On the Lefschetz properties

Fulvio Maddaloni

Introduction

The study of Artinian rings is at the outset made difficult by their apparent simplicity and therefore lack of obvious invariants. Nonetheless there are good grounds to be interested in Artinian rings, for the reasons that can be resumed in the following points:

1. many problems of local rings often "reduce" to problems of Artinian rings;

2. the classification of standard graded Artinian Gorenstein algebras was the main problem of classical invariant theory.

The viewpoint (1) is traditional and well understood, while the authors, in [16], give us less known aspects of the theory of Artinian rings encompassed by point (2).
The Weak and Strong Lefschetz properties are strongly connected to many topics in algebraic geometry, commutative algebra and combinatorics. Some of these connections are quite surprising and still not completely understood and much work remains to be done. The article [27] is an expository paper where the authors give an overview of known results on the Weak and Strong Lefschetz properties, illustrating the variety of methods and connections that have been used to bear on this problem for different families of algebras.
We can see that many results are collected by D. Cook II and U. Nagel (see [7] and [8]) by algebraic viewpoint. There is also a surprising relation to Laplace equations discovered E. Mezzetti, R. M. Mirò-Roig and G. Ottaviani (see [23]), I will talk about [Chapter 1, Section 1.3].
In algebraic geometry, the Lefschetz Properties for standard graded Artinian Gorenstein algebra are related with the vanishing of higher Hessians. These

are very important results, given by R. Gondim, who meticulously discusses about the linking between the Lefschetz properties and vanishing of higher Hessian. In his paper [13], he constructs for each pair $(N, d) \neq \{(3,3),(3,4)\}$ standard graded Artinian Gorenstein algebras A, of codimension $N+1 \geq 4$ and with socle degreee $d \geq 3$ for which the Strong Lefschetz property does not hold, failing at an arbitrary step k with $2 \leq k < \lfloor \frac{d}{2} \rfloor$.
Very recently the Lefschetz properties have been found to have new connections with combinatorics; in fact in [14] the authors have used a strategy to construct standard graded Artinian Gorenstein algebras presented by quadrics, proposing the simplest ones, whose ideal contains the complete intersection $(x_1^2, \dots, x_n^2)$. R. Gondim and G. Zappalà deal with bihomogeneous polynomials in $\mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(1,d-1)}$ of special type, called of monomial square free type. They associate to any bihomogeneous polynomial of bidegree $(1, d-1)$ of monomial square free type, bijectively, a pure simplicial complex, that determines a set of generators of the annihilator ideal. By the combinatorial structure of the simplicial complex, the Hilbert vector of such Artinian algebra is determined by the Hilbert vector of the complex.

Now I describe the contents of this thesis in more detail:

- in the chapter 1, there is a preliminary section, with basic definitions of graded rings. I will specify when an Artinian graded $\mathbb{K}-$ algebra satisfies the Weak and Strong Lefschetz property respectively. The last section of this chapter is very important since I will remind some results about them, showing some conditions when an Artinian graded $\mathbb{K}-$algebra satisfies these properties. Moreover I will describe the relation between the Weak Lefschetz property and the Sperner property. The topics are discussed widely with many examples;

- in the chapter 2, I will explain the relation between the Weak and Strong Lefschetz property and higher Hessian; in particular I will talk about GNP- hypersurfaces of type (m, n, k, e), introduced in [13] by R. Gondim. For these hypersurfaces, $\mathrm{hess}_f^k = 0$, so I will connect to problems related to the hypersurfaces with vanishing Hessian, studied by O. Hesse for the first time (see [19] and [20]). Finally I will conclude this chapter by constructing Artinian Gorenstein algebras for which the

Lefschetz properties do not hold;

- in the chapter 3, there are new main results I have found. In fact I will discuss about some problems that can be reassumed:

  Problem 1. To describe all possibles Hilbert vectors of GNP- algebras.

  Problem 2. To describe the annihilator I of GNP- algebras.

  Problem 3. To characterize the geometry of GNP- hypersurfaces.

  For the first and second problem, following the same strategy in [14], I want to characterize the GNP- algebras A. The GNP- algebras are standard bigraded Artinian Gorenstein algebra of GNP-polynomials of type (m, n, k, e). In detail I want to focus on the GNP- polynomials of type $(m, n, k, k+1)$ that are particular bihomogeneous forms in $\mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ and I deal with the case whose they are of monomial square free type. I will associate to any GNP- polynomial of this type in $\mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ a simplicial complex whose vertices are identified by the variables $u_1, \dots, u_m$. I will describe the Hilbert vector of A and the annihilator of a GNP-polynomial of type $(m, n, k, k+1)$. I will introduce standard bigraded artinian Gorenstein algebras, bihomogeneous polynomial of $\mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(d_1,d_2)}$. Moreover I will introduce the definition of simplicial complex and give a characterization of Hilbert vector and annihilator of an artinian Gorenstein graded $\mathbb{K}$-algebra. In the end of this chapter I will show some geometric properties of a GNP- hypersurface of type (m, n, k, e). In particular I will prove that GNP- hypersurfaces of type (m, n, k, e) consist of unions of residue parts, obtained by the intersection between the hypersurfaces and the linear space $\mathbb{P}^{n+1}$.

In conclusion I would thank Giovanna Ilardi and Pietro De Poi, that have followed me in this course of study. They have been zealous in their job and they have been mentors for me for their professionalism. Moreover I would thank Rodrigo Gondim for his suggestions and conversations; he has been illuminating for me in many topics of algebraic geometry and commutative algebra.

Contents

Chapter 1

Lefschetz Properties

1.1 Preliminaries

As usual in commutative algebra all rings are commutative with unity and every homomorphism sends the unity in unity.

Definition 1.1. A is a **graded ring** if A is a ring and there exists a family $(A_n)_{n\geq 0}$ of subgroups of $(A,+)$ such that $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_nA_m \subseteq A_{n+m}$, $\forall m, n \geq 0$.
We can notice that the "graduation" over a ring can be more generally defined over integers; so we have $A = \bigoplus_{n\in\mathbb{Z}} A_n$. Moreover, according to both graduations, we get that A_0 is a subring of A and each A_n is a A_0-module (see below for definition of module over a ring). In particular each $a_i \in A_i$ is said to be a **homogeneous element of degree** i.

Let $(M,+)$ be an abelian group, and let A be a (commutative) ring. We say M is an **A-module** if the pair (M,μ), where μ is a map $\mu : A \times M \to M$ defined by $\mu(a,x) = ax$, for all $a \in A$ and $x \in M$, satisfies the following axioms:

1. $a(x+y) = ax + ay$, for all $a \in A$ and $x, y \in M$;

2. $(a+b)x = ax + bx$, for all $a, b \in A$ and $x \in M$;

3. $(ab)x = a(bx)$, for all $a, b \in A$, and $x \in M$;

4. $1x = x$, for all $x \in m$.

In this thesis all rings will be assumed to be graded over positive integers; otherwise we refer, in particular cases, to other kind of graduation.

Definition 1.2. Let $f : A \to B$ be a ring homomorphism. If $a \in A$, $b \in B$ let us define the following product

$$ab = f(a)b.$$

With this definition B becomes an $A-$module. So B is a ring and it simultaneously is also an A-module and both structures coexist together. We call B an **A-algebra**.
If $A = \mathbb{K}$ is a field, B will be called $\mathbb{K}-$**algebra** (notice f is injective and $\mathbb{K}$ is identified as a subring of B).
Let A be a $\mathbb{K} - algebra$. A is called **graded**, as algebra over $\mathbb{K}$, if A is a graded ring.

Example: $R = \mathbb{K}[x_0, \dots, x_n]$ is a graded $\mathbb{K}-$algebra. Indeed let $(R_i)_{i \geq 0}$ be the subgroups of homogeneous polynomial of degree i in the variables $x_0, \dots, x_n$, then $R = \mathbb{K}[x_0, \dots, x_n] = \bigoplus_{i=0}^{\infty} R_i$.

Definition 1.3. The **Krull dimension** of a ring A is the supremum of an ascending sequence of distinct prime ideals; namely

$$\sup_{n \in \mathbb{N}} \{p_0 \subsetneq \cdots \subsetneq p_n : p_i \text{ is a prime ideal of } A\}.$$

Definition 1.4. An ideal $I \subseteq R$, where $R = \mathbb{K}[x_1, \dots, x_r]$ is the polynomial ring over the field $\mathbb{K}$, is **homogenous** if and only if $I = \bigoplus_{i \geq 0} (I \cap R_i)$; i.e. if every homogenous component of an element of I lies also in I. In particular we say I is an **artinian** ideal of R if R/I has Krull dimension 0.

Definition 1.5. Let A be a ring. We call A **artinian** if every descending sequence of ideals of A is stationary; i.e. let

$$I_0 = A \supseteq I_1 \supseteq \dots$$

be a descending sequence of ideals in A, then there exists an integer $m \in \mathbb{N}$ such that $I_n = I_m$ for every $n > m$. The same holds if we consider A, an artinian $\mathbb{K}-$algebra; in fact it is in particular an artinian ring. Moreover if it is also graded $\mathbb{K}-$algebra, all ideals in the above sequence are meant to be homogeneous ideals.

This means that a graded artinian $\mathbb{K}-$algebra has a finite graduation; i.e. $A = \bigoplus_{i=0}^{c} A_i$. From now on we set $A_c \neq 0$.

Notice that:
Every finitely generated $\mathbb{K}-$algebra A is a quotient of $\mathbb{K}[x_0, \ldots x_n]$ by an ideal I. Furthermore if A is graded, as finitely generated $\mathbb{K}-$algebra, then it is a quotient of $\mathbb{K}[x_0, \ldots x_n]$ by a homogeneous ideal I.

Remind that, by an algebraic context, in this thesis all ideals are written between $\langle \quad \rangle$. Moreover, when we refer to the homogeneous maximal ideal of a local graded ring A, we will usually denote it with $\mathfrak{m}$; while if we refer to the case of an artinian graded $\mathbb{K}-$algebra $A = \bigoplus_{i=0}^{c} A_i$ we will usually denote it with m, and it will stand for $\bigoplus_{i=1}^{c} A_i$.

Definition 1.6. A ring A is **noetherian** if every ascending sequence of ideals is stationary; i.e. there exists $m \in \mathbb{N}$ such that,

$$I_0 = \langle 0 \rangle \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \ldots,$$

$I_n = I_m = \ldots$, for all $n > m$.

Remark 1. Let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$. A graded $\mathbb{K}$-algebra $A = R/I = \bigoplus_{i=0}^{c} A_i$ is artinian if and only if I is an artinian ideal of R if and only if $\exists k \in \mathbb{N} : I \supset m^k$, where $m = \bigoplus_{i=1}^{c} A_i$ is the homogeneous maximal ideal of A (also called "irrelevant" ideal). Equivalently a ring A is artinian if and only if it is noetherian and its Krull dimension is zero; or A is artinian, as graded $\mathbb{K}-$algebra, if and only $I \subset A$ homogeneous ideal is artinian.

Definition 1.7. Let P be a prime ideal of a ring A, we call **height of** P, the sup of an ascending sequence of prime ideals contained in P; i.e.

$$\sup_n \{P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P\}.$$

While the **height of** I, $I \subset A$ a proper ideal of A, is the minimum of heights of the prime ideals containing I.

Definition 1.8. Let $I, J \subset A$ be two ideals of the ring A (commutative not necessarily with unity). Let us define a new ideal of A

$$\langle I : J \rangle := \{x \in A : xJ \subset I\},$$

it is called the **quotient ideal**. In particular the **annihilator** of J is the following ideal

$$\langle 0 : J \rangle := \{x \in A : xJ = 0\}$$

of A.

Definition 1.9. Let $A = A_0 \oplus \cdots \oplus A_n \oplus \ldots$ be a graded ring, then a **graded module** over A is a module M with a decomposition

$$M = \bigoplus_{-\infty}^{+\infty} M_i$$

as abelian groups such that $A_i M_j \subset M_{i+j}$ for all i, j.

Definition 1.10. Let A be a graded ring and let M be a graded $A-$module. We call $n-$**shift of** M, and we denote it with $M(n)$ where $n \in \mathbb{Z}$, the graded $A-$module such that $M(n)_d := M_{n+d}$.

Now we present two definitions which are the core of this thesis; namely the weak and strong Lefschetz properties. At the beginning they are just presented as notions but going on we will characterize them.

1.2 Weak and Strong Lefschetz properties (WLP & SLP)

Now let us start with two definitions, namely the weak and strong Lefschetz properties, on which we construct whole thesis. At the beginning they are merely presented as notions; after we are going to study them through some nice properties that involve the standard graded artinian $\mathbb{K}-$algebras.

Definition 1.11. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over the field $\mathbb{K}$. Let us consider an artinian graded $\mathbb{K}-$algebra $A = R/I = \bigoplus_{i=0}^{c} A_i$, where $I \subset R$ is a homogeneous ideal.
We say that A has the **weak Lefschetz property (WLP)** if there exists a linear form $L \in R_1$ such that the multiplication map

$$\times L : A_i \to A_{i+1}$$

has maximal rank for all $0 \le i \le c-1$. $L \in A_1$ is called a **weak Lefschetz element**.
We say that A has the **strong Lefschetz property (SLP)** if there exists a linear form $L \in R_1$ such that the multiplication map

$$\times L^d : A_i \to A_{i+d}$$

has maximal rank for all $0 \le i \le c-1$, and $1 \le d \le c-i$. L is called a **strong Lefschetz element**.

Remark 2.

1. It is clear the SLP is stronger than the WLP. But when $d = 1$ we have that the strong Lefschetz property coincides with the weak Lefschetz property.

2. When we consider a Lefschetz element, it is always meant to be general. If we think about the word "general", it refers to a Zariski open set in $R_1 = \mathbb{K}[x_1, \dots, x_n]_1$. More precisely we can prove that all Lefschetz elements (weak or strong) lie in a Zariski open set of R_1. In fact let us consider

$$\left\{ L \in R_1 : L \text{ is a (Weak or Strong) Lefeschetz element of } A = \bigoplus_{i=0}^{c} A_i \right\}.$$

We want to prove it is a Zariski open set in R_1. In order to do this let us consider the complement of the previous set, that is the locus of all linear elements in R_1 which are not Lefschetz elements. Each of them induces through multiplication on A_i

$$\times L : A_i \to A_{i+1}$$

matrices for all $i \leq c$. Then it follows it is the locus of zeros of determinants of the induced matrices. This means that the complement of the above set is closed in R_1, and so the original set defined above is open.

1.3 The Lefschetz properties and the link with Laplace equations of order $s \geq 2$

In a recent paper Mezzetti, Mirò-Roig, and Ottaviani [11], highlight the link between rational varieties satisfying a Laplace equation and artinian ideals failing the WLP. Continuing their work Di Gennaro, Ilardi and Vallés [11], extend results to a more general situation of the artinian ideals failing the SLP. They characterize the failure of the SLP (which includes the WLP) by the existence of special singular hypersurfaces. Other authors such as Brenner and Kaid [6] proved that, over an algebraic closed field $\mathbb{K}$ with characteristic zero, any homogeneous artinian ideal of the form $\langle x^3, y^3, z^3, f(x,y,z)\rangle$, such that $\deg(f(x,y,z)) = 3$ fails the WLP if and only if $f \in \langle x^3, y^3, z^3, xyz\rangle$, and this ideal is the only one that fails the WLP. The link with Laplace equations of order $s \geq 2$ comes from a Togliatti's work in which he proved that there is only one nontrivial example of a smooth surface $X \subset \mathbb{P}^5$ obtained by projecting the Veronese surface $V(2,3) \subset \mathbb{P}^9$ satisfying a single Laplace equation of order 2; X is projectively equivalent to the image of $\mathbb{P}^2$ via the linear system $\langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2\rangle \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$. And this is just Brenner and Kaid's example; in fact the hypersurface studied by Togliatti corresponds to the ideal failing the WLP. So in Di Gennaro, Ilardi, Vallés's work we can find out new examples of artinian ideals which do not satify the SLP and they arise just by Togliatti's example.

1.4 A short collection of some results about the WLP and SLP

We can start from [27] which focuses on approach adopted by many researchers in order to prove the failure or holding of one of these properties. Now we give some useful results characterizing the WLP and SLP properties. Let us start with a general result about the SLP, assuming the field $\mathbb{K}$ has characteristic zero. In the following we are going to see the characteristic can be assumed not necessarily being zero.

We observe that, if $R = \bigoplus_{i=0}^{\infty} R_i$ is a graded ring over a field $\mathbb{K} = R_0$, then

$$\mathfrak{m} = \bigoplus_{i \geq 1} R_i$$

is the unique homogeneous maximal ideal of R. It is often the case that a proposition, which holds for a local ring, can be translated for a graded ring and homogeneous ideals. In particular we are going to examine case by case in each assertion.

Definition 1.12. A graded algebra $R = \bigoplus_{i=0}^{\infty} R_i$ over the field $\mathbb{K}$, is called **standard** if $R = \mathbb{K}[R_1]$; i.e. if it is generated by elements of degree one as an algebra over $\mathbb{K}$.

Proposition 1.4.1. *Let $\mathbb{K}$ be a field. A noetherian ring A is a standard graded $\mathbb{K}-$algebra iff $A \simeq R/I$, where R is the polynomial ring over $\mathbb{K}$ in* $\dim_{\mathbb{K}} A_1$ *variables and I is a homogeneous ideal.*

Definition 1.13. Let $R = \mathbb{K}[x_1, \ldots, x_r] = \bigoplus_{i \geq 0} R_i$ be the graded $\mathbb{K}-$ algebra of polynomials in $r-$variables over $\mathbb{K}$, where $\mathbb{K} = R_0$ is a field, and let

$$A = R/I = \bigoplus_{i=0}^{c} A_i$$

be a standard graded artinian ring; i.e A is obtained by the quotient of R by I, where I is a homogenous ideal of R. Put

$$h(A, i) := \dim_{\mathbb{K}}(A_i) = \dim_{\mathbb{K}} R_i - \dim_{\mathbb{K}} I_i.$$

The numerical function $\mathbb{N} \to \mathbb{N}$ such that $i \mapsto h(A,i)$ is called the **Hilbert function** or **Hilbert sequence** or **Hilbert series** of A. We call

$$\mathrm{Hilb}(A,t) = \sum_{i=0}^{c} h(A,i)t^i,$$

the **Hilbert polynomial** of A.

Definition 1.14. A sequence of positive integers $(h_0, \ldots, h_n)$ is said to be **unimodal** if there exists j such that

$$\begin{cases} h_i \leq h_{i+1} & (i < j) \\ h_i \geq h_{i+1} & (i \geq j). \end{cases}$$

Furthermore we call $(h_0, \ldots, h_n)$ **strictly unimodal** if it is unimodal and in addition when it starts to decrease it is strictly decreasing until reaches zero.

Definition 1.15. An $\mathcal{O} -$ *sequence* is a sequence of positive integers that occurs as the Hilbert function of some graded algebra. We usually denote it by $\{h_i : i \geq 0\}$.

Definition 1.16. Given a non-negative series of positive integers $c := \{h_i : i \geq 0\}$, we call the **first difference** of this sequence the sequence $\Delta c := \{b_i : b_i = h_i - h_{i-1}\}, \forall i$. By convention we set $b_0 = 1$.

Theorem 1.4.2 ([27],Theorem 1.1)**.** *Let $R = \mathbb{K}[x_1, \ldots, x_r]$, where $\mathbb{K}$ is a field of characteristic zero. Let I be an artinian monomial complete intersection ideal, i.e.*

$$I = \langle x_1^{a_1}, \ldots, x_r^{a_r} \rangle.$$

Let L be a general linear form. Then for any positive integers d and i, the homomorphism induced by multiplication by L^d,

$$\times L^d : [R/I]_i \to [R/I]_{i+d}$$

has maximal rank, i.e. R/I holds the SLP. In particular, this is true for $d = 1$.

As we can see Theorem 1.4.2 is linked to characteristic of the field $\mathbb{K}$. In the following, we will prove that the WLP does not hold for a monomial artinian standard graded $\mathbb{K}-$algebras with $\operatorname{char}\mathbb{K} = p$, where p is a prime and supposing some hypothesis on p. After that we can just prove that the SLP holds whenever we consider that A has codimension two; moreover we will prove that the WLP holds for A with any characteristic. Furthermore always considering I generated by monomials we can prove the following:
if WLP holds for A in characteristic of $\mathbb{K}$ egual to zero then it holds the WLP in a sufficiently large characteristic.
Moreover of this last statement, we prove a sort of converse, that affirms:
if A holds the WLP in characteristic p, where p is a prime, then WLP also holds in characteristic zero.
Now it is time to pass at the exhibition of what we anticipated.

Definition 1.17. We call $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ a **monomial** ideal if I is generated by monomials of R.

Lemma 1. *Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over the field $\mathbb{K}$. Then the ideals J and J' generated by, respectively*

$$x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}(a_1x_1 + \cdots + a_{n-1}x_{n-1})^{j_n}$$

and

$$(a_1x_1)^{j_1} \cdots (a_{n-1}x_{n-1})^{j_{n-1}}(-a_1x_1 - \cdots - a_{n-1}x_{n-1})^{j_n},$$

assuming each a_i, $\forall i$, is not zero, are equals.

Proof. We have, by assumption, J and J' are generated by $x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}(a_1x_1 + \cdots + a_{n-1}x_{n-1})^{j_n}$, and by $(a_1x_1)^{j_1} \cdots (a_{n-1}x_{n-1})^{j_{n-1}}(-a_1x_1 - \cdots - a_{n-1}x_{n-1})^{j_n}$ respectively. The generators are equal, since each $a_i \in \mathbb{K}$ for all $i \leq n$ is not zero (since $\mathbb{K}$ is infinite) and by calculation we can transform the generators of J in generators of J' just only multiplying by $(-1)^{j_n}(a_1^{j_1} \cdots a_{n-1}^{j_{n-1}})$. $\square$

Remark 3. In Lemma 1, the characteristic can be also taken $p > 0$, with p prime. In fact the a_i, for all $i \leq n$, are in $\mathbb{Z}_p$ and we must suppose for calculations that each of them is not zero *mod.p.* While if $\operatorname{char}\mathbb{K}$ is zero then we can take a_i, for all i, is not zero since $\mathbb{K}$ is infinite.

Proposition 1.4.3 ([24], Proposition 2.2)**.** *Let $I \subset R$ be an artinian monomial ideal, with R the polynomial ring over the field $\mathbb{K}$ in variables $x_1, \ldots, x_n$, and assume that the field $\mathbb{K}$ is infinite. Then R/I has the WLP if and only if $x_1 + \cdots + x_n$ is a weak Lefschetz element for R/I.*

Proof. Set $A = R/I$ and let us consider a general linear form $L = a_1x_1 + \cdots + a_nx_n$ in $R = \mathbb{K}[x_1, \ldots, x_n]$. We may assume that each a_i is not zero (since $\mathbb{K}$ is infinite and L is general). Thus we can set $a_n = 1$. After we consider $J \subset S = \mathbb{K}[x_1, \ldots, x_{n-1}]$ an ideal generated by elements obtained by the minimal generators of I after substituting $a_1x_1 + \cdots + a_{n-1}x_{n-1}$ to x_n. In this way we get $A/LA \simeq R/(I, L) \simeq S/J$, because each element of A/LA now is in variable $x_1, \ldots, x_{n-1}$ and $x_n = a_1x_1 + \cdots + a_{n-1}x_{n-1}$, with each $a_i \neq 0$ for every $i \leq n-1$. Thus if we exhibit the minimal generators of J they are of following form

$$x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}(a_1x_1 + \cdots + a_{n-1}x_{n-1})^{j_n}.$$

Replacing each of them by $(a_1x_1)^{j_1} \cdots (a_{n-1}x_{n-1})^{j_{n-1}}(-a_1x_1 - \cdots - a_{n-1}x_{n-1})^{j_n}$, the ideal J does not change, by lemma 1. Knowing now that there exists an isomorphism between $\mathbb{K}[y_1, \ldots, y_{n-1}]$ and S which sends y_i in a_ix_i, for every $i \leq n-1$ we have $(*) A/LA \simeq S/J \simeq \mathbb{K}[y_1, \ldots, y_{n-1}]/J$. This enables us to consider $A/(x_1 + \cdots + x_n)A$, and obtaining the same Hilbert sequence of A/LA. Thus we can decide whether L is a Lefschetz element for A just looking to the Hilbert sequence of A/LA; and this is sufficient to complete the proof. $\square$

Definition 1.18. Let S be a commutative ring, with char $S = p$, with $p > 0$ a prime. We call $F : S \to S$, $s \mapsto s^p$, **Frobenius endomorphsim** of S. It has the following property: thanks to Newton's binomial theorem in prime characteristic it follows

$$F(s + \tilde{s}) = (s + \tilde{s})^p = s^p + \tilde{s}^p,$$

furthermore

$$F(s \cdot \tilde{s}) = (s \cdot \tilde{s})^p = s^p \cdot \tilde{s}^p.$$

Lemma 2 ([7], Lemma 2.5)**.** *Let $A = R/I$ be an artinian standard graded $\mathbb{K}-$algebra such that I is generated by monomials. Let a be the least integer such that $x_i^a \in I$ for each i, and suppose the Hilbert sequence of A weakly increases till to degree $s+1$. Then for any prime p, such that $a \leq p^m \leq s+1$, for some positive integer m, A does not satisfy the WLP in characteristic p.*

Proof. We can assume $L = x_1 + \cdots + x_n$ as general linear form of R thanks to proposition 1.4.3 and remark 3, i.e. we can think it as "general" because it behaves as a general linear form. If we assume now $\operatorname{char} \mathbb{K} = p$ to be prime, by Frobenius endomorphism, $L \cdot L^{p^m-1} = L^{p^m} = x_1^{p^m} + \cdots + x_n^{p^m}$. As $a \leq p^m$ (by hypothesis) and by hypothesis on a we get in A that $L^{p^m} = 0$ while $L \neq 0$. Hence this implies

$$\times L^{p^m-1} : A_1 \to A_{p^m-1}$$

is not injective since $L \in A_1 \mapsto L \cdot L^{p^m-1} = L^{p^m} = 0$. Thus A does not hold the WLP. $\square$

Theorem 1.4.4 ([7], Lemma 2.6)**.** *Let I be an artinian monomial ideal in R, the polynomial ring over the field $\mathbb{K}$. If the artinian standard graded $\mathbb{K}-$algebra R/I has the WLP in $\operatorname{char} \mathbb{K} = 0$, then it has the WLP in $\operatorname{char} \mathbb{K}$ sufficiently large.*

Proof. We can assume $L = x_1 + \cdots + x_n$ as general linear form of R thanks to proposition 1.4.3 and remark 3, i.e. we can think it as "general" because it behaves as a general linear form. As $A = R/I$ is artinian, there are finitely many maps to be checked in order to prove the WLP; this means there are finitely many determinants of the matrices associated to each map to be checked. In particular since L is $x_1 + \cdots + x_n$, the matrices are with entries one and zero; this is true since the algebra is monomial and L is $x_1 + \cdots + x_n$. Thus it implies the determinants to be computed are integers. Consider p, the smallest prime larger than all prime divisors of determinants of the matrices; in this way all determinants are non-zero modulo p and so R/I holds the WLP whenever we assume $\operatorname{char} \mathbb{K} \geq p$. $\square$

A converse of the latter result is given in this form:

Theorem 1.4.5 ([7], Lemma 2.7). *Let I be an artinian monomial ideal. If R/I satisfies the WLP in* $\operatorname{char}\mathbb{K} = p > 0$ *prime, then R/I satisfies the WLP in* $\operatorname{char}\mathbb{K} = 0$.

Proof. The proof is the same as that of Theorem 1.4.4, keeping also in mind Remark 3; except we notice that if d is an integer and it is non-zero modulo a prime p, then d is not zero. $\square$

Let us give some useful definitions which are used to prove an important concerning artinian standard graded $\mathbb{K}-$algebras in codimension two which satisfy the SLP.

Definition 1.19. Let $I \subset R$ be a monomial ideal, where $R = \mathbb{K}[x_1, \ldots, x_r]$ is the polynomial ring over the field $\mathbb{K}$. I is **stable** if it satisfies the following condition: for each monomial $u \in I$ and for each $i < m(u) := \max\{j : x_j \text{ divides } u\}$, the monomial $(x_i\ u)/x_{m(u)}$ belongs to I.
In addition I is said to be **strongly stable** if it satisfies the following condition: for each monomial $u \in I$ and for every j satisfying $x_j \mid u$, the monomial $(x_i\ u)/x_j$ belongs to I for every $i < j$. This condition is also called **Borel-fixed** when $\mathbb{K}$ has zero characteristic, and the ideal I will be called a **Borel-fixed ideal** (also in positive characteristic).

Example 1.1. The ideal $I = \langle x_1^2, x_1x_2, x_2^3\rangle \subset \mathbb{K}[x_1, x_2, x_3]$ is stable, in particular it is also strongly stable. In fact both properties are equivalent since the only one possibility of verifying the property (being stable or strongly stable) is assuming $x_1 < x_2$. So we have to verify one of them just forx_1 and x_2. Let us prove it is stable. In fact if $x_1^2 = u$ the index $m(u) = 1$ so there is nothing to prove. While for $j = 2$ we have $u = x_1x_2$ and $m(u) = 2$, so by it follows $(x_1 \cdot u)/x_2 = x_1^2 \in I$. Finally if $u = x_2^3$ $m(u) = 2$, then $(x_1 \cdot u)/x_2 = x_1x_2^2 \in I$.
While if we consider the ideal

$$J = \langle x_1^2, x_1x_2, x_2^2, x_2x_3\rangle,$$

it is stable but not strongly; in fact if we replace x_2 with x_1 in the monomial $x_2x_3 \in J$ we get $x_1x_3 \notin J$.

Definition 1.20. A **monomial order** on $R = \mathbb{K}[x_1, \ldots, x_r]$ is a total order $\leq$ on monomials of form $x^\alpha, \alpha \in \mathbb{N}^n$ such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$

- $x^\alpha \geq x^\beta \Rightarrow x^{\alpha+\gamma} \geq x^{\beta+\gamma}$;
- $\leq$ is a well ordering.

In particular a **term order** $\leq$ is an order over the terms of polynomials in R, where the word *term* refers to the monomials composing polynomials.

Definition 1.21. Let $<$ be a term order on $R = \mathbb{K}[x_1, \ldots, x_r]$. We fix one such that $x_1 > x_2 > \cdots > x_r$, e.g. the lexicographic order.

1. For an ideal I of R, the **initial ideal** $\text{in}_<(I)$ of I is the ideal of R generated by the initial terms $\text{in}_<(f)$ of $f \in I$, where the initial term $\text{in}_<(f)$ is the greatest term in f with respect to the term order $<$. We also denote the initial ideal simply by $\text{in}(I)$.

2. We have a natural action of general linear group $GL(n, \mathbb{K})$ given by: $f \in R$ and $g \in GL(n, \mathbb{K})$ (the field $\mathbb{K}$ can be finite or infinite), the action is the multiplication $gf(x_1, \ldots, x_n) = f(gx_1, \ldots, gx_n)$ where $gx_i = \sum_{j=1}^n g_{ij}x_j$. Let I be a homogeneous ideal of R. Then there exists a Zariski open subset $U \neq \emptyset$ in $GL(n, \mathbb{K})$ and a monomial ideal J ([9], chapter 15, section 15.9) of R such that for all $g \in U$ $\text{in}(gI) = J$, where gI is the ideal given by $gp, \forall p \in I$. We call J the **generic initial ideal** of I with respect to the term order $<$, written $J = \text{gin}_<(I)$. Also in this case we can assume $\text{gin}(I)$ rather than $\text{gin}_<(I)$.

Let $\text{char}\mathbb{K}$ be zero. So we have:

Theorem 1.4.6 (Galligo, Bayer-Stillman;[16])**.** *Generic initial ideals are Borel-fixed.*

Definition 1.22. The **reverse lexicographic order**, for shortness *revlex*, is defined as follow: $X^A >_{\text{revlex}} X^B$, where $X^A, X^B \in R = \mathbb{K}[x_1, \ldots, x_n]$ are monomials, and $A = (\alpha_1, \ldots, \alpha_n), B = (\beta_1, \ldots, \beta_n)$ are vectors of positive integers, if and only if either $\deg X^A > \deg X^B$ or if $\deg X^A = \deg X^B$ then the first non-zero entry of $A - B$ vector is *negative*.

Example 1.2. $>_{\text{revlex}}$ works in this way: $\mathbb{K}[x_1, x_2, x_3]$, $x_3 > x_2 > x_1$.

Remark 4. By Theorem 1.4.6 every generic initial ideal is a Borel-fixed ideal. Moreover the Hilbert function of $\text{gin}(I)$ is the same of I. Therefore, let h be the Hilbert sequence of R/I, the SLP holds for $R/(I, L)$ iff the Hilbert sequence of the quotient ring $R/(I + (L^s))$ is equal to sequence

$$\max\{h_i - h_{i-s}, 0\}, \forall i = 0, 1, 2, \dots.$$

and for all positive integers s ([16], Remark 6.12).

Theorem 1.4.7 ([16], Proposition 3.15)**.** *Let $\mathbb{K}[x, y]$ be the polynomial ring in two variables over a field $\mathbb{K}$ of characteristic zero. Every artinian $\mathbb{K}-$algebra $\mathbb{K}[x, y]/I$ with standard grading has the SLP.*

Proof. First of all suppose that I is a Borel-fixed ideal in $R = \mathbb{K}[x, y]$. Since $\operatorname{char}\mathbb{K} = 0$ and I is Borel-fixed then we may assume I is generated by monomials and

$$f \in I \Rightarrow x\frac{\partial f}{\partial y} \in I.$$

In other words, if we fix the linear order for each degree d,

$$x^d > x^{d-1}y > x^{d-2}y^2 > \cdots > xy^{d-1} > y^d,$$

then the set of monomials in I_d, for each d, consists of consecutive monomials from the first to the last one. So R/I_d is a vector space spanned by consecutive monomials from the last. Let $(h_0, \dots, h_s)$ be the Hilbert function of $A = R/I$. Then by assumed hypothesis, h is unimodal. Assume first that $h_i \leq h_{i+d}$. Then $\times y^d : (R/I)_i \to (R/I)_{i+d}$ is injective, furthermore we notice $y^d \notin I_d$ since I is Borel-fixed for the ordering on monomials in I_d. Now $\times y^d$ is injective because if a monomial $M \in (R/I)_i$ then $y^d M$ is in $(R/I)_{i+d}$. (The subtle point here is that if M is the $t-th$ monomial of $(R/I)_i$ from the last according the ordering, then $y^d M$ is also the $t-th$ monomial in $(R/I)_{i+d}$ from the last). While if we assume $h_i \geq h_{i+d}$, and M is a monomial in $(R/I)_{i+d}$, say the $t-th$ monomial from the last. Then the $t-th$ monomial of $(R/I)_i$ from the last exists since $h_i > h_{i+d}$. Let it be N. Then we have $y^d N = M$; this implies $\times y^d$ is surjective. So when I is a Borel-fixed ideal of R and $\mathbb{K}$ has characteristic zero, then R/I holds the SLP.
Now let us suppose I is not Borel-fixed. By the fact that $\text{gin}(I)$ is Borel-fixed by theorem 1.4.6, we can apply the previous reasoning, made on I, on

gin(I). Therefore it is easy to see $\mathrm{in}(I : y^d) = \mathrm{in}(I) : y^d$, for $d = 1, 2, \dots$ ([9], Proposition 15.12) where y is the last variable with respect to the given order. Now, for remark 4, we have the SLP is characterized by the Hilbert function of $R/(I, y^d) = A/(y^d)$, $d = 1, 2, \dots$ and it does not change if we pass to gin(I). So the general case reduces to Borel-fixed case (the previous one). $\square$

Theorem 1.4.8 ([7], Proposition 2.8)**.** *Let $R = \mathbb{K}[x, y]$ be the polynomial ring over the field $\mathbb{K}$. Every graded artinian algebra R/I has the WLP.*

Proof. Assume $I = \langle g_1, \dots, g_t \rangle$, and the given generators are minimal for I. Set $s = \min\{\deg\ g_i : \text{for all } i = 1, \dots, t\}$. Then $h(R/I)$, the h-sequence of R/I, strictly increases by one from h_0 to h_{s-1} and $h_{s-1} \geq h_s$, for the meaning of s and R/I that is artinian with standard grading. Thus the positive part of the first difference of $h(R/I)$, $\Delta^+ h(R/I) = \{h_{i+1} - h_i : 0 \leq i \leq s-2\}$ is $s-$times one. Moreover for a general linear form L of R, we get $R/(I, L)$ is isomorphic to $R[x]/J$, where $J = \langle x^s \rangle$; this implies that $h(R/(I, L))$ is $s-$times one. This means $\Delta^+ h(R/I) = h(R/(I, L))$ and finally R/I holds the WLP, with L weak Lefschetz element. $\square$

The following lemma give us a resume of all we got untill now.

Lemma 3 ([27], Lemma 2.8)**.** *Assume* $\mathrm{char}\,\mathbb{K} = p > 0$. *Consider the ideal*

$$I = \langle x_1^p, \dots, x_n^p \rangle \subset R = \mathbb{K}[x_1, \dots, x_n],$$

where $n \geq 2$.

1. *R/I fails the SLP, for all $n \geq 2$.*
2. *R/I fails the WLP for all $n \geq 3$.*
3. *R/I has the WLP if $n = 2$.*

As we can see by Theorem 1.4.2 and Lemma 3, the Lefschetz properties are related to the characteristic of the field. Next we will see that they are also connected to unimodality of Hilbert function of $A = R/I$. Now we give first some definitions.

Definition 1.23. A Noetherian local ring is **Cohen-Macaulay** if there is a system of parameters which is a regular sequence. That is, if S is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and Krull dimension $k \in \mathbb{N}$, we call S a Cohen-Macaulay ring if there exist $r_1, r_2, \ldots, r_k \in S$ such that r_1 is a nonzero-divisor in S, r_2 is a nonzero-divisor in $S/\langle r_1 \rangle$, r_3 is a nonzero-divisor in $S/\langle r_1, r_2 \rangle$ and so on. We must remind that this definition is improvable. In fact S can be just considered a Noetherian ring. In this case it will be called a Cohen-Macaulay ring iff every localization in a maximal ideal $\mathfrak{m}$, $S_{\mathfrak{m}}$, is Cohen-Macaulay. This latter definition is well applicable in graded case. Clearly all this makes sense also on algebras case, without it losses of meaning.

Definition 1.24. Let $A = R/I = \bigoplus_{i=0}^{c} A_i$ be a graded artinian algebra as usual and m is its homogeneous maximal ideal, we call **socle** of A, the following homogeneous ideal of A

$$\operatorname{Soc} A := \langle 0 : m \rangle = \{x \in A : m \cdot x = 0\}.$$

Furthermore we call $c = \max_i \{i : A_i \neq 0\}$ **maximal socle degree** of A. Sometimes it is clear that $A_c = \langle 0 : m \rangle$ from context (for example when we will consider the "Gorenstein" graded algebras, that have socle of dimension one.) In such a case, then, we will simply call c **socle degree** of A.

Definition 1.25. An **almost complete intersection ideal** I of $R = \mathbb{K}[x_1, \ldots, x_n]$ is an ideal which is generated by one more homogeneous form than codimension of R. I.e. $I = \langle f_1, \ldots, f_{n+1} \rangle$, where $\deg f_i = d_i \in \mathbb{N}$, $d_i \leq d_{i+1}$ and $d_{n+1} \leq \sum_{i=1}^{n} d_i - n$ (this last condition is assumed since otherwise I would to be generated by the first n-forms $f_1, \ldots, f_n$). By this definition follows $A = R/I$ is a graded artinian $\mathbb{K}-$algebra, because considering one more homogeneous form in I yelds $\dim A$ is zero.

Example 1.3. Let $A = \mathbb{K}[x, y, z]/I$ be an artinian graded $\mathbb{K}-$algebra, where I is generated by $\langle x^3, y^3, z^3, xyz \rangle$. Now following the above definition we get I is an almost complete intersection.

Definition 1.26. Given a graded artinian algebra $A = R/I = \bigoplus_{i=0}^{c} A_i$ with $A_c \neq 0$ is called a **level algebra of type t** if

$$\langle 0 : m \rangle = A_c \text{ and } \dim_{\mathbb{K}} A_c = t$$

where $m = \bigoplus_{i=1}^{c} A_i$ is the homogeneous maximal ideal in A. So this means the socle of A is concentrated in a single homogeneous degree and has dimension t. The dimension of socle of A, $\dim_{\mathbb{K}} A_c = t$, is what we called "type".

Definition 1.27. A local ring $(A, \mathfrak{m}, \mathbb{K})$, is called a **level ring** if there exists $c \in \mathbb{N}$ such that $\mathfrak{m}^c \neq 0, \mathfrak{m}^{c+1} = 0$ and

$$\langle 0 : \mathfrak{m} \rangle = \mathfrak{m}^c.$$

Definition 1.28. As usual let $A = R/I$ be a graded artinian algebra respect to the homogeneous ideal I. We call A a **Gorenstein** algebra over the field $\mathbb{K}$, if $\dim_{\mathbb{K}} \operatorname{Soc} A = 1$.

Now we give the definition of length of an $A-$module. Precisely:

Definition 1.29. Let $(A, \mathfrak{m}, \mathbb{K})$ be a Noetherian local ring and let M be an $A-$module. With length(M) we denote the **length** of a composition sequence of M; i.e.

$$M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0,$$

such that each quotient $\frac{M_{i-1}}{M_i} \simeq \mathbb{K}$ for all $i = 1, \ldots, \ell$. In this case ℓ is the length of the $A-$module M.

Now we give an important property about Cohen-Macaulay rings. But before we must give an useful definition in order to enunciate it.

Definition 1.30. A ring A is **catenary** or **has the saturated chain condition**, if given any prime ideals $P \subset Q$ of A, all the maximal sequence of primes between P and Q have the same length. In addition A is **universally catenary** if every finitely generated $A-$algebra is catenary.

So we can finally observe:

Remark 5. ([9], Corollary 18.10) Every Cohen-Macaulay ring is universally catenary and all the maximal sequence of prime ideals have the same length.

Now let us define three numbers related to A, an artinian graded $\mathbb{K}-$algebra, namely: the Dilworth, Rees and Sperner numbers. These numbers are important because through them one can prove if a given artinian graded $\mathbb{K}-$algebra holds or not the WLP.
In order to do this, let $\mu(I)$ be the minimal number of generators of an ideal $I \subset A$, where A is a ring.

Definition 1.31. Let us define three numbers related to the WLP for an artinian graded local ring A. They are, respectively, **Dilworth number**, **Rees number** and **Sperner number** of A; they are denoted by, respectively, $d(A)$, $r(A)$ and $s(A)$. They are the following:

$$s(A) = \max_i \{\mu(\mathfrak{m}^i)\},$$

$$d(A) = \max\{\mu(I) : I \subset A \text{ such that } I \text{ is homogeneous}\}$$

and

$$r(A) = \min\{\text{length}(A/LA) : L \in \mathfrak{m}\}.$$

Where $\mathfrak{m}$ is the homogeneous maximal ideal of A and $\text{length}(A/LA)$ is the length of a chain of ideals in A/LA, with $L \in \mathfrak{m}$. In general we have $d(A) \geq s(A)$. In the special case in which $d(A) = s(A)$, we say A has the **Sperner property**.

Lemma 4 ([16], Proposition 2.30)**.** *Let $(A, \mathfrak{m}, \mathbb{K})$ be an artinian local ring. For any ideal $I \subset \mathfrak{m}$ and for any element $y \in \mathfrak{m}$ we have*

$$\mu(I) \leq \text{length}(A/yA).$$

Furthermore in graded case we have

$$\mu(I) \leq \text{length}(A/yA) = \dim_{\mathbb{K}}(A/yA).$$

Now we give a result about Rees and Dilworth number of an artinian local ring.

Proposition 1.4.9 ([16], Proposition 2.33)**.** *Let* $(A, \mathfrak{m}, \mathbb{K})$ *be artinian local ring. Then we have the following*

$$r(A) \geq d(A).$$

If there are an ideal $a \subset A$ *and an element* $y \in \mathfrak{m}$ *such that*

$$\mu(a) = \text{length}(A/yA),$$

then $\mu(a) = d(A)$ *and* $\text{length}(A/yA) = r(A)$*.*

We now give an interesting definition used to construct graded artinian level $\mathbb{K}-$algebras, that is Macaulay's inverse system.

Definition 1.32. The **Macaulay's inverse system** is the following. Let $R = \mathbb{K}[x_1, \dots, x_r]$ and $S = \mathbb{K}[y_1, \dots, y_r]$ be, respectively, the polynomial ring in variables $x_1, \dots, x_r$ and the differential operators ring over the field $\mathbb{K}$. Now we can ragard R as a $S-$module considering this action $y_i \circ f = \frac{\partial f}{\partial x_i}$, with $f \in R_i$. We observe that this action is not finitely generated since it lowers the degree on R, so R as a $S-$module is not finitely generated. Furthermore we notice that there is an order reversing function from the ideals of S to $S-$submodules of R,

$$\varphi_1 : \{ \text{ Ideals of } S\} \to \{ \ S\text{−submodules of } R\},$$

where

$$\varphi_1(I) = \{f \in R : \ g \cdot f = 0 \text{ for all } g \in I\}.$$

This is a 1-1 correspondence, whose inverse φ_2 is given by $\varphi_2(M) = \text{Ann}_S(M) = \{g \in S : g \cdot f = 0 \text{ for all } f \in M\}$. In fact we denote $\varphi_1(I)$ by I^{-1}, which is called the **Macaulay's inverse system** to I. Notice that the action defined above is, trivially, preserved by $S-$submodules.

One can observe that if I is an ideal of S and I_j denotes its $j-th$ homogeneous component, then Macaulay observed:

$$(I^{-1})_j = (\text{Ann}(I))_j.$$

It immediately follows

$$\dim I_j^{-1} = \dim S_j - \dim I_j.$$

In particular it follows from this that I^{-1} is finitely generated $S-$submodule of R if and only if $A = S/I$ is artinian.
For the sake of clearness we give an example of how Macaulay's inverse system works by constructing a graded artinian level algebra, of socle degree 4, codimension 3 and type 2.
For more details about Macaulay's inverse system and its properties related to a level artinian graded $\mathbb{K}-$algebra of a given socle degree and type ([12], Chapter 5).

Proposition 1.4.10 ([16], Proposition 2.74)**.** *Let* $A = \bigoplus_{i=0}^{c} A_c$ *be a graded artinian ring;* $A_c \neq 0$*. Then the following conditions are equivalent:*

- *A is a level ring;*
- *$A \simeq S/(\mathrm{Ann}_Q(f_1, \ldots, f_m))$, where $f_1, \ldots, f_m \in R = \mathbb{K}[x_1, \ldots, x_n]$ are homogeneous forms of the same degree, namely c; and $S = \mathbb{K}[X_1, \ldots, X_n]$ is the differential operators ring over the field $\mathbb{K}$.*

Example 1.4. Let us consider $R = \mathbb{K}[x, y, z]$ the polynomial ring in three variables and let $S = \mathbb{K}[X, Y, Z]$ be the differential operators ring over the field $\mathbb{K}$, where $X = \partial_x, Y = \partial_y, Z = \partial_z$. Let J be an ideal of R generated by $f_1 = x^4$ and $f_2 = y^4 + z^4$. Let $M = J^{-1}$ be the inverse system of J and suppose it is generated by f_1, f_2. We have $M = \bigoplus_{i=0}^{4} M_i$, where $M_4 = \langle f_1, f_2 \rangle$, $M_3 = \langle x^3, y^3, z^3 \rangle$, $M_2 = \langle x^2, y^2, z^2 \rangle$, $M_1 = \langle x, y, z \rangle$ and $M_0 = \mathbb{K}$. So let $\mathrm{Ann}_S(M)$ be the annihilator in S of M, we have $A = S/\operatorname{Ann}_S M$ is a graded artinian level $\mathbb{K}-$algebra of socle degree 4, type 2 and codimension 3 by 1.4.10; in fact its $h-$vector is $(1, 3, 3, 3, 2)$.

Proposition 1.4.11 ([16], Proposition 3.5 & Proposition 3.6)**.** *Suppose that A is a graded artinian $\mathbb{K}-$algebra, with unimodal Hilbert function. Let $L \in A_1$ be a linear element. Then the following are equivalent.*

1. *L is a weak Lefschetz element for A.*
2. $\dim_{\mathbb{K}}(A/LA) = s(A) = \max_i\{h_i\}$.

Moreover, if A is a standard graded artinian $\mathbb{K}-$algebra and it holds the WLP, then has the Sperner property.

Proof: (1)⇔(2). Let $\{h_i\}$ be the Hilbert function of A, and let u be the smallest integer such that $h_u > h_{u+1}$. Then, we observe that $s(A)$ occurs as the maximum over i of $\dim_{\mathbb{K}} h_i$, $s(A) = h_u$. By this and by definition of length in case of grading, we have

$$\begin{aligned} \text{length}(A/LA) &= \text{length}(A_0) \\ &\quad + \text{length}(A_1/LA_0) + \text{length}(A_2/LA_1) + \cdots + \text{length}(A_c/LA_{c-1}) \\ &\overset{\text{grading}}{\geq} h_0 + (h_1 - h_0) + (h_2 - h_1) + \cdots + (h_u - h_{u-1}) \\ &= s(A). \end{aligned}$$

Furthermore we have the following equivalences:

L is a weak Lefschetz element.

$$\overset{\text{for meaning of u}}{\Longleftrightarrow} \begin{cases} \text{length}(A_{i+1}/LA_i) = h_{i+1} - h_i & \text{if } i = 1, 2, \ldots, u-1. \\ \text{length}(A_{i+1}/LA_i) = 0 & \text{if } i = u, u+1, \ldots, s-1. \end{cases}$$

$$\Longleftrightarrow \dim_{\mathbb{K}}(A/LA) = s(A).$$

Now let us show the last part of statement, namely we have to show the following: If A has the WLP and a standard grading, then A has the Sperner property.

Recall that $\mu(I) \leq \text{length}(A/(y))$ for any ideal $I \subset A$ and for any $y \in m$ (by lemma 4), where m is the homogeneous maximal ideal of A. Choose y to be a weak Lefschetz element. Then $\text{length}(A/(y)) = \max_k\{\dim_{\mathbb{K}} A_k\}$ by the first part just proved. Hence $\mu(I) \leq \text{length}(A/(y)) = \max_k\{\dim_{\mathbb{K}} A_k\}$. But knowing that m^k has as generating set just A_k, it implies $\mu(I) \leq \max_k\{\mu(m^k)\}$. In particular $\text{length}(A/(y)) = \max_k\{\mu(m^k)\}$. Now by definition of Sperner number we have to show that $\max_k\{\mu(m^k)\} = d(A)$, that is $s(A) = d(A)$. Finally we have

$$d(A) = \max_{I \subset A}\{\mu(I)\} \leq \text{length}(A/yA) = \max_k\{\mu(m^k)\} \overset{\text{remark 1}}{\leq} \max_{I \subset A}\{\mu(I)\},$$

completing the proof. □

Now we ask for:

How can we determine if R/I fails to have the WLP?

One possibility is the following:
Let L be a general linear form and fix an integer i. Then we have an exact sequence

$$[R/I]_{i-1} \stackrel{\times L}{\to} [R/I]_i \to [R/(I,L)]_i \to 0.$$

Thus $\times L$ fails to have maximal rank from degree $i-1$ to i if and only if

$$\dim[R/(I,L)]_i > \max\{0, \dim[R/I]_i - \dim[R/I]_{i-1}\}.$$

More precisely, if we want to show that WLP fails, it is enough to identify a degree i for which we can show one of the following assertions:

1. $\dim[R/I]_{i-1} \leq \dim[R/I]_i$ and $\dim[R/(I,L)]_i > \dim[R/I]_i - \dim[R/I]_{i-1}$; in this case we loose the WLP because $\times L$ is not *injective*; or

2. $\dim[R/I]_{i-1} \geq \dim[R/I]_i$ and $\dim[R/(I,L)]_i > 0$; in this case we loose the WLP because $\times L$ is not *surjective*.

In general, even identifying which integer i is in a correct position and consequentially find it could be very difficult. So prove which of 1 or 2 holds, and establishing both inequalities, is very hard to show. A good answer to this problem is the use of computer algebra programs for finding such an integer i. On the other hand we can surely show that R/I has the WLP using the next useful result, but before we give a definition which is involved on proof.

Definition 1.33. For a finite dimensional $\mathbb{K}-$algebra A the **$\mathbb{K}$-dual** is just $\omega_A = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$.

Proposition 1.4.12 ([24],Proposition 2.1)**.** *Let R/I be an artinian standard graded algebra and let L be a general linear form. Consider the homomorphisms $\phi_d : [R/I]_d \to [R/I]_{d+1}$ defined by multiplication by L, for $d \geq 0$.*

1. *If ϕ_{d_0} is surjective for some d_0 then ϕ_d is surjective for all $d \geq d_0$.*

2. *If R/I is level and ϕ_{d_0} is injective for some $d_0 \geq 0$ then ϕ_d is injective for all $d \leq d_0$.*

3. *In particular, if R/I is level and $\dim[R/I]_{d_0} = \dim[R/I]_{d_0+1}$ for some d_0 then R/I has the WLP if and only if ϕ_{d_0} is injective (and hence is an isomorphism).*

Proof.(1). Reasoning by induction on $d = d_0 + 1$. Now set $d = 1$ and let us consider the following exact sequence:

$$0 \to \frac{\langle I : L \rangle}{I} \to R/I \overset{\times L}{\to} (R/I)(1) \overset{\psi}{\to} (R/(I,L))(1) \to 0,$$

where $\times L$ in degree d is just ϕ_d. So for exactness of sequence follows that $\operatorname{Coker}(\phi_d) = [R/I]_{d+1}/\operatorname{Im}\phi_d = [R/I]_{d+1}/\ker(\psi) \overset{\psi \text{ is surjective}}{\simeq} [R/(I,L)]_{d+1}$, for any d. Therefore if ϕ_{d-1} is surjective, by hypothesis, follows $[R/(I,L)]_{d_0+1} = 0$. Now by the standard graduation we have the statement. In fact, taking an element f in $[R/I]_d$, knowing $A = \mathbb{K}[A_1]$ (by the standard graduation), applying $\phi_{d-1} = \times L^{d-1}$, where $L \in [R/I]_1$, and induction on $d-1$ we get :

$$f \overset{A = \mathbb{K}[A_1]}{=} \sum_{m_1+\cdots+m_r=d} a_m\, x_1^{m_1} \ldots x_n^{m_r} = L \cdot g,$$

where $g \in [R/I]_{d-1}$ and $m = (m_1, m_2, \ldots, m_r) \in \mathbb{N}^r$. This means $\operatorname{Coker}\phi_d = 0$, for any d.

(2). Knowing that $K - dual$ of the finite length module R/I is a shift of canonical module of R/I, let us denote it with $M = \operatorname{Hom}_{\mathbb{K}}(R/I, \mathbb{K})$ (see [9] for more details on canonical module). Furthermore knowing that the algebra R/I is a level, this implies M is generated by elements of first degree. So the multiplication map $\times L$ induced on M, considering the graded homomorphism of M in itself, it sometimes is surjective in some degree and, so for the point (1), is always surjective by that degree onward. Now recalling the meaning of M, that is $\operatorname{Hom}_{\mathbb{K}}(R/I, \mathbb{K})$, and knowing that the functor $\operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$ is contravariant, for duality follows the statement.

(3). For (1) we know that if ϕ_{d_0} is surjective for a given d_0, so it is also for any $d \geq d_0$. Now for hypothesis we know $\dim[R/I]_{d_0} = \dim[R/I]_{d_0+1}$, which means ϕ_{d_0} is also injective since $[R/I]_{d_0}$ and $[R/I]_{d_0+1}$ are finite $\mathbb{K}-$vector spaces and:

$$\dim_{\mathbb{K}} A_{d_0+1} = \dim_{\mathbb{K}} A_{d_0} - \dim_{\mathbb{K}} \ker\ \phi_{d_0},$$

follows

$$\dim_{\mathbb{K}} \ker\ \phi_{d_0} = 0.$$

Applying point (2), we have the isomorphism. $\square$

This result helps us to narrow down where one should look, i.e the spot d_0 of Hilbert sequence in which one has to look if wants to show the WLP holds. Now we characterize the WLP or SLP by the unimodality of Hilbert function of R/I.

Proposition 1.4.13 ([17], Proposition 3.5). *Let $h = (1, \dots, h_c)$ be a finite sequence of positive integers. Then h is the Hilbert sequence of a graded artinian $\mathbb{K}-$algebra R/I having the WLP if and only if it is an unimodal $O-$sequence such that the positive part of first difference is an $O-$sequence. Furthermore this is also a necessary and sufficient condition for h to be the Hilbert sequence of a graded artinian $\mathbb{K}-$algebra with the SLP.*

Now let us study another case, precisely the case in which WLP or SLP are held or not if we consider an ideal in complete intersection or a Gorenstein algebra.

Lemma 5 ([16], Key Lemma). *Let $\mathbb{K}$ be an algebraic closed field of characteristic zero. Let $I \subset R = [x_1, x_2, x_3]$ be a complete intersection ideal generated by homogenous elements f_1, f_2, f_3 of degrees d_1, d_2, d_3 with $d_3 \geq d_2 \geq d_1$. Let E be the kernel of the map*

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} : R(-d_1) \oplus R(-d_2) \oplus R(-d_3) \to R(0),$$

and let $\mathcal{E}$ be the sheaf associated to module E. Note that $\mathcal{E}$ is a rank 2 locally free sheaf over $\mathcal{O}_{\mathbb{P}^2}$, and there is an exact sequence:

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_3) \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

In this situation the following hold.

1. *if $d_3 \leq d_1 + d_2 + 1$, then $\mathcal{E}$ is semistable.*

2. *(Theorem of Grauert-Mülich) If (the rank two vector bundle) $\mathcal{E}$ is semistable and if $\mathcal{E}$ restricted to a general line L of $\mathbb{P}^2$, then $\mathcal{E}_{|L}$ splits as $\mathcal{E}_{|L} \simeq \mathcal{O}_{\mathbb{P}^1}(-e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-e_2)$, with $|e_1 - e_2| \leq 1$.*

Theorem 1.4.14 ([16], Theorem 3.48)**.** *Let $R = \mathbb{K}[x, y, z]$ be the polynomial ring over a field $\mathbb{K}$ of characteristic zero. Let f_1, f_2, f_3 be a homogeneous regular sequence of degrees d_1, d_2, d_3 respectively. Then $A = R/I$ has the WLP, where $I = \langle f_1, f_2, f_3 \rangle$.*

Proof. We may assume $2 \leq d_1 \leq d_2 \leq d_3$. Let $\mathrm{Hilb}(A, t)$ be the Hilbert series of A. We know that

$$\mathrm{Hilb}(A, t) = \prod_{i=1}^{3} (1 + t + \cdots + t^{d_i - 1}).$$

First assume $d_3 > d_1 + d_2 - 2$. Rewrite $\mathrm{Hilb}(A, t)$ as

$$\mathrm{Hilb}(A, t) = (1 + t + \cdots + t^{d_3 - 1}) H(t),$$

where

$$H(t) = (1 + t + \cdots + t^{d_1 - 1})(1 + t + \cdots + t^{d_2 - 1}).$$

Since $deg(H(t)) = d_1 + d_2 - 2$, we see by calculations that the maximum of the coefficients of the polynomial $\mathrm{Hilb}(A, t)$ occurs in degree $d_3 - 1$ and higher, and this coefficient is just the sum of all coefficients of $H(t)$. Thus it is just $d_1 d_2$; so for definition of Sperner number, s(A), of a graded artinian $\mathbb{K}-$algebra A it is just $d_1 d_2$. But $\mu(I + (L)/(L)) = 2$, where L is a general linear form, because $A/LA = R/(f_1, f_2, f_3, L) = R/(f_1, f_2, L)$ the last equality holds since $\deg(\mathrm{Soc}(A/LA)) = d_1 + d_2 - 2$ and $d_3 > d_1 + d_2 - 2$. Hence $r(A) = d_1 d_2 = s(A)$, so A holds the WLP by proposition 1.4.9. Next assume $d_3 \leq d_1 + d_2 - 2$. Let $L \in R$ be a linear element, and $\bar{R} = R/LR$. Consider $\bar{f}_1, \bar{f}_2, \bar{f}_3$ the natural images of f_1, f_2, f_3 in $\bar{R}$. Then there exist positive integers e_1, e_2 such that

$$0 \to \bar{R}(-e_1) \oplus \bar{R}(-e_2) \to \bar{R}(-d_1) \oplus \bar{R}(-d_2) \oplus \bar{R}(-d_3) \to \bar{R}(0) \to \bar{R}/(\bar{f}_1, \bar{f}_2, \bar{f}_3) \to 0$$

is exact, where $e_1 + e_2 = d_1 + d_2 + d_3$. Thus we have

$$\mathrm{Hilb}(A/LA, t) = \frac{1 - t^{d_1} - t^{d_2} - t^{d_3} + t^{e_1} + t^{e_2}}{(1 - t)^2}.$$

This enable us to express the number $\dim(A/LA)$ in terms of the binomial coefficients as the sum of the coefficients of $\mathrm{Hilb}(A/LA, t)$. On the other hand, the minimal free resolution of $A = R/\langle f_1, f_2, f_3\rangle$ has the form

$$0 \to F_3 \to F_2 \to F_1 \to R \to A \to 0,$$

where

$$\begin{aligned} F_3 &= R(-d_1 - d_2 - d_3), \\ F_2 &= R(-d_2 - d_3) \oplus R(-d_1 - d_3) \oplus R(-d_1 - d_2), \\ F_1 &= R(-d_1) \oplus R(-d_2) \oplus R(-d_3). \end{aligned}$$

Thus we may write the Hilbert function $\mathrm{Hilb}(A, t)$ as $\mathrm{Hilb}(R, t) - \mathrm{Hilb}(F_1, t) + \mathrm{Hilb}(F_2, t) - \mathrm{Hilb}(F_3, t)$. By the *Key Lemma* considering L, a general linear form, we have $|e_1 - e_2| \leq 1$. It is straightforward, although lengthy, to show that the sum of terms with positive coefficients of $\mathrm{Hilb}(A, t) - \mathrm{Hilb}(A, t-1)$ is equal to $\mathrm{Hilb}(A/LA, t)$, under condition $|e_1 - e_2| \leq 1$. Evaluating both polynomial at $t = 1$ we have $s(A) = r(A)$ by proposition 1.4.9. So the *Key Lemma* says that $\dim A/LA$ depends only on the degrees d_1, d_2, d_3, and not on particular choice of elements f_1, f_2, f_3. Therefore, for each triple of degrees d_1, d_2, d_3 we know at least once instance for which the WLP holds (in monomial complete intersection case). So by hypothesis on A, that is a monomial complete intersection, we have $s(A) = r(A)$. $\square$

This theorem give us a positive answer to the question, at most in three variables:

Do all artinian complete intersections have the WLP or the SLP in characteristic 0?

While for artinian Gorenstein there is another kind of question. It is the following:
Do all artinian Gorenstein algebras have the WLP? If not, which classes of these algebras do have this property?

The first question has negative answer. In fact R. Stanley [29] proved in

1978, with an example, that there is an artinian Gorenstein algebra with Hilbert function $(1, 13, 12, 13, 1)$, which is not unimodal and so not even with WLP. For instance we are going to describe it in detail. Before we give a definition.

Definition 1.34. Let S be a ring and M a $S-$module. The **idealization** of M, denoted by $S \ltimes M$, is the product set $S \times M$ in which addition and multiplication are defined as follows:

$$\begin{aligned}(a, x) + (b, y) &= (a + b, x + y)\\ (a, x)(b, y) &= (ab, ay + bx)\end{aligned}$$

Definition 1.35. Let S be a ring and M a $S-$module.
A $S-$submodule Q of M is **injective** iff there exists K, a S-submodule of M, such that $M = Q + K$ and $Q \cap K = \{0\}$.

Definition 1.36. Let S be a Noetherian ring. Any $S-$module M can be embedded in an injective module. The smallest injective module that contains M is called the **injective hull**, or **injective envelope** of the $S-$module M; it is denoted by $E(M)$. To be precise, $E(M)$ is characterized by the following properties.

1. $E(M)$ is an injective $S-$module;
2. $M \subset E(M)$;
3. If $N \subset E(M)$ is any non-trivial $S-$module, then $N \cap M \neq 0$.

Example 1.5 ([16], Stanley's example.). Now we use the principle of idealization for constructing an example, due to R. Stanley, of a graded Gorenstein artinian algebra with non-unimodal Hilbert function. Namely, let consider $A = \mathbb{K}[x_1, \ldots, x_n]/\mathfrak{m}^{d+1}$, where $\mathfrak{m}$ is the homogenous maximal ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$. Assume A has a standard grading. Thus, obviously, $\deg(\operatorname{Soc} A) = d$ and A is a level algebra. The Hilbert function is $h(A, i) = \binom{i+n-1}{n-1}, 1 \leq i \leq d$. Let $f(x)$ be a real valued polynomial function

$$f(x) = \frac{1}{(n-1)!}(x + n - 1) \ldots (x + 1).$$

Notice that $|f(x)|$ is symmetric about $x = -n/2$, the Hilbert function of the injective hull of A, E_A, is obtained by shifting degrees (as a property of injective hull of an artinian graded algebra). Namely

$$h(E_A(d+1), i) = (-1)^{n-1} f(i-n-d), i = 0, 1, 2, \ldots, d.$$

I.e. $H(A,t) = H(E_A, t^{-1})$. This shows the Hilbert function of idealization $A \ltimes E_A$ is

$$h(A \ltimes E_A, i) = \begin{cases} 1 & \text{if } i = 0, \\ f(i) + (-1)^{n-1} f(i-n-d-1) & \text{if } 0 < i < d+1, \\ 1 & \text{if } i = d+1 \end{cases}.$$

The real valued function, sum of both Hilbert functions of A and E_A, is symmetric about $x = (d+1)/2$. Assume for the moment $n > 2$. Then the Hilbert function of the idealization $h_i := h(A \ltimes E_A, i)$, has an unique minimum at $i = (d+1)/2$ if d is odd, while at $i = d/2$ and $i = d/2+1$ if d is even. Let $j = (d+1)/2$ if d is odd and $j = d/2$ if d is even. Then we have

$$1 = h_0 < h_1 > h_2 > \cdots > h_j < h_{j+1} < \cdots < h_d > h_{d+1} = 1$$

if d is odd, and

$$1 = h_0 < h_1 > h_2 > \cdots > h_j = h_{j+1} < \cdots < h_d > h_{d+1} = 1$$

if d is even. If $n = 2$, the Hilbert function is:

$$1, d+3, d+3, \ldots, d+3, 1.$$

If $n = 3$ and $d = 3$, then $A \ltimes E_A$ has Hilbert function with minimum at h_j where $j = (3+1)/2 = 2$, since $d = 3$ is odd. Using the above formula we have the following

$$h_{A \ltimes E_A} = (1, 13, 12, 13, 1) = h_A + h_{E_A},$$

where

$$h_A = (1, 3, 6, 0), h_{E_A} = (0, 6, 3, 1).$$

Though Hilbert functions of A and E_A are unimodal, the Hilbert function of idealization is not unimodal.

Even among Gorenstein algebras with unimodal Hilbert functions, WLP does not hold. For instance, Ikeda [21] in 1996 gave an example with 4 variables, in which WLP does not hold necessarily. In fact we have:

Example 1.6 ([21]). Let $R = K[w, x, y, z]$ a polynomial ring over the field K and let $Q = \mathbb{K}[W, X, Y, Z]$, where $W = \partial_w, X = \partial_x, Y = \partial_y, Z = \partial_z$. Let $I = \mathrm{Ann}_Q(F)$ be the annihilator of the form

$$F = w^3xy + wx^3z + y^3z^2.$$

We set $A = Q/I$. Then by calculation results that A is a graded Gorenstein artinian algebra with

$$s(A) = 10$$

and

$$d(A) = r(A) = 11.$$

In fact Hilbert sequence of A is $h = (1, 4, 10, 10, 4, 1)$, as calculating the partial derivatives of F till to fifth degree, since F is a homogenous form of degree five, we get the graded components of A (thanks to a property of Macaulay's inverse system). More precisely we get, via the inverse system of I, $M = \bigoplus_{i=0}^{5} M_i$, by differentiating F; namely we get:

$$\begin{aligned}
M_0 &= \mathbb{K},\\
M_1 &= \langle w, x, y, z\rangle,\\
M_2 &= \langle w^2, wx, wy, wz, x^2, xy, xz, y^2, yz, z^2\rangle,\\
M_3 &= \langle wxy, w^2y + x^2z, w^2x, x^3, wxz, w^3, yz^2, y^2z, y^3, wx^2\rangle,\\
M_4 &= \langle w^2xy + x^3z, w^3y + wx^2z, w^3x + y^2z^2, wx^3 + y^3z\rangle,\\
M_5 &= \langle F\rangle.
\end{aligned}$$

Thus we get $h = (1, 4, 10, 10, 4, 1)$, where $h_i = \dim_{\mathbb{K}} M_i, i = 0, \ldots, 5$. Now follows $s(A) = 10$, because A is standard graded artinian $K-$ algebra, and the Sperner number is just the maximum among the h_i. Now let us consider the ideal of A, J, generated by $WY, XY, Y^2, WZ, XZ, YZ, Z^2, W^3$, $W^2X, WX^2, X^3, WXY, WXZ$, $YZ^2, W^2XY + X^3Z, WX^3 + Y^3Z$. Then we have $\mu(J) = 16$, and $\text{length}(A/(w + x + y + z)A) = 16$, because the rank of the matrices associate to the following maps:

$$\times L : A_i \to A_{i+1}, \forall i \leq 4,$$

we notice the only matrices having maximal rank are for $i = 1, 2$. Hence we get $r(A) = \dim_{\mathbb{K}}(A/LA) = \text{length}(A/(w + x + y + z)A) = |h| - rk(\times L) = 30 - 14 = 16$; hence we get $d(A) = r(A) = \mu(J) = 16$. Finally we come to conclusion that, being $s(A) \neq d(A)$, A does not hold the WLP by proposition 1.4.11

Another interesting special case is the situation in which the generators of the ideal have small degree. Let us give a definition about it before the next assertion.

Definition 1.37. We say that an algebra R/I is **presented by quadrics** if the ideal I is generated by quadrics.

Now let us give a conjecture about Gorenstein algebras presented by quadrics.

Conjecture 1.4.1 ([27], Conjecture 3.6)**.** *Any artinian Gorenstein algebra presented by quadrics, over a field $\mathbb{K}$ of characteristic zero, has the WLP.*

Remark 6. This conjecture predicts that if the socle degree is at least 3, then the multiplication by a general linear form from degree 1 to 2 is injective as proved in ([26], Proposition 5.2).

Now let us discuss about monomial level algebras. Let $R/\langle x_1^{a_1}, \dots, x_r^{a_r}\rangle$ be a complete intersection monomial algebra.

Which level artinian monomial algebras fail the WLP or the SLP?

The first result give us a positive answer.

Theorem 1.4.15 (Hausel [18], Theorem 6.2)**.** *Let A be a monomial artinian level algebra of socle degree e. If the field $\mathbb{K}$ has characteristic zero, then for a general linear form L, the induced multiplication*

$$\times L : A_j \to A_{j+1}$$

is injective, for all $j = 0, \dots, \lfloor \frac{e-1}{2} \rfloor$*. In particular, over any field the sequence*

$$(1, h_1 - 1, h_2 - h_1, \dots, h_{\lfloor \frac{e-1}{2} \rfloor + 1} - h_{\lfloor \frac{e-1}{2} \rfloor})$$

is an $\mathcal{O}-$*sequence.*

Saying that the above $h-$ sequence is an $\mathcal{O}-$sequence means the first half of it satisfies WLP, but what about the second half? The first counterexample was given by Zanello ([31], Example 7), who showed that WLP does not necessarily hold for monomial level algebras even in three variables. Now let us present Zanello's example, mentioned above.

Example 1.7 ([31], Example 7). Let $M = \langle x^2y, y^3, y^2z, yz^2, z^3 \rangle \subset R = \mathbb{K}[x, y, z]$. A simple computation shows that the $h-$ vector of $A = Q/I$, where $I = \mathrm{Ann}_Q(M)$ and $Q = \mathbb{K}[X, Y, Z]$, with $X = \partial_x, Y = \partial_y, Z = \partial_z$, is $(1, 3, 5, 5)$. We want to prove A does not hold the WLP.

Since $I = \mathrm{Ann}_Q(M)$ is a monomial ideal, it easy to show that classes of monomials of degree j of the inverse system module M (written in variables $x_i's$) generate A_j as $\mathbb{K}-$vector space, for each j. Hence doing the partial derivatives of generators of M, we have:

$$\begin{aligned} A_0 &= \mathbb{K}, \\ A_1 &= \langle x, y, z \rangle, \\ A_2 &= \langle xy, x^2, y^2, yz, z^2 \rangle, \\ A_3 &= \langle x^2y, y^3, y^2z, yz^2, z^3 \rangle. \end{aligned}$$

So h-vector of A is just $(1, 3, 5, 5)$. Now supposing A holds the WLP, this implies the existence of a general linear form in R, say $L = ax + by + cz$. This means $\times L : A_2 \to A_3$ is a bijection. By calculation: $\times Lxy = ax^2y$ and $\times Lx^2 = bx^2y$, with $a, b \neq 0$ otherwise $\times L$ would not be injective. But by multiplication of the non-zero element $bxy - ax^2$, we get $abx^2y - bax^2y = 0$. This implies a loss of injectivity of $\times L$, a contradiction.

Remark 7. This example can be generalized in order to produce codimension 3 monomial level algebras of any socle degree $e \geq 3$ without WLP. In fact Zanello developed such algebras considering M as the following:

$$M = \langle x^{e-1}y, y^e, y^{e-1}z, y^{e-2}z^2, \dots, yz^{e-1}, z^e \rangle,$$

then the algebra $A = Q/\operatorname{Ann}_Q(M)$ with $h-$vector $(1, 3, 5, 6, \dots, e + 1, e + 2, e+2)$, does not hold the WLP from degree A_{e-1} to A_e, for any linear form L because the map is not injective.

Brenner and Kaid too in ([6], Example 3.1) gave an example in which they constructed a level artinian monomial almost complete intersection algebra of type 3, failing the WLP with $h-$vector $(1, 3, 6, 6, 3)$. Thus this means that the question made above give us only a half answer to the problem; surely this kind of algebras are of a great interest in several areas. One can observe that, after having adapted the definition of $\mathcal{O}-$sequence to this kind of algebras, there are solutions to the arisen problem. The answer, or better, the hope to do this is possible. In fact we can adapt definition of $\mathcal{O}-$sequence for monomial artinian level algebras in order to have a "complete" result even if, unfortunately, just only in three variables.

Definition 1.38. A **pure $\mathcal{O}-$sequence of type t in r variables** is the Hilbert function of a level artinian monomial algebra $A = \mathbb{K}[x_1, \dots, x_r]/I = \bigoplus_{i=0}^{c} A_i$ of type t.

Theorem 1.4.16 ([4], Theorem 6.2)**.** *A level artinian monomial algebra of type 2 in three variables has the WLP.*

As immediate consequence there is the following result.

Corollary 1.4.17 ([4])**.** *A pure $\mathcal{O}-$sequence of type 2 in three variables is unimodal.*

In [4], in which are present above results, there is a complete study of level artinian monomial algebras that fails the WLP. As consequence, we have the following conclusion:

Theorem 1.4.18 ([4])**.** *If $R = \mathbb{K}[x_1, \dots, x_r]$ and R/I is a level artinian monomial algebra of type t, then, for all r and t, examples exist in which the WLP fails, except if:*

1. $r = 1, 2$;
2. $t = 1$ *theorem (1.4.2);*

3. $r = 3, t = 2$ *theorem (1.4.16)*

In particular, the first case in which WLP can fail is when $r = 3, t = 3$. This happens, for example, when we consider $I = \langle x^3, y^3, z^3, xyz \rangle$; that is just the Brenner and Kaid's example in [6].
Now we give this example from an algebraic point of view. But before let us give some useful tools for explaining it as well as possible.

Definition 1.39. Let X, Y be sets. A **matching** in $X \times Y$, the cartesian product of X and Y, is a subset M of it such that

$$(x, y) \neq (x', y') \Rightarrow x \neq x', y \neq y',$$

for any $(x, y), (x', y') \in M$. In addition a matching is said to be **full** if $|M| = min\{|X|, |Y|\}$.

Example 1.8 ([16]). Let $R = \mathbb{K}[x, y, z]$ and $m = \langle x, y, z \rangle$ be the polynomial ring and its maximal ideal, respectively. For our purpose $\mathbb{K}$ is a field of characteristic zero, but all we will do works even in any other characteristic. Let $I = \langle x^3, y^3, z^3, xyz \rangle$. Then we have

$$s(A) = 6 = d(A) < r(A) = 7,$$

where $A = R/I$ is a standard artinian graded algebra. But let us show how we anticipated:

We use standard monomials of A for constructing its $h-$vector. In this way, let us consider

$$P = \{f \in R : f \text{ monomial such that } g \nmid f, g \in I\}$$

the set of all standard monomials of A. Easily we can see P decomposed in smaller pieces, say of degree s. In our instance maximal rank s is 4. So we have $P = \bigsqcup_{s=0}^{4} P_s$ such that in P_s are all standard monomials of exactly degree s. Now let us calculate each of them.

$$|P_0| = 1,\ |P_1| = 3 = |\{x, y, z\}|,\ |P_2| = |\{x^2, yz, y^2, xz, z^2, xy\}| = 6,$$

$$|P_3| = |\{x^2y, xy^2, x^2z, xz^2, y^2z, yz^2\}| = 6,\ |P_4| = |\{x^2y^2, x^2z^2, y^2z^2\}| = 3.$$

It results $s(A) = 6$, being A a standard graded artinian algebra. Now let us calculate the rank of associated matrices of following linear maps

$$\times L : A_i \to A_{i+1}, \text{ for } i = 0, \ldots, 3$$

obtained by multiplication by $L = x + y + z \in A_1$. More precisely we have:

$$M^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, M^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, M^{(4)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

where 1 stands for an affirmative divisibility between an element of P_k and P_{k-1}; 0 otherwise. Thus results that just $M^{(1)}, M^{(2)}$ and $M^{(4)}$ have maximal rank; this implies that $r(A) = \dim_{\mathbb{K}}(A/LA) = |P| - rk(\times L) = 19 - 12 = 7$. In the end we notice all the $M^{(k)}$ have full matchings for $k = 1, 2, 3, 4$, so for a result follows $d(A) = s(A)$ ([16], Theorem 1.31). Hence A, though has the Sperner property, does not hold the WLP.

Other authors nevertheless have shown that despite the failure of the WLP, all level artinian monomial algebras with $r = 3$ and $t = 3$ have strictly unimodal Hilbert function.

Theorem 1.4.19 ([27], B. Boyle)**.** *Any pure $\mathcal{O}-$sequence of type 3 in three variables is strictly unimodal.*

Another case in which WLP fails is when $r = 4$ and $t = 2$. In fact there is the following:

Theorem 1.4.20 ([27], B. Boyle)**.** *Any pure $\mathcal{O}-$sequence of type 2 in 4 variables is strictly unimodal.*

Now we have the following natural question:

Which is the smallest socle degree and the smallest socle type t for which non-unimodal pure $\mathcal{O}-$sequences exist? This is especially of interest when *r=3.*

Now we give an example, due to Boij and Zanello, regarding the case of $r = 3$ and socle degree 12 that produces a non-unimodal Hilbert function and so nevertheless holding the WLP.

Example 1.9 ([5], Example 5.1). Let us consider the inverse system module $M \subset S = \mathbb{K}[y_1, y_2, y_3]$ generated by the last (according to the lexicographic order) 36 monomials of degree 12, namely $M = \langle y_1^2 y_2^{10}, y_1^2 y_2^9 y_3, \ldots, y_3^{12} \rangle$. An easy computation shows that the $h-$vector of $R/\operatorname{Ann}_R(M)$, where $R = \mathbb{K}[x_1, x_2, x_3]$ and $x_i = \partial_{y_i}$, is

$$(1, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36).$$

Notice now that the form $F = y_1^6 y_2^3 y_3^3$ has all of its partial derivatives of order 1, 2 and 3 (which spans vector spaces of dimension 3, 6 and 10, respectively) distinct from the partial derivatives of the forms generating M, since the latter forms and their derivatives are divisible by y_1 at most twice, whereas all the derivatives of order at most 3 of F are divisible by y_1 at least thrice. Therefore, the inverse system module $M' = \langle M, F \rangle$ generates an h-vector whose last four entries are:

$$(27 + 10, 30 + 6, 33 + 3, 36 + 1) = (37, 36, 36, 37),$$

whence the h-vector h of the codimension three artinian monomial level algebra $R/\operatorname{Ann}(M')$ is non-unimodal. In fact, we have

$$h = (1, 3, 6, 10, 15, 21, 28, 33, 36, 37, 36, 36, 37),$$

as $h - vector$ that is not unimodal.

Remark 8. [5]

1. These examples produce infinitely many other examples. In fact we can generate M with the last $3e$ monomials of degree e, e is degree of the socle. Such a form F is $y_1^{e-6} y_2^3 y_3^3$. Then $M' = \langle M, F \rangle$ is, exactly as above, the inverse system module of a non-unimodal monomial level algebra of codimension 3; its $h-$vector ends with $(\ldots, 3e+1, 3e, 3e, 3e+1)$.

2. More generally the above example can be used to construct non-unimodal monomial level algebras of any codimension $r \geq 3$. Indeed, a monomial level algebra $R/\operatorname{Ann}(M)$ has $h-$vector $(1, 3, h_2, \ldots, h_e)$, then $\mathbb{K}[x_1, \ldots, x_r]/\operatorname{Ann}(\langle M, y_4^e, \ldots, y_r^e\rangle)$ has $h-$vector

$$h' = (1, r, h_2 + r - 3, \ldots, h_e + r - 3),$$

which is clearly non-unimodal if h is not.

This last observation implies the following theorem:

Theorem 1.4.21 ([5])**.** *For every integer $r \geq 3$, there exists non-unimodal monomial artinian level algebra of codimension r.*

After all this, an obvious question about artinian level monomial algebras that rise up is the following:

How many things change if we "remove" the word monomial, in order to obtain an artinian level algebra?

For example some authors, such as Zanello as well, Miró-Roig, Nagel and Boij have shown behavior of such algebras; unfortunately the answer is very bad by WLP point of view. In fact they have shown Hilbert function of that kind of algebras is just non-unimodal from early small degrees; in fact these algebras violate Hausel's theorem 1.4.15.

Now let us talk about ideal generated by powers of linear forms forming the correspondent algebra. We want to discuss about failure or holding of WLP/SLP.
In this direction let $\mathbb{K}$ be a field of characteristic zero. Since x_i is a linear form, after a change of variables, we can set $L_1, \ldots, L_n,$ with $n \geq r$, general linear forms, as follow

$$L_i = x_i, \forall i \leq r.$$

Thus theorem 1.4.2 is also a result about powers of linear forms. Now let us give this question:

Which ideals generated by powers of general linear forms define algebras failing the WLP or SLP?

As we saw in theorem 1.4.7 for just only two variables the answer is positive, in fact such algebras hold both. Surprisingly Schenck and Seceleanu showed a similar result in three variables:

Theorem 1.4.22 ([28]). *Let $R = \mathbb{K}[x, y, z]$ be a polynomial ring with* $\operatorname{char} \mathbb{K} = 0$. *Let $I = \langle L_1^{a_1}, \dots, L_m^{a_m} \rangle$ be any ideal generated by powers of linear forms. Then R/I has the WLP.*

A surprising thing about this result, is that the same is not true for SLP. For instance, if $I = \langle L_1^3, L_2^3, L_3^3, L_4^3 \rangle$ where L_i are general in R, then multiplication map $\times L_i^3$, for $i \leq 4$, fails to have maximal rank. This case acts as a border case, that means for more variables, just 4, also WLP fails. For example:

Example 1.10 ([24]). Let $r = 4$. Consider the ideal $I = \langle x_1^N, x_2^N, x_3^N, x_4^N, L^N \rangle$ for a general linear form L. It fails to have the WLP, just only with an experimental proof thanks to CoCoA, for $N = 3, \dots, 12$.

Now some natural questions rise:

1. Proving the failure of the WLP in the example 1.10 for all $N \geq 3$.

2. What happens for mixed powers?

3. What happens for almost complete intersections ideals, that is ideals generated by $r + 1$ powers of general linear forms in r variables when $r > 4$?

4. What about more than $r + 1$ powers of general linear forms?

This example motivated two different projects by Migliore, Miró-Roig, Nagel by one side in [25], and other side Harbourne, Schenck and Seceleanu in [15]. Surely both papers used ideals of powers of general linear forms and ideals of fat points in projective space, i.e.:

Definition 1.40. Let $p_i = [p_{i1}, \dots, p_{ir}] \in \mathbb{P}^{r-1}$, $I(p_i) = \wp_i \subseteq R = \mathbb{K}[y_1, \dots, y_r]$, and $\{p_1, \dots, p_n\} \subseteq \mathbb{P}^{r-1}$ be a set of distinct points. A **fat point ideal** is an ideal of the form $F = \bigcap_{i=1}^{n} \wp_i^{a_i+1} \subset R$, and $\forall i \leq n,\ a_i \geq 0$.

The following important result of Emsalem and Iarrobino describes which behavior has an almost complete intersections algebra, when we consider: mixed powers of general linear forms, more than $r+1$ powers and powers not necessary equal to N, mentioned in above example.

Theorem 1.4.23 ([10]). *Let*

$$\langle L_1^{a_1}, \dots, L_n^{a_n} \rangle \subset R = \mathbb{K}[x_1, \dots, x_r]$$

be an ideal generated by powers of n linear forms. Let $I(p_i) = \wp_i, \forall i \leq n$ be the ideals of n points in $\mathbb{P}^{r-1}$ corresponding to the linear forms, i.e. $L_i = L_{p_i} = \sum_{j=1}^{r} p_{i_j} x_j$. Then for any integer $j \geq max\{a_i\}$,

$$\dim_{\mathbb{K}} [R/\langle L_1^{a_1}, \dots, L_n^{a_n} \rangle]_j = \dim_{\mathbb{K}} [\wp_1^{j-a_1+1} \cap \dots \cap \wp_n^{j-a_n+1}]_j.$$

One important difference between the two papers is that the second assumed the powers are uniform, and usually the powers are "large enough", and it usually allows more than $r+1$ forms. On the other hand the first one allows mixed powers. Now we give here some results from both papers for a major vision of argument.

Theorem 1.4.24 ([15]). *Let*

$$\langle L_1^t, \dots, L_n^t \rangle \subset R = \mathbb{K}[x_1, x_2, x_3, x_4]$$

with $L_i \in R_1$ generic. If $n \in \{5, 6, 7, 8\}$, then the WLP fails, respectively for $t \geq \{3, 27, 140, 704\}$.

Theorem 1.4.25 ([15]). *For*

$$\langle L_1^t, \dots, L_{2k+1}^t \rangle \subset R = \mathbb{K}[x_1, \dots, x_{2k}]$$

with L_i generic linear forms, $k \geq 2$ and $t \gg 0$, then R/I fails the WLP.

Theorem 1.4.26 ([25], Four Variables)**.** *Let*

$$\langle L_1^{a_1}, \ldots, L_5^{a_5} \rangle \subset R = \mathbb{K}[x_1, x_2, x_3, x_4],$$

where all L_i are generic. Without loss of generality assume $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. Set

$$\lambda = \begin{cases} \frac{a_1+a_2+a_3+a_4}{2} - 2 & \text{if } a_1 + a_2 + a_3 + a_4 \text{ is even} \\ \frac{a_1+a_2+a_3+a_4-7}{2} & \text{if } a_1 + a_2 + a_3 + a_4 \text{ is odd} \end{cases}$$

1. *If $a_5 \geq \lambda$ then R/I has the WLP.*
2. *If $a_5 = 2$ then R/I has the WLP.*
3. *Most other cases (in terms of a_1, a_2, a_3, a_4) are proven to fail the WLP.*
4. *For the few open cases, experimentally sometimes the WLP holds and sometimes not.*

Notice that the case in which all the a_i are equal and at least 3 is contained in theorem 1.4.24. In more than four variables, it becomes progressively more difficult to obtain results for mixed powers. We have just a partial result by Migliore, Miró-Roig, Nagel.

Theorem 1.4.27 ([25]; "Five variables, almost uniform powers")**.** *Assume $r = 5$ number of variables. Let $L_1, \ldots, L_6$ be general linear forms. Let $e \geq 0$, where $e = \deg(\mathrm{Soc}(R/I))$,*

$$I = \langle L_1^d, L_2^d, L_3^d, L_4^d, L_5^d, L_6^{d+e} \rangle \subset R = \mathbb{K}[x_1, x_2, x_3, x_4, x_5].$$

1. *If $e = 0$ then R/I fails the WLP iff $d > 3$.*
2. *If $e \geq 1$ and d is odd then R/I has the WLP iff $e \geq \frac{3d-5}{2}$.*
3. *If $e \geq 1$ and d is even then R/I has the WLP iff $e \geq \frac{3d-8}{2}$.*

We also have the following improvement of theorem 1.4.25, which has the additional assumption $t \gg 0$.

Theorem 1.4.28 ([25]; "Uniform Powers")**.** *Let*

$$\langle L_1^t, \ldots, L_{2k+1}^t \rangle \subset R = \mathbb{K}[x_1, \ldots, x_{2k}]$$

with L_i generic linear forms and $k \geq 2$. Then R/I fails the WLP iff $t > 1$.

The case $k = 2$ is contained in theorem 1.4.25 one may ask for:

What happens if it considers an odd number of variables in above result?

Here is a result for seven variables by Migliore, Miró-Roig, Nagel, [25]

Theorem 1.4.29 ([25], Seven Variables)**.** *Let*

$$I = \langle L_1^t, L_2^t, L_3^t, L_4^t, L_5^t, L_6^t, L_7^t, L_8^t \rangle \subset R = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6, x_7],$$

where $L_1, \ldots, L_8$ are general linear forms.

- *If $t = 2$ the R/I has the WLP.*
- *If $t \geq 4$ then R/I fails WLP.*

Now, by the latter result, we can notice the only open case $t = 3$ does not hold the WLP using the algebraic software *CoCoA*. So if we want just to prove this case only trusting in a software we can do it, but certainly it is a little disappointing. Now give two conjectures extracted by two papers mentioned above.

Conjecture 1.4.2 ([15])**.** *For*

$$I = \langle L_1^t, \ldots, L_n^t \rangle \subset R = \mathbb{K}[x_1, \ldots, x_r]$$

with $L_i \in R_1$ generic and $n \geq r \geq 4$, the WLP fails for all $t \gg 0$.

Conjecture 1.4.3 ([25])**.** *Let $R = \mathbb{K}[x_1, \ldots, x_{2n+1}]$. Let $L_1, \ldots, L_{2n+2}$ be general linear forms and $I = \langle L_1^d, \ldots, L_{2n+2}^d \rangle$.*

- *If $n = 3$ and $d = 3$ then R/I fails the WLP. (This is the only open case in theorem 1.4.29)*
- *If $n \geq 4$ then R/I fails the WLP iff $d > 1$.*

This conjectures are supported a lot by software CoCoA.

Now let us talk about connection between Fröberg's conjecture and WLP. As usual let us consider the characteristic of our field $\mathbb{K}$ is zero.
Let us give a definition before talking on.

Definition 1.41. Consider a graded artinian algebra, or equivalently its homogeneous ideal. We say it satisfies the **MRP**, **maximal rank property** iff there exists a general form, of degree d, L such that

$$\times L : A_i \to A_{i+d}$$

has maximal rank for all i and d.

Surely at moment it is unknown if MRP implies SLP. Give now the mentioned conjecture due to Fröberg.

Conjecture 1.4.4 (Fröberg)**.** *Any ideal of general forms has the MRP. More precisely, fix positive integers $a_1, \ldots, a_s$ for $s > 1$. Let $F_1, \ldots, F_s \in R = \mathbb{K}[x_1, \ldots, x_r]$ be general forms of degrees $a_1, \ldots, a_s$ respectively and let $I = \langle F_1, \ldots, F_s \rangle$. Then for each i, $2 \leq i \leq s$, and for all t the multiplication by F_i on $R/\langle F_1, \ldots, F_{i-1} \rangle$ has maximal rank, from degree $t - a_i$ to t. As a result, the Hilbert function of R/I can be computed inductively*

This conjecture is known to be true for two variables. In fact it follows by theorem (1.4.7). In three variables it was shown to be true by Anick. Here we analyze the following questions.

What is the Hilbert function of an ideal generated by powers of general linear forms of degrees $a_1, \ldots, a_s$? In particular, is it the same as the Hilbert function predicted by Fröberg? Which, if any, is the connection to the WLP?

The theorem (1.4.2) says that when $n = r + 1$, the answer to the second question is positive. Irrobino observed and proved just in case $n = r + 2$ the answer is negative. Chandler too, gave a result in this direction. For the last question, Migliore, Miró-Roig, Nagel gave a partial answer to it.

Proposition 1.4.30 ([25]). *1. If Fröberg's conjecture is true for all ideal generated by general forms in r variables, then all ideals generated by general forms in $r+1$ variables have the WLP.*

2. Let $R = \mathbb{K}[x_1, \ldots, x_{r+1}]$, let $L \in R$ be a general linear form, and let $S = R/\langle L \rangle \simeq \mathbb{K}[x_1, \ldots, x_r]$. Fix positive integers $s, d_1, \ldots, d_{s+1}$. Let $L_1, \ldots, L_{s+1} \in R$ be linear forms. Denote by $^-$ the restriction from R to S. Make the following assumption:

(i) The ideal $I = \langle L_1^{d_1}, \ldots, L_s^{d_s} \rangle$ has the WLP.

(ii) The multiplication $\times \bar{L}_{s+1}^{d_{s+1}} : [S/\bar{I}]_{j-d_{s+1}} \to [S/\bar{I}]_j$ has maximal rank.

Then $R/\langle L_1^{d_1}, \ldots, L_{s+1}^{d_{s+1}} \rangle$ has the WLP.

Proof of point (2). Let us show the point (2). Let consider $A = R/I$. Let $f = L_{s+1}^{s+1}$. We have to prove that $A/fA \simeq R/\langle I, f \rangle$ has the WLP. In order to do this, let $d = d_{s+1}$ and consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
{[A]_{j-d-1}} \arrow[r, "\rho"] \arrow[d, "\alpha"'] & {[A]_{j-1}} \arrow[r] \arrow[d, "\beta"] & {[A/fA]_{j-1}} \arrow[r] \arrow[d, "\gamma"] & 0 \\
{[A]_{j-d}} \arrow[r, "\psi"'] \arrow[d] & {[A]_j} \arrow[r] \arrow[d] & {[A/fA]_j} \arrow[r] \arrow[d] & 0 \\
{[A/LA]_{j-d}} \arrow[r, "\varphi"'] \arrow[d] & {[A/LA]_j} \arrow[r] \arrow[d] & {[A/\langle f, L \rangle A]_j} \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0,
\end{tikzcd}$$

where α, β, γ are multiplications by L, while ρ, ψ, φ are multiplications by f. For hypothesis (i), α and β have maximal rank. We have to show γ has maximal rank too. Thus if β is surjective for exactness of diagram we have γ is surjective too. So we can assume β injective. Since the algebras in bottom row are quotients of $\bar{R} = R/LR$ by ideals generated in $\bar{R}$ by powers of general linear forms, the (ii) assumption provides also φ has maximal rank. In the end, if we assume φ is surjective, then so is γ; while φ injective also implies γ is injective because diagram is commutative and has exact rows and columns. $\square$

Remark 9. 1. Point (2) proves that any ideal of general forms in $\mathbb{K}[x_1, \ldots, x_4]$ satisfies WLP, because Anick [[1], Lemma 4.2 and Corollary 4.14] had shown before that any ideal of general forms in $\mathbb{K}[x_1, x_2, x_3]$ satisfies Fröberg's conjecture, because satisfies the MRP.

2. In addition this result provides to a short proof of theorem 1.4.22. This means the restriction of such ideals corresponds to an ideal in $\mathbb{K}[x, y]$, where in characteristic zero all such ideal have the SLP by theorem (1.4.7).

Corollary 1.4.31 ([25]). *Assume the characteristic is zero. Let R just $\mathbb{K}[x_1, \ldots, x_{r+1}]$ and let $L \in R$ be a general linear form, and let $S = R/\langle L \rangle \simeq \mathbb{K}[x_1, \ldots, x_r]$. For integers $d_1, \ldots, d_{r+2}$ if an ideal of the form $\langle L_1^{d_1}, \ldots, L_{r+2}^{d_{r+2}} \rangle \subset R$ of powers of general linear forms fails to have the WLP the an ideal of powers of general linear forms $\langle \bar{L}_1^{d_1}, \ldots, \bar{L}_{r+2}^{d_{r+2}} \rangle \subset S$ fails to have the Hilbert function predicted by Fröberg's conjecture.*

Thus the above results give additional insight to the observation of Chandler and Iarrobino about $n = r + 2$. In fact, for $n = r + 2$, there are several cases in which an ideal of powers of general linear forms does not have the same Hilbert function as that predicted by Fröberg's conjecture for general forms. Thus theorem 1.4.26 covers almost all possible choices of exponents to give general forms. It gives a much more complete answer to question of exactly which powers of five general linear forms in three variables fail to have Fröberg predicted Hilbert function, contranstig with Anick's above mentioned that says the contrary for an ideal generated by general forms of any fixed degrees in three variables.

Example 1.11. Let $R = \mathbb{K}[x_1, \ldots, x_4]$. Let $L_1, \ldots, L_5, l$ be general linear forms. Let $S/\langle L \rangle \simeq \mathbb{K}[x, y, z]$. Let $I = \langle L_1^3, \ldots, L_5^3 \rangle$. The Hilbert function of $A = R/I$ is

$$(1, 4, 10, 15, 15, 6),$$

is unimodal. In fact we have to calculate its Hilbert polynomial in standard graduation. Namely we have

$$Hilb(A,t) = \frac{(1-t^3)^5}{(1-t)^4} = 1 + 4t + 10t^2 + 15t^3 + 15t^4 + 6t^5.$$

Now, though we have unimodality of the Hilbert sequence,

$$[R/I]_3 \stackrel{\times L}{\to} [R/I]_4 \to [R/\langle I, L\rangle]_4 \to 0$$

is not holding maximal rank. In fact $\dim_{\mathbb{K}}[R/\langle I, L\rangle]_4 = 1 \neq 0 = \dim[R/I]_4 - \dim[R/I]_3 = 15 - 15$. As $R/\langle I, L\rangle \simeq S/J$, where $J = \langle \bar{L}_1^3, \ldots, \bar{L}_5^3\rangle \subset S$, $\dim_{\mathbb{K}}[S/J]_4 = 1$. Now if we follow both results, Anick's and Fröberg's conjecture, we arrive to a contradiction because for them $\dim[S/K]_4 = 0$, where K is an ideal generated in S by cubics of five general linear forms, so that J does not have the Hilbert function predicted by Fröberg. This means, whenever we prove that an ideal of n powers of general linear forms fails the WLP (for fixed exponents), then for some subsets of these powers of general linear forms, the same number and powers of general linear forms in one less variable fails to have Hilbert function predicted by Fröberg.

This concludes the description of general aspects of standard graded artinian $\mathbb{K}-$algebras and all other subspecies (such as level, almost complete intersection ones etc.). Now we want to emphasize the special role of standard graded artinian Gorenstein $\mathbb{K}-$algebras; they are good prototype algebras, for SLP to hold, even though this not always occurs, as we will see in the next chapter.

Chapter 2

Artinian Gorenstein $\mathbb{K}-$Algebras

Here we are going to study, more carefully, the artinian Gorenstein algebras. In detail we are going to analyze another technique for studying them; namely the technique of generalized Hessian. Sometimes, in several works, it is specified with another name, namely $k-$Hessian of a given homogeneous form. For the entire chapter we assume $\mathbb{K}$ being a field of characteristic zero, otherwise we specify if it is not.

2.1 Higher Hessians and Lefschetz properties

Now we analyze in detail artinian Gorenstein algebras and their behaviour about the Lefschetz properties. Analyze some of them having an unimodal Hilbert function but not necessarily enjoying WLP, and or, SLP. To show it we use higher Hessian.
As usual we consider a field $\mathbb{K}$ of characteristic zero. In this setting, let us give a definition before proceeding.

Definition 2.1. Let $A = R/I = \bigoplus_{i=0}^{c} A_i$ be an artinian graded algebra. We say that A has the SLP in a **narrow sense** if there exists an element $L \in A_1$ such that the multiplication map

$$\times L^{c-2i} : A_i \to A_{c-i}$$

is bijective for $i = 0, \ldots, \lfloor c/2 \rfloor$.

Remark 10. If a graded artinian $\mathbb{K}-$algebra A has the SLP in a narrow sense, then the Hilbert function of A is unimodal and symmetric. When a graded artinian $\mathbb{K}-$algebra A has a symmetric Hilbert function, the notions of SLP and SLP in a narrow sense coincide.

Definition 2.2. A finite dimensional graded $\mathbb{K}-$algebra A, $A = \bigoplus_{i=0}^{c} A_i$, is called **the Poincaré duality algebra** if $\dim_{\mathbb{K}} A_c = 1$ and the bilinear map

$$A_d \times A_{c-d} \to A_c \simeq \mathbb{K}$$

is non-degenerate for $d = 0, \ldots, \lfloor c/2 \rfloor$; that is if $f \in A_{c-d}$ satisfies $fg = 0$ $\forall g \in A_d$, then $f = 0$.

Proposition 2.1.1 ([16])**.** *A graded artinian $\mathbb{K}-$algebra A is a Poincaré duality algebra if and only if A is Gorenstein.*

Proof. ($\Rightarrow$)
Let A be a Poincaré duality algebra. For any homogeneous element $f \in A - \{0\}$ of degree less than c, there exists a homogeneous element $g \in A - A_0$ such that $fg \neq 0$. Hence the socle ideal $\mathrm{Soc}(A)$ of A coincides with a one-dimensional $\mathbb{K}-$subspace A_c. So A is Gorenstein. $\square$

Proof. ($\Leftarrow$)
Now, if A is Gorenstein, the socle ideal $\mathrm{Soc}(A)$ is a one-dimensional $\mathbb{K}-$vector space. Since $A_c \subset \mathrm{Soc}(A)$, it implies $A_c = \mathrm{Soc}(A)$, as for dimensional reason; thus $\dim_{\mathbb{K}} A_c = 1$. We need to show the following assertion in order to prove the Poincaré duality algebra property:

$$(*)_d \text{ if } f \in A_{c-d} \text{ satisfies } fg = 0 \text{ for all } g \in A_d, \text{ then } f = 0.$$

We prove it by induction on d (we start making induction on $d = 1$, since in the case $d = 0$ $\mathrm{Soc}(A) \simeq \mathbb{K}$; $\mathbb{K}$ is a field and it has no zero divisor except for zero). For $d = 1$, if $f \in A_{c-1}$ satisfies $fg = 0$ for all $g \in A_1$, then $fg = 0$ for all $g \in m = \langle x_0, \ldots, x_N \rangle$ as A_1 is generated by $\{x_0, \ldots, x_N\}$. This implies $f \in A_{c-1} \cap \mathrm{Soc}(A) = 0$, so for $d = 1$ is proved. Now assume $d > 1$ and assume by contradiction that $f \in A_{c-d} - \{0\}$, satisfies $fg = 0$ for all $g \in A_d$. Moreover we assume the existence of an element $h \in A_i$, $1 \leq i < d$, such that $\varphi := fh \neq 0$. By induction hypothesis $(*)_{d-i}$, we can find an element $h' \in A_{d-i}$ such that $\varphi h' \neq 0$ for the non zero element $\varphi \in A_{c-d+i}$. But this means $\varphi h' = f(hh') \neq 0$. Since $hh' \in A_d$, for $h \in A_i$ and $h' \in A_{d-i}$, follows $\varphi h' = f(hh') = 0$. This is a contradiction. Thus we proved if $f \in A_{c-d}$ satisfies $fg = 0$ for all $g \in A_d$, then we have $f = 0$ by contradiction. This completely proves $(*)_d$ for all d and consequentially implies the non-degenerateness of map $A_d \times A_{c-d} \to A_c$, for all $d = 1, \ldots, \lfloor c/2 \rfloor$ □

Remark 11. This proposition shows an equality between Poincaré duality algebras and Gorenstein ones. There actually exist examples of graded artinian non-Gorenstein algebras with SLP. For example $A = \mathbb{K}[x, y]/\langle x^2, xy, y^3\rangle$ is a non-Gorenstein algebra with SLP. Therefore Poincaré duality does not imply the SLP. Thus there exist Gorenstein algebras which do not enjoy the SLP.

From now on we set $\mathbb{K}[X_0, \ldots, X_N] = Q$ the differential operator ring in $N+1$ variables associated to the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$, in which there are homogeneous polynomials in variables $X_0 = \frac{\partial}{\partial x_0}, \ldots, X_N = \frac{\partial}{\partial x_N}$.
Now we briefly remind basics on Inverse system. For full details, the reader is referred for example to ([22], Theorem 2.1):

Theorem 2.1.2 ([22], Theorem 2.1)**.** *Let I be an ideal of $Q = \mathbb{K}[X_0, \ldots, X_N]$ and $A = Q/I$ be the quotient algebra. Denote by $\mathfrak{m} = \langle X_0, \ldots, X_N\rangle$ the maximal ideal of Q. Then $\sqrt{I} = \mathfrak{m}$ and the $\mathbb{K}-$algebra A is Gorenstein if and only if there exists a polynomial $F \in R = \mathbb{K}[x_0, \ldots, x_N]$ such that $I = \mathrm{Ann}\ F$.*

Definition 2.3. Let $R = \mathbb{K}[x_0, \ldots, x_N]$ be the polynomial ring, and let $R_d = \mathbb{K}[x_0, \ldots, x_N]_d$ be its homogenous component of degree d. As we know R_d is generated as $\mathbb{K}-$vector space by

$$\mathcal{B} = \{\prod_{i=0}^{N} x_i^{e_i} : e_0 + \cdots + e_N = d\},$$

we have $\dim_{\mathbb{K}} R_d = \binom{N+d}{d}$. Denote with $Q = \mathbb{K}[X_0, \ldots, X_N]$ the ring of differential operators of R, i.e. R is Macaulay's inverse system of Q; as usual

$X_i = \partial_{x_i}$. So for each $k \geq 1$ exist $\mathbb{K}-$bilinear maps

$$R_d \times Q_k \to R_{d-k},$$

such that $(f, \alpha) \mapsto f_\alpha := \alpha(f)$.

Remark 12. If we consider $d = k$, that is the same degree between R and Q, we have $\mathbb{K}-$bilinear maps $R_d \times Q_d \to \mathbb{K}$ which map, every set of $f_1, \ldots, f_r \in R_d$ linearly independent forms and every set of linearly independent forms of Q_d, $\alpha_1, \ldots, \alpha_r$, to $\alpha_i(f_j) = \delta_{ij}$.

Definition 2.4. Let $f \in R$ be a reduced polynomial and let $k \geq 1$. If $\mathcal{B} = \{\alpha_1, \ldots, \alpha_\nu\}$ is an ordered basis of Q_k, set $\nu = \nu(N, k) = \binom{N+k}{k}$. Define the $k - th$ **gradient** of f with respect to basis $\mathcal{B}$ by

$$\bigtriangledown^k_{\mathcal{B}} f = (\alpha_1(f), \ldots, \alpha_\nu(f)).$$

If the basis is clear from context or if it is a standard basis ordered in lexicographic order, we put $\bigtriangledown^k f$ instead of $\bigtriangledown^k_{\mathcal{B}} f$.

Example 2.1. Let $g = xy^2 \in \mathbb{K}[x, y]$. A standard basis of $Q_2 = \mathbb{K}[X, Y]_2$ is $\{X^2, XY, Y^2\}$. Then $\bigtriangledown^2 g = (0, 2y, 2x)$.

Now our intention is to identify sets of linearly independent homogenous polynomials of the same degree $g_1, \ldots, g_s \in \mathbb{K}[u_1, \ldots, u_m]_d$ whose $k - th$ gradients are linearly dependent over the fraction field $\mathbb{K}(u_1, \ldots, u_m)$. As we see below this construction is related to vanish of higher Hessians. In what follows, linearly dependent vectors of the fraction field $\mathbb{K}(u_1, \ldots, u_m)$ are denoted by $\sim$, to remind the parallelism among vectors.

Remark 13. Given $g_1, \ldots, g_s \in \mathbb{K}[u_1, \ldots, u_m]_d$, if

$$s > \binom{k+m-1}{k} = \dim_{\mathbb{K}} \mathbb{K}[U_1, \ldots, U_m]_k,$$

then it is clear that the $k-th$ gradients $\bigtriangledown^k g_1, \ldots, \bigtriangledown^k g_s$ are linearly dependent over $\mathbb{K}(u_1, \ldots, u_m)$.

Definition 2.5. Let $f \in \mathbb{K}[x_0, \ldots, x_N]$ and let $\mathcal{B}_k = \{\alpha^k_j : j = 1, \ldots, \binom{N+k}{k}\}$ be an ordered basis of $\mathbb{K}[X_0, \ldots, X_N]_k$. We call that $k-th$ **absolute Hessian matrix of f** with respect to the basis $\mathcal{B}$ the following matrix

$$\mathrm{Hess}^k_f = (\alpha_i(\alpha_j(f)))^{\nu(N,k)}_{i,j=1}.$$

While we call **the absolute $k-th$ Hessian of f** the determinant of the $k-th$ absolute Hessian matrix of f, that is

$$\text{hess}_f^k = \det(\text{Hess}_f^k).$$

The absolute $1-$ Hessian, that is the absolute Hessian for $k = 1$, with respect standard basis is just the classical Hessian. In fact $\alpha_i(f) = \partial(f)/\partial_i$, hence

$$\text{Hess}_f^1 = (\alpha_i(\alpha_j(f)))_{i,j=0}^N = \frac{\partial^2 f}{\partial_i \partial_j} = \text{Hess } f, \qquad i, j = 0, \dots, N.$$

Now we show an important fact, even if very simple, linking the absolute Hessian and the SLP in an artinian Gorenstein algebra.

Lemma 6 ([16], Lemma 3.74). *Let $A = \bigoplus_{i=0}^{c} A_i$ be a standard garded artinian Gorenstein $\mathbb{K}-$algebra. Fix an isomorphism $[\ ] : A_c \to \mathbb{K}$. For a $\mathbb{K}-$linear basis $\beta_0, \dots, \beta_N$ of A_1, define the polynomial*

$$F(x_0, \dots, x_N) := [(x_0\beta_0 + \cdots + x_N\beta_N)^c]$$

in the variables $x_0, \dots, x_N$ taking values in $\mathbb{K}$. Then we have the following presentation of A:

$$A \simeq A/\operatorname{Ann}(F).$$

Theorem 2.1.3 ([16], Theorem 3.76). *Fix an arbitrary $\mathbb{K}-$linear basis $\mathcal{B}_d$ of A_d, for $d = 1, \dots, \lfloor c/2 \rfloor$, and let us suppose* $\operatorname{char} \mathbb{K} = 0$. *An element $L = a_0X_0 + \cdots + a_NX_N \in A_1$ is a strong Lefschetz element of $A = Q/\operatorname{Ann}\ F$, where* $\deg F = c$ *homogeneously, if and only if $F(a_0, \dots, a_N) \neq 0$ and*

$$\text{Hess}_{\mathcal{B}_d}^d F\ |_{(a_0,\dots,a_N)} \neq 0, \qquad d = 1, \dots, \lfloor c/2 \rfloor.$$

Proof. Knowing by hypothesis A is Gorenstein, we have $[\ ] : A_c \to \mathbb{K}$ an isomorphism. Let us identify $[\omega(X)] := \omega(X)F(x)$ for any $\omega(X) \in A_c$. Notice that: x and X stands for, respectively, $(x_0, \dots, x_N)$ and $(X_0, \dots, X_N)$. Therefore since deg $\omega = c =$ deg F, then $\omega(X)F(x) \in \mathbb{K}$; in fact by the lemma 6 F has been chosen in the following way:

$$F(x_0, \dots, x_N) := [(x_0\beta_0 + \cdots + x_N\beta_N)^c], \qquad \beta_0, \dots, \beta_N \text{ basis of } A_1.$$

Since A is a Poincaré duality algebra, the necessary and sufficient condition for $L = a_0X_0 + \cdots + a_NX_N \in A_1$ being a strong Lefschetz element is that the bilinear pairing

$$\begin{array}{ccccc} A_d \times A_d & \longrightarrow & \mathbb{K} & \xrightarrow{[\,]^{-1}} & A_c \\ (\xi, \eta) & \mapsto & L^{c-2d}\xi\eta \mapsto [L^{c-2d}\xi\eta]. & & \end{array}$$

is non-degenerate for $d = 0, \ldots, \lfloor c/2 \rfloor$. Therefore L is a strong Lefschetz element if and only if the matrix

$$(L^{c-2d}\alpha_i^d(X)\alpha_j^d(X)F(x))_{i,j}$$

has nonzero determinant. Note that for a homogenous polynomial $G \in \mathbb{K}[x_0, \ldots, x_N]$ of degree d we have the following formula

$$(a_0X_0 + \cdots + a_NX_N)^d G(x_0, \ldots, x_N) = d!G(a_0, \ldots, a_N),$$

so

$$L^{c-2d}\alpha_i^d(X)\alpha_j^d(X)F(x) = (c-2d)!\alpha_i^d(X)\alpha_j^d(X)F(x)\mid_{(a_0,\ldots,a_N)},$$

where $\alpha_i^d(X), \alpha_j^d(X)$ are elements of a base $\mathcal{B}_d$ of Q_d. □

Corollary 2.1.4 ([16]). *1. The algebra $A = Q/\operatorname{Ann} F$ has the SLP if and only if all the absolute higher Hessians $\operatorname{Hess}^d F$, with respect to a $\mathbb{K}-$linear basis $\mathcal{B}_d$ of A_d, for $d = 1, \ldots, \lfloor c/2 \rfloor$, are non zero polynomials.*

2. Assume $\deg \operatorname{Soc} A < 5$. An element $L = a_0X_0 + \cdots + a_NX_N$ is a strong Lefschetz element if and only if

$$F(a_0, \ldots, a_N) \neq 0 \qquad \textit{and} \qquad \operatorname{Hess}\ F(a_0, \ldots, a_N) \neq 0.$$

Remark 14. One can notice that the notion of absolute higher Hessian in any degree d is related to the choice of a basis of A_d and, so, the vanishing of the absolute higher Hessian depends by this choice. Fortunately the vanishing of absolute higher Hessian does not depend by it. Namely if one changes the basis, in absolute higher Hessian there is a multiplication by a nonzero scalar of the base field $\mathbb{K}$ which does not change anything for further calculations.

Moreover it is an obvious fact that if $k-th$ partial derivatives of f are linearly dependent over $\mathbb{K}$ then the $k-th$ Hessian vanishes identically. The amazing thing would be if the converse of it was true. Hesse conjectured this problem and thought it was probably true for $k=1$. But we put the Hesse's conjecture in a more general setting.

Conjecture 2.1.1 (**Generalized Hesse's claim**). *Is the linear dependence among the $k-th$ partial derivatives of f a necessary and sufficient condition for the vanishing of absolute $k-th$ higher Hessian of f?*

Unfortunately Hesse's claim is not true in general for $k=1$ as shown by Gordan and Noether. For $d=2$ and N, number of variables, is arbitrary the claim is true; the proof is trivial if one diagonalizes the quadratic form. From now we assume, so, $d \geq 3$. If N is less or equal to 3 the claim is true and false if $N \geq 4$ and degree of f, $\deg f \geq 3$. Hence the smallest example (considering the word "small" referred to the degree of f and number of variables N at least 4) is given by $f = xu^2 + yuv + zv^2 \in \mathbb{K}[x,y,z,u,v]$. In fact $A = Q/\operatorname{Ann}(f)$ does not hold the SLP by Theorem 2.1.3, but none of variables can be eliminated by using a linear transformation. In the following we will give a result which is a connection with a geometrical viewpoint of what we mentioned before; it is due to Gordan and Noether. Before we give some definitions.

Definition 2.6. Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface given by $f \in \mathbb{K}[x_0, \ldots, x_N]$ of degree d. X is a **cone**, up to projective transformations, if the *vertex* of X, $Vert(X)$ is not empty. This means, namely

$$Vert(X) = \{x \in X : J(x,X) = X\}, \text{ where}$$

$$J(x,X) = \overline{\bigcup_{x \neq y, y \in X} \langle x, y \rangle} \subset \mathbb{P}^N$$

is the *Join of x and X.*

Theorem 2.1.5 ([13], Theorem 3.9, Gordan-Noether's theorem). *Let $X = V(f) \subset \mathbb{P}^N$, $N \leq 3$, be a hypersurface such that* $\operatorname{hess}_f = 0$. *Then X is a cone.*

As we are going to prove later, the linear dependence among partial derivatives of f is equivalent to being a cone for a certain variety generated by f, viewed in $\mathbb{P}^N$. The following result treats cases for N greater than 3. Namely:

Theorem 2.1.6 (Gordan & Noether,[13], Theorem 3.11)**.** *For each $N \geq 4$ and $d \geq 3$ there exist infinitely many irreducible hypersurfaces $X = V(f) \subset \mathbb{P}^N$, with* deg $f = d$*, not cones, such that* $\text{hess}_f = 0$*.*

Since Hesse's claim is not true in general, there are deeper conditions responsible of the vanishing of Hessian. Hence, even if we eliminate the linear dependence of partial derivatives of f, the vanishing of hess_f cannot be avoided. For studying this, we need to introduce more concepts. Let us start with a remark.

Remark 15.

- As Ann f is a homogenous ideal of Q, we can present A as
$$\frac{Q}{\text{Ann } f}.$$
In this way, we know that A is a standard graded artinian Gorenstein $\mathbb{K}-$algebra such that $A_j = 0$, for $j > d = deg\ f$, and $A_d \neq 0$. Assuming $(\text{Ann } f)_1 = 0$ is equivalent to consider the linear independence of derivatives of f. Geometrically this condition is equivalent to ask $X = V(f)$, with f reduced polynomial, is not a cone. Under this hypothesis, which we are going to assume from now on, $\{X_0, \ldots, X_N\}$ is a basis for A_1. For abuse of notation we denote by X_i the same element of Q_1, where $X_i \in A_1$. Then A_1 has dimension $N + 1$ as $\mathbb{K}-$vector space. Notice that $N + 1$ is also the codimension of A, presented as standard graded artinian Gorenstein algebra over $\mathbb{K}$.

- Since an artinian Gorenstein algebra is a Poincaré duality one, its Hilbert sequence is symmetric. Thus SLP and SLP in a narrow sense coincide as notions.

- For codimension 2, that is $N = 1$, by Theorem 1.4.7 we have that all graded artinian $\mathbb{K}-$algebras, provided char $\mathbb{K} = 0$, satisfy the SLP. Thus the $k-th$ Hessians of f, with deg $f = d$ and $f \in \mathbb{K}[x, y]$, do not vanish for all $k = 1, \ldots, \lfloor d/2 \rfloor$.

- When the codimension is greater or equal than 3, it is still an open problem. Indeed it is known neither if there exists any artinian Gorenstein algebra which satisfies the SLP or the WLP, nor if it does not satisfy the SLP or WLP.

Let us explain in more details the algebraic aspects of Gordan and Noether's theory about Theorems 2.1.5 and 2.1.6.

2.2 Classical hypersurfaces having vanishing Hessian

First of all we recall Gordan-Noether criteria. We interpret the above results by an algebraic point of view though they are of a more geometrical nature. Now recall a definition.

Definition 2.7. We say that $f_0, \dots, f_N \in \mathbb{K}[x_0, \dots, x_N]$ are **algebraically dependent** if there exists a nonzero homogeneous polynomial $\pi(y_0, \dots, y_N) \in \mathbb{K}[y_0, \dots, y_N]$ such that

$$\pi(f_0, \dots, f_N) = 0.$$

In particular, if there exists a form π of degree one, we say $f_0, \dots, f_N$ are **linearly dependent**.

Proposition 2.2.1 ([13], Proposition 3.10)**.** *Let $f \in \mathbb{K}[x_0, \dots, x_N]$ be a reduced polynomial and consider $X = V(f) \subset \mathbb{P}^N$. Then*

1. *X is a cone $\Leftrightarrow$ $f_{X_0}, \dots, f_{X_N}$ are linearly dependent;*
2. *$\mathrm{hess}_f = 0 \Leftrightarrow$ $f_{X_0}, \dots, f_{X_N}$ are algebraically dependent.*

Definition 2.8. Let $X = V(f) \subset \mathbb{P}^N, N \geq 4$, be an irreducible hypersurface not a cone. We say that X is a **Perazzo hypersurface of degree** d if $N = n + m$, with $n, m \geq 2$ and $f \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$ is a reduced polynomial of the form

$$f = x_0 g_0 + \cdots + x_n g_n + h$$

where $g_i \in \mathbb{K}[u_1, \dots, u_m]_{d-1}$, for $i = 0, \dots, n$, are algebraically dependent but linearly independent and $h \in \mathbb{K}[u_1, \dots, u_m]_d$. The polynomial f is called **Perazzo polynomial**.

Theorem 2.2.2 ([13], Theorem 3.13)**.** *Perazzo hypersurfaces are not cones and have vanishing Hessian.*

Proof. Since $f_{X_i} = g_i$, for all $i = 0, \dots, n$, the assertion follows by hypothesis on g_i, which are algebraically dependent but linearly independent being X a Perazzo hypersurfaces, and by Proposition 2.2.1. $\square$

Example 2.2 (Perazzo's Cubic Hypersurface). For example there is the Perazzo's cubic hypersurface

$$f(x_0, x_1, x_2, x_3, x_4) = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$$

which satisfies the above theorem. Indeed if we calculate the partial derivatives of f

$$f_{X_0} = x_3^2, \quad f_{X_1} = x_3x_4, \quad f_{X_2} = x_4^2,$$
$$f_{X_3} = 2x_0x_3 + x_1x_4, \quad f_{X_4} = x_1x_3 + 2x_2x_4$$

they are linearly independent which it means f does not generate a cone by Proposition 2.2.1; but they are algebraically dependent as we can see by the following calculation

$$f_{X_0}f_{X_2} - f_{X_1}^2 = x_3^2x_4^2 - (x_3x_4)^2 = 0.$$

Hence Proposition 2.2.1 still implies $\mathrm{hess}_f = 0$.

A series of examples coming from the previous one can be constructed in the following way. Let us consider the following polynomial of degree 3 and $N \geq 4$, number of variables. Namely

$$g(x_0, \dots, x_N) = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2 + x_5^3 + \dots + x_N^3.$$

Then $X = V(g) \subset \mathbb{P}^N$ is not a cone and $\mathrm{hess}_g = 0$.

Definition 2.9. Let $R = \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$. Let $Q = \sum_{i=0}^{n} x_ig_i \in R$ be a form of degree e with $g_i \in \mathbb{K}[u_1, \dots, u_m]_{e-1}$, for $i = 0, \dots, n$, algebraically dependent but linearly independent. Let $\mu = \lfloor d/e \rfloor$. Let $P_j \in \mathbb{K}[u_1, \dots, u_m]_{d-je}$, for $j = 0, \dots, \mu$. We say

$$f = \sum_{j=0}^{\mu} Q^jP_j$$

is a **Permutti polynomial of type (m,n,e)**. A **Permutti hypersurface** is given by $X = V(f) \subset \mathbb{P}^N$ with f a reduced Permutti polynomial.

Theorem 2.2.3 ([13])**.** *Permutti hypersurfaces are not cones and have vanishing Hessian.*

Proof. We have $f_{X_i} = (\sum_{j=1}^{\mu} jQ^{j-1}P_j)g_i$. Since g_i are algebraically dependent, f_{X_i} are too. Therefore by Proposition 2.2.1 we have $\mathrm{hess}_f = 0$. Moreover the linear dependence of f_{X_i}, for all i, is easy to check. □

Finally let us present the original Gordan-Noether's hypersurfaces.

Definition 2.10. Let $R = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$. For $\ell = 1, \ldots, s = n-r$ and for $i = 0, \ldots, n$ let $\Phi_{ji} \in \mathbb{K}[y_0, \ldots, y_r]$ and $\Psi_k^{\ell i} \in \mathbb{K}[u_1, \ldots, u_m]$. Let $g_{\ell j} \in \mathbb{K}[u_1, \ldots, u_m]_{e-1}$ be given by $g_{\ell i} = \Phi_{\ell i}(\Psi_0^{\ell i}, \ldots, \Psi_r^{\ell i})$, with $0 \leq i \leq n$ and $1 \leq \ell \leq s$. Let $Q_\ell = x_0 g_{0\ell} + \cdots + x_n g_{n\ell}$ with $\ell = 1, \ldots, s$. Let $d > e$ and $\mu = \lfloor d/e \rfloor$. Let $P_j(z_1, \ldots, z_s, u_1, \ldots, u_m)$ for $j = 0, \ldots, \mu$ be biforms of bi-degree $(j, d-ej)$. A **GN-hypersurface of type** (m, n, r, e) is defined by polynomial

$$f = \sum_{j=0}^{\mu} P_j(Q_1, \ldots, Q_s, u_1, \ldots, u_m).$$

Remark 16.

- It is easy to see that a Perazzo hypersurface is a Permutti hypersurface with $\mu = \lfloor d/e \rfloor = 1$, which means $d = e$. Indeed a Permutti hypersurface, with $\mu = 1$, is a hypersurface given by

  $$f = \sum_{j=0}^{1=\mu} Q^j P_j = P_0 + Q^1 P_1 = P_0 + (\sum_{i=0}^{n} x_i g_i) P_1,$$

  where $P_1 \in \mathbb{K}, P_0 \in \mathbb{K}[u_1, \ldots, u_m]_d$ and $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}, \forall i = 0, \ldots, n$. This proves that it is effectively of Perazzo.

- Consider a GN-hypersurface of type $(m, n, n-1, e)$; since $r = n-1$, the $s = n - r = 1$. Thus a GN-hypersurface is a Permutti hypersurface of type (m, n, e). Indeed a GN-hypersurface, with $s = 1$, is given by

  $$f = \sum_{j=0}^{\mu} P_j(Q_1, u_1, \ldots, u_m) = \sum_{j=0}^{\mu} Q_1^j P_j,$$

  where $Q_1 = x_0 g_{01} + \cdots + x_n g_{n1}$ and $g_{i1} \in \mathbb{K}[u_1, \ldots, u_m]_{e-1}, \forall i = 0, \ldots, n.$

Now we give an important result, always due to Gordan and Noether, that we will use later on, when we will show that all standard graded artinian Gorenstein $\mathbb{K}-$algebras of codimension 5, which means $N = 4$, and socle degree 4 satisfy the WLP.

Theorem 2.2.4 ([13], Theorem 3.19)**.** *Let $X = V(f) \subset \mathbb{P}^4$ be a reduced hypersurface, not a cone, having vanishing Hessian. Then f is a GN polynomial of type $(2, 2, 1, e)$ or equivalently a Permutti polynomial of type $(2, 2, e)$.*

2.3 Examples of hypersurfaces having vanishing $k - th$ Hessian

Here we want to generalize the Gordan and Noether's result, namely Theorem 2.1.6. To do this we have to construct two families of irreducible polynomials having $k-th$ vanishing hessian. Moreover, in constructing them we attempt to replace Proposition 2.2.1.
The unifying point of view can be summarized in the next proposition.

Proposition 2.3.1 ([13], Proposition 2.1)**.** *Let $R = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ be a polynomial ring in $N+1 = m+n+1$ variables, let $Q = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$ be the associated ring of differentials and for $f \in R_d$ let $A = A(f) = \dfrac{Q}{\mathrm{Ann}_Q\ f}$. Set deg $f = d = e+k$, with $e > k \geq 1$ and assume $(\mathrm{Ann}_Q\ f)_1 = 0$, i.e. $V(f)$ is not a cone. Consider also $\tilde{R} = \mathbb{K}[u_1, \ldots, u_m], \tilde{Q} = \mathbb{K}[U_1, \ldots, U_m]$ and $B = \tilde{Q}/(\mathrm{Ann}_Q\ f) \cap \tilde{Q}$. Suppose $\alpha_1, \ldots, \alpha_s \in A_k - B_k, \alpha_i = \alpha_i(X_0, \ldots, X_n)$ are linearly independent differential operators such that $f_{\alpha_1} = \alpha_1(f), \ldots, f_{\alpha_s} = \alpha_s(f) \in \mathbb{K}[u_1, \ldots, u_m]_e$. If $s > \binom{m+k-1}{k}$, then*

$$\mathrm{hess}_f^k = 0.$$

Proof. Choose a basis of A_k whose first $s-$vectors are $\alpha_1, \ldots, \alpha_s$ and such that the last vectors $\{\beta_1 = \beta_1(U_1, \ldots, U_m), \ldots, \beta_r = \beta_r(U_1, \ldots .U_m)\}$ consist of a basis of B_k. Notice none of $\alpha_i \in B_k$, by hypothesis. Let

$$\bigtriangledown^k \alpha_i(f) = (\beta_1(\alpha_i(f)), \ldots, \beta_r(\alpha_i(f)))$$

be the gradient of $\alpha_i(f)$ with respect to this basis. Since $\alpha_i(f) \in \mathbb{K}[u_1, \dots, u_m]$ for $i = 1, \dots, s$, the first s rows of Hess_f^k are

$$L_i = (0, \dots, 0, \bigtriangledown^k \alpha_i(f)).$$

Indeed, if $\gamma \in A_k - B_k$, then γ must depend on some of the variables $X_0, \dots, X_n$, yielding $\gamma(\alpha_i(f)) = 0$, since by hypothesis $\alpha_i(f) \in \mathbb{K}[u_1, \dots, u_m]$. By hypothesis, $s > \binom{m-1+k}{k} = \dim_{\mathbb{K}} \mathbb{K}[U_1, \dots, U_m]_k$, hence the $k-th$ gradients of the $\alpha_i(f)$, $\bigtriangledown^k \alpha_1(f), \dots, \bigtriangledown^k \alpha_s(f)$, are linearly dependent over the fractions field $\mathbb{K}(u_1, \dots, u_m)$. Therefore $L_1, \dots, L_s$ are linearly dependent over $\mathbb{K}(x_0, \dots, x_n, u_1, \dots, u_m)$, yielding $\mathrm{hess}_f^k = 0$. □

The first family we construct is a generalization of Example 1.6 due to Ikeda. Recall that Ikeda's example is given by the homogenous polynomial $f = w^3xy + wx^3z + y^3z^2 \in \mathbb{K}[x, y, z, w]_5$. We saw the algebra given by f is a Gorenstein one which does not satisfy the SLP; this has been proved by the use of Sperner number and Dilworth number. We want to show that we get the same result via the use of hessians. In fact using a basis of A_1, where $A = \mathbb{K}[X, Y, Z, W]/\operatorname{Ann} f$, consisting of $\{X, Y, Z, W\}$ we can show hess_f is not zero by Theorem 2.1.5. On the other hand, if we calculate the second Hessian, it vanishes. Indeed it is enough to use a basis of A_2, that is $\{W^2, WX, WY, WZ, X^2, XY, XZ, Y^2, YZ, Z^2\}$. Knowing $s = 4$, since we can consider in A_2 $\alpha_1 = XY, \alpha_2 = XZ, \alpha_3 = YW$ and $\alpha_4 = ZW$, then

$$f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_3}, f_{\alpha_4} \in \mathbb{K}[x, w]_3 \; and \; \binom{m+k-1}{k} = \binom{2+2-1}{2} = \binom{3}{2} = 3.$$

Since $s > 3$, by proposition 2.3.1 we get what we want.

Now we give a negative answer to the generalized Hesse's claim in arbitrary dimension $N \geq 3$, for arbitrary degree $d \geq 5$ of f and for arbitrary order $k > 1$ of higher Hessian. In order to do this we set $N \geq n \geq 2$ and $2 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Consider an irreducible hypersurface $X = V(f) \subset \mathbb{P}^N$ of degree d, where

$$(1)\ f = \sum_{i=2}^{n} x_i u^{t_i} v^{d-1-t_i} + h(x_2, \dots, x_N) \in \mathbb{K}[u, v, x_2, \dots, x_N]_d.$$

Suppose $(\mathrm{Ann}_Q\ f)_1 = 0$, i.e. $\dim A_1 = N+1$ (as we had observed before on remark 15). Let $m_i := min\{t_i, d-1-t_i\}$ and suppose that $k-1 \leq m_i$ for $i = 2, \dots, l$ with $l \geq 2$ while $k-1 > m_i$ for $i > l$. For this type of hypersurface we have.

1. For $i = 2, \dots, l$ the differentials $\alpha_{ij} = X_i U^j V^{k-1-j} \in A_k$ with $j = 0, \dots, k-1$ satisfy $f_{\alpha_{ij}} = c_{ij} u^{t_i - j} v^{d-t_i-k+j}$.

2. For $i = l+1, \dots, n$ suppose without loss of generality $m_i = t_i < k-1 \leq d-1-t_i$, then the differentials $\alpha_{ij} = X_i U^j V^{k-1-j} \in A_k$ with $j = 0, \dots, m_i$ satisfy $f_{\alpha_{ij}} = c_{ij} u^{t_i - j} v^{d-t_i-k+j}$.

Let

$$\tilde{\Lambda} = \{(i,j) \mid \text{ for } i = 2, \dots, l;\ \ j = 0, \dots, k-1 \text{ and for } i = l+1, \dots, n;\ \ j = 0, \dots, m_i\}.$$

Since the $f_{\alpha_{ij}}$ are monomials of the same degree, if we choose a maximal set

$$\Lambda = \{(i,j) \in \tilde{\Lambda} \mid\ t_i - j \neq t_{\tilde{i}} - \tilde{j},\ \forall (i,j) \neq (\tilde{i}, \tilde{j})\},$$

then the set of differentials $\{\alpha_{ij} \mid (i,j) \in \Lambda\}$ are linearly independents in A_k. The above construction motivates the following:

Definition 2.11. With the previous notation, if f is a form of type (1) and $|\Lambda| > k+1$, then $X = V(f) \subset \mathbb{P}^N$ is called an **exceptional hypersurface**. For an exceptional hypersurface $\mathrm{hess}_f^k = 0$.

Corollary 2.3.2 ([13], Theorem 2.3)**.** *For each $N \geq 3$, for each $d \geq 5$ and for $2 \leq k < \lfloor \frac{d}{2} \rfloor$ there exist infinitely many irreducible hypersurfaces $X \subset \mathbb{P}^N$ of deg $f = d$ such that*

$$\mathrm{hess}_f \neq 0 \ \ \textit{and} \ \ \mathrm{hess}_f^r = 0 \ \ \textit{for} \ \ r = 2, \dots, k.$$

Furthermore, if $k+1 \leq \lfloor \frac{d}{2} \rfloor$, then $\mathrm{hess}_f^{k+1} \neq 0$.

Proof. Consider $f = g(u, v, x_2, x_3) + h(x_2, x_3) + p(x_4, \dots, x_N)$ with $g = x_2 u^{k-1} v^{d-k} + x_3 u^{d-2} v$, and let h and p be chosen to make f irreducible. Let $\tilde{f} = g + h$ and consider $\tilde{X} \subset \mathbb{P}^3$. For a general h one can check that

$\tilde{f}$ does not define a cone in $\mathbb{P}^3$, since its first partial derivatives are linearly independent. By theorem 2.1.5 we have $\mathrm{hess}_{\tilde{f}} \neq 0$. Notice that

$$\mathrm{Hess}_f = \begin{bmatrix} \mathrm{Hess}_{\tilde{f}} & 0 \\ 0 & \mathrm{Hess}_p \end{bmatrix}$$

Since $\mathrm{hess}_{\tilde{f}} \neq 0$ and for general p $\mathrm{hess}_p \neq 0$, one concludes that $\mathrm{hess}_f \neq 0$ for a general f of this type. On the other side, for each $r \leq k$ we consider $\alpha_j = X_2U^{r-1-j}V^j$ with $J = 0, \dots, r-1$. Thus $f_{\alpha_j} = a_j u^{k-r+j}v^{d-k-j} \in \mathbb{K}[u,v]$. Consider $\beta = X_3U^{r-2}V$ and $\gamma = X_3U^{r-1}$ so that $f_\beta = bu^{d-r}$, $f_\gamma = cu^{d-r-1}v \in \mathbb{K}[u,v]$. To show that these $r+2$ differentials are linearly independent in A_r it is enough to verify neither f_β nor f_γ is a scalar multiple of f_{α_j} for $j = 0, \dots, r-1$. If, by contradiction, this was the case, one would deduce either $j = d-k \leq r-1$, yielding $d < d-1$, or $j = d-k-1$ implying $d < d$; so we get a contradiction in both cases. Since $\dim_{\mathbb{K}} \mathbb{K}[U,V]_r = \binom{r+2-1}{r} = r+1$ and since we have got $r+2$ linearly independent differentials $\{\alpha_0, \dots, \alpha_{r-1}, \beta, \gamma\} \in A_r$, by proposition 2.3.1 follows $\mathrm{hess}_f^r = 0$, as claimed. To conclude the proof we must show $\mathrm{hess}_f^{k+1} \neq 0$ for the general f if $j+1 \leq \lfloor \frac{d}{2} \rfloor$. Consider $f = g+h+p$, then

$$\mathrm{Hess}_f^{k+1} = \begin{bmatrix} \mathrm{Hess}_g^{k+1} & 0 & 0 \\ 0 & \mathrm{Hess}_h^{k+1} & 0 \\ 0 & 0 & \mathrm{Hess}_p^{k+1} \end{bmatrix}$$

Since $\mathrm{hess}_h^{k+1} \neq 0$ and $\mathrm{hess}_p^{k+1} \neq 0$ for general h, p, it is enough to prove $\mathrm{hess}_g^{k+1} \neq 0$. Let $Q = \mathbb{K}[U,V,X_2,X_3]$ be the ring of differentials and consider $A = Q/\operatorname{Ann}\ g$, for $g = u^{k-1}x_2v^{d-k} + vx_3u^{d-2}$. Notice that $\dim A_{k+1} = 2k+4$ since an ordered $\mathbb{K}-$basis for A_{k+1} is

$$\mathcal{B} = \{\alpha_1 = U^{k+1}, \alpha_2 = U^kX_3, \alpha_3 = U^kV, \alpha_4U^{k-1}VX_3, \beta_0 = V^{k+1}, \gamma_0 = V^kX_2, \beta_1 = V^kU,$$

$$\gamma_1 = V^{k-1}UX_2, \dots, \beta_i = V^{k+1-i}U^i, \gamma^i = V^{k-i}X_2U^i, \dots, \beta_{k-1} = V^2U^{k-1}, \gamma_{k-1} = VU^{k-1}X_2\}.$$

The matrix Hess_g^{k+1} can be partitioned in blocks, induced by the partition of basis $\mathcal{B}$ by choosing the first four vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $2k$ other ones.

$$\mathrm{Hess}_g^{k+1} = \begin{bmatrix} \Theta_{4\times 4} & 0_{2k\times 4} \\ 0_{4\times 2k} & \Delta_{2k\times 2k} \end{bmatrix}.$$

The zero matrices $4\times 2k$ and $2k\times 4$ follows from $\alpha_i\beta_j = U^{k+j-i+2}V^{k+1-j}X_3^{i-1} \in$ Ann g and $\alpha_i\gamma_i = U^{k+j-i+2}V^{k-j}X_2X_3^{i-1} \in$ Ann g for every $i = 1,2,3,4$ and $j = 0;\ldots,k-1$. We claim that

$$\Theta_{4\times 4} = \begin{bmatrix} * & * & * & * \\ * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}.$$

As elements on the off diagonal are not zero, det $\Theta \neq 0$. Indeed the elements of the off diagonal are $\alpha_1\alpha_4 = \alpha_2\alpha_3 = U^{2k}VX_3 \notin$ Ann g and elements of the lower triangle $\alpha_2\alpha_4, \alpha_3^2, \alpha_3\alpha_4, \alpha_4^2 \in$ Ann g. In the same way

$$\Delta_{2k\times 2k} = \begin{bmatrix} * & * & \ldots & * \\ * & \ldots & * & 0 \\ \ldots & \ldots & 0 & 0 \\ * & 0 & \ldots & 0 \end{bmatrix}.$$

In fact, the lower triangle is zero matrix because $\beta_i\gamma_j = U^{i+j}V^{2k+1-i-j}X_2 \in$ Ann g if $i+j > k-1$. On the contrary, the elements of the off diagonal are not zero, because they are $\beta_i\gamma_{k-1-i} = V^{k+2}U^{k-1}X_2 \notin$ Ann g. Therefore det $\Delta \neq 0$; hence the result follows. □

Remark 17. The previous corollary highlights the special role of exceptional hypersurfaces respect to Gordan & Noether hypersrufaces; in fact the first ones are surely of a different nature and not associated to the second ones for their different construction.

The second family of hypersurfaces we construct was inspired by the Perazzo's hypersurfaces and by Gordan & Noether polynomials. We call them **GNP-hypersurfaces of type** (m,n,k,e). In the next proposition we are going to give a characterization of them. But first we notice that they are a natural generalization of Perazzo's hypersurfaces; for instance any GNP-hypersurface of type $(m,n,1,e)$ is a Perazzo's one. Moreover they are also a generalization of some special cases of GN-polynomials, namely when $\mu = 1$, and the general case assuming $P_j = 0, j \neq 0, \mu$.

Proposition 2.3.3 ([13], Proposition 2.5)**.** *Let $x_0, \dots, x_n$ and $u_1, \dots, u_m$ be independent sets of variables with $m, n \geq 2$. For $j = 1, \dots, s$, let $f_j \in \mathbb{K}[x_0, \dots, x_n]_k$ be linearly independent forms whose sum does not define a cone and with*
g.c.d$(f_1, \dots, f_s) = 1$. Let $g_j \in \mathbb{K}[u_1, \dots, u_m]_e$ be linearly independent forms whose sum does not define a cone, let $h \in \mathbb{K}[u_1, \dots, u_m]_{e+k}$ be the a general form and let $1 \leq k < e$. If $s > \binom{m-1+k}{k} = \dim_{\mathbb{K}} \mathbb{K}[U_1, \dots, U_m]_k$, then the hypersurface $X = V(f) \subset \mathbb{P}^{m+n}$ of degree $d = e + k$ given by

$$f = f_1 g_1 + \dots + f_s g_s + h$$

is an irreducible projective hypersurface, not a cone, such that:

$$\text{hess}_f^k = 0.$$

Such a hypersurface will be called a ***GNP-hypersurface of type*** (m, n, k, e)*.*

Proof. Let $R = \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$, let $Q = \mathbb{K}[X_0, \dots, X_n, U_1, \dots, U_m]$ be the associated ring of differentials operators and let, as usual, $A = Q/\operatorname{Ann} f$ be the associated artinian Gorenstein algebra.
Let us consider a basis of A_k whose first s vectors $\alpha_1, \dots, \alpha_s$ are the dual of $f_1, \dots, f_s$, this means $\alpha_i(f_j) = \delta_{ij}$. Notice that $\alpha_j(f) = g_j \in \mathbb{K}[u_1, \dots, u_m]_e$, for $j = 1, \dots, s$, and by hypothesis $s > \binom{m-1+k}{k} = \dim_{\mathbb{K}} \mathbb{K}[U_1, \dots, U_m]_k$. Thus by proposition 2.3.1 follows. □

Now we want to prove the existence of families of GNP-hypersurfaces of type (m, n, k, e) for every codimension $N + 1 = m + n$ and for every degree $d = e + k$. The adopted strategy is to estimate the possible values of $\dim A_1$ for GNP-hypersurfaces of type (m, n, k, e) with $m \geq 2$.

Definition 2.12. Set

$\mathcal{A}^m_{k,e} = \{\dim A_1 : A = Q/\operatorname{Ann} f, f \text{ is a GNP hypersurface of type } (m, n, k, e)\}.$

Denote $a_m = a_m(k, e) = min\ \mathcal{A}^m_{k,e}$ and $b_m = b_m(k, e) = max\ \mathcal{A}^m_{k,e}$.

Lemma 7 ([13], Lemma 2.7)**.**

$$\mathcal{A}^2_{k,e} = \{5, 6, 7, \dots, e + 3\}.$$

Proof. By proposition 2.3.3, since $\dim \mathbb{K}[u,v]_k = k+1$, it is enough to exhibit $k+2$ linearly independents $g_j \in \mathbb{K}[u,v]_e$. Let

$$g_0 = u^e, g_1 = u^{e-1}v, \ldots, g_k = u^{e-k}v^k, g_{k+1} = u^{e-k-1}v^{k+1}$$

the minimal number of separated variables is 3 and we can take

$$f_0 = x^k, f_1 = x^{k-1}y, \ldots, f_k = y^k, f_{k+1} = z^k.$$

Therefore $f = x^k u^e + x^{k-1}yu^{e-1}v + \cdots + y^k u^{e-k}v^k + z^k u^{e-k-1}v^{k+1} + h(u,v)$, for a general h it is a GNP hypersurface of type $(2,2,k,e)$. Hence $\dim A_1 \geq 5$. The maximal number of linearly independents $g_j \in \mathbb{K}[u,v]_e$ is $\dim_{\mathbb{K}} \mathbb{K}[u,v]_e = e+1$ and the maximal number of separated variables is $e+1$, and we can take

$$f_0 = x_0^k, f_1 = x_1^k, \ldots, f_{k+1} = x_{k+1}^k.$$

Therefore $\dim A_1 \leq e+3$ and all intermediate values are assumed. □

Theorem 2.3.4 ([13], Theorem 2.8)**.** *For each $N \geq 4$, $d \geq 3$ and $1 \leq k < \lfloor \frac{d}{2} \rfloor$ there are infinitely many irreducible GNP-hypersurfaces $X = V(f) \subset \mathbb{P}^N$ of type (m,n,k,e) with $N = m+n$ and degf=d=e+k. For these hypersurfaces* $\text{hess}_k^f = 0$.

Proof. Following *Definition 2.12* we easily see that $\mathcal{A}_{k,d}^m = \{a_m, a_{m+1}, \ldots, b_m\}$. As for lemma 7 $\mathcal{A}_{k,d}^2 = \{5, 6, \ldots, d+3\}$, it is enough to prove that $a_{m+1} < b_m$ for $m \leq 2$. In order to verify the inequality it occurs to compute b_m and estimate a_{m+1}.

1. **Computation of** b_m.
   Let us fix m, k, e in order to maximize $\dim A_1 = m+n+1$. To do this we need to maximize n, since m is fixed. Thus we can use a complete basis, for example standard, of $\mathbb{K}[u_1, \ldots, u_m]_e$ as g_j and after that take a separated variable from each one of them. Hence we get $s = n+1 = \binom{m-1+e}{e}$ and

   $$b_m = \dim A_1 = N = m+n+1 = m+s = m + \binom{m-1+e}{e}.$$

2. **Estimation of** a_{m+1}.
   We construct an explicit example in order to get a weak estimate for a_{m+1}. Consider $g_j, h \in \mathbb{K}[u_1, \dots, u_{m+1}]$, with $j = 0, \dots, m+k$:

   $g_0 = u_1^e, g_1 = u_1^{e-1}u_2, \dots, g_k = u_1^{e-k}u_2^k, g_{k+1} = u_1^{e-1-k}u_2^{k+1}, g_{k+2} = u_3^e, \dots, g_{m+k} = u_{m+1}^e$

   and take $f_j \in \mathbb{K}[x_0, \dots, x_m]$, that is we are assuming $n = m$.

   $f_0 = x_0^k, f_1 = x_0^{k-1}x_1, \dots, f_k = x_1^k, f_{k+1} = x_2^{k-1}x_1, f_{k+2} = x_2^k, \dots, f_{m+k} = x_m^k.$

   To guarantee the irreducibility of f one can thus take $h = u_2^{e+k}$. For this explicit GNP hypersurface of type (m, n, k, d) we have $\dim A_1 = m + n + 1 \overset{m=n}{=} 2m + 1$, yielding

$$a_{m+1} \leq 2m + 1.$$

Now notice that $a_{m+1} \leq 2m + 1 < m + \binom{m-1+e}{e} = b_m$, for all $e \geq 2$ and for all $m \geq 2$, since

$$\binom{m-1+e}{e} = \binom{m+(e-1)}{(e-1)+1} > \binom{m+e-1}{1} = m + e - 1 \overset{e \geq 2}{\geq} m + 1.$$

The result follows by the fact

$$\bigcup_{m \geq 2} \mathcal{A}_{k,e}^m = \{5, 6, \dots\}$$

□

Summarizing the results of this section we proved the following generalization of Gordan and Noether theorem, namely Theorem 2.1.6.

Corollary 2.3.5 ([13], Corollary 2.10)**.** *For each pair $(N, d) \notin \{(3,3), (3,4)\}$ with $N \geq 3$ and $d \geq 3$, and for each $1 \leq k < \lfloor \frac{d}{2} \rfloor$ there exist infinitely many irreducible hypersurfaces $X = V(f) \subset \mathbb{P}^N$ not cones such that* $\mathrm{hess}_f^k = 0$.

2.4 Artinian Gorenstein algebras failing the Lefschetz properties

The goal of this section is to apply results of the last section in order to construct artinian Gorenstein algebras that do not satisfy the Lefschetz properties. The bridge between this section and the last one is the result due to Watanabe, the aforementioned Theorem 2.1.3 which has the following corollary.

Corollary 2.4.1 ([13], Corollary 3.1)**.** *Let* $A = Q/\operatorname{Ann}_Q f$ *be a standard graded artinian Gorenstein algebra with* $f \in \mathbb{K}[x_0, \dots, x_N]_d$*, suppose* $(\operatorname{Ann}_Q\ f)_1 = 0$*. Then:*

1. $A \simeq \dfrac{Q}{\operatorname{Ann}_Q\ f}$ *satisfies the SLP* $\iff$ $\operatorname{hess}_f^k \neq 0$ *for every* $k = 1, \dots, \lfloor \frac{d}{2} \rfloor$*.*
2. *If* $d \leq 4$*, then* A *satisfies the SLP* $\iff$ $\operatorname{hess}_f \neq 0$*. In particular for* $N \leq 3$*, every* A *satisfies the SLP.*
3. *For every* $N \geq 4$ *and for* $d = 3, 4$ *a polynomial* $f \in \mathbb{K}[x_0, \dots, x_N]_d$ *with vanishing Hessian and with* $(\operatorname{Ann}_Q\ f)_1 = 0$ *produces an example of a graded artinian Gorenstein algebra* $A = \displaystyle\bigoplus_{i=0}^{d} A_i$ *not satisfying the SLP.*

Corollary 2.4.2 ([13], Corollary 3.2)**.** *For each pair* $(N, d) \notin \{(3,3), (3,4)\}$ *with* $N \geq 3$ *and* $d \geq 3$*, there exist infinitely many standard graded artinian Gorenstein algebras* $A = \displaystyle\bigoplus_{i=0}^{c} A_i$ *of codimension* $\dim A_1 = N+1 \geq 4$ *and socle degree* d *that do not satisfy the SLP. Furthermore, for each* $L = \displaystyle\sum_{i=0}^{N} a_i X_i \in A_1$ *we can choose arbitrarily level* k *for which the following map*

$$\times L^{d-2k} : A_k \to A_{d-k}$$

is not an isomorphism.

Corollary 2.4.3 ([13], Corollary 3.3)**.** *For each pair* $(N, d) \neq (3, 3)$ *with* $N \geq 3$ *and odd* $d = 2q + 1 \geq 3$*, there exist infinitely many standard graded artinian Gorenstein algebras* $A = \displaystyle\bigoplus_{i=0}^{d} A_i$ *with* $\dim A_1 = N + 1$ *and socle degree* d *with unimodal Hilbert vector and that do not satisfy the WLP.*

Chapter 3

GNP-polynomials associated to a homogeneous simplicial complex

This chapter deals with some applications regarding GNP-hypersurfaces of type (m, n, k, e). In detail, it deals with, thanks to R. Gondim advice, the study of two aspects of this kind of hypersurfaces. The first one is of algebraic nature and it concernes with study and the characterization of: Hilbert vector and generators of the annihilator of GNP-polynomial using simplicial complexes. The second one deals with the characterization of GNP-hypersurface as union of residue parts, obtained by the intersection between the hypersurface and the linear space $\mathbb{P}^{n+1}$.

3.1 Hilbert Vector and Annihilator of a GNP-polynomial $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(1,d-1)}$

We deal with standard bigraded artinian Gorenstein algebras $A = \bigoplus_{i=0}^{d} A_i$, such that

$$\begin{cases} A_d \neq 0 \\ A_k = \bigoplus_{i=0}^{k} A_{(i,k-i)} \text{ for } k < d \end{cases}.$$

The pair (d_1, d_2), such that $A_{(d_1,d_2)} \neq 0$ and $d_1 + d_2 = d$, is said the socle bidegree of A.

Remark 18. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with the direct sums, we get $A^*_{(i,j)} \simeq A_{(d_1-i,d_2-j)}$.

Let $R = \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]$ be the polynomial ring viewed as standard bigraded ring in the set of variables $\{x_1, \dots, x_n\}$ and $\{u_1, \dots, u_m\}$ and let $Q = \mathbb{K}[X_1, \dots, X_n, U_1, \dots, U_m]$ be the associated ring of differential operators. A homogeneous ideal $I \subset R$ is a bihomogeneous ideal if

$$I = \bigoplus_{i,j=0}^{\infty} I_{(i,j)}$$

where $I_{(i,j)} = I \cap R_{(i,j)}$ $\forall i, j$.

If $f \in R_{(d_1,d_2)}$ is a bihomogeneous polynomial of total degree $d = d_1 + d_2$, then $I = \mathrm{Ann}(f) \subset Q$ is a bihomogeneous ideal and $A = \frac{Q}{I}$ is a standard bigraded artinian Gorenstein algebra of socle bidegree (d_1, d_2) and codimension $N = n + m$.

Remark 19. Let $f \in R_{(d_1,d_2)}$ be a bihomogeneous polynomial of degree (d_1, d_2), and let A be the associated bigraded algebra of socle bidegree (d_1, d_2), then for $i > d_1$ or $j > d_2$:

$$I_{(i,j)} = Q_{(i,j)}.$$

In fact for all $\alpha \in Q_{(i,j)}$ with $i > d_1$ or $j > d_2$ we get $\alpha(f) = 0$, so $Q_{(i,j)} = I_{(i,j)}$. As a consequence, we have the following decomposition for all the A_k:

$$A_k = \bigoplus_{i \leq d_1, j \leq d_2, i+j=k} A_{(i,j)}.$$

Furthermore for $i < d_1$ and $j < d_2$, the evaluation map $Q_{(i,j)} \to A_{(d_1-i,d_2-j)}$ given by $\alpha \to \alpha(f)$ provides the following short exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow Q_{(i,j)} \longrightarrow A_{(d_1-i,d_2-j)} \longrightarrow 0.$$

We note that all bihomogeneous polynomials $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots u_m]_{(d_1,d_2)}$ can be written

$$f = \sum_{i=1}^{s} f_i g_i,$$

where $f_i \in \mathbb{K}[x_1, \dots, x_n]_{d_1}$ and $g_i \in \mathbb{K}[u_1, \dots, u_m]_{d_2}, \forall i \leq s$ are monomials.

Definition 3.1. A bihomogeneous polynomial

$$f = \sum_{i=1}^{s} f_i g_i \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(d_1,d_2)}$$

is of *monomial square free type* if all g_i are square free monomials.

In the paper [14], the authors bijectively associate to any bihomogeneous polynomial of monomial square free type a pure simplicial complex. Their combinatoric structure determines a set of generators of the annihilator ideal. In fact Theorem 3.1.1 describes the annihilator of a bihomogeneous polynomial of bidegree $(1, d-1)$ of monomial square free type, showing that it is a binomial ideal whose generators are determined by the combinatoric of the associated simplicial complex.
Using the same technique, we will describe the annihilator of a bihomogeneous polynomial of bidegree $(k, k+1)$ of monomials square free type. We need to recall some results presented in [14], introducing combinatoric objects.

Definition 3.2. Let $V = \{u_1, \dots, u_m\}$ be a finite set. A simplicial complex Δ with vertex set V is a collection of subsets of V, i.e. a subset of the power set 2^V, such that for all $A \in \Delta$ and for all subset $B \subset A$, we have $B \in \Delta$.

The members of Δ are referred as faces and maximal faces (respect to the inclusion) are the facets. If $A \in \Delta$ and $|A| = k$, it is called a $(k-1)-$face, or a face of dimension $k-1$. If all the facets have the same dimension d the complex is said to be homogeneous of (pure) dimension d. We say that Δ is

a simplex if $\Delta = 2^V$.
In our context we identify the faces of a simplicial complex with monomials in the variables $\{u_1, \dots, u_m\}$. To any finite subset of $F \subset \{u_1, \dots, u_m\}$ we associate the monomial $M(u) = \prod_{u_i \in F} u_i$. In this way there is a natural bijection between the simplicial complex Δ and set of monomials $M(u)$, where F is a facet of Δ.

Let $f_\Delta = \sum_{i=1}^{n} x_i g_i$ in $\mathbb{K}[x_1, \dots, x_n, u_1, \dots u_m]_{(1,d-1)}$ be a bihomogeneous polynomial of monomial square free type associated to a homogeneous simplicial complex Δ of dimension $d-2$. The facets are given by monomials $g_i \in \mathbb{K}[u_1, \dots u_m]_{d-1}$. The vertex set of Δ is also called $0-$skeleton and they write $V = \{u_1, \dots, u_m\}$. In addition they identify $1-$skeleton the simple graph $\Delta_1 = (V, E)$, that is a graph which is not oriented. The $1-$faces are called edges. In the end they denote by e_k the number of $(k-1)-$faces, hence $e_0 = 1$, $e_1 = m$, $e_{d-1} = n$ and $e_j = 0$, for $j \geq d-1$. Moreover the associated algebra is $A_\Delta = Q/\operatorname{Ann}(f_\Delta)$. By abuse of notation, we will always denote f_Δ with f and A_Δ with A.

If $p \in \mathbb{K}[u_1, \dots, u_m]$ is a square free monomial, we denote by $P \in \mathbb{K}[U_1, \dots, U_m]$ the dual differential operator $P = p(U_1, \dots, U_m)$.

Theorem 3.1.1 ([14], Theorem 3.2)**.** *Let Δ be a homogeneous simplicial complex of dimension $d-2$ and let $A = Q/\operatorname{Ann}(f)$, where $f = \sum_{i=1}^{n} x_i g_i$ with $g_i \in \mathbb{K}[u_1, \dots, u_m]_{d-1}$, be the associated algebra. Then*

(1) $A = \bigoplus_{k=0}^{d} A_k$ *where* $A_k = A_{(0,k)} \oplus A_{(1,k-1)}$*;*

(2) $A_{(0,k)}$ *has a basis identified with the $(k-1)-$faces of Δ, hence* $\dim A_{(0,k)} = e_k$*;*

(3) By duality, $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$, and a basis for $A_{(1,k-1)}$ can be chosen by taking, for each $(d-k-1)-$face of Δ, a monomial $X_i\tilde{G}_i$ such that $X_i\tilde{G}_i(f)$ respresents it;

(4) the Hilbert vector of A is given by $h_k = \dim A_k = e_k + e_{d-k}$;

(5) $I = \mathrm{Ann}_Q(f)$ is generated by

(a) $\langle X_1, \ldots, X_n \rangle^2 ; U_1^2, \ldots, U_m^2$;

(b) the monomials in I respresenting non minimal faces not in Δ, that is all non minimal faces of complement of Δ, $\Delta^c = \{u_1, ..., u_m\}$;

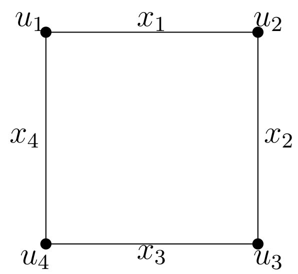
(c) the monomials $X_i F_i$ where F_i where f_i does not represent a subface of g_i in Δ, that is F_i is a dual differential operator of f_i and f_i is in Δ^c;

(d) the binomials $X_i\tilde{G}_i - X_j\tilde{G}_j$ where $g_i = \tilde{g}_i g_{ij}$ and $g_j = \tilde{g}_j g_{ij}$ and g_{ij} represents a common subface of g_i, g_j.

Proof. See [14]. □

Now we apply the above theorem to the following example:

Example 3.1. Let Δ be a square of vertices u_1, u_2, u_3 and u_4 and edges x_1, x_2, x_3 and x_4 :



Let

$$f = f_\Delta = x_1u_1u_2 + x_2u_2u_3 + x_3u_3u_4 + x_4u_1u_4$$

be the bihomogeneous polynomial associated to Δ of degree three. Since $\operatorname{hess} f = 0$, it is a GNP - polynomial of type $(4, 3, 1, 2)$ and the monomials in $u_1, \ldots, u_3$ an u_4 are square free.

So

$$A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$$

and the Hilbert vector is given by:

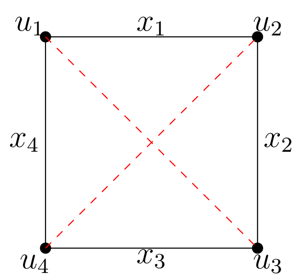
$$h_0 = 1 = h_3 \text{ and } h_1 = 8$$

We must calculate $h_2 = \dim A_2$. By theorem (3.1.1), we get

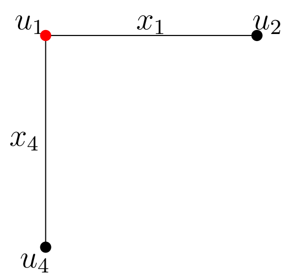
$$h_2 = \dim A_2 = e_2 + e_1 = 4 + 4 = 8.$$

The Hilbert vector is $(1, 8, 8, 1)$.
Moreover $I = Ann_q(f)$ is generated by the:

- $\langle X_1, X_2, X_3, X_4 \rangle^2$ and U_1^2, U_2^2, U_3^2 and U_4^2;
- monomials U_1U_3 and U_2U_4, representing the diagonals of square:



- binomials $X_1U_2 - X_4U_4$, $X_1U_1 - X_2U_3$, $X_2U_2 - X_3U_4$ and $X_4U_1 - X_3U_3$. In fact, the edges x_1 and x_4 have the common vertex u_1, so $g_{2,4}$ represents the vertex u_1 :



Therefore

$$\tilde{g}_2 = u_2 \text{ and } \tilde{g}_4 = u_4.$$

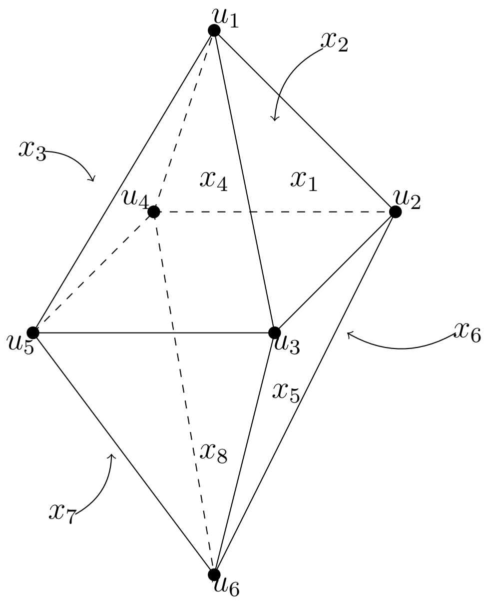
Finally we have

$$\tilde{G}_2 = \tilde{g}_2(U_1, U_2, U_3, U_4) = U_2 \text{ and } \tilde{G}_4 = \tilde{g}_4(U_1, U_2U_3U_4) = U_4.$$

We have the binomial $X_1U_2 - X_4U_4$. The same procedure holds for the other binomials.

Instead in the following example we use a GNP - polynomial of degree 4:

Example 3.2. Let Δ be a simplicial complex of vertices $u_1, \dots u_5$ and u_6 and faces $x_1, \dots, x_7$ and x_8:



Let

$$f = f_\Delta = x_1u_1u_2u_3 + x_2u_1u_2u_4 + x_3u_1u_4u_5 + x_4u_1u_3u_5 + \\ + x_5u_2u_3u_6 + x_6u_2u_4u_6 + x_7u_4u_5u_6 + x_8u_3u_5u_6$$

be the bihomogeneous polynomial of degree 4. It is a GNP-polynomial of type $(6, 8, 1, 3)$ and the monomials in u_1, u_2, u_3, u_4, u_5 and u_6 are square free.

So

$$A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$$

and the Hilbert vector is given by:

$$h_0 = 1 = h_4 \text{ and } h_1 = 14 = h_3.$$

We must calculate $h_2 = \dim A_2$.

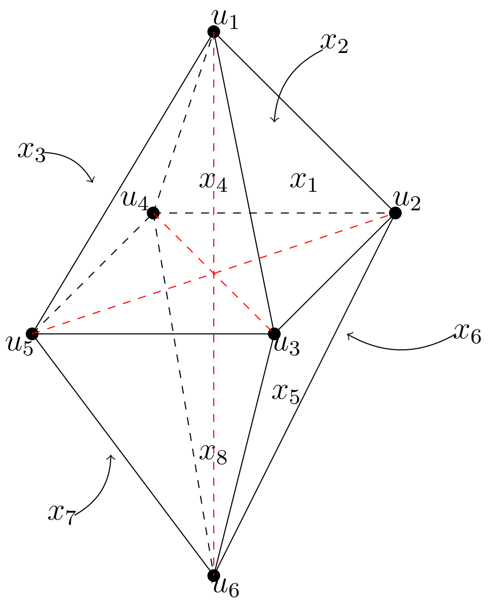
By Theorem 3.1.1, we have

$$h_2 = \dim A_2 = e_2 + e_2 = 2e_2 = 2 \cdot 12 = 24.$$

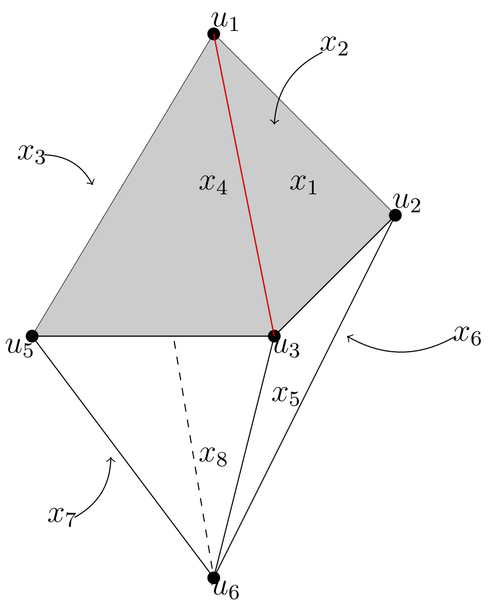
So the Hilbert vector is $(1, 14, 24, 14, 1)$.

By Theorem 3.1.1, $I = \mathrm{Ann}(f)$ is generated by:

- $\langle X_1, \ldots, X_8 \rangle^2$ and $U_1^2, \ldots, U_m^2$;
- the monomials U_1U_6, U_3U_4 and U_2U_5, representing the diagonals of the figure:



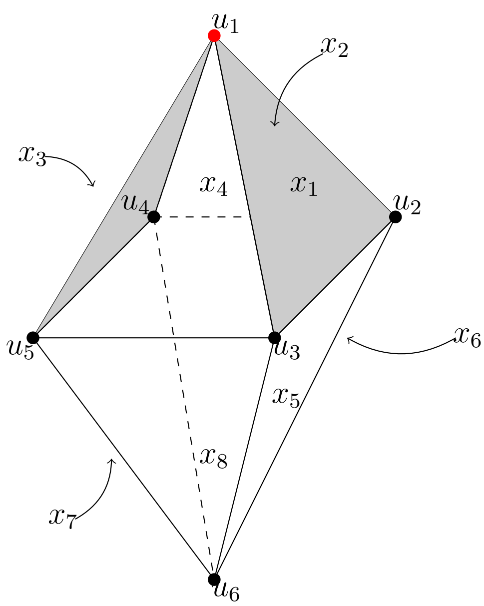
The faces x_1 and x_4 have the common edge that joins the vertices u_1u_3 :



So $g_{1,4}$ represents the edge that joins the vertices u_1 and u_3. $\tilde{g_1}$ and $\tilde{g_4}$ represent the vertices u_2 and u_5 respectively. We have:

$$\tilde{G_1} = \tilde{g_1}(U_1, \ldots, U_6) = U_2 \text{ and } \tilde{G_4} = \tilde{g_4}(U_1, \ldots, U_6) = U_5.$$

The binomial, of degree 2, $X_1U_2 - X_4U_5$, is in $I = \mathrm{Ann}(f)$. The other binomials of degree 2 are obtained with the same procedure. We note that the faces x_1 and x_3 have the common vertex u_1 :



So $g_{1,3}$ represents the vertex u_1 and therefore $\tilde{g_1}$ and $\tilde{g_3}$ represent the edges that joins the vertices u_2u_3 and u_4u_5 respectively. We have:

$$\tilde{G_1} = \tilde{g_1}(U_1, \ldots, U_6) = U_2U_3 \text{ and } \tilde{G_3} = \tilde{g_3}(U_1, \ldots, U_6) = U_4U_5.$$

The binomial, of degree 3, $X_1U_2U_3 - X_3U_4U_5$ is in $I = \mathrm{Ann}(f)$; the other binomials, of degree 3, are obtained with the same procedure.

3.2 Hilbert Vector and Annihilator of a GNP-polynomial $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$

Let $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ be a bihomogeneous polynomial of degree $d = 2k+1$, with $n \geq m \geq 3$, expressed in this way:

$$f = \sum_{r=1}^{n} x_r^k g_r \tag{3.1}$$

with g_r monomials in variables $u_1, \dots, u_m$ of degree $k+1$ and square free.

In this section we want to characterize the Hilbert vector of a standard graded artinian Gorestein algebra, generated by f of type (3.1).

Let $f_\Delta \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ be a bihomogeneous polynomial of type (3.1) associated to a homogeneous simplicial complex Δ of dimension k. The facets are given by monomials $g_i \in \mathbb{K}[u_1, \dots, u_m]_{k+1}$ and the vertex set of Δ is $V = \{u_1, \dots, u_m\}$. We denote the facets of Δ with x_i^k. Moreover we denote by e_k the number of $(k-1)-$faces, hence $e_0 = 1$, $e_1 = m$, $e_{k+1} = n$ and $e_j = 0$ for $j \geq k+1$. The associated algebra is $A_\Delta = Q/\operatorname{Ann}(f_\Delta)$. By abuse of notation, we will always denote f_Δ with f and A_Δ with A.
We remember that $p \in \mathbb{K}[u_1, \dots, u_m]$ is a square free monomial, we denote by $P \in \mathbb{K}[U_1, \dots, U_m]$ the dual differential operator $P = p(U_1, \dots, U_m)$.

Theorem 3.2.1. *Let Δ be a homogeneous simplicial complex of dimension k and let $A = Q/\operatorname{Ann}(f)$, where f is of type (3.1), be the associated algebra. Then*

$$A = \bigoplus_{i=0}^{d=2k+1} A_i \text{ where } A_i = A_{(i,0)} \oplus A_{(i-1,1)} \oplus \dots \oplus A_{(0,i)}, A_d = A_{(k,k+1)}$$

1. *for all $j = 1, \dots, k+1$:*

$$\dim A_{(i,j)} = \begin{cases} e_j & \text{for} \quad i = 0 \\ n \cdot e_j & \text{for} \quad 1 \leq i < k \\ e_{k+1-j} & \text{for} \quad i = k \end{cases}$$

2. *$I = \operatorname{Ann}_Q(f)$ is generated by*

(a) $\langle X_1, \ldots, X_n \rangle^{k+1}$*,* $U_1^2, \ldots, U_m^2$*;*

(b) the monomials in I *representing non minimal faces non contained in* Δ*, that is all non minimal faces of complement of* Δ*;*

(c) the monomials $X_i^k F_i$ *where* f_i *does not represent a subface of* g_i *in* Δ*, that is* F_i *is a dual differential operator of* f_i *and* f_i *is in* Δ^c*;*

(d) the binomials $X_r^k \tilde{G}_r - X_s^k \tilde{G}_s$ *where* $g_r = \tilde{g}_r g_{rs}$ *and* $g_s = \tilde{g}_s g_{rs}$ *and* g_{rs} *represents a common subface of* g_r, g_s*.*

Proof. 1. Let f be of type (3.1) associated to the homogeneous simplicial complex Δ of dimension $k+1$. The variables $u_1, \ldots, u_m$ represents the vertices of Δ.

We consider the following cases:

- for $i = 0$ and $j = 1, \ldots, k+1$, $A_{(0,j)}$ is generated by the monomials of degree j, that represent $(j-1)-$ faces, i.e. $\dim A_{(0,j)} = e_j$, where e_j is the number of $(j-1)-$ faces of Δ. We need to show that they are linearly independent over $\mathbb{K}$.
  For any $(j-1)-$ face ω, let Ω be the associated monomial of $Q_{(0,j)}$ and let $\Omega_1, \ldots, \Omega_\nu$ be all of them. We take any linear combination:
  $$0 = \sum_{r=1}^{\nu} c_r \Omega_r(f) = \sum_{r=1}^{\nu} c_r \sum_{s=1}^{n} x_s^k \Omega_r(g_s) = \sum_{s=1}^{n} x_s \sum_{r=1}^{\nu} c_r \Omega_r(g_i).$$
  Therefore we get $\sum_{r=1}^{\nu} c_r \Omega_r(g_s) = 0$, for all $s = 1, \ldots, n$. For each $r = 1, \ldots, \nu$ there is a $s = 1, \ldots, n$, such that $\Omega_r(g_s) \neq 0$, so $c_r = 0$ for all r.
- for $i = k$ and $j = 1, \ldots, k$, by duality $A^*_{(0,k+1-j)} \simeq A_{(k,j)}$, so we have:
  $$\dim A_{(k,j)} = \dim A^*_{(0,k+1-j)} = e_{k+1-j};$$
- for $1 \leq i < k$ and $j = 1, \ldots, k+1$, let
  $$G_j = \prod_{1 \leq i_1 < \ldots < i_j \leq m} U_{i_1} \cdots U_{i_j}$$

be a monomials of degree j of $Q_{(0,j)}$. The generators of $A_{(i,j)}$ are the monomials of type $X_s^i G_j(f)$ for $s = 1, \ldots, n$, for all j. They are linearly independent on $\mathbb{K}$.
Fixed the index j, for $s = 1, \ldots, n$, we have:

$$X_s^i G_j(f) = X_s^i G_j \left(\sum_{\mu=1}^{n} x_\mu^k g_\mu \right) = x_s^{k-i} G_j(g_s).$$

Hence $\dim A_{(i,j)} = n \cdot e_j$.

2. Let $I = \operatorname{Ann}(f)$ be the annihilator. We consider the following exactly sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow Q_{(i,j)} \longrightarrow A_{(k-i,k+1-j)} \longrightarrow 0. \tag{3.2}$$

we have the following cases:

- for $i \geq k+1$ or $j \geq k+2$, by (3.2) we have

$$\dim A_{(k-i,k+1-j)} = 0 \Rightarrow I_{(i,j)} = Q_{(i,j)}$$

and it is generated by $\langle X_1, \ldots, X_n \rangle^{k+1}$.

- for $i \leq k$ and $j = 0$, by (3.2) we have:

$$\dim A_{(k-i,k+1)} = n \Rightarrow I_{(i,0)} = \operatorname{span}(S)$$

where $S = Q_{(i,0)} - \{X_1^i, \ldots, X_n^i\}$, i.e. $I_{(i,0)}$ is generated by monomials $G = \prod_{1 \leq r_1 \leq \ldots \leq r_u < r_v \leq \ldots \leq r_i \leq n} X_{r_1} \cdots X_{r_i}$ of degree i;

- for $i = 0$ and $1 \leq j \leq k+1$, by (3.2) we have

$$\dim A^*_{(0,j)} = \dim A_{(k,k+1-j)} = e_{k+1-j}$$

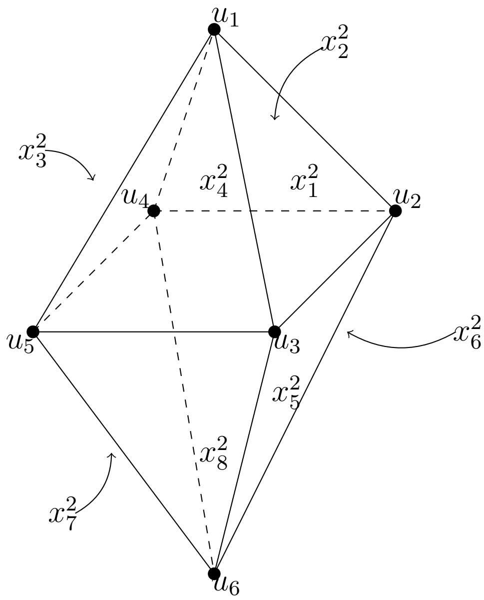
So $I_{(0,j)}$ is generated by monomials G no square free of degree j.

- for $1 \leq i < k$ and $1 \leq j \leq k+1$, by (3.2), $I_{(i,j)}$ is generated by monomials $X_r^k F_r$ where $F_r = f_r(U_1, \ldots, U_m)$, f_r does not represent a subface of g_i and by binomials $X_r^k \tilde{G}_r - X_s^k \tilde{G}_s$ where $g_r = \tilde{g}_r g_{rs}$ and $g_s = \tilde{g}_s g_{rs}$ and g_{rs} represents a common subface of g_r and g_s.

$\square$

We repropose Example 3.2, discussing it via the use of Theorem 3.2.1:

Example 3.3. Let Δ be a simplicial complex of vertices $u_1, \dots u_5$ and u_6 and faces $x_1^2, \dots, x_7^2$ and x_8^2:



Let

$$f = f_\Delta = x_1^2u_1u_2u_3 + x_2^2u_1u_2u_4 + x_3^2u_1u_4u_5 + x_4^2u_1u_3u_5 + \\ + x_5^2u_2u_3u_6 + x_6^2u_2u_4u_6 + x_7^2u_4u_5u_6 + x_8^2u_3u_5u_6$$

be the bihomogeneous polynomial of degree 5. It is a GNP-polynomial of type $(6, 8, 2, 3)$ and the monomials in u_1, u_2, u_3, u_4, u_5 and u_6 are square free.

So

$$A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$$

and the Hilbert vector is given by:

$$h_0 = 1 = h_4 \text{ and } h_1 = 14 = h_3.$$

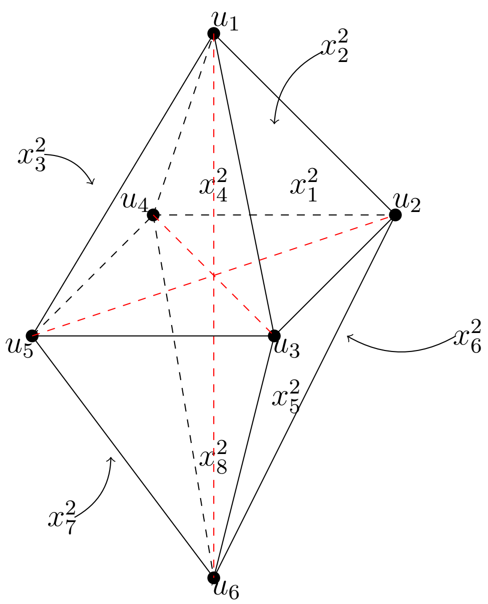
We must calculate $h_2 = \dim A_2$.
By Theorem 3.1.1, we have

$$h_2 = \dim A_2 = e_2 + e_2 = 2e_2 = 2 \cdot 12 = 24.$$

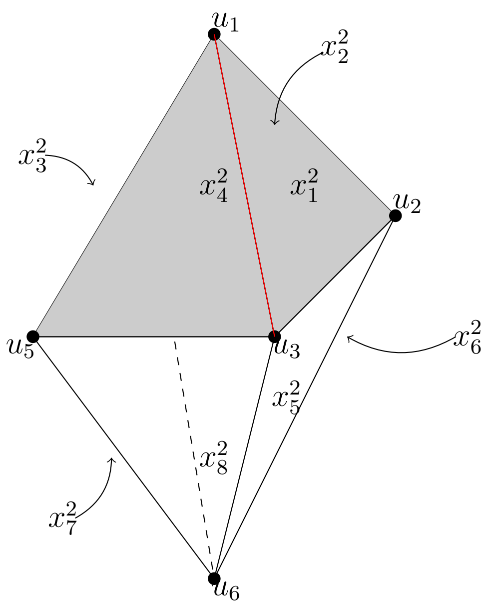
So the Hilbert vector is $(1, 14, 24, 14, 1)$.
By Theorem 3.1.1, $I = \mathrm{Ann}(f)$ is generated by:

- $\langle X_1, \ldots, X_8\rangle^3$ and $U_1^2, \ldots, U_m^2$;

- the monomials U_1U_6, U_3U_4 and U_2U_5, representing the diagonals of the figure:



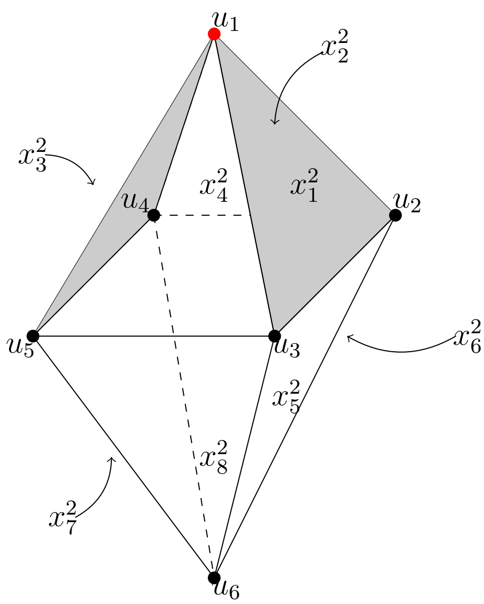
The faces x_1 and x_4 have the common edge that joins the vertices u_1u_3 :



So $g_{1,4}$ represents the edge that joins the vertices u_1 and u_3. $\tilde{g}_1$ and $\tilde{g}_4$ represent the vertices u_2 and u_5 respectively. We have:

$$\tilde{G}_1 = \tilde{g}_1(U_1, \ldots, U_6) = U_2 \text{ and } \tilde{G}_4 = \tilde{g}_4(U_1, \ldots, U_6) = U_5.$$

The binomial, of degree 3, $X_1^2 U_2 - X_4^2 U_5$, is in $I = \mathrm{Ann}(f)$. The other binomials of degree 3 are obtained with the same procedure. We note that the faces x_1 and x_3 have the common vertex u_1:



So $g_{1,3}$ represents the vertex u_1 and therefore $\tilde{g_1}$ and $\tilde{g_3}$ represent the edges that joins the vertices u_2u_3 and u_4u_5 respectively. We have:

$$\tilde{G_1} = \tilde{g_1}(U_1, \ldots, U_6) = U_2U_3 \text{ and } \tilde{G_3} = \tilde{g_3}(U_1, \ldots, U_6) = U_4U_5.$$

The binomial, of degree 4, $X_1^2U_2U_3 - X_3^2U_4U_5$, is in $I = \mathrm{Ann}(f)$; the other binomials of degree 4 are obtained with the same procedure.

3.3 Geometry of GNP-hypersurfaces

Let $R = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring in $N = n + m + 1$ variables, with $\mathbb{K}$ a field of characteristic zero. Let $f \in R$ be a GNP-polynomial of type (m, n, k, e) and $\deg f = k + e$. As above f can be written as a bihomogeneous polynomial in the following way:

$$f = \sum_{i=0}^{s} f_i g_i$$

with $f_i \in \mathbb{K}[x_0, \ldots, x_n]_k$ and $g_i \in \mathbb{K}[u_1, \ldots, u_m]_e$ monomials, for every $i \leq s$.

Let $X = V(f) \subset \mathbb{P}^N$ be a GNP-hypersurface of type (m, n, k, e). We can consider two linear spaces respectively $\mathbb{P}^{m-1}$ with coordinates $u_1, \ldots, u_m$ and $\mathbb{P}^n$ with coordinates $x_0, \ldots, x_n$. Let $p_\alpha \in \mathbb{P}^{m-1}$ be a point, let us consider the following linear space of dimension $n + 1$:

$$\mathcal{L}_\alpha := \langle p_\alpha, \mathbb{P}^n \rangle = \{\langle p_\alpha, q \rangle : q \in \mathbb{P}^n\}.$$

If we consider the intersection $\mathcal{L}_\alpha$ with X, we obtain a variety Y_α, i.e. $Y_\alpha \supset \mathbb{P}^n + \tilde{Y_\alpha}$, where $\tilde{Y_\alpha}$ is called *residue* and it is a cone with vertex p_α.

Theorem 3.3.1. *A GNP-hypersurface $X = V(f) \subset \mathbb{P}^N$ of type (m, n, k, e) consists of the union of the residue parts $\tilde{Y_\alpha}$, i.e.*

$$X = \bigcup_\alpha \tilde{Y_\alpha}.$$

Proof. Fixed a point $p_\alpha = (0 : \cdots : 0 : a_1 : \cdots : a_m) \in \mathbb{P}^{m-1}$ and let $\overline{p} = (\overline{x_0} : \cdots : \overline{x_n} : 0 \cdots : 0)$ be a point in $\mathbb{P}^n$. We consider the line that joins the points p_α and $\overline{p}$:

$$\mathscr{L}_\alpha : \begin{cases} x_0 = \lambda \overline{x_0} \\ \dots\dots\dots \\ x_n = \lambda \overline{x_n} \\ u_1 = \mu a_1 \\ \dots\dots\dots \\ u_m = \mu a_m \end{cases}$$

with $\lambda, \mu \in \mathbb{K}$.
Since X is a GNP-hypersurface of type (m, n, k, e), we have:

$$f = f_1(x_0, \dots, x_n) g_1(u_1, \dots, u_m) + \dots + f_s(x_0, \dots, x_n) g_s(u_1, \dots, u_m) = 0.$$

If we consider the intersection between the line $\mathscr{L}_\alpha$ and the GNP-hypersurface X, we get:

$$\begin{aligned} f_{\mathscr{L}_\alpha} &= f_1(\lambda\overline{x_0}, \dots, \lambda\overline{x_n}) g_1(\mu a_1, \dots \mu a_m) + \dots + f_s(\lambda\overline{x_0}, \dots, \lambda\overline{x_n}) g_s(\mu a_1, \dots, \mu a_m) \\ &= \lambda^k \mu^e \left[f_1(\overline{x_0}, \dots, \overline{x_n}) g_1(a_1, \dots, a_m) + \dots + f_s(\overline{x_0}, \dots, \overline{x_n}) g_s(a_1, \dots, a_m) \right] \\ &= \lambda^k \mu^e \sum_{i=1}^{s} f_i(\underline{\overline{x}}) g_i(\underline{a}) \end{aligned}$$

where $\underline{a}$ is the vector $(a_1, \dots, a_m)$ and $\underline{\overline{x}}$ is the vector $(\overline{x_0}, \dots, \overline{x_n})$.
Since p_α and $\overline{p}$ are points of X, then $\sum_{i=1}^{s} f_i(\underline{\overline{x}}) g_i(\underline{a}) = 0$. So

$$\tilde{Y}_\alpha = \mathcal{Z}\left(\sum_{i=1}^{s} f_i(\underline{\overline{x}}) g_i(\underline{a}) \right)$$

therefore, by arbitrariness of the points $p_\alpha \in \mathbb{P}^{m-1}$ and $\overline{p} \in \mathbb{P}^n$, we have $\bigcup_\alpha \tilde{Y}_\alpha = X$.

$\square$

As consequence of the above theorem, we can say how many linear spaces there are in X a GNP- hypersurface of type (m, n, k, e). We note that $\mathbb{P}^{m-1}$ and $\mathbb{P}^n$ are linear spaces on X. Thus we have:

Corollary 3.3.2. *Let $X = V(f) \subset \mathbb{P}^N$ be a GNP- hypersurface of type (m, n, k, e). There is a family of lines of dimension $m + n - 2$ on X.*

Proof. Let $p_\alpha \in \mathbb{P}^{m-1}$ be a point, then there is a family of lines of dimension $n - 1$ that join p_α and the linear space $\mathbb{P}^n$. This is valid for all $p_\alpha \in \mathbb{P}^{m-1}$, so we have a family of lines of dimension $(n - 1) + (m - 1) = n + m - 2$ on X. It holds viceversa too: in fact let $\overline{p} \in \mathbb{P}^n$ be a point, then there is a family of lines of dimension $m - 2$ that join $\overline{p}$ and the linear space $\mathbb{P}^{m-1}$, for all $\overline{p} \in \mathbb{P}^m$. So the proof follows. $\square$

Bibliography

[1] D. Anick, *Thin algebras of Embedding dimension three*, J. Algebra **100** (1986) no.1, 235-259.

[2] M.F. Atiyah and I.G Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading (Mass.) (1969).

[3] D. Bernstein and A. Iarrobino, *A nonunimodal graded Gorenstein Artinian algebra in codimension five.* Comm. Algebra **20** (1992) no.8, 2323-2336.

[4] M. Boij, J. Migliore, R. M. Mirò-Roig, U. Nagel and F. Zanello, *On the shape of a pure O-sequence*, Mem. Amer. Math. Soc. **218** (2012) no. 1024, viii+78 pp.

[5] M. Boij and F. Zanello, *Level algebras with bad properties*, Proc. Amer. Math. Soc. **135** (2007) no. 9, 2713-2722.

[6] H. Brenner and A. Kaid, *Syzygy bundles on* $\mathbb{P}^2$ *and the Weak Lefschetz Property*, Illinois J. Math. **51** (2007) no.4, 1299-1308.

[7] D. Cook II and U.Nagel, *Enumerations deciding the weak Lefschetz property*, **arXiv: 1105.6062**.

[8] D. Cook II and U. Nagel, *The weak Lefschetz property, monomial ideals and lozenges*, Illinois J. Math. **55** (2012) no. 1, 377-395.

[9] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York (1995).

[10] J. Emsalem and A. Iarrobino, *Inverse system of a symbolic power I*, J. Algebra **174** (1995) no. 3, 1080-1090.

[11] R. Di Gennaro, G. Ilardi, J. Vallés; *Singular hypersurfaces characterizing the Lefschetz properties*, J. Lond. Math. Soc. (2) **89** (2014), 194-212.

[12] A.V. Geramita, T. Harima, J. C. Migliore and Y. S. Shin, *The Hilbert Function of a Level Algebra*, Mem. Amer. Math. Soc. **186** (2007) no. 872, vi+139 pp.

[13] R. Gondim, *On higher Hessians and Lefschetz properties*, J. Algebra **489** (2017), 214-263.

[14] R. Gondim and G. Zappalà, *Lefschetz properties for artinian Gorenstein algebras presented by quadrics*, **arXiv: 1601.04454** (2016).

[15] B. Harbourne, H. Schenck and A. Seceleanu, *Inverse systems, Gelfand-Tsetlin patterns and the Weak Lefschetz*, J. Lond. Math. Soc. **84** (2011) no. 3, 712-730.

[16] T. Harima,T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe, *The Lefschetz properties*, Lect Notes Math. **2080** Springer (2013).

[17] T. Harima, J. Migliore, U. Nagel, and J. Watanabe. *The Weak and Strong Lefschetz properties for artinian K-algebras*, J. Algebra **262** (2003) no.1, 99-126.

[18] T. Hausel, *Quaternionic geometry of matroids*, Cent. Eur. J. Math. **3** (2005) no. 1, 26-38.

[19] O. Hesse, *Uber die bedingung, unter welche eine homoge ganze function von n unabhangingen variabeln durch linear substitutionem von n andern unabhangigen variabeln auf eine homoge functon sich zuruck - fuhren lasst, die eine variable weniger euthalt*, J. Reine Angew. Math. **42** (1851), 117 - 124.

[20] O. Hesse, *Zur theorie der ganzen homogenen functionen*, J. Reine Angew. Math. **56** (1859), 263 - 269.

[21] H. Ikeda, *Results on Dilworth and Rees numbers of artinian local rings*, Japan. J. Math. (N.S.) **22** (1996) no. 1, 147-158.

[22] T. Maeno, J. Watanabe, *Lefschetz elements of artinian Gorenstein algebras and Hessians of homogeneous polynomials*, Illinois J. Math. **53** (2009) no.2, 592-603.

[23] E. Mezzetti, R. M. Mirò- Roig and G. Ottaviani, *Laplace equations and the weaak Lefschetz property*, Can. J. Math. **65** (2012) no. 3, 634-654.

[24] J. Migliore, R. Mirò-Roig and U. Nagel. *Monomial Ideals, almost complete intersections and the Weak Lefschetz property*; Trans. Amer. Math. Soc. **363** (2011) no.1, 229-257.

[25] J. Migliore, R. M. Mirò-Roig and U. Nagel, *On the Weak Lefschetz Property for Powers of Linear Forms*, Algebra and Number Theory **6** (2012) no.3, 487-526.

[26] J. Migliore and U. Nagel, *Gorenstein algebras presented by quadrics*, Collect. Math. **64** (2013) no.2, 211-233.

[27] J. Migliore and U. Nagel. *A tour of the Weak and Strong Lefschetz properties* J. Commut. Algebra **5** (2013) no.3, 329-358.

[28] H. Schenck and A. Seceleanu, *The Weak Lefschetz Property and powers of linear forms in K[x,y,z]*, Proc. Amer. Math. Soc. **138** (2010) no.7, 2335-2339.

[29] R. Stanley, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978) no.1, 57-83.

[30] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980) no. 2, 168-184.

[31] F. Zanello, *A non-unimodal codimension 3 level h-vector*, J. Algebra **305** (2006) no.2, 949-956.