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Social Nash Equilibria: Variational Stability and Approximations

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to my parents

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Introduzione

Nel corso degli anni, la Teoria dei Giochi Non Cooperativi si è rivelata sempre più essere un efficace strumento di analisi in diversi campi, tra i quali l'Economia. Basti pensare, ad esempio, all'analisi dei mercati oligopolistici ([35]) e alla Teoria dell'Equilibrio Economico Generale ([14], [21]). Infatti, sia la competizione tra produttori in un mercato oligopolistico che la ricerca di equilibri competitivi in alcune economie fatte di agenti che perseguono il proprio interesse, possono essere analizzate studiando gli equilibri di Nash di opportuni giochi non cooperativi.
L'analisi di determinati sistemi economici può consistere, quindi, nello studio dell' esistenza di "equilibri" (intesi come stati in cui gli agenti coinvolti non hanno incentivi a deviare). Inoltre, dato un sistema economico i cui dati dipendono da parametri esogeni, può anche essere richiesto lo studio della sua stabilità variazionale. Con ciò intendiamo lo studio degli equilibri (o anche di approssimazioni di equilibri) del sistema in funzione di perturbazioni dei suoi dati.

Scopo della presente Tesi è stato quello di studiare la stabilità variazionale e l'esistenza di equilibri approssimati per *economie astratte* ([15]), anche dette *pseudo-giochi* ([21]), rispettivamente nei casi di assenza di continuità delle funzioni payoffs e di assenza di compattezza degli spazi di strategie. Sempre in assenza di continuità dei payoffs, si sono anche studiati problemi di MaxSup e di MaxInf, i quali, ad esempio, trovano applicazioni nei mercati oligopolistici in presenza di leaders ([18]).

Come è ben noto, l'esistenza di equilibri di Nash è equivalente all'esistenza di punti fissi di un'opportuna multifunzione. Al fine di applicare a tale multifunzione i classici teoremi di esistenza di punti fissi ([22], [19]), fondamentali sono state la continuità dei payoffs e la compattezza degli spazi ([40], [41], [19]). Tuttavia, vi sono numerosi giochi, anche di natura economica, in cui i payoffs non sono continui e/o gli spazi delle strategie non sono compatti.

Basti pensare al classico oligopolio di Bertrand (caso simmetrico), il quale è un gioco che esibisce discontinuità nei payoffs e spazi di strategie non compatti. Nonostante tutto, l'oligopolio di Bertrand è dotato di equilibri di Nash, anche se all'esistenza di questi non si può giungere applicando i classici teoremi di Nash ([41]) o di Glicksberg ([19]) (fondati su teoremi di punto fisso).

Al fine di illustrare la presenza di discontinuità nei giochi, nel Capitolo 1, oltre al citato oligopolio, viene richiamato qualche gioco di natura economica che presenta discontinuità nei payoffs. Inoltre, vengono riportati alcuni tra i più recenti risultati sull'esistenza di equilibri di Nash per giochi con payoffs non necessariamente continui.

Come viene mostrato nel Paragrafo 5.1, le classi di discontinuità che garantiscono l'esistenza di equilibri e che sono richiamate nel Capitolo 1, non costituiscono condizioni sufficienti per la stabilità variazionale dei giochi.
A partire da ciò, nella Tesi ci si è occupati di individuare opportune classi di giochi discontinui e stabili, fornendo, quindi, esplicite condizioni sui payoffs, più generali della continuità, atte a garantire la chiusura sequenziale della multifunzione degli equilibri corrispondenti a parametri esogeni.
Precedenti risultati nei quali è stata rilassata la continuità dei payoffs sono stati ottenuti in [11], dove si considera il caso di assenza di vincoli espliciti, e in [30], dove le ipotesi sui dati sono fatte in spazi di Banach reali, riflessivi e separabili.

Al fine di ottenere nuovi risultati sulla chiusura sequenziale della multifunzione degli equilibri di Nash sociali, con ipotesi esplicite sui dati e in spazi di convergenza del tutto generali, fondamentali strumenti si sono rivelate essere le funzioni *sequenzialmente pseudocontinue superiormente* e *inferiormente*, più generali delle funzioni sequenzialmente semicontinue superiormente e inferiormente rispettivamente.
Tali nuove classi di funzioni vengono presentate nel Capitolo 2, unitamente a qualche notevole caratterizzazione e a condizioni sufficienti. Quest'ultime esprimono il legame tra le funzioni pseudocontinue e la stretta monotonia: ogni funzione strettamente monotona è allo stesso tempo sequenzialmente pseudocontinua superiormente e inferiormente.
I nuovi risultati ottenuti sulla chiusura della multifunzione degli equilibri di Nash sociali, sono discussi al Capitolo 5.
Inoltre, sempre nel Capitolo 5, è stata studiata la buona posizione parametrica dei giochi e pseudo-giochi con payoffs *pseudocontinui.* La nozione di

buona posizione alla quale ci si riferisce è quella alla Tykhonov, il cui significato è il seguente. Supponiamo di voler calcolare l'unico equilibrio di Nash y_o di un gioco mediante un metodo composto da una successione di passi, ciascuno dei quali produce un equilibrio di un gioco i cui payoffs sono una deformazione di quelli del gioco originario. Supponiamo che le deformazioni sono rappresentate mediante un insieme di parametri X e che il gioco originario corrisponde al valore $x_o \in X$. L'essere il gioco parametricamente ben posto vuol dire che gli equilibri prodotti dai passi successivi x costitutiscono una buona approssimazione dell'equilibrio y_o tanto più quanto maggiormente x è vicino a x_o.
Anche per la buona posizione parametrica, le funzioni pseudocontinue si sono rivelate essere uno strumento con cui rilassare la continuità dei payoffs, ipotesi dominante, come esplicita sui dati, nei risultati presenti in letteratura ([43], [34]).

I corrispondenti casi di stabilità variazionale e buona posizione parametrica in ottimizzazione sono trattati al Capitolo 4.

Anche quando i payoffs di un gioco (o di un pseudo-gioco) sono continui, la non esistenza di equilibri di Nash può derivare dalla non-compattezza degli spazi delle strategie (vedi Example 6.1.2).
Partendo da queste circostanze di assenza di compattezza, altro scopo della presente Tesi è stato quello di fornire delle condizioni sufficienti per l'esistenza di equilibri approssimati in giochi e in pseudo-giochi. Precisamente, come ben noto, fissato $\varepsilon > 0$, un profilo di strategie di un gioco è detto un ε-equilibrio di Nash se ogni giocatore non può incrementare il proprio payoffs più di ε mediante deviazioni unilaterali. Per quanto concerne le economie astratte (pseudo-giochi), nel caso di vincoli di disuguaglianze, nel Paragrafo 6.3 viene proposto un concetto di equilibrio sociale approssimato.
Sia per i giochi che per le economie astratte, nel Capitolo 6 vengono stabilite condizioni sufficienti per l'esistenza di equilibri approssimati nel caso in cui gli spazi delle strategie sono sottoinsiemi limitati in spazi di Banach reali, riflessivi e separabili, e nel caso in cui gli spazi delle strategie sono sottoinsiemi totalmente limitati in spazi di Banach reali.

Gli strumenti atti ad ottenere l'esistenza di equilibri approssimati per giochi e economie astratte sono nuovi teoremi di punto fisso approssimato per multifunzioni, i quali vengono illustrati nel Capitolo 3. In tali risultati, facendo ipotesi di continuità (chiusura del grafico o semicontinuità superiore) rispetto

alla topologia debole, l'esistenza di punti fissi approssimati per multifunzioni è ottenuta in spazi di Banach reali, riflessivi e separabili per regioni convesse, limitate e con interiore non vuoto. Facendo ipotesi di continuità della multifunzione rispetto alla topologia forte, l'esistenza di punti fissi approssimati è garantita su regioni convesse, totalmente limitate e con interiore non vuoto in spazi di Banach reali.
Inoltre, viene introdotta una nuova proprietà per multifunzioni, detta *tame*, la quale, unitamente a ipotesi di continuità rispetto alla topologia debole (chiusura del grafico o semicontinuità superiore), garantisce l'esistenza di punti fissi approssimati in spazi di Banach reali, riflessivi e separabili, anche per regioni convesse e non limitate, con interiore non vuoto.

Altre situazioni in cui le discontinuità dei payoffs possono essere tali da rendere non applicabili i classici risultati di esistenza di soluzioni, sono i problemi di MaxSup e di MaxInf. Ad esempio, un giocatore avverso al rischio è portato a cercare una strategia cautelativa, cioé una strategia che gli garantisca un livello minimo per il proprio payoff. Partendo da ciò, la presente Tesi si conclude con un capitolo dedicato allo studio dell'esistenza di soluzioni per i problemi di MaxSup e di MaxInf.
Come discusso al Capitolo 7, la semicontinuità superiore delle funzioni obiettivo garantisce, sotto opportune ipotesi di compattezza, l'esistenza di soluzioni. Per quanto riguarda i problemi di MaxInf, risultati di esistenza con ipotesi più generali della semicontinuità superiore dell'obiettivo sono stati ottenuti in [1] e in [26] mediante gamma-limiti.
Nuovi risultati di esistenza sono stati ottenuti utilizzando la classe delle funzioni *sequenzialmente quasicontinue superiormente*, introdotta nel Capitolo 2, la quale contiene strettamente la classe delle funzioni sequenzialmente pseudocontinue superiormente. Così come osservato per le funzioni pseudocontinue, la monotonia è una condizione sufficiente per la quasicontinuità.
I nuovi risultati di esistenza di soluzioni per i problemi citati, sono stati ottenuti generalizzando il Teorema di Weierstrass in spazi di convergenza, in cui si rilassa la sequenziale semicontinuità superiore della funzione obiettivo, ispirandosi alle funzioni *debolmente continue superiormente per trasferimento* introdotte in [48] e che caratterizzano l'esistenza di punti di massimo in spazi topologici. Quindi, viene discussa una speculare caratterizzazione dell'esistenza di punti di massimo per funzioni definite in spazi sequenzialmente compatti.

Chapter 1

Games with Discontinuities

Aim of this chapter is to consider some remarkable economic game in which discontinuities are present. As it will be shown, several authors have studied the problem of the existence of Nash equilibria when the payoffs are not continuous. Among the others, we recall Dasgupta and Maskin in 1986 ([13]), and more recently, Baye, Tian and Zhou in 1993 ([3]) and Reny in 1999 ([44]).
Discontinuities can be also present in markets with leadership. In fact, it will be considered an oligopoly with leadership, in which the inverse of demand will be a set-valued functions. This oligopoly has been studied by Flam, Mallozzi and Morgan in [18], where the authors obtain the existence of Stackelberg-Cournot equilibria.

1.1 Existence of Equilibria in Games with Discontinuous Payoffs

In this section, we will consider games in normal form with discontinuous payoffs and we will refer to some of more recent results on existence of equilibria.

Let give m individuals (where m is a positive integer greater or equal to 2), each of them possesses a non-empty set of strategic choices Y_i ($i \in \{1, ..., m\}$) and a payoff (or utility) function f_i, which is defined on the cartesian product of all strategy sets $Y = \prod_{j=1}^{m} Y_j$. As well know, the list of data $G = \{Y_i, f_i\}_i$ is a *game in normal form*, and an element $y^* \in Y$ is a *Nash equilibrium* in *pure strategies* (see [40] and [41]) if $f_i(y_i^*, y_{-i}^*) \geq f_i(y_i, y_{-i}^*)$ for all $y_i \in Y_i$ and for all i.

In the case in which the strategy sets are compact and convex (in topological vector spaces) and the payoffs are quasiconcave, the early theorems due to Nash ([41]), Debreu ([15]) and Glicksberg ([19]) reveal that games possess a pure strategy Nash equilibria when the payoffs are continuous functions.

There are many remarkable economic games in literature which possess discontinuous payoffs, as the price competition in an oligopoly due to Bertrand ([5]), the spatial competition in linear city due to Hotelling ([20]), the pay-your-bid multi-unit auction game (see for example [44]). In the following, some examples will be given.

Example 1.1.1 (Bertrand's competition, [5]) There are two firms which produce the same good. Suppose that firms are price makers and they have the same marginal cost $c > 0$: for to produce the quantity q of good, each firm spends cq. For any fixed price p, the market of consumers asks the total quantity $d(p)$ of good and we assume that each firm can produce any quantity $d(p)$ asked by market. The competition between the producers is in to make an optimal price p in order to maximize own profit. So, the competition is represented by the game $\{Y_1, Y_2, \pi_1, \pi_2\}$ with $Y_1 = Y_2 = [0, +\infty[$ and π_i defined by:

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c)d(p_i) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c)d(p) & \text{if} \quad p_i = p_j = p \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

where $i \neq j$. This game has strategic spaces non compact and payoffs which are not continuous. Hence, the classical existence theorems cannot be applied. Now, we analyze the behavior of firms. If i wants to make a price $p_i > c$, the other firm j could make a price $p_j \in]c, p_i[$ and in this way the profit of i could be 0. So, the firm i will not make a price strictly greater than c. This game has a unique Nash equilibrium, that is the pair (c, c), even if the classical hypothesis on the existence are not satisfied.

The Bertrand competition satisfies the better replay security condition due to Reny in [44], which is a property that, together hypothesis of compactness of and convexity, recognizes discontinuous games endowed of equilibria. We recall that a game is said *better replay security* if for every non-equilibrium y^* and for every vector u^* such that the pair (y^*, u^*) belongs to the closure of the graph of the vector function $f = (f_1, ..., f_m)$, some player i has a strategy $\bar{y}_i$ such that $f_i(\bar{y}_i, y_{-i}) > u_i^*$ for all deviation y_{-i} in some neighbourhood of y_{-i}^*.

Example 1.1.2 (Hotelling's competition, [20]) On a linear city of length l, two sellers A and B of an homogeneous good are located at respective distances a and b from the ends of the city: $a+b=l$ and $a,b \geq 0$. Suppose that consumers are uniformly distributed on $[0,l]$ and that each customer consumes exactly one unit of the commodity per one unit of time. The customers have an uniform rate c of transportation. So, they will buy from the seller who make a price which minimize own total cost, that is the cost of the good and the cost of the transport. This situation is described by the game $\{Y_1, Y_2, \pi_1, \pi_2\}$ with $Y_1 = Y_2 = [0, +\infty[$ and π_i defined by:

$$\pi_1(p_1,p_2) = \begin{cases} ap_1 + 1/2(l-a-b)p_1 & \\ +(1/2c)p_1p_2 - (1/2c)p_1^2 & \text{if} \quad \mid p_1 - p_2 \mid \leq c(l-a-b) \\ lp_1 & \text{if} \quad p_1 < p_2 - c(l-a-b) \\ 0 & \text{if} \quad p_1 > p_2 - c(l-a-b) \end{cases}$$

and

$$\pi_2(p_1,p_2) = \begin{cases} bp_2 + 1/2(l-a-b)p_2 & \\ +(1/2c)p_1p_2 - (1/2c)p_2^2 & \text{if} \quad \mid p_1 - p_2 \mid \leq c(l-a-b) \\ lp_2 & \text{if} \quad p_2 < p_1 - c(l-a-b) \\ 0 & \text{if} \quad p_2 > p_1 - c(l-a-b) \end{cases}$$

The functions π_1 and π_2 are not continuous and the strategic spaces are not compact. So, the classical results on existence of equilibria cannot be applied. As established in [2], the pair $(0,0)$ is the unique equilibrium when $a+b=l$ and, if $a+b<l$, there exists an equilibrium if and only if:

$$(l + \frac{a-b}{3})^2 \geq \frac{4}{3}\, l\, (a+2b) \quad \text{and} \quad (l + \frac{b-a}{3})^2 \geq \frac{4}{3}\, l\, (b+2a). \tag{1.1}$$

A characterization of games in normal form which admit equilibria has been given by Baye, Tian and Zhou in [3]. Here the authors consider the aggregator function f defined by $f(y,z) = \sum_{i=1}^{m} f_i(y_i, z_{-i})$ for all $y, z \in Y$, and they give two condition on f (so non explicit on the data), hard to verify, called diagonal transfer continuity and diagonal transfer quasiconcavity. More precisely, a game is said *diagonal transfer continuous in* z if for every (y,z), $f(y,z) > f(z,z)$ implies that there exists a point y' and a neighbourhood I of z such that $f(y',z') > f(z',z')$ for each $z' \in I$. A game is said *diagonal transfer quasiconcave in* y if for any finite subset $\{y^1, ..., y^k\}$ there exists a subset $\{z^1, ..., z^k\}$ such that: if z^o is a convex combination of $\{z^{i_1}, ..., z^{i_h}\}$, where $i_1, ..., i_h \in \{1, ..., k\}$, it result in $f(z^o, z^o) \geq \min\{f(y^{i_1}, z^o), ..., f(y^{i_h}, z^o)\}$.

Now, these conditions characterize the existence of Nash equilibria for games with non-empty, convex strategic spaces and with a suitable "compactness assumption".
For example, one can see that the Hotelling's competition fails to have equilibria when (1.1) is not satisfy, because the game is diagonal transfer quasiconcave if and only if (1.1) holds.

Example 1.1.3 (Baye, Tian and Zhou, [3]) Two duopolists have zero cost and set prices $(p_1, p_2) \in [0, T]^2$. Let $c \in]0, T[$. Assume that each firm has committed to pay brand-loyal consumers a penalty of c if the others firm make an inferior price. So, this oligopoly is represented in the game in which the set of strategies are equal to $[0, T]$ and the payoffs are the following, where $i \neq j$:

$$f_i(p_i, p_j) = \begin{cases} p_i & \text{if} \quad p_i \leq p_j \\ p_i - c & \text{if} \quad p_i > p_j \end{cases}$$

The game does not satisfy the hypothesis on existence of equilibria of Reny ([44]), but it is diagonal transfer continuous and diagonal transfer quasiconcave. So, the game admits at least a Nash equilibria in light of the characterization due to Baye, Tian and Zhou ([3])

Others existence results on Nash equilibria without continuity of payoffs, they have been obtained by Lignola in [25] and Lignola and Morgan in [30].

In [25], the author considers strongly escaping sequences for to relax the compactness assumption on the strategic spaces. In fact, let $(C_n)_n$ be an increasing (with respect to inclusion) sequence of non-empty and compact sets such that $Y = \cup_n C_n$. A sequence $(y_n)_n \subseteq Y$ is said to be *strongly escaping from* Y *relative to* $(C_n)_n$ ([25]) if it results that the set of cluster points of $(y_n)_n$ has non-empty intersection with any C_n. Let $Y_i = \cup_n C_{i,n}$ for all $i \in \{1, 2\}$, where $(C_{i,n})_n$ has non-empty, compact and convex elements, and $C_n = C_{1,n} \times C_{2,n}$. The existence of Nash equilibria for a two person game is guaranteed if: every $f_i(\cdot, y_{-i})$ is quasiconcave for all y_{-i}; $f_1 + f_2$ is upper semicontinuous on C_n for all n; for any i, $f_i(y_i, y_{-i}) \leq \sup_{z_i \in C_{i,n}} \liminf_{z_{-i} \to y_{-i}} f_i(z_i, z_{-i})$ for all $y_i \in Y_i$, all n and for all $y_{-i} \in C_{-i,n}$; for any strongly escaping sequence $(y^n)_n$ from Y with respect $(C_n)_n$, with $y^n \in C_n$ for any n, there exists some i and a sequence $(z_i^n)_n$, with $z_i^n \in C_{i,n}$, such that either $\liminf_{n \to \infty}[f_1(z_1^n, y_2^n) - f_1(y^n)] < 0$, if $z_1^n \in C_{1,n}$ for any n, or $\liminf_{n \to \infty}[f_2(y_1^n, z_2^n) - f_2(y^n)] < 0$, if $z_2^n \in C_{2,n}$ for any n, hold.

When the strategic spaces are non-empty, bounded, convex and closed subsets of reflexive and separable real Banach spaces, under the classical assumption of quasiconcavity on payoffs, existence of Nash equilibria is obtained in [30] for function f_i which are sequentially upper semicontinuous in the weak topology, and such that: for any i, for any $y \in Y$ and any sequence (y_{-i}^n) weakly convergent to y_{-i}, there exists a sequence $(z_i^n)_n$ such that $f_i(y) \leq \liminf_{n\to\infty} f_i(z_i^n, y_{-i}^n)$.

1.2 Discontinuities in Markets with Leadership

Suppose that we have a market in which there are several individuals and one of them is a leader with respect all of others. Let f_o and Y_o be the payoff and the set of choices of the leader respectively, while the list of data $\{Y_i, f_i\}_{i=1}^m$ characterizes the followers, where all functions $f_o, f_1, ..., f_m$ are defined on $Y_o \times Y_1 \times ... \times Y_m$. Assume that all individuals want to maximize their payoffs. The leadership is as following: the leader makes own choice $y_o \in Y_o$ and, after this choice, the followers play a non-cooperative games, in which $N(y_o)$ is the set of Nash equilibria. So, the leader has a MaxSup problem to solve:

$$\text{find } \; y_o^* \in Y_o \; \text{ such that } \quad \sup_{y\in N(y_o^*)} f_o(y_o^*, y) = \max_{y_o\in Y_o} \sup_{y\in N(y_o)} f_o(y_o, y)$$

The classical results for existence of solutions to this problems ask that the multifunction $N : y_o \mapsto N(y_o)$ is closed. Now, the followers can have discontinuities in their payoffs, as we have already seen in the previous paragraph. Starting from these situations, in Chapter 5 it will be studied classes of discontinuities which guarantee the closedness of the multifunction N.

When the market here considered is an oligopoly in which the competition between the firms is on the quantities of good to put in the market, a point $(y_o^*, y_1^*, ..., y_m^*)$ is said a Stackelberg-Cournot equilibrium (see [18]) if $(y_1^*, ..., y_m^*) \in N(y_o^*)$ and y_o^* maximize the function $\sup_{y\in N(\cdot)} f_o(\cdot, y)$ over Y_o. Flam, Mallozzi and Morgan consider in [18] a case of this oligopoly in which the inverse of demand of the market is a multifunction. In fact, the authors analyze the case in which firms agree to make always the maximal price, when there are more possible prices for one quantity. Under suitable assumptions on the inverse of demand set-valued function, Flam, Mallozzi and Morgan first prove that this behaviour of firms determine an upper semicontinuous inverse of demand function. After, the authors prove that the reaction (that is the set of Cournot equilibria) of the followers, whose payoffs are upper semicontinuous, is closed over the choices of the leader. Finally, existence of Stackelberg-Cournot equilibria is obtained.

Chapter 2

New Tools I: Pseudocontinuous and Quasicontinuous Functions

In this chapter, in the setting of sequential spaces, we will present new classes of functions more general than sequential semicontinuity. The first one will be called *sequentially upper* and *lower pseudocontinuous* functions and they will be the tools able to relax the classical continuity assumptions on payoffs in order to obtain stability and well-posedness in non-cooperative games and pseudo-games. The second one will be called *sequentially upper* and *lower quasicontinuous* functions and they will allow, on one hand to enlarge the sufficient conditions for the existence of solutions to MaxSup and MaxInf Problems, on the other hand to relax the semicontinuity for the well-posedness of optimization problems.

2.1 Preliminary Notions

First, we recall some well know definitions in sequential spaces.

Let Z be a non-empty set and τ be a subset of $Z^{\mathbb{N}} \times Z$, where $\mathbb{N}$ is the set of positive integers. The subset τ is a *structure of convergence on Z* (see for example [24] and [27]) if the following axioms are satisfied:

(S_1) for each $z \in Z$ and any sequence $(z_n)_n$ in Z such that $z_n = z$ for n sufficiently large, we have $((z_n)_n, z) \in \tau$;

(S_2) if $((z_n)_n, z) \in \tau$ and if $(z_{n_k})_k$ is a subsequence of $(z_n)_n$, then we have: $((z_{n_k})_k, z) \in \tau$;

(S_3) if $((z_n)_n, z) \notin \tau$, then there exists a strictly inscreasing sequence of integers (also called selection of integers) $(n_k)_k$ such that for each its subsequence $(n_{k_m})_m$ we have: $((z_{n_{k_m}}), z) \notin \tau$.

If τ is a structure of convergence on Z, the pair (Z, τ) is termed *convergence sequential space* (or, in short, *sequential space*). Often, we will denote a sequential space (Z, τ) by Z, omitting τ. A sequence $(z_n)_n \subseteq Z$ is termed *converging to z in Z* (in short $z_n \longrightarrow z$) if $((z_n)_n, z) \in \tau$.

A subset $U \subseteq Z$ is termed: *sequentially compact*, if and only if any sequence of points of U has a subsequence converging to a point of U; *sequentially closed*, if and only if any converging sequence of points of U, it converges to a point of U.

Definition 2.1.1 ([24], [27]). *Let (Z, τ) be a sequential space and f be an extended real valued function defined on Z. The function f is said to be* sequentially *(*seq. *in short)* upper semicontinuous at $z_o \in Z$ *if and only if:*

$$\limsup_{n\to\infty} f(z_n) \leq f(z_o) \text{ for all sequence } z_n \longrightarrow z_o \text{ in } Z,$$

and f is said to be seq. lower semicontinuous at $z_o \in Z$ *if and only if:*

$$f(z_o) \leq \liminf_{n\to\infty} f(z_n) \text{ for all sequence } z_n \longrightarrow z_o \text{ in } Z.$$

Definition 2.1.2 ([24]). *Let $(A_n)_n$ be a sequence of subsets of a sequential space (Z, τ), then:*

- *$z \in Liminf A_n$ if and only if there exists a sequence $(z_n)_n$ converging to z in Z and such that $z_n \in A_n$ for n sufficiently large;*
- *$z \in Limsup A_n$ if and only if there exists a sequence $(z_k)_k$ converging to z in Z such that $z_k \in A_{n_k}$ for a subsequence (A_{n_k}) of $(A_n)_n$ and for each $k \in \mathbb{N}$.*

Definition 2.1.3 ([24]). *Let X and Y be two sequential spaces, K be a set-valued function (or multifunction) from X to Y and $x_o \in X$. Then:*

- *K is said to be* seq. closed at x_o *if and only if $Limsup K(x_n) \subseteq K(x_o)$ for all sequence $(x_n)_n$ converging to x_o in X;*

- *K is said to be* seq. lower semicontinuous at x_o *if and only if* $K(x_o) \subseteq Liminf K(x_n)$ *for all sequence* $(x_n)_n$ *converging to* x_o *in* X.

Definition 2.1.4 ([27]). *Let X and Y be two sequential spaces, K be a set-valued function from X to Y and $x_o \in X$. K is said to be* sequentially subcontinuous at x_o *if and only if for any sequence $(x_n)_n$ converging to x_o and any sequence $(y_n)_n \subseteq Y$ such that $y_n \in K(x_n)$ for n sufficiently large, there exists a subsequence of $(y_n)_n$ which converges to a point of Y.*

2.2 Pseudocontinuous Functions

In this section, following [37], we present the class of *sequentially upper pseudocontinuous functions* and the class of *sequentially lower pseudocontinuous functions*, which strictly include, respectively, the class of sequentially upper semicontinuous functions and the class of sequentially lower semicontinuous functions. Characterizations and sufficient conditions for pseudocontinuity will be also given.

Definition 2.2.1 *Let (Z, τ) be a sequential space and f be an extended real valued function defined on Z.*

- *f is said to be* seq. upper pseudocontinuous at $z_o \in Z$ *if and only if for all $z \in Z$ such that $f(z_o) < f(z)$, we have:*

  $$\limsup_{n\to\infty} f(z_n) < f(z) \text{ for all sequence } (z_n)_n \text{ converging to } z_o \text{ in } Z;$$

  f is said to be seq. upper pseudocontinuous on Z *if and only if it is seq. upper pseudocontinuous at z_o, for all $z_o \in Z$;*

- *f is said to be* seq. lower pseudocontinuous at $z_o \in Z$ *if and only if $-f$ is seq. upper pseudocontinuous at z_o and f is said to be* seq. lower pseudocontinuous on Z *if and only if it is seq. lower pseudocontinuous at z_o, for all $z_o \in Z$;*

- *f is said to be* seq. pseudocontinuous *if and only if it is both seq. upper and lower pseudocontinuous.*

As we have already said in the beginning, if (Z, τ) is a sequential space, every extended real valued sequentially upper semicontinuous function on (Z, τ) is also sequentially upper pseudocontinuous on (Z, τ) and every extended real valued sequentially lower semicontinuous function on (Z, τ) is also sequentially lower pseudocontinuous. The converse is not true, as shown by the following counterexamples.

Example 2.2.1 As a simple counterexample in an economic setting we can consider an utility function u depending on a parameter x, whose values characterize the states of the world where the agents make their choices. The value $x = 1$ corresponds to a threshold level: if $x \in [0, 1[$ the agents have positive utilities, if $x \in [1, +\infty[$ the agents have negative utilities.
So, let $\alpha \in]0, 1[$ and u be the function from $[0, +\infty[\times]0, 1]^2$ to $\mathbb{R}$ defined by:

$$u(x, y_1, y_2) = \begin{cases} (1-x)y_1^\alpha y_2^{1-\alpha} & \text{if } x \in [0, 1[\\ -1 & \text{if } x \in [1, +\infty[\end{cases}$$

The function u, which is a Cobb Douglas utility function for $x \in [0, 1[$ (see for example [35]), is not sequentially upper semicontinuous in $(1, y_1, y_2)$, for all $(y_1, y_2) \in]0, 1]^2$, but it is sequentially upper pseudocontinuous.

Example 2.2.2 The function defined by:

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \in]0, 2[\\ x & \text{if } x \in [2, +\infty[\end{cases}$$

is not seq. lower semicontinuous at $x_o = 2$ but it is sequentially lower pseudocontinuous.

We assume that (Z, σ) is a topological space, where σ denotes the topology on Z. In this setting, properties more general than semicontinuity have been introduced by Tian and Zhou in [48], where the authors used them for to characterize the existence of optimal points. Now, if we consider the convergence structure defined by the topology σ, it can be shown (see Example 2.2.3) that the classes of functions introduced in Definition 2.2.1 are not included in the classes of transfer weakly upper and lower continuous functions introduced in [48]. For example, the class of seq. upper pseudocontinuous functions are not included in the class of transfer weakly upper continuous functions, where an extended real valued function f defined on Z is called *transfer weakly upper continuous* ([48]) if and only if for every z_o and z belonging to Z such that $f(z_o) < f(z)$, there exist $z' \in Z$ and a neighbourhood I of z_o such that $f(u) \leq f(z')$ for all $u \in I$.

Example 2.2.3 Let $(\mathbb{R}, \sigma)$ be the topological space such that $A \subseteq \mathbb{R}$ is open (that is $A \in \sigma$) if and only if $\mathbb{R} \backslash A = \{z \in \mathbb{R} / z \notin A\}$ is finite or countable. In $(\mathbb{R}, \sigma)$, a sequence $(z_n)_n$ converges to a point z_o if and only if z_n coincides with z_o for n sufficiently large. So, for every extended real valued function defined on $\mathbb{R}$, every point $z_o \in \mathbb{R}$ and every $z_n \longrightarrow z_o$, we have $\lim_{n\to\infty} f(z_n) = f(z_o)$. Therefore, every extended real valued function defined on $\mathbb{R}$ is seq. upper pseudocontinuous on $\mathbb{R}$ with respect to the convergence structure induced by σ. So, if we consider the identity function i on $]0,1[$ with the relative topology induced by σ, then i is seq. upper pseudocontinuous on]0,1[but it is not transfer weakly upper continuous. In fact, let z and z_o belonging to]0,1[such that $z_o < z$. If I is an open neighbourhood of z_o in $(]0,1[, \sigma)$, there exists a sequence $(y_n)_n$ in $]0,1[$ such that $I =]0,1[\backslash \{y_n / n \in \mathbb{N}\}$. Hence, for every open neighbourhood I of z_o in $(]0,1[, \sigma)$ and for every $z' \in]0,1[$, there exists an element $y \in I$ such that $z' < y$. Therefore, i is not transfer weakly upper continuous at z_o but it is seq. upper pseudocontinuous on $]0,1[$.

For sequentially upper pseudocontinuity there are the following characterizations, which are in line with the well know characterizations of upper semicontinuity in terms of hypograph and upper level sets.

Proposition 2.2.1 *Let (Z, τ) be a sequential space and let f be an extended real valued function defined on Z. Then the following statements are equivalent:*

(i) f is sequentially upper pseudocontinuous on Z;

(ii) the set $\mathcal{L} = \{(z, \lambda) \ / \ f(z) \geq \lambda$ and $\lambda \in f(Z)\}$ is sequentially closed in $Z \times f(Z)$;

(iii) the set $S_\lambda = \{z \in Z \ / \ f(z) \geq \lambda\}$ is sequentially closed on Z for all $\lambda \in f(Z)$.

Proof. First, we prove that (i) implies (ii).
Assume that f is sequentially upper pseudocontinuous on Z. Let $(z_n, f(y_n))_n \subseteq \mathcal{L}$ (that is $f(z_n) \geq f(y_n)$) be a sequence converging to a point $(z, f(y))$ in $Z \times f(Z)$. If $(z, f(y)) \notin \mathcal{L}$, we have $f(z) < f(y)$. Then, in light of (i), we obtain that $\limsup_{n\to\infty} f(z_n) < f(y) = \lim_{n\to\infty} f(y_n)$ and, consequently, $f(z_n) < f(y_n)$ for n sufficiently large. So we get a contradiction.
Now, we prove that (ii) implies (iii).
Let $\lambda \in f(Z)$ (that is $\lambda = f(z)$ for some $z \in Z$) and let $(z_n)_n$ be a sequence of points of S_λ converging to z_o. Then $(z_n, f(z))_n$ is included in $\mathcal{L}$. Since $\mathcal{L}$ is

sequentially closed in $Z \times f(Z)$, we have that $(z_o, f(z)) \in \mathcal{L}$, that is $z_o \in S_\lambda$.
Finally, we prove that (iii) implies (i).
Assume that S_λ is sequentially closed for all $\lambda \in f(Z)$. Let $z \in Z$ and $z_o \in Z$ be such that $f(z_o) < f(z)$ and let $z_n \longrightarrow z_o$.
First, we suppose that there exists $z' \in Z$ such that $f(z_o) < f(z') < f(z)$. Let $\lambda' = f(z')$. Then $z_o \in Z \backslash S_{\lambda'}$. We claim that there exists an index n_o such that $z_n \in Z \backslash S_{\lambda'}$ for all $n \geq n_o$. In fact, if for each n_o, there exists $n \geq n_o$ such that $z_n \in S_{\lambda'}$, then there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that $(z_{n_k})_k \subseteq S_{\lambda'}$. Being $S_{\lambda'}$ sequentially closed, we obtain $z_o \in S_{\lambda'}$ and we get a contradiction. So $f(z_n) < f(z')$ for n sufficiently large. Consequently,

$$\limsup_{n \to \infty} f(z_n) \leq f(z') < f(z).$$

Otherwise, if it does not exist $z' \in Z$ such that $f(z_o) < f(z') < f(z)$, we set $\lambda = f(z)$. Being $z_o \in Z \backslash S_\lambda$ and S_λ sequentially closed, as above we can prove that $f(z_n) < f(z)$ for n sufficiently large. But $f(Z) \cap]f(z_o), f(z)[= \emptyset$, so $f(z_n) \leq f(z_o)$ for n sufficiently large. Hence

$$\limsup_{n \to \infty} f(z_n) \leq f(z_o) < f(z).$$

□

Remark 2.2.1 If (Z, τ) is a sequential space and f is an extended real valued function defined on Z, similarly, one can obtain the following result:
The following statements are equivalent:

(i) f is sequentially upper semicontinuous on Z;
(ii) the hypograph of f is sequentially closed;
(iii) the set $S_\lambda = \{z \in Z \ / \ f(z) \geq \lambda\}$ is sequentially closed on Z for all extended real valued λ.

When (Z, σ) is a topological space which satisfies the first axiom of countability, this result is nothing but the well know characterizations of topological upper semicontinuity.

We conclude the paragraph with a proposition which shows the connection between the pseudocontinuity and the monotonicity: every strictly monotonic extended real valued function is also a sequentially pseudocontinuous function.

Proposition 2.2.2 *Let f be an extended real valued function defined on $\mathbb{R}^k$ and $\mathcal{C}$ be a convex and pointed cone in $\mathbb{R}^k$ with apex at the origin and nonempty interior. If f is strictly monotonic with respect to $\mathcal{C}$, that is:*

$$y \in x + int(\mathcal{C}) \Longleftrightarrow f(x) < f(y) \qquad \textit{(strictly increasing)}$$

or

$$y \in x + int(\mathcal{C}) \Longleftrightarrow f(x) > f(y) \qquad \textit{(strictly decreasing)},$$

then f is sequentially pseudocontinuous on $\mathbb{R}^k$.

Proof. Suppose that f is strictly decreasing with respect to $\mathcal{C}$.
We first prove that f is sequentially lower pseudocontinuous. Let x_o and x be such that $f(x) < f(x_o)$ and let $x_n \longrightarrow x_o$ in X. Then, we have: $x \in x_o + \text{int}(\mathcal{C})$. So there exist an element $y \in [x - \text{int}(\mathcal{C})] \cap [x_o + \text{int}(\mathcal{C})]$ and an open neighbourhood $\mathcal{A}$ of x_o such that $y \in z + \text{int}(\mathcal{C})$ for all $z \in \mathcal{A}$. Since $x_n \longrightarrow x_o$, we have that $y \in x_n + \text{int}(\mathcal{C})$ for n sufficiently large. Then

$$f(x) < f(y) \leq \liminf_{n\to\infty} f(x_n).$$

Now, we prove that f is sequentially upper pseudocontinuous. Let x_o and x be such that $f(x_o) < f(x)$ and let $x_n \longrightarrow x_o$ in X. Then $x_o \in x + \text{int}(\mathcal{C})$. Moreover, there exist $y \in [x_o - \text{int}(\mathcal{C})] \cap [x + \text{int}(\mathcal{C})]$ and an open neighbourhood $\mathcal{B}$ of x_o such that $z \in y + \text{int}(\mathcal{C})$ for all $z \in \mathcal{B}$. Consequently, $f(x_n) < f(y)$ for n sufficiently large. So

$$\limsup_{n\to\infty} f(x_n) \leq f(y) < f(x).$$

Analogously we obtain that f is sequentially pseudocontinuous if it is stricly increasing.

□

2.3 Quasicontinuous Functions

To conclude the chapter, following [38], we present the classes of sequentially quasicontinuous functions together sufficient conditions.

Definition 2.3.1 *Let (Z, τ) be a sequential space and f be an extended real valued function defined on Z.*

- f *is said to be* seq. upper quasicontinuous at $z_o \in Z$ *if and only if for all* $z \in Z$ *such that* $f(z_o) < f(z)$*, we have:*

  $$\limsup_{n\to\infty} f(z_n) \le f(z) \text{ for all sequence } (z_n)_n \text{ converging to } z_o \text{ in } Z;$$

  f *is said to be* seq. upper quasicontinuous on Z *if and only if it is seq. upper quasicontinuous at* z_o*, for all* $z_o \in Z$*;*

- f *is said to be* seq. lower quasicontinuous at $z_o \in Z$ *if and only if* $-f$ *is seq. upper quasicontinuous at* z_o *and* f *is said to be* seq. lower quasicontinuous on Z *if and only if it is seq. lower quasicontinuous at* z_o*, for all* $z_o \in Z$*;*

- f *is said to be* seq. quasicontinuous *if and only if it is both seq. upper and lower quasicontinuous.*

Obviously, the class of seq. upper (respectively lower) quasicontinuous functions strictly includes the class of seq. upper (respectively lower) pseudocontinuous functions. The well know Dirichlet's function (which is equal to 0 on $\mathbb{Q}$ and equal to 1 on $\mathbb{R}\backslash\mathbb{Q}$) is an example of function seq. quasicontinuous which is not seq. pseudocontinuous.

In the setting of topological spaces, Campbell and Walker have introduced in [9] a property for preference relations, called weakly lower continuity, in order to relax the upper semicontinuity in the existence of solutions to optimization problems. Now, any extended real valued function which is weakly lower continuous in the sense of Campbell and Walker is also seq. upper quasicontinuous. Example 2.2.3 shows that, generally, the class of seq. upper quasicontinuous functions and the class of weakly lower continuous functions do not coincide.

Finally, as for the pseudocontinuous functions, also quasicontinuous functions have connections with monotonicity, as showed in the next proposition.

Proposition 2.3.1 *Let* f *be an extended real valued function defined on* $\mathbb{R}^k$ *and* $\mathcal{C}$ *be a convex and pointed cone in* $\mathbb{R}^k$ *with apex at the origin and nonempty interior. If* f *is monotone with respect to* $\mathcal{C}$*, that is:*

$$y \in x + int(\mathcal{C}) \Longleftrightarrow f(x) \le f(y) \qquad \textit{(increasing function)}$$

or

$$y \in x + int(\mathcal{C}) \Longleftrightarrow f(x) \ge f(y) \qquad \textit{(decreasing function)},$$

then f *is seq. quasicontinuous on* $\mathbb{R}^k$*.*

Proof. Assume that f is increasing. Let x and y be such that $f(x) < f(y)$ and $x_n \longrightarrow x$. Since f is increasing, $y \in x + int(\mathcal{C})$. Moreover there exists an open neighbourhood I of x such that $y \in z + int(\mathcal{C})$ for all $z \in I$. Consequently, $f(x_n) \leq f(y)$ for n sufficiently large. So $\limsup_{n\to\infty} f(x_n) \leq f(y)$ and f is seq. upper quasicontinuous.
Similarly, one can prove that h is seq. lower quasicontinuous.

□

Chapter 3

New Tools II: Approximate Fixed Point Theorems

In this chapter we will be interested in set-valued functions $F : X \longrightarrow 2^X$ which possess approximate fixed points.
As well know, a point $x \in X$ is said a *fixed point* of F if $x \in F(x)$. Fixed point theorems deal with sufficient conditions on X and on F which guarantee the existence of fixed points. These theorems have been abundantly used in many applied fields such as game theory, general equilibrium theory, the theory of quasi-variational inequalities.
If (X, d) is a metric space and ε is a positive real number, an *ε-fixed point of F* is a point $x^* \in X$ such that $d(x^*, F(x^*)) \leq \varepsilon$, where $d(x^*, F(x^*)) = \inf\{d(x^*, z) \ / \ z \in F(x^*)\}$. Such points will be called *approximate fixed points*. Now, following [6], when X will be a Banach space, we will present sufficient conditions on X and on F which will guarantee the existence of approximate fixed points. Weak and strong topologies will play here a role and both bounded and unbounded regions will be considered. Application of fixed point theorems to game theory will be showed in a next chapter.

3.1 Fixed Point Theorems

We recall same of well know fixed point theorems. The first one is due to Brouwer ([8]). This theorem deal the case of a function from a simplex to itself.

Theorem 3.1.1 ([8]) *Let f be a continuous function from $\Delta_{n-1} = \{x \in \mathbb{R}^n_+ \ / \ x_1 + ... + x_n = 1\}$ to itself. Then, f admits at least one fixed point.*

Using Brouwer's theorem applied to continuous selections, Kakutani ([22]) proved a fixed point theorem for set-valued functions.

Theorem 3.1.2 ([22]) *Let F be a set-valued function from Δ_{n-1} to itself. If $F(x)$ is non-empty, closed and convex for all $x \in \Delta_{n-1}$ and F is sequentially closed on Δ_{n-1}, then F admits at least one fixed point.*

Remark 3.1.1 The hypothesis of F sequentially closed in Theorem 3.1.2 is said "upper semicontinuous" in [22]. Precisely, if X, Y are topological spaces, a multifunction $F : X \longrightarrow 2^Y$ is said *upper semicontinuous at $x_o \in X$* if for all open set $B \subseteq Y$ such that $F(x_o) \subseteq B$, there exists an open neighbourhood A of x_o such that: $F(z) \subseteq B$ for all $z \in A$. Moreover, the multifunction F is said (topological) *closed at $x_o \in X$* if for all $y \notin F(x_o)$, there exist a neighbourhood I of x_o and a neighbourhood J of y such that: $F(z) \cap J = \emptyset$ for all $z \in I$. Closedness (in topological sense) and upper semicontinuity are equivalent when: Y is an Hausdorff compact space and F has compact values (see for example [4] and [1]). Now, if X and Y are topological space which satisfy the first axiom of countability, sequential closedness and (topological) closedness are equivalent. Hence, for multifunctions which satisfy the hypothesis of Theorem 3.1.2, sequential closedness is equivalent to upper semicontinuity.

Kakutani's theorem has been extended in infinite dimensional case by Glicksberg in [19].

Theorem 3.1.3 ([19]) *Let S be a non-empty, convex and compact subset of an Hausdorff and locally convex topological vector space. Let $F : S \longrightarrow 2^S$. If F is closed on S and it has non-empty and convex values, then there exists at least one fixed point of F.*

We note that for the existence of fixed points is necessary a compactness assumption on the domain in which the multifunction is defined. In the next section, we will deal with sets in which the compactness will be dropped.

3.2 Approximate Fixed Point Theorems on Bounded and Totally Bounded Sets

In this section, following [6], V will be a real Banach space and for $F : X \longrightarrow 2^X$ with $X \subseteq V$, the set $\{x \in V \ / \ d(x, F(x)) = \inf_{y \in F(x)} \| y - x \| \leq \varepsilon\}$ of the

ε-fixed points of the multifunction F on X will be denoted by $FIX^{\varepsilon}(F)$.

As noted in the previous paragraph, for existence of fixed points, the hypothesis of compactness of the set X is crucial. In Banach spaces, the compactness with respect the strong topology is characterized by completeness and totally boundedness (see for example [23]). We recall that a set $X \subseteq V$ is *complete* if every Cauchy-sequence converges, while it is *totally bounded* if for any $\eta > 0$ there exists a finite set $T \subseteq X$ such that $X \subseteq \cup_{x \in T} \overset{o}{B}(x, \eta)$, where $\overset{o}{B}(x, \eta) = \{y \in V \ / \ \| y - x \| < \eta\}$. So, if V is a Banach space, every subset closed and totally bounded is compact. Instead, if the space V is also reflexive, every subset closed, convex and bounded is also compact with respect to the weak topology on V (see for example [7]).

Using the weak topology in the following two theorems, sufficient conditions for existence of approximate fixed points on *bounded* sets are given.

Theorem 3.2.1 *Let V be a reflexive real Banach space and let X be a bounded and convex subset of V with non-empty interior. Assume that $F : X \longrightarrow 2^X$ is a weakly closed multifunction (that is a multifunction closed with respect to the weak topology) such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Suppose without loss of generality that $0 \in intX$. Let $\alpha = \sup\{\| x \| \ / \ x \in X\}$. Take $\varepsilon > 0$ and $0 < \delta < 1$ such that $\delta\alpha \leq \varepsilon$. Let Y be the weakly compact and convex subset of X defined by $Y = (1 - \delta)\overline{X}$, where $\overline{X}$ is the closure of X (note that the closure in strong and weak topologies are the same for convex set). Define the multifunction $G : Y \longrightarrow 2^Y$ by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$. Then G is a weakly closed multifunction with non empty, convex and weakly compact values. But, with respect to the weak topology, V is an Hausdorff locally convex topological vector space, so, in view of Theorem 3.1.3, G has at least one fixed point on Y. So there is an $x^* \in Y$ such that $x^* \in G(x^*) = (1 - \delta)F(x^*)$. Then there is a $z \in F(x^*)$ such that $x^* = (1 - \delta)z$, so $\| z - x^* \| = \delta \| z \| \leq \delta\alpha \leq \varepsilon$. Hence x^* is an ε-fixed point of F.

□

For proving Theorem 3.2.2, we need of the following lemma.

Lemma 3.2.1 ([12]) *Let V be a reflexive and separable real Banach space and let X be a non-empty bounded subset of V. Then, there exists a metric d_X on V which induces the weak topology on X.*

Theorem 3.2.2 *Let V be a reflexive and separable real Banach space and let X be a bounded and convex subset of V with non-empty interior. Assume that $F : X \longrightarrow 2^X$ is a weakly upper semicontinuous multifunction (that is a multifunction upper semicontinuous with respect to the weak topology) such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. As in the proof of Theorem 3.2.1, we assume that $0 \in intX$ and $\alpha = \sup\{\| x \| \ / \ x \in X\}$. Take $\varepsilon > 0$, $0 < \delta < 1$ such that $\delta\alpha \leq \frac{\varepsilon}{2}$ and $Y = (1-\delta)\overline{X}$. Define the multifunction $G : Y \longrightarrow 2^Y$ by $G(x) = (1-\delta)\overline{F(x)}$ for all $x \in Y$. G is weakly upper semicontinuous. In fact, we consider the metric d_X by Lemma 3.2.1. Let $x \in Y$ and assume that A is a weakly open neighbourhood of $G(x)$. For $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \ / \ d_X(y, G(x)) < \sigma\}$. Since $G(x)$ is weakly compact, we have that $d_X(Y \backslash A, G(x)) = \inf\{d_X(y,z) \ / \ y \in Y \backslash A, \ z \in G(x)\} > 0$, where $Y \backslash A = \{y \in Y \ / \ y \notin A\}$. So, if $0 < \sigma' < \sigma < d_X(Y \backslash A, G(x))$, we have $G(x) \subset A_{\sigma'} \subset \{y \in Y \ / \ d_X(y, G(x)) \leq \sigma'\} \subset A_\sigma \subset A$. In view of the weakly upper semicontinuity of the multifunction $(1-\delta)F$, there exists an open neighbourhood I of x such that $(1-\delta)F(z) \subset A_{\sigma'}$ for all $z \in I$. Therefore $G(z) = (1-\delta)\overline{F(z)} \subseteq \{y \in Y \ / \ d_w(y, G(x)) \leq \sigma'\} \subset A$ for all $z \in I$. So G is a weakly upper semicontinuous multifunction at x. As noted in Remark 3.1.1, G is also a weakly closed multifunction at x. Therefore, in view Theorem 3.1.3, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$. Hence, there exists $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$, so $\| z - x^* \| = \delta \| z \| \leq \delta\alpha \leq \frac{\varepsilon}{2}$. Moreover, there is $z' \in F(x^*)$ such that $\| z' - z \| < \frac{\varepsilon}{2}$. Hence $\| z' - x^* \| < \varepsilon$, that is $x^* \in FIX^{\varepsilon}(F)$.

□

When the hypothesis on the multifunctions are given with respect to the strong topology, existence of approximate fixed points is guaranteed on *totally bounded* sets, as showed in the next theorem.

Theorem 3.2.3 *Let V be a real Banach space and let X be a convex and totally bounded subset of V with non-empty interior. Assume that $F : X \longrightarrow 2^X$ is a closed or upper semicontinuous multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Assume without loss of generality that $0 \in intX$. Take $\varepsilon > 0$ and $\eta > 0$. Since X is totally bounded there exists $m \in \mathbb{N}$ and $x_1, ..., x_m \in X$ such that $X \subseteq \cup_{i=1}^m \overset{o}{B} (x_i, \eta)$. If $0 < \delta < 1$ the set $Y = (1-\delta)\overline{X}$ is a non-empty, convex and totally bounded subset of V. Since Y is also closed, it is compact. Let $h = \max\{\| x_r \| \ / \ r = 1, ..., m\}$.

- First, we assume that F is a closed multifunction and we take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \varepsilon$. Then the multifunction $G : Y \longrightarrow 2^Y$ defined by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$ is closed. This implies by Theorem 3.1.3 that G possesses a fixed point x^*. Then there is a point $z \in F(x^*)$ such that $x^* = (1 - \delta)z$. Since $X \subseteq \cup_{i=1}^m \overset{o}{B} (x_i, \eta)$, there exists an $r \in \{1, ..., m\}$ such that $z \in \overset{o}{B} (x_r, \eta)$. So $\| x^* - z \| = \delta \| \ z \| \leq \delta(\| z - x_r \| + \| x_r \|) < \delta(\eta + h) \leq \varepsilon$. Hence $x^* \in FIX^{\varepsilon}(F)$.
- Assume now that F is an upper semicontinuous multifunction. We take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \frac{\varepsilon}{2}$. Let $G : Y \longrightarrow 2^Y$ defined by $G(x) = (1 - \delta)\overline{F(x)}$ for all $x \in Y$. We claim that G is upper semicontinuous. Let $x \in Y$ and assume that A is an open neighbourhood of $G(x)$. For each $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \ / \ \inf_{z \in G(x)} \| z - y \| < \sigma\}$. As in the proof of Theorem 3.2.2, we obtain that G is an upper semicontinuous multifunction at x and is also a closed multifunction at x. In view of Theorem 3.1.3, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$ and $z \in \overline{F(x^*)}$ such that $x^* = (1 - \delta)z$. Since $X \subseteq \cup_{i=1}^m \overset{o}{B} (x_i, \eta)$, there exists $s \in \{1, ..., m\}$ such that $z \in \overset{o}{B} (x_s, \eta)$, so $\| z - x^* \| = \delta \| z \| \leq \delta(\| z - x_s \| + \| x_s \|) < \delta(\eta + h) \leq \frac{\varepsilon}{2}$. Moreover there exists a point $z' \in F(x^*)$ such that $\| \ z' - z \ \| < \frac{\varepsilon}{2}$, so $\| z' - x^* \| < \varepsilon$, that is $x^* \in FIX^{\varepsilon}(F)$.

□

3.3 Approximate Fixed Point Theorems on Unbounded Sets

Following [6], the next theorems deal the existence of approximate fixed points for multifunctions on convex regions which are not necessarily bounded. Useful here is the notion of a tame multifunction.

Definition 3.3.1 *Let U be a normed space and $X \subseteq U$. A multifunction $F : X \longrightarrow 2^X$ is called a* tame multifunction *if, for each $\varepsilon > 0$, there is an $R_\varepsilon > 0$ such that $B(0, R_\varepsilon) \cap X \neq \emptyset$ and for each $x \in B(0, R_\varepsilon) \cap X$ the set $F(x) \cap B(0, R_\varepsilon + \varepsilon)$ is non-empty, where $B(0, R_\varepsilon) = \{z \in U \mid \ \| z \| \leq R_\varepsilon\}$.*

Tame multifunctions are shown in the following examples.

Example 3.3.1 The map $F : [0, \infty[\longrightarrow 2^{[0,\infty[}$ defined by

$$F(x) = [x + (x + 1)^{-1}, \infty[\quad \text{for all} \quad x \in [0, \infty[$$

is a tame multifunction on the unbounded set $[0, \infty[$. Moreover, F has ε-fixed points for each $\varepsilon > 0$ (see the following Theorems 3.3.1 and 3.3.2).

Example 3.3.2 Let U be a normed space and $F : U \longrightarrow 2^U$ be a multifunction with $F(x) \neq \emptyset$ for each $x \in U$. Suppose that the image $F(U) = \{y \in U \ / \ y \in F(x)$ for some $x \in U\}$ of F is a bounded set. Then F is a tame multifunction (for each $\varepsilon > 0$, take $R_\varepsilon = 1 + \sup\{\| y \|,\ y \in F(U)\}$).

It follows from Example 3.3.2 that each $F : X \longrightarrow 2^X$, where X is a bounded subset of a normed space U and $F(x)$ is non-empty for all $x \in X$, is a tame multifunction.

Example 3.3.3 Let U be a normed linear spece. The translation $T : U \longrightarrow U$ given by $T(x) = x + a$, where $a \in U\backslash\{0\}$, is not *tame* and for small $\varepsilon > 0$, T has no ε-fixed points.

Example 3.3.3 shows that the *tame* property for multifunction in the next theorems is a non-superfluous condition for the existence of ε-fixed points.

Theorem 3.3.1 *Let X be a convex subset with non-empty interior of a reflexive real Banach space. Assume that $F : X \longrightarrow 2^X$ is a tame and weakly closed multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^\varepsilon(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Since F is tame, let $\varepsilon > 0$ and $R_\varepsilon > 0$ such that $F(x) \cap B(0, R_\varepsilon + \frac{\varepsilon}{2}) \neq \emptyset$ for each $x \in B(0, R_\varepsilon) \cap X$, and let $C = B(0, R_\varepsilon) \cap X$. C is a non-empty, bounded and convex set. Then $G : C \longrightarrow 2^C$, defined by

$$G(x) = R_\varepsilon (R_\varepsilon + \frac{\varepsilon}{2})^{-1} F(x) \cap B(0, R_\varepsilon + \frac{\varepsilon}{2}) \quad \text{for all} \quad x \in C$$

satisfies the conditions of Theorem 3.2.1. Hence there is $x^* \in FIX^{\frac{\varepsilon}{4}}(G)$ such that $d(x^*, G(x^*)) \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$ and there exists $x' \in G(x^*)$ such that $\| x' - x^* \| < \frac{\varepsilon}{2}$. Moreover there exists an element $z \in F(x^*)$ such that $z = R_\varepsilon^{-1}(R_\varepsilon + \frac{\varepsilon}{2})x'$. This implies that

$$\| z - x^* \| \leq \| R_\varepsilon^{-1}(R_\varepsilon + \frac{\varepsilon}{2})x' - x' \| + \| x' - x^* \| < \frac{\varepsilon}{2} R_\varepsilon^{-1} \| x' \| + \frac{\varepsilon}{2} \leq \varepsilon$$

So $x^* \in FIX^\varepsilon(F)$.

□

Theorem 3.3.2 *Let X be a convex subset with non-empty interior of a reflexive and separable real Banach space. Assume that $F : X \longrightarrow 2^X$ is a tame and weakly upper semicontinuous multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Using the same arguments of the proof of Theorem 3.3.1, we can show that the multifunction G defined on $B(0, R_\varepsilon) \cap X$ by

$$G(x) = R_\varepsilon (R_\varepsilon + \frac{\varepsilon}{2})^{-1} F(x) \cap B(0, R_\varepsilon + \frac{\varepsilon}{2})$$

satisfies the conditions of Theorem 3.2.2 and the conclusion follows as in Theorem 3.3.1.

□

Chapter 4

Stability and Well-Posedness in Optimization

Let X and Y be non-empty sequential spaces, f be an extended real valued function defined on $X \times Y$ and K be a set-valued function defined on X with values in Y. In this chapter, in the setting of sequential spaces, we are interested in *stability* and *well-posedness* of the following parametric maximum problem:

$$\mathcal{M}(x) : \begin{cases} \text{find an element } y \in Y \text{ such that :} \\ \ y \in K(x) \text{ and } f(x,y) \geq f(x,z) \ \forall z \in K(x) \end{cases}$$

Let $\mathcal{M} = \{\mathcal{M}(x) \ / \ x \in X\}$ and $M : x \in X \longrightarrow M(x) \in 2^Y$ be the set-valued function such that, for all $x \in X$, $M(x)$ is the set of solutions to $\mathcal{M}(x)$.

The family $\mathcal{M}$ will be said *stable* at a point x_o (see, for example [1]) if whenever one takes a perturbation $(x_n)_n$ of the value x_o, that is $x_n \longrightarrow x_o$ in X, and a converging sequence $(y_n)_n \subseteq Y$, where y_n is a solution to $\mathcal{M}(x_n)$ for all n, then the limit point of $(y_n)_n$ is a solution to $\mathcal{M}(x_o)$. So, the stability at x_o is nothing but the sequential closedness of the multifunction M at a point x_o.

The family $\mathcal{M}$ is said *parametrically well-posed at* x_o (see [51] for the case $K(x) = Y$ for all $x \in X$, and [28] for the case in which the constraint K is described by variational inequalities) if:

(i) there exists a unique solution to $\mathcal{M}(x_o)$;

(ii) $\sup_{y \in K(x)} f(x,y) < +\infty$ for all $x \in X$;

(iii) for any $x_n \longrightarrow x_o$ and any sequence $(y_n)_n \subseteq Y$, with $y_n \in K(x_n)$ for n sufficiently large, such that:

$$\sup_{z \in K(x_n)} f(x_n, z) - f(x_n, y_n) \longrightarrow 0,$$

then the sequence $(y_n)_n$ converges to the unique solution of $\mathcal{M}(x_o)$.

If $(x_n)_n$ is a sequence in X, a sequence $(y_n)_n$ in Y is said approximating sequence (with respect to $(x_n)_n$) if $y_n \in K(x_n)$ for n sufficiently large and $\sup_{z \in K(x_n)} f(x_n, z) - f(x_n, y_n) \longrightarrow 0$.
Following [37] and [39], aim of the chapter is to give new sufficient conditions, *explicit on the data* and weaker than continuity, for the stability and the well-posedness of $\mathcal{M}$.

4.1 Some Previous Results on Closedness of the Multifunction *M*

In the setting of topological spaces, one of the first results on closedness of the multifunction defined by solutions to parametric maximum problem is the Maximum Theorem due to Berge in [4], under recalled.

Theorem 4.1.1 ([4]) *Let X and Y be Hausdorff topological spaces. If f is continuous and K is lower semicontinuous and upper semicontinuous with compact values, then the multifunction M is upper semicontinuous with compact values (and so also closed).*

More recently, Maximum Theorem has been generalized by Tian and Zhou in [48], where the authors deal the case in which f is not necessarily continuous. In fact, they introduce properties on the set of data $\{X, Y, f, K\}$, called quasi-transfer upper continuity and transfer upper continuity with respect to K. Precisely, f is said *quasi-transfer upper continuous with respect to* K if for any $(x_o, y_o) \in X \times Y$ with $y_o \in K(x_o)$ and $f(x_o, z_o) > f(x_o, y_o)$ for some $z_o \in K(x_o)$, there exists a neighborhood N of (x_o, y_o) such that: for any $(x, y) \in N$ with $y \in K(x)$, there exists $z \in K(x)$ such that $f(x, z) > f(x, y)$. Moreover, f *is said transfer upper continuous in* y *on* K if for any $(x_o, y_o) \in X \times Y$ with $y_o \in K(x_o)$ and $f(x_o, z_o) > f(x_o, y_o)$ for some $z_o \in K(x_o)$, there exists a point $z' \in Y$ and a neighborhood J of y_o such that: for any $y \in J$ with $y \in K(x_o)$, $f(x_o, z') > f(x_o, y)$ and $z' \in K(x_o)$. So, the following theorem holds.

Theorem 4.1.2 ([48]) *Let X and Y be topological spaces and K closed with compact values. Then the multifunction M is closed with non-empty and compact values if and only if the function f is transfer upper continuous in y on K and quasi-transfer upper continuous with respect to K. If, in addition, K is upper semicontinuous, then M is upper semicontinuous.*

In the setting of gamma limits ([16], [12]), an exhaustive study on the closedness of the multifunction M has been given in the book of Dal Maso ([12]), where one can find numerous applications.

When the spaces are sequential spaces, sufficient conditions for sequential closedness of the set-valued function M have been introduced by Zolezzi in [50], where the set of parameters is $\mathbb{N} \cup \{\infty\}$. If the set of parameters is a general sequential space X, the following result holds.

Theorem 4.1.3 ([50]) *Let X and Y sequential spaces and $x_o \in X$. Assume that the following statements are satisfy:*

(C1) $f + \delta_K$ is sequentially upper semicontinuous in (x_o, y);

(C2) for all $(x_n)_n$ converging to x_o in X and for all $y \in Y$, there exists a sequence $(y_n)_n \subseteq Y$ such that:

$$(f + \delta_K)(x_o, y) \leq \liminf_{n\to\infty}(f + \delta_K)(x_n, y_n);$$

where δ_K is the function defined by $\delta_K(x, y) = 0$ if $y \in K(x)$ and $\delta_K(x, y) = -\infty$ if $y \notin K(x)$.
Then, the set-valued function M is sequentially closed at x_o.

4.2 Sequential Closedness of Solutions to Parametric Maximum Problems

In this section, following [37], we present new sufficient conditions for sequential closedness of the set-valued function M in the setting of sequential upper and lower pseudocontinuous functions. Let $v(x) = \sup_{z\in K(x)} f(x, z)$ for any $x \in X$.

Theorem 4.2.1 *Let $x_o \in X$. Assume that:*

(i) the set-valued function K is sequentially closed at x_o;

(ii) the function f is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$ such that $y \in K(x_o)$;

(iii) the function v is sequentially lower semicontinuous at x_o.

Then, the set-valued function M is sequentially closed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $y \in \text{Limsup} M(x_n)$ such that $y \notin M(x_o)$. Then there exists a sequence $(y^k)_k$ converging to y, with $y^k \in M(x_{n_k})$ for all k. Since $y^k \in K(x_{n_k})$ and K is sequentially closed, we have that $y \in K(x_o)$. So, for some $z \in K(x_o)$: $f(x_o, y) < f(x_o, z)$. Since f is seq. upper pseudocontinuous, we have:

$$\limsup_{k\to\infty} f(x_{n_k}, y^k) < f(x_o, z) \leq v(x_o).$$

By (iii), we have

$$\limsup_{k\to\infty} f(x_{n_k}, y^k) < \liminf_{k\to\infty} v(x_{n_k}).$$

So, there exists k_o such that $f(x_{n_{k_o}}, y^{k_o}) < v(x_{n_{k_o}})$ and we get a contradiction. □

Now, we present explicit assumptions on the data in order to obtain the sequential closedness of M.

Corollary 4.2.1 *Let $x_o \in X$. Assume that:*

(i) the set-valued function K is sequentially closed and sequentially lower semicontinuous at x_o;

(ii) the function f is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y \in K(x)$.

Then, the set-valued function M is sequentially closed at x_o.

Proof. It is sufficient to observe that sequential lower semicontinuity of K and sequential lower semicontinuity of f imply condition (iii) of the Theorem 4.2.1, as one can see by Proposition 3.2.1 in [27]. □

Theorem 4.2.2 *Let $x_o \in X$. Assume that:*

(i) the set-valued function K is sequentially closed and sequentially lower semicontinuous at x_o;

(ii) the function f is sequentially lower pseudocontinuous and sequentially upper semicontinuous at (x_o, y), for any $y \in Y$ such that $y \in K(x)$.

Then, the set-valued function M is sequentially closed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $y \in \text{Limsup} M(x_n)$ such that $y \notin M(x_o)$. As in the proof of Theorem 4.2.1, there exists $y^k \longrightarrow y$ such that $y^k \in M(x_{n_k})$ for all k and $z \in K(x_o)$ such that $f(x_o, y) < f(x_o, z)$. Since K is sequentially lower semicontinuous at x_o, there exists a sequence $z^k \longrightarrow z$ with $z^k \in K(x_{n_k})$ for all k. Then, in light of (ii), we have:

$$\limsup_{k\to\infty} f(x_{n_k}, y^k) \leq f(x_o, y) < \liminf_{k\to\infty} f(x_{n_k}, z^k)$$

and we get a contradiction.

□

We note that the class of sequentially lower pseudocontinuous functions does not coincide with the class of functions described by condition (C2). In fact, in Example 4.2.1, f is a sequentially lower pseudocontinuous function which does not verify condition (C2), and in Example 4.2.2, f is a function which satisfies (C2) but it is not sequentially lower pseudocontinuous.

Example 4.2.1 Let $f : [1, 2] \times [0, 1] \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1 - x & \text{if } (x,y) \in]1,2] \times [0,1] \\ y - 1 & \text{if } (x,y) \in \{1\} \times [0,1[\\ 1 & \text{if } (x,y) = (1,1) \end{cases}$$

and $K(x) = [0, 1]$ for all $x \in [0, 1]$. The function f is sequentially lower pseudocontinuous at $(1, y)$ for all $y \in [0, 1]$, but it does not satisfy the condition (C2) at $x_o = 1$.

Example 4.2.2 Let $f : [0, 1]^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x(1-y) & \text{if } (x,y) \in]0,1] \times [0,1[\\ 1 & \text{if } (x,y) \in [0,1] \times \{1\} \\ 0 & \text{if } (x,y) \in \{0\} \times [0,1[\end{cases}$$

and $K(x) = [0, 1]$ for all $x \in [0, 1]$. The function f satisfies (C2) at $x_o = 0$ but it is not sequentially lower pseudocontinuous at $(0, 1)$.

Finally, we observe that if X and Y are topological spaces, the sequential hypothesis, explicit on the data, introduced in the present paragraph are not always included in hypothesis of Theorem 4.1.2, as it is shown by the function f defined on $]0,1[^2$ as follows:

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in]0,\frac{1}{2}[^2 \\ -xy & \text{if } (x,y) \notin]0,\frac{1}{2}[^2 \end{cases}$$

where the topology on $]0,1[^2$ is the product with itself of the topology on]0,1[considered in Example 2.2.3.

4.3 Previous Result on Well-Posedness

We start the section recalling the definition of well-posedness due to Tykhonov ([49]). Let Y be a sequential space and f be an extended real valued function defined on Y. The problem [maximize $f(\cdot)$ over Y], that is:

$$\text{find } y_o \in Y \text{ such that } f(y_o) = \max_{y \in Y} f(y),$$

is said *Tykhonov well-posed* if:

(i) there exists a unique maximizer y_o ;

(ii) every maximizer sequence $(y_n)_n$ (that is $f(y_n) \longrightarrow \max_{y \in Y} f(y)$) converges to y_o.

Let us note that in order to obtain Tykhonov well-posedness of a maximum problem, the only uniqueness of the maximizer is not sufficient. In fact, we can see at the following very simple example.

Example 4.3.1 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(y) = -y^2 e^{-y}$ for all y. The point $y_o = 0$ is the only one maximizer of f over $\mathbb{R}$. The sequence $(n)_n$, which does not converge to $y_o = 0$, is such that $f(n) \longrightarrow \max f = f(0) = 0$. So, the maximum problem of f over $\mathbb{R}$ is not Tykhonov well-posed.

Moreover, the problem [maximize $f(\cdot)$ over Y] is said *generalized Tykhonov well-posedness* ([49]) if:

(i) the set of maximizer points of f over Y is non-empty;

(ii) every maximizer sequence $(y_n)_n$ has a subsequence which converges to a maximizer point.

Tykhonov well-posedness is a concept very helpful, for example, in cases in which one wants to solve optimization problems using algorithms. In these situations, algorithms develop maximizer sequences and it is fundamental to know if every maximizing sequence converges to the unique solutions of the optimization problem. Many applications and examples of Tykhonov well-posed problems can be find in the book of Dontchev and Zolezzi [17].

The concept of Tykhonov well-posedness deals with optimization problems in which there are not perturbations of the objective function. When the objective function f is depending also on a parameter x, an other concept of well-posedness has been introduced by Zolezzi in [51]: let X, Y be sequential spaces and f be an extended real valued function defined on $X \times Y$. We consider the family of maximum problems:

$$\{ \text{ [maximize } f(x, \cdot) \text{ over } Y] \ / \ x \in X\}$$

Let $x_o \in X$ and $x_n \longrightarrow x_o$. So, ([maximize $f(x_n, \cdot)$ over Y] $)_n$ is a sequence of maximum problems which correspond to the perturbation $(x_n)_n$ of the problem [maximize $f(x_o, \cdot)$ over Y].
Let $v(x) = \sup_{y \in Y} f(x, y)$, the problem [maximize $f(x_o, \cdot)$ over Y] is said *parametric well-posed* (or *well-posed by pertubations) at* x_o ([51]) if the following statements hold:

(i) there is an unique maximizer $y(x_o)$ to [maximize $f(x_o, \cdot)$ over Y];

(ii) $v(x) < +\infty$ for all $x \in X$;

(iii) if $(x_n)_n \longrightarrow x_o$ and if $(y_n)_n \subseteq Y$ is such that $v(x_n) - f(x_n, y_n) \longrightarrow 0$, then: $y_n \longrightarrow y(x_o)$.

The meaning of parametric well-posedness is the following: suppose that one wants to solve the problem [maximize $f(x_o, \cdot)$ over Y] by a method performed on a sequence of steps, each of them develops a deformation of the objective function with respect the original one. So, if the problem is parametric well-posedness, steps produce a sequence of good approximations of the solution to [maximize $f(x_o, \cdot)$ over Y], when the perturbation are sufficiently close to x_o.
We emphasize that parametric well-posedness depends on the method used to produce the approximations: formally, parametric well-posedness is depending on the function $f(\cdot, \cdot)$, as shown by the following example.

Example 4.3.2 We want to determine sequences converging to the maximum point of the function $-y^2$ (defined over $\mathbb{R}$), using the methods f_1 and f_2 defined on $\mathbb{R}^2$ by:

$$f_1(x,y) = -y^2 \log(e + x^2) \quad \text{and} \quad f_2(x,y) = \begin{cases} -y^2 & \text{if} \quad x = 0 \\ - \mid x \mid y^2 & \text{if} \quad x \neq 0 \end{cases}$$

Now, $v_1(x) = \sup_y f_1(x,y) = 0$, $v_2(x) = \sup_y f_2(x,y) = 0$ for all $x \in \mathbb{R}$ and $f_1(0,y) = f_2(0,y) = -y^2$. Let $x_n \longrightarrow 0$ and $(y_n)_n \subseteq Y$.
If $v_1(x_n) - f_1(x_n, y_n) = y_n^2 \log(e + x_n) \longrightarrow 0$, then $y_n \longrightarrow 0$, so the problem [maximize $f_1(0,\cdot)$ over $\mathbb{R}$] is parametric well-posed. On the contrary for the problem [maximize $f_2(0,\cdot)$ over $\mathbb{R}$]. In fact, if $x_n \longrightarrow 0^+$, we have that $v_2(x_n) - f_2(x_n, y_n) = x_n y_n^2 \longrightarrow 0$ for any bounded sequence $(y_n)_n$. So [maximize $f_2(0,\cdot)$ over $\mathbb{R}$] is not parametric well-posed.

About sufficient conditions for Tykhonov and parametric well-posedness, there are the following theorems.

Theorem 4.3.1 ([49]) *Let Y be a sequential compact space and f be an extended real valued function defined on Y. If f is sequentially upper semicontinuous, then* [maximize $f(\cdot)$ over Y] *is Tykhonov well-posed.*

Theorem 4.3.2 ([33]) *Let X and Y be sequential spaces, f be an extended real valued function defined on $X \times Y$ and $x_o \in X$. If: Y is sequentially compact, f is sequentially upper semicontinuous at (x_o, y) for all $y \in Y$, v is sequentially lower semicontinuous at x_o, then the family:*

$$\{ \text{ [maximize } f(x,\cdot) \text{ over } Y] \ / \ x \in X\}$$

is parametrically well-posed at x_o.

With explicit assumptions on the data, there is the following result.

Theorem 4.3.3 ([33]) *Let X and Y be sequential spaces, f be an extended real valued function defined on $X \times Y$ and $x_o \in X$. If Y is sequentially compact and f is sequentially continuous at (x_o, y) for all $y \in Y$, then the family:*

$$\{ \text{ [maximize } f(x,\cdot) \text{ over } Y] \ / \ x \in X\}$$

is parametrically well-posed at x_o.

4.4 Parametric Well-Posedness of Parametric Maximum Problems

Following [39], this section is devoted to parametric well-posedness of constrained (or not) maximum problems.
In addition to the definition of parametric well-posedness already given in the introduction to the chapter, $\mathcal{M}$ is said *generalized parametrically well-posed at* $x_o \in X$ if: the set of maximum points of $f(x_o, \cdot)$ over $K(x_o)$ is non-empty; $v(x) = \sup_{y \in K(x)} f(x, y) < +\infty$ for all $x \in X$; for any $x_n \longrightarrow x_o$ and any approximating sequence $(y_n)_n$ (with respect to $(x_n)_n$), there exists a subsequence of $(y_n)_n$ which converges to a maximum point of $f(x_o, \cdot)$ over $K(x_o)$.

Using seq. upper pseudocontinuity, we can relax the hypothesis on the objective function of the results quoted in the previous section. In particular, we can obtain a sufficient condition, explicit on the data, weaker than continuity. In the next results we deal with parametric and constrained maximum problems.

Theorem 4.4.1 *Let* $x_o \in X$. *Assume that there is a unique maximum point* y_o *for* $f(x_o, \cdot)$ *over* $K(x_o)$. *If* Y *is sequentially compact and:*

(i) the set-valued function K *is sequentially closed at* x_o,

(ii) the function f *is sequentially upper pseudocontinuous at* (x_o, y), *for any* $y \in Y$ *such that* $y \in K(x_o)$,

(iii) the function v *is sequentially lower semicontinuous at* x_o,

then, the family $\mathcal{M}$ *is parametrically well-posed at* x_o.

Proof. Let $x_n \longrightarrow x_o$ and $(y_n)_n$ be an approximating sequence (with respect to $(x_n)_n$) such that $y_n \not\longrightarrow y_o$. In light of axiom (S_3) in Paragraph 2.1 and by seq. compactness of Y, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ converging to a point $y \in Y \backslash \{y_o\}$ and $y_{n_k} \in K(x_{n_k})$ for k sufficiently large. Since K is seq. closed at x_o, we have that $y \in K(x_o)$. Now, y_o is the unique maximum point of f on $K(x_o)$ and $y \neq y_o$. So, there exists $z \in K(x_o)$ such that $f(x_o, y) < f(x_o, z)$. Since f is seq. upper pseudocontinuous, we have:

$$\limsup_{k \to \infty} f(x_{n_k}, y_{n_k}) < f(x_o, z) \leq v(x_o)$$

and by seq. lower semicontinuity of v, we obtain:

$$\limsup_{k \to \infty} f(x_{n_k}, y_{n_k}) < \liminf_{k \to \infty} v(x_{n_k}).$$

Let α be a real number such that:

$$\limsup_{k\to\infty} f(x_{n_k}, y_{n_k}) < \alpha < \liminf_{k\to\infty} v(x_{n_k}), \tag{4.1}$$

So, we obtain that there exists $k_o \in \mathbb{N}$ such that:

$$\alpha - f(x_{n_k}, y_{n_k}) < v(x_{n_k}) - f(x_{n_k}, y_{n_k}) \tag{4.2}$$

for all $k \geq k_o$. So, by (4.1) and (4.2), we obtain:

$$0 < \alpha - \limsup_{k\to\infty} f(x_{n_k}, y_{n_k}) \leq \lim_{k\to\infty} [v(x_{n_k}) - f(x_{n_k}, y_{n_k})] = 0$$

and we get a contradiction.

□

Under explicit assumptions on the data, we have the following result, which easy follows by Theorem 4.4.1.

Corollary 4.4.1 *Let $x_o \in X$. Assume that there is a unique maximum point y_o for $f(x_o, \cdot)$ over $K(x_o)$. If Y is sequentially compact and:*

(i) the set-valued function K is sequentially closed and sequentially lower semicontinuous at x_o,

(ii) the function f is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y \in K(x_o)$,

then, the family $\mathcal{M}$ is parametrically well-posed at x_o.

We remark that the hypothesis of Theorem 4.4.1 and Corollary 4.4.1 are sufficient conditions also for parametric well-posedness in generalized sense. If X, Y are metric space, the following theorem holds, where:

$$\varepsilon - M(x) = \{y \in K(x) \ / \ f(x, y) > v(x) - \varepsilon\}$$

Theorem 4.4.2 *Let $x_o \in X$. Assume that X, Y are metric spaces, with Y also complete and:*

$$\lim_{\varepsilon\downarrow 0} diam\{\cup_{x\in B(x_o,\varepsilon)}[\varepsilon - M(x)]\} = 0 \tag{4.3}$$

If:

(i) the set-valued function K is sequentially closed at x_o,

(ii) the function f is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$,

(iii) the function v is sequentially lower semicontinuous at x_o,

then, the family $\mathcal{M}$ is parametrically well-posed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $(y_n)_n$ be an approximating sequence (with respect to $(x_n)_n$). Fixed $\varepsilon > 0$, we have $y_n \in \varepsilon - M(x_n)$ for n sufficiently large. So, by (4.3), $(y_n)_n$ is a Cauchy sequence. Therefore $(y_n)_n$ converges to an element y_o and, by seq. closedness of K, $y_o \in K(x_o)$. If y_o is not a solution of $\mathcal{M}(x_o)$, proceeding as in the proof of Theorem 4.4.1, we obtain a contradiction. Hence $y_o \in M(x_o)$. Moreover, in light of (4.3), we have that $M(x_o) = \{y_o\}$ and thesis follow.

□

For explicit assumption on the data, we have the following result.

Corollary 4.4.2 *Let $x_o \in X$. Assume that X, Y are metric spaces, with Y also complete and:*

$$\lim_{\varepsilon \downarrow 0} diam\{\cup_{x \in B(x_o,\varepsilon)}[\varepsilon - M(x)]\} = 0 \tag{4.4}$$

If:

(i) the set-valued function K is sequentially closed and sequentially lower semicontinuous at x_o,

(ii) the function f is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$,

then, the family $\mathcal{M}$ is parametrically well-posed at x_o.

Finally, in the case of Tykhonow well-posedness of an unparametrized and unconstrained maximum problem, it is possible to weak the seq. upper pseudocontinuity by seq. upper quasicontinuity, as shown in the following theorem.

Theorem 4.4.3 *Assume that Y is sequentially compact and the problem* [maximize $f(\cdot)$ over Y] *has a unique solution y_o. If f is sequentially upper quasicontinuous on Y, then* [maximize $f(\cdot)$ over Y] *is Tykhonov well-posedness.*

Proof. Suppose that [maximize $f(\cdot)$ over Y] is not Tykhonov well-posed: let $(y_n)_n$ be a sequence such that $f(y_n) \longrightarrow f(y_o) = \max_Y f$ and $y_n \not\longrightarrow y_o$. Then, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ which converges to a point $\bar{y} \neq y_o$. So $f(\bar{y}) < f(y_o) = \lim_{k\to\infty} f(y_{n_k})$ and consequently we have that $f(y_{n_k}) \in]f(\bar{y}), f(y_o)] \cap f(Y)$ for $k \geq k_o$. Now, if $f(y_{n_k}) = f(y_o)$ for all $k \geq k_o$, then $y_{n_k} = y_o$ for all $k \geq k_o$ because y_o is the unique maximum point. So $\bar{y} = y_o$, that is a contradiction. Hence there exists $f(y') \in]f(\bar{y}), f(y_o)[\cap f(Y)$. Since f is seq. upper quasicontinuious at $\bar{y}$, we have $f(y_o) = \limsup_{k\to\infty} f(y_{n_k}) \leq f(y')$ and we get a contradiction.

□

We note that transfer upper continuity ([48]) cannot be substituted for upper quasicontinuity in Theorem 4.4.3. In fact, in the following example is given a function f transfer upper continuous over a compact set Y such that the problem [maximize $f(\cdot)$ over Y] is not Tykhonov well-posed.

Example 4.4.1 Let $Y = [0, 2]$ and $f : Y \longrightarrow \mathbb{R}$ such that:

$$f(y) = \begin{cases} -(y-1)^2 & \text{if } y \in [0,1[\\ y-2 & \text{if } y \in [1,2] \end{cases}$$

f is not seq. upper quasicontinuous at $y_o = 1$ but it is transfer upper continuous. Now, if $y_n \longrightarrow 1^-$, we have $f(y_n) \longrightarrow 0 = \max f$, but $(y_n)_n$ does not converge to the unique maximum point $y_o = 2$. So, the maximum problem corresponding to f is not Tykhonov well-posed.

We conclude the chapter observing that using the class of pseudocontinuous functions, the continuity of payoffs can be relaxed in the same way for both problems of stability and well-posedness of parametric and constrained maximum problems.

Chapter 5

Stability and Well-Posedness in Non-Cooperative Game Theory

In this chapter, following [37] and [39], we present sufficient conditions in order to obtain stability and well-posedness of social Nash equilibria problems, when the *data* are depending also on an exogenous parameter. In particular, we are interested in social equilibria problems in which the payoff functions have some discontinuities. Upper pseudocontinuity and lower pseudocontinuity will play a central role to obtain explicit assumptions on the data, weaker than continuity of payoffs, for both problems of stability and well-posedness.

5.1 Unstable Discontinuous Games

As we have already noted in Paragraph 1.1, existence of Nash equilibria has been enough studied for games with discontinuous payoffs. We recall again that a game is a list of data $\{Y_i, f_i\}_{i=1}^m$, in which $\{1, ..., m\}$ is the set of player (individuals), Y_i is the set of all strategies for i and f_i is its payoff (utility). A profile of strategies $y^* = (y_1^*, ..., y_m^*)$ is a *Nash equilibrium* ([40], [41]) if $f_i(y_1^*, ..., y_i^*, ..., y_m^*) \geq f_i(y_1^*, ..., y_i, ..., y_m^*)$ for all $y_i \in Y_i$ and for all $i \in \{1, ..., m\}$.
Now, the aim of this chapter is to study the stability of equilibria in presence of discontinuities for games and pseudo-games (also called abstract economies). In pseudo-games, we consider the concept of "feasible strategies" in a game in normal form: any individual i has a set-valued function K_i defined on $Y_{-i} = \prod_{j \neq i} Y_j$ with values in Y_i. The values of K_i are the "feasible strategies" that the individual i will be able to use after that all of others have chosen a profile of their strategies in Y_{-i}. Hence, in this case,

the others individuals can influence i also indirectly, restricting his feasible strategies to $K_i(y_{-i})$. The list of data $\{Y_i, K_i, f_i\}_i$ is called *abstract economy* (see [15]) or also *pseudo-game* (see [21]). We emphasize that no one of the individuals can play individually this "game", since he must to know the choices of others in order to know his feasible strategies. So, a profile of strategies y^* is said to be a *social Nash equilibrium* (see [15]) if for all i: $y_i^* \in K_i(y_{-i}^*)$ and $f_i(y_i^*, y_{-i}^*) \geq f_i(y_i, y_{-i}^*)$ for all $y_i \in K_i(y_{-i}^*)$.

In section 1.1, several properties, able to recognize games with discontinuous payoffs endowed of Nash equilibria, have been recalled. In particular, the better replay security ([44]) has been presented, which is not the more general. Even if the better replay security is a property sufficient for existence of equilibria in a game, it does not guarantee the stability when there is a perturbation. For example, we can look at the Bertrand competition in an duopoly considered in section 1.1. Suppose, now, that there are several states of the world, represented on the interval $[0, +\infty[$. Let the demand function be constant on the states of the world, and assume that the two firms have equal constant marginal costs $c(\cdot)$, which are depending on the states of the world. Assume that the value $x = 1$ is a threshold level: $c(x) = c$ if $x \in [0, 1]$, $c(x) = c + 1$ if $x \in]1, +\infty[$. The payoffs of the two firms i, j (with $i \neq j$) are:

$$\pi_i(x, p_i, p_j) = \begin{cases} (p_i - c(x))d(p_i) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c(x))d(p) & \text{if} \quad p_i = p_j = p \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

For any fixed state of the world x, there is a only one Nash equilibrium pair: $(p_i^*(x), p_j^*(x)) = (c(x), c(x))$. The above perturbed duopoly is better replay security, but the multifunction N, whose values are the set of Nash equilibria for any state x, it is not closed at $x = 1$. In fact, the set of equilibria at $x = 1$ is $N(1) = \{(c, c)\}$, and the set of equilibria at $x > 1$ is $N(x) = \{(c+1, c+1)\}$. Hence, if $x_n \longrightarrow 1^+$, we have $LimsupN(x_n) = \{(c + 1, c + 1)\} \not\subseteq N(1)$. So the set-valued function N is not seq. closed at $x = 1$.
Therefore, following [37], our purpose will be to give sufficient conditions, *weaker than continuity*, which will be able to guarantee the stability of solutions in perturbed pseudo-games.

5.2 Previous Results on Closedness of Nash Equilibria

In this section we recall some previous result on closedness of equilibria of parametric games and pseudo-games, when no hypothesis of convexity are necessary.

A first result on closedness of Nash equilibria is due to Cavazzuti and Pacchiarotti in [11]. In their work, the authors considered the case in which the set of perturbations on each payoff f_i is a sequence of functions $(f_{i,n})_n$. They gave the following conditions, weaker than continuity.

Theorem 5.2.1 ([11]) *Assume that the following statements are satisfied for all $i \in \{1, ..., m\}$:*

(C1) for any (y_i, y_{-i}) and any $(y_i^n, y_{-i}^n) \longrightarrow (y_i, y_{-i})$ one has:

$$\limsup_{n\to\infty} f_{i,n}(y_i^n, y_{-i}^n) \leq f_i(y_i, y_{-i});$$

(C2) for all (y_i, y_{-i}) and all $y_{-i}^n \longrightarrow y_{-i}$, there exists a sequence $(\bar{y}_i^n)_n$ converging to y_i such that $f_i(y_i, y_{-i}) \leq \liminf_{n\to\infty} f_{i,n}(\bar{y}_i^n, y_{-i}^n)$.

Then, denoting with N_n the set of Nash equilibria of $\{Y_i, f_{i,n}\}_i$ for any $n \in \mathbb{N}$, one has: $LimsupN_n \subseteq N$.

In the case in which Y_i is a subset of a reflexive and separable real Banach spaces E_i, Lignola and Morgan (see [30]) gave a result on closedness of social Nash equilibria, where strong and weak topologies are involved and an Hausdorff topological vector space X of parameters is considered. Let τ be the topology on X. Moreover, let s_i and w_i be the strong and the weak topology on E_i respectively.

Theorem 5.2.2 ([30]) *Assume that the following statements are satisfy:*

(i) f_1+f_2 is sequentially upper semicontinuous on $(X\times Y_1\times Y_2, \tau\times w_1\times w_2)$;

(ii) f_1 is sequentially lower semicontinuous on $(X\times Y_1\times Y_2, \tau\times s_1\times w_2)$ and f_2 is sequentially lower semicontinuous on $(X \times Y_1 \times Y_2, \tau \times w_1 \times s_2)$;

(iii) K_1 is sequentially lower semicontinuous set-valued function from $(X \times Y_2, \tau \times w_2)$ to (Y_1, s_1) and K_2 is sequentially lower semicontinuous set-valued function from $(X \times Y_1, \tau \times w_1)$ to (Y_2, s_2);

(iv) for $i \in \{1,2\}$, K_i is sequentially closed from $(X \times Y_j, \tau \times w_j)$ to (Y_i, w_i), where $i \neq j$.

Then, the multifunction N, whose values are social Nash equilibria, is sequentially closed from (X, τ) to $(Y_1 \times Y_2, w_1 \times w_2)$.

5.3 Sequential Closedness of Solutions to Parametric Social Nash Equilibrium Problems

Let X, Y_1,...,Y_m be sequential spaces and, for $i \in \{1, ..., m\}$, let f_i be an extended real valued function defined on $X \times Y$ and K_i be a set-valued function from $X \times Y_{-i}$ to Y_i, where, as in the previous paragraph, $Y = \prod_{j=1}^{m} Y_j$ and $Y_{-i} = \prod_{j \neq i} Y_j$. For any fixed $x \in X$, we consider the following parametric problem:

$$\mathcal{N}(x) : \begin{cases} \text{find an element } y \in Y \text{ such that :} \\ \forall\, i \in \{1, ..., m\},\ y_i \in K_i(x, y_{-i}) \text{ and} \\ f_i(x, (y_i, y_{-i})) \geq f_i(x, (z_i, y_{-i}))\ \forall z_i \in K_i(x, y_{-i}) \end{cases}$$

The set of solutions to $\mathcal{N}(x)$, denoted by $N(x)$, is the set of all social Nash equilibrium of the parametric abstract economy $\Gamma(x) = \{Y_i, K_i(x, \cdot), f_i(x, \cdot)\}_i$. Following [37], aim of this section is to give sufficient conditions on the payoff functions f_i, weaker than seq. continuity, in order to obtain the seq. closedness of the following *social Nash equilibria set-valued function*:

$$N : x \in X \longrightarrow N(x) \in 2^Y$$

In the following, for all i, we will denote by v_i the marginal function of the individual i, defined by $v_i(x, y_{-i}) = \sup_{z_i \in K_i(x, y_{-i})} f_i(x, (z_i, y_{-i}))$ for any $(x, y_{-i}) \in X \times Y_{-i}$.

Theorem 5.3.1 *Let $x_o \in X$. Assume that:*

(i) for all i, the set-valued function K_i is sequentially closed at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$;

(ii) for all i, the function f_i is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$;

(iii) for all i, the function v_i is sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y$.

Then, the set-valued function N is sequentially closed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $y \in \text{Limsup}N(x_n)$ such that $y \notin N(x_o)$. Then there exists a sequence $(y^k)_k$ converging to y, with $y^k \in N(x_{n_k})$ for all k. Since $y_j^k \in K_j(x_{n_k}, y_{-j}^k)$ and K_j is sequentially closed, we have that $y_j \in K_j(x_o, y_{-j})$ for all j. Therefore, there exist $i \in \{1, ..., m\}$ and $z_i \in K_i(x_o, y_{-i})$ such that

$$f(x_o, (y_i, y_{-i})) < f(x_o, (z_i, y_{-i})).$$

By (ii) we obtain

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^k, y_{-i}^k)) < f(x_o, (z_i, y_{-i})) \leq v_i(x_o, y_{-i}).$$

Then, in light of (iii), we have

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^k, y_{-i}^k)) < \liminf_{k\to\infty} v_i(x_{n_k}, y_{-i}^k).$$

So, there exists k_o such that $f_i(x_{n_{k_o}}, (y_i^{k_o}, y_{-i}^{k_o})) < v_i(x_{n_{k_o}}, y_{-i}^{k_o})$ and we get a contradiction.

□

Under explicit assumptions on the data, we have the following two results.

Corollary 5.3.1 *Let $x_o \in X$. Assume that:*

(i) for all i, the set-valued function K_i is sequentially closed and sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$;

(ii) for all i, the function f_i is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x, y_{-i})$.

Then, the set-valued function N is sequentially closed at x_o.

Proof. It is sufficient to observe that, in light of Proposition 3.2.1 in [27], sequential lower semicontinuity of K_i and sequential lower semicontinuity of f_i imply assumption (iii) of Theorem 5.3.1.

□

The next theorem supplies alternative conditions to hypothesis in Theorem 5.3.1.

Theorem 5.3.2 *Let $x_o \in X$. Assume that:*

(i) for all i, the set-valued function K_i is sequentially closed and sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$;

(ii) for all i, the function f_i is sequentially lower pseudocontinuous and sequentially upper semicontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$.

Then, the set-valued function N is sequentially closed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $y \in \text{Limsup} N(x_n)$ such that $y \notin N(x_o)$. As in the proof of Theorem 5.3.1, there exists $y^k \longrightarrow y$ such that $y^k \in N(x_{n_k})$ for all k and $z_i \in K_i(x_o, y_{-i})$ such that $f_i(x_o, (y_i, y_{-i})) < f_i(x_o, (z_i, y_{-i}))$ for some $i \in \{1, ..., m\}$. Since K_i is sequentially lower semicontinuous at (x_o, y_{-i}), there exists a sequence $z_i^k \longrightarrow z_i$ with $z_i^k \in K_i(x_{n_k}, y_{-i}^k)$ for all k. Then, in light of (ii), we have:

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^k, y_{-i}^k)) \le f_i(x_o, (y_i, y_{-i})) < \liminf_{k\to\infty} f_i(x_{n_k}, (z_i^k, y_{-i}^k))$$

and we get a contradiction.

□

About explicit assumptions on the data in the above results, we point out that the sequential continuity of payoffs has been relaxed through two way: in the first one (Corollary 5.3.1), the continuity has been replaced by *seq. upper pseudocontinuity* and seq. lower semicontinuity; in the second one (Theorem 5.3.2), the continuity has been replaced by seq. upper semicontinuity and *seq. lower pseudocontinuity.*

Sufficient conditions for seq. closedness of the set-valued N introduced in [11] and recalled in the previous section (Theorem 5.2.1), become the following when the set of parameter is a general sequential space:

(C1)' f_i is sequentially upper semicontinuous at (x_o, y) for all $y \in Y$;

(C2)' $\forall\, y \in Y$, $\forall\, x_n \longrightarrow x_o$ and $\forall\, y_{-i}^n \longrightarrow y_{-i}$, there exists a sequence $(\bar{y}_i^n)_n$ converging to y_i such that $f_i(x, y_i, y_{-i}) \le \liminf_{n\to\infty} f_i(x_n, \bar{y}_i^n, y_{-i}^n)$.

Sequential lower pseudocontinuity of functions (Theorem 5.3.2) is a condition independent of condition (C2)'. In fact sequential lower pseudocontinuity does not imply condition (C2)' and condition (C2)' does not imply sequential lower pseudocontinuity, as shown by the following examples.

Example 5.3.1 Let $K_1(x, y_2) = [0, 1]$ for all $(x, y_2) \in [0, 2] \times [0, 1]$ and $K_2(x, y_1) = [0, 1]$ for all $(x, y_1) \in [0, 2] \times [0, 1]$. The functions $f_1, f_2 : [1, 2] \times [0, 1]^2 \longrightarrow \mathbb{R}$, defined by:

$$f_i(x, y_i, y_j) = \begin{cases} 1 - x & \text{if} \quad y_i \in [0, 1] \text{ and } (x, y_j) \in \,]1, 2] \times [0, 1] \\ y_i - 1 & \text{if} \quad y_i \in [0, 1[\text{ and } (x, y_j) \in \{1\} \times [0, 1] \\ 1 & \text{if} \quad y_i = 1 \text{ and } (x, y_j) \in \{1\} \times [0, 1] \end{cases}$$

(where $i \neq j$) do not satisfy condition (C2)' but they are seq. lower pseudocontinuous.

Example 5.3.2 Let $K_1(x, y_2) = [0, 1]$ for all $(x, y_2) \in [0, 1]^2$ and $K_2(x, y_1) = [0, 1]$ for all $(x, y_1) \in [0, 1]^2$. The functions $f_1, f_2 : [0, 1]^3 \longrightarrow \mathbb{R}$, defined as following (where $i \neq j$):

$$f_i(x, y_i, y_j) = \begin{cases} x(1 - y_i) & \text{if} \quad y_i \in [0, 1[\text{ and } (x, y_j) \in \,]0, 1] \times [0, 1] \\ 1 & \text{if} \quad y_i = 1 \text{ and } (x, y_j) \in [0, 1] \times [0, 1] \\ 0 & \text{if} \quad x = 0,\ y_i \neq 1 \text{ and } y_j \in [0, 1] \end{cases}$$

satisfy the above condition (C2)' but they are not seq. lower pseudocontinuous.

Besides, in the case of coupled constraints (that is when K_i is a general multifunction), (C1)' and (C2)' are not explicit sufficient conditions for the sequential closedness of the set-valued function N, as shown in Example 5.3.3.

Example 5.3.3 We consider the functions defined in Example 5.3.2 and the constraints defined by:

$$K_i(x, y_j) = [0, 1 - x] \text{ for all } (x, y_j) \in [0, 1]^2$$

(where $i \neq j$).
K_1 and K_2 satisfy condition (i) in Theorem 5.3.2 but the set-valued function N is not sequentially closed at $x = 0$. In fact, if $x_n \longrightarrow 0^+$, we have $N(x_n) = \{(0, 0)\}$ for any n but $N(0) = \{(1, 1)\}$.

We conclude observing that sequential upper pseudocontinuity of payoffs f_i is not connected with assumption (ii) in Theorem 5.2.2 ([30]), as it can be seen by the following easy example. Let $f_1, f_2 : [0, 2] \longrightarrow \mathbb{R}$ defined by:

$$f_1(y) = \begin{cases} y - 1 & \text{if } y \in [0, 1] \\ y + 1 & \text{if } y \in]1, 2] \end{cases}$$

and

$$f_2(y) = \begin{cases} 1 - y & \text{if } y \in [0,1] \\ -y & \text{if } y \in]1,2] \end{cases}$$

f_1 and f_2 are seq. upper pseudocontinuous but $f_1 + f_2$ is not seq. upper semicontinuous at $y = 1$.

5.4 Tykhonov Well-Posedness in Game Theory

Tykhonov well-posedness has been extended to zero-sum games (see [10]) and to non-zero sum games (see [43]).
For a unparametrized game $G = \{Y_i, f_i\}_{i=1}^m$, a sequence $(y^n)_n \subseteq Y$ is said an *approximating equilibria sequence* if:

$$\sup_{z_i \in Y_i} f_i(z_i, y_{-i}^n) - f_i(y_i^n, y_{-i}^n) \longrightarrow 0 \quad \text{for all} \quad i \in \{1, ..., m\}.$$

Hence, G is *Tykhonov well-posed* if there exists a unique Nash equilibrium y^o of G and any approximating equilibria sequence converges to y^o.

Moreover, G is *generalized Tykhonov well-posed* if the set of Nash equilibria of G is non-empty and any approximating equilibria sequence has a subsequence which converges to some equilibrium of G.

In the case in which $Y_1, ..., Y_m$ are metric spaces, characterizations of Tykhonov well-posedness and generalized Tykhonov well-posedness have been given by Margiocco, Patrone and Pusillo Chicco in [34]. In fact, let Ω_ε be the set of all ε-Nash equilibria, that is $y \in Y$ such that $f_i(y_i, y_{-i}) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}) - \varepsilon$ for all i. In the following, for a metric space (Z, d), $\delta(A)$ denotes the diameter of A and d_H denotes the Hausdorff metric.

Theorem 5.4.1 ([34]) *The game G is Tykhonov well-posed if and only if there is a Nash equilibria and* $\lim_{\varepsilon \to 0} \delta(\Omega_\varepsilon) = 0$.

Theorem 5.4.2 ([34]) *The game G is generalized Tykhonov well-posed if and only if Ω_0 is non-empty, compact and* $\lim_{\varepsilon \to 0} d_H(\Omega_0, \Omega_\varepsilon) = 0$.

Until now, the explicit assumptions on the data for well-posedness of a game are the continuity of all payoffs. In the next paragraph, sufficient conditions, explicit on the data and weaker than continuity of payoffs, will be given.

5.5 Parametric Well-Posedness of Social Nash Equilibria Problems

Let $\Gamma(x)$ be the parametric abstract economy defined in Paragraph 5.3, with $x \in X$. We set:

$$\mathcal{N} = \{\mathcal{N}(x) \ / \ x \in X\}.$$

In this section, following [39], we consider the concept of Tykhonov well-posedness in the setting of abstract economies.
First, we introduce the notion of approximating equilibria sequence for the problem $\mathcal{N}(x_o)$. Let $(x_n)_n$ be a sequence converging to x_o. A sequence $(y^n)_n \in Y$ is said to be an *approximating equilibria sequence* for $\mathcal{N}(x_o)$ (with respect to $(x_n)_n$) if the following conditions are satisfied for all i:

- $y_i^n \in K_i(x_n, y_{-i}^n)$ for n sufficiently large;
- $v_i(x_n, y_{-i}^n) - f_i(x_n, y^n) \longrightarrow 0$.

(where $v_i(x, y_{-i}) = \sup_{z_i \in K_i(x, y_{-i})} f_i(x, (z_i, y_{-i}))$).

Definition 5.5.1 *The family $\mathcal{N}$ is said to be parametrically well-posed at $x_o \in X$ if and only if:*

- *there is a unique solution y^o to $\mathcal{N}(x_o)$;*
- $v_i(x, y_{-i}) < +\infty$ *for all* $(x, y_{-i}) \in X \times Y_{-i}$*;*
- *for all $x_n \longrightarrow x_o$ and for all approximating equilibria sequence $(y^n)_n$ (with respect to $(x_n)_n$), we have $y^n \longrightarrow y^o$.*

$\mathcal{N}$ is said parametrically well-posed if it is parametrically well-posed at x_o for all $x_o \in X$.

Definition 5.5.1 extends to parametric social Nash equilibrium problems the definitions of well-posedness recalled in the previous section and well-posedness concepts for Nash equilibria presented in [36], where the more general case of multicriteria Nash equilibria is considered.

Now, we give sufficient conditions for parametric well-posedness of $\mathcal{N}$ in which the continuity of payoffs is dropped using sequentially upper pseudo-continuous functions.

Theorem 5.5.1 *Let $x_o \in X$. Assume that there is a unique solution to $\mathcal{N}(x_o)$. If, for all i, Y_i is sequentially compact and:*

(i) the set-valued function K_i is sequentially closed at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

(iii) the function v_i is sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y$,

then, the family $\mathcal{N}$ is parametrically well-posed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $(y^n)_n$ be an approximating equilibria sequence (with respect to $(x_n)_n$) such that $y^n \not\longrightarrow y^o$. In light of axiom (S_3) in Paragraph 2.1 and by seq. compactness of $Y_1, ..., Y_m$, there exists a subsequence $(y^{n_k})_k$ of $(y^n)_n$ converging to a point $y \in Y \backslash \{y^o\}$ and, for all j, $y_j^{n_k} \in K_j(x_{n_k}, y_{-j}^{n_k})$ for k sufficiently large. Since K_j is seq. closed at (x_o, y_{-j}), we have that $y_j \in K_j(x_o, y_{-j})$ for all j. Then, there exists $i \in \{1, ..., m\}$ and $z_i \in K_i(x_o, y_{-i})$ such that $f_i(x_o, (y_i, y_{-i})) < f_i(x_o, (z_i, y_{-i}))$. Since f_i is seq. upper pseudocontinuous, we have:

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) < f_i(x_o, (z_i, y_{-i})) \leq v_i(x_o, y_{-i})$$

and by seq. lower semicontinuity of v_i, we obtain:

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) < \liminf_{k\to\infty} v_i(x_{n_k}, y_{-i}^{n_k}).$$

Let α be a real number such that:

$$\limsup_{k\to\infty} f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) < \alpha < \liminf_{k\to\infty} v_i(x_{n_k}, y_{-i}^{n_k}), \tag{5.1}$$

So, we obtain that there exists $k_o \in \mathbb{N}$ such that:

$$\alpha - f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) < v_i(x_{n_k}, y_{-i}^{n_k}) - f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) \tag{5.2}$$

for all $k \geq k_o$. So, by (5.1) and (5.2), we obtain:

$$0 < \alpha - \limsup_{k\to\infty} f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k})) \leq \lim_{k\to\infty} [v_i(x_{n_k}, y_{-i}^{n_k}) - f_i(x_{n_k}, (y_i^{n_k}, y_{-i}^{n_k}))] = 0$$

and we get a contradiction.

□

In the case in which $X, Y_1, ..., Y_m$ are metric spaces, denoted with $B(x_o, \varepsilon)$ the open sphere in X of center x_o and ray ε and set:

$$\varepsilon - N(x) = \{y \in Y \; / \; y_i \in K_i(x, y_{-i}) \text{ and } v_i(x, y_{-i}) - \varepsilon < f_i(x, y) \; \forall \; i\},$$

we have the following result.

Theorem 5.5.2 *Let $x_o \in X$. Assume that $X, Y_1, ..., Y_m$ are metric spaces, with $Y_1, ..., Y_m$ also complete and:*

$$\lim_{\varepsilon \downarrow 0} diam\{\cup_{x \in B(x_o,\varepsilon)} [\varepsilon - N(x)]\} = 0. \quad (5.3)$$

If for all i:

(i) the set-valued function K_i is sequentially closed at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

(iii) the function v_i is sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y$,

then, the family $\mathcal{N}$ is parametrically well-posed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $(y^n)_n$ be an asymptotically equilibria sequence (with respect to $(x_n)_n$). Fixed $\varepsilon > 0$, we have $y^n \in \varepsilon - N(x_n)$ for n sufficiently large. So, by (5.4), $(y^n)_n$ is a Cauchy sequence. Therefore $(y^n)_n$ converges to an element y^o and, by seq. closedness of K_j, $y^o_j \in K_j(x_o, y^o_{-j})$ for all j. If y^o is not a solution of $\mathcal{N}(x_o)$, proceeding as in the proof of Theorem 5.5.1, we obtain a contradiction. Hence $y^o \in N(x_o)$. Moreover, in light of (5.4), we have that $N(x_o) = \{y^o\}$ and thesis follow.

□

Concerning parametric well-posedness under explicit assumptions on the data, we obtain the following results.

Corollary 5.5.1 *Let $x_o \in X$. Assume that there is a unique solution to $\mathcal{N}(x_o)$. If, for all i, Y_i is sequentially compact and:*

(i) the set-valued function K_i is sequentially closed and sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

then, the family $\mathcal{N}$ is parametrically well-posed at x_o.

Corollary 5.5.2 *Let $x_o \in X$. Assume that $X, Y_1, ..., Y_m$ are metric spaces, with $Y_1, ..., Y_m$ also complete and:*

$$\lim_{\varepsilon \downarrow 0} diam\{\cup_{x \in B(x_o,\varepsilon)}[\varepsilon - N(x)]\} = 0. \tag{5.4}$$

If for all i:

(i) the set-valued function K_i is sequentially closed and sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

then, the family $\mathcal{N}$ is parametrically well-posed at x_o.

Now, for well-posedness in the generalized sense, we have the following definition.

Definition 5.5.2 *The family $\mathcal{N}$ is said to be generalized parametrically well-posed at $x_o \in X$ if and only if:*

- *the set of solutions to $\mathcal{N}(x_o)$ is non-empty;*
- *$v_i(x, y_{-i}) < +\infty$ for all $(x, y_{-i}) \in X \times Y_{-i}$;*
- *for all $x_n \longrightarrow x_o$ and for all approximating equilibria sequence $(y^n)_n$ (with respect to $(x_n)_n$), there exists a subsequence of $(y^n)_n$ which converges to a solution of $\mathcal{N}(x_o)$.*

$\mathcal{N}$ is said generalized parametrically well-posed if it is generalized parametrically well-posed at x_o for all $x_o \in X$.

In the next theorems, sufficient conditions for generalized parametric well-posedness are given.

Theorem 5.5.3 *Let $x_o \in X$. Assume that the set of solutions to $\mathcal{N}(x_o)$ is non-empty. If, for all i, Y_i is sequentially compact and:*

(i) the set-valued function K_i is sequentially closed at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

(iii) the function v_i is sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y$,

then, the family $\mathcal{N}$ is generalized parametrically well-posed at x_o.

Proof. Let $x_n \longrightarrow x_o$ and $(y^n)_n$ be an approximating equilibria sequence (with respect to $(x_n)_n$) such that every convergent subsequence converges to a point which is not a solution to $\mathcal{N}(x_o)$. Let $(y^{n_k})_k$ be a such sequence converging to a point y. By seq. closedness of K_j, we obtain that $y_j \in K_j(x_o, y_{-j})$ for each j. Now, we can proceed as in the proof of Theorem 5.5.1 and thesis follow.

□

Under explicit assumption on the data, we obtain the following corollary.

Corollary 5.5.3 *Let $x_o \in X$. Assume that the set of solutions to $\mathcal{N}(x_o)$ is non-empty. If, for all i, Y_i is sequentially compact and:*

(i) the set-valued function K_i is sequentially closed and sequentially lower semicontinuous at (x_o, y_{-i}), for any $y_{-i} \in Y_{-i}$,

(ii) the function f_i is sequentially upper pseudocontinuous and sequentially lower semicontinuous at (x_o, y), for any $y \in Y$ such that $y_i \in K_i(x_o, y_{-i})$,

then, the family $\mathcal{N}$ is generalized parametrically well-posed at x_o.

To conclude the chapter, we note that, using pseudocontinuous functions, the continuity of payoffs has been relaxed in the same way for both closedness of solutions and parametric well-posedness of parametric social Nash equilibria problems.

Chapter 6

Approximate Nash and Social Nash Equilibria

As well known, existence of Nash and social Nash equilibria are equivalent to existence of fixed point of suitable set-valued functions. So, for applying fixed point theorems, it is crucial to make compactness assumptions on strategic spaces. In the classical results (among others, see [40], [41], [15], [14], [19], [21]), in order to obtain existence of equilibria, the compactness of strategic space cannot be weaken. In fact, there are several games and pseudo-game in which all assumptions of quoted results on payoffs and on constraints are satisfy, but in which there are not equilibria, because the strategic spaces are not compact.
The intention of this chapter is to study games and pseudo-games in the cases of lake of compactness on sets of strategies. In fact, following [6] and [46], first it will be presented existence results of approximate Nash equilibria for games in which the sets of strategies are bounded or totally bounded in Banach spaces. Finally, following [46], a suitable concept of approximate equilibria for pseudo-games will be introduced and existence result will be given in Banach spaces, when the strategic spaces are bounded and totally bounded.

6.1 Approximate Nash Equilibria for Games in Normal Form

In Nash [40], Nash equilibria for n-person non-cooperative games have been introduced and using Kakutani's fixed point theorem ([22]) it has been shown

that mixed extensions of finite n-person non-cooperative games possess at least one Nash equilibrium. The aggregate best response multifunction on the Cartesian product of the strategy spaces constructed with the aid of the best response multifunctions for each player possesses fixed points which coincide with the Nash equilibria of the game. Of course, for many non-cooperative games Nash equilibria do not exist. For example, this may be the situation for games which possess not compact strategic sets. Interesting are games for which ε-Nash equilibria exist for each $\varepsilon > 0$. Here a strategy profile is called an ε-Nash equilibrium if unilateral deviation of one of the players does not increase his payoff with more than ε. More precisely, let $G = \{Y_i, f_i\}_i$ be the game in normal form considered in Paragraph 5.1 and $\varepsilon > 0$. A profile of strategies $y^* \in Y$ is said *ε-Nash equilibrium* if $f_i(y_i^*, y_{-i}^*) + \varepsilon \geq f_i(y_i, y_{-i}^*)$ for all $y_i \in Y_i$ and for all i. Such profile of strategies will be also called *approximate Nash equilibria.*
The problem of existence of ε-Nash equilibria is equivalent to a fixed point problem. In fact, for any player i, let $B_i^\varepsilon : Y_{-i} \longrightarrow 2^{Y_i}$ be the *ε-best response multifunction* defined by:

$$B_i^\varepsilon(y_{-i}) = \{y_i \in Y_i \ / \ f_i(y_i, y_{-i}) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}) - \varepsilon\},$$

and let $B^\varepsilon : Y \longrightarrow 2^Y$ be the *aggregate ε-best response multifunction*, defined by:

$$B^\varepsilon(y) = \prod_{i=1}^{m} B_i^\varepsilon(y_{-i}).$$

It is easy to prove that a profile of strategies y^* is a ε-Nash equilibrium if and only if y^* is a fixed point of the multifunction B^ε. So, using approximate fixed point theorems, we will be able to solve the fixed point problem $y^* \in B^\varepsilon(y^*)$ without compactness assumptions on strategic sets. To make that, following [6], we proceed as in the next scheme:
(i) develop ε-fixed point theorems and find conditions on strategy spaces and payoff functions of the game such that the aggregate ε-best response multifunction satisfies conditions in an ε-fixed point theorem;
(ii) add extra conditions on the payoff-functions, guaranteeing that points in the cartesian product of the strategy spaces nearby each other have payoffs sufficiently nearby.

The next results are called the key propositions because they open the door to obtain different ε-equilibrium point theorems, using as inspiration source the existing literature on Nash equilibrium point theorems. Many of them contain collections of sufficient conditions on the strategy spaces and payoff functions, guaranteeing that the aggregate best response multifunction has a fixed point. To guarantee the existence of ε-fixed points one has to modify, often in an obvious way, the conditions guaranteeing the existence of δ-fixed points for the aggregate ε-best response multifunction and to replace the condition (iii) or (iii)' in the key propositions by the obtained conditions.

Key Proposition 6.1.1 *Let $G = \{Y_i, f_i\}_{i=1}^m$ be an m-person game in normal form with the following three properties:*

(i) for each $i \in \{1, ..., m\}$, the strategy space Y_i is endowed with a metric d_i;

(ii) the payoff functions $f_1, ..., f_m$ are uniformly continuous on $Y = \prod_{i=1}^m Y_i$, where Y is endowed with the metric d, defined by:

$$d(y, z) = \sum_{i=1}^{m} d_i(y_i, z_i) \quad \textit{for all } y, z \in Y;$$

(iii) for each $\varepsilon > 0$ and $\delta > 0$, the aggregate ε-best response multifunction B^ε possesses at least one δ-fixed point, i. e. $FIX^\delta(B^\varepsilon) \neq \emptyset$.

Then, for each $\varepsilon > 0$, the set $NE^\varepsilon(G)$ of all ε-Nash equilibria is non-empty.

Proof. Take $\varepsilon > 0$. By (ii) we can find $\eta > 0$ such that for all $y, y' \in Y$ with $d(y, y') < \eta$ we have $| f_i(y) - f_i(y') | < \frac{1}{2}\varepsilon$ for all i. We will prove that:

$$y^* \in FIX^{\frac{1}{2}\eta}(B^{\frac{1}{2}\varepsilon}) \Longrightarrow y^* \in NE^\varepsilon(G).$$

Take $y^* \in FIX^{\frac{1}{2}\eta}(B^{\frac{1}{2}\varepsilon})$, which is possible by (iii). Then there exists $\hat{y} \in B^{\frac{1}{2}\varepsilon}(y^*)$ such that $d(y^*, \hat{y}) < \eta$, and, consequently, for each i: $d((y_i^*, y_{-i}^*), (\hat{y}_i, y_{-i}^*)) < \eta$. This implies that:

$$f_i(y_i^*, y_{-i}^*) \geq f_i(\hat{y}_i, y_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all } i \in \{1, ..., m\}. \tag{6.1}$$

Further $\hat{y} \in B^{\frac{1}{2}\varepsilon}(y^*)$ implies:

$$f_i(\hat{y}_i, y_{-i}^*) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all } i \in \{1, ..., m\}. \tag{6.2}$$

Combining (6.1) and (6.2) we obtain:

$$f_i(y_i^*, y_{-i}^*) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}^*) - \varepsilon \quad \text{for all } i \in \{1, ..., m\}, \tag{6.3}$$

that is $y^* \in NE^\varepsilon(G)$.

□

Now, following [46], a variant of Key Proposition 6.1.1 is here presented.

Key Proposition 6.1.2 *Let $G = \{Y_i, f_i\}_{i=1}^m$ be an m-person game in normal form with the following three properties:*

(i)' for each $i \in \{1, ..., m\}$, the strategy space Y_i is endowed with two metrics d_i and d'_i such that:

$$d'_i(y_i, z_i) < \eta \Longrightarrow d_i(y_i, z_i) < \eta;$$

(ii) the payoff functions $f_1, ..., f_m$ are uniformly continuous functions on $Y = \prod_{i=1}^m Y_i$, where Y is endowed with the metric d, defined by:

$$d(y, z) = \sum_{i=1}^{m} d_i(y_i, z_i) \quad \text{for all } y, z \in Y;$$

(iii)' for each $\varepsilon > 0$ and $\delta > 0$, the aggregate ε-best response multifunction B^ε possesses at least one δ-fixed point with respect to metric (on Y) d' defined by $d'(y, z) = \sum_{i=1}^m d'_i(y_i, z_i)$.

Then, for each $\varepsilon > 0$, the set $NE^\varepsilon(G)$ of all ε-Nash equilibria is non-empty.

Proof. Let ε and η as in the previous proof and $y^* \in FIX^{\frac{1}{2}\eta}(B^{\frac{1}{2}\varepsilon})$. So, there exists $\hat{y} \in B^{\frac{1}{2}\varepsilon}(y^*)$ such that $d'(y^*, \hat{y}) < \eta$, and $d'((y_i^*, y_{-i}^*), (\hat{y}_i, y_{-i}^*)) < \eta$ for all i. This implies $d((y_i^*, y_{-i}^*), (\hat{y}_i, y_{-i}^*)) < \eta$. From this point, we can proceed as in the above proof and the thesis follows.

□

It will be clear that using key propositions, many approximate Nash equilibrium theorems can be obtained. In the following, we give three examples of existence of approximate Nash equilibria.

Example 6.1.1 (Games on the open unit square). Let $\{]0, 1[,]0, 1[, f_1, f_2\}$ be a game with uniform continuous payoff functions f_1 and f_2. Suppose that f_1 is concave in the first coordinate and f_2 is concave in the second coordinate. Then for each $\varepsilon > 0$, the game has an ε-Nash equilibrium point. In fact, apply the key proposition to the above game and note that (i) and (ii) are satisfied by taking the standard metric on]0,1[. Further, (iii) follows from Theorem 3.2.3 applied to the set-valued function B^ε.

Example 6.1.2 (Completely mixed approximate Nash equilibria for finite games). Let A and B be $m \times n$-matrices of real numbers. Consider the two-person game $\{\overset{o}{\Delta}_h, \overset{o}{\Delta}_k, f_1, f_2\}$, where:

$$\overset{o}{\Delta}_h = \{p \in \mathbb{R}^h \mid p_i > 0 \text{ for each } i \in \{1, ..., h\}, \sum_{i=1}^{h} p_i = 1\},$$

$$\overset{o}{\Delta}_k = \{q \in \mathbb{R}^k \mid q_j > 0 \text{ for each } j \in \{1, ..., k\}, \sum_{j=1}^{k} q_j = 1\},$$

$$f_1(p, q) = p^T A q, \;\; f_2(p, q) = p^T B q \text{ for all } p \in \overset{o}{\Delta}_h, \;\; q \in \overset{o}{\Delta}_k.$$

Then for each $\varepsilon > 0$ this game has an ε-Nash equilibrium. Such an ε-Nash equilibrium is called *completely mixed*, because both players use each of their pure strategies with a positive probability. The proof follows from the key proposition and Theorem 3.2.3 taking the standard Euclidean metric.

Example 6.1.3 Let Y be a normed linear space such that there exists $a \in Y \backslash \{0\}$. Let $G = \{Y, Y, f_1, f_2\}$ be the two-person game with $f_1(y_1, y_2) = -\parallel y_1 - y_2 \parallel$, $f_2(y_1, y_2) = -\parallel y_1 - y_2 - \frac{a}{1+\|y_1\|} \parallel$ for all $(y_1, y_2) \in Y \times Y$. Then $B_1(y_2) = \{y_2\}$ and $B_2(y_1) = \{y_1 - \frac{a}{1+\|y_1\|}\}$. So $B(y_1, y_2) = \{(y_2, y_1 - \frac{a}{1+\|y_1\|})\}$ for each $(y_1, y_2) \in Y \times Y$. Hence, $FIX(B) = \emptyset$. However, for each $\delta > 0$, $FIX^\delta(B) \neq \emptyset$ since one can take $y \in Y$ with $\parallel y \parallel \geq \delta^{-1} \parallel a \parallel$ and, then, $(y, y) \in FIX^\delta(B)$ because $\parallel (y, y) - (y, y - \frac{a}{1+\|y\|}) \parallel = \frac{a}{1+\|y\|} \leq \frac{\|a\|}{\|y\|} \leq \delta$. Moreover f_1 and f_2 are uniform continuous functions on $Y \times Y$. In fact: $\mid f_2(y_1, y_2) - f_2(z_1, z_2) \mid \leq \parallel (y_1 - z_1) - (y_2 - z_2) + \frac{\|y_1\| - \|z_1\|}{(1+\|y_1\|)(1+\|z_1\|)} a \parallel \leq (\parallel y_1 - z_1 \parallel + \parallel y_2 - z_2 \parallel)(1 + \parallel a \parallel)$. Therefore, in light of the Key Proposition 6.1.1 we can conclude that $NE^\varepsilon(G) \neq \emptyset$ for each $\varepsilon > 0$. In fact, for $\parallel y \parallel$ sufficiently large, $(y, y) \in NE^\varepsilon(G)$, since $f_2(y, y_2) - f_2(y, y) \leq \frac{\|a\|}{1+\|y\|}$.

Following [46], in the next two theorems, using approximate fixed point theorems, explicit assumptions on the data are given in order to obtain existence of approximate Nash equilibria on bounded and totally bounded sets. For the first one we need of Remark 6.1.1.

Remark 6.1.1 Assume that E is a reflexive and separable real Banach space and X is a non-empty bounded subset of E. As we have recalled in Lemma 3.2.1, the weak topology on X is induced by a metric d_X. This metric can be obtain in the following way (see for example [7]). Let E' be the topological dual space of E, B' be the closed unit ball in E' and let $(\phi_n)_n \subseteq E'$ be a sequence dense in B'. It can be proved that:

$$d_X(x, y) = \sum_{n=1}^{\infty} 2^{-n} \mid \langle \phi_n, x - y \rangle \mid .$$

So, the relationship on X between d_X and the norm $\| \cdot \|$ is the following:

$$C^{-1} d_X(x, y) \leq \| x - y \|,$$

where C is a positive number.

Theorem 6.1.1 *Let $G = \{Y_i, f_i\}_{i=1}^m$ be an m-person game in normal form. Let $Y_1, ..., Y_m$ be convex and bounded subsets, with non empty interior, of reflexive and separable real Banach spaces. Assume that the following hypothesis are satisfied for all $i \in \{1, ..., m\}$:*

(i) the payoff function f_i is uniformly continuous on $Y = \prod_{j=1}^m Y_j$ with respect the metric d_Y defined by $d_Y(y, z) = \sum_{j=1}^m d_{Y_j}(y_j, z_j)$, where d_{Y_j} is the metric which induces the weak topology on Y_j;

(ii) the function $f_i(\cdot, y_{-i})$ is quasi concave and bounded above for each $y_{-i} \in Y_{-i} = \prod_{j \neq i} Y_j$.

Then, for each $\varepsilon > 0$, the set $NE^{\varepsilon}(G)$ of all ε-Nash equilibria is non-empty.

Proof. First, we will prove that the aggregate ε-best response multifunction is closed in the weak topology. It will be sufficient to prove that the ε-best response multifunction B_i^{ε} of any player i is weakly closed. Let i be fixed. The set Y_{-i} is endowed of the topology τ_{-i} product of the weak topologies on all Y_j with $j \neq i$. So, in light of Lemma 3.2.1, τ_{-i} coincides with the topology induced by the metric $d_{-i}(y_{-i}, z_{-i}) = \sum_{j \neq i} d_{Y_j}(y_j, z_j)$. Hence, for the multifunction B_i^{ε}, the weak closedness is equivalent to the sequential closedness with respect to d_{-i}. Let $(y^n)_n$ a sequence in Y. As usual, the notation $y_j^n \rightharpoonup y_j$ indicates that the sequence $(y_j^n)_n$ is converging to y_j in the weak topology on Y_j. Now, let: $y_{-i}^n \rightharpoonup y_{-i}$, $y_i^n \in B_i^{\varepsilon}(y_{-i}^n)$ for n sufficiently large and $y_i^n \rightharpoonup y_i$. So:

$$f_i(y_i^n, y_{-i}^n) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}^n) - \varepsilon \quad \text{for all } n \tag{6.4}$$

Being f_i continuous with respect the weak topology induced by d_Y, the marginal function:

$$z_{-i} \mapsto \sup_{z_i \in Y_i} f_i(z_i, z_{-i}) \tag{6.5}$$

is sequentially lower semicontinuous in the weak topology (see [4] and [27]), and in light of (6.4) we obtain:

$$f_i(y_i, y_{-i}) \geq \liminf_{n \to \infty} \sup_{z_i \in Y_i} f_i(z_i, y_{-i}^n) - \varepsilon \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}) - \varepsilon.$$

So $y_i \in B_i^\varepsilon(y_{-i})$, which proves the closedness of B_i^ε.
Moreover, it is easy to show that hypothesis (ii) implies $B_i^\varepsilon(y_{-i})$ convex and non-empty for all y_{-i}.
Hence, B^ε is weakly closed and it has non-empty and convex values. Applying Theorem 3.2.1, we deduce that $FIX^\delta(B^\varepsilon) \neq \emptyset$ for all $\varepsilon, \delta \in \mathbb{R}_+$. So, remembering Remark 6.1.1, the thesis follows by Key Proposition 6.1.2.

□

Theorem 6.1.2 *Let $G = \{Y_i, f_i\}_{i=1}^m$ be an m-person game in normal form. Let $Y_1, ..., Y_m$ be convex and totally bounded subsets, with non empty interior, of real Banach spaces. Assume that the following hypothesis are satisfied for all $i \in \{1, ..., m\}$:*

(i) the payoff function f_i is uniformly continuous on $Y = \prod_{j=1}^m Y_j$ with respect the norm $\| \cdot \| = \sum_{j=1}^m \| \cdot \|_j$, where $\| \cdot \|_j$ is the norm of the space which includes Y_j;

(ii) the function $f_i(\cdot, y_{-i})$ is quasi concave and bounded above for each $y_{-i} \in Y_{-i} = \prod_{j \neq i} Y_j$.

Then, for each $\varepsilon > 0$, the set $NE^\varepsilon(G)$ of all ε-Nash equilibria is non-empty.

Proof. Let $y_{-i}^n \longrightarrow y_{-i}$, $y_i^n \in B_i^\varepsilon(y_{-i}^n)$ for n sufficiently large and $y_i^n \longrightarrow y_i$. So $f_i(y_i^n, y_{-i}^n) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}^n) - \varepsilon$ for all n. Moreover, the marginal function defined by (6.5) is still lower semicontinuous. Hence $f_i(y_i, y_{-i}) \geq \sup_{z_i \in Y_i} f_i(z_i, y_{-i}) - \varepsilon$, which proves that B_i^ε is closed. It is easy to check that B_i^ε has non-empty and convex values. So, in light of Theorem 3.2.3, $FIX^\delta(B^\varepsilon) \neq \emptyset$ for all $\varepsilon, \delta \in \mathbb{R}_+$ and the thesis follows by Key Proposition 6.1.1.

□

6.2 Determinateness of Two-Person Games

In this section, we want briefly to show a question in which approximate Nash equilibria are strongly involved.
Let $G = \{Y_1, Y_2, f_1, f_2\}$ be a two person game and $\varepsilon_1, \varepsilon_2 > 0$. A pair (y_1^*, y_2^*) is called an $(\varepsilon_1, \varepsilon_2)$-*equilibrium point* (see [32]) if:

$$f_1(y_1^*, y_2^*) \geq f_1(y_1, y_2^*) - \varepsilon_1 \quad \text{for all} \;\; y_1 \in Y_1$$

$$f_2(y_1^*, y_2^*) \geq f_2(y_1^*, y_2) - \varepsilon_2 \quad \text{for all} \;\; y_2 \in Y_2$$

In the case in which G is a zero-sum game, that is $f_1 + f_2$ identically equal to zero, $(\varepsilon_1, \varepsilon_2)$-equilibrium points are related to the determinateness of the game. More precisely, a zero-sum game is *determined* ([42]) if it possesses a *value*, which is an element v such that:

$$v = \inf_{y_1 \in Y_1} \sup_{y_2 \in Y_2} f(y_1, y_2) = \sup_{y_2 \in Y_2} \inf_{y_1 \in Y_1} f(y_1, y_2),$$

where $f = f_2$. Now, we have the following proposition.

Proposition 6.2.1 ([32]) *Let $\{Y_1, Y_2, f_1, f_2\}$ be a zero-sum game. Then the following assertions are equivalent:*

(i) The game has finite value.

(ii) The game possesses an $(\varepsilon_1, \varepsilon_2)$-equilibrium point for each $\varepsilon_1, \varepsilon_2 > 0$.

If a game possesses ε-Nash equilibria for any $\varepsilon > 0$, then it has also $(\varepsilon_1, \varepsilon_2)$-equilibrium points for each $\varepsilon_1, \varepsilon_2 > 0$. In fact, fixed $\varepsilon_1, \varepsilon_2 > 0$, it is sufficient to choose $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$. Hence, Theorems 6.1.1 and 6.1.2 are sufficient conditions in order to recognize zero-sum game with finite value.

The idea of determinateness for zero-sum games has been carried in general two-person game by Lucchetti, Patrone and Tijs in [32]. In fact, set $E^{\varepsilon_1,\varepsilon_2}$ the set of all $(\varepsilon_1, \varepsilon_2)$-equilibrium points and

$$E^{\varepsilon,k} = \{(y_1^*, y_2^*) \;/\; f_1(y_1^*, y_2^*) \geq \sup\nolimits_{y_1 \in Y_1} f_1(y_1, y_2^*) - \varepsilon \;\text{ and }\; f_2(y_1^*, y_2^*) \geq k\},$$

$$E^{k,\varepsilon} = \{(y_1^*, y_2^*) \;/\; f_1(y_1^*, y_2^*) \geq k \;\text{ and }\; f_2(y_1^*, y_2^*) \geq \sup\nolimits_{y_2 \in Y_2} f_2(y_1^*, y_2) - \varepsilon\},$$

$$E^{k_1,k_2} = \{(y_1^*, y_2^*) \;/\; f_1(y_1^*, y_2^*) \geq k_1 \;\text{ and }\; f_2(y_1^*, y_2^*) \geq k_2\},$$

a general two person game is said to be *determined* if at least one of the next properties is satisfied:

- $E^{\varepsilon_1,\varepsilon_2} \neq \emptyset$ for all $\varepsilon_1, \varepsilon_2 > 0$;
- $E^{\varepsilon,k} \neq \emptyset$ for all $\varepsilon > 0$ and all $k \in \mathbb{R}$;
- $E^{k,\varepsilon} \neq \emptyset$ for all $k \in \mathbb{R}$ and all $\varepsilon > 0$;
- $E^{k_1,k_2} \neq \emptyset$ for all $k_1, k_2 \in \mathbb{R}$.

As well for general two-person game, Theorems 6.1.1 and 6.1.2 are sufficient conditions also for determinateness in nonzero sum games.

6.3 Approximate Social Nash Equilibria

In this paragraph, following [46], a suitable concept of approximate social Nash equilibrium for abstract economies will be considered and existence results on non-compact sets will be given.

Let $\Gamma = \{Y_i, K_i, f_i\}_i$ be the abstract economy considered in Paragraph 5.1. Assume that the constraints are described by inequalities: any individual i has a function $g_i : Y = \prod_{j=1}^m Y_j \longrightarrow \mathbb{R}$ (called *constraint function*) such that $K_i(y_{-i}) = \{y_i \in Y_i \ / \ g_i(y_i, y_{-i}) \leq 0\}$ for all $y_{-i} \in Y_{-i}$.
For every positive real number σ, we set $K_i^\sigma(y_{-i}) = \{y_i \in Y_i \ / \ g_i(y_i, y_{-i}) \leq \sigma\}$.

Definition 6.3.1 *Let ε and σ be two positive real numbers. A point $y^* \in Y$ is said to be an (ε, σ)*-social Nash equilibrium *for Γ if, for every $i \in \{1, ..., m\}$:*

$$y_i^* \in K_i^\sigma(y_{-i}^*) \qquad \textit{and} \qquad f_i(y_i^*, y_{-i}^*) \geq \sup_{y_i \in K_i(y_{-i}^*)} f_i(y_i, y_{-i}^*) - \varepsilon.$$

These (ε, σ)-social Nash equilibria will be also said approximate social Nash equilibria.

As for social Nash equilibria, the existence of (ε, σ)-social Nash equilibria is equivalent to the existence of fixed points of a suitable set-valued function. In fact, defining the (ε, σ)-*aggregate best response set-valued function* B_σ^ε as follows:

$$B_\sigma^\varepsilon : y \in Y \longrightarrow \prod_{i=1}^m B_{\sigma,i}^\varepsilon(y_{-i}) \in 2^Y$$

where

$$B_{\sigma,i}^\varepsilon(y_{-i}) = \{z_i \in K_i^\sigma(y_{-i}) \ / \ f_i(z_i, y_{-i}) \geq \sup_{t_i \in K_i(y_{-i})} f_i(t_i, y_{-i}) - \varepsilon\},$$

it is easy to prove that a point y^* is a (ε, σ)-social Nash equilibrium for Γ if and only if $y^* \in B^{\varepsilon}_{\sigma}(y^*)$.

Using approximate fixed point theorems, we can solve the fixed point problem $y^* \in B^{\varepsilon}_{\sigma}(y^*)$ without compactness assumptions on the strategic spaces. As made in the previous section and following [46], first we give two "key propositions".

Key Proposition 6.3.1 *Assume that the following statements are satisfied:*

(i) for any $i \in \{1, ..., m\}$, the strategy space Y_i is endowed with a metric d_i;

(ii) the payoff functions $f_1, ..., f_m$ and the constraint functions $g_1, ..., g_m$ are uniformly continuous on Y with respect to the metric d, defined by:

$$d(y, z) = \sum_{i=1}^{m} d_i(y_i, z_i) \quad \text{for all } y, z \in Y;$$

(iii) $FIX^{\delta}(B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}})$ is non empty for all $\varepsilon > 0$, all $\sigma > 0$ and all $\delta > 0$.

Then, for all $\varepsilon > 0$ and all $\sigma > 0$, there exists an (ε, σ)-social Nash equilibrium for Γ.

Proof. By uniform continuity on Y of payoffs and constraints, there exists $\delta > 0$ such that if $y \in Y$ and $y' \in Y$, with $d(y, y') < \delta$, then we have for all i:

$$f_i(y) \geq f_i(y') - \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{\sigma}{2} + g_i(y) \geq g_j(y')$$

Let $y^* \in FIX^{\frac{\delta}{2}}(B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}})$, which exists by (iii). Then there exists $\hat{y} \in B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}}(y^*)$ such that $d(y^*, \hat{y}) < \delta$. So, we have for all i:

$$f_i(\hat{y}_i, y^*_{-i}) \geq \sup_{t_i \in K_i(y^*_{-i})} f_i(t_i, y^*_{-i}) - \frac{\varepsilon}{2} \tag{6.6}$$

and

$$g_i(\hat{y}_i, y^*_{-i}) \leq \frac{\sigma}{2} \tag{6.7}$$

Moreover $d((\hat{y}_i, y^*_{-i}), (y^*_i, y^*_{-i})) < \delta$, therefore, for each i, we have:

$$f_i(y^*_i, y^*_{-i}) \geq f_i(\hat{y}_i, y^*_{-i}) - \frac{\varepsilon}{2} \tag{6.8}$$

and

$$g_i(y_i^*, y_{-i}^*) \leq g_i(\hat{y}_i, y_{-i}^*) + \frac{\sigma}{2} \tag{6.9}$$

Combining (6.6) with (6.8) and (6.7) with (6.9), we obtain respectivelly:

$$f_i(y_i^*, y_{-i}^*) \geq \sup_{t_i \in K_i(y_{-i}^*)} f_i(t_i, y_{-i}^*) - \varepsilon \tag{6.10}$$

and

$$g_i(y_i^*, y_{-i}^*) \leq \sigma \tag{6.11}$$

for all $i \in \{1, ..., m\}$.
Finally, (6.10) and (6.11) implies that $y_i^* \in B_{\sigma,i}^{\varepsilon}(y_{-i}^*)$ for all $i \in \{1, ..., m\}$. So $y^* \in B_{\sigma}^{\varepsilon}(y^*)$, that is: y^* is a (ε, σ)-social Nash equilibrium for Γ. □

With the same arguments used in the proof of Key Proposition 6.3.1 and as made in the proof of Key Proposition 6.1.2, one can prove the following Key Proposition 6.3.2.

Key Proposition 6.3.2 *Assume that the following statements are satisfied:*

(i)' for each $i \in \{1, ..., m\}$, the strategy space Y_i is endowed with two metrics d_i and d_i' such that:

$$d_i'(y_i, z_i) < \eta \Longrightarrow d_i(y_i, z_i) < \eta;$$

(ii) the payoff functions $f_1, ..., f_m$ and the constraint functions $g_1, ..., g_m$ are uniformly continuous on Y with respect to the metric d, defined by:

$$d(y, z) = \sum_{i=1}^{m} d_i(y_i, z_i) \quad \text{for all } y, z \in Y;$$

(iii)' for all $\varepsilon > 0$, all $\sigma > 0$ and all $\delta > 0$, the (ε, σ)-aggregate best response multifunction B_{σ}^{ε} possesses at least one δ-fixed point with respect to metric (on Y) d' defined by $d'(y, z) = \sum_{i=1}^{m} d_i'(y_i, z_i)$.

Then, for all $\varepsilon > 0$ and all $\sigma > 0$, there exists an (ε, σ)-social Nash equilibrium for Γ.

Now, by these key propositions, it is possible to obtain existence results for approximate social Nash equilibria in the case in which strategic spaces are *bounded* sets in separable and reflexive real Banach spaces, and in the case in which strategic spaces are *totally bounded* sets in real Banach spaces. In fact, we have the following two theorems ([46]). First, we need of Lemma 6.3.1.

Lemma 6.3.1 ([27]) *Let U_1 and U_2 be two sequential spaces, $x_o \in U_1$ and $g : U_1 \times U_2 \longrightarrow \mathbb{R}$. Assume that the following statements are satisfied:*

(i) if $y_m = \lim_{n\to\infty} y_{m,n}$ and $y = \lim_{m\to\infty} y_m$, then there exists a selection of integers $\{m(n) \ / \ n \in \mathbb{N}\}$ such that $y = \lim_{n\to\infty} y_{m(n),n}$;

(ii) there exists $y_o \in U_2$ such that $g(x_o, y_o) < 0$ (Slater condition);

(iii) for any $x_n \longrightarrow x_o$ and any $y \in U_2$, there exists a sequence $(\bar{y}_n)_n$ converging to y such that $\limsup_{n\to\infty} g(x_n, \bar{y}_n) \leq g(x_o, y)$;

(iv) the function $g(x_o, \cdot)$ is strictly quasi convex.

Then, the multifunction K, defined by $K(x) = \{y \in U_2 \ / \ g(x, y) \leq 0\} \ \forall \ x \in U_1$, is sequentially lower semicontinuous at x_o.

Theorem 6.3.1 *Let $\Gamma = \{Y_i, K_i, f_i\}_{i=1}^m$ be an abstract economy. Let $Y_1, ..., Y_m$ be convex and bounded subsets, with non empty interior, of reflexive and separable real Banach spaces. Assume that the following hypothesis are satisfied for all $i \in \{1, ..., m\}$:*

(i) the payoff function f_i and the constraint function g_i are uniformly continuous on $Y = \prod_{j=1}^m Y_j$ with respect the metric d_Y defined by $d_Y(y, z) = \sum_{j=1}^m d_{Y_j}(y_j, z_j)$, where d_{Y_j} is the metric which induces the weak topology on Y_j;

(ii) the function $f_i(\cdot, y_{-i})$ is quasi concave and bounded above for each $y_{-i} \in Y_{-i} = \prod_{j\neq i} Y_j$;

(iii) for any $y_{-i} \in Y_{-i}$, there exists $y_i \in Y_i$ such that $g_i(y_i, y_{-i}) < 0$ (Slater condition);

(iv) the function $g_i(\cdot, y_{-i})$ is strictly quasi convex for all $y_{-i} \in Y_{-i}$.

Then, for each $\varepsilon, \sigma \in \mathbb{R}_+$, Γ has at least an (ε, σ)-social Nash equilibrium.

Proof. Let ε and σ be positive real number. In order to apply the approximate fixed point theorem 3.2.1, we show that the $(\frac{\varepsilon}{2}, \frac{\sigma}{2})$-best response multifunction

$$B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i} : y_{-i} \in Y_{-i} \longrightarrow B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i}(y_{-i}) \in 2^{Y_i}$$

is closed in the weak topology on Y_{-i}, for all individual i. In light of the Lemma 3.2.1, it is sufficient to prove that $B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i}$ is closed with respect the metric on Y_{-i} defined by $d_{-i}(y_{-i}, z_{-i}) = \sum_{j \neq i} d_{Y_j}(y_j, z_j)$. Let: $y^n_{-i} \rightharpoonup y_{-i}$, $y^n_i \in B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i}(y^n_{-i})$ for n sufficiently large and $y^n_i \rightharpoonup y_i$. So:

$$f_i(y^n_i, y^n_{-i}) \geq \sup_{z_i \in K_i(y^n_{-i})} f_i(z_i, y^n_{-i}) - \frac{\varepsilon}{2} \quad \text{for all } n. \tag{6.12}$$

All hypothesis of Lemma 6.3.1 are verified (condition (i) holds because Y_{-i} is a metric space), so the set-valued function K_i is sequentially lower semicontinuous and the marginal function:

$$z_{-i} \mapsto \sup_{z_i \in K_i(z_{-i})} f_i(z_i, z_{-i}) \tag{6.13}$$

is sequentially lower semicontinuous (see [4] and [27]). By (6.12) one obtain:

$$f_i(y_i, y_{-i}) \geq \liminf_{n\to\infty} \sup_{z_i \in K_i(y^n_{-i})} f_i(z_i, y^n_{-i}) - \frac{\varepsilon}{2} \geq \sup_{z_i \in K_i(y_{-i})} f_i(z_i, y_{-i}) - \frac{\varepsilon}{2}.$$

Moreover, by $g_i(y^n_i, y^n_{-i}) \leq \frac{\sigma}{2}$ follows that $y_i \in K_i^{\frac{\sigma}{2}}(y_{-i})$, and so that the multifunction $B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i}$ is closed for all i. So, the multifunction $B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}}$ is closed. It is easy to check that $B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}}$ has convex and non-empty values. Hence, Theorem 3.2.1 guarantees that $FIX^{\delta}(B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}})$ is non empty for all $\delta > 0$. Remembering Remark 6.1.1, the thesis follows by Key Proposition 6.3.2.

□

If the strategic spaces are included in real Banach spaces, we have the following result.

Theorem 6.3.2 *Let $\Gamma = \{Y_i, K_i, f_i\}_{i=1}^m$ be an abstract economy. Let $Y_1, ..., Y_m$ be convex and totally bounded subsets, with non empty interior, of real Banach spaces. Assume that the following hypothesis are satisfied for all $i \in \{1, ..., m\}$:*

(i) the payoff function f_i and the constraint function g_i are uniformly continuous on $Y = \prod_{j=1}^m Y_j$ with respect the norm $\| \cdot \|_Y = \sum_{j=1}^m \| \cdot \|_{Y_j}$;

(ii) the function $f_i(\cdot, y_{-i})$ is quasi concave and bounded above for each $y_{-i} \in Y_{-i} = \prod_{j \neq i} Y_j$;

(iii) for any $y_{-i} \in Y_{-i}$, there exists $y_i \in Y_i$ such that $g_i(y_i, y_{-i}) < 0$ (Slater condition);

(iv) the function $g_i(\cdot, y_{-i})$ is strictly quasi convex for all $y_{-i} \in Y_{-i}$.

Then, for each $\varepsilon, \sigma \in \mathbb{R}_+$, Γ has at least a (ε, σ)-social Nash equilibrium.

Proof. Let $y^n_{-i} \longrightarrow y_{-i}$, $y^n_i \in B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2},i}(y^n_{-i})$ and $y^n_i \longrightarrow y_i$. So $f_i(y^n_i, y^n_{-i}) \geq \sup_{z_i \in K_i(y^n_{-i})} f_i(z_i, y^n_{-i}) - \frac{\varepsilon}{2}$ for all n. With the same arguments used in the above proof, one can prove that the marginal function defined by (6.13) is still lower semicontinuous. So, all hypothesis of Theorem 3.2.3 are satisfied for $B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}}$ and $FIX^{\delta}(B^{\frac{\varepsilon}{2}}_{\frac{\sigma}{2}}) \neq \emptyset$ for all $\delta > 0$ and the thesis follows by Key Proposition 6.3.2.

□

Chapter 7

Marginal functions and Existence in MaxSup and MaxInf Problems

In the present chapter, properties (weaker than semicontinuity) on marginal functions are considered in order to obtain new existence results for MaxSup and MaxInf problems. In fact, in the setting of sequential spaces, following [38], first we will give a characterization of functions which possess maximum points on sequentially compact set. Hence, using sequentially upper quasi-continuous function, we will be able to relax sequential upper semicontinuity of objective function for existence in both MaxSup and MaxInf problems.

7.1 Existence of Maximum Points in Topological Spaces

In this section, we will recall existence results of maximum points for extended real valued function defined on topological spaces.
We recall that an extended real valued function h defined on a topological space Z is said *upper semicontinuous* at a point $z_o \in Z$ (see for example [1],[4]) if for all $\varepsilon > 0$ there exists a neighborhood I of z_o such that $h(z) \leq h(z_o) + \varepsilon$ for all $z \in I$.

In first we have the famous Theorem of Weierstrass.

Theorem 7.1.1 *Let Z be a topological space and h be an extended real valued function defined on Z. If Z is compact and h is upper semicontinuous, then* *there exists at least a maximum point of h on Z.*

The compactness assumption on Z has been relaxed by Rockafellar in [45] and by Aubin in [1], where authors used upper semicompact functions. An extended real valued function h is said *upper semicompact* if $\{z \in Z \ / \ h(z) \geq \lambda\}$ is relatively compact for all extended real valued λ.

Theorem 7.1.2 ([1]) *Let Z be a topological space and h be an extended real valued function defined on Z. If h is upper semicontinuous and upper semicompact, then there exists at least a maximum point of h on Z.*

Finally, in order to obtain existence of maximum points, the upper semicontinuity of objective functions has been relaxed by Tian and Zhou in [48]. In this paper, using the class of transfer weakly upper continuous functions (which definition has been recalled in Paragraph 2.2) the authors characterize the functions endowed of maximum points on compact sets.
For reasons of comfort, we recall the definition of transfer weakly upper continuity. An extended real valued function h is said *transfer weakly upper continuous on Z* if: $h(z_o) < h(z)$ implies that there exists an element $z' \in Z$ and a neighborhood I of z_o such that $h(z) \leq h(z')$ for all $z \in I$.

Theorem 7.1.3 ([48]) *Let Z be a compact topological space and h be an extended real valued function defined on Z. Then, h admits at least a maximum point if and only if h is transfer weakly upper continuous on Z.*

7.2 Existence of Maximum Points in Sequential Spaces

In this paragraph, following [38], we study the existence of maximum points for functions defined on sequential spaces.
First, we recall the sequential version of the Weierstrass' Theorem.

Theorem 7.2.1 *Let Z be a sequential space and h be an extended real valued function defined on Z. If Z is sequentially compact and h is sequentially upper semicontinuous, then there exists at least a maximum point of h on Z.*

Transfer weakly upper continuity has a natural extension in sequential spaces. In fact, we give the following definition.

Definition 7.2.1 *Let Z be a sequential space and h be an extended real valued function defined on Z. The function h is said to be seq. transfer weakly upper continuous at $z_o \in Z$ if and only if for any $z \in Z$ with $h(z_o) < h(z)$, there exists $z' \in Z$ such that:*

$$\limsup_{n\to\infty} h(z_n) \le h(z') \text{ for all sequence } (z_n)_n \text{ converging to } z_o \text{ in } Z.$$

The function h is said to be seq. transfer weakly upper continuous on Z if and only if it is seq. transfer weakly upper continuous at every point of Z.

Note that the class of seq. transfer weakly upper continuous functions strictly includes the classes of seq. upper pseudocontinuous and quasicontinuous functions.
Obviously, if Z is a topological space, any transfer weakly upper continuous function is also sequentially transfer weakly upper continuous with respect to the convergence structure induced by topology but the converse is not always true. In fact, one can see at Example 2.2.3. The equivalence between the two classes holds in topological spaces which satisfy the first axiom of countability, as showed in the next proposition.

Proposition 7.2.1 *Let Z be a topological space and h be an extended real valued function defined on Z. If h is a transfer weakly upper continuous function on Z then h is seq. transfer weakly upper continuous on Z. Moreover, if Z satisfies the first axiom of countability, then the converse is also true.*

Proof. The first statement is obvious. Assume now that:
- h is seq. transfer weakly upper continuous on Z;
- h is not transfer weakly upper continuous on Z;
- Z satisfies the first axiom of countability.

Let $z_o \in Z$ and $z_1 \in Z$ be such that $h(z_o) < h(z_1)$ and that, for any $z \in Z$ and for any neighbourhood I of z_o, there exists $z_I \in I$ such that $h(z) < h(z_I)$.
Then:
- there exists $z' \in Z$ such that $\limsup_{n\to\infty} h(z_n) \le h(z')$ for all sequence $(z_n)_n$ converging to z_o;
- the function h has not maximum points (see Theorem 7.1.3);
- there exists a countable local base $(I_n)_n$ of z_o decreasing with respect to the inclusion.

Therefore, there exists $z'' \in Z$ such that $h(z') < h(z'')$ and for any $n \in \mathbb{N}$, there exists $\overline{z}_n \in I_n$ such that: $h(z'') < h(\overline{z}_n)$ and $h(z') < h(z'') \le \limsup_{n\to\infty} h(\overline{z}_n)$. Since the sequence $(\overline{z}_n)_n$ converges to z_o, we get a contradiction.

□

Now, we have the following generalization of Weierstrass' Theorem in sequential spaces.

Theorem 7.2.2 *Let Z be a sequential space and let h be an extended real valued function defined on Z. The following statements hold:*

(i) If h is seq. transfer weakly upper continuous on Z and Z is seq. compact, then h has at least a maximum point on Z.

(ii) If h has at least a maximum point on Z, then h is seq. transfer weakly upper continuous on Z.

Proof. First, let us show the statement (i). Assume that h does not have a maximum point.
If $\alpha = \sup_{z \in Z} h(z) < \infty$, there exists a maximizing sequence $(z_n)_n$ such that $\alpha - \frac{1}{n} < h(z_n)$ for all $n \in \mathbb{N}$. Being Z seq. compact, there exists a subsequence $(z_{n_k})_k$ converging in Z to a point z_o. Since the function h does not have maximum points, there exists $z \in Z$ such that $h(z_o) < h(z)$. The function h is seq. transfer weakly upper continuous at z_o, therefore there exists $z' \in Z$ such that $\alpha = \limsup_{k \to \infty} h(x_{n_k}) \leq h(z')$, and we get a contradiction, since z' become a maximum point.
If $\alpha = \infty$, the same arguments can be used with the maximizing sequence $(z_n)_n$ such that $h(z_n) \geq n$ for all n.
The statement (ii) is obvious.

□

Moreover, we have the following corollary.

Corollary 7.2.1 *Let Z be a sequential space and h be an extended real valued function defined on Z. If h is sequentially upper quasicontinuous (or pseudocontinuous) on Z and Z is sequentially compact, then h admits at least a maximum point on Z.*

7.3 MaxSup and MaxInf Problems

Let X and Y be non-empty sets, f be an extended real value function defined on $X \times Y$ and K be a set-valued function from X to Y with non-empty values. With respect the data $\{X, Y, f, K\}$, we call: *sup-marginal function* the function v_{sup} defined on X by:

$$v_{sup}(x) = \sup_{y \in K(x)} f(x, y); \tag{7.1}$$

and *inf-marginal function* the function v_{inf} defined on X by:

$$v_{inf}(x) = \inf_{y \in K(x)} f(x, y). \tag{7.2}$$

In this section, we are interested in existence results for the following MaxSup and MaxInf problems:

$$\text{MaxSup} \ : \begin{cases} \text{find } x_o \in X \text{ such that:} \\ v_{sup}(x_o) = \max_{x \in X} v_{sup}(x) \end{cases}$$

and

$$\text{MaxInf} \ : \begin{cases} \text{find } x_o \in X \text{ such that:} \\ v_{inf}(x_o) = \max_{x \in X} v_{inf}(x) \end{cases}$$

In the setting of topological spaces, first existence results for these two problems are two theorems due to Berge.

Theorem 7.3.1 ([4]) *Let X and Y be topological spaces, f be an extended real valued function defined on $X \times Y$ and K be a set-valued function from X to Y with non-empty values. If:*

(i) K is upper semicontinuous with compact values;

(ii) f is upper semicontinuous;

then, the sup-marginal function v_{sup} is upper semicontinuous.

Theorem 7.3.2 ([4]) *Let X and Y be topological spaces, f be an extended real valued function defined on $X \times Y$ and K be a set-valued function from X to Y with non-empty values. If:*

(i) K is lower semicontinuous;

(ii) f is upper semicontinuous;

then, the inf-marginal function v_{inf} is upper semicontinuous.

Adding the hypothesis of compactness of X to previous theorems and using Weierstrass' theorem, one has existence results of solutions to MaxSup and MaxInf problems respectively.

About the existence of solutions to MaxInf problems, the hypothesis of upper semicontinuity on f (and also the compactness of X) has been relaxed in Aubin [1], when $K(x) = Y$ for all $x \in X$, and in Lignola and Morgan [26], when the constraint K is general. In [26], authors used a *gamma limit*. An exhaustive study on gamma limits is in De Giorgi and Franzoni [16] and in Dal Maso [12]. Let τ be the topology on X, σ be the topology on Y and let $(x, y) \in X \times Y$. We denote with $\tau(x)$ the family of all neighbourhood of x and with $\sigma(y)$ the family of all neighbourhood of y. The gamma limit $\Gamma(\tau^+, \sigma^-)$ of f at (x, y) used in [26] is defined by:

$$\Gamma(\tau^+, \sigma^-)f(x, y) = \sup_{J \in \sigma(y)} \inf_{I \in \tau(x)} \sup_{x' \in I} \inf_{y' \in J} f(x', y').$$

Theorem 7.3.3 ([1]) *Let X and Y be topological spaces and f be an extended real valued function defined on $X \times Y$. If:*

(i) the function $f(\cdot, y)$ is upper semicontinuous for all $y \in Y$;

(ii) there exists $y_o \in Y$ such that the function $f(\cdot, y_o)$ is upper semicompact;

then, MaxInf problem admits at least a solution.

Theorem 7.3.4 ([26]) *Let X and Y be topological spaces, f be an extended real valued function defined on $X \times Y$ and K be a set-valued function from X to Y with non-empty values. If:*

(i) $\Gamma(\tau^+, \sigma^-)f(x, y) \leq f(x, y)$ for all $x \in X$ and all $y \in K(x)$;

(ii) there exists $y_o \in Y$ such that the function $f(\cdot, y_o)$ is upper semicompact;

(iii) K has open graph;

then, MaxInf problem admits at least a solution.

If X and Y are sequential spaces, Lignola and Morgan present in [27] a characterization of seq. upper semicontinuity of the inf-marginal function v_{inf} when $K(x) = Y$ for all $x \in X$. In fact, the next theorem holds.

Theorem 7.3.5 ([27]) *Let X and Y be sequential spaces, f be an extended real valued function defined on $X \times Y$ and $x_o \in X$. Then, the inf-marginal function v_{inf} is sequentially upper semicontinuous at x_o if and only if the following statement holds:*
for all $y \in Y$ and all $x_n \longrightarrow x_o$, there exists a sequence $(y_n)_n \subseteq Y$ such that

$$\limsup_{n \to \infty} f(x_n, y_n) \leq f(x_o, y) \qquad (7.3)$$

When the constraint K is general, again in [27], a sufficient condition on f, weaker than seq. upper semicontinuity, is given in order to obtain v_{inf} seq. upper semicontinuous.

Theorem 7.3.6 ([27]) *Let X and Y be sequential spaces, f be an extended real valued function defined on $X \times Y$, K be a set-valued function from X to Y with non-empty values and $x_o \in X$.*
If K is sequentially open graph and if f satisfies the following property:
for all $y \in K(x_o)$ and all $x_n \longrightarrow x_o$, there exists a sequence $(y_n)_n \subseteq Y$ converging to y such that (7.3) holds,
then, the inf-marginal function v_{inf} is seq. upper semicontinuous at x_o.

Hypothesis in Theorems 7.3.5 and 7.3.6 and sequential compactness of X give sufficient conditions for existence of solutions to MaxInf problems in sequential spaces.

About the existence of solution to MaxSup problems, in topological or sequential spaces, the sufficient condition on f used until now is the upper semicontinuity (see [27] for sequential spaces). So, some question rises spontaneous: *Is it possible the obtain sufficient conditions for existence of solutions to MaxSup problems using classes of functions more general than upper semicontinuity? What is it happen for MaxInf problems?*

For to answer to above questions, first we observe that the minimal condition for existence of maximum points is not a sufficient condition neither for MaxSup nor for MaxInf problems, as shown in the following examples.

Example 7.3.1 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \frac{1}{2}(x+y) & \text{if } (x,y) \in [0,1[\times[0,1] \\ x-y & \text{if } (x,y) \in \{1\} \times [0,1] \end{cases}$$

and $K : [0,1] \longrightarrow 2^{[0,1]}$ defined by:

$$K(x) = [x,1] \qquad \text{for all } x \in [0,1].$$

The function f is transfer weakly upper continuous on $[0,1] \times [0,1]$ but the sup-value function v_{sup} does not have a maximum on [0,1]. Then, the corresponding MaxSup Problem does not have solutions.

Example 7.3.2 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the transfer weakly upper continuous function considered in Example 7.3.1 and $K(x) = [0,1]$ for all $x \in [0,1]$. The inf-value function v_{inf} does not have a maximum on $[0,1]$. Then, the corresponding MaxInf Problem does not have solutions.

Affirmative answers to the previous questions will be given in the next two sections, where the class of sequentially upper quasicontinuous functions (introduced in Paragraph 2.3) will be the tool able to relax the semicontinuity of f in both MaxSup and MaxInf problems.

7.4 Sequential Upper Quasicontinuity of v_{sup} and MaxSup Problems

In this section, following [38], in the setting of sequential spaces, we present sufficient conditions of minimal character on the data $\{X, Y, f, K\}$, first for the seq. upper quasicontinuity of the sup-value function v_{sup} defined by (7.1) and second for the existence of solutions to the MaxSup problem. So, we start with the following two theorems on the sequentially upper quasicontinuity of v_{sup}.

Theorem 7.4.1 *If K is sequentially closed and sequentially subcontinuous on X and f is sequentially upper quasicontinuous on $X \times Y$, then v_{sup} is sequentially upper quasicontinuous on X.*

Proof. We assume that $v_{sup}(x_o) < v_{sup}(x)$ and $\limsup_{n\to\infty} v_{sup}(x_n) > v_{sup}(x)$ for some sequence $(x_n)_n$ converging to x_o. Taken $\varepsilon > 0$ such that $\limsup_{n\to\infty} v_{sup}(x_n) > v_{sup}(x) + \varepsilon > v_{sup}(x)$, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a sequence $(y_k)_k$ such that: $v_{sup}(x) + \varepsilon < f(x_{n_k}, y_k)$ and $y_k \in k(x_{n_k})$ for all k. Since K is seq. subcontinuous and seq. closed at x_o, we have that there exists a subsequence $(y_{k_i})_i$ of $(y_k)_k$ that converges to a point $y_o \in K(x_o)$ and we obtain:

$$v_{sup}(x) + \varepsilon \leq \limsup_{i\to\infty} f(x_{n_{k_i}}, y_{k_i}) \tag{7.4}$$

On the other hand, $y_o \in K(x_o)$ and $v_{sup}(x_o) < v_{sup}(x)$ imply that there exists $\hat{y} \in k(x)$ such that $f(x_o, y_o) < f(x, \hat{y})$. By seq. upper quasicontinuity of the function f at (x_o, y_o), we have:

$$\limsup_{i\to\infty} f(x_{n_{k_i}}, y_{k_i}) \leq f(x, \hat{y}) \tag{7.5}$$

Combining (7.4) and (7.5) we obtain:

$$\limsup_{i\to\infty} f(x_{n_{k_i}}, y_{k_i}) < v_{sup}(x) + \varepsilon \leq \limsup_{i\to\infty} f(x_{n_{k_i}}, y_{k_i})$$

and we get a contradiction.

□

Theorem 7.4.2 *If K is sequentially closed and sequentially subcontinuous on X and f satisfies the following condition $\mathcal{C}_{sup}$:*

$$\left\{\begin{array}{c} f(x_o,y_o) < f(x,y) \\ y_o \in K(x_o), y \in K(x) \\ (x_n,y_n) \longrightarrow (x_o,y_o) \end{array}\right. \Rightarrow \left\{\begin{array}{l} \exists\, (y'_n)_n \subseteq K(x) \text{ and } n_o \in \mathbb{N} \text{ such that:} \\ f(x_n,y_n) \leq f(x,y'_n)\ \forall\, n \geq n_o \end{array}\right.$$

then v_{sup} is sequentially upper quasicontinuous on X.

Proof. We assume that $v_{sup}(x_o) < v_{sup}(x)$ and $\limsup_{n\to\infty} v_{sup}(x_n) > v_{sup}(x)$ for some sequence $(x_n)_n$ converging to x_o. Then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $v_{sup}(x) < v_{sup}(x_{n_k})$ for all k. Therefore there exists a sequence $(y_k)_k$ such that, for all k, $y_k \in K(x_{n_k})$ and:

$$f(x,y) < f(x_{n_k},y_k) \text{ for all } y \in K(x). \tag{7.6}$$

Since K is seq. subcontinuous at x_o, there exists a subsequence $(y_{k_i})_i$ of $(y_k)_k$ which converges to a point $y_o \in Y$. Being K seq. closed at x_o we have $y_o \in K(x_o)$ and, since $v_{sup}(x_o) < v_{sup}(x)$, $f(x_o,y_o) < f(x,y')$ for some $y' \in K(x)$. By condition $\mathcal{C}_{sup}$, there exists a sequence $(y'_i)_i$ in $K(x)$ such that:

$$f(x_{n_{k_i}},y_{k_i}) \leq f(x,y'_i) \text{ for } i \text{ sufficiently large.} \tag{7.7}$$

Combining (7.6) and (7.7), we get a contradiction. □

Note that the class of seq. upper quasicontinuous functions and the class of functions which satisfy condition $\mathcal{C}_{sup}$ are not the same, as shown in next two examples 7.4.1 and 7.4.2. Moreover, Example 7.4.2 shows also that the class of functions for which the condition $\mathcal{C}_{sup}$ holds does not coincide with the class of seq. transfer weakly upper continuous functions.

Example 7.4.1 Let $f : [0,1] \times [0,2] \longrightarrow \mathbb{R}$ defined by:

$$f(x,y) = \left\{\begin{array}{ll} 0 & \text{if } (x,y) \in [0,1]^2 \\ 1 & \text{if } x \in [0,1] \text{ and } y \in]1,2] \cap \mathbb{Q} \\ y & \text{if } x \in [0,1] \text{ and } y \in]1,2] \cap (\mathbb{R}\backslash\mathbb{Q}) \end{array}\right.$$

where $\mathbb{Q}$ denotes the set of all rational numbers and let $K(x) = [2x,2]$ for all $x \in [0,1]$ (note that the set-valued function K satisfies the assumptions of theorems 7.4.1 and 7.4.2). The function f is seq. upper quasicontinuous but it does not satisfy condition $\mathcal{C}_{sup}$ for $(x_o,y_o) = (0,1)$, $(x,y) = (1,2)$ and $(x_n,y_n) \longrightarrow (0,1)$ with $(y_n)_n \subseteq \mathbb{R}\backslash\mathbb{Q}$ and $y_n \longrightarrow 1^+$.

Example 7.4.2 Let $f : [0,2] \times [0,1] \longrightarrow \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,2] \times \{0\} \\ 2-y & \text{if } (x,y) \in [0,2] \times]0, \frac{1}{2}] \\ 1-xy & \text{if } (x,y) \in [0,2] \times]\frac{1}{2}, 1] \end{cases}$$

and $K(x) = [0,1]$ for all $x \in [0,2]$. Then f satisfies condition $\mathcal{C}_{sup}$ but it is not seq. transfer weakly upper continuous on $[0,2] \times [0,1]$ and so it is not seq. upper quasicontinuous either.

In light of Theorem 7.4.1 and Theorem 7.4.2, upper quasicontinuity and condition $\mathcal{C}_{sup}$ are sufficient conditions for existence of solutions to MaxSup Problems. In fact, we have the following result.

Theorem 7.4.3 *If X is sequentially compact, K is sequentially closed and sequentially subcontinuous on X and if f is sequentially upper quasicontinuous on $X \times Y$ (or f satisfies the condition $\mathcal{C}_{sup}$), then the MaxSup Problem has solutions.*

Proof. It is sufficient to apply Theorem 7.4.1 (or Theorem 7.4.2) and Corollary 7.2.1.

□

7.5 Sequential Upper Quasicontinuity of v_{inf} and MaxInf Problems

In this section, following [38], we present, in sequential spaces, sufficient conditions of minimal character on the data $\{X, Y, f, K\}$ for the sequential upper quasicontinuity of the inf-marginal function v_{inf} defined by (7.2) and for the existence of solutions to the MaxInf Problem. First, we consider a generic set-valued function K and we show that the sequential upper quasicontinuity of f implies the sequential upper quasicontinuity of the inf-marginal function v_{inf} (under a suitable condition on K). Then, we improve the previous result when $K(x) = Y$ for any $x \in X$.

Theorem 7.5.1 *Assume that f is sequentially upper quasicontinuous on $X \times Y$ and K is sequentially lower semicontinuous on X. Then, the inf-marginal function v_{inf} is sequentially upper quasicontinuous on X.*

Proof. Let x_o and x in X such that $v_{inf}(x_o) < v_{inf}(x)$ and $x_n \longrightarrow x_o$. There exists $y_o \in K(x_o)$ such that $f(x_o, y_o) < f(x, y)$ for any $y \in K(x)$. Since K is seq. lower semicontinuous at x_o, there exists a sequence $(y_n)_n$ converging to y_o with $y_n \in K(x_n)$ for n sufficiently large. Being f seq. upper quasicontinuous at (x_o, y_o), we have $\limsup_{n\to\infty} f(x_n, y_n) \leq f(x, y)$. Therefore, $\limsup_{n\to\infty} v_{inf}(x_n) \leq f(x, y)$ for any $y \in K(x)$ and the thesis follows. □

We note that the condition on the function f considered in Theorem 7.3.6 ([27]) is not connected with seq. upper quasicontinuity, as it is shown in the next example.

Example 7.5.1 Let $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} 2(1-x) & \text{if } (x,y) \in [0,1[\times[0,1/2] \\ 2y(1-x) & \text{if } (x,y) \in [0,1[\times]1/2,1] \\ -1 & \text{if } (x,y) \in \{1\} \times [0,1] \end{cases}$$

and $K(x) = [0, 1]$ for any $x \in [0, 1]$. The assumptions of Theorem 7.5.1 are satisfied but the inf-marginal function v_{inf} is not seq. upper semicontinuous, so the condition on f in Theorem 7.3.6 is not verified.

Concerning existence of solutions to the MaxInf Problem, one has the following result.

Theorem 7.5.2 *Assume that f is sequentially upper quasicontinuous on $X \times Y$ and K is sequentially lower semicontinuous on X sequentially compact. Then, the MaxInf Problem has at least a solution.*

Proof. It is sufficient to apply Theorem 7.5.1 and Corollary 7.2.1. □

Now we analyze the case in which $K(x) = Y$ for all $x \in X$. We have the following theorem.

Theorem 7.5.3 *Assume that the following condition $\mathcal{C}_{inf}$ is satisfied:*

$$\begin{cases} f(x_o, y_o) < f(x, y) \\ x_n \longrightarrow x_o \end{cases} \Rightarrow \begin{cases} \exists\ (\bar{y}_n)_n \subseteq Y \text{ such that:} \\ \limsup_{n\to\infty} f(x_n, \bar{y}_n) \leq f(x, y) \end{cases}$$

then v_{inf} is sequentially upper quasicontinuous on X.

Proof. Let x_o and x be two elements of X such that $v_{inf}(x_o) < v_{inf}(x)$ and $(x_n)_n$ be a sequence converging to x_o. There exists $y_o \in Y$ such that $f(x_o, y_o) < f(x, y)$ for all $y \in Y$. Let $y \in Y$. In light of $\mathcal{C}_{inf}$, there exists a sequence $(\bar{y}_n)$ in Y such that $\limsup_{n\to\infty} f(x_n, \bar{y}_n) \leq f(x, y)$. So $\limsup_{n\to\infty} v_{inf}(x_n) \leq f(x, y)$ for all $y \in Y$ and the thesis follows.

□

The condition on f used in Theorem 7.3.5 implies $\mathcal{C}_{inf}$ but the converse is not true, as shown by the following example.

Example 7.5.2 Let $f : [0, 2] \times [0, 1] \longrightarrow \mathbb{R}$ defined by:

$$f(x, y) = \begin{cases} y - x & \text{if } (x, y) \in [0, 1[\times[0, 1] \\ xy - 3 & \text{if } (x, y) \in [1, 2] \times [0, 1] \end{cases}$$

Then the function f verifies $\mathcal{C}_{inf}$ but not the condition used in Theorem 7.3.5 ([27]).

Finally, to conclude the section, one can apply Theorem 7.5.3 and Corollary 7.2.1 in order to obtain an other existence result for the solutions to the MaxInf problem.

Theorem 7.5.4 *If f satisfies condition $\mathcal{C}_{inf}$, X is sequentially compact and $K(x) = Y$ for all $x \in X$, then the MaxInf Problem has at least a solution.*

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