with $\gamma_3(G)$ and with the next proposition we are going to show a similar behavior in DC_{snn}-groups.

Proposition 5.3.5. *Let G be a periodic soluble group with the DC_{snn} condition and let L be the hypocentre of G. Then G/L is nilpotent and G'/L is finite. In particular, if G is hypocentral, G is nilpotent.*

Proof. Assume by a contradiction that G/L is not nilpotent, then G/γ_ω is not nilpotent, too, so we can suppose G being residually nilpotent and hence it is the product of its Sylow subgroups [see [49], p. 8]. Since G is clearly not Černikov, by Proposition 5.3.4 each subnormal subgroup has finite index in its normal closure and moreover we can easily see that the abelian divisible radical D of G is in the centre of G. Hence G is nilpotent since its Sylow subgroups are all nilpotent [see [10], Theorem 3.2].
If G/L is a T-group, then it is also a Dedekind group and has finite commutator subgroup, while, if that is not the case, it has to be Černikov, and then again G'/L is finite. □

Lemma 5.3.6. *Le G be a periodic soluble DC_{snn}-group and let L be the hypocentre of G. If $L \neq L^2$, then G is either a Černikov group or a T-group.*

Proof. Let us assume the thesis is invalid. By Proposition 5.3.5, we know that G/L is nilpotent. Moreover, by Proposition 5.3.5 and Lemma 5.3.3 G/L is Černikov, so L is not Černikov. Take now $H \leq L$ such that $|L : H| \leq 2$. By Lemma 5.3.2, H is normal in G since it is contained in the Fitting subgroup of G and G/H is a Baer group and then it is also nilpotent, implying that $L = L^2$, and this is a contradiction. □

Theorem 5.3.7. *Let G be an primary soluble DC_{snn}-group. If G is not Černikov, then the following hold*

(a) If G is a p-group for some odd prime p, then G is abelian;

(b) if G is a 2-group, then it has finitely many subnormal non-normal subgroups.

Proof. By Proposition 5.3.4, every subnormal non-normal subgroup of G has finite index in its normal closure, so we can apply Theorem 3.2 of [10]. Then, if p is odd, G is abelian and if $p = 2$ the Fitting subgroup of G has index in G at most 2. Clearly we may suppose the latter statement holds. Since G is not Černikov, by Lemma 5.3.2 every subgroup of G is normal in G, so we may suppose $|G : F| = 2$ and $G = \langle F, z\rangle$ for any element $z \in G \setminus F$. Since by Proposition 5.3.5 and Lemma 5.3.6 the last term L of the lower central series of G is a non-trivial divisible group, we know that z acts as the inversion on each element of F. Thus there exists a natural number n such that $L = F^{2^n}$, which means G/L has finite exponent and G/L is finite by Lemma 5.3.3. Let now H a subnormal non-normal subgroup of G. Clearly H is not contained in F, so $[L, H] = L^2 = L$ and $L \leq H$. Therefore the group G has finitely many subnormal non-normal subgroups. □

The general case

In this section we are going to prove general results on DC_{snn}-groups, particularly in connection with other already known properties on subnormal non-normal subgroups, such as the obvious property of being a T-group, that of having no infinite subnormal non-normal subgroups and that of having no infinite subnormal non-normal subgroups of infinite index. Groups satisfying the latter properties are said IT-groups and LT-groups, respectively and have being studied in [23] and in [24], respectively, by Franciosi and de Giovanni.

Lemma 5.3.8. *Let G be a soluble DC_{snn}-group and let F be the Fitting subgroup of G. If F is not periodic and every subgroup of F is normal in G, then G is either a T-group or an LT-group.*

Proof. Since every subgroup of F is normal in G and F is not periodic, $G = \langle F, z\rangle$ where z acts trivially on every element of F or $x^z = x^{-1}$ for each $x \in F$ and $z^2 \in F$. We can suppose that z acts as the inversion on every element of F. If G/F^4 is finite, then G is an LT-group [see [24], Theorem 3.3] so we can suppose that G/F^4 is infinite. We know that G/F^4 is nilpotent [see [7], Lemma 3.8] and it has finite exponent, so by Lemma 5.3.2 it is Dedekind. Let us define $H = \langle z^2, F^4\rangle$ and consider the Dedekind group G/H. Clearly zH has order 2 and lies in the centre of G/H so F/H is in the centre of G/H, too, and this means that F/H has exponent 2. So we have found that $\langle z^2, F^2\rangle = \langle z^2, F^4\rangle$ and G is a T-group [see [45], Theorem 3.1.1]. □

Lemma 5.3.9. *Let G be a soluble DC_{snn}-group and let A be a torsion-free abelian subnormal subgroup of G which is not finitely generated. Then every subgroup of A is normal in G.*

Proof. Assume by contradiction the existence of a finitely generated subgroup H of A which is not normal in G. So, for a proper choice of $k_1, \ldots, k_n, \ldots$,

$$\ldots < H^{k_n} < \ldots < H^{k_1} < H$$

is a descending chain of subnormal non-normal subgroups of G. Hence we can take H as a subgroup being maximal with respect to the condition of not being

G-invariant. So A is normal in G and we can still suppose that H is finitely generated. Let us suppose that the torsion-free rank $r_0(H)$ of H is strictly less than the torsion-free rank $r_0(A)$ of A. Hence there exists an element $x \in A \setminus H$ such that $H \cap \langle x \rangle = \{1\}$. By the maximality of H we have then that $H = \bigcap \langle H, x^n \rangle$ is normal in G, so we can suppose that $r_0(H) = r_0(A)$ and A has finite rank. Since A/H cannot be generated by two proper subgroups, it is isomorphic to a group of type p^∞ for a prime p. If $r_0(A) = 1$, H is contained in a cyclic normal subgroup of G and hence is clearly normal in G, so we can take $H = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_m \rangle$ with $\langle a_1 \rangle$, for instance, being non-normal in G. Let us fix a prime $q \neq p$ and put $H_q = \langle a_1 \rangle \times \langle a_2^q \rangle \times \ldots \times \langle a_m^q \rangle$. It is easy to see that $A/H_q = H/H_q \times B_q/H_q$ where B_q/H_q is a group of type p^∞. Since each B_q is not finitely generated and contains a descending chain of subnormal non-normal subgroups, for instance $\ldots < \langle a_1^{l_n} \rangle < \ldots \langle a_1^{l_1} \rangle < \langle a_1 \rangle$ for a proper choice of such integers, it is normal in G and hence $B = \bigcap\limits_{q \neq p} B_q$ is normal in G, too. But we have that

$$B \cap H = (\bigcap_{q \neq p} B_q) \cap H = \bigcap_{q \neq p} (B_q \cap H) = \bigcap_{q \neq p} H_q = \langle a_1 \rangle.$$

Since, for each $q \neq p$, $r_0(B_q) = r_0(B_q \cap H)$, then $r_0(B) = 1$ and $\langle a_1 \rangle$ is contained in a cyclic G-invariant subgroup of B and hence it is normal in G, which is a contradiction. □

Lemma 5.3.10. *Let G be a DC_{snn}-group, let F be the Fitting subgroup of G and let*

$$H_1 < \ldots < H_n < \ldots$$

be an ascending chain of subgroups of F which are not normal in G. Then H_n is periodic for each n.

Proof. Assume the result is false, so let n be the minimum natural number such that H_n is non-periodic and let a be an aperiodic element of H_n. Clearly $\langle a \rangle$ has to be normal in G so every aperiodic element of H_n is normal in G. This means that H_n is normal in G, but that is a contradiction. □

Lemma 5.3.11. *Let G be a soluble non-polycyclic DC_{snn}-group and let F be the Fitting subgroup of G. If F is reduced, then every subgroup of F is normal in G. In particular, if F is not periodic, G is either a T-group or an LT-group.*

Proof. Let T be the torsion subgroup of F. If T is infinite, as F is reduced, it does not satisfy Min, and hence contains a direct product of infinite cycles. By Lemma 5.3.2 every subgroup of F is normal in G, so we may suppose that

T is finite. Since G is not polycyclic, F cannot be periodic and hence not Černikov so, by Proposition 5.0.8, it satisfies the maximal condition on non-normal subgroups. By Theorem 3.1.6, F can be an abelian group or the direct product of $\mathbb{Q}_2$ with a finite group. So in every case $F = T \times A$ where A is an abelian torsion-free not finitely generated group. By Lemma 5.3.9 every subgroup of A is normal in G so if we say x a periodic element of F we can construct a double chain of subnormal subgroups of G, for instance

$$\ldots < A_{-n}\langle x\rangle < \ldots < A_0\langle x\rangle < \ldots < A_n\langle x\rangle < \ldots$$

where the collection $\{A_i | i \in \mathbb{N}\}$ forms a double chain in A. Then there exists an integer k such that $A_k\langle x\rangle$ is normal in G and hence $\langle x\rangle = T \cap A_k\langle x\rangle$ is normal in G and so every subgroup of F is normal in G. □

Corollary 5.3.12. *Let G be a soluble residually finite DC_{snn}-group. Then G is either polycyclic or metabelian and hypercyclic.*

Proof. Supposing that G is not polycyclic, by Lemma 5.3.11 $G/C_G(F)$ is abelian and $G' \leq C_G(F) = F$, so G' is abelian and every subgroup of G' is normal in G. So G is metabelian and hypercyclic. □

Proposition 5.3.13. *Let G be a soluble torsion-free DC_{snn} group. Then G is either polycyclic or abelian.*

Proof. Supposing that G is not polycyclic, by Lemma 5.3.11 F is abelian, so G/F is a group of power automorphisms of an abelian torsion-free group and so $|G/F| \leq 2$ (see [15], Corollary 4.2.3). Let us now suppose the existence of a non-trivial element x of $G \setminus F$. Since $x^2 \in F$, x has to act trivially on every element of F and so it belongs to F itself, which cannot be. This implies that G is abelian. □

Lemma 5.3.14. *Let G be a soluble non-polycyclic group satisfying the DC_{snn} and let F be the Fitting subgroup of G. If the torsion subgroup T of F is not a finite extension of a group of type p^∞ and F is not periodic, then G is either a T-group or an LT-group.*

Proof. Since G is not polycyclic F cannot be finitely generated and by hypothesis it is neither Černikov, so, by Proposition 5.0.8, it satisfies the maximal condition on non-normal subgroups. Since we can suppose that T is Černikov, F is the direct product of T by an abelian torsion-free group, say A. If A is not finitely generated, then similarly to what we proved in Lemma 5.3.11 we can prove that every subgroup of F is normal in G. So suppose that T is not

finitely generated. Since it is not a finite extension of a group of type p^∞, it has to contain a subgroup B such that $B = B_1 \times B_2$ where B_1 and B_2 are both not finitely generated. So, if we take a as an aperiodic element of F, we have that $\langle B_1, a\rangle$ and $\langle B_2, a\rangle$ are both normal in G by Corollary 5.0.4 and so $\langle a\rangle = \langle B_1, a\rangle \cap \langle B_2, a\rangle$ is normal in G and hence every subgroup of F is normal in G.
The conclusion follows from Lemma 5.3.8. □

Lemma 5.3.15. *Let G be a soluble DC_{snn}-group and let F be the Fitting subgroup of G. If F is torsion-free, then G is either polycyclic or an LT-group.*

Proof. Let us assume that G is not polycyclic, so F is not polycyclic as well. By Lemma 5.3.11 every subgroup of F is normal in G and by Lemma 5.3.8 we have that G is an LT-group. □

Lemma 5.3.16. *Let G be a soluble DC_{snn}-group. If G is residually finite and F, the Fitting subgroup of G, is not periodic, then G is either polycyclic or an LT-group.*

Proof. Let us assume that G is not polycyclic, so F is not polycylic as well. F is also reduced, so by Lemma 5.3.11 every subgroup of F is normal in G. So we have that G is an LT-group by Lemma 5.3.8. □

Lemma 5.3.17. *Let G be a soluble non-periodic DC_{snn}-group and let F be the Fitting subgroup of G. If F is a Prüfer p-group for a prime p, then G is either an IT-group or nilpotent.*

Proof. Since F is a Prüfer group, G/F is an abelian, residually finite group, so if we call J the finite residual of G we must have $J = F$. Moreover $G' \leq F$. If we assume that $G' = F$ then G is an IT-group (see [23], Theorem 1.11). So let G' be a finite subgroup of F. Then G is a BFC-group and $F = J \leq Z(G)$. So G is nilpotent. □

Lemma 5.3.18. *Let G be a soluble non-periodic DC_{snn}-group and let F be the Fitting subgroup of G. If F is the direct product of infinitely many cyclic subgroups, then G satisfies the maximal condition on subnormal non-normal subgroups.*

Proof. Let $F = \mathrm{Dr}_{i\in I} C_i$, where I is an infinite set and C_i is cyclic for each $i \in I$. By Lemma 5.0.1 every subgroup of F is normal in G and G is metabelian. Let

$$H_0 < H_1 < \ldots < H_n < \ldots$$

be an ascending chain of subnormal non-normal subgroups with H_0 being a minimal subnormal non-normal subgroup of G. Then we have that H_0' is normal in G and H/H_0' is a cyclic subnormal non-normal subgroup of G/H_0'. Since, by hypothesis, only a finite number of subgroups of F can live out of H_0, the commutator subgroup of G/H_0' is finite and abelian and the Fitting subgroup of G/H_0' coincides with the Hirsch-Plotkin radical of G/H_0' by Lemma 5.0.5. On the other hand the Fitting subgroup of G/H_0' is clearly reduced and not periodic, since G itself is not periodic and G/H_0' is an FC-group, so by Lemma 5.3.11 every subgroup of the Fitting subgroup of the non-polycyclic group G/H_0' is normal in G/H_0', which is a contradiction. □

Finally, we conclude this chapter proving a result which shows a good behavior of DC_{snn}-groups in accordance with what will be seen for DC_{np}-groups, which are strictly related with the present ones.

Proposition 5.3.19. *Let G be a soluble DC_{snn}-group and let $\langle x\rangle$ be an infinite subnormal non-normal subgroup of G. Then G is minimax.*

Proof. Firstly assume G being locally nilpotent. Let F be the Fitting subgroup of G and T be the torsion subgroup of G. If T is not Černikov, then in F we can find an infinite direct product of cyclic subgroups, which, combined with $\langle x\rangle$, would make it normal, which is not. Hence T is Černikov and particularly, since by Lemma 5.3.11 we can assume F is not reduced, T is a finite extension of a Prüfer group, since $\langle x\rangle$ cannot normalize two periodic subnormal subgroups with trivial intersection. Let now assume that G/T is not polycyclic. As G is locally nilpotent, G/T is torsion-free and by Proposition 5.3.13 G/T is abelian and $G' \leq T$. Thus G is a CC-group, namely a group with Černikov conjugacy classes. Then $G/Z_2(G)$ is periodic (see [11], Theorem 2.4.7) and G/F is periodic, too. F cannot be polycyclic, but because of $\langle x\rangle$ it is neither Černikov, so is satisfies the maximal condition on non-normal subgroups by Proposition 5.0.8 and then it is central-by-finite by Corollary 3.1.7. On the other hand, $Z(F) = A \times P$ where P is a finite extension of a Prüfer group and A is torsion-free and finitely generated by Lemma 3.1.2. Since it is impossible that $\langle x\rangle A^k$ is normal for every natural number k, otherwise we would have that

$$\langle x\rangle = \bigcap_{k\in\mathbb{N}} \langle x\rangle A^k$$

is normal and this is not the case, we know that there exists a natural number k such that G/A^k is still periodic and contains a finite subnormal non-normal subgroup, so it is Černikov by Lemma 5.3.2 and our lemma is proved for the locally nilpotent case. Now let G be a soluble DC_{snn}-group and H its Hirsch-Plotkin radical. By the first part of the proof, H is minimax and hence every ascendant abelian subgroup of G is minimax. The result follows from Corollary 6.3.9 in [34] □

Chapter 6

Double chain condition on non-pronormal subgroups

This chapter is devoted to groups satisfying the maximal, the minimal and the double chain condition on non-pronormal subgroups.

Pronormal subgroups has been studied for a long time in relationship with the behavior of finite groups. In the late 80s, however, Kuzennyi and Subbotin began studying the effect of having *many* pronormal subgroups in infinite groups. In their paper [31] in 1987 they gave a complete description of periodic locally graduated groups whose subgroups are all pronormal, showing, in the same paper, that locally soluble non-periodic groups whose subgroups are all pronormal are abelian. After this, in their paper [32] in 1988 they investigated the structure of groups all of whose infinite subgroups are pronormal, proving that such groups, if locally soluble, either have all of their subgroups being pronormal or are finite extensions of a Prüfer group.

More recently, a paper [27] of 1995 by de Giovanni and Vincenzi and then a paper [56] in 1998 by Vincenzi have dealt with maximal and minimal conditions on non-pronormal subgroups. Finally, in 2001 a survey [28] about the theory of pronormal subgroups has been published by the same authors.

We should previously remark that the examples shown by Ol'šanskiĭ play their role here, too. In fact, simple groups exist which are easily shown to have each proper subgroup being cyclic and pronormal. From this consideration it follows that many results in this chapter will have to bring along a suitable hypothesis to avoid those simple groups. The hypothesis in question is that of having no infinite simple sections, which is of wide use throughout the study of pronormality. This class of groups is easily shown to contain every hyper-(finite or locally soluble) group.

Lemma 6.0.1. *Let G be a group having no infinite simple sections. Then G is locally graded.*

Proof. Le H be a finitely generated non-trivial subgroup of G. Then by Zorn's Lemma H contains a maximal normal subgroup K and H/K is simple. Then H/K is finite and G is locally graded. □

Here we are going to give account of some basic and easy properties which we will use abroad throughout the present chapter.

Lemma 6.0.2. *Let G be a group and H a subgroup of G. Then H is normal in G if and only if H is pronormal and ascendant in G.*

The above lemma underlines an interesting, maybe unexpected connection between non-pronormal and subnormal (ascendant) non-normal subgroups. In fact, as we will see, many are the correlations between the two properties and the methods used in attaching both problems overlap many times. Anyway, one of the most pivotal difference is, from the very beginning, the fact that the DC_{np} condition is inherited by subgroups, while the DC_{snn} condition is inherited only by subnormal subgroups.

Lemma 6.0.3. *Let G be a group and let H and K be pronormal subgroups of G such that $H^K = K$. Then HK is a pronormal subgroup of G.*

Next lemma will show a strict connection between the property of having *many* pronormal subgroups and that of being a group with every subgroup being T-groups, namely the so-called $\overline{T}-$ *groups*, firstly studied in [45]. This will do as our extremal case most of the times in this chapter.

Lemma 6.0.4. *Let G be a group whose cyclic subgroups are pronormal. Then G is a $\overline{T}$-group.*

Another lemma which has a good balance of ease and strength is the following, which reduces the study of the locally nilpotent case to that of DC_{nn}. Notice that here we are meeting, among many similarities, one of the crucial differences between DC_{np} and DC_{snn}.

Lemma 6.0.5. *Let G be a locally nilpotent group and H a pronormal subgroup of G. Then H is normal in G.*

Proof. Let x be an element of G and put $K = N_G(H)$. Since H is pronormal there exists an $y \in \langle H, H^x\rangle$ such that $H^x = H^y$. We have clearly that $x = xy^{-1}y$ belongs to $\langle K, K^x\rangle$ and so there exists a finitely generated subgroup F of K such that $x \in \langle F, F^x\rangle$. But $\langle F, F^x\rangle = \langle F, x\rangle$ is nilpotent and F is obviously pronormal in $\langle F, F^x\rangle$, so it is also normal in it and we have $x \in F \leq K$. Thus H is normal in G. □

Now we present a useful lemma which consists of the transposition of Lemma 0.0.1 to pronormal subgroups with the addition of some related consequences. Notice that in this case the intersection closure of the property is granted by the consideration of ascendant pronormal subgroups, which are hence normal by Lemma 6.0.2.

Lemma 6.0.6. *Let G be a DC_{np}-group and let H/K be a section of G which is a direct product of infinitely many non-trivial cyclic subgroups. Suppose that H is ascendant in G, then K and H are normal in G and every cyclic subgroup of G/H is pronormal. Moreover G/K is a $\overline{T}$-group.*

Proof. H/K is a direct product of infinitely many non-trivial cyclic subgroups if and only if it contains a direct product of countably many non-trivial cyclic subgroups so without loss of generality we can let $\{H_n | n \in \mathbb{Z}\}$ be a countably infinite collection of subgroups of H properly containing K and such that H_i/K is cyclic for all $i \in \mathbb{Z}$ and that $H/K = \mathrm{Dr}_{i\in\mathbb{Z}} H_i/K$. We can then split this collection into two infinite collections, namely $\{U_i | i \in \mathbb{Z}\}$ and $\{V_j | j \in \mathbb{Z}\}$, such that

$$H/K = \mathrm{Dr}_{i\in\mathbb{Z}} U_i/K \times \mathrm{Dr}_{j\in\mathbb{Z}} V_j/K$$

and for each integer n define

$$U_n^* = \langle U_k | k < n \rangle$$

$$V_n^* = \langle V_k | k < n \rangle.$$

Since the group satisfies the double chain condition on non-pronormal subgroups, there exist two integers s and r such that $U_r^* prG$ and $V_s^* prG$ the both of them being normal in G since H is ascendant. Obviously, we have that $U_r^* \cap V_s^* = K$ and K is normal in G. On the other hand, by the same reasoning we see that each direct term of H/K is normal in G/K, so H/K is normal in G/K, too, and H is normal in G.

In order to prove that every cyclic subgroup of G/H is pronormal and then that G/H is a $\overline{T}$-group we can suppose, without loss of generality, that $K = \{1\}$. Let now x be an element of G and notice that $\langle x \rangle \cap H$ can be contained only in a finite number of direct terms of H, so we can assume that $\langle x \rangle \cap H = \{1\}$. This way, again by the double chain condition, we have that there exists an integer r such that, for instance, $\langle x \rangle U_r^* prG$. Thus the subgroup $\langle x \rangle H = (\langle x \rangle U_r^*)H$ is pronormal in G.

Still assuming without loss of generality that $K = \{1\}$, let now E be a subgroup of G and F be a subnormal subgroup of E. Call I the set of indices such that $H = \mathrm{Dr}_{i\in I} H_i$ and define $I_E = \{i \in I | H_i \cap E \neq \{1\}\}$, $I_F = \{i \in$

$I|H_i \cap F \neq \{1\}\}$, $L_E = \mathrm{Dr}_{i\in I_E}(H_i \cap E)$ and $L_F = \mathrm{Dr}_{i\in I_F}(H_i \cap F)$. If I_F is infinite countable, then by the first part of the proof E/L_F is a $\overline{T}$-group and F is normal in E. If otherwise I_F is not infinite we can suppose that $F \cap H = \{1\}$. By the double chain condition on non-pronormal subgroups, there exist two integer k and l such that FU_k^* and FV_l^* are pronormal in G. So they are pronormal and hence normal in EU_k^* and in EV_l^*, respectively. If we now assume that I_E is infinite and countable, we may suppose that $EU_k^* = EV_l^* = E$ and hence that $F = FU_k^* \cap FV_l^*$ is normal in E. Finally, if I_E is finite we can assume that $E \cap H = \{1\}$ and so we have that $F = E \cap FU_k^*$ is normal in E. □

Lemma 6.0.7. *Let G be a periodic DC_{np}-group. If the Hirsch-Plotkin radical H of G is not Černikov, then all of its subgroups are normal in G and every finite subgroup of G is a T-group.*

Proof. By Lemma 6.0.5, H is a DC_{nn}-group and hence by Theorem 3.3.8 and by Corollary 3.1.7 it is nilpotent. Let A be a maximal abelian normal subgroup of H, then it is not Min and thus contains a subgroup B such that $B = \mathrm{Dr}_{i\in\mathbb{Z}} B_i$. Since H is nilpotent and B is subnormal in G, as we have already seen B_i is normal in G for each $i \in \mathbb{Z}$ and by Lemma 6.0.6 G/B is a $\overline{T}$-group. Let X be a cyclic subgroup of H. Clearly we can suppose that $X \cap B = \{1\}$. We know that G/B is a T-group, so XB is normal in G. On the other hand, by the double chain condition on non-pronormal subgroups, there exist two integers k and l such that XB_k and XB_l are both pronormal and hence normal in G. This way we have that $X = XB_l \cap XB_k$ is normal in G.

Let now E be a finite subgroup of G and L be a subnormal subgroup of E. Clearly we can suppose that $E \cap B = \{1\}$. There exists an integer k such that LB_k is pronormal in G. So it is pronormal (and hence normal) in EB_k and so we have that $L = E \cap LB_k$ is normal in E. □

6.1 Maximal condition

This section is based on the paper of Vincenzi [56]. Some results are here extended from the same hypotheses while some others are reported in their double chain (hence still extended) form in the dedicated section.

Proposition 6.1.1. *Let G be a finitely generated radical group satisfying the maximal condition on non-pronormal subgroups. Then G is polycyclic.*

Proof. By Lemma 6.0.5, the Hirsch-Plotkin radical H of G satisfies Max-nn. In particular, by Theorem 3.1.6, it is a central extension of an abelian group by a polycyclic group.

Assume by a contradiction that G is not polycyclic, from which it follows that H is not finitely generated by Theorem 1.1.2, so G/H cannot be periodic. Since H is not finitely generated, it follows easily that G/H is a $\overline{T}$-group, and hence it is abelian. Then G is abelian-by-polycyclic by Theorem 2.1.3, G is Max-n and so every infinite ascending chain of subgroups of H is definitively not G-invariant. Since every subgroup of H is subnormal in G, by Lemma 6.0.2 we have our contradiction. □

Theorem 6.1.2. *Let G a hyper-(abelian or finite) group satisfying Max-np. Then G is soluble-by-finite. Moreover, if G is not Max, then G is soluble.*

Proof. Let

$$\{1\} = G_0 < G_1 < \ldots < G_\tau = G$$

be an ascending series of G such that $G_{\beta+1}/G_\beta$ is abelian or finite for each $0 \leq \beta < \tau$, by hypothesis and Lemma 6.0.2 there exists a natural number n such that G_α is normal in G for each $n \leq \alpha \leq \tau$. Since clearly G_n is soluble-by-finite we can assume that G_α is normal for each $\alpha \leq \tau$. Now, by induction on τ, we have directly our claim if τ is not a limit ordinal, so suppose this is not the case. Then G, being the union of infinitely many soluble-by-finite subgroups, is locally(soluble-by-finite). If we suppose by a contradiction that G is not soluble-by-finite, there is in G a cyclic non-pronormal subgroup E_1 and it has to be contained in a proper term of the ascending series of G, say G_{α_1}. Then G/G_{α_1} is not soluble-by-finite and contains a cyclic non-pronormal subgroup E_2/G_{α_1}. This way we can construct an infinite ascending chain of non-pronormal subgroups of G. Hence G is soluble-by-finite.
Suppose now that G is not Max and let K be a soluble subgroup of finite index of G. Then clearly there is a natural number n such that K^n/K^{n+1} is not Max. Then it is easily seen that every cyclic subgroup of G/K^n is pronormal and then G/K is soluble (see Peng [42], Theorem p.233) and so is G □

Making use of an interesting, technical lemma, which follows, we will be able to prove an optimal characterization for periodic soluble groups satisfying Max-*np*.

Lemma 6.1.3. *Let G be a periodic soluble group, N a normal π-subgroup of G and H a π'-subgroup of G. Suppose either N or H is finite. If x is an element of G such that the subgroups HN and H^xN are conjugated in $\langle H, H^x\rangle N$, then H and H^x are conjugated in $\langle H, H^x\rangle$. In particular, if HN is a pronormal subgroup of G, then so is H.*

Proposition 6.1.4. *Let G be a periodic soluble group satisfying the maximal condition on non-pronormal subgroups. If G is not a Černikov group, then all subgroups of G are pronormal.*

Proof. Say A the last term of the lower central series of G. By Lemma 6.0.7, every cyclic subgroup of G is pronormal and by Lemma 6.0.4 G is a $\overline{T}$-group, so, by the results of [45], we have that every subgroup of A is normal in G, that A has no elements of period 2, hence it is abelian, G/A is a Dedekind group and $\pi(A) \cap \pi(G/A) = \emptyset$. Let $\pi \subseteq \pi(G/A)$ and P be a π-subgroup of G. If A is finite we can apply Lemma 6.1.3, so let

$$A_1 < \ldots < A_n < \ldots$$

an infinite chain of finite subgroups of A. Since $P \cap A = \{1\}$,

$$A_1P < \ldots < A_nP < \ldots$$

is strictly ascending and then there exists a natural number k such that A_kP is pronormal in G. Then P is pronormal in G by Lemma 6.1.3. Now we have our claim, by applying Lemma 5, Lemma 6 and, easily, the main theorem of [31]. □

Though Theorem 6.3.8 below obviously applies also to periodic groups satisfying Max-*np* and is an optimal split into extremal cases, Vincenzi [56] showed a sharp characterization for this groups, too, which is worth to be stated here.

Theorem 6.1.5. *Let G be a periodic hyper-(abelian or finite) group. Then G satisfies Max-np if and only if one of the following conditions holds:*

- *(a) G is a finite group;*
- *(b) every subgroup of G is pronormal;*
- *(c) G is an extension of a subgroup P of type p^∞ for a prime p by a finite $\overline{T}$-group, and the Sylow p-subgroups of G are nilpotent.*

Finally, we state the main result of [56], which is a full characterization of hyper-(abelian or finite) groups satisfying the maximal condition on non-pronormal subgroups.

Theorem 6.1.6. *Let G be a hyper-(abelian or finite) group. Then G satisfies Max-np if and only if one of the following conditions holds:*

- *(a) G is Max;*

(b) every subgroup of G is pronormal;

(c) G contains a normal subgroup P of type p^∞ for a prime p and a finitely generated torsion-free central subgroup A of rank r such that, for every subgroup N of A with rank r, G/PN is a finite $\overline{T}$-group and the Sylow p-subgroups of G/N are nilpotent;

(d) every finite homomorphic image of G is a T-group, G is soluble and contains a subgroup L of finite index, with all its subgroups being normal in G and which is extension of a finitely generated group by a Prüfer group.

6.2 Minimal condition

This section is based on the paper of de Giovanni and Vincenzi [27]. Some results are here extended from the same hypotheses while some others are reported in their double chain (hence still extended) form in the dedicated section. The aim of the paper was that of proving a result of "Phillips and Wilson" type, namely a result of the type explored in Section 3.2.

We are firstly dealing with the periodic case, but before that we are going to see how the imposition for a group to have only finite descending chains of non-pronormal subgroups steps in when coming to chief factors.

Lemma 6.2.1. *Let G be a group satisfying Min-np. If G has no infinite simple sections, then every chief factor of G is finite.*

Proof. Let us take H/K as a chief factor of G. By Min-*np*, H/K is Min-*n* and then by Corollary 2.2.7 it is product of finitely many isomorphic simple groups. Therefore H/K is finite. □

Here is a useful lemma, namely Lemma 13 of [27], which we state here.

Lemma 6.2.2. *Let G be a periodic countable group satisfying Min-np. If G has no infinite simple sections, then G is hyperabelian-by-finite.*

Lemma 6.2.3. *Let G be a periodic hyperabelian-by-finite group satisfying Min-np. If G has no infinite simple sections, then either G is a Černikov group or all subgroups of G are pronormal.*

Proof. Assume that G is not Černikov, say K a hyperabelian normal subgroup of finite index of G and say L the Fitting subgroup of K. Clearly L cannot be Černikov and by Lemma 6.0.7 every finite subgroup of G is pronormal in G.

Assume that G contains non-pronormal subgroups and let M be a minimal non-pronormal subgroup of G. Clearly M/M' is a Prüfer p-group for a prime p, since it cannot be generated by two proper subgroups. But M is a T-group, so $M/C_M(M')$ is isomorphic with a group of power automorphisms of M' and so it is residually finite. Then $M' \leq Z(M)$ and M is nilpotent, thus Dedekind, which means that M' has to be trivial in this case. So M is a Prüfer group. Now, we know that G itself is a T group and so, if we say F the fitting subgroup of G, $G/C_G(F)$ is residually finite, so that $M \leq C_G(F)$. This way we have that M is normal in MF, which is normal in G, a contradiction. □

Now we are ready to prove the main result for the periodic case.

Theorem 6.2.4. *Let G be a periodic group satisfying Min-np. If G has no infinite simple sections, then either H is a Černikov group or all subgroups of G are pronormal.*

Proof. Assume G is not a Černikov group. Then it contains a countable subgroup X which is not Černikov (see, for instance, [49], p. 107). If we take in G a finitely generated subgroup H, then $\langle H, X\rangle$ is still countable and hence hyperabelian-by-finite by Lemma 6.2.2. Now, by Lemma 6.2.3 all subgroups of $\langle H, X\rangle$ are pronormal and in particular $\langle H, X\rangle$ is metabelian. By the arbitrary choice of H we deduce that every finitely generated subgroup of G is metabelian, hence G is metabelian and every subgroup of G is pronormal by Lemma 6.2.3. □

Now we prove a lemma which will help us proving the general result of this section.

Lemma 6.2.5. *Let G be a group satisfying Min-np and let $\langle x\rangle$ be an infinite cyclic subnormal subgroup of G. Then $\langle x\rangle \leq Z(G)$.*

Proof. By Min-*np*, $\langle x\rangle$ satisfies Min-G and hence $\langle x\rangle$ is normal in G. Assume the existence in G of an element y such that $y^{-1}xy = x^{-1}$. Hence $\langle x\rangle \cap \langle y\rangle = \{1\}$ and

$$\ldots < \langle x^{2^i}, y\rangle < \ldots < \langle x^2, y\rangle < \langle x, y\rangle$$

is a strictly descending chain. So there exists a $k > 1$ such that $\langle x^{2^k}, y\rangle$ is pronormal in G. Since $y^2 \in Z(\langle x, y\rangle)$, $\langle x^{2^k}, y\rangle$ is normal in $\langle x, y\rangle$ and $\langle x, y\rangle/\langle x^{2^k}, y^2\rangle$ is a finite 2-group. In particular, $\langle x^{2^k}, y\rangle$ is subnormal and hence normal in $\langle x, y\rangle$, so that $\langle x, y\rangle' \leq \langle x^{2^k}, y\rangle$. Therefore $x^{-2} \in \langle x^{2^k}, y\rangle \cap \langle x\rangle = \langle x^{2^k}\rangle$, which is impossible. □

From the previous lemma and from the fact that a non-periodic nilpotent group is generated by its element of infinite order follows that

Corollary 6.2.6. *Let G be a group satisfying Min-np and let N be a nilpotent non-periodic normal subgroup of G. Then N is contained in $Z(G)$.*

Lemma 6.2.7. *Let G be a (locally soluble)-by-finite non-periodic group satisfying Min-np. Then G is abelian.*

Proof. Assume first that the lemma is false for finitely generated soluble groups and say G such a group of minimal derived length k. If G' is not periodic then by Corollary 6.2.6 it is contained in $Z(G)$ and G is nilpotent and hence abelian by 6.2.6. Then G' is periodic. Hence G/G^{k-1} is not periodic and hence abelian, so $k = 2$ and G' is abelian. If we take $x \in G'$, then $\langle x\rangle^G$ is an abelian group of finite exponent such that $\langle x\rangle^G/\langle x\rangle$ satisfies the minimal condition on subgroups, so $\langle x\rangle^G$ is finite. Let a be an element of infinite order of G and n a positive integer such that $[\langle a^n\rangle, \langle x\rangle^G] = \{1\}$. Then $\langle a^n, \langle x\rangle^G\rangle$ is an abelian normal subgroup of $\langle a\rangle\langle x\rangle^G$ and by Corollary 6.2.6 it is central. From this $\langle a\rangle\langle x\rangle^G$ is nilpotent and hence abelian. Therefore $\langle a, G'\rangle$ is abelian and by Corollary 6.2.6, again, it lies in $Z(G)$. So G is nilpotent and hence abelian, a contradiction.

Suppose now that G is a (locally soluble)-by-finite group and let R be a locally soluble normal subgroup of finite index of G. Then R is not periodic and is abelian by the first part of the proof. So G is abelian-by-finite. Assume the lemma is false and take G as a minimal counterexemple with regard to the index of an abelian normal subgroup A of finite index. Then, clearly, if $A \leq H < G$, H is abelian. Therefore a well-known result by Miller and Moreno [39] shows that G/A is soluble, so G is soluble and hence is abelian by the first part of the proof. □

Next we finally state the non-periodic part of the result of [27].

Theorem 6.2.8. *Let G be a non-periodic group satisfying Min-np. If G has no infinite simple sections, then G is abelian.*

Proof. Clearly we can assume that G is finitely generated. By Lemma 6.0.1 there is in G a descending normal series with finite factors, say

$$G = H_0 > H_1 > \ldots > H_n > \ldots,$$

and there exists a natural number k such that H_n/H_{n+1} is a finite $\overline{T}$, and hence metabelian group, for each $n \geq k$. If we now say $N = \bigcap_{n\in\mathbb{N}} H_n$, then H_m/N is hypoabelian and then soluble [see [27], Lemma 4], So G/N is soluble-by-finite and finitely generated, so it cannot be periodic and it is abelian by Lemma 6.2.7. Then $G' \leq N$ and G/G' is not periodic. Assume that $G' \neq \{1\}$. Since

G is finitely generated, clearly G' satisfies Max-G and we can take K as a maximal G-invariant subgroup of G'. Then G'/K is a chief factor of G and by Lemma 6.2.1 it is finite. But this way we have shown that G/K is finite-by-abelian and finitely generated, hence abelian-by-finite and by Lemma 6.2.7 it is abelian, which is a contradiction. □

Theorem 6.2.4, together with the theorem just proved, contribute to form the main result of [27].

Theorem 6.2.9. *Let G be a group satisfying Min-np. If G has no infinite simple sections, then either G is a Černikov group or all subgroups of G are pronormal. In particular, if G is not periodic, G is abelian.*

6.3 Double chain condition

Notice that the class of DC_{np}-groups is closed under **S** and **H**. If we take A as the locally dihedral 2-group and B as the semidirect product between $\mathbb{Z} \times \mathbb{Z}$ and the automorphism $x : (a, b) \mapsto (a, ab)$, then $A \times B$ does not belong to the class DC_{np} while both A and B do, so our class is not closed under $\mathbf{N_0}$.

Lemma 6.3.1. *Let G be a DC_{np}-group and let H be a locally nilpotent subgroup of G. If H is not Černikov, then H is nilpotent. Equivalently, H satisfies either the minimal or the maximal condition on non-pronormal subgroups.*

Proof. By Lemma 6.0.5 we have that H satisfies the double chain condition on non-normal subgroups and so by Theorem 3.3.8 it satisfies either the minimal or the maximal condition on non-normal subgroups. Since H is not a Černikov group it satisfies the maximal condition on non-normal subgroups and hence by Corollary 3.1.7 it is nilpotent. □

Since there exists, for instance, the locally dihedral 2-group which is locally nilpotent, Černikov, but is not nilpotent, then the non-Černikov hypothesis cannot be removed from the previous Lemma.

Lemma 6.3.2. *Let G be a DC_{np}-group and let H be the Hirsch-Plotkin radical of G. If every subgroup of H is normal in G, if H is not periodic and not finitely generated, then G/H is a $\overline{T}$-group.*

Proof. Let us take an element of infinite order a and construct the following infinite double chain in H

$$\ldots < \langle a^{2^n} \rangle < \ldots < \langle a^2 \rangle < \langle a \rangle < F_1 < \ldots < F_n < \ldots$$

where $F_1, \ldots F_n$ are finitely generated subgroups of H. Let b be an element of $G \setminus H$ and suppose by a contradiction that $H\langle b \rangle$ is not pronormal in G. If b is of infinite order, then there is an increasing sequence of integers $k_1, \ldots, k_n, \ldots$ such that $\langle b^{k_n} \rangle$ is not pronormal in G for each n. So we can take into account

$$\ldots < \langle b^{k_n} \rangle < \ldots < \langle b^{k_1} \rangle < \langle b \rangle \leq F_1\langle b \rangle \leq \ldots \leq F_n\langle b \rangle \leq \ldots$$

which is a descending chain.

On the other hand, if we suppose that b has finite order, $\langle a^{2^n} \rangle \cap \langle b \rangle = \{1\}$ for any positive integer n and so we can consider

$$\ldots < \langle a^{2^n} \rangle\langle b \rangle < \ldots < \langle a^2 \rangle\langle b \rangle < \langle a \rangle\langle b \rangle \leq F_1\langle b \rangle \leq \ldots \leq F_n\langle b \rangle \leq \ldots$$

as a chain in G, which is certainly descending. In both cases, if we assume that there exists an $l > 0$ such that $F_l\langle b \rangle = F_n\langle b \rangle$ for each $n > l$, we have that the polycyclic subgroup $F_l\langle b \rangle$ contains an ascending chain of subgroups, and that is impossible. So we have shown that both of the chains taken into account are infinite double chains and so at least one of their terms is pronormal in G. From this it follows that in both cases we have found that $H\langle b \rangle prG$, but this is a contradiction, and hence G/H is a $\overline{T}$-group. □

Proposition 6.3.3. *Let G be a finitely generated radical DC_{np}-group. Then G is polycyclic.*

Proof. If G is a Černikov group, then it would be soluble and finite and thus trivially polycyclic so we may suppose that G is not Černikov. Hence we know that also the Hirsch-Plotkin radical H of G cannot be Černikov by Theorem 1.2.2, and by Lemma 6.3.1 it is nilpotent and satisfies the maximal condition on non-normal subgroups. In particular it is a central extension of an abelian group by a polycyclic group by Theorem 3.1.6.

Assume by a contradiction that G is not polycyclic, from which it follows that H is not finitely generated. Assume that H is periodic. H clearly does not satisfy Min-*ab* and therefore it contains an abelian subgroup A which is product of infinitely many non-trivial cycles and so, by Lemma 6.0.6, G is a $\overline{T}$-group and hence polycyclic.

Let then a be an element of infinite order of H and suppose that $\langle a \rangle$ is not pronormal in G. It contains an infinite descending chain of non-pronormal subgroups and so G/H satisfies Max-*np* which means, by Proposition 6.1.1, that

G/H is polycyclic and G is abelian-by-polycyclic. Now we know that G has Max-n and so every infinite ascending chain of subgroups of H is definitively not G-invariant. Since every subgroup of H is subnormal in G, we have that the terms of every infinite ascending chain of subgroups of H are definitively not pronormal in G and this leads to a contradiction properly choosing in H some elements $h_1, h_2, \dots, h_n, \dots$ and constructing, for instance, the following infinite double chain of subgroups of H

$$\dots < \langle a^{k_n} \rangle < \dots < \langle a^{k_1} \rangle < \langle a \rangle < \langle a, h_1 \rangle < \dots < \langle a, h_1, h_2, \dots, h_n, \dots \rangle < \dots$$

for a sequence $k_1, \dots, k_n, \dots$ of positive integers.

Hence every element of infinite order of H is normal in G and being H generated by such elements it turns out to be a Dedekind group with all of its subgroups normal in G.

By Lemma 6.3.2, G/H is an abelian $\overline{T}$-group. We also have this way shown that G is abelian-by-polycyclic, thus satisfying Max-n, which is our final contradiction, since H is not finitely generated. □

This proposition has an immediate consequence in the following corollary, whose proof is straightforward.

Corollary 6.3.4. *Let G be a finitely generated radical-by-finite group the DC_{np} condition. Then G is polycyclic-by-finite.*

We are now on the way to prove that every radical DC_{np}-group is indeed soluble. So we prove a couple of lemmas first.

Lemma 6.3.5. *Let G be a hyper-(locally nilpotent or finite)-group with the DC_{np} condition. Then G is hyper-(abelian or finite).*

Proof. Let

$$\{1\} = G_0 < G_1 < \dots < G_\tau = G$$

be a series of G such that $G_{\beta+1}/G_\beta$ is locally nilpotent or finite for each $0 \leq \beta < \tau$. By Lemma 6.3.1, $G_{\beta+1}/G_\beta$ is nilpotent or abelian-by-finite for each β and hence G is hyper-(abelian or finite). □

Proposition 6.3.6. *Let G be a radical group with the DC_{np} condition. Then G is soluble.*

Proof. Clearly we can suppose that the Hirsch-Plotkin radical H of G is not Černikov, so it does not satisfy Min-ab and furthermore it is abelian-by-polycyclic. If H is periodic, then it has an abelian subgroup which is product of infinitely many cycles and is obviously subnormal in G, so by Lemma 6.0.6 G is soluble. Hence, let a be an element of infinite order of H and suppose that $\langle a \rangle$ is not pronormal in G. This way G/H satisfies Max-np and we can assume it does not satisfy Max, otherwise we would have our thesis. By Lemma 6.3.5, G/H is hyper-(abelian or finite) and so G/H is soluble by Theorem 6.1.2. Therefore we can assume that every element of infinite order of H is normal in G, having that H is a Dedekind group with all subgroups normal in G. Then using Lemma 6.3.2 we have that G/H is soluble. □

The periodic case

Let now come to our results concerning periodic DC_{np} groups which we will obtain using the well-known Theorem 1.2.5 by Zaicev on periodic locally soluble groups.

Lemma 6.3.7. *Let G be a periodic locally soluble DC_{np}-group. Then G is soluble and, in particular, if $Z(G)$ is trivial G is metabelian.*

Proof. Assume that G is not Černikov, take a finitely generated (hence finite and soluble) subgroup F of G and say E a subnormal subgroup of F. If the hypercentre of G is not Černikov we can find in it a maximal normal abelian subgroup which is direct sum of infinitely many cyclic subgroups and so, by Lemma 6.0.6, G is soluble. So we may suppose that the hypercentre of G is soluble end hence even trivial. By Theorem 1.2.5, there is an abelian subgroup A of G such that $A = \mathrm{Dr}_{i\in\mathbb{Z}} A_i$ where A_i is a E-invariant subgroup for each integer i. Clearly we can suppose that $A \cap F = \{1\}$. By the double chain condition there exists a subgroup K of A such that EK is pronormal in G. So it is pronormal (and hence normal) in FK and so we have that $E = F \cap EK$ is normal in F. Hence the finite soluble subgroup F is metabelian, and G is metabelian, too. □

Theorem 6.3.8. *Let G be a periodic locally radical DC_{np}-group. Then either G is Černikov or all subgroups of G are pronormal.*

Proof. By Proposition 6.3.6 and Lemma 6.3.7 we know that G is soluble. Suppose H, the Hirsch-Plotkin radical of G, being not Černikov and so, by Lemma

6.0.7, every subgroup of H is normal in G and every finite subgroup of G is a T-group. By Proposition 6.3.3, G is locally polycyclic, hence locally finite and G is a $\overline{T}$-group [see [45] Lemma 2.1.1, Corollary 2]. Let E be a finite subgroup of G and g an element of G, then $\langle E, g\rangle$ is still a finite subgroup of G, this way being a finite T-group, which is equal to have all subgroups pronormal. Hence there exists in $\langle E, g\rangle$ an element x such that $E^x = E^g$ so obtaining that E is pronormal in G.

Now assume by a contradiction that G contains non-pronormal subgroups. Clearly G does not satisfy Max-np, for instance by Theorem 6.1.6, and therefore it possesses a minimal non-pronormal subgroup M. M cannot be finite and so M/M' is infinite, too, from which it follows that $M/M' \simeq C_{p^\infty}$ for a prime p. Furthermore, since M is a T-group, $M/C(M')$ is residually finite and then $M' \leq Z(M)$, implying that M is nilpotent and hence of Dedekind type. Since M/M' is isomorphic with C_{p^∞}, M' has to be trivial. We notice that H clearly coincide with the Fitting subgroup of G. So, as we already pointed out, $G/C_G(H)$ is residually finite and M has to lie in $C_G(H)$. But thus we have that M is normal in MH, hence normal in G and this is a contradiction. □

The general case

Now that the periodic case is proved, we can move on to the general case. In the remainder, we are going to show some relevant properties pertaining DC$_{np}$-groups eventually concluding by proving that a radical group with our condition is either a $\overline{T}$ or a minimax group.

To now deal with the general case some inspection is needed for the case in which the presence of a particular torsion-free subgroup cannot be avoided. Here we give a slightly different version of two lemmas in Vincenzi [56].

Lemma 6.3.9. *Let G be a soluble DC$_{np}$-group and let A be a torsion-free abelian ascendant subgroup of G which is not finitely generated. Then every subgroup of A is normal in G.*

Proof. Assume by a contradiction the existence of a finitely generated subgroup H of A which is not normal in G. Clearly it is not even pronormal and so, for a proper choice of natural numbers $k_1, \dots, k_n, \dots$,

$$\dots < H^{k_n} < \dots < H^{k_1} < H$$

is a descending chain of non-pronormal subgroups of G. Hence, we can take H as a subgroup being maximal with respect to the condition of being non

G-invariant. So A is normal in G and we can still suppose that H is finitely generated. Suppose that the torsion-free rank of H, say $r_0(H)$, is strictly less then the torsion-free rank $r_0(A)$ of A. Hence, there exists an element $x \in A \setminus H$ such that $H \cap \langle x \rangle = \{1\}$. By the maximality of H we have then that

$$H = \bigcap_{n \in \mathbb{N}} \langle H, x^n \rangle$$

is normal in G, so we have, indeed, that $r_0(H) = r_0(A)$ and A has finite rank. Since A/H cannot be generated by two proper subgroups it is isomorphic to a group of type p^∞ for a prime p. If $r_0(A) = 1$, H is contained in a cyclic normal subgroup of G and hence is clearly normal in G, so we can take $H = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_m \rangle$ with $\langle a_1 \rangle$, for instance, being non-pronormal in G. Let us fix a prime $q \neq p$ and say $H_q = \langle a_1 \rangle \times \langle a_2^q \rangle \times \ldots \times \langle a_m^q \rangle$. It is easy to see that $A/H_q = H/H_q \times B_q/H_q$ where B_q/H_q is a group of type p^∞. Since each B_q is not finitely generated and contains a descending chain of non-pronormal subgroups, for instance $\ldots < \langle a_1^{k_n} \rangle < \ldots \langle a_1^{k_1} \rangle < \langle a_1 \rangle$, it is normal in G and hence $B = \bigcap_{q \neq p} B_q$ is normal in G, too. But we have that

$$B \cap H = (\bigcap_{q \neq p} B_q) \cap H = \bigcap_{q \neq p} (B_q \cap H) = \bigcap_{q \neq p} H_q = \langle a_1 \rangle.$$

Since, for each $q \neq p$, $r_0(B_q) = r_0(B_q \cap H)$, then $r_0(B) = 1$ and $\langle a_1 \rangle$ is contained in a cyclic G-invariant subgroup of B and hence it is normal in G, which is our final contradiction. □

Lemma 6.3.10. *Let G be a soluble DC_{np}-group and let A be a torsion-free abelian normal subgroup of G which is not finitely generated. Then A is contained in $Z(G)$.*

Proof. By a contradiction suppose that there exists an $x \in G$ such that $[A, x] \neq \{1\}$; this means, since by Lemma 6.3.9 every subgroup of A is normal in G, that x acts as the inversion on each element of A, then $x^2 \in C_G(A)$ and $\langle x \rangle \cap A = \{1\}$.

Take now the elements $a_1, a_2, \ldots, a_k, \ldots$ in A and construct the following chain

$$\ldots < \langle a_1^{2^k}, x \rangle < \ldots < \langle a_1^2, x \rangle < \langle a_1^2, a_2^2, x \rangle < \ldots < \langle a_1^2, a_2^2, \ldots, a_k^2, x \rangle < \ldots$$

which is a double chain for a proper choice of the elements of A. For the double chain condition on non-pronormal subgroups we find a subgroup B of A and a positive integer n such that $\langle B^{2^n}, x\rangle$ is pronormal in G. Since x^2 centralizes $\langle B, x\rangle$, then $\langle B^{2^n}, x^2\rangle$ is normal in $\langle B, x\rangle$, hence $\langle B, x\rangle/\langle B^{2^n}, x^2\rangle$ is a finite 2-group and $\langle B^{2^n}, x\rangle$ is subnormal and hence normal in $\langle B, x\rangle$. But this way we have found that

$$\langle B, x\rangle/\langle B^{2^n}\rangle \simeq \langle B^{2^n}, x\rangle/\langle B^{2^n}\rangle \times \langle B\rangle/\langle B^{2^n}\rangle,$$

which is impossible, since x acts as the inversion on each element of $\langle B\rangle/\langle B^{2^n}\rangle$. □

Lemma 6.3.11. *Let G be a DC_{np}-group with no infinite simple sections. Then either G is a $\overline{T}$-group or each ascendant abelian subgroup of G is minimax.*

Proof. Suppose G is not a $\overline{T}$-group and let A be an abelian ascendant subgroup of G and B a free abelian subgroup of A such that A/B is periodic. By Lemma 6.0.6, B is finitely generated. Suppose by a contradiction that A/B is not Min, then, if we take an integer n, A/B^{2^n} has an infinite socle which is product of infinitely many subgroups. Again by Lemma 6.0.6, A and B^{2^n} are normal in G and G/B^{2^n} is a $\overline{T}$-group; then, since G has no infinite simple sections, G/B^{2^n} is metabelian for every n, and hence G is metabelian. Let H be the Hirsch-Plotkin radical of G, which is not minimax since it contains A. By Lemma 6.3.1, it satisfies the maximal condition on non-normal subgroups and hence it is a Dedekind group, but H is not periodic and so it is abelian. If we say T the torsion subgroup of H, it has to satisfy Min, otherwise G would be abelian, and hence $H = T \times F$ where F is a torsion-free not finitely generated subgroup of H. By Lemma 6.3.10, $F \leq Z(G)$. If each subgroup of H were normal in G, then G/H as a group of power automorphisms of an abelian aperiodic group H would have order 2, containing hence an element inverting every element of H, which is impossible, since F is contained in the centre of G. Thus, there is an infinite cyclic subgroup $\langle x\rangle$ of H which is not pronormal in G. Since it contains a descending chain of non-pronormal subgroups and H is not finitely generated, we can take a subgroup M of H which is finitely generated, non-pronormal in G and maximal with respect to this condition. Clearly H/M cannot be generated by two proper non-trivial subgroups, otherwise M would be normal in G, so H/M satisfies Min and H is minimax, which is a contradiction. □

Theorem 6.3.12. *Let G be a radical DC_{np}-group with no infinite simple sections. If G is not a $\overline{T}$-group, then G is minimax.*

Proof. G is soluble by Proposition 6.3.6. The result now follows from Lemma 6.3.11 and from the fact that a soluble groups all of whose abelian ascendant subgroups are minimax is minimax [see [34] Corollary 6.3.9]. □

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