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Chapter 1

Introduction

The subject of this thesis is the space $\mathcal{S}$ of quantum states, and the language in which (hopefully) it is written is that of differential geometry. The aim of this thesis is to present a geometrical analysis of the structural properties of $\mathcal{S}$, being them of “kinematical” or “dynamical” character. Of course, it will be impossible to cover all these aspects in full generality, rigour, and completeness, hence, this presentation should be thought of as a preliminary step of a (hopefully long lasting) research journey centered around the geometrical aspects of quantum theories.

We will focus on finite-dimensional systems, so that, from the mathematical point of view, we can introduce geometrical structures without the need to deal with all the technical difficulties related to the differential geometry of infinite dimensions. From the physical point of view, finite-level systems are extensively studied, for example, in quantum information theory, quantum optics, and quantum cryptography, both from the theoretical and from the experimental point of view. In this context, recent developments in problems like quantum state estimation or quantum hypotesis testing (see [73, 26, 92, 94]), have led to the need of considering nonlinear parametric families of quantum states (similarly to what happens in quantum optics with coherent states), and nonlinear functions on quantum states, e.g., quantum (relative) entropies. Because of the emergence of these kind of “nonlinearities” in the realm of quantum mechanics, we believe that a study of the geometrical structures present in quantum theories will prove to be useful in order to deal with the above-mentioned nonlinearities. Indeed, by means of the mathematical formalism of differential geometry it is possible to deal with arbirtrary changes of coordinates in a natural and intrinsic fashion. Furthermore, the use of differential geometry allows us to exploit all the mathematical constructions and results developed in different contexts, e.g., classical mechanics, in the framework of quantum theories, thus enalarging the arsenal of mathematical tools available. For instance, we will see in chapter 3 how a geometrization of the GKLS dynamics of open quantum systems makes possible to exploit LaSalle's invariance principle, which is a classical result on the asymptotic behaviour of dynamical systems, in the quantum context. On the other hand, the intrinsic language of differential geometry will be used in chapter 4 to deal with the nonlinearities associated with the use of relative quantum entropies (highly nonlinear functions) in the context of quantum information theory. Specifically, we will develop an abstract setting in which the study of quantum relative entropies, quantum metric tensors, as well as their symmetry properties, may be handled without the need to resort to specific coordinate systems. As a byproduct, in this scheme we will be able to perform computations in arbitrary (finite) dimensions in a coordinate-free fashion.

The physical/mathematical background in which we will move is that of the algebraic approach to quantum mechanics, which is unavoidably associated with the names of Heisenberg, Born, von Neumann, and Jordan. In this approach, the principal object of concern is an algebra $\mathcal{A}$ which is related to the space $\mathfrak{O}$ of observables (see [76]). Specifically, $\mathcal{A}$ is what is known as a C^*-algebra, while $\mathfrak{O} \subset \mathcal{A}$ is the space of self-adjoint elements in $\mathcal{A}$. From the mathematical point of view, a C^*-algebra is defined as follows (see [28, 54, 56]):

Definition 1. *A C^*-algebra $\mathcal{A}$ is a complex Banach space $(\mathcal{A}\,,+\,,||\cdot||)$ possessing the following additional structures:*

- *an associative product structure $\mathcal{A}\times\mathcal{A} \ni (\mathbf{A}\,,\mathbf{B}) \to \mathbf{A}\,\mathbf{B} \in \mathcal{A}$ such that:*

$$||\mathbf{A}\,\mathbf{B}|| \leq ||\mathbf{A}||\,||\mathbf{B}||\,; \tag{1.1}$$

- *an involution operation $\dagger\colon \mathcal{A} \to \mathcal{A}$ such that:*

$$(a\mathbf{A} + b\mathbf{B})^\dagger = \bar{a}\mathbf{A}^\dagger + \bar{b}\mathbf{B}^\dagger \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{A}\,, \tag{1.2}$$

  where $a, b \in \mathbb{C}$ and $\bar{a}, \bar{b} \in \mathbb{C}$ denote their complex conjugate, and such that:

$$(\mathbf{A}\mathbf{B})^\dagger = \mathbf{B}^\dagger\,\mathbf{A}^\dagger\,, \quad (\mathbf{A}^\dagger)^\dagger = \mathbf{A}\,, \quad ||\mathbf{A}\,\mathbf{A}^\dagger|| = ||\mathbf{A}^2|| \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{A}\,. \tag{1.3}$$

The element $\mathbf{A}^\dagger \in \mathcal{A}$ is called the adjoint of $\mathbf{A} \in \mathcal{A}$. An element $\mathbf{A} \in \mathcal{A}$ is called self-adjoint if $\mathbf{A} = \mathbf{A}^\dagger$, and it is called normal if $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$. If $\mathcal{A}$ admits an identity element $\mathbb{I}$ for the associative product structure, then, we may define unitary elements in $\mathcal{A}$ as all those $\mathbf{U} \in \mathcal{A}$ such that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbb{I}$. In the following, *we will always consider C^*-algebras possessing an identity element $\mathbb{I}$. According to a theorem in [56], this is not a very restrictive request since we can always add an identity element to a C^*-algebra.*

As said before, the space $\mathfrak{O}$ of observables is identified with the subset in $\mathcal{A}$ consisting of self-adjoint elements, that is:

$$\mathfrak{O} := \left\{ \mathbf{a} \in \mathcal{A} \colon \quad \mathbf{a}^\dagger = \mathbf{a} \right\} . \tag{1.4}$$

Since $(a\mathbf{A})^\dagger = \bar{a}\mathbf{A}^\dagger$, it follows that $\mathfrak{O}$ is not a complex vector subspace of $\mathcal{A}$. However, $\mathfrak{O}$ can be endowed with the structure of real Banach space in such a way that the Banach space structure underlying $\mathcal{A}$ becomes the Banach space complexification of $\mathfrak{O}$ (see [39, 54, 56, 60, 99]). Furthermore, we will observe that $\mathfrak{O}$ possesses a richer structure known as Lie-Jordan-Banach algebra structure.

In order to identify the space of quantum states, we need to introduce the notion of state on a C^*-algebra (see [28, 54, 56]):

Definition 2. *Given a C^*-algebra $\mathfrak{A}$ with, we define a state ρ on $\mathcal{A}$ to be a continuous linear functional in the topological dual $\mathcal{A}^*$ of $\mathcal{A}$ such that:*

$$\rho(\mathbf{a}) \in \mathbb{R}\; \forall \mathbf{a} \in \mathfrak{O}\,, \quad \rho(\mathbf{A}\mathbf{A}^\dagger) \geq 0\; \forall \mathbf{A} \in \mathcal{A}\,, \quad \rho(\mathbb{I}) = 1\,. \tag{1.5}$$

Since ρ is linear and $\rho(\mathbf{a}) \in \mathbb{R}$ for all observables in $\mathfrak{O}$, it follows that ρ may be seen an element of the topological dual $\mathfrak{O}^* \subset \mathcal{A}^*$ of $\mathfrak{O} \subset \mathcal{A}$.

Then, the space $\mathcal{S}$ of states of the physical system is identified with the space of states of the C^*-algebra $\mathcal{A}$ associated with the physical system, that is:

$$\mathcal{S} := \left\{ \rho \in \mathfrak{O}^* \subset \mathcal{A}^* \colon \rho(\mathbf{A}\,\mathbf{A}^\dagger) \geq 0\; \forall \mathbf{A} \in \mathcal{A}\,,\;\; \rho(\mathbb{I}) = 1 \right\} . \tag{1.6}$$

The pairing map $\mu\colon \mathfrak{O} \times \mathcal{S}$ given by the evaluation of a state ρ on the self-adjoint element $\mathbf{a}$ representing an observable, that is:

$$\mu(\mathbf{a}\,,\rho) := \rho(\mathbf{a})\,, \tag{1.7}$$

is interpreted as the mean value for the outcome of the measure of the observable $\mathbf{a}$ when the system is in the quantum state ρ.

It is immediate tho see that $\mathcal{S}$ is a convex subset in $\mathcal{A}^*$ ($\mathfrak{O}^*$), that is, $\rho_1, \rho_2 \in \mathcal{S}$ implies $\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{S}$ for all $\lambda \in [0\,,1]$. The convex structure reflects the mixture property of states of a physical system, that is, the possibility of mixing different preparation procedures with respect to a definite mixture. Since $\mathcal{S}$ is a convex set, the Krein-Milman theorem assures us that it is the closed convex hull of its extreme points. These are all those elements in $\mathcal{S}$ that can not be written as the convex combination of more than one element. In the context of quantum mechanics, they are called pure quantum states.

Since $\mathcal{A}^*$ is the dual of a Banach space, it is a Banach space itself with respect to the norm:

$$||\xi|| := \sup_{\mathbf{A}\in\mathcal{A}:\ ||\mathbf{a}||\leq 1} |\xi(\mathbf{a})| \,, \tag{1.8}$$

where $\xi \in \mathcal{A}^*$. Consequently, it is easy to show that a state ρ on $\mathcal{A}$ is such that $||\rho|| = \rho(\mathbb{I}) = 1$. The Banach space structure of $\mathcal{A}^*$ determines a topology on $\mathcal{A}^*$ itself. Since $\mathcal{S}$ is a subset of $\mathcal{A}^*$, it can be endowed with the so-called relative topology of $\mathcal{S}$ in $\mathcal{A}^*$ (see [2] page 17, [90] page 88). In this topology, a set $E \subset \mathcal{S}$ is defined to be open (closed) if it is the intersection $F \cap \mathcal{S}$ of an open (closed) subset $F \subset \mathfrak{O}_n^*$ with $\mathcal{S}$. The relative topology of $\mathcal{S}$ in $\mathcal{A}^*$ clearly depends on the topology on the ambient space $\mathcal{A}^*$. In the finite-dimensional case, we do not have to worry because all the topologies on $\mathcal{A}^*$ are equivalent. On the other hand, this is no longer true in the infinite-dimensional case, and we must pay attention to what topology we put on $\mathcal{A}^*$. For instance, it can be proved that $\mathcal{S}$ is compact subset of $\mathcal{A}^*$ (see [5] page 68, [28] page 53) with respect to the so-called w^*-topology on $\mathcal{A}^*$. In the finite-dimensional case, this topology is equivalent to the one induced by the Banach space structure of $\mathcal{A}^*$, while, this is not true in the infinite-dimensional case (see [28, 53, 54, 56, 71, 96, 101]).

Once we select a quantum state $\rho \in \mathcal{S}$ we may recover a Hilbert space by means of the so-called GNS construction (see [37], and theorem I.9.6 in [54]). This is the way in which the Hilbert space formulation of quantum mechanics (see [55]) is shown to be equivalent to the algebraic one. What is known as the equivalence of Heisenberg picture and Schrödinger picture. However, it is necessary to point out that this equivalence does depend on the choice of a state ρ on $\mathcal{A}$, and this is particularly relevant in the case of algebraic quantum field theory (see [14, 67, 68]).

Quantum dynamical evolutions are described by one-parameter family of linear maps $\Phi_\tau \colon \mathcal{A}^* \to \mathcal{A}^*$ such that $\Phi_t(\mathcal{S}) \subseteq \mathcal{S}$. Depending on the explicit form of the family Φ_τ different kind of dynamics can be described. For instance, if Φ_τ is such that there exists a one-parameter group $\{\mathbf{U}_\tau\}_{\tau\in\mathbb{R}}$ of unitary operators in $\mathcal{A}$ such that:

$$\left(\Phi_\tau(\rho)\right)(\mathbf{A}) = \rho\left(\mathbf{U}_\tau^\dagger\,\mathbf{A}\,\mathbf{U}_\tau\right) \quad \forall \mathbf{A}\in\mathcal{A}\,, \tag{1.9}$$

then the resulting quantum dynamical evolution is interpreted as the evolution of a closed system. If an associative condition is satisfied but maps are not necessarily invertible, we may have a semigroup. In chapter 3 we will give a geometrical description of the generators of the Markovian semigroups of evolutions of finite-dimensional open quantum systems (see also [6, 17, 18, 19, 28, 29, 51, 63, 71, 82, 101]).

Finally, the decomposition of a physical system S_{AB} into its components S_A and S_B is described by giving a decomposition of the C^*-algebra $\mathcal{A}_{AB}$ of the composite system into the tensor product $\mathcal{A}_A \otimes \mathcal{A}_B$ of the C^*-algebras of the subsystems. Note that, in general, given $\mathcal{A}_{AB}$, $\mathcal{A}_A$, and $\mathcal{A}_B$, the decomposition:

$$\mathcal{A}_{AB} \cong \mathcal{A}_A \otimes \mathcal{A}_B \tag{1.10}$$

is not unique, and different decomposition should be thought of as different composite systems. Furthermore, we point out that, when the C^*-algebras considered are noncommutative, the composition/decomposition law of physical systems encoded in the tensor product operation is at the heart of the genuinely quantum feature of entanglement in composite systems.

Recently, a new line of research on quantum mechanics has been developed which is concerned with a geometrization process of this theory (see [16, 32, 47, 48, 49, 50, 58, 64, 65, 66, 85, 95]). The appearence of a geometrical attitude in quantum mechanics should not be surprising. On the contrary, we could say that, since the very beginning of quantum mechancis, interpretational issues were calling for a geometrical (nonlinear) attitude towards the theory. Indeed, let us recall that, in the Hilbert space formulation of quantum mechanics (see [55]), a correct pysical interpretationof the (pure) states of the systems is achieved only when we realize that they are not vectors in the Hilbert space of the system, but rays (usually represented by normalized vectors). This means that the space of interest is not the linear space $\mathcal{H}$, rather, it is the nonlinear manifold $\mathcal{P}(\mathcal{H})$ known as the complex projective space associated with $\mathcal{H}$. This space is a Kähler manifold and the geometrical structures available on it encode all the essential features of quantum mechanics. The Hilbert space $\mathcal{H}$ may be seen as a sort of convenient computational tool by means of which we are able to perform explicit calculations without the need to introduce local nonlinear coordinates on $\mathcal{P}(\mathcal{H})$. In this geometrical context, it is possible to appreciate some of the distinguishing features of quantum mechanics from an abstract point of view, that is, without the need to refer to some linear algebraic structure that is lost as soon as we perform nonlinear change of coordinates. For example, the Hermitian scalar product on $\mathcal{H}$ is "replaced" by a Hermitean tensor field $\mathfrak{h}$ on $\mathcal{P}(\mathcal{H})$ such that its symmetric part g defines a metric tensor on $\mathcal{P}(\mathcal{H})$, while its antisymmetric part ω defines a symplectic structure on $\mathcal{P}(\mathcal{H})$. The metric tensor g, known as the Fubini-Study metric tensor, is intimately connected to a statistical notion of distance between pure quantum states (see [100]), and this turns out to be relevant in the context of quantum information theory as we will see in chapter 4. On the other hand, the symplectic form ω allows to write unitary evolutions (Schrödinger equation) as Hamiltonian vector fields both in the finite-dimensional and the infinite-dimensional case (see [48, 49]). Since $\mathfrak{h}, g$ and ω are tensor fields they are covariant with respect to arbitrary changes of coordinates on $\mathcal{P}(\mathcal{H})$.

As said before, the scope of this thesis is to present a study of the differential geometry of the space of quantum states. Of course, the subject is incredibly vast, and presents a very huge amount of conceptual and technical difficulties. For these reasons, and because deadlines are very intransigent entities, some choices have been made. First of all, we decided to work only with finite-level quantum systems, that is, systems whose associated observables are self-adjoint elements of a finite dimensional C^*-algebra. The theories of infinite-dimensional C^*-algebras and infinite-dimensional manifolds would bring in some highly nontrivial technical difficulties related essentially to the infinite-dimensionality, and these technicalities would somehow obscure the main structural ideas we want to expose. Roughly speaking, the work presented here may be thought as a sort of test-drive for our ideas in view of the generalization to the infinite-dimensional case of both quantum mechanical systems described by infinite-dimensional C^*-algebras and quantum field theories described by nets of von Neumann's algebras. Nevertheless, some considerations on the case of a quantum mechanical systems described by the infinite-dimensional C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a complex infinite-dimensional separable Hilbert space $\mathcal{H}$ will be exposed in chapter 5.

Since we will deal essentially with finite-dimensional C^*-algebras, it is good to review some of the main results on the structures of such algebras. First of all, in [54], page 74, the following theorem is proved:

Theorem 1. *Every finite-dimensional C^*-algebra $\mathcal{A}$ is $*$-isomorphic to the **universal model C^*-algebra**:*

$$\mathfrak{A} = \bigoplus_{j=1}^{N} \mathcal{M}_{n_j}(\mathbb{C}) \,, \tag{1.11}$$

where $n_j < +\infty$ for all j, $N < +\infty$, and $\mathcal{M}_{n_j}(\mathbb{C})$ denotes the C^*-algebra of $(n_j \times n_j)$ complex matrices. In particular, every finite-dimensional C^*-algebra has an identity. The $*$-isomorphism between $\mathcal{A}$ and $\mathfrak{A}$ is denoted by $\mathfrak{I}_{\mathcal{A}}$.

We will focus on studying the geometry of the space of states of the ***model C^*-algebra*** $\mathfrak{A}_n = \mathcal{M}_n(\mathbb{C})$ with $n \geq 2$. The system described by $\mathfrak{A}_n$ may be thought of as a sort of universal model for finite-level quantum systems without superselection sectors. It is clear that, because of theorem 1, our choice is not a dramatic limitation. Indeed, whenever we have a finite-dimensional quantum system without superselection sectors, it will be described, in the algebraic approach, by a C^*-algebra $\mathcal{A}$ which is $*$-isomorphic to $\mathfrak{A}_n$. Consequently, all the geometrical structures available on $\mathfrak{A}_n$ can be immediately "transported" on $\mathcal{A}$ by means of the $*$-isomorphism $\mathfrak{I}_{\mathcal{A}}$. Furthermore, again because of theorem 1, the C^*-algebras $\mathfrak{A}_n$ with arbitrary n are the building block of a generic finite-dimensional C^*-algebra $\mathcal{A}$, and thus, an understanding of their geometrical structures is necessary to obtain an understanding of the geometry of a generic $\mathcal{A}$.

Let us take now a closer look at the structure of the thesis. The first step we will take is to analyze the differential geometry of $\mathcal{S}$ in chapter 2. We will see that $\mathcal{S}$ is not a differential manifold as a whole, the only exception being the two-dimensional case in which it becomes a compact manifold with boundary, namely, the Bloch ball ([64]). However, we will be able to exhibit a partition of $\mathcal{S}$ into the disjoint union of differential manifolds of different dimensions. Furthermore, all these manifolds will turn out to be homogeneous spaces for an action of the Lie group $SL(\mathfrak{A}_n)$ made up of invertible elements in $\mathfrak{A}_n$ with determinant equal to 1. The explicit action α_{g} of $\mathrm{g} \in SL(\mathfrak{A}_n)$ on $\rho \in \mathcal{S}$ is such that:

$$(\alpha_{\mathrm{g}}(\rho))\,(\mathbf{A}) = \frac{\rho(\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathrm{g})}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})} \quad \forall \mathbf{A} \in \mathfrak{A}_n\,. \tag{1.12}$$

To see things more clearly, let us recall that the $\mathfrak{A}_n$ carries the natural Hilbert product:

$$\langle \mathbf{A}\,,\mathbf{B}\rangle := Tr\left(\mathbf{A}^{\dagger}\,\mathbf{B}\right)\,. \tag{1.13}$$

This product is compatible with the structure of C^*-algebra, and makes $\mathfrak{A}_n$ a complex Hilbert space. Then, because of Riezs theorem, we have that for every $\xi \in \mathfrak{A}_n^*$ we can find an element $\bar{\xi} \in \mathfrak{A}_n$ such that:

$$\xi(\mathbf{A}) = \langle \bar{\xi}\,,\mathbf{A}\rangle \;\; \forall \mathbf{A} \in \mathfrak{A}_n\,. \tag{1.14}$$

When $\xi \equiv \rho$ is $\mathcal{S} \subset \mathfrak{A}_n^*$, that is, it is a quantum state, it is easy to see that $\bar{\rho} \in \mathfrak{A}_n$ is a self-adjoint positive semidefinite matrix such that $Tr(\rho) = 1$, i.e., a density matrix. The image in $\mathfrak{O}_n \subset \mathfrak{A}_n$ of the space $\mathcal{S}$ of quantum states will be denoted by[1] $\bar{\mathcal{S}}$.

[1] In the rest of the thesis, in particular in chapter 4, we will make extensive use of the isomorphism between $\mathfrak{A}_n$ ($\mathfrak{O}_n$) and $\mathfrak{A}_n^*$ ($\mathfrak{O}_n^*$), and of the one-to-one correspondence between quantum states in $\mathcal{S}$ and density matrices in $\bar{\mathcal{S}}$.

Remark 1. *There is a "canonical basis"* $\{\mathbf{E}_{jk}\}_{j,k=1,\ldots,n^2}$ *in* $\mathfrak{A}_n$*, where* $\mathbf{E}_{jk}$ *is the* $(n \times n)$ *matrix with* 1 *in the* (j, k) *position and* 0 *elsewhere. A direct calculation shows that this is an orthonormal basis in* $\mathfrak{A}_n$*. From the computational point of view, this basis will prove to be very useful, and we will exploit it often in the following chapters.*

Now, the action $\alpha_{\mathrm{g}}(\rho)$ may be written in terms of the density matrix $\bar{\rho} \in \bar{\mathcal{S}}$ associated with $\rho \in \mathcal{S}$ as:

$$\bar{\rho}_{\mathrm{g}} = \frac{\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}}{Tr(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger})}\,. \tag{1.15}$$

From this we deduce that $\bar{\rho}$ and $\bar{\rho}_{\mathrm{g}}$ have the same matrix rank. More specifically, we will see that for every density matrices $\bar{\rho}_0, \bar{\rho}_1$ with equal rank k there is $\mathrm{g} \in SL(\mathfrak{A}_n)$such that $\bar{\rho}_1 = (\bar{\rho}_0)_{\mathrm{g}}$. The space $\mathcal{S}$ of quantum states is then partitioned as:

$$\mathcal{S} = \bigsqcup_{k=1}^{n} \mathcal{S}_k\,, \tag{1.16}$$

where the quantum states in $\mathcal{S}_k$ are such that their associated density matrices have fixed rank equal to k, and $SL(\mathfrak{A}_n)$ acts transitively on each $\mathcal{S}_k$. The space $\mathcal{S}_1$ of quantum states of rank 1 is precisely the space of pure quantum states.

Since the Lie group $SL(\mathfrak{A}_n)$ acts transitively on each $\mathcal{S}_k$, and we will see that the associated isotropy subgroup G_k of the action is closed in $SL(\mathfrak{A}_n)$, we can use the closed subgroup theorem 2 to endow $\mathcal{S}_k$ with the differential structure coming from the homogeneous space $SL(\mathfrak{A}_n)/G_k$. Accordingly, we may look at $\mathcal{S}$ as being stratified by the differential manifolds $\mathcal{S}_k$ whose dimensions depend on k.

The manifold structure on each $\mathcal{S}_k$ is such that the canonical action of $SL(\mathfrak{A}_n)$ is smooth. Consequently, every subgroup of $SL(\mathfrak{A}_n)$ acts smoothly on $\mathcal{S}_k$. In particular, we will focus on the action of the subgroup $SU(\mathfrak{A}_n)$ made up of invertible elements $\mathbf{U}$ with determinant equal to 1 and such that $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$. This is a compact Lie group (the special unitary group in n dimensions), and thus its orbits on every $\mathcal{S}_k$ are embedded submanifolds. These orbits are identified by the eigenvalues of the associated density matrices. Specifically, all the density matrices associated with quantum states lying on the same orbit of $SU(\mathfrak{A}_n)$ will have the same eigenvalues. We refer to these manifolds as the manifolds of isospectral states. In the particular case of density matrices having a single eigenvalue equal to 1 we recover the manifold $\mathcal{S}_1$ of pure quantum states. Therefore, pure quantum states form a homogeneous space with respect to both $SL(\mathfrak{A}_n)$ and its subgroup $SU(\mathfrak{A}_n)$.

What is very interesting is that we may enlarge the action of $SU(\mathfrak{A}_n)$ to be defined not only on $\mathcal{S} \subset \mathfrak{O}_n^*$ but on the whole $\mathfrak{O}_n^*$. In doing so, we will be able to relate this action with the coadjoint action of $SU(\mathfrak{A}_n)$ on the dual of its Lie algebra (see [86] chapter 14). Accordingly, we will see that there is a natural structure of Kähler manifold on each orbit of $SU(\mathfrak{A}_n)$, and, in particular, on the manifolds of isospectral states. In the case of pure quantum states, this Kähler structure coincides with the Kähler structure discussed above. By means of this structure we will be able to define Hamiltonian and gradient vector fields associated with observables in $\mathfrak{O}_n$, and to show that they close on a realization of the Lie algebra of the Lie group $SL(\mathfrak{A}_n)$ of which $SU(\mathfrak{A}_n)$ is a subgroup. Since the orbits of a compact group are compact (see theorem 3) the vector fields realizing the Lie algebra of $SL(\mathfrak{A}_n)$ are complete, and thus the realization "integrates" to a group action.

Chapter 3 will be dedicated to the geometrical formulation of the GKLS master equation:

$$\mathbf{L}(\bar{\rho}) = -\imath\,[\mathbf{H}\,,\bar{\rho}] - \frac{1}{2}\sum_{j=1}^{N}\left\{\mathbf{v}_j^{\dagger}\mathbf{v}_j\,,\bar{\rho}\right\} + \sum_{j=1}^{N}\mathbf{v}_j\,\bar{\rho}\,\mathbf{v}_j^{\dagger}\,, \tag{1.17}$$

where $\bar{\rho}$ is the density matrix associated with the quantum state $\rho\in\mathcal{S}$, $\mathbf{H}\in\mathfrak{O}_n$, $\mathbf{v}_j\in\mathfrak{A}_n$, and $\mathbf{V}=\sum_{j=1}^{N}\mathbf{v}_j^{\dagger}\mathbf{v}_j$. According to [63, 82], this is the dynamical equation[2] for the most general Markovian comletely-positive trace-preserving evolution of an open quantum system described by the C^*-algebra $\mathfrak{A}_n$. Following the work in [42, 44], the aim of the chapter is to present this equation of motion as an affine vector field on the affine hyperplane $\mathfrak{T}_n^1\subset\mathfrak{O}_n^*\subset\mathfrak{A}_n^*$ of self-adjoint linear functionals ξ such that $\xi(\mathbb{I})=1$. In order to do this, we will first exploit the Lie-Jordan algebra structure on $\mathfrak{O}_n$ (see [76, 5, 4]) to define an antysimmetric bivector field $\widetilde{\Lambda}$ and a symmetric bivector field G on $\mathfrak{O}_n^*$. Let us recall that, denoting with $[\,,]$ and $\{\,,\}$, respectively, the commutator and the anticommutator in $\mathfrak{A}_n$, the Lie-Jordan algebra structure of $\mathfrak{O}_n$ is encoded in the two product structures:

$$[[\mathbf{a}\,,\mathbf{b}]] := \frac{\imath}{2}\,[\mathbf{a}\,,\mathbf{b}]\,,\qquad \mathbf{a}\odot\mathbf{b} := \frac{1}{2}\,\{\mathbf{a}\,,\mathbf{b}\}\,. \tag{1.18}$$

Then, to every quantum observable $\mathbf{a}\in\mathfrak{O}_n$ we associated a linear function $f_{\mathbf{a}}(\xi):=\xi(\mathbf{a})$ on the dual space $\mathfrak{O}_n^*$ so that the two bivector fields are defined according to:

$$\widetilde{\Lambda}\,(\mathrm{d}f_{\mathbf{a}}\,,\mathrm{d}f_{\mathbf{b}}) = f_{[[\mathbf{a}\,,\mathbf{b}]]}\,,\qquad G\,(\mathrm{d}f_{\mathbf{a}}\,,\mathrm{d}f_{\mathbf{b}}) = f_{\mathbf{a}\odot\mathbf{b}}\,. \tag{1.19}$$

These definitions are well-posed because the linear functions $f_{\mathbf{a}}$ with $\mathbf{a}\in\mathfrak{O}_n$ generate the cotangent space at each point of $\mathfrak{O}_n^*$, that is, their differentials form an overcomplete basis of the module of differential one-forms on $\mathfrak{O}_n^*$. By means of these

[2] A very nice account of the genesis of this equation can be found in [38].

tensor fields we define Hamiltonian and gradient-like vector fields. Hamiltonian vector fields are tangent to $\mathfrak{T}_n^1$ (actually they are tangent to the manifolds of quantum states with fixed rank, and to the manifolds of isospectral states seen as susbsets in $\mathfrak{O}_n^*$), while gradient-like vector fields are not tangent to $\mathfrak{T}_n^1$. Furthermore, we find that the Hamiltonian and gradient-like vector fields associated with quantum observables by means of the expectation value functions $f_{\mathbf{a}}$ close on a realization of the Lie algebra of the Lie group $GL(\mathfrak{A}_n)$ made up of invertible elements of $\mathfrak{A}_n$ that integrates to a group action.

Then, we perform a reduction of the tensor fields $\widetilde{\Lambda}$ and G from $\mathfrak{O}_n^*$ to $\mathfrak{T}_n^1$, so that we obtain an antisymmetric bivector field Λ and a symmetric bivector field $\mathcal{R}$ on $\mathfrak{T}_n^1$ such that their associated Hamiltonian and gradient-like vector fields are by construction tangent to $\mathfrak{T}_n^1$. Quite interestingly, Hamiltonian and gradient-like vector fields associated with traceless quantum observables (again by means of the expectation value functions) close on a realization of the Lie group $SL(\mathfrak{A}_n)$. Contrarily to what happens for the realization of the Lie algebra of $GL(\mathfrak{A}_n)$ on $\mathfrak{O}_n$, this realization does not integrate to an action of the Lie group $SL(\mathfrak{A}_n)$. The reason is that gradient-like vector fields are "quadratic" and not complete. However, the integral curves of both Hamiltonian and gradient-like vector fields are complete when the initial condition is a quantum state $\rho \in \mathcal{S}$. In this case, it turns out that the integral curves of this vector fields are tangent to the manifold $\mathcal{S}_k$ to which the initial state ρ belongs (however, gradient-like vector fields change the spectrum of ρ and thus are not tangent to the manifolds of isospectral states). This means that Hamiltonian and gradient-like vector fields associated with quantum observables are tangent to the manifolds of quantum states with fixed rank.

The last step is to show that the GKLS equation describing Markovian open quantum system can be written by means of an affine vector field Γ which may be decomposed as:

$$\Gamma = X_{\mathbf{a}} + Y_{\mathbf{b}} + Z_{\mathbf{K}} \,, \tag{1.20}$$

where $X_{\mathbf{a}}$ is a Hamiltonian vector field on $\mathfrak{T}_n^1$ which is associated with the Hamiltonian part of the GKLS generator, $Y_{\mathbf{b}}$ is a gradient-like vector field on $\mathfrak{T}_n^1$ which is connected to the dissipative part of the GKLS generator, and Z_K is a "quadratic" related to the dissipative part of the generator GKLS. What is particularly interesting is that the gradient-like vector field $Y_{\mathbf{b}}$ and the (Kraus) vector field Z_K are not independent from one another, they must be fine-tuned so that their nonlinear parts cancel in the sum leaving an affine vector field.

Once we obtain the geometrical formulation of the GKLS dynamics, we can immediately start to exploit all the technical tools characteristics of classical dynamical systems in the quantum context. This is done at the end of chapter 3 where we will see how to exploit LaSalle invariance principle (see [81, 2]) to inquire about the asymptotic behaviour of open quantum dynamics. By means of LaSalle invariance principle it is possible to identify a LaSalle function for a given dynamics, and this function is essentially all we need to characterize the asymptotic behaviour of the dynamics, that is, find the possible asymptotic equilibrium states. In particular, we will analyze the so-called quantum random unitary semigroups, of which quantum Poisson semigroups and quantum Gaussian semigroups are particular instances (see [82, 78, 11]), and we will show that the purity function is a LaSalle function for the GKLS vector field associated with these semigroups in every dimension.

In chapter 4 we will consider quantum information geometry. We will start with a brief survey of the main ideas behind the classical theory of information geometry, that is, the concept of parametrized manifold of probability distributions, the Fisher-Rao metric and its uniqueness property, and the role of divergence functions as a mean to extract geometrical structures. This will lead us to consider the case of the information geometry of the manifold of pure quantum states $\mathcal{S}_1$. In this case, we are still able to select a sort of quantum analogue of the Fisher-Rao metric tensor, namely the Fubini-Study metric tensor. According to the results reviewed in chapter 2, this metric tensor is the unique (up to a multiplicative constant factor) metric tensor which is invariant with respect to the canonical action of the unitary group on the space of pure quantum states. From the point of view of information theory, the relevance of the Fubini-Study metric is outlined in [100], where it is shown that the geodesical distance function associated with the Fubini-Study metric coincide with the notion of statistical distance between pure quantum states based on distinguishability and statistical fluctuations in the outcomes of measurements[3]. Furthermore, we will briefly review the connection between the Fubini-Study metric tensor and the Fisher-Rao metric tensor on a submanifold of parametrized pure quantum states exposed in [59].

Next, we will focus on the manifold $\mathcal{S}_n$ of invertible quantum states on $\mathfrak{A}_n$. Here, the requirement of invariance with respect to the canonical action of the unitary group is not enough to single out a unique metric tensor as it happens on the manifold of pure quantum states or in the classical case. There is an infinite number of metric tensors satisfying unitary invariance and a more general property known as the monotonicity property (see [89, 93]). This has to do with the behaviour of a metric tensor with respect to the class of quantum stochastic maps. Petz gave a classification of this kind of metric tensors in terms of positive operator monotone functions (see [93]), specifically, they found a one-to-one correspondence between

[3]A dynamical characterization of this divergence function in terms of the Hamilton-Jacobi theory is given in [40, 46].

this kind of functions and the quantum metric tensors on $\mathcal{S}_n$.

As it happens in classical information geometry, it is possible to extract metric tensors on $\mathcal{S}_n$ from two-point functions, that is, functions defined on $\mathcal{S}_n \times \mathcal{S}_n$. In particular, some well-known examples of quantum relative entropies like von Neumann-Umegaki's relative entropy (see [98], and chapter 12 of [26]) $S^n(\bar{\rho}\,,\bar{\varrho}) = Tr\,(\bar{\rho}(\log(\bar{\rho}) - \log(\bar{\varrho})))$ give rise to metric tensors on $\mathcal{S}_n$ satisfying the monotonicity property. Following the ideas and results exposed in [40, 41, 45, 46], we will consider the problem of characterizing those two-point functions on $\mathcal{S}_n \times \mathcal{S}_n$ from which it is possible to extract quantum metric tensors satisfying the monotonicity property. At this purpose, we will first introduce a coordinate-free version of the algorithm used in information geometry to extract covariant tensor fields from two-point functions. In doing so, we will be able to introduce a class of two-point functions, which we call ***potential functions***, for which the usual procedures of classical information geometry are well-posed. Furthermore, we will clarify the relationship between potential functions and smooth maps between manifolds, showing that the pullback of a potential function is a again a potential function, and that taking the pullback of a potential function and then extract the covariant tensor fields from the resulting potential function is equivalent to extracting the covariant tensor fields from the starting potential function and then pull back the resulting covariant tensor fields. All this abstract formalism, which is presented in section 4.1, will be of capital importance in section 4.2. There, we will give a precise definition of the monotonicity property for quantum metric tensors, and show that it is intimately connected to the so-called Data Processing Inequality (DPI) for quantum divergence functions (i.e., non-negative potential functions on $\mathcal{S}_n \times \mathcal{S}_n$). Specifically, we will show that the DPI for quantum divergence functions implies the monotonicity property for the associated quantum metric tensors.

Finally, in section 4.3, we will apply all the abstract results of the chapter to the study of the quantum metric tensors associated with a two-parameter family of quantum potential functions, namely, the so-called $\alpha - z$-*Rényi Relative Entropies* ($\alpha - z$-RRE) inroduced in [20]. This two-parameter family of quantum relative entropies encloses some of the well-known notions of quantum relative entropies and information functions. For instance, von Neumann-Umegaki relative entropy, the α-*Rényi Relative Entropies*, the q-*quantum Rényi divergences*, Tsallis quantum relative entropies, the divergence function for the Bures metric tensor, and the divergence function for the Wigner-Yanase metric tensor.

Finally, in chapter 5 we will review the results and ideas described in the thesis and point out some possible future developments.

Chapter 2

Geometry of the Space of Quantum States

The content of this chapter is the geometry of the space $\mathcal{S}$ of quantum states of the finite-dimensional model C^*-algebra $\mathfrak{A}_n$ introduced in chapter 1. The scope of the chapter is to introduce a partition of $\mathcal{S}$ into the disjoint union of differential manifolds of different dimensions, namely, the manifolds of quantum states of fixed rank, and to study the differential structure of these manifolds. Then, we will provide a finer decomposition of $\mathcal{S}$ into the disjoint union of Kähler manifolds, specifically, the so-called manifolds of isospectral states. The point of view adopted here heavily relies on the theory of group actions and homogeneous space.

Recall that the space of states $\mathcal{S}$ of $\mathfrak{A}_n$ is defined as:

$$\mathcal{S} := \left\{\rho \in \mathfrak{O}_n^* \subset \mathfrak{A}_n^* \colon \rho(\mathbf{A}\,\mathbf{A}^\dagger) \geq 0 \;\forall \mathbf{A} \in \mathfrak{A}_n\,, \;\; \rho(\mathbb{I}) = 1\right\}\,, \tag{2.1}$$

where $\mathbb{I} \in \mathfrak{A}_n$ is the identity element. As explained in chapter 1, there is a one-to-one correspondence between elements in $\mathfrak{O}_n^*$ and elements in $\mathfrak{O}_n$, and we use this one-to-one correspondence to give the following definition:

Definition 3. *Given $\xi \in \mathfrak{A}_n^*$, we define the* ***spectrum*** $\mathrm{sp}(\xi)$ *of ξ, the* ***rank*** $\mathrm{rk}(\xi)$ *of ξ, and the* ***determinant*** $\det(\xi)$ *of ξ to be, respectively, the matrix spectrum, the matrix rank, and the matrix determinant of $\bar{\xi} \in \mathfrak{O}_n \subset \mathfrak{A}_n$.*

It follows from definition 3 that $\mathcal{S}$ may be decomposed as:

$$\mathcal{S} = \bigsqcup_{k=1}^{n} \mathcal{S}_k\,, \tag{2.2}$$

with:

$$\mathcal{S}_k := \{\rho \in \mathcal{S} \colon \mathrm{rk}(\rho) = k\} \; . \tag{2.3}$$

We will prove that every $\mathcal{S}_k$ is a homogeneous space for the Lie group $SL(\mathfrak{A}_n)$ (defined below), and that every $\mathcal{S}_k$ admits a structure of differential manifold. Successively, we will see that on each manifold $\mathcal{S}_k$ there is an action of the compact Lie subgroup $SU(\mathfrak{A}_n) \subset SL(\mathfrak{A}_n)$ (defined below) and that the orbits of this action are embedded compact Kähler manifolds of different dimensions. In particular, the manifold of pure quantum states (the extremal points of the convex set $\mathcal{S}$), turns out to be a homogeneous space for both $SL(\mathfrak{A}_n)$ and its compact subgroup $SU(\mathfrak{A}_n)$.

Now, let us introduce the relevant Lie groups naturally emerging once we have the C^*-algebra $\mathfrak{A}_n$. First of all, we consider the group of invertible elements of $\mathfrak{A}_n$:

$$GL(\mathfrak{A}_n) := \left\{ \mathrm{g} \in \mathfrak{A}_n \colon \exists\, \mathrm{g}^{-1} \in \mathfrak{A}_n \text{ such that } \mathrm{g}\,\mathrm{g}^{-1} = \mathbb{I} \right\} \; , \tag{2.4}$$

where $\mathbb{I} \in \mathfrak{A}_n$ is the identity operator. It is clear that $GL(\mathfrak{A}_n)$ may be thought of as the complex general linear group $GL(n\,,\mathbb{C})$, and thus it is a Lie group. Next, we can introduce the Lie subgroup $SL(\mathfrak{A}_n) \subset GL(\mathfrak{A}_n)$ given by:

$$SL(\mathfrak{A}_n) := \{\mathrm{g} \in GL(\mathfrak{A}_n) \colon \det(\mathrm{g}) = 1\} \; . \tag{2.5}$$

It is clear that $SL(\mathfrak{A}_n)$ may be thought of as the complex special linear group $SL(n\,,\mathbb{C})$. Moreover, we can introduce the Lie subgroups:

$$U(\mathfrak{A}_n) := \left\{ \mathbf{U} \in GL(\mathfrak{A}_n) \colon \mathbf{U}\,\mathbf{U}^\dagger = \mathbb{I} \right\} \; , \tag{2.6}$$

$$SU(\mathfrak{A}_n) := \{ \mathbf{U} \in U(\mathfrak{A}_n) \colon \det(\mathbf{U}) = 1 \} \; , \tag{2.7}$$

and it is clear that $U(\mathfrak{A}_n)$ coincides with the unitary group $U(n)$, while $SU(\mathfrak{A}_n)$ coincides with the special unitary group $SU(n)$.

Remark 2. *If we consider an infinite-dimensional C^*-algebra $\mathcal{A}$, we can always define the group $GL(\mathcal{A})$ of invertible elements of $\mathcal{A}$, and it can be proved that this is a Banach-Lie group (see for instance [39] page* 81*, [99] page* 96*), that is, a (complex analytical) Banach manifold with a compatible group structure.*

2.1 The space of quantum states with fixed rank as a homogeneous space

Here we will introduce the differential structure on the space $\mathcal{S}_k \subset \mathcal{S}$ of quantum states with fixed rank by showing that each $\mathcal{S}_k$ is a homogeneous space for the Lie group $SL(\mathfrak{A}_n)$ introduced above. Let us recall the definition of orbit and isotropy group of an action of a group on a set:

Definition 4 (***Orbit and isotropy group*** **).** *Let G be a group, M a set, and let $\alpha\colon G \times M \to M$ be an action of G on M. The orbit $Orb(m_0)$ of α at $m_0 \in M$ is the set:*

$$Orb(m_0) := \{m \in M \colon m = \alpha_g(m_0)\,, g \in G\}\ . \tag{2.8}$$

The isotropy group G_{m_0} of m_0 is:

$$G_{m_0} := \{g \in G \colon \alpha_g(m_0) = m_0\}\ . \tag{2.9}$$

An orbit of G on M is then a subset of M. The collection of all orbits of G on M gives a partition of M into disjoint subsets. Let us now take $G = SL(\mathfrak{A}_n)$ and $M = \mathcal{S}$ and define an action of $SL(\mathfrak{A}_n) = SL(n\,,\mathbb{C})$ on $\mathcal{S}$ (see [65] section 6):

Definition 5. *Consider the Lie group $SL(\mathfrak{A}_n)$. Then, for every $\mathrm{g} \in SL(\mathfrak{A}_n)$ and for every quantum state $\rho \in \mathcal{S} \subset \mathfrak{O}_n^* \subset \mathfrak{A}_n^*$, we define an action $\rho \mapsto \alpha_{\mathrm{g}}(\rho) \equiv \rho_{\mathrm{g}}$ of $SL(\mathfrak{A}_n)$ on $\mathcal{S}$ setting:*

$$\left(\alpha_{\mathrm{g}}(\rho)\right)(\mathbf{A}) \equiv \rho_{\mathrm{g}}(\mathbf{A}) := \frac{\rho(\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathrm{g})}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})}\ \ \forall \mathbf{A} \in \mathfrak{A}_n\,. \tag{2.10}$$

This is a left action (see definition 8 in [2] at page 328), that is:

$$\alpha_{\mathrm{g}_1} \circ \alpha_{\mathrm{g}_2}(\rho) = \alpha_{\mathrm{g}_1\mathrm{g}_2}(\rho)\,. \tag{2.11}$$

To be sure that the action α is well defined for all g and ρ, we must show that the denominator $\rho(\mathrm{g}^{\dagger}\,\mathrm{g})$ does not blow up and that $\alpha_{\mathrm{g}}(\rho) \equiv \rho_{\mathrm{g}}$ is a quantum state. At this purpose, we note that we can write:

$$\rho(\mathrm{g}^{\dagger}\,\mathrm{g}) = \langle\bar{\rho}\,, \mathrm{g}^{\dagger}\,\mathrm{g}\rangle = Tr\left(\bar{\rho}\,\mathrm{g}^{\dagger}\,\mathrm{g}\right)\ , \tag{2.12}$$

where $\langle\,,\rangle$ denotes the inner product on $\mathfrak{A}_n$ introduced in chapter 1. Now, since $\bar{\rho}$ is positive semidefinite, there is a nonzero element $\mathbf{A} \in \mathfrak{A}_n$ such that $\bar{\rho} = \mathbf{A}\mathbf{A}^{\dagger}$, and thus:

$$\rho(\mathrm{g}^{\dagger}\,\mathrm{g}) = Tr\left(\mathbf{A}\,\mathbf{A}^{\dagger}\,\mathrm{g}^{\dagger}\,\mathrm{g}\right) = \langle\mathrm{g}\,\mathbf{A}\,, \mathrm{g}\mathbf{A}\rangle = |\mathrm{g}\mathbf{A}|^2 > 0\,, \tag{2.13}$$

where the last inequality follows because $\mathrm{g}\,\mathbf{A} \neq \mathbf{0}$ and $\langle\,,\rangle$ is an inner product on $\mathfrak{A}_n$. It is easy to see that if ρ is in $\mathcal{S}$ then $\alpha_{\mathrm{g}}(\rho) = \rho_{\mathrm{g}}$ is again in $\mathcal{S}$. Indeed:

$$
\rho_{\mathrm{g}}(\mathbf{A}\,\mathbf{A}^{\dagger}) = \frac{\rho(\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathbf{A}^{\dagger}\,\mathrm{g})}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})} = \frac{\rho(\mathbf{B}\,\mathbf{B}^{\dagger})}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})} \geq 0\,, \tag{2.14}
$$

where $\mathbf{B} = \mathrm{g}^{\dagger}\,\mathbf{A}$, and:

$$
\rho_{\mathrm{g}}(\mathbb{I}) = \frac{\rho(\mathrm{g}^{\dagger}\,\mathbb{I}\,\mathrm{g})}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})} = 1\,. \tag{2.15}
$$

The one-to-one correspondence between quantum states and density matrices implies that every $\rho_{\mathrm{g}} \in \mathcal{S}$ is associated with $\bar{\rho}_{\mathrm{g}}$ given by:

$$
\bar{\rho}_{\mathrm{g}} = \frac{\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}}{Tr\,(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger})}\,. \tag{2.16}
$$

Indeed:

$$
\begin{aligned}
\rho_{\mathrm{g}}(\mathbf{A}) &= \frac{\langle\bar{\rho}\,,\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathrm{g}\rangle}{\langle\bar{\rho}\,,\mathrm{g}^{\dagger}\,\mathrm{g}\rangle} = \frac{Tr\left(\bar{\rho}\,\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathrm{g}\right)}{Tr\,(\bar{\rho}\,\mathrm{g}^{\dagger}\,\mathrm{g})} = \\
&= \frac{Tr\left(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}\,\mathbf{A}\right)}{Tr\,(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger})} = \left\langle \frac{\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}}{Tr\,(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger})}\,,\mathbf{A}\right\rangle \quad \forall\mathbf{A}\in\mathfrak{A}_n\,,
\end{aligned} \tag{2.17}
$$

and $\bar{\rho}_{\mathrm{g}} = \frac{\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}}{Tr\left(\mathrm{g}\,\bar{\rho}\,\mathrm{g}^{\dagger}\right)}$ is clearly a positive semidefinite matrix with trace equal to 1, that is, a density matrix.

Remark 3. *Note that the action α is not well defined on the whole $\mathfrak{O}_n^*$. When $\xi \in \mathfrak{O}_n^*$ is not positive, then there is $\mathrm{g} \in SL(\mathfrak{A}_n)$ such that $\xi(\mathrm{g}\,\mathrm{g}^{\dagger}) = 0$, and thus the denominator in the definition of α blows up. Indeed, let us look at consider a concrete example of a two-dimensional quantum system. The density matrix $\bar{\xi}$ associated with an element $\xi \in \mathfrak{O}_2^*$ may be written as:*

$$
\xi = x_0\,\sigma^0 + x_1\sigma^1\,, \tag{2.18}
$$

where the $\{\sigma^{\mu}\}$ are the Pauli matrices with $\sigma^0 = \mathbb{I}$ the identity operator. Note that the Pauli matrices are a basis for the Lie algebra of $GL(\mathfrak{A}_2)$.

Let $\mathrm{g} \in SL(\mathfrak{A}_2)$ be of the form

$$
\mathrm{g} = \mathrm{e}^{\mathbf{A}} = \mathrm{e}^{a\sigma^1} = \cosh(a)\mathbb{I} + \sinh(a)\sigma^1\,. \tag{2.19}
$$

It is clear that $det(\bar{\xi}\,) = \det(\mathrm{g}\,\bar{\xi}\,\mathrm{g}^{\dagger})$. Furthermore, it is $det(\bar{\xi}\,) = (x_0)^2 - (x_1)^2$. Let us now perform the following change of coordinates $y_0 = x_0 + x_1$, $y_1 = x_0 - x_1$, so that

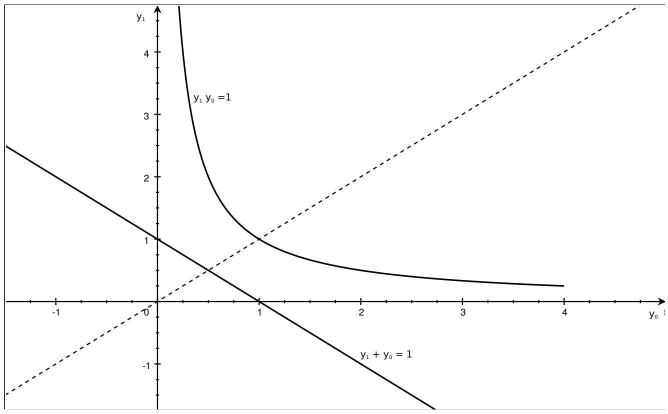

Figure 2.1: *The dotted line passing through the origin gives a graphical realization of the one-to-one correspondence between points on the hyperboloid* $y_0 y_1 = 1$ *and points on the affine hyperplane* $y_0 + y_1 = 1$.

$\det(\bar{\xi}\,) = y_0\, y_1$. *Accordingly, in the* $(y_0\,, y_1)$*-plane, every element* ξ *with* $\det(\bar{\xi}\,) = c$ *is represented as a point on the hyperboloid* $y_0\, y_1 = c$*, and the action* $\bar{\xi} \mapsto \mathbf{g}\,\bar{\xi}\,\mathbf{g}^{\dagger}$ *moves the point on this hyperboloid.*

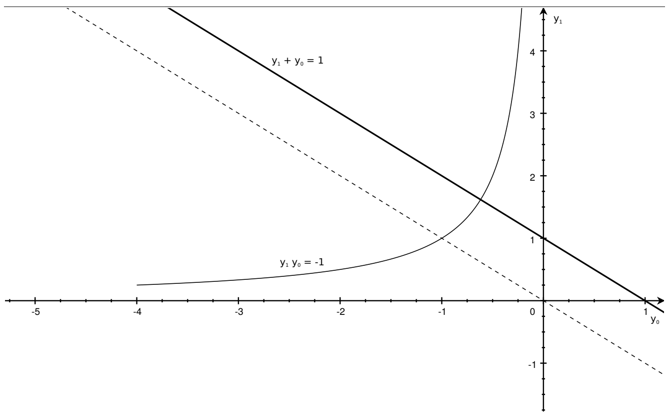

Figure 2.2: *There is a point on the hyperboloid* $y_0 y_1 = -1$ *for which dotted line through the origin intersecting the hyperboloid becomes parallel to the affine hyperplane* $y_0 + y_1 = 1$*. Consequently, the one-to-one correspondence between points on the hyperboloid and points on the affine hyperplane breaks down.*

In this framework, to divide $\bar{\xi}_{\mathrm{g}} = \mathrm{g}\,\bar{\xi}\,\mathrm{g}^{\dagger}$ *by its trace* $Tr(\mathrm{g}\,\bar{\xi}\,\mathrm{g}^{\dagger})$ *corresponds to move the point* $\bar{\xi}_{\mathrm{g}}$ *on the hyperboloid to the point* $\widetilde{\bar{\xi}}_{\mathrm{g}}$ *which is the the intersection between the straight line* $y_0 + y_1 = 1$ *and the straight line connecting the origin* $(0\,,0)$

with the $\bar{\xi}_{\rm g}$. Then, as long as det$(\bar{\xi}\,) = c > 0$, we see that for every $\bar{\xi}_{\rm g}$ there is one and only one such $\tilde{\bar{\xi}}_g$ (see figure 2.1). However, if det$(\bar{\xi}\,) = c < 0$, there will always be a point $\bar{\xi}^*_{\rm g}$ for which the straight line connecting it with the origin $(0\,,0)$ becomes parallel to the straight line $y_0 + y_1 = 1$ (see figure 2.2). Consequently, $\tilde{\bar{\xi}}^*_{\rm g}$ does not exist, and to divide $\bar{\xi}_{\rm g} = {\rm g}\,\bar{\xi}\,{\rm g}^\dagger$ by its trace $Tr({\rm g}\,\bar{\xi}\,{\rm g}^\dagger)$ is forbidden.

We want to show that $\mathcal{S}_k$ is a homogeneous space for $SL(\mathfrak{A}_n)$ with respect to the action α restricted to $\mathcal{S}_k \subset \mathcal{S}$. This is equivalent to prove that this action is transitive on $\mathcal{S}_k$:

Proposition 1. *Let $\alpha\colon SL(\mathfrak{A}_n)\times\mathcal{S} \to \mathcal{S}$ be as in definition 5, then, it is $\alpha(\mathcal{S}_k) = \mathcal{S}_k$.*

Proof. *Take ρ in $\mathcal{S}_k \subset \mathcal{S}$, and consider $\rho_{\rm g}$. Associated with the quantum state $\rho_{\rm g}$ there is the density matrix $\bar{\rho}_{\rm g} \in \mathfrak{A}_n$. Recall that the rank ${\rm rk}(\rho_{\rm g})$ of $\rho_{\rm g}$ is defined to be the rank of $\bar{\rho}_{\rm g}$ (see definition 3). According to equation (2.16), it is:*

$$\bar{\rho}_{\rm g} = \frac{{\rm g}\,\bar{\rho}\,{\rm g}^\dagger}{Tr({\rm g}\,\bar{\rho}\,{\rm g}^\dagger)}\,, \tag{2.20}$$

and, being ${\rm g}$ and ${\rm g}^\dagger$ of maximal rank, we have that the matrix rank of $\bar{\rho}_{\rm g}$ is equal to the matrix rank of $\bar{\rho}$, and thus ${\rm rk}(\rho) = {\rm rk}(\rho_{\rm g})$. This means that $\alpha(\mathcal{S}_k) \subseteq \mathcal{S}_k$.

To complete the proof, we must show that, given a fixed $\rho_0 \in \mathcal{S}_k$, for every $\rho \in \mathcal{S}_k$ there is ${\rm g} \in SL(\mathfrak{A}_n)$ such that $\rho = \alpha_{\rm g}(\rho_0)$. Let us consider the element $\bar{\rho}_0 \in \mathfrak{D}_n$ associated with ρ_0, and the element $\bar{\rho} \in \mathfrak{D}_n$ associated with ρ. Then, there are $\mathbf{U}_0, \mathbf{U} \in \mathcal{SU}(\mathfrak{A})$ such that:

$$\begin{aligned}\bar{\rho}_0 &= \mathbf{U}_0^\dagger \left(\sum_{j=1}^n p_0^j\,|j\rangle_0\langle j|\right)\mathbf{U}_0 \\ \bar{\rho} &= \mathbf{U}^\dagger \left(\sum_{j=1}^n p^j\,|j\rangle\langle j|\right)\mathbf{U}\,.\end{aligned} \tag{2.21}$$

Where $\{|j\rangle_0\}_{j=1,\ldots,n}$ and $\{|j\rangle\}_{j=1,\ldots,n}$ denote, respectively, the orthonormal basis of eigenvectors of $\bar{\rho}_0$ and $\bar{\rho}$, and $\vec{p}_0$ and $\vec{p}$ denote the n-dimensional vectors the elements of which are, respectively, the eigenvalues of $\bar{\rho}_0$ and the eigenvalues of $\bar{\rho}$. Without loss of generality, we may take $\mathbf{U}_0$ and $\mathbf{U}$ such that $\vec{p}_0$ and $\vec{p}$ have the first k elements different form 0, and the remaining $(n-k)$ equal to 0. The element in $\mathcal{SU}(\mathfrak{A})$ connecting the orthonormal basis $\{|j\rangle_0\}_{j=1,\ldots,n}$ with the orthonormal basis $\{|j\rangle\}_{j=1,\ldots,n}$ will be denoted as $\mathbf{V}$.

Now, being p_0^j and p^j strictly greater than 0 for all $j \leq k$, there exists $g_j > 0$ such that:

$$p^j = (g_j)^2 \, p_0^j \;\; \forall j \leq k \,. \tag{2.22}$$

With the help of these numbers, we define the element $\mathbf{G} \in SL(\mathfrak{A})$ given by:

$$\mathbf{G} = \sum_{j=1}^{k} g_j \, |j\rangle_0\langle j| + \sum_{j=k+1}^{n} a_j |j\rangle_0\langle j| \,, \tag{2.23}$$

where $a_j > 0$ for all $(k+1) \leq j \leq n$ and:

$$\left(\prod_{j=1}^{k} g_j \right) \cdot \left(\prod_{j=k+1}^{n} a_j \right) = 1 \,. \tag{2.24}$$

A direct calculation shows that:

$$\mathbf{G} \left(\sum_{j=1}^{n} p_0^j \, |j\rangle_0\langle j| \right) \mathbf{G}^\dagger = \sum_{j=1}^{n} p^j \, |j\rangle_0\langle j| \,. \tag{2.25}$$

Eventually, we have:

$$\bar{\rho} = \mathbf{V} \, \mathbf{G} \, \mathbf{U}_0 \, \bar{\rho}_0 \, \mathbf{U}^\dagger \, \mathbf{G}^\dagger \, \mathbf{V}^\dagger \,, \tag{2.26}$$

and it is clear that $\mathrm{g} \equiv \mathbf{V} \, \mathbf{G} \, \mathbf{U}_0$ is in $SL(\mathfrak{A})$. This means that:

$$\rho_{\mathrm{g}}(\rho_0) = \omega \tag{2.27}$$

as required.

From this proposition it naturally follows:

Proposition 2. *Let α^k denotes the action $\rho \mapsto \alpha_{\mathrm{g}}^k(\rho) \equiv \rho_{\mathrm{g}}$ of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$ given by:*

$$\rho_{\mathrm{g}}(\mathbf{A}) := \frac{\rho(\mathrm{g}^\dagger \, \mathbf{A} \, \mathrm{g})}{\rho(\mathrm{g}^\dagger \, \mathrm{g})} \;\; \forall \mathbf{A} \in \mathfrak{A} \,, \tag{2.28}$$

where $\rho \in \mathcal{S}_k$ and $\mathrm{g} \in SL(\mathfrak{A}_n)$. Then, $\mathcal{S}_k$ is a homogeneous space of $SL(\mathfrak{A}_n)$ for every $1 \leq k \leq n$, that is, $SL(\mathfrak{A}_n)$ acts transitively on $\mathcal{S}_k$ for every $1 \leq k \leq n$.

Since $\mathcal{S}_k$ is a homogeneous space of $SL(\mathfrak{A}_n)$, there is a one-to-one correspondence between points in $\mathcal{S}_k$ and points in the coset space $SL(\mathfrak{A}_n)/G_\rho$, where G_ρ is the isotropy subgroup of $\rho \in \mathcal{S}_k$. The actual choice of ρ is irrelevant because isotropy subgroups are conjugate. This one-to-one correspondence is what we need to endow $\mathcal{S}_k$ with a differential structure. Indeed, the so-called closed subgroup theorem implies that, when G_ρ is a closed Lie subgroup of $SL(\mathfrak{A}_n)$, then $SL(\mathfrak{A}_n)/G_\rho$ inherits a differential structure such that the canonical projection $\pi_\rho \colon SL(\mathfrak{A}_n) \to SL(\mathfrak{A}_n)/G_\rho$ is a smooth submersion. We can use the one-to-one correspondence between $\mathcal{S}_k$ and $SL(\mathfrak{A}_n)/G_\rho$ to transport the differential structure from $SL(\mathfrak{A}_n)/G_\rho$ to $\mathcal{S}_k$. For the sake of completeness, we recall here the statement of the closed subgroup theorem (see [1] page 264):

Theorem 2 (Closed subgroup theorem)**.** *Let H be a closed subgroup of the Lie group G. Then the left coset space $G/H := \{\mathrm{g}H \colon \mathrm{g} \in G\}$ is a smooth manifold and the canonical projection $\pi \colon G \to G/H$ is a submersion.*

We will now show that G_ρ is actually a closed Lie subgroup of $SL(\mathfrak{A}_n)$. At this purpose, recall that the actual choice of $\rho \in \mathcal{S}_k$ is irrelevant because all the isotropy subgroups of points in a homogeneous space are conjugate. Consequently, we will take a particular $\rho_k \in \mathcal{S}_k$ for which the computations are easy. Specifically, we take the element $\rho_k \in \mathcal{S}_k$ such that its associated density matrix is:

$$\bar{\rho}_k = \sum_{j=1}^{k} \mathbf{E}_{jj} \,, \tag{2.29}$$

where $\{\mathbf{E}_{jk}\}_{j,k=1,\dots,n}$ is the canonical basis in $\mathfrak{A}_n$ introduced in chapter 1. Now, the isotropy subgroup $G_k \equiv G_{\rho_k}$ is the subgroup of $SL(\mathfrak{A}_n)$ defined by:

$$G_k := \left\{\mathrm{g} \in SL(\mathfrak{A}_n) \colon \; \alpha^k_{\mathrm{g}}(\rho_k) = \rho_k \right\} \,, \tag{2.30}$$

and we can prove the following:

Proposition 3. *Every element* g *in the isotropy subgroup G_k of ρ_k is of the form:*

$$\mathrm{g} = \begin{pmatrix} \mathbf{U} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \,, \tag{2.31}$$

where $\mathbf{U} \in \mathfrak{A}_k$, $\mathbf{D} \in \mathfrak{A}_{n-k}$ and $\mathbf{B}$ is a $k \times (n-k)$ complex matrix. Furthermore, G_k is a closed subgroup of $SL(\mathfrak{A}_n)$.

Proof. *We start proving that G_k is a closed subgroup of $SL(\mathfrak{A}_n)$. Consider the smooth map $\Phi_k \colon SL(\mathfrak{A}_n) \to \mathcal{S}_k$ defined by:*

$$\Phi_k(\mathrm{g}) := \alpha^k_{\mathrm{g}}(\rho_k)\,. \tag{2.32}$$

It is clear that $G_k = \Phi_k^{-1}(\rho_k)$, that is, G_k is the preimage of the closed set $\{\rho_k\}$ by means of a continuous map, which means that it is a closed subset of $SL(\mathfrak{A}_n)$, and thus a closed subgroup.

Now, every element $\mathrm{g} \in G_k$ must be such that:

$$\frac{\mathrm{g}\,\bar{\rho}_k\,\mathrm{g}^\dagger}{Tr\,(\mathrm{g}\,\bar{\rho}_k\,\mathrm{g}^\dagger)} = \bar{\rho}_k\,, \tag{2.33}$$

therefore, writing:

$$\bar{\rho}_k = \frac{1}{k}\begin{pmatrix} \mathbb{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \textit{and} \quad \mathrm{g} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\,, \tag{2.34}$$

equation (2.33) *becomes:*

$$\mathrm{g} = \frac{1}{\beta}\begin{pmatrix} \mathbf{A}\,\mathbf{A}^\dagger & \mathbf{A}\,\mathbf{C}^\dagger \\ \mathbf{C}\,\mathbf{A}^\dagger & \mathbf{C}\,\mathbf{C}^\dagger \end{pmatrix} = \frac{1}{k}\begin{pmatrix} \mathbb{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\,, \tag{2.35}$$

where:

$$\beta = Tr_k(\mathbf{A}\,\mathbf{A}^\dagger) + Tr_{(n-k)}(\mathbf{C}\,\mathbf{C}^\dagger)\,. \tag{2.36}$$

It is clear that it must be $\mathbf{C} = \mathbf{0}$, and that:

$$\frac{\mathbf{A}\,\mathbf{A}^\dagger}{Tr_k(\mathbf{A}\,\mathbf{A}^\dagger)} = \frac{\mathbb{I}_k}{k}\,, \tag{2.37}$$

which is equivalent to $\mathbf{A} \in SU(\mathfrak{A}_k)$, and thus we will write it as $\mathbf{U}$. Now, $\mathbf{B}$ and $\mathbf{D}$ must be such that g is an element of $SL(\mathfrak{A}_n)$, that is, it is invertible and has determinant equal to 1. According to the rules of matrix algebras, we have:

$$\det(\mathrm{g}) = \det(\mathbf{U})\det(\mathbf{D})\,, \tag{2.38}$$

and thus, it must be $\mathbf{D} \in SL(\mathfrak{A}_{(n-k)})$, while $\mathbf{B}$ is completely arbitrary.

Remark 4. *In the finite-dimensional case, we have the isomorphism between $\mathcal{I} \colon \mathfrak{A}_n \to \mathfrak{A}_n^*$. This allows us to "transport" all the structure available on the space $\mathcal{S} \subset \mathfrak{O}_n^* \subset$*

$\mathfrak{A}_n^*$ of quantum states to the space $\bar{\mathcal{S}} \subset \mathfrak{O}_n \subset \mathfrak{A}_n$ of density matrices. Consequently, we have a partition of $\bar{\mathcal{S}}$ into the disjoint union $\bigsqcup \bar{\mathcal{S}}_k$ of manifolds $\bar{\mathcal{S}}_k$ of density matrices with fixed rank. The differential structure on each $\bar{\mathcal{S}}_k$ is the only one making the one-to-one map $\mathcal{I}_k \colon \bar{\mathcal{S}}_k \to \mathcal{S}_k$, derived from the isomorphism $\mathcal{I}$, a diffeomorphism. It is clear that, from an operational point of view, the possibility of working with density matrices is highly desirable since we can use all the tools coming from matrix calculus in the context of quantum states. Indeed, in chapter 4 we will make use of this instance in the context of quantum information geometry, where the primary concern is on finite-dimensional systems, and it is customary to work directly in terms of density matrices rather than quantum states (positive normalized linear functionals on $\mathfrak{A}_n$).

Once we have dressed every $\mathcal{S}_k$ with its own differential structure coming from the group $SL(\mathfrak{A}_n)$, we naturally have a topology on every $\mathcal{S}_k$ underlying these differential structures (see [2] page 130, and [80] page 22). However, since the space $\mathcal{S}$ of quantum states is a subset of $\mathfrak{O}_n^*$, we may look at every $\mathcal{S}_k$ as a subset of $\mathfrak{O}_n^*$, and ask about its topological properties as a subset of $\mathfrak{O}_n^*$. Specifically, we will show that, for $1 < k \leq n$, $\mathcal{S}_k$ is not closed in $\mathfrak{O}_n^*$, and that, for $1 \leq k \leq n$, $\mathcal{S}_k$ is not open in $\mathfrak{O}_n^*$.

Let us start with the closedness. Consider $1 < k \leq n$, and the sequence $\{\rho_j^{(k)}\}_{j\in\mathbb{N}} \in \mathcal{S}_k$ where $\rho_j^{(k)} \in \mathcal{S}_k$ has the following density matrix representation:

$$\bar{\rho}_j^{(k)} = \sum_{r=1}^{n} \mathrm{p}^r \, \mathbf{E}_{rr} \,, \tag{2.39}$$

where:

$$\begin{aligned} \mathrm{p}_r &= \frac{j+1}{(j+2)(k-1)} \quad \text{if } r = 1, ..., k-1 \,, \\ \mathrm{p}_k &= \frac{1}{2+j} \,, \\ \mathrm{p}_r &= 0 \quad \text{if } r = k+1, ..., n \,. \end{aligned} \tag{2.40}$$

Since $\mathfrak{O}_n$ and $\mathfrak{O}_n^*$ are isomorphic as Banach spaces, we will work with the sequence of density matrices $\{\bar{\rho}_j^{(k)}\}_{j\in\mathbb{N}} \in \mathfrak{O}_n$ rather than with the sequence of linear functionals $\{\rho_j^{(k)}\}_{j\in\mathbb{N}} \in \mathcal{S}_k$. A direct computation shows that $\{\bar{\rho}_j^{(k)}\}_{j\in\mathbb{N}} \in \mathfrak{O}_n$ converges to the element:

$$\bar{\rho}_{+\infty}^{(k)} = \sum_{r=1}^{n} \mathrm{P}^r \, \mathbf{E}_{rr} \,, \tag{2.41}$$

with:

$$
\begin{aligned}
&\mathrm{P}_r = \frac{1}{k-1} \ \text{ if } r = 1, ..., k-1\,, \\
&\mathrm{P}_r = 0 \ \text{ if } r = k, ..., n\,,
\end{aligned}
\tag{2.42}
$$

which is clearly a density matrix with rank $(k-1)$. This means that the linear functional $\rho_{+\infty}^{(k)}$ uniquely associated with $\bar{\rho}_{+\infty}^{(k)}$ can not be in $\mathcal{S}_k$, and thus $\mathcal{S}_k$ is not closed in $\mathfrak{O}_n^*$.

Remark 5. *We will see in the next section that the space $\mathcal{S}_1$ of pure quantum states is a homogeneous space with respect to an action of the special unitary group $SU(\mathfrak{A}_n)$ which is well-defined on the whole $\mathfrak{O}_n^*$. Since $SU(\mathfrak{A}_n)$ is a compact group, it follows from theorem 3 that $\mathcal{S}_1$ is an embedded submanifold of $\mathfrak{O}_n^*$, and thus, it is a closed compact subset*[1] *of $\mathfrak{O}_n^*$.*

Now, we will prove that $\mathcal{S}_k$ is not open in $\mathfrak{O}_n^*$ for $1 \leq k \leq n$. Again, we prefer to work in $\mathfrak{O}_n$ rather than with $\mathfrak{O}_n^*$. This amount to consider the image of $\mathcal{S}_k$ in $\mathfrak{O}_n$ by means of the isomorphism between $\mathfrak{O}_n^*$ and $\mathfrak{O}_n$. A set $E \subset \mathfrak{O}_n$ is open if, given any $\mathbf{e} \in E$, there exists a real number $\epsilon > 0$ such that every point $\mathbf{a} \in \mathfrak{O}_n$ for which:

$$
Tr\left((\mathbf{e} - \mathbf{a})^2\right) < \epsilon \tag{2.43}
$$

is in E. Consequently, we take a rank k density matrix $\bar{\rho}$ and the element $\mathbf{a} = \bar{\rho} - c\mathbb{I}$ in $\mathfrak{O}_n$, where $0 \neq c \in \mathbb{R}$ and $\mathbb{I}$ is the identity operator. It is clear that $\mathbf{a}$ is not in $\mathcal{S}_k$, for instance, $Tr(\mathbf{a}) = 1 - n\,c \neq 1$. Then:

$$
Tr\left((\bar{\rho} - \mathbf{a})^2\right) = n\,c^2\,. \tag{2.44}
$$

Since c is arbitrary, given any real number $\epsilon > 0$, we can always take c such that $n\,c^2 < \epsilon$. This means that the image of $\mathcal{S}_k$ in $\mathfrak{O}_n$ is not open, and thus $\mathcal{S}_k$ is not open in $\mathfrak{O}_n^*$.

2.1.1 Fundamental vector fields and tangent spaces

By construction, the action α^k of $SL(\mathfrak{A}_n)$ on the differential manifold $\mathcal{S}_k$ is a smooth action, and, being $\mathcal{S}_k$ a homogeneous space of $SL(\mathfrak{A}_n)$, there is only one orbit for α_k which is precisely the whole $\mathcal{S}_k$.

[1] For the notion of embedded submanifold see [2] chapter 3 section 2, [80] chapter 2 section 2, and [84] chapter 7.

We may represent a tangent vector $\mathbf{v}_\rho \in T_\rho \mathcal{S}_k$ by means of an element in the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ of $SL(\mathfrak{A}_n)$. In order to do this, we consider the curve $\alpha^k(\mathrm{g}_t\,,\rho) \equiv \rho_t \in \mathcal{S}_k$ starting at ρ defined by:

$$\rho_t(\mathbf{A}) := \frac{\rho\left(\mathrm{g}_t^\dagger\,\mathbf{A}\,\mathrm{g}_t\right)}{\rho(\mathrm{g}_t^\dagger\,\mathrm{g}_t)}\,, \tag{2.45}$$

where:

$$\mathrm{g}_t := \mathrm{e}^{(\mathbf{a}+\imath\mathbf{b})t} \quad \text{with} \;\; (\mathbf{a}+\imath\mathbf{b}) \in \mathfrak{sl}(\mathfrak{A}_n)\,. \tag{2.46}$$

Clearly, being $SL(\mathfrak{A}_n)$, we have that $\mathbf{a}$ and $\mathbf{b}$ are self-adjoint matrices. If we consider the density matrix $\bar{\rho}_t$ associated with ρ_t, a direct computation shows that:

$$\bar{\rho}_t = \frac{\mathrm{e}^{(\mathbf{a}+\imath\mathbf{b})t}\,\bar{\rho}\,\mathrm{e}^{(\mathbf{a}-\imath\mathbf{b})t}}{Tr\left(\mathrm{e}^{\mathbf{a}t}\,\bar{\rho}\,\mathrm{e}^{\mathbf{a}t}\right)}\,, \tag{2.47}$$

where $\bar{\rho}$ is the density matrix associated with $\rho_0 = \rho$. Consequently:

$$\left(\frac{\mathrm{d}\bar{\rho}_t}{\mathrm{d}t}\right)_{t=0} = \{\mathbf{a}\,,\bar{\rho}\} + \imath\,[\mathbf{b}\,,\bar{\rho}] - \bar{\rho}\;Tr(\{\mathbf{a}\,,\bar{\rho}\}) \equiv \mathbf{V}_{\bar{\rho}} \in \mathfrak{sl}(\mathfrak{A}_n)\,, \tag{2.48}$$

where $\{\,,\}$ and $[\,,]$ denote, respectively, the matrix anticommutator and the matrix commutator. Eventually, we have the correspondence:

$$\mathbf{v}_\rho \longleftrightarrow \mathbf{V}_{\bar{\rho}} \equiv \{\mathbf{a}\,,\bar{\rho}\} + \imath\,[\mathbf{b}\,,\bar{\rho}] - \bar{\rho}\;Tr\left(\{\mathbf{a}\,,\bar{\rho}\}\right)\,. \tag{2.49}$$

Obviously, given $\bar{\rho}$, the couple $(\mathbf{a}\,,\mathbf{b})$ of self-adjoint matrices associated with $\mathbf{V}_{\bar{\rho}} \in \mathfrak{sl}(\mathfrak{A}_n)$ is not unique. In the following, we will use both $\mathbf{v}_\rho$ and $\mathbf{V}_{\bar{\rho}}$ to denote a tangent vector to ρ.

According to the theory of actions of Lie groups we can introduce the notion of ***fundamental vector field*** associated with the action of a Lie group on a differential manifold:

Definition 6 (***Fundamental vector field***)**.** *Let G be a Lie group, and let $\alpha\colon G \times M \to M$ be a smooth left action of G on the differential manifold M. Given $\mathbf{A} \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, we define the fundamental vector field $X_{\mathbf{A}} \in \mathfrak{X}(M)$ as the infinitesimal generator of the flow $\alpha^{\mathbf{A}}\colon \mathbb{R} \times M \to M$ given by*

$$\alpha^{\mathbf{A}}(t\,,m) := \alpha(\exp(t\mathbf{A})\,,m)\,, \tag{2.50}$$

that is:

$$X_{\mathbf{A}}(m) = \left(\frac{d}{dt}\alpha_{\exp(t\mathbf{A})}(m)\right)_{t=0} \tag{2.51}$$

Recalling how equation 2.49 has been obtained, we realize that:

$$X_{\mathbf{A}}(\rho) = \mathbf{V}_{\bar{\rho}} = \{\mathbf{a}\,,\bar{\rho}\} + \imath\,[\mathbf{b}\,,\bar{\rho}\,] - \bar{\rho}\,Tr(\{\mathbf{a}\,,\bar{\rho}\,\}) \tag{2.52}$$

with $\mathfrak{sl}(\mathfrak{A}_n) \ni \mathbf{A} = \mathbf{a} + \imath\mathbf{b}$. Interestingly, it can be proved (see [2] page 333) that:

Proposition 4. *Let G be a Lie group, and let $\alpha\colon G\times M \to M$ be a smooth action of G on the differential manifold M. Then, the map $\mathfrak{g} \ni \mathbf{A} \to X_{\mathbf{A}} \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism, that is:*

$$X_{a\mathbf{A}+b\mathbf{B}} = aX_{\mathbf{A}} + bX_{\mathbf{B}}\ ,\ a,b\in\mathbb{R}\,, \tag{2.53}$$

$$[X_{\mathbf{A}}\,, X_{\mathbf{B}}] = -X_{[\mathbf{A}\,,\mathbf{B}]}\,. \tag{2.54}$$

Referring to equation (2.45), we see that the fundamental vector fields associated with elements in the Lie algebra of $\mathfrak{A}_n$ of the form $\mathbf{A} = \imath\mathbf{a}$ with $\mathbf{a} \in \mathfrak{O}_n$ generate an action of the unitary group $SU(\mathfrak{A}_n) \subset SL(\mathfrak{A}_n)$. In the next section we will study the orbits of this action, which we refer to as manifolds of isospectral

There is a geometro-dynamical interpretation of the fundamental vector fields of the action of $SL(\mathfrak{A}_n)$ on each $\mathcal{S}_k$ we will see in chapter 3 where we will present a geometrical formulation for the GKLS master equation governing the dynamical evolution of open quantum systems (see [63, 82]). In that context, the infinitesimal generator of the dynamical evolution is written as an affine vector field Γ on the affine hyperplane $\mathfrak{T}_n^1 \subset \mathfrak{O}_n^*$ of self-adjoint linear functionals ξ such that $\xi(\mathbb{I}) = 1$. The vector field Γ is naturally decomposed as the sum $\Gamma = X_{\mathbf{a}} + Y_{\mathbf{b}} + Z_K$ of three vector fields, where $\mathbf{a},\mathbf{b} \in \mathfrak{O}_n$ and K is a completely positive trace preserving map on $\mathfrak{O}_n$. The relevant fact is that the restriction of the vector fields $X_{\mathbf{a}}$ and $Y_{\mathbf{b}}$ to the submanifolds $\mathcal{S}_k \subset \mathfrak{T}_n^1$ of quantum states with fixed rank coincides, respectively, with the fundamental vector fields $X_{\mathbf{A}}$ and $X_{\mathbf{B}}$ on $\mathcal{S}_k$ associated with $\mathbf{A} = \imath\mathbf{a}$ and $\mathbf{B} = \mathbf{b}$ for all k. The affine vector field $X_{\mathbf{a}}$ will be seen to be related to the "Hamiltonian part" of the dynamical evolution, specifically, we will see that there are a Poisson tensor and a smooth function $f_{\mathbf{a}}$ on $\mathfrak{T}_n^1$ such that $X_{\mathbf{a}}$ is the Hamiltonian vector field associated with $f_{\mathbf{a}}$ by means of Λ for every $\mathbf{a} \in \mathfrak{O}_n$. Furthermore, $X_{\mathbf{a}}$ turns out to be tangent to the submanifold $\mathcal{S}_\sigma \subset \mathcal{S}_k \subset \mathfrak{T}_n^1$ of isospectral quantum states we will introduce in the next section, and this means that the flow of $X_{\mathbf{a}}$ preserves the spectrum of quantum states. In particular, when $Y_{\mathbf{b}} = Z_K = 0$ so that the GKLS vector field is $\Gamma = X_{\mathbf{a}}$, the resulting dynamical evolution is precisely the unitary evolution of equation (1.9).

On the other hand, we will see that there is a symmetric bivector field $\mathcal{R}$ on $\mathfrak{T}_n^1$ such that $Y_{\mathbf{b}}$ is the gradient-like vector field associated with $f_{\mathbf{b}}$ by means of $\mathcal{R}$ for all $\mathbf{b} \in \mathfrak{O}_n$ (see definition 9), and the flow of the gradient-like vector field $Y_{\mathbf{b}}$ does not preserve the spectrum of quantum states. Furthermore, the vector fields $Y_{\mathbf{b}}$ and Z_K will be seen to be related to one another. Specifically, both of them are non-affine vector fields on $\mathfrak{T}_n^1$, and thus they must be fine-tuned so that their sum is an affine vector field on $\mathfrak{T}_n^1$.

2.2 The manifolds of isospectral states

In the previous section, it was proved that there is a transitive action α^k of the group $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$, and thus, by the closed subgroup theorem 2, we gave $\mathcal{S}_k$ the unique differential structure such that the action α^k is a smooth action. If we consider the restriction of the smooth action α^k of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$ to the maximal compact subgroup $SU(\mathfrak{A}_n) \subset SL(\mathfrak{A}_n)$, we obtain a smooth action of $SU(\mathfrak{A}_n)$ on $\mathcal{S}_k$. It is clear that this action is not transitive on $\mathcal{S}_k$, and thus we obtain a partition of each $\mathcal{S}_k$ into orbits of $SU(\mathfrak{A}_n)$. These orbits present a very rich geometrical structure that we will now explore.

When everything is finite-dimensional, a crucial result on the differential structure of orbits of smooth actions of Lie groups is proved, for example, in [1] (page 265):

Theorem 3. *Let G be a Lie group, and let $\alpha\colon G \times M \to M$ be a smooth action of G on the differential manifold M. Then, the stabilizer G_{m_0} at $m_0 \in M$ is a closed subgroup of G, and the orbit $Orb(m_0) \subset M$ is an immersed submanifold of M diffeomorphic to the coset space G/G_{m_0}. Furthermore, if G is compact, then $Orb(m_0)$ is a closed embedded submanifold*[2] *of M.*

In our case, it is $M = \mathcal{S}_k$, and $G = SU(\mathfrak{A}_n)$ is a compact Lie group. Consequently, we have that every orbit is a compact closed embedded submanifold of $\mathcal{S}_k$ which is diffeomorphic to $SU(\mathfrak{A}_n)/G_\rho$, where the isotropy subgroup G_ρ is easily seen to be:

$$G_\rho = \{\mathbf{U} \in SU(\mathfrak{A}_n)\colon\ [\bar{\rho}\,, \mathbf{U}] = 0\}\ , \tag{2.55}$$

[2] For the notion of embedded submanifold see [2] chapter 3 section 2, [80] chapter 2 section 2, and [84] chapter 7.

with $\bar{\rho} \in \mathfrak{A}_n$ denoting the density matrix associated with ρ. The next step is to characterize $Orb(\rho)$. The action of $SU(\mathfrak{A}_n)$ on ρ can be represented on $\bar{\rho}$ as a matrix similarity transformation:

$$\bar{\rho} \mapsto \bar{\rho}_{\mathbf{U}} = \mathbf{U}\,\bar{\rho}\,\mathbf{U}^{\dagger} = \mathbf{U}\,\bar{\rho}\,\mathbf{U}^{-1}\,. \tag{2.56}$$

Consequently, $\bar{\rho}$ and $\bar{\rho}_{\mathbf{U}}$ have the same eigenvalues for all $\mathbf{U} \in SU(\mathfrak{A}_n)$, even if the n-uples of such eigenvalues can be arbitrarily ordered. Conversely, if ρ_1 and ρ_2 have the same spectrum, up to permutations, the spectral theorem for self-adjoint matrices assures us that there is $\mathbf{U} \in SU(\mathfrak{A}_n)$ such that $\rho_2 = \alpha_{\mathbf{U}}(\rho_1)$.

Now, being $\bar{\rho}$ positive-semidefinite, the n-uple of eigenvalues of $\bar{\rho}$ and $\bar{\rho}_{\mathbf{U}}$ may be thought of as elements in the n-dimensional simplex Δ_n, that is, the set of n-uple $(p_1, ..., p_n)$ of non-negative real numbers such that $\sum_{j=1}^{n} p_j = 1$. On Δ_n, the permutation group acts naturally, and we denote with σ the equivalence class $[\vec{p}]$ of $\vec{p} \in \Delta_n$ with respect to this action, and write Π_n for the set of all these equivalence classes. It is then clear that each orbit $\mathrm{Orb}(\rho)$ can be labelled by a point $\sigma \in \Pi_n$, and we will write:

$$\mathcal{S}_\sigma := \{\rho \in \mathcal{S}\colon\ \ [\mathrm{sp}(\rho)] = \sigma\}\ , \tag{2.57}$$

and thus $\mathrm{Orb}(\rho) = \mathcal{S}_\sigma$.

Remark 6. *When we consider the manifold of pure quantum states, that is $\mathcal{S}_k$ with $k = 1$, we have that every $\rho \in \mathcal{S}_1$ is such that $\sigma = [\mathrm{sp}(\rho)] = [(1, 0, ..., 0)]$, and thus $\mathcal{S}_1 = \mathcal{S}_\sigma$. This means that the space of pure quantum states of $\mathfrak{A}_n$ is a homogeneous space of both the special linear group $SL(\mathfrak{A}_n)$ and the special unitary group $SU(\mathfrak{A}_n)$.*

Remark 7. *When $k = n$, there is an orbit of $SU(\mathfrak{A}_n)$ which is degenerate, that is, it consists of a single point. This is the orbit associated with the maximally mixed state ρ_n. This follows from the fact that $\bar{\rho}_n = \frac{\mathbb{I}}{n}$, where $\mathbb{I}$ is the identity operator in $\mathfrak{A}_n$. In the following, when working with the differential manifold $\mathcal{S}_\sigma$ we will always implicitely assume that $\mathcal{S}_\sigma$ is never the (degenerate) unitary orbit through the maximally mixed state ρ_n.*

Just as we did when we derived equation 2.49, we can provide a matrix realization $\mathbf{V}_{\bar{\rho}} \in \mathfrak{su}(\mathfrak{A}_n)$ for a tangent vector $\mathbf{v}_\rho \in T_\rho\mathcal{S}_\sigma$:

$$\mathbf{v}_\rho \longleftrightarrow \mathbf{V}_{\bar{\rho}} \equiv \imath\,[\mathbf{b}\,,\bar{\rho}]\ , \tag{2.58}$$

where $\imath\mathbf{b}$ is a self-adjoint matrix. Furthermore, it is possible to characterize the tangent space $T_\rho\mathcal{S}_\sigma$ to $\rho \in \mathcal{S}_\sigma$ by means of the fundamental vector fields of the unitary action according to the following proposition (see, for example, [2] page 331):

Proposition 5. *Let G be a Lie group, and let $\alpha\colon G \times M \to M$ be a smooth action of G on the differential manifold M. Then, the tangent space $T_m Orb(m_0)$ is:*

$$T_m Orb(m_0) = \{X_{\mathbf{A}}(m)\,, \mathbf{A} \in \mathfrak{g}\}\ , \tag{2.59}$$

where $X_{\mathbf{A}}$ is the fundamental vector field defined in 6.

Proposition 5 tells us that the fundamental vector fields on $\mathcal{S}_k$ are tangent to the unitary orbits $\mathcal{S}_\sigma \subset \mathcal{S}_k$. Consequently, every fundamental vector field $X_{\mathbf{A}}$ on $\mathcal{S}_k$ defines a vector field on $\mathcal{S}_\sigma$ (its restriction to $\mathcal{S}_\sigma$), and all these vector field are precisely the fundamental vector fields of the canonical transitive action of $SU(\mathfrak{A}_n)$ on $\mathcal{S}_\sigma$. For the sake of notational simplicity, we shall write $X_{\mathbf{A}}$ for the fundamental vector fields of the action of $SU(\mathfrak{A}_n)$ on $\mathcal{S}_\sigma$. Choosing a basis $\{\tau_j\}$ in $\mathfrak{su}(\mathfrak{A}_n)$, we obtain the set $\{X_j\}$ of fundamental vector fields associated with $\{\tau_j\}$, and this set provides an (in general) overcomplete basis for the module of all vector fields on $\mathcal{S}_\sigma$.

2.2.1 The Kähler structure

The manifolds of isospectral states have some interesting additional geometric structures. The emergence of such geometrical structures is connected with the fact that every manifold $\mathcal{S}_\sigma$ is diffeomorphic to a coadjoint orbit of the special unitary group $SU(\mathfrak{A}_n) = SU(n)$. Indeed, the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ is the Lie algebra of traceless anti-self-adjoint matrices with the matrix commutator acting as Lie product, and we may identify the dual $\mathfrak{su}^*(n)$ of $\mathfrak{su}(n)$ with the vector space of traceless self-adjoint matrices by using the trace functional $Tr(\cdot)$ as follows:

$$\xi(\mathbf{A}) = Tr\left(\bar{\xi}\,\mathbf{a}\right)\ , \tag{2.60}$$

where $\xi \in \mathfrak{su}^*(n)$ and $\imath\mathbf{a} = \mathbf{A} \in \mathfrak{su}(n)$. Then, the coadoint action of $SU(n)$ on $\mathfrak{su}^*(n)$ reads:

$$(\mathbf{U}\,, \bar{\xi}) \mapsto \bar{\xi}_{\mathbf{U}} = \mathbf{U}\,\bar{\xi}\,\mathbf{U}^\dagger\,. \tag{2.61}$$

When the self-adjoint matrix $\bar{\xi}$ is actually the density matrix $\bar{\rho}$ associated with the quantum state ρ, it is clear that the coadjoint action coincides with the α action of $SU(\mathfrak{A}_n) = SU(n)$ on the density matrices.

The one-to-one correspondence between manifolds of isospectral states and coadjoint orbits of $SU(\mathfrak{A}_n)$ allows us to exploit the theory of coadjoint orbits (see for instance [86] chapter 14) in this quantum mechanical context. For example, it is well known that coadjoint orbits of a Lie group G are always symplectic manifolds, that is, they always admit a symplectic form (nondegenerate closed differential two-form). This symplectic form is known as the Konstant-Kirillov-Souriau symplectic form, and it is invariant with respect to the natural action of G on its coadjoint orbit (see [86] page 455). Furthermore, in our quantum case, it is possible to prove that every coadjoint orbit $\mathcal{S}_\sigma$ (except for the degenerate one) of $SU(\mathfrak{A}_n)$ is a Kähler manifold (see [58]), that is, apart from the Konstant-Kirillov-Souriau symplectic form ω^σ, there are a Riemannian metric tensor g^σ and a complex structure J^σ satisfying a compatibility condition. Specifically, the following theorem can be proved to hold (see [64] theorem 7, theorem 8 and the paragraph just after theorem 8):

Theorem 4. *Let $\sigma \neq [\frac{1}{n}(1,...,1)]$, then, the manifold $\mathcal{S}_\sigma \subset \mathcal{S}_k$ of isospectral quantum states is a Kähler manifold. This means that there are a symplectic form ω^σ, a Riemannian metric tensor g^σ and a complex structure*[3] *such that:*

$$g^\sigma\left(X\,,Y\right)=\omega^\sigma\left(X\,,J^\sigma(Y)\right)\,,\quad \forall\ X,Y\in\mathfrak{X}(\mathcal{S}_\sigma). \tag{2.62}$$

All of these tensors are invariant with respect to the canonical action of $SU(\mathfrak{A}_n)$ on the coadjoint orbit, that is:

$$\phi_{\mathbf{U}}^*\omega^\sigma=0\,,\quad \phi_{\mathbf{U}}^*g^\sigma=0\,,\quad \phi_{\mathbf{U}}^*J^\sigma=0\,, \tag{2.63}$$

where $\phi_{\mathbf{U}}$ is the diffeomorphism of $\mathcal{S}_\sigma$ associated with the action of $\mathbf{U}\in SU(\mathfrak{A}_n)$. From the infinitesimal point of view, denoting with $X_{\mathbf{A}}$ the generic fundamental vector field of the action of $SU(\mathfrak{A}_n)$ on $\mathcal{S}_\sigma$, we have:

$$\mathcal{L}_{X_{\mathbf{A}}}\left(\omega^\sigma\right)=0\,,\quad \mathcal{L}_{X_{\mathbf{A}}}\left(g^\sigma\right)=0\,,\quad \mathcal{L}_{X_{\mathbf{A}}}\left(J^\sigma\right)=0\,, \tag{2.64}$$

where $\mathcal{L}$ denotes the Lie derivative.

The symplectic form on every $\mathcal{S}_\sigma$ is given by:

$$(\omega^\sigma(X\,,Y))(\rho)=\omega^\sigma_\rho\left(\imath[\mathbf{a}\,,\bar{\rho}]\,,\imath[\mathbf{b}\,,\bar{\rho}]\right):=\imath\;Tr\left([\mathbf{a}\,,\bar{\rho}]\,\mathbf{b}\right)=\imath\;Tr\left(\bar{\rho}\,[\mathbf{b}\,,\mathbf{a}]\right)\,, \tag{2.65}$$

where $\bar{\rho}$ is the density matrix associated with ρ, and $X(\rho)=\imath[\mathbf{a}\,,\bar{\rho}\,]$, $Y(\rho)=\imath[\mathbf{b}\,,\bar{\rho}\,]$. Let us denote with $\{|j\rangle\}_{j=1,\dots,n}$ the basis of eigenvectors of the density matrix $\bar{\rho}$ associated with $\rho\in\mathcal{S}_\sigma$, and let us order the basis elements so that $\lambda^1\geq\lambda^2\geq\cdots\geq\lambda^n$, where λ^j is the j-th eigenvalue of $\bar{\rho}$. Define[4] *$\mathbf{M}_{kl}:=|k\rangle\langle l|$ with $k<l$, and $\mathbf{D}_j:=|j\rangle\langle j|$. Then, the complex structure on every $\mathcal{S}_\sigma$ is given by:*

[3] A complex structure J on a manifold M is a $(1,1)$ tensor field such that $J\circ J=\mathrm{Id}\;J^\sigma$, and such that its Nijenhuis tensor vanishes. For more details, we refer to [26, 88, 91].

[4] This may be seen as a sort of generalization of the canonical basis in $\mathfrak{A}_n$ defined in chapter 1.

$$
\left(J^{\sigma}(X)\right)(\rho) = J^{\sigma}_{\rho}(\imath[\mathbf{a}\,,\bar{\rho}]) := \sum_{k<l}\,\left(\lambda^{k}-\lambda^{l}\right)\,\left(a^{kl}\mathbf{M}_{kl} + \bar{a}^{kl}\mathbf{M}_{lk}\right)\,, \tag{2.66}
$$

where $X(\rho) = \imath[\mathbf{a}\,,\bar{\rho}]$, and $\mathbf{a} = a^{kl}\mathbf{M}_{kl} + \bar{a}^{kl}\mathbf{M}_{lk} + a^{j}\mathbf{D}_{j}$. Accordingly, the metric tensor on every $\mathcal{S}_{\sigma}$ is given by:

$$
\begin{aligned}
(g^{\sigma}(X\,,Y))(\rho) = g^{\sigma}_{\rho}\left(\imath[\mathbf{a}\,,\bar{\rho}]\,,\imath[\mathbf{b}\,,\bar{\rho}]\right) = \omega^{\sigma}_{\rho}\left(\imath[\mathbf{a}\,,\bar{\rho}]\,,J^{\sigma}_{\rho}\left(\imath[\mathbf{b}\,,\bar{\rho}]\right)\right) = \\
= \sum_{k<l}\,\left(\lambda^{k}-\lambda^{l}\right)\,\left(a^{kl}\bar{b}^{kl} + \bar{a}^{kl}b^{kl}\right)\,,
\end{aligned} \tag{2.67}
$$

where $X(\rho) = \imath[\mathbf{a}\,,\bar{\rho}]$, $\mathbf{a} = a^{kl}\mathbf{M}_{kl} + \bar{a}^{kl}\mathbf{M}_{lk} + a^{j}\mathbf{D}_{j}$, $Y(\rho) = \imath[\mathbf{b}\,,\bar{\rho}]$, and $\mathbf{b} = b^{kl}\mathbf{M}_{kl} + \bar{b}^{kl}\mathbf{M}_{lk} + b^{j}\mathbf{D}_{j}$.

Remark 8. *An explicit computation shows that, when we consider the space $\mathcal{S}_{\sigma} = \mathcal{S}_{1}$ of pure quantum states, the complex structure J^{σ} is:*

$$
\left(J^{\sigma}(X)\right)(\rho) = [[\mathbf{a}\,,\bar{\rho}]\,,\bar{\rho}] = \{\mathbf{a}\,,\bar{\rho}\} - 2\bar{\rho}\,\mathbf{a}\,\bar{\rho}\,, \tag{2.68}
$$

where $\bar{\rho}$ is the density matrix associated with ρ, $\{\cdot\,,\cdot\}$ is the matrix anticommutator, and $X(\rho) = \imath[\mathbf{a}\,,\bar{\rho}\,]$. Consequently, the metric tensor g^{σ} is:

$$
\left(g^{\sigma}\left(X\,,Y\right)\right)(\rho) = Tr\left([\mathbf{a}\,,\bar{\rho}]\,[\mathbf{b}\,,\bar{\rho}]\right)\,, \tag{2.69}
$$

where $\bar{\rho}$ is the density matrix associated with ρ, and $X(\rho) = \imath[\mathbf{a}\,,\bar{\rho}\,]$, $Y(\rho) = \imath[\mathbf{b}\,,\bar{\rho}\,]$. Clearly, g^{σ} is proportional to the Fubini-Study metric tensor because it is invariant with respect to the canonical action of $SU(\mathfrak{A}_{n})$ (see chapter 4 in [26], and [58]).

2.2.2 Hamiltonian and gradient vector fields

Here we will exploit the Kähler structure of $\mathcal{S}_{\sigma}$ to introduce the notions of Hamiltonian and gradient vector fields associated with functions on $\mathcal{S}_{\sigma}$. Consider $f \in \mathcal{F}(\mathcal{S}_{\sigma})$, then, we define its associated Hamiltonian vector field X_{f} and its associated gradient vector field Y_{f} to be, respectively, the vector fields given by:

$$
X_{f} := (\omega^{\sigma})^{-1}\left(\mathrm{d}f\,,\cdot\right)\,, \tag{2.70}
$$

$$
Y_{f} := (g^{\sigma})^{-1}\left(\mathrm{d}f\,,\cdot\right)\,, \tag{2.71}
$$

where ω^{σ} and g^{σ} are, respectively, the canonical symplectic structure and canonical metric tensor on $\mathcal{S}_{\sigma}$.

Associated with every $\mathbf{a} \in \mathfrak{O}_n$ there is a function $e_{\mathbf{a}} \colon \mathfrak{O}_n^* \to \mathbb{R}$:

$$e_{\mathbf{a}}(\xi) := \xi(\mathbf{a}) = Tr\left(\bar{\xi}\,\mathbf{a}\right) . \tag{2.72}$$

We refer to $e_{\mathbf{a}}$ as the expectation value function associated with $\mathbf{a}$. Since $\mathcal{S}_\sigma$ is a subset of $\mathfrak{O}_n^*$, we may consider the canonical immersion $i \colon \mathcal{S}_\sigma \to \mathfrak{O}_n^*$, and take the pullback $i^*(e_{\mathbf{a}})$ of the expectation value function $e_{\mathbf{a}}$ associated with $\mathbf{a} \in \mathfrak{O}_n$. In the following, for the sake of notational simplicity, we shall write $i^*(e_{\mathbf{a}}) \equiv e_{\mathbf{a}}$. Expectation value functions are intimately connected to the action of $SU(\mathfrak{A}_n)$ on $\mathcal{S}_\sigma$, indeed, we can prove the following:

Proposition 6. *Let $\mathbf{a} \in \mathfrak{O}_n$ so that $\mathbf{A} := \imath\mathbf{a}$ is in the Lie algebra of $SU(\mathfrak{A}_n)$, and consider the fundamental vector field $X_{\mathbf{A}}$ on $\mathcal{S}_\sigma$ associated with the element $\mathbf{A}$ according to definition 6. Then, for every expectation value function $e_{\mathbf{b}}$ associated with $\mathbf{b} \in \mathfrak{O}_n$ as defined above, the following equality holds:*

$$\mathcal{L}_{X_{\mathbf{A}}}\left(e_{\mathbf{b}}\right) = e_{\imath[\mathbf{b}\,,\mathbf{a}]} \,, \tag{2.73}$$

from which it follows that $X_{\mathbf{A}}$ is the Hamiltonian vector field associated with $-e_{\mathbf{a}}$ by means of the symplectic structure ω^σ, that is:

$$\omega^\sigma\left(X_{\mathbf{A}}\,,\cdot\right) = -\mathrm{d}e_{\mathbf{a}} \,. \tag{2.74}$$

Then, every cotangent vector in $T^*_\rho\mathcal{S}_\sigma$ may be represented as $\mathrm{d}e_{\mathbf{a}}(\rho)$ for some $\mathbf{a} \in \mathfrak{O}_n$.

Proof. *By the very definition of Lie derivative of a function with respect to a vector field we have:*

$$\begin{aligned}\left(\mathcal{L}_{X_{\mathbf{A}}}\left(e_{\mathbf{b}}\right)\right)(\rho) = \left(X_{\mathbf{A}}\,e_{\mathbf{b}}\right)(\rho) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e_{\mathbf{b}}\left(\alpha_{\exp(t\mathbf{A})}(\rho)\right)\right)\right)_{t=0} = \\ = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(Tr(\exp(\imath t\mathbf{a})\,\bar{\rho}\,\exp(-\imath t\mathbf{a})\,\mathbf{b})\right)\right)_{t=0} = \\ = Tr(\imath\,[\mathbf{a}\,,\bar{\rho}]\;\mathbf{b}) = Tr(\bar{\rho}\,\imath\,[\mathbf{b}\,,\mathbf{a}]) = e_{\imath[\mathbf{b}\,,\mathbf{a}]}(\rho)\,,\end{aligned} \tag{2.75}$$

and the first assertion is proved. The fact that $X_{\mathbf{A}}$ is the Hamiltonian vector field associated with $e_{\mathbf{a}}$, where $\mathbf{A} = \imath\mathbf{a}$ with $\mathbf{a} \in \mathfrak{O}_n$, can be seen as follows. Because of proposition 5, every tangent vector $v_\rho \in T_\rho\mathcal{S}_\sigma$ can be represented by $X_{\mathbf{B}}(\rho) = \imath[\mathbf{b}\,,\bar{\rho}]$ for some $\mathbf{B} = \imath\mathbf{b}$ with $\mathbf{b} \in \mathfrak{O}_n$. Consequently, according to equations (2.75) *and* (2.65)*, the cotangent vector $\mathrm{d}e_{\mathbf{a}}(\rho) \in T^*_\rho\mathcal{S}_\sigma$ acts on v_ρ as follows:*

$$\left(\mathrm{d}e_{\mathbf{a}}(\rho)\right)\left(X_{\mathbf{B}}(\rho)\right) = Tr(\bar{\rho}\,\imath\,[\mathbf{a}\,,\mathbf{b}]) = -\omega^\sigma_\rho(X_{\mathbf{A}}(\rho)\,,X_{\mathbf{B}}(\rho))\,. \tag{2.76}$$

Since $\mathbf{B} = \imath\mathbf{b}$ *is arbitrary, we conclude that:*

$$\omega^{\sigma}\left(X_{\mathbf{A}}\,,\cdot\right) = -\mathrm{d}e_{\mathbf{a}}\,, \tag{2.77}$$

which means that $X_{\mathbf{A}}$ *is the Hamiltonian vector field associated with the expectation value function* $-e_{\mathbf{a}}$*, where* $\mathbf{A} = \imath\mathbf{a}$ *with* $\mathbf{a} \in \mathfrak{O}_n$*. Clearly, the symplectic structure* ω^{σ}_{ρ} *provides us with an isomorphism between the tangent space* $T_{\rho}\mathcal{S}_{\sigma}$ *and the cotangent space* $T^{*}_{\rho}\mathcal{S}_{\sigma}$ *for every* $\rho \in \mathcal{S}_{\sigma}$*, therefore, recalling proposition 5, it follows from*

$$\omega^{\sigma}_{\rho}(X_{\mathbf{A}}(\rho)\,,\cdot) = -\mathrm{d}e_{\mathbf{a}}(\rho)$$

that every cotangent vector in $T^{*}_{\rho}\mathcal{S}_{\sigma}$ *may be represented as* $\mathrm{d}e_{\mathbf{a}}(\rho)$ *for some* $\mathbf{a} \in \mathfrak{O}_n$*.*

According to proposition 6, we will write $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$ to denote, respectively, the Hamiltonian and gradient vector field associated with the expectation value function $e_{\mathbf{a}}$. Note that $X_{\mathbf{a}}$ is equal to the fundamental vector field $X_{-\mathbf{A}}$ associated with $-\mathbf{A} = -\imath\mathbf{a}$. Analogously, the gradient vector field $Y_{\mathbf{a}} = J(X_{\mathbf{a}})$ is equal to the gradient vector field $-Y_{\mathbf{A}} = -J(X_{\mathbf{A}})$. As said at the end of section 2.1, the flow of Hamiltonian vector fields may be dynamically interpreted as the unitary evolution of the quantum system. These dynamical evolutions describe the behaviour of a closed quantum system, that is, a system which is isolated from its environment. Consequently, from the point of view of the dynamics of closed quantum systems, Hamiltonian vector fields on the manifolds of isospectral states provide the correct geometrical framework for a complete treatment of the subject. In the case of pure quantum states, this geometrical picture can be generalized to the infinite-dimensional case according to the works [48, 49]. On the other hand, the necessity to describe the dynamics of open quantum systems will bring us out of the manifolds of isospectral states. As it will become clear in chapter 3, the correct way to achieve a geometrization of such dynamical processes is to consider the geometry of the space $\mathcal{S}$ of quantum states developed here from what may be called an "extrinsic point of view". Specifically, we will exploit the fact that $\mathcal{S}$ may be seen as a convex body sitting in an affine hyperplane in $\mathfrak{O}_n^{*}$, and see how the geometrical structures of $\mathcal{S}$ fit into the geometrical structures available on this affine hyperplane in such a way that the open quantum evolutions governed by the GKLS master equation can be described by an affine vector field Γ.

We will now compute the commutator among the vector fields $X_{\mathbf{A}}, Y_{\mathbf{B}}$ with $\mathbf{A}, \mathbf{B} \in \mathfrak{su}(\mathfrak{A}_n)$. In order to do so, we must recall some important properties of the complex structure J^{σ} on $\mathcal{S}_{\sigma}$. First of all, we recall the definition of the Nijenhuis tensor N_T associated with a $(1,1)$ tensor field T on a manifold M (see definition 2.10, and equation 2.4.26 in [88]):

$$N_T(X,Y) = \left(\mathcal{L}_{T(X)}(T)\right)(Y) - (T \circ \mathcal{L}_X(T))(Y)\,, \tag{2.78}$$

where X, Y are arbitrary vector fields on M. A fundamental result in the theory of complex manifold is that (see [91]) the $(1,1)$ tensor field defining the complex structure of a complex manifold must have vanishing Nijenhuis tensor. This means that, on every manifold of isotropic quantum states, the complex structure J^σ is such that $N_{J^\sigma} = 0$, which means:

$$\left(\mathcal{L}_{J^\sigma(X)}(J^\sigma)\right)(Y) = (J^\sigma \circ \mathcal{L}_X(J^\sigma))(Y)\,, \tag{2.79}$$

where X, Y are arbitrary vector fields on $\mathcal{S}_\sigma$. In particular, if we consider the Hamiltonian vector field $X_{\mathbf{A}}$, theorem 4 assures us that $\mathcal{L}_{X_{\mathbf{A}}} J^\sigma = 0$, and thus:

$$\left(\mathcal{L}_{J^\sigma(X_{\mathbf{A}})}(J^\sigma)\right)(Y) = 0 \tag{2.80}$$

for every vector field Y on $\mathcal{S}_\sigma$. Eventually, we can prove the following:

Proposition 7. *Let* $\mathbf{A}, \mathbf{B}$ *be generic elements in the Lie algebra* $\mathfrak{su}(\mathfrak{A}_n)$ *of* $SU(\mathfrak{A}_n)$*, then, the following commutation relations among Hamiltonian and gradient vector fields hold:*

$$[X_{\mathbf{A}}\,, X_{\mathbf{B}}] = -X_{[\mathbf{A}\,,\mathbf{B}]}\,, \qquad [X_{\mathbf{A}}\,, Y_{\mathbf{B}}] = -Y_{[\mathbf{A}\,,\mathbf{B}]}\,, \qquad [Y_{\mathbf{A}}\,, Y_{\mathbf{B}}] = \ . \tag{2.81}$$

Proof. *The first commutator follows directly from proposition 4. Regarding the second commutator, we recall that* $Y_{\mathbf{A}} = J^\sigma(X_{\mathbf{A}})$*, and that theorem 4 assures us that* $\mathcal{L}_{X_{\mathbf{A}}} J^\sigma = 0$*, so that:*

$$\begin{aligned}
[X_{\mathbf{A}}\,, Y_{\mathbf{B}}] =& \mathcal{L}_{X_{\mathbf{A}}}\left(J^\sigma(X_{\mathbf{B}})\right) = \\
=& \left(\mathcal{L}_{X_{\mathbf{A}}} J^\sigma\right)(X_{\mathbf{B}}) + J^\sigma\left(\mathcal{L}_{X_{\mathbf{A}}} X_{\mathbf{B}}\right) = \\
=& J^\sigma\left([X_{\mathbf{A}}\,, X_{\mathbf{B}}]\right) = -Y_{[\mathbf{A}\,,\mathbf{B}]}
\end{aligned} \tag{2.82}$$

as claimed. Finally, using equation (2.80) *together with the fact that* $J^\sigma \circ J^\sigma = -\mathrm{Id}$ *because it is a complex structure, we obtain:*

$$\begin{aligned}
[Y_{\mathbf{A}}\,, Y_{\mathbf{B}}] =& \mathcal{L}_{J^\sigma(X_{\mathbf{A}})}\left(J^\sigma(X_{\mathbf{B}})\right) = \\
=& \left(\mathcal{L}_{J^\sigma(X_{\mathbf{A}})}(J^\sigma)\right) X_{\mathbf{B}} + J^\sigma\left(\mathcal{L}_{J^\sigma(X_{\mathbf{A}})} X_{\mathbf{B}}\right) = \\
=& J^\sigma\left([Y_{\mathbf{A}}\,, X_{\mathbf{B}}]\right) = X_{[\mathbf{A}\,,\mathbf{B}]}
\end{aligned} \tag{2.83}$$

as claimed.

From the commutator just computed we obtain that the family of Hamiltonian and gradient vector fields associated with elements in the Lie algebra $\mathfrak{su}(\mathfrak{A}_n)$ of the Lie group $SU(\mathfrak{A}_n)$ provide a realization of the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ which is the complexification of the Lie algebra $\mathfrak{su}(\mathfrak{A}_n)$. Since every manifold of isospectral quantum states is a compact manifold, Hamiltonian and gradient vector fields are complete, and thus the realization of $\mathfrak{sl}(\mathfrak{A}_n)$ "integrates" to an action of $SL(\mathfrak{A}_n)$ on every manifold of isospectral quantum states. Clearly, this action of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_\sigma \subset \mathcal{S}_k$ does not coincide with the action α^k of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$ introduced in the previous section because α^k does not preserve the spectrum of quantum states.

Chapter 3

Geometry of the GKLS equation

In this chapter we will review the ideas sketched in [44] and developed then in [42]. The central topic will be the dynamical evolutions of finite-level quantum systems according to the GKLS master equation (see [38, 63, 82]). The main result will be a geometrical formulation of these kind of dynamics in terms of vector fields (dynamical systems) on a suitable affine manifold. The perspective adopted will be that of relating the geometrical structures of the space of quantum states $\mathcal{S}$ introduced in chapter 2 with the geometry of $\mathcal{S}$ seen as a convex body in the affine ambient space $\mathfrak{T}_n^1$ defined below. The necessity of considering the space of quantum states as sitting in some ambient space comes from the dynamics of (genuinely) open quantum systems. Indeed, the description of dissipation phenomena leads to consider quantum evolutions that are transversal to the manifolds of quantum states with fixed rank introduced in chapter 2 in the sense that the rank of a quantum states may change in time (see proposition 3 in [23], or proposition 4 in [24]). We can manage to overcome this difficulty by enlarging our perspective looking at $\mathcal{S}$ as living in some suitable ambient space which is a differential manifold. From the mathematical point of view this is possible because $\mathcal{S}$ is a convex body, and thus, at least in finite-dimensions, there are always a vector space, and an affine subspace of it, of which $\mathcal{S}$ is a convex subset. In the specific case of the space of quantum states of a quantum system with C^*-algebra $\mathfrak{A}_n$, the vector space is the dual $\mathfrak{O}_n^*$ of the space of observables $\mathfrak{O}_n$ and the affine subspace $\mathfrak{T}_n^1 \subset \mathfrak{O}_n^*$ is the space of self-adjoint linear functionals ρ such that $\rho(\mathbb{I}) = 1$, where $\mathbb{I} \in \mathfrak{A}_n$ is the identity operator.

If $\bar{\rho}$ denotes the density matrix associated with the quantum state $\rho \in \mathcal{S} \subset \mathfrak{O}_n^*$, then, the generator $\mathbf{L}$ of the GKLS equation may be written as a linear map on density matrices:

$$\mathbf{L}(\bar{\rho}) = -\imath\,[\mathbf{H}\,,\bar{\rho}] - \frac{1}{2}\sum_{j=1}^{N}\left\{\mathbf{v}_j^{\dagger}\mathbf{v}_j\,,\bar{\rho}\right\} + \sum_{j=1}^{N}\mathbf{v}_j\,\bar{\rho}\,\mathbf{v}_j^{\dagger} \tag{3.1}$$

From the mathematical point of view, equation (3.1) has a clear algebraic flavour. This follows from the fact that the most used mathematical tools in quantum mechanics are algebraic. However, in the last decades, something changed, and the geometrical picture of quantum mechanics has started to grow [16, 26, 32, 36, 48, 58]. In this picture, a rich geometrical structure associated with finite-level quantum systems naturally emerges. We have seen in chapter 2 that the space $\mathcal{S}_k$ of quantum state with rank equal to k is a homogeneous space for an action of the Lie group $SL(\mathfrak{A}_n)$ (see also [64, 65]). Furthermore, denoting with σ the spectrum of the density matrix $\bar{\rho}$ associated with the quantum state ρ, the set $\mathcal{S}_\sigma$ of all quantum states with the same spectrum σ turns out to be a Kähler manifold. In particular, the space of pure states is the complex projective space[1] $P(\mathcal{H})$, which is a well-known Kähler manifold. Unitary evolutions are realized by means of Hamiltonian vector fields on the manifold $\mathcal{S}_\sigma$ of isospectral states. However, open quantum dynamics may change both the spectrum and the rank of a quantum state, and thus the geometrical description of such dynamical processes can not be accomplished resorting to the differential structure of $\mathcal{S}_\sigma$ or $\mathcal{S}_k$. We must be able to describe the motion across orbits of quantum states of different rank.

At this purpose, we note that the dynamical trajectories of quantum states under open quantum dynamics lies entirely in the set $\mathfrak{T}_n^1 \subset \mathfrak{O}_n^*$ given by self-adjoint linear functional ρ such that $\rho(\mathbb{I}) = 1$, where $\mathbb{I} \in \mathfrak{A}_n$ is the identity operator. Consequently, we will rely on the differential structure of $\mathfrak{T}_n^1$ in order to describe open quantum dynamics by means of a vector field $\Gamma \in \mathfrak{X}(\mathfrak{T}_n^1)$ which will turn out to be a fine-tuned combination of geometrically meaningful vector fields. Specifically, we will get a decomposition of Γ as (compare with the three terms in the r.h.s. of equation (3.1)):

$$\Gamma = X + Y + Z\,, \tag{3.2}$$

where X is a Hamiltonian vector field the flow of which preserves the spectrum of quantum states, Y is a gradient-like vector field whose flow changes the spectrum but preserves the rank of quantum states, and Z is a vector field the flow of which is responsible for the change in rank of quantum states. Interestingly, X will turn out to be an affine vector field which need not be correlated with Y and Z. On the other hand, Y and Z will turn out to be highly related. They will be nonaffine vector fields such that their sum is an affine vector field.

[1]The set $P(\mathcal{H})$ is a Kähler manifold even in the infinite-dimensional case [47].

To accomplish this task, we will make use of the Lie-Jordan algebra structure on the space of linear functions on the dual $\mathfrak{O}_n^*$ of the space of observables $\mathfrak{O}_n$, and exploit a symmetric and an anti-symmetric product structure on the algebra $\mathcal{F}(\mathfrak{T}_n^1)$ of smooth functions on $\mathfrak{T}_n^1$. These products allow us to define, respectively, the gradient-like and the Hamiltonian vector field by means of the affine functions on $\mathcal{F}(\mathfrak{T}_n^1)$ associated with elements of $\mathfrak{O}_n$. By construction, these vector fields will be precisely the vector fields generating the nonlinear action of $SL(\mathfrak{A}_n)$ of which all spaces $\mathcal{S}_k$ are homogeneous spaces (see chapter 2). Consequently, the trajectory of a quantum state $\rho \in \mathcal{S}_k$ by means of the flow of these vector fields will be completely contained in $\mathcal{S}_k$. The vector field Z will be constructed with the help of an affine map on $\mathfrak{T}_n^1$.

Once we have this geometrical formulation of open quantum dynamics, some interesting possibile applications arise. We believe that the geometrical reformulation of open quantum dynamics could provide some new insights on the mathematical structure of open quantum systems, as well as the possibility of replenish the arsenal of useful mathematical tools bringing in elements from the classical theory of dynamical systems. Indeed, the mathematical results of the theory of dynamical systems, which are mainly related to the geometrical structure of classical mechanics, become immediately available in the quantum case because of the common mathematical language in which classical physics and open quantum dynamics are here formulated, namely, using vector fields on differential manifolds. This interplay between the mathematical methods of classical physics and the quantum theory could help to better understand the structure underlying quantum physics, and to provide some useful tools in the computation of specific physical situations. Of course, this should not lead to think that classical physics should drive our understanding of quantum physics. It is simply a way to point out how casting physical problems pertaining to the quantum domain into a mathematical formalism which is common to classical physics leads us to benefit of all the mathematical results available in that formalism. Clearly, the physical interpretation of these results must be consistent with the quantum nature of the system at hand. A similar attitude, but in the opposite direction, was pursued by Koopman [77] who reformulated the dynamical problem of classical physics in the mathematical formalism of Hilbert spaces characteristic of quantum mechanics.

At the end of the chapter, the geometrical formalism presented is applied to the case of quantum Poisson semigroups, of quantum Gaussian semigroups, and quantum random unitary semigroups (see [11, 78, 82]. It is shown that every such dynamics admits an accumulating set, that is, the dynamical evolution of every quantum state ρ tends to a non-equilibrium steady state ρ_∞. Concrete examples show that ρ_∞ may or may not depend on the initial state ρ.

3.1 Symmetric and skew-symmetric tensors on $\mathfrak{O}_n^*$

In this section we will exploit the Lie-Jordan algebra structure of the space $\mathfrak{O}_n$ (see [76, 5, 4]) of observables in order to introduce symmetric and skew-symmetric tensor fields on $\mathfrak{O}_n^*$. By means of these tensor fields, we will introduce the notions of gradient-like and Hamiltonian vector fields associated with functions on $\mathfrak{O}_n^*$. In particular, we will see that the gradient-like and Hamiltonian vector fields associated with linear functions close on a realization of the Lie algebra $\mathfrak{gl}(\mathfrak{A}_n)$ that integrates to an action of the Lie group $GL(\mathfrak{A}_n)$ on $\mathfrak{O}_n^*$.

We start with the following definition:

Definition 7 (Lie-Jordan algebra)**.** *Let* $(A\,,\odot)$ *denote a real Jordan algebra, and* $(A\,,[[\,,]])$ *a real Lie algebra. Then* $(A\,,\odot\,,[[\,,]])$ *is called a Lie-Jordan algebra iff the following conditions hold:*

- $[[\mathbf{a}\,;\cdot]]$ *is a derivation of* $\odot$*:*

$$[[\mathbf{a}\,,\mathbf{b}\odot\mathbf{c}]]] = [[[\mathbf{a}\,,\mathbf{b}]]\odot\mathbf{c} + \mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]]\;; \tag{3.3}$$

- *the associator of* $\odot$ *is proportional to the Lie product:*

$$(\mathbf{a}\odot\mathbf{b})\odot\mathbf{c} - \mathbf{a}\odot(\mathbf{b}\odot\mathbf{c}) = [[\mathbf{b}\,,[[\mathbf{c}\,,\mathbf{a}]]\,]]\;. \tag{3.4}$$

The space $\mathfrak{O}_n$ of self-adjoint elements in $\mathfrak{A}_n$ is naturally endowed with a Jordan product $\odot$ and a Lie product $[[\,,]]$ given by:

$$\mathbf{a}\odot\mathbf{b} := \frac{(\mathbf{ab}+\mathbf{ba})}{2}\,, \tag{3.5}$$

$$[[\mathbf{a}\,,\mathbf{b}]] := \frac{\imath(\mathbf{ab}-\mathbf{ba})}{2}\,. \tag{3.6}$$

These product structures make $\mathfrak{O}_n$ a Lie-Jordan algebra according to the previous definition.

Remark 9. *In the infinite-dimensional case (see [28, 29, 39, 54, 56, 67]) we must pay attention to all those aspects connected with the Banach space structure of the* C^**-algebra* $\mathcal{A}$ *of the system. Indeed,* $\mathcal{A}$ *carries a norm* $||\cdot||$ *which is compatible with its algebra structure and its involution, so that the space* $\mathfrak{O}$ *of observables becomes a Banach space (see [3, 53, 101]). In this context, the relevant object is a Lie-Jordan-Banach algebra (see [5, 4]), that is, a Lie-Jordan algebra* $(A\,,\odot\,,[[\,,]])$ *which is itself a Banach space with a norm* $||\cdot||$ *such that:*

- $[[\,,]]$ *is continuous;*
- $||\mathbf{a} \odot \mathbf{b}|| \leq ||\mathbf{a}||\,||\mathbf{b}||$ *for all* $\mathbf{a}, \mathbf{b} \in A$*;*
- $||\mathbf{a} \odot \mathbf{a}|| = ||\mathbf{a}||^2$ *for all* $\mathbf{a} \in A$*;*
- $||\mathbf{a} \odot \mathbf{a}|| \leq ||\mathbf{a} \odot \mathbf{a} + \mathbf{b} \odot \mathbf{b}||$ *for all* $\mathbf{a}, \mathbf{b} \in A$*.*

What we want to do now is to use the Lie-Jordan algebra structure of $\mathfrak{O}_n \subset \mathfrak{A}_n$, together with the duality relation between $\mathfrak{O}_n \subset \mathfrak{A}_n$ and $\mathfrak{O}_n^* \subset \mathfrak{A}_n^*$, in order to build up a symmetric and a skewsymmetric contravariant tensor field on the linear manifold $\mathfrak{O}_n^*$. At this purpose, we note that every element $\mathbf{a} \in \mathfrak{O}_n$ may be represented as a linear function $f_{\mathbf{a}}$ on $\mathfrak{O}_n^*$ as follows:

$$f_{\mathbf{a}}(\xi) := \xi(\mathbf{a})\,. \tag{3.7}$$

Conversely[2], any linear function on $\mathfrak{O}_n^*$ is associated with an element $\mathbf{a} \in \mathfrak{O}_n$. Defining the following product structures among linear functions:

$$f_{\mathbf{a}} \odot f_{\mathbf{b}} := f_{\mathbf{a}\odot\mathbf{b}}\,, \tag{3.8}$$

$$[[f_{\mathbf{a}}\,, f_{\mathbf{b}}]] := f_{[[\mathbf{a}\,,\mathbf{b}]]}\,, \tag{3.9}$$

a direct computation shows that:

Proposition 8. *The set* $(\mathcal{F}_l(\mathfrak{O}^*)\,, \odot\,, [[\,,]])$*, where* $\mathcal{F}_l(\mathfrak{O}^*) \subset \mathcal{F}(\mathfrak{O}^*)$ *is the space of real linear functions on* $\mathfrak{O}^*$*, and* $\odot$*,* $[[\,,]]$ *are given by (3.8) and (3.9), provides a realization of the Lie-Jordan algebra* $(\mathfrak{O}\,, \odot\,, [[\,,]])$*.*

We can now define two contravariant $(2,0)$ tensor fields on $\mathfrak{O}_n^*$ using the product structures $\odot$ and $[[\,,]]$ just introduced as follows. We first inroduce an orthornormal basis[3] $\{\mathbf{e}^\mu\}_{\mu=0,\dots,n^2-1}$ of $\mathfrak{O}_n$ having $\mathbf{e}^0 = \frac{\mathbb{I}}{\sqrt{n}}$. Next, we denote with f_μ the linear function associated with $\mathbf{e}^\mu$ by means of equation (3.7). Since the differentials of the linear functions generate the cotangent space at each point of $\mathfrak{O}_n^*$, we have that every differential one-form θ on $\mathfrak{O}_n^*$ can be written as:

$$\theta = \theta^\mu \,\mathrm{d}f_\mu\,. \tag{3.10}$$

[2]This is always true in finite dimensions where every linear space is isomorphic to its dual and double dual, while it is no longer true for general infinite dimensional spaces which are not reflexive.

[3]Throughout the rest of the paper, greek indexes will run from 0 to (n^2-1), while latin indexes will run from 1 to (n^2-1).

Consequently, for every differential one-form θ_1, θ_2 on $\mathfrak{O}_n^*$ we define:

$$G(\theta_1\,,\theta_2) := \theta_1^\mu\,\theta_2^\nu\,f_\mu \odot f_\nu\,, \tag{3.11}$$

$$\widetilde{\Lambda}(\theta_1\,,\theta_2) := \theta_1^\mu\,\theta_2^\nu\,[[f_\mu\,,f_\nu]]\,. \tag{3.12}$$

It is clear that G is actually a symmetric tensor field because $\odot$ is symmetric, and $\widetilde{\Lambda}$ is an antisymmetric tensor field because $[[\,,]]$ is antisymmetric. Now, let $f, g \in \mathcal{F}(\mathfrak{O}_n^*)$, and let $\widetilde{\Lambda}$ and G be the tensor fields in equations (3.12) and (3.11). We define the following bilinear, binary product structures among functions on $\mathfrak{O}_n^*$:

$$\langle f\,;g\rangle := G\,(\mathrm{d}f\,;\mathrm{d}g)\ . \tag{3.13}$$

$$\{f\,;g\} := \widetilde{\Lambda}\,(\mathrm{d}f\,;\mathrm{d}g)\ , \tag{3.14}$$

The second product is a Poisson bracket, while the first one is commutative but it does not possess additional properties, unless we restrict it to a properly chosen subspace of functions, namely, linear functions. In that case, we recover the Jordan structure of equation (3.8).

By means of the linear functions, we may introduce a Cartesian coordinate system[4] $\{x^\mu\}_{\mu=0,\ldots,n^2-1}$ associated with $\{\mathbf{e}^\mu\}_{\mu=0,\ldots,n^2-1}$ setting:

$$x^\mu(\xi) := f_{\mathbf{e}^\mu}(\xi) = \xi(\mathbf{e}^\mu)\,, \tag{3.15}$$

where $\{\mathbf{e}^\mu\}_{\mu=0,\ldots,n^2-1}$ is an orthonormal basis of $\mathfrak{O}_n$ having $\mathbf{e}^0 = \frac{\mathbb{I}}{\sqrt{n}}$. It is a matter of straightforward calculations to show that the coordinate expressions of the tensor fields G and $\widetilde{\Lambda}$ are:

$$G = d^{\mu\nu}_\sigma x^\sigma\,\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\,. \tag{3.16}$$

$$\widetilde{\Lambda} = c^{\mu\nu}_\sigma x^\sigma\,\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}\,, \tag{3.17}$$

The coefficients $c^{\mu\nu}_\sigma$ are the structure constants of the Lie product $[[\,,]]$ in $\mathfrak{O}_n$. Note that $2c^{\mu\nu}_\sigma$ are the structure constants of the Lie algebra $\mathfrak{u}(\mathfrak{A}_n) = \mathfrak{u}(n)$ of the unitary group $U(\mathfrak{A}_n) = U(n)$, and thus, they are antisymmetric in all indices. Analogously, the coefficients $d^{\mu\nu}_\sigma$ are the structure constants of the Jordan product $\odot$ in $\mathfrak{O}_n$. The structure constants $d^{\mu\nu}_\sigma$ are symmetric in μ, ν, and we have $d^{00}_j = 0$ and $d^{\mu\nu}_0 =$

[4] Throughout the rest of the paper, greek indexes will run from 0 to (n^2-1), while latin indexes will run from 1 to (n^2-1).

$\frac{\delta^{\mu\nu}}{\sqrt{n}}$. Furthermore, the structure constants are invariant with respect to unitary transformations, that is, the structure constants of the basis $\{\mathbf{e}^{\mu}\}$ equal those of the basis $\{\mathbf{e}'^{\mu}\}$, where $\mathbf{e}'^{\mu} = \mathbf{U}\,\mathbf{e}^{\mu}\,\mathbf{U}^{\dagger}$ with $\mathbf{U}\mathbf{U}^{\dagger} = \mathbb{I}$.

Remark 10. *In [43] the antisymmetric and symmetric product among quantum observables are used to introduce, respectively, an antisymmetric bivector field Λ and a symmetric bivector field G on a finite-dimensional submanifold $\mathcal{M}$ of parametrized pure states in $\mathcal{P}(\mathcal{H})$, where $\mathcal{H}$ is the complex separable infinite-dimensional Hilbert space of the system. It turns out that, if some compatibility conditions between the immersion of the submanifold $\mathcal{M}$ of parametrized pure states and the quantum unitary dynamical evolution considered are satisfied, it is possible to define a classical-like dynamics on $\mathcal{M}$ which is Hamiltonian with respect to Λ.*

3.1.1 Hamiltonian and gradient-like vector fields on $\mathfrak{O}_n^*$

Having at our disposal a symmetric and an antisymmetric bivector field, we may introduce the notions of gradient-like and Hamiltonian vector fields:

Definition 8 (Gradient-like and Hamiltonian vector fields on $\mathfrak{O}_n^*$)**.** *Let f be a smooth function on $\mathfrak{O}_n^*$, and let G and $\widetilde{\Lambda}$ be as in equations (3.11) and (3.12). Then, the gradient-like vector field $\mathbb{Y}_f$ and the Hamiltonian vector field $\widetilde{\mathbb{X}}_f$ associated with f are defined as:*

$$\mathbb{Y}_f := G(\mathrm{d}f\,,\cdot)\,, \tag{3.18}$$

$$\widetilde{\mathbb{X}}_f := \widetilde{\Lambda}(\mathrm{d}f\,,\cdot)\,. \tag{3.19}$$

For the sake of notational simplicity, we will write $\mathbb{Y}_{\mathbf{a}}$ and $\widetilde{\mathbb{X}}_{\mathbf{a}}$ for the gradient-like and the Hamiltonian vector field associated with the linear function $f_{\mathbf{a}}$, where $\mathbf{a} \in \mathfrak{O}_n$.

Interestingly enough, we can prove that the gradient-like and Hamiltonian vector fields associated with linear functions on $\mathfrak{O}_n^*$ close on a realization of the Lie algebra $\mathfrak{gl}(\mathfrak{A}_n)$:

Proposition 9. *Let $\mathbf{a}, \mathbf{b} \in \mathfrak{O}_n$, then the associated gradient-like and Hamiltonian vector fields satisfy the following commutation relations:*

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\widetilde{\mathbb{X}}_{\mathbf{b}}] = \widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = \mathbb{Y}_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [\mathbb{Y}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = -\widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.20}$$

This means that the gradient-like and Hamiltonian vector fields associated with linear functions close on the Lie algebra $\mathfrak{gl}(\mathfrak{A}_n)$ of the general linear group $GL(\mathfrak{A}_n)$.

Proof. *Since the differentials of the linear functions generate the cotangent space at each point of $\mathfrak{O}_n^*$, it suffices to check the commutation relations by computing the action of the commutators on a generic $f_{\mathbf{c}}$ with $\mathbf{c} \in \mathfrak{O}_n$. We start with the following computation:*

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\widetilde{\mathbb{X}}_{\mathbf{b}}](f_{\mathbf{c}}) = \widetilde{\mathbb{X}}_{\mathbf{a}}(\widetilde{\mathbb{X}}_{\mathbf{b}}(f_{\mathbf{c}})) - \widetilde{\mathbb{X}}_{\mathbf{b}}(\widetilde{\mathbb{X}}_{\mathbf{a}}(f_{\mathbf{c}})) =$$
$$= \widetilde{\mathbb{X}}_{\mathbf{a}}(f_{[[\mathbf{b}\,,\mathbf{c}]]}) - \widetilde{\mathbb{X}}_{\mathbf{b}}(f_{[[\mathbf{a}\,,\mathbf{c}]]}) = f_{[[\mathbf{a}\,,[[\mathbf{b}\,,\mathbf{c}]]]]} - f_{[[\mathbf{b}\,,[[\mathbf{a}\,,\mathbf{c}]]]]}$$

According to the Jacobi identity of the Lie product we have:

$$[[\mathbf{a}\,,[[\mathbf{b}\,,\mathbf{c}]]\,]] - [[\mathbf{b}\,,[[\mathbf{a}\,,\mathbf{c}]]\,]] = [[\,[[\mathbf{a}\,,\mathbf{b}]]\,,\mathbf{c}]]\,, \tag{3.21}$$

and thus:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\widetilde{\mathbb{X}}_{\mathbf{b}}](f_{\mathbf{c}}) = f_{[[[[\mathbf{a}\,,\mathbf{b}]]\,,\mathbf{c}]]} = \widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]}(f_{\mathbf{c}})\,, \tag{3.22}$$

which means:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\widetilde{\mathbb{X}}_{\mathbf{b}}] = \widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.23}$$

Next, we have:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}](f_{\mathbf{c}}) = \widetilde{\mathbb{X}}_{\mathbf{a}}(\mathbb{Y}_{\mathbf{b}}(f_{\mathbf{c}})) - \mathbb{Y}_{\mathbf{b}}(\widetilde{\mathbb{X}}_{\mathbf{a}}(f_{\mathbf{c}})) =$$
$$= \widetilde{\mathbb{X}}_{\mathbf{a}}\,(f_{\mathbf{b}\odot\mathbf{c}}) - \mathbb{Y}_{\mathbf{b}}(f_{[[\mathbf{a}\,,\mathbf{c}]]}) = f_{[[\mathbf{a}\,,\mathbf{b}\odot\mathbf{c}]]} - f_{\mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]]}$$

Recalling that $[[\mathbf{a}\,,\cdot]]$ is a derivation of the Jordan product $\odot$ for all $\mathbf{a} \in \mathfrak{O}_n$, we have:

$$[[\mathbf{a}\,,\mathbf{b}\odot\mathbf{c}]] - \mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]] = [[\mathbf{a}\,,\mathbf{b}]]\odot\mathbf{c}\,, \tag{3.24}$$

and thus:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}](f_{\mathbf{c}}) = f_{[[\mathbf{a}\,,\mathbf{b}]]\odot\mathbf{c}}\,, \tag{3.25}$$

which implies:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = \mathbb{Y}_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.26}$$

Finally:

$$[\mathbb{Y}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}](f_{\mathbf{c}}) = \mathbb{Y}_{\mathbf{a}}(\mathbb{Y}_{\mathbf{b}}(f_{\mathbf{c}})) - \mathbb{Y}_{\mathbf{b}}(\mathbb{Y}_{\mathbf{a}}(f_{\mathbf{c}})) =$$
$$= \mathbb{Y}_{\mathbf{a}}\,(f_{\mathbf{b}\odot\mathbf{c}}) - \mathbb{Y}_{\mathbf{b}}(f_{\mathbf{a}\odot\mathbf{c}}) = f_{\mathbf{a}\odot(\mathbf{b}\odot\mathbf{c})} - f_{\mathbf{b}\odot(\mathbf{a}\odot\mathbf{c})}\,.$$

Using equations (3.4) and (3.21) we get:

$$\mathbf{a} \odot (\mathbf{b} \odot \mathbf{c}) - \mathbf{b} \odot (\mathbf{a} \odot \mathbf{c}) = -[[\,[[\mathbf{a}\,,\mathbf{b}]]\,\mathbf{c}]]\ , \tag{3.27}$$

and thus:

$$[\mathbb{Y}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}](f_{\mathbf{c}}) = -f_{[[\,[[\mathbf{a}\,,\mathbf{b}]]\,\mathbf{c}]]}\ , \tag{3.28}$$

which implies:

$$[\mathbb{Y}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = -\widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.29}$$

Collecting the results, we have:

$$[\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\widetilde{\mathbb{X}}_{\mathbf{b}}] = \widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [\widetilde{\mathbb{X}}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = \mathbb{Y}_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [\mathbb{Y}_{\mathbf{a}}\,,\mathbb{Y}_{\mathbf{b}}] = -\widetilde{\mathbb{X}}_{[[\mathbf{a}\,,\mathbf{b}]]}\ , \tag{3.30}$$

which means that the Hamiltonian and gradient-like vector fields associated with linear functions close a representation of the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ of the general linear group $GL(\mathfrak{A}_n)$ as claimed.

We will now show that the realization of the Lie algebra $\mathfrak{gl}(\mathfrak{A}_n)$ given in proposition 9 integrates to a right action of the Lie group $GL(\mathfrak{A}_n)$ on $\mathfrak{O}_n^*$. At this purpose, we consider the right action $GL(\mathfrak{A}_n) \times \mathfrak{O}_n^* \ni (\mathrm{g}\,,\xi) \mapsto \widetilde{\alpha_{\mathrm{g}}}(\xi) \equiv \xi_{\mathrm{g}} \in \mathfrak{O}_n^*$ given by:

$$\xi_{\mathrm{g}}(\mathbf{a}) := \xi(\mathrm{g}\,\mathbf{a}\,\mathrm{g}^{\dagger}) \quad \forall \mathbf{a} \in \mathfrak{O}_n\,. \tag{3.31}$$

Recalling that for every $\xi \in \mathfrak{O}_n^*$ there is $\bar{\xi} \in \mathfrak{O}_n$ such that $\xi(\mathbf{a}) = Tr(\bar{\xi}\,\mathbf{a})$, we can write the action $\widetilde{\alpha}$ on $\bar{\xi}$ as:

$$\bar{\xi}_{\mathrm{g}} = \mathrm{g}^{\dagger}\,\bar{\xi}\,\mathrm{g}\,. \tag{3.32}$$

We are now ready to prove the following:

Proposition 10. *The vector fields of the form $2\left(\mathbb{Y}_{\mathbf{a}} + \widetilde{\mathbb{X}}_{\mathbf{b}}\right)$ with $\mathfrak{gl}(\mathfrak{A}_n) \ni \mathbf{A} = \mathbf{a} + \imath\mathbf{b}$ and $\mathbf{a}, \mathbf{b} \in \mathfrak{O}_n$, are the fundamental vector fields (see definition 6) of the right action $\widetilde{\alpha}$ of $GL(\mathfrak{A}_n)$ on $\mathfrak{O}_n^*$ given by equations (3.31) and (3.32).*

Proof. *Let $\mathrm{g} \in GL(\mathfrak{A}_n)$ and take $\mathfrak{gl}(\mathfrak{A}_n) \ni \mathbf{A} = \mathbf{a} + \imath\mathbf{b}$, with $\mathbf{a}, \mathbf{b} \in \mathfrak{O}_n$, such that:*

$$\mathrm{g} = \mathrm{e}^{\mathbf{A}} = \mathrm{e}^{\mathbf{a} + \imath\mathbf{b}}\,. \tag{3.33}$$

Consider the one parameter group $\mathrm{g}_t = \mathrm{e}^{\mathbf{A}t}$, and let us compute:

$$
\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\xi_{\mathrm{g}_t}(\mathbf{e}^{\mu})\right)_{t=0} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\,Tr\left(\mathrm{e}^{(\mathbf{a}-\imath\mathbf{b})t}\,\bar{\xi}\,\mathrm{e}^{(\mathbf{a}+\imath\mathbf{b})t}\,\mathbf{e}^{\mu}\right)\right)_{t=0} = \\
&= 2\,Tr\left(\bar{\xi}\,\left(\mathbf{e}^{\mu}\odot\mathbf{a}-[[\mathbf{e}^{\mu}\,,\mathbf{b}]]\right)\right) = \\
&= 2\left(d^{\mu\nu}_{\sigma}\,x^{\sigma}\,a_{\nu}+c^{\nu\mu}_{\sigma}\,x^{\sigma}\,b_{\nu}\right)\,.
\end{aligned}
\tag{3.34}
$$

Then, the proposition follows after confronting this expression with the expression of the components of $2\left(\mathbb{Y}_{\mathbf{a}}+\widetilde{\mathbb{X}}_{\mathbf{b}}\right)$ *obtained using equations* (3.18) *and* (3.19)*.*

From this proposition follows that the integral curves of the Hamiltonian vector field $\widetilde{\mathbb{X}}_{\mathbf{a}}$may be represented by:

$$
\gamma_t(\bar{\xi}) = \mathbf{U}_t^{\dagger}\,\bar{\xi}\,\mathbf{U}_t\,,
$$

with $\mathbf{U}_t=\exp(\imath\frac{t}{2}\mathbf{a})$, therefore, if ξ is a quantum state, that is ξ is in $\mathcal{S}$, the integral curves starting at ξ remain in $\mathcal{S}$ for all $t\in\mathbb{R}$. On the other hand, the integral curves of the gradient vector field $\mathbb{Y}_{\mathbf{a}}$ may be represented by:

$$
\gamma_t(\bar{\xi}) = \mathbf{A}_t^{\dagger}\,\bar{\xi}\,\mathbf{A}_t\,,
$$

with $\mathbf{A}_t=\exp(\frac{t}{2}\mathbf{a})$, and thus the integral curves starting at $\xi\in\mathcal{S}$ will exit from $\mathcal{S}$ because the trace of $\bar{\xi}_t$ is not preserved. This means that Hamiltonian vector fields are tangent to the affine submanifold $\mathfrak{T}^1_n\subset\mathfrak{O}^*_n$ containing the space $\mathcal{S}$ of quantum states, while gradient-like vector fields are not tangent to $\mathfrak{T}^1_n$.

3.2 Symmetric and skew-symmetric tensors on $\mathfrak{T}^1_n$

As we have seen in the previous section, the Jordan-Lie algebra structure of $\mathfrak{O}_n$ allows us to construct two particular bivector fields on $\mathfrak{O}^*_n$. Starting with these bivector fields, we are able to introduce the notions of Hamiltonian and gradient-like vector fields, and these vector fields turn out to be the fundamental vector fields associated with a smooth action of $GL(\mathfrak{A}_n)$ on $\mathfrak{O}^*_n$. However, this group action does not preserve the space of quantum states $\mathcal{S}\subset\mathfrak{O}^*_n$, nor it preserves the affine subspace $\mathfrak{T}^1_n\subset\mathfrak{O}^*_n$ of which $\mathcal{S}$ is a convex subset. Consequently, we want to perform a reduction procedure for the bivector fields G and $\widetilde{\Lambda}$ in order to obtain bivector fields on $\mathfrak{T}^1_n$. Then, we will define gradient-like and Hamiltonian vector fields on $\mathfrak{T}^1_n$, and it will turn out that these vector fields close on a realization of the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ which is closely related to the action of $SL(\mathfrak{A}_n)$ on $\mathcal{S}$ seen in chapter 2.

The set $\mathfrak{T}_n^1$ is an affine subspace of $\mathfrak{O}_n^*$, and thus its algebra of smooth functions can be identified with the quotient algebra[5] $\mathcal{F}(\mathfrak{O}_n^*)/\mathcal{I}_n^1$, where $\mathcal{F}(\mathfrak{O}_n^*)$ is the algebra of smooth functions on $\mathfrak{O}_n^*$, and $\mathcal{I}_n^1 \subset \mathcal{F}(\mathfrak{O}_n^*)$ is the closed linear subspace consisting of smooth functions vanishing on $\mathfrak{T}_n^1$. What we will do is to perform a reduction procedure for the additional product structures $\{\,,\}$ and $\langle\,,\rangle$ on the algebra $(\mathcal{F}(\mathfrak{O}_n^*), +, \cdot)$ given by, respectively, equations (3.13) and (3.14), and then use the affine structure on $\mathfrak{T}_n^1$ to extend the reduced product structures on functions to bivector fields on $\mathfrak{T}_n^1$.

In order to perform the reduction procedure, let us briefly recall how product structures may pass to a quotient space. First of all, let us consider a vector space V, and a closed linear subspace $W \subset V$. It is well-known that the quotient space $E \equiv V/W$ inherits the structure of a vector space:

$$[v_1] + [v_2] := [v_1 + v_2]\,. \tag{3.35}$$

Now, let us endow the vector space V with a product structure $\cdot$ compatible with $+$ so that V becomes an algebra $A \equiv (V\,, +\,, \cdot)$. It is clear that W is again a closed linear subspace of A, however, as it stands it carries no information on the algebra structure of A. This means that, in general, $E \equiv A/W$ will not be an algebra. If we want $E \equiv A/W$ to inherit an algebra structure, we must select W so that it is an ideal of A. In this case we can define the following product structure on E:

$$[v_1] \cdot_E [v_2] := [v_1 \cdot v_2]\,. \tag{3.36}$$

Indeed, expressing $[v_1]$ and $[v_2]$ as the sum of a representative of the equivalence class with a generic element in W we have:

$$(v_1+w_1)\cdot(v_2+w_2) = v_1\cdot v_2+v_1\cdot w_2+w_1\cdot v_2+w_1\cdot w_2 \equiv v_1\cdot v_2+w_{12} = [v_1\cdot v_2]\,, \tag{3.37}$$

where $w_{12} = v_1 \cdot w_2 + w_1 \cdot v_2 + w_1 \cdot w_2$ is in W for all v_1 and v_2 if and only if W is an ideal of $\mathcal{A}$.

Now, let us consider the algebra $(\mathcal{F}(\mathfrak{O}_n^*)\,, +\,, \{\,\})$. It is a matter of straightforward calculation to show that $\mathcal{I}_n^1$ is an ideal with respect to the product structure

[5] In general, if V is the vector space $\mathcal{F}(M)$ of smooth functions on a differential manifold M, and W is the closed subspace $\mathcal{I}_\Sigma$ of smooth functions vanishing on a submanifold $\Sigma \subset M$, we obtain $E \equiv \mathcal{F}(M)/\mathcal{I}_\Sigma \cong \mathcal{F}(\Sigma)$ [33].

$\{\,,\}$ given by equation (3.13), that is, $\{f\,,g\} \in \mathcal{I}_n^1$ whenever f is in $\mathcal{I}_n^1$. This means that on the quotient space $\mathcal{F}(\mathfrak{D}_n^*)/\mathcal{I}_n^1 \cong \mathcal{F}(\mathfrak{T}_n^1)$ we have the product structure $\{\,\}_1$:

$$\{[f]\,,[g]\}_1 := [\{f\,,g\}]\,. \tag{3.38}$$

We can now use the product structure $\{\,,\}_1$ to construct a contravariant bivector field on $\mathfrak{T}_n^1$. In order to do so, we identify the elements of the quotient space $\mathcal{F}(\mathfrak{D}_n^*)/\mathcal{I}_n^1$ with functions in $\mathcal{F}(\mathfrak{T}_n^1)$. If $\{x^\mu\}_{\mu=0,\dots,n^2-1}$ is the Cartesian coordinates in $\mathfrak{D}_n^*$ associated with the orthonormal basis $\{\mathbf{e}^\mu\}_{\mu=0,\dots,n^2-1}$ introduced before, then the affine subspace $\mathfrak{T}_n^1$ may be identified with all those elements in $\mathfrak{D}_n^*$ having $x^0 = \frac{1}{\sqrt{n}}$.

Remark 11. *What we have done here, is to select an origin in the affine subspace $\mathfrak{T}_n^1$, namely, the point ξ such that $x^0(\xi) = \frac{1}{\sqrt{n}}$ and $x^j(\xi) = 0$ for all $j \neq 0$. Interestingly, this point corresponds to the maximally mixed state.*

Denoting with $i\colon \mathfrak{T}_n^1 \to \mathfrak{D}_n^*$ the canonical immersion, we note that the pullback $f = \frac{A_0}{\sqrt{n}} + A_j\,x^j$ of a linear function $\widetilde{f} = A_\mu\,x^\mu \in \mathcal{F}(\mathfrak{D}_n^*)$ by means of i is an affine function on $\mathfrak{T}_n^1$. Consequently, we can select (n^2-1) of them, say $f^j = x^j$, such that their differentials form a basis of the cotangent space $T_\xi^*\,\mathfrak{T}_n^1$ at each point $\xi \in \mathfrak{T}_n^1$. An explicit calculation shows that:

$$\{f^j\,,f^k\}_1 = c_l^{jk}\,f^l\,. \tag{3.39}$$

Since $\{\mathrm{d}f^j\}_{j=1,\dots,n^2-1}$ is a basis of the cotangent space, we can define a contravariant bivector field Λ setting:

$$\Lambda(\mathrm{d}f^j\,,\mathrm{d}f^k) := \{f^j\,,f^k\}_1 = c_l^{jk}\,f^l\,. \tag{3.40}$$

The explicit expression of Λ in the coordinate system associated with $\{f^j\}_{j=1,\dots,n^2-1}$ is:

$$\Lambda = c_l^{jk}\,x^l\,\frac{\partial}{\partial x^j}\wedge\frac{\partial}{\partial x^k}\,. \tag{3.41}$$

When we try to proceed similarly for G, we immediately find that $\mathcal{I}_n^1$ is not an ideal for the product structure $\langle\,,\rangle$ by equation (3.14). Our proposal to deal with this situation is to modify G so that $\mathcal{I}_n^1$ becomes an ideal for the product structure associated with the new tensor field. In doing so, we will lose the Jordan-Lie algebra structure on the linear functions on $\mathfrak{D}_n^*$, Indeed, when we modify G, the resulting symmetric product will no longer be the Jordan product realized on linear functions, nor will it be compatible with the antisymmetric product $\{\,,\}$ associated with $\widetilde{\Lambda}$.

Let us consider the following tensor field on $\mathfrak{D}_n^*$:

$$\widetilde{\mathcal{R}} := G - \widetilde{\Delta}\otimes\widetilde{\Delta} = d^{\mu\nu}_{\sigma}\, x^{\sigma}\,\frac{\partial}{\partial x^{\mu}}\otimes\frac{\partial}{\partial x^{\nu}} - \widetilde{\Delta}\otimes\widetilde{\Delta}\,, \tag{3.42}$$

where $\widetilde{\Delta} = x^{\mu}\,\frac{\partial}{\partial x^{\mu}}$ is the Euler vector field representing the linear structure of $\mathfrak{O}^{*}$. A direct calculation shows that $\mathcal{I}^{1}_{n}$ is an ideal for the product structure $\langle\langle\,,\rangle\rangle$ on functions associated with $\widetilde{\mathcal{R}}$ by means of:

$$\langle\langle f\,, g\rangle\rangle := \widetilde{\mathcal{R}}\,(\mathrm{d}f\,,\mathrm{d}g)\ . \tag{3.43}$$

For future reference, we note that the gradient-like vector fields associated with $\widetilde{\mathcal{R}}$ are:

$$\widetilde{\mathbb{Y}}_{f} := \widetilde{\mathcal{R}}\,(\mathrm{d}f\,,\cdot) = G\,(\mathrm{d}f\,,\cdot) - \widetilde{\Delta}(f)\,\widetilde{\Delta} = \mathbb{Y}_{f} - \widetilde{\Delta}(f)\,\widetilde{\Delta}\,. \tag{3.44}$$

Remark 12. *If we focus on affine functions $f_{\mathbf{a}}$ with $\mathbf{a}\in\mathfrak{O}_{n}$, we immediately see that:*

$$\langle\langle f_{\mathbf{a}}\,, f_{\mathbf{b}}\rangle\rangle = f_{\mathbf{a}\odot\mathbf{b}} - f_{\mathbf{a}}\,f_{\mathbf{b}}\,. \tag{3.45}$$

If $\mathbf{a}=\mathbf{b}$, then $\langle\langle f_{\mathbf{a}}\,, f_{\mathbf{a}}\rangle\rangle$ is the variance of $\mathbf{a}$. Moreover, it is clear that linear functions are no longer an algebra with respect to the product structure associated with $\widetilde{\mathcal{R}}$, let alone a Jordan algebra.

Now, we can proceed in complete analogy with what has been done for $\{\,\}$ and $\widetilde{\Lambda}$ in order to obtain a symmetric product structure $\langle\,,\rangle_{1}$ on $\mathcal{F}(\mathfrak{T}^{1}_{n})$, and the symmetric contravariant tensor field $\mathcal{R}$ on $\mathfrak{T}^{1}_{n}$ given by:

$$\mathcal{R} = \left(d^{jk}_{l}\, x^{l} + \frac{\delta^{jk}}{n}\right)\frac{\partial}{\partial x^{j}}\otimes\frac{\partial}{\partial x^{k}} - \Delta\otimes\Delta \tag{3.46}$$

where $\Delta = x^{j}\,\frac{\partial}{\partial x^{k}}$.

3.2.1 Hamiltonian and gradient-like vector fields on $\mathfrak{T}^{1}_{n}$

Now that we have the tensor fields Λ and $\mathcal{R}$ on $\mathfrak{T}^{1}_{n}$, we can proceed and define gradient-like and Hamiltonian vector fields in analogy with definition 8:

Definition 9 (Gradient-like and Hamiltonian vector fields on $\mathfrak{T}^{1}_{n}$)**.** *Let $f\in\mathcal{F}(\mathfrak{T}^{1}_{n})$, and $\mathcal{R}$ and Λ be as in equation (3.46) and (3.41). Then, the gradient-like vector field Y_{f} and the Hamiltonian vector field X_{f} associated with f are defined as:*

$$Y_{f} := \mathcal{R}\,(\mathrm{d}f\,;\cdot)\ , \tag{3.47}$$

$$X_f := \Lambda\left(\mathrm{d}f\,;\cdot\right)\,. \tag{3.48}$$

For the sake of notational simplicity, we will write $Y_{\mathbf{a}}$ and $X_{\mathbf{a}}$ for the gradient-like and the Hamiltonian vector field associated with the affine function $f_{\mathbf{a}}$, where $\mathbf{a}\in\mathfrak{O}_n$.

Writing $\xi=\frac{1}{\sqrt{n}}\mathbf{e}_0+x^j\mathbf{e}_j$, the explicit expressions of the gradient-like and Hamiltonian vector fields associated with the affine function $f_{\mathbf{a}}=\frac{a_0}{\sqrt{n}}+a_jx^j$, where $\mathbf{a}\in\mathfrak{O}_n$, are:

$$Y_{\mathbf{a}}=Tr\left(\mathbf{a}\odot\bar{\xi}\,\bar{\mathbf{e}}^k\right)\,\frac{\partial}{\partial x^k}-f_{\mathbf{a}}\Delta=\left(d_l^{jk}\,x^l a_j+\frac{\delta^{jk}a_j}{n}\right)\,\frac{\partial}{\partial x^k}-x^j a_j\,\Delta\,, \tag{3.49}$$

$$X_{\mathbf{a}}=c_l^{jk}\,x^l a_j\,\frac{\partial}{\partial x^k}\,. \tag{3.50}$$

Note that the gradient-like vector fields contain a quadratic term with respect to the coordinate system $\{x^k\}_{k=1,\dots,n^2-1}$ adapted to $\mathfrak{T}_n^1$. Moreover, note that the 0-component of $\mathbf{a}$ does not play any role in the definition of $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$. In particular, the Hamiltonian and gradient-like vector fields associated with $f_{\mathbf{a}}$ are everywhere vanishing whenever $\mathbf{a}=a_0\mathbf{e}^0$.

There is a very interesting relation between the vector fields $\widetilde{\mathbb{X}}_{\mathbf{a}}$, $\widetilde{\mathbb{Y}}_{\mathbf{a}}$, $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$. To see this, recall that the vector fields on $\mathfrak{O}_n^*$ are derivations of the pointwise product of $\mathcal{F}(\mathfrak{O}_n^*)$. Any such derivation, say $\widetilde{D}$, defines a derivation D of the quotient algebra (with respect to the pointwise product) if and only if $D(\mathcal{I}_n^1)\subset\mathcal{I}_n^1$ [33], indeed, we can define:

$$D([f]):=[\widetilde{D}(f)]\,. \tag{3.51}$$

It is then clear that Hamiltonian vector fields $\widetilde{\mathbb{X}}_{\mathbf{a}}$ and gradient-like vector fields $\widetilde{\mathbb{Y}}_{\mathbf{a}}$ define derivations of $\mathcal{F}(\mathfrak{O}_n^*)/\mathcal{I}_n^1$. Once we identify $\mathfrak{T}_n^1\subset\mathfrak{O}_n^*$ as the affine subspace characterized by $x^0(\xi)=\frac{1}{\sqrt{n}}$, it follows from a direct computation that the derivation associated with $\widetilde{\mathbb{X}}_{\mathbf{a}}$ is the Hamiltonian vector field $X_{\mathbf{a}}$ associated with $\mathbf{a}$ by means of Λ, and the derivation associated with $\widetilde{\mathbb{Y}}_{\mathbf{a}}$ is the gradient-like vector field $Y_{\mathbf{a}}$ associated with $\mathbf{a}$ by means of $\mathcal{R}$.

We will now prove that Hamiltonian and gradient-like vector fields on $\mathfrak{T}_n^1$ associated with traceless elements in $\mathfrak{O}_n$ close on a realization of the Lie algebra of $SL(\mathfrak{A}_n)$:

Proposition 11. *Let* $\mathbf{a}, \mathbf{b} \in \mathfrak{O}_n$ *be such that* $Tr(\mathbf{a}) = Tr(\mathbf{b}) = 0$*. Then the associated gradient-like and Hamiltonian vector fields on* $\mathfrak{T}_n^1$ *satisfy the following commutation relations:*

$$[X_{\mathbf{a}}\,, X_{\mathbf{b}}] = X_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [X_{\mathbf{a}}\,, Y_{\mathbf{b}}] = Y_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [Y_{\mathbf{a}}\,, Y_{\mathbf{b}}] = -X_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.52}$$

This means that the gradient-like and Hamiltonian vector fields associated with affine functions close on the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ *of the special linear group* $SL(\mathfrak{A}_n)$*.*

Proof. *Since the affine functions* $f_{\mathbf{a}}$ *are enough to generate the cotangent space at each point, we will compute the commutators simply evaluating them on the affine functions themselves. For the Hamiltonian vector fields we have:*

$$\begin{aligned}[X_{\mathbf{a}}\,, X_{\mathbf{b}}](f_{\mathbf{c}}) &= X_{\mathbf{a}}(X_{\mathbf{b}}(f_{\mathbf{c}})) - X_{\mathbf{b}}(X_{\mathbf{a}}(f_{\mathbf{c}})) = \\ &= X_{\mathbf{a}}(f_{[[\mathbf{b}\,,\mathbf{c}]]}) - X_{\mathbf{b}}(f_{[[\mathbf{a}\,,\mathbf{c}]]}) = f_{[[\mathbf{a}\,,[[\mathbf{b}\,,\mathbf{c}]]]]} - f_{[[\mathbf{b}\,,[[\mathbf{a}\,,\mathbf{c}]]]]}\,,\end{aligned}$$

where we have used:

$$X_{\mathbf{a}}(f_{\mathbf{b}}) = \Lambda(\mathrm{d}f_{\mathbf{a}}\,, \mathrm{d}f_{\mathbf{b}}) = \{f_{\mathbf{a}}\,, f_{\mathbf{b}}\}_1 = f_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.53}$$

It is easy to see that:

$$[[\mathbf{a}\,, [[\mathbf{b}\,, \mathbf{c}]]\,]] - [[\mathbf{b}\,, [[\mathbf{a}\,, \mathbf{c}]]\,]] = [[\,[[\mathbf{a}\,, \mathbf{b}]]\,, \mathbf{c}]]\,, \tag{3.54}$$

from which it follows that:

$$[X_{\mathbf{a}}\,, X_{\mathbf{b}}](f_{\mathbf{c}}) = f_{[[[[\mathbf{a}\,,\mathbf{b}]]\,,\mathbf{c}]]} = X_{[[\mathbf{a}\,,\mathbf{b}]]}(f_{\mathbf{c}})\,, \tag{3.55}$$

and thus:

$$[X_{\mathbf{a}}\,, X_{\mathbf{b}}] = X_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.56}$$

Before computing the commutator between Hamiltonian and gradient-like vector fields, let us note that:

$$Y_{\mathbf{a}}(f_{\mathbf{b}}) = \mathcal{R}(\mathrm{d}f_{\mathbf{a}}\,, \mathrm{d}f_{\mathbf{b}}) = d_l^{jk} a_j b_k + \frac{\delta^{jk} a_j b_k}{n} - x^j a_j\, x^k b_k\,. \tag{3.57}$$

Now, the Jordan product $\mathbf{a} \odot \mathbf{b}$ *reads:*

$$\mathbf{a} \odot \mathbf{b} = d_\sigma^{\mu\nu} a_\mu b_\nu\, \mathbf{e}^\sigma = d_0^{jk} a_j b_k\, \mathbf{e}^0 + d_l^{jk} a_j b_k\, \mathbf{e}^l = \frac{\delta^{jk} a_j b_k}{\sqrt{n}}\, \mathbf{e}^0 + d_l^{jk} a_j b_k\, \mathbf{e}^l\,, \tag{3.58}$$

where we used $d_j^{00}=0$, $d_0^{\mu\nu}=\frac{\delta^{\mu\nu}}{\sqrt{n}}$, and the fact that $\mathbf{a}$ *and* $\mathbf{b}$ *are traceless. Comparing equation (3.57) with equation (3.58) it follows that*

$$Y_{\mathbf{a}}(f_{\mathbf{b}}) = f_{\mathbf{a}\odot\mathbf{b}} - f_{\mathbf{a}} f_{\mathbf{b}}\,. \tag{3.59}$$

Computing the commutator, we have:

$$[X_{\mathbf{a}}\,, Y_{\mathbf{b}}](f_{\mathbf{c}}) = X_{\mathbf{a}}(Y_{\mathbf{b}}(f_{\mathbf{c}})) - Y_{\mathbf{b}}(X_{\mathbf{a}}(f_{\mathbf{c}})) = X_{\mathbf{a}}\left(f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{b}} f_{\mathbf{c}}\right) - Y_{\mathbf{b}}(f_{[[\mathbf{a}\,,\mathbf{c}]]}) =$$
$$= f_{[[\mathbf{a}\,,\mathbf{b}\odot\mathbf{c}]]} - f_{\mathbf{c}}\, f_{[[\mathbf{a}\,,\mathbf{b}]]} - f_{\mathbf{b}}\, f_{[[\mathbf{a}\,,\mathbf{c}]]} - f_{\mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]]} + \; f_{\mathbf{b}} f_{[[\mathbf{a}\,,\mathbf{c}]]} =$$
$$= f_{[[\mathbf{a}\,,\mathbf{b}\odot\mathbf{c}]]} - f_{\mathbf{c}}\, f_{[[\mathbf{a}\,,\mathbf{b}]]} - f_{\mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]]}\,.$$

A direct computation shows that:

$$[[\mathbf{a}\,, \mathbf{b}\odot\mathbf{c}]] - \mathbf{b}\odot[[\mathbf{a}\,,\mathbf{c}]] = [[\mathbf{a}\,,\mathbf{b}]]\odot\mathbf{c}\,, \tag{3.60}$$

and thus:

$$[X_{\mathbf{a}}\,, Y_{\mathbf{b}}](f_{\mathbf{c}}) = f_{[[\mathbf{a}\,,\mathbf{b}]]\odot\mathbf{c}} - f_{[[\mathbf{a}\,,\mathbf{b}]]}\, f_{\mathbf{c}}\,, \tag{3.61}$$

which means:

$$[X_{\mathbf{a}}\,, Y_{\mathbf{b}}] = Y_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.62}$$

Finally, noting that:

$$Y_{\mathbf{a}}\left(f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{b}} f_{\mathbf{c}}\right) = f_{\mathbf{a}\odot(\mathbf{b}\odot\mathbf{c})} - f_{\mathbf{a}} f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{c}}\left(f_{\mathbf{a}\odot\mathbf{b}} - f_{\mathbf{a}} f_{\mathbf{b}}\right) - f_{\mathbf{b}}\left(f_{\mathbf{a}\odot\mathbf{c}} - f_{\mathbf{a}} f_{\mathbf{c}}\right)$$

we have:

$$[Y_{\mathbf{a}}\,, Y_{\mathbf{b}}](f_{\mathbf{c}}) = Y_{\mathbf{a}}(Y_{\mathbf{b}}(f_{\mathbf{c}})) - Y_{\mathbf{b}}(Y_{\mathbf{a}}(f_{\mathbf{c}})) = Y_{\mathbf{a}}\left(f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{b}} f_{\mathbf{c}}\right) - Y_{\mathbf{b}}\left(f_{\mathbf{a}\odot\mathbf{c}} - f_{\mathbf{a}} f_{\mathbf{c}}\right) =$$

$$= f_{\mathbf{a}\odot(\mathbf{b}\odot\mathbf{c})} - f_{\mathbf{a}} f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{c}}\left(f_{\mathbf{a}\odot\mathbf{b}} - f_{\mathbf{a}} f_{\mathbf{b}}\right) - f_{\mathbf{b}}\left(f_{\mathbf{a}\odot\mathbf{c}} - f_{\mathbf{a}} f_{\mathbf{c}}\right) -$$
$$- f_{\mathbf{b}\odot(\mathbf{a}\odot\mathbf{c})} + f_{\mathbf{b}} f_{\mathbf{a}\odot\mathbf{c}} + f_{\mathbf{c}}\left(f_{\mathbf{a}\odot\mathbf{b}} - f_{\mathbf{a}} f_{\mathbf{b}}\right) + f_{\mathbf{a}}\left(f_{\mathbf{b}\odot\mathbf{c}} - f_{\mathbf{b}} f_{\mathbf{c}}\right) = -f_{[[\,[[\mathbf{a}\,,\mathbf{b}]]\,\mathbf{c}]]}$$

where, in the last equality, we used equation (3.27). Eventually, we get:

$$[Y_{\mathbf{a}}\,, Y_{\mathbf{b}}] = -X_{[[\mathbf{a}\,,\mathbf{b}]]}\,. \tag{3.63}$$

Collecting the results we have:

$$[X_{\mathbf{a}}\,, X_{\mathbf{b}}] = X_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [X_{\mathbf{a}}\,, Y_{\mathbf{b}}] = Y_{[[\mathbf{a}\,,\mathbf{b}]]} \qquad [Y_{\mathbf{a}}\,, Y_{\mathbf{b}}] = -X_{[[\mathbf{a}\,,\mathbf{b}]]}\,, \tag{3.64}$$

which defines a realization of the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ of the special linear group $SL(\mathfrak{A}_n)$ as claimed.

One could be tempted to say that the realization of $\mathfrak{sl}(\mathfrak{A}_n)$ by means of Hamiltonian and gradient-like vector fields associated with affine functions, integrates to a right action of $SL(\mathfrak{A}_n)$ on $\mathfrak{T}_n^1$ just as it happens for the realization of $\mathfrak{gl}(\mathfrak{A}_n)$ on $\mathfrak{O}_n^*$ (see equation (3.20)). However, this is not the case. What happens is that the gradient-like vector fields in equation (3.52) are, in general, not complete, and thus the Lie algebra realization does not integrate to an action of the Lie group (see remark 3). What is very interesting though, is that on the positive elements of $\mathfrak{T}_n^1$, that is, the quantum states, these vector fields are complete, and their flow is precisely the action α of definition 5:

Proposition 12. *Let $\mathbf{a}, \mathbf{b} \in \mathfrak{O}$ be such that $Tr(\mathbf{a}) = Tr(\mathbf{b}) = 0$. Then, the evolution ρ_t of the quantum state $\rho \in \mathcal{S}$ along the flow of $X_{\mathbf{a}} + Y_{\mathbf{b}}$ is defined by:*

$$\rho_t(\mathbf{e}^j) = \frac{\rho(\mathbf{g}_t\, \mathbf{e}^j\, \mathbf{g}_t^\dagger)}{\rho(\mathbf{g}_t\, \mathbf{g}_t^\dagger)}\,, \tag{3.65}$$

where $\mathbf{A} = \mathbf{a} + \imath\mathbf{b} \in \mathfrak{sl}(\mathfrak{A}_n)$ so that $\mathbf{g}_t = \exp(\frac{t}{2}\mathbf{A})$ is in $SL(\mathfrak{A}_n)$ for all t.

Proof. *Let us start recalling that $\rho = \frac{1}{\sqrt{n}}\mathbf{e}_0 + x^l\mathbf{e}_l$, and compute:*

$$\begin{aligned}\left.\frac{\mathrm{d}\rho_t(\mathbf{e}^j)}{\mathrm{d}t}\right|_{t=0} &= \rho\left(\mathbf{a}\odot\mathbf{e}^j\right) - \rho\left([[\mathbf{e}^j\,,\mathbf{b}]]\right) - \rho(\mathbf{e}^j)\,\rho(\mathbf{a}) = \\ &= d_l^{kj}\,x^l a_k + \frac{\delta^{kj}a_k}{n} + c_l^{kj}\,x^l b_k - a_k x^k\,x^j\,. \end{aligned} \tag{3.66}$$

The proposition follows from the comparison between equation (3.66) with equations (3.49) and (3.50). Note that if ρ is not a quantum state ($\bar{\rho}$ is not positive), we can not use equation (3.65) to define ρ_t because the denominator may explode (see remark 3).

It is clear that Hamiltonian and gradient vector fields on $\mathfrak{T}_n^1$ are tangent to every manifold $\mathcal{S}_k$ of quantum states with fixed rank. Furthermore, comparing proposition 12 with the results of subsection 2.1.1 on the fundamental vector fields of the action of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$, we realize that the restriction of the vector field $(Y_{\mathbf{a}} + X_{\mathbf{b}})$ to the submanifold $\mathcal{S}_k \subset \mathfrak{T}_n^1$ coincides with the fundamental vector field $X_{\mathbf{A}}$ associated with $\mathbf{A} = \frac{\mathbf{a} - \imath \mathbf{b}}{2}$. Consequently, the integral curves $\gamma_t(\rho)$ of $(Y_{\mathbf{a}} + X_{\mathbf{b}})$ starting at $\rho \in \mathcal{S}$ are clearly complete, and they lie entirely in the space $\mathcal{S}$ of quantum states in the sense that if $\rho \in \mathcal{S}_k$ then $\gamma_t(\rho)$ is in $\mathcal{S}_k$ for all $t \in \mathbb{R}$. In particular, the integral curves of $X_{\mathbf{b}}$ lie entirely in the set of isospectral states, and thus $X_{\mathbf{b}}$ may be thought of as the vector field generating the unitary part of a quantum dynamical process. On the other hand, the integral curves of $Y_{\mathbf{a}} + X_{\mathbf{b}}$ are generically transversal to the set of isospectral states, however remaining entirely in the set of quantum states with fixed rank. Clearly, since $Y_{\mathbf{a}}$ contains nonlinear terms, its integral curves can not represent linear quantum dynamical processes. Indeed, in the next section we will see that if we combine $Y_{\mathbf{a}}$ with a properly defined vector field $Z_{\mathcal{K}}$, then the vector field $Y_{\mathbf{a}} + Z_{\mathcal{K}}$ may be thought of as representing the dissipative part of a quantum dynamical process.

Example 1 (Two-level quantum system)**.** *To illustrate our general arguments we consider the example of a two-level quantum system. To make contact with the widespread notation for qubit, we will here drop the requirement of orthonormality for the basis $\{\mathbf{e}^{\mu}\}_{\mu=0,\ldots,3}$ and $\{\mathbf{e}_{\mu}\}_{\mu=0,\ldots,3}$ and consider the orthogonal basis generated by the Pauli matrices:*

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.67}$$

$$\sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.68}$$

This choice will affect some numerical factors in the coefficients c_l^{jk} and $d_{\alpha}^{\mu\nu}$. However, from the practical point of view, it is a convenient choice because of the peculiar properties of the Pauli matrices. The self-adjoint element $\bar{\xi} \in \mathfrak{O}_n$ associated with $\xi \in \mathfrak{O}_n^*$ is written as:

$$\bar{\xi} = \frac{1}{2}\left(\sigma^0 + \mathbf{x} \cdot \boldsymbol{\sigma}\right) , \tag{3.69}$$

and it is the density matrix associated with a quantum state if and only if $|\mathbf{x}|^2 \leq 1$. In this case, $\mathcal{S}$ has only two strata, namely, $\mathcal{S}_1$ and $\mathcal{S}_2$, and it is a proper manifold with boundary. As shown in [64] this is the only case in which $\mathcal{S}$ is a differential manifold with a smooth boundary. Specifically, $\mathcal{S}$ is the 3-dimensional solid ball and the two strata are the surface of the ball, that is, the pure states; and the open interior of the ball, that is, the mixed states. It should be noticed that while pure states are represented by a compact manifold without boundary, the stratum of mixed states is bounded but not compact, and its closure is the whole space of quantum states $\mathcal{S}$.

It is easy to work out the explicit expressions for the structure constants of the anti-symmetric product $[[\sigma^\mu\,,\sigma^\nu]]$ and the symmetric product $\sigma^\mu \odot \sigma^\nu$, and, once this has been done, the expressions for $\mathcal{R}$ and Λ read:

$$\mathcal{R} = \delta^{jk}\,\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} - \Delta \otimes \Delta\,, \tag{3.70}$$

$$\Lambda = -\epsilon_l^{jk}\,x^l\,\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}\,, \tag{3.71}$$

where ϵ_l^{jk} is the Levi-Civita symbol. Gradient vector fields associated with the affine function f_{σ_j} are:

$$Y_j = \frac{\partial}{\partial x^j} - x^j\,\Delta\,, \tag{3.72}$$

while Hamiltonian ones read:

$$X_j = -\epsilon_l^{jk}\,x^l\,\frac{\partial}{\partial x^k}\,. \tag{3.73}$$

Together they close on the Lie algebra of $SL(\mathfrak{A}_2) = SL(2\,,\mathbb{C})$. Furthermore, the Hamiltonian ones are tangent to the sphere of radius r for all $r > 0$:

$$\mathcal{L}_{X_j} r^2 = -\epsilon_l^{jk}\,x^l x^k = 0\,, \tag{3.74}$$

while the gradient ones are tangent only to the sphere of radius $r = 1$ (the pure states), in fact we get:

$$\mathcal{L}_{Y_j} r^2 = (1 - r^2) x^j\,. \tag{3.75}$$

This instance reflects the fact that the manifold of pure quantum states is a homogeneous space for both $SU(\mathfrak{A}_n)$ and $SL(\mathfrak{A}_n)$ for every n.

3.3 Evolution of open quantum systems as an affine dynamical system

The mathematical description of (Markovian) open quantum systems was initiated in the pioneriing works [63] and [82]. In these papers, the explicit form of the most general master equation governing the Markovian dynamics of a finite-level quantum system was found. The theoretical and experimental richness of the theory of open quantum systems is continuously growing, and the important increase in the level of experimental control on quantum systems has led to a wide number of experimental realizations of open quantum systems in different fields of physical applications. For instance, in quantum optics; in atomic and molecular physics; and in mesoscopic physics.

An open system can be thought of as a physical system S which is not closed, that is, it is interacting in some way with an environment E. From the conceptual point of view, one may hope to be able to consider a new physical system T, which is the sum of S and E, so that T becomes a closed system. This conceptual attitude is corroborated by a number of mathematical results, both in classical, and quantum physics. Indeed, given a classical system S the dynamical evolution of which is described by means of a vector field X on some carrier manifold M, it is always possible to find a symplectic lift $\tilde{X}$ of X to the cotangent bundle T^*M, so that $\tilde{X}$ becomes a Hamiltonian vector field, that is, $\tilde{X}$ describes a closed classical system [15, 1]. On the other hand, the state ρ of a (finite-level) quantum system S with Hilbert space $\mathcal{H}_S$, may be described by a density operator in the space $\mathcal{B}(\mathcal{H}_S)$ of bounded linear operators on $\mathcal{H}_S$. Here, the evolution of a closed system corresponds to the unitary evolution:

$$\Phi_\tau(\rho) = \mathbf{U}_\tau \, \rho \, \mathbf{U}_\tau^\dagger \,, \tag{3.76}$$

with $\mathbf{U}_\tau$ a unitary operator for all τ. An open system is described by a semigroup Φ_τ of completely-positive trace preserving (CPTP) maps from $\mathcal{B}(\mathcal{H}_S)$ into itself. In this context, Stinespring theorem [97] states that every completely-positive trace-preserving map $\mathcal{K}(\rho)$ from $\mathcal{B}(\mathcal{H}_S)$ to itself can be obtained in three steps. First, we have to consider the tensor product $\mathcal{H}_S$ with an auxiliary Hilbert space $\mathcal{H}_E$. According to the general postulates of quantum mechanics [55], $\mathcal{H}_S \otimes \mathcal{H}_E$ represents the Hilbert space of the composite system $S + E$. Once we have the composite system, we let it evolve by means of a unitary evolution depending on the explicit form of $\mathcal{K}$. Finally, we project the evolved state back from $\mathcal{H}_S \otimes \mathcal{H}_E$ to $\mathcal{H}_S$ to define an evolution from $\mathcal{H}_S$ into $\mathcal{H}_S$.

Although these prescription seems clear cut, its practical implementation suffers of some limitations. Indeed, it is often the case that the environment is so complicated that a complete knowledge of the actual state describing it is impossible. Consequently, it is impossible to determine the evolution of the composite system because we do not know the initial state of the composite system. Furthermore, it is very likely that our knowledge of the explicit form of the interaction between the system and its environment is uknown to us. What we actually have, is only an effective dynamics on the subsystem S. The Gorini-Kossakowski-Sudarshan-Linbland equation (or GKLS for short) [63, 82] describes precisely the most general form for the generator of a finite-level open quantum system from the perspective of the effective dynamics. For an interesting exposition of the technical and conceptual development of the ideas that have led to the discovery of the GKLS equation we refer to [38].

Recalling that, in the finite-dimensional case, every $\xi \in \mathfrak{O}_n^*$ is uniquely related to an element $\bar{\xi} \in \mathfrak{O}_n$, the GKLS generator $\mathbf{L}$ of a linear quantum dynamical process can be expressed as the linear operator (see [63, 82]):

$$\mathbf{L}(\bar{\xi}) = -2\left[\left[\mathbf{H}\,,\bar{\xi}\,\right]\right] - \mathbf{V} \odot \bar{\xi} + \mathcal{K}(\bar{\xi})\,, \tag{3.77}$$

where $\mathbf{H} \in \mathfrak{O}_n$, $\mathbf{v}_j \in \mathfrak{A}_n$, $\mathbf{V} = \sum_{j=1}^{N} \mathbf{v}_j^\dagger \mathbf{v}_j$, and the linear map $\mathcal{K}$ is a completely-positive map:

$$\mathcal{K}(\bar{\xi}) = \sum_{j=1}^{N} \mathbf{v}_j\,\bar{\xi}\,\mathbf{v}_j^\dagger \qquad \text{with} \;\; N \leq (n^2 - 1)\,. \tag{3.78}$$

In the finite-dimensional case, according to [63, 82], this is the most general form for the generator of a dynamical process which is linear, completely positive and trace preserving (CPTP).

The integration of the equations of motion associated with $\mathbf{L}$ gives a one-parameter semigroup $\{\Phi_\tau\}$ of completely-positive maps $\Phi_\tau : \mathcal{S} \to \mathcal{S}$ for $\tau \geq 0$, such that Φ_0 is the identity transformation. Actually, $\{\Phi_\tau\}$ is well-defined and differentiable for all $\tau \in \mathbb{R}$ on the whole $\mathfrak{O}^*$, but, for $\tau < 0$ it fails to preserve positivity, hence, it maps quantum states out of $\mathcal{S}$.

We will now analyze the vector field $\widetilde{\Gamma} \equiv Z_{\mathbf{L}}$ associated with the GKLS generator $\mathbf{L}$. For this purpose, let $\{\mathbf{e}_\mu\}_{\mu=0,\ldots,n^2-1}$ denote the basis in $\mathfrak{O}_n^*$ which is dual to the orthonormal basis $\{\mathbf{e}^\mu\}_{\mu=0,\ldots,n^2-1}$ of $\mathfrak{O}_n$ introduced before.

Definition 10 (Linear vector field associated with a linear map)**.** *Let $A(\xi) = A^\mu_\nu \xi^\nu\,\mathbf{e}_\mu$ be a linear map from $\mathfrak{O}_n^*$ to itself, and let $\{x^\mu\}$ be the Cartesian coordinates system associated with $\{\mathbf{e}^\mu\}_{\mu=0,\ldots,n^2-1}$. We define a linear vector field $\mathbb{Z}_A$ on $\mathfrak{O}_n^*$ associated with A as follows (see [33] chapter 2 and 3):*

$$\mathbb{Z}_A := A^\mu_\nu\, x^\nu\, \frac{\partial}{\partial x^\mu}\,. \tag{3.79}$$

Its action on linear functions reads:

$$Z_A(f_{\mathbf{b}})(\xi) = f_{\mathbf{b}}(A(\xi)) = A^\mu_\nu\, b_\mu x^\nu\,. \tag{3.80}$$

It is a matter of straightforward calculation to prove that:

Proposition 13. *Let A, B be linear maps from $\mathfrak{O}_n^*$ to itself, then:*

$$Z_{A+B} = Z_A + Z_B \,. \tag{3.81}$$

The GKLS generator $\mathbf{L}$ is a linear map from $\mathfrak{O}_n^*$ to itself, therefore, we may define its associated linear vector field $\widetilde{\Gamma} \equiv Z_L$ on $\mathfrak{O}_n^*$ by means of definition 10.

Proposition 14 (GKLS vector field on $\mathfrak{O}_n^*$)**.** *Let $\mathbf{L}$ be the GKLS generator of equation (3.77). Then:*

$$\widetilde{\Gamma} = \widetilde{\mathbb{X}}_{\mathbf{a}} + \mathbb{Y}_{\mathbf{b}} + \mathbb{Z}_{\mathcal{K}} \,, \tag{3.82}$$

where $\widetilde{\mathbb{X}}_{\mathbf{a}}$ is the Hamiltonian vector field associated with $\mathbf{a} = -2\mathbf{H}$ by means of $\widetilde{\Lambda}$, the gradient-like vector field $\mathbb{Y}_{\mathbf{b}}$ is the one associated with $\mathbf{b} = -\mathbf{V}$ by means of G, and $\mathbb{Z}_{\mathcal{K}}$ is the linear vector field associated with the CPTP map $\mathcal{K}$ by means of (3.79).

Proof. *Let us start writing:*

$$\mathbf{L}(\bar{\xi}) = 2\left[\left[\mathbf{H}\,,\bar{\xi}\,\right]\right] - \mathbf{V}\odot\bar{\xi} + \mathcal{K}(\bar{\xi}) \equiv -2C_{\mathbf{H}}(\bar{\xi}) - A_{\mathbf{V}}(\bar{\xi}) + \mathcal{K}(\bar{\xi})\,, \tag{3.83}$$

where the linear maps $C_{\mathbf{H}}$ and $A_{\mathbf{V}}$ are given by:

$$C_{\mathbf{H}}(\xi) := [[\mathbf{H}\,,\bar{\xi}\,]] = c_{\sigma}^{\mu\nu} h_{\nu} x_{\mu}\, \mathbf{e}^{\sigma}\,,$$

and

$$A_{\mathbf{V}}(\bar{\xi}) := \mathbf{V}\odot\bar{\xi} = d_{\sigma}^{\mu\nu} V_{\nu} x_{\mu}\, \mathbf{e}^{\sigma}\,.$$

According to definition 10, and recalling that $\mathfrak{O}_n^* \ni \xi = x^{\mu}\,\mathbf{e}_{\mu}$ and $\mathfrak{O}_n \ni \bar{\xi} = x_{\mu}\,\mathbf{e}^{\mu}$ with $x^{\mu} = \delta^{\mu\nu}\,x_{\nu}$, we have:

$$Z_{C_{\mathbf{H}}} = c_{\sigma}^{\mu\nu} h_{\nu} x^{\sigma}\,\frac{\partial}{\partial x^{\mu}} = \widetilde{\mathbb{X}}_{\mathbf{H}}\,, \tag{3.84}$$

$$Z_{A_{\mathbf{V}}} = d_{\sigma}^{\mu\nu} V_{\nu} x^{\sigma}\,\frac{\partial}{\partial x^{\mu}} = \mathbb{Y}_{\mathbf{V}}\,, \tag{3.85}$$

where we have used the coordinate expressions of Hamiltonian and gradient-like vector fields given by equations (3.50) and (3.49). Since $\mathbf{L}$ is a linear combination of the three linear maps $C_{\mathbf{H}}(\bar{\xi})$, $A_{\mathbf{V}}(\bar{\xi})$, and $\mathcal{K}(\bar{\xi})$, the results follows from proposition 13.

Our aim is to define a vector field on $\mathfrak{T}_n^1$ representing the GKLS generator $\mathbf{L}$. We will construct such a vector field by means of a reduction procedure applied to Γ_L.

Proposition 15 (GKLS vector field on $\mathfrak{T}_n^1$)**.** *The linear vector field $\widetilde{\Gamma}$ defines a derivation, say Γ, of the algebra $\mathcal{F}(\mathfrak{T}_n^1)$. Furthermore, Γ will decompose into the sum of three vector fields all of which, spearately, define derivations of $\mathcal{F}(\mathfrak{T}_n^1)$. Specifically, we have:*

$$\Gamma = X_{\mathbf{a}} + Y_{\mathbf{b}} + Z_{\mathcal{K}}\,, \tag{3.86}$$

where $X_{\mathbf{a}}$ is the Hamiltonian vector field associated with $\mathbf{a} = -2\mathbf{H}$, $Y_{\mathbf{b}}$ is the gradient-like vector field associated with $\mathbf{b} = -\mathbf{V} = \sum_j \mathbf{v}_j^\dagger \mathbf{v}_j$, and $Z_{\mathcal{K}}$ is a vector field associated with the linear map $\mathcal{K}(\bar{\xi}) = \sum_j \mathbf{v}_j\,\bar{\xi}\,\mathbf{v}_j^\dagger$.

Proof. *We have to prove that $\mathcal{L}_{\widetilde{\Gamma}}\mathcal{I}_{\mathfrak{T}_n^1} \subset \mathcal{I}_{\mathfrak{T}_n^1}$, where $\mathcal{I}_{\mathfrak{T}_n^1} \subset \mathcal{F}(\mathfrak{O}_n^*)$ is the ideal of smooth functions vanishing on $\mathfrak{T}_n^1$.*

Let us start recalling equation (3.44), so that we can introduce the gradient-like vector field $\widetilde{\mathbb{Y}}_{\mathbf{b}}$ associated with the symmetric bivector field $\widetilde{\mathcal{R}}$:

$$\widetilde{\mathbb{Y}}_{\mathbf{b}} = \mathbb{Y}_{\mathbf{b}} - f_{\mathbf{b}}\,\widetilde{\Delta}\,. \tag{3.87}$$

From this, it follows that $\mathbb{Y}_{\mathbf{b}} = \widetilde{\mathbb{Y}}_{\mathbf{b}} + f_{\mathbf{b}}\,\widetilde{\Delta}$, and thus:

$$\widetilde{\Gamma} = \widetilde{\mathbb{X}}_{\mathbf{a}} + \mathbb{Y}_{\mathbf{b}} + \mathbb{Z}_{\mathcal{K}} = \widetilde{\mathbb{X}}_{\mathbf{a}} + \widetilde{\mathbb{Y}}_{\mathbf{b}} + f_{\mathbf{b}}\,\widetilde{\Delta} + \mathbb{Z}_{\mathcal{K}} \equiv \widetilde{\mathbb{X}}_{\mathbf{a}} + \widetilde{\mathbb{Y}}_{\mathbf{b}} + \widetilde{\mathbb{Z}}_{\mathcal{K}}\,. \tag{3.88}$$

We already know that $\widetilde{\mathbb{X}}_{\mathbf{a}}$ and $\widetilde{\mathbb{Y}}_{\mathbf{b}}$ define, separately, derivations of $\mathcal{F}(\mathfrak{T}_n^1)$. In particular, we know that the derivation associated with $\widetilde{\mathbb{X}}_{\mathbf{a}}$ is the Hamiltonian vector field $X_{\mathbf{a}}$, while the derivation associated with $\widetilde{\mathbb{Y}}_{\mathbf{b}}$ is the gradient-like vector field $Y_{\mathbf{b}}$.

In order to better understand $\widetilde{\mathbb{Z}}_{\mathcal{K}}$, we start writing the map $\mathcal{K}(\bar{\xi})$ as:

$$\mathcal{K}(\bar{\xi}) = Tr\left(\mathcal{K}(\bar{\xi})\bar{\mathbf{e}}_0\right)\mathbf{e}^0 + Tr\left(\mathcal{K}(\bar{\xi})\bar{\mathbf{e}}_k\right)\mathbf{e}^k \equiv A(\bar{\xi}) + B(\bar{\xi})\,, \tag{3.89}$$

where we have used the fact that $\mathbf{e}_\mu(\mathbf{e}^\nu) = Tr(\bar{\mathbf{e}}_\mu\,\mathbf{e}^\nu)$. From this equation it follows that:

$$\mathbb{Z}_{\mathcal{K}} = \mathbb{Z}_A + \mathbb{Z}_B\,. \tag{3.90}$$

Next, we look at the map A:

$$A(\bar{\xi}) = \sum_j Tr\left(\mathbf{v}_j\,\bar{\xi}\,\mathbf{v}_j^\dagger\,\bar{\mathbf{e}}_0\right)\mathbf{e}^0 = \sum_j Tr\left(\mathbf{v}_j^\dagger\mathbf{v}_j\,\bar{\xi}\right)\frac{\mathbf{e}^0}{\sqrt{n}} = \frac{f_{\mathbf{V}}(\xi)}{\sqrt{n}}\,\mathbf{e}^0\,. \tag{3.91}$$

Recalling that $\widetilde{\mathbb{Z}}_{\mathcal{K}} = \mathbb{Z}_{\mathcal{K}} - f_{\mathbf{V}}\,\widetilde{\Delta}$, we have:

$$\widetilde{\mathbb{Z}}_{\mathcal{K}} = \mathbb{Z}_A + \mathbb{Z}_B - f_{\mathbf{V}}\,\widetilde{\Delta} = f_{\mathbf{V}}\left(\frac{1}{\sqrt{n}} - x^0\right)\frac{\partial}{\partial x^0} - f_{\mathbf{V}}\,x^k\frac{\partial}{\partial x^k} + \mathbb{Z}_B\,. \tag{3.92}$$

The first term in the RHS clearly vanishes when we are on the hyperplane $x^0 = \frac{1}{\sqrt{n}}$ representing $\mathfrak{T}^1_n$ in $\mathfrak{O}^*_n$. This means that it defines a derivation of $\mathcal{F}(\mathfrak{T}^1_n)$ corresponding to the zero vector field. Furthermore, since the second and third terms in the RHS have no component along $\frac{\partial}{\partial x^0}$, they define, separately, derivations of the algebra $\mathcal{F}(\mathfrak{T}^1_n)$, that is, vector fields on $\mathfrak{T}^1_n$. We denote with Z_B the vector field on $\mathfrak{T}^1_n$ which is associated with the vector field $\mathbb{Z}_B$ on $\mathfrak{O}^*_n$, and with $f_{\mathbf{V}}\Delta$ the vector field on $\mathfrak{T}^1_n$ which is associated with the vector field $f_{\mathbf{V}}\,x^k\frac{\partial}{\partial x^k}$ on $\mathfrak{O}^*$. Recalling that $\mathfrak{O}^*_n \ni \xi = x^\mu\,\mathbf{e}_\mu$ and $\mathfrak{O}_n \ni \bar{\xi} = x_\mu\,\mathbf{e}^\mu$ with $x^\mu = \delta^{\mu\nu}\,x_\nu$, the coordinate expression of Z_B reads:

$$Z_B = \mathcal{K}^k_\mu\,x^\mu\,\frac{\partial}{\partial x^k} = Tr\left(\mathcal{K}(\mathbf{e}_\mu)\bar{\mathbf{e}}^k\right)\,x^\mu\,\frac{\partial}{\partial x^k}\,, \tag{3.93}$$

where $\mathbf{e}_\mu = \delta_{\mu\nu}\mathbf{e}^\mu$, $\bar{\mathbf{e}}^k = \delta_{jk}\bar{\mathbf{e}}_j$, and $x^0 = \frac{1}{\sqrt{n}}$ is implicitely assumed. In the end, $\widetilde{\mathbb{Z}}_{\mathcal{K}}$ defines a derivation of $\mathcal{F}(\mathfrak{T}^1_n)$ given by:

$$Z_{\mathcal{K}} = Z_B - f_{\mathbf{V}}\Delta\,. \tag{3.94}$$

Now:

$$\widetilde{\Gamma} = \widetilde{\mathbb{X}}_{\mathbf{a}} + \widetilde{\mathbb{Y}}_{\mathbf{b}} + \widetilde{\mathbb{Z}}_{\mathcal{K}}$$

is the sum of three vector fields defining, separately, derivations of $\mathcal{F}(\mathfrak{T}^1_n)$, and thus, $\widetilde{\Gamma}$ itself defines a derivation of $\mathcal{F}(\mathfrak{T}^1_n)$ which we denote with Γ.

Eventually, we find that the quantum dynamical evolution generated by the GKLS generator $\mathbf{L}$ of equation (3.77) is described by the following GKLS vector field Γ on $\mathfrak{T}^1_n$:

$$\Gamma = X_{\mathbf{a}} + Y_{\mathbf{b}} + Z_{\mathcal{K}}\,. \tag{3.95}$$

By construction, the integral curves of $X_{\mathbf{a}} + Y_{\mathbf{b}}$ starting at $\rho \in \mathcal{S}_k \subset \mathcal{S}$ remain in $\mathcal{S}_k$, and thus, it is the vector field $Z_{\mathcal{K}}$ which is responsible for the change of the rank of a quantum state.

Note that $Z_{\mathcal{K}}$, as well as $Y_{\mathbf{b}}$, contains a quadratic term with respect to the coordinate system $\{x^k\}_{k=1,\ldots,n^2-1}$ adapted to $\mathfrak{T}^1_n$ given by $f_{\mathbf{V}}\Delta$. Interestingly, these quadratic terms cancel out in the sum $Y_{\mathbf{b}} + Z_{\mathcal{K}}$, and thus, the GKLS vector field Γ representing a linear quantum dynamical process is an affine vector field on $\mathfrak{T}_1$.

Remark 13. *Inspired by the explicit form of $Z_{\mathcal{K}}$, we will give a general prescription to associate a vector field on $\mathfrak{T}_n^1$ with a CPTP map on $\mathfrak{O}_n^*$. Let*

$$A(\bar{\xi}) = \sum_j \mathbf{a}_j \, \bar{\xi} \, \mathbf{a}_j^\dagger$$

be a CPTP map from $\mathfrak{O}_n$ to $\mathfrak{O}_n$. Next, define $A^\sharp \colon \mathfrak{O}_n \to \mathfrak{O}_n$ as follows:

$$A^\sharp(\bar{\xi}) = \sum_j \mathbf{a}_j^\dagger \, \bar{\xi} \, \mathbf{a}_j \, . \tag{3.96}$$

It is clear that $A^\sharp$ is a completely-positive map according to Choi's theorem [35]. Now, we set:

$$Z_A := \left(A^k_\mu \, x^\mu - x^k \, f_{A^\sharp(\mathbf{e}^0)} \right) \frac{\partial}{\partial x^k} \, , \tag{3.97}$$

where $x^0 = \frac{1}{\sqrt{n}}$ is implicitely assumed. Note that this way of associating a vector field Z_A on $\mathfrak{T}_n^1$ with a CPTP map A on $\mathfrak{O}_n$ is completely unrelated with A being the CPTP map of some GKLS generator.

As said before, both $Y_{\mathbf{b}}$ and Z_A contain, in general, nonaffine parts with respect to the coordinate system $\{x^k\}_{k=1,\ldots,n^2-1}$ adapted to $\mathfrak{T}_n^1$. This means that their sum $Y_{\mathbf{b}} + Z_A$ is, in general, a nonaffine vector field on $\mathfrak{T}_n^1$. However, if we take

$$\mathbf{b} = - \sum_{j=1}^N \mathbf{v}_j^\dagger \, \mathbf{v}_j \, , \tag{3.98}$$

and

$$A(\bar{\xi}) = \sum_{j=1}^N \mathbf{v}_j \, \bar{\xi} \, \mathbf{v}_j^\dagger \, , \tag{3.99}$$

then the nonaffine terms in $Y_{\mathbf{b}}$ and Z_A cancel each other, and $Y_{\mathbf{b}} + Z_A$ becomes an affine vector field. This is precisely what happened in the construction of the GKLS vector field Γ of equation (3.86). We can not describe a linear quantum dynamical evolution using the vector field $\Gamma = X_{\mathbf{a}} + Y_{\mathbf{b}} + Z_A$, where $\mathbf{a}, \mathbf{b}$ and A are completely arbitrary. The linearity requirement for the evolution, which is equivalent to Γ being an affine vector field on $\mathfrak{T}_n^1$, forces us to fine-tune $Y_{\mathbf{b}}$ and Z_A using equations (3.98) and (3.99).

Example 2 (Phase damping of a qubit)**.** *We will now give the explicit expression of the GKLS vector field associated with the quantum dynamical process known as the phase damping of a qubit. For the notation, we refer to example 1.*

The GKLS generator for the phase damping is given by equation (3.77) with $\mathbf{H} = 0$*,* $N = 1$*, and* $\mathbf{v} \equiv \mathbf{v}_1 = \sqrt{\gamma}\,\sigma^3$*:*

$$\mathbf{L}(\bar{\xi}) = -\gamma\left(\bar{\xi} - \sigma^3\,\bar{\xi}\,\sigma^3\right)\,. \tag{3.100}$$

To compute the GKLS vector field Γ*, note that* $\mathbf{a} = 2\mathbf{H} = 0$ *implies* $X_{\mathbf{a}} = 0$*, and, since* $-\mathbf{b} = \mathbf{V} = \mathbf{v}^\dagger\,\mathbf{v} = \gamma\sigma_0$*, it is* $Y_{\mathbf{b}} = 0$*. It follows that* $\Gamma = Z_{\mathcal{K}}$*.*

Now:

$$Tr\left(\mathcal{K}(\bar{\xi})\bar{\sigma}_k\right) = \gamma Tr\left(\sigma^3\bar{\xi}\,\sigma^3\,\bar{\mathbf{e}}_k\right) = 2\gamma\,x_3\,\delta^3_k - \gamma\,x_l\,\delta^l_k\,, \tag{3.101}$$

and $f_{\mathbf{V}} = \gamma$*, which means: :*

$$\Gamma = Z_{\mathcal{K}} = -2\gamma\left(x^1\,\frac{\partial}{\partial x^1} + x^2\,\frac{\partial}{\partial x^2}\right)\,. \tag{3.102}$$

The flow Φ_τ *generated by* Γ *reads:*

$$\Phi_\tau(\xi) = \frac{1}{2}\,\left(\sigma_0 + \exp(-2\gamma\tau)\left(x^1\sigma_1 + x^2\sigma_2\right) + x^3\sigma_3\right)\,, \tag{3.103}$$

and it is clear that this dynamics only affects the phase terms (off-diagonal terms) of ξ *represented by its components along* σ_1 *and* σ_2*. All the quantum states lying on the* x^3 *axis are fixed points of the dynamics, and it is clear that every initial state will evolve towards its projection on the* x^3 *axis. Indeed, from the geometrical point of view, the integral curves of* Γ *are radial lines in a two-dimensional plane orthogonal to the* x^3*-axis. Therefore, the dynamical evolution of every initial state is always transversal to the spheres centered in* $x^1 = x^2 = x^3 = 0$*. These spheres represent isospectral quantum states, hence, the dynamics will change the spectrum of the quantum states giving rise to dissipation.*

Note that, for $\tau \leq 0$*, the flow of* Γ *takes a quantum state* ρ *out of* $\mathcal{S}$*, and thus, from the point of view of the space of states* $\mathcal{S}$*,* Γ *generates a one-parameter semigroup of transformations.*

Example 3 (Energy damping of a qubit)**.** *Let us now look at the dynamical evolution of a qubit associated with a GKLS generator having* $\mathbf{H} = 0$*,* $N = 1$*, and* $\mathbf{v} \equiv \mathbf{v}_1 = \sqrt{\gamma}\,(\sigma^1 + \imath\sigma^2)$*:*

$$\mathbf{L}(\bar{\xi}) = -\mathbf{V}\odot\bar{\xi} + \gamma\left(\sigma^1 + \imath\sigma^2\right)\,\bar{\xi}\,\left(\sigma^1 - \imath\sigma^2\right)\,. \tag{3.104}$$

Now, $\mathbf{a} = 2\mathbf{H} = 0 \Rightarrow X_{\mathbf{a}} = 0$ *as for the phase damping, however,* $\mathbf{b} = -\mathbf{V} = \mathbf{v}^{\dagger}\mathbf{v} = -2\gamma(\sigma^0 - \sigma^3)$*, and thus the gradient-like vector field* $Y_{\mathbf{b}}$ *reads:*

$$Y_{\mathbf{b}} = 2\gamma \left(\frac{\partial}{\partial x^3} - x^3\,\Delta \right) , \tag{3.105}$$

with Δ *the dilation vector field, and we see that it has a quadratic term. As stated before, we will see that the vector field* $Z_{\mathcal{K}}$ *contains a quadratic term which will cancel the quadratic term of* $Y_{\mathbf{b}}$*. This is a concrete instance of the fine-tuning between* $Y_{\mathbf{b}}$ *and* $Z_{\mathcal{K}}$ *imposed by the requirement of linearity for the quantum dynamics.*

Now, we have:

$$\begin{aligned} Tr\left(\mathbf{v}\,\bar{\xi}\,\mathbf{v}^{\dagger}\,\bar{\sigma}_k\right) &= \frac{1}{2}\left(Tr\left(\mathbf{v}\,\mathbf{v}^{\dagger}\,\bar{\sigma}_k\right) + x_l\,Tr\left(\mathbf{v}\,\sigma^l\,\mathbf{v}^{\dagger}\,\bar{\sigma}_k\right)\right) = \\ &= 2\gamma\left(x_l(\delta^{1l}\delta_{k1} + \delta^{2l}\delta_{k2} - \delta^l_k) + \delta_{k3}\right) \end{aligned} \tag{3.106}$$

and $f_{\mathbf{V}} = 2\gamma\,(1 - x^3)$*. Therefore:*

$$Z_{\mathcal{K}} = 2\gamma\left(1 - x^3\right)\,\frac{\partial}{\partial x^3} - 2\gamma\,\left(1 - x^3\right)\Delta\,. \tag{3.107}$$

Collecting the results we obtain the following form for the GKLS vector field:

$$\begin{aligned} \Gamma = Z_{\mathcal{K}} + Y_{\mathbf{b}} &= -2\gamma\left(x^1\,\frac{\partial}{\partial x^1} + x^2\,\frac{\partial}{\partial x^2}\right) + 4\gamma\left(1 - x^3\right)\,\frac{\partial}{\partial x^3} \\ &= -2\gamma\,\Delta + 2\gamma\left(2 - x^3\right)\,\frac{\partial}{\partial x^3}\,, \end{aligned} \tag{3.108}$$

where $\Delta = x^1\,\frac{\partial}{\partial x^1} + x^2\,\frac{\partial}{\partial x^2} + x^3\,\frac{\partial}{\partial x^3}$ *is the dilation vector field on* $\mathbb{R}^3$*. The quadratic terms in* $Z_{\mathcal{K}}$ *and* $Y_{\mathbf{b}}$ *canceled out, and we are left with an affine vector field as it should be. We stress the fact that this cancellation occurs because of the fine-tuning between* $Z_{\mathcal{K}}$ *and* $Y_{\mathbf{b}}$ *imposed by the linearity requirement for the quantum evolution.*

Looking at Γ*, we immediately see that it has a single fixed point, specifically, the pure state* $\rho_{\infty} = \frac{1}{2}(\sigma_0 + \sigma_3)$*. Furthermore, we realize that* Γ *is the sum of the GKLS vector field of the phase damping with the vector field* $4\gamma\,(1 - x^3)\,\frac{\partial}{\partial x^3}$*. These two vector fields commute, and thus the flow* Φ_{τ} *of their sum can be written as the composition of their flows. The specific expression is:*

$$\Phi_{\tau}(\xi) = \frac{1}{2}\,\left(\sigma_0 + \mathrm{e}^{-\gamma\tau}\left(x^1\sigma_1 + x^2\sigma_2\right) + \left(\mathrm{e}^{-4\gamma\tau}\left(x^3 - 1\right) + 1\right)\sigma_3\right)\,. \tag{3.109}$$

The asymptotic behaviour of this dynamics is quite interesting. Indeed, every initial state evolves toward the common asymptotic pure quantum state $\rho_\infty = \frac{1}{2}(\sigma_0 + \sigma_3)$. On the one hand, if we start from an initial mixed quantum state, the dynamical evolution may be read as a "purification" process, which, however, is accomplished only in the limit $\tau \to +\infty$. On the other hand, if we start from an initial pure quantum state, the dynamics will immediately destroy its purity turning it into a mixed state, and then it will start to "purify" it again. We see here a "collapse" and "revival" phenomenon.

3.3.1 LaSalle principle and asymptotic behaviour

Now that we have written the GKLS generator of a linear quantum dynamical process in terms of a vector field Γ on the affine manifold $\mathfrak{T}^1_n$, we are able to use all the tools from the theory of dynamical systems in the quantum context. This will help us, for example, in analyzing the stability properties of a given quantum dynamical process.

First of all, let us recall that the fixed points of the dynamical evolution associated with the GKLS vector field Γ are all those points ξ_f such that $\Gamma(\xi_f) = 0$. Denoting with Φ_τ the flow of Γ, we have that $\Phi_\tau(\xi_f) = \xi_f$ for all τ. Fixed points are of primary interest in the stability theory of fixed points of dynamical systems. Essentially, given a fixed point ξ_f, stability theory consists of understanding the long time behaviour of the dynamical trajectories with initial conditions belonging to a neighbourhood of ξ_f. The literature on the subject is mainly focused on classical physical systems. What is interesting, is that the geometric reformulation of quantum dynamics we have achieved in the previous section allows us to make good use of the results of stability theory directly in the quantum case. We do not want to enter into a detailed and exhaustive discussion of the stability theory of quantum dynamical evolutions. Our main scope is to show how the geometric formulation of the GKLS dynamics can be used to gain physical intuition on quantum situations using mathematical tools coming from classical physics.

Let us start recalling the so-called LaSalle invariance principle (see [81, 2]):

Theorem 5. *Let B be a finite-dimensional Banach space, and let $M \subseteq B$ be a finite-dimensional differential manifold. Consider the smooth dynamical system associated with the complete vector field Γ. Let Ω be a compact set in M that is invariant under the flow Φ_τ of Γ for $\tau \geq 0$. Let $F\colon M \to \mathbb{R}$ be a smooth function such that $F \geq 0$ on Ω, and assume that:*

$$\mathcal{L}_\Gamma F \leq 0 \tag{3.110}$$

on Ω. Let S_∞ be the largest invariant set in Ω, for $\tau \in \mathbb{R}$, where $\mathcal{L}_\Gamma F = 0$. If $m \in \Omega$, then:

$$\lim_{\tau \to +\infty} \left(\inf_{m^* \in S_\infty} ||\Phi_\tau(m) - m^*|| \right) = 0 \tag{3.111}$$

where $||\cdot||$ is the norm of B . In particular, if S_∞ is an isolated fixed point of Γ, it is asymptotically stable.

We call a function F satisfying the hypotesis of theorem 5 a LaSalle function for the vector field Γ. In the following, we will see how theorem 5 may be applied to the quantum case at hand, namely, how to find LaSalle functions for the GKLS dynamical evolutions. Referring to theorem 5, we will take $B = \mathfrak{O}_n^*$, $M = \mathfrak{T}_n^1$, Γ the GKLS vector field of the semigroup at hand, and $\Omega = \mathcal{S}$, where $\mathcal{S}$ is the space of quantum states. As usual, we will denote with ρ a generic quantum state in $\mathcal{S}$.

Example 4 (Energy damping of a qubit II)**.** *Let us consider the GKLS evolution of example 3:*

$$\Gamma = -2\gamma \left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} \right) + 4\gamma \left(1 - x^3\right) \frac{\partial}{\partial x^3} \,. \tag{3.112}$$

We already know that every initial quantum state ρ tends asymptotically to the pure state $\rho_\infty = \frac{1}{2}(\sigma_0 + \sigma_3)$. Consequently, it is reasonable to suppose that the distance between a quantum state and ρ_∞ decreases along the dynamical trajectories, that is, it is a LaSalle function for Γ. The notion of distance we are referring to is that associated with the Banach space structure of $\mathfrak{O}_2$ and obsp$_2^*$, specifically, the so-called trace distance:

$$D(\xi \,, \eta) := Tr\left((\bar{\xi} - \bar{\eta})(\bar{\xi} - \bar{\eta})\right) \,. \tag{3.113}$$

We will now show that, in accordance with the results of example 3, the function:

$$F(\xi) := D(\xi \,, \rho_\infty) \tag{3.114}$$

is a LaSalle function for Γ. At this purpose, let us note that:

$$F(\xi) := D(\xi \,, \rho_\infty) = \frac{\delta_{jk}\, x^j\, x^k - 2x^3}{4} \,, \tag{3.115}$$

and let us compute:

$$\mathcal{L}_\Gamma\, F = \frac{\partial F}{\partial x^j}\, \Gamma^j = -\gamma \left((x^1)^2 + (x^2)^2 + 2(1 - x^3)^2\right) \,. \tag{3.116}$$

It is clear that $\mathcal{L}_\Gamma F \leq 0$ for every $\xi \in \mathfrak{O}_2^*$, and thus, in particular, for every quantum state $\rho \in \mathcal{S}$. According to theorem 5, the function F is a LaSalle function for Γ, and the set S_∞ is the largest invariant set in $\mathcal{S}$ characterized by the fact that:

$$\mathcal{L}_\Gamma F|_{S_\infty} = 0\,. \tag{3.117}$$

It is immediate to see that $\mathcal{L}_\Gamma F = 0$ if and only if $x^3 = 1$, which means that $S_\infty = \{\rho_\infty\}$, with $\rho_\infty = \frac{1}{2}(\sigma_0 + \sigma_3)$ as expected.

We will now consider the function:

$$F(\xi) := \frac{\chi(\xi)}{2} = \frac{Tr(\bar{\xi}^2)}{2}\,, \tag{3.118}$$

where $\chi(\xi) = Tr(\bar{\xi}^2)$ is the purity function. This is a smooth function on $\mathfrak{T}_1$ such that $F(\rho) \geq 0$. It is connected to the so-called linearized entropy function:

$$S_L(\xi) = 1 - \chi(\xi) = 1 - 2F(\xi)\,. \tag{3.119}$$

We will analyze the so-called quantum random unitary semigroups, of which quantum Poisson semigroups and quantum Gaussian semigroups are particular instances (see [82, 78, 11]), and we will show that the purity function is a LaSalle function for the GKLS vector field associated with these semigroups in every dimension[6]. The GKLS generator $\mathbf{L}$ for the quantum random unitary semigroups is characterized by the following form [11] :

$$\mathbf{L}(\bar{\xi}) = -2[[\mathbf{H}\,,\bar{\xi}\,]] - \mathbf{V}\odot\bar{\xi} + \sum_{j=1}^{n^2-1} \alpha_j \mathbf{e}^j\,\bar{\xi}\,\mathbf{e}^j + \beta \sum_{j=1}^{r\leq n^2-1} p_j \mathbf{U}_j\,\xi\,\mathbf{U}_j^\dagger - \beta\xi\,, \tag{3.120}$$

where $\mathbf{H} \in \mathfrak{A}_n$ is self-adjoint, α, β are non-negative real numbers, $\{p_j\}_{j=1,..r}$ is a probability vector, $\{\mathbf{e}^j\}_{j=1,\dots,n^2-1}$ is an orthonormal set of self-adjoint operators in $\mathfrak{A}_n$, $\mathbf{V} = \sum_{j=1}^{n^2-1}\alpha_j\mathbf{e}_j^2$, and U_j is unitary for all j.

We will break the problem in steps. We will start analyzing the so-called quantum Poisson semigroups. These form a subclass of the quantum random unitary semigroups for which $\mathbf{H} = 0$, $\alpha_j = 0$, $\beta = 1$, and $r = 1$:

$$\mathbf{L}(\bar{\xi}) = -\bar{\xi} + \mathbf{U}\,\bar{\xi}\,\mathbf{U}^\dagger\,. \tag{3.121}$$

[6] Note that the energy damping of the qubit studied in examples 3 and 4 does not belong to this class of GKLS evolutions.

Next, we will consider "positive linear combinations" of quantum Poisson semigroups, and add to them a Hamiltonian term. After the quantum Poisson semigroups, we will focus on the so-called quantum Gaussian semigroups. Again, these semigroups form a subclass of the quantum random unitary semigroups. Their GKLS generator is characterized by $\mathbf{H} = 0$, $p_j = 0$, $\alpha_1 = 1$ and $\alpha_j = 0$ for $j \geq 2$:

$$\mathbf{L}(\bar{\xi}) = -\mathbf{v}^2 \odot \bar{\xi} + \mathbf{v}\,\bar{\xi}\,\mathbf{v}\,. \tag{3.122}$$

We will proceed considering "positive linear combinations" of quantum Gaussian semigroups, and adding to them a Hamiltonian term. Finally, we will consider the general case of quantum random unitary semigroups.

From this result we conclude that quantum random unitary semigroups always minimize the (trace) distance between a given quantum state and the maximally mixed state ρ_m, indeed, it is easy to see that the distance between the maximally mixed state ρ_m and every quantum state ρ always decreases on the dynamical trajectories of the quantum random unitary semigroups because F is a LaSalle function for these evolutions. To see this, consider the distance function $D(\xi\,,\rho_m)$, where ρ_m is the maximally mixed quantum state. Recalling that $\bar{\rho}_m = \frac{\mathbb{I}}{n}$, we have:

$$D(\xi\,,\rho_m) = Tr\left((\bar{\xi} - \bar{\rho}_m)^2\right) = Tr\left(\bar{\xi}^2\right) - \frac{2}{n}\,Tr\left(\bar{\xi}\right) + \frac{1}{n}\,, \tag{3.123}$$

from which it follows that on quantum states:

$$D(\rho\,,\rho_m) = 2F(\rho) - \frac{1}{n}\,, \tag{3.124}$$

and thus:

$$\mathcal{L}_\Gamma\left(D(\xi\,,\rho_m)\right)\big|_{\mathcal{S}} = \mathcal{L}_\Gamma\left(2F(\xi)\right)\big|_{\mathcal{S}} \leq 0\,, \tag{3.125}$$

where Γ is the GKLS vector field associated with a quantum random unitary semigroup.

Quantum Poisson semigroups

Quantum Poisson semigroups are characterized by a GKLS generator $\mathbf{L}$ with $\mathbf{H} = 0$, and with a single, unitary Kraus operator $\mathbf{v} = \mathbf{U}$ [82, 78, 11]:

$$\mathbf{L}(\bar{\xi}) = -\bar{\xi} + \mathbf{U}\,\bar{\xi}\,\mathbf{U}^\dagger\,. \tag{3.126}$$

Since $\mathbf{V} = \mathbf{U}^\dagger\,\mathbf{U} = \mathbb{I}$, it follows from equation (3.49) that the gradient-like vector field $Y_\mathbf{b}$ in the GKLS vector field Γ describing $\mathbf{L}$ is zero. Being $\mathbf{H} = \mathbf{0}$, the Hamiltonian vector field $X_{\mathbf{a}}$ is zero too, and we are left only with the vector field $Z_{\mathcal{K}}$. Concerning this vector field, we note that $f_{\mathbf{V}} = 1$, so that:

$$Z_K = Z_B - \Delta \,. \tag{3.127}$$

Then, we note that:

$$B(\bar{\xi}) = Tr\left(\mathbf{U}\,\bar{\xi}\,\mathbf{U}^{\dagger}\,\bar{\mathbf{e}}_k\right)\mathbf{e}^k \ , \tag{3.128}$$

and thus (see equation (3.93)):

$$\Gamma = Z_K = Z_B - \Delta = Tr\left(\mathbf{U}\,\mathbf{e}_l\,\mathbf{U}^{\dagger}\,\bar{\mathbf{e}}^k\right)x^l\,\frac{\partial}{\partial x^k} - \Delta \,. \tag{3.129}$$

An easy calculation, shows that the fixed points for the dynamical system associated with Γ are all those $\bar{\xi}_f$ such that $[\mathbf{U}\,,\bar{\xi}_f] = 0$, in particular, the maximally mixed state $\rho = \frac{\mathbf{e}_0}{\sqrt{n}} = \frac{\mathbb{I}}{n}$ is a fixed point for every choiche of the operator $\mathbf{U}$. We will show that $F(\xi)$ is a LaSalle function for Γ.

Proposition 16. *The function:*

$$F(\xi) = \frac{Tr\left(\bar{\xi}^2\right)}{2} \tag{3.130}$$

is a LaSalle function for the GKLS vector field Γ of equation (3.129) representing a quantum Poisson semigroup.

Proof. *Writing*

$$\rho = \frac{1}{\sqrt{n}}\mathbf{e}_0 + \rho_* = \frac{1}{\sqrt{n}}\mathbf{e}_0 + x^j\,\mathbf{e}_j \,, \tag{3.131}$$

it is:

$$F(\rho) = \frac{1}{2n} + \frac{\delta_{jk}x^j x^k}{2} \,. \tag{3.132}$$

Therefore:

$$\left.\frac{\partial F}{\partial x^k}\right|_{\rho} = \delta_{jk}x^j \,. \tag{3.133}$$

Consequently:

$$\mathcal{L}_\Gamma F\big|_\rho = \frac{\partial F}{\partial x^k}\,\Gamma^k = \delta_{jk}x^j\left(Tr\left(\mathbf{U}\,\mathbf{e}_l\,\mathbf{U}^\dagger\,\bar{\mathbf{e}}^k\right)x^l - x^k\right) =$$
$$= Tr\left(\mathbf{U}\,\bar{\rho}_*\,\mathbf{U}^\dagger\,\bar{\rho}_*\right) - Tr\left(\bar{\rho}_*\,\bar{\rho}_*\right)\,. \tag{3.134}$$

By direct computation, we see that:

$$Tr\left(\mathbf{U}\,\bar{\rho}\,\mathbf{U}^\dagger\,\bar{\rho}\right) - Tr\left(\bar{\rho}\,\bar{\rho}\right) = Tr\left(\mathbf{U}\,\bar{\rho}_*\,\mathbf{U}^\dagger\,\bar{\rho}_*\right) - Tr\left(\bar{\rho}_*\,\bar{\rho}_*\right)\,. \tag{3.135}$$

Writing $\bar{\rho}_U = \mathbf{U}\,\bar{\rho}\,\mathbf{U}^\dagger$*, we have:*

$$\mathcal{L}_\Gamma F\big|_\rho = Tr\left(\bar{\rho}_U\,\bar{\rho}\right) - Tr\left(\bar{\rho}\,\bar{\rho}\right)\,. \tag{3.136}$$

The expression $Tr\left(\bar{\rho}_U\,\bar{\rho}\right)$ *on the RHS is nothing but the Euclidean scalar product, in the Euclidean vector space* $\mathfrak{O}^*$*, between the vectors* ρ_U *and* ρ*. Analogously,* $Tr\left(\bar{\rho}\,\bar{\rho}\right) \equiv |\rho|^2$ *is the scalar product between* ρ *and itself. Therefore:*

$$\mathcal{L}_\Gamma F\big|_\rho = |\rho_U|\,|\rho|\,\cos(\theta) - |\rho|^2\,, \tag{3.137}$$

where θ *is the angle between* ρ *and* ρ_U*. Being* $\bar{\rho}_U = \mathbf{U}\,\bar{\rho}\,\mathbf{U}^\dagger$*, it follows that* $|\rho_U| = |\rho|$ *and thus:*

$$\mathcal{L}_\Gamma F\big|_\rho = |\rho|^2\,\left(\cos(\theta) - 1\right) \leq 0\,, \tag{3.138}$$

where the equality holds if and only if $\rho_U = \rho$*. This means that theorem 5 applies, and thus* F *is a LaSalle function for* Γ *as claimed.*

According to theorem 5, the accumulating set S_∞ is the largest invariant subset in:

$$E := \left\{\rho \in \mathcal{S}:\; \mathcal{L}_\Gamma F\big|_\rho = 0\right\}\,. \tag{3.139}$$

From equation (3.138), we have that E coincide with the intersection of the space of states $\mathcal{S}$ with the set of fixed points of the GKLS vector field Γ. Since every $\rho \in E$ is a fixed point, the set E is an invariant set for the dynamics, and thus $E = S_\infty$. From the practical point of view, we can see that S_∞ is the intersection of the commutant[7] $\mathcal{C}_\mathbf{U}$ of $\mathbf{U}$ with the space of states $\mathcal{S}$. The comutant $\mathcal{C}_\mathbf{U}$ is a vector space (actually, an algebra with respect to the operator product) the dimension d_U of which depends on the degenerancy of the spectrum of $\mathbf{U}$. Specifically, it is:

[7]The commutant $\mathcal{C}_\mathbf{A}$ of $\mathbf{A} \in \mathcal{B}(\mathcal{H})$ is the set of all elements in $\mathcal{B}(\mathcal{H})$ commuting with $\mathbf{A}$.

$$d_U = \sum_{j=1}^{m} (d_j)^2 \,, \tag{3.140}$$

where m is the number of different eigenvalues of $\mathbf{U}$, and d_j denotes the degenerancy of the j-th eigenvalue of $\mathbf{U}$. Consequently, the more degenerancy in the spectrum of $\mathbf{U}$, the bigger is the accumulating set S_∞.

Example 5 (Phase damping of a qubit revisited I)**.** *Let us come back to the phase damping studied in example 2. Because of the peculiar properties of the Pauli matrices, we have that* $\mathbf{v}$ *is unitary when we set* $\gamma = 1$*. The GKLS vector field in equation (3.102) becomes:*

$$\Gamma = -2\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}\right) \,, \tag{3.141}$$

and its associated flow Φ_τ *is:*

$$\Phi_\tau(\xi) = \frac{1}{2}\,\left(\sigma_0 + \exp(-2\tau)\left(x^1\sigma_1 + x^2\sigma_2\right) + x^3\sigma_3\right) \,. \tag{3.142}$$

It is easy to see that the set S_∞ *is precisely the intersection between the* x^3*-axis and the Bloch-ball. Furthermore, from the explicit form of* Φ_τ *it follows that the initial state* ρ *tends to the state* ρ_∞ *which is the projection of* ρ *onto the* x^3*-axis.*

We can go a little further and analyze the "positive linear combinations" of quantum Poisson semigroups with a Hamiltonian term. These are all those quantum dynamics characterized by a GKLS generator $\mathbf{L}$ for which $\mathbf{H} \neq \mathbf{0}$, $\mathbf{v}_j = \alpha_j \mathbf{U}_j$ with $\mathbf{U}_j$ unitary and $\alpha_j \in \mathbb{C}$ for all j.

$$\mathbf{L}(\bar{\xi}) = -2\left[\left[\mathbf{H}\,, \bar{\xi}\,\right]\right] - \sum_{j=1}^{N} |\alpha_j|^2 \bar{\xi} + \sum_{j=1}^{N} |\alpha_j|^2 \mathbf{U}_j\,\bar{\xi}\,\mathbf{U}_j^\dagger \,. \tag{3.143}$$

Again, there is no gradient-like contribution in the GKLS vector field Γ, however, being $\mathbf{H} \neq \mathbf{0}$, there is a Hamiltonian contribution. An explicit calculation shows that (see equation (3.93)):

$$\Gamma = \sum_j |\alpha_j|^2\, \Gamma_j - 2X_{\mathbf{H}} =$$

$$= \sum_j |\alpha_j|^2 \left(Tr\left(\mathbf{U}_j\,\mathbf{e}_l\,\mathbf{U}_j^\dagger\,\bar{\mathbf{e}}^k\right) x^l \frac{\partial}{\partial x^k} - \Delta \right) - 2Tr\left(\left[\left[\mathbf{H}\,, \mathbf{e}_l\right]\right]\bar{\mathbf{e}}^k\right) x^l \frac{\partial}{\partial x^k} \,. \tag{3.144}$$

where Γ_j is the GKLS vector field of the quantum Poisson semigroup associated with $\mathbf{U}_j$. The fixed points of Γ are now all those ξ_f such that:

$$\sum_j |\alpha_j|^2 \left(\mathbf{U}\,\bar{\xi}_f\,\mathbf{U}^\dagger - \bar{\xi}_f\right) = 2[[\mathbf{H}\,,\bar{\xi}_f\,]]\,. \tag{3.145}$$

Proposition 17. *The function:*

$$F(\xi) = \frac{Tr\left(\bar{\xi}^2\right)}{2} \tag{3.146}$$

is a LaSalle function for the GKLS vector field Γ of equation (3.144).

Proof. *Because of the linearity of the Lie derivative, we have:*

$$\mathcal{L}_\Gamma F\big|_\rho = \sum_j |\alpha_j|^2 \left(\mathcal{L}_{\Gamma_j} F\big|_\rho\right) - 2\,\mathcal{L}_{X_\mathbf{H}} F\big|_\rho\,. \tag{3.147}$$

The Lie derivative of the function F with respect to $X_\mathbf{H}$ is easily seen to be zero. To see this, recall that the flow of the Hamiltonian vector field $X_\mathbf{H}$ is given by

$$\bar{\Phi}_\tau(\bar{\xi}) = \exp(-\imath\tau\mathbf{H})\,\bar{\xi}\,\exp(\imath\tau\mathbf{H})\,, \tag{3.148}$$

and thus ξ and $\Phi_\tau(\xi)$ have the same spectrum (see definition 3) for all τ. Now, we can write

$$F(\xi) = \sum_k \lambda_k^2\,,$$

with λ_k the k-th eigenvalue of $\bar{\xi}$, from which it follows that

$$F(\xi) = F(\Phi_\tau(\xi))$$

for all τ, which is equivalent to $\mathcal{L}_{X_\mathbf{H}} F = 0$. Consequently:

$$\begin{aligned}\mathcal{L}_\Gamma F\big|_\rho &= \sum_j |\alpha_j|^2 \left(\mathcal{L}_{\Gamma_j} F\big|_\rho\right) - 2\,\mathcal{L}_{X_\mathbf{H}} F\big|_\rho = \\ &= \sum_j\, |\alpha_j|^2\,|\rho|^2\,\left(\cos(\theta_j) - 1\right) \leq 0\,,\end{aligned} \tag{3.149}$$

where θ_j is the angle between $\bar{\rho}$ and $\bar{\rho}_{U_j} = \mathbf{U}_j\,\bar{\rho}\,\mathbf{U}_j^\dagger$. This means that theorem 5 applies, and the proposition is proved.

Note that the Lie derivative in equation (3.149) is zero if and only if $\bar{\rho}$ commutes with $\mathbf{U}_j$ for all j, and thus:

$$E := \left\{\rho \in \mathcal{S} :\ \mathcal{L}_\Gamma F\big|_\rho = 0\right\}\,, \tag{3.150}$$

coincides with the intersection of $\mathcal{S}$ with the intersection of the commutants $\mathcal{C}_{\mathbf{U}_j}$. The accumulating set S_∞ is then the largest invariant set in E, and the maximally mixed state is always in S_∞. Clearly, S_∞ highly depends on the spectral properties of the unitary operators $\mathbf{U}_j$. One could be tempted to say that the Hamiltonian part plays no role in this discussion since the Lie derivative of the LaSalle function with respect to the Hamiltonian vector field vanishes. However, we know that $S_\infty \subseteq E$ must be an invariant set with respect to the total dynamics of the system, and the Hamiltonian part of Γ obviously takes part in determing the explicit form of the dynamical trajectories.

Remark 14. *In the qubit case, it is enough to take $N = 2$ with $\mathbf{v}_1, \mathbf{v}_2$ any couple of different Pauli matrices (except the identity) in order for S_∞ to coincide with the singleton represented by the maximally mixed state.*

Example 6 (Phase damping of a qubit, revisited II)**.** *Let us come back to the phase damping studied in example 5, and denote with Γ_U its GKLS vector field. We want to understand what happens when we add a Hamiltonian term to the GKLS vector field Γ_U. The resulting GKLS vector field is:*

$$\Gamma = X_{\mathbf{a}} + \Gamma_U\,. \tag{3.151}$$

Now, let us write the most general expression for the Hamiltonian vector field $X_{\mathbf{a}}$:

$$X_{\mathbf{a}} = \left(h_3 x^2 - h_2 x^3\right)\frac{\partial}{\partial x^1} + \left(h_3 x^1 - h_1 x^3\right)\frac{\partial}{\partial x^2} + \left(h_1 x^2 - h_2 x^1\right)\frac{\partial}{\partial x^3}\,, \tag{3.152}$$

where $\mathbf{a} = -2\mathbf{H} = -2h_\mu\,\sigma^\mu$. Recall that the fixed points of Γ_U are all those points ξ such that $[\mathbf{U}\,,\bar{\xi}\,] = 0$, while the fixed points of $X_{\mathbf{a}}$ are all those points ξ such that $[\mathbf{H}\,,\bar{\xi}\,] = 0$. A direct computation shows that:

$$[X_{\mathbf{a}}\,,\Gamma_U] = h^2 x^3\,\frac{\partial}{\partial x^1} + h^1 x^3\,\frac{\partial}{\partial x^2}\,. \tag{3.153}$$

From this, it follows that $[X_{\mathbf{a}}\,,\Gamma_U] = 0$ if and only if $\mathbf{H} = h_3\sigma^3$. In this case, all the points in E are fixed points of Γ because$[\mathbf{U}\,,\mathbf{H}] = \mathbf{0}$. Consequently, $E = S_\infty$ exactly as in example 5.

On the other hand, if $\mathbf{H} = h_1\sigma^1 + h_2\sigma^2$*, the situation changes drastically. Indeed, let us take* $h_1 = 1$*,* $h_2 = h_3 = 0$*. The GKLS vector field becomes:*

$$\begin{aligned}\Gamma &= -2x^1 \frac{\partial}{\partial x^1} - 2\left(x^2 + x^3\right) \frac{\partial}{\partial x^2} + 2x^2 \frac{\partial}{\partial x^3} \\ &= -2\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}\right) - 2x^3 \frac{\partial}{\partial x^2} + 2x^2 \frac{\partial}{\partial x^3} \,.\end{aligned} \tag{3.154}$$

This vector field has a single fixed point, namely, the maximally mixed state $\rho_m = \frac{\mathbb{I}}{2}$*. It is clear that* Γ *is the sum of two vector fields, one representing a rotation in the* $x^2 - x^3$ *hyperplane, the other representing a dilation in the* $x^1 - x^2$ *hyperplane. However, since these vector fields do not commute, the flow of* Γ *is not the simple composition of the flows of these two vector fields. The explicit form of* Φ_τ *reads:*

$$\Phi_\tau(\xi) = \begin{cases} x^1(\tau) &= x^1\,\mathrm{e}^{-2\tau} \\ x^2(\tau) &= -\mathrm{e}^{-\tau}\left(x^2\left(\frac{\sin(\sqrt{3}\tau)}{\sqrt{3}} - \cos(\sqrt{3}\tau)\right) + \frac{2x^3}{\sqrt{3}}\sin(\sqrt{3}\tau)\right) \\ x^3(\tau) &= \mathrm{e}^{-\tau}\left(\frac{2x^2}{\sqrt{3}}\sin(\sqrt{3}\tau) + x^3\left(\frac{\sin(\sqrt{3}\tau)}{\sqrt{3}} + \cos(\sqrt{3}\tau)\right)\right) \end{cases} . \tag{3.155}$$

Now, the set E *consits of all those quantum states* ρ *commuting with* $\mathbf{U}$*. It is clear that, given* $\rho \in E$*, it is* $\Phi_\tau(\rho) \in \rho$ *if and only if* $\rho = \rho_m$ *wiht* $\rho_m = \frac{1}{2}\sigma_0$ *the maximally mixed state. Otherwise,* ρ *is mapped outside* E *by the dynamical evolution* Φ_τ*. This means that the largest invariant set in* E *is the singleton* $\{\rho_m\}$*, and we conclude that* $S_\infty = \{\rho_m\}$*. The Hamiltonian term* $X_{\mathbf{a}}$ *has thus changed the long-time behaviour of the dynamics making the maximally mixed state the limiting point of the dynamical evolution of every quantum state* ρ*.*

Quantum Gaussian semigroups

A quantum Gaussian semigroup [82, 78, 11] is characterized by GKLS generator $\mathbf{L}$ having $\mathbf{H} = \mathbf{0}$, $N = 1$ and $\mathbf{v} \equiv \mathbf{v}_1$ self-adjoint:

$$\mathbf{L}(\bar{\xi}) = -\mathbf{v}^2 \odot \bar{\xi} + \mathbf{v}\,\bar{\xi}\,\mathbf{v}\,. \tag{3.156}$$

An explicit calculation shows that the GKLS vector field is (see equations (3.49) and (3.93)):

$$\Gamma = Z_{\mathcal{K}} + Y_{\mathbf{b}} = \left(Tr\left(\mathbf{v}\,\bar{\xi}_*\,\mathbf{v}\,\bar{\mathbf{e}}^k\right) - Tr\left(\mathbf{v}^2\,\bar{\xi}_*\,\bar{\mathbf{e}}^k\right)\right) \frac{\partial}{\partial x^k}\,, \tag{3.157}$$

where we have used the notation:

$$\xi = \frac{1}{\sqrt{n}}\mathbf{e}_0 + \xi_* = \frac{1}{\sqrt{n}}\mathbf{e}_0 + x^j\mathbf{e}_j \,. \tag{3.158}$$

Since:

$$Tr\left(\mathbf{v}\,\bar{\xi}\,\mathbf{v}\,\bar{\mathbf{e}}^k\right) = Tr\left(\mathbf{v}\,\bar{\xi}_*\,\mathbf{v}\,\bar{\mathbf{e}}^k\right) + \frac{1}{n}Tr\left(\mathbf{v}^2\,\bar{\mathbf{e}}^k\right)\,, \tag{3.159}$$

and:

$$Tr\left(\mathbf{v}^2\,\bar{\xi}\,\bar{\mathbf{e}}^k\right) = Tr\left(\mathbf{v}^2\,\bar{\xi}_*\,\bar{\mathbf{e}}^k\right) + \frac{1}{n}\,Tr\left(\mathbf{v}^2\,\bar{\mathbf{e}}^k\right)\,, \tag{3.160}$$

we may write the GKLS vector field Γ as follows:

$$\Gamma = \left(Tr\left(\mathbf{v}\,\bar{\xi}\,\mathbf{v}\,\bar{\mathbf{e}}^k\right) - Tr\left(\mathbf{v}^2\,\bar{\xi}\,\bar{\mathbf{e}}^k\right)\right)\,\frac{\partial}{\partial x^k}\,. \tag{3.161}$$

The fixed points of Γ are all those ξ_f such that:

$$Tr\left(\mathbf{v}\,\left[\bar{\xi}_f\,,\mathbf{v}\right]\,\bar{\mathbf{e}}^k\right) = 0 \quad \forall k = 1,..n^2-1\,. \tag{3.162}$$

Proposition 18. *The function:*

$$F(\xi) = \frac{Tr\left(\bar{\xi}^2\right)}{2} \tag{3.163}$$

is a LaSalle function for the GKLS vector field Γ of equation (3.161).

Proof. *By direct computation we have:*

$$\mathcal{L}_\Gamma F\big|_\rho = \delta_{jk}x^j\,\left(Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\mathbf{e}}^k\right) - Tr\left(\mathbf{v}^2\,\bar{\rho}\,\bar{\mathbf{e}}^k\right)\right) = Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\rho}_*\right) - Tr\left(\mathbf{v}^2\,\bar{\rho}\,\bar{\rho}_*\right)\,, \tag{3.164}$$

where we have used equation (3.133) for the derivative of F. Since:

$$Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\rho}\right) = Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\rho}_*\right) + \frac{1}{n}\,Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\right)\,, \tag{3.165}$$

and:

$$Tr\left(\mathbf{v}^2\,\bar{\rho}\,\bar{\rho}\right) = Tr\left(\mathbf{v}^2\,\bar{\rho}\,\bar{\rho}_*\right) + \frac{1}{n}\,Tr\left(\mathbf{v}^2\,\bar{\rho}\right)\,, \tag{3.166}$$

we may write:

$$\mathcal{L}_{\Gamma}F\big|_{\rho} = Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\rho}\right) - Tr\left(\mathbf{v}^2\,\bar{\rho}^2\right) \,. \tag{3.167}$$

Note that the first and second terms in the RHS are both positive real numbers. Now, the first term in the RHS is the inner product

$$\langle\bar{\rho}\,\mathbf{v}|\mathbf{v}\,\bar{\rho}\rangle_{\mathfrak{A}_n} = Tr\left((\bar{\rho}\mathbf{v})^{\dagger}\,\mathbf{v}\,\bar{\rho}\right) = Tr\left(\mathbf{v}\,\bar{\rho}\,\mathbf{v}\,\bar{\rho}\right)$$

between $\bar{\rho}\,\mathbf{v}$ and $\mathbf{v}\,\bar{\rho}$. Analogously, the second term in the RHS is the inner product $\langle\bar{\rho}\,\mathbf{v}|\bar{\rho}\,\mathbf{v}\rangle_{\mathfrak{A}_n} = |\bar{\rho}\,\mathbf{v}|^2$ between $\bar{\rho}\,\mathbf{v}$ and itself. Being $\langle\bar{\rho}\,\mathbf{v}|\mathbf{v}\,\bar{\rho}\rangle_{\mathfrak{A}_n}$ real and positive, we may write:

$$\langle\bar{\rho}\,\mathbf{v}|\mathbf{v}\,\bar{\rho}\rangle_{\mathfrak{A}_n} = |\bar{\rho}\,\mathbf{v}|\,|\mathbf{v}\,\bar{\rho}|\,\cos(\theta)\,, \tag{3.168}$$

where θ is the angle between $\bar{\rho}\,\mathbf{v}$ and $\mathbf{v}\,\bar{\rho}$. Furthermore, since:

$$|\mathbf{v}\,\bar{\rho}|^2 = Tr\left((\mathbf{v}\,\bar{\rho})^{\dagger}\,\mathbf{v}\,\bar{\rho}\right) = Tr\left(\bar{\rho}\,\mathbf{v}^2\,\bar{\rho}\right) = Tr\left(\mathbf{v}^2\,\bar{\rho}^2\right) = |\bar{\rho}\,\mathbf{v}|^2\,, \tag{3.169}$$

we have:

$$\mathcal{L}_{\Gamma}F\big|_{\rho} = |\bar{\rho}\,\mathbf{v}|\,|\mathbf{v}\,\bar{\rho}|\,\cos(\theta) - |\bar{\rho}\,\mathbf{v}|^2 = |\bar{\rho}\,\mathbf{v}|^2\left(\cos(\theta) - 1\right) \leq 0\,, \tag{3.170}$$

and thus, theorem 5 applies, and the proposition is proved.

If we want to consider the quantum semigroup with GKLS generator:

$$\mathbf{L}(\bar{\xi}) = -2\left[\left[\mathbf{H}\,,\bar{\xi}\,\right]\right] - \mathbf{V}\odot\bar{\xi} + \sum_{j=1}^{N}|\alpha_j|^2\mathbf{v}_j\,\bar{\xi}\,\mathbf{v}_j\,, \tag{3.171}$$

with $\mathbf{V} = \sum_{j=1}^{N}|\alpha_j|^2\mathbf{v}_j^2$, we may proceed in complete analogy with what has been done for the GKLS generator of equation (3.143), to obtain:

Proposition 19. *The function:*

$$F(\xi) = \frac{Tr\left(\bar{\xi}^2\right)}{2} \tag{3.172}$$

is a LaSalle function for the GKLS vector field Γ associated with the GKLS generator of equation (3.171).

Essentially, the Hamiltonian term will not contribute to the Lie derivative of the function F, while the vector field $Y_{\mathbf{b}} + Z_{\mathcal{K}}$ will be decomposed as a positive linear combination of vector fields representing the GKLS vector fields of Gaussian semigroups, and thus, the Lie derivative of F with respect to Γ will be always negative when evaluated on the space $\mathcal{S}$ of quantum states. This means that F is a LaSalle function according to theorem 5.

Similarly to what happens for quantum Poisson and quantum Gaussian semigroups, the explicit form of the accumulating set S_∞ requires a case by case analysis. However, the maximally mixed state will always be in S_∞.

Quantum random unitary semigroups

As said before, the GKLS generator $\mathbf{L}$ for a quantum random unitary semigroups is characterized by the following form [11] :

$$\mathbf{L}(\bar{\xi}) = -2[[\mathbf{H}\,,\bar{\xi}\,]] - \mathbf{V}\odot\bar{\xi} + \sum_{j=1}^{n^2-1}\alpha_j \mathbf{e}^j\,\bar{\xi}\,\mathbf{e}^j + \beta\sum_{j=1}^{r\leq n^2-1} p_j \mathbf{U}_j\,\bar{\xi}\,\mathbf{U}_j^\dagger - \beta\xi\,, \tag{3.173}$$

where $\mathbf{H}\in\mathfrak{A}_n$ is self-adjoint, α,β are non-negative real numbers, $\{p_j\}_{j=1,..r}$ is a probability vector, $\{\mathbf{e}^j\}_{j=1,\ldots,n^2-1}$ is an orthonormal set of self-adjoint operators in $\mathfrak{A}_n$, $\mathbf{V}=\sum_{j=1}^{n^2-1}\alpha_j\mathbf{e}_j^2$, and U_j is unitary for all j. It is clear that we may write $\mathbf{L}$ as:

$$\mathbf{L}(\bar{\xi}) = -2[[\mathbf{H}\,,\bar{\xi}\,]] + \sum_{j=1}^{n^2-1}\alpha_j L_G^j(\bar{\xi}) + \beta\sum_{j=1}^{r\leq n^2-1} p_j L_P^j(\bar{\xi}) \tag{3.174}$$

where L_G^j is the GKLS generator of a quantum Gaussian semigroup, and L_P^j is the GKLS generator of a quantum Poisson semigroup. From this decomposition, it follows that the GKLS vector field Γ may be written as:

$$\Gamma = X_{\mathbf{a}} + \sum_{j=1}^{n^2-1}\alpha_j\Gamma_G^j + \beta\sum_{j=1}^{r\leq n^2-1} p_j\Gamma_P^j \tag{3.175}$$

where Γ_G^j is the GKLS vector field of the quantum Gaussian semigroup with generator L_G^j, and Γ_P^j is the GKLS vector field of the quantum Poisson semigroup with generator L_P^j. From this, it naturally follows that the maximally mixed state is a fixed point for every such Γ being it a fixed point for $X_{\mathbf{a}}$, Γ_G^j and Γ_P^j. Furthermore, we immediately have the following:

Proposition 20. *The function:*

$$F(\xi) = \frac{Tr\left(\bar{\xi}^2\right)}{2} \tag{3.176}$$

is a LaSalle function for the GKLS vector field Γ *of equation (3.175)*

Proof. *It follows from proposition 17 and proposition 19.*

Again, the explicit form of S_∞ requires a case by case analysis, but the maximally mixed state will always be in S_∞.

Let us close this chapter with the following observation. Consider the case in which $\mathfrak{A}_n$ decomposes as the tensor product of N C^*-algebras, that is, the quantum system at hand is the composition of N quantum systems. Regardless of the explicit form of the decomposition of $\mathfrak{A}_n$, the maximally mixed state ρ_m is always a separable state (precisely, a product state), furthermore, it is in the set S_∞ of asymptotic quantum states for every quantum random unitary evolution on $\mathfrak{A}_n$. In all those cases in which the maximally mixed state is the only asymptotic quantum state, that is, S_∞ becomes the singleton $\{\rho_m\}$, we have that every initial quantum state ρ, being it a separable or an entangled state, evolves towards the separable state ρ_m. Consequently, we conclude that the dynamics of the system destroys quantum entanglement.

Chapter 4

Information geometry of the manifold of invertible quantum states

In this chapter we will explore the differential geometry of the manifold $\mathcal{S}_n$ of invertible quantum states from the point of view of information geometry. The ideas and results presented here may be seen as a reformulations of those exposed in [40, 41, 45, 46]. Recent experimental and theoretical developments in situations like quantum optics and quantum computing have led to an increasing attention towards the geometry of information theory for finite-level quantum systems. In this context, the geometrical picture of the space $\mathcal{S}$ of quantum states presented in chapter 2 can be exploited to give an abstract (coordinate-free) presentation of quantum information geometry. Roughly speaking, we will first introduce a coordinate-free algorithm to extract covariant tensor fields of order $(0,2)$ and $(0,3)$ from two-point functions. Then, we will analyze how these two-point functions behave under smooth maps between manifolds. These will help us formalize the notion of monotone quantum metrics on the manifold of invertible quantum states, and its connection with the so-called data processing inequality for quantum divergence functions. The use of quantum relative entropies as divergence functions in quantum information geometry is now well-estabilished (see [20, 26, 70]), and the geometrical methods developed in chapter 2 and expanded here allow us to consider a coordinate-free theoretical framework to work with these functions. Indeed, at the end of the chapter we will compute the family of monotone quantum metric tensors associated with the family of $(q-z)$-Rényi relative entropies[1], introduced in [20], in any finite dimension with-

[1] Differently from the notation used in [20], in this work we use the parameter q instead of α, following the notation adopted in [83]. This relabelling of the parameter helps to compare our

out introducing particular coordinate charts on the manifold of invertible quantum states. In the following, we denote with $\bar{\mathcal{S}}_n$ the image of $\mathcal{S}_n$ in $\mathfrak{O}_n \subset \mathfrak{A}_n$ by means of the isomorphism between $\mathfrak{O}_n$ and $\mathfrak{O}_n^*$. According to remark 4, all the geometrical structures on $\mathcal{S}_n$ can be perfectly transported on $\bar{\mathcal{S}}_n$. To make contact with the existing literature on finite-dimensional quantum information geometry, we will work here directly in terms of density matrices in $\bar{\mathcal{S}}_n$ rather than on quantum states in $\mathcal{S}_n$. In this way, computations will be easier since we will perform them by means of matrix algebras.

The principal object of interest in classical information geometry is a manifold $M \subset \mathcal{P}(\chi)$ of parametrized probability distributions on a sample space χ. This manifold is naturally endowed with a distinguished metric tensor g_{FR}, the so-called Fisher-Rao metric tensor (see [9] chapter 2, [34] chapter II section 11). This metric tensor is the only metric tensor which is equivariant with respect to the category of Markov morphisms characteristics of classical probability theory. Among other things, classical information geometry deals with the problem of state estimation, statistical inference, and decision theory. The Riemannian geometry of M associated with the Fisher-Rao metric tensor g_{FR} plays an important role in all these problems.

In this context, the Fisher-Rao metric tensor may be extracted starting from a ***divergence function*** S. This is a smooth[2] function on $M \times M$ (a two-point function) such that $D(m_1\,, m_2) \geq 0$ for all $(m_1\,, m_2) \in M \times M$, and such that $D(m_1\,, m_2) = 0$ is equivalent to $m_1 = m_2$, that is, D vanishes only on the diagonal. A paradigmatic example of classical divergence function is the so-called Kullback-Leibler divergence function $S_{KL}(\xi\,,\eta)$ (see [9] chapter 3):

$$S_{KL}(\xi\,,\eta) := \int_{\chi} p(\xi\,,x)\, \ln\left(\frac{p(\xi\,,x)}{q(\eta\,,x)}\right)\,\mathrm{d}\mu(x)\,, \tag{4.1}$$

where $\xi\,,\eta \in M$, $p, q \in \mathcal{P}(\chi)$, and μ is a measure on χ such that p and q are absolutely continuous with respect to it. Once we have a divergence function $S(\xi\,,\eta)$, the components of the Fisher-Rao metric tensor are:

$$g_{jk}(\xi) = -\frac{\partial S}{\partial\xi^j\partial\xi^k}\bigg|_{\xi=\eta} = -\frac{\partial S}{\partial\eta^j\partial\eta^k}\bigg|_{\xi=\eta} = -\frac{\partial S}{\partial\xi^j\partial\eta^k}\bigg|_{\xi=\eta}\,, \tag{4.2}$$

where $\{\xi^j\,,\eta^k\}$ is a cordinate chart on $M \times M$ which is adapted to the product structure of the manifold. In general, equation (4.2) need not define the components of a tensor, however, as it will be thoroughly discussed in the following section, this

results with those in [83].

[2] According to [87] it suffices for D to be of class C^2.

is always the case when S is a divergence function[3]. By using third derivatives of S it is possible to introduce a covariant $(0,3)$ tensor T, called the ***skewness tensor***, from which a family, parametrized by a real number α, of dually-related affine connections can be defined (see [8] chapter 4). From a local point of view, the Christoffel symbols Γ^{α}_{jkl} of an affine connection Γ^{α} of the parametrized family are expressed as:

$$\Gamma^{\alpha}_{jkl} = \Gamma^{g}_{jkl} - \frac{\alpha}{2}\, T_{jkl}\,, \tag{4.3}$$

where Γ^{g}_{jkl} are the Christoffel symbols of the Levi-Civita connection associated with the Fisher-Rao metric tensor. The duality relation is expressed by the fact that:

$$\mathcal{L}_Z\left(g_{FR}(X\,,Y)\right) = g_{FR}\left(\nabla^{\alpha}_{Z}X\,,Y\right) + g_{FR}\left(X\,,\nabla^{-\alpha}_{Z}Y\right)\,, \quad \forall X,Y,Z\in\mathfrak{X}(M)\,, \tag{4.4}$$

where ∇^{α} and $\nabla^{-\alpha}$ are, respectively, the covariant derivatives of Γ^{α} and $\Gamma^{-\alpha}$.

Divergence functions often arise as suitable generalizations of the concept of distance between probability distributions. They also appear in connection with explicit problems of statistical inference, and some of them may be thought of as being relative-entropies. This latter observation will be of capital importance in the context of the information geometry of the invertible quantum states.

In the quantum case, things are more complicated. First of all, the relevant object is the space $\bar{\mathcal{S}}$ of quantum density matrices. According to the results of the previous sections, $\bar{\mathcal{S}}$ is not a differential manifold as a whole, rather, it is the disjoint union of differential manifolds of different dimension, that is, the manifolds of quantum states with fixed rank. Among all of them, two are the most studied, specifically, the manifold $\bar{\mathcal{S}}_1\cong CP(n)$ of density matrices associated with pure quantum states, and the manifold $\bar{\mathcal{S}}_n$ of invertible density matrices.

Regarding the manifold of pure quantum states, we have seen that it is a homogeneous space for two different groups, the complex special linear group $SL(\mathfrak{A}_n) = SL(n\,,\mathbb{C})$, and the special unitary group $SU(\mathfrak{A}_n) = SU(n)$. In the latter case, we have seen that $\bar{\mathcal{S}}_1$ can be endowed with the structure of a Kähler manifold, that is, there are a metric tensor g, a symplectic form ω, and a complex structure J on $\bar{\mathcal{S}}_1$ such that:

$$\omega\left(X\,,Y\right) = g\left(X\,,J(Y)\right)\,,\quad \forall\, X,Y\in\mathfrak{X}(\mathcal{S}_1). \tag{4.5}$$

[3]In section 4.1 we will introduce the class of ***potential functions*** as a more general class of two-point functions from which it is possible to extract covariant tensors using a coordinate-free algorithm.

In this case, the metric tensor g is known as the Fubini-Study metric tensor, and it is the unique (up to a constant factor) metric tensor on $\bar{\mathcal{S}}_1$ which is invariant with respect to the action of the special unitary group on $\bar{\mathcal{S}}_1$. In some sense, the Fubini-Study metric tensor on $\bar{\mathcal{S}}_1$ plays a role similar to that of the Fisher-Rao metric tensor of classical information geometry. For instance, Wotters (see [100]) has shown that an operational definition of statistical distance between pure quantum states based on distinguishability and statistical fluctuations in the outcomes of measurements naturally leads to the geodesical distance function associated with the Fubini-Study metric. A dynamical characterization of this statistical distance function in terms of the Hamilton-Jacobi theory is given in [40, 46], where it is shown that Wotter's statistical distance is precisely Hamilton principal function (see chapter VII of [79]) for the metric Lagrangian associated with the Fubini-Study. An abstract characterization of this distance-like function which is based only on purely operational concepts and which is independent of the quantum mechanical framework can be found in [30, 31]. Unfortunately, there is no canonical notion of skewness tensor on $\bar{\mathcal{S}}_1$, and the topological nature of $\bar{\mathcal{S}}_1$, which is completely determined by the fact that it is a compact Kähler manifold of constant curvature, points out the fact that the manifold of pure quantum states does not admit flat dual connections (see [21, 22]).

Remark 15. *An interesting connection between Fubini-Study and Fisher-Rao metric tensors has been given in [59]. There, a quantum system is described by means of a Hilbert space $\mathcal{H}$ and the pure states of the system are then rays in $\mathcal{H}$ conveniently parametrized by rank-one projectors operators. Given a nonzero $|\psi\rangle \in \mathcal{H}$ we may associate with it the rank-one projector operator by means of the map:*

$$\pi\colon\ |\psi\rangle \mapsto \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\,. \tag{4.6}$$

The Fubini-Study metric tensor on pure states may be pulled-back on $\mathcal{H}_0$ (the Hilbert space without the zero vector) by means of the map π. The result is a symmetric tensor, say G.

Now, let $\mathcal{H}$ be realized as the space of square-integrable functions on some configuration space χ, and consider a subset of wave functions parametrized by means of points in M, that is, $\psi(x\,,m)$ with $m \in M$. Writing the wave function in its polar form, we obtain:

$$\psi(x\,,m) = p(x\,,m)\,\mathrm{e}^{\imath\alpha(x\,,m)}\,, \tag{4.7}$$

where $p(x\,,m)$ is a probability distribution on χ for every $m \in M$. The main result of the paper, is then to show that the pullback of the symmetric tensor G on the *parameter manifold M is made up of three different contributions. Two of them depend on the phase function $\alpha(x\,,m)$, while the third one depends only on the probability function $p(x\,,m)$. Quite interestingly, the term depending only on $p(x\,,m)$ coincides with the classical Fisher-Rao metric tensor on M that we would have obtained starting directly with the immersion of $m \mapsto p(x\,,m)$ as an immersion of M into the space $\mathcal{P}(\chi)$ of probability functions on χ, and proceeding as is done in classical information geometry (see [9] chapter 2). In a certain sense, the quantum contribution to the classical Fisher-Rao metric tensor is encoded in the phase part of the wave functions representing pure quantum states (see [41] for an application of this ideas to some simple cases of geodesical motion on parametrized manifolds of pure quantum states).*

On the other hand, the manifold $\bar{\mathcal{S}}_n$ of invertible density matrices does not have a preferred notion of metric tensor. Specifically, while in the case of pure states the requirement of invariance with respect to the canonical action of the special unitary group $SU(\mathfrak{A}_n)$ unavoidably leads to a unique (up to a constant factor) metric tensor g, the Fubini-Study metric tensor, this is no longer true for invertible quantum states. Indeed, it is possible to prove (see [89, 93]) that there is an infinite number of metric tensors on $\bar{\mathcal{S}}_n$ satisfying the invariance requirement with respect to the special unitary group $SU(\mathfrak{A}_n)$. Actually, all these metric tensors do satisfy a more general property known as ***monotonicity property***. This property is related with the behaviour of metric tensors with respect to a class of maps which plays a role analogous to the Markov maps in classical probability theory. Roughly speaking, these maps represent quantum stochastic transformations, and are often associated with coarse graining procedures or randomization procedures (see [72] and [73]). Then, to require that a metric g satisfies the monotonicity property means, essentially, to require that the notion of geodesic distance between quantum states naturally associated with g does not increase under quantum stochastic maps. In section 4.2 we will introduce the mathematical definition of the quantum stochastic maps, and the rigorous formulation of the monotonicity property for a quantum metric tensor g on $\bar{\mathcal{S}}_n$. What we will find is that these notions acquire a well-defined meaning only when we consider not a single quantum system, but a family of quantum systems of increasing dimensions, very much like it happens in the finite-dimensional classical case (see [34] chapters I and II).

The existence of an infinite number of quantum metric tensors satisfying the monotonicity property may be thought of as a concrete instance in which the complexity of the physics of quantum systems manifests itself. Contrarily to the classical case, the relevant notion of metric tensor is no longer unique, and the explicit choice of some specific metric tensor should be made in accordance with the specific problem at hand. Nevertheless, it is important to give a sort of general classification of the quantum metric tensors satsfying the monotonicity property. This is done in [93], where it is provided a one-to-one correspondence between positive operator monotone functions and quantum metric tensors satisfying the monotonicity property. This correspondence is constructive in the sense that, given a positive operator monotone function f, an explicit expression of the associated metric tensor g_f is given.

The fact that, both in the classical and quantum case, it is possible to extract a metric tensor from a divergence function plays no role in the analysis given in [93]. However, quantum divergence functions play a central role in quantum information theory (see [20, 92]). An example of such quantum divergence is given by the so-called von Neumann-Umegaki relative entropy (see [98], and chapter 12 of [26]):

$$S^n(\bar{\rho}\,,\bar{\varrho}) = Tr\left(\bar{\rho}(\log(\bar{\rho}) - \log(\bar{\varrho}))\right) \tag{4.8}$$

This quantum divergence function may be thought of as the quantum generalization of the Kullback-Leibler divergence function used in classical information geometry. In the asympotic, memoryless setting, it yields fundamental limits on the performance of information-processing tasks (see [70]). Analogously to what happens for the classical case, it is possible to compute the metric tensor associated with this quantum divergence, and the result is a quantum metric tensor satisfying the monotonicity property. A natural question is then what are the properties a quantum divergence function must satisfy in order to extract from it a quantum metric tensor satisfying the monotonicity property. We will provide a partial answer to this question in section 4.2, where we will use the results of section 4.1 to express a connection between the so-called ***data processing inequality*** (***DPI***) for a quantum divergence function S and the monotonicity property of the metric tensor g extracted from S. Specifically, we will prove that *DPI implies monotonicity property.* Quite interestingly, the very definition of the DPI is centered around the behaviour of quantum divergence functions with respect to the class of quantum stochastic maps with respect to which the monotonicity property of metric tensors is formulated. We will give an explicit definition of the DPI in section 4.2, where we will find that a correct definition of DPI can be done only when we consider a family of quantum systems of increasing dimensions.

As a concrete application of the abstract scheme presented in section 4.1 and 4.2, we will compute the covariant metric-like tensor associated with the two-parameter family of two-point functions introduced in [20], and known as *q-z-Rényi Relative Entropies* (q-z-RRE):

$$S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q-1}\log\left(Tr\left(\bar{\rho}^{\frac{q}{2z}}\,\bar{\varrho}^{\frac{1-q}{z}}\,\bar{\rho}^{\frac{q}{2z}}\right)^z\right) = \frac{1}{q-1}\log\left(Tr\left(\bar{\rho}^{\frac{q}{z}}\,\bar{\varrho}^{\frac{1-q}{z}}\right)^z\right)\,. \tag{4.9}$$

According to [20], by suitably chosing the values of the parameters, it is possible to recover well-known examples of quantum divergence functions. For instance, the *q-Rényi Relative Entropies* are recovered when $z \to 1$:

$$S^n_{q,1}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z\to 1} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{RRE}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q-1}\log\left(Tr\left(\bar{\rho}^q\,\bar{\varrho}^{1-q}\right)\right)\,, \tag{4.10}$$

the *q-quantum Rényi divergences* are recovered when $z \to q$:

$$S^n_{q,q}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z\to q} S^n_{q,q}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{QRD}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q-1}\log\left(Tr\left(\bar{\varrho}^{\frac{1-q}{q}}\,\bar{\rho}\right)\right)\,, \tag{4.11}$$

the von Neumann-Umegaki relative entropy is recovered when $q, z \to 1$:

$$S^n_{1,1}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z=q\to 1} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{vN}(\bar{\rho}\,,\bar{\varrho}) = Tr\left(\bar{\rho}(\log(\bar{\rho}) - \log(\bar{\varrho}))\right)\,. \tag{4.12}$$

The data processing inequality for the q-z-RRE was studied in [25, 57, 61, 69], and it is not established yet in full generality. The results of these analysis are well summarized in [20].

For computational purposes, it is convenient to consider the following regularization of the logarithm (q-logarithm):

$$\log_q(x) = \frac{1}{1-q}(x^{1-q} - 1) \qquad \text{with} \qquad \lim_{q\to 1}\log_q(x) = \log(x)\,. \tag{4.13}$$

Inspired by Petz (see [93]), we will consider this function rescaled by a factor $1/q$, so that the resulting function will be symmetric under the exchange of $q \to (1-q)$. With a slight abuse of notation, we will denote the resulting two-point function with the same symbol of the q-z-RRE, that is:

$$S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q(1-q)}\left[1 - Tr\left(\bar{\rho}^{\frac{q}{z}}\,\bar{\varrho}^{\frac{1-q}{z}}\right)^z\right]\,, \tag{4.14}$$

and we will call it modified q-z-RRE. Since the analysis of the DPI involves only the trace functional (see [20] and references therein), we are ensured that the DPI for the modified q-z-RRE holds whenever it holds for the q-z-RRE.

Again, by suitably varying the parameters q and z, we are able to recover well-known examples of quantum divergences. In the limit $z \to 1$, it is possible to recover the expression for the Tsallis relative entropy in [83]:

$$S^n_{q,1}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z\to 1} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{Ts}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q(1-q)}\left[1 - \mathrm{Tr}\left(\bar{\rho}^q\,\bar{\varrho}^{1-q}\right)\right], \tag{4.15}$$

in the limit $z = q \to 1$, we recover the von Neumann-Umegaki relative entropy:

$$S^n_{1,1}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z=q\to 1} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{vN}(\bar{\rho}\,,\bar{\varrho}) = Tr\left(\bar{\rho}(\log(\bar{\rho}) - \log(\bar{\varrho}))\right)\,, \tag{4.16}$$

in the limit $z = q = 1/2$, we recover the divergence function of the Bures metric tensor:

$$S^n_{\frac{1}{2},\frac{1}{2}}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z=q\to\frac{1}{2}} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{B}(\bar{\rho}\,,\bar{\varrho}) = 4\big[1 - \mathrm{Tr}\left(\bar{\rho}\,\bar{\varrho}\right)^{\frac{1}{2}}\big], \tag{4.17}$$

and, finally, in the limit $z = 1$ and $q \to 1/2$, we recover the divergence function of the Wigner-Yanase metric tensor (see [62]):

$$S^n_{\frac{1}{2},1}(\bar{\rho}\,,\bar{\varrho}) := \lim_{z=1,q\to\frac{1}{2}} S^n_{q,z}(\bar{\rho}\,,\bar{\varrho}) \equiv S^n_{WY}(\bar{\rho}\,,\bar{\varrho}) = 4\big[1 - \mathrm{Tr}\left(\bar{\rho}^{\frac{1}{2}}\,\bar{\varrho}^{\frac{1}{2}}\right)\big]\,. \tag{4.18}$$

All these special cases belong to the range of parameters for which $S^n_{q,z}$ is actually a quantum divergence function satisfying the DPI. Consequently, in accordance with the result of sections 4.2, the family of associated quantum metric tensors satisfies the monotonicity property.

In order to perform all the calculations in a coordinate-free way, and regardless of the dimension of the quantum system considered, in section 4.3 we will introduce an unfolding space $\mathcal{M}_n$ for the manifold $\bar{\mathcal{S}}_n$ of invertible density matrices. This unfolding space is the direct product of the unitary group $U(\mathfrak{A}_n)$ and the open interior Δ^0_n of the n-dimensional simplex Δ_n. We will show that there is a surjective submersion $\pi_n \colon \mathcal{M}_n \to \bar{\mathcal{S}}_n$ so that, according to the results of section 4.1, we obtain a potential function on $\mathcal{M}_n$ by taking the pullback of $S_{q,z}$ to $\mathcal{M}_n$ by means of π_n. Then, since $\mathcal{M}_n$ is a parallelizable manifold, we will exploit its global differential calculus to perform computations, in any dimension, without introducing coordinates. The final result will be a symmetric tensor on $\mathcal{M}_n$ which will be decomposed in the sum of two symmetric tensors, one which "lives" on the unitary group $U(\mathfrak{A}_n)$, and the other one which "lives" on the open interior Δ^0_n of the n-dimensional simplex Δ_n and coincides with the Fisher-Rao metric tensor on Δ^0_n.

4.1 Covariant tensors from two-point functions

Here we will provide an intrinsic definition of the coordinate-based formulae used in information geometry to derive a metric tensor and a skewness tensor from a divergence function (see [7] chapter 1, [9] chapter 2 and [87]) according to the results in [45]. Essentially, we will recast most of the well-known material on divergence functions and their symmetry properties using the intrinsic language of differential geometry. This will provide very useful when dealing with quantum information geometry, where we have to take in consideration nonlinear manifolds. The first step will be that of introducing the notions of left and right lift of a vector field, along with the diagonal immersion of a manifold into its double. Then, after some important properties connecting the diagonal immersion with the left and right lifts are proved, we introduce a coordinate-free algorithm to extract covariant tensors of order $(0,2)$ and $(0,3)$ from a two-point function. This will lead us to define the class of ***potential functions***. These are two-point functions generalizing the concept of divergence functions of classical information geometry. Finally, we will analyze how potential functions behave with respect to smooth maps between differential manifolds.

Let M be a differential manifold, TM its tangent bundle, and $\tau\colon TM \to M$ the canonical projection. A point in the tangent bundle TM is a couple $(m\,, v_m)$, where $m \in M$, and $v_m \in T_mM$ is a tangent vector at m. Note that, in general, TM is not a cartesian product, hence, the notation $(m\,, v_m)$ should be treated with care because the second factor v_m is not independent from the first one. A vector field $X \in \mathfrak{X}(M)$ may be thought of as a derivation of the associative algebra $\mathcal{F}(M)$ of smooth functions on M, or as a section of the tangent bundle TM, that is, a map $X\colon M \to TM$ such that $\tau \circ X = id_M$ (see [2] chapter 4). In the latter case, we may write the evaluation of a vector field on $m \in M$ as $X(m) = (m\,, v_m^X)$.

Let $M \times M$ denote the so-called double manifold of M. There are two canonical projections $pr_l\colon M \times M \to M$ and $pr_l\colon M \times M \to M$ acting as:

$$
\begin{aligned}
pr_l(m_1\,, m_2) &:= m_1 \\
pr_r(m_1\,, m_2) &:= m_2\,.
\end{aligned}
\tag{4.19}
$$

Given $f \in \mathcal{F}(M)$, we may define the following functions on $M \times M$ by means of pr_l and pr_r:

$$
\begin{aligned}
f_l\colon M \times M \to \mathbb{R}\,, &\quad f_l := pr_l^* f \\
f_r\colon M \times M \to \mathbb{R}\,, &\quad f_r := pr_r^* f\,.
\end{aligned}
\tag{4.20}
$$

This means that on $M\times M$ we have identified two different subalgebras of $\mathcal{F}(M\times M)$, the left and the right subagebras:

$$
\begin{aligned}
&\mathcal{F}_l(M\times M) := \{f_l \in \mathcal{F}(M\times M)\colon\ \exists f\in\mathcal{F}(M) \text{ such that } \ f_l = \pi_l^* f\} \\
&\mathcal{F}_r(M\times M) := \{f_r \in \mathcal{F}(M\times M)\colon\ \exists f\in\mathcal{F}(M) \text{ such that } \ f_r = \pi_r^* f\}\ .
\end{aligned}
\tag{4.21}
$$

The tangent space $T_{(m_1,m_2)}M\times M$ at $(m_1\,,m_2)\in M\times M$ splits into the direct sum $T_{m_1}M\oplus T_{m_2}M$ (see [2] page 147). Accordingly, we may write the evaluation of a vector field $\mathbb{X}\in\mathfrak{X}(M\times M)$ at $(m_1\,,m_2)$ as:

$$
\mathbb{X}(m_1\,,m_2) = (m_1\,,v^{\mathbb{X}}_{m_1}\,;m_2\,,v^{\mathbb{X}}_{m_2})\,. \tag{4.22}
$$

This motivates the following:

Definition 11. *Let $X\in\mathfrak{X}(M)$ be a smooth vector field. We defined the left and right lift of X to be, respectively, the vector fields $\mathbb{X}_l,\mathbb{X}_r\in\mathfrak{X}(M\times M)$ defined as:*

$$
\mathbb{X}_l(m_1\,,m_2) = (m_1\,,v^{X}_{m_1}\,;m_2\,,0)\,, \tag{4.23}
$$

$$
\mathbb{X}_r(m_1\,,m_2) = (m_1\,,0\,;m_2\,,v^{X}_{m_1})\,. \tag{4.24}
$$

By direct computation, it is possible to prove the following:

Proposition 21. *Let $X,Y\in\mathfrak{X}(M)$, and $f\in\mathcal{F}(M)$, and denote with L the Lie-derivative. The following equalities hold:*

$$
[\mathbb{X}_l\,,\mathbb{Y}_l] = ([X\,,Y])_l\ , \qquad [\mathbb{X}_r\,,\mathbb{Y}_r] = ([X\,,Y])_r\ , \qquad [\mathbb{X}_l\,,\mathbb{Y}_r] = 0\,, \tag{4.25}
$$

$$
(fX)_l = f_l\mathbb{X}_l\,, \qquad (fX)_r = f_r\mathbb{X}_r\,, \qquad L_{\mathbb{X}_l}f_r = L_{\mathbb{X}_r}f_l = 0\,. \tag{4.26}
$$

There is a natural immersion i_d of M into its double $M\times M$ given by:

$$
M\ni m\mapsto i_d(m) = (m\,,m)\in M\times M\,. \tag{4.27}
$$

The map i_d allows us to immerse M in the diagonal of its double, and, by means of the pullback operation, gives an intrinsic and coordinate-free definition of the procedure of "restricting to the diagonal" used in information geometry. Indeed, the pullback of a function to a submanifold can be identified with the restriction of the function to the submanifold. Note that the same is not true for covariant tensors of higher order for which a "restriction" in the sense of evaluation at specific points is always possible, however this does not coincide with the value that the pulled-back covariant tensor will take at the same point as an element of the submanifold.

By using the tangent functor it is possible to associate vector fields on M with vector fields on $M \times M$ along the immersion i_d of M into $M \times M$. We have the following proposition:

Proposition 22. *Let $X \in \mathfrak{X}(M)$, then X is i_d-related to $\mathbb{X}_l + \mathbb{X}_r$, that is (see [2] page 235):*

$$Ti_d \circ X = \mathbb{X}_{lr} \circ i_d, , \tag{4.28}$$

where $\mathbb{X}_{lr} \equiv (\mathbb{X}_l + \mathbb{X}_r)$, and Ti_d denotes the tangent map of i_d.

Proof. *By direct computation, we have:*

$$Ti_d \circ X(m) = Ti_d(m\,, v_m^X) = (m\,, v_m^X\,; m\,, v_m^X)\,, \tag{4.29}$$

$$\mathbb{X}_{lr} \circ i_d(m) = \mathbb{X}_{lr}(m\,, m) = (m\,, v_m^X\,; m\,, v_m^X)\,, \tag{4.30}$$

and the proposition follows.

Now, we are ready to introduce the coordinate-free algorithm to extract covariant $(0, 2)$ tensor from a two-point function. In order to do so, we define the following maps:

Definition 12. *Let $S \in \mathcal{F}(M \times M)$. We define the following bilinear, $\mathbb{R}$-linear maps from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathcal{F}(M)$:*

$$g_{ll}\,(X\,, Y) := i_d^*\,(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} S)\;, \qquad g_{rr}\,(X\,, Y) := i_d^*\,(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} S)\;, \tag{4.31}$$

$$g_{lr}\,(X\,, Y) := i_d^*\,(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} S)\;, \qquad g_{rl}\,(X\,, Y) := i_d^*\,(L_{\mathbb{X}_r} L_{\mathbb{Y}_l} S)\;. \tag{4.32}$$

Notice that, at the moment, these maps do not have definite symmetry properties. To prove that these maps give a coordinate-free version of the formulae for metric-like tensors used in information geometry, we start with the following proposition:

Proposition 23. *Consider the maps in definition 12. Then:*

1. *g_{lr}, g_{rl} are covariant $(0\,, 2)$ tensors on M, and $g_{lr}(X\,, Y) = g_{rl}(Y\,, X)$;*

2. g_{ll} is a symmetric covariant $(0\,,2)$ tensor on M if and only if:

$$i_d^*\left(L_{\mathbb{X}_l}S\right)=0\ \ \forall X\in\mathfrak{X}(M)\,; \tag{4.33}$$

3. g_{rr} is a symmetric covariant $(0\,,2)$ tensor on M if and only if:

$$i_d^*\left(L_{\mathbb{X}_r}S\right)=0\ \ \forall X\in\mathfrak{X}(M)\,. \tag{4.34}$$

Proof. *To show 1 we have to show that g_{lr} and g_{rl} are bilinear with respect to vector fields, and $\mathcal{F}(M)$-linear. We start with g_{lr}. According to proposition 21, we have:*

$$g_{lr}(fX+hY\,,Z)=i_d^*\left(L_{f_l\mathbb{X}_l+h_l\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)\,. \tag{4.35}$$

The linearity of the pullback, together with the properties of the Lie derivative, imply:

$$i_d^*\left(L_{f_l\mathbb{X}_l+h_l\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)=i_d^*\left(f_lL_{\mathbb{X}_l}L_{\mathbb{Z}_r}S\right)+i_d^*\left(h_lL_{\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)\,. \tag{4.36}$$

Since $L_{\mathbb{X}_l}L_{\mathbb{Z}_r}S$ and $L_{\mathbb{Y}_l}L_{\mathbb{Z}_r}S$ are smooth functions, we have that

$$\begin{aligned} i_d^*\left(f_l\,L_{\mathbb{X}_l}L_{\mathbb{Z}_r}S\right)&=i_d^*f_l\,i_d^*\left(L_{\mathbb{X}_l}L_{\mathbb{Z}_r}S\right)\,,\\ i_d^*\left(f_l\,L_{\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)&=i_d^*f_l\,i_d^*\left(L_{\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)\,,\end{aligned}$$

and thus:

$$g_{lr}(fX+hY\,,Z)=f\,i_d^*\left(L_{\mathbb{X}_l}L_{\mathbb{Z}_r}S\right)+h\,i_d^*\left(L_{\mathbb{Y}_l}L_{\mathbb{Z}_r}S\right)=f\,g_{lr}(X\,,Z)+h\,g_{lr}(Y\,,Z)\,. \tag{4.37}$$

According to last equality of proposition 21, we have $L_{\mathbb{X}_l}f_r=L_{\mathbb{X}_r}f_l=0$ for all X and f. Taking this equality into account, we may proceed as above, and show that:

$$g_{lr}(Z\,,fX+hY)=f\,g_{lr}(Z\,,X)+h\,g_{lr}(Z\,,Y)\,. \tag{4.38}$$

This proves that g_{lr} is a covariant $(0\,,2)$ tensor field on M. With exactly the same procedure, we can prove that g_{rl} is a covariant $(0\,,2)$ tensor field on M. Finally, the equality $g_{lr}(X\,,Y)=g_{rl}(Y\,,X)$ follows from direct computation.

To show 2, again, we have to show that g_{ll} is bilinear with respect to vector fields, and $\mathcal{F}(M)$-linear. The linearity and $\mathcal{F}(M)$-linearity on the first factor are proved analogously to the previous case. Concerning the second factor, we start with the following chain of equalities:

$$g_{ll}(Z\,,fX+hY)=i_d^*\left(L_{\mathbb{Z}_l}L_{f_l\mathbb{X}_l+h_l\mathbb{Y}_l}S\right)=i_d^*\left(L_{\mathbb{Z}_l}\left(f_lL_{\mathbb{X}_l}S\right)\right)+i_d^*\left(L_{\mathbb{Z}_l}\left(h_lL_{\mathbb{Y}_l}S\right)\right)=$$

$$= i_d^*\left(L_{\mathbb{Z}_l} f_l\, L_{\mathbb{X}_l} S\right) + i_d^*\left(f_l L_{\mathbb{Z}_l} L_{\mathbb{X}_l} S\right) + i_d^*\left(L_{\mathbb{Z}_l} h_l\, L_{\mathbb{Y}_l} S\right) + i_d^*\left(h_l L_{\mathbb{Z}_l} L_{\mathbb{Y}_l} S\right) =$$
$$= i_d^*\left(L_{\mathbb{Z}_l} f_l\right)\, i_d^*\left(L_{\mathbb{X}_l} S\right) + f\, g_{ll}(Z\,, X) + i_d^*\left(L_{\mathbb{Z}_l} h_l\right)\, i_d^*\left(L_{\mathbb{Y}_l} S\right) + h\, g_{ll}(Z\,, Y)\,. \quad (4.39)$$

It is then clear that:

$$g_{ll}(Z\,, fX + hY) = f\, g_{ll}(Z\,, X) + h\, g_{ll}(Z\,, Y) \quad (4.40)$$

is equivalent to:

$$i_d^*\left(L_{\mathbb{Z}_l} f_l\right)\, i_d^*\left(L_{\mathbb{X}_l} S\right) + i_d^*\left(L_{\mathbb{Z}_l} h_l\right)\, i_d^*\left(L_{\mathbb{Y}_l} S\right) = 0\,. \quad (4.41)$$

Being f and h arbitrary functions, equation 4.41 is satisfied if and only if:

$$i_d^*\left(L_{\mathbb{X}_l} S\right) = 0 \;\; \forall X \in \mathfrak{X}(M) \quad (4.42)$$

as claimed. Now, we prove that g_{ll} is a symmetric tensor:

$$g_{ll}\,(X\,, Y) = i_d^*\left(L_{\mathbb{X}} L_{\mathbb{Y}} S\right) = i_d^*\left(L_{\mathbb{Y}} L_{\mathbb{X}} S\right) + i_d^*\left(L_{[\mathbb{X}\,,\mathbb{Y}]} S\right) = i_d^*\left(L_{\widetilde{Y}} L_{\widetilde{X}} S\right) = g_{ll}\,(Y\,, X)\;, \quad (4.43)$$

where, in the last passage, we have used the first equality of proposition 21. With exactly the same procedure we can prove 3.

Interestingly, when S satisfies condition (4.33) and condition (4.34), the covariant tensor fields are all related to one another. In order to clearly see this, we recall the following proposition (see [2] page 239):

Proposition 24. *Let $\phi\colon N \to M$ be a smooth map between smooth manifolds. Let $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ be ϕ-related, that is $T\phi \circ X = Y \circ \phi$, then:*

$$L_X\, \phi^*(f) = \phi^*\left(L_Y\, f\right) \quad \forall f \in \mathcal{F}(M)\,. \quad (4.44)$$

This means that X and Y agree along the image of N into M. In particular, since $X \in \mathfrak{X}(M)$ is i_d-related to $\mathbb{X}_l + \mathbb{X}_r$, we have that:

$$L_X\, i_d^*(f) = i_d^*\left(L_{\mathbb{X}_l + \mathbb{X}_r}\, f\right) \quad \forall f \in \mathcal{F}(M \times M)\,. \quad (4.45)$$

Now, we are ready to prove:

Proposition 25. *Let S be a smooth function on $M \times M$ satisfying condition* (4.33) *and condition* (4.34)*. Then, all the maps in definition 12 define covariant* $(0, 2)$ *tensor fields on M, and it holds:*

$$g_{ll} = g_{rr} = -g_{lr} = -g_{rl}\,. \quad (4.46)$$

In particular, all these tensor fields are symmetric.

Proof. *According to definition 12, we have:*

$$g_{ll}\,(X\,,Y)+g_{lr}\,(X\,,Y)=i_d^*\,(L_{\mathbb{X}_l}L_{\mathbb{Y}_l+\mathbb{Y}_r}\,S)=i_d^*\,(L_{\mathbb{Y}_l+\mathbb{Y}_r}L_{\mathbb{X}_l}\,S)+i_d^*\left(L_{[\mathbb{X}_l\,,\mathbb{Y}_l+\mathbb{Y}_r]}\,S\right)$$
$$=i_d^*\,(L_{\mathbb{Y}_l+\mathbb{Y}_r}L_{\mathbb{X}_l}\,S)=L_Y\,i_d^*\,(L_{\mathbb{X}_l}\,S)=0\,, \tag{4.47}$$

where we used the fact that $[\mathbb{X}_l\,,\mathbb{Y}_r]=0$ *because of the third equality in proposition 21, that* S *satisfies condition* (4.33)*, and equation* (4.45)*. This proves that* $g_{ll}=-g_{lr}$*. Proceeding analogously, we obtain* $g_{rr}=-g_{rl}$*. Then, being* g_{ll} *and* g_{rr} *symmetric (see proposition 23), and being* $g_{lr}\,(X\,,Y)=g_{rl}\,(Y\,,X)$ *(see proposition 23), we obtain* $g_{lr}=g_{rl}$*, and thus* $g_{ll}=g_{rr}$*.*

Remark 16. *Note that conditions* (4.33) *and* (4.34) *are are not equivalent to* $i_d^*(\mathrm{d}S)=0$*. For instance, take* $M=\mathbb{R}$*, and*

$$S=\frac{x^2-y^2}{2}\,.$$

We have:

$$\mathrm{d}S=x\mathrm{d}x-y\mathrm{d}y\,,$$

and thus $i_d^*\mathrm{d}S=0$*, while, an easy calculation shows that*

$$i_d^*(L_{\mathbb{X}_l}S)=x\neq 0\quad \textit{if}\;\mathbb{X}_l=\frac{\partial}{\partial x}\,,$$

and

$$i_d^*(L_{\mathbb{X}_r}S)=-y\neq 0\quad \textit{if}\;\mathbb{X}_r=\frac{\partial}{\partial y}\,.$$

It can be checked that the maps g_{ll} *and* g_{rr} *associated with* S *do not define tensor fields because they are not* $\mathcal{F}(M)$*-linear in the second factor.*

Motivated by proposition 25, we give the following definition:

Definition 13 (Potential function). *Let* S *be a smooth function on* $M\times M$*. We call* S *a* **potential function** *if it satisfies condition* (4.33) *and condition* (4.34)*, that is:*

$$
\begin{aligned}
i_d^*\left(L_{\mathbb{X}_l} S\right) &= 0 \quad \forall X \in \mathfrak{X}(M)\,, \\
i_d^*\left(L_{\mathbb{X}_r} S\right) &= 0 \quad \forall X \in \mathfrak{X}(M)\,.
\end{aligned} \tag{4.48}
$$

We denote with g the symmetric covariant $(0,2)$ tensor field associated with S (see proposition 25).

We stress that proposition 25 gives necessary and sufficient conditions for S to give rise to a (unique) symmetric covariant $(0,2)$ tensor field on M. This gives a formal and intrinsic characterization of potential functions.

To make contact with the coordinate-based formulae of information geometry, we introduce coordinate chart $\{x^j\}$ on M, and a coordinate chart $\{q^j\,, Q^j\}$ on $M\times M$. Then, we have:

$$
X = X^j(\mathbf{x})\,\frac{\partial}{\partial x^j}\,, \qquad \mathbb{X}_l = X^j(\mathbf{q})\,\frac{\partial}{\partial q^j}\,, \qquad \mathbb{X}_r = X^j(\mathbf{Q})\,\frac{\partial}{\partial Q^j}\,. \tag{4.49}
$$

Consequently, it is easy to see that:

$$
g = \left.\left(\frac{\partial^2 S}{\partial q^j \partial q^k}\right)\right|_d \mathrm{d}x^j \otimes_s \mathrm{d}x^k = \left.\left(\frac{\partial^2 S}{\partial Q^j \partial Q^k}\right)\right|_d \mathrm{d}x^j \otimes_s \mathrm{d}x^k = -\left.\left(\frac{\partial^2 S}{\partial q^j \partial Q^k}\right)\right|_d \mathrm{d}x^j \otimes_s \mathrm{d}x^k\,, \tag{4.50}
$$

and these expressions are in complete accordance with the ones used in information geometry (see [7] chapter 1, [9] chapter 2 and [87]).

Remark 17. *If S is not a potential function, we can not define the tensor g_{ll}, or the tensor g_{rr}, or both. However, we can always define the tensors g_{lr} and g_{rl}. These tensors will not be symmetric, and we can decompose them into symmetric and anti-symetric part. For example, let $M=\mathbb{R}^2$, let $\{x^j\}_{j=0,1}$ be a global Cartesian coordinates system on M, and let $\{q^j\,, Q^j\}_{j=0,1}$ be a global Cartesian coordinates on $M\times M$. Consider the function:*

$$
S(q^0\,, q^1\,; Q^0\,, Q^1) = -\frac{1}{2}\left((q^0 - Q^0)^2 + (q^1 - Q^1)^2 + q^0 Q^1 - q^1 Q^0\right)\,. \tag{4.51}
$$

An explicit calculation shows that:

$$
g_{lr} = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^0 \wedge \mathrm{d}x^1\,. \tag{4.52}
$$

The coordinate expressions in equation (4.50) allow us to give a "local" characterization of potential functions:

Proposition 26. *A function $S \in \mathcal{F}(M \times M)$ is a potential function according to definition 13 if and only if every point $(m\,, m)$ on the diagonal of $M \times M$ is a critical point for S.*

Proof. *The proof follows upon comparing the local expression for $(m\,, m)$ to be a critical point for S with the coordinate expressions of condition* (4.33) *and condition* (4.34) *in the coordinate system $\{q^j\,, Q^j\}$ introduced before.*

This characterization of potential functions allows us to better understand what kind of tensor field is g. Specifically, resorting to the theory of multivariable calculus it is possible to prove the following proposition:

Proposition 27. *Let S be a potential function on $M \times M$. Then:*

1. *g is positive-semidefinite if and only if every point on the diagonal is a local minimum for S. In particular, g is a metric if and only if every point of the diagonal is a nondegenerate local minimum for S;*

2. *g is negative-semidefinite if and only if every point on the diagonal is a local maximum for S.*

It is now easy to see the relation between the class of potential functions intriduced here and the class of divergence functions of classical information geometry:

Definition 14 (**Divergence function**)**.** *A smooth function S on $M \times M$ such that:*

$$S(m_1\,, m_2) \geq 0\,, \quad S(m_1\,, m_2) = 0 \iff m_1 = m_2\,, \tag{4.53}$$

is called a **divergence function**.

According to proposition 26 S is a potential function (see definition 13) and thus it gives rise to a symmetric covariant $(0\,, 2)$ tensor field g on M. According to proposition 27, the tensor field g is positive-semidefinite.

In information geometry, divergence (contrast) functions give rise to metric tensors by means of the second derivatives, and to symmetric covariant $(0, 3)$ tensors by means of third derivatives. These tensors are referred to as skewness tensors. We will now give an intrinsic definition for these skewness tensors using again Lie derivatives. Let S be a potential function on $M \times M$. For $j = 1, ..., 8$, define the following maps $T_j : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$:

$$T_1(X, Y, Z) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} L_{\mathbb{Z}_l}\, S\right)\,, \quad T_2(X, Y, Z) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} L_{\mathbb{Z}_r}\, S\right)\,, \tag{4.54}$$

$$T_3(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} L_{\mathbb{Z}_r}\, S\right)\,, \qquad T_4(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} L_{\mathbb{Z}_l}\, S\right)\,, \tag{4.55}$$

$$T_5(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} L_{\mathbb{Z}_r}\, S\right)\,, \qquad T_6(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_l} L_{\mathbb{Z}_l}\, S\right)\,, \tag{4.56}$$

$$T_7(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} L_{\mathbb{Z}_l}\, S\right)\,, \qquad T_8(X,Y,Z) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_l} L_{\mathbb{Z}_r}\, S\right)\,, \tag{4.57}$$

Following the line of reasoning developed above, patient but simple calculations show that:

$$T_{12}(X\,,Y\,,Z) := T_1(X\,,Y\,,Z) - T_2(X\,,Y\,,Z)\,, \tag{4.58}$$

$$T_{34}(X\,,Y\,,Z) := T_3(X\,,Y\,,Z) - T_4(X\,,Y\,,Z)\,, \tag{4.59}$$

$$T_{56}(X\,,Y\,,Z) := T_5(X\,,Y\,,Z) - T_6(X\,,Y\,,Z)\,, \tag{4.60}$$

$$T_{78}(X\,,Y\,,Z) := T_7(X\,,Y\,,Z) - T_8(X\,,Y\,,Z) \tag{4.61}$$

are actually tensors fields on M. Recalling that X is i_d related to $\mathbb{X}_l + \mathbb{X}_r$, and applying equation (4.45), we have:

$$T_1(X\,,Y\,,Z) + T_6(X\,,Y\,,Z) = i_d^*\left(L_{\mathbb{X}_l+\mathbb{X}_r} L_{\mathbb{Y}_l} L_{\mathbb{Z}_l}\, S\right) = L_X g(Y\,,Z)\,, \tag{4.62}$$

$$T_2(X\,,Y\,,Z) + T_5(X\,,Y\,,Z) = i_d^*\left(L_{\mathbb{X}_l+\mathbb{X}_r} L_{\mathbb{Y}_r} L_{\mathbb{Z}_r}\, S\right) = L_X g(Y\,,Z)\,, \tag{4.63}$$

and thus $T_{12} = T_{56}$. Similarly, it can be shown that $T_{34} = T_{78}$. Furthermore:

$$T_3(X\,,Y\,,Z) + T_5(X\,,Y\,,Z) = i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l+\mathbb{Y}_r} L_{\mathbb{Z}_r}\, S\right) = L_Y\, g(X\,,Z) + g\left([X\,,Y]\,,Z\right)\,, \tag{4.64}$$

$$T_4(X\,,Y\,,Z) + T_6(X\,,Y\,,Z) = i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_l+\mathbb{Y}_r} L_{\mathbb{Z}_r}\, S\right) = L_Y\, g(X\,,Z) + g\left([X\,,Y]\,,Z\right)\,, \tag{4.65}$$

and thus $T_{34} = -T_{56}$, from which it follows that

$$T_{12} = T_{56} = -T_{34} = -T_{78}\ . \tag{4.66}$$

This means that we can define a single symmetric tensor field T of order 3 on M starting with a potential function S. For instance, we set:

$$T(X\,,Y\,,Z) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} L_{\mathbb{Z}_r}\, S - L_{\mathbb{X}_r} L_{\mathbb{Y}_r} L_{\mathbb{Z}_l}\, S\right)\ . \tag{4.67}$$

We have thus proved the following proposition:

Proposition 28. *Let S be a potential function on $M \times M$. Then, all the maps defined in equations* (4.58), (4.59), (4.60), *and* (4.61) *define the same symmetric covariant* $(0\,,3)$ *tensor field T on M.*

For the sake of simplicity, we write T as in equation (4.67). In the coordinate charts $\{x^j\}$ and $\{q^j\,,Q^j\}$ introduced above, we have:

$$T=\left(\frac{\partial^3 S}{\partial q^j\partial q^k\partial Q^l}-\frac{\partial^3 S}{\partial Q^j\partial Q^k\partial q^l}\right)\Big|_d \mathrm{d}x^j\otimes_s \mathrm{d}x^k\otimes_s \mathrm{d}x^l\,, \tag{4.68}$$

and this expression is the one coventionally used in information geometry for the skewness tensor in information geometry (see [9] chapter 2 and [87]).

4.1.1 Potential functions and smooth mappings

Now that we have a formal intrinsic characterization for potential functions, we may ask what happens to a potential function through the pullback operation. This will be of capital importance when we analyze the monotonicity properties of metric tensors on the space of invertible quantum states.

Suppose $\phi\colon N\to M$ is a smooth map between differential manifolds. Let i_N and i_M denote, respectively, the diagonal immersions of N and M into their doubles $N\times N$ and $M\times M$. Let $\Phi\colon N\times N\to M\times M$ be the map defined by:

$$(n\,,n)\mapsto\Phi(n\,,n):=(\phi(n)\,,\phi(n))\,. \tag{4.69}$$

A direct calculation shows that:

$$\Phi\circ i_N=i_M\circ\phi\,. \tag{4.70}$$

Furthermore:

Proposition 29. *Let $X\in\mathfrak{X}(N)$ be ϕ-related to $Z\in\mathfrak{X}(M)$, that is, $T\phi\circ X=Z\circ\phi$. Then $\mathbb{X}_l$ is Φ-related to $\mathbb{Z}_l$, that is, $T\Phi\circ\mathbb{X}_l=\mathbb{Z}_l\circ\Phi$, and $\mathbb{X}_r$ is Φ-related to $\mathbb{Z}_r$, that is, $T\Phi\circ\mathbb{X}_r=\mathbb{Z}_r\circ\Phi$*

Proof. *By hypotesis, it is $T\phi\circ X=Z\circ\phi$. We want to cast this equality in a more useful form. We start noting that:*

$$T\phi\circ X(n)=T\phi(n\,,v_n^X)=(\phi(n)\,,T_n\phi(v_n^x))\;, \tag{4.71}$$

$$Z\circ\phi(n)=Z(\phi(n))=\left(\phi(n)\,,v^Z_{\phi(n)}\right)\;, \tag{4.72}$$

from which it follows that $T\phi\circ X=Z\circ\phi$ implies:

$$T_n\phi(v_n^X)=v^Z_{\phi(n)}\,. \tag{4.73}$$

Now, we have

$$T\Phi(n_1\,,v_{n_1}\,;n_2\,,v_{n_2})=(\phi(n_1)\,,T_{n_1}\phi(v_{n_1})\,;\phi(n_2)\,,T_{n_2}\phi(v_{n_2}))\;, \tag{4.74}$$

and thus:

$$T\Phi\circ\mathbb{X}_l(n_1\,,n_2)=T\Phi(n_1\,,v^X_{n_1}\,,n_2\,,0)=\big(\phi(n_1)\,,T_{n_1}\phi(v^X_{n_1})\,;\phi(n_2)\,,0\big)\ . \tag{4.75}$$

On the other hand:

$$\mathbb{Z}_l\circ\Phi(n_1\,,n_2)=\mathbb{Z}_l(\phi(n_1)\,,\phi(n_2))=\big(\phi(n_1)\,,v^Z_{\phi(n_1)}\,;\phi(n_2)\,,0\big)\ . \tag{4.76}$$

Plugging equation (4.73) *into equation* (4.76)*, and then comparing equation* (4.76) *with equation* (4.75) *we obtain:*

$$T\Phi\circ\mathbb{X}_l=\mathbb{Z}_l\circ\Phi \tag{4.77}$$

as claimed. Proceeding analogously, we prove that $\mathbb{X}_r$ is Φ-related to $\mathbb{Z}_r$. This completes the proof.

With the help of proposition 29 we are able to analyze the behaviour of potential functions with respect to smooth maps. Specifically, we have the following:

Proposition 30. *Let $\phi\colon N\to M$ be a smooth map between smooth manifold, and let $\Phi\colon N\times N\to M\times M$ be defined as $\Phi(n_1\,,n_2):=(\phi(n_1)\,,\phi(n_2))$. Let S be a potential function on $M\times M$ then Φ^*S is a potential function on $N\times N$, and the symmetric covariant tensor extracted from Φ^*S is equal to the pullback by means of ϕ of the symmetric covariant tensor extracted from S.*

Proof. *Suppose that S is a potential function on $M\times M$. Take a generic $X\in\mathfrak{X}(N)$, and consider a vector field $Z\in\mathfrak{X}(M)$ which is ϕ-related to X. Then:*

$$\begin{aligned} i^*_N\left(L_{\mathbb{X}_l}\Phi^*S\right)&=i^*_N\Phi^*\left(L_{\mathbb{Z}_l}S\right)=\left(\Phi\circ i_N\right)^*\left(L_{\mathbb{Z}_l}S\right)=\\ &=\left(i_M\circ\phi\right)^*\left(L_{\mathbb{Z}_l}S\right)=\phi^*i^*_M\left(L_{\mathbb{Z}_l}S\right)=0\,, \end{aligned} \tag{4.78}$$

where we used equation (4.45)*, proposition 29, equation* (4.70)*, and condition* (4.33)*. In a similar way, it is possible to show that $i^*_N\left(L_{\mathbb{X}_r}\Phi^*S\right)=0$, and this means that Φ^*S is a potential function on $N\times N$.*

Denote with g_N the symmetric covariant tensor field on N generated by the potential function Φ^*S, and with Z (W) the vector field on M which is ϕ-related to X (Y). Recalling equation (4.45)*, proposition 29, and equation* (4.70)*, we have:*

$$\begin{aligned} g_N(X\,,Y)&=i^*_N\left(L_{\mathbb{X}_l}L_{\mathbb{Y}_l}\Phi^*S\right)=i^*_N\Phi^*\left(L_{\mathbb{Z}_l}L_{\mathbb{W}_l}S\right)=\\ &=\left(\Phi\circ i_N\right)^*\left(L_{\mathbb{Z}_l}L_{\mathbb{W}_l}S\right)=\left(i_M\circ\phi\right)^*\left(L_{\mathbb{Z}_l}L_{\mathbb{W}_l}S\right)= \end{aligned}$$

$$= \phi^* i_M^* \left(L_{\mathbb{Z}_l} L_{\mathbb{W}_l} S \right) = \phi^* \left(g_M(Z\,,W) \right) \,.$$

Being $g_N(X\,,Y)$ a function, we may evaluate it at n:

$$\begin{gathered} \left(g_N(X\,,Y) \right)(n) = \left(\phi^* \left(g_M(Z\,,W) \right) \right)(n) = \\ = \left(g_M(Z\,,W) \right)(\phi(n)) = g_M|_{\phi(n)} \left(Z|_{\phi(n)}\,, W|_{\phi(n)} \right) \end{gathered} \tag{4.79}$$

Now, by the very definition of the pullback $\phi^* g_M$ we have:

$$\left((\phi^* g_M)(X\,,Y) \right)(n) = g_M|_{\phi(n)} (\phi_* X|_{\phi(n)}\,, \phi_* Y|_{\phi(n)}) \,. \tag{4.80}$$

Being Z and W ϕ-related to, respectively X and Y, we have:

$$Z|_{\phi(n)} = \phi_* X|_{\phi(n)}\,, \qquad W|_{\phi(n)} = \phi_* Y|_{\phi(n)}\,, \tag{4.81}$$

and thus the symmetric covariant tensor we can extract from $\Phi^* S$ coincides with the pullback $\phi^* g_M$ we can extract from S.

We are now in the position to say something about the connection between the symmetry properties of g and the symmetry properties of the potential function S with which it is associated. Let G be a Lie group acting on M by means of diffeomorphisms ϕ_{g} with $\mathrm{g} \in G$. Then G acts on $M \times M$ by means of the maps $\Phi_{\mathrm{g}}(m_1\,, m_2) := (\phi_{\mathrm{g}}(m_1)\,, \phi_{\mathrm{g}}(m_2))$. Let S be a potential function on $M \times M$. It then follows from proposition 30 that:

$$\left(\phi_{\mathrm{g}}^* g - g \right)(X\,,Y) = -i_d^* \left(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} \left(S - \Phi_{\mathrm{g}}^* S \right) \right) = 0 \,. \tag{4.82}$$

From this equation we conclude that if S is invariant under the action of G on $M \times M$ associated with the action of G on M, then

$$\left(\phi_{\mathrm{g}}^* g - g \right)(X\,,Y) = 0 \quad \forall X, Y \in \mathfrak{X}(M) \,, \tag{4.83}$$

and thus G is a symmetry group for the metric-like tensor g associated with S, that is:

$$\phi_{\mathrm{g}}^* g = g \quad \forall \mathrm{g} \in G \,. \tag{4.84}$$

4.2 Quantum divergence functions and monotonicity

We will now use the geometric tools developed in the previous section in order to define the monotonicity property for quantum metric tensors, to define the data processing inequality (DPI) for quantum divergence functions, and to prove that quantum divergence functions satsifying the data processing inequality give rise to quantum metric tensors satisfying the monotonicity property. Essentially, the monotonicity property is a quantum version of the so-called invariance criterion of classical information geometry (see [7, 34]), where classical stochastic mappings are replaced with quantum stochastic mappings. Consequently, we will introduce the notion of quantum stochastic mapping according to [93]. These class of maps plays a prominent role not only in the definition of the monotonicity property for quantum metric tensors, but also for the definition of the data processing inequality for quantum divergence functions.

Let us start denoting with $\mathbb{N}_2$ the set of natural number without $\{0\}$ and $\{1\}$, and considering the family $\{\bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$ of manifolds of invertible density matrices. The notion of quantum stochastic map is then formulated in terms of completely-positive trace preserving (CPTP) maps on $\mathfrak{A}_n$:

Definition 15. *Let $\phi\colon \mathfrak{A}_n \to \mathfrak{A}_m$ be a CPTP map with $n, m \in \mathbb{N}_2$. According to [93], we say that ϕ is a* ***quantum stochastic map*** *if:*

$$\phi\left(\bar{\mathcal{S}}_n\right) \subseteq \bar{\mathcal{S}}_m\,. \tag{4.85}$$

Note that the family of quantum stochastic map form a category precisely as the family of classical stochastic map (see [34], chapter I and II).

In Holevo's books [72] and [73] there is an interesting discussion on the theoretical and operational relevance of the class of quantum stochastic maps. Once we have fixed this class of maps between invertible density matrices, we are ready to give a definition of the monotonicity property for quantum Riemnnian metric tensors. Clearly, since the family of quantum stochastic maps may connect systems with different dimensions, we must not consider a single tensor field defined on the manifold of invertible density matrices of a single quantum system, but, rather, a family of tensor fields.

Definition 16. *The family $\{g^n\}_{n\in\mathbb{N}_2}$, where each g^n is a covariant tensor on $\bar{\mathcal{S}}_n$ for each $n \in \mathbb{N}_2$, satisfies the monotonicity property if:*

$$g^n(X\,,X) \geq \left(\phi^* g^m\right)(X\,,X)\,, \tag{4.86}$$

for all $X \in \mathfrak{X}(\bar{\mathcal{S}}_n)$ and for all quantum stochastic maps ϕ. By the very definition of the pullback operation, the monotonicity property is equivalent to:

$$g^n|_\rho\left(X|_\rho\,,\,X|_\rho\right) \geq g^m|_{\phi(\rho)}\left(\phi_* X|_{\phi(\rho)}\,,\,\phi_* X|_{\phi(\rho)}\right)\,, \tag{4.87}$$

where $\rho \in \bar{\mathcal{S}}_n$.

Remark 18. *Note that, in principle, the elements in the family* $\{g^n\}_{n\in\mathbb{N}_2}$ *are generic covariant tensor fields, that is, they need not be symmetric, nor invertible, nor having positivity properties.*

Roughly speaking, the monotonicity property for a family of quantum Riemannian metric tensors ensures that the notion of geodesical distance between invertible density matrices, as encoded in the family of quantum Riemannian metric tensors, does not increase under quantum stochastic maps. We will now rephrase the monotonicity property of the family $\{g^n\}_{n\in\mathbb{N}_2}$ in terms of the behaviour, with respect to quantum stochastic maps, of the family $\{S^n\}_{n\in\mathbb{N}_2}$ of two-point functions from which $\{g^n\}_{n\in\mathbb{N}_2}$ is extracted.

Proposition 31. *Consider the family of two-point functions* $\{S^n\}_{n\in\mathbb{N}_2}$ *on* $\{\bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$*, and suppose that the family* $\{g^n\}_{n\in\mathbb{N}_2}$ *of covariant tensors on* $\{\bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$*, extracted from* $\{S^n\}_{n\in\mathbb{N}_2}$ *according to the third map in definition 12, satisfies the monotonicity property of definition 16. Let* $\phi\colon \bar{\mathcal{S}}_n \to \bar{\mathcal{S}}_m$ *be a stochastic map, and let* $\Phi\colon \bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n \to \bar{\mathcal{S}}_m \times \bar{\mathcal{S}}_m$ *be defined as:*

$$\Phi(\bar{\rho}\,,\bar{\varrho}) := (\phi(\bar{\rho})\,,\phi(\bar{\varrho}))\,. \tag{4.88}$$

Then, setting $S^{nm}_{\Phi} = (S^n - \Phi^* S^m)$*, the monotonicity property of* $\{g^n\}_{n\in\mathbb{N}_2}$ *is equivalent to:*

$$g^{nm}_{\phi}(X\,,X) := -i_n^*\left(L_{\mathbb{X}_l} L_{\mathbb{X}_r} S^{nm}_{\Phi}\right) \geq 0 \tag{4.89}$$

for all $X \in \mathfrak{X}(\bar{\mathcal{S}}_n)$ *and for all stochastic maps* ϕ.

Proof. *According to proposition 30, we know that* $\phi^* g^m$ *is the covariant tensor field generated by the divergence function* $\Phi^* S^m$*. This means that we may write:*

$$\left(\phi^* g^m\right)(X\,,Y) = -i_m^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} \Phi^* S^m\right) \tag{4.90}$$

where we used the third map in definition 12. Again, using definition 12, we write:

$$g^n(X\,,Y) = -i_n^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} S^n\right)\,. \tag{4.91}$$

Comparing these two equations, it then follows that:

$$g^n(X\,,X) \geq \left(\phi^* g^m\right)(X\,,X) \tag{4.92}$$

is equivalent to:

$$g_{\phi}^{nm}(X\,,X) := -i_n^*\left(L_{\mathbb{X}_l} L_{\mathbb{X}_r} S_{\Phi}^{nm}\right) \equiv -i_n^*\left(L_{\mathbb{X}_l} L_{\mathbb{X}_r}\left(S^n - \Phi^* S^m\right)\right) \geq 0 \tag{4.93}$$

as claimed.

As anticipated before, there is a very interesting connection between this result and the so-called data processing inequality (DPI) for quantum divergences:

Definition 17. *Let $\{S^n\}_{n\in\mathbb{N}_2}$ be a family of two-point funcions on $\{\bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$. We say that $\{S^n\}_{n\in\mathbb{N}_2}$ satisfies the DPI if:*

$$S^n(\bar{\rho}\,,\bar{\varrho}) \geq S^m(\phi(\bar{\rho})\,,\phi(\bar{\varrho})) \tag{4.94}$$

for all $\bar{\rho}, \bar{\varrho}$ and for all stochastic maps ϕ.

The operational meaning of this inequality is to ensure that the information-theoretical content encoded in the family of quantum two-point functions does not increase under quantum stochastic maps. Then, the following proposition shows that *the DPI "implies" the monotonicity property*:

Proposition 32. *If the family $\{S^n\}_{n\in\mathbb{N}_2}$ of quantum divergence functions on $\{\bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$ satisfies the DPI, then, the family $\{g^n\}_{n\in\mathbb{N}_2}$ of symmetric covariant tensor fields we can extract from $\{S^n\}_{n\in\mathbb{N}_2}$ satisfies the monotonicity property.*

Proof. *The function $\Phi^* S^m$ is a potential function because S^m is a potential function (see proposition 30). According to the DPI, we have:*

$$\begin{aligned} S_{\Phi}^{nm}(\bar{\rho}\,,\bar{\varrho}) &:= S^n(\bar{\rho}\,,\bar{\varrho}) - \Phi^* S^m(\bar{\rho}\,,\bar{\varrho}) = \\ &= S^n(\bar{\rho}\,,\bar{\varrho}) - S^m(\phi(\bar{\rho})\,,\phi(\bar{\varrho})) \geq 0\,. \end{aligned} \tag{4.95}$$

From this, we conclude that S_{Φ}^{jk} is a non-negative potential function vanishing on the diagonal of $\bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n$. This means that every point on the diagonal of $\bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n$ is a local minimum for S_{Φ}^{nm}. Then, according to proposition 27 the symmetric covariant tensor g_{ϕ}^{nm} it generates is positive-semidefinite. In particular it is:

$$g_{\phi}^{nm}(X\,,X) \geq 0\,. \tag{4.96}$$

According to proposition 31 this is equivalent to the monotonicity property for the family $\{g^n\}_{n\in\mathbb{N}_2}$, and the proposition is proved.

The result expressed in proposition 32 may be seen as a sort of generalization to the quantum case of the invariance criterion of classical information geometry (see [7], chapter 3 and 6). Furthermore, the abstract coordinate-free framework in which proposition 32 is contextualized may prove to be useful for a generalization to the infinite dimensional case.

4.3 Quantum metric tensors from modified $(q-z)$-Rényi relative entropies

Here we will actually compute the symmetric covariant tensors associated with the modified $(q-z)$-Rényi relative entropy of equation (4.14). We want to perform calculations without referring to explicit coordinate systems, therefore, we will unfold the manifold $\bar{\mathcal{S}}_n$ of invertible density matrices to the more gentle manifold $\mathcal{M}_n = SU(\mathfrak{A}_n) \times \Delta_n^0$, where Δ_n^0 is the open interior of the n-dimensional simplex Δ_n, that is:

$$\Delta_n^0 := \left\{ \vec{p} \in \mathbb{R}^n : \, p^j > 0 \, , \, \sum_{j=1}^n p^j = 1 \right\} . \tag{4.97}$$

This manifold is parallelizable since it is the Cartesian product of parallelizable manifolds, and thus, we have global basis of vector fields and differential one-forms at our disposal. We will use these basis to perform coordinate-free computations in any dimension. However, before entering the description of these basis, we want to explain why $\mathcal{M}_n$ may be thought of as an unfolding manifold for $\bar{\mathcal{S}}_n$. To do this, let us consider an invertible density matrix $\bar{\rho} \in \bar{\mathcal{S}}_n$. It is well known that $\bar{\rho}$ can be diagonalized, and that its eigenvalues are strictly positive and sum up to one. This means that, denoting with $\vec{p} \in \Delta_n^0$ a vector the components of which coincide with the eigenvalues of $\bar{\rho}$, we can find a $\mathbf{U} \in SU(\mathfrak{A}_n)$ such that:

$$\bar{\rho} = \mathbf{U} \, \bar{\rho}_0 \, \mathbf{U}^\dagger \, , \tag{4.98}$$

where $\bar{\rho}_0$ is a diagonal matrix in the sense that its only nonzero components with respect to the canonical basis $\{\mathbf{E}_{jk}\}_{j,k=0,\dots,(n-1)}$ of $\mathfrak{A}_n$ introduced in chapter 1 are those relative to $\{\mathbf{E}_{jj}\}_{j=0,\dots,(n-1)}$. It is clear that every $\bar{\rho}_0$ can be identified with a point $\vec{p}$ in Δ_n^0 and viceversa. This one-to-one correspondence is given by the map $\bar{\rho}_0 = p^j \mathbf{E}_{jj}$.

Remark 19. *It is important to point out that the correspondence between $\bar{\rho}_0$ and $\vec{p}$ explicitely depends on the choice of the basis $\{\mathbf{E}_{jk}\}_{j=0,\dots,(n-1)}$ as a reference basis. For instance, if we consider a multipartite system for which:*

$$\mathfrak{A}_N = \bigotimes_{j=1}^r \mathfrak{A}_{n_j} \, , \tag{4.99}$$

where $N = n_1 \, n_2 \cdots n_r$, the canonical basis in $\mathfrak{A}_N$ is made up of separable elements with respect to the decomposition of $\mathfrak{A}_N$. Consequently, the reference density matrix

$\bar{\rho}_0$ *associated with the probability vector* $\vec{p}$ *is separable, and this clearly has consequences with respect to the entanglement properties of the system. Specifically, when we unfold the density matrix* $\bar{\rho}$ *into the couple* $(\mathbf{U}\,,\vec{p})$*, all the information regarding the entanglement properties of* $\bar{\rho}$ *will be encoded in* $\mathbf{U}$ *because* $\vec{p}$ *is associated with the separable density matrix* $\bar{\rho}_0$*.*

The diagonalization procedure for $\bar{\rho} \in \bar{\mathcal{S}}_n$ provides us with a map:

$$\begin{aligned} \pi_n \colon SU(\mathfrak{A}_n)\times\Delta_n^0 &\to \bar{\mathcal{S}}_n \\ (\mathbf{U}\,,\vec{p}) \mapsto\ &\pi_n(\mathbf{U}\,,\vec{p}) = \mathbf{U}\,\bar{\rho}_0\,\mathbf{U}^\dagger \ \text{ with } \bar{\rho}_0 = p^j\,\mathbb{E}_{jj}\,. \end{aligned} \tag{4.100}$$

Obviously, the map π_n is a surjection because for a given $\bar{\rho} \in \bar{\mathcal{S}}_n$ there is an infinite number of elements $(\mathbf{U}\,,\vec{p}) \in \mathcal{M}_n$ such that $\pi_n(\mathbf{U}\,,\vec{p}) = \bar{\rho}$. It is in this sense that we think of $\mathcal{M}_n$ as an unfolding manifold for $\bar{\mathcal{S}}_n$. Now that we have the map π_n, we proceed to prove the following:

Proposition 33. *The map* $\pi_n \colon \mathcal{M}_n \to \bar{\mathcal{S}}_n$ *is a surjective submersion, and the kernel of its differential at* $(\mathbf{U}\,,\vec{p}) \in \mathcal{M}_n$ *is given by* $(\imath\mathbf{H}\,,\vec{0})$*, where* $\mathbf{H}$ *is a self-adjoint matrix such that* $[\mathbf{H}\,,\bar{\rho}_0] = \mathbf{0}$*.*

Proof. *The surjectivity of the map* π_n *follows from the spectral decomposition for every density matrix* $\bar{\rho}$*. To prove that* π_n *is a submersion, we consider the following curve* γ_t *on* $\mathcal{M}_n$*:*

$$\gamma_t(\mathbf{U}\,,\vec{p}) = (\mathbf{U}\exp(\imath t\mathbf{H}), \vec{p}_t)\,, \tag{4.101}$$

where $\mathbf{H}$ *is self-adjoint and traceless, and* $\vec{p}_t$ *is any curve in the interior of the n-simplex* Δ_n^0 *starting at* $\vec{p}_0 = \vec{p}$ *and such that* $\left.\frac{\mathrm{d}\vec{p}_t}{\mathrm{d}t}\right|_{t=0} = \vec{a}$ *with* $\sum_j a^j = 0$*. The differential:*

$$(T\pi_n)_{(\mathbf{U},\vec{p})} : T_{(\mathbf{U},\vec{p})}\mathcal{M}_n \to T_\rho\mathcal{S}_n$$

of π *at* $(\mathbf{U}\,,\vec{p})$ *is then:*

$$\begin{aligned} (T\pi_n)_{(\mathbf{U},\vec{p})}\left(\left.\frac{\mathrm{d}\gamma(\mathbf{U},\vec{p})}{\mathrm{d}t}\right|_{t=0}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{U}\exp(\imath t\mathbf{H})\,\bar{\rho}_0(t), \exp(-\imath t\mathbf{H})\mathbf{U}^\dagger\right)_{t=0} = \\ &= \mathbf{U}\,\left(\imath\,[\mathbf{H}\,,\bar{\rho}_0] + a^j\mathbf{E}_{jj}\right)\,\mathbf{U}^\dagger\,. \end{aligned} \tag{4.102}$$

The tangent space $T_{\bar{\rho}}\bar{\mathcal{S}}_n$ *at* $\bar{\rho} = \mathbf{U}\,\bar{\rho}_0\,\mathbf{U}^\dagger$ *is the space of traceless self-adjoint matrices, and it is clear that every such element can be written in the form*

$$\mathbf{U}\,\left(\imath\,[\mathbf{H}\,,\bar{\rho}_0] + a^j\mathbf{E}_{jj}\right)\,\mathbf{U}^\dagger\,.$$

This means that π_n is a submersion at every $(\mathbf{U},\vec{p}) \in \mathcal{M}_n$. Furthermore, it follows that the tangent vector $(\imath\mathbf{H},\vec{a})$ at $(\mathbf{U},\vec{p})$ is sent to the null vector $\mathbf{0}$ at $\bar{\rho} = \mathbf{U}\,\bar{\rho}_0\,\mathbf{U}^\dagger$ if and only if $\vec{a} = \vec{0}$ and $[\mathbf{H}\,,\bar{\rho}_0] = \mathbf{0}$.

The global differential calculus on $\mathcal{M}_n$ is easily defined considering the projection maps:

$$\begin{aligned} pr_{SU(\mathfrak{A}_n)} \colon \mathcal{M}_n = SU(\mathfrak{A}_n) \times \Delta_n^0 \to SU(\mathfrak{A}_n)\,,\ (\mathbf{U}\,,\vec{p}) \mapsto pr_{SU(\mathfrak{A}_n)}(\mathbf{U}\,,\vec{p}) = \mathbf{U}\,, \\ pr_{\Delta_n^0} \colon \mathcal{M}_n = SU(\mathfrak{A}_n) \times \Delta_n^0 \to \Delta_n^0\,,\ (\mathbf{U}\,,\vec{p}) \mapsto pr_{\Delta_n^0}(\mathbf{U}\,,\vec{p}) = \vec{p}\,. \end{aligned} \tag{4.103}$$

Then, since $SU(\mathfrak{A}_n)$ is a Lie group, we have, for instance, a basis of gloabally defined left-invariant differential one-forms $\{\theta^j\}_{j=1,\ldots,n^2-1}$ and a basis of globally defined left-invariant vector fields $\{X_j\}_{j=1,\ldots,n^2-1}$ which is dual to $\{\theta^j\}_{j=1,\ldots,n^2-1}$. Consequently, we can take the pullback of every θ^j by means of $pr_{SU(\mathfrak{A}_n)}$ and obtain a set of globally defined differential one-forms on $\mathcal{M}_n$. With an evident abuse of notation, we will keep writing $\{\theta^j\}_{j=1,\ldots,n^2-1}$ for this set of one-forms. Regarding Δ_n^0, we will construct an "overcomplete" basis of differential one-forms as follows. First of all, we define n functions $\mathrm{P}^j \colon \Delta_n^0 \to \mathbb{R}$:

$$\vec{p} \mapsto \mathrm{P}^j(\vec{p}) = p^j\,. \tag{4.104}$$

This functions are globally defined and smooth, and thus their differential $\mathrm{dP}^j = \mathrm{d}p^j$ are globally defined differential one-forms. Clearly, we have n of them, and since $\dim(\Delta_n^0) = n-1$, these one-forms are not functionally independent. Indeed, it holds:

$$\sum_{j=1}^n \mathrm{P}^j(\vec{p}) = \sum_{j=1}^n p^j = 1\,, \tag{4.105}$$

and thus:

$$\sum_{j=1}^n \mathrm{dP}^j = \sum_{j=1}^n \mathrm{d}p^j = 0\,. \tag{4.106}$$

Now, the set $\{\mathrm{d}p^j\}_{j=1,\ldots,n}$ of globally defined differential one-forms on Δ_n^0 is a basis of the module of one-forms on Δ_n^0, that is, for every differential one-form ω on Δ_n, it always exists a decomposition:

$$\omega = \omega_j\,\mathrm{d}p^j\,, \tag{4.107}$$

where $\omega_j \in \mathcal{F}(\Delta_n^0)$. This is the sense in which $\{\mathrm{d}p^j\}_{j=1,\dots,n}$ is an overcomplete basis for the space of differential one-forms on Δ_n^0. Now, similarly to what we have done for $SU(\mathfrak{A}_n)$, we consider the pullback of $\mathrm{d}p^j$ by means of $pr_{\Delta_n^0}$, and we obtain a set of globally defined differential one-forms on $\mathcal{M}_n$. Again with an abuse of notation, we will keep writing this set as $\{\mathrm{d}p^j\}_{j=1,\dots,n}$. Eventually, the set $(\{\theta^j\}_{j=1,\dots,n^2-1}\,,\{\mathrm{d}p^j\}_{j=1,\dots,n})$ is a basis of the module of differential one-forms on $\mathcal{M}_n$.

Now that we have the global basis for a differential calculus on $\mathcal{M}_n$ (for all n), we may proceed with the explicit computations. First of all, we consider the modified $(q-z)$-Rényi relative entropy of equation (4.14):

$$S_{q,z}(\bar{\rho}\,,\bar{\varrho}) = \frac{1}{q(1-q)}\left[1 - Tr\left(\bar{\rho}^{\frac{q}{z}}\,\bar{\varrho}^{\frac{1-q}{z}}\right)^{z}\right]\,, \tag{4.108}$$

and take its pullback to $\mathcal{M}_n \times \mathcal{M}_n$ by means of the map:

$$\begin{aligned} \Pi_n \colon \mathcal{M}_n \times \mathcal{M}_n &\to \bar{\mathcal{S}}_n \times \bar{\mathcal{S}}_n \\ (\mathbf{U}\,,\vec{p}_1\,;\mathbf{V}\,,\vec{p}_2) &\mapsto (\pi_n(\mathbf{U}\,,\vec{p}_1)\,;\pi_n(\mathbf{V}\,,\vec{p}_2))\,. \end{aligned} \tag{4.109}$$

The result is the following function on $\mathcal{M}_n \times \mathcal{M}_n$:

$$D^n_{q,z}(\mathbf{U}\,,\vec{p}_1\,;\mathbf{V}\,,\vec{p}_2) = \frac{1}{q(1-q)}\left(1 - Tr\left[\left((\mathbf{U}\,\bar{\rho}_0\,\mathbf{U}^\dagger)^{\frac{q}{z}}(\mathbf{V}\,\bar{\varrho}_0\,\mathbf{V}^\dagger)^{\frac{1-q}{z}}\right)^{z}\right]\right)\,, \tag{4.110}$$

where $\bar{\rho}_0 = p_1^j\mathbf{E}_{jj}$ and $\bar{\varrho}_0 = p_2^j\mathbf{E}_{jj}$. At the moment, we do not know if $D^n_{q,z}$ is a potential function, but we can always extract a covariant tensor field from it by computing (see proposition 23):

$$g^n_{q,z}(X\,,Y) := -i_d^*\left(L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}\,D^n_{q,z}\right)\,. \tag{4.111}$$

Here, we will discuss the result of the computations and simply refer to subsection 4.3.1 for all the details. The final result is:

$$g^n_{q,z} = g^{n\perp}_{q,z} + g^{n\parallel}_{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)}\sum_{j,k=1}^{n^2-1}{}' \mathcal{C}_{jk}\,\theta^j\otimes\theta^k\,, \tag{4.112}$$

where $\{p_\alpha\}_{\alpha=1,..n}$ denote the eigenvalues of $\bar{\rho}$, the coefficients C_{jk} are given by:

$$\mathcal{C}_{jk} = \sum_{\alpha,\beta=1}^{n}{}' \, \mathcal{E}_{\alpha\beta} \, \Re\big[M_j^{\alpha\beta} M_k^{\beta\alpha}\big] \, , \tag{4.113}$$

with $M_j^{\alpha\beta}$ being numerical coefficients depending on the choice of a basis in the Lie algebra of $U(\mathfrak{A}_n)$, and with:

$$\mathcal{E}_{\alpha\beta} := \frac{(p_\alpha - p_\beta)(p_\alpha^{\frac{q}{z}} - p_\beta^{\frac{q}{z}})(p_\alpha^{\frac{1-q}{z}} - p_\beta^{\frac{1-q}{z}})}{(p_\alpha^{\frac{1}{z}} - p_\beta^{\frac{1}{z}})} \, . \tag{4.114}$$

It turns out that the coefficients C_{jk} are symmetric in j and k, and thus $g_{q,z}^n$ is a symmetric tensor. Furthermore, the sum over j and k in Eq. (4.112) does not involve the basic left-invariant 1-forms dual to the Cartan subalgebra. Indeed, the only terms which contain the left-invariant 1-forms associated with the Cartan subalgebra are those with $\alpha = \beta$ which have vanishing coefficients $\mathcal{C}_{jk}$.

Whenever the parameters q and z are such that the modified q, z-RRE $S_{q,z}^n$ of equation (4.14) is a divergence function in the sense of definition 14, we have that $D_{q,z}^n$ is a non-negative potential function, and that $g_{q,z}^n$ is a positive-semidefinite symmetric covariant tensor field which is the pullback of the positive-semidefinite symmetric covariant tensor field on $\bar{\mathcal{S}}_n$ extracted from $S_{q,z}^n$ (see proposition 30 and proposition 27). Recalling that $g_{q,z}^n$ does not contain the basic left-invariant 1-forms dual to the Cartan subalgebra, and since $\mathrm{d}p_j$ and θ^j are basis elements, we conclude that the kernel of $g_{q,z}^n$ is given by the span of the vector fields dual to the left-invariant 1-forms associated with the Cartan subalgebra. According to proposition 33, these vector fields are π_n related with the null vector field on $\bar{\mathcal{S}}_n$. This means that $g_{q,z}^n$ is the pullback to $\mathcal{M}_n$ of a symmetric invertible tensor on $\bar{\mathcal{S}}_n$, that is, a Riemannian metric tensor on the space of invertible density matrices.

If the values of q and z for which $S_{q,z}^n$ is a divergence function are such that $\{S_{q,z}^n\}_{n\in\mathbb{N}_2}$ satisfies the DPI, then the family of symmetric tensors $\{g_{q,z}^n\}_{n\in\mathbb{N}_2}$, where $g_{q,z}^n$ is given by equation (4.112), is the pullback to $\{\mathcal{M}_n\}_{n\in\mathbb{N}_2}$ of a family of quantum Riemannian metric tensors on $\{\bar{\mathcal{S}}_n\}_{n\in\mathbb{N}_2}$ satisfying the monotonicity property. In particular, according to the formulae in the introduction of this chapter, the family of metric tensors associated with the von Neumann-Umegaki relative entropies, with the Tsallis relative entropies, with the Wigner-Yanase skew informations, and with the Bures divergences, all satisfies the monotonicity property.

Equation (4.112) points out another interesting fact. The first term in the expression of $g_{q,z}^n$ is precisely the Fisher-Rao metric tensor related to the component of the "classical" probability vector $\vec{p} = (p_1, \ldots, p_n)$ identified with the diagonal elements of the invertible density matrix. Consequently, since the monotonicity property is connected to the DPI, since the DPI depends on the explicit values of q and z, and since the Fisher-Rao contribution to $g^n_{q,z}$ does not depend on (q,z), the "obstruction" to the monotonicity property is completely encoded in the unitary contribution to $g^n_{q,z}$.

4.3.1 Explicit computations

Here we will perform the detailed computation of the covariant tensor field:

$$g^n_{q,z}(X\,,Y) := -i_d^*\left(L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}\,D^n_{q,z}\right)\,, \tag{4.115}$$

where:

$$D^n_{q,z}\left(\mathbf{U}\,,\vec{p}_1\,;\mathbf{V}\,,\vec{p}_2\right) = \frac{1}{q(1-q)}\left(1 - Tr\left[\left((\mathbf{U}\,\bar{\rho}_0\,\mathbf{U}^\dagger)^{\frac{q}{z}}(\mathbf{V}\,\bar{\varrho}_0\,\mathbf{V}^\dagger)^{\frac{1-q}{z}}\right)^z\right]\right)\,, \tag{4.116}$$

with $\bar{\rho}_0 = \mathrm{diag}(\vec{p}_1)$ and $\bar{\varrho}_0 = \mathrm{diag}(\vec{p}_2)$. At this purpose, we start setting:

$$A = \bar{\rho}^{\frac{q}{z}} = \mathbf{U}\,\bar{\rho}_0^{\frac{q}{z}}\,\mathbf{U}^\dagger \qquad , \qquad B = \bar{\varrho}^{\frac{1-q}{z}} = \mathbf{V}\,\bar{\varrho}_0^{\frac{1-q}{z}}\,\mathbf{V}^\dagger\,. \tag{4.117}$$

Since $z \in \mathbb{R}_+$, it can take both integer and noninteger values. Therefore, in order to have a well defined expression, we consider the analytical expansion of the function $(AB)^z$ with respect to the identity, say:

$$(AB)^z = \sum_{n=0}^{\infty} c_n(z)(AB - \mathbb{1})^n\,. \tag{4.118}$$

Let us notice that, as stressed in [20], even if $AB = \bar{\rho}^{\frac{q}{z}}\,\bar{\varrho}^{\frac{1-q}{z}}$ is not Hermitian, the spectrum of AB coincides with the spectrum of $BA = (AB)^\dagger$ for A and B Hermitian operators as in (4.117). This ensures that the spectrum of AB is real and hence $(AB)^z$ as a function of z does not have nonanalyticity branches and can be expanded as in (4.118).

Next, we consider:

$$L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}\,\mathrm{Tr}\left[(A\,B)^z\right] = \sum_{m=0}^{\infty} c_m(z)\,L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}\,\mathrm{Tr}\left[(AB-\mathbb{1})^m\right]\,. \tag{4.119}$$

Using the Leibniz rule together with the cyclic property of the trace and with the relation $L_{\mathbf{Y}_r} = i_{\mathbf{Y}_r}\mathrm{d}$ which is valid on functions, we have:

$$L_{\mathbb{Y}_r}\,\mathrm{Tr}\left[(A\,B)^z\right] = \sum_{m=0}^{\infty} c_m(z)\, L_{\mathbb{Y}_r}\,\mathrm{Tr}\left(\underbrace{(AB-\mathbb{1})\ldots(AB-\mathbb{1})}_{m}\right) =$$

$$= \sum_{m=0}^{\infty} c_m(z)\,\mathrm{Tr}\big(A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\ (AB-\mathbb{1})^{m-1} + (AB-\mathbb{1})\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\ (AB-\mathbb{1})^{m-2} +$$

$$+\cdots+ (AB-\mathbb{1})^{m-1}\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big) =$$

$$= \sum_{m=0}^{\infty} m\,c_m(z)\,\mathrm{Tr}\big((AB-\mathbb{1})^{m-1}\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big) \tag{4.120}$$

where we used the fact that $i_{\mathbf{Y}_r}\mathrm{d}A = 0$ because A depends only on the elements of the left factor of $\mathcal{M}_n \times \mathcal{M}_n$. Then:

$$L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}\,\mathrm{Tr}\left[(A\,B)^z\right] = \sum_{m=0}^{\infty} m\,c_m(z)\,L_{\mathbb{X}_l}\,\mathrm{Tr}\left(\underbrace{(AB-\mathbb{1})\ldots(AB-\mathbb{1})}_{m-1}A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\right) =$$

$$= \sum_{m=0}^{\infty} m\,c_m(z)\ \big[\mathrm{Tr}\big((AB-\mathbb{1})^{m-1}\ (i_{\mathbb{X}_l}\mathrm{d}A)\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big) + \mathrm{Tr}\big((i_{\mathbb{X}_l}\mathrm{d}A)\ B\,(AB-\mathbb{1})^{m-2}\ (i_{\mathbb{Y}_r}\mathrm{d}B) +$$

$$+\ (AB-\mathbb{1})\ (i_{\mathbb{X}_l}\mathrm{d}A)\ B\,(AB-\mathbb{1})^{m-2}\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B) + \cdots + (AB-\mathbb{1})^{m-2}\ (i_{\mathbb{X}_l}\mathrm{d}A)\ B\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big)\big] =$$

$$= z\,\mathrm{Tr}\big((AB)^{z-1}\ (i_{\mathbb{X}_l}\mathrm{d}A)\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big) +$$

$$+ \sum_{m=0}^{\infty} m\,c_m(z) \sum_{a=0}^{m-2} \mathrm{Tr}\big((AB-\mathbb{1})^a\ (i_{\mathbb{X}_l}\mathrm{d}A)\ B\,(AB-\mathbb{1})^{m-a-2}\,A\ (i_{\mathbb{Y}_r}\mathrm{d}B)\big)\ , \tag{4.121}$$

where we used the fact that $i_{\mathbf{X}_l}\mathrm{d}B = 0$ because B depends only on the elements of the right factor of $\mathcal{M}_n \times \mathcal{M}_n$, and, in the first term of the last equality, we have used the relation

$$z(AB)^{z-1} = \sum_{m=0}^{\infty} m\,c_m(z)(AB-\mathbb{1})^{m-1} \tag{4.122}$$

for the first-order derivative of a analytical function. The metric is then:

$$g^n_{q,z}(X\,,Y) = \frac{z}{q(1-q)} i^*_d \left[\mathrm{Tr}\left((AB)^{z-1}\ (i_{\mathbb{X}_l}\mathrm{d}A)\ (i_{\mathbb{Y}_r}\mathrm{d}B)\right)\right] +$$

$$+\frac{1}{q(1-q)}i_d^*\left[\sum_{m=0}^{\infty} m\,c_m(z)\sum_{a=0}^{m-2}\mathrm{Tr}\left((AB-\mathbb{1})^a\;(i_{\mathbb{X}_l}\mathrm{d}A)\;B\,(AB-\mathbb{1})^{m-a-2}\,A\;(i_{\mathbb{Y}_r}\mathrm{d}B)\right)\right]\;. \quad (4.123)$$

In order to perform computations for a generic N-level system it is useful to use the canonical basis $\{\mathbf{E}_{jk}\}_{j,k=1,\ldots,n}$ of $\mathfrak{A}_n$ introduced in chapter 1. The left-invariant Maurer-Cartan 1-form $U^{-1}dU$ can be then written as:

$$U^{-1}dU = i\sigma_k\theta^k = i\mathbf{E}_{\alpha\beta}\theta^{\alpha\beta} \quad (4.124)$$

where $\{\sigma_k\}_{k=0,\ldots,n^2-1}$ denote the basis for the $\mathfrak{u}(\mathfrak{A}_n)$ algebra with $\sigma_0=\mathbb{I}$, $\{\theta^k\}_{k=0,\ldots,n^2-1}$ the dual basis of left-invariant 1-forms. The matrices σ can be expressed into the standard basis as a linear combination of the $\mathbf{E}_{\alpha\beta}$ matrices with complex coefficients[4], say:

$$\sigma_k = \sum_{\alpha,\beta=1}^{N} M_k^{\alpha\beta}\mathbf{E}_{\alpha\beta} \qquad , \qquad M_k^{\alpha\beta}\in\mathbb{C} \quad (4.125)$$

from which, according to Eq. (4.124), it follows that:

$$\theta^{\alpha\beta} = \sum_{k=0}^{N^2-1} M_k^{\alpha\beta}\theta^k\;. \quad (4.126)$$

The complex coefficients $M_k^{\alpha\beta}$ have to satisfy the following property[5]:

$$M_k^{\beta\alpha} = \overline{M_k^{\alpha\beta}} \qquad \forall\; k=0,\ldots,N^2-1 \quad (4.127)$$

as can be seen by taking the Hermitian conjugate of Eq. (4.125) which yields:

[4]For instance, in the $U(\mathfrak{A}_2)$ case we have:

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\tau_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \tau_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \tau_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \tau_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\;.$$

Therefore

$$\sigma_0=\tau_{11}+\tau_{22},\quad \sigma_1=\tau_{12}+\tau_{21},\quad \sigma_2=i(\tau_{21}-\tau_{12}),\quad \sigma_3=\tau_{11}-\tau_{22}$$

from which, by imposing the equality (4.124), it is easy to see that

$$\theta^{11}=\theta^0+\theta^3,\quad \theta^{12}=\theta^1-i\theta^2,\quad \theta^{21}=\theta^1+i\theta^2,\quad \theta^{22}=\theta^0-\theta^3\;.$$

[5]Here $\overline{M_k^{\alpha\beta}}$ denotes the complex conjugate of $M_k^{\alpha\beta}$.

$$\sigma_k = \overline{M_k^{\alpha\beta}}\,(\tau_{\alpha\beta})^\dagger = \overline{M_k^{\alpha\beta}}\tau_{\beta\alpha} = M_k^{\beta\alpha}\tau_{\beta\alpha}\ , \tag{4.128}$$

where we have used the fact that the σ's are Hermitian and the property of the real matrices $\mathbf{E}$ according to which $(\mathbf{E}_{\alpha\beta})^\dagger = (\mathbf{E}_{\alpha\beta})^T = \mathbf{E}_{\beta\alpha}$.

Remark 20. *More precisely, when we write the left-invariant 1-form $U^{-1}dU$ in the standard basis as in (4.124), we are exploiting the fact that the σ matrices provide a basis both of the vector space underlying the $\mathfrak{u}(\mathfrak{A}_n)$ Lie algebra and of its complexification. Consequently, we are able to consider suitable complex linear combinations to express the basis matrices of $\mathfrak{u}(\mathfrak{A}_n)$ in terms of the $\mathbf{E}_{\alpha\beta}$ and to recast then the matrix-valued Maurer-Cartan 1-form $U^{-1}dU$ in terms of the $\mathbf{E}_{\alpha\beta}$ and the 1-forms $\theta^{\alpha\beta}$ given in (4.126). This is just a computational trick which allows to simplify the calculation of the metric and in the end we should check that it does not introduce any additional information by rewriting the resulting expression of the metric in terms of the basic left-invariant 1-forms on the group.*

The diagonal density matrix $\bar{\rho}_0$ can be written in terms of the $\mathbf{E}$ basis as:

$$\bar{\rho}_0 = \sum_{\alpha=1}^{n} p_\alpha \mathbf{E}_{\alpha\alpha}\ , \tag{4.129}$$

where the p_α denote the n eigenvalues of $\bar{\rho}_0$ satisfying the constraint $\mathrm{Tr}(\bar{\rho}_0) = \sum_\alpha p_\alpha = 1$, and the sum (4.129) involves only the diagonal matrices $\mathbf{E}_{\alpha\alpha}$. Moreover, by using the decomposition (4.129) and the commutation relations:

$$[\mathbf{E}_{\alpha\beta}, \mathbf{E}_{\alpha'\beta'}] = \delta_{\beta\alpha'}\mathbf{E}_{\alpha\beta'} - \delta_{\beta'\alpha}\mathbf{E}_{\alpha'\beta}\ , \tag{4.130}$$

we have:

$$\left[U^{-1}dU, \bar{\rho}_0^r\right] = i\sum_{\alpha,\beta}(p_\beta^r - p_\alpha^r)\mathbf{E}_{\alpha\beta}\theta^{\alpha\beta}\ , \tag{4.131}$$

for any power $\bar{\rho}_0^r$ of $\bar{\rho}_0$. Consequently, we have:

$$\begin{aligned} L_{\mathbb{X}_l}\,\bar{\rho}^{\frac{q}{z}} &= \mathrm{d}\bar{\rho}^{\frac{q}{z}}(\mathbb{X}_l) = \mathbf{U}\,\left(\mathrm{d}\bar{\rho}^{\frac{q}{z}}(\mathbb{X}_l)\right)\,\mathbf{U}^\dagger + \mathbf{U}\,\left[\left(\mathbf{U}^\dagger\mathrm{d}\mathbf{U}((\mathbb{X}_l))\right)\,,\bar{\rho}^{\frac{q}{z}}\right]\,\mathbf{U}^\dagger = \\ &= \frac{q}{z}\,\sum_\alpha \mathbf{U}\,\mathbf{E}_{\alpha\alpha}\,\mathbf{U}^\dagger\,p_\alpha^{\frac{q-z}{z}}\,\mathrm{d}p_\alpha(\mathbb{X}_l) + \mathbf{U}\,i\sum_{\alpha,\beta}(p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\mathbf{E}_{\alpha\beta}\,\theta^{\alpha\beta}(\mathbb{X}_l)\,\mathbf{U}^\dagger\,, \end{aligned} \tag{4.132}$$

$$\begin{aligned} L_{\mathbb{Y}_r}\,\bar{\varrho}^{\frac{1-q}{z}} &= \mathrm{d}\bar{\varrho}^{\frac{1-q}{z}}(\mathbb{Y}_r) = \mathbf{V}\,\left(\mathrm{d}\bar{\varrho}^{\frac{1-q}{z}}(\mathbb{Y}_r)\right)\,\mathbf{V}^\dagger + \mathbf{V}\,\left[\left(\mathbf{V}^\dagger\mathrm{d}\mathbf{V}(\mathbb{Y}_r)\right)\,,\bar{\varrho}^{\frac{1-q}{z}}\right]\,\mathbf{V}^\dagger = \\ &= \frac{1-q}{z}\,\sum_\alpha \mathbf{V}\,\mathbf{E}_{\alpha\alpha}\,\mathbf{V}^\dagger\,\widetilde{p}_\alpha^{\frac{1-q-z}{z}}\,\mathrm{d}\widetilde{p}_\alpha(\mathbb{Y}_r) + \mathbf{V}\,i\sum_{\alpha,\beta}(\widetilde{p}_\beta^{\frac{1-q}{z}} - \widetilde{p}_\alpha^{\frac{1-q}{z}})\mathbf{E}_{\alpha\beta}\,\eta^{\alpha\beta}(\mathbb{Y}_r)\,\mathbf{V}^\dagger\,, \end{aligned} \tag{4.133}$$

where $\bar{\varrho}_0 = \sum_{\alpha=1}^{n} \widetilde{p}_\alpha \mathbf{E}_{\alpha\alpha}$. Now, coming back to the expression (4.123) for the tensor:

$$
g^n_{q,z}(X\,,Y) = \frac{z}{q(1-q)} i_d^* \left[\mathrm{Tr}\left((AB)^{z-1}\, (i_{\mathbb{X}_l}\mathrm{d}A)\, (i_{\mathbb{Y}_r}\mathrm{d}B) \right) \right] +
$$

$$
+\frac{1}{q(1-q)} i_d^* \left[\sum_{m=0}^{\infty} m\, c_m(z) \sum_{a=0}^{m-2} \mathrm{Tr}\left((AB-\mathbb{1})^a\, (i_{\mathbb{X}_l}\mathrm{d}A)\, B\, (AB-\mathbb{1})^{m-a-2}\, A\, (i_{\mathbb{Y}_r}\mathrm{d}B) \right) \right] ,
$$

we focus on the first term in the RHS, and, using (4.133) and (4.133) and performing the pullback along i_d, we obtain:

$$
\begin{aligned}
&\frac{z}{q(1-q)} \left[\mathrm{Tr}\left(\bar{\rho}_0^{\frac{z-1}{z}} \left(\frac{q}{z} \sum_{\alpha} \mathbf{E}_{\alpha\alpha}\, p_\alpha^{\frac{q-z}{z}}\, \mathrm{d}p_\alpha(X) + i \sum_{\alpha,\beta} (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}}) \mathbf{E}_{\alpha\beta}\, \theta^{\alpha\beta}(X) \right) \cdot \right.\right. \\
&\left.\left. \cdot \left(\frac{1-q}{z} \sum_{\alpha} \mathbf{E}_{\gamma\gamma}\, p_\gamma^{\frac{1-q-z}{z}}\, \mathrm{d}p_\gamma(Y) + i \sum_{\gamma,\mu} (p_\mu^{\frac{1-q}{z}} - p_\gamma^{\frac{1-q}{z}}) \mathbf{E}_{\gamma\mu}\, \theta^{\gamma\mu}(Y) \right) \right) \right] .
\end{aligned}
\tag{4.134}
$$

It is easy to see that the terms with $\mathrm{d}p_\alpha(X) \cdot \theta^{\gamma\mu}(Y)$ and $\theta^{\alpha\beta}(X) \cdot \mathrm{d}p_\gamma(Y)$ vanish, indeed:

$$
\frac{i}{1-q} \sum_{\alpha\,\beta\,\gamma\,\mu} p_\beta^{\frac{z-1}{z}}\, p_\alpha^{\frac{q-z}{z}}\, (p_\mu^{\frac{1-q}{z}} - p_\gamma^{\frac{1-q}{z}})\, \underbrace{\mathrm{Tr}\left(\mathbf{E}_{\beta\beta}\, \mathbf{E}_{\alpha\alpha}\, \mathbf{E}_{\gamma\mu} \right)}_{\delta_{\beta\mu}\delta_{\beta\alpha}\delta_{\alpha\gamma}} \mathrm{d}p_\alpha(X) \cdot \theta^{\gamma\mu}(Y) = 0\,, \tag{4.135}
$$

$$
\frac{i}{q} \sum_{\alpha\,\beta\,\gamma\,\mu} p_\mu^{\frac{z-1}{z}}\, p_\gamma^{\frac{1-q-z}{z}}\, (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, \underbrace{\mathrm{Tr}\left(\mathbf{E}_{\mu\mu}\, \mathbf{E}_{\alpha\beta}\, \mathbf{E}_{\gamma\gamma} \right)}_{\delta_{\mu\gamma}\delta_{\mu\alpha}\delta_{\beta\gamma}} \mathrm{d}p_\gamma(Y) \cdot \theta^{\alpha\beta}(X) = 0\,. \tag{4.136}
$$

On the other hand, the terms with $\mathrm{d}p_\alpha(X) \cdot p_\gamma(Y)$ and $\theta^{\alpha\beta}(X) \cdot \theta^{\gamma\mu}(Y)$ are:

$$
\frac{1}{z} \sum_{\alpha} p_\alpha^{-1}\, \mathrm{d}p_\alpha(X) \cdot \mathrm{d}p_\alpha(Y)\,, \tag{4.137}
$$

$$
\frac{z}{q(1-q)} \sum_{\alpha\beta} p_\alpha^{\frac{z-1}{z}}\, (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\beta^{\frac{1-q}{z}} - p_\alpha^{\frac{1-q}{z}})\, \theta^{\alpha\beta}(X) \cdot \theta^{\gamma\mu}(Y)\,. \tag{4.138}
$$

Concerning the second term in the RHS of (4.123), using (4.133) and (4.133) and performing the pullback along i_d, we obtain:

$$
\begin{aligned}
&\frac{1}{q(1-q)}\sum_{m=0}^{\infty} m\, c_m(z) \sum_{a=0}^{m-2} \mathrm{Tr}\left((\bar{\rho}_0^{\frac{1}{z}} - \mathbb{1})^a \left(\frac{q}{z} \sum_{\alpha} \mathbf{E}_{\alpha\alpha}\, p_\alpha^{\frac{q-z}{z}}\, \mathrm{d}p_\alpha(\mathbb{X}_l) + i \sum_{\alpha,\beta} (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}}) \mathbf{E}_{\alpha\beta}\, \theta^{\alpha\beta}(X) \right) \cdot \right. \\
&\left. \cdot \bar{\rho}_0^{\frac{1-q}{z}} (\bar{\rho}_0^{\frac{1}{z}} - \mathbb{1})^{m-a-2}\, \bar{\rho}_0^{\frac{q}{z}} \left(\frac{1-q}{z} \sum_{\gamma} \mathbf{E}_{\gamma\gamma}\, p_\gamma^{\frac{1-q-z}{z}}\, \mathrm{d}p_\gamma(Y) + i \sum_{\gamma,\mu} (p_\mu^{\frac{1-q}{z}} - p_\gamma^{\frac{1-q}{z}}) \mathbf{E}_{\gamma\mu}\, \theta^{\gamma\mu}(Y) \right) \right) .
\end{aligned}
\tag{4.139}
$$

Again, it is easy to see that the terms with $\mathrm{d}p_\alpha(X) \cdot \theta^{\gamma\mu}(Y)$ and $\theta^{\alpha\beta}(X) \cdot \mathrm{d}p_\gamma(Y)$ vanish, and we are left with the term in $\mathrm{d}p_\alpha(X) \cdot \mathrm{d}p_\alpha(Y)$:

$$
\begin{aligned}
&\frac{1}{z^2} \sum_{m=0}^{\infty} \sum_{\alpha} m(m-1)\, c_m(z)\, (p_\alpha^{\frac{1}{z}} - 1)^{m-2}\, p_\alpha^{2\frac{1-z}{z}}\, \mathrm{d}p_\alpha(X) \cdot \mathrm{d}p_\alpha(Y) = \\
&= \frac{z-1}{z} \sum_{\alpha} p_\alpha^{-1}\, \mathrm{d}p_\alpha(X) \cdot \mathrm{d}p_\alpha(Y) \, ,
\end{aligned}
\tag{4.140}
$$

where we have used the expression:

$$
z(z-1)(p_\alpha^{\frac{1}{z}})^{z-2} = \sum_{m=0}^{\infty} m(m-1)\, c_m(z) (p_\alpha^{\frac{1}{z}} - 1)^{m-2} \tag{4.141}
$$

for the second-order derivative of an analytical function, and the term in $\theta^{\alpha\beta}(X) \cdot \theta^{\gamma\mu}(Y)$:

$$
\begin{aligned}
&\frac{-1}{q(1-q)} \sum_{m=0}^{\infty} \sum_{a=0}^{m-2} \sum_{\alpha,\beta,\gamma,\mu} m\, c_m(z)\, \mathrm{Tr}\left((\bar{\rho}_0^{\frac{1}{z}} - \mathbb{1})^a\, \mathbf{E}_{\alpha\beta}\, \bar{\rho}_0^{\frac{1-q}{z}}\, (\bar{\rho}_0^{\frac{1}{z}} - \mathbb{1})^{m-a-2}\, \bar{\rho}_0^{\frac{q}{z}}\, \mathbf{E}_{\gamma\mu} \right) \cdot \\
&\cdot (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\mu^{\frac{1-q}{z}} - p_\gamma^{\frac{1-q}{z}})\, \theta^{\alpha\beta}(X) \cdot \theta^{\gamma\mu}(Y) = \\
&= \frac{1}{q(1-q)} \sum_{m=0}^{\infty} \sum_{a=0}^{m-2} \sum_{b=0}^{a} \sum_{c=0}^{m-a-2} \sum_{\alpha,\beta} m\, c_m(z)\, (-1)^{b+c} \binom{a}{b} \binom{m-a-2}{c} \cdot \\
&\cdot p_\alpha^{\frac{a-b}{z}}\, p_\beta^{\frac{m-a-1-c}{z}}\, (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\beta^{\frac{1-q}{z}} - p_\alpha^{\frac{1-q}{z}})\, \theta^{\alpha\beta}(X) \cdot \theta^{\beta\alpha}(Y) = \\
&= \frac{1}{q(1-q)} \sum_{m=0}^{\infty} \sum_{a=0}^{m-2} \sum_{\alpha,\beta} m\, c_m(z)\, (p_\alpha^{\frac{1}{z}} - 1)^a\, p_\beta^{\frac{1}{z}}\, (p_\beta^{\frac{1}{z}} - 1)^{m-a-2} \cdot \\
&\cdot (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\beta^{\frac{1-q}{z}} - p_\alpha^{\frac{1-q}{z}}) \theta^{\alpha\beta}(X) \cdot \theta^{\beta\alpha}(Y) = \\
&= \frac{1}{q(1-q)} \sum_{m=0}^{\infty} \sum_{\alpha,\beta} m\, c_m(z)\, \frac{(p_\beta^{\frac{1}{z}} - 1)^{m-1} - (p_\alpha^{\frac{1}{z}} - 1)^{m-1}}{p_\beta^{\frac{1}{z}} - p_\alpha^{\frac{1}{z}}}\, p_\beta^{\frac{1}{z}} \\
&(p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\beta^{\frac{1-q}{z}} - p_\alpha^{\frac{1-q}{z}})\, \theta^{\alpha\beta}(X) \cdot \theta^{\beta\alpha}(Y) = \\
&= \frac{z}{q(1-q)} \sum_{\alpha,\beta} \left[\frac{p_\beta^{\frac{1}{z}}\, (p_\beta^{\frac{z-1}{z}} - p_\alpha^{\frac{z-1}{z}})\, (p_\beta^{\frac{q}{z}} - p_\alpha^{\frac{q}{z}})\, (p_\beta^{\frac{1-q}{z}} - p_\alpha^{\frac{1-q}{z}})}{p_\beta^{\frac{1}{z}} - p_\alpha^{\frac{1}{z}}} \right] \theta^{\alpha\beta}(X) \cdot \theta^{\beta\alpha}(Y)\,.
\end{aligned}
\tag{4.142}
$$

where we have used equation (4.129) and the binomial expansions:

$$
(\bar{\rho}_0^{\frac{1}{z}} - \mathbb{I})^a = \sum_{b=0}^{a} \binom{a}{b} (-1)^b \bar{\rho}_0^{\frac{a-b}{z}} \tag{4.143}
$$

$$
(\bar{\rho}_0^{\frac{1}{z}} - \mathbb{I})^{m-a-2} = \sum_{c=0}^{m-a-2} \binom{m-a-2}{c} (-1)^c \bar{\rho}_0^{\frac{m-a-2-c}{z}}, \tag{4.144}
$$

in the first equality; the relations:

$$
p_\alpha^{\frac{1}{z}} (p_\alpha^{\frac{1}{z}} - 1)^{m-a-2} = \sum_{c=0}^{m-a-2} \binom{m-a-2}{c} (-1)^c\, p_\alpha^{\frac{m-a-1-c}{z}} \tag{4.145}
$$

$$
(p_\alpha^{\frac{1}{z}} - 1)^a = \sum_{b=0}^{a} \binom{a}{b} (-1)^b\, p_\alpha^{\frac{a-b}{z}} \tag{4.146}
$$

in the second equality; the expression for the finite sum of a geometric series:

$$\sum_{a=0}^{m-2} x^a = \frac{1-x^{m-1}}{1-x}\,, \quad \text{with } x = \frac{p_\alpha^{\frac{1}{z}}-1}{p_\beta^{\frac{1}{z}}-1} \tag{4.147}$$

in the third equality, and the expression:

$$\sum_{m=0}^{\infty} m\, c_m(z)(p_\alpha^{\frac{1}{z}}-1)^{m-1} = z\, p_\alpha^{\frac{z-1}{z}} \tag{4.148}$$

for the first-order derivative of an analytical function in the last equality.

Collecting the terms in $\mathrm{d}p_\alpha(X)\cdot\mathrm{d}p_\alpha(Y)$ (equations (4.137) and (4.140)) we obtain:

$$\begin{aligned} g_{q,z}^{n\perp}(X\,,Y) &= \frac{1}{z}\sum_\alpha p_\alpha^{-1}\,\mathrm{d}p_\alpha(X)\cdot\mathrm{d}p_\alpha(Y) + \frac{z-1}{z}\sum_\alpha p_\alpha^{-1}\,\mathrm{d}p_\alpha(X)\cdot\mathrm{d}p_\alpha(Y) = \\ &= \sum_{\alpha=1}^{n}\frac{1}{p_\alpha}\mathrm{d}p_\alpha(X)\cdot\mathrm{d}p_\alpha(Y)\,. \end{aligned} \tag{4.149}$$

From this it follows that:

$$g_{q,z}^{n\perp} = \sum_{\alpha=1}^{n}\frac{1}{p_\alpha}\mathrm{d}p_\alpha\otimes\mathrm{d}p_\alpha = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha \tag{4.150}$$

which is the Fisher-Rao metric related to the component of the “classical” probability vector $\vec{p} = (p_1,\dots,p_n)$ identified with the diagonal elements of the invertible density matrix.

Collecting the terms in $\theta^{\alpha\beta}(X)\cdot\theta^{\gamma\mu}(Y)$ (equations (4.138) and (4.142)) we obtain:

$$\begin{aligned} g_{q,z}^{n\|}(X\,,Y) &= \frac{z}{q(1-q)}\sum_{\alpha\beta} p_\alpha^{\frac{z-1}{z}}\,(p_\beta^{\frac{q}{z}}-p_\alpha^{\frac{q}{z}})\,(p_\beta^{\frac{1-q}{z}}-p_\alpha^{\frac{1-q}{z}})\,\theta^{\alpha\beta}(X)\cdot\theta^{\gamma\mu}(Y)+ \\ &+\frac{z}{q(1-q)}\sum_{\alpha,\beta}\left[\frac{p_\beta^{\frac{1}{z}}\,(p_\beta^{\frac{z-1}{z}}-p_\alpha^{\frac{z-1}{z}})\,(p_\beta^{\frac{q}{z}}-p_\alpha^{\frac{q}{z}})\,(p_\beta^{\frac{1-q}{z}}-p_\alpha^{\frac{1-q}{z}})}{p_\beta^{\frac{1}{z}}-p_\alpha^{\frac{1}{z}}}\right]\theta^{\alpha\beta}(X)\cdot\theta^{\beta\alpha}(Y) = \\ &=\frac{z}{q(1-q)}\sum_{\alpha,\beta}\left[\frac{(p_\beta-p_\alpha)\,(p_\beta^{\frac{q}{z}}-p_\alpha^{\frac{q}{z}})\,(p_\beta^{\frac{1-q}{z}}-p_\alpha^{\frac{1-q}{z}})}{p_\beta^{\frac{1}{z}}-p_\alpha^{\frac{1}{z}}}\right]\theta^{\alpha\beta}(X)\cdot\theta^{\beta\alpha}(Y)\,. \end{aligned} \tag{4.151}$$

Let us now rewrite this expression in the basis of left-invariant $\mathfrak{su}(\mathfrak{A}_n)$-valued 1-forms. In order to do this, let us introduce the shorthand notation for the coefficients:

$$\mathcal{E}_{\alpha\beta} := \frac{(p_\alpha - p_\beta)(p_\alpha^{\frac{q}{z}} - p_\beta^{\frac{q}{z}})(p_\alpha^{\frac{1-q}{z}} - p_\beta^{\frac{1-q}{z}})}{(p_\alpha^{\frac{1}{z}} - p_\beta^{\frac{1}{z}})} \in \mathbb{R} \qquad \text{s.t.} \qquad \mathcal{E}_{\alpha\beta} = \mathcal{E}_{\beta\alpha} \,. \tag{4.152}$$

Then, $g^{n\|}_{q,z}$ can be written as

$$\begin{aligned} g^{n\|}_{q,z} &= \frac{z}{q(1-q)} \sum_{\alpha,\beta=1}^{n}{}' \, \mathcal{E}_{\alpha\beta}\, \theta^{\alpha\beta} \otimes \theta^{\beta\alpha} \\ &= \frac{z}{q(1-q)} \sum_{\alpha,\beta=1}^{n}{}' \, \frac{1}{2}\mathcal{E}_{\alpha\beta}(\theta^{\alpha\beta} \otimes \theta^{\beta\alpha} + \theta^{\beta\alpha} \otimes \theta^{\alpha\beta}) \,. \end{aligned} \tag{4.153}$$

Using now the expression (4.126) of the $\theta^{\alpha\beta}$ in terms of the θ^j we have:

$$g^{n\|}_{q,z} = \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1} \sum_{\alpha,\beta=1}^{n}{}' \, \frac{1}{2}\mathcal{E}_{\alpha\beta} \left(M_j^{\alpha\beta} M_k^{\beta\alpha} + M_j^{\beta\alpha} M_k^{\alpha\beta} \right) \theta^j \otimes \theta^k \,, \tag{4.154}$$

from which, according to the property (4.127), it follows that:

$$g^{n\|}_{q,z} = \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1}{}' \, \mathcal{C}_{jk}\, \theta^j \otimes \theta^k \,, \tag{4.155}$$

with:

$$\mathcal{C}_{jk} = \sum_{\alpha,\beta=1}^{n}{}' \, \mathcal{E}_{\alpha\beta}\, \Re\big[M_j^{\alpha\beta} M_k^{\beta\alpha}\big] \,. \tag{4.156}$$

The coefficients $\mathcal{C}_{jk}$ in Eq. (4.156) depend only on the eigenvalues of the density matrix $\bar{\rho}$ and on the transformation matrices relating the two basis. Moreover, the $\mathcal{C}_{jk}$ are symmetric with respect to the exchange of j and k (i.e., $\mathcal{C}_{jk} = \mathcal{C}_{kj}$) and, being the $\mathcal{E}_{\alpha\beta}$ defined in Eq. (4.152) real, they are also real.

Eventually, summing equations (4.150) and (4.155) we obtain the expression of the tensor $g^n_{q,z}$:

$$g^n_{q,z} = g^{n\perp}_{q,z} + g^{n\|}_{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)} {\sum_{j,k=1}^{n^2-1}}' \mathcal{C}_{jk}\, \theta^j \otimes \theta^k \ . \qquad (4.157)$$

Note that the sum over j and k in Eq. (4.157) does not involve the basic left-invariant 1-forms dual to the Cartan subalgebra. Indeed, the only terms which contain the left-invariant 1-forms associated with the Cartan subalgebra are those with $\alpha = \beta$ which have vanishing coefficients $\mathcal{C}_{jk}$.

Chapter 5

Conclusions

Now that we have reached the final chapter of the thesis, it is time to pause and comment on the ideas exposed in view of future developments. We will focus on two main branches of possible future developments, specifically, "the geometry of entanglement of quantum states in finite dimensions", and "the geometry of quantum states in infinite dimensions". There would be a third branch, namely, "the geometry of entanglement of quantum states in infinite dimensions", but we will simply ignore it because it presents a huge amount of technical difficulties, and, what is more, it necessarily depends on the previous development of the first two branches.

Before discussing possible future developments, let us briefly recall the content of the previous chapters. The overall focus of the thesis was the study of the geometrical structures available on the space $\mathcal{S}$ of quantum states of a quantum system the observables of which are the self-adjoint elements in the finite-dimensional C^*-algebra $\mathfrak{A}_n$ introduced in chapter 1. In chapter 2 we presented the geometry of the space $\mathcal{S}$ of quantum states of the model C^*-algebra $\mathfrak{A}_n$ introduced in chapter 1. The attitude towards the problem was of group-theoretical character. According to the work in [64, 65, 66], we selected the Lie group $SL(\mathfrak{A}_n)$ of invertible elements with unit determinant in $\mathfrak{A}_n$, and showed that it acts on the space $\mathcal{S}$ of quantum states in such a way that $\mathcal{S}$ is partitioned into the disjoint union:

$$\mathcal{S} = \bigsqcup_{k=1}^{n} \mathcal{S}_k$$

of orbits $\mathcal{S}_k$ of $SL(\mathfrak{A}_n)$. The orbit $\mathcal{S}_k$ consists of quantum states with fixed rank equal to k (see definition 3), and it is a homogeneous space for $SL(\mathfrak{A}_n)$ admitting the structure of differential manifold for every $k = 1, ..., n$. Except the case $n = 2$, in which it is a manifold with boundary (the so-called Bloch ball), the space $\mathcal{S}$ of quantum states does not admit the structure of differential manifold as a whole. In particular, all the orbits $\mathcal{S}_k$ have different dimension (increasing as k increases). From the topological point of view, it is worth to mention that the manifold $\mathcal{S}_1$ of pure quantum states is a compact submanifold[1] of $\mathfrak{O}_n^*$, while the manifold $\mathcal{S}_n$ of invertible quantum states is an open submanifold[2] of the affine hyperplane $\mathfrak{T}_n^1$ of self-adjoint linear functionals ξ in $\mathfrak{O}_n^*$ such that $\xi(\mathbb{I}) = 1$. By construction, the action of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$ is smooth, and we used this fact to analyze the orbits in $\mathcal{S}_k$ of the compact subgroup $SU(\mathfrak{A}_n) \subset SL(\mathfrak{A}_n)$ made up of unitary elements with unit determinant in $\mathfrak{A}_n$. All the orbits of $SU(\mathfrak{A}_n)$, except the one passing through the maximally mixed state, are Kähler manifolds. In particular, the manifold $\mathcal{S}_1$ of pure quantum states turns out to be a homogeneous space for both $SL(\mathfrak{A}_n)$ and $SU(\mathfrak{A}_n)$, and its Kähler structure is widely studied in the literature.

In chapter 3 we presented here a geometric formulation of the dynamics of open quantum systems governed by the GKLS master equation (3.77) (see [63, 82]). The geometrization consists in describing the infinitesimal generator of these dynamical processes as an affine vector field Γ on the affine hyperplane $\mathfrak{T}_n^1$ in $\mathfrak{O}_n^*$ consisting of self-adjoint linear functionals ξ on $\mathfrak{A}_n$ such that $\xi(\mathbb{I}) = 1$. By construction, Γ is written as the affine combination of three vector fields, namely, a Hamiltonian vector field $X_{\mathbf{H}}$ associated with the Hamiltonian operator $\mathbf{H}$ in equation (3.77) by means of a Poisson tensor Λ on $\mathfrak{T}_n^1$, a gradient-like vector field $Y_{\mathbf{V}}$ associated with the positive operator $\mathbf{V}$ in equation (3.77) by means of a symmetric bivector field $\mathcal{R}$, and a vector field $Z_{\mathcal{K}}$ associated with the completely-positive map $\mathcal{K}$ in equation (3.77). The decomposition of Γ is adapted to the geometry of the space $\mathcal{S}$ of quantum states when we thought of as a compact convex body in $\mathfrak{T}_1$. In particular, Hamiltonian and gradient-like vector fields provide a realization of the Lie algebra $\mathfrak{sl}(\mathfrak{A}_n)$ of the special linear group $SL(\mathfrak{A}_n)$ on $\mathfrak{T}_n^1$, "integrating" to a nonlinear action only on the space $\mathcal{S}$ of quantum states. In particular, every Hamiltonian and gradient-like vector fields on $\mathfrak{T}_n^1$ are tangent to the manifolds $\mathcal{S}_k$ of quantum states with fixed rank, and their restrictions are related to the fundamental vector fields of the canonical action of $SL(\mathfrak{A}_n)$ on $\mathcal{S}_k$ constructed in chapter 2. The flow of the vector field $Z_{\mathcal{K}}$ turns out to be responsible for the change of rank of quantum states characteristic of dynamical evolutions of open quantum systems. Interestingly, it is found that $X_{\mathbf{H}}$ is completely unrelated to the vector fields $Y_{\mathbf{V}}$ and $Z_{\mathcal{K}}$. On the other hand, the linearity of the GKLS generator $\mathbf{L}$ implies that the vector field Γ is affine, and this

[1] This follows from the fact that it is an orbit of the canonical action of the compact group $SU(\mathfrak{A}_n)$ on $\mathfrak{O}_n^*$ (see section 2.2).

[2] This follows from the fact that it is the complement of the inverse image of the closed set $\{0\} \in \mathbb{R}$ by means of the smooth function det (see definition 3).

requires a fine tuning between $Y_{\mathbf{V}}$ and $Z_{\mathcal{K}}$ so that their sum preserves the affine character of Γ. After having introduced the vector field Γ, we proceded to analyze the long-time behaviour of the so-called quantum random unitary semigroups, of which quantum Poisson semigroups and quantum Gaussian semigroups are particular instances (see [11, 78, 82]). The analysis is carried out exploiting LaSalle's invariance principle (see [2, 81]), an exquisitely classical tool that proves to be particularly useful in the quantum context thanks to the geometrical structures introduced on $\mathcal{S}$ in chapter 2 and chapter 3. The explicit result is that, in any finite dimension, the purity function $\chi(\rho) := Tr(\bar{\rho}^2)$ always decreases along the dynamical trajectories of quantum random unitary semigroups, and allows for a characterization of the set of asymptotic quantum states of the dynamics generated by Γ as the set of quantum states for which the Lie derivative of χ with respect to Γ vanishes. Interestingly, the maximally mixed state is always an element of the set of asymptotic quantum states for every possible Γ associated with a quantum random unitary semigroup in any finite dimension, and, in some cases, it is the only element in that set. From the point of view of entanglement this has an interesting consequence. Indeed, given any decomposition of the quantum system described by $\mathfrak{A}_n$ in terms of its quantum subsystems, the maximally mixed state is always a separable state (actually, it is a product state). Consequently, the dynamics of the quantum random unitary semigroups destroys the entanglement of the initial quantum state, and this could have interesting consequences at the level of experimental applications. A more thorough analysis of this and related istances will be the subject of future work.

In chapter 4 we studied the information geometry of the manifold $\mathcal{S}_n$ of invertible quantum states. The main idea was to exploit the geometrical structures on $\mathcal{S}$ and introduced in chapter 2 to provide a coordinate-free treatment of quantum information theory which is well-suited in order to deal with situations where nonlinear changes of coordinates are naturally used like, for instance, when considering nonlinear parametrized submanifolds of quantum states in problems of state estimation. We started introducing an abstract framework for extracting, in a coordinate-free way, covariant tensor fields on a generic differential manifold M starting from two-point functions on $M \times M$. This led us to define the class of ***potential functions*** as all those two-point functions for which the coordinate-free extraction algorithm works. This family of functions turns out to generalize the class of divergence (contrast) functions of classical information geometry (see [9]) used as a mean to extract metric tensors. Next, we reviewed the notion of quantum stochastic maps, the so-called monotonicity property for quantum metric tensors, and the so-called Data Processing Inequality (DPI) for quantum divergence functions in the abstract framework just introduced. Both the monotonicity property and the DPI are formulated in terms of the behaviour of a family of relevant objects (metric tensors in one case, and quantum divergence functions in the other) with respect to the action of quantum stochastic maps. The coordinate-free formalism introduced allows us to point out a deep connection between the monotonicity property for quantum metric tensors and the so-called Data Processing Inequality (DPI) for quantum divergence functions. Specifically, we showed that *"DPI implies monotonicity property"* (see proposition 32). Eventually, we computed the quantum metric tensors associated with the family of $(\alpha - z)$-Rényi relative entropies introduced in [20]. In order to perform calculations in any dimension without introducing coordinates, we decided to exploit the diagonalization procedure for density matrices to unfold the manifold $\mathcal{S}_n$ of invertible quantum states to the better-behaved manifold $\mathcal{M}_n = U(\mathfrak{A}_n) \times \Delta_n^0$, where Δ_n^0 is the open interior of the n-dimensional simplex. This manifold is parallelizable and we can exploit its global differential calculus to perform coordinate-free calculations. The final result is a symmetric covariant tensor on $\mathcal{M}_n$ which is the pullback of a metric tensor on $\mathcal{S}_n$ satisfying the monotonicity property. The symmetric covariant tensor field on $\mathcal{M}_n$ decomposes as the sum of two objects, one which "lives on" $U(\mathfrak{A}_n)$ and depends also on Δ_n^0, and the other one which "lives" and depends only on Δ_n^0. Quite interestingly, the latter coincides with the Fisher-Rao metric tensor on Δ_n^0 characteristic of classical information geometry.

Now, we will collect some final comments and perspectives on future work. Let us start with some comments on the geometry of entanglement of quantum states in finite dimensions. Even though in this thesis we have not dealt with entanglement and separability, we feel that the geometrical approach presented will be particularly useful because entangled states constitute a "nonlinear" subset of the space of quantum states. From the experimental point of view, entanglement seems to be a particularly useful resource for example in quantum computation and quantum cryptography, and we believe that a more geometrical understanding of the phenomenon could reflect in a sensible broadening of the set of tools by means of which it is possible to manipulate and exploit entanglement.

For the sake of simplicity, let us consider a bipartite system of distinguishable particles. From the mathematical point of view, we are chosing a decomposition of the C^*-algebra of the total system into the tensor product:

$$\mathfrak{A}_{nm}^{AB} = \mathfrak{A}_n^A \otimes \mathfrak{A}_m^B \,, \tag{5.1}$$

where $\mathfrak{A}_n^A$ and $\mathfrak{A}_m^B$ are the C^*-algebras of the two subsystems. The map:

$$\mathfrak{A}_n^A \ni \mathbf{A} \mapsto \mathbf{A} \otimes \mathbb{I}_B \in \mathfrak{A}_{nm}^{AB}$$

gives a realization of $\mathfrak{A}_n^A$ as a sub-C^*-algebra of $\mathfrak{A}_{nm}^{AB}$, and the observables of the form

$\mathbf{A} \otimes \mathbb{I}_B$ with $\mathbf{A} \in \mathfrak{A}_n$ are called local observables on the subsystem A, and similarly for $\mathfrak{A}_m^B$. From the physical point of view, we may think of local observables as all those observables pertaining exclusively to a single subsystem.

A quantum state $\rho \in \mathcal{S}^{AB}$ on $\mathfrak{A}_{nm}^{AB}$ is said to be a product state if:

$$\rho(\mathbf{A} \otimes \mathbf{B}) = \rho(\mathbf{A} \otimes \mathbb{I}_B)\, \rho(\mathbb{I}_A \otimes \mathbf{B}) \quad \forall \mathbf{A} \in \mathfrak{A}_n^A\,, \;\; \forall \mathbf{B} \in \mathfrak{A}_m^B\,. \tag{5.2}$$

It is easy to see that this is equivalent to the density matrix $\bar{\rho}$ associated with ρ being decomposable as:

$$\bar{\rho} = \bar{\rho}_A \otimes \bar{\rho}_B\,, \tag{5.3}$$

where $\bar{\rho}_A$ and $\bar{\rho}_B$ are, respectively, density matrices on $\mathfrak{A}_n^A$ and $\mathfrak{A}_m^B$. Note that this decomposition is unique. Considering convex combinations of product states we obtain the so-called separable states, that is, a state ρ is said to be separable if it can be decomposed as a convex combination of product states. In particular, this is easily seen to be equivalent to the density matrix $\bar{\rho}$ of ρ admitting a decomposition as:

$$\bar{\rho} = \sum_j p_j\, \bar{\rho}_A^j \otimes \bar{\rho}_B^j\,, \tag{5.4}$$

where $\bar{\rho}_A^j$ and $\bar{\rho}_B^j$ are, respectively, density matrices on $\mathfrak{A}_n^A$ and $\mathfrak{A}_m^B$ for all j, and $p_j > 0$ with $\sum_j p_j = 1$. Clearly, the space of separable quantum states is a convex set. Note that, unlike for product states, the decomposition of a separable state into a convex combination is in general not unique[3]. If ρ is a pure quantum state, the notions of separable state and product state coincide, while, if ρ is not pure, then, all product states are separable but the converse does not hold. A quantum state $\rho \in \mathcal{S}^{AB}$ which is not separable is called entangled. The set of entangled states is highly nonlinear being it the topological complement of the convex set of separable states in the convex set $\mathcal{S}^{AB}$. From the operational point of view, the measure of local observables on separable states dos not exhibit correlations between subsystems, while the measure of local observables on entangled states does present correlations among subsystems. These kind of "nonlocal" correlations between subsystems are characteristic of quantum mechanics, and led Schrödinger to point out entanglement as "the" characteristic feature of quantum mechanics. A proper discussion on the physical meaning of entanglement requires the introduction of a mathematical model for measurement operations, and this is out of the scope

[3]The only convex sets with the property that all their elements admit a unique decomposition as convex combinations are the simplexes.

of this brief chapter, hence, for a detailed discussion on the operational and physical meaning of entanglement we refer to [26] (in particular chapter 15), and [74], as well as references therein.

Geometrical methods have already proven to be useful in the context of quantum entanglement, see for instance [10, 64, 65, 75], and references therein. One interesting result is the geometrical decomposition of the manifold $\mathcal{S}_1^{AB}$ of pure quantum states of a bipartite system of distinguishable particles into submanifolds of separable and entangled states. It turns out that the manifold of separable pure states is the Cartesian product $\mathcal{S}_1^A \times \mathcal{S}_1^B$ of the manifold of pure quantum states of the subsystems. The pullback to $\mathcal{S}_1^A \times \mathcal{S}_1^B$ of the canonical symplectic form on $\mathcal{S}_1^{AB}$ coincides with the product of the symplectic forms of the pure quantum states of the subsystems. The pullback of the canonical symplectic form to the manifolds of entangled states presents a kernel, and the dimension of this kernel may be used as a sort of geometrical measure of entanglement. Indeed, the manifold of maximally entangled states turns out to be a Lagrangian submanifold of the manifold of pure quantum states of the composite system. Furthermore, similar results hold for the Fubini-Study metric tensor.

Since the space of pure quantum states is the only manifold of quantum states with fixed rank possessing a canonical symplectic structure and a canonical metric tensor, it is difficult to directly extend the line of reasoning leading to these results to the case of mixed quantum states. Nevertheless, we could try to apply the Lie group theory approach to the geometry of quantum states in the context of entanglement in order to understand the structure of the space of separable and entangled mixed states of fixed rank. As an illustrative example, let us briefly consider what happens for product states. The map:

$$SL(\mathfrak{A}_n^A) \times SL(\mathfrak{A}_m^B) \ni (\mathrm{g}_A\,,\mathrm{g}_B) \mapsto \mathrm{g}_A \otimes \mathrm{g}_B \in SL(\mathfrak{A}_{nm}^{AB}) \tag{5.5}$$

gives a realization of the Lie group $SL(\mathfrak{A}_n^A)\times SL(\mathfrak{A}_m^B)$ as a Lie subgroup of $SL(\mathfrak{A}_{nm}^{AB})$. By means of the canonical action α of $SL(\mathfrak{A}_{nm}^{AB})$ on $\mathcal{S}^{AB}$ we obtain an action of $SL(\mathfrak{A}_n^A) \times SL(\mathfrak{A}_m^B)$ on the quantum states of the composite system:

$$\begin{gathered}\rho \mapsto \rho_{\mathrm{g}_A\otimes\mathrm{g}_B} \\ \rho_{\mathrm{g}_A\otimes\mathrm{g}_B}(\mathbf{A}) := \frac{\rho(\mathrm{g}_A^\dagger \otimes \mathrm{g}_B^\dagger\,\mathbf{A}\,\mathrm{g}_A \otimes \mathrm{g}_B)}{\rho(\mathrm{g}_A^\dagger \otimes \mathrm{g}_B^\dagger\,\mathrm{g}_A \otimes \mathrm{g}_B)} \quad \forall \mathbf{A} \in \mathfrak{A}_{nm}^{AB}\,.\end{gathered}$$

In terms of density matrices we have:

$$\bar{\rho} \mapsto \bar{\rho}_{\mathrm{g}_A\otimes\mathrm{g}_B}$$

$$\rho_{\mathrm{g}_A\otimes\mathrm{g}_B} := \frac{\mathrm{g}_A\otimes\mathrm{g}_B\,\bar{\rho}\,\mathrm{g}_A^\dagger\otimes\mathrm{g}_B^\dagger}{Tr\left(\mathrm{g}_A\otimes\mathrm{g}_B\,\bar{\rho}\,\mathrm{g}_A^\dagger\otimes\mathrm{g}_B^\dagger\right)}\,.$$

It is clear that this action preserves product states. If we define the product state $\rho\in\mathcal{S}_k$ with density matrix $\bar{\rho}=\bar{\rho}_A\otimes\bar{\rho}_B$ to have (A,B)-rank equal to (k_A,k_B) when $\mathrm{rk}(\bar{\rho}_A)=k_A$ and $\mathrm{rk}(\bar{\rho}_B)=k_B$, it is immediate to see that, according to the results of chapter 2, the action of $SL(\mathfrak{A}_n^A)\times SL(\mathfrak{A}_m^B)$ on $\mathcal{S}_k^{AB}$ is transitive on product states with fixed (A,B)-rank. In particular, since the rank and the (A,B)-rank coincide when we consider pure product states or invertible product states, we conclude that pure product states and invertible product states are homogeneous spaces of $SL(\mathfrak{A}_n^A)\times SL(\mathfrak{A}_m^B)$.

Concerning the geometry of the GKLS dynamical evolutions in the context of composite systems, we already pointed out that the analysis pursued in chapter 3 has shown that, in some cases, the the long-term behaviour of the so-called quantum random unitary groups destroys entanglement because all the initial states evolve towards the maximally mixed state, which is the only asymptotic quantum state of the dynamics and it is a product state. Accordingly, a characterization of the asymptotic states of open quantum dynamics in relation to the geometry of separable and entangled states could present some useful insight on possible experimental implementations of open system dynamics as a way to manipulate entanglement and/or separability of quantum states. This, in turns, may reveal useful in the context of quantum information theory and quantum computation. A possible way to proceed would be to analyze the bivector fields Λ^{AB} and $\mathcal{R}^{AB}$ associated with the Lie-Jordan algebra structure of $\mathfrak{O}_{nm}^{AB}=\mathfrak{O}_n^A\otimes\mathfrak{O}_m^B$ in terms of the bivector fields Λ^A, $\mathcal{R}^A$ and Λ^B, $\mathcal{R}^B$ associated, respectively, with the Lie-Jordan algebra structures of $\mathfrak{O}_n^A$ and $\mathfrak{O}_m^B$.

From the point of view of information geometry, an iteresting development would be to analyze the relation between quantum divergence functions, their associated metric tensors, and the submanifolds of separable/entangled states. For instance, it is customary to require quantum divergence functions (quantum relative entropies) to satisfy the additivity property. Specifically, given a family $\{S^N\}_{N\in\mathbb{N}_2}$ of quantum divergece functions and two product states $\rho\,,\omega\in\mathcal{S}_{nm}^{AB}$ with associated density matrices $\bar{\rho}=\bar{\rho}_A\otimes\bar{\rho}_B$ and $\bar{\omega}=\bar{\omega}_A\otimes\bar{\omega}_B$, then $\{S^N\}_{N\in\mathbb{N}_2}$ is required to satisfy (see [20]):

$$S^{nm}(\bar{\rho}\,,\bar{\omega})=S^n(\bar{\rho}_A\,,\bar{\omega}_A)+S^m(\bar{\rho}_B\,,\bar{\omega}_B)\,. \tag{5.6}$$

According to the abstract framework exposed in chapter 4, the additivity property implies that on the manifold of invertible product states the pullback of the metric tensor associated with the metric tensor extracted from S^{nm} is the "sum" of the metric tensors extracted frm S^n and S^m. To see this, let us consider the product manifold $\bar{\mathcal{S}}_n^A \times \bar{\mathcal{S}}_m^B$ of invertible density matrices on the two subsystems. Being a product manifold, we may decompose its tangent bundle into the direct sum of the tangent bundle of $\bar{\mathcal{S}}_n^A$ and $\bar{\mathcal{S}}_m^B$. A generic vector field X on $\bar{\mathcal{S}}_n^A \times \bar{\mathcal{S}}_m^B$ can be written as:

$$X = f^A\, X_A + f^B\, X_B\,, \tag{5.7}$$

where X_A and X_B are, respectively, vector fields along $\bar{\mathcal{S}}_n^A$ and $\bar{\mathcal{S}}_m^B$, while f^A and f^B are functions on $\bar{\mathcal{S}}_n^A \times \bar{\mathcal{S}}_m^B$. Now, let us denote with $\mathfrak{I}_{nm}$ the immersion of the manifold $\bar{\mathcal{S}}_n^A \times \bar{\mathcal{S}}_m^B$ in the manifold $\bar{\mathcal{S}}_{nm}^{AB}$ of invertible density matrices of the composite system given by:

$$(\bar{\rho}_A\,, \bar{\rho}_B) \mapsto \mathfrak{I}_{nm}(\bar{\rho}_A\,, \bar{\rho}_B) := \bar{\rho}_A \otimes \bar{\rho}_B\,. \tag{5.8}$$

Because of the additivity property for S^{nm}, its pullback D^{nm} to $\bar{\mathcal{S}}_n^A \times \bar{\mathcal{S}}_m^B$ by means of the double of the map $\mathfrak{I}_{nm}$ (see equation (4.69)) reads:

$$D^{nm}(\bar{\rho}_A\,, \bar{\rho}_B\,; \bar{\omega}_A\,, \bar{\omega}_B) = S^n(\bar{\rho}_A\,, \bar{\omega}_A) + S^m(\bar{\rho}_B\,, \bar{\omega}_B)\,. \tag{5.9}$$

Consequently we can extract from D^{nm} the following symmetric covariant tensor G (see definition 12 and proposition 23):

$$\begin{aligned} G(X\,, Y) &:= -i_d^*\,(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} D^{nm}) = \\ &= -i_d^*\,(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} S^n + L_{\mathbb{X}_l} L_{\mathbb{Y}_r} S^m) = \\ &= g^n(X_A\,, Y_A) + g^m(X_B\,, Y_B)\,, \end{aligned} \tag{5.10}$$

where g^n and g^m are the metric tensors extracted, respectively, from S^n and S^m. This is structurally similar to what happens with the Fubini-Study metric tensor in the case of pure quantum states, and it would be interesting to understand what happens in the case of separable and entangled states. On the one hand, quantum divergence functions and their associated metric tensors may be used to investigate the geometry of composite systems. On the other hand, the geometry of composite systems may help in characterizing particular information-theoretic properties of quantum divergence functions and quantum metric tensors.

Since we considered only finite-dimensional systems, it is natural to ask what can be generalized to the infinite-dimensional case, and how can we perform this generalization. Concerning the geometrical structures on the space of quantum states presented in chapter 2, we point out that the group $GL(\mathcal{A})$ of invertible elements of an infinite-dimensional C^*-algebra $\mathcal{A}$ is a Banach-Lie group, that is, a (complex and real analytical) Banach manifold with a compatible group structure, and its Banach-Lie algebra is precisely $\mathcal{A}$ (see for instance [39] page 81, [99] page 96). Furthermore, the map $\rho \mapsto \alpha_{\mathrm{g}}(\rho) \equiv \rho_{\mathrm{g}}$, where $\mathrm{g} \in GL(\mathcal{A})$ and with ρ_{g} defined by (see equation (2.10)):

$$\rho_{\mathrm{g}}(\mathbf{A}) := \frac{\rho\left(\mathrm{g}^{\dagger}\,\mathbf{A}\,\mathrm{g}^{\dagger}\right)}{\rho(\mathrm{g}^{\dagger}\,\mathrm{g})} \quad \forall\,\mathbf{A} \in \mathcal{A}\,, \tag{5.11}$$

can be proved to be an action of $GL(\mathcal{A})$ on $\mathcal{S} \subset \mathfrak{O}^* \subset \mathcal{A}^*$. Having an action of the Banach-Lie group $GL(\mathcal{A})$ on $\mathcal{S}$, we may try to exploit the differential geometry of Banach-Lie groups (see [27, 39, 80, 99]) to endow the orbits of $GL(\mathcal{A})$ in $\mathcal{S}$ with the structure of a Banach manifold. At this purpose, we note that, in the infinite-dimensional setting, the closed subgroup theorem 2 is not valid. Specifically, even if it is true that the orbit $\mathrm{Orb}(\rho)$ of α through ρ is in one-to-one correspondence with the coset space $GL(\mathcal{A})/G_{\rho}$, where G_{ρ} is the isotropy subgroup of ρ with respect to α, the coset space $GL(\mathcal{A})/G_{\rho}$ is a real Banach manifold if and only if G_{ρ} is a real Banach-Lie split subgroup of $GL(\mathcal{A})$ (see [27] page 105, and [99] page 136), and this, in infinite dimensions, is a more stringent requirement than being a closed subgroup. If $GL(\mathcal{A})/G_{\rho}$ is a real Banach-Lie split subgroup of $GL(\mathcal{A})$, then we can endow $\mathrm{Orb}(\rho)$ with the structure of a real Banach manifold. Obviously, it is not necessarily true that the isotropy subgroup of every quantum state $\rho \in \mathcal{S}$ is a Banach-Lie split subgroup of $GL(\mathcal{A})$ when $\mathcal{A}$ is a generic infinite-dimensional C^*-algebra.

As an illustrative example, let us consider the case where $\mathcal{A}$ is the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all the bounded linear operators on the complex separable (infinite-dimensional) Hilbert space $\mathcal{H}$. Let us denote with $\mathcal{T}_N$ the set of bounded linear operators on $\mathcal{H}$ having fixed finite rank N. Note that $\mathcal{T}_N$ is a subset of the space $\mathfrak{TC}(\mathcal{H})$ of trace-class operators on $\mathcal{H}$. For every positive $\bar{\rho} \in \mathcal{T}_N$ such that $Tr(\bar{\rho}) = 1$ (density operator) there is a normal[4] state $\rho \in \mathcal{S}$ such that:

$$\rho(\mathbf{A}) = Tr\,(\bar{\rho}\,\mathbf{A}) \quad \forall \mathbf{A} \in \mathcal{B}(\mathcal{H})\,. \tag{5.12}$$

Note that ρ is faithful, that is, $\rho(\mathbf{A}) = 0 \iff \mathbf{A} = \mathbf{0}$. According to the spectral theory of compact operators (see [96]), there is a decomposition:

[4]This functional is continuous in both the norm topology and the ultraweak topology on $\mathcal{B}(\mathcal{H})$. Furthermore, $||\rho|| = ||\bar{\rho}||_1$, where $||\bar{\rho}||_1 = tr(|\bar{\rho}|)$.

$$\mathcal{H} = \mathcal{H}_\rho \oplus \mathcal{H}_\rho^\perp$$

with $dim(\mathcal{H}_\rho) = N < +\infty$ and a countable orthonormal basis $\{|e_j\rangle\}$ adapted to this decomposition such that $\bar{\rho}$ can be written as:

$$\bar{\rho} = \sum_{j=1}^{N} p^j \, |e_j\rangle\langle e_j| \,, \tag{5.13}$$

with $p^j > 0$ and $\sum_j p^j = 1$. We will now show that the action of $GL(\mathcal{A})$ on $\mathcal{S}$ is transitive on the set of normal states associated with density operators in $\mathcal{T}_N$. Indeed, let ρ_0 be associated with the density operator $\bar{\rho}_0 \in \mathcal{T}_N$ such that $p_j = \frac{1}{n}$ for all $j = 1, ..., N$, and let $\bar{\rho}_1 \in \mathcal{T}_N$ be an arbitrary density operator. Let $\{|e_j^0\rangle\}$ and $\{|e_j^1\rangle\}$ be the orthonormal basis adapted to the decompositions $\mathcal{H} = \mathcal{H}_{\rho_0} \oplus \mathcal{H}_{\rho_0}^\perp$ and $\mathcal{H} = \mathcal{H}_{\rho_1} \oplus \mathcal{H}_{\rho_1}^\perp$ associated with, respectively, $\bar{\rho}_0$ and $\bar{\rho}_1$. The operator $\bar{\rho}_1$ may be written as:

$$\bar{\rho}_1 = \sum_{j=1}^{N} p_1^j \, |e_j^1\rangle\langle e_j^1| \,. \tag{5.14}$$

Now, let us define the following map[5] T:

$$|e_j^0\rangle \mapsto a^j |e_j^1\rangle \,, \tag{5.15}$$

with $a^j = \sqrt{p_1^j}$ for $j = 1, ..., N$, and $a^j = 1$ for $j \geq N+1$. Being $\{|e_j\rangle\}$ and $\{|e_j^1\rangle\}$ orthonormal basis, we can extend this map by linearity so that it becomes a linear map from $\mathcal{H}$ to itself. A straightforward calculation shows that T is bounded, and thus there is an element $\mathrm{g} \in \mathcal{A} = \mathcal{B}(\mathcal{H})$ representing T. Since $p_1^j > 0$ for all j, it is immediate to check that g admits an inverse, and thus it is an element of $GL(\mathcal{A})$. Now, if $\mathrm{g} \in \mathcal{A}$, then the range of $\mathrm{g}\,\bar{\rho}_0$ is certainly contained in the range of $\bar{\rho}_0$. Being g invertible, its action on the range of $\bar{\rho}_0$ does not have kernel, and thus the rank of $\mathrm{g}\,\bar{\rho}_0$ is the same as that of $\bar{\rho}_0$. With a similar reasoning, we conclude that the rank of $\mathrm{g}\,\bar{\rho}_0\,\mathrm{g}^\dagger$ is the same as that of $\bar{\rho}_0$. Furthermore, a direct calculation shows that:

$$\frac{\mathrm{g}\,\bar{\rho}_0\,\mathrm{g}^\dagger}{Tr(\mathrm{g}\,\bar{\rho}_0\,\mathrm{g}^\dagger)} = \bar{\rho}_1 \,, \tag{5.16}$$

which means $\rho_1 = (\rho_0)_{\mathrm{g}}$, and we find that the action of $GL(\mathcal{A})$ is transitive on the spaces of normal states associated with density operators with fixed rank.

[5]There is no summation on the index j.

We will now show that the set $\mathcal{S}_k$ of quantum states associated with density operators with rank equal to k can be endowed with the structure of a real Banach manifold. In accordance with the above discussion we will do so by showing that the isotropy subgroup G_ρ of ρ is a Banch-Lie split subgroup of $GL(\mathcal{A})$. An element $\mathbf{K}$ in the isotropy subgroup G_ρ is characterized by the fact that:

$$\bar{\rho} = \frac{\mathbf{K}\,\bar{\rho}\,\mathbf{K}^\dagger}{Tr(\mathbf{K}\,\bar{\rho}\,\mathbf{K}^\dagger)} \,. \tag{5.17}$$

Now, bounded linear operators on $\mathcal{H}$ are completely and uniquely determined by their matrix elements with respect to an orthonormal basis of $\mathcal{H}$ ([3] chapter $II.26$ p. 48). This means that we can look at them as infinite-dimensional matrices, and manipulate them using the rules of matrix algebra. Consequently, proceeding as in proposition 3, it is easy to see that every $\mathbf{K} \in G_\rho$ is of the form:

$$\mathbf{K} = \begin{pmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} , \tag{5.18}$$

where $\mathbf{U} \in \mathcal{U}(\mathcal{H}_\rho)$, $\mathbf{B} \in \mathcal{B}(\mathcal{H}_\rho^\perp)$ and $\mathbf{C} \in \mathcal{W}$ with $\mathcal{W} \cong (\mathcal{H}_\rho^\perp)^{\otimes N}$. A direct computation shows that every element $\mathbf{k}$ in the Lie algebra $\mathfrak{g}_\rho$ of G_ρ can be written as:

$$\mathbf{k} = \begin{pmatrix} \imath\mathbf{H} & \mathbf{c} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} , \tag{5.19}$$

where $\mathbf{H}$ is a self-adjoint matrix, while $\mathbf{c}$ and $\mathbf{A}$ are complex, infinite-dimensional matrices[6]. According to [99] (page 129) the closed Banach-Lie subgroup G_ρ of the Banach-Lie subgroup $GL(\mathcal{A})$ is a Banach-Lie split subgroup if and only if the closed subalgebra :

$$\mathfrak{g}_\rho = \{\mathbf{A} \in \mathcal{A} : \; \exp(t\mathbf{A}) \in G_\rho \; \forall t \in \mathbb{R}\} \,, \tag{5.20}$$

is a split subspace of $\mathcal{A}$, and for every neighbourhood V of $\mathbf{0} \in \mathfrak{g}_\rho$, $\exp(V)$ is a neighbourhood of the identity element in G_ρ. Direct inspection shows that for every neighbourhood V of $\mathbf{0} \in \mathfrak{g}_\rho$, $\exp(V)$ is a neighbourhood of the identity element in G_ρ. Therefore, in order for $GL(\mathcal{A})/G_\rho$ to carry a manifold structure, we must show that $\mathfrak{g}_\rho$ is a split subspace of $\mathcal{A}$.

At this purpose, let us define $\mathfrak{f}_\rho$ to be the subset of $\mathcal{A}$ made up by all those elements $\mathbf{F}$ such that their matrix representation with respect to the orthonormal basis $\{|e_j\rangle\}$ is:

[6] Clearly, the matrices $\mathbf{H}, \mathbf{c}$ and $\mathbf{A}$ must be such that $\mathbf{k}$ is a bounded linear operator on $\mathcal{H}$ ([3] chapter $II.26$ p. 48).

$$\mathbf{F} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{d} & \mathbf{0} \end{pmatrix} , \tag{5.21}$$

where $\mathbf{B}$ is a self-adjoint matrix, and $\mathbf{d}$ is a complex, infinite-dimensional matrix[7]. It is clear that $\mathfrak{f}_\rho$ is a vector subspace of $\mathcal{A}$, and that $\mathfrak{f}_\rho \cap \mathfrak{g}_\rho = \emptyset$. We will now show that $\mathfrak{f}_\rho$ is closed. To see this, let us note that for every $\mathbf{F}$ is in $\mathfrak{f}_\rho$ if and only if for every $|\psi_\rho^\perp\rangle \in \mathcal{H}_\rho^\perp$ we have $\mathbf{F}|\psi_\rho^\perp\rangle = \mathbf{0}$. Now, let us take sequence $\{\mathbf{F}_n\} \in \mathfrak{f}_\rho$ such that it converges to $\mathbf{F}$ in $\mathcal{A}$. This means that for every $\epsilon > 0$ there exists a $\bar{n} \in \mathbb{N}$ such that:

$$||\mathbf{F}_n - \mathbf{F}|| < \epsilon \tag{5.22}$$

for all $n > \bar{n}$. Now, suppose $\mathbf{F} \notin \mathfrak{f}_\rho$. Then, there is $|\psi_\rho^\perp\rangle \in \mathcal{H}_\rho^\perp$ such that $\mathbf{F}|\psi_\rho^\perp\rangle = |\phi\rangle \neq \mathbf{0}$. Consequently:

$$||\mathbf{F}_n - \mathbf{F}|| > \left|\left|(\mathbf{F}_n - \mathbf{F})|\psi_\rho^\perp\rangle\right|\right| = ||\phi|| \,. \tag{5.23}$$

From the arbitrariness in the modulus of $|\psi_\rho^\perp\rangle$ and $|\phi\rangle$, we conclude that either $\{\mathbf{F}_n\}$ does not converge to $\mathbf{F}$, or $\mathbf{F} \in \mathfrak{f}_\rho$. This means that $\mathfrak{f}_\rho$ is closed in $\mathcal{A}$. The whole Lie algebra $\mathcal{A}$ of $GL(\mathcal{A})$ can thus be decomposed as $\mathcal{A} = \mathfrak{g}_\rho \oplus \mathfrak{f}_\rho$, where $\mathfrak{g}_\rho$ and $\mathfrak{f}_\rho$ are closed subspaces of $\mathcal{A}$ having null intersection, which means that $\mathfrak{g}_\rho$ is a split subspace of $\mathcal{A}$, and $\mathfrak{f}_\rho$ is a complement of $\mathfrak{g}_\rho$ in $\mathcal{A}$.

Eventually, according to [27] (page 105) and [99] (page 136), $GL(\mathcal{A})/G_\rho$ acquires the structure of a real Banach manifold modelled on the Banach space $\mathfrak{f}_\rho$, and we can transport this differential structure on the orbit $\mathcal{S}_k$ of $GL(\mathfrak{A})$ in $\mathcal{S}$ as claimed.

We hope to be able to extend these results to orbits through generic quantum states on $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and to be able to work out some results in cases in which $\mathcal{A}$ is no longer $\mathcal{B}(\mathcal{H})$ but some other example of infinite-dimensional C^*-algebra. For instance, the case of factors of type III_1 would be relevant for quantum field theory[8].

Once we have a more thorough understanding of the geometry of the space of quantum states in infinite dimensions we could start to study the infinite-dimensional analogue of quantum information geometry using some of the well-known examples of quantum relative entropies that are naturally formulated in the context of C^*-algebras without any assumption of finite-dimensionality (see [12, 13, 70, 92, 98]).

[7] Again, the matrices $\mathbf{B}$ and $\mathbf{d}$ must be such that $\mathbf{F}$ is a bounded linear operator on $\mathcal{H}$ ([3] chapter II.26 p. 48).

[8] At this purpose, some topological results have been presented in [52], where the action of the unitary group $U(\mathcal{A})$ on the space of states of a factor of type III_1 von Neumann algebra $\mathcal{A}$ has been analyzed.

The geometrical structures introduced in chapter 3 in relation with the dynamical evolutions of open quantum systems depend on the Lie-Jordan algebra structure of $\mathfrak{O}_n$. In the infinite-dimensional case, the space $\mathfrak{O}$ of observables is a Lie-Jordan-Banach algebra, which may be thought of as an infinite-dimensional counterpart of a finite-dimensional Lie-Jordan algebra. Consequently, a natural direction of investigation would be to understand to what extent the constructions of chapter 3 generalize to the infinite-dimensional case. Furthermore, a careful analysis of LaSalle's invariance principle (see [81]) suggests that a generalization to the case of arbitrary Banach spaces may be feasible. This would give us a very important tool for the study of the asymptotic behaviour of infinite-dimensional dynamical systems, both quantum and classical.

To conclude, what the author has learned during his PhD experience may be summarized by saying that "the search for an ever increasing level of geometrization of quantum mechanics (and quantum theories in general) seems to be a promising route to take".

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