A solvable model of a tracking chamber

Claudio Cacciapuoti[1], Raffaele Carlone[2], Rodolfo Figari[2]

[1]Institut für Angewandte Mathematik der Universität Bonn.
Wegelerstrasse 6, D-53115 Bonn, Germany.

[2]Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli.
Dipartimento di Scienze Fisiche, Università di Napoli Federico II.
Via Cintia 80126 Napoli, Italy.

E-mail: caccia@na.infn.it, carlone@na.infn.it, figari@na.infn.it.

Abstract

We analyze a quantum system consisting of a particle travelling in a model-environment made up of a localised two-level subsystems. We examine the dynamics of the entanglement through which the environment acquires information about the direction of propagation of the particle. Our study reproduces, within a non perturbative investigation, an old result obtained by N. F. Mott in the early days of Quantum Mechanics.

Keywords: quantum mechanics, decoherence, dynamics of the entanglement.

Introduction

The subject of our investigation was first analysed in a paper by Sir Nevill Francis Mott published in 1929 [14].
Two years after the famed Solvay Congress of 1927, the 24 years old physicist, N. F. Mott, was wondering about the appearance of classical-like tracks in a Wilson chamber (the first tracking chamber for particle physics experiments).
According to the theory of radiating nuclei, developed by Gamow in 1928, the decay rate of an atom emitting an α-particle is well fitted by assuming that the α-particle is described by a spherical wave function. On the other hand an α-particle emitted by a nucleus manifests itself as a straight track in a Wilson chamber. The words of Mott clearly express his thought: "It is a little difficult to picture how it is that an outgoing spherical wave function can produce a straight track; we think intuitively that it should ionise atoms at random throughout the space.".
Mott guessed that the solution of the problem could be found considering "the α-particle and the gas [in the Wilson chamber] together as one system". To validate his conjecture he proposed a very simplified model in which the gas within the Wilson chamber was composed by only two hydrogen atoms. The nuclei of the atoms were considered fixed and the electrons interacted with the α-particle via Coulomb forces. Using time independent perturbation theory he showed that "the atoms cannot both be ionised unless they lie in a straight line with the radioactive nucleus".
His approach, even though not entirely rigourous from a mathematical point of view, was undoubtedly pioneering and largely outside the mainstream of ideas about the nature of the interaction of a quantum

particle with a macroscopic system. His work can be considered as the first attempt to investigate the emergence of the phenomenon nowadays referred to as *decoherence induced by the environment* (for a review see [9]).
We present a model of a tracking chamber in which the detectors are realised with spins (equivalently any two level subsystem) placed in fixed positions of space and the interaction between the α-particle and the spins is modelled by a zero range potential. The Hamiltonian is chosen among the ones characterised in [5] for the three dimensional case. The knowledge of the resolvent and of the spectrum allows to avoid perturbation theory.
In the same spirit of many works in the field (see e.g. [1, 2, 4, 6, 7, 8, 10, 11, 12]) our aim is to use solvable or almost solvable models to show that the interaction with the environment drives a quantum object to behave more classically.
We prove that given an initial state with the particle described by an outgoing spherical wave function centred in the origin and the spins both in the state down, the probability to find, as t goes to infinity, the spins both in the state up has a maximum when the positions of the spins are aligned with the origin.

1 The model

The system we consider consists of one quantum particle in $\mathbb{R}^3$ and two spins 1/2 placed in fixed positions of space, we indicate with $y_1, y_2 \in \mathbb{R}^3$ the positions of the two spins.
A spin 1/2 is described by a vector in $\mathbb{C}^2$. Then the natural Hilbert space for our system is

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 . \tag{1}$$

To define the Hamiltonian we follow what was done, in a more general setting, in [5] (see in particular the Hamiltonians defined in Example 2). For the sake of clearness in this section we recall some notation and we rephrase the results of [5] in the framework of a two spin system.
We indicate with a capital Greek letter a vector in $\mathcal{H}$; given $\Psi \in \mathcal{H}$ the following decomposition formula holds

$$\Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} , \tag{2}$$

where $\underline{\sigma}$ indicates the two-components vector $\underline{\sigma} = (\sigma_1, \sigma_2)$ with σ_1, $\sigma_2 = \pm$ and the sum runs over all the possible choices of σ_1 and σ_2. The vector $\mathcal{X}_{\underline{\sigma}}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is defined by

$$\mathcal{X}_{\underline{\sigma}} = \chi_{\sigma_1} \otimes \chi_{\sigma_2} , \tag{3}$$

where we chose χ_{σ_1} and χ_{σ_2} to be the normalised eigenvectors of the Pauli matrices, $\hat{\sigma}_j^{(1)}$, associated to the first components of the two localised spins

$$\hat{\sigma}_j^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \hat{\sigma}_j^{(1)} \chi_{\sigma_j} = \sigma_j \chi_{\sigma_j} \qquad \sigma_j = \pm , \quad j = 1, 2 . \tag{4}$$

Functions $\psi_{\underline{\sigma}}$ belong to $L^2(\mathbb{R}^3)$ for each $\underline{\sigma}$.
We indicate with $\langle \cdot , \cdot \rangle$ the scalar product in $\mathcal{H}$; it is defined, in a standard way, as

$$\langle \Psi_1, \Psi_2 \rangle := \sum_{\underline{\sigma}} (\psi_{1\underline{\sigma}}, \psi_{2\underline{\sigma}})_{L^2} ; \qquad \Psi_1,\, \Psi_2 \in \mathcal{H} . \tag{5}$$

In formula (2) the choice of $\chi_{\sigma_1} \otimes \chi_{\sigma_1}$ as a basis of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is obviously arbitrary and it was made in accordance with the particular Hamiltonian that we will use as generator of the "free" dynamics.
As it was done in [5] we start defining the self-adjoint operator H generating the free dynamics i.e. the evolution when there is neither interaction between the particle and the spins nor interaction between the spins.

Let $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined as follows

$$D(H) := H^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{6}$$

$$H := -\frac{\hbar^2}{2m}\Delta + \alpha(\hat{\sigma}_1^{(1)} + \hat{\sigma}_2^{(1)})\,; \qquad \alpha > 0\,. \tag{7}$$

here m is the mass of the quantum particle and α is a constant with the dimension of an energy. In formula (7) the Laplacian is intended to act on $\mathbb{C}^2 \times \mathbb{C}^2$ as the identity operator and the spin operators $\hat{\sigma}_1^{(1)}$ and $\hat{\sigma}_2^{(1)}$ act as the identity on $H^2(\mathbb{R}^3)$
Using the decomposition formula (2) the action of H on its domain is easily obtained

$$H\Psi = \sum_{\underline{\sigma}} \left[\left(-\frac{\hbar^2}{2m}\Delta + \underline{\sigma}\,\underline{\alpha}\right)\psi_{\underline{\sigma}}\right] \otimes \mathcal{X}_{\underline{\sigma}}\,; \qquad \underline{\sigma}\,\underline{\alpha} = \alpha(\sigma_1 + \sigma_2)\,. \tag{8}$$

The resolvent of H, $R(z) = (H - z)^{-1}$, is found to be

$$R(z)\Psi = \sum_{\underline{\sigma}} \left[\left(-\frac{\hbar^2}{2m}\Delta - (z - \underline{\sigma}\,\underline{\alpha})\right)^{-1}\psi_{\underline{\sigma}}\right] \otimes \mathcal{X}_{\underline{\sigma}}\,; \qquad z \in \rho(H)\,. \tag{9}$$

The spectrum of H can be derived from the spectrum of the free Laplacian

$$\sigma_{pp}(H) = \varnothing\,; \qquad \sigma_{ess}(H) = \sigma_{ac}(H) = [-2\alpha, \infty)\,. \tag{10}$$

The strongly continuous unitary group $e^{-i\frac{H}{\hbar}t}$ generated by H (see, e.g., Th. VIII.7 [15]) can be explicitly computed, and the solution of the Schrödinger equation

$$i\hbar\frac{d\Psi^t}{dt} = H\Psi^t \tag{11}$$

with initial data

$$\Psi^{t=0} = \Psi^0\,; \qquad \Psi^0 \in \mathcal{H} \tag{12}$$

is

$$\Psi^t = e^{-i\frac{H}{\hbar}t}\Psi^0 = \sum_{\underline{\sigma}} \left(U^t\psi^0_{\underline{\sigma}}\right) \otimes e^{-\frac{i}{\hbar}\underline{\sigma}\,\underline{\alpha}t}\chi_{\underline{\sigma}}\,, \tag{13}$$

where U^t is the integral kernel of the generator of the dynamics of a free particle of mass m

$$\left(U^t f\right)(x) = \left(\frac{2m}{\hbar}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3} \frac{e^{i\frac{m}{\hbar}\frac{|x-x'|^2}{2t}}}{(4\pi i t)^{\frac{3}{2}}} f(x')dx'\,. \tag{14}$$

In order to simplify the notation, we will fix in the following $2m = 1$ and $\hbar = 1$.
In our model of a tracking chamber the spins are detectors for the position of the particle. For this reason, among all the Hamiltonians that are point perturbations of H we chose the simplest ones generating dynamics where the interaction affects the evolution of both spins and the particle (see Example 2 of [5]). As it was shown in [5] the following operator is the resolvent of a self-adjoint operator that we will indicate with H_γ

$$R_\gamma(z) := R(z) + \sum_{j,\underline{\sigma},j',\underline{\sigma}'} (\Gamma_\gamma(z))^{-1}_{j\underline{\sigma},j'\underline{\sigma}'} \langle \Phi^{\bar{z}}_{j'\underline{\sigma}'}, \cdot\,\rangle \Phi^z_{j\underline{\sigma}}\,; \qquad z \in \mathbb{C}\backslash\mathbb{R}\,. \tag{15}$$

where the action of $R(z)$ defined in (9), can be explicitly written as

$$R(z)\Psi = \sum_{\underline{\sigma}} \int_{\mathbb{R}^3} G^{z-\underline{\sigma}\,\underline{\alpha}}(\cdot - x')\psi_{\underline{\sigma}}(x')dx' \otimes \mathcal{X}_{\underline{\sigma}} \tag{16}$$

with

$$G^z(x) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|}\,; \qquad z \in \mathbb{C}\backslash\mathbb{R}^+\,, \quad \mathrm{Im}\,\sqrt{z} > 0\,. \tag{17}$$

Vectors $\Phi^z_{j\underline{\sigma}}$ in (15) are defined by

$$\begin{aligned} \Phi^z_{j\underline{\sigma}} = G^{z-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}}\,; \qquad & j = 1,2 \\ & \underline{\sigma} = (\sigma_1, \sigma_2), \quad \sigma_1, \sigma_2 = \pm\,, \end{aligned} \tag{18}$$

it is easy to check that they are in $\mathcal{H}$. Functions $(\Gamma_\gamma(z))^{-1}_{j\underline{\sigma},j'\underline{\sigma}'}$ are the elements of a 8×8 matrix whose inverse is

$$\begin{array}{lr} (\Gamma_\gamma(z))_{j\underline{\sigma},j'\underline{\sigma}'} = 0 & j \neq j'\,; \underline{\sigma} \neq \underline{\sigma}' \\ (\Gamma_\gamma(z))_{j\underline{\sigma},j'\underline{\sigma}} = -G^{z-\underline{\sigma}\,\underline{\alpha}}(y_j - y_{j'}) & j \neq j' \\ (\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}'} = 0 & \sigma_k \neq \sigma'_k \text{ for } k \neq j \\ (\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}'} = \sigma'_j\, i\gamma & \sigma'_j \neq \sigma_j \text{ and } \sigma_k = \sigma'_k \text{ for } k \neq j \\ (\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}} = \dfrac{\sqrt{z - \underline{\sigma}\,\underline{\alpha}}}{4\pi i} & \end{array} \tag{19}$$

where γ is a positive constant. Notice that for $\gamma \to \infty$ the resolvent $R_\gamma(z)$ converges to the free resolvent, $R(z)$. In this sense γ defines the inverse coupling strength between the particle and the spins. In particular it is possible to show that $1/\gamma$ is proportional to the inverse of the effective scattering length of the potential between the particle and the spins.
With $\rho(H_\gamma)$ we indicate the resolvent set of the self-adjoint operator H_γ defined via the resolvent $R_\gamma(z)$. The domain of H_γ is

$$\begin{aligned} D(H_\gamma) := \mathrm{Ran}[R_\gamma(z)] = \Big\{ \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \chi_{\underline{\sigma}} \in \mathcal{H} : \\ \Psi = \Psi^z + \sum_{j\underline{\sigma},j'\underline{\sigma}'} (\Gamma_\gamma(z))^{-1}_{j\underline{\sigma},j'\underline{\sigma}'} \psi^z_{\underline{\sigma}'}(y_{j'}) G^{z-\underline{\sigma}\,\underline{\alpha}}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}}; \\ \Psi^z = \sum_{\underline{\sigma}} \psi^z_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in D(H)\,; z \in \rho(H_\gamma) \Big\}\,. \end{aligned} \tag{20}$$

The action of H_γ on its domain is given by the relation

$$(H_\gamma - z)\Psi = (H - z)\Psi^z\,; \qquad z \in \rho(H_\gamma)\,. \tag{21}$$

The essential spectrum of H_γ coincides with the (absolutely continuous) spectrum of H (see, e.g., Th. 4.1.4 in [3]), $\sigma_{ess}(H_\gamma) = \sigma_{ac}(H)$. The point spectrum of H_γ is given by the real solutions of the equation

$$\mathrm{Det}\Big[\Gamma_\gamma(z)\Big] = 0\,. \tag{22}$$

We stress that the Krein's resolvent formula implies that all the entries of the inverse of each $\Gamma_\gamma(z)$-matrix are analytic functions of $z \in \rho(H_\gamma)$.
From formulas (20) and (21) it is easily seen that a vector $\Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \chi_{\underline{\sigma}} \in D(H)$ such that $\psi_{\underline{\sigma}}(y_j) = 0$ $\forall \underline{\sigma}$ and $\forall j$ is also in the domain of H_γ, $\Psi \in D(H_\gamma)$, and $H_\gamma \Psi = H\Psi$. For this reason we call H_γ a point perturbation of H.
The following alternative characterisation of the domain of H_γ and of its action clarifies some details of

the interaction between the particle and the spins

$$
\begin{aligned}
D(H_\gamma) = \Big\{ \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in \mathcal{H} : \Psi = \Psi^z + \sum_{j,\underline{\sigma}} q_{j\underline{\sigma}} \frac{e^{i\sqrt{z-\underline{\alpha}\,\underline{\sigma}}|\cdot - y_j|}}{4\pi|\cdot - y_j|} \otimes \mathcal{X}_{\underline{\sigma}}; \\
\Psi^z \in D(H),\ z \in \rho(H_\gamma),\ \mathrm{Im}\sqrt{z-\underline{\alpha}\,\underline{\sigma}} > 0,\ q_{j\underline{\sigma}} \in \mathbb{C}, \\
\lim_{|x-y_1|\to 0} \Big(\psi_{(\pm,\sigma_2)}(x) - \frac{q_{1(\pm,\sigma_2)}}{4\pi|x-y_1|} \Big) = \pm i\gamma q_{1(\mp,\sigma_2)} \\
\lim_{|x-y_2|\to 0} \Big(\psi_{(\sigma_1,\pm)}(x) - \frac{q_{2(\sigma_1,\pm)}}{4\pi|x-y_2|} \Big) = \pm i\gamma q_{2(\sigma_1,\mp)} \Big\}
\end{aligned} \tag{23}
$$

$$
H_\gamma \Psi = H\Psi^z + z \sum_{j,\underline{\sigma}} q_{j\underline{\sigma}} \frac{e^{i\sqrt{z-\underline{\alpha}\,\underline{\sigma}}|\cdot - y_j|}}{4\pi|\cdot - y_j|} \otimes \mathcal{X}_{\underline{\sigma}}\,; \qquad \Psi \in D(\hat{H})\,. \tag{24}
$$

Following the standard terminology used for the point perturbations of the Laplacian we refer to the constants $q_{j\underline{\sigma}}$ as charges. Notice that

$$
q_{j\underline{\sigma}} = \lim_{|x-y_j|\to 0} 4\pi|x-y_j|\psi_{\underline{\sigma}}(x) = \sum_{j',\underline{\sigma}'} (\Gamma_\gamma(z))^{-1}_{j\underline{\sigma},j'\underline{\sigma}'}\, \psi^z_{\underline{\sigma}'}(y_{j'}) \tag{25}
$$

then $q_{j\underline{\sigma}}$ is related to the coefficient of the singular term in the point y_j of the wave function part of the state Ψ relative to the configuration of the spins defined by $\underline{\sigma}$. It is easy to convince oneself that the charges $q_{j\underline{\sigma}}$ do not depend on z.
The limits for $|x-y_j| \to 0$ in (23) define a sort of boundary conditions for the wave function part of vectors in $D(H_\gamma)$. The particular form the boundary conditions take in (23) indicate that the interaction is local, inasmuch as both conditions refer to each y_j separately.

2 Scattering theory

To analyse the state of the system when t goes to infinity we will make use of the scattering theory. In this section we introduce some notation and we state the main results about scattering theory for the pair of Hamiltonians H_γ and H.
Since $R_\gamma(z) - R(z)$ is a finite rank operator the wave operators

$$
W_\pm := \text{s - }\lim_{t\to\pm\infty} e^{iH_\gamma t} e^{-iHt} \tag{26}
$$

exist and are complete (see, e.g., [13]).

Proposition 1. *Assume that* $(|y_1-y_2|\gamma)^{-1} \ll 1$ *and* $\gamma^2 \gg \alpha$ *then there are not eigenvalues embedded in the continuous spectrum.*

Proof. By a direct calculation one can show that

$$
\mathrm{Det}\Big[\Gamma_\gamma(z)\Big] = \left(\gamma^2 + \frac{\sqrt{z}\sqrt{z+2\alpha}}{(4\pi)^2}\right)^2 \left(\gamma^2 + \frac{\sqrt{z}\sqrt{z-2\alpha}}{(4\pi)^2}\right)^2 + O\big((|y_1-y_2|\gamma)^{-2}\big)\,. \tag{27}
$$

Then the following series expansions hold for the eigenvalues, λ_1 and λ_2, of H_γ

$$
\begin{aligned}
\lambda_1 &= \alpha - \sqrt{\alpha^2 + (4\pi\gamma)^4} + O\big((|y_1-y_2|\gamma)^{-2}\big) \\
\lambda_2 &= -\alpha - \sqrt{\alpha^2 + (4\pi\gamma)^4} + O\big((|y_1-y_2|\gamma)^{-2}\big)\,,
\end{aligned} \tag{28}
$$

for $\gamma^2 \gg \alpha$ both λ_1 and λ_2 are less than -2α. □

In the following we will assume that the hypothesis of proposition 1 are satisfied, in this way we will avoid the occurrence of eigenvalues embedded in the continuous spectrum for the Hamiltonian H_γ.
With $L^2([\underline{\sigma}\,\underline{\alpha},\infty),\Omega)$ we indicate the Hilbert space with scalar product

$$(\psi_1,\psi_2)_{L^2([\underline{\sigma}\,\underline{\alpha},\infty),\Omega)} := \int_{\underline{\sigma}\,\underline{\alpha}}^{\infty} d\lambda \int_\Omega d\omega \overline{\psi_1}(\lambda,\omega)\psi_2(\lambda,\omega) \tag{29}$$

where Ω is the solid angle.
Define the map

$$\mathcal{F}_\gamma : \mathcal{H} \to \bigoplus_{\underline{\sigma}} L^2([\underline{\sigma}\,\underline{\alpha},\infty),\Omega) \tag{30}$$

$$\mathcal{F}_\gamma \Psi := \bigoplus_{\underline{\sigma}} \langle \Phi_\gamma^{\underline{\sigma}}, \Psi\rangle = \bigoplus_{\underline{\sigma}} \tilde{\psi}_\gamma^{\underline{\sigma}} \tag{31}$$

where

$$\begin{aligned}\Phi_\gamma^{\underline{\sigma}}(\lambda,\omega) =& \frac{(\lambda-\underline{\alpha}\,\underline{\sigma})^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}}\Bigg[e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega\,\cdot} \otimes \mathcal{X}_{\underline{\sigma}} + \\ &+ \sum_{j',\underline{\sigma}',j} (\Gamma_\gamma(\lambda))^{-1}_{j'\underline{\sigma}',j\underline{\sigma}} e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega y_j} \frac{e^{-i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}'}|\cdot-y_{j'}|}}{4\pi|\cdot-y_{j'}|} \otimes \mathcal{X}_{\underline{\sigma}'}\Bigg]; \quad \lambda \geq \underline{\sigma}\,\underline{\alpha}\,,\ \lambda \geq \underline{\sigma}'\,\underline{\alpha}\,,\end{aligned} \tag{32}$$

and

$$\begin{aligned}\Phi_\gamma^{\underline{\sigma}}(\lambda,\omega) =& \frac{(\lambda-\underline{\alpha}\,\underline{\sigma})^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}}\Bigg[e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega\,\cdot} \otimes \mathcal{X}_{\underline{\sigma}} + \\ &+ \sum_{j',\underline{\sigma}',j} (\Gamma_\gamma(\lambda))^{-1}_{j'\underline{\sigma}',j\underline{\sigma}} e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega y_j} \frac{e^{-\sqrt{\underline{\alpha}\,\underline{\sigma}'-\lambda}|\cdot-y_{j'}|}}{4\pi|\cdot-y_{j'}|} \otimes \mathcal{X}_{\underline{\sigma}'}\Bigg]; \quad \lambda \geq \underline{\sigma}\,\underline{\alpha}\,,\ \lambda \leq \underline{\sigma}'\,\underline{\alpha}\,,\end{aligned} \tag{33}$$

with

$$(\Gamma_\gamma(\lambda))_{j'\underline{\sigma}',j\underline{\sigma}} = \lim_{\varepsilon\to 0^+} (\Gamma_\gamma(\lambda - i\varepsilon))_{j'\underline{\sigma}',j\underline{\sigma}}\,. \tag{34}$$

Under the assumptions of proposition 1 the essential spectrum of H_γ is only absolutely continuous and coincides with $[\underline{\alpha}\,\underline{\sigma},+\infty)$. We denote by $P_{\sigma_{ac}(H_\gamma)}$ the projector on the continuous part of the spectrum of H_γ. The map $\mathcal{F}_\gamma$ is unitary on $\mathcal{H}_{ac}(H_\gamma)$, where $\mathcal{H}_{ac}(H_\gamma) = P_{\sigma_{ac}(H_\gamma)}\mathcal{H}$, and its inverse is

$$\mathcal{F}_\gamma^{-1} : \bigoplus_{\underline{\sigma}} L^2([\underline{\sigma}\,\underline{\alpha},\infty),\Omega) \to \mathcal{H}_{ac}(H_\gamma) \tag{35}$$

$$\mathcal{F}_\gamma^{-1} \bigoplus_{\underline{\sigma}} \tilde{\psi}_\gamma^{\underline{\sigma}} = \sum_{\underline{\sigma}} \int_{\underline{\sigma}\,\underline{\alpha}}^{\infty} d\lambda \int_\Omega d\omega \Phi_\gamma^{\underline{\sigma}}(\lambda,\omega)\tilde{\psi}_\gamma^{\underline{\sigma}}(\lambda,\omega)\,. \tag{36}$$

Define the map $\mathcal{F} : \mathcal{H} \to \bigoplus_{\underline{\sigma}} L^2([\underline{\sigma}\,\underline{\alpha},\infty),\Omega)$

$$\mathcal{F}\Psi := \bigoplus_{\underline{\sigma}} \langle \Phi^{\underline{\sigma}}, \Psi\rangle = \bigoplus_{\underline{\sigma}} \tilde{\psi}^{\underline{\sigma}} \tag{37}$$

where

$$\Phi^{\underline{\sigma}}(\lambda,\omega) = \frac{(\lambda-\underline{\alpha}\,\underline{\sigma})^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}} e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega\,\cdot} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \lambda > \underline{\sigma}\,\underline{\alpha}\,, \tag{38}$$

the map $\mathcal{F}$ is unitary on $\mathcal{H}$ and its inverse is $\mathcal{F}^{-1} : \bigoplus_{\underline{\sigma}} L^2([\underline{\sigma}\,\underline{\alpha}, \infty), \Omega) \to \mathcal{H}$

$$\mathcal{F}^{-1} \bigoplus_{\underline{\sigma}} \tilde{\psi}^{\underline{\sigma}} = \sum_{\underline{\sigma}} \int_{\underline{\sigma}\,\underline{\alpha}}^{\infty} d\lambda \int_{\Omega} d\omega \Phi^{\underline{\sigma}}(\lambda, \omega) \tilde{\psi}^{\underline{\sigma}}(\lambda, \omega) \,. \tag{39}$$

The wave operator W_+^{-1} is given by

$$W_+^{-1} = \mathcal{F}^{-1} \mathcal{F}_\gamma \tag{40}$$

and it is unitary from $\mathcal{H}_{ac}(H_\gamma)$ to $\mathcal{H}$. Given $\Psi \in \mathcal{H}_{ac}(H_\gamma)$, W_+^{-1} satisfies

$$\lim_{t \to +\infty} \| e^{-iH_\gamma t} \Psi - e^{-iHt} W_+^{-1} \Psi \| = 0 \,. \tag{41}$$

3 Asymptotic estimates

Consider the initial state

$$\Psi^0 = \psi^0 \otimes \mathcal{X}_{(-,-)} \,; \quad \text{with } \psi^0(x) = \psi^0(|x|). \tag{42}$$

In our setting $\psi^0(|x|)$ is a spherical wave function travelling out from the origin O.
In the spirit of the result of Mott we want to show that the large-time probability of having both spins flipped is maximal if the spins lie on a straight line passing through the origin. We assume that the initial state is orthogonal to the eigenfunctions of H_γ, i.e.,

$$P_{\sigma_{ac}(H_\gamma)} \Psi_0 = \Psi_0 \,. \tag{43}$$

Define $\Psi^t := e^{-iH_\gamma t} \Psi^0$, we denote by $\psi^t_{(+,+)}$ the function

$$\psi^t_{(+,+)} := (\Psi^t, \mathcal{X}_{+,+})_{\mathbb{C}^2 \otimes \mathbb{C}^2} \,. \tag{44}$$

The asymptotic probability to find both the spin in the state up is

$$\mathscr{P} := \lim_{t \to +\infty} \| \psi^t_{(+,+)} \|_{L^2} \,. \tag{45}$$

Let us denote by $\mathscr{B}_0(R)$ the open ball in $\mathbb{R}^3$, with center in the origin and radius R.

Proposition 2. *Take Ψ^0 like in* (42)*. Assume that Ψ^0 satisfies condition* (43) *and that* $\mathrm{supp}[\psi^0] \subset \mathscr{B}_0(R)$*, with $R < |y_j|$, $j = 1, 2$. Assume moreover that $(|y_1 - y_2| \gamma)^{-1} \ll 1$, $\gamma^2 \gg \alpha$ and $|y_2| = |y_1| + \delta$, with $\delta > 0$ and $\delta \ll |y_2|$. Then $\mathscr{P}$ has its maximum in correspondence of the minimum of $|y_1 - y_2|$.*

Proof. Define $\Psi^0_a := W_+^{-1} \Psi^0$ and $\Psi^t_a := e^{-iHt} \Psi^0_a$.
From formula (41) one obtains

$$\lim_{t \to +\infty} \| \Psi^t - \Psi^t_a \| = \lim_{t \to +\infty} \sum_{\underline{\sigma}} \| \psi^t_{\underline{\sigma}} - \psi^t_{a, \underline{\sigma}} \|_{L^2} = 0 \tag{46}$$

then

$$\lim_{t \to +\infty} \| \psi^t_{\underline{\sigma}} - \psi^t_{a, \underline{\sigma}} \|_{L^2} = 0 \qquad \forall \underline{\sigma} \,. \tag{47}$$

Since

$$e^{-iHt} \Psi^0 = \sum_{\underline{\sigma}} \left(U^t \psi^0_{\underline{\sigma}} \right) \otimes e^{-i \underline{\alpha}\, \underline{\sigma}} \mathcal{X}_{\underline{\sigma}} \,, \tag{48}$$

with $U^t = e^{-i(-\Delta)t}$, and U^t is unitary in $L^2(\mathbb{R}^3)$, by using the result stated in equation (47), we have that

$$\mathscr{P} = \lim_{t \to +\infty} \| \psi^t_{(+,+)} \|_{L^2} = \lim_{t \to +\infty} \| \psi^t_{a,(+,+)} \|_{L^2} = \| \psi^0_{a,(+,+)} \|_{L^2} = \| (W_+^{-1} \Psi^0)_{(+,+)} \|_{L^2} \,. \tag{49}$$

From the definition of W_+^{-1}

$$\|(W_+^{-1}\Psi^0)_{(+,+)}\|_{L^2} = \int_{\mathbb{R}^3} dx \left| \int_{2\alpha}^{+\infty} d\lambda \int_\Omega d\omega \phi^{(+,+)}(x;\lambda,\omega) \langle \Phi_\gamma^{(+,+)}(\lambda,\omega), \Psi^0\rangle \right|^2 \tag{50}$$

$$= \int_{2\alpha}^{+\infty} d\lambda \int_\Omega d\omega \left| \langle \Phi_\gamma^{(+,+)}(\lambda,\omega), \Psi^0\rangle \right|^2 \tag{51}$$

where $\phi^{(+,+)}$ comes from the definition of $\mathcal{F}^{-1}$, see equation (39), and

$$\phi^{(+,+)}(x;\lambda,\omega) = \frac{(\lambda-2\alpha)^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}} e^{i\sqrt{\lambda-2\alpha}\omega x}\,. \tag{52}$$

To obtain equation (51) from (50) we used the fact that

$$\int_{\mathbb{R}^3} dx \left| \int_{2\alpha}^{+\infty} d\lambda \int_\Omega d\omega \phi^{(+,+)}(x;\lambda,\omega) \tilde{f}(\lambda,\omega) \right|^2 = \int_{2\alpha}^{+\infty} d\lambda \int_\Omega d\omega \left| \tilde{f}(\lambda,\omega) \right|^2 . \tag{53}$$

From the definition of the generalised eigenfunctions (32), we have that for $\lambda \geq 2\alpha$

$$\langle \Phi_\gamma^{(+,+)}(\lambda,\omega), \Psi^0\rangle = \frac{(\lambda-2\alpha)^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}} \sum_{j,j'} \overline{\Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{j'--,j++}} e^{-i\sqrt{\lambda-2\alpha}\omega y_j} \int_{\mathbb{R}^3} dx \frac{e^{i\sqrt{\lambda+2\alpha}|x-y_{j'}|}}{4\pi|x-y_{j'}|} \psi^0(|x|)\,. \tag{54}$$

Let us pose

$$F(\lambda;|y_{j'}|) := \int_{\mathbb{R}^3} dx \frac{e^{i\sqrt{\lambda+2\alpha}|x-y_{j'}|}}{4\pi|x-y_{j'}|} \psi^0(|x|)\,. \tag{55}$$

By a direct calculation one can verify that $\Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{1--,1++} = \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{2--,2++}$ and $\Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{1--,2++} = \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{2--,1++}$. Let us denote by

$$A(\lambda) := \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{1--,1++} = \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{2--,2++} \tag{56}$$

$$B(\lambda) := \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{1--,2++} = \Big(\Gamma_\gamma(\lambda)\Big)^{-1}_{2--,1++} \tag{57}$$

Under the assumptions, $\mathrm{supp}[\psi_0] \subset \mathscr{B}_0(R)$ and $R < |y_j|$, $j = 1,2$, we obtain

$$F(\lambda;|y_{j'}|) = \frac{e^{i\sqrt{\lambda+2\alpha}|y_{j'}|}}{|y_{j'}|} f(\lambda)\,, \tag{58}$$

where

$$f(\lambda) := \frac{1}{\sqrt{\lambda+2\alpha}} \int_0^R d|x|\,|x| \sin(\sqrt{\lambda+2\alpha}|x|)\psi^0(|x|)\,. \tag{59}$$

Then

$$\begin{aligned} |\langle \Phi_\gamma^{(+,+)}(\lambda,\omega), \Psi^0\rangle|^2 = & \frac{\sqrt{\lambda-2\alpha}}{16\pi^3} |f(\lambda)|^2 \bigg| A(\lambda) \bigg(\frac{e^{-i\sqrt{\lambda-2\alpha}\omega y_1 + i\sqrt{\lambda+2\alpha}|y_1|}}{|y_1|} + \frac{e^{-i\sqrt{\lambda-2\alpha}\omega y_2 + i\sqrt{\lambda+2\alpha}|y_2|}}{|y_2|} \bigg) + \\ & + B(\lambda) \bigg(\frac{e^{-i\sqrt{\lambda-2\alpha}\omega y_1 + i\sqrt{\lambda+2\alpha}|y_2|}}{|y_2|} + \frac{e^{-i\sqrt{\lambda-2\alpha}\omega y_2 + i\sqrt{\lambda+2\alpha}|y_1|}}{|y_1|} \bigg) \bigg|^2 \end{aligned} \tag{60}$$

Under the assumption $(|y_1 - y_2|\gamma)^{-1} \ll 1$ one can see that $A(\lambda) = O\big((|y_1 - y_2|\gamma)^{-2}\big)$ while $B(\lambda) = O\big((|y_1 - y_2|\gamma)^{-1}\big)$, in particular

$$B(\lambda) = -\frac{e^{-i\sqrt{\lambda}|y_1-y_2|}(4\pi\gamma)^3}{\big((4\pi\gamma)^2 - \sqrt{\lambda}\sqrt{\lambda+2\alpha}\big)\big((4\pi\gamma)^2 - \sqrt{\lambda}\sqrt{\lambda-2\alpha}\big)} \frac{1}{|y_1 - y_2|\gamma} + O\big((|y_1 - y_2|\gamma)^{-3}\big)\,. \tag{61}$$

By assuming that $|y_2| = |y_1| + \delta$, with $\delta > 0$ and $\delta \ll |y_2|$, and taking into account (61) one obtains the following estimate for the probability $\mathscr{P}$

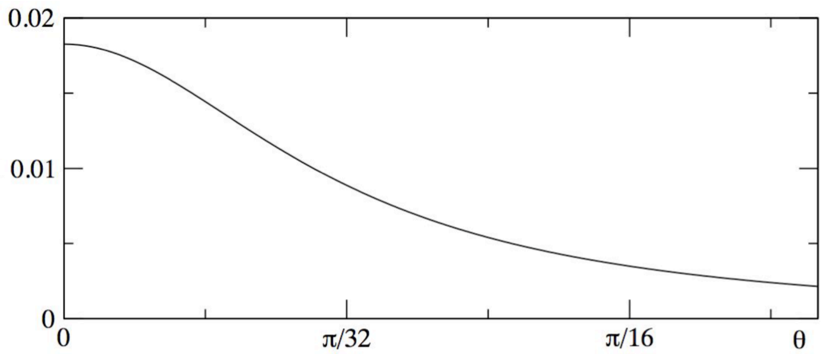
$$\begin{aligned}\mathscr{P} =& \frac{8(4\pi\gamma)^4}{|y_2|^2|y_1-y_2|^2} \int_{2\alpha}^{\infty} \frac{\sqrt{\lambda-2\alpha}\,|f(\lambda)|^2}{\big((4\pi\gamma)^2 + \sqrt{\lambda}\sqrt{\lambda+2\alpha}\big)^2\big((4\pi\gamma)^2 + \sqrt{\lambda}\sqrt{\lambda-2\alpha}\big)^2} \times \\ &\times \left(1 + \cos(\sqrt{\lambda+2\alpha}\delta)\frac{\sin(\sqrt{\lambda-2\alpha}|y_1-y_2|)}{\sqrt{\lambda-2\alpha}|y_1-y_2|}\right) d\lambda + O\big((|y_1-y_2|\gamma)^{-3}, (\delta/|y_2|)\big)\,.\end{aligned} \tag{62}$$

The statement of the proposition follows from the fact that the function

$$\frac{1}{L^2}\left(1 + \cos(\sqrt{\lambda+2\alpha}\delta)\frac{\sin(\sqrt{\lambda-2\alpha}L)}{\sqrt{\lambda-2\alpha}L}\right) \tag{63}$$

is decreasing in L. □

Proposition 2 indicates that if the positions of the spins are such that $|y_2| = |y_1| + \delta$ with $\delta > 0$ and $\delta \ll |y_2|$ then the probability to find both the spins in the state up has a maximum when their distance $|y_1 - y_2|$ is minimum. In particular this indicates that if the distances $|y_1|$ and $|y_2|$ are fixed the configuration in which the probability $\mathscr{P}$ has a maximum corresponds to the configuration in which the spin are aligned with the origin.



In figure it is plotted the probability $\mathscr{P}$ when the initial state is of the form

$$\psi^0(|x|) = N\frac{e^{-\frac{|x|^2}{2s^2}+ik_0|x|}}{|x|} \qquad N = \frac{1}{\sqrt{2s}\pi^{\frac{3}{4}}}\,, \tag{64}$$

with $s > 0$ and $k_0 > 0$, in a setting in which $|y_1|$ and $|y_2|$ are fixed. On the x-axis of the plot there is the angle θ between y_1 and y_2, the plot clearly shows that the probability has a maximum when $\theta = 0$.

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