Listing Price and Non-Price-Taking behavior in market mechanisms with differential information

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Abstract

This work investigates the optimal initial public offering mechanism (IPO) comparing the book-building, the fixed price offer and the auction to understand if the supremacy of the first mechanism is economically justified or if there are other motivations behind.
In the first chapter, a survey of the theories concerning the underpricing and a general overview of the market evolution are presented. Later, let us present a theoretical model in which the listing methods are compared basis on the firm's profit. It examines what are the key variables studied by the issuer to decide the mechanism maximizing its revenue. The information cost and the reservation utility of the investors turn out the most significant variables.

Introduction

Despite the economic and political uncertainty, the global Ipo market never stops to expand. This trend makes it interesting for the economic field such that the researchers' attention is focused on this phenomenon and any aspect: the reason to become public, the underpricing's shares and, mostly, they would crown the best method among the three popular mechanisms: the book-building, the fixed price offering and the auction. A controversial aspect that influences the mechanisms is the phenomenon of underpricing. It could be perceived as a market anomaly because it takes place when the offer price is lower than the market price. In this way, the new issues underperform the market, bringing substantial high returns on the first trading day for investors, but theoretically a lower listing profit for the firm. Some economists suggest that the auction process might be more accurate in terms of price definition and more fair in the allocation of new shares [20], but, despite that, at the end of 1990's, the book-building emerged, becoming the dominant mechanism.

The auction method returned renowned with Google, in 2004 ([9]). The company announced that it would become public through the auction, instead of the U.S. trend. At the time, Google was a famous and attractive firm, already before the listing. Therefore, the expectation on the auction was a strong overpricing for the optimistic circumstances. Indeed, initially, the estimated price range for share was \$108 to \$135, but its high evaluation decreased the price of the other technological companies. Under these new conditions, Google had to review downwards its range as consequence an underpricing. With this background, even if Google could be an important example of the auction's efficiency, it is excluded by literature for its anomalous history.

The question generally accompanies the book building method and so the natural question is: *Do the advantages of the Book Building justify the progressive disappearance of the auction mechanism?*.

Now, let us describe the structure of the thesis. The analysis begins with the main theories that explain the underpricing phenomena: Asymmetric information, Control and Ownership, Procedural theory and Behavioral explanation. Each theory focuses on a particular step of the listing procedure or on a particular group of

agents involved in the mechanism. Then, for having a complete overview, the process of an initial public offering is accurately described, for each procedure, there is a detailed description of strengths and weaknesses point.
Later, a dataset of listed firms is described and commented. It collects 903 observations across several financial markets: Nasdaq, Milan, London, Paris, Hong Kong, Thailand and Nyse and it is organized by the offer price, by the first price after the listing, by the underpricing and by the number of shares. All these variables are taken into account for understanding if there are some aspects which influence incisively the underpricing and consequently the profit of the issuer.
The last chapter, on the basis of the conclusion of the first one, elaborates a theoretical model. In this last one, the mechanisms are divided into two groups: the insider and the outsider method of price's definition, comparing the book-building and the fixed price offering with the auction. The involved agents are the firm and the investors, leaving aside the other figures: market authorities, banks and so on. The issuer has the goal to maximize its profit representing by function increasing in price and quantity, while the investors have the aim to access to the shares acquisition, if and only if their participation constraint holds.
The key variable is the information cost, sustained by the firm or the investors according to the mechanisms.

Chapter 1

The evolution of Ipo: theory

The first chapter focuses on the evolution of the financial market and on a draft of the theories about the underpricing nature. Its goal is finding the best method among the book- building, the fixed price offering, the auction in terms of underpricing and of interests for the several agents involved in the listing process. It is structured as follows:

- In the first section, there is a description of the main theories:
  - Asymmetric information ([7], [26]). It is conceived as the classical problem of unbalance information among the agents involved in an economic process (Akerlof, 1970). Therefore, as usual, the insider members (the firms and underwriter banks) have an informational advantage over outside members (the investors);
  - Control and Ownership ([8]). The separation between ownership and control produces a condition with conflict of interest between the owner and the manager. Their purpose may diverge because the first one pursues the profit maximization, while the second one often has the aim to improve his reputation also through unscrupulous strategies;
  - Procedural theory ([17]). This theory considers how the mandatory disclosure and the regulation of the financial market impact on the issuer firm decision;
  - Behavioral explanation ([25]). It is a new line of thought in economic theory. Usually, the financial world assumes that the agents are wealth maximizers and rational. However, they are human beings with emotion and feeling that influence their decision. This aspect translates into an unexpected first valuation of the shares;
- Then, the most diffused methods of listing: Fixed price offer, Auction and

BookBuilding are exposed and analyzed, underlying similarities and differences.

1.1 Under-pricing phenomenon: Uniform Price Auction for Ipo

Ritter (1998) stated an initial public offering (IPO) occurs when a security is sold to the general public for the first time, with the expectation that a liquid market will develop. It is a placement of financial securities, in particular, equity shares, aimed at the first flotation of a company on a stock exchange market.
The academic theory [5] suggests the following motivations for going public:

1. Cost of capital. The Ipo becomes a strategic mechanism to finance immediate growth projects both inside and outside the company, as well as to feed net working capital or fixed capital, depending on the precise needs of the company. In this way the company can strengthen its financial structure, allowing it to become more solid and structured. Likewise, it may at least partially diversify its sources of capital and at the same time reduce dependence on expensive bank financing. It should be noted that the Floatation process presents the way for a possibility of continuous capital procurement, not only relegated to the initial phase: a company in difficulty may indeed return to the capital market even later;

2. Creation of an exit route. Through the listing, the shares become not only hypothetical transaction objects, but they allow the possibility to liquidate their investment when the shareholders consider it appropriate, but can also act as a currency. The price of the same ones indeed gives liquidity to shares that have a precise market value. In particular, this last aspect translates into the possibility, given to the newly quoted company, to acquire other companies, or in any case to conduct transactions of various kinds, using an alternative source to money in the payment of the transaction: the share unit;

3. A Reassurance for the stakeholders. A listed company has stringent disclosure requirements that make it more transparent to the public. The several steps that must necessarily be overcome to gain the entry to the capital market, such as the delicate Due Diligence process and official approval by of the competent institution, reassure customers and suppliers about its business soundness and the correctness of the disclosed data;

4. Skills. The passage from a not listed list on a listed company increases its prestige and reinforces its image. Obviously, this can be a trigger for attracting qualified professionals, eager to achieve greater visibility through a prominent role in the newly-listed company

At the same time, the decision to become public is accompanied by some disadvantages:

1. Influence of the market. The trend, both negatively and positively, influences the company's listing and the firm becomes vulnerable to exogenous variables. In this way, the price, which should be an objective representation of the company's value since decreed by the market, it could actually assume values that do not reflect the quality of the company;

2. Disclosure requirements. It will equip with expensive management systems information and monitoring of compliance, subtracting huge resources that could be employed in the process of investment in a more profitable way. In addition. as already mentioned, a company that is willing to be listed must disclose some information that, until privately managed, should not be disclosed. This Disclosure and loss of privacy actually occur not only when it is for the first time listed on the market, but also continuously, as an essential requirement to maintain "listed" status;

3. Loss of control. It is implied that if a firm decides to take advantage with the investing public as a capital procurement channel, this action is subject to a sequential loss of control and dilution of ownership. What manifests itself is a natural redefinition of the ownership structure, in particular with a reformulation of the majority shareholder who, following the listing, may have lost their role as *"blockholders"*. In this sense we often witness situations in which, driven by the need to satisfy the requests of the new substitutive shareholders, managers make more projected decisions to achieve significant performance in the short term, which however prove unsuccessful if analyzed over a longer time horizon. Furthermore, this greater fragmentation of ownership can constitute an obstacle to normal and fluent business management: for example, to undertake activities such as the acquisition of another company, it is now necessary to obtain approval from the various shareholders. At the same time, an overly widespread and decision-free ownership can facilitate the arrival of hostile takeovers;

4. Deployment of forces and time. The IPO involves significant costs that the firm is forced to support both in the phase of preparation for listing both subsequently. In particular, the direct costs refer to the standard costs that

must be incurred from the issuer: sales commissions, fees for legal advice as well as for auditing, expenses for accounting service, transaction costs and last but not least, the time spent by management in assisting step by step the whole procedure. This last one qualifies as opportunity cost and is much more difficult to quantify if compared with other costs, purely monetary.

Taking into account all these aspects and much more, the shares are put on the market at an offer price, showing, in the most of the case, an under price. The issuer undervalues, more or less consciously, its shares positioning himself in a highly unfavorable situation because, in absence of underpricing, he could collect the same amount of funds by issuing a lower quantity of shares.

The underpricing is computed as the difference between the Ipo price and the market price of the first trading day after the closure of the initial public offering. Associated with this concept, there is the notion of *Money left on the table*. It is defined as the first-day price increase multiplied by the number of shares issued and constitutes opportunity cost of going public for issuing firms.

$$\textit{Money left on the table} = (P_{ipo} - P_{1day}) * (\text{Number of shares}).$$

The underpricing goes to "clash" with the concept of an efficient market. It affects the IPO worldwide: this is not a circumscribed phenomenon, although the magnitude of the same is affected by multiple factors related to the geographical element. The aim is to identify the causes of this phenomenon. In this regard, four explanatory theories are exposed:

1. Asymmetric information theory;
2. Control and ownership theory;
3. Procedural theory;
4. Behavioral theory.

Besides looking through these ones, the accent will be placed on the listing mechanism. The focus changes because the point is to understand if the issuer company, choosing the listing method, chooses also the underpricing level. The most important and diffused ones are:

1. Book-building method (BM);
2. Fixed Price offering (FPO);
3. Uniform Price auction (UPA)[1].

[1] Uniform-price auctions are often referred to in the financial press, less descriptively, as Dutch auctions and elsewhere as nondiscriminatory auctions, competitive auctions, or single-price auctions.

1.1.1 Asymmetric information Theory

The basic assumption is extremely intuitive: what is theorized here is that one of the actors has more information than the remaining ones. Rock (1986) [26], starting from the famous concept of "The Market for Lemons" by Akerlof (1970), argues that the information disparity exists within the same group of investors, thus excluding in this scenario the considerations on the figure of the issuer that on the underwriter. In his opinion, some investors are more informed about the real value of the shares than other investors. In this sense, a double-track situation is revealed: on the one hand, in the underpriced IPOs, the demand comes from both the group of participants and it is, therefore, necessary to carry out a rationing of the shares offered. On the contrary, in situations of overpriced IPOs, obviously, the most informed investors don't participate, leaving instead the totality of the "beneficial" share allotment of the less informed counterpart.

Therefore, the "*Winner's Curse*" is revealed because it might seem a victory for the less informed about the informed when the reality is very different.

Obviously, the less informed investors once suffered the overpricing, decide to not participate in further IPO. Rock (1986) however considers the presence of less informed investors numerically fundamental. It is precisely from this observation that the concept of underpricing is inserted. It is described in the Rock model (1986) as a necessary tool to simultaneously attract less informed investors and thus preserve the liquidity of the financial market: it is therefore used to guarantee an expected positive in any case.

Related to Informational Asymmetry, it is the model, that of the Principal-Agent, which finds application not only in the context of the IPOs, but also in all those situations in which there is a delegation relationship: an agent that works for a principal in a context of information asymmetry that may occur in a pre and post-contractual form. Normally it refers to the model of Jensen Meckling (1976) which, however, looks as reference the manager-owner relationship, in this case, the model is applied to the first quotation process: specifically to the information asymmetry between issuer and underwriter. The last one could use the allocative mechanism to his own advantage, giving the shares mainly to those subjects who hold key roles within the various companies, the so-called Executives, in the hope that they would in future turn to them to start a possible own investment activity. To avoid this problem of incorrect behavior, it is necessary to implement a contract between Principal and Agent that pivots on the incentive scheme. Specifically, the underwriter makes appropriate use of the more information that he divulges in favor of all investors. In this contract, the underpricing reveals like a gift for the executive that buys shares at a lower price.

In the asymmetric information theories, two phenomena reveal

1. Winner's curse[11]. Assuming a bad deal and an information asymmetry, the

demand is reduced to that of less informed investors. It, therefore, revealed a "Winner's curse" because, although at first view, it could be considered a victory for the less informed about the informed, the reality was very different. The first ones buy shares of an unsuccessful firm and if they know about it, they would keep away from the IPO.

2. Free rider problem [21]. It has like consequence a moral hazard problem. Higher is the information production, higher is the incentive for new bidders to not collect any information. This happens because the new investors hope that the previous bidders have just decided the clearing price, making the correct and fair analysis about the firm. On average, the effect of this problem is the reduction of the incentive of other investors to produce information and in this way, the IPO becomes less efficient in terms of price's determination. The fact that a less amount of information would be produced in an equilibrium with free riders means that less underpricing would be necessary to compensate the expense of the informed bidders. The free rider problem leads to a hypothetical overpricing.

The fundamental difference between the winner's curse and the free rider problem is that the winner's curse does not lead people to bid more than they really believe. On the contrary, with the free rider problem, the investors could deliberately bid an excessive amount, since the point is to make an offer blindly high enough to be first in line for the shares, rather than devoting time and resources to come up with a reasonable bid.

1.1.2 Control and Ownership theory

Another theory, which explains the underpricing phenomenon, is that one of *Control and Ownership*, which focuses on the conflict of interests among the parties in the underwriting process.
The starting idea from which the theory of Brennan and Franks starts (1997) [8] is extremely intuitive: according to it, the managers of the companies deliberately underpriced the actions of their company in order to maintain the control over it and avoiding any monitoring action by external parties. The underpricing is a fundamental tool used in this specific case in order to create an excess on the demand side: in fact, proposing to the primary market with an extremely advantageous offer price, managers distribute the totality of the shares among a plurality of subjects, all particularly interested in seeing the appealing deal. Auspiciously, it is difficult that many small shareholders implement a monitoring action. In this situation, emerges the problem of free riding: being the act of monitoring a public good whose implementation benefits all shareholders, no one would employ their own assets at this purpose, aware that all the investors may have benefits from it.

On the basis of these considerations, this theory asserts not only that the underpricing is strategic, since it allows the creation of an excess of demand, but at the same time that the managers have the possibility to shrewdly allocate the shares among the different candidates, selecting the more harmless, the less relevant. They propose also an alternative way to control the company: the issue of non-voting share for not losing the power. However, the Stock Exchange Market, in general, discourages this issuance to avoid discrimination among shareholders.

1.1.3 Procedural theory

The third theory is the Procedural one. As the name suggests, it deals with all the aspects regarding the procedure. It is consistent with the assumption that many companies are subject to legal liability and lawsuits.
It is basically a theoretical theory that interprets the underpricing as a tool adopted by companies, to reduce the probability of being denounced by their own investors, dissatisfied with the post IPO performances recorded by the shares. Proposed by Lougue (1973) and Ibbotson (1975), they study the phenomenon almost as if it were an insurance form adopted by the company, hence the further appellation *Insurance* or *Lawsuit Avoidance Hypothesis*.

1.1.4 Behavioral explanations

It focuses the accent on the feeling of investors, on the emotional aspect. Not being robots, their drivers are the emotions, from that it asserts the existence of irrational investors (or sentiment or noise trader) which follows the feeling in the economic decision. The context of financial markets, especially the process of first quotation, represents an extremely fertile situation for the proliferation of behavioral irrationality: already the impossibility of evaluating the company and its shares based on its history (since non-existent) or on the few and fragmented information. In this view, it compensates, in the worst of the case, a failed investment. The irrationality refers to the fact that the trader may be optimistic without any good economic reason for being it. The starting point of these theories is, as the name implies, refusing one of the main features of all the models mentioned so far: the rationality of the various subjects involved in the first listing process. This refers to the investors, motivated by logically irrational motivations when they have to choose how to invest. Starting from all that, the Welch (1992) [25] theory of the Information Cascades seems to be potentially able to explain the phenomenon. It would be optimal for the firms to exploit the sentimental investors' surplus. It is a very intuitive theoretical vein, based on a process of taking the cascade decisions. Welch, through his studies, argues that investors who enter into the transaction only after the time of first listing, consider the work performed by their predecessors, regardless of opinions and information about the firm. With this assumption, it is clear that the role played by early investors is extremely important, as well as strategic, for the company itself. At the light of what has been illustrated the underpricing is explained as a "Premium" that the "first" investors demand as a reward for starting up, through their equity acquisitions, to the positive cascade.

1.2 The process of quotation

The focus of my research is not on the previous theory, but it is on the listing procedure and how it can affect the underpricing, so if there exists a relation between the flotation's method of share and the devaluation of the company.
The common procedure for an initial public offering, whatever the mechanism is, is composed of three step[4]:

1. Flotation process of shares and meeting between the firm and possible investors through an underwriting bank.

2. Initial listing and shares allocation at the determined price. The mechanisms that lead to the setting of an offer price are multiple and crucial since it is crucial to the success of the IPO: in particular, in the price determination process, the underwriter figure can be the undisputed protagonist.

   This phase differs according to the mechanism:

   a. Fixed price offer;

   b. Auction[2] or offer for subscription by tender;

   c. Book building.

3. Allocation of the shares among the several investors.

In the first step, there is often an important role of the underwriters. They usually are investment banks that have IPO specialists on staff. These organizations collaborate with the issuer to ensure, first of all, that all regulatory requirements are satisfied. It organizes the flotation process of the firm's share and puts the issuer firm in contact with the various parties concerned: market authorities, communication agency, investment company, auditors and possibly legal advisers. Its commitment is contractually defined.
According to the bank, the firm decides the number of the share and the type of procedure.
This is the first step common to all procedures.

The definition of the price and the allocation of shares among the investors underlines the difference between the book building, auction and fixed price offering.

a. Starting from the fixed price listing method, called fixed price regime, it reveals a situation in which the price is previously set arbitrarily, even before the collection of orders from investors and their intentions, both institutional and retail. The allocation of the shares needs to implement adequate allocation mechanisms:

    1. There may be cases in which a certain profile of fairness is maintained, applying the so-called fair rules. These mechanisms take the form of a pro-rated allocation, that is, one opts to satisfy the requests of all the participants, within certain limits and not for the whole until the offer is exhausted. However, investors, aware of the aforementioned mechanism, could anticipate this move and adopt incorrect behavior: that is, they could do request for a quantity much higher than that actually desired so that, into the allocation phase, they obtain the amount really desired;
    2. however, there may be cases in which the shares are assigned according to mechanisms that are not strictly transparent: in some countries, for example, when securities are placed, retail investors are privileged, which, as is well known, represent the weak and minority link, without no possibility of monitoring action;
    3. there are also situations in which the institutional actors are favorite.

b. In the book building method, the figure of the underwriting bank has a role more central than in the other two procedures. Indeed, in this case, the bank conducts all the steps of the listing and of the share allocation. At the beginning, the investment bank, styles a list of investors admitted to participation: these will be invited to take part in the procedure with the official mode. Usually, this phase involves exclusively institutional investors: the reasons for excluding the retail counterpart are indeed several. In particular, there could be a great problem of management to administer a plurality of requests coming from this category of investors, often small in amount and not even particularly representative of the actual corporate value. Once identified the subjects admitted to the book-building procedure is the duty of the underwriter, always in agreement with the issuer, identify a price range within which the price should fall the offering price. After having selected the subjects admitted to the listing procedure and identified the relative price range, the marketing phase begins (in this case the so-called road shows, which indeed are only of this type of issuing method). The underwriter bank, for determining the price of a unit share, makes several and precise analysis. The offer price is normally set in an amount up to a maximum of 20% in rising or falling compared to the last price range.

Concretely, the most used valuation methods are the discounted future cash flows system (Discounted Cash), the market multiples method and the EVA (Economic Value Added).

- Starting from the first computation method (DCF), it is not only one of the most widespread and accredited methods in the financial field, but it also one of the more rational and exhaustive in the definition of an appropriate business value. It computes the current value of the net operating cash flows that the company will be able to obtain in the future discounted by an interest rate equal to the weighted average cost of capital for that specific company, representative of both its financial and operating risk (so-called Weighted Average Cost of Capital or WACC).
- The second method is an approach that is completely out of long and complex mathematical formulas, as is the case with the DCF method. It computes the value of the listed company comparing it with similar companies, already present in the regulated market, and similar to the issuer on the basis of characteristics pertaining to the risk, profitability, growth opportunities, size and industrial sector.
- The last method is the EVA. It is defined like the *"economic profit"* and it aims to identify the extra value produced by the firm in favor of shareholders. This performance parameter is very close to the true wealth produced by the company, in terms not only of accounting profit, but of the economic value actually achieved: at the base of this method there is the idea that the company in addition to covering the costs with the revenues, it must also be able to remunerate the risk supported by the different stakeholders involved, and be able to repay the interest on the invested capital.

During the phase of price definition, the underwriter publishes the purchasing intention of the interested investors in an order book (from that, the name *"book building"*).
Once the price has been fixed, the bank distributes the shares. Unlike what is described for fixed price regime, only discretionary rules are applied; no residual space therefore for fair rules.

c. The last listing method is the auction. It is a system in which, after setting a minimum price below which it is not permissible to go down, the investors are required to indicate their quantity and their maximum price: subsequently, on the basis of the different offers collected, decreed the actual final price, such as to exhaust the available offer The introduction of a reservation price will affect three aspects.

1. The issuing firm can protect itself from selling at a very low price in the IPO, avoiding a devaluation. It makes sure that the offering price is higher than the reservation price, deterring the investors;
2. There is a possibility of IPO failure when the firm sets a high reservation price;
3. The information production is less because there is less attention of the bidders to analyze the firm and the market, thus they have a starting point.

The auction method can take place without the intervention of external subjects, but on the basis of easy automated computer systems that allow the meeting between supply and demand, in compliance with the price-quantity parameters set by rules specified ex-ante : in this case it is commonly referred to as Internet-based auction mechanism, a very diffuse reality.

Table 1.1: Bid in Uniform Price Auction

Bidders	**Share**	**Bid**	$\sum q$
Investor A	200	25	200
Investor B	250	18	450
Investor C	150	20	600
Investor D	200	20	**800**
Investor E	150	30	950

Table 1.2: Share Allocation in Uniform Price Auction

Bidders	**Share**	**Final Allocation**	$\sum q$
Investor A	200	168.42	168.42
Investor B	250	210.53	378.95
Investor C	150	126.32	505.27
Investor D	200	168.42	673.69
Investor E	150	126.32	800

The determination of price in the uniform price auction can be easily described by an example with only 5 bidders and an available number of shares equal to 800.

In table 1.1, there is the description of the pair quantity-price offered for each investor.

In table 1.2, it is shown how takes place the final allocation. The auction finishes with the bidder D.

Each bidder obtains a percentage of shares computed on the maximal available quantity:

$$800 : 950 = 0,84\%.$$

Of course, this last procedure shows clearly the allocation and the determination of price.
The announcement of price-quantity takes place on a technological platform created for the event. The procedure is open to everyone and it is very easy to participate thus it is totally automated.

Using an easy system, it is possible to explain the mechanism. If the demand is greater or equal than the number of the shares, the market price will be p_m, otherwise the price will be zero and there is no market, in this way the auction procedure will re-begin with other parameters:

$$\begin{cases} D(p) \geq 1, p = p^{\mathrm{m}} \\ D(p) < 1, p = 0 \end{cases}$$

The number of shares is normalized to 1.

The Mise en Vente

A particular Ipo auction is diffused in France: *Mise en Vente*.
It operates as follows: five days prior to the initial public offering, the offered quantity of share and the reservation price are set jointly by the bank, the broker and the firm. On the day of the IPO, investors submit limit orders to their brokers. The latter transmits these orders to the stock exchange market. The total demand function is computed and graphically plotted by the auctioneer, who is the *"Bourse officielle"*. As a function of this demand, the auctioneer sets the IPO price. As in the Book Building method, there is no formal explicit algorithm mapping demand into prices. But, the price adjustment, in the *Mise en Vente*, exhibits strong empirical regularities, with a very low underpricing. So, in this modified auction, it seems that the power is equally distributed between the firm and the bank and it is respected the intrinsic value of firm.
These results derive from an empirical study of 92 Mise en Vente happened between 1983 and 1996. The Ipo's present an underpricing[2] in average equal to 13% while in the United States with the Book Building method, it was on 17,35% ([22]).

1.2.1 Agents of the Initial public offering

After having described all the features of the procedure, each method should be evaluated with respect to the different point of view of all the participant to the Ipo mechanism. The figures are various and numerous: sellers, underwriters, institutional investors, retail investors, market authorities and so on. The literature states the superiority of one method considering one of this point of view, but not all simultaneously, considering the presence of

[2] In this case, it is computed as

$$\log \frac{\text{stock market clearing price}}{\text{IPO price}}.$$

interests' conflicts. Each candidate profile has a best specific mechanism in response to their own needs:

- Issuing Firm. The book building method is suited for young, unknown and risk companies thus these companies need someone accompanies them in the listing and furthermore it is difficult to acquire information for investors under these conditions, getting into a under-subscription. Auction and FPO mechanisms are more suited for mature companies (Chemmanur and Liu, 2003)[10]. The auction is preferred by well-known firms having sufficient attraction not to have recourse to a firm commitment or a fixed priced contract. In this case, in fact, the issuer has the sureness that the auction will be fine and the investors, both retail and institutional, will be able to evaluate the firm value and so given a price for the shares.

- Investors. The institutional investors are better advised to use the fixed price offering or the book-building method. With a FPO, the flotation price is often more undervalued in relation to the basic share value than with an auction procedure and, since all orders are taken and rationed in proportion to the quantity demanded, large investors are assured of getting shares with good profit potential. With the book-building method, professional investors could hope to be rewarded for services provided to the underwriter (e.g. regular share subscription, honest disclosure of their buying intentions, price support).
  For the retail investors, in the majority of the case, the book-building procedure is closed and in the fixed price offering they are penalized for the amount of order.
  At the same time, they need enough information to evaluate the shares. So, for them, the best method is a mix between auction and bb in the way in which they minimize the risk of overvaluation and a fair allocation rate.

- The underwriter. The book-building method results to be the best procedure for the underwriter to which it gives great control over the IPO process as well as larger placing and brokerage commissions. At the contrary, in the other two case, its role is marginal and also its commission.

Conclusion

This chapter, as already amply stressed in several points, aims to provide first of all an overview of the different strands of historically formulated explanatory theories to explain the phenomenon of IPO Underpricing. Secondly, the objective is to understand if they can find an unequivocal response to the phenomenon of underpricing and the evolution of the book building.

To proceed in this direction, we have provided a clear and orderly overview of theories and a description of the listing process. It has shown a plurality of theories: information asymmetry, control ownership, procedural theory, behavioral theory. In the next chapter, through a descriptive analysis of the data, we will try to understand if the theories apply to reality and if they can give an answer the initial questions.

Chapter 2

The evolution of Ipo Process

Up to this point of the present work we have focused on illustrating in detail the characteristics of the IPO Process, while at the same time offering an in-depth overview of the studies that have investigated the causes. The contribution of external academics and researchers, who have attempted to unravel Underpricing through their papers, has undoubtedly been significant, but not sufficient. So, here we analyze the trend of the domestic and foreign market.

The chapter is organized as follows:

- The first section describes the Ipo in the history from 1994 to 2004, with a particular accent on the Italian situation. In this description, there is the evidence of the Italian market to follow the trends of the US market;
- In the second section. there is a description of the main financial market both Western and Eastern, considering the main peculiarities of each of them. Here, there is a careful analysis of the overpricing and underpricing peaks.

Certainly, to analyze the various IPO procedures used in Europe, it was difficult to identify the identical listing mechanisms for which the terminology differed from one country to another. A clear codification of IPO procedures, available to broadcasters in Europe, would facilitate the process of opening up the public on a European scale.

2.1 The IPO in the history

Auctions were used in Italy, Portugal, Sweden, Switzerland and the UK in the 1980s, and in Argentina, Malaysia, Singapore and Turkey in the 1990s, but they were abandoned in all these countries well before book building was introduced. Auctions were required for many years in Japan, yet quickly vanished once book building was allowed. In France, auctions were popular in the first half of the 1990s. On the regulated exchanges, they gradually lost market share to a restricted form of book building over several years, then dried up quickly in 1999 when a more standard form of book building was allowed.

In Latin America, auctions have been used in the majority of the case in Brazil and Peru. This last market was very quiet for much of the last decade, with delisting outnumbering listings in Brazil, Argentina and Chile. Thus, out of 46 countries, auctions have been tried in more than 20, and yet all except France, Israel[1], Taiwan and the US[2] seem to have abandoned them entirely, and auctions are rare events in these last four countries. Book building gains in popularity and it is the dominant method in 34 of the 46[3]. Fixed price public offer is still used in smaller countries and for smaller offerings, and is common for the retail tranche of hybrids.

The panel shows the context from 1994 to 2004.

It underlines the increasing popularity of the book-building method and the market decline in varying degrees of the other two procedures. From 1995 to 1997 a little more than 60% of flotation was carried out through the book-building method, while this percentage rose to over 90% for the 1998-2000 and 2001-2004 periods. Conversely, since 1995, fixed price offerings and auctions have fallen increasingly out of favor: together they represent less than 5% of flotations for the 2001-2004 period as against nearly 40% for 1995-1997. However, the auction method seems to have held up better than fixed price offerings against book-building during the speculative phase. Its capac-

[1] In Israel, the auction was the only method until 2003.

[2] In 1999, WR Hambrecht introduced the OpenIPO auction mechanism in the United States to compete with the book-building approach, which effectively had complete control over IPO issuance. Between 1999 and 2007, WR Hambrecht was the lead underwriter in 19 auctioned IPOs.

[3] The sample is composed by Austria, Belgium, Czech Rep., Denmark, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Luxembourg, Netherlands, Norway, Poland, Portugal, Spain, Sweden, Switzerland, U.K., Argentina, Barbados, Brazil, Canada, Chile, Mexico, Paraguay, Peru, U.S., Australia, Bangladesh, China, Hong Kong, India, Indonesia, Japan, Korea, Malaysia, New Zealand, Philippines, Singapore, Sri Lanka, Taiwan, Thailand, Kenya, Israel.

Table 2.1: Ipo in the history

				Total Period		*Sub-Period*		
	1995-2004		*1995-1997*		*1998-2000*	*2001-2004*		
	Number	%	Number	%	Number			
FPOs	139	7.67	71	23.28	60	5.31	8	2.12
Auctions	96	5.30	46	15.08	45	3.98	5	1.33
Book-building	1,577	87.03	188	61.64	1,025	90.71	364	96.55
Total	1,812	100	305	16.83	1,130	63,36	377	20.81

ity to limit initial underpricing during hot market periods could explain the fact that, without offering a serious challenge to the book-building method, it remained in relatively frequent use in the 1998-2000 period[4], even if the method has been abandoned. In Italy, there are no auctions in the last years. Many fixed price offerings were used in Turkey and many auctions in France. Particular is the result of Portugal and United Kingdom where the auction Ipo's are equal to zero.
For the other countries, the most popular method is Book Building.
The sample concerns the period of analysis before the Euronext and so the requirements to became public were different by countries. The differences in Turkey and France can be explained by the different legislation conditions of the market. For example, in both countries the auctioneer can be the stock market company and not only the lead bank. So this table isn't credible because the data are not harmonized.
Since 2003, European countries can be easily compared because they are subject to the same legislation. It is Directive 2003/71/EC transposed by the countries belonging to the EU. Called European passport, it indicates the prospectus to show to the public in the event of a public offering. In general, it envisaged new rules which would allow raising capital easily. Also, it has strengthened the protection for investors by a clear offer and complete statements to take rational decisions.

2.1.1 Italian situation

The Italian market is very poor of information about these procedures and its evolution. Indeed, the unique Italian studies are given by *Borsa Italiana* in few Bit Notes, [16], [15], [24]. This aspect is linked to another one: the majority of the Italian firms declared to have available scarce information on the characteristics of the markets, on the societies and on the operation of the process of quotation (4 enterprises on 5)[15].

Revision	1989-1994	1995-1997	1998-2000	Total sample
(a) binding range (99 IPOs)				
The IPO price is the maximum price of the range	-	9	19	28
The IPO price is comprised between the maximum price and the midpoint price of the range	3	7	21	31
The IPO price is the midpoint price of the range	1	2	3	6
The IPO price is comprised between the midpoint price of the range and the minimum price	3	13	11	27
The IPO price is the minimum price of the range	-	2	5	7

Revision	1998-2000	2001-2004	Total sample
(b) not binding range (57 IPOs)			
The IPO price is the maximum price of the range	2	-	2
The IPO price is comprised between the maximum price and the midpoint price of the range	6	2	8
The IPO price is the midpoint price of the range	-	-	-
The IPO price is comprised between the midpoint price of the range and the minimum price	7	12	19
The IPO price is the minimum price of the range, or lower than the minimum price	9	19	28

(1) Sample: 156 IPOs on the Italian Exchange from 1995 to 2004, with a price range filed in the prospectus

Figure 2.1: Process of pricing

Despite the unclear information, the underpricing has been decreasing showing a better efficiency of the pricing process. The price process has deeply changed during the last years, from the auctioned Ipo disappeared in the late 1980s and the fixed price offering disappeared in the mid-1990s until the listing of Enimont[4] in 1989. With this firm, there was the publication of a range price in a document with a lower and upper bound. It was a pioneer of the Book-building method. In 1999, there was another innovation: for the first time, a firm was listed without the publication of an offer price, it was the Banca Monte dei Paschi di Siena. The retail investors didn't know the price was binding.

The Italian market shows a tendency towards that one of United State market where the going public process occurs with the book-building method. Indeed, Del Favero (1986), Faema (1986) and Finarte (1986) were the last Ipo auction.

[4]Enimont S.p.A. was a very important Italian firm, result of the most relevant fusion and alliance between the public chemistry and that one private.

2.2 Description of Markets

The principal exchange markets have different mechanisms. Following, for each one, there is a description of the procedure:

- Hong Kong. The permitted listed methods are:
    * Offer for Subscription: An offer to the public by, or on behalf of, an issuer of its own securities for subscription.
    * Offer for Sale: An offer to the public by, or on behalf of, the holders of securities already in issue or agreed to be subscribed.
    * Placing: The obtaining of subscriptions for, or the sale of securities by, an issuer or intermediary primarily from or to persons selected or approved by the issuer or the intermediary.
    * Introduction: An application for listing of securities already in issue where no marketing arrangements are required.
- London. The permitted listed methods are:
    * Public offering: GDRs[5] are offered to institutional investors and listed on an international stock exchange outside the issuer's home market. The offering will be underwritten by an investment bank.
    * Listing: Issuers can list their GDRs without raising capital known as an introduction. By listing, issuers will have access to a wider group of institutional investors as well as increasing their visibility and name recognition
    * Placing (or private placement): A placing is usually a more selective process whereby GDRs are offered to a small number of selected institutions. While this route gives the issuer more control over the distribution, it can restrict the shareholder base and naturally limit liquidity.
- Thailand. There are two main methods of underwriting, they are as follow:
    * Firm underwriting: The underwriter must sell all securities. Failing to do so, the underwriter shall buy all unsold/undistributed securities. This way, the company can ensure that the full amount of capital can be sought.

[5] Global Depositary Receipts (GDRs) are negotiable certificates issued by depositary banks which represent ownership of a given number of a company's shares which can be listed and traded independently from the underlying shares. These instruments are typically used by companies from emerging markets and marketed to professional investors only.

    * Best effort underwriting: The underwriter uses its best efforts to sell as many securities as possible. When the underwriter is unable to distribute/sell all shares, the underwriter isn't obliged to buy the unsold securities. Normally, the fee for this method is lower than that for the first method.

- Euro Paris. The sale of shares in an IPO may take several forms:
    * Initial Public Offering: The equity securities of the company are offered for subscription to institutional and retail investors. The IPO process is used when the company seeks to raise capital. This type of listings is used when the company offers its equity securities to the public. For a Public Offering of equity securities, a prospectus approved by the Competent Authority is required.
    * Private placement: It involves an offering of equity securities to a selected base of professional/qualified investors. An approved prospectus is not required except if the equity securities are admitted to listing and/or trading on a Regulated Market. For the first admission to trading on Alternext, the company must provide the Listing Execution team or the EMS CA team with an information document, which must be drafted in accordance with the same standards as those for a prospectus.
    * Direct listing: A direct listing simply consists in making securities tradable (no capital raising). An approved prospectus is not required except if the company seeks for a listing on a Regulated Market. Notice that when the company's equity securities are already admitted to trading on another EU-regulated Market for more than 18 months, the EU Prospectus Directive's pass-porting process (a facility for issuers wishing to publicly offer securities or admit securities to trading on a Regulated Market in a Member State other than their home Member State's Regulated Market) is available.
- Nasdaq. The common methods include:
    * Best efforts contract: The underwriter agrees to sell as many shares as possible at the agreed-upon price. All-or-none contract: contract the underwriter agrees either to sell the entire offering or to cancel the deal.
    * Firm commitment contract: The underwriter guarantees the sale of the issued stock at the agreed-upon price. For the issuer, it is the safest but the most expensive type of the contracts, since the underwriter takes the risk of sale.

        * All-or-none contract: Contract the underwriter agrees either to sell the entire offering or to cancel the deal.
        * Bought deal: Form of financial arrangement often associated with anInitial Public Offering. It occurs when an underwriter, such as aninvestment bankor a syndicate, purchasessecuritiesfrom anissuerbefore a preliminary prospectus is filed. The investment bank (orunderwriter) acts asprincipalrather thanagentand thus actually *"goes long"* in the security. The bank negotiates a price with the issuer (usually at a discount to the currentmarket price, if applicable).
    - Nyse. The kind of listings are:
        * Listing in conjunction with an initial public offering (a classical Book-building procedure).
        * Listing in conjunction with a carve-out transaction: A carve-out is the partial divestiture of a business unit in which a parent company sells minority interest of a child company to outside investors. A company undertaking a carve-out is not selling a business unit outright but, instead, is selling an equity stake in that business or spinning the business off on its own while retaining an equity stake itself. A carve-out allows a company to capitalize on a business segment that may not be part of its core operations.
        * Listing in conjunction with a spin-off transaction: A spinoff is the creation of an independent company through the sale or distribution of new shares of an existing business or division of a parent company. A spinoff is a type of divestiture. Businesses wishing to streamline their operations often sell less productive or unrelated subsidiary businesses as spinoffs.

All these markets are described by a panel collected the date of the interval 2010-2014. In this way, the Italian scenery can be analyzed with respect the evolution of the principal market of the world, both Western and Eastern and analyzing both the vibrancy of the national market that the proceeds are used.
The period is the interval 2010 - 2014. The decision about the period is linked to the fact that I would drop the influence of the economic events of the crisis in 2007 which represents a period of high uncertainty on the financial market. It is large enough to obtain an amount of information such that the anomalies will be negligible.
The sample derives from the database of Thomson Reuters Eikon and by the stock market websites.

The operations will be divided according to:

- Equity Deal Number;
- Offer Price;
- Offering Technique;
- Price 1 Day After Offer;
- Listing Date;
- Exchange of Listing.

2.2.1 Description of the data

I have micro panel data with a number of observations $N = 903$ and with T=5 (t_0=2010,t_1=2011,t_2=2012,t_3=2013,t_4=2014) subdivided as follows:

- In 2014, we have 324 observations;
- in 2013, we have 262 observations;
- in 2012, we have 140 observations;
- in 2011, we have 169 observations;
- in 2010, we have 222 observations;

From the data of Thomson Reuters Eikon, the initial public offerings are subdivided in:

- Offer for Subscription: A method of selling shares in new issue. The issue is aborted if the offer does not raise sufficient interest from investors.
- Offer for Sale: A method of bringing a company to the stock market by selling shares in a new issue. The company sponsor offers shares to the public by inviting subscriptions from investors:

  a. Offer for sale by fixed price - the sponsor fixes the price prior to the offer.
  b. Offer for sale by tender - investors state the price they are willing to pay. A strike price is established by the sponsors after receiving all the bids. All investors pay the strike price.

  A prospectus, containing details of the sale, must be printed in a national newspaper.

- Firm Commitment: In a firm commitment, underwriters act dealers and are responsible for any unsold inventory. The dealer profits from the spread between the purchase price and the offering price Also known as a "firm commitment underwriting".
- Placement: A placement is the sale of securities to a small number of private investors that is exempt from registration with the Securities and Exchange Commission under the Regulation D as are fixed annuities. This exemption makes a placement a less expensive way for a company to raise capital compared to a public offering. A formal prospectus is not necessary for a private placement, and the participants in a private placement are usually large, sophisticated investors such as investment banks, investment funds and insurance companies.
- Best efforts: Best efforts relieve underwriters from responsibility for any unsold inventory if they are unable to sell all the securities.
- Negotiated Sale: In a negotiated sale, the underwriter, selected by the issuing entity before the sale date, will perform the financing for the issue. Lower quality issues generally reap the greatest benefit from this type of underwriting technique as the underwriter works with the company to sell the issue to the market. When the underwriter and the issuer work together to clearly explain the issue, they will often receive a better rate in the market for the issuer. Negotiated sales allow for greater flexibility to when the issue is released so that it can be better timed in the market to get the best rate.
- Open Offer: Open IPO offers to sell unseasoned shares through a Dutch auction. In its advertising (for example on its website: www.openipo.com), it emphasizes that the use of this standard uniform price, market clearing auction ensures that IPO offering prices are fixed by the market and reflect what people are truly willing to pay for the stock. It also advises that investors should make a bid at the maximum price at which they are comfortable owning shares of the issue.
- Auction: The seller sets a reservation price p, the investors submit demand functions, and the IPO price is set in order to equate supply and demand.
- Wit Capital: It follows a somewhat more standard strategy. It seeks to entice individual investors to bid in IPOs, but it does not contribute in a major way to the price discovery mechanism. The latter is in large part operated by the lead managers of the IPOs in which Wit Capital participates. These lead managers, which are major investment banks, such as Goldman Sachs, Morgan Stanley, or Merrill Lynch, determine

the IPO price based on the traditional Book Building process, which is more focused on large institutional investors.

The several methods have to be classified or almost reconnected to the two principal mechanisms: BookBuilding and Auction. The fixed price offering is included in the first one type. This decision is made in order to divide the procedure following the discriminant of insider/outsider method:

- It is an insider method when the formation of the price is defined by the agents involved in the decision to go public (bank and issuer).
- It is an outsider method when the formation of the price is defined by the market: agents not involved in the decision to go public, but only affected.

The UNDERPRICING is defined:

$$\frac{(p_{i1} - p_{i0})}{p_{i0}}$$

where p_{i1} is the price the first day after the Ipo and p_{i0} is the offer price.

2.2.2 Analysis of data

Table 2.2: Summary statistics

Variable	**Mean**	**Std. Dev.**	**N**
Underpricing	0.163	1.384	903
Number	$2.4*10^8$	$1.9*10^8$	903
Offer Price	2.016	5.608	903
Price 1Day Af	2.179	6.144	903

Table 2.2 collects the key variables described through the mean and the standard deviation,, showing a strong volatility of the First Price Day After. This value is clearly correlated to the underpricing and so it is more significant with respect to the volatility of the shares' number.

The BoxPlot describes the trend of the underpricing of all 903 observations. It shows an anomalous value in correspondence of an underpricing equal to 35$. It coincides with the Ipo of FSBancorp. Inc. Its issuer price was 10$ against the price of the first day after 45$. The firm listed on Nasdaq with the BookBuilding procedure. FS Bancorp, Inc. operates as a bank holding company for 1st Security Bank of Washington that provides banking and financial services to families, businesses, and industry niches in Puget Sound area communities, Washington. The company offers various deposit instruments, including checking accounts, money market deposit accounts, savings accounts, and certificates of deposits.

The Italian Market is less active with respect the others ones and it underlines only two cases of a strong underpricing:

1. The *Powszechny Zaklad Ubezpieczen SA* is a traded insurance company and is one of the largest financial institutions in Poland. It is also one of the top insurance companies in Central and Eastern Europe. Its offer price has been 105,975 against the 1 day after price 113,373. The strong difference can easily be explained: the company had many interests in the success of the operation and putting the investors in a favorable situation given his position of financial intermediary. Indeed, usually, the financial institutions are eliminated from the analysis. The reason is that the depositors often have the right to purchase the shares with priority. More, being customers of the bank, they have a lower Ipo price.
2. The *Brunello Cucinelli* is an Italian fashion firm. Its great underpricing is justified with the strong relationship that Brunello Cucinelli had developed with the banks. Indeed, also as private company, he illustrated the business budgets to the banks that sustained him [12].

The Hong Kong market shows a large number of initial public offering, with a very tight range $[-0.3; 1]$. Indeed, the strongest underpricing reaches the value 1, very distant from the maximum of the Western world.

The Nasdaq exhibits a significant underpricing with the RocketFuelInc company in 2013 [28], [27]. Its offer price was \$29 against \$59, 95 for share the first day after the initial public offering. In this case, the success didn't explain through particular relationship with banks, indeed the unique possible reason was that the interesting market of the public company: the A.I. and the strong growth of the firm in the last years.

The Euronext is a quite equilibrated stock market. It presents a small range in which there are underpricing and overpricing situation in the same size. Mauna Kea Technologies obtained an underpricing equal to $6,186$. It is a medical device company and it was at the forefront of medical innovations, a true pioneer. These features explain why there was an underpricing so strong. The market exhibits also an overpricing with a French start up Viadeo, the biggest online network for job seekers in China and in France. Its competition with LinkedIn could have generated a failure and so, for having 171 million euros to capitalization, it decided to increase the final price.

Thailand is the more stable market of all presented. The range is $[-0,089; 0,797]$ for all the operation.

The London Exchange Market has an underpricing very moderate. The greater underpricing was only of $\$3,323$ for Bet-fair Group Ltd. It was an online gambling company founded in 2000.

In the New York Stock Exchange, the OaktreeCapitalGroupLLC lived a particular situation: an undersubscription of shares and an underpricing. It is specialized in alternative investment and its peculiarity maybe scared the investors. This event shows that a strong underpricing is not an assurance against the unsold shares.

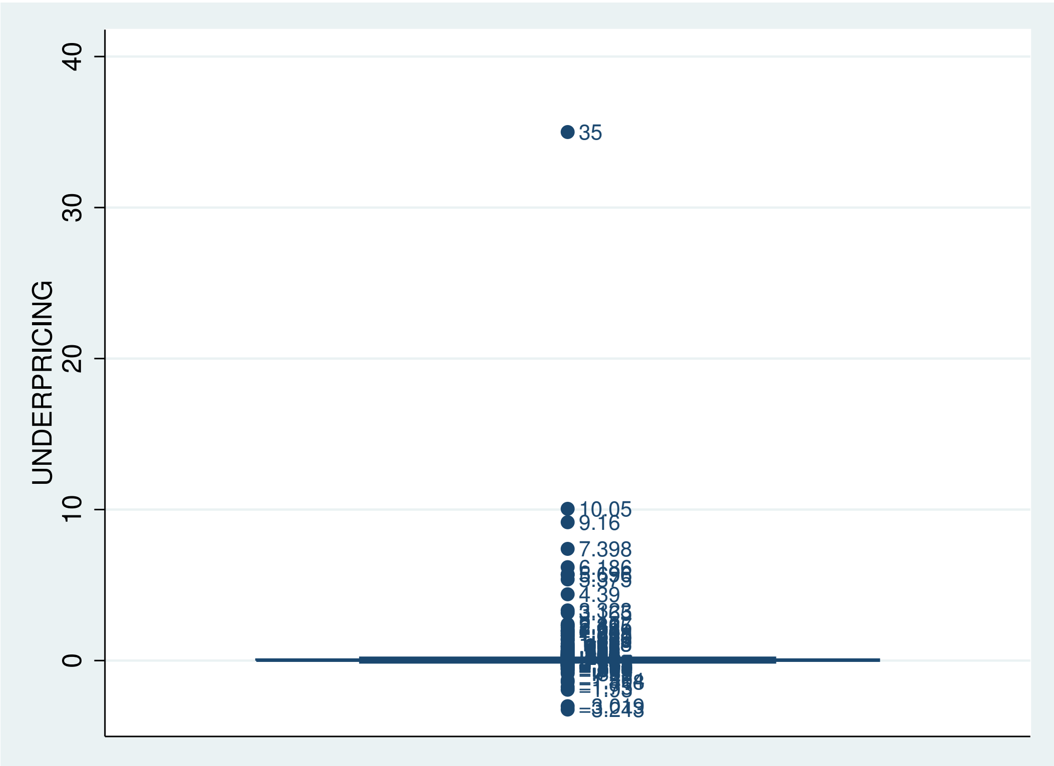

Figure 2.2: Box Plot Underpricing

Figure 2.3: Underpricing in Milan

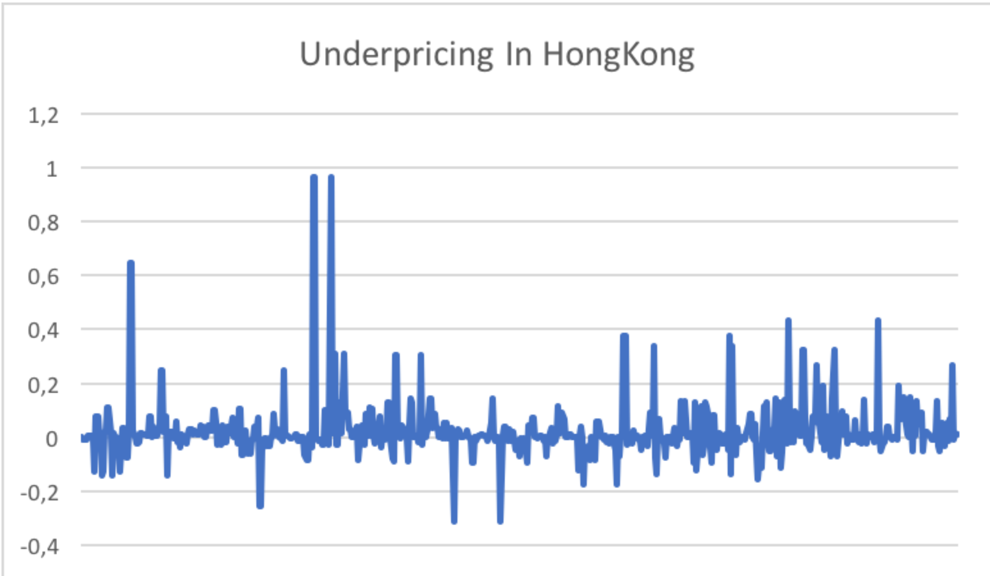

Figure 2.4: Underpricing in HongKong

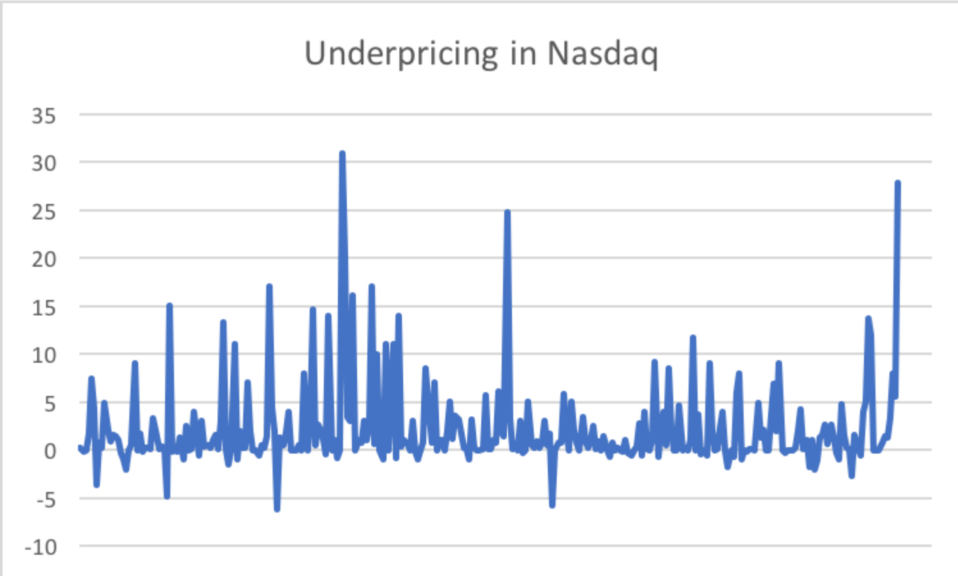

Figure 2.5: Underpricing in Nasdaq

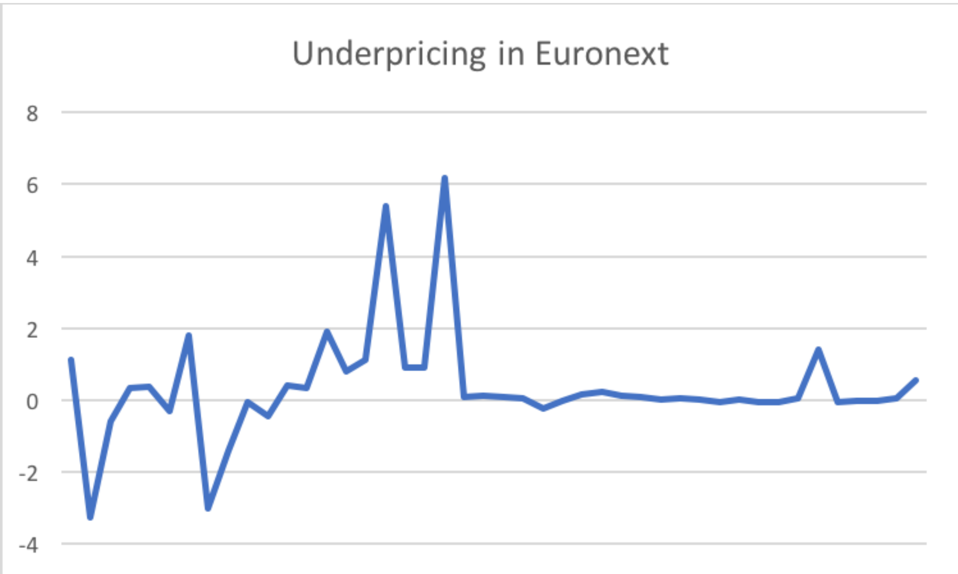

Figure 2.6: Underpricing in Euronext

Figure 2.7: Underpricing in Thailand

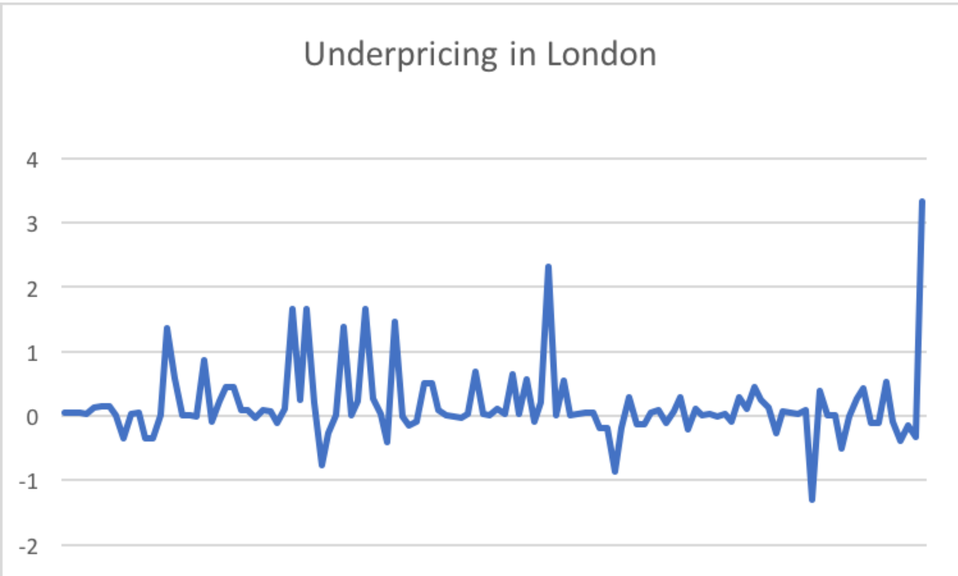

Figure 2.8: Underpricing in London

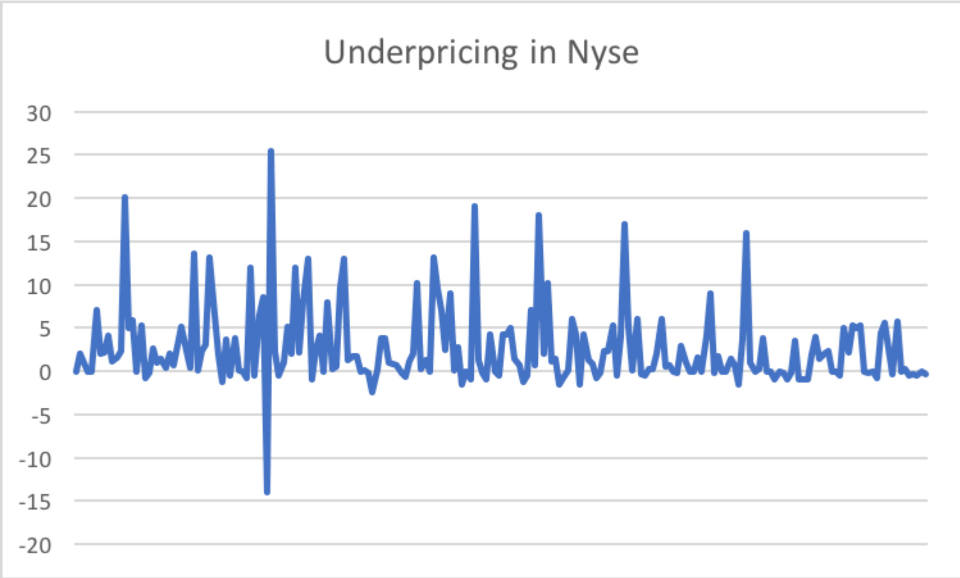

Figure 2.9: Underpricing in Nyse

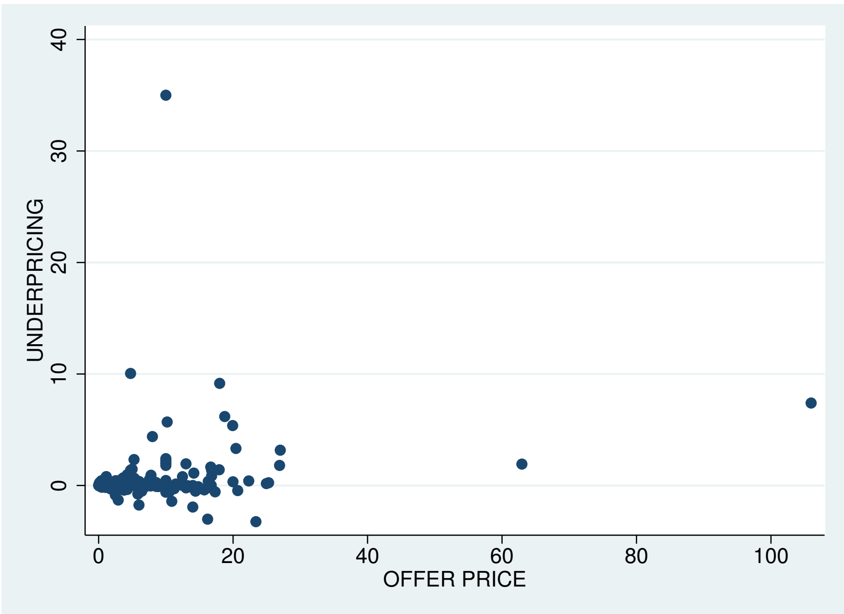

Figure 2.10: Underpricing in relation with Offerprice

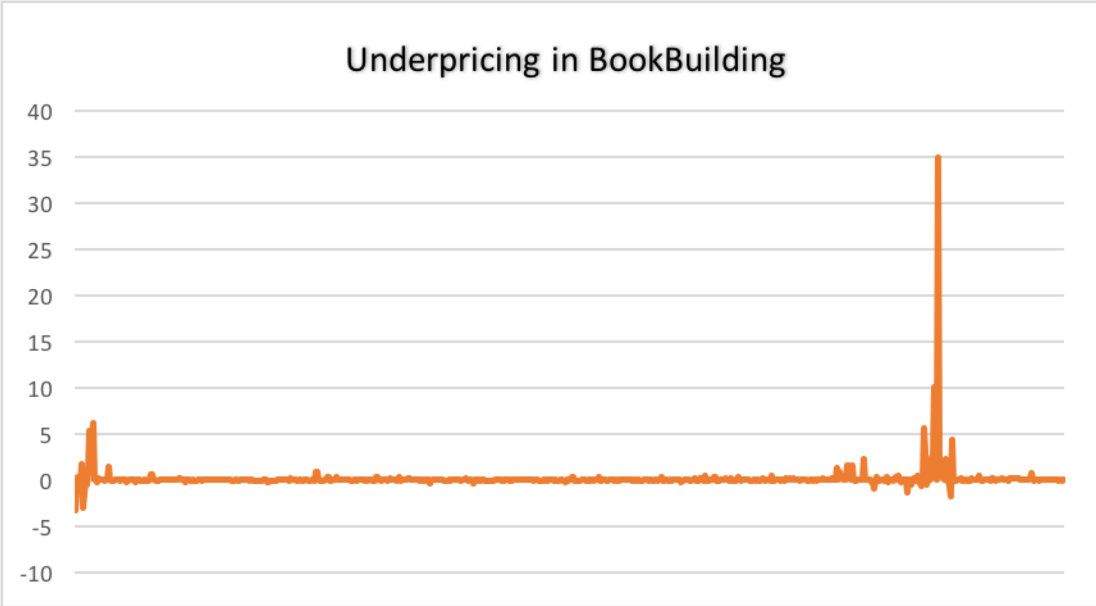

Figure 2.11: Underpricing in Bookbuilding

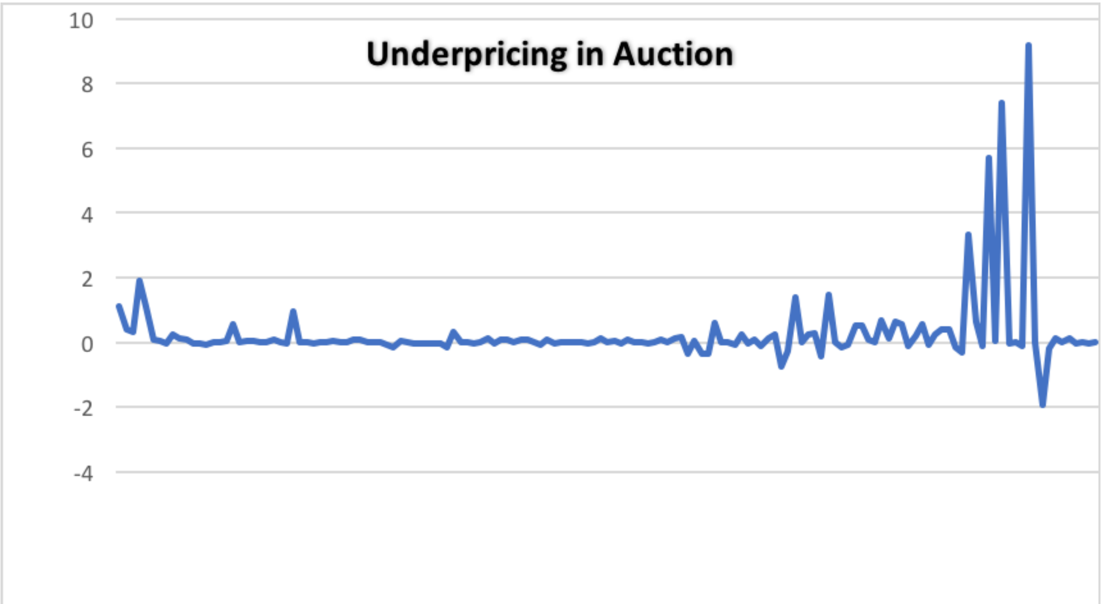

Figure 2.12: Underpricing in Auction

Conclusion

This chapter shows that the topic is very controversial. The literature doesn't show any final statement in favor of a procedure, indeed, each theory has some lacks because none, at the same time, considers all aspects.
Unfortunately, our data doesn't explain what is the best mechanism thus the number of auction is very limited in proportion to the book-building procedure. On 903 observation, only 149 firms use the auction mechanism to become public, the others use the placement method.
The table 2.3 that the book-building method is better in terms of mean and standard deviation, even if it shows some strong maximum. The problem of the data is the low number of auctions in the last years. For this reason, we overcome the empirical analysis to a theoretical model, explained in the following chapter.

Table 2.3: Summary description

Underpricing	**Mean**	**Std. Dev.**	**N**
Auction	1.373	1.182	149
Book-building	0.149	0.477	754

The analysis of the theories and the evolution of the Ipo did not provide any significant information on the phenomenon of underpricing and on the superiority of one procedure over another one. For this reason, the next chapter will try to give an answer on what is the best method using another approach.

Chapter 3

The best Ipo mechanism

The aim of this chapter is to develop the resolution to the problem of the best Ipo method, based on the theoretical analysis of the previous literature review. There are two principal problems of the listing process: the maximization of the proceeds of the firm and the efficient sale of the shares to the investors.

The chapter is subdivided in the following key points:

– The first section describes the features of the revenue function for the firms in each mechanism. The decision variable is p the offer price for a unit share. The revenue function is analyzed also for what concerns the risk degree. Each mechanism has a particular function deriving from the loving or averse risk function. The risk degree is evaluated both from the supply side and from the demand side, considering, obviously, their own peculiarities;

– The second section incorporates the cost of an initial public offering and it works out the profit function. The key configuration cost considered is the informational one, keeping in the dark the others kind of cost;

– The third section re-works the considerations of the first and second section and it presents the model. The focus is on the construction of a model which makes endogenous the information cost structure. It refers to an optimization problem with the participation constraint of the investors. From its resolution, we get the *indifference condition*;

– The last section focuses on the main open questions and the results of the model.

At the end of this chapter, the success of the book-building will assume another lecture with respect to the previous theories.

3.1 The revenue function

Several academics, as seen in the previous paragraph ([13], [19], [18]), studied and are studying the Ipo mechanism. A critical analysis of their works allows me to find some significant differences between the fixed price offering/book-building[1] and the auction.
The main difference lies in the price determination. Its definition is an essential phase. Indeed, it must satisfy a plurality of subjects, unfortunately bearers of conflicting interests: the investors, the issuer and the underwriter. Furthermore, it represents the value of the company that is going to be listed, a value that encompasses the profitability of its future strategies and the ability to generate sufficient income to satisfy potential shareholders.

For the previous considerations, I'll decide to build a model in which the price has the central role and the several listing mechanisms can be divided into two groups: the insider and outsider ones, according to the price setting. The determinant is the endogeneity or the exogeneity of the final price, i.e. if it is defined by an internal subject or by the market. Based on this criterium, the procedures are subdivided as follows:

- The book-building procedure and the fixed price offering are considered an insider method: the price is specified by actors internal like the issuer firm or hired by it. Indeed, it is not a question of the easy equilibrium price resulting from the meeting between the demand and the supply as usually occurred in the goods' market. The decision could be due to the fact that only the internal actors like the issuer firm and the underwriter bank know the real fair value of the shares, contrary the demand side could consider some aspects misleading of the reality[2].
- The auction is considered an outsider method since the price is specified by the market. The mechanism conducts a transparent definition of price. On the other hand, it leads the investors versus two negative phenomena: the winner's curse and the free-riding problem and therefore the price could suffer some situation totally unrelated to what concerns the firm's value.

Starting from these conditions, let see in more depth way, the price definition and how it influences the revenue value.

[1] In the following pages, the two methods are joined under the same word: book-building.

[2] In a period of crisis, investors could underestimate the company, with the risk for it to obtain a too low profit. Or, at favorable times of the market, they could overestimate the company, remaining subsequently disappointed causing a fall in the price of the shares.

In the insider procedure, the revenue is given by $E = p * Q$ where p is the unitary price of a share and Q is the number of the shares listed on the market. The price is a determined value, deriving from analytical methods such as the Discounted Cash Flow, the Eva or in a totally discretionary mechanism.

Unlike in the auction method, the revenue depends on the value V that the bidders assign to the firm and it is equal to $E(V, p) = p(V) * Q$. So, in this case, the determination of the price depends on the valuation of the investors, external subjects that carry out personal analyzes on the solidity and profitability of the deal and then, they identify a monetary value for the share.

What however does not change within the revenue analysis, is the number of shares proposed to the market. There is no indication that the listing method can justify a variation on the shares' quantity.

In both cases, investors have the same treatment. Everyone pays the same final price p and it is a blind procedure in which everyone moves simultaneously assuming the impossibility of tacit or explicit collusion.
Given these features, the revenue of the issuer in the two cases is respectively:

$$E_{bb} = p * Q \quad \text{for BB.}$$

$$E_a = p(V) * Q \quad \text{for Auction.}$$

Sometimes, the auctioned firm may impose a reservation price p_r to protect itself from a too low profit: $p(V) \geq p_r$. That means the revenue structure becomes:

$$\begin{cases} p(V) * q & \text{if} \quad p(V) \geq p_r \\ 0 \quad \text{otherwise} \end{cases}$$

The valuation V of investors is the same in the two procedure because the mechanism doesn't affect their beliefs on the firm.

3.1.1 Risk propensity

The risk degree of the two procedure is different The decision-making process of actors, combining different behaviors, leads to a different risk attitude On this aspect, there are two lines of thought.
A first line, with main exponents Chemmanuer and Liu[10], considers the two mechanisms as follows:

i. Risk averse in the auction mechanism. Auction will be preferred by well known firms having enough attraction for not recurring to a book runner or a fixed price contract. Indeed, if an issuer isn't enough famous on the financial markets, there is a high probability to fail.

ii. Risk loving in the book-building mechanism. This method is suitable for young and high risk companies, needing to be guided and accompanied during the listing procedure.

On the other hand, the second line of thought, given by Sherman[18], makes different assertions:

i. Risk loving in auction method. The firm doesn't have references about the investors, their liquidity and solvency and there is a lack of information about the number of participants. All these aspects increase the uncertainty of the success and so the connected risk. Therefore, the firms, use this listing method, are risk seeking;

ii. Risk adverse in book-building mechanism. The firm has all the information about the investors and the number of participants. The analysis is conducted by an investment bank which possesses all the capabilities to make an accurate elaboration of truthful data. This analysis has a cost $C > 0$ that may be identified with the contract cost.

Aligned with the firms' vision, there is a change also in the investors' perceived risk degree. Their perceptions changes with respect the quantity and quality of information about the firms:

i. In the BB method, the information about the solidity of the company are fixed up from the book-runner, after numerous studies of the market and an analysis of the balances sheet. The bank has all the abilities to pick up a truthful analysis thanks to a specialist Ipo group;

ii. In the auction method, the analysis of firm's solidity must be conducted by the investors, which, with difficulty, have to elaborate data for understanding the real value of the issuer. Notice that, the bank, in this case, has a marginal role (see the first chapter) and the investors make autonomously an evaluation. This self-analysis has a cost $C_a \in [\underline{c}, \overline{c}]$ and it is strictly positive.

At the light of that, let us conclude that the auction mechanism has a bigger risk to fail for the uncertainty about the investors evaluation. So, summarizing my conclusion about the risk propensity, let us obtain:

- The auction mechanism is more hazardous. It is a method intrinsically influenced by the information of the bidders: the winner's curse and the free rider are the biggest problems of the procedure success. But, in the Ipo auction, this risk is amplified because of financial markets volatility. It has the power to modify easily and quickly the price for share.
- The BB mechanism has a lower risk. The firms and the investors are accompanied by all the steps by a specialized figure. In this way, both agents are confidants: the issuer firm knows that its offer price reflects the market demand and it is protected by unsold shares with particular underwriting contract; while the investors believe that the Ipo price reflects the real solidity of the firm.

3.2 The profit function

To formulate our problem, in addition to the revenue function, it's necessary to design the features of the profit function. Indeed, previously, the focus was only on the revenues leaving out the Ipo's cost.
The issuing firm, both in auction and in book building, has to support several costs which can be resumed into three groups [14]:

- Advisory counsel. An Ipo is a very complex process. It involves interaction with stock markets and securities agencies to change the firm's legal incorporation and to apply for the shares' listing.
- Filing and listing fees. Any firm must pay a fee to the stock exchange market. More, it is obliged to pay an annual fee to remain listed.
- Corporate resources. The initial public offering is the result of strong and hard work of several months prior to the offer date. The resources for directing public companies are higher than those ones of private companies in terms of management's skills, time and wage.

The previous expense items won't be included in the analysis of the profit function because they are common to any firm whatever it is the listing procedure[3].

[3] The cost can be separated also in two kinds: direct and indirect. The direct cost are standard cost and they are represented by fees, sales commissions, fees for legal advice as well as for auditing, expenses for accounting service, transaction costs. On the other side, there are the indirect cost. They are much more difficult to quantify, if compared with other costs, purely monetary. They are correlated to the the time spent by management in assisting step by step the whole procedure.

For our analysis, the relevant cost is the information cost which differs between the two kinds of procedure in the figure that sustains it. The information concerns the features of the public firms. It includes the balance sheet, the rating, the main operations, the structure of ownership, the solidity, the liquidity, the competitive position, the corporate purpose, the consumer target and the forecast on the profit after the listing. So, they are all the information considered relevant not only related on the characteristics of the offer, but also on the economic-financial position of the company.

In the book-building, the issuer firms recruit a staff to make these analyses, signing a contract with a bank-runner under the payment of a fee C. The underwriter draws up a document with all the tax, legal and financial information necessary for fulfilling the Due Diligence function and then he makes available the information prospectus. The cost C is a parachute for the issuer. The greatest risk: unsold shares falls on the underwriter. The issuer firm won't then worry about the subsequent disposal of the shares to the public of investors: if the sale fails in this sense, the weight of this failure will fall completely on the underwriter.

In the auction, the procedure information is not charged to the issuing firm, but to the investors. They must independently collect information on issuer financial soundness to propose an offer that represents in a truthful and realistic way the value of the company.

However, some information is made publicly available. It is a practice that the business strategies are announced as well as the motivations underlying these choices in a document. It must be submitted to the supervisory authority that manages and controls the market (in the case of US, the judgment is entrusted to the SEC, in the domestic context it is the Consob) and must be made public before the offer and the issue of the security, so that investors can use it as a decision tool [Borsa Italiana S.p.A. (2001)].

So, for the auction case, the firm has only to define the shares whole, the reservation price and to respect the legal obligations, it avoids to spend other resources[4]. To the light of these considerations, the profit for the bookbuilding firm and for the auctioned one are respectively:

$$\pi_{bb} = p * Q - C$$

$$\pi_a = p * Q$$

[4]Notice that in our assumption, the auctioned firm is more risky because it doesn't hire any bank to support itself along the procedure.

3.3 The dual problem

The model considers the views' point of the issuer firm. It assumes homogeneity of the firms that participate to the Ipo's and a homogeneity within the investors. The maximization problem consists in an objective function, represented by the firm's revenue and in an investors' participation constraint[5]. They participate to the listing mechanism if their benefits are greater or equal than the cost of participation.

$$p \leq V - \underline{u} \quad \textit{BB participation constraint.}$$

$$p \leq V - C_a - \overline{u} \quad \textit{Auction participation constraint.}$$

The requested above notations represent:

- p is the unitarian price. It is the monetary expenditure for buying one unit share;
- V is the evaluation of the shares from the bidders (or investors);
- u is the reservation utility[6]. In BB method, the notation is $\underline{u}$, while in the other one, the notation is $\overline{u}$;
- C_a is the information cost. It is inserted in the participation constraint because it may be prevented ex-ante, using its expected value. It can move into an interval $[\underline{c}, \overline{c}] \subset R_+$. The information cost is uniformly distributed and its probability density function and cumulative distribution function are respectively:

$$P(C_a) = \begin{cases} 0 & \text{for} \quad C_a < \underline{c} \\ \frac{1}{(\overline{c} - \underline{c})} & \text{for} \quad \underline{c} \leq C_a \leq \overline{c} \\ 0 & for \quad C_a > \overline{c} \end{cases} .$$

$$D(C_a) = \begin{cases} 0 & \text{for} \quad C_a < \underline{c} \\ \frac{C_a - \underline{c}}{(\overline{c} - \underline{c})} & \text{for} \quad \underline{c} \leq C_a \leq \overline{c} \\ 0 & for \quad C_a > \overline{c} \end{cases} .$$

[5]The participation constraint represents the constraint that must be satisfied for an agent to decide to participate or not to an economic process. Participation takes place when the agent obtains a utility at least equal to that which he would have if he did not participate in the economic process.

[6]The minimum level of utility that must be guaranteed by a contract to make it acceptable to an agent in contract theory

The profit of the firm is:

$$\pi = \begin{cases} p(V) * Q & \text{for auction} \\ p * Q - C & \text{for BB} \end{cases}.$$

In the auction, there are no cost or at least no relevant costs for the firms, in the BB method, the firm has to sustain the contract cost C with the underwriter bank.
The assumption about the investors' reservation utility captures the fact that the auction is riskier and the investors require a higher utility when they participate to the last listing method.

$$\overline{u} = \underline{u} + k \quad \text{with} \quad k \in R_{++}.$$

This specification is explained by the fact that the auction choice is riskier and so they claim a higher utility. Substituting in the constraint:

$$p_a \leq V - C_a - (\underline{u} + k).$$

The optimization problem of the firm in BookBuilding procedure is:

$$\max_{p_{bb}} log(p_{bb} * Q - C)$$
$$\text{subj. to} \qquad p_{bb} \leq V - \underline{u}$$

The profit π_{bb} is equal to the difference between the revenue $E_{bb} = p_{bb} * Q$ and the contract's cost C. Notice that we consider a concave utility function, risk averse.
The optimization problem of the firm in Auction procedure is:

$$\max_{p_a} (p_a * Q)^2$$
$$\text{subj. to} \qquad p_a \leq V - (\underline{u} + k) - C_a$$

Here, the profit π_a is represented only by the revenue $E_a = p_a * Q$ because we consider the other costs supported by the firm very low and so irrelevant for the analysis. Notice that it is a convex function, risk loving.
However, the form of functions is nevertheless irrelevant because both functions are increasing in p. This means that the optimization could be analyzed also under a common utility function, instead of the differentiation between concave and

convex form.

$$\begin{cases} log(p_a * Q) & \text{for auction} \\ log(p_{bb} * Q - C) & \text{for BB} \end{cases}$$

The solutions of the problems are respectively:

$$p_{bb} = V - \underline{u}.$$

$$p_a = V - (\underline{u} + k) - C_a.$$

The firm's revenue in the book-building is higher than in auction when:

$$E_{bb} > E_a \quad \text{if} \quad C_a + k > 0.$$

Since $C_a + k$ is always greater than zero, the firm always gains more with the first method than with the second one.
The information cost C_a is the crucial value to understand what profit is higher. It is linked to k, that is the difference of reservation utility $\underline{u}$ and $\overline{u}$. Wider is the difference, greater is the risk in the auction procedure.
Indeed, substituting the C_a with its expected value $\frac{\underline{c}+\overline{c}}{2}$ and omitting the real number, the inequality becomes:

$$\overline{c} + \underline{c} > \underline{u} - \overline{u}.$$

What is important for each firm, it isn't the revenue E, but the net profit π. Under this point, the conclusion could be different:

$$\pi_{bb} = V - \underline{u} - C.$$

$$\pi_a = V - (\underline{u} + k) - C_a,$$

The choice of firm is not more clearly unequivocal. Indeed, the firm choices the book-building procedure when:

$$\pi_{bb} > \pi_a \leftrightarrow C_a > k + C.$$

or the auction method when:

$$\pi_{bb} < \pi_a \leftrightarrow C_a < k + C.$$

The firm's profit is greater in the first case when the cost of the auction for the investors is greater than the difference of reservation utility and the contract's cost.
The two procedure are indifferent for the issuer when:

$$C_a = k + C \quad \textit{Indifference condition} \tag{3.1}$$

that is:

$$C_a - \overline{u} = C - \underline{u}$$

$$C_a - C = \overline{u} - \underline{u}$$

So, substituting the expected value of the cost of the information cost in the indifference condition, let's obtain:

$$\overline{c} + \underline{c} = k + C.$$

In any case, the information cost has to be compensated by the reservation utility, falling in the investors' cost. The model explains when a procedure is better than the other one. The key variables are the costs and the reservation utility. When the information's cost is too high, the firm will choose the book-building. This is a consequence of the fear to fail. If the auction has a high possibility to fail for lack of investors, the issuer will decide to list with a more reassurance method.

Conclusion

Under the indifference condition, we can conclude that any information cost is paid by the investors. Indeed, even if the firm pays the contract cost, it moves the price upward for covering it. Rearranging the indifference condition let us get the relationship between the prices:

$$k = p_{bb} - p_a - C.$$

Thus k is a strictly positive number for hypothesis:

$$p_{bb} \geq p_a + C.$$

The book-building price results higher than the auction price. The investors situation is unfavorable in both cases: or they pay their own information cost or they pay the cost for the issuer and the revenue of the bank runner: it is a negative externality of the relationship between the firm and the underwriter. Furthermore, it is an inefficiency of the market because the price doesn't reflect the value of the share as should be in a financial market.
Some implications of our model are exposed below:

- The model predicts that, under indifference condition, the equilibrium price of the firm is higher in the book building method. The gap between the placement price and the auction price is wider, higher is the contract cost of the bank runner.
- The model shows that the book-building offering price aggregates the information production cost and so it doesn't reflect the intrinsic value of the firm, but it involves some indirect procedural cost. On the other hand, a higher price reflects a strong positive signal of the firm to the market because the issuer availed himself of a very competent bank (considering the contract cost proportional to the bank's reputation).
- The firm doesn't have particular benefits of selling equity with the auction rather than the book-building method.
- The investors don't have incentive to acquire a low amount of shares in the book-building procedure because they prefer to buy a great amount to spread the information cost on the quantities. Therefore, the investors that take part in the placement method have a strong financial capacity.

Under these explanations, the exponential trend of book-building mechanism isn't justified because the firms would have the same profit also under the auctioned Ipo given some conditions. Thus, at a first glance, the book-building procedure is useful when the information asymmetry is significant. However, in this way the

price doesn't fulfill its role anymore: valorizing what it sells.
The best choice for the firm and the investors would be to transfer the information cost with a specific function to a third subject: a financial authority.

Unfortunately, this figure is still alive. Organizations such as Consob in Italy, SEC in U.S. are not able to ensure an efficient performance to the market. They shall have the responsibility of a sufficient information production.

Is it necessary that investors support a cost of information in the auction?

Probably, everything lies in the distrust of the market towards the supervisory bodies. Hence, since they are not reassured by the controls and analyzes carried out by the appropriate organisms, they decide to spend monetary resources, time and skills in the collection of information that should theoretically already be available.

Indeed, if the cost that the company sustains for the contract with the underwriter is justifiable in the fear of the bankruptcy of the company that intends to list, the information cost of investors cannot be justified in objective situations. It resides in their fear of being "cheated" by the institutions and they decide to "work on their own".

We could conclude that therefore the bookbuilding method is nowadays predominant with respect to the method of the auction, which instead pursues the law of the market, as a response of economic agents to fear. It overcomes the problem of unsold shares of the company and solves the problem of the lack of investor information by offering a complete and careful overview of the economic and financial situation of the company.

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Contents

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Abstract

This work considers an exchange economy with a finite number of agents and a finite number of goods where the agents are arbitragers. The general equilibrium theory borrows the notion of arbitrage to the finance for showing how the consumers move in an economy closer to the real world with uncertainty and differential information. The first chapter describes the notion of arbitrage, its evolution during the years through the eyes of several authors. The second chapter analyses the main aspects of a differential information economy, studying the concept of core and competitive equilibrium in the ex-ante, in the interim and in the ex-post stage. In the last chapter, the arbitrage condition becomes the guideline for defining new notions of competitive equilibrium when the economy is characterized by uncertainty and differential information.

Introduction

The Walrasian equilibrium has a long tradition in economic theory. One crucial concept is the auctioneer. Under this figure, the agents cannot influence the price system. The assumption implies that firms and consumers are price-takers and they accept the equilibrium passively. My work is directed to identify a new mechanism for arriving to a competitive equilibrium. To reach this goal, I'll use the concept of *re-contracting*.

One of the scholars discussing the limits of competitive equilibrium notion is Morgenstern. He pointed out 13 points of criticism in the classical economic theory [26]:

1. Control of economic variables. The equilibrium is obtained only through the control of some variables such as the profit, the utility, the cost. The economic theory maximizes the previous functions, not considering the interaction among agents and considering a complete control in any situation. But, empirically, it is clear that there are some decision makers that influence the maximization problem, i.e. the State.

2. Revealed Preference Theory. It asserts that it is possible to understand, by observing the goods' choice of individuals under a budget constraints, their preferences, i.e. if an agent has to compare the goods x, y, z and he chooses sequentially y, x, z, he reveals $y \succ x \succ z$. But, this assumption says nothing about the nature of the three goods. Indeed, if one of the previous items is a durable good and the other two ones are not, the situation is totally different because the choice is influenced by this aspect: for instance, one item may be acquired for first only because it is cheaper than the durable good, not because it is preferred. Furthermore, if the agent buys all the goods, the order of acquisition could be a consequence of his proximity to these ones. For these reasons, the revealed preference theory has some lacks if it is applied to the real world.

3. Pareto Optimum. The idea of Pareto is that an optimum of a society or of a group of individuals may be reached when, by improving the utility of one

individual, no one else's position is deteriorated. So, Pareto compares the utilities of different individuals and the exchange of commodities. The limits are:

a. No information is given about the nature and the amount of trades, i.e. some goods could be available only until a maximum amount or there could be some restrictions about their trades;

b. The agents could not be honest about their own preferences. Therefore, the reached optimum doesn't maximize the utility of all actors involved in the economic process.

4. Tâtonnement[1] (French word for "trial and error"). It indicates a process by which the equilibrium prices are determined through the matching between demand and supply. This means that, starting from an initial agreement, the process will change until the market clearing. In this way, the goods are exchanged many time, forgetting that infinite trade is not possible for any kind of good (i.e. services or perishable goods don't have a reversible supply). Besides this criticism, there will be other unsolved questions such as: how many trades are permitted, how many commodities and so on.

5. The Walras Pareto Fixation. With the last two points, the doctrine asserts that an economic system reaches a general equilibrium by a process of "free competition" producing the optimum equilibrium. But the term "competition" is misleading. It doesn't consider human interactions and the agents are helpless in the face of reality. The term *fixation* refers to the set of given prices, wages, interest rates which are not affected by agents' behavior and so, they don't follow the law of market.

6. Allocation of Resources. The theorists of competitive equilibrium don't consider the existence of monopoly, duopoly or monopolistic competition. This is a great leak empirically, no other market forms are admitted besides the perfect competition. The figure of governments is not considered and so the possibility of public procurement. More, not all exchanges take place under monetary disbursement. Indeed, other allocations can take place in a dynamic way i.e. inheritance and gift. The Walrasian theory reaches the optimum through a static process and it is blocked to the present with any view on the future.

7. Substitution. When the goods are substitutable, they have the same value for the agents. In reality, the value of an object is not defined by its sub-

[1]This point will be analyzed in the following paragraph.

stitutability, but especially for the relation that it has with the other commodities. This problem is expressed by *all swans are white by finding one black swan*[2]. To follow this statement, Morgenstern produces some mathematically escamotage proving that not all substitutable goods have the same value.

8. Demand and Supply. These curves are constructed with no time period for which they are valid. However, time period may be different due to the different nature of goods. Sometimes, the consumption time is regular, in other cases, it doesn't follow a cyclic interval. In addition, the demand functions have the same elasticity. The market demand cannot be calculated by adding individual demand because there are some unexplainable human behavior not captured by this methodology, i.e. two women want to buy a dress, but if one woman buys it before the other one, the second woman doesn't want to purchase it anymore. The two individual demands cannot be summed to compute the market demand.

9. Indifference Curve. These lines are simply given. No explanation is given about the formation of the preferences on which they are based. Furthermore, they don't involve some political processes changing the order of curves, i.e. law, voting, authority. Also, some individuals can have the same preferences, but not the same utility representation, i.e. someone could have finer utility scales than others.

10. Theory of the firm. In this case, the gap between the reality and the theory may be very huge. For example, the productivity is linked only to physical notion and so it is not possible to compute that one of a school, a law firm or of an artist. Many other aspects are absents: the location of an enterprise, the quality control process, the management role and so on.

11. Back to Cantillon[3]. The effect of monetary policy is not contemplated. More, the intermediaries function is not present and this is a strong limit when the agents are faced with the purchase of a durable good.

12. Personal and Function Distribution. The traditional equilibrium theory asserts that there are not undistributed resources and that the marginal productivity explains how the goods are distributed. So, it presents a connection

[2]It derives from a latin expression. It refers to an unexpected event.

[3]Richard Cantillon was an eighteenth-century economist who raised the idea of monetary non-neutrality also known as *"Cantillon Effects"*. Cantillon and his followers emphasized that changes in the quantity of money progress over the economy in a step-by-step fashion and lead to wealth redistributions and to real effects on production processes.

between the supply (productivity) and the demand (allocation). The hypothesis is not realistic, as it emerges by distinguishing between the personal and functional distribution. The first one refers to the individual income or households, the second one to the productivity of the inputs such as labor, capital and so on. The two kinds of distribution derive from two different processes and they can't be overlapped. Indeed, the actual prices of goods depend on the productivity of the previous period, the actual personal income determines the input prices of the next period. The price formation is not static and so, it is impossible to explain it with the productivity theory.

13. Theory relevance. All the previous points underline that reality is clouded by the theory. The new theory should underline the relevance of the empirical data also in the mathematical model.

These critical points emerged during the 1970's accompanied by several criticisms of other authors. As suggested above, the more interesting for the aim of this work is the one concerning the term *"competition"* and its identification with *"price-taking"* behavior. Morgestern underlines as the term *"competition"* refers, in common meaning, to a situation in which there is a fight to get ahead or at least to hold one's place; while in economic theory it is used in the opposite way: a quiet situation in which everyone always arrives to the equilibrium.

Precisely, agents choose their strategies independently one to each other only considering their budget constraints. Under a *"price-taking"* behavior assumption, they passively accept trade conditions and prices imposed by the market.

Arrow summarizes this idea as follows: *"There exists a logical gap in the usual formulation of the theory of perfectly competitive economy, namely, that there is no place for a rational decision with respect to prices as there is with respect to quantities"*(1959). According to Arrow, the identification of perfectly competitive markets with price-taking assumption leads to no other alternative than the existence of the so-called Walrasian auctioneer as an explanation of the source of equilibrium prices.

Therefore, in order to describe reality, first of all, any model has to take into account imperfect competition. This suggestion, due to Arrow again, has lead to the questioning of general equilibrium approach. This is, for example, the case of applied economic models. In order to deal with a systematic treatment of imperfectly competitive markets, one is logically led to get rid of the existence of the auctioneer in favor of a strategic behavior [2]. The perfect competition, understood as a market in which the price is formed through an invisible hand that matches demand and supply, is reconsidered giving rise to new theories.

The important difference between the two lines of thought is the abandonment of the hypothesis that the agents are price-takers.

The theorists of imperfect competition criticize the classical view of competitive equilibrium for two main reasons: the explicit treatment of decentralized exchange and money's function. These criticisms show that there is no institution in the economy that could participate to encourage the equilibrium and that the money is not present in the sense that the agents rely also on their own endowments and allocations without no possibility of leading or borrowing money. Walras expresses clearly that the money is only a pure economic device and therefore, he decides to exclude it from his theory.

So, the Walrasian (competitive) equilibrium is based on two main hypotheses: price-taking behavior and no exchange out of equilibrium. It is founded on two elements:

1. The numéraire: It is the common measure of relative prices of the goods. It is equal to the marginal rate of substitution in equilibrium. The price never changes and the consumers always acquire a bundle of goods, it never happens that they acquire only one good otherwise it would not be possible to construct the marginal rate.

2. The tâtonnement: As explained before, it is the process of exchange related to find the market clearing price for all commodities, giving rise to general equilibrium.

Walras affirms that, if the process takes place under perfect competition, agents cannot exchange at a different numéraire. For this reason, he justifies the existence of a calculator (auctioneer or others[4]) which determines the quantities and the market price.

The opposition between strategic behavior and the presence of auctioneer, that underlines the characteristic approach of imperfect competitive models, resumes the criticisms to Walras theory on prices formulated by Edgeworth.

The Edgeworth's approach to economic theory is totally different from the one of Walras. With the aim of describing economies and markets, he moves from the analysis of agents' behavior. Then, he considers perfect competition not as an assumption, but as a special consequence of agents' interactions. The Edgeworth's criticism focuses on the absence of agents' strategic behavior and on their price-taking attitude. His theory does not assume these premises, but it considers perfect competition as the result of neutralization of strategic behavior [2]. In particular,

[4]The Walrasian auctioneer is the person conducting the Walrasian auction who effectively matches supply and demand based upon the bids provided by the economic agents. Such an auctioneer renders the process of finding trading opportunities perfect and cost-free, which results in perfect competition in the market. In contrast, outside of a Walrasian auction, there might exist a "search problem" in which there is a cost for finding a partner to trade with and additional transaction costs when one meets such a partner.

Edgeworth develops the concept of the *"final settlement"*. It is a situation in which there is no way for at least two agents of establishing a new better contract. In the final settlement, the emphasis is on the outcomes which are immune to recontracting.

Later, it is accepted, during the 1970's, that the Edgeworth's analysis and view on perfect competition can be reconciled with the Walras's theory through the connection between the final settlement and the core. This reconciliation is formalized by the Debreu-Scarf theorem.

0.1 Reconciliation between Walras and Edgeworth

Quoting Hildebrand: *"Since an economy with a finite number of agents is not perfectly competitive, Edgeworth introduced a new concept of equilibrium, the "final settlement" or in a today's language, the core. The "limit theorem" expresses that under " perfect competition" these two concepts coincide"*. Under this different interpretation, Edgeworth's approach makes explicit the conditions, only implicit in Walras, under which a perfectly competitive allocation emerges from the agents' interactions. A task that is left to the auctioneer by Walras.

The Debreu and Scarf's idea has been resumed by Aumann: *"The definition of competitive equilibrium assumes that the traders allow market pressures to determine prices, and that they then trade in accordance with these prices, whereas that of core ignores the price mechanism and involves only direct trading between the participants. Intuitively, one feels that money and prices are no more than a device to simplify trading, and therefore the two concepts should lead to the same allocations. It is to be expected that this will not happen in finite markets, where the notion of competitive equilibrium is not really applicable; and, indeed, though any equilibrium allocation is always in the core, the core of a finite market usually contains points that are not equilibrium allocations. But, when the notion of perfect competition is built into the model, that is, in a continuous market, one may expect that the core equals the set of equilibrium allocations. That this is indeed the case is the main result of this paper. It has long been conjectured that some such theorem holds; the basic idea dates back at least to Edgeworth. The usual rough statement is that the core approaches the set of equilibrium allocations as the number of traders tends to infinity. Unfortunately, it is extremely difficult to lend precise meaning to this kind of statement, to say nothing of proving it. Very recently, Debreu and Scarf did succeed in stating and proving a theorem of this kind in a brilliant and elegant fashion"* Under these assertions, the convergence theorem is interpreted as the proof that perfect competition can be obtained as a limit economic situation in which there are many agents. Linked to the concept of great number, the replica economy occurs. It is an expedient to have a large economy as an illustration of

Edgeworth's mechanism of recontracting. The convergence of an economy with a great number of agents to a perfect competitive situation has been developed in the 20th century economics as the general Edgeworthian framework.

In contrast with the commonly accepted reconciliation between the two approaches built on Debreu Scarf equivalence theorem, the confrontation between Edgeworth's and Walras's theories has been considered as a methodological misunderstanding by many scholars (see [2], [13]). Even if the two theories have been placed side by side, there are some significant differences between them. The first difference is given by the point of view: the French economist has an individual point of view while the second one has the focus on the actions of a group introducing the blocking coalitions. In this way, the set of actions available to individuals is replaced by the actions of a coalition the view. of cooperative games. Furthermore, the very notion of cooperative games implies that the coalition may be interpreted as a collective decision body. Within this collective body, individual strategies are subject to the distributional rule imposed by the coalition. However, even if there are many analogies with game theory, the Edgeworth's idea does not really follow it in any aspect. He establishes a difference between a group of agents entering voluntarily in a series of bilateral contracts and a collective decision body that he calls a combination.

In opposition to the purely individual logic of these bilateral arrangements, a combination depends on a collective logic: the distributional rules established among their members impose a form of collectivization of private property. He explains the way in which a combination may be a source of imperfect competition. So, he introduces only the concept of blocking coalition, but he recognizes the possibility of the imperfect competition. Another breaking point between the two academics is represented by the conception of the price system. In Walras' theory, there is a unique price system determined by a sort of calculator, in the Edgeworth's studies, there are several exchanges bilaterally defined and different one from the others. The final settlement is the result of a contracting process, in which the price will be defined only at a second time.

The richness of Edgeworth's theoretical background falls down in modern theory with the above mentioned theorem. If there exists a convergence in terms of the result (namely the final allocation) of the perfect competitive situation between Edgeworth and Walras, there is no possible reconciliation in terms of the exchange mechanism that allows obtaining these results.

Edgeworth's theory of re-contracting cannot give content to the elusive notion of tâtonnement as an actual economic process. The only way to match the two theories is considering the unique price as an assumption and not as a consequence of the market. A large number of agents allows the possibility to neutralize the arbitrage and the conclusion is that the general equilibrium price vector is given by the institutional framework. Finally, there is the construction of the core concept: it is a mathematical representation of final settlement, a solution with few data and a less level of reality description, without an institutional structure. A result with many mistakes: Walras has a market with perfect information, an auctioneer that imposes a price vector and no conjectures of agents about the behavior of others, while Edgeworth underlines the importance of information flow[5] for arriving to the equilibrium and of cooperation among agents.

This is the background of competitive equilibrium theory with price taking behavior. It is strongly influenced by an overlapping created by the Convergence Theorem that connects the core concept with the competitive equilibrium.

The evolution of imperfect competition

The evolution of the literature that we are presenting, moves from an economy with a given price system [28], inserts the concept of re-contracting process to determine the equilibrium until to obscure the price system in favor of the interaction among the agents. The agents' active role determines the equilibrium price: from passive, their behavior becomes active under a no - price taking attitude.

Schmeidler and Vind describe a finite exchange economy with a given price system (exogenous as in classical competitive model), but also with the concept of *fair net trade* that, in view of the budget set, maximizes the agents' utility. The *fairness* refers to the increments in the commodities of two (or more) agents deriving from an exchange process between them. They define a net trade as a function $f : N \to R^L$ where N is the set of the agents and L the set of commodities. The net trade is any change in the holdings of some or all agents that leaves each of them with a non-negative quantity of each good $f(i) + e_i$[6] ≥ 0 The net trade is balanced when the sum of the increments for each good is zero, i.e.: if the function f is such that $\sum_{i \in N} f(i) = 0$.

The net trade is said to be *fair* when each agent considers his net trade at least as good as the net trade of any other agent. A fair net trade is a *Walrasian net trade* if there exists $p \in R^L \setminus \{0\}$ such that:

$$pf(i) = 0$$

for all $i \in N$.

So, a net trade f is competitive if there exists a price p in R^L such that for all agent i in N, the net trades are balanced and fair. They use this concept instead of the classical allocation, considering the possibility of recontracting for the agents.

[5]The information flow is important because under it individuals are in condition to re-contract and so to communicate.

[6]The initial endowment of agent $i \in N$.

Makowski and Ostroy [21] introduce another way to get the competitive equilibrium: the sequential elimination of arbitrage opportunities of the agents that we briefly describe here. They think that for those prices in which the agents cannot improve their utility, the market has its own equilibrium and the vector price system is that arising in equilibrium. So, they consider a model of perfect competition in which the prices are constructed endogenously, by the actions of each agent. To implement this procedure, they consider that the possibilities of arbitrage are based on the differences of the marginal rate of substitution. The result is called *"the flattening effect of large number"*.

They represent the arbitrage by means of the formation of an opportunity set S which is a convex cone while the price emerges as the supporting hyperplane to a convex cone[7] .

For any set S and points $x, y \in S$, y is visible from x if the line segment $[x, y] \subset S$. S is *star shaped (with center x)* if any point in it is visible from x. All convex sets are star shaped, but not conversely. The concept is represented in the following figure.

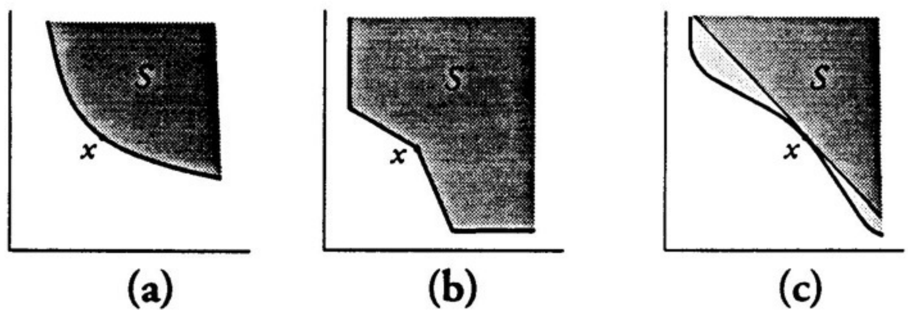

Figure 1: Star shaped

The panel (a), (b) are star shaped with center x, (c) is not star shaped because only the darker shaded portion is visible from x.

Given an agent of type $t = (u, e, x)$, where u is the utility function, e the initial endowment and x the allocation consumption, he will accept any exchange's offer that will leave him at least at the previous utility level. But the *arbitraggists* are only able *"to see"* visible exchanges from x, and accordingly, from their perspective, the set of the possible exchanges with an individual type t is given

[7]The cone property distinguishes the arbitrage approach to competitive equilibrium from Walrasian equilibrium: price emerges as the supporting hyperplane to a convex cone rather than the supporting hyperplane to a convex set.

by:

$$A(t) = \{z : u(x - z) \geq u(x) \quad \& \quad x - z \quad \text{is visible from} \quad x\}.$$

$A(t)$ is the set of all the possible trades of type t that prefers the allocation x given his own preferences and endowment. It is the set of the arbitrage opportunities of the individual of type t, while z represents the arbitrager's point of view: if $z^h > 0$, the arbitrager receives z^h units of commodity h from an individual of type t. Otherwise, is $z^h < 0$, the arbitrager doesn't lead to exchange.

The crucial concept is that the agents have only "local knowledge" about others' preferences, i.e. each agent observes only a part of all the possibilities of arbitrage.

A *group* is a vector of types $(t_1, ...t_i, ...t_n)$. $\mathcal{G}(\mu)$ represents the set of possible groups that any arbitrager can form, given the allocation μ. So, it is composed by all vectors $(t_1, ..., t_i, ..t_n)$ that satisfy the following features:

1. The number of arbitragers is finite;
2. for any individual $i, t_i \in supp(\mu)$;
3. for any type $t, \mu(i : t_i = t) > 1$ only if $\mu(t) > 0$.

The interpretation is that an arbitrager can make offers only to a finite number of agents and he can exchange with "several" individuals of the same type only if that type is an atom in μ. The trading possibilities of an arbitrager at μ are given by:

$$K(\mu) = \{z : \text{for some group} \quad (t_1, ..t_i, ..t_n) \in \mathcal{G}(\mu)$$
$$z = \sum_{i=1}^{n} z_i, z_i \in A(t_i)\}.$$

An individual i can reach the bundle z if and only if there is a group of individuals willing to trade z with him in aggregate. The exchange is acceptable if each individual trade z_i is visible to the arbitrager starting from the initial allocation μ.

At an arbitrary allocation μ, agents may have several reservation price and so they may get a gain from the trade: buying low and selling high as in the finance field.

The allocation μ is arbitrage free, if:

$$K(\mu) \cap R^L_{++} = \emptyset.$$

The arbitrage possibilities set $K(\mu)$ contains no "free lunches".

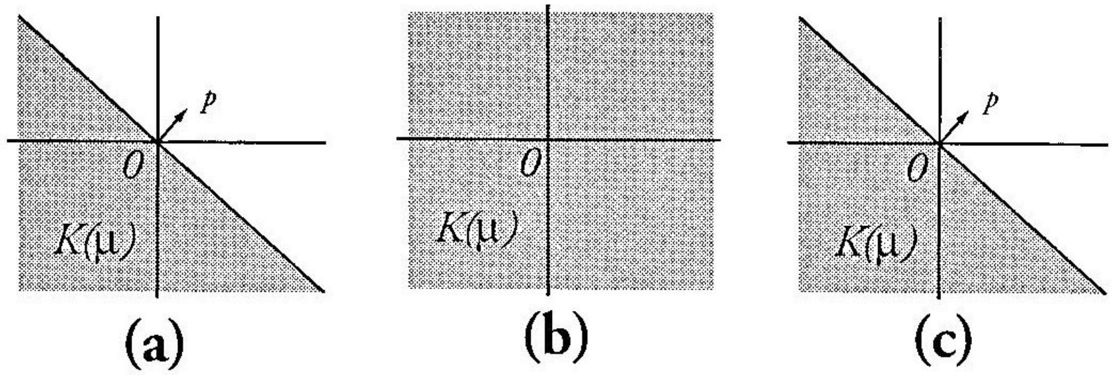

Figure 2: Trading possibilities form a convex cone.

This theory evolves with McLennan and Sonnenschein [24]. They study the abstract equilibrium in which the choice sets are generated from additive sets of net trades. Let $W \subseteq R^L$ be the set of net trades that can be obtained, let Z be the set of finite sums of elements of W, alternatively, Z is the smallest superset of W that is closed under vector addition.

They consider an economy $\xi = (X_i, \succeq_i, e_i)_{i \in N}$ where X_i is the commodity space, $\succeq_i$ is the preference relation, e_i is the initial endowment and N the set of the agents.

Theorem 0.1.1. Let the preference relation $\succeq_i$. For $i \in N$, let $X_i \subseteq R^L_+$ be open, let $u_i : X_i \to R$ be an utility function. Suppose that x is an allocation and e is the initial endowment such that $\sum_{i \in N} x_i = \sum_{i \in N} e_i$. Assume that there is a set $Z \subseteq R^L$ with the following properties:

i. $0 \in Z = Z + Z$: possibility not to trade;

ii. $x_i - z \succeq_i x_i$ implies $z \in Z$;

iii. $z_i \succeq_i (e_i + Z) \cap X_i$.

Then x is a Walrasian equilibrium.

z is a net trade that has the same features of a bundle and $z \in W \subset Z$.

These authors propose the possibility to reach an equilibrium obtained through the additive net trade. The additivity on Z is imposed to justify the possibility that *agents are not prevented from trading each other repeatedly.*

All these studies fall under the breakup with the past following trends:

- The rise of game theory. It analyzes interactive decision making. Unfortunately, they don't have the features of the general classical equilibrium model. Indeed, in a strategic game, all the endogenous variables are chosen by the players and any strategy gives a unique feasible outcome. These features are absents in Walras theory, thus the prices are exogenous or, at least, determined by an auctioneer. Moreover, it may happen that some strategy profiles (as the bundles chosen when there is an excess of demand) don't lead to a feasible outcome.

- Founding a rational background for perfect competition. The competitive equilibrium theory does not define in which situation it can be appropriate, i.e. it is applied when there are two agents as well as there are million agents. This means that the equilibrium is forced because any situation is enforced to be directed towards the perfect competition also when its features are not present. The last aspect is justified in the literature with the distinction between *limit theorem* and *theorems in the limit*. The perfect competition is an idealized state, but all the markets are approximated to it when the number of agents grows up to a significant amount. In the limit, there is an insignificant weight of individuals' actions. The theorem in the limit characterizes the precise condition under which the perfect competition is used.

The most relevant works to support the rationality of a competitive equilibrium and a cooperation among agents which are of interest for our work, are represented by:

- *"Recontracting and Competition"* of Dagan N. (1996);
- *"Walrasian Allocations without Price-Taking Behavior"* of Serrano R. and Volij O. (2002).

The authors show that an allocation is competitive if and only if the agents choose the final allocation in an admissible set that is constructed endogenously by their arbitrage desire. The concept is presented introducing the idea of interactive choice set (*ICS*).

Dagan, [5], starting from a review of the theories of the Edgeworth's box and Walrasian equilibrium, eliminates the price assumption and imposes a recontracting condition, similar to the concept of final settlement. He shows that the core of this arbitrage economy matches to the shrunk core of replica economies, using the large number expedient as Debreu and Scarf with the convergence theorem. His results are retaken by Serrano and Volij [31] which introduce the concept of interactive choice set, an introspective way to determine the final allocation: with Dagan, the agents are choice-set takers, with Serrano-Volij, they are actively engaged in the construction of the best allocation. The new focus is on the introspective approach of the equilibrium construction.

We notice that both approaches by Serrano-Volij and Dagan assume that there is no uncertainty about the state of nature in which the consumption takes place. Moreover, it is assumed that each agent perfectly knows the other agents' features. So in the next pages, the first part is focused on the review of the principal aspects of the above mentioned theories to make the thesis self-contained. Then, uncertainty and differential information will be introduced, as in [6], together with interactive algorithm for the construction of competitive equilibria.

Chapter 1

An introspective way for the competitive equilibrium

This chapter focuses on a new way to build the competitive equilibrium (the arbitrage free-equilibrium). Its goal is an exposition of theories which put the emphasis on the active role of the economic agents as opposed to those based on a price-taker behavior. Under this approach, almost all the authors focus on the pure exchange economies except Dagan ([5]). Therefore we will introduce an extension to production economies of the arbitrage free equilibrium.

Chapter 1 focuses on these topics:

a. The representation of the abstract and arbitrage free equilibrium with a particular attention to the interactive choice set ([31], [5]) in the first section. Within this section, we shall present a characterization of equilibria with and without short sales.

b. The relationship of the arbitrage free-equilibrium with the equivalence result and the replica economies ([4], [34], [15]) in the final part. This is a crucial point because it drives the main results of the thesis.

Notice that all the equilibrium notions introduced in Section 1.1 assume that the choice set is given and not constructed.

1.1 The abstract equilibrium and the choice set

In [5], Dagan proposes a new notion: the abstract equilibrium. This kind of equilibrium is based on the hypothesis of:

1. Market clearing;

2. optimization of agent's utility;

3. re-contracting condition among agents.

The third hypothesis is new when compared with classical approach to competitive equilibrium concept and expresses the freedom of agents to re-contract.

Let ξ be a finite exchange economy $(X_i, \succeq_i, e_i)_{i\in N}$ where N is a finite non-empty set of agents; for all $i \in N$, $X_i \subseteq R^L_+$ is the consumption set of agent i, $\succeq_i$ (denoted also by R_i) is the preference relation of agent i on X_i and e_i is the initial endowment of agent i.

An allocation $(x_i)_{i\in N}$ is a list where for all $i, x_i \in X_i$ and

$$\sum_{i\in N} e_i = \sum_{i\in N} x_i.$$

For each agent $i \in N$, let us define the strict preference relation P_i as follows: for all $x, x^{'} \in X_i, xP_ix^{'}$ if and only if $xR_ix^{'}$ and not $x^{'}R_ix$. The construction of P_i makes it irreflexive. Let us define the weakly preferred set by $R_i(x) = \{x' \in X_i : x^{'}R_ix\}$ and the strictly preferred set by $P_i(x) = \{x' \in X_i : x^{'}P_ix\}, \forall x \in X_i$. An *abstract equilibrium* is a list $(x_i, C_i)_{i\in N}$ where $(x_i)_{i\in N}$ is an allocation and $(C_i)_{i\in N}$ denotes the family of the choice sets of consumers, that is $C_i \subset R^L$ satisfies the following properties for all $i \in N$:

1. $e_i \in C_i$, interpreted as the possibility of no trade;

2. $x_i \in C_i$, i.e. the feasibility of actual trade;

3. $C_i \cap P_i(x_i) = \emptyset$ that is the optimality of x_i among the consumption choices for agent i in the choice set C_i.

Then, an abstract equilibrium fixes for each agent i a consumption bundle and a choice set.

Notice that a Walrasian equilibrium can be considered as a particular abstract equilibrium. Given a price $p \in R^L \setminus \{0\}$, it is enough to define for all $i \in N$, $C_i = \{x \in R^L : px \leq pe\}$ the budget set of agent i. We shall say that equilibrium, $(x_i, C_i)_{i\in N}$ satisfies the re-contracting condition if for any $j \in N, i \neq j$, the choice sets respect the following condition:

$$C_i \supset C_i + C_j - R_j(x_j). \tag{1.1}$$

The re-contracting condition (RC) states that agent i may offer any bundle in the choice set C_i to other agent j in order to obtain by agent j any bundle in the choice set C_j. Of course, agent j, which expects to receive x_j, will accept any other bundle in $R_j(x_j)$. The meaning of this condition in equilibrium is that all the agents have the possibility to trade (or re-contract), but they do not decide to use this possibility because they are already maximizing.

The following theorem gives a new interpretation of Walrasian equilibrium.

Theorem 1.1.1. ([5]) Let $\xi = (X_i, \succeq_i, e_i)_{i \in N}$ be an economy with at least two agents that satisfy the following assumptions:

A1. For all $i \in N$, X_i is closed and convex.

A2. For all $i \in N$, for all $x \in X_i$, $P(x_i)$ is open relative to X_i.

A3. For all $i \in N$, for all $x \in X_i$, for all $\delta > 0$
$[x' \in X_i : |x' - x| < \delta] \cap P_i(x_i) \neq \emptyset$,(local non satiation)

A4. For all $i \in N$, $e_i \in int(X_i)$.

or alternatively, the assumptions A1- A2 and the following hypothesis A5:

A5. For all $i \in N$, $X_i = R^L_+$ and for all $x, x \in X_i$
$x < x'$ implies $x' \in P_i(x)$ and $\sum_{i \in N} e_i \gg 0$.

Then an allocation $(x_i)_{i \in N}$ of ξ is a Walrasian allocation if and only if it is an abstract equilibrium that satisfies the re-contracting condition.

It is therefore clear that we are no more in a situation in which prices are quoted and agents are price takers, but it is a model in which agents are choice set takers. Under this point of view, the auctioneer doesn't decide the final price, but his role is to let the agents communicate with each other.

Theorem 1.1.2. stated below is the link between the equilibria expressed like sets of net trades and the abstract equilibrium. The sets of net trades are those studied by Andrew McLennan and Hugo Sonnenschein in [24].

Theorem 1.1.2. ([5]) Let $\xi = (X_i, \succeq_i, e_i)_{i \in N}$ be an economy.

1.1.2.1 Let $(x_i)_{i \in N}$ be an allocation and $Z \subset R^L$ be a set of net trades that satisfies the conditions:

i. $0 \in Z = Z + Z$;

ii. $x_i - e_i \in Z$;

iii. $(x_i + Z) \cap P_i(x_i) = \emptyset$;

iv. $x_i - R_i(x_i) \subset Z$.

For all $i \in N$, let $C_i = e_i + Z$. Then $(x_i, C_i)_{i\in N}$ is an equilibrium that satisfies the re-contracting condition:

$$C_i \supset C_i + C_j - R_j(x_j).$$

1.1.2.2. Let $(x_i, C_i)_{i\in N}$ be an equilibrium that satisfies the re-contracting condition. If there are at least two agents in the economy, then there exists a set Z such that for all $i \in N, C_i = e_i + Z$ and the conditions i-iv are satisfied

The conditions used by McLennan and Sonnenschein [24] can be interpreted as follows: the first condition expresses the possibility of not contracting, the second one is the feasibility, the third one expresses the optimality of the net trade Z and the last condition asserts the re-contracting condition. This theorem, called *"completeness of the auctioneer"*, implies that all agents have the same opportunities represented by the set Z. It is possible to get access to trade opportunities which are cost-free like in the perfect competition of Walrasian framework.

1.1.1 Characterization of the core

The concept of core is based on the one of blocking coalition. A coalition S is a non-empty subset of N. An S -allocation is a list $(x_i)_{i\in S}$ where for all $i \in S, x_i \in X_i$ and $\sum_{i\in S} e_i = \sum_{i\in S} x_i$.

An allocation $(x_i)_{i\in S}$ is a *core allocation* if there does not exist a coalition S and S- allocation $(x'_i)_{i\in S}$ such that for all $i \in S$, $x'_i \in R(x_i)$ and for some $i \in S$, $x_i \in P_i(x_i)$.

The coalition concept and the blocking mechanism enter in the model because of the following new formulation of re-contracting condition.

An abstract equilibrium $(x_i, C_i)_{i\in N}$ satisfies the weak-recontracting condition if for all $i \in S$ and for all coalitions S with $i \notin S$:

$$C_i \supset e_i + \sum_{j\in S} e_j - \sum_{j\in S} R_j(x_j). \tag{1.2}$$

The weak re-contracting condition *WRC* presents clearly some differences with respect to the re-contracting condition *RC*.

1. It focuses on multilateral trades while the *RC* on bilateral trades. The concept of *group* emerges like in Edgeworth box.

2. It involves the notion of initial endowments while the first one only uses the choice set. So, the idea is to move from the initial situation to a better situation.

However, using equation 1.1, it is possible to arrive to the following statement:

$$C_i \supset C_i + \sum_{j \in S} C_j - \sum_{j \in S} R_j(x_j). \tag{1.3}$$

In this way, the two conditions become more similar.

The following theorem connects the core notion with the *WRC*.

Theorem 1.1.3. [5] Let $\xi = (x_i, \succeq_i, e_i)_{i \in N}$ be an exchange economy. An allocation $(x_i)_{i \in N}$ may be supported by an equilibrium that satisfies the weak re-contracting condition if and only if it is a core allocation.

The weak re-contracting condition considers the agent i which trades with all the agents $j \in S$. In analogy, the core refers to a coalition S in which all the members trade among them for arriving to the equilibrium. Both notions build on the coalition concept.

1.1.2 Production economies

Dagan translates the concept of abstract equilibrium with *RC* to the case of economies with production.

A private ownership production economy is a triple $\xi = [(X_i, \succeq_i, e_i)_{i \in N}, (Y_j)_{j \in J}, (\theta_{ij})_{(ij) \in N \times J}]$, where $(X_i, \succ_i, e_i)_{i \in N}$ is an exchange economy, J is a finite set of producers, $(Y_j)_{j \in J}$ is the family of production sets where $Y_j \subset R^L$ and θ_{ij} is the share distribution of profits to the consumers in the firm where $\forall (i,j) \in N \times J, \theta_{ij} \geq 0, \sum_{i \in N} \theta_{ij} = 1, \forall j \in J$ and $J \cap N = \emptyset$ (no consumer is a firm and no firm is a consumer).

An allocation is a pair $[(x_i)_{i \in N}, (y_j)_{j \in J}]$ where $x_i \in X_i, \forall i \in N, y_j \in Y_j, \forall j \in J$ and

$$\sum_{j \in J} y_j + \sum_{i \in N} e_i = \sum_{i \in N} x_i.$$

Let $\xi = [(X_i, \succeq_i, e_i)_{i \in N}, (Y_j)_{j \in J}, (\theta_{ij})_{ij \in N \times J}]$ be an economy. An *abstract equilibrium* is a triple $[x_i, y_j, C_i]$ where $[(x_i)_{i \in N}, (y_i)_{j \in J}]$ is an allocation; for all $i \in N, C_i \subset R^L$ is the agent i's choice set that satisfies:

1. $e_i + \sum_{j \in J} (\theta_{ij} y_j) \in C_i$;
2. $x_i \in C_i$;
3. $C_i \cap P_i(x_i) = \emptyset$ or $C_i \cap X_i$ has an empty interior.

With respect to the exchange economy, the only condition modified is the first one which, now involves the share of profits of consumer i in the firm j.

The three conditions can be explained as follows:

- Condition 1. is interpreted as the possibility of not trading;
- condition 2. expresses feasibility;
- the last condition 3. implies that the choice set maximizes the agent i's utility.

A particular case of abstract equilibrium $[(x_i)_{i\in N}, (y_j)_{j\in J}, (C_i)_{i\in N}]$ is again represented by the Walrasian equilibrium. There exists a price vector $p \in R^L \setminus \{0\}$ such that for all $i \in N, C_i = \{x \in R^L : px \leq p(e_i + \sum_{j\in J} \theta_{ij} y_j)\}$ and for all $j \in J$ and for all $y_j^{'} \in Y_j, py_j \geq py_j^{'}$.

In the case in which $[(x_i)_{i\in N}, (y_j)_{j\in J}, (C_i)_{i\in N}]$ represents a quasi Walrasian equilibrium, the condition 3. is satisfied in the form $C_i \cap X_i$ has an empty interior.

The re-contracting condition with firm is given for all $i \in N$ and $j \in J$ by:

$$C_i \supset C_i + Y_j - y_j. \tag{1.4}$$

It asserts that each consumer re-contracts with each firm by offering it a plan in exchange of another feasible plan.

The re-contracting condition with firms and with consumers can be resumed as follows:

$$C_i \supset C_i + \sum_{k\in S} R_k(x_k) + \sum_{j\in J}(Y_j - y_j). \tag{1.5}$$

$\forall i \in N, \forall S \subset N, i \notin S.$

Theorem 1.1.4. [5] Let $\xi = [(x_i, \succ_i, e_i)_{i\in N}, (Y_j)_{j\in J}, (\theta_{ij})_{ij\in N\times J}]$ be a production economy. Let $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ be an allocation and $Z \subset R^L$ be a set of net trades that satisfies the following assumptions:

1. Possibility not to trade and additivity: $0 \in Z = Z + Z$.
2. Feasibility of actual trade: $x_i - (e_i + \sum_{j\in J} \theta_{ij}) \in Z$.
3. Optimality: $x_i - (e_i + \sum_{j\in J} \theta_{ij}) \bigcap P_i(x_i) = \emptyset$.
4. Ex post recontracting in consumption: $x_i - R(x_i) \subset Z$.
5. Ex post recontracting in production: $Y_j - y_j \subset Z$ [1].

[1]This last condition may represent the maximization profit condition, in the same light of the condition 4 for the maximization of utility. Any production plan can be bought in exchange of the actual production plan through the market represented by Z. Z is the net trade inspired by [24].

Let the choice set be defined as follows $\forall i \in N$:

$$C_i = e_i + \sum_{j \in J} \theta_{ij} y_j + Z. \tag{1.6}$$

Then $[(x_i)_{i \in N}, (y_j)_{J \in J}, (C_i)_{i \in N}]$ is an equilibrium that satisfies the re-contracting condition.

Notice that in exchange economies, the re-contracting condition is among consumers, here it involves both consumer i and firm j. So, each consumer trades with each firm offering a feasible plan.

Then, Dagan translates Theorem 1.1.1 that connects the Walrasian equilibrium and the abstract equilibrium to the production economies through the following theorem.

Theorem 1.1.5. Let $\xi = [(X_i, \succ_i, e_i)_{i \in N}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in N \times J}]$ be a production economy let $[(x_i)_{i \in N}, (y_j)_{j \in J}]$ be an abstract equilibrium. Given the following assumptions:

A1 For all $i \in N, X_i$ is convex and closed.

A2 For all $i \in N$, for all $x \in X_i, P(x_i)$ is open relative to X_i.

A3 For all $i \in N$, for all $x \in X_i$, for all $\delta > 0, x' \in X_i : |x' - x| < \delta \cap P_i(x_i) = \emptyset$.

A6 For all $i \in N, X_i = R^L_+$ and $e_i \in X_i$ for all $j \in J$, there exists a production plan $(y_j)_{j \in J}$ that satisfies:
$\sum_{j \in J} y_j + \sum_{i \in N} e_i \gg 0$;

if an allocation $[(x_i)_{i \in N}, (y_j)_{j \in J}]$ is an equilibrium that satisfies the re-contracting condition with firms and with consumers, then it is a quasi Walrasian allocation.

1.2 Evolution of abstract equilibrium

Serrano and Volij [31] make a leap forward with respect to Dagan [5] because they don't use the initial concept of choice set as given, but they start from the concept of interactive choice set. This is the way they use to replace the auctioneer with the construction of the competitive equilibrium through an introspective behavior of the agents. They imagine an Edgeworth box and they suppose that an allocation x is subject to an arbitrage-free test. Indeed, each agent, starting from his own endowment, may propose an arbitrage-freeexchange to the others. The trades are considered achievable in the sense that any contract doesn't worsen the previous situation. This interactive algorithm may move forward again until the final consumption. In this section, we shall denote the preferred and the weakly preferred sets as follows.

For any $i \in N$ and $x_i \in X_i$, the preferred and the weakly preferred sets are defined by:

$$P_i(x_i) = \{z_i \in X_i : z_i \succ x_i\}.$$
$$W_i(x_i) = \{z_i \in X_i : z_i \succeq x_i\}.$$

The following definition introduces the interactive choice sets.

Definition 1

Let ξ be an exchange economy. Fix an allocation x and an agent $i \in N$. The interactive choice set (ICS) of i is the set Z_i recursively defined by:

$$Z_i^{*x}(0) = \{e_i\}.$$

For $t > 0$

$$Z_i^{*x}(t) = Z_i^{*x}(t-1) \cup \bigcup_{k \in N \setminus \{i\}} [Z_i^{*x}(t-1) + Z_k^{*x}(t-1) - W_k(x_k)].$$

And finally

$$Z_i^{*x} = \cup_{t \in \mathbb{N}} Z_i^{*x}(t).$$

An example clarifies the interactive choice set construction. Let us consider two players $i = 1, 2$, two commodities and the initial endowments $e_1 = (0, 2)$ and $e_2 = (2, 0)$ with a utility function $u(x_i) = min\{x_i^1, x_i^2\}$. The allocation x is constant and it gives $(1, 1)$ to each agent.

The interactive choice sets are defined as follows:

At $t = 0$

$$Z_1^{*x}(0) = \{(0, 2)\},$$
$$Z_2^{*x}(0) = \{(2, 0)\}.$$

At $t = 1$

$$Z_1^{*x}(1) = \{(0, 2)\} \cup ((1, 1) - R_+^2),$$
$$Z_2^{*x}(1) = \{(2, 0)\} \cup ((1, 1) - R_+^2),$$

where $(1, 1)$ is the bundle assigned to each agent, while R_+^2 is the set of all weakly preferred bundles to e_2 for consumer 2.

At $t = 2$

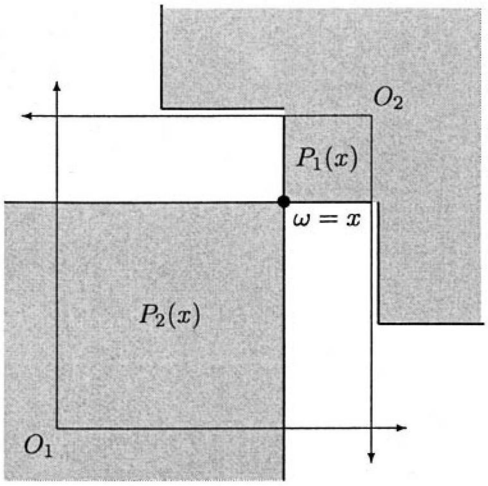

Figure 1.1: Short Sales in ICS [31]

$$Z_1^{*x}(2) = Z_2^{*x}(2) = \{(0,2) - R_+^2\} \cup \{(2,0) - R_+^2\} \cup ((1,1) - R_+^2).$$

Definition 2

Let ξ be an exchange economy. Fix an allocation x and an agent $i \in N$. The interactive choice set (ICS) of i is the set Z_i defined recursively as follows:

$$Z_i^x(0) = \{e_i\}.$$

For $t > 0$

$$Z_i^x(t) = \cup_{S \subseteq N, i \in S}[\sum_{k \in S} Z_k^x(t-1) - \sum_{k \in S \setminus \{i\}} W_k(x_k)].$$

$$Z_i^x = \cup_{t \in \mathbb{N}} Z_i^x(t).$$

In this case, the trades are not only between two agents, but an agent may exchange with any other agent in the economy. The following result shows that the two processes are equivalent for infinite t.

Proposition 1.2.1. ([30]) Let ξ be an economy and let $x = (x_1, x_2, ...x_n)$ be an allocation.

Then, for all $i \in N$:

$$Z_i^{*x} = Z_i^x.$$

From now on, let us use the second process without loss of generality.

We shall impose the following assumptions on the economy $\xi = (x_i, \succeq_i, e_i)_{i \in N}$:

A1 For all $i \in N$, the consumption set $X_i = R^L_+$;

A2 For all $i \in N$, for all $x_i \in X_i$, the preferred set $P_i(x_i)$ is open relative to X_i;

A3 For all $i \in N$, for all $x_i, y_i \in X_i, y_i \gg x_i$ implies $y_i \in P_i(x_i)$;

A4 For all $i \in N, e_i \in R^L_{++}$.

Let us define the arbitrage-free equilibrium.

An *arbitrage-free equilibrium* is a pair $[(x_i)_{i\in N}, (C_i)_{i\in N}]$ satisfying the following conditions $\forall i \in N$:

1. $(x_i)_{i\in N}$ is an allocation;
2. $e_i, x_i \in C_i$;
3. $\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i, \forall i \in S, \forall S \subseteq N$;
4. $P_i(x_i) \cap C_i = \emptyset$.

An arbitrage free - equilibrium is a pair $[(x_i)_{i\in N}, (C_i)_{i\in N}]$ in which x_i is the consumer i's bundle of goods and it belongs to the choice set C_i. By the point 2, we derive that the choice set is also feasible and by point 4, it respects the utility maximization. The more innovative point is the third one. It is the re-contracting or arbitrage freeness condition. It states that the agent i builds his choice set making exchange with agents of coalitions.

Proposition 1.2.2. Let $[(x_i, C_i)_{i\in N}$ be an arbitrage free- equilibrium, then:

- $Z^x_i \subseteq C_i$ for each $i \in N$;
- $(x_i, Z^x_i)_{i\in N}$ is an arbitrage free - equilibrium.

Proof. The proposition affirms that if $(x_i, C_i)_{i\in N}$ is an arbitrage free - equilibrium, $[x_i, Z^x_i]_{i\in N}$ is an arbitrage free - equilibrium too and the interactive choice set is a subset of the choice set. Serrano and Volij suggest a proof by induction. Since, $Z^x_i(0) = \{e_i\}, e_i \in C_i, \forall i \in N$. Then, let us assume that $Z^x_i(t-1) \subseteq C_i, \forall i \in N$ for some t. Starting from the assumption, $(x_i, C_i)_{i\in N}$ is an arbitrage free equilibrium and so it respects the arbitrage free condition:

$$\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i, \forall i \in S, \forall S \subseteq N.$$

By the induction hypothesis:

$$\sum_{k\in S} Z^x_k(t-1) - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i, \forall i \in S, \forall S \subseteq N.$$

Then, taking the union over coalitions $S \subseteq N$, we have:

$$\bigcup_{S\subseteq N, i\in S} \Big[\sum_{k\in S} Z_k^x(t-1) - \sum_{k\in S\setminus\{i\}} W_k(x_k)\Big] \subseteq C_i, \forall i \in N.$$

Therefore, applying the definition 2 of the interactive choice set, we arrive to the first statement of the Proposition 1.2.2.:

$$Z_i^x(t) \subseteq C_i,$$

$\forall i \in N$.

To prove the second statement of Proposition 1.2.2., we check the four conditions of an arbitrage free - equilibrium. The first one is obvious because $(x_i)_{i\in N}$ is an allocation by definition. The second condition implies the feasibility and the possibility of actual trade. The feasibility is given by the first stage of construction of the interactive choice set. The condition 4. is trivially proved because thus $Z_i(t)^x \subseteq C_i, P_i(x_i) \cap Z_i$ is an empty set too. The condition 3. is less immediate. Notice that if:

$$z \in \sum_{k\in S} Z_k^x - \sum_{k\in S\setminus\{i\}} W_k(x_k) \quad \text{for some} \quad t \in \mathbb{N}.$$

Therefore, by definition of $Z_i^x(t+1)$, we have $z \in Z_i^x(t+1) \subseteq Z_i^x$. □

So, following Serrano and Volij, the equilibrium will be defined as *"arbitrage free"*. The aim is to underline the possibility of arbitrage[2] of the agents. Proposition 1.2.2 ensures that here is no loss of generality when the choice set C_i is substituted by Z_i.

1.2.1 Connection between ICS and Replica economies

Let m be a natural number. The m fold replica economy ξ^m of ξ is an economy with $N \times m$ agents x for which:

- $e_i = e_{ij}$;
- $X_i = X_{ij}$;
- $\succeq_i = \succeq_{ij}$;

[2]This is a parallelism with the finance where the term "arbitrage" stands for buying low and selling high. Makowski and Ostroy [21] extend this notion to explore the differences in marginal rates of substitution.

for each $(i, j) \in N \times \overline{m}$. Here $\overline{m} = \{1, ...m\}$. Fix an allocation x.

Let us define the following sets for $i \in N$ and for $m \in \mathbb{N}$:

$$A_i^x(0) = \{e_i\},$$

$$A_i^x(m) = \bigcup_{S \subseteq N \times \overline{m}, (i,1) \notin S} \left[e_i + \sum_{(k,l) \in S} (e_k - W_k(x_k))\right]$$

$$A_i^x = \bigcup_{m \in \mathbb{N}} A_i^x(m).$$

The core of the m-fold replica is represented by means of the following condition: the allocation $x \in A(\xi)$ is in the core of the r-replica economy ξ^m of ξ if for all $i \in N$:

$$P_i(x_i) \cap A_i^x(m) = \emptyset.$$

Moreover, by definition of the set A_i^x

$$a \in A_i^x(m) \Leftrightarrow a \in e_i + n_k \sum_{k \in S} (e_k - W_k(x_k)),$$

with $n_k \in \mathbb{N}$ and $0 < n_k \leq m$ for $k \in S \setminus \{i\}$, for some $S \subseteq N$ with $i \notin S$. The m-fold replica and the interactive choice share a common feature: in the replica economies, they are interactions among copies while in under the arbitrage economy they are interactions represented by trades.

When the interactions are infinite, both processes determine the same result according to [31].

Theorem 1.2.3. Let ξ be an economy and let $x = (x_i, ..., x_n)$ be an allocation in ξ.

For all $i \in N, Z_i^x = A_i^x$.

Proof. The connection between the interactive choice sets and the replica economy expresses the analogy between large coalitions and large contracts.

Following [31], the theorem can be proved by two lemmas.

The first lemma states that the interactive choice set is a subset of the set interactive set of replica economies.

Lemma 1.2.4. Let ξ be an economy and let $x = (x_i, ..., x_n)$ be an allocation in ξ. For all $i \in N, Z_i^x \subseteq A_i^x$.

Starting from the assumption $Z_i^x = A_i^x, \forall i \in N$, let us take a bundle $z \in Z_i(t+1)$. By definition of interactive choice set:

$$Z_i(t+1) = \sum_{k \in F} Z_k(t) - \sum_{k \in F \setminus \{i\}} W_k(x_k).$$

And so:

$$z \in \sum_{k\in F} Z_k(t) - \sum_{k\in F\setminus\{i\}} W_k(x_k).$$

For some $F \subseteq N, i \in F$.

By the assertion of lemma $Z_k(t) \subseteq A_k$ for all $k \in F$. Let us take another bundle k and for the statement of lemma, let us obtain:

$$Z_k(t) \subseteq A_k(m_k) \quad \text{for some} \quad m_k, \quad \text{for all} \quad k \in F.$$

Taking m as the $max\{m_k : k \in F\}$ and recalling that $\{A_k(m)\}_{m\in\mathbb{N}}$ is an increasing sequence, let us substitute $Z_k(t)$ with $A_k(m_k)$ in the definition of interactive choice set:

$$z \in \sum_{k\in F} A_k(m) - \sum_{k\in F\setminus\{i\}} W_k(x_k) \quad \text{for some} \quad F \subseteq N, i \in F.$$

There must be sets $S_k \subseteq N \times \overline{m}$ with $(k, 1) \in S_k$ such that:

$$z \in \sum_{k\in F}(\sum_{(j,l)\in S_k} e_j - \sum_{(j,l)\in S_k\setminus\{(k,1)\}} W_j(x_j)) - \sum_{k\in F\setminus\{i\}} W_k(x_k)),$$

for some $F \subseteq N$, with $i \in F$.

But this can be written without the summation by definition of set A, abandoning the integer n_k, $e_i + \sum_{k\in F}(e_k - W_k(x_k))$ for some integers $n_k \geq 0$ which is the definition of A_i and so $z \in A_i$.

The second lemma proves that A_i^x is a subset of Z_x. In this way, at infinite, we prove that the two sets coincide.

Lemma 1.2.5. Let ξ be an economy and let $x = (x_i, ..., x_n)$ be an allocation in ξ. For all $i \in N, A_i^x \subseteq Z_i(x)$.

Proof. Let $a \in A_i$. By definition of A_i:

$$a \in e_i + \sum_{k\in F}(e_k - W_k(x_k))n_k.$$

for some $F \subseteq N$ and some positive integers n_k.

By definition 1 of Z_i, we have that:

$$Z_k + Z_{k+1} - W_{k+1}(x_{k+1}) \subseteq Z_k, \forall k \in N. \tag{1.7}$$

where $k + 1$ means $k + 1 mod|N|$. Applying the same inclusion interactively n_k times we have:

$$Z_k + (n_k + 1)(Z_{k+1} - W_{k+1}(x_{k+1})) \subseteq Z_k, \forall k \in N. \tag{1.8}$$

Consider an agent $j \in N \setminus \{i\}$:

$$Z_j + Z_i - W_i(x_i) \subseteq Z_i. \tag{1.9}$$

Substituting (1.10) into (1.9) we have:

$$Z_k + \sum_{k \in N} (Z_k - W_k(x_k)) \subseteq Z_i. \tag{1.10}$$

Substituting (1.8) into (1.11) we get:

$$Z_i + \sum_{k \in N} [Z_k + (Z_{k+1} - W_{k+1}(x_{k+1}))(n_k + 1) - W_k(x_k)] \subseteq Z_i, \tag{1.11}$$

which can be written as

$$Z_i + \sum_{k \in N} [Z_k - W_k(x_k)](n_k + 1) \subseteq Z_i.$$

$$Z_i + \sum_{k \in F} [Z_k - W_k(x_k)(n_k) - W_k(x_k)] \subseteq Z_i.$$

Since $0 \in \sum_{k \in N} (Z_k - W_k(x_k))$ and $e_k \in Z_k, \forall k \in N$ we get:

$$e_i + \sum_{k \in F} (e_k - W_k(x_k)) n_k \subseteq Z_i.$$

From the right side of the last equation, we obtain $A_i(m)$. □

Therefore, $A_i(m) \subseteq Z_i$. □

The previous theorem holds true only for a number of interactions close to infinity. Re-considering the initial example ([30]) with interactive choice set and the utility function $u(x_i) = min\{x_i^1, x_i^2\}$, it is possible to prove that replica sets and interactive choice sets do not coincide in any case.

With two players $i = 1, 2$ and the endowments $e_1 = (0, 2), e_2 = (2, 0)$, the interactive choice set is built as follows. At $t = 0$

$$Z_1^{*x}(0) = (0, 2); \quad Z_2^{*x}(0) = (2, 0).$$

At $t = 1$

$$Z_1^{*x}(1) = (0, 2) \cup ((1, 1) - R_+^2)$$
$$Z_2^{*x}(1) = (2, 0) \cup ((1, 1) - R_+^2).$$

At $t = 2$

$$Z_1^{*x}(2) = Z_2^{*x}(2) = (2,0) \cup (0,2) \cup ((1,1) - R_+^2).$$

For the replica economies, it is obvious that $A_1^x(0) = Z_1^x(0)$ and $A_2^x(0) = Z_2^x(0)$, $A_1^x(1) = Z_1^x(1)$ and $A_2^x(1) = Z_2^x(1)$.

However, the sets at t=2 and for m=2 are different:

$$A_1^x(2) = ((-1,3) - R_+^2) \cup ((2,0) - R_+^2) \cup ((1,1) - R_+^2) \cup ((0,2) - R_+^2),$$

$$A_2^x(2) = ((3,-1) - R_+^2) \cup ((2,0) - R_+^2) \cup ((1,1) - R_+^2) \cup ((0,2) - R_+^2).$$

The sets $A_i^x(m)$ and $Z_i^x(t)$ are different because, in the first set, agent i cooperates with agent j exchanging his initial endowments e_j, at contrary in the interactive choice set, when agent i and agent j cooperate, the object of the trade is the own set $Z_j^x(t)$. These differences explain why the two sets don't coincide for a finite number of interactions.

1.2.2 The equivalence result

The definition of replica economies given through the sets A_i^x creates the connection with the convergence theorem of Debreu and Scarf (see [7]).

Theorem 1.2.6. An allocation $(x_i)_{i \in N}$is Walrasian if and only if $[(x_i)_{i\in N}, (Z_i^x)_{i \in N}]$ is an arbitrage free equilibrium.

Proof. We have already observed that a WE is an arbitrage free-equilibrium. Starting from the second claim of the theorem, assume that $[(x_i)_{i\in N}, (Z_i^x)_{i \in N}]$ is an arbitrage free equilibrium. Therefore, by the point 4. of its definition, $P(x_i) \cap Z_i^x = \emptyset, \forall i \in N$. By Theorem 1.2.3, $P(x_i) \cap A_i^x = \emptyset, \forall i \in N$, so that x is a shrunk allocation. From this conclusion, let us follow the proof of the Debreu-Scarf (see appendix at the end of the chapter) and the theorem is proved. □

1.2.3 Short sales and convexity

The definition of the interactive choice sets assumes the existence of negative bundles, that is the assumption that there exists the possibility of short sales. An example makes it clearer ([31]).

Given two agents 1 and 2 and their initial endowment $e_i = (30,30)$ and $e_2 = (9,9)$, let us define their preferred sets:

$$\begin{aligned} P_1(30,30) = \{&(x_1,x_2) : (x_1,x_2) \gg (30,30)\} \\ \cup\{&(x_1,x_2) : (x_1,x_2) \gg (10,40)\} \\ \cup\{&(x_1,x_2) : (x_1,x_2) \gg (40,10)\} \end{aligned}$$

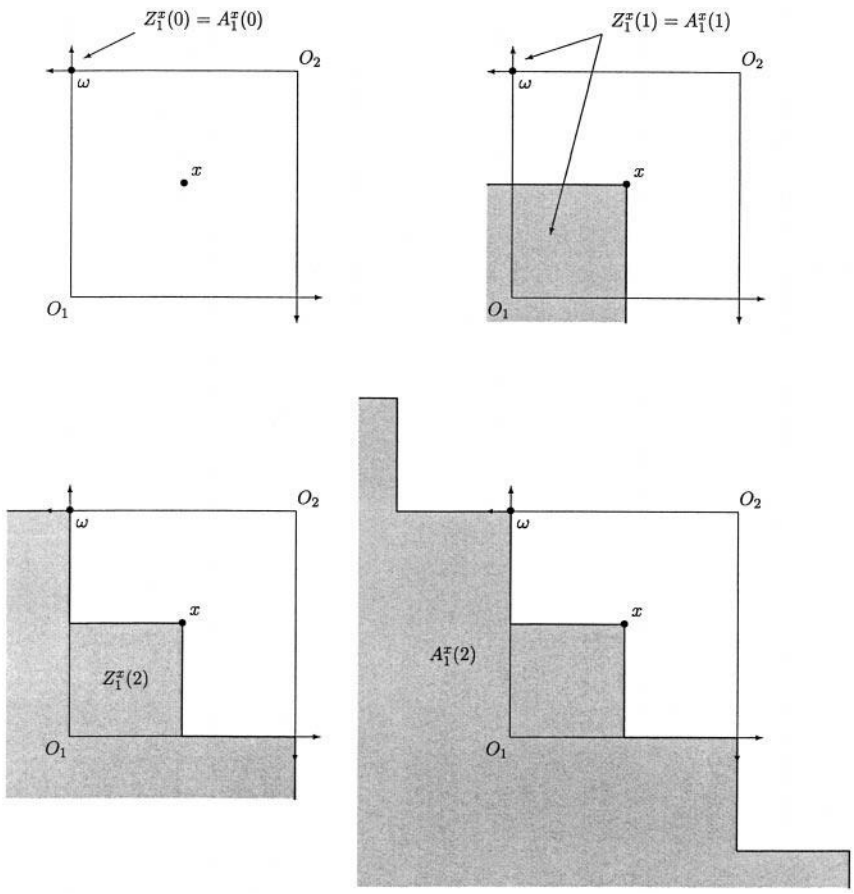

Figure 1.2: The difference between Replica economies and Interactive choice set as reported in [31]

$$P_2(9,9) = \{(x_1,x_2) : (x_1,x_2) \gg (9,9)\}.$$

Thus $Z_1(0) = (30,30)$ and $Z_2(0) = (9,9)$, let us define:

$$Z_1(1) = (30,30) + (9,9) - (9,9) = (30,30)$$

and

$$\begin{aligned} Z_2(1) &= (9,9) + (30,30) - (30,30) = (9,9), \\ Z_2(1) &= (9,9) + (30,30) - (40,10) = (-1,29), \\ Z_2(1) &= (9,9) + (30,30) - (10,40) = (29,-1). \end{aligned}$$

We see that the interactive choice set $Z_2(1)$ has a negative amount of commodities. This example shows the meaning of shorts sales, i.e. the agents pay their bundle by using the upper contour sets of the agents' preferences.

To overcome this restriction, the convexity of preferences is added as assumption:

A5 For all $i \in N$, for all $x_i \in X_i$, the preferred set $P_i(x_i)$ is convex.

In this way, the definition of arbitrage free - equilibrium is converted in that one without short sales.
Arbitrage free- equilibrium without short sales
An arbitrage free-equilibrium is a pair $[(x_i)_{i\in N}, (C_i)_{i\in N}]$ satisfying the following conditions $\forall i \in N$:

i $(x_i)_{i\in N}$ is an allocation;

ii $e_i, x_i \in C_i \subseteq R^L_+$;

iii $\left(\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k)\right) \cap R^L_+ \subseteq C_i, \forall i \in S, \forall S \subseteq N$;

iv $P_i(x_i) \cap C_i = \emptyset$.

We add the constraint of non negative sets on the condition [ii] and [iii]. With the convexity, the equilibrium is ensured because the agents have the possibility to find endless opportunities along a direction, even if they don't have the possibility of short sales.

1.3 Arbitrage free equilibrium of production economies

In analogy with the previous sections, let us give a notion of arbitrage free-equilibrium in a production economy.

A production economy is a triple $\xi = [(x_i, \succeq_i, e_i)_{i\in N}, (Y_j)_{j\in J}, (\theta_{ij})_{(ij)\in N\times J}]$ where $(x_i, \succeq_i, e_i)_{i\in N}$ is an exchange economy, $(Y_j)_{j\in J}$ is the production set and θ_{ij} is the share distribution of profits to the consumers in the firm. An allocation is a pair $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ where for all $i \in N$, for all $j \in J$, for all $x_i \in X_i$, for all $y_j \in Y_j$ one has $\sum_{j\in J} y_j + \sum_{i\in N} e_i = \sum_{i\in N} x_i$.
The interactive choice set with the firms may be defined recursively:

Definition 1

Let $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ be an allocation in the production economy ξ. Let i's ICS Z_i^{*x} be defined recursively. At $t = 0$:

$$Z_i^{*x}(0) = e_i + [\sum_{j\in J} \theta_{ij} y_j]_{j\in J}.$$

For $t > 0$

$$Z_i^{*x}(t) = Z_i^{*x}(t-1) \cup \bigcup_{k\in N\setminus\{i\}} [Z_i^{*x}(t-1) + Z_k^{*x}(t-1) - W_k(x_k)],$$

and finally

$$Z_i^{*x} = \cup_{t\in\mathbb{N}} Z_i^{*x}(t).$$

Definition 2

At $t = 0$:

$$Z_i^{x}(0) = e_i + [\sum_{j\in J} \theta_{ij} y_j]_{j\in J}.$$

For $t > 0$

$$Z_i^{x}(t) = \bigcup_{S\subset N, i\in S} [\sum_{k\in S} Z_k^{x}(t-1) - W_k(x_k)],$$

and finally

$$Z_i^{x} = \cup_{t\in\mathbb{N}} Z_i^{x}(t).$$

As for the case of pure exchange economies, we can prove that:

Proposition 1.3.1. Let ξ be an economy and let $[x_i, y_j]$ be an allocation. Then:

$$Z_i^{*x} = Z_i^{x}.$$

Definition of arbitrage free equilibrium in production economy

Let $\xi = [(x_i, \succeq_i, e_i)_{i\in N}, (Y_j)_{j\in J}, (\theta_{ij})_{ij\in N\times J}]$ be a production economy. An arbitrage free - equilibrium is a pair $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ and $(C_i)_{i\in N}$ that satisfies:

- $(x_i)_{i\in N}$ is an allocation;
- $e_i + \sum_{j\in J} \theta_{ij} y_j \in C_i$ (the agents have the possibility to not re-contract);

- $x_i \in C_i$ (the choice set is feasible);
- $C_i \cap P_i(x_i) = \emptyset$ (optimality) or $C_i \cap X_i$ has an empty interior (both the assertions express the optimality of the choice set);
- $\sum_{k \in S} C_k - \sum_{k \in S \setminus \{i\}} W_k(x_k) \subset C_i, \forall S \subseteq N$ (a new re-contracting condition with the same logic of [5]).

Proposition 1.3.2. Let $[(x_i)_{i \in N}, (y_j)_{j \in J}, (C_i)_{i \in N}]$ be an arbitrage free-equilibrium, then:

- $Z_i^x \subseteq C_i$;
- $[(x_i)_{i \in N}, (y_j)_{j \in J}, (Z_i)_{i \in N}]$ is an arbitrage free-equilibrium.

The proof of this proposition is trivial, following the same steps of the proposition 1.2.2..

1.3.1 Replica in production economies

A replica economy of the economy ξ, denoted with ξ^m, consists of a replication by m-times of ξ, each of whom has the same composition of individuals, firms, endowments, and corporate ownership for $k = 1, ...m$. Individuals or firms of the same type have the same characteristics:

- $u_i = u_{ik}$;
- $e_i = e_{ik}$;
- $Y_i = Y_{ik}$;
- $\theta_{ij} = \theta_{ij,k}$.

Let us define the set of replica economies in terms of recursive sets also in the production case.
Definition of replica production economies

$$A_i^x(0) = \{e_i + \sum_{j \in J} \theta_{ij} y_j\} \quad \forall i \in N,$$

$$A_i^x(m) = \cup_{S \subseteq N \times \overline{m}, (i,1) \notin S} [e_i + \sum_{j \in J} \theta_{ij} y_j + \sum_{(k,l) \in S} (e_k + \sum_{j \in J} \theta_{kj} y_j - W_k(x_k))].$$

$$A_i^x = \cup_{m \in \mathbb{N}} A_i^x(m).$$

The core of a replica production economy is given by allocations x such that:

$$P_i(x_i) \cap A_i^x(m) = \emptyset, \quad \text{for each} \quad i \in N.$$

Arbitrage free equilibrium and competitive equilibrium

Theorem 1.3.3. Let ξ be a production economy and let $(x, y) = [(x_i)_{i\in N}, (y_j)_{j\in J}]$ be an allocation in ξ. For all $i \in N$, $Z_i^x = A_i^x$.

Proof. The proof follows the same steps of the Theorem 4. of Serrano- Volij (see [31]). □

Let us consider the following assumptions:

A1 For all $i \in N$, the consumption set $X_i = R_+^L$;

A2 For all $i \in N$, for all $x_i \in X_i$, the preferred set $P_i(x_i)$ is open relative to X_i;

A3 For all $i \in N$, for all $x_i, y_i \in X_i, y_i \gg x_i$ implies $y_i \succ x_i$, i.e. the preference are strictly monotone;

A6 For all $i \in N, e_i \in X_i, \forall j \in J$, there exists a production plan $(y_j)_{j\in J}$ such that $\sum_{j\in J} y_j + \sum_{i\in N} e_i \gg 0$.

Theorem 1.3.4. An allocation $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ of the production economy is a quasi Walrasian equilibrium if and only if $[(x_i)_{i\in N}, (y_j)_{j\in J}, (Z_i^x)_{i\in N}]$ is an arbitrage free-equilibrium.

Proof. Only an implication must be proved. Let $[(x_i)_{i\in N}, (y_j)_{j\in J}, (Z_i^x)_{i\in N}]$ be an arbitrage free equilibrium. Then, it satisfies the condition for which:

$$P_i(x_i) \cap Z_i^x = \emptyset \quad \text{for each} \quad i \in N.$$

In addition, by Theorem 1.3.3., we know that for all $i \in N, Z_i^x = A_i^x$, so $P_i(x_i) \cap A_i^x = \emptyset$. Therefore, $[(x_i)_{i\in N}, (y_j)_{(j\in J)}]$ is a shrunk core allocation.

Thus $[(x_i)_{i\in N}, (y_j)_{j\in J}]$ satisfies the assumptions of Theorem 3. of Debreu and Scarf (see [7] and the appendix) and by the limit theorem of the core, it is a Walrasian allocation. □

Conclusion

The first chapter should be interpreted as a general overview of concepts of competitive equilibrium. This notion is based on the idea of interactive exchange of bundles among agents. The notion of abstract equilibrium proposed by Dagan in 1996, has been used to build the arbitrage free equilibrium by Serrano and Volji in 1998, moving from Dagan's choice set. In the last paragraph, an extension of arbitrage free equilibrium to production economies has been proposed. All these approaches have in common the idea that when the number of agents becomes large, the core shrinks to the set of competitive equilibria which respects the arbitrage free - equilibrium.

Even if the new approach overcomes a limit of the individualistic view of Walrasian equilibrium, a further limit persists: the construction considered so far always assumes perfect communication and information among agents. To construct a notion of arbitrage free-equilibrium under uncertainty and differential information, will be the main aim of the next chapters. We shall overview economies with uncertainty and under differential information ([1], [20], [27], [23], [33], [34], [29]) their competitive equilibria and the relative notion of core (ex-ante, interim and ex-post) and the equivalence result.

In the third chapter, we propose an extension of the concept of AFE to an economy with uncertainty and differential information.

Appendix

The concept of replica economy is introduced to extend the Walrasian theory to a more general setting. It was mentioned also by Edgeworth to prove that the contract curve shrinks to the competitive equilibrium when the number of agents is close to infinite. In general, the replica economy is useful to compare two realities. Indeed, a comparison between two different situations, adding new preferences and new endowments, would result more complicated with respect to the comparison between two situations in which one is the copy of the other one. The construction of a replica, instead, is easy to handle. It can be considered like a technical expedient.

If ξ is an exchange economy with n consumer the m-fold replica economy ξ^m of ξ is a new exchange economy with mn consumers indicated with $(i, j), i = 1, ..., n; j = 1, ..., m$. The first index refers to the type of individual and the second index distinguishes different individuals of the same type.

For any consumer (i, j), it is assumed that:

- The preference relation $\succeq_{ij}$ is equal to $\succeq_i$.
- The initial endowment $e_{ij} = e_i$ and so the total endowment of the economy is represented by:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij} = me.$$

Any allocation of the economy ξ can be considered as an allocation for each m-fold economy ξ^m and any Walrasian equilibrium of the initial economy is an equilibrium for the m-fold replica economy.

Theorem 1.3.5. Equal treatment property [4].

Let us assume that preferences are strictly monotonic, strictly convex and continuous. In the core, an allocation x assigns the same consumption bundle to all consumers of the same type.

Let $(x_{11}, x_{1i}, ...x_{nm})$ be a core allocation of ξ^m. Then for each type i, and each j, $k = 1, .., m$ we have $x_{ij} = x_{ik}$.

Proof. Let us prove the theorem by contradiction and let us assume $x = (x_{ij})_{(i=1,...,n;j=1,..m)}$ is not an equal treatment allocation, i.e. the consumption differs within types.

We suppose that the allocation x_i less preferred than the $j-$ replica x_{ij} for the agent i. Let us take an allocation $y_i = \frac{1}{m}\sum_{j=1}^{m} x_{ij}$ and by the convexity, $y_i \succeq_i x_i$ for each agent, with the strict preference for at least one agent. The allocation y_i satisfies the feasibility:

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} e_i.$$

In this way, the coalition formed by one consumer of each type receives at least one preferred bundle and so x cannot belong to the core. This is a contradiction. $\square$

The equal treatment theorem implies that when the economy is enlarged to m participants of each type, the resulting allocation is competitive for the larger economy and consequently it is in the core. From that, the convergence theorem follows.

Theorem 1.3.6. Limit of the core [7].

Assume that preferences are continuous, strictly convex, strictly monotone and that there are strictly positive initial endowments. Any allocation in the core of ξ^m, x is Walrasian equilibrium.

Proof. Assume that $x^* \in C(\xi^m)$. Let us define a set P_i as follows:

$$P_i(x_i^*) = \{z_i \in R_+^L : z_i + e_i \succeq x_i^*\}$$

for each i.

Since $P(x^*)$ is a nonempty, open and convex set, it may be written as $\sum_{i=1}^m \alpha_i z_i$ with$\alpha_i \geq 0$ and $\sum_{i=1}^m \alpha_i = 1$ and with $0 \notin P(x^*)$.

By contradiction, assume that $0 \in P(x^*)$. So, for any i and for any α, we can obtain a bundle $\alpha_i z_i \in P_i(x^*)$ such that $\sum_{i=1}^m \alpha_i z_i = 0$.

Taking a $k \in \mathbb{N}$ that tends to infinity and let α_i^k the smallest integer greater than or equal to k and let define for any i an allocation $z_{ij} = \frac{m\alpha_i}{\lambda_{ij}} z_i$.

T The allocation $z_{ij} + e_i$ belongs to the commodity set X_i since it's possible to define it as: $z_i + e_i = t(z_i + e_i) + (1-t)e_i$ with $t = \frac{m\alpha_i}{\lambda_{ij}} z_i$. Since the preferences are continuous, for t which tends to 1 and for a large number m, we have that $z_{ij} + e_i \succ_i x_i^*$. In this way, we have constructed a coalition that blocks x^* and this is impossible thus the allocation is in the core of ξ^m.

The origin doesn't belong to the set and consequently there exists a vector $p^* \in R^L \setminus \{0\}$ such that $p^* z \geq 0$ for any $z \in P(x^*)$.

Let us take a bundle $x_i^{'} \succ_i x_i^*$. Since $x_i^{'} - e_i \in P_i(x_i^*)$, we obtain $p^* x_i^{'} > p^* e_i$. Thus in any neighborhood of x_i^* there exists allocation strictly preferred to x_i^*, we obtain $p^* x_i^* \geq p^* e_i$. But $\sum_{i=1}^m (x_i^* - e_i) = 0$, then $p^* x_i^* = p * e_i$. $\square$

This theorem asserts that for economies with a large number of agents, core allocations are approximately competitive. Both the previous theorems use the assumption of convexity of preferences that is an assumption fully independent from the number of agents in the economy. This means that, without this assumption, in a non convex economy, when there is a replica, there are new equilibria. Only a number of replication near to infinity can avoid this problem and so ensure the existence of an equilibrium (see [15]).

1.3.2 Production economy

An allocation is a distribution of commodities $x = (x_1, ...x_n)$ such that there is a production plan for which the market is clear:

$$\sum_{i \in N} x_i = y + \sum_{i \in N} e_i \quad \text{such that} \quad \sum_{i \in N} (x_i - e_i) \in Y.$$

The core of a production economy is defined as the collection of allocations which cannot be blocked. An allocation $x^{'}$ is blocked by an allocation x , if there exists a coalition S for which:

- $\sum_{i \in S} (x_i^{'} - e_i) \in Y$;
- $x_i^{'} \succeq_i x_i$ for all $i \in S$ and for at least one agent the relation is strictly preferred.

Theorem 1.3.7. Limit of the core in production economy[7].

Assume that preferences are continuous, strictly convex, strictly monotone and that there are strictly positive initial endowment, assume that Y is a convex cone. Then, any allocation in the core of the economy ξ^m is an equal treatment allocation and if x is in the core of any replicated economy, it is a Walrasian equilibrium.

Proof. The first step is to prove that a competitive allocation is in the core. So, let us take a blocking set S and a blocking allocation x_i^* such that it respects:

- $\sum_{i \in S} (x_i^* - e_i) \in Y$;
- $x_i^* \succeq_i x_i$ for all $i \in S$ and for at least one agent the relation is strictly preferred;
- $py \leq 0$.

Since, $py_i^* > pe_i, \forall i \in S$, we have $\sum_{i \in S} x_i^* \geq \sum_{i \in S} pe_i$ or $py > 0$, but this last one is a contradiction so a competitive allocation is in the core. Let us describe a replica economy with n types of consumers i and m copies j. The allocation of this new economy is described by a collection of nonnegative bundles x_{ij} such that:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} - m \sum_{i=1}^{n} e_i = y.$$

The term $m \sum_{i=1}^{n} e_i$ derives from the fact that an allocation in the core assigns the same consumption to all agents of the same type.

Let us define a set $P_i(x_i^*) = \{z \in R_+^L | z + e_i \succeq_i x_i\}$. This set is disjointed from the set Y. If not, we have that $\sum_{i=1}^{m} \alpha_i z_i = y$ with $\alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1$ and

$z_i + e_i \succ_i x_i$. Let us assume that $\alpha_i z_i \in Y$ such that for $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i z_i = y$. For any copy j, let us define an integer λ_{ij} such that $z_{ij} = \frac{m\alpha_i}{\lambda_{ij}} z_i$. Since the economy respects the strong convexity, it is possible to write $z_{ij} + e_i = t(z_i + e_i) + (1-t)e_i$ with $t = \frac{m\alpha_i}{\lambda_{ij}} z_i$, remembering that $\frac{m\alpha_i}{\lambda_{ij}} z_i \in Y$. In this way, x^* is blocked because $z_{ij} + e_i \succ_i x_i^*$, which is a contradiction. From now on, the proof follows the same steps of the exchange economies's case. Let us take a bundle $x_i^{'} \succ_i x_i^*$. Then, $x_i^{'} - e_i \in P_i(x_i^*)$, by continuity $p^* x_i^{'} > p^* e_i$. Therefore, $p^* x_i^* = p^* e_i$ and the allocation is a Walrasian equilibrium for the economy ξ. $\square$

Chapter 2

The uncertainty and the differential information economy

Chapter 2 explores the important connection between several notions of core and of competitive equilibrium, under uncertainty and differential information. The reference model will be the one of a pure exchange economy with a finite number of commodities and a finite number of agents.

The chapter is organized as follows:

- Section 2.1 presents the concept of the ex-ante core under uncertainty ([15], [27], [33]).
- Section 2.2 introduces the asymmetric information of the agents, dealing with the case of ex-ante differential information economies ([34], [1], [20], [17]).
- Section 2.3 focuses on the interim core and the constrained market equilibria ([6], [8]).
- Section 2.4 studies the ex-post core and the rational expectations equilibria ([9], [3]).

The different notions of core analyzed turn out to be equivalent when the uncertainty is solved and the economy is one with complete information ξ. The core notions may differ for two reasons: the meaning of feasibility; the meaning of improvement.

The allocation could be physically and "*informationally*" feasible:

i. When the private information becomes public and so verifiable, no other restrictions are necessary to define the feasibility.

ii. When the private information is unverifiable, the allocation has to be measurable with respect to the information of the agents.

The mechanism of *"improving upon"* an allocation changes depending on the configuration of utility:

i. If the agents enter into a coalition at an ex-ante stage, the ex-ante expected utility is the right configuration to compare allocations.

ii. If the agents form the coalition at the interim stage, the utility is represented through the conditional expected utility (conditional to the information).

iii. If the agents form the coalition at the ex-post stage, the utility will depend on the realized state of nature.

2.1 The Arrow-Debreu model under uncertainty

Arrow and Debreu [27] introduced uncertainty in the analysis of competitive equilibrium by means of an exogenous space of states of nature and the concept of contingent commodities. The idea of their model is that commodities can be distinguished not only by their physical features, but also by the environmental event in which the items are available (such as a particular date or place). They considered the possibility that goods may depend on the states of nature in which they are consumed.

In accordance with these observations, they describe the economic environment through three variables:

1. Decision variables. They are controlled and chosen by the economic agents.

2. Environmental variables. They are not controlled by any economic agent.

3. All others variables. They are a mix of the previous ones.

A state of the environment is a complete specification of the environmental variable, an event is a set of states. In this type of economy, the contract consists of the purchase (or sale) of a good at a specified date and location if and only if the event occurs.

The states of nature are identified with the possible outcomes of uncertainty. The set of all possible states of nature Ω has the following characteristics:

- All the elements in Ω are mutually exclusive;
- Ω is exhaustive;
- Ω is known to everybody;
- Ω is a finite set.

The state of nature is denoted by $\omega \in \Omega$.

For any physical commodity $l = 1, ..., L$ and for each state of nature $\omega \in \Omega$, a unit of (state) contingent commodity $x_l(\omega)$ is a title to receive a unit of physical good l if and only if the state ω occurs. Accordingly, a state-contingent consumption bundle is specified by:

$$x = (x_1(\omega), .., x_L(\omega)) \in R_+^{L\Omega}.$$

The consumption set of consumer i is $X_i \subseteq R_+^{L\Omega}$.

The agent i's preference $\succeq_i$ has an expected utility representation. For any $x_i, y_i \in X_i, x_i \succeq_i y_i$ if and only if

$$\sum_{\omega \in \Omega} \pi_i(\omega) u_i(\omega, x_i(\omega)) \geq \sum_{\omega \in \Omega} \pi_i(\omega) u_i(\omega, y_i(\omega)),$$

where $\pi_i(\omega)$ is the probability that consumer i assigns to the state ω.

Consumer i's utility function for actual consumption in each state ω is:

$$u_i(\omega, \cdot) : R_+^L \to R.$$

The utilities have an ex-ante nature because the variables describe the possible consumption before the revelation of uncertainty [15].

The initial endowment of consumer $i \in N$ is:

$$e_i = (e_1(\omega), ..., e_L(\omega)) \in R_+^{L\Omega}.$$

A price system p is a function defined by:

$$p : \Omega \to R_+^L.$$

The agent i's budget set at price p is defined by:

$$B_i(p, e) = \{x_i \in X_i; \sum_{\omega \in \Omega} p(\omega) x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) e_i(\omega)\},$$

so that, the budget set depends on contingent goods.

The market with uncertainty takes place at two time periods, date 0 and date 1. The sequence of events is:

1. At $t = 0$. All markets are open. Individuals choose $x \in R_+^{L\Omega}$ and pay prices $p \in R_+^{L\Omega}$, but they ignore the state of nature that will be realized at the time of consumption;

2. At $t = 1$. Once uncertainty is revealed and the state of nature becomes known, the contracts contingent to state ω are executed i.e. the goods are delivered and consumed while the contract contingent to void states are destroyed.

There are markets for any contingent commodity. Their features are:

- Markets open before the realization of uncertainty.
- Each contingent commodity has a price.
- Deliveries are contingent.
- All individuals have the same information and this symmetry is common knowledge.

Formally, the market economy under uncertainty represents a particular case of the Walrasian economy because the information is symmetric across economic agents. Therefore, the Walrasian concept can be applied to this economy. A *"State Contingent Economy"* (or an Economy under Uncertainty) is characterized by:

$\xi = \{(\Omega, \pi); (X_i, u_i, e_i)_{i \in N}\}$ where N represents the set of the agents, Ω the set of all possible states of nature, π the common prior, X_i represents the consumption set and the remaining elements the utility function and the initial endowment of agent i.

For this economy, it's possible to define the Arrow-Debreu Equilibrium under uncertainty.

A pair (x^*, p^*) with $x^* \in R_+^{L\Omega}$ and $p^* \in R_+^{L\Omega}$ represents an *Arrow-Debreu Equilibrium* [1] if:

- For all $i \in N, x_i^*$ maximizes $\succeq_i$ on $B_i(p, e)$:

$$B_i(p, e) = \{x_i \in X_i; \sum_{\omega \in \Omega} p(\omega) x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) e_i(\omega)\}.$$

- x^* is feasible in each state $\omega \in \Omega$:

$$\sum_{i \in N} x_i^*(\omega) \leq \sum_{i \in N} e_i(\omega).$$

This is the specification of a market in which all the features are settled before the beginning of the trade and so there is no incentive to modify the plan or to reopen the market. In the Arrow- Debreu economy at time $t = 0$, agents have incomplete information about the state of environment, but all the agents have the same information. In the differential information model that we are going to introduce, there may be differences about the capacity of agents to discern the states.

[1] Also called Walrasian expectations equilibrium.

2.2 The model under differential information

In this section, we take into account the possibility that traders have different information structures, presenting an overview of models with asymmetric information. Radner [27] was the first author that criticized the previous economy introducing the information structure concept.

Indeed, his framework is not only affected by the uncertainty about the states of nature, but also by the phenomenon of differential information.

From his theories forward, several models of differential information have been developed. To analyze the new framework, it is necessary to consider the timing of process: ex-ante, interim and ex-post stage. In the ex-ante stage, the agents take decision only using their private information; in the interim stage, the agents know their own type and they update their information; in the ex-post stage, the state of nature is realized and agents consume according to contracts subscribed at the ex-ante or interim stage.

2.2.1 The ex-ante differential information economy

The Radner economy is an ex-ante differential information economy because the agents decide what to consume based on their private information, before the opening of the market. A pure exchange economy with differential information consists of a finite number of agents each of whom is characterized by a random utility function, a random initial endowment, a prior as before and a private information structure. So it is given by $\xi = ((\Omega, \mathcal{F},), x_i, u_i, e_i, \mathcal{F}_i, \pi), i \in N$ where:

- $(\Omega, \mathcal{F})$ is a measurable space. Ω denotes the set of states of nature, $\mathcal{F}$ represents the set of all events. An information correspondence for the agent i is a nonempty valued correspondence:

  $$P_i : \Omega \to \mathcal{F}.$$

  $P_i(\omega)$ represents the event in $\mathcal{F}$ observed by i when ω is realized. This means that the agent i, in the state ω, only knows that the state is in $P_i(\omega)$, but he is not able to distinguish two states in $P_i(\omega)$. Usually, the information correspondence matches with an agent's knowledge having the following characteristics:

  i. The agent always includes the true state in the set of states. So for any $\omega \in \Omega, \omega \in P_i(\omega)$;

  ii. The agent uses his information to make inferences about the state. If $\omega^{'} \in P_i(\omega), P_i(\omega^{'}) = P_i(\omega)$.

$\mathcal{F}$ represents the events (a subset of Ω) to which the agents assign a probability of occurrence;

- $\mathcal{F}_i$ is a (measurable) partition of $(\Omega, \mathcal{F})$ in the sense given before. The consumer i isn't able to discriminate among the states of nature belonging to any element $P_i(\omega)$ of $\mathcal{F}_i$;
- $u_i : \Omega \times R^L_+ \to R$ is the random utility function of agent i. The utility function respects the invariance property ($\mathcal{F}_i$ measurability functions of ω);
- π is the common prior of all agents.

The (ex-ante) expected utility of the agent i:

$$V_i(x_i) = \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\pi(\omega).$$

An allocation x assigns a commodity bundle $x_i(\omega)$ to each agent i in each state ω.

It is a function $x_i : \Omega \to R^L_+, \mathcal{F}_i$-measurable and physically feasible:

$$\sum x_i(\omega) = \sum e_i(\omega), \forall \omega \in \Omega.$$

Let us define the Radner competitive equilibrium.

Denote by X_i the set of all functions $x_i : \Omega \to R^L_+$ which are $\mathcal{F}_i$-measurable, i.e. informationally feasible from the point of view of agent i. Given a price system $p : \Omega \to R^L_+$, the budget set of agent i is given by:

$$B_i(p) = \{x_i \in X_i : \sum_{\omega \in \Omega} p(\omega)x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega)e_i(\omega)\}.$$ [2]

A pair (p^*, x^*), where p is the system price and $x = (x_1, ...x_n)$ is an allocation, is a *Radner equilibrium* if:

i. For all $i \in N, x_i \in X_i$ and x_i maximizes the ex-ante expected utility of i on $B_i(p)$;

ii. $\sum_{i \in N} x_i(\omega) \leq \sum_{i \in N} e_i(\omega), \forall \omega \in \Omega$;

iii. $\sum_{\omega \in \Omega} p(\omega)x_i(\omega) = \sum_{\omega \in \Omega} p(\omega)e_i(\omega), \forall i \in N$.

The interpretation of the Radner equilibrium follows the same logic of the Walrasian equilibrium. The agents maximize their ex-ante expected utility under their budget constraint. The main difference with respect to the WE is that the budget set includes the informational constraints. Notice that when each trader has perfect information, the model gives back the Arrow-Debreu model under uncertainty.

[2] Notice that consumption bundles in the budget set are constants over the states of nature that the agent i is not able to distinguish.

2.2.2 Ex-ante core in differential information economies

Many of the core notions for an economy with asymmetric information share the same basic assumptions about the impact of information on an agent's consumption decisions: if an agent is partially informed about the reality, this imperfect information restricts his choice of consumption plans, according to his private information (see [29]). The Radner economy is characterized by the concept of private core, in which the agents do not communicate each other when they join a coalition (the name private derives from this kind of sharing information rule). Formally this means that an allocation in the private core will be informationally measurable with respect to each agent private information and so is also the allocation improving the status quo.

An allocation $x = (x_1, ...x_n)$ is said to be a private core allocation for the economy if the following conditions are satisfied:

- Each x_i is $\mathcal{F}_i$-measurable, i.e. $x_i \in X_i, \forall i \in N$;
- $\sum_{i \in N} x_i(\omega) \leq \sum_{i \in N} e_i(\omega), \forall \omega \in \Omega$;
- there does not exist a coalition S and an assignment $(y_i)_{i \in S}$ such that:
    1. $\sum_{i \in S} y_i(\omega) \leq \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega$;
    2. y_i is $\mathcal{F}_i$ - measurable (or $y_i \in X_i$), $\forall i \in S$;
    3. $V_i(y_i) > V_i(x_i), \forall i \in S$.

Each x_i is $\mathcal{F}_i$- measurable $\forall i \in N$ and has the property that no groups of agents within a coalition can redistribute their initial endowments, based on their own private information, making each better off within the coalition. The private core of the economy ξ is the set of all feasible allocations which are not private blocked by any coalition. Notice that when agents have perfect information, the private core is the ex-ante core of an Arrow - Debreu economy.

The equivalence theorem for the private core

The equivalence result of Debreu and Scarf may be applied also in the differential information economy with some differences [17].

Let us denote by $WE(\xi)$ and $C(\xi)$ respectively the set of all Radner Equilibria and the private core.

Theorem 2.2.1. Let x be an allocation in the economy ξ. If $x \in WE(\xi)$, then $x \in C(\xi)$.

Proof. Let (p, x) a Radner equilibrium. Assume by contradiction that $x \notin C(\xi)$. So there exists a coalition S and an assignment $(y_i)_{i \in S}$ such that:

- Each y_i is $\mathcal{F}_i$-measurable;
- $\sum y_i(\omega) \leq \sum e_i(\omega), \forall \omega \in \Omega$;
- $V_i(y_i) > V_i(x_i) \forall i \in S$.

By definition of Radner equilibrium, x_i is feasible and maximizes $V_i(x_i)$ on $B_i(p, \omega)$. Thus $y_i \notin B_i(p, \omega)$. Knowing that y_i is $\mathcal{F}_i$-measurable, we have that for each $i \in S$:

$$\sum_{\omega \in \Omega} p(\omega) y_i(\omega) > \sum_{\omega \in \Omega} p(\omega) e_i(\omega).$$

This implies:

$$\sum_{i \in S} \sum_{\omega \in \Omega} p(\omega) y_i(\omega) > \sum_{i \in S} \sum_{\omega \in \Omega} p(\omega) e_i(\omega).$$

This is a contradiction of the feasibility of y on S, so the theorem is proved. $\square$

To prove the inverse inclusion, we need to introduce the replicas.

2.2.3 Replica in differential information economy

If m is any positive integer, then the m-fold replica economy of ξ, denoted by ξ^m, is a new differential information economy with mn agents indexed by $(i, j), i = 1, ..., n, j = 1, ..., m$ such that each agent (i, j) has the same characteristics of agent i; that is, $\forall j \in \{1, ...m\}$:

- $\mathcal{F}_{ij} = \mathcal{F}_i$;
- $e_{ij} = e_i$;
- $u_{ij} = u_i$;
- $\pi_{ij} = \pi_i$.

Notice that all the agents $(i, j), j = 1, ...m$ are copies of agent i.

Any allocation $x = (x_1, ..., x_n)$ of ξ gives rise to an allocation $x = (x_{11}, ..., x_{nm})$ for the m-fold replica economy ξ^m with $x_{ij} = x_i$ for $j = 1, ..., m$ and $i = 1, ..., n$.

The equivalence result of Debreu and Scarf has validity also in the presence of asymmetric information. For this part, we will refer to the section 2.2.2 of Meo dissertation (see [25]).

2.3 The type-agent economy

In this section, we analyze a model with differential information in which agents take their decision at the interim stage. Following De Clippel [6], let us introduce the interim core using the notion of type-agent economy. The agents are still partially informed when they decide what to consume but, when they consume, the state of nature is commonly known and publicly announced. This timing avoids any problem of incentive compatibility and measurability constraint[3].

Let us denote the economy by $\xi = (\Omega, u_i, e_i, \mathcal{F}_i, \pi)$. The conditional probability $\pi(\omega|E)$ of an event E is:

$$\pi(\omega|E) = 0 \quad \text{if } \omega \notin E \quad \text{or}$$
$$\pi(\omega|E) = \left(\frac{\pi(\omega)}{\pi(E)}\right) \quad \text{iff} \quad \omega \in E.$$

The interim expected utility $V_i(x_i|P_i(\omega))$ is:

$$V_i(x_i|P_i(\omega)) = \sum_{\omega' \in P_i(\omega)} u_i(x_i(\omega), \omega')\pi(\omega'|P_i(\omega)) = V_i(x_i|\omega).$$

We introduce now market conditions and constrained market equilibria (De Clippel and Wilson).

An allocation rule a is a *constrained market equilibrium* if it is feasible and there exists a price system $p : \Omega \to R^L_+$ such that for each $\omega \in \Omega$:

$$a_i \in argmax \quad V_i(a_i'|P_i(\omega)) \quad \text{on the set}$$

$$B_i(p, \omega) := \{a_i' \in R_+^{L\Omega}| \sum_{\omega' \in P_i(\omega)} p(\omega')a_i'(\omega') \leq \sum_{\omega' \in P_i(\omega)} p(\omega')e_i(\omega')\}.$$

$B_i(\omega)$ is the budget set of the agent i in the state $\omega \in \Omega$. So, a constrained market equilibrium allocation is a feasible allocation for which there exists a price system such that, for each ω and for each agent i, it maximizes the interim expected utility function. Intuitively, each agent, using only his own information, maximizes his expected utility function under the additional constraint that he may not sell

[3]Problems of incentive compatibility constraint occur whenever the agents take decisions at the ex-ante stage, but they receive private information which is not publicly verifiable before the consumption takes place.

Notice that the measurability constraint, like the one introduced in the previous section, refers to the case in which the state doesn't become public. In this hypothesis, the trade of an agent has to be measurable with respect his private information ([12]).

contingent commodities associated with states that he knows. In this case, the invisible hand specifies a price for each commodity in each state in order to clear all the markets.

Then, let us introduce the set $D(a, a^{'}, \omega)$ of agents that prefers to receive, at the interim stage given their private information, the allocation $a^{'}$ instead of keeping allocation a. Given the allocation rules $a, a^{'}$, let $D(a, a^{'}, \omega)$ denote the set of deviators:

$$D(a, a^{'}, \omega) := \{i \in N \quad | \quad V_i(a_i^{'}|P_i(\omega)) > V_i(a_i|P_i(\omega))\},$$

for each $\omega \in \Omega$.

An allocation rule $a^{'}$ is strictly feasible when proposed against a if:

$$\sum_{i \in D(a,a^{'},\omega)} a_i^{'}(\omega) \leq \sum_{i \in D(a,a^{'},\omega)} e_i(\omega),$$

for each $\omega \in \Omega$, the inequality being strictly for some $\omega \in \Omega$. This notion will be used to define the type agent core in the next section.

2.3.1 Type-agent Core and the other interim core notions

Before to define the De Clippel type-agent core, we recall other definitions of interim core notions. The first two are given by Wilson (1978):

- *The coarse core.* The re-allocation of initial endowment can happen only through contingent events which are common knowledge for everyone in the coalition. So, the CC is the set of all feasible allocations which cannot be blocked by any coalition when agents do not need to communicate. The coarse core is based on the assumption that a coalition can focus its potential objection on an event if and only if the event is commonly known to all the members of the coalition. An allocation $x : \Omega \to R_+^{LN}$ is said to be a coarse core allocation for the economy if the following conditions hold:
  - Each x_i is $\wedge_{i \in N} \mathcal{F}_i$ - measurable;
  - $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega;$
  - there does not exist a coalition S and a function $y : \Omega \to R_+^{LN}$ such that:
    - $\sum_{i \in S} y_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega, y_i$ is $\wedge_{i \in S} \mathcal{F}_i$ - measurable and
    - $V_i(y_i|\omega) > V_i(x_i|\omega), \forall i \in S, \forall \omega \in \Omega.$

- *The fine core.* It is the extreme case in which a coalition can make an objection choosing any informational event which can be discerned by pooling the private information. The FC is the set of all feasible allocations that cannot be blocked by any coalition given a perfect communication system. An allocation $x : \Omega \to R_+^{LN}$ is said to be a fine core allocation for the economy if the following conditions hold:
  - Each x_i is $\vee_{i \in N} \mathcal{F}_i$ - measurable;
  - $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega$;
  - there does not exist a coalition S and a function $y : \Omega \to R_+^{LN}$ such that:
    - $\sum_{i \in S} y_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega, y_i$ is $\vee_{i \in S} \mathcal{F}_i$ - measurable and
    - $V_i(y_i|\omega) > V_i(x_i|\omega) \forall i \in S, \forall \omega \in \Omega$.
- We introduce now a general definition of *interim core*, without deviators. The interim core is the set of all feasible allocations rule that cannot be blocked in the following sense: an allocation rule a is blocked if there exists another allocation rule $a^{'}$ and a family of coalitions $S = \{S(\omega)\}$ in each state of nature ($\{S(\omega)\} \subseteq N, \forall \omega \in \Omega$) such that:
  - The coalition $S(\omega)$ is non-empty for at least one $\omega \in \Omega$;
  - $i \in S(\omega) \Rightarrow i \in S(\omega^{'})$ for each $\omega^{'} \in P_i(\omega)$;
  - $\sum_{i \in S} a_i^{'}(\omega) = \sum_{i \in S} e_i(\omega)$;
  - $V_i(a_i^{'}|P_i(\omega)) > V_i(a_i|P_i(\omega)), \forall \omega \in \Omega, \forall i \in S$.

Notice again that measurability constraints are ignored because the state becomes public when consumption takes place.

Let us analyze deeply the type-agent core. An allocation rule a is blocked if there exists an allocation rule $a^{'}$ that is strictly feasible when proposed against a.

The *type agent core* is the set of feasible allocation rules that are not blocked. We recall below the well-known relationship between the type-agent core, the coarse core and the fine core. The equivalence between the interim core and the type agent core will be presented later with the notion of fictitious economy.

The coarse core and type-agent core

Let S be a coalition, an event $E \subseteq \Omega$ is common knowledge among the members of S if it can be written as a union of elements of P_i for each $i \in S$. An allocation rule is feasible for the coalition S if:

$$\sum_{i \in S} a_i(\omega) \leq \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega.$$

The coalition S has a coarse objection against an allocation rule a if there exists an allocation rule $a^{'}$ feasible for S and an event E that is commonly known to each member of the coalition S such that:

$$V_i(a_i^{'}|P_i(\omega)) > V_i(a_i|P_i(\omega)).$$

for each $i \in S$, for each $\omega \in \Omega$.

We have seen that the coarse core is the set of feasible allocation rules against which no coalition has a coarse objection[4].

Theorem 2.3.1. The type-agent core is a subset of the coarse core.

The fine core and the type - agent core

Let a be an allocation rule. A coalition has a fine objection against the allocation a if there exists an event E and an allocation rule $a^{'}$ feasible for S such that the following properties are true at each $\omega \in E$, for each $i \in S$:

1. $\cap_{i \in S} P_i(\omega) \subseteq E$;
2. $V_i(a_i^{'}|E \cap P_i(\omega)) > V_i(a_i|E \cap P_i(\omega))$.

The first point defines the fine core as the intersection of all private states $P_i(\omega)$ of each agent i that, shared, becoming common, and so a subset of the common event E.

The second point indicates that the coalition is not blocked.

The fine core is the set of all feasible allocation rules against which no coalition has a fine objection.

2.3.2 The type core equivalence

We report some main results which are valid in a type agent economy (see [6]).

Theorem 2.3.2. Suppose that each agent has a strictly positive amount of each good in each state of the economy. Then the set of constrained market equilibria is not empty.

Theorem 2.3.3. The set of constrained market equilibria is a subset of the type-agent core.

[4]The coarse objections are based on events that are common knowledge among the members of the coalition instead the fine objections are based on events that can be discerned by pooling the information of the members of the coalition.

Theorem 2.3.4. Suppose that each agent's utility function is strictly concave in each state of the economy and that each agent is endowed with a strictly positive amount of each good in each state of the economy. Then, the type-agent core shrinks to the set of constrained market equilibria as the number of replicas m tends to infinity.

These theorems have been proved by De Clippel (see [6]). We recall below the construction of the fictitious economy functional to the equivalence result.

The construction of a fictitious economy

In order to prove the previous theorems concerning the equivalence result, De Clippel uses a fictitious economy defined with uncertainty and symmetric information constructed as follows.

Let $\overline{\xi}$ be the fictitious economy $(\Omega, \overline{u}_i, \overline{e}_i, \mathcal{F}, \pi)$ where Ω is the set of the states of nature, $\overline{u}_i$ is the utility function of type-agent (i, E), $\overline{e}_i$ is the initial endowment function of type-agent (i, E). The set of type agents is $\mathcal{N}$, the set of good is R^L_+. The type-agent (i, E) is represented by a couple where i is any agent of the original economy and E an atom of his information partition. All the agents have the same beliefs regarding the probability of occurrence and they don't have private information.

The utility of type agent (i, E) for the bundle $x \in R^L_+$ at ω equals $u_i(x, \omega)$ if $\omega \in E$ and zero otherwise. Since we are in an Arrow-Debreu economy, type-agents decide today how to redistribute their endowments when the state will be common knowledge, i.e. the fictitious economy is like an economy under uncertainty. Each agent in the original economy ξ gives rise to as many type-agents as are the events in his information partition.

An allocation rule in the fictitious economy is denoted by:

$$\overline{a} : \Omega \to R^{L\mathcal{N}}_+.$$

There is a natural correlation between allocations of the original economy and allocations of type agent economy. Given an allocation rule $\overline{a}$ in the type agent economy, its corresponding allocation in the original economy is:

$$a_i(\omega) = \overline{a}_{(i, P_i(\omega))}(\omega), \forall i \in N, \forall \omega \in \Omega.$$

Similarly, the allocation rule $\overline{a}$ in the type-agent representation, associated to the allocation rule a of the initial economy, is $\forall (i, E) \in \mathcal{N}$:

$$\overline{a}_{(i,E)}(\omega) = a_i(\omega) \quad \text{if} \quad \omega \in E,$$
$$\overline{a}_{(i,E)}(\omega) = 0 \quad \text{if} \quad \omega \in \Omega \setminus E.$$

For the type-agent representation, the initial endowment function is:

$$\overline{e}_{(i,E)} : \Omega \to R^L_+.$$

where the endowment of type agent (i, E) is defined $\forall (i, E) \in \mathcal{N}$ by:

$$\overline{e}_{(i,E)}(\omega) := e_i(\omega) \quad \text{if} \quad \omega \in E,$$

$$\overline{e}_{(i,E)}(\omega) := 0 \quad \text{if} \quad \omega \notin E.$$

Given the initial endowment $\overline{e}$ in the type-agent economy, its corresponding in the original economy is:

$$e_i(\omega) = \overline{e}_{(i,P_i(\omega))}(\omega), \forall i \in N, \forall \omega \in \Omega.$$

An allocation $\overline{a}$ is feasible for coalition $\mathcal{S} \subseteq \mathcal{N}$ if as usual:

$$\sum_{(i,E)\in\mathcal{S}} \overline{a}_{(i,E)}(\omega) \leq \sum_{(i,E)\in\mathcal{S}} \overline{e}_{(i,E)}(\omega), \quad \forall \omega \in \Omega.$$

Notice that in the fictitious economy, the expected utility has the ex-ante configuration because we are in an Arrow-Debreu model. The core is the set of allocation rules $\overline{a}$ feasible for $\mathcal{N}$ and such that there do not exist a coalition $\mathcal{S}$ and allocation rule $\overline{a}^{'}$ that dominates $\overline{a}$ over $\mathcal{S}$.

Proposition 2.3.5. Let a be an allocation rule that belongs to the type-agent core. Then the associated allocation rule $\overline{a}$ belongs to the core of the type-agent representation of the economy.

Proof. The allocations are feasible in both economies. Indeed, for each $\omega \in \Omega, \sum_{i\in N} a_i(\omega) \leq \sum_{i\in N} e_i(\omega)$ implies that:

$$\sum_{i\in N} \overline{a}_{(i,P_i(\omega))}(\omega) \leq \sum_{i\in N} \overline{e}_{(i,P_i(\omega))}(\omega),$$

and so

$$\sum_{(i,E)\in\mathcal{N}} \overline{a}_{(i,E)}(\omega) \leq \sum_{(i,E)\in\mathcal{N}} \overline{e}_{(i,E)}(\omega).$$

Suppose that there exists an allocation rule $\overline{a}^{'}$ that dominates an allocation rule $\overline{a}$ in a coalition $\mathcal{S}$. If we consider the relation between allocation rule of original economy and fictitious economy, we know that $a_i^{'} := \overline{a}_{(i,P_i(\omega))}(\omega)$ for each couple $(i, \omega) \in N \times \Omega$ such that $\omega \in P_i(\omega)$ and $a_i^{'} := 0$ for each other couple $(i, \omega) \in N \times \Omega$.

Notice that, for each $\omega \in \Omega, i \in D(a, a', \omega)$ if and only if $(i, P_i(\omega)) \in \mathcal{S}$. So, we have:

$$\begin{aligned}
\sum_{i \in D(a,a',\omega)} a_i'(\omega) &= \sum_{i \in N s.t. (i,P_i(\omega)) \in \mathcal{S}} a_i'(\omega) \\
&= \sum_{i \in N s.t. (i,P_i(\omega)) \in \mathcal{S}} \overline{a}'_{(i,P_i(\omega))}(\omega) \\
&= \sum_{(i,E) \in \mathcal{S}} \overline{a}'_{(i,E)}(\omega) \\
&\leq \sum_{(i,E) \in \mathcal{S}} \overline{e}_{(i,E)}(\omega) \\
&= \sum_{i \in N s.t. (i,P_i(\omega)) \in \mathcal{S}} \overline{e}_{(i,P_i(\omega))}(\omega) \\
&= \sum_{i \in N s.t. (i,P_i(\omega)) \in \mathcal{S}} e_i(\omega) \\
&= \sum_{i \in D(a,a',\omega)} e_i(\omega).
\end{aligned}$$

and a contradiction. □

With similar arguments one can prove the following.

Proposition 2.3.6. Let $\overline{a}$ be an allocation rule that belongs to the core of the type-agent representation of the economy. Then the associated rule a belongs to the type-agent core.

A similar correspondence holds true for the competitive equilibria as proved in the two following propositions:

Proposition 2.3.7. Let a be a constrained market equilibrium. Then, the associated allocation rule $\overline{a}$ is an Arrow-Debreu equilibrium in the type-agent representation of the economy.

Proof. We know, as in Proposition 2.3.2, that $\overline{a}$ is feasible for $\mathcal{N}$. Let p be the price system associated to a. Now, let us consider the type-agent $(i, E) \in \mathcal{N}$ noticing that $\overline{a}_{(i,E)} \in \overline{B}_{(i,E)}(p)$ as the allocation rule $a_i \in B_i(p)$ for each $\omega \in E$. Let us take another allocation rule $\overline{a}'_{(i,E)} \in \overline{B}_{(i,E)}(p)$ and let us prove that the expected utility associated to $\overline{a}_{(i,E)}$ is greater or at least equal to the expected utility associated to $\overline{a}'_{(i,E)}$:

$$V_{(i,E)}(\overline{a}_{(i,E)}) \geq V_{(i,E)}(\overline{a}'_{(i,E)}).$$

Given the preferences of the type-agent (i, E), $\overline{a}_{(i,E)}(\omega) := 0$ for each $\omega \in \Omega \setminus E$. Let $\omega \in E$ and let $\overline{a}'_{(i,E)} \in R_+^{L\Omega}$ defined as $a_i'(\omega') := \overline{a}'_{(i,\mathcal{F}_i(\omega))}(\omega')$ for each $\omega' \in \Omega$.

Then, $a_i^{'} \in B(p, \omega)$. For the initial hypothesis a_i maximizes the utility of the agents under the budget constraint:

$$V(a_i^{'}|P_i(\omega)) < V(a_i|P_i(\omega)).$$

Since $V(a_i^{'}|P_i(\omega))\pi(P_i(\omega)) = V_{(i,E)}(\overline{a}_{(i,E)}^{'})$ and $V(a_i|P_i(\omega))\pi(P_i(\omega)) = V_{(i,E)}(\overline{a}_{(i,E)})$, we can prove that:

$$V_{(i,E)}(\overline{a}_{(i,E)}) \geq V_{(i,E)}(\overline{a}_{(i,E)}^{'}).$$

□

Proposition 2.3.8. Let $\overline{a}$ be an Arrow-Debreu equilibrium in the type-agent representation of the economy. Then the associated allocation rule a is a constrained market equilibrium.

Proof. It's analogous to the previous one. □

The presence of symmetric uncertainty in the fictitious economy allows us to apply the standard notion of the ex-ante core of an Arrow Debreu economy and to verify the validity of the previous theorems 2.3.2, 2.3.3 and 2.3.4. In the type-agent economy, the set of Arrow-Debreu equilibria coincides with the core of replica economies.

2.4 The ex-post core and the rational expectations equilibria

Let us define the equilibria in the ex-post economy ξ. We consider again a measurable space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ represents the set of all events and Ω denotes the set of states of nature. The commodity space is R_+^L. Each agent $i \in N$ is described as before by a measurable partition of $(\Omega, \mathcal{F})$: $\mathcal{F}_i$ is the partition information of agent i and π is the common prior that represents the relative probability of each state. The common prior is strictly positive in each state $\omega \in \Omega$.

The utility function is state-dependent:

$$u_i : \Omega \times R_+^L \to R$$

and such that for any $x \in R_+^L$, the function $u_i(\cdot, x)$ is $\mathcal{F}$-measurable. It represents the (ex-post) preference of agent i.

The initial endowment function of agents is described by:

$$\mathbf{e} : \Omega \times N \to R_+^L$$

such that $\mathbf{e}(\omega, i)$ represents the initial endowment of agent i in the state of nature ω.

An assignment is a function x:

$$\mathbf{x} : \Omega \times N \to R_+^L$$

such that for any $\omega \in \Omega$, the function $\mathbf{x}(\cdot, i)$ is $\mathcal{F}$- measurable.

An assignment is a feasible allocation when:

$$\sum_{i \in N} \mathbf{x}(\omega, i) \leq \sum_{i \in N} \mathbf{e}(\omega, i),$$

for each $\omega \in \Omega$.

For studying the ex-post equilibria, we need to deal with two economies: ξ and $\xi(\omega)$. The economy ξ is the differential information economy until now described, while $\xi(\omega)$ for each $\omega \in \Omega$ is a complete information economy in which the initial endowment of each agent is $\mathbf{e}(\omega, i)$ and the utility function is $u_i(\omega, \cdot)$.

2.4.1 The ex-post core

Let us define the ex-post core of the economy ξ and then let us prove that it represents the core corresponding to the families of complete information economies $(\xi(\omega))_{\omega \in \Omega}$. Let $\mathbf{x}$ be an allocation, let $S \subseteq N$ be a coalition and let $\omega_0 \in \Omega$. An assignment $\mathbf{y}$ is an ex-post improvement of $\mathbf{x}$ for $S^{'}$ at ω_0 if:

- The assignment $\mathbf{y}$ is feasible: $\sum_{i \in S} \mathbf{y}(\omega_0, i) \leq \sum_{i \in S} \mathbf{e}(\omega_0, i)$;
- $u_i(\omega_0, \mathbf{y}(\omega_0, i)) > u_i(\omega_0, \mathbf{x}(\omega_0, i)), \forall i \in S.$

An allocation is an *ex-post core* allocation if no coalition $S \subseteq N$ has an ex-post improvement upon $\mathbf{x}$ at any $\omega \in \Omega$. The ex-post core of $\xi, C(\xi)$ is the set of all ex-post core allocations of ξ.

Theorem 2.4.1. [10] $C(\xi)$ is non-empty if the economy ξ satisfies the following conditions:

i. For any $\omega \in \Omega, \mathbf{e}(\omega, i) \gg 0$ (no commodity is absent from the market in each state).

ii. For each $i \in N$, for any $x \in R_+^L, \mathbf{e}(\cdot, i)$ is $\mathcal{F}_i$ - measurable (the agents know their own endowment in each state).

iii. For each $i \in N, u_i(\cdot, x)$ is $\mathcal{F}_i$-measurable (the utility is state dependent and known by each agent in any state).

Moreover, the ex-post core of ξ coincides with the following set:

$$C(\xi) = \{\mathbf{x}|\mathbf{x} \quad \text{is an assignment and} \quad \mathbf{x}(\omega, \cdot) \in C(\xi(\omega)) \quad \text{for all} \quad \omega \in \Omega\}.$$

Proof. Let X be defined as:

$$X = \{\mathbf{x}|\mathbf{x} \quad \text{is an assignment and} \quad \mathbf{x}(\omega, \cdot) \in C(\xi(\omega)) \quad \text{for all} \quad \omega \in \Omega\}.$$

First of all, we prove that the set X is non-empty and then that the core of the economy with complete information coincides with the core of the ex-post economy.

It is true that $C(\xi(\omega))$ is non empty for all $\omega \in \Omega$ (see, e.g., Aumann, 1964, 1966, and Hildenbrand, 1968, 1974) if the following conditions hold:

i. For any $\omega \in \Omega, \mathbf{e}(\omega, i) \gg 0$.

ii. For each $i \in N, u_i(\cdot, x)$ is $\mathcal{F}_i$-measurable.

Let $A_1, ...A_k$ all the atoms partition of the field $\mathcal{F}$. For any $1 \leq j \leq k$, let $\omega_j \in A_j$ and $\mathbf{x}_j \in C(\xi(\omega_j))$.

Define $\mathbf{x} : \Omega \times N \to R^L_+$ by $\mathbf{x}(\omega, i) = \mathbf{x}_j(i)$ for all $\omega \in A_j$ and for all $i \in N$. Then $\mathbf{x}$ is a well-defined assignment in ξ. Let $1 \leq j \leq k$ and $\omega \in A_j$, then $\mathbf{e}(\omega, i) = \mathbf{e}_j(\omega_j, \cdot)$ and $u_i(\omega, i) = u_i(\omega_j, \cdot)$. Therefore, the economies $\xi(\omega)$ and $\xi(\omega_j)$ coincide and $\mathbf{x}(\omega, \cdot) = \mathbf{x}(\omega_j, \cdot)$ which implies that $\mathbf{x}(\omega, \cdot) \in C(\xi(\omega)), \forall \omega \in \Omega, \text{i.e.} \mathbf{x} \in X$. The proof that $X \subseteq C(\xi)$ is satisfied.

Let's show now that $C(\xi) \subseteq X$. To prove our claim, we suppose by contradiction that $\mathbf{x} \in C(\xi)$ and $\mathbf{x} \notin X$.

So, there exists another state of nature ω_0 for which $\mathbf{x}(\omega_0, \cdot) \in X$ and $\mathbf{x}(\omega_0, \cdot) \notin C(\xi(\omega_0))$. From here, there exists a coalition S and an allocation $\mathbf{y}$ feasible and for which $u_i(\omega_0, \mathbf{y}(i)) > u_i(\omega_0, \mathbf{x}(\omega_0, i))$ for all $i \in S$. Let us consider the atom information partition containing $A(\omega_0)$ and a function $z : \Omega \times N \to R^L_+$ defined by:

$$\mathbf{z}(\omega, i) = \begin{cases} \mathbf{y}(i) & \text{if} \quad \omega \in A(\omega_0) \\ \mathbf{e}(\omega, i) & \text{otherwise.} \end{cases}$$

So $\mathbf{z}$ is an assignment in ξ. Moreover, $\mathbf{z}$ is an ex-post improvement of S upon $\mathbf{x}$ on ω_0, which contradicts that $\mathbf{x} \in C(\xi)$.

□

2.4.2 The rational expectations equilibria

The rational expectations equilibrium is defined by a pair $(p, \mathbf{x})$ where p is the price system and $\mathbf{x}$ is the allocation, consistent with the information, the budget constraint and the utility's maximization, as formalized below.

In the ex-post framework, the price assumes a great importance. It becomes the variable that the agents analyze to make conjectures for taking a decision. It is described as a non zero function:

$$p : \Omega \to R^L_+.$$

Let us introduce $\sigma(p)$ the smallest subfield $\mathcal{G}$ of $\mathcal{F}$ for which p is $\mathcal{G}$-measurable. It is the subfield generated by p and it represents the additional information that agents may infer from the prices. The budget set of agent i is still defined in each state ω and it is given by:

$$B_i(p,\omega) = \{x \in R^L_+ \quad | \quad p(\omega)x \leq p(\omega)\mathbf{e}(\omega,i)\}.$$

A *rational expectations equilibrium* is a pair $(p, \mathbf{x})$ such that:

- For all $i \in N$, $\mathbf{x}(\cdot, i)$ is $\sigma(p) \vee \mathcal{F}_i$-measurable.
- For any $\omega \in \Omega$ and for all $i \in N$, $\mathbf{x}(\omega, i) \in B_i(p,\omega)$.
- For all $i \in N$, if $\mathbf{x} : \Omega \to R^L_+$ is $\sigma(p) \vee \mathcal{F}_i$-measurable and satisfies $\mathbf{x}(\omega) \in B_i(p,\omega)$ for each $\omega \in \Omega$, then

$$V_i(\cdot, \mathbf{x}(\cdot, i)|\sigma(p) \vee \mathcal{F}_i) \geq V_i(\mathbf{x}(\cdot)|\sigma(p) \vee \mathcal{F}_i).$$

This is the notion of rational expectations equilibrium for the economy ξ and it coincides with a selection of the Walrasian equilibrium correspondence generated by the family of economies $(\xi(\omega))_{\omega\in\Omega}$ with perfect information in each state ω.

Theorem 2.4.2. [10] The set of rational expectations equilibrium allocations of ξ coincides with the set of competitive equilibrium allocation of ξ[5]:

$$RE(\xi) = \{\mathbf{x}|\mathbf{x} \quad \text{is an assignment and} \quad \mathbf{x}(\omega,\cdot) \in W(\xi(\omega)) \quad \forall \omega \in \Omega\}.$$

Proof. Let a set $Y(\xi)$:

$$Y(\xi) = \{\mathbf{x}|\mathbf{x} \quad \text{is an assignment and} \quad x(\omega,\cdot) \in W(\xi(\omega)) \forall \omega \in \Omega\}.$$

[5]When the following assumptions are satisfied:

i. For any $\omega \in \Omega, \mathbf{e}(\omega, i) \gg 0$.

ii. For any $\omega \in \Omega, i \in N, \mathbf{e}(\cdot.i)$ is $\mathcal{F}_i$-measurable.

iii. For any $\omega \in \Omega, i \in N, u_i(\cdot, \mathbf{x})$ is $\mathcal{F}_i$-measurable.

iv. For any $\omega \in \Omega, i \in N, u_i(\cdot, \mathbf{x})$ is continuous, increasing and strictly quasi concave on R^L_+.

Let us prove that the set $Y \subseteq RE(\xi)$ and $RE(\xi) \subseteq Y$. Taking a state $\omega^{'} \in \Omega$, we prove that $\mathbf{x}(\omega^{'}, \cdot) \in W(\xi(\omega^{'}))$.

We consider a rational expectation equilibrium $(p, \mathbf{x})$ of ξ and we prove that $(p, \mathbf{x}(\omega)^{'})$ is a competitive equilibrium for the economy $\xi(\omega^{'})$[6]. Let S the set of agents i for which the theorem is proved and let an allocation $a \in B_i(\omega^{'}, p)$. Denoting by $A_i(\omega^{'})$ the atom $\sigma(p) \vee \mathcal{F}_i$ that contains the state of nature $\omega^{'}$, let us define $\mathbf{y} : \Omega \to R^L_+$ by:

$$\mathbf{y} = \begin{cases} a, & \text{if } \omega \in A_i(\omega^{'}) \\ 0, & \text{otherwise.} \end{cases}$$

So, $\mathbf{y}$ is $\sigma(p) \vee \mathcal{F}_i$-measurable function. As $p(\omega) = p(\omega^{'}), \mathbf{e}(\omega, i) = e(\omega^{'}, i)$ for $\omega \in A_i(\omega^{'})$, then $\mathbf{y} \in B_i(\omega, p), \forall \omega \in \Omega$. This means that, since $i \in S$ and each agent i in S follows the theorem 2.4.2:

$$V_i(\cdot, \mathbf{x}(\cdot, i)|\sigma(p) \vee \mathcal{F}_i) \geq V_i(\mathbf{y}(\cdot)|\sigma(p) \vee \mathcal{F}_i).$$

Thus the utility function $u_i(\cdot, i)$ is continuous, strictly increasing, and quasi-concave on R^L_+:

$$V_i(\mathbf{y}(\cdot)|\sigma(p) \vee \mathcal{F}_i) = u_i(\cdot, \mathbf{y}(\cdot)),$$

and

$$V_i(\mathbf{x}(\cdot, i)|\sigma(p) \vee \mathcal{F}_i) = u_i(\cdot, \mathbf{x}(\cdot, i)).$$

So:

$$u_i(\omega^{'}, a) = u_i(\omega^{'}, \mathbf{y}(\omega^{'})) \leq u_i(\cdot, \mathbf{x}(\cdot, i)).$$

The result is that $\mathbf{x}(\omega^{'}, i)$ maximizes $u_i(\omega^{'}, \cdot)$ on $B_i(\omega^{'}, p)$, for all $i \in S$. Now, it remains to prove that $Y \subseteq RE(\xi)$.

Let an allocation $\mathbf{x} \in Y$. Then, $\mathbf{x}$ is an allocation in ξ and $\mathbf{x}(\omega, \cdot) \in W(\xi(\omega)), \forall \omega \in \Omega$. Therefore, there exists an equilibrium price $p(\omega) \in R^L_+$ such that $(p(\omega), \mathbf{x}(\omega, \cdot))$ is a competitive equilibrium for $\xi(\omega)$. Since, we suppose that the utility is strictly increasing, strictly concave and continuous on R^L_+, the equilibrium price is strictly positive for all states in Ω.

Let $\mathcal{F}$ composed by the atom informations $A_1, ...A_k$ and for every $1 \leq j \leq k, \omega_j \in A_j$. Then, let define the price function $\overline{p} : \Omega \to R^L_+$ such that $\overline{p}(\omega) = p(\omega_j)$ and $\overline{p}$ is $\mathcal{F}$-measurable. Hypothesizing that for any $i \in N, \mathbf{e}(\cdot, i)$ is $\mathcal{F}_i$-measurable and $\forall x \in R^L_+, \forall \omega \in \Omega, u_i(\cdot, x)$ is $\mathcal{F}_i$-measurable, we have $\xi(\omega) = \xi(\omega_j)$ and so $\mathbf{x}(\omega, \cdot) = \mathbf{x}(\omega_j, \cdot)$. From these conditions, we obtain that $(\overline{p}(\omega), \mathbf{x}(\omega, \cdot))$ is a competitive equilibrium for $\xi(\omega)$.

[6] Remember that $\mathbf{x}$ belongs to the budget set:

$$\mathbf{x}(\omega^{'}, i) \in B_i(\omega^{'}, p) \quad \text{for} \quad i \in N.$$

To prove the theorem, it's necessary also to prove that $\mathbf{x}(\cdot, i)$ is $\sigma(\overline{p}) \vee \mathcal{F}_i$-measurable[7].

Let suppose that $\omega_1, \omega_2 \in A$, then $\overline{p}(\omega_1) = \overline{p}(\omega_2)$. Therefore:

$$B_i(\omega_1, \overline{p}) = B_i(\omega_2, \overline{p}).$$

That means

$$u_i(\omega_1, \overline{p}) = u_i(\omega_2, \overline{p}).$$

In this way, $\overline{p}(\omega_1)$ and $\overline{p}(\omega_2)$ are competitive equilibria prices of $\xi(\omega_1)$ and of $\xi(\omega_2)$, but thus the utility is strictly quasi concave, the economy has a unique competitive equilibrium and $\mathbf{x}(\omega_1, i) = \mathbf{x}(\omega_2, i)$ which affirms that $\mathbf{x}(\cdot, i)$ is $\sigma(\overline{p}) \vee \mathcal{F}_i$ - measurable.

In this way, the pair $(\overline{p}, \mathbf{x})$ is a rational expectation equilibrium for the economy ξ, thus respects the price measurability and the optimality. It remains to show that if an assignment is a rational expectations equilibrium, it is also a competitive equilibrium assignment for the economy $\xi(\omega)$[8].

Let consider a coalition $S(\omega)$ such that $\mathbf{x}(\omega, i)$ maximizes $u_i(\omega, \cdot)$ on $B_i(\omega, \overline{p})$. Then $\pi(S(\omega)) = \pi(N)$. Let us consider an allocation $\mathbf{y}$ as before. We have that for all $\omega \in \Omega, u_i(\omega, \mathbf{y}(\omega)) \leq u_i(\omega, \mathbf{x}(\omega, i))$ and therefore:

$$V_i(\cdot, \mathbf{x}(\cdot, i) | \sigma(p) \vee \mathcal{F}_i) \geq V_i(\mathbf{y}(\cdot) | \sigma(p) \vee \mathcal{F}_i)$$

on Ω. This is the inequality that satisfies the theorem. □

The Rational expectations equilibria can assume three configurations. They are said to be:

1. Fully revealing if the price function reveals to each agent the complete information: $\sigma(p) = \mathcal{F}$.

2. Partially revealing if the price function reveals partial information.

3. Non-revealing if it does not disclose any particular information.

Proposition 2.4.3. Let ξ be an economy and (p, x) a fully revealing rational expectations equilibrium. Then, x is an ex-post core allocation of ξ.

Proof. Since the equilibrium is fully revealing, it coincides with the case of complete information and uncertainty. Therefore, the proof is analogous to that one used for the equivalence between $WE(\xi)$ and $C(\xi)$. □

[7] $\mathbf{x}(\cdot, i)$ is constant on the atoms of $\sigma(\overline{p}) \vee \mathcal{F}_i$.

[8] The thesis of the theorem is the last thing to prove, after having proved the hypothesis.

Conclusion

This chapter presents a general overview of models of exchange economies with uncertainty and differential information. These models introduce new aspects in the classical Walrasian framework.

The symmetric and perfect information of the environment fall down and make space for several theories, closer to the real world.

The survey begins with the classical Arrow and Debreu theory of economies under uncertainty. They introduce the possibility to have several states of nature and a probability to occur.

Then, Radner introduced a model with uncertainty and with differential information. The model introduced by De Clippel is presented. In this case, choices take place at the interim stage. Finally, we present the rational expectations equilibria and the ex-post core. For each of these economies, the equivalence result is analyzed and presented.

In the next chapter, a new equilibrium concept is described. It involves uncertainty and differential information, but, simultaneously, a construction of the equilibrium inspired by the choice set notion, instead of the price-taking behavior.

Chapter 3

Arbitrage condition in differential information economies

It is the aim of this chapter to join the theories presented in Chapter 1 and 2 under a new notion of equilibrium. I shall define an economy in which the main concept of equilibrium is based on trade without prices, following in this way the ICS construction. Furthermore, this trade takes place under uncertainty and differential information. Therefore, the new equilibrium concept is characterized by the following main features:

- The possibility of arbitrage (see [5], [31], [24]);
- Uncertainty and differential information (see [27], [23], [34]).

The structure of the chapter is as follows:

- Section 3.1 introduces the model, with the definition of arbitrage condition, weakly preferred set and preferred set. The last part is focused on the configuration of the core and the possible equivalence result in an ex-ante stage setting. In this section, the uncertainty and asymmetric information scenario is formalized like in the Radner model;
- section 3.2 describes arbitrage free condition using the concept of deviators (see [6]) and so focusing on the interim stage of a differential economy;
- section 3.3 deals with the ex post core and rational expectations equilibrium concept (see [10]);
- the last section analyses the incentive compatibility constraint issues (see [32], [22], [14], [11]).

The main proofs of this section follow [31], [6].

3.1 The model

Let us consider an economy with a finite set of agents N and a finite number of commodities L. Each agent will be characterized by a random utility function, a random initial endowment, a private information structure and a prior. Let us denote the economy by $\xi = ((\Omega, \mathcal{F}), X_i, u_i, e_i, \mathcal{F}_i, \pi)$ with $i \in N$, where:

- $(\Omega, \mathcal{F})$ is the information structure in which Ω is a finite set of states of nature and $\mathcal{F}$ is the subset of Ω that represents the events to which the consumers assign a probability π, with $\pi(\omega) > 0, \forall \omega \in \Omega$;

- $X_i = R^L_+$ is the commodity space of consumer i;

- $e_i : \Omega \to R^L_+$ is the random initial endowment of agent i;

- $u_i : \Omega \times R^L_+ \to R_+$ is the state-dependent utility function of agent i. It is monotone, continuous and concave $\forall \omega \in \Omega$;

- $\mathcal{P}(\Omega)$ is the family of all partitions of Ω and $\mathcal{F}_i \in \mathcal{P}(\Omega)$ is the private information of agent i. It is assumed that when the future state of the economy is ω, agent i cannot discriminate the states and so he only knows that ω belongs to the event in $\mathcal{F}_i(\omega)$ of his partition containing ω.

To describe the arbitrage condition, let us introduce the ex-ante preferred and weakly preferred sets:

$$P_i(x_i) = \{z_i \in R^{L\Omega}_+ | V_i(z_i) > V_i(x_i)\}$$

$\forall i \in N$.

$$W_i(x_i) = \{z_i \in R^{L\Omega}_+ | V_i(z_i) \geq V_i(x_i)\}$$

$\forall i \in N$.

Notice that the notion of preferred sets, in this section, is defined considering the ex-ante expected utility.

In the economy ξ, the behavior of agents is described by means of the ex-ante interactive choice set Z (see [31].) The set Z is affected by the timing of the process for two reasons: the sequence of trades according to the arbitrage free equilibrium concept and to the revelation of uncertainty according to the Radner model.

Precisely:

At $\tau = 0$ (ex-ante stage) the consumer doesn't know the state in which he will consume at time $\tau = 1$. At time $\tau = 0$, a sequence of trades takes place (for which the timing is $t = 0, 1, 2, \ldots$) like in the arbitrage free scheme. At $\tau = 1$, when uncertainty is solved, he only knows that $\omega \in \mathcal{F}_i(\omega)$. Therefore, the agent i exchanges his own bundle with the others proposing a trade which improves all the parts involved in the process. His preferences are defined ex-ante and so the notion of ICS given an allocation $x : N \to R^L_+$, involves $W_j(x_j)$:

$$Z_i^x(0) = \{e_i\}.$$

$$Z_i^x(1) = \bigcup_{j\in N}[Z_i^{*x}(0) + Z_j^{*x}(0) - W_j(x_j)] \cap \mathcal{G}_i.$$

$\forall i, j \in N, i \neq j$, with $x_j \in \mathcal{G}_i$, with $\mathcal{G}_i$ defined by:

$$\mathcal{G}_i = \{z : \Omega \to R^L_+ \quad \text{s.t.} \quad z \quad \text{is } \mathcal{F}_i \text{ -measurable}\}.$$

This exchange procedure can be extended to infinite defining:

$$Z_i^x(t) = \bigcup_{S\subseteq N, i\in S} \big[\sum_{j\in S} Z_j(x_j)(t-1) - \sum_{j\in S\setminus\{i\}} W_j(x_j)\big] \cap \mathcal{G}_i, \quad t \in \mathbb{N}.$$

The uncertainty is revealed at $\tau = 1$ and the trades are fulfilled in the realized states. Trades in the ICS of i are required to be measurable and compatible with the private information of agent i.

3.1.1 The ex-ante arbitrage free condition

An ex-ante arbitrage free - equilibrium in the economy with differential information is composed by a pair $(x_i, C_i)_{i\in N}$, where $x_i \in \mathcal{G}_i$ is the bundle assigned by the private allocation x to agent i and C_i is the choice set of the agent i.

C_i is a function:

$$C_i : \Omega \to R^L_+$$

and also a subset of $\mathcal{G}_i, \forall i \in N$. As before, the choice set allows the optimization of the expected utility and it supports the ex-ante arbitrage free condition. It is feasible and contains the actual trade. Then, formally, an ex-ante arbitrage free - equilibrium is a pair $(x_i, C_i)_{i\in N}$ satisfying the following conditions:

1. $(x_i)_{i\in N}$ is an allocation;
2. $e_i, x_i \in C_i, \forall i \in N$;
3. $C_i \supseteq \big[\sum_{j\in S} C_j - \sum_{j\in S\setminus\{i\}} W_j(x_j)\big] \cap \mathcal{G}_i$ with $j \in S, S \subseteq N$;
4. $P_i(x_i) \cap C_i = \emptyset$.

Notice that, according to the definition of choice set, agent i proposes only trades which are compatible with his own private information.

A particular case of choice set is represented by the ex-ante budget set which defines Walrasian expectations equilibrium:

$$B_i = \{x_i \in \mathcal{G}_i | px_i \leq pe_i\} \forall i \in N$$

with $p : \Omega \to R^L_+$.

Taking the features of the choice set, let us show this property. By conditions defining a Walrasian expectations equilibrium, we have $e_i, x_i \in B_i = C_i$. Let us prove that:

$$C_i \supseteq \big[C_i + \sum_{j \in S, j \neq i} C_j - \sum_{j \in S, j \neq i} W_j(x_j)\big] \cap \mathcal{G}_i.$$

Consider a bundle z s.t.

$$z = \gamma_i + \sum_{j \in S, j \neq i} \gamma_j - \sum_{j \in S, j \neq i} w_j \in \mathcal{G}_i,$$

with $\gamma_i \in B_i, \gamma_j \in B_j, w_j \in W_j(x_j)$. Since $p\gamma_j \leq pe_j$ and $p\gamma_i \leq pe_i$, we have:

$$pz = p\gamma_i + \sum_{j \in S, j \neq i} (p\gamma_j - pe_j) \leq pe_i + \sum_{j \in S, j \neq i} pe_j - pw_j. \tag{3.1}$$

But, $p(e_j - w_j) \leq 0$ since, for monotonicity of preferences,

$$w_j \succeq_j x_j \Rightarrow pw_j \geq pe_j.$$

So, the equation 3.1 gives:

$$pz \leq pe_i \quad \text{and} \quad z \in B_i = C_i.$$

The choice set and the interactive choice sets are subsets of privately measurable functions $\mathcal{G}_i$ and moreover, they are interrelated by the following Proposition.

Proposition 3.1.1. Let ξ be a differential information economy and let the pair $(x_i, C_i)_{i \in N}$ satisfies conditions 1. – 4. . Then:

i. $Z^x_i \subseteq C_i, \forall i \in N$;

ii. $(x_i, Z^x_i)_{i \in N}$ is an ex-ante arbitrage free- equilibrium.

Proof. Let us prove by induction that $Z^x_i \subseteq C_i, \forall i \in N$. $(x_i, C_i)_{i \in N}$ is an ex-ante arbitrage free equilibrium, then $Z^x_i(0) = \{e_i\} \subseteq C_i, \forall i \in N$. Now, assume that $Z^x_i(t-1) \subseteq C_i$ for some $t, \forall i \in N$.

Since (x_i, C_i) satisfies the condition 3.:

$$C_i \supseteq \big[\sum_{j \in S} C_j - \sum_{j \in S \setminus \{i\}} W_j(x_j)\big] \cap \mathcal{G}_i$$

with $j \in S, S \subseteq N$, by induction, the condition becomes:

$$C_i \supseteq \big[\sum_{j \in S} Z_j^x(t-1) - \sum_{j \in S \setminus \{i\}} W_j(x_j)\big] \cap \mathcal{G}_i$$

with $j \in S, S \subseteq N$. Consequently:

$$C_i \supseteq \bigcup_{S \subseteq N} \big[\sum_{j \in S} Z_j^x(t-1) - \sum_{j \in S \setminus \{i\}} W_j(x_j)\big] \cap \mathcal{G}_i.$$

The right side implies, by definition of $Z_i^x(t)$, that $Z_i^x(t) \subseteq C_i$.

The first point of the proposition is proved.

Now, let us show that Z_i^x satisfies the conditions $1.-4$. The first and the second conditions are trivially satisfied. The fourth condition derives from the proof of [i.] The third condition is satisfied thanks to the definition of interactive choice set. Indeed, taking a bundle z:

$$z \in Z_i^x(t-1) = \big[\sum_{j \in S} Z_j^x(t) - \sum_{j \in S \setminus \{i\}} W_j(x_j)\big] \cap \mathcal{G}_i.$$

Thus $Z_i^x(t)$ is an increasing sequence in $t, z \in Z_i^x(t+1) \subseteq Z_i^x$. □

Replica economies in an ex-ante differential information economy

The m-fold replica economy of a differential information economy ξ, denoted by ξ^m is a new differential information economy with mn agents, such that each agent $(i,j), j = 1,..m$ has the same characteristics of agent i:

- $\mathcal{F}_{ij} = \mathcal{F}_i$;
- $e_{ij} = e_i$;
- $u_{ij} = u_i$;
- $\pi_{ij} = \pi_i$.

The core of a replica economy can be expressed by means of feasible sets very close to the interactive choice sets.

Fix an allocation x and an agent $i \in N$. Let us define the following sets:

$$A_i^x(0) = \{e_i\} \quad \forall i \in N.$$

$$A_i^x(m) = \bigcup_{S \subseteq N \times \overline{m}, (i,1) \notin S} \big[e_i + \sum_{(k,l) \in S} (e_k - W_k(x_k))\big] \cap \mathcal{G}_i.$$

$$A_i^x = \bigcup_{m \in \mathbb{N}} A_i^x(m).$$

The core of the m-fold replica is represented by the following condition: the allocation $x \in A(\xi)$ is in the core of the m-replica economy ξ^m of ξ if for all $i \in N$:

$$P_i(x_i) \cap A_i^x(m) = \emptyset.$$

We remark that the previous condition refers to a strong private core notion. In differential information economies, due to the measurability restriction imposed on allocations, the strong and weak core are not equivalent.

Theorem 3.1.2. Let ξ be a differential information economy and let $(x_i)_{i\in N}$ be an allocation. For all $i \in N, Z_i^x = A_i^x$.

Proof. The proof follows the same steps of the theorem 1.2.3. □

3.1.2 Ex-ante arbitrage differential core

Let us consider the coalition S and an assignment $(x_i)_{i\in S}$. We say that x is an S-allocation if it is ex-ante feasible:

$$\sum_{i \in S} x_i(\omega) \le \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega.$$

The feasible allocation is an ex-ante core allocation if there does not exist a coalition S and an S-allocation $(x_i^{'})_{i\in S}$ such that for all $i \in S, x_i^{'} \in W(x_i)$ and for some $i \in S, x_i^{'} \in P_i(x_i)$.

3.1.3 The equivalence result

If the differential information economy ξ satisfies the assumption listed before, i.e.

1. For all $i \in N$, for all $\omega \in \Omega$, the consumption set $X_i = R_+^L$;

2. upper semicontinuity of state-dependent utilities;

3. strict monotonicity of state-dependent utilities;

4. strictly positivity of the initial endowment in each state.

Theorem 3.1.3. An allocation is a Walrasian expectations equilibrium if and only if $(x_i, Z_i^x)_{i\in N}$ is an ex-ante arbitrage-free equilibrium.

Proof. We start from the assumptions that $(x_i, Z_i^x)_{i\in N}$ is an ex-ante arbitrage free equilibrium. By condition 4. of the definition of ex-ante arbitrage free equilibrium:

$$P_i(x_i) \cap Z_i^x = \emptyset, \forall i \in N.$$

By theorem 3.1.2:

$$P_i(x_i) \cap A_i = \emptyset, \forall i \in N.$$

which implies that x_i is an ex-ante core allocation of the replica economy. By Debreu-Scarf theorem for the ex -ante economy, it is a Walrasian expectations equilibrium.

So, we have proved that $(x_i, Z_i^x)_{i\in N}$ is a Walrasian expectations equilibrium. □

3.2 Arbitrage condition in a type-agent economy

In this section, we introduce a notion of arbitrage free equilibrium for an economy at the interim stage. Let us recall that the type agent economy refers to an economy in which each agent i is characterized by the signal that he receives. So, a type-agent is defined by particular agent i together with an event in his information partition.

In this contest, let us introduce the deviators concept. Given two allocation rules $a, a^{'} : \Omega \to R_+^{LN}$, the deviators are agents which prefer in each state to have $a^{'}$ instead of a, therefore the set of deviators is given by:

$$D(a_i, a_i^{'}, \omega) := \{i \in N | V_i(a_i^{'}|P_i(\omega)) > V_i(a_i|P_i(\omega))\}.$$

The interim expected utility, with $P_i(\omega)$ denoting the event in $\mathcal{F}_i$ containing the realized state of nature ω, is expressed as follows:

$$V_i(a_i|P_i(\omega)) = \frac{1}{\pi(P_i(\omega))} \sum_{\omega^{'} \in P_i} u_i(\omega^{'}, a_i(\omega^{'})).$$

The concepts of weakly preferred sets and preferred sets need to be re-formulated.

The sets are described as follows $\forall i \in N$ and $\omega \in \Omega$, and can be considered as interim concepts:

$$P_i(a_i, \omega) = \{a_i^{'} \in R_+^{L\Omega} | V_i(a_i^{'}|P_i(\omega)) > V_i(a_i|P_i(\omega))\},$$

$$W_i(a_i, \omega) = \{a_i^{'} \in R_+^{L\Omega} | V_i(a_i^{'}|P_i(\omega)) \geq V_i(a_i|P_i(\omega))\}.$$

3.2.1 The constrained or interim arbitrage free condition

Let define an interim choice set as a function $C_i : \Omega \to R^L_+$. A pair $[a_i(\omega), C_i(\omega)]_{i \in N}$ is an interim arbitrage free equilibrium if it satisfies the following properties:

i. There is the possibility of no trade for each agent i in each state, so $e_i(\omega) \in C_i(\omega), \forall \omega \in \Omega, \forall i \in N$.

ii. The allocation rule $a_i(\omega) \in C_i(\omega)$ for all $i \in N$, for each $\omega \in \Omega$.

iii. $\sum_{k \in S} C_k(\omega) - \sum_{k \in S \setminus \{i\}} W_k(x_k, \omega) \subseteq C_i(\omega), \forall i \in N, \forall \omega \in \Omega, \forall S \in S_i(\omega)$ where $S_i(\omega) = \{S \subseteq N : P_i(\omega) = P_k(\omega) \forall k \in S\}$.

iv. The agent optimizes his own utility in the choice set in each state i.e. $C_i(\omega) \cap P_i(a_i(\omega)) = \emptyset, \forall i \in N, \forall \omega \in \Omega$.

Notice that condition $[iii.]$ in the definition that we propose is defined only with respect to those coalitions whose members receive in the state ω the same information of agent i.

A particular $C_i(\omega)$ is represented by a budget set on which the agent i maximizes his interim expected utility at a vector price $p \in R^{L\Omega}_+$ in a given constrained market equilibrium.

Let

$$B_i(\omega) = \{a_i \in R^{L\Omega}_+ | \sum_{\omega \in P_i(\omega)} p(\omega) a_i(\omega) \leq \sum_{\omega \in P_i(\omega)} p(\omega) e_i(\omega)\}.$$

Proposition 3.2.1. Let (a, p) be a constrained market equilibrium, then $[a_i(\omega), C_i(\omega)]_{i \in N, \omega \in \Omega}$ with $B_i(\omega) = C_i(\omega)$ is an interim arbitrage free-equilibrium.

Proof. Given the condition $[iii.]$ the proof is similar to the one for a complete information economy. □

For the next step, it's important to consider the fictitious economy, i.e. the above mentioned type -agent economy.

Let $\overline{\xi} = (\overline{a}_{(i,E)}, \overline{u}_{(i,E)}, \overline{e}_{(i,E)}, \mathcal{F}_{(i,E)})$. Recall that the set of type agents is $\mathcal{N}$, the set of goods is L, the set of possible states is Ω and each agent i is associated with an event in his information partition. The difference with respect to the economy ξ is that it is characterized by uncertainty and symmetric information.

Given an interim arbitrage free equilibrium $[a_i(\omega), C_i(\omega)]_{i \in N}$ of ξ the choice set $C_i(\omega)$ is associated to the state contingent choice set of type agent (i, E) $\overline{C}_{(i,E)}$ which we can define as follows;

$$\overline{C}_{(i,E)}(\omega) := C_i(\omega) \quad \text{if} \quad \omega \in E.$$

$$\overline{C}_{(i,E)}(\omega) := R^L_+ \quad \text{if} \quad \forall \omega \in \Omega \setminus E.$$

The allocation for the type-agent (i, E) is:

$$\overline{a}_{(i,E)}(\omega) : \Omega \to R^L_+;$$

defined as follows:

$$\overline{a}_{(i,E)}(\omega) := a_i(\omega) \quad \text{if} \quad \omega \in E.$$

$$\overline{a}_{(i,E)}(\omega) := 0 \quad \text{if} \quad \omega \in \Omega \setminus E.$$

The initial endowment function for the type-agent (i, E) is:

$$\overline{e}_{(i,E)}(\omega) : \Omega \to R^L_+$$

where the endowment for type agent (i, E) is:

$$\overline{e}_{(i,E)}(\omega) := e_i(\omega) \quad \text{if} \quad \omega \in E.$$

$$e_{(i,E)}(\omega) := 0 \quad \text{if} \quad \omega \notin E.$$

An allocation is feasible for a coalition S if:

$$\sum_{(i,E)\in\mathcal{S}} \overline{a}_{(i,E)}(\omega) \leq \sum_{(i,E)\in\mathcal{S}} \overline{e}_{(i,E)}(\omega), \quad \forall\omega \in \Omega, \forall\mathcal{S} \subseteq \mathcal{N}.$$

An allocation rule $\overline{a}'$ dominates an allocation rule $\overline{a}$ for a coalition $\mathcal{S}$ if $\overline{a}'$ is feasible for $\mathcal{S}$ and

$$V_{(i,E)}(\overline{a}'_{(i,E)}) > V_{(i,E)}(\overline{a}_{(i,E)}) \quad \text{for each} \quad (i, E) \in \mathcal{S},$$

where in the type-agent economy, we refer to the standard ex-ante core notion of a model with only uncertainty.

Conversely, if $[\overline{a}_{(i,E)}(\omega), \overline{C}_{(i,E)}(\omega)]$ is an arbitrage free - equilibrium of the type-agent economy, the choice set $\overline{C}_{(i,E)}(\omega)$ is associated to the following choice sets $(C_i(\omega))_{i\in N,\omega\in\Omega}$:

$$C_i(\omega) := \overline{C}_{(i,P_i(\omega))}(\omega);$$

where $\overline{C}_{(i,P_i(\omega))}(\omega)$ is the projection of $\overline{C}_{(i,P_i(\omega))}$ on the state $\omega \in \Omega$.

The next proposition extends to interim equilibrium the correspondence already known (see [6]) for constrained market equilibria.

Proposition 3.2.2. Let $[a_i(\omega), C_i(\omega)]_{i\in N,\omega\in\Omega}$ an interim arbitrage free equilibrium in ξ, then $[\overline{a}_{(i,E)}(\omega), \overline{C}_{(i,E)}(\omega)]_{(i,E)\in\mathcal{N},\omega\in E}$ is an arbitrage free equilibrium of $\overline{\xi}$ and conversely.

Replica in fictitious economy

The m-fold replica economy of a fictitious economy $\overline{\xi}$ denoted by $\overline{\xi}^m$ is a new fictitious economy with nm type-agents such that $\forall j \in \{1, ...m\}$:

- $\overline{e}_{(i,E)} = \overline{e}_{(ij,E)}$;
- $V_{(i,E)}(\cdot) = V_{(j,E)}(\cdot)$.

Fixing an allocation $\overline{a}$ and a type agent (i, E), let us define the following sets:

$$\overline{A}^x_{(i,E)}(\omega)(0) = \{e_{(i,E)}(\omega)\} \quad \text{if} \quad \omega \in E;$$

$$\overline{A}^x_{(i,E)}(\omega)(m) = \bigcup_{\mathcal{S} \subseteq \mathcal{N} \times m, (i1,E) \notin \mathcal{S}} [\overline{e}_{(i,E)}(\omega) + \sum_{(j,F) \in \mathcal{S}} (\overline{e}_{k,F)}(\omega) - \overline{W}_{(j,F)}(\overline{a}_{(j,F)}(\omega))].$$

$$\overline{A}^x_{(i,E)} = \bigcup_{m \in \mathbb{N}} \overline{A}^x_{(i,E)}(m).$$

The core of a replica economy can be expressed using suitable intersections. The allocation $\overline{a}_{(i,E)}$ is a shrunk core if and only if:

$$P_{(i,E)}(\overline{a}_{(i,E)}) \cap \overline{A}_{(i,E)}(m) = \emptyset, \forall (i, E) \in \mathcal{S}.$$

3.2.2 The characterization of the core in the fictitious economy

Let us denote by $\overline{xi}$ the fictitious economy. Given an allocation $\overline{a}$, we say that it is a core allocation when it cannot be blocked by a coalition in the ex-ante sense.

For showing the limit of the core, let us retrace the same steps made for the Radner economy in the arbitrage scheme.

Definition of the interactive choice set

At $t = 0$:

$$\overline{Z}_{(i,E)}(0) = \{\overline{e}_{(i,E)}\}, \forall (i, E) \in \mathcal{N}, \forall \omega \in E.$$

$$\overline{Z}_{(i,E)}(0) = \emptyset, \forall \omega \in \Omega \setminus E.$$

For $t > 0$:

$$\overline{Z}_{(i,E)}(t) = \bigcup_{\mathcal{S} \subseteq \mathcal{N}, (i,E) \in \mathcal{S}} \big[\sum_{(j,F) \in \mathcal{S}} \overline{Z}_{(j,F)}(t-1) - \sum_{(j,F) \in \mathcal{S} \setminus \{(i,E)\}} \overline{W}_{(j,F)}(\overline{a}_{(j,F)}) \big],$$

for $t \in \mathbb{N}$.

Where the set $\overline{W}$ is defined using the ex-ante utility of the type-agent (j, F).

Proposition 3.2.3. Let $(\overline{a}_{(i,E)}(\omega), \overline{C}_{(i,E)})_{(i,E) \in \mathcal{N}}$ an arbitrage free equilibrium of the fictitious economy $\overline{\xi}$. Then:

i. $\overline{Z}^x_{(i,E)} \subseteq \overline{C}_{(i,E)}, \forall (i,E) \in \mathcal{N}$;

ii. $[\overline{a}_{(i,E)}, \overline{Z}_{(i,E)}]_{(i,E)\in\mathcal{N}}$ is an arbitrage free- equilibrium.

Theorem 3.2.4. Let $\overline{\xi}$ be the fictitious economy and let $(\overline{a}_{(i,E)})_{(i,E)\in\mathcal{N}}$ be an allocation. For all $(i,E) \in \mathcal{N}$, $Z^x_{(i,E)} = A^x_{(i,E)}$.

Proof. The proof follows the same steps of the theorem 1.2.3. □

3.2.3 The equivalence result

The equivalence result between the interim (or constrained) arbitrage free equilibrium and the constrained market equilibrium introduced by Wilson and studied by De Clippel, can be proved using the usual steps. In particular:

- We first show the equivalence in the fictitious economy $\overline{\xi}$ between arbitrage free equilibrium and Arrow-Debreu equilibrium;
- We deduce the equivalence in the original economy ξ.

3.3 The arbitrage free rational expectations equilibrium

In this section, we introduce a notion of arbitrage free equilibrium at the ex post stage and prove its equivalence with Rational expectations equilibrium (REE) introduced in section 2.4. We shall consider the economy $\xi(\omega)$ for each $\omega \in \Omega$, instead of ξ. In this way, we deal with an economy with complete information for each state of nature ω.

The new economy is defined by $\xi(\omega) = (N, L, \mathbf{e}(\omega, i), u_i(\omega, i))$ where N is the finite set of agents, L the finite set of goods, $u_i(\omega, \cdot)$ for each agent is the utility function, the initial endowment for agent i is $\mathbf{e}(\omega, i)$; that is the initial endowment of agent i in the state ω. The function $u_i(\cdot, x)$ is $\mathcal{F}_i$-measurable, continuous, strictly increasing and quasi-concave on R^L_+. The choice set function of agent i in each state $\omega \in \Omega$ is:

$$\mathbf{C} : \Omega \times N \to R^L_+.$$

The allocation function $\mathbf{x}$ of agent i in each state $\omega \in \Omega$ is:

$$\mathbf{x} : \Omega \times N \to R^L_+,$$

where $\mathbf{x}(\cdot, i)$ is $\mathcal{F}$-measurable, $\forall i \in N$. We shall use the notation $\mathbf{x}(\omega, i) = x_i(\omega)$ and $\mathbf{C}(\omega, i) = C_i(\omega)$ to denote the bundle of commodities and the choice set assigned to i in the state $\omega \in \Omega$. The feasibility is expressed ad follows:

$$\sum_{i \in N} \mathbf{x}(\omega, i) \leq \sum_{i \in N} \mathbf{e}(\omega, i);$$

for all agent $i \in N$ and for each $\omega \in \Omega$.

The weakly and the preferred sets are defined ex post. In each state $\omega \in \Omega$:

$$\mathbf{W}_i(x_i(\omega)) = \{z \in R_+^L | u_i(z, \omega) \geq u_i(x_i(\omega), \omega)\}.$$

$$\mathbf{P}_i(x_i(\omega)) = \{z \in R_+^L | u_i(z, \omega) > u_i(x_i(\omega), \omega)\}.$$

An ex post arbitrage free-equilibrium is a pair $[\mathbf{x}, \mathbf{C}]$ for each $\omega \in \Omega, \forall i \in N$ such that:

1. $\mathbf{x}$ is an allocation;
2. $x_i(\omega), e_i(\omega) \in C_i(\omega)$;
3. $\mathbf{P}_i(x_i(\omega)) \cap \mathbf{C}_i(\omega) = \emptyset$;
4. The ex post arbitrage free condition holds:

$$\sum_{j \in S} \mathbf{C}(\omega, j) - \sum_{j \in S \setminus \{i\}} \mathbf{W}(x(\omega, j)) \subseteq \mathbf{C}(\omega, i)$$

   $\forall S \subseteq N, j \neq i$;

All these conditions follow the same logic of Serrano, Volij and Dagan (see [5], [30]).

Let us define the interactive choice sets.

Definition of Interactive choice sets

The interactive choice set is recursively defined as usual, but with reference to $\omega \in \Omega$. At $t = 0$:

$$\mathbf{Z}(\omega, i)(0) = \{\mathbf{e}(\omega, i)\} \quad \text{for each} \quad \omega \in \Omega, \forall i \in N.$$

At $t > 0$:

$$\mathbf{Z}(\omega, i)(t) = \bigcup_{j \in S, S \subseteq N} \big[\sum_{j \in S} \mathbf{Z}(\omega, j)(t-1) - \sum_{j \in S \setminus \{i\}} \mathbf{W}(\mathbf{x}(\omega, j))\big]. \quad \text{and} \quad \mathbf{Z}(\omega, i) = \cup_{t \in N} \mathbf{Z}(\omega, i)(t).$$

The choice set and the interactive choice are linked by the following proposition.

Proposition 3.3.1. Let $[\mathbf{x}, \mathbf{C}]$ be an arbitrage free rational expectations equilibrium. Then:

- $\mathbf{Z}(\omega, i) \subseteq \mathbf{C}(\omega, i), \forall i \in N, \forall \omega \in \Omega$;
- $[\mathbf{x}, \mathbf{Z}]$ is an arbitrage free rational expectations equilibrium too.

Notice that the Proposition holds true simply because the economy corresponding to each $\omega \in \Omega$ has the same characteristics of those studied in the first chapter for the case of complete information. So, the attention can be limited to the arbitrage free REE in which the choice set is actually interactive.

3.3.1 Replica in ex-post economy

The m-fold replica economy of the state dependent economy $\xi(\omega)$, denoted by $\xi^m(\omega)$ is a new perfect information economy with nm agents, such that each agent $(i, j), j = 1, ..m$ has the same features of agent $i, \forall j \in \{1, ..m\}$, for each $\omega \in \Omega$:

- $u_i(\omega, x) = u_{ij}(\omega, x)$;
- $\mathbf{e}(\omega, i) = \mathbf{e}(\omega, ij)$.

The core of the m - fold replica can be expressed by means of feasible sets close to the interactive choice set. Fix an allocation $\mathbf{x}$ and an agent $i \in N$.

Define the following sets $\forall i \in N$ and for each $\omega \in \Omega$:

$$A_i^x(\omega)(0) = \{\mathbf{e}(\omega, i)\}.$$

$$A_i^x(\omega)(m) = \bigcup_{S \subseteq N \times \{1,..m\}, (i,1) \notin S} \big[\mathbf{e}(\omega, i) + \sum_{(k,l) \in S} \mathbf{e}(\omega, k_l) - \mathbf{W}(\mathbf{x}(\omega, k_l))\big].$$

$$A_i^x(\omega) = \bigcup_{m \in \mathbb{N}} A_i^x(m), \forall \omega \in \Omega.$$

The core of the m-fold replica is represented by the following condition: the allocation $\mathbf{x} \in A(\xi(\omega))$ is in the core of the r-replica economy $\xi(\omega)^m$ of $\xi(\omega)$ if for all $i \in N$, for each $\omega \in \Omega$:

$$\mathbf{P}(\mathbf{x}(\omega, i)) \cap A_i^x(\omega)(m) = \emptyset.$$

Theorem 3.3.2. Let ξ be a differential information economy and let $\mathbf{x}$ be an allocation. For all $i \in N$ and for each $\omega \in \Omega, \mathbf{Z}_i^x(\omega) = A_i^x(\omega)$.

Proof. The proof follows the same steps of the theorem 1.2.3., since the considered economies have the same features of a complete information economy, for each $\omega \in \Omega$. $\square$

Proposition 3.3.3. Each rational expectations equilibrium $(\mathbf{x}, \mathbf{p})$ is an ex-post arbitrage free equilibrium when we choose as choice set the state dependent budget set, i.e. when $\forall i \in N, \forall \omega \in \Omega$:

$$C_i(\omega) = B_i(p, \omega) = \{z \in R^L_+ : p(\omega)z \leq p(\omega)e_i(\omega)\}.$$

Proof. Conditions 1 and 2 are obviously satisfied. Fix $i \in N$ and $\overline{\omega} \in \Omega$. We want to show condition 3 for $\mathbf{P}_i(x_i(\overline{\omega}))$ and $\mathbf{C}_i(\overline{\omega})$. Assume by contradiction that there exists:

$$z \in \mathbf{P}_i(x_i(\overline{\omega})) \cap \mathbf{C}_i(\bar{(\omega)}).$$

Then $z \in B_i(p, \overline{\omega}) = \mathbf{C}_i(\overline{\omega})$ and $u_i(\overline{\omega}, z) > u_i(\overline{\omega}, x_i(\overline{\omega}))$.
Consider the atom $A_i(\overline{\omega})$ in the σ-algebra $\sigma(p) \vee \mathcal{F}_i$ containing $\overline{\omega}$. Define $\overline{z} : \Omega \to R^L_+$ as follows

$$\overline{z}(\omega) = \begin{cases} z & \text{if} \quad \omega \in A_i(\overline{\omega}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{z}(\cdot)$ is $\sigma(p) \vee \mathcal{F}_i$-measurable by definition and from $p(\omega) = p(\overline{\omega})$ and $\mathbf{e}(\omega, i) = \mathbf{e}(\overline{\omega}, i), \forall \omega \in A_i(\overline{\omega})$ we have, by definition of rational expectations equilibria:

$$V_i(u_i(\cdot, \overline{z}(\cdot))|\sigma(p) \vee \mathcal{F}_i) \leq V_i((\cdot, x(\cdot, i))|\sigma(p) \vee \mathcal{F}_i).$$

Since u_i is $\mathcal{F}_i$ - measurable, we see that: $V_i(u_i(\cdot, \overline{z}(\cdot))|\sigma(p) \vee \mathcal{F}_i) = u_i(\cdot, \overline{z}(\cdot))$ and $V_i(u_i(\cdot, \overline{x}(\cdot, i))|\sigma(p) \vee \mathcal{F}_i) = u_i(\cdot, \overline{x}(\cdot, i))$. So we conclude that for the agent i in the state $\overline{\omega}$:

$$u_i(\overline{\omega}, \overline{z}) \leq u_i(\overline{\omega}, x(\overline{\omega}, i))$$

which is a contradiction. Finally, to prove condition 4, fix $\overline{\omega} \in \Omega$ and consider the vector:

$$z \in \sum_{k \in S} B_k(\overline{\omega}, p) - \sum_{k \in S, k \neq i} W_k(x(\overline{\omega}, k))$$

Then $z = \sum_{k \in S} g_k - \sum_{k \in S, k \neq i} w_k$, with

$$pg_k \leq pe_k(\overline{\omega}) \quad \text{and} \quad u_k(\overline{\omega}, w_k) \geq u_k(\overline{\omega}, x(\overline{\omega}, k)).$$

Define $\overline{w}^n_k = w_k + \frac{1}{n}v$, with $v > 0$. By monotonicity of state dependent utility functions, we have that

$$u_k(\overline{\omega}, w^n_k) \geq u_k(\overline{\omega}, x(\overline{\omega}, k)).$$

Hence, with a construction similar to the previous one, we can see that $p\overline{w}^n_k > pe_k(\overline{\omega}) \forall k, \forall \omega$ and so, in the limit,

$$p.w_k = lim_n p\overline{w}^n_k \geq e_k(\overline{\omega})$$

for each $k \in S$. As a consequence

$$pz = \sum_{k \in S} pg_k - \sum_{\substack{k \in S \\ k \neq i}} p_k w_k \leq \sum_{k \in S} pe_k(\overline{\omega}) - \sum_{k \in S k \neq i} p_k e_k(\overline{\omega}) = p_i e_i(\overline{\omega}).$$

which implies $x \in C_i(\omega) = B_i(p, \omega)$ and the conclusion is proved. □

3.3.2 Arbitrage free rational expectations equilibrium

The main result of this section follows. It gives a new characterization of REE which does not defined on prices.

Theorem 3.3.4. A pair $(\mathbf{x}, p)$ is a rational free expectations equilibrium if and only if $[\mathbf{x}, \mathbf{Z}]$ for each $\omega \in \Omega$ and $\forall i \in N$ is an ex post arbitrage equilibrium.

Proof. Assume that $[\mathbf{x}, \mathbf{Z}]$ is an arbitrage free expectations equilibrium.We know that $\mathbf{Z}_i^x(\omega) = A_i^x(\omega), \forall \omega, \forall i$ by theorem 3.3.2.. Then $\mathbf{P}(\mathbf{x}(\omega, i)) \cap A_i^x(\omega)(m) = \emptyset, \forall i, \forall \omega \in \Omega$, and consequentially the allocation $\mathbf{x}$ is in the core of each replica economy $\xi^m(\omega)$, i.e. $\mathbf{x} \bigcap_{m>1}^{\infty} C(\xi^m(\omega))$.

By [10], $\mathbf{x}$ belongs to the set $RE(\xi), \mathbf{x}$ is an allocation, $x \in W(\xi(\omega)) = REE$. So $\mathbf{x}$ is a rational expectations equilibrium. □

Conclusions, further remarks and open questions

This chapter analyzes several cases of differential information economies in which competitive equilibria can be characterized through arbitrage free conditions. In the first section, the case of an ex-ante differential information economy is studied, The main issue of this part consists in the measurability constraint imposed on agent's trades, Indeed, also if the reasoning of proof follows the same logic of Serrano and Volji ([31]) we use privately measurable trades at each step of the recontracting process by introducing the set $\mathcal{G}_i$.

It is worth to point out that, assuming that feasibility of allocations is defined with no free disposal, each no-free disposal ex ante arbitrage free equilibrium satisfies the following Coalition Incentive Compatibility Property:

(CIC): It is not true that there exists a coalition $S \subset N$ and two states $\omega, \omega', \omega \neq \omega'$ with $\omega' \in P_i(\omega), \forall i \in S$ and such that

$$u_i(\omega, e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(\omega, x_i(\omega)), \forall i \in S$$

This condition, introduced in [18] states that the members of a coalition S cannot benefit by announcing to the members of the complementary coalition a false state that they cannot distinguish from the true one. This property of arbitrage free

Walrasian expectations equilibrium follows from previous theorem 3.1.3 and from Theorem 5.1 in [18].

In the second section, the framework is that of an interim differential information economy in which the arbitrage free condition is satisfied in the fictitious economy and then extended to the original interim differential information economy as in De Clippel ([6]).

The third section shows the case of the ex-post differential information economies and provides, under measurability of utility functions with respect to private information characterization of Rational expectations equilibria as arbitrage free rational expectation equilibria. In a recent paper, [16], He and Yannelis define competitive equilibria for ambiguous asymmetric information economies, economies in which agents maximize Maxmin expected utilities. It would be interesting for future research to investigate characterization of Maxmin expectations equilibria in terms of non arbitrage conditions.

Incentive compatibility constraint

One final remark is still relative to incentive compatibility. Until now, the described models of differential information are enriched by the asymmetric information and at the same time, they don't need incentive compatibility constraint or because the lack of information is symmetric, or thanks to the measurability or because the event is announced. From now on, our framework is characterized by a situation in which each agent only knows his information and the system is characterized by null communication. It describes a world in which the agent i could lie to obtain a profit from the asymmetric information, i.e. he may mystify the reality to have an advantage. For this reason, an incentive constraint (see [32], [19], [20]) is necessary to arrive to a state of equilibrium. The incentive constraint should induce the players to be honest with respect to their knowledge about the event E. It is not necessary of course when the event is measurable by each agent $i \in N$ and for all $\omega \in \Omega$. The new challenge is to allow the exchange among the agents to respect the arbitrage free- condition, following the previous logic.

A way to express the incentive compatible constraint may be to link the trade to the credibility of each agent. To do that, we insert a variable β_i which resumes the credibility degree of each agent i. If it is positive, the ith agent has credibility and so all other agents $j \in N, i \neq j$ make an exchange with him. If the variable β_i assumes a value equal 0, the agent i is alone in the market, he is marginalized from the other players because they don't trust him and so they decide to make no trade.

The utility of agent i changes, incorporating his own β_i.

$$V_i(x_i(\omega)|E) = \pi(\omega|E)u_i(x_i(\omega), \beta_i, \omega).$$

where

$$\begin{cases} u_i(x_i(\omega), \beta_i, \omega) = u_i(x_i(\omega), \omega) \quad \text{if} \quad \beta_i \geq 0 \\ u_i(x_i(\omega), \beta_i, \omega) = 0 \quad \text{if} \quad \beta_i = 0 \end{cases} \tag{3.2}$$

With this definition, all the agent have an incentive to be honest for having a positive utility. The new notation allows us to obtain the re-formulation of the weakly and preferred set.

$$P_i(x_i(\omega), \beta_i, \omega) = \{x_i \in R_+^{L\Omega} | V_i(z_i(\omega), \beta_i, \omega) > V_i(x_i(\omega), \beta_i, \omega)\}.$$

$\forall j, i \in N$.

$$W_i(x_i(\omega), \beta_i, \omega) = \{x_i(\omega) \in R_+^{L\Omega} | V_i(z_i(\omega), \beta_i, \omega) \geq V_j(x_j(\omega), \beta_j, \omega)\}$$

$\forall j, i \in N$.

If the agent i lies, he doesn't have a positive expected utility and his bundle loses any value.

The interactive choice set is again recursively defined:

$$Z_i^{*x} = e_i(\omega), \forall i \in N, \forall \omega \in \Omega.$$

Then at $t = 1$, the agents make conjectures about nature's states with the common knowledge of the economy at $t = 0$. Therefore, they exchange their own bundle with the other ones in a convenient trade for all the parts of the process. The preference is defined with respect to the common prior π and the state of nature ω.

$$Z_i^{*x}(1) = Z_i^{*x}(0) \bigcup_{j \in N} [Z_i^{*x}(0) + Z_j^{*x}(0) - W_j(x_j(\omega), \beta_j, \omega)]$$

$\forall i, j \in N, i \neq j$, with $x_j(\omega) \in R_+^{L\Omega}$.

The liar will decide to say the truth because, otherwise, his expected utility will be zero and no one makes exchange with him.

The characterization of the core and of the arbitrage condition is the same because the incentive credibility constraint avoids the problem of a mystification of the reality. Indeed, if an agent decides to lie his credibility decreases and the β_i becomes equal zero, giving an expected utility equal zero. So, he won't lie for avoiding this risk.

Appendix

Correspondance

A correspondence $\phi : X \to 2^Y$ is a function from X to the family of all subsets of Y.

The graph of a correspondence $\phi : X \to 2^Y$ is defined by:

$$G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}.$$

A correspondence $\phi : X \to 2^Y$ has open (closed) graph if the set $G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$ is open (closed) in $X \times Y$.

Let $\phi : X \to 2^Y$ be a correspondence.

1. ϕ is closed at x if $(x_n, y_n) \to (x, y)$ and $y_n \in \phi(x_n)$ for any n imply $y \in \phi(x)$. It is closed (has a closed graph) if it is closed at any $x \in X$.

2. ϕ is upper hemi-continuous (u.h.c.) at x if, for any open set V containing $\phi(x)$, there exists a neighborhood U of x such that $\phi(x^{'}) \subset V$ for any $x^{'} \in U$. ϕ is upper hemi continuous if it is upper hemi -continuous at any $x \in X$.

3. ϕ is lower hemi-continuous (l.h.c.) at x if, for any open set V with $\phi(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $\phi(x^{'}) \cap V \neq \emptyset$ for any $x^{'} \in U$. A correspondence ϕ is lower hemi continuous if it is lower hemi-continuous at any $x \in X$.

4. ϕ is continuous at x if ϕ is both upper hemi-continuous and lower hemi-continuous at x. It is continuous if it is continuous at any $x \in X$.

Concavity

Let X be a convex subset of R^L and let $f : X \to R$ be a function.

1. A function f is concave if for $x, x^{'} \in X, f(\alpha x + (1 - \alpha)x^{'}) \geq \alpha f(x) + (1 - \alpha)f(x^{'})$ for $\alpha \in [0, 1]$.

2. A function f is convex if $-f$ is concave.

3. A function f is quasi-concave if $\{x \in X : f(x) \geq \alpha\}$ is convex for any $\alpha \in R$.

4. A function f is quasi-convex if $(-f)$ is quasi-concave.

Preferences

A relation R is a correspondence from X to 2^X. The properties of R are defined as follows:

1. R is reflexive if, for any $x \in X, x \in R(x)$;

2. R is irreflexive if, for any $x \in X, x \notin R(x)$;

3. R is complete if, for any $x, x^{'} \in X, x^{'} \in R(x)$ or $x \in R(x^{'})$;

4. R is transitive if, $x^{''} \in R(x)^{'}$ and $x^{'} \in R(x)$, implies $x^{''} \in R(x)$;

5. R is negatively transitive if, $x^{''} \notin R(x^{'})$ and $x^{'} \notin R(x)$, implies $x^{''} \notin R(x)$;

6. R is symmetric if $x^{'} \in R(x)$ implies $x \in R(x^{'})$;

7. R is asymmetric if $x^{'} \in R(x)$ implies $x \notin R(x^{'})$;

8. R is anti-asymmetric if $x^{'} \in R(x)$ and $x \in R(x^{'})$ implies $x^{'} = x$.

Define a relation $R : X \to 2^X$ by $R(x) = \{x^{'} \in X : x^{'} \succeq x\}$. The properties of $\succeq$(or R) are defined as follows:

1. $\succeq$ is reflexive if, for any $x \in X, x \succeq x$.

2. $\succeq$ is complete if, for any $x, x^{'} \in X, x^{'} \succeq x$ or $x \succeq x^{'}$.

3. $\succeq$ is transitive if $x \succeq x^{'}$ and $x^{'} \succeq x^{''}$ implies that $x \succeq x^{''}$.

4. $\succeq$ is weakly monotonic if $x^{'} \geq x$ implies $x^{'} \succeq x$.

5. $\succeq$ is monotonic if $x^{'} \gg x$ implies $x^{'} \succ x$.

6. $\succeq$ is strongly monotonic if $x^{'} > x$ implies $x^{'} \succ x$.

7. $\succeq$ is non-satiated if, for any $x \in X$, there is $x^{'} \in X$ such that $x^{'} \succ x$.

8. $\succeq$ is locally non-satiated if, for any $x \in X$, for any $\varepsilon > 0$ there is a $x^{'} \in B_\varepsilon(x) \cap X$ such that $x^{'} \succ x$.

9. $\succeq$ is convex if $x^{'} \succeq x$ and $x^{''} \succeq x$ implies $\alpha x^{'} + (1-\alpha)x^{''} \succeq x$ for any $\alpha \in [0,1]$.

10. $\succeq$ is semi-strictly convex if $x^{'} \succ x$ implies $\alpha x^{'}+(1-\alpha)x \succ x$ for any $\alpha \in (0,1]$ and $x \sim x^{'}$ implies $\alpha x^{'} + (1-\alpha)x \succeq x$ for any $\alpha \in [0,1]$.

11. $\succeq$ is strictly convex if $x^{'} \succeq x$ and $x^{''} \succeq x$ and $x^{'} \neq x^{''}$ implies $\alpha x^{'}+(1-\alpha)x^{''} \succ x$ for any $\alpha \in (0,1)$.

Boundary condition

Each individual's utility function $u : X \rightarrow R$ satisfies the boundary condition for any $x \in X$:

$$\{y : u(y) \geq u(x)\}\text{is closed relative to} \quad R^L$$

That is, all the indifference curves are asymptotic to the axes. The interpretation of the boundary condition is that some amount of each commodity is required for subsistence.

Existence of Walrasian Equilibrium

Let $\xi = \{(X_i, u_i, e_i) : i \in N\}$ be an exchange economy satisfying the following assumptions for each $i \in N$.

a. X_i is a nonempty, compact and convex subset of R^L;

b. $u_i : X_i \rightarrow R_+$ is quasi-concave and continuous;

c. $e_i \in intX_i$.

Then ξ has a free disposal equilibrium, i.e.there exist $(p^*, x^*) \in \Delta \times X$ with $X = \Pi_{i\in N} X_i$ such that:

(1) $\forall i \in N, x_i^* \in \pi_i(p^*) := \{u_i(x_i) \geq u_i(x_i^{'}), \forall x_i^{'} \in \mathcal{B}_i(p^*)\}$, where $\mathcal{B}_i(p^*) := \{x_i \in X_i : p^* x_i \leq p^* e_i\}$;

(2) $\sum_{i\in N} x_i^* \leq \sum_{i\in N} e_i$.

Optimality of Walrasian equilibrium

- A pair $(p, x) \in \Delta \times X$ is a Walrasian equilibrium if x is feasible and $x_i^{'} \succ_i x_i$ implies that $px_i^{'} > pe_i \geq px_i, \forall i \in N$;

- a pair $(p, x) \in \Delta \times X$ is a quasi Walrasian equilibrium if x is feasible and $x_i^{'} \succeq_i x_i$ implies $px_i^{'} \geq pe_i, \forall i \in N$.

Equilibrium of abstract economy

An equilibrium for the abstract economy is $x^* \in X$ such that, for any $i \in N$

1. $x^* \in A_i(x^*)$;
2. $P_i(x^*) \cap A_i(x^*) = \emptyset$.

Core allocation

An allocation $x \in X$ is a core allocation for $\xi = \{(X_i, u_i, e_i) : i \in N\}$ if:

1. $\sum_{i \in N} x_i = \sum_{i \in N} e_i$;
2. There does not exist a coalition $S \subseteq N$ and $(x_i^{'})_{i \in S} \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} x_i^{'} = \sum_{i \in S} e_i$ and $x_i^{'} \succ_i x_i, \forall i \in S$.

and $C(\xi)$ denotes the set of all core allocations for ξ.

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