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SOME RESULTS ON SUBNORMAL-LIKE SUBGROUPS

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CONTENTS

Introduction

A group G is said to be a *Dedekind group* if all its subgroups are normal. Obviously, every abelian group has this property, and each Dedekind group is not too far from being abelian. In fact, a classical result of R. Baer [1] and R. Dedekind [14] proves that a non-abelian group G has only normal subgroups if and only if $G \simeq Q \times A$, where Q is the quaternion group of order 8 and A is a periodic abelian group with no elements of order 4. In these groups normality is transitive, which is not so for an arbitrary group, the alternating group of order 4 being an example. This remark led H. Wielandt [62] to introduce in 1939 the concept of "subnormal subgroup": a subgroup X of a group G is said to be *subnormal* in G if there is a finite series of subgroups of the form

$$X = X_0 \triangleleft X_1 \triangleleft \ldots \triangleleft X_k = G;$$

the smallest non-negative integer k for which such a series exists being called the *subnormal defect* of X in G. Thus a subgroup is normal if and only if it is subnormal with defect at most 1. Subnormality is a concept of highest importance in group theory; for instance, a finite group is nilpotent if and only if all its subgroups are subnormal.

The structure of infinite groups in which every subgroup is subnormal can be much more complicated, and in fact H. Heineken and I.J. Mohamed (see [35]) in 1968 were the first to construct an example of an infinite group with trivial centre in which all proper subgroups are subnormal. On the other hand, it was proved by J.E. Roseblade [58] in 1965 that if G is a group, and there is a positive integer k such that all subgroups of G are subnormal with defect at most k, then G is nilpotent and its nilpotency class is bounded by a function of k. Although the example of H. Heineken and I.J. Mohamed shows that groups with all subgroups subnormal are in general far from being nilpotent, a fundamental result of W. Möhres [46] of 1990 states that these groups are at least soluble.

In 1972, R.E. Phillips [52] generalizes the subgroup property of "being subnormal" introducing *f*-subnormality: a subgroup H of a group G is said to be *f-subnormal* in G if there is a finite *f-series* from

H to G, that is a finite chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant ... \leqslant H_n = G$$

such that $|H_i : H_{i-1}| < \infty$ or $H_{i-1} \trianglelefteq H_i$ for every $i \in \{1, \ldots, n\}$.

Obviously this definition represents a substantial generalization of subnormality only for infinite groups and the subgroup

$$H = \langle 123 \rangle \times S_4 \times S_4 \times \ldots$$

is an example of an *f*-subnormal subgroup of the infinite group

$$G = S_4 \times S_4 \times S_4 \times \ldots$$

which is not subnormal.

Groups with all *f*-subnormal subgroups have been characterized by C. Casolo and M. Mainardis (see [8]) in 2001; their work is based on a generalization of the result of Möhres mentioned above. Indeed, their main theorem proves that a group with all subgroups *f*-subnormal is finite-by-soluble and their Theorem 1.4 proves that this result is the best possible.

Furthermore, it's easy to verify that a finitely generated group having only subnormal subgroups is nilpotent, and J.C. Lennox and S.E. Stonehewer proved ([44], Theorem 6.3.3) that a finitely generated group is finite-by-nilpotent if and only if all its subgroups are *f*-subnormal. It's important to stress that in a finite-by-nilpotent group all subgroups satisfy a stronger condition than *f*-subnormality. To be more precise, in a finite-by-nilpotent group G, each subgroup has finite index in a subnormal subgroup of G. This property has been considered in the second chapter of this dissertation, with respect to some "large subgroups".

Let G be a group, H a subgroup of G and n, m two fixed non-negative integers. If there is a subgroup H_0 containing H such that $|H_0 : H| \leqslant n$ and H_0 is subnormal in G with subnormal defect at most m, then we say that H is (n, m)-*subnormal* in G. We order the pairs (n, m) lexicographically and, with this ordering, if H is (n, m)-subnormal in G for some pair (n, m), then the least such pair is called the *near defect* of H in G (see [22, 42]).

The second chapter of this thesis deals with a generalization of Roseblade's theorem mentioned above. For fixed non-negative integers n, m we define the group class $S(n, m)$ to be the class of all groups of infinite rank (and the trivial groups) whose subgroups of infinite rank are (n, m)-subnormal in G and we prove that $S(n, m)$-groups are finite-by-nilpotent in a large class of generalized soluble groups. Recall that a group G is said to have *finite (Prüfer) rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exist, we will say that the group G has *infinite rank*.

In 1958 H. Wielandt [63] introduces the *Wielandt subgroup* $w(G)$ of a group G as the intersection of all normalizers of subnormal subgroups of G. It is clear that $w(G) = G$ if and only if every subnormal subgroup of G is normal in G and the consideration of the infinite dihedral group proves that $w(G)$ may as well be trivial; however, this cannot happen in a non-trivial group satisfying the minimal condition on subnormal subgroups (see [63]). Under this hypothesis D.J.S. Robinson [54] and J.E. Roseblade [57] proved that $w(G)$ has finite index in G.

In a nilpotent group G, the Wielandt subgroup obviously coincides with the *norm* $N(G)$ of the group, that was defined by R. Baer [2] in 1935 as the intersection of the normalizers of all subgroup of G. In 1960 E. Schenkman [59] showed that the norm of a group is contained in the second centre of the group and so for the case of nilpotent groups, the basic properties of the Wielandt subgroup are known.

Using the *f*-subnormality we define the *f-Wielandt subgroup* $\overline{w}(G)$ of a group G as the intersection of all normalizers of *f*-subnormal subgroups of G. It is clear that

$$Z(G) \leqslant N(G) \leqslant \overline{w}(G) \leqslant w(G)$$

where $Z(G)$ is the centre of the group G, and an example in chapter one will prove that $\overline{w}(G)$ can be strictly contained between $N(G)$ and $w(G)$. In addition we study the behavior of the *f*-Wielandt subgroup in a residually finite group and we prove that in a group G satisfying the minimal condition on subnormal subgroups $\overline{w}(G)$ has finite index in G.

As already noted, normality is not a transitive relation in an arbitrary group. Groups in which normality is a transitive relation are called T-*groups* or said to have the T-*property*. Obviously, every simple group has the T-property, but soluble T-groups have a restricted structure, that was studied by W. Gaschütz [27] in the finite case in 1957 and by D.J.S. Robinson [53] in the general case in 1964. In particular, it turns out that every soluble T-group is metabelian and that a finitely generated soluble group with the T-property is either finite or abelian.

Inspired by a relevant result of B.H. Neumann [47] which shows that a group has a finite commutator subgroup if and only if each of its subgroups has finite index in its normal closure, C. Casolo [7] in 1989 investigated the class T^* of all groups in which every subnormal subgroup has finite index in its normal closure. In particular, he proved that a soluble T^*-group is finite-by-metabelian and that a finitely generated soluble T^*-group is abelian-by-finite.
(Note that we will denote the normal closure and the core of a subgroup H of a group G with H^G and H_G respectively.)

In a dual way, in 1995, S. Franciosi, F. de Giovanni and M. Newell (see [26]) studied T_*-groups, namely groups in which each subnormal subgroup X is *normal-by-finite*, or, that is the same, $|X : X_G|$ is finite for each subnormal subgroup X of G. Using a famous result of J. Buckley, J.C. Lennox, B.H. Neumann and H. Smith [5] proving that a locally finite group whose all subgroups are normal-by-finite is abelian-by-finite, it is proved that a *subsoluble* group, that is a group having an ascending series with abelian factors consisting of subnormal subgroups, satisfying the property T_*, is metabelian-by-finite and that in a subsoluble T_*-group every finitely generated subgroup is abelian-by-finite and belongs to the class T_*.

In 2014 F. de Giovanni, M. Martusciello and C. Rainone defined the *normal oscillation* of a subgroup X of a group G as the cardinal number

$$\min\{|X : X_G|, |X^G : X|\}.$$

It was proved in [28] that a locally finite group in which each subgroup has finite normal oscillation, contains a nilpotent subgroup of finite index. The third chapter of this dissertation deals with the study of the class $T(*)$ of groups consisting of all groups whose subnormal subgroups have finite normal oscillation. We prove that a periodic subsoluble $T(*)$-group is metanilpotent-by-finite and that a finitely generated soluble $T(*)$-group contains an abelian subgroup of finite index.

Finally, in last chapter we show that the class of groups whose subgroups are *f*-subnormal and the group classes T^*, T_*, $T(*)$ are countably recognizable. Recall that a class of groups $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by R. Baer [3]. In his paper, Baer produced many interesting examples of countably recognizable group classes, and later many other relevant classes of groups with such a property were discovered (see for instance [20],[48],[50],[51],[60] and the more recent papers [29], [30], [31], [32]). We will here give a general method to prove that some *f*-subnormality related properties have countable character (see [25]).

Most of the notation used in this dissertation is standard and can for instance be found in [56].

Chapter 1

A Wielandt-like subgroup

In 1972 R.E. Phillips [52] introduces a generalization of subnormality in this way: a subgroup H of a group G is said to be *f-subnormal* in G if there exists an *f-series* from H to G, that is, a finite chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant ... \leqslant H_n = G$$

such that either $|H_i : H_{i-1}| < \infty$ or $H_{i-1} \trianglelefteq H_i$ for every $i \in \{1, \dots, n\}$[1]. Clearly, all subgroups of a finite group are *f*-subnormal and therefore it makes sense to study this property only in infinite groups.

Recall that the *norm* $N(G)$ of a group G was defined by R. Baer [2] in 1935 as the intersection of the normalizers of all subgroups of G and, in a similar way, H. Wielandt [63] in 1958 gave the definition of the *Wielandt subgroup* $w(G)$ of a group G as the intersection of the normalizers of all subnormal subgroups of G. He, among other results, proved that any minimal normal subgroup satisfying the minimal condition on normal subgroups is contained in $w(G)$, but the Wielandt subgroup can also be trivial as happens in the infinite dihedral group.

Using the *f*-subnormality instead of subnormality we can define the *f-Wielandt subgroup* $\overline{w}(G)$ of a group G as the intersection of the normalizers of all *f*-subnormal subgroups of G. It is clear that

$$Z(G) \leqslant N(G) \leqslant \overline{w}(G) \leqslant w(G)$$

where $Z(G)$ is the centre of G and the following example proves that $\overline{w}(G)$ can be strictly contained between $Z(G)$ and $w(G)$.

Let C be a cyclic group of order 7, let θ be an automorphism of order 3 of C and put B be the locally dihedral 2-group, that is, $B = \langle x \rangle \ltimes A$ where $A = C_{2^\infty}$ and x is such that $a^x = a^{-1}$ and $x^2 = 1$. Let $\Theta = \langle \theta \rangle$, $D = C \ltimes \Theta$ and consider the group $G = B \times D$.
The norm of the group B coincides with the center of B, while

$$\overline{w}(B) = w(B) = B.$$

[1] Note that n is a non-negative integer.

For the group D, instead, we have that

$$Z(D) = N(D) = \overline{w}(D) = \{1\},$$

while $w(D) = D$.
It follows that for the group G, $Z(G) = N(G) = Z(B)$, $\overline{w}(G) = B$ and $w(G) = G$, namely, the norm, the *f*-Wielandt subgroup and the Wielandt subgroup are three different subgroups of G.

In the first section of this chapter we examine some nice elementary properties of the *f*-Wielandt subgroup, while in the second section we examine the behaviour of $\overline{w}(G)$ when G is a group satisfying the minimal condition on subnormal subgroups. Our main result will show that under such hypothesis, the *f*-Wielandt subgroup has finite index in G.

1.1 Elementary results

In 1960 E. Schenkman [59] showed that the norm of a group G is contained in the second centre of G and hence for the case of nilpotent groups the basic properties of the Wielandt subgroup are known; for instance, the following easy lemma shows that the Wielandt subgroup of a torsion-free nilpotent group coincides with the centre of the group.

Lemma 1.1.1 *Let* G *be a torsion-free nilpotent group. Then* $w(G) = Z(G)$.

Proof — Suppose by contradiction that there is an element x contained in $w(G) \setminus Z(G)$. Then there exists an element $y \in G$ such that $[x, y] \neq 1$. The element x normalizes each subgroup of G and in particular $\langle y\rangle^x = \langle y\rangle$. It follows that $y^x = y^{-1}$ and hence $[x^2, y] = 1$. However $1 = [x^2, y] = [x, y]^2$ and therefore $[x, y] = 1$. □

In 1990 J. Cossey [13] proved that even in a residually nilpotent group G, $w(G) \leqslant Z_2(G)$ and the first new result of this section provides the same result for the *f*-Wielandt subgroup in a residually finite group. Note first that if G is a group and H is *f*-subnormal in G, then $\overline{w}(G) \cap H \leqslant \overline{w}(H)$ and if N is a normal subgroup of G, it is easy to prove that $\overline{w}(G)N/N \leqslant \overline{w}(G/N)$. Clearly in a group with all subgroups *f*-subnormal the *f*-Wielandt subgroup coincides with the norm of the group and hence in the case of finite group $\overline{w}(G) \leqslant Z_2(G)$.

Lemma 1.1.2 *In a residually finite group* G *the f-Wielandt subgroup* $\overline{w}(G)$ *is contained in the second centre of* G *and, in particular* $\overline{w}(G)$ *is a Dedekind group.*

Proof — Let N be a normal subgroup of G such that G/N is finite. Since $\overline{w}(G/N) \leqslant Z_2(G/N)$ we have that

$$\left[\frac{\overline{w}(G)N}{N}, \frac{G}{N}, \frac{G}{N}\right] = \{1\}$$

and hence

$$[\overline{w}(G), G, G] \leqslant N.$$

This relation is true for all normal subgroups of G of finite index and so $[\overline{w}(G), G, G] = \{1\}$. It follows that $\overline{w}(G) \leqslant Z_2(G)$.

Since any subgroup of $\overline{w}(G)$ is subnormal in G, any subgroup of $\overline{w}(G)$ will be normalized by $\overline{w}(G)$. Therefore $\overline{w}(G)$ is a Dedekind group. □

In a torsion-free residually finite group we can said more; in fact the *f*-Wielandt subgroup is abelian.

Proposition 1.1.3 *Let* G *be a torsion-free residually finite group. Then* $\overline{w}(G)$ *is abelian.*

Proof — Let a and b be elements of $\overline{w}(G)$ and let N be a normal subgroup of finite index of G. The factor group $\overline{w}(G/N)$ is a Dedekind group by Lemma 1.1.2 and so it is abelian or of the type $Q \times A$, where Q is the quaternion group of order 8 and A is a periodic abelian group with no elements of order 4. In the first case we have $[a, b]N/N = 1$ since $\overline{w}(G)N/N \leqslant \overline{w}(G/N)$. In the second one, instead, we have $[a^2, b]N/N = 1$, that is $[a^2, b] \in N$. In any case, $[a^2, b] \in M$ for all normal subgroups M of finite index of G and so $[a^2, b] = 1$. By Lemma 1.1.2

$$1 = [a^2, b] = [a, b]^2,$$

and since G is torsion-free we have $[a, b] = 1$. □

Furthermore the *f*-Wielandt subgroup of a torsion-free polycyclic group coincides with the centre of the group. To prove this we need to recall that in a polycyclic group G, the centralizer in G of $\overline{w}(G)$ has finite index in G as a corollary of Theorem 1 of [13].

Proposition 1.1.4 *Let* G *be a torsion-free polycyclic group. Then the f-Wielandt subgroup of* G *coincides with the centre of* G.

Proof — Let x be an element of $\overline{w}(G)$ and $y \in G$, then there exists a positive integer n such that $y^n \in C_G(\overline{w}(G))$. It follows by Lemma 1.1.2 that

$$1 = [x, y^n] = [x, y]^n$$

and hence $[x, y] = 1$ since the group G is torsion-free. □

1.2 Conjugacy classes of *f*-subnormal subgroups

J.E. Roseblade [44, Theorem 1.7.10] and D.J.S. Robinson [54] have proved that a subnormal subgroup has only finitely many conjugates in a group satisfying the minimal condition on subnormal subgroups. To prove a similar result for a *f*-subnormal subgroup we first need to prove that for a group is equivalent to satisfy the minimal condition on subnormal subgroups or the minimal condition on *f*-subnormal subgroups.

Proposition 1.2.1 *Let* G *be a group that satisfies the minimal condition on subnormal subgroups. Then* G *satisfies the minimal condition on f-subnormal subgroups.*

Proof — Let

$$G_1 > G_2 > G_3 > \dots$$

be a descending chain of *f*-subnormal subgroups of G.
By [8, Proposition 3.1], G_1 contains a subgroup R_1 of finite index such that R_1 is subnormal in G and certainly we may assume that $R_1 \triangleleft G_1$. Then $G_2R_1/R_1 \simeq G_2/(G_2 \cap R_1)$ is finite and $G_2 \cap R_1$ is *f*-subnormal in G. Applying [8, Proposition 3.1] again, we obtain a normal subgroup R_2 of finite index of G_2 which is subnormal in G and such that $R_2 \leqslant G_2 \cap R_1$. Continue in this way and suppose that for some positive integer i we have constructed a normal subgroup R_i of finite index of G_i which is subnormal in G and such that $R_i \leqslant G_i \cap R_{i-1}$. It follows that

$$G_{i+1}R_i/R_i \simeq G_{i+1}/(G_{i+1} \cap R_i)$$

is finite. Since $G_{i+1} \cap R_i$ is *f*-subnormal in G there is a normal subgroup R_{i+1} of G_{i+1} such that G_{i+1}/R_{i+1} is finite, R_{i+1} is subnormal in G and $R_{i+1} \leqslant G_{i+1} \cap R_i$ by [8, Proposition 3.1].
This construction gives us a descending chain

$$R_1 \geqslant R_2 \geqslant R_3 \geqslant \dots$$

of subnormal subgroups of G and the minimal condition on subnormal subgroups then implies that there is a positive integer m such that $R_i = R_{i+1}$ for all $i \geqslant m$. In particular, G_m/R_m is a finite group and contains the descending chain

$$G_{m+1}/R_m \geqslant G_{m+2}/R_m \geqslant \dots$$

and it follows that for some positive integer $l \geqslant m$ we have $G_i = G_{i+1}$ for all $i \geqslant l$.
This completes the proof. □

Since the converse implication is obvious, a group G satisfies the minimal condition on subnormal subgroups if and only if it satisfies the minimal condition on *f*-subnormal subgroups.

Theorem 1.2.2 *Let* G *be a group satisfying the minimal condition on subnormal subgroups and let* H *be a f-subnormal subgroup of* G*. Then* H *has only finitely many conjugates in* G*.*

Proof — Suppose that this result is false and let G be a group satisfying the minimal condition on subnormal subgroups containing an *f*-subnormal subgroup which has infinitely many conjugates.
By [44, Theorem 1.7.10] this subgroup is not subnormal in G. We choose an *f*-subnormal subgroup of G, X say, minimal subject to containing an *f*-subnormal subgroup H such that $|X : N_X(H)|$ is infinite, and without loss of generality, we may assume that $X = G$. Since G satisfies the minimal condition on subnormal subgroups, it has the finite residual R of finite index. Put $Y = HR$ and note that Y is *f*-subnormal in G. If $Y < G$, then the minimal choice of G shows that $|Y : N_Y(H)|$ is finite and since $|G : Y|$ is finite we obtain the contradiction that $|G : N_Y(H)|$ is finite. It follows that $G = HR$.
Let

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_{n-1} \leqslant H_n = G$$

be a series of *f*-subnormality of H in G. Then H_{n-1} is *f*-subnormal in G and we may assume that $H_{n-1} \neq G$. The minimal choice of G shows that $|H_{n-1} : N_{H_{n-1}}(H)|$ is finite. If $|G : H_{n-1}|$ is finite, then we obtain a contradiction as before. Hence we may assume that H_{n-1} in normal in G. Let S be the finite residual of H_{n-1}, so that $S \leqslant N_{H_{n-1}}(H)$. Then $[S, H] \leqslant H$.
The subgroup S is normal in G, since it is characteristic in H_{n-1} and hence the factor group $G/C_G(H_{n-1}/S)$ is finite. Thus R is contained in $C_G(H_{n-1}/S)$ and $[R, H_{n-1}] \leqslant S$. It follows that

$$[R, H, H] \leqslant [R, H_{n-1}, H] \leqslant [S, H] \leqslant H.$$

Hence $H \triangleleft H[R, H] \triangleleft RH = G$, so H is subnormal in G and in this case $|G : N_G(H)|$ is finite by [44, Theorem 1.7.10], giving us a final contradiction. □

The following result is now immediate.

Corollary 1.2.3 *Let* G *be a group satisfying the minimal condition on subnormal subgroups. Then* $|G : \overline{w}(G)|$ *is finite.*

Chapter 2

Groups with subgroups of bounded near defect

A non-empty collection $\mathfrak{X}$ of groups is a *group class* if every group isomorphic to a group in $\mathfrak{X}$ belongs itself to $\mathfrak{X}$ and $\mathfrak{X}$ contains a trivial group.

A property θ pertaining to subgroups of a group is called *absolute* if in any group G all subgroups isomorphic to some θ-subgroup are likewise θ-subgroups. Thus θ is absolute if and only if there exists a group class $\mathfrak{X} = \mathfrak{X}(\theta)$ such that in any group G a subgroup X has the property θ if and only if X belongs to $\mathfrak{X}$. Thus among the most natural absolute properties we have those of being an abelian subgroup, a nilpotent subgroup, a finite subgroup.

A subgroup property θ, instead, is called an *embedding* property if in any group G all images of θ-subgroups under automorphisms of G likewise have the property θ. Of course, any absolute property is trivially an embedding property, but the most relevant embedding properties, like normality and subnormality, are embedding properties which are not absolute.

If θ is an embedding property for subgroups, a group class $\mathfrak{X}$ is said to *control* θ if it satisfies the following condition: if G is any group containing some $\mathfrak{X}$-subgroup, and all $\mathfrak{X}$-subgroups of G have the property θ, then θ holds for all subgroups of G.
This definition can of course be given also inside a fixed universe $\mathfrak{U}$.

Clearly, the class of cyclic groups controls periodicity, and the class of finitely generated groups controls every local property but it is well-known that the class of finitely generated groups neither controls nilpotency nor solubility and although normality is controlled by the class of finitely generated groups it is easy to see that most of the significant embedding properties cannot be controlled by the class of finitely generated groups.
For instance, it is well-known that there exist unsoluble groups in which all finitely generated subgroups are subnormal, while an important result by W. Möhres [46] shows that every group in which all subgroups are subnormal is soluble. Therefore subnormality cannot be controlled by the class of finitely generated groups. This failure depends on the fact that finitely generated groups are "too small".

Therefore it is natural to consider the problem of "how large" should be $\mathfrak{X}$-groups in order to obtain that a group class $\mathfrak{X}$ controls the main embedding properties, at least within an appropriate universe. M. De Falco, F. de Giovanni and C. Musella give [16] in 2014 the following definition of class of large groups.

A group class $\mathfrak{X}$ is called a class of *large groups* if it satisfies the following conditions:

- if a group G contains an $\mathfrak{X}$-subgroup, then G belongs to $\mathfrak{X}$;
- if N is a normal subgroup of an $\mathfrak{X}$-group G, then at least one of the groups N and G/N belongs to $\mathfrak{X}$;
- $\mathfrak{X}$ contains no non-trivial finite cyclic groups.

Let $\mathfrak{X}$ be now a class of large groups, and let θ be a subgroup property. Since every group containing an $\mathfrak{X}$-subgroup likewise belongs to $\mathfrak{X}$, it follows that $\mathfrak{X}$ controls θ if and only if whenever in an $\mathfrak{X}$-group G all $\mathfrak{X}$-subgroups have the property θ, then θ holds for all subgroups of G.
The easiest non-trivial example of a class of large groups is provided by the class $\mathfrak{I}$ consisting of all infinite groups (and the trivial groups); however, the consideration of the locally dihedral 2-group shows that normality, and therefore also the subnormality, cannot be controlled by the class $\mathfrak{I}$, even in the universe of periodic metabelian groups.

A group G is said to have *finite (Prüfer) rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exist, we will say that the group G has *infinite rank*.
In particular, a group has rank 1 if and only if it is locally cyclic; it is easy to see that the class of groups of finite rank is closed with respect to subgroups, homomorphic images and extensions, and hence groups of infinite rank (and the trivial groups) form a class of large groups.

In a series of recent papers (see for instance [15, 18, 21, 23, 39]) it has been proved that in a (generalized) soluble group of infinite rank the behaviour of subgroups of finite rank with respect to an embedding property can be neglected in many cases, so that the class of groups of infinite rank controls such embedding property in a suitable universe of (generalized) soluble groups.

The first relevant theorem of this topic was obtained by M.J. Evans and Y. Kim [23] in 2004, and deals with the control of the property of being subnormal with defect at most k, for some fixed positive integer k, by the class of groups of infinite rank in the universe of $\mathfrak{X}$-groups.
This group class $\mathfrak{X}$ was defined by N.S. Černikov [11] in 1990 as the closure of the class of all periodic locally graded groups $\mathfrak{D}$ by the group theoretical operators $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$, $\mathbf{L}$ (see the first chapter of [56]).

Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. The class of locally graded groups is rather large, it contains for instance the locally finite groups, the locally soluble groups, the residually finite groups, the locally (soluble-by-finite) groups and the radical groups, and it is often considered in order to avoid finitely generated simple groups, and in particular the so-called *Tarski monsters*, that are infinite groups in which all proper subgroups are cyclic of the same order.

It is easy to prove that any $\mathfrak{X}$-group is locally graded, and that the class $\mathfrak{X}$ is also closed with respect to forming subgroups; however it is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank but this is true in the case of $\mathfrak{X}$-groups and this is an easy consequence of the following result by M. Dixon, M.J. Evans and H. Smith (see [19]).

Proposition 2.1 *Let* G *be a locally soluble group and suppose that every proper subgroup of* G *has finite rank. Then* G *has finite rank.*

In addition A.I. Mal'cev [45] proved the existence of an abelian subgroup of infinite rank in a locally nilpotent group and a corresponding result for locally finite groups was later proved by V.P. Šunkov [61]. Mal'cev's theorem was extended by R. Baer and H. Heineken [4] to the case of radical groups of infinite rank and recently the study of generalized radical groups of infinite rank was carried out in the papers [15, 37] (here a group is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite).

The structure of groups in which every subgroup of infinite rank is subnormal has been investigated by L.A. Kurdachenko and H. Smith (see [39, 40]) in 2004, who proved that locally (soluble-by-finite) groups with this property are soluble and all its finitely generated subgroups are subnormal. Moreover, they were able to show that the class of groups of infinite rank (and the trivial groups) controls subnormality in the universe of torsion-free locally (soluble-by-finite) groups and also in that of locally finite groups. On the other hand, a their example proves that the class of groups of infinite rank (and the trivial groups) cannot control subnormality even in the universe of metabelian locally nilpotent groups.

Let G be a group, H a subgroup of G and n, m two fixed non-negative integers. If there is a subgroup H_0 containing H such that $|H_0 : H| \leqslant n$ and H_0 is subnormal in G with subnormal defect at most m, then we say that H is (n, m)-*subnormal* in G. We order the pairs (n, m) lexicographically and, with this ordering, if H is (n, m)-subnormal in G for some pair (n, m), then the least such pair is called the *near defect* of H in G (see [42]). This chapter is concerned with subgroups with near defect at most (n, m), that is, subgroups whose near defect (n_0, m_0) is such that $n_0 \leqslant n$ and $m_0 \leqslant m$. Clearly if H is subnormal in G of defect at most d, then H is $(1, d)$-subnormal in G and every subgroup of a finite group G is $(|G|, 0)$-subnormal in G. J.C. Lennox [42, 43] called a subgroup *almost subnormal* if it is (n, m)-subnormal for certain non-negative integers n, m, a terminology is also used in [8, 9, 17]. In [15], $(n, 1)$-subnormal subgroups are called *nearly normal*.

Let G be a finite-by-nilpotent group and let N be a finite normal subgroup of G such that G/N is nilpotent. Then it is clear that every subgroup of G is of near defect at most (n, m), where $n = |N|$ and m is the nilpotency class of G/N. Lennox [42] proved the following theorem (where we let $\gamma_i(G)$ denote the ith term of the lower central series of G).

Theorem 2.2 *Let G be a group and let m, n be non-negative integers. Suppose that every finitely generated subgroup is of near defect at most (n, m). Then there is a function μ depending only on n and m such that $|\gamma_{\mu(m+n)}(G)| \leqslant n!$.*

A fuller discussion of this theorem is contained in [44].

The bounds here cannot be omitted. The Heineken-Mohamed groups (see [35]) are not finite-by-nilpotent, even though every subgroup is subnormal and, as Lennox points out, FC-groups have the property that each finitely generated subgroup is of finite index in its normal closure, but there are FC-groups that are not finite-by-nilpotent as the group

$$\mathrm{Dr}_{n \in \mathbb{N}} S_n$$

where S_n is the symmetric group of order n.
In the absence of these bounds however something can be salvaged and C. Casolo and M. Mainardis [8] have shown that if G is a group in which every subgroup has finite index in a subnormal subgroup, then G is finite-by-soluble and besides, E. Detomi was able to partly extend the Theorem 2.2 in [17].

For the fixed non-negative integers n, m we define the group class $S(n, m)$ to be the class of all groups of infinite rank (and the trivial groups) whose subgroups of infinite rank are (n, m)-subnormal in G. It was already shown in [15, Theorem B] that a generalized radical group of infinite rank in which every subgroup of infinite rank is nearly normal in G is finite-by-abelian.

The aim of this chapter is to prove that $S(n, m)$-groups are finite-by-nilpotent in the class $\mathfrak{X}$, and this a generalization of the theorem of Evans and Kim mentioned above.

The following section deals with some preliminary results showing in particular that $\mathfrak{X}$-groups in the class $S(n, m)$ are soluble-by-finite. We shall often implicitly use the facts that subgroups and factor groups of $S(n, m)$-groups which are themselves of infinite rank are also $S(n, m)$-groups.

2.1 Preliminary results

The first result of this section is well-known, here there is a proof.

Lemma 2.1.1 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. Then* G *contains a proper subgroup of infinite rank.*

Proof — Suppose, for a contradiction, that every proper subgroup of G has finite rank. If G is finitely generated then, since G is locally graded, there exists $N \triangleleft G$ such that G/N is a non-trivial finite group and clearly N has infinite rank, a contradiction. Thus G is not finitely generated and it follows that every finitely generated subgroup of G has finite rank. A result of N.S. Černikov [11] implies that G is locally (soluble-by-finite) so, by [18, Theorem], G has a locally soluble subgroup H of infinite rank. Using [19, Lemma 1] H is a proper subgroup of G, a contradiction which completes the proof. □

Corollary 2.1.2 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank whose proper subgroups of infinite rank are* (n, m)*-subnormal for some non-negative integers* n, m*. Then* G *contains a proper normal subgroup* N *of infinite rank.*

Proof — Let H be a proper subgroup of G of infinite rank, which exists by Lemma 2.1.1. Then either H^G or H_G are proper normal subgroups of infinite rank. □

Lemma 2.1.3 *Let* G *be an* $\mathfrak{X}$*-group in the class* $S(n, m)$*. Then* G *is not perfect.*

Proof — Suppose for a contradiction that G is perfect. By Corollary 2.1.2 there is a proper normal subgroup N of G of infinite rank and every subgroup of G containing N is (n, m)-subnormal. Hence G/N is finite-by-nilpotent by Theorem 2.2 and indeed there is a function $\mu = \mu(m + n)$ such that $\gamma_\mu(G/N)$ has order at most $n!$. Since G is perfect this implies that G/N has order at most $n!$, for each such normal subgroup N. Consequently the finite residual, R, of G satisfies $|G/R| \leqslant n!$.

Since R has infinite rank and inherits the hypotheses, it contains a proper normal subgroup of infinite rank, K say. Since R has no proper normal subgroups of finite index R/K is finite-by-nilpotent and certainly not perfect, so $R' \neq R$. Then G/R' is divisible-by-finite. If R' has infinite rank, then G/R' is finite-by-nilpotent and since G is perfect it follows that G/R' is finite, so $R = R'$, a contradiction.

Hence R' has finite rank and R/R' is divisible of infinite rank. Let T/R' be the torsion subgroup of R/R'. If T/R' has infinite rank, then so does the socle S/R'. Therefore G/S is finite-by-nilpotent and hence is finite, since G is perfect. This is a contradiction, so T/R' has finite rank as does T and R/T is a torsion-free divisible group of infinite rank.

Factoring, we may assume that $T = 1$ and that R is an infinite direct product of copies of the additive group of rationals. Since G/R is finite, there is a finitely generated subgroup F such that $G = RF$. Then $F \cap R$ is normal in G and FR/R is finite, so $F \cap R$ is a finitely generated abelian group and hence has finite rank. Factoring by $F \cap R$ we may assume that $G = R \rtimes F$, where F is now finite. By Maschke's theorem R is completely reducible as a $\mathbb{Q}F$-module (see [36, Corollary 5.15]), so $R = A \times B$, where A, B are normal subgroups of G of infinite rank. In particular, G/A inherits the hypotheses on G, so G/A is finite-by-nilpotent and clearly not finite. Hence G cannot be perfect, a final contradiction. □

Theorem 2.1.4 *Let G be an $\mathfrak{X}$-group in the class $S(n, m)$. Then G is soluble-by-finite.*

Proof — Let $\mu = \mu(m + n)$ as defined in Theorem 2.2. Suppose first that $\gamma_\mu(G)$ has infinite rank and let

$$\Gamma = \bigcap \{N \triangleleft G \mid N \text{ has infinite rank}\}.$$

Then $\Gamma \leqslant \gamma_\mu(G)$. Now if M has infinite rank and $M \triangleleft G$, then by Theorem 2.2 $|\gamma_\mu(G/M)| \leqslant n!$, so $|\gamma_\mu(G) : \gamma_\mu(G) \cap M| \leqslant n!$. If also $N \triangleleft G$ has infinite rank, then by Remak's theorem $\gamma_\mu(G)/(\gamma_\mu(G) \cap M \cap N)$ is isomorphic to a subgroup of

$$\gamma_\mu(G)/(\gamma_\mu(G) \cap M) \times \gamma_\mu(G)/(\gamma_\mu(G) \cap N)$$

so is finite. Hence $M \cap N$ has infinite rank also, so

$$|\gamma_\mu(G) : \gamma_\mu(G) \cap M \cap N| \leqslant n!.$$

From this we deduce that $|\gamma_\mu(G) : \Gamma| \leqslant n!$. Hence Γ has infinite rank so, by Lemma 2.1.3, it is not perfect. Consequently Γ' has finite rank, otherwise $\Gamma \leqslant \Gamma'$. By the theorem of N.S. Černikov [11] the group $F = \Gamma'$ is almost locally soluble. If S is the locally soluble radical of F, then $G/C_G(F/S)$ is finite, so $C_G(F/S)$ has infinite rank, whence $\Gamma \leqslant C_G(F/S)$. Hence $[F,\Gamma] \leqslant S$ so, in particular, F/S is abelian and hence $F = S$. Consequently F is locally soluble of finite rank. By [56, Lemma 10.39] there is a non-negative integer k such that $F^{(k)}$ is a periodic hypercentral group and a direct product of Černikov p-groups.

Let R be the divisible part of $F^{(k)}$ and note that without loss of generality we may assume $R=1$, so the p-primary components of $F^{(k)}$ are finite. If P is such a p-primary component, then $G/C_G(P)$ is finite. Thus $C_G(P)$ has infinite rank, so $P \leqslant F \leqslant \Gamma \leqslant C_G(P)$ and hence P is abelian. Consequently $F^{(k)}$ is abelian, so F is soluble. Then Γ is soluble. Since G/Γ is finite-by-nilpotent it is nilpotent-by-finite by a well-known result of P. Hall (see [56, Theorem 4.25] for example). Hence G is soluble-by-finite.

If $\gamma_\mu(G)$ has finite rank, then it is almost locally soluble by [11]. Let F be the locally soluble radical of $\gamma_\mu(G)$ and note, using the result [56, Lemma 10.39], that F has a periodic hypercentral G-invariant subgroup L with Černikov primary components such that F/L is soluble. Again, we may suppose, without loss of generality, that L has finite primary components. If X is a finite characteristic subgroup of L, then $G/C_G(X)$ is finite and Theorem 2.2 implies that

$$|\gamma_\mu(G)/\gamma_\mu(G) \cap C_G(X)| \leqslant n!.$$

It is then easy to see that also $|\gamma_\mu(G) : \gamma_\mu(G) \cap C_G(L)| \leqslant n!$. Since $G/F \cap C_G(L)$ is finite-by-nilpotent, it follows that it is also soluble-by-finite. Then G is soluble-by-finite, since clearly $F \cap C_G(L)$ is soluble. □

We next prove a very elementary result which nevertheless is very useful.

Lemma 2.1.5 *Let G be a group and suppose that A, B are (n, m)-subnormal subgroups of G. Then $A \cap B$ is an (n^2, m)-subnormal subgroup of G.*

Proof — There are chains of subgroups

$$A \leqslant A_m \lhd A_{m-1} \lhd \cdots \lhd A_1 \lhd A_0 = G$$

and

$$B \leqslant B_m \lhd B_{m-1} \lhd \cdots \lhd B_1 \lhd B_0 = G,$$

where $|A_m : A|, |B_m : B| \leqslant n$. It is clear that for each i we have $A_{i+1} \cap B_{i+1} \lhd A_i \cap B_i$ and also that $|A_m \cap B_m : A \cap B| \leqslant n^2$, so the result follows. □

The next lemma is very useful for the proof of the main result of this chapter and we shall quite often use it without explicitly mentioning it.

Lemma 2.1.6 *Let* G *be a group, let* H *be a subgroup of* G *and let*

$$A = A_1 \times A_2 \times \dots$$

be a subgroup of G *of infinite rank that is direct product of infinitely many* H*-invariant subgroups of finite rank. Suppose that there exists* k *such that*

$$A \cap H \leqslant A_1 \times A_2 \times \dots \times A_k.$$

Then there is a subgroup $C = C_1 \times C_2$ *of* A *such that* C_1, C_2 *are* H*-invariant subgroups, each of infinite rank, such that* $H \cap C = 1$.

Recall that a group G is called a *Baer group* if it is generated by abelian subnormal subgroups. It is easy to show that the property of being a Baer group is equivalent to the requirement that all cyclic subgroups (or even all finitely generated subgroups) of G are subnormal. Any Baer group is locally nilpotent, and in any group G, there is a largest Baer subgroup, the so-called *Baer radical* of G.

The final part of this section deals with the locally finite version of our main result, whose proof being somewhat simpler than that of the general case. Notice that we shall say that the subgroup H of a group G is *finite-by-subnormal* when it is (n, m)-subnormal for some non-negative integers n, m.

Lemma 2.1.7 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank in which every subgroup of infinite rank is finite-by-subnormal. If* G *contains a periodic abelian subgroup* C *of infinite rank, then the finite subgroups of* G *are finite-by-subnormal. Furthermore, if* G *is an* $S(n, m)$*-group, then every finite subgroup of* G *is* (n^2, m)*-subnormal in* G.

Proof — By hypothesis the subgroup C has finite index in a subnormal subgroup X of G and so its core in X, C_X, is an abelian subnormal subgroup of infinite rank of G, so is contained in the Baer radical B of G.

The torsion subgroup T of B has infinite rank since it contains C. Let X be a finite subgroup of G. By [33, Theorem 1], T contains a subgroup H that is an infinite direct sum of non-trivial X-invariant subgroups. Using Lemma 2.1.6, there are X-invariant subgroups H_1, H_2 of H of infinite rank such that $H_1 \cap H_2 = 1$ and $X \cap (H_1 \times H_2) = 1$.

Consider $X \ltimes (H_1 \times H_2)$ and let $S_1 = X \ltimes H_1$ and $S_2 = X \ltimes H_2$. Then S_1 (respectively S_2) has finite index in a subnormal subgroup K_1 (respectively K_2) of G. It follows that $X = S_1 \cap S_2$ has finite index in the subnormal subgroup $K_1 \cap K_2$. The last part of the statement follows easily using Lemma 2.1.5. □

Corollary 2.1.8 *Let G be a periodic $\mathfrak{X}$-group in the class $S(n, m)$. Then G is finite-by-nilpotent. Indeed, $|\gamma_{\mu(m+n^2)}(G)| \leqslant (n^2)!$.*

Proof — By Theorem 2.1.4, G is soluble-by-finite, while by [4, Theorem 6.3] G has a proper abelian subgroup of infinite rank. Let H be a finitely generated subgroup of G, so H is finite and, by Lemma 2.1.7, is also (n^2, m)-subnormal in G. Theorem 2.2 now implies that G is finite-by-nilpotent. □

2.2 Proof of the Main Theorem

This section starts showing that the Baer radical of an $\mathfrak{X}$-group in the class $S(n, m)$ is nilpotent.

Lemma 2.2.1 *Let G be a locally nilpotent $S(n, m)$-group. Then there is a function f such that G is nilpotent of class at most $f(m+n)$.*

Proof — Let Y be a subgroup of G of infinite rank, so there is a subgroup X such that $|X : Y| \leqslant n$ and X is subnormal of defect at most m in G. Let $N = Y_X$, the core of Y in X. Then X/N is a finite nilpotent group, so Y/N is a subnormal subgroup of X/N of defect at most n, since $|X : Y| \leqslant n$. Hence Y is a subnormal subgroup of G of defect at most $m + n$. By the theorem of Evans and Kim [23] G is nilpotent of which is a function of $m + n$. □

Lemma 2.2.2 *Let G be an $\mathfrak{X}$-group in the class $S(n, m)$. Then the Baer radical of G is nilpotent.*

Proof — The group G is soluble-by-finite, by Theorem 2.1.4, so G has an abelian subgroup of infinite rank, by [4, Theorem 6.3] and hence an abelian subnormal subgroup of infinite rank. Since the Baer radical B of G is a locally nilpotent it follows that B is nilpotent by Lemma 2.2.1. □

Lemma 2.2.3 *Let G be a group and F a finitely generated subgroup. Suppose that A and B are subgroups of G such that $A \leqslant Z(B)$, A has finite rank, B/A has infinite rank and $[B, F] \leqslant A$. Then $C_B(F)$ has infinite rank.*

Proof — Let F be generated by elements $g_1, \dots, g_n$ and consider the maps

$$\phi_i : b \in B \mapsto [b, g_i] \in A.$$

By hypothesis ϕ_i is a homomorphism whose kernel is $C_B(g_i)$. Then $B/C_B(g_i)$ has finite rank and therefore

$$B/C_B(F) = B/\bigcap_{i=1}^{n} C_B(g_i)$$

has finite rank. It follows that $C_B(F)$ has infinite rank. □

A simple induction now proves the following corollary.

Corollary 2.2.4 *Let* G *be a group, let* F *a finitely generated subgroup and* N *an* F*-invariant subgroup of* G *of infinite rank. Let* A *and* B *be two central terms of a finite central* F*-invariant series of* N *such that* A *has finite rank while* B *has infinite rank. If* $[B, F] \leqslant A$*, then* $C_N(F)$ *has infinite rank.*

The proof of the following proposition is based on that of [41, Lemma 1.16].

Proposition 2.2.5 *Let* G *be a* $S(n, m)$*-group, let* g *be an element of* G *and let* A *be a torsion-free abelian* $\langle g\rangle$*-invariant subgroup of* G *of infinite rank. If* gA *has infinite order, then* $\langle g^q\rangle$ *is* (n^2, m)*-subnormal in* G*, for some prime* q *or* $q = 1$*.*

Proof — Let $E = A \otimes_{\mathbb{Z}} \mathbb{Q}$. We regard E as a right $\mathbb{Q}\langle g\rangle$-module. It is well-known that $A \simeq A \otimes_{\mathbb{Z}} \mathbb{Z}$. Let $1 \neq c \in A$. Set

$$C = \langle c\rangle^{\langle g\rangle} = c\mathbb{Z}\langle g\rangle$$

and

$$D = c\mathbb{Q}\langle g\rangle.$$

Suppose first that $Ann_{\mathbb{Z}\langle g\rangle}(c) = \langle 0\rangle$. Then $C \simeq \mathbb{Z}\langle g\rangle$ has infinite rank. Let q be a prime and write

$$\mathbb{Z}\langle g\rangle = \mathbb{Z}\langle g^q\rangle \oplus \mathbb{Z}\langle g^q\rangle g \oplus \ldots \oplus \mathbb{Z}\langle g^q\rangle g^{q-1}.$$

It follows that

$$C = A_q \times (A_q)^g \times \ldots \times (A_q)^{g^{q-1}}$$

where $A_q = \langle c\rangle^{\langle g^q\rangle}$. The subgroups A_q and A_q^g are $\langle g^q\rangle$-invariant subgroups of infinite rank. Since

$$\langle g^q\rangle = \langle g^q\rangle A_q \cap \langle g^q\rangle(A_q)^g$$

it follows that $\langle g^q\rangle$ is (n^2, m)-subnormal in G, using Lemma 2.1.5.

Hence we may suppose that $Ann_{\mathbb{Z}\langle g\rangle}(c) \neq \langle 0\rangle$, so $Ann_{\mathbb{Q}\langle g\rangle}(c) \neq \langle 0\rangle$, for all $1 \neq c \in A$. Since $D \simeq \mathbb{Q}\langle g\rangle/Ann_{\mathbb{Q}\langle g\rangle}(c)$ then $dim_{\mathbb{Q}}D$ is finite, so C has finite rank and is a finitely generated $\mathbb{Z}\langle g\rangle$-module. Hence [56, Corollary 1 of Lemma 9.53] may be applied to deduce that C contains a free abelian subgroup L such that C/L is periodic and $\pi(C/L)$ is finite. Hence C is minimax and if $a_1, \ldots, a_n \in A$, then $\langle a_1, \ldots, a_n\rangle^{\langle g\rangle}$ is minimax.

Let $A_1 = \langle a_1 \rangle^{\langle g \rangle}$ for some element $a_1 \in A$. Then A/A_1 has infinite torsion-free rank so we may choose two infinite cyclic subgroups $\langle a_2 A_1 \rangle$ and $\langle a_3 A_1 \rangle$ such that

$$\langle a_2 A_1, a_3 A_1 \rangle = \langle a_2 A_1 \rangle \times \langle a_3 A_1 \rangle.$$

Let $A_2 = \langle A_1, a_2, a_3 \rangle^{\langle g \rangle}$, so $r_0(A_2/A_1) \geqslant 2$. In this way we can construct an ascending chain of $\langle g \rangle$-invariant subgroups

$$\langle 1 \rangle = A_0 \leqslant A_1 \leqslant \dots$$

such that A_n is a finitely generated $\mathbb{Z}\langle g \rangle$-module and

$$r_0(A_{n+1}/A_n) \geqslant n + 1.$$

A result of P. Hall [56, Lemma 5.35] implies that the factors A_{n+1}/A_n satisfy Max-$\langle g \rangle$ and so have finite torsion subgroups.

Clearly, A_1 contains a free abelian subgroup C_1 such that A_1/C_1 is Černikov. We let π be the (finite) set of prime divisors of $(n^2)!$ and let σ be the complement of $\pi(A_1/C_1) \cup \pi$ in the set of all primes. Let $p_1 \in \sigma$. Then $A_1/C_1^{p_1}$ is the direct product of its p-component $P_1/C_1^{p_1}$ and its p'-component $Q/C_1^{p_1}$. It follows that $A_1/A_1^{p_1}$ is a non-trivial Černikov group of finite exponent and hence is finite. Set $B_1 = A_1^{p_1}$.

Since A_2/A_1 has finite torsion subgroup, it follows that

$$A_2/B_1 = S/B_1 \oplus U/B_1,$$

where S/B_1 is the (finite) p_1-component of A_2/B_1, containing A_1/B_1. Let $s = |S/B_1|$, so $(A_2/B_1)^s$ is contained in U/B_1. Arguing as before, but in $(A_2/B_1)^s$, we can find a prime $p_2 \neq p_1$ with $p_2 \in \sigma$ such that A_2/B_2 is finite of rank greater than 2, where we set $B_2 = (A_2)^{sp_2} B_1$. Moreover $B_2 \cap A_1 = B_1$. Continuing in this way, we can construct an ascending series

$$\langle 1 \rangle = B_0 \leqslant B_1 \leqslant B_2 \leqslant \dots \leqslant B_n \leqslant \dots$$

of $\langle g \rangle$-invariant subgroups satisfying the conditions:

(i) $B_n \leqslant A_n$ and A_n/B_n is finite of rank at least n;

(ii) $B_{n+1} \cap A_n = B_n$;

(iii) $|\pi(A_n/B_n)| \geqslant n$.

Let $E = \cup_{n \geqslant 1} A_n$ and $B = \cup_{n \geqslant 1} B_n$. Then $E/B = \cup_{n \geqslant 1} A_n B/B$ and

$$A_n B/B \simeq A_n/A_n \cap B = A_n/B_n.$$

Consequently, E/B is a periodic group of infinite rank such that $\pi(E/B)$ is infinite, so it is a direct sum of infinitely many $\langle g \rangle$-invariant subgroups. Set $H = E\langle g \rangle$. Using Lemma 2.1.6 and Lemma 2.1.5, if L/B is a finitely generated subgroup of H/B of finite rank, we deduce that L/B is (n^2, m)-subnormal in H/B. By Theorem 2.2, H/B has a finite normal subgroup F/B of order at most $(n^2)!$ such that H/F is nilpotent of class bounded by a function of m and n only. Since $A_1 B/B \simeq A_1/B_1$, an elementary abelian p_1-group, has order relatively prime to that of F/B, it follows that $A_1 B/B \cdot \langle gB \rangle$ is nilpotent, so $[A_{1,t}\ g] \leqslant B \cap A_1 = B_1 = A_1^{p_1}$ for some integer t depending only on m and n. We note that p_1 was an arbitrary prime in σ so, setting

$$L_1 = \bigcap_{q \in \sigma} (A_1)^q,$$

we have $[A_{1,t}\ g] \leqslant L_1$. Then, from $A_1^q \cap C_1 = C_1^q$, we have

$$L_1 \cap C_1 = \bigcap_{q \in \sigma} C_1^q = \langle 1 \rangle.$$

Since A_1/C_1 is periodic, it follows that $L_1 = \langle 1 \rangle$ and, as

$$[A_{1,t}\ g] \leqslant L_1 = \langle 1 \rangle,$$

it follows that $\langle A_1, g \rangle$ is nilpotent. In particular, $C_E(g) \neq \langle 1 \rangle$, so the centre of H is non-trivial.

It is easy to prove that $H/C_E(g)$ is torsion-free. If $C_E(g)$ has infinite rank, then $H/C_E(g)$ is nilpotent by Theorem 2.2, whence so is H. If $C_E(g)$ has finite rank, then $H/C_E(g)$ satisfies the hypotheses of the proposition so, as above, either $H/C_E(g)$ is nilpotent or the centre of $H/C_E(g)$ is not trivial. Continuing in this way, we can prove that H is hypercentral and hence locally nilpotent. By Lemma 2.2.1, H is nilpotent in any case.

Using Corollary 2.2.4 we find that $C_E(g)$ has infinite rank in any case. Finally, using Lemmas 2.1.6 and 2.1.5 it follows that $\langle g \rangle$ is (n^2, m)-subnormal in G. □

Lemma 2.2.6 *Let* G *be a torsion-free* $S(n,m)$*-group. Suppose that* G *has a nilpotent subgroup* N *such that* G/N *is infinite cyclic. If* $G = N\langle g\rangle$*, then* $C_N(g^s)$ *has infinite rank for some natural number* s*.*

Proof — Choose a non-negative integer l such that $Z_l(N)$ has finite rank while $Z_{l+1}(N)$ has infinite rank. Using Proposition 2.2.5, there is a natural number s such that $\langle g^s Z_l(N)\rangle$ is (n^2,m)-subnormal in the group $Z_{l+1}(N)/Z_l(N)\cdot\langle g^s Z_l(N)\rangle$. Since a torsion-free almost cyclic group is abelian it follows that $Z_{l+1}(N)/Z_l(N)\cdot\langle g^s Z_l(N)\rangle$ is locally nilpotent and so even nilpotent by Lemma 2.2.1. Hence

$$Z_{l+1}(N)/Z_l(N)$$

has a subgroup $A/Z_l(N)$ of infinite rank such that $[A, g^s] \leqslant Z_l(N)$, by Corollary 2.2.4. A further application of Corollary 2.2.4 now shows that $C_N(g^s)$ has infinite rank. □

Lemma 2.2.7 *Let* G *be an* $\mathfrak{X}$*-group in the class* $S(n,m)$ *and suppose that the torsion subgroup* T *of the Baer radical* B *of* G *has finite rank. Then* G *is finite-by-nilpotent. Indeed,* $|\gamma_{\mu(m+n^2)}(G)| \leqslant (n^2)!$.

Proof — First recall from Lemma 2.2.2 that the Baer radical B of an $S(n,m)$-group is nilpotent. We prove that G/B is periodic, and so locally finite, by Theorem 2.1.4. Suppose, for a contradiction, $\langle g\rangle$ is an arbitrary infinite cyclic group such that $B\cap\langle g\rangle = \langle 1\rangle$.
By Lemma 2.2.6 B/T contains an abelian subgroup A/T of infinite rank such that $[A, g^l] \leqslant T$ for some natural number l.

We apply Corollary 2.2.4 to $A\langle g^l\rangle$. Then $C_A(g^l)$ has infinite rank. Therefore it contains an abelian subgroup of infinite rank and using Lemma 2.1.6 and Lemma 2.1.5 we see that $\langle g^l\rangle$ is (n^2,m)-subnormal in G, so there is a natural number t such that $\langle g^t\rangle$ is subnormal in G. In this way we have proved that G/B is periodic.

Now let F be an arbitrary finitely generated subgroup of finite rank. Let c be the nilpotency class of B and consider the series

$$1 \leqslant Z_1(B)\cap T \leqslant \ldots \leqslant Z_c(B)\cap T = T = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_c = B,$$

where C_1/T is the centre of B/T and so on. There is an index i such that C_i has finite rank while C_{i+1} has infinite rank. Since FB/B is finite each subgroup of C_{i+1}/C_i has only finitely many conjugates in FC_{i+1}/C_i and [15, Lemma 6] implies that FC_{i+1}/C_i contains an infinite direct sum of F-invariant subgroups of finite but unbounded ranks. Since F is polycyclic-by-finite, Lemmas 2.1.6 and 2.1.5 may be applied to deduce that FC_i/C_i is (n^2, m)-subnormal in G. This clearly holds for all finitely generated subgroups of finite rank of FC_{i+1}/C_i, so FC_{i+1}/C_i is finite-by-nilpotent, using Theorem 2.2. Hence there is an FC_i/C_i-central finite series in the torsion-free subgroup C_{i+1}/C_i. Using Corollary 2.2.4 we find that there is an abelian subgroup A_1/C_i of infinite rank which is FC_i/C_i-central and so $[A_1, F] \leqslant C_i$. A further application of Corollary 2.2.4 shows that F is (n^2, m)-subnormal in G and since F is an arbitrary finitely generated subgroup of finite rank, it follows that G is finite-by-nilpotent, again by Theorem 2.2. □

If all primary components of the torsion subgroup T have finite rank, then we can obtain bounds similar to those given in Lemma 2.2.7.

Lemma 2.2.8 *Let G be an $\mathfrak{X}$-group in the class $S(n, m)$ and suppose that the torsion subgroup T of the Baer radical B of G has infinite rank. If G is not finite-by-nilpotent, then T contains a primary component of infinite rank.*

Proof — By Lemma 2.2.2, B is nilpotent and, by Theorem 2.1.4, G is soluble-by-finite. We suppose that the result is false and for each prime p let T_p be the p-component of T. Let F be a finitely generated subgroup of G of finite rank and let L be a soluble subgroup of F of finite index. Then L is a finitely generated soluble subgroup of G of finite rank, so it is minimax. Hence F is also minimax. Thus $T \cap F$ is a periodic soluble minimax group, so is a Černikov group.

Let π be the set of primes not dividing the orders of the elements of $T \cap F$ and let K be the direct product of the T_p with $p \in \pi$. Then $K \cap F = \{1\}$. It follows that K is a direct product of two subgroups, each of infinite rank, so Lemma 2.1.5 implies that F is (n^2, m)-subnormal in G. Theorem 2.2 now shows that G is finite-by-nilpotent. This proves the result. □

For the next result we need some information about modules over Principal Ideal Domains and we here give a brief discussion of the ideas we need (see [38], for example, for a more account).

Let R be a Principal Ideal Domain, which is not a field, and let A be an R-module. For each maximal ideal P of R, let

$$A_P = \{a \in A \,:\, aP^n = 0, \text{ for some } n \in \mathbb{N}\}$$

and for each natural number n, let

$$A_P[n] := \{a \in A \,:\, aP^n = 0\}.$$

Then $A_P[n] \leqslant A_P[n+1]$, for all $n \in \mathbb{N}$ and $A_P = \bigcup_{n\in\mathbb{N}} A_P[n]$. If $\{P_i\}_{i\in I}$ is the set of distinct maximal ideals of R, then it is well-known that $A_{P_i} \cap (\sum_{j\neq i} A_{p_j}) = 0$. If C is a simple R-module, in which the action of R is not trivial, then $C \simeq R/P_i$, for some $i \in I$ and the R-injective envelope of C is denoted by $C_{P_i^\infty}$, the Prüfer P_i-module. As with abelian group theory

$$C_{P_i^\infty} \simeq \varinjlim \{R/P_i^n \mid n \in \mathbb{N}\}$$

and every proper R-submodule of $C_{P_i^\infty}$ is isomorphic with the cyclic module R/P_i^n, for some $n \in \mathbb{N}$.

If $P_i = yR$, then $C_{P_i^\infty}$ has a subset of elements, $\{a_n \mid n \in \mathbb{N}\}$, such that $a_1 y = 0, a_{n+1} y = a_n$, for each $n \in \mathbb{N}$ and $C_{P_i^\infty} \simeq \bigcup_{n\in\mathbb{N}} a_n R$. As in abelian group theory, if $A_P[1] = C_1 \oplus \cdots \oplus C_n$, where C_i is a simple R-module for $1 \leqslant i \leqslant n$, then $A_{P_i} = E_1 \oplus \cdots \oplus E_n$, where E_j is a Prüfer P_i-module, or $E_j \simeq R/P_i^{m_j}$, for some $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant n$.

Let $\mathbb{F}_p$ denote the field with p elements. It is well-known that the group ring $\mathbb{F}_p\langle g\rangle$ is a Principal Ideal Domain with infinitely many maximal ideals, when $\langle g\rangle$ is an infinite cyclic group.

Proposition 2.2.9 *Let G be an $\mathfrak{X}$-group in the class $S(n,m)$-group and let A be a normal elementary abelian p-subgroup of G of infinite rank. Then G is finite-by-nilpotent.*

Proof — Let B be the Baer radical of G. Note that B is nilpotent using Lemma 2.2.2. We prove that G/B is locally finite. Let $g \in G$ be such that $\langle gB\rangle$ is infinite and consider the group $H = A \rtimes \langle g\rangle$. If the centre of H has infinite rank, then using Lemma 2.1.5, we deduce that $\langle g\rangle$ is (n^2, m)-subnormal in G. Then there is a natural number k such that $\langle g^k\rangle$ is subnormal in G and hence $g^k \in B$. Therefore we may suppose that the centre of H has finite rank. In this case, if g^l is central in H, for some $l \neq 0$, then we may use Lemmas 2.1.6 and 2.1.5 to show that $\langle g^l\rangle$ is (n^2, m)-subnormal in G. It follows that $g^k \in B$, for some $k \neq 0$. Hence we may assume the centre of H is finite.

Consider A as a $\mathbb{F}_p\langle g\rangle$-module. If $A_P \neq 0$ for infinitely many maximal ideals P of $\mathbb{F}_p\langle g\rangle$, then Lemmas 2.1.6 and 2.1.5 imply that $\langle g\rangle$ is

(n^2, m)-subnormal in G. Then, as earlier, $\langle gB\rangle$ is finite, a contradiction.

Hence we may assume that there are only finitely many non-trivial subgroups of type A_P. Suppose that all elements of A have finitely many conjugates in H. Then there is a prime ideal Q such that A_Q has infinite rank. If $A_Q[n]$ has infinite rank, then each of its subgroups has finitely many conjugates in H and [15, Lemma 6] shows that A contains an infinite direct sum of $\langle g\rangle$-invariant subgroups X_i of increasing ranks. By Lemma 2.1.5 $\langle g\rangle$ is (n^2, m)-subnormal and, as earlier, $\langle gB\rangle$ is finite, again a contradiction.

Suppose none of the subgroups $A_Q[n]$ has infinite rank. By our comments above $A_Q[1] = C_1 \oplus \ldots \oplus C_n$, where C_i is a simple $\mathbb{F}_p\langle g\rangle$-submodule and $A_Q = E_1 \oplus \ldots \oplus E_n$ where either E_i is a Prüfer module or is cyclic. Since A_Q is infinite, then at least one of the E_i, E_1, say, is a Prüfer module. Hence there are elements $a_1, a_2, a_3, \ldots$ of A and an element $y \in \mathbb{F}_p\langle g\rangle$ such that $a_1 y = 0$ and $a_{n+1}y = a_n$. Then $C = \langle a_i \mathbb{F}_p\langle g\rangle \mid i \in \mathbb{N}\rangle$ is $\mathbb{F}_p\langle g\rangle$-divisible. The $\langle g\rangle$-invariant subgroup $A_1 = \langle a_1, a_2\rangle^{\langle g\rangle}$ is finite so $|\langle g\rangle : C_{\langle g\rangle}(A_1)|$ is also finite.
Let $\langle x\rangle = C_{\langle g\rangle}(A_1)$ and note that C is an $\mathbb{F}_p\langle x\rangle$-module that is $\mathbb{F}_p\langle x\rangle$-divisible. Since $a_1, a_2 \in C_A(x)$ we have

$$C_{Q\cap\mathbb{F}_p\langle x\rangle}[1] = D_1 \oplus D_2 \oplus \ldots \oplus D_k$$

where $k \geqslant 2$. Then $C_{Q\cap\mathbb{F}_p\langle x\rangle}$ is a direct sum of at least two Prüfer submodules, say $E_{1,0}$ and $E_{1,1}$, both of which have infinite rank, being infinite elementary abelian subgroups. Using Lemma 2.1.5, $\langle x\rangle$ is (n^2, m)-subnormal in G and again $\langle gB\rangle$ is finite.

Suppose now that there exists $a \in A$ such that $M = \langle a\rangle^{\langle g\rangle}$ has infinite rank, and set $L = \langle a\rangle^{\langle g\rangle}\langle g\rangle$. We consider M as an $\mathbb{F}_p\langle g\rangle$-module. Then $M \simeq \mathbb{F}_p\langle g\rangle/I$ where $I = Ann_{\mathbb{F}_p\langle g\rangle}(a)$. If $I \neq \langle 0\rangle$, then $\mathbb{F}_p\langle g\rangle/I$ is finite, so a has only finitely many conjugates in L, which is a contradiction. Therefore we may assume that $I = \langle 0\rangle$ and in this case $M \simeq \mathbb{F}_p\langle g\rangle$. Let q be a prime. Then

$$\mathbb{F}_p\langle g\rangle = \mathbb{F}_p\langle g^q\rangle \oplus \mathbb{F}_p\langle g^q\rangle g \oplus \ldots \oplus \mathbb{F}_p\langle g^q\rangle g^{q-1}.$$

It follows that

$$M = M_q \times (M_q)^g \times \ldots \times (M_q)^{q^{q-1}}$$

where $M_q = \langle a \rangle^{\langle g^q \rangle}$. The subgroups M_q and M_q^g are $\langle g^q \rangle$-invariant and infinite, so $\langle g^q \rangle M_q$ and $\langle g^q \rangle (M_q)^q$ are subgroups of infinite rank. Since their intersection is $\langle g^q \rangle$, we have that $\langle g^q \rangle$ is (n^2, m)-subnormal in G, using Lemma 2.1.5 and once again $\langle gB \rangle$ is finite.

Consequently G/B is periodic, so is locally finite since G is soluble-by-finite, by Theorem 2.1.4. Furthermore, B is nilpotent and A has infinite rank. There exists i such that $(Z_{i+1}(B) \cap A)/(Z_i(B) \cap A)$ has infinite rank while $Z_i(B) \cap A$ has finite rank (and so it is also finite). It is now easily proved, using [15, Lemma 6] and Theorem 2.2 that $G/(Z_i(B) \cap A)$ is finite-by-nilpotent, since each subgroup of

$$(Z_{i+1}(B) \cap A)/(Z_i(B) \cap A)$$

has finitely many conjugates in any finite extension of it. Consequently G is finite-by-nilpotent. □

Lemma 2.2.10 *Let* G *be an* $\mathfrak{X}$*-group in the class* $S(n, m)$ *and let* T *be a nilpotent* p*-subgroup of* G *of infinite rank. Then* G *is finite-by-nilpotent.*

Proof — There are G-invariant subgroups $A \leqslant B$ of T such that B/A is an elementary abelian p-group and A has finite rank. Then A is Černikov and its divisible part D is central in A of finite index p^r, say. Hence A' is finite, as is $C/A' = (A/A')[p^r]$. It suffices to show that G/C is finite-by-nilpotent, so we may assume that C is trivial.

In this case A is divisible of finite rank, so it is contained in the centre of B. If $b \in B$, then $[b, B] \simeq B/C_B(b)$. Since $A \leqslant Z(B)$, $B/C_B(b)$ is elementary abelian and $[b, B] \leqslant A$, so $|B/C_B(b)| \leqslant p^l$, where l is the rank of A. Hence B is a BFC-group, so B' is finite. Factoring, we may assume B is abelian and note that then $B[p]$ is infinite. The fact that G is finite-by-nilpotent follows using Proposition 2.2.9. □

As corollary of these results follows the main result of this chapter.

Corollary 2.2.11 *Let* G *be an* $\mathfrak{X}$ *in the class* $S(n, m)$. *Then* G *is finite-by-nilpotent.*

Finally, note that in Corollary 2.1.8 and Lemma 2.2.7 we can specify bounds for the order of the finite subgroups obtained and the nilpotency class of the corresponding factor group. We can also specify bounds when T does not contain a primary component of infinite rank. However the proof of Proposition 2.2.9 makes such a bound appear unlikely in general.

Chapter 3

Groups whose subnormal subgroups have finite normal oscillation

Let G be a group, and let X be a subgroup of G. The *normal oscillation* of X in G is the cardinal number

$$\min\{|X : X_G|, |X^G : X|\}.$$

Clearly, X is normal in G if and only if it has normal oscillation 1, and in particular, a group has the T-property if and only if all its subnormal subgroups have normal oscillation 1. Moreover, X has finite normal oscillation in G if and only if either X has finite index in its normal closure X^G or it is finite over its core X_G; in particular, finite subgroups and subgroups of finite index have finite normal oscillation. It has recently been proved that any locally finite group whose subgroups have finite normal oscillation contains a nilpotent subgroup of finite index (see [28]).

The aim of this chapter is to study the class $T(*)$ consisting of all groups in which every subnormal subgroup has finite normal oscillation. Of course, $T(*)$ contains both group classes T^* and T_* (see [7, 26]). In the following section, the behavior of certain groups of automorphisms which play a central role in the study of the class $T(*)$ will be studied.

Note that in this chapter we will denote the finite residual of a group G with $\rho^*_{\mathfrak{F}}(G)$.

3.1 Automorphisms

Recall that an automorphism of a group G is called a *power automorphism* if it maps every subgroup of G onto itself. The set of all power automorphisms $\mathrm{PAut}(G)$ of a group G is an abelian normal subgroup of the full automorphism group $\mathrm{Aut}(G)$ of G, which is naturally involved in the study of soluble groups with the T-property; in fact, if G is a T-group and A is any abelian normal subgroup of G, then G induces by conjugation on A a group of power automorphisms. We refer to [12] for a detailed description of the behavior of power automorphisms.

Let G be a group, and let Γ be a group of automorphisms of G. If X is any subgroup of G, the Γ-*oscillation* of X is defined as the cardinal number

$$\min\{|X : X_\Gamma|, |X^\Gamma : X|\},$$

where

$$X_\Gamma = \bigcap_{\gamma \in \Gamma} X^\gamma \qquad \text{and} \qquad X^\Gamma = \langle X^\gamma \mid \gamma \in \Gamma \rangle.$$

Of course, Γ is contained in the group $\mathrm{PAut}(G)$ if and only if all subgroups of G have Γ-oscillation 1. Moreover, if $\mathrm{Inn}(G)$ denotes the group of all inner automorphisms of G, the normal oscillation of X in G coincides with the $\mathrm{Inn}(G)$-oscillation of X. In particular, a group G has the $T(*)$-property if and only if all its subnormal subgroups have finite $\mathrm{Inn}(G)$-oscillation.

In this section, there are some lemmas on groups of automorphisms which determine finite oscillation for all subgroups of the group; they will be relevant for the proofs of our main results on generalized soluble $T(*)$-groups.
This first elementary lemma shows in particular that if a subgroup P of type p^∞ of a group G has finite oscillation with respect to a subgroup Γ of $\mathrm{Aut}(G)$, then P is fixed by Γ.

Lemma 3.1.1 *Let G be a group, and let Γ be a group of automorphisms of G. If X is a subgroup of G which has finite Γ-oscillation, then the finite residual of X is Γ-invariant. In particular, if X has no proper subgroups of finite index, then it is Γ-invariant.*

PROOF — Obviously, the finite residual of a group coincides with the finite residual of any of its subgroups of finite index. Therefore either

$$\rho^*_{\mathfrak{F}}(X) = \rho^*_{\mathfrak{F}}(X^\Gamma) \qquad \text{or} \qquad \rho^*_{\mathfrak{F}}(X) = \rho^*_{\mathfrak{F}}(X_\Gamma),$$

and hence the finite residual of X is Γ-invariant. □

The main results of this section deal with the case of automorphism groups of abelian p-groups (where p is a prime number).

Lemma 3.1.2 *Let A be an abelian p-group (where p is a prime number) and let B be a residually finite subgroup of A. If Γ is a locally finite group of automorphisms of A such that every subgroup of finite index of B contains a Γ-invariant subgroup on which Γ does not act as a group of power automorphisms, then there exists a subgroup of A which has infinite Γ-oscillation.*

Proof — By hypothesis, Γ is not a group of power automorphisms of A, and so there exists an element a_1 of A and an automorphism $\gamma_1 \in \Gamma$ such that $\langle a_1\rangle^{\gamma_1} \neq \langle a_1\rangle$. Put $\Gamma_1 = \langle \gamma_1\rangle$ and consider the finite subgroup $E_1 = \langle a_1\rangle^{\Gamma_1}$ of A. Since B is residually finite, it has a subgroup of finite index B_1 such that $E_1 \cap B_1 = \{1\}$ and Γ does not act as a group of power automorphisms on the largest Γ-invariant subgroup C_1 of B_1. Thus there exist elements a_2 of C_1 and γ_2 of Γ such that $\langle a_2\rangle^{\gamma_2} \neq \langle a_2\rangle$. Of course, $\Gamma_2 = \langle \gamma_1, \gamma_2\rangle$ is a finite subgroup of Γ, and $E_2 = \langle a_1, a_2\rangle^{\Gamma_2}$ is a finite subgroup of A of order at least p^{k+3}, where p^k is the order of a_1. Let B_2 be a subgroup of finite index of B_1 such that $E_2 \cap B_2 = \{1\}$; clearly, Γ does not act as a group of power automorphisms on the largest Γ-invariant subgroup C_2 of B_2. The iteration of this argument allows to construct two increasing sequences $(E_n)_{n\in\mathbb{N}}$ and $(\Gamma_n)_{n\in\mathbb{N}}$ of finite subgroups of A and Γ, respectively, and a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of subgroups of finite index of B, satisfying the following conditions:

- $\Gamma_n = \langle \gamma_1, \dots, \gamma_n\rangle$
- $E_n = \langle a_1, \dots, a_n\rangle^{\Gamma_n}$ and $|E_n| \geqslant p^{k+2n-1}$
- $\langle a_n\rangle^{\gamma_n} \neq \langle a_n\rangle$
- $E_n \cap B_n = \{1\}$ and $a_{n+1} \in (B_n)_\Gamma = C_n$.

Put

$$E = \mathrm{Dr}_{n\in\mathbb{N}}\langle a_n\rangle,$$

and suppose first that the Γ-invariant subgroup E_Γ has finite index in E, so that there exist positive integers r and s with $r < s$ and $E_\Gamma a_r = E_\Gamma a_s$. Thus $a_r a_s^{-1}$ belongs to the Γ-invariant subgroup $E_\Gamma \cap C_{r-1}$, so that also $(a_r a_s^{-1})^{\gamma_r}$ lies in

$$E_\Gamma \cap C_{r-1} \leqslant E \cap C_{r-1} = Dr_{n \geqslant r}\langle a_n \rangle,$$

and hence

$$(a_r a_s^{-1})^{\gamma_r} = a_r^\lambda a_s^\mu a_{i_1}^{\varepsilon_1} \cdots a_{i_t}^{\varepsilon_t},$$

where $i_1, \dots, i_t$ are integers larger than r and different from s, and the exponents $\lambda, \mu, \varepsilon_1, \dots, \varepsilon_t$ are suitable integers. Therefore,

$$a_r^{\gamma_r} a_r^{-\lambda} = a_s^{\gamma_r} a_s^\mu a_{i_1}^{\varepsilon_1} \cdots a_{i_t}^{\varepsilon_t}$$

belongs to $E_r \cap B_r = \{1\}$, which is impossible as $\langle a_r \rangle^{\gamma_r} \neq \langle a_r \rangle$. It follows that E_Γ has infinite index in E. Assume now that the index $|E^\Gamma : E|$ is finite, p^m say. Clearly,

$$E \cap E_{m+1} = \langle a_1 \rangle \times \cdots \times \langle a_m \rangle \times \langle a_{m+1} \rangle,$$

and so

$$|E_{m+1} : \langle a_1, \dots, a_m, a_{m+1} \rangle| \leqslant |E^\Gamma : E| = p^m,$$

which is impossible because E_{m+1} has order at least p^{k+2m+1}. This contradiction shows that also the index $|E^\Gamma : E|$ is infinite, and hence the subgroup E has infinite Γ-oscillation. □

Lemma 3.1.3 *Let A be a reduced abelian p-group, and let Γ be a group of automorphisms of A such that all subgroups of A have finite Γ-oscillation. If B is an infinite subgroup of A which has no non-trivial Γ-invariant subgroups, then there exists a finite Γ-invariant subgroup C of B^Γ such that Γ induces a group of power automorphisms on B^Γ/C. In particular, each element of B^Γ has finitely many images under the action of Γ and $B^\Gamma = BC$.*

Proof — Since B has finite Γ-oscillation and $B_\Gamma = \{1\}$, the index $|B^\Gamma : B|$ must be finite. Assume for a contradiction that B contains an element b admitting infinitely many images under the action of Γ, so that the subgroup $\langle b \rangle^\Gamma$ is infinite. Then $U = \langle b \rangle^\Gamma \cap B$ is an infinite group of finite exponent, and so there exist subgroups V and W such that $U = VW$, $V \cap W = \langle b \rangle$ and both indices $|U : V|$ and $|U : W|$ are infinite. Obviously, $V_\Gamma = W_\Gamma = \{1\}$ and hence V and W have finite index in $\langle b\rangle^{\Gamma} = V^{\Gamma} = W^{\Gamma}$. It follows that $\langle b\rangle^{\Gamma} = V^{\Gamma} \cap W^{\Gamma}$ is a finite Γ-invariant subgroup of A, and this contradiction shows that each element of B has only finitely many images under the action of Γ. Clearly, the same property also holds for all elements of B^{Γ}.

Let X be any infinite subgroup of B^{Γ}. Then also $X \cap B$ is infinite, and so it has finite index in $(X \cap B)^{\Gamma}$. Moreover, there is a finite subgroup Y of B^{Γ} such that $X = \langle X \cap B, Y\rangle$, and Y^{Γ} is finite by Dietzmann's Lemma, so that also the index $|X^{\Gamma} : X|$ is finite. Application of Lemma 2.9 of [7] yields now that B^{Γ} contains a finite Γ-invariant subgroup C such that Γ induces a group of power automorphisms on B^{Γ}/C. □

The following result will be used in the study of $T(*)$-groups, in order to produce elements admitting only finitely many conjugates.

Lemma 3.1.4 *Let A be an abelian p-group (where p is a prime number) and let Γ be a locally finite group of automorphisms of A such that all subgroups of A have finite Γ-oscillation. Then A contains a non-trivial element which has only finitely many images under the action of Γ.*

Proof — Assume for a contradiction that the statement is false. Then the group A cannot satisfy the minimal condition on subgroups, and so its socle S is an infinite Γ-invariant subgroup. Clearly, A can be replaced by S, and hence it can be assumed without loss of generality that A has exponent p. Since A has no finite non-trivial Γ-invariant subgroups, it follows from Lemma 3.1.3 that every infinite subgroup of A contains a non-trivial Γ-invariant subgroup. On the other hand, all subgroups of A have finite Γ-oscillation, and so an application of Lemma 3.1.2 yields that A contains a non-trivial Γ-invariant subgroup W on which Γ induces a group of power automorphisms. Clearly, each element of W has only finitely many images under the action of Γ, and this contradiction proves the statement. □

The next step is the study of automorphism groups of torsion-free locally nilpotent groups. Note that the next lemma is non-trivial only in the case of elements of infinite order.

Lemma 3.1.5 *Let G be a group, and let Γ be a group of automorphisms of G. If x is an element of G such that the cyclic subgroup $\langle x\rangle$ has finite Γ-oscillation, then the index $|\langle x\rangle : \langle x\rangle_{\Gamma}|$ is finite.*

Proof — It can obviously be assumed that the subgroup $\langle x\rangle$ has finite index in $\langle x\rangle^{\Gamma}$. Then there is a positive integer m such that

$(\langle x\rangle^\Gamma)^m$ is contained in $\langle x\rangle$, and hence also in $\langle x\rangle_\Gamma$.
As $\langle x\rangle^\Gamma$ is cyclic-by-finite, the factor group $\langle x\rangle^\Gamma/(\langle x\rangle^\Gamma)^m$ is finite, and so also the index $|\langle x\rangle : \langle x\rangle_\Gamma|$ is finite. □

Lemma 3.1.6 *Let* G *be a torsion-free locally nilpotent group. If* x *is an element of* G *and* γ *is an automorphism such that* $\langle x^n\rangle^\gamma = \langle x^n\rangle$ *for some positive integer* n*, then* $\langle x\rangle^\gamma = \langle x\rangle$*.*

Proof — As $\langle x^n\rangle$ is infinite cyclic, we have $(x^n)^\gamma = x^{\epsilon n}$, where $\epsilon = \pm 1$ and so x^n belongs to the center of the group $\langle x, x^\gamma\rangle$. Then $\langle x, x^\gamma\rangle/Z(\langle x, x^\gamma\rangle)$ is finite, so that the commutator subgroup $\langle x, x^\gamma\rangle'$ is finite by the celebrated Schur's theorem (see for instance [56] Part 1, Theorem 4.12) and hence $\langle x, x^\gamma\rangle$ is abelian.
Therefore,

$$(x^{-\epsilon}x^\gamma)^n = x^{-\epsilon n}(x^n)^\gamma = 1,$$

and so $x^\gamma = x^\epsilon$. The lemma is proved. □

Lemma 3.1.7 *Let* G *be a torsion-free locally nilpotent group, and let* Γ *be a group of automorphisms of* G *such that every subgroup of* G *has finite* Γ*-oscillation. Then* Γ *is a group of power automorphisms of* G*.*

Proof — Let γ be any element of Γ. If x is an arbitrary element of G, the index $|\langle x\rangle : \langle x\rangle_\Gamma|$ is finite by Lemma 3.1.5, and so in particular $\langle x\rangle^\gamma = \langle x\rangle^n$ for some positive integer n. It follows from Lemma 3.1.6 that $\langle x\rangle^\gamma = \langle x\rangle$, and hence γ is a power automorphisms of G. □

In the next section we will analyze the structure of $T(*)$-groups. Among other results, we shall prove that any periodic soluble $T(*)$-group contains a metanilpotent subgroup of finite index, and that all finitely generated soluble groups with the $T(*)$-property are abelian-by-finite; moreover, it will be shown that torsion-free soluble $T(*)$-groups are abelian.

3.2 $T(*)$-groups

Let G be a group, and let X be a subgroup of G which has finite normal oscillation. Then it is clear that X has finite normal oscillation in every subgroup Y of G such that $X \leqslant Y$, and that XN/N has finite normal oscillation in G/N for each normal subgroup N of G. It follows that the class $T(*)$ is closed with respect to subnormal subgroups and homomorphic images.

As a direct consequence of Lemma 3.1.1, we have the following information concerning Prüfer subgroups of $T(*)$-groups.

Lemma 3.2.1 *Let G be a $T(*)$-group. Then the finite residual of any subnormal subgroup of G is normal. In particular, all subnormal subgroups of type p^∞ of G are normal.*

Lemma 3.2.2 *Let G be a $T(*)$-group, and let*

$$K = Dr_{i \in I} K_i$$

be a periodic normal subgroup of G, where $\pi(K_i) \cap \pi(K_j) = \emptyset$ if $i \neq j$. Then there are only finitely many indices $i \in I$ such that K_i contains a subnormal subgroup which is not normal in G.

Proof — Let I_0 be the set of all indices i such that K_i contains a subnormal subgroup which is not normal in G, and for each $i \in I_0$, let X_i be a subnormal subgroup of defect 2 of K_i. Put

$$X = \langle X_i \mid i \in I_0 \rangle = Dr_{i \in I_0} X_i,$$

so that

$$X_G = Dr_{i \in I_0} (X_i)_G \qquad \text{and} \qquad X^G = Dr_{i \in I_0} X_i^G.$$

As X is subnormal in G, and

$$(X_i)_G < X_i < (X_i)^G$$

for all i in I_0, it follows that the set I_0 is finite, because X has finite normal oscillation. □

Of course, all subgroups of a Baer T-group are normal, and it is also known that any Baer group in the class $T^* \cup T_*$ is nilpotent (see [7, 26]). The next statement shows that this result can be generalized to the case of $T(*)$-groups.

Lemma 3.2.3 *Let G be a Baer group with the $T(*)$-property. Then G is nilpotent.*

Proof — Assume for a contradiction that the group G is not nilpotent. If x is any element of infinite order of G, the cyclic normal subgroup $\langle x\rangle_G$ is infinite by Lemma 3.1.5 and so it is contained in the center $Z(G)$, because G is locally nilpotent. Thus the factor group $G/Z(G)$ is a periodic counterexample to the statement, and hence without loss of generality it can be assumed that G is periodic. In particular, G is the direct product of its Sylow subgroups, and it follows from Lemma 3.2.2 that all but finitely many of such factors have the T-property, and so are Dedekind groups. Therefore, there exists a prime number p such that the Sylow p-subgroup of G is not nilpotent, and hence we may suppose that G is a p-group.

As the Baer group G is not nilpotent, it cannot be a Černikov group, and so it does not satisfy the minimal condition on abelian subnormal subgroups (see [55], Theorem E). Thus G contains an infinite abelian subnormal subgroup U of exponent p, and the subgroup U can be chosen even normal in G, because it has finite normal oscillation. Application of Lemma 3.1.4 with $\Gamma = G/C_G(U)$ yields that U contains a non-trivial element u which has only finitely many conjugates in G. Then the normal subgroup $\langle u\rangle^G$ is finite, and hence $Z(G) \neq \{1\}$. Since the hypotheses are inherited by homomorphic images, it follows that each non-trivial homomorphic image of G has a non-trivial center, and so G is hypercentral.

Now, let A be a maximal abelian normal subgroup of G, so that $C_G(A) = A$, and let B be a basic subgroup of A. As the subnormal subgroup B of G has finite normal oscillation, there exists a residually finite G-invariant subgroup B_0 of A such that A/B_0 is finite-by-divisible, and so also divisible-by-finite. On the other hand, every divisible normal subgroup of a periodic Baer group lies in the center (see for instance [56] Part 1, Lemma 3.13), and hence the subgroup

$$Z/B_0 = A/B_0 \cap Z(G/B_0)$$

has finite index in A/B_0.

It follows from Lemma 3.1.2 that B_0 contains a subgroup V such that the index $|B_0 : V|$ is finite and G induces a group of power automorphisms on the core V_G of V. Moreover, by Lemma 3.1.3 the normal closure V^G contains a G-invariant subgroup N such that NV_G/V_G is finite and G induces a group of power automorphisms on V^G/NV_G. The intersection

$$C = C_G(V_G) \cap C_G(NV_G/V_G) \cap C_G(V^G/NV_G) \cap C_G(B_0/V^G) \cap C_G(A/Z)$$

is a normal subgroup of finite index of G, and C/A is nilpotent, because it is isomorphic to a group of automorphisms of A stabilizing a finite series. On the other hand, A is contained in $Z_6(C)$, so that C is nilpotent and G is nilpotent-by-finite, which is impossible. This contradiction proves that all Baer $T(*)$-groups are nilpotent. □

Recall that a group G is *subsoluble* if it has an ascending series with abelian factors consisting of subnormal subgroups. Although for a finite group (or even for a group satisfying the maximal condition) the properties of being soluble or subsoluble are equivalent, it turns out that there exist infinite subsoluble groups which have no abelian non-trivial normal subgroups. It is easy to prove that a group G is subsoluble if and only if the upper Baer series of G (i.e., the ascending series defined by transfinite induction of the consecutive Baer radicals) terminates with G. Note also that if G is a subsoluble group and A is its Baer radical, then $C_G(A) \leqslant A$.

In [7] C. Casolo proved that a soluble T^*-group is finite-by-metabelian and in [26] S. Franciosi, F. de Giovanni and M.L. Newell proved that a subsoluble T_*-group is metabelian-by-finite; the aim of the following theorems is to generalized these results obtaining a similar theorem for a subsoluble $T(*)$-group.

Theorem 3.2.4 *Let* G *be a periodic subsoluble* $T(*)$*-group, and let* A *be the Baer radical of* G*. Then* A *is nilpotent and* $G'A/A$ *is finite, so that* G *is nilpotent-by-finite-by-abelian.*

Proof — The Baer radical A of G is nilpotent by Lemma 3.2.3, and so A/A' is the Baer radical of G/A' by Philip Hall's nilpotency criterion (see for instance, [56] Part 1, Theorem 2.27). Moreover, it is clearly enough to show that the statement holds for the factor group G/A', and hence without loss of generality it can be assumed that A is abelian.

It follows from Lemma 3.2.2 that there are finitely many prime numbers $p_1, \dots, p_t$ such that G induces by conjugation a group of power automorphisms on the q-component A_q of A for each prime

$q \neq p_1, \ldots, p_t$, and in particular $[A_q, G'] = \{1\}$ for all such q.
Let B_i be a basic subgroup of A_{p_i}, for $i = 1, \ldots, t$, so that A_{p_i}/B_i is divisible. Since B_i has finite normal oscillation in G, it follows that there exists a residually finite G-invariant subgroup B_i^* of A_{p_i} such that A_{p_i}/B_i^* is finite-by-divisible, and so it contains a divisible characteristic subgroup D_i/B_i^* such that A_{p_i}/D_i is finite. Notice here that D_i/B_i^* is a direct product of G-invariant subgroups of type p_i^∞ by Lemma 3.1.1, and hence G' acts trivially on it. Application of Lemma 3.1.2 yields that B_i^* has a subgroup of finite index X_i such that G acts as a group of power automorphisms on the core $Y_i = (X_i)_G$. Moreover, it follows from Lemma 3.1.3 that X_i^G contains a finite G-invariant subgroup N_i such that G induces a group of power automorphisms on X_i^G/N_iY_i. In particular, G' acts trivially on X_i^G/N_iY_i. As the groups $A_{p_i}/D_i, B_i^*/X_i^G$, and N_i are finite, it follows that G' contains a normal subgroup of finite index K_i stabilizing the series

$$\{1\} \leqslant Y_i \leqslant N_iY_i \leqslant X_i^G \leqslant B_i^* \leqslant D_i \leqslant A_{p_i},$$

and so the group $K_i/C_{K_i}(A_{p_i})$ is nilpotent. Of course, the intersection

$$K = \bigcap_{i=1}^{t} K_i$$

is a normal subgroup of finite index of G'. Moreover,

$$\bigcap_{i=1}^{t} C_{K_i}(A_{p_i}) = C_K(A) \leqslant A$$

and so the factor group K/A is nilpotent. But A is contained in a term with finite ordinal type of the upper central series of K, so that K itself is nilpotent and hence it is contained in A. Therefore,

$$G'A/A \simeq G'/G' \cap A$$

is finite. □

It is well known that if G is any group with a finite commutator subgroup, then also the index $|G : Z_2(G)|$ is finite (see for instance, [56] Part 1, Theorem 4.25). Therefore, Theorem 3.2.4 has the following consequence.

Corollary 3.2.5 *Let* G *be a periodic subsoluble* T(∗)*-group. Then* G *is metanilpotent-by-finite, and so also soluble.*

Now, it will be considered the case of non-periodic subsoluble T(∗)-groups. Here, a group is said to be of *dihedral type* if it is a semidirect product $\langle x \rangle \ltimes A$, where A is a non-trivial torsion-free abelian group and x is an element of order 2 such that $a^x = a^{-1}$ for all $a \in A$.

Theorem 3.2.6 *Let* G *be a subsoluble* T(∗)*-group, and let* T *be the largest periodic normal subgroup of* G*. Then the factor group* G/T *is either abelian or of dihedral type.*

Proof — The replacement of G by the factor group G/T allows to assume that G has no periodic non-trivial normal subgroups, so that it follows from Lemma 3.2.3 that the Baer radical A of G is a torsion-free nilpotent group. Then all subgroups of A have finite normal oscillation in G, and an application of Lemma 3.1.7 yields that G induces a group of power automorphisms on A. It follows that $G/C_G(A)$ has order at most 2. As $C_G(A) \leqslant A$, we have in particular that A is abelian and so $C_G(A) = A$.

Suppose that G is not abelian, so that $|G : A| = 2$, and we may consider an element $g \in G \setminus A$. Then $a^g = a^{-1}$ for all $a \in A$, and hence $g^2 = 1$ and $G = \langle g \rangle \ltimes A$ is a group of dihedral type. □

Corollary 3.2.7 *Let* G *be a subsoluble torsion-free* T(∗)*-group. Then* G *is abelian.*

From the last corollary, it follows that both a subsoluble torsion-free T^*-group and a subsoluble torsion-free T_*-group are abelian, as noted by S. Franciosi, F. de Giovanni and M.L. Newell in [26] for T_*-groups; furthermore combining Theorems 3.2.4 and 3.2.6, we obtain the following information, where $\mathfrak{N}$ denotes the class of nilpotent groups.

Corollary 3.2.8 *Let* G *be a subsoluble* T(∗)*-group. Then* G *is soluble and contains a subgroup of finite index which is in the class* $\mathfrak{N}^3$.

The last result shows that finitely generated soluble T(∗)-groups are almost abelian, as is the case of the soluble finitely generated T^*-groups and the subsoluble finitely generated T_*-groups (see respectively, [7] and [26]), and should be seen in relation to the fact that a finitely generated soluble T-group is either abelian or finite.

Theorem 3.2.9 *Let* G *be a finitely generated soluble* $T(*)$*-group. Then* G *contains an abelian subgroup of finite index.*

Proof — Assume for a contradiction that the statement is false, and let Ω be an arbitrary chain of normal subgroups of G such that G/N is not abelian-by-finite for each element N of Ω. Put

$$W = \bigcup_{N \in \Omega} N$$

and suppose that the factor group G/W is abelian-by-finite. Then G/W is finitely presented, and so W is the normal closure of a finite subset, which is of course impossible. Therefore, G/W is not abelian-by-finite, and an application of Zorn's lemma yields that G contains a normal subgroup M which is maximal with respect to the condition that G/M is not abelian-by-finite. As G/M is a counterexample to the statement, it can be assumed without loss of generality that all proper homomorphic images of G are abelian-by-finite.

Let T be the largest periodic normal subgroup of G. Then G/T is abelian-by-finite by Theorem 3.2.6, so that $T \neq \{1\}$ and hence the Baer radical A of T is a non-trivial nilpotent normal subgroup of G. Assume that $A' \neq \{1\}$, so that the factor group G/A' is abelian-by-finite and G is nilpotent-by-finite; it follows that G is polycyclic, so that T is finite and hence G is finite-by-abelian-by-finite and so even abelian-by-finite. This contradiction shows that A is an infinite abelian group. As G/A is finitely generated and abelian-by-finite, we have that A is the normal closure in G of a finite subgroup X. In particular, A has finite exponent, and so it contains two infinite subgroups H and K such that $H \cap K = X$. Then $H^G = K^G = A$, and hence both indices $|H^G : H|$ and $|K^G : K|$ are infinite. On the other hand, all subgroups of A have finite normal oscillation in G, so that H_G has finite index in H and K_G has finite index in K. In particular, the normal subgroups H_G and K_G of G are not trivial, and hence the factor groups G/H_G and G/K_G are abelian-by-finite. Moreover, the intersection $H_G \cap K_G$ is finite, so that G is finite-by-abelian-by-finite, and hence also abelian-by-finite. This last contradiction completes the proof. □

3.3 An open problem

In [27] W. Gaschütz proved that a subgroup of a finite soluble T-group is still a T-group. In the infinite case D.J.S. Robinson [53] showed that each soluble T-group of type 1 (namely, a non-abelian group such that $C_G(G')$ is not periodic) has a subgroup which is not a T-group. However H. Heineken and J.C. Lennox in [34] proved that a subgroup H of finite index of a T-group G is still a T-group if it contains some term of the derived series of G. Generalizing this result, C. Casolo proved that a subgroup of finite index in a T^*-group is again a T^*-group (see [6]).

Theorem 3.3.1 *Let* G *be a group* T^**-group and let* H *and* L *be subgroups of* G *such that* $H \leqslant L$*. If* H *is subnormal in* L *and* L *has finite index in* G*, then* $|H^G : H|$ *is finite.*

Proof — Proceed by induction on the subnormal defect n of H in L. For $n = 0$ the assert is obvious, so let $n = 1$. The subgroup $H \cap L_G$ is subnormal in G and then $|(H \cap L_G)^G : H \cap L_G|$ is finite. If $W = (H \cap L_G)^G$ we have also $W \leqslant L_G$. Now, $|HW/W|$ is finite and $N_G(HW) \geqslant L$, it follows that $|G : N_G(HW)|$ is finite too. By Dietzmann's Lemma, $(HW/W)^{G/W}$ is finite. In particular, $|(HW)^G : HW|$ is finite. Now, since the index $|HW : H| = |W : H \cap L_G|$ is finite, $|(HW)^G : H|$ is finite too and it follows that $|H^G : H|$ is finite.

Let now $n > 1$, and $T = H^L$; then, by the case discussed above, $|T^G : T|$ is finite. The subgroup H is subnormal of defect $n-1$ in T and by inductive hypothesis and the fact that T^G is a T^*-group, the index $|H^{T^G} : H|$ is finite. Moreover the subgroup H^{T^G} is subnormal in G and then $|(H^{T^G})^G : H^{T^G}|$ is finite; it follows that $|(H^{T^G})^G : H|$ is finite and so $|H^G : H|$ is finite too. □

Corollary 3.3.2 *A subgroup of finite index in a* T^**-group is again a* T^**-group.*

In a dual way, S. Franciosi, F. de Giovanni and M.L. Newell in [26] have easily proved an analogous result for the T_*-groups.

Theorem 3.3.3 *Let* G *be a* T_**-group, and let* X *be a subgroup of finite index of* G*. Then* X *is a* T_**-group.*

Proof — Let H be a subnormal subgroup of X. Then $H \cap X_G$ is subnormal in X_G and so also in G, so that $(H \cap X_G)/(H \cap X_G)_G$ is finite. Since $H \cap X_G$ has finite index in H, it follows that H/H_G is finite. In particular H/H_X is finite, and X is a T_*-group. □

It seems to be reasonable that a similar result should indeed be obtained for the $T(*)$-groups, however, we did not manage to prove it.

Open problem: Let G be a $T(*)$-group, and let X be a subgroup of finite index of G. Is X still a $T(*)$-group?

Chapter 4

Countable character of subnormal-like subgroups

A class of groups $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by R. Baer [3].

The so-called local classes are of course countably recognizable: a group class $\mathfrak{X}$ is *local* if it contains all groups in which every finite subset lies in some $\mathfrak{X}$-subgroup. It is clear that any variety of groups is itself a local class, and so the property of being soluble of bounded length and that of being nilpotent of bounded class are both local. Although the class $\mathfrak{N}$ of nilpotent groups and the class $\mathfrak{S}$ of soluble groups are not local, it is easy to see that they are at least countably recognizable (see for instance [29, Lemma 2.1]).

A famous theorem of A.I. Mal'cev may be applied to prove that many relevant group classes are local (see [56, Chapter 8], for a description of these methods). In particular, starting from suitable series of the members of a local system of a group G, Mal'cev's result allows to construct a new series of G. For instance, it follows that if $\mathfrak{V}$ is any variety, then the class of all groups admitting a series whose factors are in $\mathfrak{V}$ is local.

On the other hand, Mal'cev theorem does not allow to control the order type of the new series, and the aim of this chapter is to provide a general method to construct finite series of a group G based on suitable finite series of the countable subgroups of G.

Let $\mathfrak{V}_1, \ldots, \mathfrak{V}_t$ be finitely many varieties of groups and consider a subset M of $\{-1, -2, \ldots, -t\}$. Put

$$Q = \bigcup_{k \in \mathbb{N}} (M \cup \{0\})^k$$

and let $\mathfrak{q} \in Q$. If Σ is a non-empty initial segment of $\mathbb{N}$, a subgroup H of a group G is said to be $(\mathfrak{q}, \Sigma)$-*subnormal* in G if there exists a $(\mathfrak{q}, \Sigma)$-*chain* from H to G, i.e., a finite chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G,$$

where $\mathfrak{q} = (q_1, \ldots, q_n)$ and for each $i \in \{1, \ldots, n\}$ we have that

$$|H_i : H_{i-1}| \in \Sigma$$

if $q_i = 0$, while H_{i-1} is normal in H_i and $H_i/H_{i-1} \in \mathfrak{B}_{-q_i}$ when $q_i \neq 0$.

Define a partial order $\prec$ in Q by setting

$$(q_1, \ldots, q_m) = \mathfrak{q} \prec \mathfrak{q}' = (q'_1, \ldots, q'_n) \quad (n, m \in \mathbb{N})$$

if and only if $m \leqslant n$ and there is a strictly increasing function

$$\varphi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$$

such that $q_i = q'_{\varphi(i)}$ for $i \in \{1, \ldots, m\}$. This means that, $\mathfrak{q} \prec \mathfrak{q}'$ if and only if one can go from $\mathfrak{q}'$ to $\mathfrak{q}$ by removing some components. Note that every subset of Q has an element which is $\prec$-minimal.

Fix now a non-empty initial segment Σ of $\mathbb{N}$. We will then speak of $\mathfrak{q}$-*subnormality* and $\mathfrak{q}$-*chains* instead of, respectively, $(\mathfrak{q}, \Sigma)$-subnormality and $(\mathfrak{q}, \Sigma)$-chains.

The main result of this chapter is the following:
Theorem *Let* G *be a group,* H *a subgroup of* G *and* $\mathfrak{q} = (q_1, \ldots, q_k) \in Q$. *If* $H \cap C$ *is* $\mathfrak{q}$*-subnormal in* C *for every countable subgroup* C *of* G*, then* H *is* $\mathfrak{p}$*-subnormal in* G*, for some* $\mathfrak{p} \prec \mathfrak{q}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be group classes. We shall denote by $\mathfrak{X}\mathfrak{Y}$ the *product* of $\mathfrak{X}$ and $\mathfrak{Y}$, i.e. the class consisting of all groups G containing a normal $\mathfrak{X}$-subgroup N such that the factor group G/N belongs to $\mathfrak{Y}$. It seems to be unknown under which hypotheses the product of two countably recognizable classes is likewise countably recognizable. On the other hand, this problem has a positive solution in the case of varieties, since it is well-known that the product of two varieties is again a variety (see for instance [49]). Moreover, if $\{\mathfrak{B}_n\}_{n\in\mathbb{N}}$ and $\{\mathfrak{C}_n\}_{n\in\mathbb{N}}$ are sequences of group varieties, then [29, Lemma 2.1] implies that the class of groups

$$\Big(\bigcup_m \mathfrak{B}_m\Big)\Big(\bigcup_n \mathfrak{C}_n\Big) = \bigcup_{m,n} \Big(\mathfrak{B}_m \mathfrak{C}_n\Big)$$

is countably recognizable. Furthermore, the class of groups with a finite series (of bounded length) whose factors belong to $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$

is also countably recognizable. It follows for instance that the class of metanilpotent groups, and more generally that of soluble groups of bounded Fitting length is countably recognizable. Notice also that the class of poly-$\mathfrak{B}$ groups is countably recognizable for any group variety $\mathfrak{B}$.

As a consequence of the Theorem, we generalize the above results proving, for instance, that the class of all groups with a finite series whose factors are either finite or belongs to a given variety is countably recognizable.

The range of applicability of the Theorem and its method is not limited to properties of this type. In fact, in the last section of the chapter we prove that many different properties defined by subnormality-like conditions can be countably detectable. In particular, it turns out that all group classes and subgroup properties considered in [7, 8, 24, 26] have countable character: this is, for instance, the case of *f*-subnormality, the property T^*, the property T_* and the property $\mathsf{T}(*)$.

4.1 Proof of the Theorem

This section deals with the proof of the main result, that is labelled as Theorem 4.1.1.

Theorem 4.1.1 *Let* H *be a subgroup of a group* G *and* $\mathfrak{q} = (q_1, \dots, q_k) \in Q$. *If* $H \cap C$ *is* $\mathfrak{q}$*-subnormal in* C *for every countable subgroup* C *of* G, *then* H *is* $\mathfrak{p}$*-subnormal in* G, *for some* $\mathfrak{p} \prec \mathfrak{q}$.

Proof — Let $\mathcal{C}$ be the set of all countable subgroups of the group G. For each $C \in \mathcal{C}$, there is a $\mathfrak{q}_C \in Q$ such that $\mathfrak{q}_C \prec \mathfrak{q}$, $H \cap C$ is $\mathfrak{q}_C$-subnormal in C and $\mathfrak{q}_C$ is $\prec$-minimal with respect to these properties. Set

$$\mathcal{C}_{\mathfrak{p}} = \{C \in \mathcal{C} \,:\, \mathfrak{q}_C = \mathfrak{p}\},$$

for all $\mathfrak{p} \in Q$. Suppose that, for each $\mathfrak{p} \prec \mathfrak{q}$, there is a $C_{\mathfrak{p}} \in \mathcal{C}$ which is not contained in any element of $\mathcal{C}_{\mathfrak{p}}$. Then, the countable subgroup $\langle C_{\mathfrak{p}} \,:\, \mathfrak{p} \prec \mathfrak{q}\rangle$ is not contained in any element of

$$\bigcup_{\mathfrak{p} \prec \mathfrak{q}} \mathcal{C}_{\mathfrak{p}} = \mathcal{C},$$

which is a contradiction. Therefore, there exists $\mathfrak{p} \prec \mathfrak{q}$ in Q such that $\mathcal{C}_{\mathfrak{p}}$ is a countable system of G.

For each $C \in \mathcal{C}_{\mathfrak{p}}$, there is a $\mathfrak{p}$-chain from $H \cap C$ to C with a smallest number of infinite jumps, say $s(C)$. If $C_1 \leqslant C_2$ are elements of $\mathcal{C}_{\mathfrak{p}}$, then $s(C_1) \leqslant s(C_2)$, and hence, the set

$$\{s(C) \,:\, C \in \mathcal{C}_{\mathfrak{p}}\}$$

has a largest element $s = s(C^1)$. Thus, whenever $C \in \mathcal{C}_{\mathfrak{p}}$ and $C \geqslant C^1$, it follows that $s(C^1) = s(C)$, which also means that the number of finite jumps is the same, say f_j. Let now for convenience

$$\mathcal{C}^1_{\mathfrak{p}} = \{C \geqslant C^1 \,:\, C \in \mathcal{C}_{\mathfrak{p}}\}.$$

Suppose that $f_j \neq 0$. For each $C \in \mathcal{C}^1_{\mathfrak{p}}$, there is a $\mathfrak{p}$-chain from $H \cap C$ to C having f_j finite jumps and, under this condition, such that the sum $j(C)$ of the orders of its finite jumps is the smallest possible. Again, it can be easily proved that, if $C_1 \leqslant C_2$ are elements of $\mathcal{C}^1_{\mathfrak{p}}$, then $j(C_1) \leqslant j(C_2)$. Suppose that the set

$$J = \{j(C) \,:\, C \in \mathcal{C}^1_{\mathfrak{p}}\}$$

does not contains a largest element. Then there is a strictly increasing sequence of numbers

$$j(C_1) < j(C_2) < \ldots < j(C_i) < \ldots$$

and the countable subgroup $\langle C_i \,:\, i \in \mathbb{N}\rangle$ is contained in a a suitable element C_∞ of $\mathcal{C}^1_{\mathfrak{p}}$. However, this is a contradiction, since it should be $j(C_\infty) \geqslant j(C_i)$ for each $i \in \mathbb{N}$. Therefore, J has a largest element $j = j(C^2)$. Notice that we have $j(C^2) = j(C)$, whenever $C \in \mathcal{C}^1_{\mathfrak{p}}$ and $C \geqslant C^2$. Clearly,

$$\mathcal{C}^2_{\mathfrak{p}} = \{C \geqslant C^2 \,:\, C \in \mathcal{C}^1_{\mathfrak{p}}\}$$

is still a countable system of G and for every countable subgroup C of $\mathcal{C}^2_{\mathfrak{p}}$, there exists a $\mathfrak{p}$-chain

$$\mathcal{S}_C \,:\, H \cap C = H_{0,C} \leqslant H_{1,C} \leqslant \ldots \leqslant H_{n,C} = C$$

in which the orders of the finite jumps corresponding to the 0-components of $\mathfrak{p}$ are bounded by

$$l = \min\{j, \sup(\Sigma)\}.$$

Given $\mathcal{S}_C$, we define a binary relation $\mathcal{R}_C$ on C by setting $x\,\mathcal{R}_C\,y$ if and only if

$$\bigcap_{i\,:\,x\in H_{i,C}} H_{i,C} \leqslant \bigcap_{i\,:\,y\in H_{i,C}} H_{i,C}.$$

The relation $\mathcal{R}_C$ can be encoded as a function

$$f_C \,:\, C \times C \longrightarrow \{0,1\}$$

such that $f_C(x,y) = 1$ if and only if $x\,\mathcal{R}_C\,y$.

Applying now Lemma 8.22 of [56], it follows that there is a function

$$f \,:\, G \times G \longrightarrow \{0,1\}$$

having the property that, for every finite subset

$$\{x_1, \ldots, x_m\}$$

of $G \times G$, there exists $C \in \mathcal{C}_p^2$ such that $x_i \in C \times C$ and $f(x_i) = f_C(x_i)$ for $i = 1, \ldots, m$. From this function we go back to a binary relation $\mathcal{R}$ setting $x\mathcal{R}y$ whenever $f(x, y) = 1$, for each $x, y \in G$. Our next step in the proof is to describe some properties of $\mathcal{R}$ in order to construct a suitable chain from H to G.

We claim that $\mathcal{R}$ is a total and transitive relation. In fact, if x, y are elements of G, then there is a $C \in \mathcal{C}_p^2$ such that $f_C(x, y) = f(x, y)$ and $f_C(y, x) = f(y, x)$. However, the construction of $\mathcal{R}_C$ shows that either $f_C(x, y) = 1$ or $f_C(y, x) = 1$. Therefore $\mathcal{R}$ is total. The transitivity can be proved in a similar way.

Another relevant property of $\mathcal{R}$ is that, given $n+2$ arbitrary elements $x_1, \ldots, x_{n+2}$ of G, there are two of them which are each other in relation. In fact, assume for a contradiction that $x_i \mathcal{R} x_{i+1}$ and $x_{i+1} \not\mathcal{R} x_i$, for each $i \in \{1, \ldots, n-2\}$. Then there is $C \in \mathcal{C}_p^2$ such that $f_C(x_k, x_h) = f(x_k, x_h)$ for all $h, k \in \{1, \ldots, n+2\}$ and hence

$$x_i \mathcal{R}_C x_{i+1} \quad \text{and } x_{i+1} \not\mathcal{R}_C x_i,$$

for $i \in \{1, \ldots, n-2\}$, which cleary is a contradiction. Since we have already shown that $\mathcal{R}$ is total and transitive, it follows that the above property holds.

Finally, it can be proved that, for $x, y, z \in G$ with $x\mathcal{R}z$ and $y\mathcal{R}z$, one has $xy^{-1}\mathcal{R}z$. As before, there is a $C \in \mathcal{C}_p^2$ such that

$$f_C(xy^{-1}, z) = f(xy^{-1}, z),\ f_C(x, z) = f(x, z) = 1,\ f_C(y, z) = f(y, z) = 1.$$

Again, the costruction of $\mathcal{R}_C$ shows that $f(xy^{-1}, z) = f_C(xy^{-1}, z) = 1$, which is what was claimed.

We can now proceed to construct the quoted chain from H to G. Define by recursion a sequence of elements $\{x_i\}_{i \in \mathbb{N}_0}$ of G by putting $x_0 = 1$, and, by choosing x_{i+1} as an $\mathcal{R}$-minimal element of G such that $x_i \mathcal{R} x_{i+1}$ and $x_{i+1} \not\mathcal{R} x_i$, if there exists such an element and by setting $x_{i+1} = x_i$ otherwise. By the above properties of $\mathcal{R}$, this sequence stops after at most n steps, and for each $i = 0, \ldots, n$ the set

$$H_i = \{x \in G \mid x\mathcal{R}x_i\}$$

is a subgroup of G. Notice that $H_n = G$. If h_1, h_2 are arbitrary elements of H, there is $C \in \mathcal{C}^2_{\mathfrak{p}}$ such that

$$f_C(h_1, h_2) = f(h_1, h_2).$$

On the other hand, by the construction of $\mathcal{S}_C$, it follows that $h_1 \mathcal{R}_C h_2$, and so $f(h_1, h_2) = 1$, which means that $h_1 \mathcal{R} h_2$. Therefore H is contained in H_0. Suppose by contradiction that there exist $g \in G \backslash H$ such that $g\mathcal{R}1$. Then $f_C(g, 1) = 1$ for some $C \in \mathcal{C}^2_{\mathfrak{p}}$. However, by construction, no element of $C \backslash (H \cap C)$ is in relation with an element of $H \cap C$. This contradiction proves that $H_0 = H$.

Assume that $H < G$ and let

$$\mathcal{S}_G \,:\, H = H_0 < \ldots < H_m = G \quad (m \leqslant n)$$

be the above constructed chain (here $m \leqslant n$). Take $e_i \in H_i \backslash H_{i-1}$ for $i = 0, \ldots, m$, with the convention that $H_{-1}{=}\emptyset$. Suppose by contradiction that $\mathcal{S}_G$ does not correspond to any $\mathfrak{p}'$-chain with

$$\mathfrak{p}' = (p'_1, \ldots, p'_m) \prec \mathfrak{p}.$$

Then, for each $\mathfrak{p}' \prec \mathfrak{p}$, the jump (H_{i-1}, H_i) does not correspond to p'_i for some positive integer $i \leqslant n$. If $p'_i = 0$, we take elements

$$y_{1,\mathfrak{p}'}, \ldots, y_{l+1,\mathfrak{p}'} \in H_i$$

such that

$$y_{h,\mathfrak{p}'} y^{-1}_{k,\mathfrak{p}'} \notin H_{i-1} \quad \forall\, h, k \in \{1, \ldots, l+1\},$$

and define

$$V_{\mathfrak{p}'} = \{y_{j,\mathfrak{p}'},\, y_{h,\mathfrak{p}'} y^{-1}_{k,\mathfrak{p}'} \mid j, h, k \in \{1, \ldots, l+1\}\}.$$

Suppose instead that $p'_i < 0$ and that H_{i-1} is not normal in H_i. Then there are elements $w_{\mathfrak{p}',1}$ and $w_{\mathfrak{p}',2}$ in H_i such that

$$w_{\mathfrak{p}',2}, w_{\mathfrak{p},'1}^{w_{\mathfrak{p}',2}} \notin H_{i-1} \quad \text{and} \quad w_{\mathfrak{p}',1} \in H_{i-1}.$$

In this case, we put

$$V_{\mathfrak{p}'} = \{w_{\mathfrak{p}',1},\, w_{\mathfrak{p}',2},\, w_{\mathfrak{p}',1}^{w_{\mathfrak{p}',2}}\}.$$

Finally, if $p'_i < 0$ and H_{i-1} is normal in H_i, there is a word $\theta_{\mathfrak{p}'}$ defining $\mathfrak{B}_{-p'_i}$ and elements

$$z_{1,\mathfrak{p}'}, \dots, z_{t_{\mathfrak{p}'},\mathfrak{p}'}$$

in H_i such that

$$\theta_{\mathfrak{p}'}(z_{1,\mathfrak{p}'}, \dots, z_{t_{\mathfrak{p}'},\mathfrak{p}'})$$

does not belong to H_{i-1}. In this case, define $V_{\mathfrak{p}'}$ to be the set

$$\{z_j, \theta_{\mathfrak{p}'}(z_{1,\mathfrak{p}'}, \dots, z_{t_{\mathfrak{p}'},\mathfrak{p}'}) \mid j = 1, \dots, t_{\mathfrak{p}'}\}.$$

Let

$$V = \bigcup_{\mathfrak{p}' \prec \mathfrak{p}} V_{\mathfrak{p}'}$$

and put

$$U = V \cup \{e_1, \dots, e_m\}.$$

There exists $C \in \mathcal{C}^2_{\mathfrak{p}}$ such that f and f_C act in the same way on U. All elements of U which are in relation one another, are also in relation one another in relation with a unique e_k, for some $k = 0, \dots, m$. It follows that all these elements lie in a set of the form $K^2 \backslash K^1$, where (K^1, K^2) is a jump of $\mathcal{S}_C$. If we take the components of $\mathfrak{p}$ corresponding to these jumps ordered from $H \cap C$ to C, we obtain a new element $\mathfrak{p}'' \in Q$ such that

$$(p''_1, \dots, p''_b) = \mathfrak{p}'' \prec \mathfrak{p}.$$

Therefore, there is $i \leqslant m$ such that the jump (H_{i-1}, H_i) does not correspond to p''_i and $V_{\mathfrak{p}''} \subseteq U$. However, all elements of $H_i \backslash H_{i-1}$ are doubly in relation one another and also with e_i, and hence they are contained in the set $L^2 \backslash L^1$, where (L^1, L^2) is the jump of $\mathcal{S}_C$ corresponding to p''_i. On the other hand, the relations between the elements of U show that this is impossible. The statement is proved. □

4.2 Main consequences

Notice first that if we choose $H = \{1\}$ in the statement of Theorem 4.1.1, and with a suitable choice of the varieties defining Q, we obtain that the property of being finite-by-abelian-by-finite is countably recognizable (see also [29], where other proofs of this fact are discussed).

The following statement is instead a special case of a more general result proved in [29].

Corollary 4.2.1 *Let $\mathfrak{X}$ be a variety of groups. Then the class $\mathfrak{X}\mathfrak{F}$ of all groups containing an $\mathfrak{X}$-subgroup of finite index and the class $\mathfrak{F}\mathfrak{X}$ of all groups containing a finite normal subgroup with $\mathfrak{X}$-factor group are countably recognizable.*

In order to extend the range of applicability of the Theorem 4.1.1 we need the following result, in which Q is the set defined in the first section of this chapter.

Corollary 4.2.2 *Let G be a group, and let H be a subgroup of G such that for each countable subgroup C of G there exists $\mathfrak{q} \in Q$ such that $H \cap C$ is $\mathfrak{q}$-subnormal in C. Then H is $\mathfrak{p}$-subnormal in G for some $\mathfrak{p} \in Q$.*

Proof — Suppose by contradiction that the statement is false. Then it follows from the Theorem 4.1.1 that for each $\mathfrak{q} \in Q$ there is a countable subgroup $C_\mathfrak{q}$ of G such that $H \cap C_\mathfrak{q}$ is not $\mathfrak{q}$-subnormal in $C_\mathfrak{q}$. Let C be the countable subgroup generated by all $C_\mathfrak{q}$'s with $\mathfrak{q} \in Q$. By hypothesis, there is a $\mathfrak{q}' \in Q$ such that $H \cap C$ is $\mathfrak{q}'$-subnormal in C, which is a contradiction since $H \cap C_{\mathfrak{q}'}$ is not $\mathfrak{q}'$-subnormal in $C_{\mathfrak{q}'}$. □

As an immediate consequence of Corollary 4.2.2, we obtain the following result.

Theorem 4.2.3 *Let G be a group, and let H be a subgroup of G such that $H \cap C$ is f-subnormal in C, for each countable subgroup C of G. Then H is f-subnormal in G.*

As an application of Corollary 4.2.2 for $H = \{1\}$ and of [29, Lemma 2.1] we have the following result.

Theorem 4.2.4 *Let $\{\mathfrak{B}\}_{n \in \mathbb{N}}$ be a sequence of varieties of groups. Then the class of all groups admitting a finite series whose factors either are finite or belong to $\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n$ is countably recognizable.*

A $\mathfrak{q}$-chain is said to be *normal* if all its terms are normal in the group. In these circumstances a normal subgroup H of a group G will be said $\mathfrak{q}$-*normal* if there is a normal $\mathfrak{q}$-chain from H to G. It is easy to see that, with minor changes in the proofs, in the above statement normality can be replaced by $\mathfrak{q}$-normality, obtaining thus the following result.

Theorem 4.2.5 *Let $\{\mathfrak{B}_n\}_{n\in\mathbb{N}}$ be a sequence of varieties of groups, then the class of all groups admitting a finite normal series whose factors either are finite or belong to $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ is countably recognizable.*

4.3 Subgroup properties

Let Θ be a subgroup property. In the following, it will be often written $H_{\Theta}G$ or "H is a Θ-subgroup of G" whenever H is a subgroup of a group G and H has the property Θ in G. Following [29], we say that Θ has *countable character* if a subgroup Y of an arbitrary group G is a Θ-subgroup of G whenever Θ holds in G for all countable subgroups of Y.

Suppose now that Θ is such that $H_{\Theta}K$ follows from $H_{\Theta}G$, for an arbitrary group G and two its subgroups $H \leqslant K$. In this case, it can be easily proved that Θ has countable character if, given a group G and a subgroup H, we have $H_{\Theta}G$ whenever $H \cap C_{\Theta}C$ for all countable subgroups C of G. If Θ satisfies this latter property, we shall say that Θ has *strong countable character*.

It is clear that, if Θ is actually an absolute property, then the concepts of countable character, strong countable character and countable recognizability coincide. However, in general, they may not coincide. In fact, let Θ be the embedding property defined in the following way: $H_{\Theta}G$ if and only if $|G : H| \leqslant \aleph_0$. Obviously, Θ has countable character, but the consideration of any uncountable group shows that this character is not strong.

It was recently proved in [30] that the property of being closed in the profinite topology has strong countable character. The Theorem 4.1.1 gives directly a further contribution to the list of properties of strong countable character, adding the property of being q-subnormal for a given set Q. In particular, choosing $\mathfrak{B}_1$ to be the class of all groups, $\Sigma = \{1\}$ and $M = \{-1\}$ we get that the property of being subnormal and that of being subnormal of bounded defect have both strong countable character (see also [29, Theorem 2.4]). On the other hand, Theorem 4.2.3 shows the strong countable character of *f*-subnormality. The aim of this section is to prove that many other subgroup properties have strong countable character. We first prove some corollaries of Theorem 4.1.1.

Corollary 4.3.1 *Let* G *be a group, and let* H *be a subgroup of* G*. If* $H \cap C$ *has finite index in a subnormal subgroup of* C*, for each countable subgroup* C *of* G*, then* H *has finite index in a subnormal subgroup of* G*.*

Proof — For each countable subgroup C of G, denote by $l(C)$ the smallest subnormality defect of a subnormal subgroup L of C such that $H \cap C \leqslant L$ and $H \cap C$ has finite index in L. Clearly, $l(C_1) \leqslant l(C_2)$ whenever C_1 and C_2 are countable subgroups of G such that $C_1 \leqslant C_2$. Therefore the set of all $l(C)$'s ranging on all countable subgroups C of G has a largest element, $l = l(C*)$ say. If C is any countable subgroup of G, also the subgroup $\langle C, C* \rangle$ is countable and clearly $l(\langle C, C* \rangle) = l(C*) = l$. It follows that $H \cap C$ has finite index in a subnormal countable subgroup C of subnormal defect at most l. An easy application of the Theorem 4.1.1 now gives the result. □

The following result can be proved similarly. Note that in both corollaries it is possible to add restrictions on the subnormality defect and on the finite index.

Corollary 4.3.2 *Let* G *be group and let* H *be a subgroup of* G*. If* $H \cap C$ *is subnormal in a subgroup of finite index of* C*, for each countable subgroup* C *of* G*. Then* H *is subnormal in a subgroup of finite index of* G*.*

We prove now that both the property of having finite index in the normal closure and that of having a finite number of conjugates have strong countable character.

Corollary 4.3.3 *Let* G *be a group and* H *a subgroup of* G *such that* $H \cap C$ *has finite index (has index at most* m*, for some fixed positive integer* m*) in its normal closure in* C*, for each countable subgroup* C *of* G*. Then* H *has finite index (has index at most* m*) in its normal closure in* G*.*

Proof — Fix $\mathfrak{B}_1$ to be the class of all groups, $\Sigma = \mathbb{N}$ and $M = \{-1\}$. Then, applying the Theorem 4.1.1 for $q = (0, -1)$, we get that H is either of finite index in G, or is normal in G, or has finite index in a normal subgroup of G. In every case, H has finite index in its normal closure.

If m is any positive integer, and $\Sigma = \{1, \dots, m\}$, the same argument proves the other point of the statement. □

Corollary 4.3.4 *Let* G *be a group and* H *a subgroup of* G *such that* $H \cap C$ *has a finite number of conjugates in* C*, for each countable subgroup* C *of* G*. Then* H *has a finite number of conjugates in* G*. Moreover, if* $H \cap C$ *has at most* m *conjugates in* C*, for each countable subgroup* C *of* G*, and for a fixed positive integer* m*, then* H *has at most* m *conjugates in* G*.*

Next lemmas deal with the countable character of some further embedding properties.

Lemma 4.3.5 *Let* G *be a group and let* m *be an element of* $\mathbb{N} \cup \{\aleph_0\}$*. If* H *is a subgroup of* G *such that* $|(H \cap C)^C : (H \cap C)_C| < m$ *for each countable subgroup* C *of* G*, then* $|H^G : H_G| < m$*.*

Proof — Suppose first that $m \neq \aleph_0$ and assume by contradiction that $|H^G : H_G| \geqslant m$. Then there are elements $x_1, \dots, x_m$ of H^G such that $x_i x_j^{-1} \notin H_G$ for all $i \neq j \in \{1, \dots, m\}$. Hence there exist elements $g(i, j)$ such that $x_i x_j^{-1} \notin H^{g(i,j)}$. Therefore we can find a countable subgroup L of G containing the elements $g(i, j)$, for $i \neq j$, and such that $x_1, \dots, x_m$ belong to $(H \cap L)^L$. This clearly implies that

$$|(H \cap L)^L : (H \cap L)_L| \geqslant m,$$

a contradiction. The proof is similar for $m = \aleph_0$. □

Notice that part of the above proof can be used to show that the property of being finite (of bounded order) over the core has strong countable character. The next lemma proves that the property of having finite normal oscillation has strong countable character.

Lemma 4.3.6 *Let* G *be a group and let* m *be an element of* $\mathbb{N} \cup \{\aleph_0\}$*. If* H *is a subgroup of* G *such that, for each countable subgroup* C *of* G*, the subgroup* $H \cap C$ *has normal oscillation strictly smaller than* m *in* C*. Then* H *has normal oscillation strictly smaller than* m *in* G*.*

Proof — We assume that m is finite (the proof is similar for $m = \aleph_0$). Suppose for a contradiction that $|H^G : H| \geqslant m$ and $|H : H_G| \geqslant m$. Let $x_1, \dots, x_m$ be elements of H such that $x_i H_G \neq x_j H_G$ if $i \neq j$, and put

$$X = \langle x_1, \dots, x_m \rangle.$$

For all elements i and j of $\{1, \dots, m\}$ such that $i \neq j \in \{1, \dots, m\}$ there exists an element $g(i, j)$ of G such that $x_i^{-1} x_j$ does not belong to the subgroup $H^{g(i,j)}$. On the other hand, as $|H^G : H| \geqslant m$, there are countable subgroups Y of H and Z of G such that $X \leqslant Y$ and the normal closure Y^Z contains a subset $W = \{w_1, \dots, w_m\}$ for which $w_i H \neq w_j H$, whenever $w_i \neq w_j$. Then

$$C = \langle Y, Z, g(i, j);\ i \neq j \in \{1, \dots, m\} \rangle$$

is a countable subgroup of G, and it is obvious that the normal oscillation of $H \cap C$ in C larger than m, a contradiction. □

Let G be a group. We say that a subgroup H has the χ *property* in G if there is a subnormal subgroup H_0 of G such that $H_0 \leqslant H$ and the index $|H : H_0|$ is finite. Groups in which all proper subgroups have the χ property have been studied by C. Casolo and M. Mainardis [8]. This section ends by sketching how to use the method of the Theorem 4.1.1 in order to prove that χ has strong countable character.

Lemma 4.3.7 *Let* G *be a group and let* H *be a subgroup of* G *such that, for each countable subgroup* C *of* G, *there is a subnormal subgroup* $H_{0,C}$ *of* C, *such that* $H_{0,C} \leqslant H \cap C$ *and* $H_{0,C}$ *has finite index in* $H \cap C$. *Then* G *has a subnormal subgroup* H_0 *such that* $H_0 \leqslant H$ *and* $|H : H_0| < \infty$.

Proof — Let $\mathcal{C}$ be the set of all countable subgroups of G, and let $C \in \mathcal{C}$. There exists a subnormal subgroup $H^1_{0,C}$ of C such that $H^1_{0,C}$ is contained in $H \cap C$ and $H_{0,C}$ has smallest subnormal defect, $s(C)$ say, in C, and among these the smallest index in $H \cap C$, $f(C)$ say. As in the proof of the Theorem 4.1.1, we can find a countable system $\mathcal{C}^1$ of G such that, for each $C_1, C_2 \in \mathcal{C}^1$ we have

$$s(C_1)=s(C_2) \quad \text{and} \quad f(C_1)=f(C_2).$$

For each $C \in \mathcal{C}^1$, consider the series

$$H_{0,C} \leqslant H \cap C = H_{1,C} \leqslant H_{2,C} = C$$

and the series of normal closures of $H_{0,C}$ in C

$$H_{0,C} = K_{0,C} < \ldots < K_{n,C} = C.$$

Define now two binary relations $\mathcal{R}_{1,C}$ and $\mathcal{R}_{2,C}$ on C by putting $x\mathcal{R}_{1,C}y$ if and only if

$$\bigcap_{i\,:\,x\in H_{i,C}} H_{i,C} \leqslant \bigcap_{i\,:\,y\in H_{i,C}} H_{i,C},$$

and, we set $x\mathcal{R}_{2,C}y$ if and only if

$$\bigcap_{i\,:\,x\in K_{i,C}} K_{i,C} \leqslant \bigcap_{i\,:\,y\in K_{i,C}} K_{i,C}.$$

We can encode these two relations in a function

$$f:\ C \times C \longrightarrow \{0,1,2,3\},$$

in such a way that $f(x, y) = 1$ whenever $x\mathcal{R}_{1,C}y$ and $x\not\mathcal{R}_{2,C}y$. Now, applying Lemma 8.22 of [56], it follows that there is a function

$$f : G \times G \longrightarrow \{0, 1, 2, 3\}$$

having the property that, for every finite subset $\{x_1, \ldots, x_m\}$ of $G \times G$, there exists a $C \in \mathcal{C}^1$ such that $x_i \in C \times C$ and $f(x_i) = f_C(x_i)$ for all $i = 1, \ldots, m$. From this function we go back to two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on G. Each of these relations has the analogous of the properties mentioned in the proof of the Theorem 4.1.1. Hence, we can define two series in G:

$$H_0 \leqslant H_1 \leqslant G$$

and

$$K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n = G.$$

Since

$$H_0 = \{g \in G : g\mathcal{R}_1 1\} \quad \text{and} \quad K_0 = \{g \in G : g\mathcal{R}_2 1\},$$

it is easy to show that $H_0 = K_0$. It can be also proved that K_i is normal in K_{i+1}, and that H_0 is a subgroup of finite index in H. This completes the proof. □

4.4 Group properties

In [8], C. Casolo and M. Mainardis called *S-groups*, the groups in which every subgroup is *f*-subnormal. Here we show that the class of S-groups, the class of T^*-groups, the class of T_*-groups and, the class of $T(*)$-groups, are countably recognizable, as well as the other classes of groups defined below (see also [7], where they were introduced).

- The class of L-groups: a group G is said to be a L-*group* if for every subgroup H of G there is a subnormal subgroup H_0 of G with $H_0 \leqslant H$ and $|H : H_0|$ finite.

- The class of T_m-groups, for $m \in \mathbb{N}$: a group G is said to be a T_m-*group* if every subnormal subgroup of G has finite index at most m in its normal closure.

- The class of V-groups: a group G is said to be a V-*group* if every subnormal subgroup H of G has finitely many conjugates. This clearly is equivalent to require that the normalizer of H has finite index in G.

- The class of V_m-groups, for $m \in \mathbb{N}$: a group G is said to be a V_m-*group* if every subnormal subgroup H of G has at most m conjugates. This clearly is equivalent to require that the normalizer of H in G has index at most m.

- The class of U-groups: a group G is said to be an U-*group* if $|H^G : H_G|$ is finite for every subnormal subgroup H of G.

- The class of U_m-groups, for m in $\mathbb{N}$: a group G is said to be an U_m-*group* if $|H^G : H_G|$ is at most m for every subnormal subgroup H of G.

- The class of T^m-groups, for $m \in \mathbb{N}$: a group G is said to be a T^m-*group* if $|H : H_G|$ is at most m for every subnormal subgroup H of G.

- The class of $T(m)$-groups, for m in $\mathbb{N}$: a group G is said to be a $T(m)$-*group* if either $|H : H_G| \leqslant m$ or $|H^G : H| \leqslant m$ for every subnormal subgroup H of G.

First, we introduce a lemma which enables us to pass from the strong countable character of the embedding properties to the countable recognizability of some group classes.

Lemma 4.4.1 *Let* Ξ *be an embedding property with strong countable character and* Θ *any subgroup property such that* $X \cap H$ *is a* Θ*-subgroup of* H *whenever* X *is a* Θ*-subgroup of a group* G *and* $H \leqslant G$. *Then the class of groups with all* Θ*-subgroups satisfying* Ξ *is countably recognizable.*

Proof — Let G be a group and suppose that each countable subgroup of G has all its Θ-subgroups satisfying Ξ. Take an arbitrary Θ-subgroup H of G. Then $H \cap C$ is both a Θ-subgroup and a Ξ-subgroup of C for each countable subgroup C of G. The strong countable character of Ξ now implies that $H \Xi G$. The statement is proved. □

Our final corollary is a trivial application of Lemma 4.4.1 and results of the previous section.

Corollary 4.4.2 *The group classes* $S, L, T^*, T_m, V, V_m, U, U_m, T_*, T^m$, $T(*)$ *and* $T(m)$ *are all countably recognizable, for* $m \in \mathbb{N}$.

The above corollary should be compared with some analogous results in the last part of [31]. Finally, we remark that if $\mathfrak{X}$ is a subgroup closed class of groups, it follows from the same results that, for instance, the class of groups in which all $\mathfrak{X}$-subgroups are *f*-subnormal is countably recognizable. In particular, notice that the class of groups with all abelian subgroups *f*-subnormal is countably recognizable.

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