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To my parents, my sisters, my family, my friends and my colleagues...
To all those who made this journey magical!

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> «Showing gratitude is one of the simplest yet most powerful things humans can do for each other.»
>
> \- *Randy Pausch*

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Konstantinos F. Dialektopoulos
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Preface

> «The answer to all these questions may not be simple. I know there are some scientists who go about preaching that Nature always takes on the simplest solutions. Yet the simplest by far would be nothing, that there would be nothing at all in the universe. Nature is far more interesting than that, so I refuse to go along thinking it always has to be simple.»
>
> - *Richard Feynman*

Undoubtedly, the two great pillars of modern physics are General Relativity and Quantum Mechanics. Even though they were both proposed in the beginning of the 20th century by Albert Einstein and Max Planck respectively, they are still considered modern for two reasons. The first one is that, they brought a revolution in the way we see and describe nature, and thus we wanted to distinguish them from the more *classical* and established ideas. The second one is that, they are both describing concepts and events that are quite unfamiliar to us in everyday life and in this sense, they feel modern.

General Relativity suggests that gravity is no longer a force, as Newton proposed, but instead, it is just the effect of geometry, i.e. the curvature of spacetime that causes objects to fall down to earth, planets to move around stars and stars to form larger structures, such as galaxies. In addition, time is not anymore an absolute notion; the measure of time strictly depends on the position in a gravitational field, and together (time and position) they form the notion of spacetime. Quantum Mechanics on the other hand, tells us, among others, that there is no way to know simultaneously the position and the momentum of an object. Both theories have been stringently tested from experiments and observations and, even if they present some shortcomings and they are completely incompatible with each other, they are considered the most convincing explanations describing the physical world.

The topic of this thesis however, has to do with gravitational theories, so we will focus on such theories from now on. Even from the very beginning, i.e. when Einstein introduced his theory, he proposed several tests that would verify its validity. These were the "anomalous" precesion of the perihelion of Mercury, the deflection of light by the sun and the gravitational redshift of light. Later, many were those that followed:

gravitational lensing, light travel time delay, frame-dragging effects, as well as binary pulsars, X-ray spectroscopy and finally the direct detection of gravitational waves two years ago.

Far beyond astrophysical scales, general relativity is still very successful, as part of the so-called standard model in cosmology or better the *concordance model*, Λ Cold Dark Matter (ΛCDM). This model has also passed all the cosmological observation tests with flying colors. The accelerating expansion of the Universe through supernovætype Ia, the polarization of the Cosmic Microwave Background radiation (CMB) and the large scale structure of the Universe are only some of them.

It seems though, that apart from its great success, it is not the final theory of gravity. There are hints both from the theoretical and from the observational point of view, that prove the malfunction of general relativity. First and foremost, the inability to reconcile gravity with general relativity to a single theory of the fundamental interactions, is puzzling the physical community for many years. This is mainly because, Quantum Field Theory assumes the spacetime to be flat or, at least non-dynamical[1], while in general relativity the spacetime is unavoidably dynamical and the quantum nature of matter fields is not taken into account.

Furthermore, the existence of black holes is predicted by general relativity and even though, physicists where certain for their existence (because of many indirect observations) for more than a century, they were directly observed two years ago, through the detection of gravitational waves. However, in the very center of a black hole, the theory predicts the existence of a singularity: a "point" in spacetime where the gravitational field becomes infinite. Apart from the black hole singularity, there exist some cosmological singularities too, with the most important being the one at the beginning of the Universe, the Big Band singularity. Obviously, physicists are not comfortable dealing with infinite quantities, since most of the time they are unphysical.

At cosmological scales now, there exists the so called problem of the dark sector, i.e. the nature of dark matter and dark energy. It is known from various observations, that the Universe is undergoing a phase of accelerating expansion. The ΛCDM model assumes the existence of the cosmological constant, Λ, which is associated with the energy density of space (vacuum energy). However, the observed value of Λ disagrees with the theoretical prediction by 120 orders of magnitude, consisting the biggest open problem in cosmology. Furthermore, the value of the Hubble constant according to Planck measurements [1] is about $H_0 = 67.8 \pm 0.9$km/s/Mpc. However, according to Hubble Space Telescope [2] its value is about $H_0 = 73.24 \pm 1.74$km/s/Mpc. Obviously, there is a discrepancy of

[1] In Quantum Field Theory in curved spacetime, the spacetime is, obviously, non-flat, but it is still considered a fixed arena where quantum fields live.

almost 3σ between them and many cosmologists think that this number will grow in the presence of new data [3]. In addition, the model suggests that approximately 85% of the matter content in the Universe consists of an unknown form of matter, that does not interact with the baryonic matter that we see. No interaction with photons or any other known particles means that it is very difficult to observe. Obviously, there are many proposals in the literature of what the nature of dark matter is, but to date no particle candidate has been found. More details about the current problems of general relativity will be presented in the introduction.

Based on the above, many scientists started pursuing a better explanation for the gravitational interactions in nature. But this is not new; Einstein himself wanted to encompass his theory with Electromagnetism and thus started looking for unifications, i.e. more general theories of gravity. Since then, many attempts for modifications have been made; Eddington, Weyl, Kaluza and Klein, Dirac, Stelle, Brans and Dicke are only few of the ones tried to find a more complete theory of gravity. Some examples of the theories proposed (not in a chronological order) are: string theories, nonlocal theories, scalar-tensor theories, teleparallel theories, higher order theories, emergent approaches (Causal dynamical triangulation, Padmanabhan thermodynamic approach, Verlinde's entropic gravity) are more. The point of this thesis is not to review all the theories that have been studied. This has been done extensively in the literature. What we do here is to discuss a point of view that is not so well studied yet. This is the three conceptually different, but equivalent formulations of gravity.

The thesis is split into three parts. The first one tries to focus in the fundamental geometric structure on which theories of gravity are built. We start by considering the most general connection on a 4-dimensional manifold and from that we derive the three equivalent descriptions of the gravitational interactions, i.e. General Relativity, Teleparallel Equivalent of General Relativity and Symmetric Teleparallel of General Relativity. The first one describes gravity as the effect of the curvature of spacetime, the second one of its torsion and the third one of its non-metricity.

The last decade a plethora of theories has been proposed in the literature and we have to start discriminating between those. Symmetries play a very important role in field theories and can certainly help us with that task. In the second part of the thesis, I present a geometric criterion using Lie and Noether symmetries of differential equations, to select those theories of gravity that are invariant under point transformations. Using the invariant functions of these symmetries one can reduce the dynamics of the system and find exact solutions.

Many applications in the cosmological minisuperspace are also included in the second part. However, every new theory should be consistent with observations at astrophysical scales too. For this reason, in the last part of the thesis, we use the maximum turnaround radius of large scale structures as a stability criterion to constrain theories of gravity.

The author's ultimate goal is, that this thesis will contribute as a guide to future students and researchers who want to study the foundations of gravity. It is more than certain, that this thesis will contain trivial mistakes and typos that the author did. If the reader finds something like this, he is adviced to contact the author directly. Nevertheless, I hope the reader will enjoy these pages and/or to be inspired by them, as much as the author did writing them.

List of publications

During this PhD thesis I published several papers with different groups. However, not all of them are included in this dissertation, since some of them are not relevant to its topic. For a full list of publications check my profile in inspire-hep.

The following list contains the papers that are published or are still under review, are relevant to this PhD thesis and were written during the PhD program.

1. *Noether symmetries in Gauss-Bonnet-teleparallel cosmology.*
   Salvatore Capozziello, Mariafelicia de Laurentis, and **Konstantinos F. Dialektopoulos**, Eur.Phys.J. C76 (2016) no.11, 629 ; [arXiv:1609.09289]

2. *The maximum sizes of large scale structures in alternative theories of gravity.*
   Sourav Bhattacharya, **Konstantinos F. Dialektopoulos**, Antonio Enea Romano, Constantinos Skordis and Theodore N. Tomaras, JCAP 1707 (2017) no.07, 018 ; [arXiv:1611.05055]

3. *Constraining Generalized Non-local Cosmology from Noether Symmetries.*
   Sebastian Bahamonde, Salvatore Capozziello and **Konstantinos F. Dialektopoulos**, Eur.Phys.J. C77 (2017) no.11, 722 ; [arXiv:1708.06310]

4. *Classification of the Horndeski cosmologies via Noether Symmetries.*
   Salvatore Capozziello, **Konstantinos F. Dialektopoulos** and Sergey V. Sushkov, Eur.Phys.J. C78 (2018) no.6, 447; [arXiv:1803.01429]

5. *Maximum turnaround radius in f(R) gravity.*
   Salvatore Capozziello, **Konstantinos F. Dialektopoulos** and Orlando Luongo, Int. Jour. Mod. Phys. D ; [arXiv:1805.01233]

6. *Noether symmetries as a geometric criterion to select theories of gravity.*
   Konstantions F. Dialektopoulos and Salvatore Capozziello, IJGMMP ; [arXiv:1808.03484]

7. *Generalized Symmetric Teleparallel Theories of Gravity.*
   Salvatore Capozziello, **Konstantinos F. Dialektopoulos** and Tomi Koivisto, *in preparation.*

8. *Spherically Symmetric Solutions in pure $f(\mathcal{G})$ gravity.*
   Francesco Bajardi, **Konstantinos F. Dialektopoulos** and Salvatore Capozziello, *in preparation.*

Finally, I was part of the *white paper* for the Cost Action "Gravitational Waves, Black Holes and Fundamental Physics".

1. *Black holes, gravitational waves and fundamental physics: a roadmap.*
   Leo Barack, *et. al.*, arXiv:1806.05195

Contents

Chapter 1

Introduction

«If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things.»

- René Descartes, Principia Philosophiæ

1.1 Gravity in the march of history

Gravity is one of the four fundamental interactions in nature, among electromagnetism, and the weak and strong nuclear forces. Even if, it is the weakest of all, gravity (and electromagnetism) is directly observed in everyday life. Due to this, it was the first of the four that was studied experimentally. However, until today, it still remains the least understood and most puzzling compared to the other three.

It was Aristotle who first thought that objects with different masses should fall at different rates, but Galileo disproved this hypothesis, showing experimentally that all the objects accelerate toward the Earth uniformly (if we neglect resistant forces). He may not have conducted the well-known experiment from the Tower of Pisa, however, he did many other similar ones using inclined planes, pendulums and even telescopes, making himself the "father of the scientific method".

A consistent theory of gravity though, was proposed by Newton in 1687. His inverse-square law of universal gravitation will be the best description of the gravitational interactions for more than 300 years, until Einstein introduces the general theory of relativity. What Newton's theory says is that, every object attracts every other object with a force, which is inversely proportional to their distance squared and proportional to the product of their masses. This description is so accurate (and also simpler, compared to general relativity), that even nowadays, is used in many applications where the masses, the speeds and the energies of the objects studied, are sufficiently far from the relativistic limit.

The most important thing regarding Newton's theory of gravity, is the conceptual ideas introduced with it, behind its mathematical formulation. To be more specific, Newton thought that *space* and *time* are two absolute entities, which practically means that, all the physical phenomena take place in an unaffected (non-dynamical) background. Moreover, he also believed that gravitational and inertial masses coincide, which later became known as the Weak Equivalence Principle.

Seemingly any theory that is consistent and can describe physical phenomena is correct; so whether or not Newton's theory of gravitation is right, would not be so well-posed question. It would be more correct to ask how much of the physical world a theory can describe; and if there exist more than one theories describing the same phenomena, then the one with the fewest assumptions is the right one (Occam's razor).

As far as Newton's theory is concerned, it established itself soon after its appearance in the scientific community, but Newton was not so comfortable with what his theory was implying. Using his own words to Bentley, "that one body may act upon another, at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it". Indeed, the inability of his theory to comply with Mach's principle[1], *local physical laws are determined by the large-scale structure of the universe* [4], was one of the biggest problems of the theory.

The idea of the absolute space, together with the failure of the theory to explain the excess precession of Mercury's orbit, were some of the reasons that made Einstein want to find another description for the gravitational interactions. He became sure after 1905, when he completed the Special Theory of Relavity, which explained a series of phenomena, but was not compatible with Newton's theory. It was then, when he started working intensely towards a gravity theory consistent with Special Relativity, and he managed to do it in 1915.

It is remarkable how good it fitted the observations (Mercury's precession, Lense-Thirring effect, deflection of light by the Sun and more) and how successful it still is. At cosmological scales it passed with flying colours all the observetional tests; from the redshift of type Ia supernovæ, to the Hubble rate, to the cosmic microwave background and the large scale structure in the Universe. This, however, does not make Newton's theory wrong. It is indeed less valid compared to General Relativity, but even Einstein's theory in the weak field limit, i.e. where the masses/energies and the speeds of particles/objects are sufficiently small, reduces to Newtonian gravity.

[1] Nowadays, when we refer to Mach's principle, we usually mean the physical sense that Einstein made out of an imprecise idea of Ernst Mach. R. H. Dicke reformulated Einstein's statement too.

The whole idea we had about gravity may have changed thanks to Einstein, but some of the vital elements of Newton's theory are still valid and have been incorporated to the new theory.

It may be the case, that today, we are facing similar problems as in the beginning of the 20th century. Back then, it was Newton's theory under the microscope of investigation, today it is General relativity. Even though successful with many observations, it contains some shortcomings and we are called to explain them, either in the framework of this theory or by changing fundamentally the picture we have about gravitational interactions. In the following sections of this chapter, we will review the success of General Relativity (GR from now on) and we will stress the questions we are supposed to answer.

1.2 General Relativity and its mysteries

GR is a metric theory and it gives a geometric description of gravitational interactions in the Universe. It generalizes Special Relativity (SR) and Newton's theory of gravity and it provides a general picture of gravity as a geometric property of spacetime. Specifically, gravity is mediated through the curvature, i.e. the deviation of spacetime from flatness, and this is directly related with the matter and energy content of the Universe. Mathematically, this relation is described by the Einstein field equations. In this sense, i.e. the spacetime structure is in relation with the matter in the Universe, GR is in accordance with some of Mach's ideas.

It has been more than 100 years since Einstein first introduced GR and since then it has passed many tests at different scales. Specifically, at astrophysical scales, it explains the orbits of the planets and other self-gravitating systems, it passes the Solar System Tests and reduces to the Newtonian theory in the weak field limit. At larger scales, it addresses the galactic dynamics as well as the clustering of galaxies and it explains, in a consistent way, the cosmological observations, from the cosmic microwave background radiation, to the late-time acceleration of the Universe and the redshift of type Ia supernovæ. Moreover, it implies the existence of black holes, predicts gravitational waves, explains the gravitational lensing and redshift and more.

In what follows, we will review the success of GR at astrophysical and cosmological scales, in more detail; we will discuss the possibility and the need to extend its validity at smaller scales and we will point out its weak points, i.e. those phenomena and concepts that are still under investigation.

1.2.1 Astrophysical implications

It was only two months after Einstein introduced his theory, and specifically on January 1916, when Karl Schwarzschild, a German physicist, managed to solve Einstein's field equations in vacuum for a spherically symmetric non-rotating mass. When Einstein read Schwarzschild's paper, he wrote to him: "[...] I had not expected that one could formulate the exact solution of the problem in such a simple way. I liked very much your mathematical treatment of the subject [...]". For many years physicists thought that Schwarzschild solution is nothing more but a mathematical construction. From the beginning of the sixties though, and especially after the discovery of pulsars and their identification with rapidly rotating neutron stars, not only black holes stopped being considered as theoretical curiosities, but also the whole theory of general relativity entered its "Golden age" [5] and became part of theoretical physics.

It was then, between 1960-1975, that many exact solutions of the Einstein's equations had been found. R. Kerr found an exact solution of a rotating black hole; E. Newman found a solution of a rotating and electically charged black hole. In addition, W. Israel, B. Carter and D. Robinson came up with the, later called, no-hair theorem which states that any stationary black hole can be described only by three parameters, its mass, its angular momentum and its charge. In practice, this means that any two black holes having these parameters in common, cannot be distinguished.

The black holes are created in the end of the life circle of a massive star, and are regions in the Universe where spacetime is so curved that not even light can escape. So a good question would be, if nothing can escape, then how are we able to know the three parameters mentioned above? Because of the fact that, the above properties correspond to long-range gauge fields, they can be observed indirectly from the outside of the black hole. More explicitly, the mass can be calculated from the so-called ADM mass (Arnowitt, Desser, Misner), which is the gravitational analog of the Gauss's law [6]. Moreover, as any other object with electromagnetic charge, the black hole would repel other objects with the same charge and finally, the angluar momentum of a rotating black hole could be measured by the frame-dragging of the gravitational field, which is the distortion of the spacetime around a rotating mass.

We already said that nothing, not even light can escape from the black hole, but what is the boundary that disconnects the outer spacetime, from the black hole. This surface (or to be more precise the 3-dimensional hypersurface) that acts as a boundary between the inner and the outer regions of spacetime, is called the event horizon of a black hole and it got its name from the fact that if an event happens inside it, an observer outside from this will never know it occured, because the information cannot escape. An interesting phenomenon that takes place near a black hole is that, clocks appear to tick more slowly for an observer far away from it; this is known as time dilation. Of course, observers moving towards the black hole do not notice any difference on the ticking of their clocks and they cross the event horizon without noticing anything peculiar.

For many years, it was believed that Schwarzschild black hole, i.e. the simplest form of a black hole, which is spherically symmetric has a singularity at the event horizon. However, it was proven in the mid-twenties that this was only a coordinate artifact and there was no singularity there; this can be seen by making a proper coordinate transformation. On the contrary, at the center of the black hole, there exists a singularity which cannot be removed by a mathematical trick. For a static black hole the singularity has a point shape while for a rotating black hole its shape is a ring. In both cases though, this region contains the total mass of the black hole and zero volume, thus having infinite density [6, 7].

The existence of singularities in GR is still an open issue in theoretical physics and it is under investigation. The fact that the curvature and the density at a specific region is infinite does not seem so physical and thus this is usually seen as a breakdown of the theory [8]. It is believed though, that in a theory where quantum effects could be incorporated in a theory of gravity, singularities would not exist.

Apart from black holes, GR implies also the existence of new phenomena that its predecessor did not. At astrophysical scales, GR can explain how matter could bend the light of a distant source, as it travels towards an observer. We call this effect gravitational lensing, since the matter distribution, i.e. galaxies or clusters of galaxies, acts as a lens between the source and the observer. How much the beam of light deviates from its initial direction, or as we call it, the deflection of light can be calculated in the context of GR, but unlike an optical lens, in gravity, the closer the light passes to the center of the structure the more it will be deflected. Depending on the size of the structure there exist three types of gravitational lensing: the strong, the weak and the microlensing.

What is more, GR predicts the existence of gravitational waves (GW). These are ripples in the fabric of spacetime that are produced by accelerated masses and propagate as waves with the speed of light. Just as in electromagnetism, GW's transport energy in the form of radiation and they were not detected before 2016, i.e. 101 years after their prediction. They come in two polarization modes (in the framework of GR), plus and cross and they can penetrate regions of spacetime the electromagnetic waves cannot.

The first gravitational wave signal, GW150914, was observed on February 2016 [9] and it was produced by a binary black hole merger that happened a billion years ago. When it crossed the Earth, it changed the length of a 4km arm of LIGO (Laser

Interferometer Gravitational-Wave Observatory) by a thousandth of the width of a proton. Since then, we observed six more signals such as this one. Interestingly, the last one, GW170817, was produced a binary neutron star merger [10] and it was the only one which was observed also in its electromagnetic spectrum, initiating thus the field of multi-messenger astronomy [11].

1.2.2 Cosmological implications

Apart from the role at astrophysical scales, gravity is at great importance especially at cosmic scales. We expect that the larger the scale, the more powerful gravity becomes, since all the other interactions are tend to vanish at large scales. The cornerstone of modern cosmology is the idea that the position in which we are in the Universe is in no way special. This is known as the *cosmological principle* and practically, it states that the spatial distribution of matter in the Universe is homogeneous and isotropic at large scales. Of course this is by no means exact, but it holds as a very good approximation the larger the scales we consider (let alone the whole Universe) and it breaks down at local phenomena. In the rest of this section we will see how cosmology is described in the framework of GR under the ΛCDM (Λ Cold Dark Matter) concordance model.

The cosmological principle has a straightforward mathematical description. Regarding the geometry of the Universe, we have to assume that the curvature thoughout it is the same and the matter content is described by a perfect fluid, and the only parameters needed for its description are the uniform energy density ρ and the uniform pressure p. Let us now see, how the equations describing the dynamics of the Universe, or else Friedmann equations look like. More details about the geometric structure of the theory will be given in the next chapter and also could be found in numerous books in the literature [4,6–8]. However, for what follows basic notions of tensor calculus, differential geometry and general relativity are considered known.

The energy-momentum tensor of a perfect fluid is given by

$$T^{\mu\nu} = (p + \rho)\, u^\mu u^\nu + p g^{\mu\nu}\,, \tag{1.1}$$

where u^μ is the 4-velocity of an observer comoving with the fluid and $g_{\mu\nu}$ is the metric of the spacetime. The Einstein field equations are

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}\,, \tag{1.2}$$

with G being the Newton's constant, Λ being the cosmological constant and $G^{\mu\nu}$ the Einstein tensor, which is given by

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\,, \tag{1.3}$$

where $R^{\mu\nu}$ and R are the Ricci tensor and scalar respectively. $T^{\mu\nu}$ is the energy-momentum tensor for the matter and in our case it is given by (1.1).

The metric that describes the homogeneity and isotropy of the spacetime is called Friedmann-Lemaître-Robertson-Walker and in spherical coordinates it is given by

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right] . \tag{1.4}$$

$a(t)$ is the scale factor and denotes the relative expansion of the Universe, $d\Omega^2$ is the metric of a 2-sphere, i.e. $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ and k is a constant representing the curvature of space; it can be taken either as a dimensionless number taking the values $k = -1$ if the spatial hypersurface of constant t has negative curvature, $k = 0$ if it is flat and $k = 1$ if it has positive curvature or to have units of $[L]^{-2}$.

By plugging (1.1) and (1.4) into (1.2) we get the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} , \tag{1.5a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} . \tag{1.5b}$$

Considering that the Universe is homogeneous and isotropic at large scales, and this stays unaffected in time, the only parameter left to evolve is its size, i.e. the scale factor $a(t)$. Indeed, if we impose a specific relation between the energy density and the pressure of the perfect fluid, or else an equation of state, we can solve the equations (1.5a), (1.5b) for the scale factor and find the evolution of the Universe.

It is very often considered that, cosmological fluids obey the following equation of state $p = w\rho$, where w is a dimensionless parameter. Hence, the continuity equation yields

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \frac{\dot{\rho}}{\rho} + 3(w+1)\frac{\dot{a}}{a} = 0 \Rightarrow \rho \propto a^{-3(w+1)} . \tag{1.6}$$

A matter-dominated Universe (or else *dust*) consists of non-relativistic, collisionless particles, whose pressure, compared to their energy density, is negligible. This means that $w = 0$ and thus $p_M = 0$. Moreover, from Eq. (1.6) we see that its energy density decreases as

$$\rho_M \propto a^{-3}, \tag{1.7}$$

or else that as the Universe expands the number density of the particles decreases. In radiation-dominated universes, where radiation could be either electromagnetic or any other particle moving close to the speed of light, it is $w = -1/3$ and $p_R = -\rho_R/3$. From Eq. (1.6) we find that $\rho_R \propto a^{-4}$, which practically means that photons' energy density falls off faster compared to dust; this happens because they redshift. Last but not least, the equation of state $p_\Lambda = -\rho_\Lambda$, i.e. $w = -1$ describes the vacuum energy of the universe and from the continuity equation we find that (in this case we consider k in (1.4) to have units of length^{-2})

$$\rho_\Lambda \propto 1\,. \tag{1.8}$$

This means that as long as the universe expands, the energy density of matter and radiation decreases, but vacuum energy wins in the long run and we call these universe, vacuum-dominated, or in the context of the concordance model, where Λ is associated with dark energy, dark energy-dominated.

Nowadays the consensus in modern cosmology is that, only (less than) 5% of the total energy density of an FLRW Universe, in the framework of ΛCDM model, is the matter-energy that we see, i.e. the baryonic matter. Everything else, namely the rest 95% must be something unknown, *dark*, in order to fit the observations. More explicitly, the 25% of this dark sector is identified as *dark matter*, a non-interacting and non-relativistic form of matter, while the rest 70% is called *dark energy*, a non-clustering form of energy density with a negative equation of state, as we saw above, causing accelerated the expansion of the Universe.

Chronologically, dark matter was proposed much earlier compared to dark energy [12]. Fritz Zwicky was examining the Coma cluster in the early thirties when he used the virial theorem[2] to infer the existence of existent matter which could not be seen, dark matter. Specifically, he calculated the graviational mass of the cluster and after comparing it with what he expected from its luminosity, he realized it should be 400 times larger. Thus, he deduced that the missing matter must be dark [13].

Another hint for the existence of dark matter is the rotation curves of galaxies. Galaxies can be thought of as point masses in the center and test particles orbiting around these point masses. If so, from Kepler's second law, we can deduce that the rotation velocities of the test particles will decrease the further they are from the center. However, what we observe is that the galactic rotation curves remain flat as we approach the "edge" of galaxies. Effectively, this means that apart from the luminous mass of the galaxy, there should exist another form of *dark* mass, in the sense that it does not interact with baryonic matter, in the outskirts of the galaxy.

Furthermore, as we already discussed before, one of the predictions of GR is the gravitational lensing, which is the phenomenon where a massive object between an observer and a source, acts as a lens bending the light coming from the source. Both

[2]For a stable system of N particles, bound by potential forces, the average kinetic energy over time is related with its average potential energy under the following folrmula

$$\langle T \rangle = -\frac{1}{2}\sum_{k=1}^{N} \langle \mathbf{F}_k \cdot \mathbf{r}_k \rangle\,,$$

with $\mathbf{F}_k$ being the force on the kth particle, located at $\mathbf{r}_k$.

weak and strong graviatational lensing [14–17] indicate the presence of dark matter in galaxies and galaxy clusters.

There are other hints for the existence of dark matter too. From the angular power spectrum of the CMB anisotropies we can deduce the density of dark matter, since the imprints that it leaves on the CMB are different from those of baryonic matter [1]. In addition, the effect of dark matter not to interact with radiation, makes its existence more probable, because it can allow density perturbations in the early universe to have enough time, in order for structures, such as galaxies and cluster of galaxies, to be formed. The collision of two clusters of galaxies resulted the, well-known, Bullet cluster; however, its apparent center of mass is far displaced from the baryonim center mass [18] and dark matter can explain this observation.

Dark matter has been around for almost a century now and it was integrated in the Big Bang evolutionary model of the Universe, ΛCDM, in the beginning of the eighties. Since then, there have been many proposals about its nature. The candidate scenarios are three: hot, warm and cold dark matter. The best particle candidate for hot dark matter is the neutrino, and for warm dark matter the sterile neutrino. The most successful scenario, however, seems to be that of Cold Dark Matter, i.e. a very weakly interacting, non-relativistic form of matter, and the possible candidates are the axions, the MACHO's (MAssive Compact Halo Objects) and the WIMP's (Weakly Interacting Massive Particles). It is not our task to review neither the properties nor the success of each one of the candidates. For a review of the current status of dark matter the interested reader should check [19, 20].

Apart from its nature another deficit of the theory, is its inability to explain the observed *coincidence*, between the baryonic energy density and the dark matter energy density. Why two totally different forms of matter, with different production mechanisms, have almost the same densities? Other problems, related to dark matter are: the cusp problem [29], i.e. the fact that galaxies are observed to have a flat energy density of CDM ρ_{CDM}, but N-body and other simulations give $\rho_{CDM} \sim r^{-\alpha}$ with $\alpha \sim 0.7-1.5$; the missing satelite problem, i.e. the fact that simulations signify the existence of about 500 sattelite dwarf galaxies around the Milky Way, but only 30 such galaxies have been observed. Just as important is, the apparent relation between dark matter and baryonic matter in galaxies, dictated in spiral galaxies by the Tully-Fischer [31] and in elliptical galaxies by the Faber-Jackson relation [32], since in the framework of the concordance model such relations should not be observed.

It is indeed believed in the community, that in the near future one of the particle candidates will be detected either directly or indirectly. However, expectations are not always fullfilled and thus many skeptic scientists, based on all the above shortcomings of the theory, try to find an alternative explanation for the "missing matter" problem.

As far as dark energy is conserned, its notion has been around since the eighties [21] and it is the most accepted hypothesis to explain the accelerated expansion of the Universe. Supposing that GR is correct, then the role of dark energy is played by the cosmological constant Λ [22,23], which acts like an intrinsic, fundamental energy of space. Einstein thought that the Universe must be static and since his equations dictated a gravitational collapse from the concentration of mass, he added by hand a Λ term to oppose to the gravitational force and keep the Universe static. However, not many years after, Edwin Hubble discovered that the Universe is expanding, and Einstein called the cosmological constant term "the biggest blunder of his life".

An expanding universe is very well described by the de Sitter solution of the Einstein field equations in vacuum, named after Willem de Sitter who was probably the first to study an acceleratingly expanding universe [33]. A de Sitter space is the analog in Minkowski space, of a sphere in Euclideal space and therefore, it is a Lorentzian manifold with positive curvature, which is maximally symmetric and simply connected. From the Friedmann equations (1.5a), (1.5b) we see that if we set $k = 0$, i.e. a flat universe for simplicity, and also $T_{\mu\nu} = 0$, then the scale factor takes the form

$$a(t) = e^{Ht}\,, \tag{1.9}$$

with the constant $H = \sqrt{\frac{\Lambda}{3}}$ being the Hubble expansion rate. But even if, we do not neglect the matter content in the Friedmann equations, i.e. $p \neq 0 \neq \rho$ in (1.5a), (1.5b), we notice that at late times the cosmological constant will become dominant, since its energy density remains constant (1.8), while that of matter falls off as a^{-3} (1.7). Practically, this means that asymptotically the Universe will approach a de Sitter space.

The presence of an energy density with negative equation of state, i.e. $w = -1$, is favoured [24–26] compared to a flat, matter dominated or a universe with negative curveture, by supernovæ type Ia measurements. Moreover, such a model is also consistent with CMB anisotropies [1], combined with SDSS data [27]; with the Integrated-Sachs-Wolfe effect of CMB [28], and other observations.

What's more, one can move the cosmological constant term at the right hand side of Einstein's field equations (1.2). If one considers a perfect fluid with energy momentum tensor

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left[\Lambda, -\Lambda, -\Lambda, -\Lambda\right]\,,$$

namely, $w = -1$, then the cosmological constant can be in a way regarded as matter term, thus serving as the vacuum energy of matter fields. More precisely, it would be effectively

$$\Lambda = 8\pi G \rho_{\Lambda}\,,$$

and from the Standard Model of particle physics one could specify its value. Omitting the details [34], by taking the bound of validity of classical effects of gravity to be at the Planck scale, we find that the energy density of the cosmological constant is

$$\rho_\Lambda{}^{\text{theoretical}} \sim 10^{108} eV^4 \,,$$

while its observed value is

$$\rho_\Lambda{}^{\text{observed}} \sim 10^{-12} eV^4 \,.$$

This 120 orders of magnitude discrepancy, apart from almost embarassing, is the biggest open problem in modern cosmology and it is called the *cosmological constant problem.*

This is, sadly, not the only problem related to the cosmological constant. Apart from the dark matter coincidence problem that we already talked about, there is another coincidence probem. It seems [35] that the energy density of the cosmological constant, as indicated by cosmological observations, has the same order of magnitude with the energy density of the matter content of the Universe, i.e. $\rho_\Lambda|_{\text{today}} \sim \rho_{mat.}|_{\text{today}}$. This is rather unexpected because the scaling of these two quantities with the size of the Universe is totally different.

1.2.3 Gravity at smaller scales

Even though it is not related to the subject of this thesis, the introduction would not be complete, if we would not mention the problems of GR at lower scales, i.e. high energies. At those scales the theory that describes the physical world is the second great pillar of modern physics, Quantum Field Theory (QFT). In the context of GR, matter fields are treated classically, with their quantum nature being ignored. So it was more than expected for someone to ask, how gravity behaves at very small scales, where the quantum phenomena cannot be ignored.

Indeed, very soon after the formulation of GR, attempts have been made to incorporate it in a more unified theory. Even Einstein was affected by the work of Eddington, Weyl and Kaluza and Klein. Eddington was followed by Dirac and himself by Brans and Dicke who formulated their theory claiming that Newton's constant may not be a constant, but it may vary with time. Indeed, Brans and Dicke substituted the running Newton's "constant" with a scalar field and they were the first, in 1961, who introduced a scalar-tensor theory of gravity.

However, it was not before the '70's and '80's when the idea of quantizing gravity became very popular. Even though, there have been numerous proposals since then, such as the supergravity and the superstring theory, none of them has achieved to overcome all the problems. It seems that the riddles scientists have to solve, are more fundamental and have to do with conceptual issues rather than mathematical. Specifically, the background (in)dependence is one of those. In QFT Heisenberg's principle is a key concept and introduces an uncertainty for the position/momentum of a particle. On the other hand, in GR is of fundamental importance the fact that the spacetime encompasses all the necessary information of the past, present and future. Moreover, the fact that time is not a dynamical concept in QFT, while in GR it is, creates another conceptual barrier between the two theories.

All in all, attempts for a quantum description of the gravitational interactions have been made for at least half a century and curiosity was probably the main motivation. Indeed, a neat theory that helps us understand better both gravity and the quantum world would be a great success. However, up to now, we are led by the "beauty" that nature should have and we try to formulate a theory with this criterion. It could be the case though, that gravity has no quantum representation at small scales and this whole hunt is futile.

1.3 Summary

An brief introduction to the implications of General Relativity has been attempted in the previous sections. Apart from its great success, GR presents some intrinsic limitations, as well as some notions not so well explained in its framework.

Putting them all together, these problems are: at high energies, the inability of gravity to be integrated in a more general, unified theory with Quantum Field Theory. In addition, the existence and the nature of singularities, either inside black holes or the cosmological ones (e.g. Big Bang) seems to puzzle the scientific community for a while now. Apart from these, the ΛCDM model presents its own riddles: regarding dark energy, the most important one, is the cosmological constant, which shows 120 orders of magnitude difference between its theoretical and observed value. Furthermore, the coincidene that the energy density of the cosmological constant appears to be the same as that one of matter today, creates more confusion in the field. As far as dark matter is conserned, there exist even more problems, with the biggest one being that no particle-candidate has been observed yet. Another coincidence problem exists also here, between the energy density of baryonic matter with this of dark matter, even though the two quantities are formed in different ways. Finally, the missing satellite problem, the distribution of dark matter in galaxies, as well as the relation between baryonic matter and dark matter in spiral and elliptic galaxies complete the picture of open questions in modern cosmology.

Either one by one or as in groups, all of these problems have been studied thoroughly in the literature and individual or collective solutions have been proposed. It is not our task to present them here and the interested reader is directly refered to the associated literature, see e.g. [79, 171, 274] and references therin. However, none of the above is solved completely in the context of GR. This is considered by many scientists as a signal to go beyond Eisntein's theory.

The quest for an alternative description of gravitational interactions on cosmological scales started to take a front seat in physics almost twenty years ago, in the beginning of 2000. There has been both theoretical progress including attempts for renormalizing gravity as well as higher dimensional theories, as well as phenomenological one, trying to fix the theory to fit observations. The list is humongous: there are scalar-tensor theories, such as Brans-Dicke, which is the simplest, or as Horndeski's, which is the most general scalar-tensor theory with second order field equations. In these theories, apart from the metric, gravity is also mediated by another (dynamical) field which transforms as a scalar. There are Tensor-Vector-Scalar theories (TeVeS), which include a metric (rank-2 tensor), a vector and a scalar. These theories were constructed by Bekenstein, as a relativistic formulation of M. Milgrom's Modified Newtonian Dynamics (MOND). What Milgrom thought [36] is to modify Newton's laws at galactic scales to explain their rotation curves. Indeed, he proposed that there should exist an acceleration limit, a_0, below which Newton's second law would become $F_N = ma^2/a_0$. Precisely, the Newtonian force is given by

$$F_N = m\mu(\frac{a}{a_0})a\,, \tag{1.10}$$

where

$$\mu(x) = \begin{cases} 1 & x \gg 1\,, \\ x & x \ll 1 \end{cases} . \tag{1.11}$$

is an interpolating function. Moreover, there exist bimetric theories, either with a dynamical metric and another fixed, background metric, or with two dynamical metrics. Generalizations of the Einstein-Hilbert action have been also considered, with the most common one being $f(R)$ theory. What one does, is instead of considering the E-H action, which is linear in the Ricci scalar, R, one takes a general function of R and constructs a fourth order theory. Even more general actions have been studied, including contraction of the Ricci tensor $R_{\mu\nu}R^{\mu\nu}$, of the Riemann tensor $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, Gauss-Bonnet terms $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, as well as derivatives of these. Numerous are the higher dimensional theories as well: Kaluza-Klein, Randall-Sundrum, Einstein-Gauss-Bonnet, Dvali-Gabadadze-Porrati are only some of them.

In this thesis, the aim is not to deal with extensions of General Relativity, such as the addition of an extra field, or the generalization of an action. The goal is to discuss theories that are equivalent to GR, but have totally different conceptual formulation. For this reason, we will start by writing down the most general affine connection in a four-dimensional manifold. This contains not only the Levi-Civita connection, which is the case for GR, but also an anti-symmetric part with non-vanishing torsion and non-metricity. If curvature describes how the tangent spaces "roll" at each point along a curve, the torsion shows how they are twisted about a curve and non-metricity shows how they scale. More details will be given in the following chapters. Apart from the mathematical description, we will devote some time to discuss the physical properties of these theories and how they could help to tackle the problems discussed above.

In the second part of this thesis, we will use Noether's theorem to classify different theories of gravity in several spacetimes. Specifically, we will use the Noether Symmetry Approach in order to classify the models that are invariant under point-transformations. What is more, we will use these symmetries to find exact solutions in each spacetime. In more detail, Lie and Noether point symmetries of differential equations have some geometric properties. Effectively, this means that, there is a connection between the point symmetries of differential equations that are of second order, with the collinations of the underlying manifold where the motion occurs. Even though this might be known for a while, it has only been a couple of years since this geometric property has been used as a criterion to select theories of gravity. Specifically, we use this method to determine the symmetries of different theories (i.e. dynamical systems) and the invariant function of the systems are used to reduce their dynamics and find analytical solutions.

Finally, in the last part, we will see some astrophysical applications. In particular, we will use the turnaround radius, which is the radius at which initially expanding, gravitationally bound structures will halt their expansion, turnaround and collapse, as a stability criterion to constrain alternative theories of gravity. The maximum size of a large scale structure with a given mass M can be estimated using the maximum turnaround radius. Theories that predict an estimate for the maximum turnaround radius of a structure, that is smaller than its actual observed size, will be severely constrained.

Part I

Geometric Foundations and the Geometric Trinity of Gravity

Chapter 2

Building blocks of a gravitational theory

«Let noone who is ignorant of Mathematics enter here»

- Inscription above the doorway of Plato's Academy

«When the objects of investigation, in any subject, have first principles, foundational conditions, or basic constituents, it is through acquaintance with these that knowledge, scientific knowl- edge, is attained. For we cannot say that we know an object before we are acquainted with its conditions or principles, and have carried our analysis as far as its most elementary constituents.»

- Aristotle, Physics I.1

Apart from the phenomenological requirements (explain astrophysical and cosmological observations) that any self-consistent theory of gravity should have, it must contain some basic theoretical properties, such as the universality of free fall, the local lorentz invariance and more. This is the subject of this chapter; we will try to discuss the minimum theoretical requirements that a relativistic gravity theory should have and we will formulate them mathematically. Finally, we will built general relativity from these and we will study what happens if we loosen up some of its assumptions.

2.1 Necessities for validity

One of the great pillars of General Relativity is the Equivalence Principle. Based on the works of Galileo and Kepler, Newton formulated a version of it in Principia as that the inertial mass of any body, $m_{\rm I}$, i.e. this property of a body that governs its response to a given force, $\mathbf{F} = m_{\rm I}\mathbf{a}$, is equal to its gravitational mass, $m_{\rm g}$, the property that dictates its response to gravitation, $\mathbf{F} = m_{\rm g}\mathbf{g}$. This is known today as the *Weak Equivalence Principle* (WEP) and is better stated as "if an uncharged test body is placed at an initial trajectory will be independent of its internal structure and composition" [57].

The WEP was verified with great accuracy by the Eötvös experiment [58]. He used a rod with two masses on its edges and he hung it from a thin fiber. He attached a mirror on the rod, which in turn reflected light into a telescope. In this way, if the rod would rotate, the light beam would be deflected and this would seen by the telescope. If the inertial mass was different from the gravitational, then gravity and the centrifugal force would not act in the same way on the two bodies and eventually the rod would rotate.

However, Einstein was the one who thought that if the universality of free fall holds, then an observer in a freely falling elevator in the same gravitational field, will not understand the effect of gravity. So what he did is, generalize this idea to all the laws of physics, not only the mechanical ones. It was this idea that opened the path to GR and it is the foundation of metric theories of gravity. It is called the *Einstein Equivalence Principle* and it states that: "(i) WEP is valid, (ii) the outcome of any local non-gravitational test experiment is independent of the velocity of the (freely falling) apparatus (which is the local lorentz invariance) and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (which is the local position invariance)". This was more difficult to be tested than the WEP, since it required experiments for testing both the local lorentz invariance and the local position invariance. The Hughes and Drever experiments [59, 60] managed to show that there is no preferred rest frame in which the laws of physics change and thus local lorentz invariance holds. In addition, local position invariance has been tested by gravitational redshift experiments and also by measuring the constancy of fundamental non-gravitational constants, showing that their values do not change with time.

EEP is fully embodied only in the so-called *metric theories of gravity* and this is because its validity constrains gravitation to be a "curved spacetime" phenomenon. The metric theories satisfy the following two postulates [61]:

1. There exists a metric $g_{\mu\nu}$, which is a rank-2 tensor and captures all the geometric and causal structure of spacetime throught the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \, .$$

2. If $T^{\mu\nu}$ is the energy-momentum tensor of all matter fields and ∇_μ a covariant

derivative, derived by the Levi-Civita connection of the above metric $g_{\mu\nu}$[1], then
$$\nabla_\mu T^{\mu\nu} = 0\,.$$

Using the above postulates, Thorne and Will, but also Nordtvedt, Baierlein and others after them, were able to construct the *Parametrized Post-Newtonian* (PPN) formalism, which is now used to test *metric* theories of gravity. In particular, this formalism contains parameters that observational gravitational physicists can constrsain from the data and theorists can use these bounds on their new theories. For more details see [274].

Apart from these two forms of the Equivalence Principle, there exists another one which distincts itself from WEP and EEP by including self-gravitating bodies and also local gravitational experiments. In particular, it is called the *Strong Equivalence Principle* (SEP) and it states that: (i) WEP is valid not only for test bodies but also for bodies with self-interactions (planets, stars), (ii) the outcome of any local test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed". It is worth noting that SEP includes EEP in the limit where gravitational forces are ignored. The extension, however, of the local lorentz invariance and the local position invariance to experiments involving gravitational forces, is quite a strong requirement. Up to now, there is no other theory satisfying the SEP, but general relativity.

Schiff and Dicke were the first who realized [62, 63] that the gravitational experiments are in a way, probes of the foundations of gravitation theory and not of general relativity itself. This particular point of view led Dicke formulate a framework [64] in which one can discuss the nature of spacetime and gravity. He formulated thus, a set of assumptions and constraints any theory of gravity should satisfy and we summarize them below:

- Geometric points are to be associated with physical events and the only geometric properties a spacetime should have *a priori* are those of a four-dimensional differentiable manifold; there shall be neither a metric nor an affine connection.

- All the mathematical quantities should be expressed in a coordinate invariant, *covariant*, form.

[1] The fact that in the covariant derivative ∇_μ enters only the metric $g_{\mu\nu}$, does not necessarily mean that the metric is the only gravitational field. As it is in Brans-Dicke theory [77], the other fields could help in the generation of spacetime, but they cannot directly couple to matter.

- Gravitational effects should be described by one or more long-range fields, having a tensorial form (scalar, vector, 2-rank tensor or even higher).
- The dynamical equations will be obtained from an invariant action principle.
- Last but not least, nature likes things as simple as possible, meang that Ockam's razor should be considered as a guiding principle [61].

However Dicke's framework is very strict and confines a lot the theories that could be accepted. This led theorists throughout the years to formulate a set of fundamental criteria that, any viable gravitation theory should respect and they emanate not only from a theoretical viewpoint, but also from experimental evidence. These are the following:

- it must be complete, in the sense that, it should be able to analyze from "first principles" the outcome of any experiment,
- it must be self-consistent, i.e. predictions for the outcomes of experiments should be unique and independrent of the calculating method,
- it must be relativistic, that means to be able to recover Special Relativity at low energies and,
- it must have the correct Newtonian limit, when the masses/energies are sufficiently weak.

It is obvious that the last two criteria are based on the great success of both Special Relativity and Newtonian theory of gravity at their range of validity.

Based on the above discussion, it is now time to study in more mathematical detail the necessary tools that one needs to construct a theory of gravity. This is what we do in what follows.

2.2 Gravity as Geometry: affine structure of spacetime

As we already discussed in the previous section, the fundamental object to construct a gravity theory is the metric tensor and/or spacetime, which is associated with a 4-dimensional differentiable manifold. All these notions are part of the Riemannian geometry. All the physical laws up to 1900 were built on a flat and non-dynamical space. Because of the fact that, Einstein perceived gravity as the curvature of spacetime, he should build his theory on a different geometry than the Euclidean, and this is the reason it took him ten years, after SR, to formulate GR. The necessary mathematical tools and a mere introduction to the geometric description of a 4-dimensional manifold will be attempted in this chapter. We will deal mostly, if not exclusively, with quantities that are useful from a physicist's point of view. For more details one should check the literature on the subject [6–8].

Spacetime is described by a 4-dimensional differentiable manifold $\mathcal{M}$, and the metric $g_{\mu\nu}$ is a symmetric, rank-2, covariant tensor on $\mathcal{M}$. It is used to define notions such as time, distance, volume, curvature, angle and it is directly related to the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,.$$

The fact that it is symmetric means that $g_{\mu\nu} = g_{\nu\mu}$. In n dimensions it can be written as a $n \times n$ matrix, which means that in 4 dimensions it is supposed to have 16 independent coefficients. However, because of the symmetricity it has only 10. It is also non-degenerate, meaning that the matrix is non-singular, $g = \det g_{\mu\nu} \neq 0$ and thus invertible $g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho$, which allows it to be used to raise or lower indices. It has either one positive and three negative, or one negative and three positive eigenvalues which means that its signature can be chosen by convention to be either $(-+++)$ or $(+---)$. Such a metric is called Lorentzian or pseudo-Riemannian metric.

Apart from the metric, there exists another characteristic quantity of $\mathcal{M}$ and this is the connection $\Gamma^\lambda{}_{\mu\nu}$. This is related to the parallel transport of a vector and therefore it defines the covariant derivative as follows

$$\nabla_\mu T^\rho = \partial_\mu T^\rho + \Gamma^\rho{}_{\mu\alpha} T^\alpha \,. \tag{2.1}$$

Respectively, for rank-2 mixed tensors it is given by

$$\nabla_\mu T^\rho{}_\sigma = \partial_\mu T^\rho{}_\sigma + \Gamma^\rho{}_{\mu\alpha} T^\alpha{}_\sigma - \Gamma^\alpha{}_{\mu\sigma} T^\rho{}_\alpha \,. \tag{2.2}$$

As we will see later on, in GR the connection is symmetric and it coincides with the Levi-Civita one, $\{^\lambda{}_{\mu\nu}\}$, which depends on the metric. This is not always the case; the ∇_μ in Eq. (2.1), (2.2) is defined through the general connection $\Gamma^\lambda{}_{\mu\nu}$ which does not necessarily depend on the metric. Making our notation clear: $\Gamma^\alpha{}_{\mu\nu}$ is a generic affine connection, while $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\}$ is the Levi-Civita one, that depends only on the metric through

$$\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} \right) \,. \tag{2.3}$$

It is known from differential geometry [65] that a generic affine connection can be decomposed into the following three parts

$$\Gamma^\lambda{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu} \,, \tag{2.4}$$

where the second term is the contorsion tensor, defined through the torsion tensor as

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(T_{\mu\rho\nu} + T_{\nu\rho\mu} + T_{\rho\mu\nu} \right) = -K_{\nu\mu}{}^{\rho} \,, \tag{2.5}$$

and the third term is the disformation tensor, defined through the nonmetricity tensor as

$$L^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(-Q_{\mu\rho\nu} - Q_{\nu\rho\mu} + Q_{\rho\mu\nu} \right) = L^{\lambda}{}_{\nu\mu} \,. \tag{2.6}$$

The torsion and the nonmetricity tensor are given by

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = \Gamma^{\lambda}{}_{[\mu\nu]} \,, \tag{2.7}$$

$$Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\mu} g_{\kappa\nu} - \Gamma^{\kappa}{}_{\rho\nu} g_{\mu\kappa} \,, \tag{2.8}$$

where $[\mu\nu]$ denotes the anti-symmetric part. In addition, through this connection, one can define the Riemann tensor as

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_{\beta} \Gamma^{\alpha}{}_{\nu\mu} - \partial_{\nu} \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\beta\lambda} \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\beta\mu} \,. \tag{2.9}$$

This tensor is used to express the curvature of the manifold $\mathcal{M}$ and it does not *a priori* depend on the metric. As it is easily seen, it is anti-symmetric in the last two indices. Contraction of the first with the third index of the Riemann tensor gives the Ricci tensor, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, from which another geometric quantity is defined, the Ricci scalar, by contracting it with the (inverse of the) metric, $R = g^{\mu\nu} R_{\mu\nu}$. For completeness, we should mention that one could define another Ricci tensor as $\bar{R}_{\mu\nu} = R^{\alpha}{}_{\alpha\mu\nu} = \partial_{\mu} \Gamma^{\alpha}{}_{\alpha\nu} - \partial_{\nu} \Gamma^{\alpha}{}_{\alpha\mu}$, however, because this is anit-symmetric and the metric is symmetric, their contraction vanishes.

Before we proceed, let us pause to discuss the properties of all the above tensors. Strictly speaking, the curvature, the torsion and the non-metricity are all properties of the connection and not of the spacetime. The curvature tensor measures the failure of a vector to return in its initial position when parallely transported along a curve. Moreover, the inability of a vector to form a closed parallelogram is described by the torsion tensor. Finally, the non-metricity tensor measures the changes in the norm of a vector when parallely transported. Mathematically, a parallel transport of a vector along a curve in a spacetime with connection $\Gamma^{\lambda}{}_{\mu\nu}$ is given by

$$\nabla_{[\mu} \nabla_{\nu]} u^{\rho} = \frac{1}{2} \left(R^{\rho}{}_{\nu\mu\sigma} u^{\sigma} + T^{\sigma}{}_{\mu\nu} \nabla_{\sigma} u^{\rho} \right) \,. \tag{2.10}$$

It should be obvious by now, that if we "switch off" the torsion and the non-metricity of the connection, i.e. $T^{\rho}{}_{\mu\nu} = 0$ and $Q_{\alpha\mu\nu} = 0$, then identically the connection becomes the Levi-Civita one and the theory reduces from *metric-affine*, where both the metric and the connection of the manifold are independent, to GR, where all the necessary information to describe the gravitational field is encoded in the metric. However, it turns out that there exist three equivalent descriptions of GR by "playing" with the properties of the connection, and these are analytically presented in the following three chapters.

Specifically, the theory of gravity which is based on the curvature of spacetime (or better, of the connection) is General Relativity. Gravity is mediated by curvature and the spacetime is torsionless and metric compatible. Furthermore, all the mathematical quantities are calculated by the Levi-Civita connection, which is symmetric in its two lower indices.

The Teleparallel theory of gravity (or else TEGR: Teleparallel Equivalent to General Relativity), is an equivalent description of GR, with (almost) the same action and field equations, but with different conceptual basis. In particular, gravity is mediated by torsion on a flat spacetime (thus the curvature is zero) and the dynamical field is the tetrad, which is covariantly conserved (meaning that the non-metricity tensor vanishes). Moreover, its mathematical quantities are constructed by the Weitzenböck connection, which is anti-symmetric in its two lower indices.

Finally, there is a third equivalent description of gravity. It is called the Symmetric Teleparallel Equivalent to General Relativity (STEGR); teleparallel because the curvature vanishes and symmetric because the torsion vanishes. Gravity is thus mediated through non-metricity and through a gauge freedom (called the coincident gauge), the general connection can be chosen to be zero, ending up with $\mathring{\Gamma}^{\lambda}{}_{\mu\nu} = -L^{\lambda}{}_{\mu\nu}$. We will study all of them, one by one, in the following chapters.

2.3 Conclusions

In this chapter, we studied the necessary tools and the viability criteria needed to construct a relativitic theory of gravity. Specifically, we discussed the differences between the different forms of the equivalence principle and we presented the so-called Dicke's framework in which one can discuss the nature of spacetime and gravity. Furthermore, we listed a set of fundamental criteria, taking into account both theoretical and experimental arguments, that any viable theory of gravity should obey.

In addition, we described the affin structure of spacetime. We showed that a general affine connection can be split into three parts: the Levi-Civita connection, the contorsion tensor and the disformation tensor. Depending on whether the connection is purely symmetric or anti-symmetric, whether the torsion is zero or the metric of the manifold is covariantly conserved, the geometry of the spacetime changes and we can discuss the different properties of each theory. This is going to be the subject of the following three chapters.

Chapter 3

General Relativity: The curvature of spacetime

> «I was sitting in a chair in the patent office at Bern when all of a sudden a thought occurred to me: “If a person falls freely he will not feel his own weight.” I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation.»
>
> *- Albert Einstein (Lecture in Japan, 1922)*

After describing the picture of the affine structure of spacetime in the previous chapter, it is time to start formulating gravity; meaning how is gravity mediated in spacetime, how do different particles behave, what equations do they obey and so on. In this chapter we are going to present the basic geometric setup of General Relativity and we will attempt to answer the above questions in its framework.

3.1 General Relativity

When Einstein formulated his theory, he started using the analogy with the Poisson equation,

$$\nabla^2 \phi = 4\pi G_N \rho \,, \tag{3.1}$$

which describes the dynamics of Newton's theory. ρ is the density of a massive object, G_N is Newton's constant and ϕ is the gravitational potential. In addition, the fact that GR is a classical theory makes the notion of the action unnecessary. Only the field equations could perfectly do the job. However, since one needs to directly compare with alternative field theories and also it is easier to study the quantum behaviour of the theory, it is better to adopt the Lagrangian formulation.

As already mentioned in the previous chapter, the spacetime is described by a manifold M, which is endowed with a metric $g_{\mu\nu}$ and a connection $\Gamma^{\alpha}{}_{\mu\nu}$, that are in principle independent. In GR however, the connection is assumed to be the Levi-Civita one, i.e.

$$\Gamma^{\alpha}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu} \right) . \tag{3.2}$$

It is easily seen that, the above connection is symmetric in its two lower indices, i.e. $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$, and in addition that, the metric is covariantly conserved,

$$\overset{\circ}{\nabla}_{\alpha} g_{\mu\nu} = 0 . \tag{3.3}$$

Before we proceed to the essentials of the theory, let us make the comparison with the discussion in the previous chapter. The fact that the connection is the Levi-Civita one, i.e. it is symmetric, means that the torsion tensor vanishes

$$T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{[\mu\nu]} = 0 . \tag{3.4}$$

So in GR the spacetime (or more correctly, the connection) is assumed to be torsionless. In addition, the metric is compatible and thus the non-metricity tensor vanishes too, $Q_{\alpha\mu\nu} = 0$. This means, the only property of the connection, and thus of the spacetime, that survives is the curvature, which is now given by

$$\overset{\circ}{R}{}^{\alpha}{}_{\mu\beta\nu} = \partial_{\beta} \overset{\circ}{\Gamma}{}^{\alpha}{}_{\nu\mu} - \partial_{\nu} \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\mu} + \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\lambda} \overset{\circ}{\Gamma}{}^{\lambda}{}_{\nu\mu} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\nu\lambda} \overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\mu} . \tag{3.5}$$

One can easily verify that the above Riemann tensor is symmetric in the exchange of the first and last pair of indices and anti-symmetric in the flipping of a pair. In addition, it satisfies the first and second Bianchi identities which respectively read

$$\overset{\circ}{R}_{\alpha\beta\mu\nu} + \overset{\circ}{R}_{\alpha\nu\beta\mu} + \overset{\circ}{R}_{\alpha\mu\nu\beta} = 0 , \tag{3.6}$$

$$\overset{\circ}{\nabla}_{\mu} \overset{\circ}{R}{}^{\nu}{}_{\alpha\beta\gamma} + \overset{\circ}{\nabla}_{\gamma} \overset{\circ}{R}{}^{\nu}{}_{\mu\alpha\beta} + \overset{\circ}{\nabla}_{\beta} \overset{\circ}{R}{}^{\nu}{}_{\gamma\mu\alpha} = 0 . \tag{3.7}$$

Furthermore, one can now define uniquely the Ricci tensor as $\overset{\circ}{R}_{\mu\nu} = \overset{\circ}{R}{}^{\alpha}{}_{\mu\alpha\nu}$, which is symmetric and from this one, the Ricci scalar by contracting with the metric, $\overset{\circ}{R} = g^{\mu\nu} \overset{\circ}{R}_{\mu\nu}$.

After introducing the necessary geometric tools let us proceed by examining the Einstein-Hilbert action. In particular, this reads

$$\mathcal{S} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \overset{\circ}{R} + \mathcal{S}_m . \tag{3.8}$$

The constant term $1/(16\pi G_N)$ is introduced in order for the theory to have the correct weak field limit, i.e. Newtonian. g is the determinant of the metric and it acts as a measure and $\overset{\circ}{R}$ is the Ricci scalar. This action was introduced by Hilbert and it is the simplest action that gives second order covariant equations of motion for the dynamical field, the metric. The second term in the right hand side is given by

$$\mathcal{S}_m = \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu}, \psi)\,, \tag{3.9}$$

and it is called the matter action. It contains the matter Lagrangian $\mathcal{L}_m$ in which all matter fields denoted for simplicity ψ, couple directly to the metric.

As it is known from classical field theory, in order to derive the equations of motion, one has to vary the action with respect to the metric. This gives

$$\begin{aligned}
\delta\mathcal{S} &= \frac{1}{16\pi G_N}\int d^4x\left[\delta\sqrt{-g}\overset{\circ}{R} + \sqrt{-g}\delta\overset{\circ}{R}\right] + \delta\mathcal{S}_m \\
&= \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\overset{\circ}{R}\delta g^{\mu\nu} + \left(\overset{\circ}{R}_{\mu\nu} + \overset{\circ}{\nabla}_\alpha(g^{\mu\nu}\delta\overset{\circ}{\Gamma}{}^\alpha{}_{\nu\mu} - g^{\mu\alpha}\delta\overset{\circ}{\Gamma}{}^\beta{}_{\beta\mu})\right)\right] + \delta\mathcal{S}_m \\
&= \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[\overset{\circ}{G}_{\mu\nu}\delta g^{\mu\nu} + \overset{\circ}{\nabla}_\alpha(g^{\mu\nu}\delta\overset{\circ}{\Gamma}{}^\alpha{}_{\nu\mu} - g^{\mu\alpha}\delta\overset{\circ}{\Gamma}{}^\beta{}_{\beta\mu})\right] + \delta\mathcal{S}_m\,.
\end{aligned} \tag{3.10}$$

The first term in the integrand is the Einstein tensor and reads $\overset{\circ}{G}_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - \overset{\circ}{R}g_{\mu\nu}/2$. The second term multiplied by $\sqrt{-g}$ becomes a total derivative and thus by Stoke's theorem, yields a boundary term when integrated. The fact that the Ricci scalar contains also the second order derivatives of the metric is responsible for this boundary term, which in general does not vanish, because we are allowed to fix only the number of degrees of freedom of the metric and not of its first order derivatives. In order to properly define an action which will give the Einstein equations after variations, we have to subtract such a boundary term in the very definition of the action, in order for these two terms to cancel each other. Even though it may seem trivial, it is not, and in addition it gives rise to interesting properties such as the black hole entropy. For more details see [66–68]. Finally, the last term in Eq. (3.10) will give the energy-momentum tensor of all the matter fields in the universe as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}}\,. \tag{3.11}$$

Thus, the field equations for the gravitational field, or else the Einstein equations read

$$\overset{\circ}{G}_{\mu\nu} = 8\pi G_N T_{\mu\nu}\,. \tag{3.12}$$

A comment here is necessary. In 1925 Einstein discovered what is today known as *Palatini's method* [69]. That is, if one considers the metric and the affine connection of a manifold to be independent, then the connection acts as a rank-3 gravitational tensor field and all the curvature invariants are defined through the connection and not the metric. However, if one varies the Einstein-Hilbert action $\left(\mathcal{S} \sim \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g} d^4x\right)$ with respect to $\Gamma^{\alpha}{}_{\mu\nu}$, the associated equation of motion is

$$\nabla_{\mu} g_{\alpha\beta} = 0\,, \tag{3.13}$$

which is the metric compatibility condition, that yiels $\Gamma^{\alpha}{}_{\mu\nu} = \{^{\alpha}{}_{\mu\nu}\}$. However, this is only a coincidence; for other Lagrangians, the field equations in metric and Palatini formalisms are, in general, different. We will study this difference in more detail, later in this chapter.

Up to now, we have seen that GR is based on several assumptions and has some fundamental properties. Both the field equations and the action are covariant, this means that they do not depend on the choice of the coordinates. In addition, the only field that mediates gravity is the metric and thus contains all the necassary information to describe the gravitational interactions. The connection is symmetric, $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{(\mu\nu)}$, i.e. it is torsionless and the relation between the metric and the connection is given by the non-metricity tensor, which in GR vanishes, $Q_{\alpha\mu\nu} = 0 \Rightarrow \mathring{\Gamma}^{\alpha}{}_{\mu\nu} = \{^{\alpha}{}_{\mu\nu}\}$. Finally, all the matter fields couple directly and only to the metric. If someone changes one or more of the above assumptions then the theory changes completely and this is going to be the subject of study in the rest of this chapter.

3.2 Extending General Relativity

As we mentioned before, there are several ways to modify the theory of gravity. We will focus only on theories that are covariant. That is because, there are a lot of cases where non-covariant equations can be brought in a covariant form by imposing some constraints. However, these constraints are responsible for structures in the theory, such as preferred coordinate systems, which make the theory background dependent; and this is not wanted. Thus, in what follows we will allow gravity to be mediated through other fields (beyond the metric), the equations to be of higher order, as well as the connection to have an anti-symmetric part and the metric to be non-compatible. Specifically, we will focus on theories with extra fields (scalar, vector or tensor), as well as higher order theories. There are many theories that we omit, especially the higher dimensional ones (e.g. Kaluza-Klein, Randall-Sundrum, DGP and more), but it is not our task to review all of them. We will deal only with theories that we are going to use later on the thesis.

3.2.1 Addition of extra fields

One way to modify GR is to allow other fields, apart from the metric, to be responsible for the gravitational interactions. This can be done, depending the field of study, using scalar, vector, tensor (rank-2) or even higher order tensors as extra degrees of freedom. In the literature, there have been numerous attempts to change the dynamics of gravity, however, we will stress only those that are still viable, i.e. they are not excluded from observations.

Obviously, the effect of all those additional degrees of freedom should be suppressed in the regimes where GR is well tested, e.g. solar system. To do this, either we consider the coupling to be weak or we coscript a screening mechanism [70] such as, the Vainshtein [71], the chameleon [72, 73] and the symmetron [74, 75]. Briefly, the Vainshtein mechanism (mostly essential for massive gravity, brane-induced theories, galileons and more) is based on the non-linearities introduced by the derivative coupling of a scalar field. Specifically, these become large enough as they approach massive objects and they increase the kinetic term of perturbations, resulting weaker interactions with matter. The chameleon mechanism (mostly needed in $f(R)$ theories[1]) suggests that the mass of a scalar field depends on the mass of its local environment. Effectively, in the solar system experiments, where the local density is high, the scalar effects are suppressed; while deeper in space, where the mass density is low, the scalar field becomes lighter and plays a significant role to gravity. Finally, the symmetron mechanism suggests that, a scalar field couples to matter proportionally to the vacuum expectation value, so that the coupling be higher in lower density regions, because of the fact that, the vacuum expectation value (VEV) of the scalar field depends on the local mass density. Having said all that, let us proceed by discussing specific theories.

Scalar-Tensor theories

The scalar-tensor theories are considered by many scientists the simplest (and the first) extension of general relativity. Furthermore, they arise naturally in the dimensional reduction of higher dimensional theories (e.g. Kaluza-Klein). The first such theory was introduced by Brans and Dicke in 1961 [77], who followed the work of Jordan in 1955 [76]. They wanted their theory to be more satisfactory (compared to GR) from the point of view of Mach's principle. Dicke's formulation of Mach's principle says that *the gravitational constant G_N should be a function of the mass distribution of the universe*. For this reason, they replaced the constant fraction in the EH action by a dynamical scalar field. Since the field is dynamical, it should

[1]Even though $f(R)$ theories are not scalar tensor theories, there is a dynamical equivalence between $f(R)$ and O'Hanlon theory; that is Brans-Dicke theory with no kinetic term.

also have a kinetic term and therefore, the Brans-Dicke action reads

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[\phi\overset{\circ}{R} - \frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi\right] + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu},\psi)\,. \tag{3.14}$$

ω is a coupling paratemer between the metric and the scalar and the denominator is there to make it dimensionless. Notice that the scalar field does not couple with any matter field. Thus, the equivalence principle is not violated.

A more general form of scalar-tensor theory of gravity is the following

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[\phi\overset{\circ}{R} - \frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + \mathcal{S}_m(g_{\mu\nu},\psi)\,, \tag{3.15}$$

where $\omega(\phi)$ is now an arbitrary function of the scalar field and we added also a potential term, of the form $V(\phi)$. In the $\omega\to\infty$, $\omega'/\omega^2\to 0$ and $\phi\to$ constant, $V(\phi)\to 2\Lambda$ limit, the above action reduces to the Einstein-Hilbert action with a cosmological constant.

As one can notice, matter fields couple only to the metric. Thus the role of the scalar field is purely to generate the spacetime associated with the metric. This means that scalar-tensor theories are also metric theories of gravity.

Varying (3.15) with respect to the metric and the scalar field we obtain the equations of motion of the theory which are

$$\phi\overset{\circ}{G}_{\mu\nu} + \frac{1}{2}V(\phi)g_{\mu\nu} = 8\pi G_N T^M_{\mu\nu} - \frac{\omega(\phi)}{\phi}\left(\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\right) + \overset{\circ}{\nabla}_\mu\partial_\nu\phi - g_{\mu\nu}\overset{\circ}{\Box}\phi\,, \tag{3.16}$$

$$(2\omega(\phi)+3)\,\overset{\circ}{\Box}\phi = 8\pi G_N T^M - \omega'(\phi)\partial_\mu\phi\partial^\mu\phi + \phi V'(\phi) - 2V(\phi)\,. \tag{3.17}$$

$T^M_{\mu\nu}$ is the matter stress-energy tensor and T^M its trace.

The following comment is necessary here. Consider the action (3.15) and for simplicity set $\omega(\phi)\to$constant. If we perform a conformal transformation of the form

$$g_{\mu\nu}\to\tilde{g}_{\mu\nu} = G_N\phi g_{\mu\nu}\,, \tag{3.18}$$

and we redefine our scalar field as

$$\phi\to\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi G_N}}\ln\phi\,, \tag{3.19}$$

(for $\omega > -3/2$ and $\phi > 0$) then the action takes the form

$$\mathcal{S} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\overset{\circ}{\tilde{R}}}{16\pi G_N} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\tilde{\phi})\right] + \exp\left(-8\sqrt{\frac{\pi G_N}{2\omega+3}}\tilde{\phi}\right)\mathcal{S}_m(\tilde{g}_{\mu\nu},\psi)\,. \tag{3.20}$$

The new potential U is defined as

$$U(\tilde{\phi}) = V(\phi(\tilde{\phi}))\exp\left(-8\sqrt{\frac{\pi G_N}{2\omega+3}}\tilde{\phi}\right) = \frac{V(\phi)}{(G_N\phi)^2}\,. \tag{3.21}$$

As it is easily seen, this is Einstein's gravity minimally coupled to a scalar field. For this reason this is called the *Einstein frame* while the conformal one *Jordan frame*. The significant difference between the two frames is that matter fields in the Einstein frame couple, apart from the metric, also to the scalar field. This implies changes to the geodesic motion and thus the Equivalence Principle is violated in the Einstein frame. This is the reason that makes many scientists to claim that the Jordan frame is the physical one [78, 79].

Before we proceed to other theories, we should mention the most general scalar-tensor theory with second order field equations proposed by Horndeski [80] and was later written in a covariant form by several authors [81, 82]. The action is given by the sum of the integrals of four different Lagrangians, i.e.

$$\mathcal{S}_{Horndeski} = \sum_{i=2}^{5}\int d^4x\sqrt{-g}\mathcal{L}_i\,, \tag{3.22}$$

where

$$\mathcal{L}_2 = G_2(\phi, X)\,, \tag{3.23}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\,\mathring{\Box}\phi\,, \tag{3.24}$$

$$\mathcal{L}_4 = G_4(\phi, X)\,\mathring{R} + G_{4X}\left[\left(\mathring{\Box}\phi\right)^2 - \left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\right)^2\right], \tag{3.25}$$

$$\mathcal{L}_5 = G_5(\phi, X)\,\mathring{G}_{\mu\nu}\mathring{\nabla}^\mu\mathring{\nabla}^\nu\phi - \frac{1}{6}G_{5X}\left[\left(\mathring{\Box}\phi\right)^3 - 3\mathring{\Box}\phi\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\right)^2 + 2\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\right)^3\right]. \tag{3.26}$$

The functions $G_2(\phi, X)$, $G_3(\phi, X)$, $G_4(\phi, X)$ and $G_5(\phi, X)$ are arbitrary functions of the scalar field ϕ and its kinetic term $X = -\frac{1}{2}\left(\mathring{\nabla}\phi\right)^2 = -\frac{1}{2}\mathring{\nabla}^\mu\phi\mathring{\nabla}_\mu\phi$. In addition, G_{iX} is the derivative of G_i with respect to X, $\mathring{R}$ is the Ricci scalar, $\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R}$ is the Einstein tensor, and the remaining kinetic terms are

$$\mathring{\Box}\phi = g^{\mu\nu}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\,, \tag{3.27}$$

$$\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\right)^2 = \mathring{\nabla}^\mu\mathring{\nabla}^\nu\phi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\,, \tag{3.28}$$

$$\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\right)^3 = \mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi\mathring{\nabla}^\nu\mathring{\nabla}^\lambda\phi\mathring{\nabla}_\lambda\mathring{\nabla}^\mu\phi\,. \tag{3.29}$$

If we vary the action with respect to the metric and the scalar field, we get the field equations for the Horndeski theory [83]. The variation is

$$\delta\mathcal{S} = \delta\left(\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i\right) = \sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{G}^i_{\mu\nu}\delta g^{\mu\nu} + \sum_{i=2}^{5}\left(P^i_\phi - \overset{\circ}{\nabla}{}^\mu J^i_\mu\right)\delta\phi\right] + \text{total derivatives}, \tag{3.30}$$

and thus the equations of motion are given by

$$\sum_{i=2}^{5}\mathcal{G}^i_{\mu\nu} = 0\,, \quad \overset{\circ}{\nabla}{}^\mu\left(\sum_{i=2}^{5}J^i_\mu\right) = \sum_{i=2}^{5}P^i_\phi\,, \tag{3.31}$$

for the metric and the scalar field respectively. The components are

$$P^2_\phi = G_{2\phi}\,, \tag{3.32a}$$

$$P^3_\phi = \overset{\circ}{\nabla}_\mu G_{3\phi}\overset{\circ}{\nabla}{}^\mu\phi\,, \tag{3.32b}$$

$$P^4_\phi = G_{4\phi}\overset{\circ}{R} + G_{4\phi X}\left[(\overset{\circ}{\Box}\phi)^2 - (\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi)^2\right]\,, \tag{3.32c}$$

$$P^5_\phi = -\overset{\circ}{\nabla}_\mu G_{5\phi}\overset{\circ}{G}{}^{\mu\nu}\overset{\circ}{\nabla}_\nu\phi - \frac{1}{6}G_{5\phi X}\left[(\overset{\circ}{\Box}\phi)^3 - 3\overset{\circ}{\Box}\phi(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi)^2 + 2(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi)^3\right]\,, \tag{3.32d}$$

and

$$J^2_\mu = -\mathcal{L}_{2X}\overset{\circ}{\nabla}_\mu\phi\,, \tag{3.33a}$$

$$J^3_\mu = -\mathcal{L}_{3X}\overset{\circ}{\nabla}_\mu\phi + G_{3X}\overset{\circ}{\nabla}_\mu X + 2G_{3\phi}\overset{\circ}{\nabla}_\mu\phi\,, \tag{3.33b}$$

$$\begin{aligned} J^4_\mu &= -\mathcal{L}_{4X}\overset{\circ}{\nabla}_\mu\phi + 2G_{4X}\overset{\circ}{R}_{\mu\nu}\overset{\circ}{\nabla}{}^\nu\phi - 2G_{4XX}\left(\overset{\circ}{\Box}\phi\overset{\circ}{\nabla}_\mu X - \overset{\circ}{\nabla}{}^\nu X\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi\right) \\ &\quad - 2G_{4\phi X}(\overset{\circ}{\Box}\phi\overset{\circ}{\nabla}_\mu\phi + \overset{\circ}{\nabla}_\mu X)\,, \end{aligned} \tag{3.33c}$$

$$\begin{aligned} J^5_\mu &= -\mathcal{L}_{5X}\overset{\circ}{\nabla}_\mu\phi - 2G_{5\phi}\overset{\circ}{G}_{\mu\nu}\overset{\circ}{\nabla}{}^\nu\phi - \\ &- G_{5X}\left[\overset{\circ}{G}_{\mu\nu}\overset{\circ}{\nabla}{}^\nu X + \overset{\circ}{R}_{\mu\nu}\overset{\circ}{\Box}\phi\overset{\circ}{\nabla}{}^\nu\phi - \overset{\circ}{R}_{\nu\lambda}\overset{\circ}{\nabla}{}^\nu\phi\overset{\circ}{\nabla}{}^\lambda\overset{\circ}{\nabla}_\mu\phi - \overset{\circ}{R}_{\alpha\mu\beta\nu}\overset{\circ}{\nabla}{}^\nu\phi\overset{\circ}{\nabla}{}^\alpha\overset{\circ}{\nabla}{}^\beta\phi\right] + \\ &+ G_{5XX}\{\frac{1}{2}\overset{\circ}{\nabla}_\mu X\left[(\overset{\circ}{\Box}\phi)^2 - (\overset{\circ}{\nabla}_\alpha\overset{\circ}{\nabla}_\beta\phi)^2\right] - \overset{\circ}{\nabla}_\nu X\left(\overset{\circ}{\Box}\phi\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}{}^\nu\phi - \overset{\circ}{\nabla}_\alpha\overset{\circ}{\nabla}_\mu\phi\overset{\circ}{\nabla}{}^\alpha\overset{\circ}{\nabla}{}^\nu\phi\right)\} + \\ &+ G_{5\phi X}\{\frac{1}{2}\overset{\circ}{\nabla}_\mu\phi\left[(\overset{\circ}{\Box}\phi)^2 - (\overset{\circ}{\nabla}_\alpha\overset{\circ}{\nabla}_\beta\phi)^2\right] + \overset{\circ}{\Box}\phi\overset{\circ}{\nabla}_\mu X - \overset{\circ}{\nabla}{}^\nu X\overset{\circ}{\nabla}_\nu\overset{\circ}{\nabla}_\mu\phi\}\,, \end{aligned} \tag{3.33d}$$

as well as

$$\mathcal{G}^2_{\mu\nu} = -\frac{1}{2}G_{2X}\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi - \frac{1}{2}G_2 g_{\mu\nu}\,, \tag{3.34a}$$

$$\mathcal{G}^3_{\mu\nu} = \frac{1}{2}G_{3X}\mathring{\Box}\phi\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi + \mathring{\nabla}_{(\mu}G_3\mathring{\nabla}_{\nu)}\phi - \frac{1}{2}g_{\mu\nu}\mathring{\nabla}_\lambda G_3\mathring{\nabla}^\lambda\phi\,, \tag{3.34b}$$

$$\begin{aligned}
\mathcal{G}^4_{\mu\nu} = {} & G_4\mathring{G}_{\mu\nu} - \frac{1}{2}G_{4X}\mathring{R}\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi - \frac{1}{2}G_{4XX}\left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2\right]\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi - \\
& - G_{4X}\mathring{\Box}\phi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi + G_{4X}\mathring{\nabla}_\lambda\mathring{\nabla}_\mu\phi\mathring{\nabla}^\lambda\mathring{\nabla}_\nu\phi + 2\mathring{\nabla}_\lambda G_{4X}\mathring{\nabla}^\lambda\mathring{\nabla}_{(\mu}\phi\mathring{\nabla}_{\nu)}\phi - \\
& - \mathring{\nabla}_\lambda G_{4X}\mathring{\nabla}^\lambda\phi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi + g_{\mu\nu}\left(G_{4\phi}\mathring{\Box}\phi - 2XG_{4\phi\phi}\right) + \\
& + g_{\mu\nu}\{-2G_{4\phi X}\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi + G_{4XX}\mathring{\nabla}_\alpha\mathring{\nabla}_\lambda\phi\mathring{\nabla}_\beta\mathring{\nabla}^\lambda\phi\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi + \\
& + \frac{1}{2}G_{4X}\left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2\right]\} + 2\Big[G_{4X}\mathring{R}_{\lambda(\mu}\mathring{\nabla}_{\nu)}\phi\mathring{\nabla}^\lambda\phi - \mathring{\nabla}_{(\mu}G_{4X}\mathring{\nabla}_{\nu)}\phi\mathring{\Box}\phi\Big] - \\
& - g_{\mu\nu}\left[G_{4X}\mathring{R}^{\alpha\beta}\mathring{\nabla}_\alpha\phi\mathring{\nabla}_\beta\phi - \mathring{\nabla}_\lambda G_{4X}\mathring{\nabla}^\lambda\phi\mathring{\Box}\phi\right] + G_{4X}\mathring{R}_{\mu\alpha\nu\beta}\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi - \\
& - G_{4\phi}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi - G_{4\phi\phi}\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi + 2G_{4\phi X}\mathring{\nabla}^\lambda\phi\mathring{\nabla}_\lambda\mathring{\nabla}_{(\mu}\phi\mathring{\nabla}_{\nu)}\phi - \\
& - G_{4XX}\mathring{\nabla}^\alpha\phi\mathring{\nabla}_\alpha\mathring{\nabla}_\mu\phi\mathring{\nabla}^\beta\phi\mathring{\nabla}_\beta\mathring{\nabla}_\nu\phi\,,
\end{aligned} \tag{3.34c}$$

$$\begin{aligned}
\mathcal{G}^5_{\mu\nu} = {} & G_{5X}\mathring{R}_{\alpha\beta}\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\mathring{\nabla}_{(\mu}\phi\mathring{\nabla}_{\nu)}\phi - G_{5X}\mathring{R}_{\alpha(\mu}\mathring{\nabla}_{\nu)}\phi\mathring{\nabla}^\alpha\phi\mathring{\Box}\phi - \frac{1}{2}G_{5X}\mathring{R}_{\mu\alpha\nu\beta}\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi\mathring{\Box}\phi - \\
& - \frac{1}{2}G_{5X}\mathring{R}_{\alpha\beta}\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi + G_{5X}\mathring{R}_{\alpha\lambda\beta(\mu}\mathring{\nabla}_{\nu)}\phi\mathring{\nabla}^\lambda\phi\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi - \\
& - \frac{1}{2}\mathring{\nabla}_{(\mu}\left[G_{5X}\mathring{\nabla}^\alpha\phi\right]\mathring{\nabla}_\alpha\mathring{\nabla}_{\nu)}\phi\mathring{\Box}\phi + \frac{1}{2}\mathring{\nabla}_{(\mu}\left[G_{5\phi}\mathring{\nabla}_{\nu)}\right]\mathring{\Box}\phi - \mathring{\nabla}_\lambda\left[G_{5\phi}\mathring{\nabla}_{(\mu}\phi\right]\mathring{\nabla}_{\nu)}\mathring{\nabla}^\lambda\phi + \\
& + \frac{1}{2}\left[\mathring{\nabla}_\lambda\left(G_{5\phi}\mathring{\nabla}^\lambda\phi\right) - \mathring{\nabla}_\alpha\left(G_{5X}\mathring{\nabla}_\beta\phi\right)\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi\right]\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi + \mathring{\nabla}^\alpha G_5\mathring{\nabla}^\beta\phi\mathring{R}_{\alpha(\mu\nu)\beta} + \\
& + \frac{1}{2}\mathring{\nabla}_{(\mu}G_{5X}\mathring{\nabla}_{\nu)}\phi - \left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2\right] + G_{5X}\mathring{R}_{\alpha\lambda\beta(\mu}\mathring{\nabla}_{\nu)}\mathring{\nabla}^\lambda\phi\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi - \\
& - \mathring{\nabla}^\lambda G_5\mathring{R}_{\lambda(\mu}\mathring{\nabla}_{\nu)}\phi + \mathring{\nabla}_\alpha\left[G_{5X}\mathring{\nabla}_\beta\phi\right]\mathring{\nabla}^\alpha\mathring{\nabla}_{(\mu}\phi\mathring{\nabla}^\beta\mathring{\nabla}_{\nu)}\phi - \mathring{\nabla}_{(\mu}G_5G_{\nu)\lambda}\mathring{\nabla}^\lambda\phi -
\end{aligned}$$

$$
\begin{aligned}
&-\mathring{\nabla}_\beta G_{5X}\left[\mathring{\Box}\phi\mathring{\nabla}^\beta\mathring{\nabla}_{(\mu}\phi-\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi\mathring{\nabla}_\alpha\mathring{\nabla}_{(\mu}\phi\right]\mathring{\nabla}_{\nu)}\phi+\\
&+\frac{1}{2}\mathring{\nabla}^\alpha\phi\mathring{\nabla}_\alpha G_{5X}\left[\mathring{\Box}\phi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi-\mathring{\nabla}_\beta\mathring{\nabla}_\mu\phi\mathring{\nabla}^\beta\mathring{\nabla}_\nu\phi\right]+\frac{1}{2}\mathring{\nabla}_\lambda G_5\mathring{G}_{\mu\nu}\mathring{\nabla}^\lambda\phi-\\
&-\frac{1}{2}G_{5X}\mathring{G}_{\alpha\beta}\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi\mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi-\frac{1}{2}G_{5X}\mathring{\Box}\phi\mathring{\nabla}_\alpha\mathring{\nabla}_\mu\phi\mathring{\nabla}^\alpha\mathring{\nabla}_\nu\phi+\\
&+\frac{1}{2}G_{5X}(\mathring{\Box}\phi)^2\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi+\frac{1}{12}G_{5XX}\left[(\mathring{\Box}\phi)^3-3\mathring{\Box}\phi(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2+2(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^3\right]\mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi+\\
&+g_{\mu\nu}\{-\frac{1}{6}G_{5X}\left[(\mathring{\Box}\phi)^3-3\mathring{\Box}\phi(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2+2(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^3\right]+\mathring{\nabla}_\alpha G_5\mathring{R}^{\alpha\beta}\mathring{\nabla}_\beta\phi-\\
&-\frac{1}{2}\mathring{\nabla}_\alpha\left(G_{5\phi}\mathring{\nabla}^\alpha\phi\right)\mathring{\Box}\phi+\frac{1}{2}\mathring{\nabla}_\alpha\left(G_{5\phi}\mathring{\nabla}_\beta\phi\right)\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi-\frac{1}{2}\mathring{\nabla}_\alpha G_{5X}\mathring{\nabla}^\alpha X\mathring{\Box}\phi+\\
&+\frac{1}{2}\mathring{\nabla}_\alpha G_{5X}\mathring{\nabla}_\beta X\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi-\frac{1}{4}\mathring{\nabla}^\lambda G_{5X}\mathring{\nabla}_\lambda\phi\left[(\mathring{\Box}\phi)^2-(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\phi)^2\right]+\\
&+\frac{1}{2}G_{5X}\mathring{R}_{\alpha\beta}\mathring{\nabla}^\alpha\phi\mathring{\nabla}^\beta\phi\mathring{\Box}\phi-\frac{1}{2}G_{5X}\mathring{R}_{\alpha\lambda\beta\rho}\mathring{\nabla}^\alpha\mathring{\nabla}^\beta\phi\mathring{\nabla}^\lambda\phi\mathring{\nabla}^\rho\phi\}\,,
\end{aligned}
\tag{3.34d}
$$

It is easy to see that, from (3.22), one can derive several already known models. For example, if $G_2=\frac{\omega}{\phi}X$, $G_3=0$, $G_4=\phi$, and $G_5=0$, we obtain the Brans-Dicke theory and so on.

Vector-Tensor theories

Theories that, apart from the metric, include also a vector field, are probably the less studied of all, since they contain serious pathogenies. However, there are two interesting examples that, even if they are not viable (for different reasons) are both worth mentioning. The first is the Einstein-æther theories and the second the Tensor-Vector-Scalar theories. The mathematical structure of both is complicated enough and therefore we will not present it here. However, the interested reader is referred to e.g. [274] and references therein.

The Einstein-æther theory was proposed by Jacobson and Mattingly [84] and it includes the metric and a vector field. The vector field introduces a preferred reference frame, dubbed *æther*, and thus the theory violates Lorentz invariance. This plays a key role in cosmology, in the sense that, it can lead to a renormalization of the Newton constant, it can help explaining the dynamics of the early universe, since there are many Lorentz-violating scenarios during inflation, as well as affect the growth rate of structure in the Universe.

Such theories however, even though they arise as effective field theories in quantum gravity, suffer from instabilities at the classical level. Moreover, when æther is studied to play the role of dark matter, it is impossible to fit simultaneously the CMB and the large scale structure data.

There is another gravity theory that contains a vector field responsible for gravity. It was proposed by Bekenstein in 2004 [85] as a relativistic extension of Milgrom's MOND, that we already mentioned in the introduction. Briefly, MOND proposes a modification of Newton's second law at low accelerations, in order to explain the rotation curves of galaxies without the need for dark matter. Bekenstein's theory contains a metric (rank-2 tensor), a vector and a scalar field, all of which participate in the gravitational sector. It is called TeVeS and it has been extensively studied in the literature. A very comprehensive review can be found here [86]. However, besides its successes it contains also some shortcomings, such as the un-stable spherically symmetric solutions, and these are the reasons that is not studied anymore.

Bimetric theories

In this section we will discuss theories that involve two rank-2 tensor fields and are called either tensor-tensor theories, or shorter, bimetric theories, even if there is no second metric in the strict geometric sense. Although in the literature, there have been several proposals including two metrics in the gravitational sector, such as Rosen's and Drummond's theory [87,88], as well as bigravity [89], we will present one such theories that is of interest today.

This is massive gravity. In this theory, gravity is described by a massive spin-2 field, $g_{\mu\nu}$ which can be decomposed to a non-dynamical background, $\tilde{g}_{\mu\nu}$ and a dynamical fluctuation $h_{\mu\nu}$. If we consider for simplicity, the background to be Minkowski, we can generate a mass for the spin-2 $h_{\mu\nu}$ field by adding a Fierz-Pauli term in the Einstein-Hilbert action. This gives

$$\mathcal{S} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left[\overset{\circ}{R} + \frac{m^2}{4} \left(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} \right) h_{\mu\nu} h_{\alpha\beta} \right] , \tag{3.35}$$

where m is the mass of the graviton $h_{\mu\nu}$. However, when dealing with a theory of massive gravity, there are two problems that tackle the progress since the seventies. The first one is related to the so-called vDVZ (van Dam-Veltman-Zakharov) discontinuity, and relies on the three extra degrees of freedom that the massive spin-2 field propagates, compared to GR. This however, is solved by the Vainshtein mechanism; the non-linearities are screened by their own interactions which dominate over the linear terms in the massless limit. The second problem that arise is the Bouldware-Deser (BD) ghost. This ghost-like term can be avoided either in the framework of the DGP (Dvali-Gabadadze-Porrati) model, or in the dRGT (de-Rham, Gabadadze, Tolley) massive gravity. However, after the recent observation of the gravitational wave signal and its electromagnetic part, of the binary neutron star merger, these theories tend to become obsolete.

3.2.2 Higher order theories

Another way to extend the theory of gravity is to allow the field equations to be higher than second order. In this way, the graviton propagator will decrease faster in the high energy regime, thus improving the renormalizability properties. However, it could also introduce instabilities in the classical regime, in the form of ghosts [90]. In the rest of this section, we will present the $f(R)-$theories of gravity both in the metric and in the Palatini formalism.

$f(R)-$theories

If instead of the Ricci scalar in the Einstein-Hilber action, we consider an arbitrary function of it, $f(R)$, we obtain the action of $f(R)-$theories

$$\mathcal{S} = \frac{1}{16\pi G_N} \int d^4x\sqrt{-g}f(\overset{\circ}{R}) + \mathcal{S}_m[g_{\mu\nu}, \psi] \,. \tag{3.36}$$

Varying this action with respect to the metric we obtain the field equations

$$f'(\overset{\circ}{R})\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}f(\overset{\circ}{R})g_{\mu\nu} + \left(g_{\mu\nu}\overset{\circ}{\Box} - \overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\right) f'(\overset{\circ}{R}) = 8\pi G_N T^M_{\mu\nu} \,, \tag{3.37}$$

where $T^M_{\mu\nu}$ is the energy-momentum tensor of all the matter fields, defined as usual

$$T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}} \,. \tag{3.38}$$

Unlike GR, the arbitrariness of $f(R)$ introduces an extra degree of freedom, and thus if we take the trace of the equation (3.37) we obtain

$$f'(\overset{\circ}{R})\overset{\circ}{R} - 2f(\overset{\circ}{R}) + 3\Box f'(\overset{\circ}{R}) = 8\pi G_N T^M \,, \tag{3.39}$$

which gives the relation between $\overset{\circ}{R}$ and T^M, being differential and not algebraical as in GR. This equation shows explicitly that $\overset{\circ}{R} = 0$ is no longer true when $T^M = 0$ and in addition the Birkhoff's theorem is violated, having much reacher phenomenology in this case.

After some manipulations, one can rewrite the equation (3.37) as

$$\overset{\circ}{G}_{\mu\nu} = \frac{8\pi G_N}{f'(\overset{\circ}{R})}\left[T^M_{\mu\nu} + \frac{g_{\mu\nu}\left(f(\overset{\circ}{R}) - \overset{\circ}{R}f'(\overset{\circ}{R})\right)}{2} + \overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu f'(\overset{\circ}{R}) - g_{\mu\nu}f'(\overset{\circ}{R})\right] \,, \tag{3.40}$$

and we can define $G_{\rm eff} = G_N/f'(\overset{\circ}{R})$ to be the effective gravitational coupling ($G_{\rm eff} > 0 \Rightarrow f'(\overset{\circ}{R}) > 0$), while

$$T^{\rm eff}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(f(\overset{\circ}{R}) - \overset{\circ}{R}f'(\overset{\circ}{R})\right) + \overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu f'(\overset{\circ}{R}) - g_{\mu\nu}f'(\overset{\circ}{R})\,, \tag{3.41}$$

is an effective energy-momentum tensor and it can be considered as a “geometrical fluid”. By doing so, one can find cosmological solutions and also set constraints to the form of $f(R)$ and its derivatives.

Moving on to the Palatini formalism, we consider the connection to be independent from the metric and thus all the curvature invariants are calculated only from the connection without the need of a metric. The new action takes now the form

$$\mathcal{S}_{\rm Palatini} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}f(R) + \mathcal{S}_m[g_{\mu\nu},\psi]\,. \tag{3.42}$$

In this approach, all the matter fields, ψ, couple uniquely to the metric $g_{\mu\nu}$ and not the connection. For an extended discussion about the significance of matter coupling in $f(R)$ theories see [91].

Varying the action with respect to the metric and the connection we get respectively,

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G_N T^M_{\mu\nu}\,, \tag{3.43}$$

$$\nabla_\alpha\left(\sqrt{-g}f'(R)g^{\mu\nu}\right) = 0\,, \tag{3.44}$$

It is easily seen that if we set $f(R) = R$, Eq. (3.44) becomes the non-metricity condition, leading the connection to be the Levi-Civita one. Respectively, $R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu}$ and $R = \overset{\circ}{R}$, meaning that we reduce th Palatini GR.

If we consider a conformal metric of the form $h_{\mu\nu} = f'(R)g_{\mu\nu}$, it is easy to see that in four dimensions

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}f'(R)g^{\mu\nu} \tag{3.45}$$

and Eq. (3.44) yields

$$\begin{aligned}\Gamma^\alpha{}_{\mu\nu} &= \frac{1}{2}h^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right)\,,\\ &= \frac{1}{2f'(R)}g^{\alpha\beta}\left[\partial_\mu(f'(R)g_{\mu\beta}) + \partial_\nu(f'(R)g_{\beta\nu}) - \partial_\beta(f'(R)g_{\mu\nu})\right]\,,\end{aligned} \tag{3.46}$$

which is the Levi-Civita connection for $h_{\mu\nu}$. From this we can also deduce the relations $R_{\mu\nu}(\Gamma) = R_{\mu\nu}(h)$, $R(\Gamma) = R(h)$ and thus $G_{\mu\nu}(\Gamma) = G_{\mu\nu}(h)$. In addition, from the trace of Eq. (3.43) we obtain

$$Rf'(R) - 2f(R) = 8\pi G_N T^M \Rightarrow R = R(T^M)\,. \tag{3.47}$$

By combining these, we end up with only one equation depending only on the metric ($g_{\mu\nu}$) and the matter fields ($T^M_{\mu\nu}$ and T^M) and the theory resembles GR with a modified source.

Last but not least, there is even another formulation of $f(R)$ theories, which in a sense generalizes the Palatini one. As we already discussed, the matter fields in the Palatini approach do not couple to the independent connection (by assumption), but only to the metric, satisfying thus the Equivalence Principle. In this way, we managed to treat the independent connection as an auxiliary field and we wrote the equations in a GR-like form. However, in principle, the independent connection should also define the covariant derivatives of the matter fields, meaning that the action of this theory would take now the form

$$\mathcal{S}_{\text{Metric-affine}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} f(R) + \mathcal{S}_m[g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}, \psi] \,. \tag{3.48}$$

We will not elaborate more on the structure of this formalism, since it is out of the topic of this thesis. However, we refer the interested reader to [274] and [92].

3.3 Conclusions

Summing up, what we presented in this chapter is the fundamental construction of General Relativity, i.e. Einstein's theory. We discussed the background geometry of the theory, the action that describes it and we derived its equations of motion. Even though, this was not the way the theory was historically built, it gives a better picture of the basics of a relativistic theory.

In the rest of the chapter, we discussed modifications of the GR paradigm. In particular, we studied theories with extra fields, i.e. scalar-tensor, vector-tensor and tensor-tensor (or bimetric) theories. Of course, this was only an overview of the most important theories that are still viable, and of the theories that we will study in detail later on. For a review on all the possible modifications, the reader can see the suggested literature thoughout the chapter.

The main purpose of the chapter, was to construct a theory of gravity using only the symmetric part of a general affine connection (see discussion in chapter 2), which is the Levi-Civita connection. In the next two chapters we will discuss two conceptually different theories from GR, but completely equivalent.

Chapter 4

Torsion based Gravity: the teleparallel theory

«I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect.»

- Carl Friedrich Gauss

« (...) geometry (which is the only science that it hath pleased God hitherto to bestow on mankind) (...)»

- Thomas Hobbes, Leviathan

In the previous chapter, we described the theory of gravity which is based on the curvature of spacetime. In this chapter, we will study another formulation of gravity, whihc lead to the exact same results as GR, but has totally different phenomenology. Specifically, we will present the Teleparallel Equivalent of General Relativity which at the level of equations is the same as GR, however, the spacetime is flat, but endowed with torsion.

4.1 Teleparallel theory

As we already have discussed so far, shortly after the formulation of GR, alternatives were being pursued. H. Weyl attempted to unify gravitation and electromagnetism in 1918 [93], but he was unsuccessful. Einstein himself [94] tried to do the same, using *teleparallelism*. The idea is to introduce a tetrad field as the dynamical field in the theory, instead of the metric. The tetrad fields are orthonormal bases on the tangest space of each point of a four-dimensional manifold. The have sixteen components, while metric has only ten. So what Einstein thought is, to use the abondant six degrees of freedom to describe the components of the electromagnetic field. However, it turns out that these extra degrees of freedom are constrained by the 6-parameter local Lorentz invariance of the theory.

Even though unsuccessful, Weyl's and Einstein's attempts introduced the notion of gauge theories and thus the hunt for a gauge theory of gravitation begun [95, 96]. In 1979, K. Hayasko and T. Shirafuji [97] proposed a "new general relativity" which is the gauge theory of the translation group, using the idea of teleparallelism. This new GR contains three free parameters to be fixed by experiment, and instead of curvature, gravity was mediated by torsion.

It turns out that, for a specific choice of these three parameters, their theory is completely equivalent to GR and that is why it is known as the *Teleparallel Equivalent to General Relativity* (TEGR). However, there is a fundamental conceptual difference between the two theories: In the context of GR, curvature is used to treat gravitational interactions as geometry, meaning that there is no notion of a gravitational *force* and test-particles move on geodesics. On the contrary, TEGR uses torsion to describe gravity, but here torsion plays the role of a force, just like the Lorentz force in electromagnetism. Consequently, there are no geodesics in TEGR, but only force equations.

At the level of equations, the two theories are identical and at the level of actions they differ by a total derivative term. However, even though conceptually different, the two theories cannot be distinguished by experiments. In this chapter we are going to present all the necessary mathematical tools needed, as well as the action and the equations of motion for the dynamical field of the theory, the tetrad. Furthermore, as we did in the previous chapter, we will present extensions and modifications proposed in the literature. We will use the book by Aldrovandi and Pereira [98] as well as the review [99], but we will stick to the essentials and details are not going to be discussed. So the interested reader is strongly referred to these.

4.2 Basic Notions

Even though we already discussed in chapter 2 the affine structure of spacetime, we have to devote some time to explain the tetrad formalism, needed for the description of TEGR. As every theory of gravitation, TEGR is also built on a four-dimensional manifold. At each point of the manifold there is a tangent space of Minkwoski type, represented by the metric

$$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)\,. \tag{4.1}$$

For clarity, we will denote the spacetime indices with Greek letters, $\alpha, \beta, ..., \mu, \nu, ...$ and the tangent space indices with Latin letters $a, b, ..., i, j, ...$. Spacetime coordinates will be $\{x^\mu\}$, while tangent space coordinates $\{x^a\}$. Thus, the local basis in each space will be given by $\{\partial_\mu\} = \{\partial/\partial x^\mu\}$ and $\{\partial_a\} = \{\partial/\partial x^a\}$ respectively. The dynamical fields of the theory are the four linearly independent *vierbeins*, or else *tetrads* and their relation with the spacetime metric and its inverse is given by

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \quad g^{\mu\nu} = \eta^{ab} E^\mu_a E^\nu_b \,, \tag{4.2}$$

where η_{ab} is the flat Minkowski metric of the tangent space and e^a_μ, E^μ_a are the tetrads and the inverse tetrads. Obviously, we used the orthonoraml conditions

$$e^a_\mu E^\nu_a = \delta^\nu_\mu \text{ and } e^a_\mu E^\mu_b = \delta^a_b \,. \tag{4.3}$$

The basis is written as

$$e_a = E^\mu_a \partial_\mu \quad e^a = e^a_\mu dx^\mu \,, \tag{4.4}$$

and conversely

$$\partial_\mu = e^a_\mu e_a \quad dx^\mu = E^\mu_a e^a \,. \tag{4.5}$$

4.3 TEGR as a gauge theory

As we already mentioned, the TEGR is a gauge theory for the translation group. The theory is constructed on the tangent bundle, i.e. the tangent-Minkowski space, defined at each point p of the base-Rimannian space. If we translate the coordinates x^a of the tangent bundle, by $\epsilon^a(x^\mu)$ (a point-dependent parameter), or mathematically,

$$x^a \to x'^a = x^a + \epsilon^a \,, \tag{4.6}$$

then we say, we have performed a gauge transformation. The generators of infinitesimal translations are the differential operators, $P_a = \partial_a$, which satisfy the commutation relations

$$[P_a, P_b] = 0 \,. \tag{4.7}$$

An arbitrary scalar field $\psi = \psi(x^a(x^\mu))$ transforms under (4.6) as

$$\delta\psi = \epsilon^a \partial_a \psi \,, \tag{4.8}$$

that is covariant, but its ordinary derivative transforms according to

$$\delta\left(\partial_\mu \psi\right) = \epsilon^a \partial_a \left(\partial_\mu \psi\right) + \left(\partial_\mu \epsilon^a\right) \partial_a \psi \,. \tag{4.9}$$

It is a usual prescription in the theory of gauge fields [100], to replace ordinary derivatives by covariant derivatives involving a connection. In this case, we have to introduce a gauge potential B_μ, which belongs to the Lie algebra of the translation group

$$B_\mu = B^a{}_\mu P_a \,. \tag{4.10}$$

It is now easy to see that the quantity

$$e_\mu \psi = \partial_\mu \psi + B^a{}_\mu \partial_a \psi \tag{4.11}$$

transforms covariantly under (4.6)

$$\delta(e_\mu \psi) = \epsilon^a \partial_a (e_\mu \psi) \,, \tag{4.12}$$

if the gauge potential $B^a{}_\mu$ transforms as $\delta B^a{}_\mu \to -\partial_\mu \epsilon^a$. So finally, we have the coupling prescription for the translations

$$\partial_\mu \psi = \partial_\mu x^a \partial_a \psi \to e_\mu \psi = \partial_\mu \psi + B_\mu \psi = (\partial_\mu x^a + B^a{}_\mu) \partial_a \psi = e^a_\mu \partial_a \psi \,. \tag{4.13}$$

In gravity, however, things are a bit more complicated due to the background dependence. There is a first necessary replacement, that is universal for all matter fields, of the Minkowski metric by a general pseudo-Riemannian metric

$$\eta_{\mu\nu} \to g_{\mu\nu} \,,$$

but there is another one too, related to the coupling of the spins of matter fields to gravity (thus not universal) and is related to the requirement of covariance under local Lorentz transformations.

So let us now move on, to a general Lorentz frame, and consider the local Lorentz transformation $x^a \to x'^a = \Lambda^a_b x^b$. The scalar field ψ transforms under this transformation as $\psi \to U(\Lambda)\psi$, where $U(\Lambda)$ is an element of the Lorentz group. Taking into account that $B^a{}_\mu \to \Lambda_b{}^a B^b{}_\mu$, it is stragithforward to see that the covariant derivative (4.13) transforms covariantly

$$e_\mu \psi \to U(\Lambda) e_\mu \psi \,, \tag{4.14}$$

where now

$$e^a_\mu = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu \,, \tag{4.15}$$

where we have defined a purely inertial Lorentz connection $\overset{\bullet}{\omega}{}^b_{c\mu} = \Lambda^b_d \partial_\mu \Lambda^{d}_c$[1]. We can introduce now the Lorentz covariant derivative

$$\overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b \,, \tag{4.16}$$

[1]We remind here that the quantities with a bullet, • on top, are defined through the antisymmetric part of the connection, i.e. the Weitzenböck connection, to be distinguished from those with a circle, ◦, which are calculated by the Levi-Civita connection.

and thus, the tetrad becomes

$$e^a_\mu = \dot{\mathcal{D}}_\mu x^a + B^a_\mu\,, \tag{4.17}$$

and the gauge potential transforms according to

$$\delta B^a_\mu = -\dot{\mathcal{D}}_\mu \epsilon^a\,. \tag{4.18}$$

In this way the tetrad remains gauge invariant.

As far as the spin connection is conserned, in order to obtain its coupling prescription, we have to use the principle of general covariance. For details one can check the book [98], but for a general source field ψ we have

$$\partial_a\psi \to \mathcal{D}_a\psi = e_a\psi - \frac{i}{2}\left(\dot{\omega}^{bc}{}_a - \dot{K}^{bc}{}_a\right) S_{bc}\psi\,, \tag{4.19}$$

where S_{ab} are the generators of the Lorentz group and

$$\dot{K}^a{}_{bc} = \frac{1}{2}\left(\dot{T}_b{}^a{}_c + \dot{T}_c{}^a{}_b - \dot{T}^a{}_{bc}\right)\,, \tag{4.20}$$

is the contorsion tensor in the tetrad frame.

So finally, the gravitational coupling prescription is

$$\partial_\mu\psi \to \mathcal{D}_\mu\psi = \partial_\mu\psi - \frac{i}{2}\left(\dot{\omega}^{ab}{}_\mu - \dot{K}^{ab}{}_\mu\right) S_{ab}\psi\,. \tag{4.21}$$

The curvature of the teleparallel connection $\dot{\omega}$ is given by

$$\dot{R}^a{}_{b\nu\mu} = \partial_\nu\dot{\omega}^a{}_{b\mu} - \partial_\mu\dot{\omega}^a{}_{b\nu} + \dot{\omega}^a{}_{e\nu}\dot{\omega}^e{}_{b\mu} - \dot{\omega}^a{}_{e\mu}\dot{\omega}^e{}_{b\nu}\,, \tag{4.22}$$

and as it is easily seen it vanishes identically, $\dot{R}^a{}_{b\nu\mu} = 0$. This is called the *teleparallel condition* and means that the spacetime of TEGR is flat. However, the torsion

$$\dot{T}^a{}_{\nu\mu} = \partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu + \dot{\omega}^a{}_{e\nu}e^e{}_\mu - \dot{\omega}^a{}_{e\mu}e^e{}_\nu\,, \tag{4.23}$$

is not zero, meaning that gravity is mediated through torsion, in a globally flat spacetime.

The linear connection corresponding to the spin connection $\dot{\omega}^a{}_{b\mu}$ is

$$\dot{\Gamma}^\rho{}_{\nu\mu} = E_a{}^\rho\left(\partial_\mu e^a{}_\nu + \dot{\omega}^a{}_{b\mu}e^b{}_\nu\right) = E_a{}^\rho\dot{\mathcal{D}}_\mu e^a{}_\nu\,. \tag{4.24}$$

This is the *Weitzenböck connection*, which is anti-symmetric in its two lower indices and its relation with the Levi-Civita one is given by the Ricci theorem

$$\dot{\Gamma}^\rho{}_{\mu\nu} = \mathring{\Gamma}^\rho{}_{\mu\nu} + \dot{K}^\rho{}_{\mu\nu}\,. \tag{4.25}$$

In gauge theories, we can find the field strength by the commutation relation of gauge covariant derivatives. In this case Eq. (4.15) will give

$$[e_\mu, e_\nu] = e_\mu e_\nu - e_\nu e_\mu = e^a_\mu \partial_a e^b_\nu \partial_b - e^a_\nu \partial_a e^b_\mu \partial_b = \left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} B^b_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} B^b_\mu\right) \partial_a \,. \tag{4.26}$$

We define the field strength as

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} B^b_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} B^b_\mu = \overset{\bullet}{\mathcal{D}}_\mu B^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a_\mu \,, \tag{4.27}$$

the covariant rotational of the gauge potential. Using now $[\overset{\bullet}{\mathcal{D}}_\mu, \overset{\bullet}{\mathcal{D}}_\nu] x^a = 0$ we get[2]

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} e^b_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} e^b_\mu = \overset{\bullet}{\mathcal{D}}_\mu e^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu e^a_\mu = \overset{\bullet}{\Gamma}{}^a{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^a{}_{\nu\mu} \,. \tag{4.28}$$

Summarizing, in TEGR instead of geometrizing gravity, we obtain the gravitational interactions as a gauge theory for the translation group. Its field strength is torsion, meaning that gravity is mediated through torsion in a flat spacetime, and the connection of the spacetime is purely anti-symmetric and it's called Weitzenböck connection.

4.4 Action and field equations

We have already mentioned that the Teleparallel theory is completely equivalent to General Relativity, even though conceptually they differ. In this section, we will prove this equivalence at a mathematical level, discussing the action and field equations of TEGR.

In order to relate the topic of this chapter with the discussion in chapter 2, let us pause for a moment, to make some comments. Eq. (2.4) shows how a general affine connection can be split into three parts. We already studied the case where the connection is torsionless and metric compatible, i.e. $K^\lambda{}_{\mu\nu} = 0 = L^\lambda{}_{\mu\nu}$, and this is GR. GR is built on the Levi-Civita connection, has the metric as the dynamical field and the only property of the connection that does not vanish is curvature. In this chapter, we discuss the theory in which the spacetime is flat and metric compatible (the tetrad is also covariantly conserved), but has non-zero torsion. In TEGR all the quantities (including torsion) are built on the Weitzenböck connection and the dynamical field is the tetrad. In the next chapter, we will study the Symmetric Teleparallel theory where both the curvature and the torsion vanish, but the non-metricity does not.

[2] From the next section we omit the bullet for the torsion and the contorsion tensors and their contractions, since they are non-zero only in the Teleparallel theory.

Coming back to the previous discussion: using the Eq. (2.9) together with Eq. (4.25), it is easy to see that

$$\overset{\bullet}{R} = \overset{\circ}{R} + T - \frac{2}{e}\partial_\mu(eT^\mu)\,, \tag{4.29}$$

where

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\nu\mu\lambda} - T^\mu T_\mu\,, \tag{4.30}$$

is the torsion scalar and $T_\mu = T^\lambda{}_{\lambda\mu}$ is the torsion vector. For simplicity, we can also define the *superpotential*

$$S_\lambda{}^{\mu\nu} = \frac{1}{2}\left(K_\lambda{}^{\mu\nu} - \delta^\mu_\lambda T^\nu + \delta^\nu_\lambda T^\mu\right)\,, \tag{4.31}$$

in order to re-express the torsion scalar in a more compact form, $T = S_\lambda{}^{\mu\nu}T^\lambda{}_{\mu\nu}$.

Going back to Eq. (4.29), we know from the teleparallel condition that $\overset{\bullet}{R} = 0$ and thus

$$\overset{\bullet}{R} = 0 \Rightarrow \overset{\circ}{R} = -T + B\,, \tag{4.32}$$

where we defined the *boundary term*, $B = \frac{2}{e}\partial_\mu(eT^\mu)$. The above equation, (4.32), gives the relation between the Ricci scalar and the torsion scalar. As we notice, their difference is only the boundary term, B.

Constructing the action for the Teleparallel theory by its field strength [98], we see that it reads

$$\mathcal{S}_{\text{TEGR}} = \frac{1}{16\pi G_N}\int d^4xeT + \mathcal{S}_m[e^a{}_\mu, \psi]\,, \tag{4.33}$$

where $e = \det(e^a{}_\mu) = \sqrt{-g}$ is the determinant of the tetrad field, T is the torsion scalar and $\mathcal{S}_m$ a matter action. Moreover, $1/16\pi G_N$ is there to verify that Newtonian theory will be recovered in the weak field limit.

A comment here is appropriate: the Ricci scalar differs from the torsion scalar by a total derivative; moreover, the Einstein-Hilbert action is linear to the Ricci scalar and the TEGR action is linear to the torsion scalar. This means that the two actions differ only by a boundary term

$$\mathcal{S}_{\text{EH}} - \mathcal{S}_{\text{TEGR}} = \frac{1}{16\pi G_N}\int d^4xe(\overset{\circ}{R}+T) = \frac{1}{16\pi G_N}\int d^4xeB = \frac{1}{16\pi G_N}\int d^4x\partial_\mu(eT^\mu)\,. \tag{4.34}$$

This means that, by varying the action (4.33) with respect to the tetrads one will get the exact same equations as the Einstein equations, proving that the two theories are equivalent. It turns out that, it is a matter of choice to believe that gravity is mediated through curvature or through torsion. Having the same equations of motion, the two theories cannot be experimentally distinguished.

Varying this action with respect to the tetrad[3], we get

$$\frac{4}{e}\partial_\mu(eS_a{}^{\mu\nu}) - 4T^\rho{}_{\mu a}S_\rho{}^{\nu\mu} - TE_a^\nu = 16\pi G_N T_a^{\rho,M}\,, \tag{4.35}$$

which are the tetrad field equations, with the stress-energy tensor of the matter fields defined as

$$T_a^{\rho,M} = \frac{1}{e}\frac{\delta(e\mathcal{L}_m)}{\delta e_\rho^a}\,. \tag{4.36}$$

Finally, before we proceed discussing modifications, we should check how particles move in the gravitational field. Specifically, just as in GR, the motion of test-particles of mass m and spin-0 is described by

$$\mathcal{S} = m\int_a^b ds\,. \tag{4.37}$$

After some straightforward manipulations and variation of the action with respect to the orbit in the tangent space, δx^a, we find that the motion of the particle is governed by

$$\frac{du_\mu}{ds} - \overset{\bullet}{\Gamma}{}^\lambda{}_{\mu\nu}u_\lambda u^\nu = -K^\lambda{}_{\mu\nu}u_\lambda u^\nu\,, \tag{4.38}$$

which is known as the *Teleparallel force equation*, since the contorsion tensor acts as a force. Using (4.25), it is easy to see that the force equation yields the geodesic equation of GR. However, the similarity is just dynamical; in TEGR there is no geodesic motion since the covariant derivative of the four-velocity of the particle is not conserved.

4.5 Extending Teleparallel gravity: The case of $f(T)$

Up to now in this chapter, we presented the Teleparallel Equivalent of General Relativity, which, as its name denotes, is a theory of gravitation equivalent to GR. At the level of the action and field equations the two theories are the same, but conceptually they differ. In addition, in the first chapter 1, we mentioned several reasons why an alternative description of the gravitational interactions should be pursued. Based on these, there have been proposed in the literature, modifications of the TEGR. This is going to be the subject of study for the rest of this chapter.

[3]For computational reasons mostly, Teleparallel gravity (up to [101]) was formulated in the so-called *pure tetrad formalism*, where the spin connection is set to zero. As we already discussed, the spin connection is related to inertial effects and thus it is like choosing a specific frame to perform our calculations. This will lead to Lorentz violations of the torsion scalar and more details will be given in the next section.

Specifically, we are going to present the well-known $f(T)$ gravity, which is an extension of TEGR, just like $f(\overset{\circ}{R})$ in GR. There are many other modifications of TEGR, such as teleparallel scalar-tensor theories [102,103], which are very good candidates for the late-time acceleration of the Universe, theories including couplings between the torsion scalar and the boundary term, $f(T,B)$ [104,105], theories including the Gauss-Bonnet teleparallel term, $f(T,T_{\mathcal{G}})$ [106,107], theories with decomposition of the torsion tensor to its axial, tensorial and vectorial parts, $f(T_{\rm ax},T_{\rm ten},T_{\rm vec})$ [108], and more. As we already mentioned, our task is not to review all of them, but to give a general idea of the field. For more details one can check the references given. Finally, apart from the $f(T)$ gravity, in the chapter 10 we are going to discuss also teleparallel non-local theories [109,110].

The most well studied and also very straightforward generalization of TEGR, is the so-called $f(T)$ theory. Just as in GR with $f(\overset{\circ}{R})$ gravity, generalizing the TEGR action from a linear term in the torsion scalar, to an arbitrary function of it, we allow much more phenomenology to the theory. However, as we will see, even though GR is equivalent to TEGR, $f(T)$ is not equivalent to $f(\overset{\circ}{R})$ as one might expect. As before, we will insist on the basics of the theory, i.e. action, equations of motion and phenomenology. However, we refer the reader to the extended review on the subject [99].

The action of $f(T)$ theory reads [111]

$$\mathcal{S}_{f(T)} = \frac{1}{16\pi G_N}\int d^4x e f(T) + \mathcal{S}_m \tag{4.39}$$

Before we proceed by varying this action, let us discuss the problem regarding Lorentz invariance, already mentioned in the footnote[3]. For an arbitrary ($\equiv$ non-zero) spin connection, the torsion tensor (4.28) is generally covariant, i.e. invariant under infinitesimal coordinate transformations $x^\mu \to x^\mu + \epsilon^\mu$, which is expected being a tensor, as well as local Lorentz invariant, $T^a{}_{\mu\nu} \to \Lambda^a_b T^b{}_{\mu\nu}$. Therefore, the torsion scalar T is also generally covariant and invariant under local Lorentz transformations.

As we mentioned however, for computational reasons TEGR was formulated in the vanishing spin-connection frame, where the torsion tensor is no longer local Lorentz invariant

$$T^a{}_{\mu\nu} \to \Lambda^a_b T^b{}_{\mu\nu} + \Lambda^a_b(e^c_\nu\partial_\mu\Lambda^b_c - e^c_\mu\partial_\nu\Lambda^b_c)\,. \tag{4.40}$$

This means that the torsion scalar is Lorentz violating. It turns out though [112], that the TEGR Lagrangian and thus the field equations, are invariant under local Lorentz transformations, because the Lorentz violating term in the torsion scalar is just a boundary term, which does not contribute when integrated,

$$T(e^a_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}) = T(e^a_\mu, 0) + \frac{4}{e}\partial_\mu(e\overset{\bullet}{\omega}{}^\mu)\,. \tag{4.41}$$

In practice, this means that any linear combination of the torsion scalar in the action will be Lorentz invariant. However, when one moves to generalizations of the form of $f(T)$ things change [113–115]. The surface term is no longer a total derivative and thus it will contribute in the action integral. It seems that there are two alternatives to overcome this problem.

The most well known up to now, is to continue working in the pure tetrad formalism, where the spin connection vanishes. Indeed, the field equations will violate Lorentz invariance, meaning that different tetrads will give different field equations, which consecutively might give different solutions. However, some of these solutions will not match with GR solutions, while others will do in the appropriate limit. This means that we should carefully choose those tetrads, the *good tetrads* [116], which do not constrain the form of $f(T)$ and one can always consider the limit $f(T) \to T$.

The other alternative is to consider a general spin connection different from zero. This means to formulate $f(T)$ gravity in a frame-independent way, where one uses both the tetrad and the spin connection, in such a way that, for every tetrad choice, a suitably constructed connection makes the theory covariant and Lorentz invariant. This was first discussed in [101]. However, it is still an open problem [112,115,117], since we do not know yet, how to find the appropriate spin connection for each tetrad choice. This is why, we are going to stick with the first approach, i.e. the pure tetrad formalism, for lack of a better choice.

Coming back to varying the action (4.39) with repsect to the tetrad we obtain the field equations

$$4ef''(T)(\partial_\mu T)S_\nu{}^{\mu\lambda} + 4e^a_\nu\partial_\mu\left(eS_a{}^{\mu\lambda}\right)f'(T) - 4ef'(T)T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef(T)\delta^\lambda_\nu = 16\pi G_N e T^{\lambda,M}_\nu\,, \tag{4.42}$$

where the energy-momentum tensor of the matter fields, $T^{\lambda,M}_\nu$ is defined as before (4.36). In addition, as expected, replacing $f(T) \to T$ we recover the field equations of TEGR.

Before we close this chapter, it is interesting to notice that, even though the Teleparallel theory is completely equivalent to GR, since the Ricci scalar and the torsion scalar differ only by a total derivative term, it seems that the same does not happen for $f(\overset{\circ}{R})$ and $f(T)$ theories. One would expect the boundary term to behave completely arbitrarily, for non-linear terms of the torsion tensor. However, one might say that $f(T)$ has more similarities with TEGR, than $f(\overset{\circ}{R})$ with GR, since the equations of motion in the former are of second order just as in TEGR, in GR and in all physical theories, while in $f(\overset{\circ}{R})$ the equations are of fourth order.

4.6 Conclusions

An equivalent to GR description of gravitational interactions was presented in this chapter. Specifically, we showed that, if one allows the spacetime (or better, the connection) to have torsion and imposes the teleparallel condition, i.e. $R^{\alpha}{}_{\mu\beta\nu} =$, then, in the case where the gravitational field (either the metric or the tetrad) is compatible, the connection becomes purely anti-symmetric, i.e. Weitzenböck.

In this case, gravity is mediated in a flat spacetime through torsion and the test-particles do not follow geodesics, as in GR, but rather, obey a force equation, as in electrodynamics. This theory is called Teleparallel theory or Teleparallel Equivalent of General Relativity (TEGR) and it is a gauge theory of the translations. Its action and thus its equations is exactly the same with the Einstein-Hilbert action, but unplagued by boundary terms. This means that the two theories are completely equivalent.

Last but not least, we discussed the most straightforward generalization of TEGR, which is the $f(T)$ theories. It is of great interest, the fact that $f(T)$ theories consist of a better extension of TEGR than $f(R)$ theories of GR, because of the fact that they have second order field equations.

Chapter 5

Symmetric Teleparallel Gravity

«(...) by natural selection our mind has adapted itself to the conditions of the external world. It has adopted the geometry most advantageous to the species or, in other words, the most convenient. Geometry is not true, it is advantageous.»

- Henri Poincaré, Science and Method

In the previous two chapters we studied two equivalent descriptions of gravity. Specifically, we presented General Relativity, Einstein's theory based on the curvature of spacetime to mediate the effect of gravity and Teleparallel gravity (or Teleparallel Equivalent of General Relativity) which is a gauge theory for the group of translations and uses torsion to describe the gravitational interactions. As we mentioned though, in chapter 2, there is a third equivalent description that uses non-metricity in a torsionless (thus symmetric), flat (thus teleparallel) spacetime. It comes with the name Symmetric Teleparallel gravity or Symmetric Teleparallel Equivalent of General Relativity (STEGR) and was firstly discussed in [118] and later developed in [119–121]. This is going to be the subject of this chapter. We are going to discuss its basic notions, as well as possible modifications that can be or have been studied.

5.1 Symmetric Teleparallel theory

Unlike General Relativity, both TEGR and STEGR came after years of work of many people and not just one. However, as we already mentioned in the beginning of the previous chapter, the work of many scientists was based especially on the work of Weyl, who among other things, introduced the notion of gauge theory, and also Einstein who tried to unify gravity with electrodynamics, using teleparallelism.

The major advantage in reformulating GR in non-Riemannian and specifically in teleparallel geometries relies on the fact that, teleparallel formulations can be regarded as gauge theories of translations and in addition, in their context the energy momentum is defined properly [122]. Apart from the standard teleparallel representation of GR, where the metric is compatible, curvature vanishes and torsion does not, there exist other possibilities too.

If one considers a general, non-metric compatible, non-symmetric, teleparallel (with vanishing curvature) connection, they can always choose a case geometry in which the torsion vanishes, but the non-metricity does not. This is the so-called *Symmetric Teleparallel* formulation of GR. Symmetric because the torsion is zero and teleparallel because the curvature is zero.

Let us be mathematically precise. In chapter 2, we showed that the curvature is defined through the connection as

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}{}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\beta\mu} \,. \tag{5.1}$$

and the torsion and non-metricity tensors

$$T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]} \,, \quad Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} \,. \tag{5.2}$$

A general affine connection can be decomposed as

$$\Gamma^{\alpha}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu} \tag{5.3}$$

where

$$\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \,, \tag{5.4}$$

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(T_{\mu\beta\nu} + T_{\nu\beta\mu} + T_{\beta\mu\nu}\right) \,, \tag{5.5}$$

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}\right) \,, \tag{5.6}$$

are the Levi-Civita connection, the contorsion tensor and the disformation tensor respectively.

The curvature (5.1) can be rewritten using covariant derivatives of the Levi-Civita connection as

$$\begin{aligned} R^{\alpha}{}_{\mu\beta\nu} = \overset{\circ}{R}{}^{\alpha}{}_{\mu\beta\nu} &+ \overset{\circ}{\nabla}_\beta (K^{\alpha}{}_{\nu\mu} + L^{\alpha}{}_{\nu\mu}) - \overset{\circ}{\nabla}_\nu (K^{\alpha}{}_{\beta\mu} + L^{\alpha}{}_{\beta\mu}) + \\ &+ (K^{\rho}{}_{\nu\mu} + L^{\rho}{}_{\nu\mu})(K^{\alpha}{}_{\beta\rho} + L^{\alpha}{}_{\beta\rho}) - (K^{\rho}{}_{\beta\mu} + L^{\rho}{}_{\beta\mu})(K^{\alpha}{}_{\nu\rho} + L^{\alpha}{}_{\nu\rho}) \,, \end{aligned} \tag{5.7}$$

and after contractions we get

$$\begin{aligned} R = \overset{\circ}{R} &+ (K^{\rho}{}_{\nu\mu} + L^{\rho}{}_{\nu\mu})(K^{\beta}{}_{\beta\rho} + L^{\beta}{}_{\beta\rho}) g^{\nu\mu} - (K^{\rho}{}_{\beta\mu} + L^{\rho}{}_{\beta\mu})(K^{\beta}{}_{\nu\rho} + L^{\beta}{}_{\nu\rho}) g^{\nu\mu} + \\ &+ \overset{\circ}{\nabla}_\beta \left((K^{\beta}{}_{\nu\mu} + L^{\beta}{}_{\nu\mu}) g^{\nu\mu} - (K^{\nu}{}_{\nu\mu} + L^{\nu}{}_{\nu\mu}) g^{\nu\mu} \right) \,. \end{aligned} \tag{5.8}$$

The teleparallel condition sets $R^{\alpha}{}_{\mu\beta\nu} = 0$ and the symmetry condition $T^{\alpha}{}_{\mu\nu} = 0$ so (5.8) takes the form

$$\overset{\circ}{R} = Q - \overset{\circ}{\nabla}_{\alpha}(Q^{\alpha} - \tilde{Q}^{\alpha})\,. \tag{5.9}$$

This is the relation between the Ricci scalar (in GR) and the non-metricity scalar defined as

$$\mathcal{Q} = -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\gamma\beta\alpha} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\tilde{Q}^{\alpha}\,, \tag{5.10}$$

with $Q_{\alpha} = Q_{\alpha}{}^{\mu}{}_{\mu}$ and $\tilde{Q}^{\alpha} = Q_{\mu}{}^{\mu\alpha}$.

The relation (5.9) is the counterpart of the Eq. (4.32) and by putting them all together we have

$$\overset{\circ}{R} = \mathcal{Q} - B_{Q} = -T + B_{T}\,. \tag{5.11}$$

where B_T and B_Q are the two boundary terms

$$B_{T} = \frac{2}{e}\partial_{\mu}(eT^{\mu}) \quad \text{and} \quad B_{Q} = \overset{\circ}{\nabla}_{\alpha}(Q^{\alpha} - \tilde{Q}^{\alpha})\,. \tag{5.12}$$

This is the relation which proves that both Teleparallel theory and Symmetric Teleparallel theory are equivalent descriptions to GR. They both differ from the E-H action by a boundary term, meaning that they will all yield the same field equations. Furthermore, TEGR and STEGR may have an advantage compared to GR, that appears in the variation of the action. The Einstein-Hilber action contains the curvature scalar which is generally covariant but it depends not only on the metric and its first derivatives (which after variations would lead to second order equations) but also on the second derivatives of the metric. Apart from that, it diverges asymptotically as $O(1/r^3)$. If we subtract a total derivative term, this is improved to a convergent $O(1/r^4)$ but then the energy-momentum from Noether arguments, will lead to pseudotensors. On the other hand, in TEGR and STEGR the action is covariant and asymptotically convergent that generates a covariant energy-momentum tensor.

Coming back to STEGR. The counterpart of the superpotential (4.31) in TEGR is the non-metricity conjugate in STEGR and is given by

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{4}Q^{\alpha}{}_{\mu\nu} + \frac{1}{2}Q_{(\mu}{}^{\alpha}{}_{\nu)} + \frac{1}{4}Q^{\alpha}g_{\mu\nu} - \frac{1}{4}\left(\tilde{Q}^{\alpha}g_{\mu\nu} + \delta^{\alpha}_{(\mu}Q_{\nu)}\right)\,. \tag{5.13}$$

So by constructing the quadratic form $\mathcal{Q} = Q_{\alpha}{}^{\mu\nu}P^{\alpha}{}_{\mu\nu}$ we can consider the theory

$$\mathcal{S}_{\text{STEGR}} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,\mathcal{Q} + \mathcal{S}_m\,, \tag{5.14}$$

whose equations are obtained by varying the action with respect to the metric

$$\overset{\circ}{\nabla}_\alpha P^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\circ}{\nabla}_\mu Q_\nu + Q_{[\alpha\beta](\mu}Q^{\alpha\beta}{}_{\nu)} - \frac{1}{2}Q_\alpha P^\alpha{}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left[\mathcal{Q} + \overset{\circ}{\nabla}_\alpha(Q^\alpha - \tilde{Q}^\alpha)\right] = -T^M_{\mu\nu}\,, \tag{5.15}$$

where $T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$,.

The theory contains some symmetries which we can use in order to simplify it [123–125]. The teleparallel condition constrains the connection to be purely inertial, i.e. it is just a general linear gauge transformation of $GL(4,\Re)$, parametrized by $\Lambda^\alpha{}_\mu$

$$\Gamma^\alpha{}_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\beta \partial_{[\mu}\Lambda^\beta{}_{\nu]}\,. \tag{5.16}$$

In addition, the torsionless condition makes it even simpler, constraining the transformation matrix to satisfy $(\Lambda^{-1})^\alpha{}_\nu\partial_{[\mu}\Lambda^\nu{}_{\beta]} = 0$ so that it can be parametrized as $\Lambda^\alpha{}_\mu = \partial_\mu\xi^\alpha$, with ξ^α some arbitrary vector field and the connection is now

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\xi^\lambda}\partial_\mu\partial_\nu\xi^\lambda\,. \tag{5.17}$$

This is however, how the trivial connection transforms under a change of coordinates and thus, the connection of STEGR can be cancelled by a diffeomorphism. We call this gauge the *coincident gauge* and now the vector field ξ^α plays the role of a Stückelberg field of the diffeomorphism.

5.2 Extending STEGR

There is no need in repeating the reasons why we should modify the theory of gravity. Since Symmetric Teleparallel theory is equivalent to GR, the same arguments will apply to it too.

The most straightforward modification is the generalization of the action to an arbitrary function of the non-metricity scalar (5.10), just as with $f(\overset{\circ}{R})$ and $f(T)$. The action of the theory will look like

$$\mathcal{S}_{f(\mathcal{Q})} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}f(\mathcal{Q})\,, \tag{5.18}$$

and variations with respect to the metric will give

$$\begin{aligned}8\pi G_N T^M_{\mu\nu} =& \frac{1}{2}f''(\mathcal{Q})Q_{,\alpha}\left(Q_{(\mu}{}^\alpha{}_{\nu)} - \delta^\alpha{}_{(\mu}\tilde{Q}_{\nu)} - \frac{1}{2}Q^\alpha{}_{\mu\nu} + \frac{1}{2}Q^\alpha g_{\mu\nu}\right) + \\ &+ f'(\mathcal{Q})\left[\left(\overset{\circ}{\nabla}_\alpha - \frac{1}{2}Q_\alpha\right)L^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\circ}{\nabla}_\mu Q_\nu - L^\alpha{}_{\beta\mu}L^\beta{}_{\alpha\nu}\right] - \\ &- \frac{1}{2}g_{\mu\nu}\left[f(\mathcal{Q}) + f'(\mathcal{Q})\overset{\circ}{\nabla}_\alpha(Q^\alpha - \tilde{Q}^\alpha)\right]\,,\end{aligned} \tag{5.19}$$

which are the equations of motion. It seems [123,125] that there are some interesting models, at least in cosmology that need further investigation.

In addition, from the non-metricity tensor we can construct five quadratic invariants from different contractions. These are

$$A = Q_{\alpha\mu\nu}Q^{\alpha\mu\nu}\,,\ B = Q_{\alpha\mu\nu}Q^{\mu\alpha\nu}\,,\ C = Q_{\alpha}Q^{\alpha}\,,\ D = \tilde{Q}_{\alpha}\tilde{Q}^{\alpha} \text{ and } E = \tilde{Q}_{\alpha}Q^{\alpha}\,. \tag{5.20}$$

Using them, one can construct a much general theory than $f(\mathcal{Q})$, whose action reads

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(f(A,B,C,D,E) + \mathcal{L}_m)\,. \tag{5.21}$$

This theory is under investigation by the author (with collaborators) in the cosmological (flat FRW) minisuperspace, in order to see which of these theories are invariant under point transformations. A classification of these models will be presented in the *Generalized Symmetric Teleparallel Theories of Gravity*, in the List of Publications .

Finally, a very interesting extension of non-metricity theories including scalar fields has appeared recently [126, 127]. In particular, they consider the action

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\,(\mathcal{L}_{\rm g} + \mathcal{L}_{\rm l}) + \mathcal{S}_m \tag{5.22}$$

, where the gravitational Lagrangian density is given by

$$\mathcal{L}_{\rm g} = \mathcal{A}(\phi)\mathcal{Q} - \mathcal{B}(\phi)\partial_\mu\phi\partial^\mu\phi - 2\mathcal{V}(\phi)\,, \tag{5.23}$$

the Lagrange multiplier terms are

$$\mathcal{L}_{\rm l} = 2\lambda_\mu{}^{\beta\alpha\gamma}R^\mu{}_{\beta\alpha\gamma} + 2\lambda_\mu{}^{\alpha\beta}T_{\mu\alpha\beta}\,, \tag{5.24}$$

and the matter action depends only on the metric and matter fields coupled on it, i.e. $\mathcal{S}_m = \mathcal{S}_m[g_{\mu\nu},\psi]$.

It is easy to see that, when $\mathcal{A} = 1$ and $\mathcal{B} = 0 = \mathcal{V}$ the theory (5.24) reduces to STEGR. Furthermore, the Lagrange multipliers impose the teleparallel and the torsionless condition and are taken to have to antisymmetries of the associated geometrical objects, i.e. $\lambda_\mu{}^{\beta\alpha\gamma} = \lambda_\mu{}^{\beta[\alpha\gamma]}$ and $\lambda_\mu{}^{\alpha\beta} = \lambda_\mu{}^{[\alpha\beta]}$. For more details on the equations of motion, conformal transformations, applications in cosmology, as well as a more general family of theories, see [126, 127].

5.3 Conclusions

In this last chapter of the first part, we studied the least known, but completely equivalent formulation of gravity. If one considers, the teleparallel condition, $R^\alpha{}_{\mu\beta\nu} =$

0 and in the same time the torsionless condition, $T^{\alpha}{}_{\mu\nu} = 0$, they arrive at the so-called Symmetric Teleparallel theory of gravity.

The action of this theory is differs both from the Einstein-Hilbert and from the TEGR action, by a boundary term. Since this boundary term does not contribute in the dynamics of the theory, all three theories are completely equivalent, with the same equations of motion. In the framework of STEGR (Symmetric Teleparallel Equivalent of General Relativity), gravitational interactions are mediated through the scaling of the non-metricity tensor.

Finally, we discussed extensions of STEGR including not only generalizations of the form of $f(\mathcal{Q})$, but also other invariants constructed by the contractions of the non-metricity tensor. All of these theories are less than a year old and need to be studied in greater detail. However, their mathematical structure, i.e. the fact that they can be expressed in a gauge where the connection vanishes, make them very appealing.

Part II

Classifying theories of gravity with Noether Symmetries

Chapter 6

Symmetries: a selection criterion

> «If measure and symmetry are absent from any composition in any degree, ruin awaits both the ingredients and the composition... Measure and symmetry are beauty and virtue the world over.»
>
> - *Socrates*

In the first part of this thesis, we presented three conceptually different, but completely equivalent descriptions of the gravitational interactions; the General Theory of Relativty, the Teleparallel and the Symmetric Telepallel Theory of gravity, as well as their modifications. In these theories the notion of symmetry plays a very important role. This is the reason why, in this chapter we will study the one parameter point transformations, which leave the differential equations invariant. We will focus on the Lie and Noether point symmetries and we will present the *Noether Symmetry Approach*, which is a method that can be used as a *geometric criterion* to select theories of gravity.

6.1 Introduction

As we already discussed in the previous chapter, General Relativity (GR) and the ΛCDM cosmological model match very well current observations, however, they present some shortcomings both at cosmological and astrophysical scales. The discrepancy between the observed value for the cosmological constant with the theoretically calculated vacuum energy of gravitational field is maybe the biggest open problem in modern physics. But it is not the only one; the inability of experiments or observations to find a convincing particle candidate for dark matter, the existence of singularities in the theory, as well as the inefficiency to find a quantum description of gravitational interactions, made the scientific community pursue for alternative or extensions to GR.

There have been numerous proposals in the literature in order not only to extend, but also to modify the current picture we have for gravitational interactions. Introducing extra fields, higher order derivatives, new degrees of freedom in the form of invariants (e.g. $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $\mathcal{G}$, etc.), torsion, non-metricity and other ingredients are some examples of these modifications. Some attempts rely on adopting arbitrary functions, such as $f(R)$, $f(T)$, $f(\mathcal{G})$, etc. where R, T, $\mathcal{G}$ are curvature, torsion, Gauss-Bonnet scalar invariants respectively.

Such theories have to be confronted with data in order to be constrained and then give rise to physically reliable models. However, apart from phenomenology, they can also be theoretically constrained and, searching for symmetries, is a straightforward way to do so. Specifically, Noether symmetries can be used as a geometric criterion to choose among modified theories of gravity, because the presence of symmetries identifies conserved quantities that, in many cases, have also a straightforward physical meaning. It is possible to use this theoretical constraint as a sort of selection criterion based on the geometric symmetries of spacetime [37]. This can be obtained by expressing the Lie/Noether symmetry conditions of second order differential equations of the form

$$\ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = F^i \,, \tag{6.1}$$

in terms of collineations (i.e. symmetries) of the metric. When this is done, one can use the properties of collineations in differential geometry to find general solutions of the Lie/Noether symmetry problem [195].

The *Noether Symmetry Approach* is outlined in [166] and in [195] and we are going to review it in this chapter. However, it has been extensively studied in the literature and some examples are the following: applications to scalar-tensor cosmologies are reported in [190, 196–198]. The most general scalar-tensor theory, giving second order field equations, the so-called Horndeski gravity, is discussed in [38]. $f(R)$ theories are studied in [200, 203], teleparallel gravity and its modifications are discussed in [105, 162, 170] as well as a class of non-local theories in [110]. Apart from cosmology, the method has also been used in spherically and axially symmetric space-time to find exact solutions [39, 40, 176, 203, 255].

Before we proceed, it is worth noticing that, apart from the Lie point symmetries[1], which are the simplest kind of symmetries, there exist several other types of transformations to search for symmetries in differential equations. In particular, in [41], S. Hojman proposed a conservation theorem, where one uses directly the equations of motion, rather than the Lagrangian or the Hamiltonian of a system and, in general, the conserved quantities can be different from those derived

[1]Noether point symmetries are a subclass of Lie symmetries applied to dynamical systems that are described by a point Lagrangian and leave invariant the action integral.

from the Noether Symmetry Approach [42,43]. In addition, there exist higher order symmetries, such as contact symmetries; that is, when the equation of motion are invariant under contact transformations, which are defined as one parameter transformations, in the tangent bundle of the associated dynamical system [44]. Another type of symmetries are the Cartan symmetries [45], which are point transformations with generators in the tangent bundle, that leave the Cartan 1-form invariant. It has been shown [46, 47], that the Cartan symmetries for holonomic dynamical systems are equivalent to the generalized Noether symmetries. In cosmology there are many studies in this direction (see, for example [48–50] and references therein). The interested reader can consider the recent review on symmetries in differential equations [51].

6.2 Point Transformations

What we aim here is to discuss how to find symmetries of differential equations and how to use them to derive analytic solutions. This cannot be done before we introduce the notion of a symmetry, which in turn, leads to define the point transformations and their generators.

The mapping of points (x, y) into points $(\bar{x}, \bar{y})$, where x is the independent and y is the dependent variable, is called *point transformations.* The one parameter point transformations,

$$\bar{x} = \bar{x}(x, y, \epsilon) \quad \bar{y} = \bar{y}(x, y, \epsilon)\,, \tag{6.2}$$

are a specific class of point transformations, which have the following properties [52]:

- they are invertible,
- repeated applications yield a transformation of the same family,

  $$\bar{\bar{x}} = \bar{\bar{x}}(\bar{x}, \bar{y}, \bar{\epsilon}) = \bar{\bar{x}}(x, y, \bar{\bar{\epsilon}})\,, \tag{6.3}$$

  for some $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}(\epsilon, \bar{\epsilon})$,
- the identity is contained for, say, $\epsilon = 0$

  $$\bar{x}(x, y, 0) = x\,, \quad \bar{y}(x, y, 0) = y\,. \tag{6.4}$$

Consider now the one-parameter point transformations (6.2). If we expand around $\epsilon = 0$ ($\equiv$ the identity), we get

$$\bar{x}(x, y, \epsilon) = x + \epsilon\frac{\partial\bar{x}}{\partial\epsilon}|_{\epsilon=0} + ... = x + \epsilon\xi(x, y) + ... \tag{6.5}$$

$$\bar{y}(x, y, \epsilon) = y + \epsilon\frac{\partial\bar{y}}{\partial\epsilon}|_{\epsilon=0} + ... = y + \eta(x, y) + ... \,. \tag{6.6}$$

The tangent vector

$$\mathbf{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}\,, \tag{6.7}$$

is called the *infinitesimal generator* of the transformation.

Since our goal is to see how differential equations are affected by these transformations, we have first to extend/prolong them to the derivatives. The transformed derivatives are defined as

$$\bar{y}' = \frac{d\bar{y}(x,y,\epsilon)}{d\bar{x}(x,y,\epsilon)} = \frac{y'(\partial\bar{y}/\partial y) + (\partial\bar{y}/\partial x)}{y'(\partial\bar{x}/\partial y) + (\partial\bar{x}/\partial x)} = \bar{y}'(x,y,y',\epsilon)\,, \tag{6.8}$$

$$\bar{y}'' = \frac{d\bar{y}'}{d\bar{x}} = \bar{y}''(x,y,y',y'',\epsilon)\,, etc.\,. \tag{6.9}$$

By Taylor expanding around $\epsilon = 0$, as we did in (6.5),(6.6), and substitute these into (6.8),(6.9), we obtain

$$\bar{y}' = y' + \epsilon\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + ... = y' + \epsilon\eta^{[1]} + ...\,, \tag{6.10}$$

$$\bar{y}^{(n)} = y^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{dx} - y^{n}\frac{d\xi}{dx}\right) + ... = y^{(n)} + \epsilon\eta^{[n]}\,, \tag{6.11}$$

where

$$\eta^{[n]} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx} = \frac{d^n}{dx^n}(\eta - y'\xi) + y^{(n+1)}\xi\,, \tag{6.12}$$

is the n^{th} prolongation function of η. Thus, the n^{th} prolongation of the generator $\mathbf{X}$ (6.7) is

$$\mathbf{X}^{[n]} = \mathbf{X} + \eta^{[1]}\partial_{y'} + ... + \eta^{[n]}\partial_{y^{(n)}}\,. \tag{6.13}$$

Until now, we referred only to one parameter point transformations. However, the procedure followed to define multiparameter point transformations on variables, their derivatives, as well as their generators, is the same. Since we will use it later on, it is worth mentioning, what happens to the generating vector if, both the dependent and the independent variables, are more than one.

Suppose that, a differential equation depends on r independent and s dependent variables, that is $\{x_i : i = 1,...,r\}$ and $\{y^j : j = 1,...,s\}$, where $y = y(x)$. If we consider the following one parameter point transformation [53]

$$\bar{x}_i = \Xi_i(x,y,\epsilon) = x_i + \epsilon\xi_i(x,y) + ... \quad , \quad \bar{y}^j = H^j(x,y,\epsilon) = y^j + \epsilon\eta^j(x,y) + ...\,, \tag{6.14}$$

with ξ_k and η^j being

$$\xi_k(x,y) = \frac{\partial\Xi_k(x,y,\epsilon)}{\partial\epsilon}|_{\epsilon\to 0} \quad , \quad \eta^j(x,y) = \frac{\partial H^j(x,y,\epsilon}{\partial\epsilon}|_{\epsilon\to 0}\,. \tag{6.15}$$

the generating vector is given by

$$\mathbf{X} = \xi^k(x^i, y^j)\partial_k + \eta^l(x^i, y^j)\partial_l \,. \tag{6.16}$$

Following the same procedure as before, we will see how the derivatives of the dependent variables are transformed. At first, we note that $y_i^j = \partial y^j/\partial x_i$ and also

$$D_i = \frac{\partial}{\partial x_i} + y_i^j \frac{\partial}{\partial y^j} + y_{ik}^j \frac{\partial}{\partial y_k^j} + ... + y_{ii_1 i_2 ... i_n}^j \frac{\partial}{\partial y_{i_1 i_2 ... i_n}^j} + ... \,. \tag{6.17}$$

Now the derivatives take the form

$$\bar{y}_i^j = H_i^j(x, y, \partial y, \epsilon) = y_i^j + \epsilon \eta_i^{[1]j}(x, y, \partial y) + ... \tag{6.18}$$

$$\vdots$$

$$\bar{y}_{i_1 i_2 ... i_k}^j = H_{i_1 i_2 ... i_k}^j(x, y, \partial y, ..., \partial^k y, \epsilon) = y_{i_1 i_2 ... i_k}^j + \epsilon \eta_{i_1 i_2 ... i_k}^{[k]j}(x, y, \partial y, ..., \partial^k y) + ... \,, \tag{6.19}$$

where $\eta_i^{[1]j} = D_i \eta^j - (D_i \xi_k)\, y_k^j$ and $\eta_{i_1 i_2 ... i_k}^{[k]j} = D_{i_k} \eta_{i_1 i_2 ... i_{k-1}}^{[k-1]j} - (D_{i_k} \xi_l)\, y_{i_1 i_2 ... i_{k-1} l}^j$. Thus the prolongation of the generator of the point tranformations (6.14) is given by

$$\mathbf{X}^{[n]} = \mathbf{X} + \eta_i^{[1]j} \partial_{y_i^j} + ... + \eta_{i_1 i_2 ... i_k}^{[k]j} \partial_{y_{i_1 i_2 ... i_k}^j} \,. \tag{6.20}$$

These tools will be used to seek for symmetries of differential equations.

6.3 Symmetries of differential equations

Now we are ready to study the behaviour of differential equation under the action of point transformations. We already mentioned in the introduction 6.1 that, apart from Lie/Noether symmetries, which are point symmetries, there exist non-point-like symmetries and higher order symmetries (contact, non-local, Cartan), with which we will not deal here.

A group of point transformations that maps solutions into solutions, i.e. the mapping $\bar{y}(\bar{x})$ of any solution $y(x)$ is again a solution, is called a *symmetry* of the differential equations. Mathematically formulated it is: the differential equation

$$H(x, y, y', ..., y^{(n)}) = 0 \,, \tag{6.21}$$

remains invariant, under the point transformations (or else symmetry)

$$\bar{x} = \bar{x}(x, y) \,, \quad \bar{y} = \bar{y}(x, y) \,. \tag{6.22}$$

Consider now a one parameter point transformation and re-express the differential equation (6.21) in the transformed variables, i.e. $H(\bar{x}, \bar{y}, \bar{y}', ..., \bar{y}^{(n)}) = 0$. It is easily seen that,

$$\frac{\partial H(\bar{x}, \bar{y}, \bar{y}', ..., \bar{y}^{(n)})}{\partial \epsilon}|_{\epsilon=0} = \lambda H \Rightarrow \mathbf{X}^{[n]} H = \lambda H \,, \tag{6.23}$$

where λ are eigenvalues. The converse is also true and this can be seen only by considering the fact that the existence of symmetries is independent of the choice of variables. Thus we end up with the following theorem:

Theorem: *A differential equation, $H = 0$, admits a group of symmetries with generator $\mathbf{X}$, if and only if there exist a function λ such that $\mathbf{X}^{[n]} H = \lambda H$.*

6.3.1 Noether point symmetries

A specific class of Lie point symmetries are the so-called Noether symmetries. They are restricted to dynamical systems coming from a Lagrangian. The Lagrangian function $L = L(t, q^i, \dot{q}^i)$[2], is a function of the affine parameter t, the generalized coordinates $q^i = q^i(t)$ and the generalized velocities $\dot{q}^i(t)$. It contains information about the dynamics of a system. The equations of motion of the system are given by the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \,. \tag{6.24}$$

When the point transformations, that are Lie symmetries for a system of differential equations, transform the Lagrangian in such a way that the action integral remains invariant, they are called Noether symmetries.

As already known from Lagrangian mechanics, Emmy Noether proved that if a Lagrangian admits a symmetry, then this symmetry is associated with a conserved quantity. The most well known examples are

- the conservation of the total energy of a system, when the Lagrangian is time independent, i.e. it is not affected by transformations of the form $\bar{t} \rightarrow t + \delta t$,
- the momentum conservation is associated with the translational invariance, i.e. when a Lagrangian has an ignorable variable, then the associated momentum is conserved,
- and also the angular momentum conservation is related to the rotational symmetry of the Lagrangian, i.e. the orientation of the physical system in space does not affect the Lagrangian.

[2]The index i takes the values $1, 2, ..., n$ and denotes the number of dimensions of the configuration space.

In a more precise mathematical language:

Theorem: *Suppose a dynamical system described by the Lagrangian* $L = L(t, q^i, \dot{q}^i)$. *The generator of the infinitesimal point transformations*

$$\bar{t} = t + \epsilon\xi(t, q^i) + \dots , \tag{6.25}$$

$$\bar{q}^i = q^i + \epsilon\eta^i(t, q^j) + \dots , \tag{6.26}$$

is

$$\mathbf{X} = \xi(t, q^i)\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} , \tag{6.27}$$

while its first prolongation, according to (6.13), *is given by*

$$\mathbf{X}^{[1]} = \mathbf{X} + \eta^{i,[1]}\frac{\partial}{\partial \dot{q}^i} . \tag{6.28}$$

If there exists a function $g(t, q^i)$ *such that*

$$\mathbf{X}^{[1]}L + L\frac{d\xi(t, q^i)}{dt} = \frac{dg(t, q^i)}{dt} , \tag{6.29}$$

then the Euler Lagrange equations remain invariant under the action of the transformations (6.25),(6.26) *and* $\mathbf{X}$ *is the Noether symmetry vector of the system.*

The associated first integral of motion is given by the function

$$\phi(t, q^i, \dot{q}^i) = \xi\left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right) - \eta^i\frac{\partial L}{\partial q^i} + g . \tag{6.30}$$

This is known as the *Noether's second theorem.* In general, Noether symmetries are valid also for non-point transformations [54].

Invariant functions

The condition (6.29) is equivalent to the following Lagrange system

$$\frac{dt}{\xi} = \frac{dq^i}{\eta^i} = \frac{d\dot{q}^i}{\eta^{[1],i}}. \tag{6.31}$$

The Lagrangian in physical systems contains up to first order derivatives in the canonical variables, yielding up to second order differential equations. In more general systems with n order differential equations, the above system can be defined with fractions of n^{th} order derivatives of the canonical variables over n^{th} prolongations of the generator coordinates.

The above Lagrange system (6.31) can give us the zero and first order invariants (n^{th} order in general) respectively

$$W^{[0]}(t, q^i) = \frac{dt}{\xi} - \frac{dq^i}{\eta^i} \quad \text{and} \quad W^{[1]}(t, q^i, \dot{q}^i) = \frac{dt}{\xi} - \frac{dq^i}{\eta^i} - \frac{d\dot{q}^i}{\eta^{[1]i}}. \tag{6.32}$$

By using these invariants, we can reduce the order of the Euler-Lagrange equations and thus solve them in an easier way.

6.3.2 Finding symmetries

In the last section of this chapter, let us see how to find symmetries. We give here a general description of the procedure in the form of an algorithm.

In order to determine Noether symmetries, what we need is to find the coefficients of the generator $\mathbf{X}$, $\xi(t,q)$ and $\eta^i(t,q)$, such that the symmetry condition (6.29) is satisfied. So, if we have a dynamical system described by a Lagrangian $L = L(t, q^i, \dot{q}^i)$ then:

1. We write an *ansatz* for the generator of the form (6.27) defined on the configuration space.

2. We expand the symmetry condition (6.29) to obtain a polynomial depending on $\xi(t,q)$, $\eta^i(t,q)$ and products of the generalized velocities, i.e. $(\dot{q}^a\dot{q}^b...)$.

3. Since the unknown coefficients ξ, η depend only on (t,q), in order for the polynomial to vanish, the coefficients of the products $(\dot{q}^a\dot{q}^b...)$ have to vanish. Thus we end up with a set of partial differential equations for ξ and η, which, most of the times, can be solved in a straightforward way.

4. Once we calculate the generating vector $\mathbf{X}$, we can easily find the first integral ϕ from (6.30), and thus obtain a better insight into the physical meaning of these integrals.

5. Finally, from the generator we can construct the associated Lagrange system, find the zero and first order invariants and reduce the order of the Euler-Lagrange equations.

Depending on the number of symmetries, one can achieve the complete integrability of the dynamical system.

6.4 Conclusions

Summing up, symmetries can be considered a general criterion to select physical models. In this chapter, we discussed a specific class of symmetries, the so-called Noether symmetries, which are Lie symmetries for dynamical systems derived from a Lagrangian. Specifically, we presented the one-parameter point transformations that maintain second order differential equations invariant. It is long known that there is a connection between the point symmetries of second order differential equations with the collinations of the Riemannian manifold where the motion occurs. However, it is really new that this geometric property has been used to classify theories of gravity that admit Noether symmetries, and those in turn give invariant functions which we can use to reduce the dynamics of the dynamical system and find exact solutions. As we saw in the first part of this thesis, there are numerous theories with arbitrary functions in their actions, and Noether Symmetries can help us discriminate between those. Applications will be seen in the following chapters of this thesis.

Chapter 7

Classification of Horndeski cosmologies through Noether Symmetries

Horndeski's theory is the most general scalar tensor theory with second order field equations. It was proposed in 1974 by G. Horndeski but it was reincarnated in the late 2000's and was rewritten in a covariant from. The form of the theory, i.e. action and field equations are presented in chapter 3. In this chapter, we will discuus the cosmology of the theory and we will classify the arbitrary functions appearing in the action using the Noether Symmetry Approach. Finally, we will correlate the results of this method to known scalar-tensor theories, such as Brans-Dicke, cubic galileon, the scalar-tensor representation of $f(R)$ gravity and more.

7.1 Introduction

The inability of General Relativity (GR), together with the ΛCDM model, to constitute a complete theory capable of describing the gravitational interactions at all scales led the scientific community to pursue new approaches by which GR should be modified or extended at infrared and ultraviolet scales [79, 99, 171–175, 273, 349]. In 1974, Horndeski developed [80] the most general scalar-tensor theory (with a single scalar field) that leads to second order field equations[1]. In [82, 178], the Horndeski theory has been reconsidered according to a generalization of the covariant galileon models, already proposed in [179], as the decoupling limit of the graviton in the Dvali-Gabadadze-Porrati model.

[1] Theories with higher than second order equations of motion are, in most cases, plagued by the so called Ostrogradski instability and thus give rise to ghost degrees of freedom.

A lot of progress has been done and the Horndeski theory can now be considered as a general theory from which several modified theories of gravity can be recovered. Scalar-tensor models, such as Brans-Dicke, k-essence, kinetic braiding, as well as the scalar-tensor analogue of $f(R)$ gravity, are nothing else but special cases of the Horndeski action. Apart from cosmology, significant progress has been done at smaller scales in this theory. Specifically, charged black hole solutions have been studied in the context of this theory [180–183, 330]; numerical simulations for neutron stars in specific subclasses of this theory have also been developed [184, 185]. Recently, in [186], the authors reviewed the Horndeski cosmologies that have asymptotically de Sitter critical point. In [83], generalized galileons are considered as the most general framework to develop single-field inflationary models. Moreover, in [187], the author proves that Horndeski theory is part of the effective field theory of cosmological perturbations, which is also a useful framework to develop inflation. Finally, in [188], the authors considered possible breaking of the Vainshtein mechanism, in a generalized Horndeski theory (or generalized galileon model), and they claim that such a breaking could be responsible for gravitational effects attributed to dark matter.

Even though a lot of ink has been spilled on the fact that, scalar fields may or may not couple with matter, the predominant opinion is that matter-fields do couple, with the field being "screened" (=hidden) at small scales. This screening mechanisms could solve several problems and, among them, the Cosmological Constant problem. Three such mechanisms are known; the chameleon, the symmetron and the Vainshtein mechanism [70, 189]. Although, all of them emerge in scalar-tensor theories, the latter is explicitly seen in massive gravity, in galileon and thus in Horndeski theory. Simply, this mechanism "hides" the effects of the non-linear kinetic terms inside the so called Vainshtein radius, allowing them to play an important role only at large infrared scales, that is in cosmology as pointed out in [71]. For more details see the discussion in chapter 3.

The Horndeski theory contains a lot of arbitrariness encoded in the functions of the action: $G_i(\phi, X)$, where $i = 2, ..., 5$, where ϕ is the scalar field and $X = -1/2(\partial_\mu\phi\partial^\mu\phi)$ its kinetic term. The aim of this chapter is to classify the Horndeski models according to the *Noether Symmetry Approach* [166]. This method helps to find exact solutions for a given theory, once a symmetry exists. Besides, the existence of a symmetry "selects" the integrable form of a model in a given class of theories. Finally, the symmetries of a theory are always connected to conserved quatities, according to the Noether's Theorem, and thus observables. Here, we classify the Horndeski models according to the specific forms of functions $G_i(\phi, X)$ assuming the only criterion that the field equations are invariant under Noether point symmetries. There exist similar approaches in the literature [190]; however, the similarity is only the fact that they discuss a general family of scalar-tensor Lagrangian. They study a part of the cosmological Horndeski Lagrangian and their results are very interesting, however, we consider the whole Horndeski action.

As in the previous chapter, the same holds here too. All the quantities we will use (curvature tensors, covariant derivatives, etc.) are calculated with respect to the Levi-Civita connection and the use of $\circ$ is redundant, so we omit it.

7.1.1 The Horndeski Cosmology

The essentials of the gravity are discussed in chapter 3. Here, we want to study the cosmology related to the Horndeski theory, so we suppose that the spacetime is described by a spatially flat Friedmann-Robertson-Walker (FRW) metric, which reads

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,. \tag{7.1}$$

The Ricci scalar takes the form

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \,. \tag{7.2}$$

It is $\phi = \phi(t)$ and thus the scalars in the Lagrangians become[2]

$$X = \frac{1}{2}\dot{\phi}^2 \,,\ \Box\phi = -\left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}\right) \,,\ (\nabla_\mu\nabla_\nu\phi)^2 = \ddot{\phi}^2 + 3\frac{\dot{a}^2}{a^2}\dot{\phi}^2 \,,\ (\nabla_\mu\nabla_\nu\phi)^3 = -\ddot{\phi}^3 - \frac{3\dot{a}^3}{a^3}\dot{\phi}^3 \,. \tag{7.3}$$

If we substitute all these quantities into (3.22), the Lagrangian assumes a point-like form

$$\begin{aligned}\mathcal{L} = a^3G_2 + 3a^2G_3\dot{a}\dot{\phi} + a^3G_3\ddot{\phi} + 6aG_{4X}\dot{a}^2\dot{\phi}^2 + 6aG_4\dot{a}^2 + 6a^2G_{4X}\dot{a}\dot{\phi}\ddot{\phi} + 6a^2G_4\ddot{a} + \\ + 3aG_5\dot{a}^2\ddot{\phi} + 6aG_5\dot{a}\dot{\phi}\ddot{a} + 3G_5\dot{a}^3\dot{\phi} + G_{5X}\dot{a}^3\dot{\phi}^3 + 3aG_{5X}\dot{a}^2\dot{\phi}^2\ddot{\phi} \,.\end{aligned} \tag{7.4}$$

As we see, there are second order derivatives in the Lagrangian. We can integrate all of them out with integration by parts, except from the term $a^3G_3\ddot{\phi}$. Specifically,

$$\begin{aligned}a^3G_3\ddot{\phi} &= (a^3G_3\dot{\phi})_{,t} - 3a^2G_3\dot{a}\dot{\phi} - a^3G_{3\phi}\dot{\phi}^2 - a^3G_{3X}\dot{\phi}^2\ddot{\phi} \\ &= (a^3G_3\dot{\phi})_{,t} - 3a^2G_3\dot{a}\dot{\phi} - a^3G_{3\phi}\dot{\phi}^2 + a^2G_{3X}\dot{a}\dot{\phi}^3 + \frac{1}{3}a^3G_{3X\phi}\dot{\phi}^4 + \frac{1}{3}a^3G_{3XX}\dot{\phi}^4\ddot{\phi} \,,\end{aligned}$$

and it goes on like this, since G_3 depends on $X(t)$ and $\dot{X}(t) = \dot{\phi}\ddot{\phi}$. Hence, if we want the Lagrangian to be canonical and to depend only on first derivatives of the

[2] This is explained if we assume that the matter fields and scalar field inherit the isometries of the FRW spacetime.

variables of the configuration space[3], we have to choose where to stop and just set one derivative of G_3 over X equal to zero. We choose to set

$$G_{3XX}=0 \Rightarrow G_3(\phi,X)=g(\phi)X+h(\phi)\,. \tag{7.5}$$

This choice seems arbitrary, but also with this limitation, it is possible to formulate most of the scalar-tensor theories known in literature, such as kinetic braiding, cubic galileons and others containing interaction terms like $\sim \nabla_\mu\phi\nabla^\mu\phi\Box\phi$. Finally, the Lagrangian (7.4) becomes

$$\begin{aligned}\mathcal{L}=a^3G_2+a^2g(\phi)\dot{a}\dot{\phi}^3-\frac{1}{6}a^3g'(\phi)\dot{\phi}^4-a^3h'(\phi)\dot{\phi}^2-6aG_4\dot{a}^2-6a^2G_{4\phi}\dot{a}\dot{\phi}+\\+3a\left(2G_{4X}-G_{5\phi}\right)\dot{a}^2\dot{\phi}^2+G_{5X}\dot{a}^3\dot{\phi}^3\,.\end{aligned} \tag{7.6}$$

The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{a}}\right)-\frac{\partial\mathcal{L}}{\partial a}=0\,,\qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)-\frac{\partial\mathcal{L}}{\partial\phi}=0\,, \tag{7.7}$$

and the energy condition

$$E_{\mathcal{L}}=\frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a}+\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi}-\mathcal{L}=0\,, \tag{7.8}$$

constitute the dynamical system derived from the Lagrangian (7.6). We do not find necessary to include them in their general form since they can be easily derived from the Lagrangian (7.6). We will derive them for the specific cases that we are going to discuss below.

7.1.2 Noether Symmetries in Horndeski Cosmology

As we already mentioned, the configuration space of the Lagrangian (7.6) is $\mathcal{Q}=\{a,\phi\}$ and the independent variable is the cosmic time t. The generator of an infinitesimal transformation is

$$\mathbf{X}=\xi\left(t,a,\phi\right)\partial_t+\eta_a\left(t,a,\phi\right)\partial_a+\eta_\phi\left(t,a,\phi\right)\partial_\phi\,. \tag{7.9}$$

By applying the Noether's condition

$$\mathbf{X}^{[1]}L+L\frac{d\xi}{dt}=\frac{df}{dt}\,, \tag{7.10}$$

[3]In our case the configuration space is the minisuperspace $\mathcal{Q}=\{a,\phi\}$ and the tangent space is $\mathcal{TQ}=\{a,\dot{a},\phi,\dot{\phi}\}$.

to (7.6), we get a system of 28 equations for the coefficients of the Noether vector $\xi(t,a,\phi)$, $\eta_a(t,a,\phi)$, $\eta_\phi(t,a,\phi)$, $f(t,a,\phi)$ and the arbitrary functions of the Lagrangian $G_2(\phi,X)$, $G_3(\phi,X)$, $G_4(\phi,X)$, $G_5(\phi,X)$, which, of course, are not all each other independent (see also [110,166]). A comment is necessary at this point; if we consider given forms for the unknown functions of the Lagrangian, i.e. the G_i, as it has been done in several papers [190,196–198], we can specify in detail all the functions, as well as the Noether vector coefficients. What we are doing here is to consider the most general Horndeski Lagrangian and try to constrain its unknown functions and, at the same time, to find out symmetries in the most general way. Clearly, particular models are recovered by specific choices of the above functions, as we will show below with some examples.

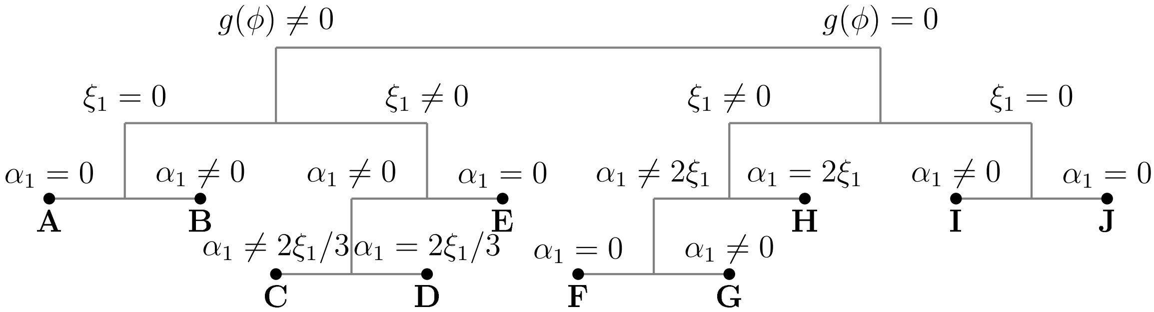
It is straightforward to notice that the Noether vector takes immediately the form

$$\mathbf{X} = (\xi_1 t + \xi_2)\partial_t + \eta_a(a)\partial_a + (\xi_1\phi + \phi_1)\partial_\phi\,, \tag{7.11}$$

with ξ_1, ξ_2, ϕ_1 being integration constants. In addition, the function f of Eq. (7.10) is forced to be a constant, $f(t,a,\phi) = f_1$.

Now, depending on whether the function $g(\phi)$ in Eq. (7.5) vanishes or not, there are different solutions. In the class of solutions with $g(\phi) \neq 0$, the Noether vector, and specifically the η_a coefficient, becomes $\eta_a(a) = \alpha_1 a$. In the other case, where $g(\phi) = 0$, we get $\eta_a(a) = \frac{1}{3}(\alpha_1 + 2\xi_1)a$. It might seem that a redefinition of the constants would equate the two cases, but this is not the case. As we show in Table 7.1, the Horndeski functions take different forms.

The following graph summarizes the 10 different symmetry classes we get depending on the values of the constants. By changing ξ_1 and α_1, the form of symmetry, i.e. the Noether vector, changes in a straightforward way. In the graph, any different case is assigned to a capital letter; the Horndeski functions, for each case, are given in Table 7.1. The cases **A**, **J** and **B**, **I** coincide by redefining the constants and by setting $c_2 = 0$ in **A** and **B**. However, the other cases are different.



This is the main result of this chapter. Before we move to the next section, let

	$\mathbf{G_2(\phi,X)}$	$\mathbf{G_3(\phi,X)}$	$\mathbf{G_4(\phi,X)}$	$\mathbf{G_5(\phi,X)}$
A	$g_2(X)$	$c_1+c_2X+c_3\phi$	$g_4(X)$	$g_5(X)+c_4\phi$
B	$e^{-\frac{3\alpha_1\phi}{\phi_1}}g_2(X)$	$c_1+e^{-\frac{3\alpha_1\phi}{\phi_1}}\left(c_2X-\frac{c_3\phi_1}{3\alpha_1}\right)$	$e^{-\frac{3\alpha_1\phi}{\phi_1}}g_4(X)$	$c_4-\frac{\phi_1 e^{-\frac{3\alpha_1\phi}{\phi_1}}g_5(X)}{3\alpha_1}$
C	$g_2(X)\left(\xi_1\phi+\phi_1\right)^{-\frac{3\alpha_1}{\xi_1}-1}$	$\left(\xi_1\phi+\phi_1\right)^{-\frac{3\alpha_1}{\xi_1}}\left(c_1X-\frac{c_2}{3\alpha_1}\right)+c_3$	$g_4(X)\left(\xi_1\phi+\phi_1\right)^{1-\frac{3\alpha_1}{\xi_1}}$	$c_4+\frac{g_5(X)(\xi_1\phi+\phi_1)^{2-\frac{3\alpha_1}{\xi_1}}}{2\xi_1-3\alpha_1}$
D	$\frac{g_2(X)}{(\xi_1\phi+\phi_1)^3}$	$c_1-\frac{c_2-2c_3\xi_1X}{2\xi_1(\xi_1\phi+\phi_1)^2}$	$\frac{g_4(X)}{\xi_1\phi+\phi_1}$	$\frac{c_4\ln(\xi_1\phi+\phi_1)}{\xi_1}+g_5(X)+c_4$
E	$\frac{g_2(X)}{\xi_1\phi+\phi_1}$	$\frac{c_1\ln(\xi_1\phi+\phi_1)}{\xi_1}+c_2X+c_3$	$\left(\xi_1\phi+\phi_1\right)g_4(X)$	$c_4+\frac{g_5(X)(\xi_1\phi+\phi_1)^2}{2\xi_1}$
F	$\frac{g_2(X)}{(\xi_1\phi+\phi_1)^3}$	$c_1-\frac{c_2}{2\xi_1(\xi_1\phi+\phi_1)^2}$	$\frac{g_4(X)}{(\xi_1\phi+\phi_1)}$	$\frac{c_3\ln(\xi_1\phi+\phi_1)}{\xi_1}+g_5(X)$
G	$g_2(X)\left(\xi_1\phi+\phi_1\right)^{-\frac{\alpha_1}{\xi_1}-3}$	$c_1-\frac{c_2(\xi_1\phi+\phi_1)^{-\frac{\alpha_1}{\xi_1}-2}}{\alpha_1+2\xi_1}$	$g_4(X)\left(\xi_1\phi+\phi_1\right)^{-\frac{\alpha_1}{\xi_1}-1}$	$c_3-\frac{g_5(X)(\xi_1\phi+\phi_1)^{-\frac{\alpha_1}{\xi_1}}}{\alpha_1}$
H	$\frac{g_2(X)}{\xi_1\phi+\phi_1}$	$c_1+\frac{c_2\log(\xi_1\phi+\phi_1)}{\xi_1}$	$\frac{g_4(X)(\xi_1\phi+\phi_1)}{\xi_1}$	$c_3+\frac{g_5(X)(\xi_1\phi+\phi_1)^2}{2\xi_1}$
I	$e^{-\frac{\alpha_1\phi}{\phi_1}}g_2(X)$	$c_1-\frac{c_2\phi_1e^{-\frac{\alpha_1\phi}{\phi_1}}}{\alpha_1}$	$e^{-\frac{\alpha_1\phi}{\phi_1}}g_4(X)$	$c_3-\frac{\phi_1e^{-\frac{\alpha_1\phi}{\phi_1}}g_5(X)}{\alpha_1}$
J	$g_2(X)$	$c_1+c_2\phi$	$g_4(X)$	$c_3\phi+g_5(X)$

Table 7.1: We summarize the Horndeski functions with respect to the Noether Symmetries; $g_i(X)$ are arbitrary functions of X, the kinetic term; c_i are arbitrary constants and ξ_1, ϕ_1 and α_1 are the constants coming from the Noether vector.

us shortly discuss the above classification. The arbitrariness of the functions $g_i(X)$ makes this classification broad enough, as far as the restrictions are concerned. By choosing specific classes (and thus symmetries) and playing with the form of the function g_i, we can map modifications of GR to the Horndeski theory and see if they are invariant or not under the action of Noether point symmetries. In this perspective, the Noether Symmetry Approach is a selection criterion discriminating among integrable models. As discussed in [167], the Noether symmetries select "physical" models in the sense that the related conserved quantities result physical observables of the theory.

Moreover, from the Noether vector (7.11), we can define the Lagrange system

$$\frac{dt}{\xi_1 t+\xi_2}=\frac{da}{\eta_a(a)}=\frac{d\phi}{\xi_1\phi+\phi_1}\,. \tag{7.12}$$

Without loss of generality, we can set $\xi_2=0$. As we already mentioned before, for $g(\phi)\neq 0$ the η_a coefficient becomes $\eta_a(a)=\alpha_1 a$, while for $g(\phi)=0$, it is $\eta_a(a)=\frac{1}{3}(\alpha_1+2\xi_1)a$. By solving the system (7.12) for each case, we get the zero-order invariants which are solutions of the system of the E-L equations

$$a(t)=\alpha_0 t^{\alpha_1/\xi_1}\,,\ \phi(t)=\phi_0 t-\frac{\phi_1}{\xi_1}\text{ for } g(\phi)\neq 0\,,$$
$$a(t)=\alpha_0 t^{(\alpha_1+2\xi_1)/3\xi_1}\,,\ \phi(t)=\phi_0 t-\frac{\phi_1}{\xi_1}\text{ for } g(\phi)=0\,.$$

There are two E-L equations, one for a and one for ϕ, but we also have the constraint equation. By plugging these solutions in the E-L equations we can get constraints for the arbitrary functions $g_i(X)$ in the table 7.1.

7.2 From Horndeski to specific modified theories of gravity

By choosing specific forms of the arbitrary functions $g_2(X)$, $g_4(X)$ and $g_5(X)$, as well as by fixing the constants ξ_1, ϕ_1, α_1 and c_i, we can recast the Horndeski Lagrangian, to Lagrangians coming from modified theories. For each theory, if Noether symmetries exist, we can find out exact cosmological solutions. In what follows, we match theories that show symmetries (the different classes are presented in Table I), with some extended theories of gravity (Brans-Dicke, $f(R)$, etc). For these theories, cosmological solutions exist and we present them. In principle, the approach consists in finding out the conserved quantities for each case (if they exist), in reducing the dynamics of the system, and in obtaining exact solutions.

7.2.1 Brans-Dicke gravity

Let us start with the simplest, and one of the first considered modification of gravity, the Brans-Dicke theory. The action is [77]

$$\mathcal{S} \sim \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] + \mathcal{S}_m \tag{7.13}$$

where ω is the Brans-Dicke parameter, i.e. the coupling constant between the scalar field and the metric. In this theory, the Newton constant, G_N, is not constant, but it varies according to the evolution of a scalar field $\phi \sim 1/G_N$. The reasons for this choice are several. In particular, Brans and Dicke considered a theory which is in more agreement with Mach's principle, compared to GR, assuming that the gravitational coupling can depend on space and time. In cosmology, the point-like, canonical Lagrangian takes the form

$$\mathcal{L} = -6a\phi\dot{a}^2 - 6a^2\dot{a}\dot{\phi} - \omega\frac{a^3}{\phi}\dot{\phi}^2\,, \tag{7.14}$$

where we considered that the potential $V(\phi) = 0$. In order to match this Lagrangian to the Horndeski theory, we have to set in the case **E** of table 7.1,

$$c_1 = c_2 = c_3 = c_4 = 0\,,\; g_4(X) = 1\,,\; g_5(X) = 0\,,\; \phi_1 = 0\,,\; \xi_1 = 1,\text{ and } g_2(X) = -2\omega X\,. \tag{7.15}$$

The fact that the two Lagrangians coincide, means that, our Lagrangian inherits also the cosmological solutions found in [77] and [201], i.e.

- For $\omega \geq -\frac{3}{2}$ and $\omega \neq -\frac{4}{3}$,

$$a(t) = a_0\left(\frac{t}{t_0}\right)^q\,, \quad \phi(t) = \phi_0\left(\frac{t}{t_0}\right)^r\,, \tag{7.16}$$

- For $\omega \geq -\frac{3}{2}$ and $\omega = -\frac{4}{3}$,

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}} , \quad \phi(t) = \phi_0 \left(\frac{t}{t_0}\right)^{-1} , \tag{7.17}$$

where a_0, ϕ_0 are constants and $q = \frac{1}{3}(1-r)$, $r = \frac{1}{4+3\omega}\left(1 \pm \sqrt{3(3+2\omega)}\right)$. This means that the equations of motion of Brans-Dicke theory remain invariant under the point transformations described by the Noether vector

$$\mathbf{X} = (t + \xi_2)\partial_t + \phi\partial_\phi . \tag{7.18}$$

In addition, there is an integral of motion, which is given by

$$I = f_1 + a^2 \left(6\phi\dot{a} + 2\omega a\dot{\phi}\right) . \tag{7.19}$$

String motivated gravity

Let us now consider a string-motivated Lagrangian of the form [210, 212]

$$\mathcal{S} \sim \int d^4x\sqrt{-g}e^{-2\phi}\left[R + 4\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] . \tag{7.20}$$

It turns out that this theory is actually a Brans-Dicke-like theory for specific forms of the coupling, the self-interaction potential, and a redefinition of the scalar field acting as the string-dilaton field. It interesting to include also this model in the discussion of the Horndeski theory and search for its Noether symmetries since it has been extensively studied in literature for several physical implications[4].

Assuming a FRW cosmology (7.1), the above Lagrangian becomes

$$\mathcal{L} = e^{-2\phi}\left[12a^2\dot{a}\dot{\phi} - 6a\dot{a}^2 - a^3\left(4\dot{\phi}^2 + V(\phi)\right)\right] . \tag{7.21}$$

Besides, the Horndeski Lagrangian, with the Noether symmetry

$$\mathbf{X} = \xi_2\partial_t + \frac{2}{3}\phi_1\partial_a + \phi_1\partial_\phi , \tag{7.22}$$

i.e. $\xi_1 = 0$ and $\alpha_1 = \frac{2}{3}\phi_1 \neq 0$, becomes[5], after adopting the symmetry class **B** from Table 7.1,

$$\begin{aligned}\mathcal{L} &= a^3e^{-2\phi}g_2(X) + \frac{c_2}{3}a^3e^{-2\phi}\dot{\phi}^4 - c_3a^3e^{-2\phi}\dot{\phi}^2 + 3ae^{-2\phi}\left(2g_4'(X) - g_5(X)\right)\dot{a}^2\dot{\phi}^2 - \\ &\quad -\frac{1}{2}e^{-2\phi}g_5'(X)\dot{a}^3\dot{\phi}^3 - 6ae^{-2\phi}g_4(X)\dot{a}^2 + c_2a^2e^{-2\phi}\dot{a}\dot{\phi}^3 + 12a^2e^{-2\phi}g_4(X)\dot{a}\dot{\phi}\end{aligned} \tag{7.23}$$

[4]Starting from a D-dimensional theory, e.g. the so called Polyakov action, after compactification, we remain with only four macroscopic dimensions ending up with the action (7.20). This is a simplification that allows us to study the dynamics of the degrees of freedom associated to the four macroscopic dimensions. For details, see [218–221].

[5]We set $\alpha_1 = \frac{2}{3}\phi_1$ in order to recover the dilaton coupling from (7.20).

The two actions (7.21) and (7.23) become the same, if we identify

$$g_2(X) = -V(\phi) = V_0\,,\ c_1 = c_2 = c_4 = 0\,,\ c_3 = 4\,,\ g_4(X) = 1\,,\ g_5(X) = 0\,. \quad (7.24)$$

In this way, the Horndeski functions take the following form,

$$G_2(\phi, X) = V_0 e^{-2\phi}\,,\ G_3(\phi, X) = -8\phi_1 e^{-2\phi}\,,\ G_4(\phi, X) = e^{-2\phi}\,,\ G_5(\phi, X) = 0\,. \quad (7.25)$$

As we see, the form of $V(\phi)$ is not arbitrary, and specifically, it is the constant V_0. Solutions in 4 dimensions are discussed in [210–212]. Solutions in D dimensions are discussed in [218].

7.2.2 $f(R)$ gravity

Another class of modified theories is the $f(R)$ gravity. If one replaces the Ricci scalar in the Einstein-Hilbert action, with an arbitrary function f of the Ricci scalar, the family of $f(R)$ theories arise. In some sense, this is the most straightforward generalization of GR. The arbitrariness of the function f allows, in specific cases, to explain lingering problems in cosmology and astrophysics, such as the accelerated expansion, the structure formation, the inflation, etc, without including exotic forms of matter/energy in the stress-energy tensor. For the interested reader, there is a large amount of literature on this topic. For reviews see [79, 204, 321, 337].

As already shown in [79] and references therein, by setting $\phi \equiv f'(R) \Rightarrow R = \mathcal{R}(\phi)$ and $V(\phi) = \phi\mathcal{R}(\phi) - f(\mathcal{R}(\phi))$ we obtain the following equivalence,

$$\mathcal{S} \sim \int d^4x\sqrt{-g}f(R) \Leftrightarrow \mathcal{S} \sim \int d^4x\sqrt{-g}\left(\phi R - V(\phi)\right)\,. \quad (7.26)$$

This scalar-tensor form of $f(R)$ theories is similar to the Brans-Dicke theory, without the kinetic term, i.e. with $\omega = 0$ and with an arbitrary potential $V(\phi)$ (see [202]). The point like Lagrangian of this action is given by

$$\mathcal{L} = -6a\phi\dot{a}^2 - 6a^2\dot{a}\dot{\phi} - a^3V(\phi)\,, \quad (7.27)$$

which means that in order to match it with the Horndeski Lagrangian (7.6) we have to set

$$G_2(\phi, X) = -V(\phi)\,,\ g(\phi) = 0\,,\ h(\phi) = \text{const.}\,,\ G_4(\phi, X) = \phi \text{ and } G_5(\phi, X) = 0\,. \quad (7.28)$$

By comparing with the different classes of symmetries from the table 7.1, we can see that $f(R)$ can be recovered only from the **C, E, G** or **H** class. For example, in the **E**-class we can set

$$\xi_1 = 1\,,\ \phi_1 = 0\,,\ g_2(X) = V_0\,,\ c_1 = c_2 = c_3 = c_4 = 0\,,\ g_4(X) = 1 \text{ and } g_5(X) = 0\,, \quad (7.29)$$

with V_0 an arbitrary constant and get that $V(\phi) = V_0/\phi$. This potential corresponds to the $f(R) = \sqrt{R}$ model. In fact, if we force the coupling of the scalar field with curvature to be of the form ϕR, we always end up with this potential and thus only with $f(R) = R^{1/2}$. However, we know from the literature [166, 200, 203, 205], that $f(R)$ accepts more Noether symmetries. Specifically, the power law model $f(R) = R^n$ accepts the Noether vector

$$\mathbf{X} = 2t\partial_t + \frac{a}{3}(4n-2)\,\partial_a - 4R\partial_R\,. \tag{7.30}$$

In order for this to be the same with the vector (7.11) we have to set $\xi_1 = 2\,,\ \xi_2 = 0$ and $\eta_a = a(4n-2)/3$ or better $a_1 = (4n-2)/3$ in the $\mathbf{C}$ class of symmetries and $a_1 = 4n-6$ in the $\mathbf{G}$ class. As an example, let us check the $n = 3/2$ case, which accepts a symmetry [203]. For $n = 3/2$ it is $a_1 = 4/3$ (if we consider the $\mathbf{C}$ class of symmetries) and thus the Horndeski functions should be

$$G_4(\phi, X) = (2\phi)^{-1} \text{ and } G_2(\phi, X) = \frac{V_0}{8\phi^3}\,, \tag{7.31}$$

where for simplicity we set $\phi_1 = 0$. Now the Lagrangian density looks like $\mathcal{L} \sim R/(2\phi) - V_0/(8\phi^3)$, but if we redefine the scalar field as $\psi = 1/(2\phi)$ it becomes

$$\mathcal{S} \sim \int d^4x\sqrt{g}\left(\psi R - V_0\psi^3\right)\,. \tag{7.32}$$

In this way we can recover the power-law $f(R)$ models that admit symmetries.

As discussed in [176] for spherical symmetry, the power n is related to the conserved quantities that have physical meaning [175, 177]. It is straightforward to solve the Euler-Lagrange equations produced by (7.27) to get

$$a(t) = a_0\left(\frac{t}{t_0}\right)^m\,, \quad \phi(t) = \pm i\sqrt{\frac{V_0}{48m^2 - 24m}}\,t\,. \tag{7.33}$$

In order for the scalar field solutions to be real, we have two branches: 1) $V_0 < 0$ and $0 < m < 1/2$ and 2) $V_0 > 0$ and $m < 0$ or $m > 1/2$. There exist also exponential solutions for the scale factor, which lead to constant scalar field.

7.2.3 Cubic Galileon model

The galileon theories have also been proposed as an natural explanation of the accelerated expansion of the Universe, without the need of dark energy and, as such, a lot of progress has been made in the last few years in this direction. The name comes from the fact that, in galileon gravity theories, the action is invariant under the shift symmetry in flat spacetime, $\partial_\alpha\phi \to \partial_\alpha\phi + \upsilon_\alpha$. They pass the Solar-System tests [298] and applications of MOND have been studied in this context [206]. Inflationary and self-accelerating solutions have been also been considered [213–217] and, moreover, gravitational waves have also been taken into account [207,208]. We will focus on the cubic galileon theory with the action, in the Einstein frame, given by

$$\mathcal{S} \sim \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{k_1}{2}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi - \frac{k_2}{2M^2}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi\tilde{\Box}\psi\right] + \mathcal{S}[\chi_m, g_{\mu\nu}]\,. \tag{7.34}$$

The spacetime metric is described by $\tilde{g}_{\mu\nu}$, k_1, k_2 are coupling parameters and M is a mass scale of the galileon field, ψ. Matter fields, χ_m, couple minimally to a physical metric (in the Jordan frame) $g_{\mu\nu} = e^{2\alpha\psi}\tilde{g}_{\mu\nu}$, with α the matter-galileon coupling parameter [327].

Matching the Einstein-cubic galileon and the Horndeski theory, i.e symmetry class **A** in the Table 7.1, we have to set

$$g_2(X) = k_1X\,,\; g_4(X) = 1\,,\; g_5(X) = 0\,,\; c_1 = 0\,,\; c_2 = \frac{k_2}{M^2}\,,\; c_3 = 0\,,\; c_4 = 0\,, \tag{7.35}$$

where the Noether vector takes the form

$$\mathbf{X} = \xi_2\partial_t + \phi_1\partial_\phi\,, \tag{7.36}$$

and the integral of motion becomes

$$I = f_1 - \phi_1 a^2\dot{\psi}\left(k_1 a - \frac{3k_2}{M^2}\dot{a}\dot{\psi}\right)\,, \tag{7.37}$$

since the point-like cosmological Lagrangian coming from (7.34) is

$$\mathcal{L} = -6a\dot{a}^2 + \frac{k_1}{2}a^3\dot{\psi} - \frac{k_2}{M^2}a^2\dot{a}\dot{\psi}\,. \tag{7.38}$$

This model is very well studied in the literature and there have been found both cosmological as well as spherically symmetric solutions [209, 327]. For example, if one considers the linear *ansatz*

$$\phi(t) = \phi_0 + \phi_1 t\,, \tag{7.39}$$

where ϕ_0 and ϕ_1 are constants, for the scalar field, they get that $H = k_2M^2/(3k_3\phi_1)$, which is an expanding solution as long as $k_2k_3\phi_1 > 0$.

7.2.4 Non-minimal kinetic coupling

An interesting subclass of Horndeski theory is represented by scalar-tensor models where the scalar kinetic term has non-minimal coupling to curvature. Theories with the non-minimal kinetic coupling lead to a rich variety of solutions for different cosmological epochs, particularly for late time acceleration, as shown in [222–228].

The action of the theory of gravity with non-minimal kinetic coupling reads

$$S=\int d^4x\sqrt{-g}\left\{\frac{R}{16\pi}-\frac{1}{2}\left[g^{\mu\nu}+\eta G^{\mu\nu}\right]\nabla_\mu\phi\nabla_\nu\phi-V(\phi)\right\}, \tag{7.40}$$

where η is a coupling parameter with the dimension of inverse mass-squared. Comparing this with the Horndeski action (3.22), we find

$$G_2(\phi,X)=X-V(\phi)\,,\ G_3(\phi,X)=0\,,\ G_4(\phi,X)=\frac{1}{16\pi}\,,\ G_5(\phi,X)=\frac{1}{2}\eta\phi. \tag{7.41}$$

Since we assume that $G_3(\phi,X)=0$, then from Eq.(7.5) we get $g(\phi)=0$ and $h(\phi)=0$. In addition, the coupling to the Einstein tensor is derived by integrating out a total derivative. The theory (7.40) possesses the Noether symmetry iff $V(\phi)\sim\Lambda=const$, and the configuration providing the Noether symmetry belongs to the symmetry-class **J** in Table 7.1, where

$$c_1=0=c_2\,,\ c_3=\frac{1}{2}\eta\ \ g_2(X)=X-2\Lambda\,,\ g_4(X)=\frac{1}{16\pi}\,,\ g_5(X)=0. \tag{7.42}$$

Now, the Lagrangian (7.6) takes the form

$$\mathcal{L}=a^3(\tfrac{1}{2}\dot\phi^2-2\Lambda)-\frac{3a\dot a^2}{8\pi}-\frac{3}{2}\eta a\dot a^2\dot\phi^2. \tag{7.43}$$

After solving the Euler-Lagrange equations for the above Lagrangian, we get, e.g. for $\Lambda>0$ and $\eta>0$,

$$a(t)=H_\Lambda t\,,\ \phi(t)=\phi_0=\text{const.} \tag{7.44}$$

$$a(t)=\frac{t}{\sqrt{3\eta}}\,,\ \phi(t)=\sqrt{\frac{3\eta H_\Lambda^2-1}{16\pi\eta}}\,t\,, \tag{7.45}$$

where $H_\Lambda\geq 1/\sqrt{3\eta}$. For different combinations of Λ and η signs, as well as for a discussion on solutions (e.g. with $\Lambda=0$), see [223] and references therein.

7.3 Discussion and Conclusions

The Horndeski gravity is the most general scalar-tensor theory giving rise to second order field equations. In principle, any theory of gravity containing scalar-tensor terms can be mapped onto the action (3.22). In this chapter, we presented a systematic classification of scalar-tensor models coming from the Horndeski theory, which are invariant under infinitesimal point transformations. Specifically, using the so-called Noether Symmetry Approach, we were able to find theories that possess symmetries and thus, integrals of motion. When symmetries exist, the related dynamical systems are reducible and integrable. In other words, the presence of symmetries fixes the functional form of the theory, gives conserved quantities and allows to find out exact solutions.

In Table 7.1, we reported all the possible Horndeski functions that have Noether symmetries in the minisuperspace of cosmology. As it appears evident, the existence of Noether symmetries fixes the classes of models and their mathematical and physical properties.

The paradigm is twofold: *i*) couplings and scalar-field potentials of a given theory can be derived from the general Horndeski action (3.22); *ii*) the invariance under point infinitesimal transformations gives rise to the Noether symmetries and then allows to exactly integrate the system. Furthermore, the most popular alternative gravities come out from this approach and can be worked out under the standard of Noether symmetries. In particular, we considered Brans-Dicke gravity, $f(R)$ gravity, galileon gravity, string motivated gravity and non-minimal derivative coupling gravity. They are five specific models of theories belonging to the four classes of the Noether symmetry: **A**, **B**, **E**, and **J**. In principle, all symmetry classes can be discussed under the present standard.

An important remark may be necessary at this point; in the last two years, significant progress has been done in gravitational wave astronomy. Specifically, the observation of black hole-black hole mergers, as well as the binary neutron star merger GW170817 [230], have provided the possibility to test GR in the strong field regime. The last observed event (binary neutron stars), together with its electromagnetic counterpart, started the so-called the *multi-messenger astrophysics* setting severe constraints on the propagation of tensor modes. Since the Horndeski theory shows, besides the standard $+$ and $\times$ polarization modes of GR, an extra mode excited by a massive scalar field [191], it means that the theory can be severely constrained by the mass of the graviton [193, 194]. Besides, the motion of stars as well as the energy radiated away as gravitational radiation are different if compared to GR: this means that more constraints can be obtained and several Horndeski models can be ruled out by the observations [192]. In particular, some models (such as the non-minimal derivative coupling) are presently excluded by gravitational wave observations and then G_4 and G_5 functions are strictly constrained. However, also considering observational limitations, our approach goes beyond because it is aimed to classify the general Horndeski action.

As we already mentioned, the purpose of this chapter was to classify all the possible models originating from the general Horndeski action (3.22), that present Noether symmetries. Clearly the zero-order invariants, derived from symmetries, can be used to construct general exact solutions. For example, in Refs. [231] and [232], cosmology coming from scalar-tensor theories of gravity have been discussed in detail deriving exact solutions from zero-order invariants. In particular, in Tables I and II of [232], the specific forms of gravitational coupling and self-interaction potential are given allowing to achieve the general exact solutions for the scalar-tensor dynamics related to their action (1). Such an action, can be derived, from our approach, specifying, for example, the form of function G_2. In other words, our Table I can be compared to Tables I and II in [232] deriving the same results. Similar considerations hold for [231].

Chapter 8

Noether Symmetries in $f(R,\mathcal{G})$ gravity

There are many modifications of gravity with one of the most well studied being $f(R)$. However, we can also introduce other curvature scalars in the action, such as $R_{\mu\nu}R^{\mu\nu}$ or $R^{\alpha}{}_{\beta\mu\nu}R_{\alpha}{}^{\beta\mu\nu}$, in order to exhaust the possible curvature budget. In this chapter we will consider the $f(R,\mathcal{G})$ theory of gravity, where $\mathcal{G}$ is the Gauss-Bonnet topological invariant. Specifically, we will apply the Noether Symmetry Approach developed in chapter 6 and we will classify those models that are invariant under point-transformations. We will use their symmetries to find exact solutions in the cosmological minisuperspace. Finally, we will study the Noether symmetries of pure $f(\mathcal{G})$ gravity in a spherically symmetric spacetime and we will find exact solutions.

8.1 Introduction

Numerous are the times we mentioned in the first part of this thesis, the need to modify gravity. Emanating mostly from cosmology, but also from quantum field theory and astrophysics, there are several shortcomings in the current framework of general relativity. Attempts to construct a renormalizable theory of gravity started to include higher order terms of curvature invariants, such as contractions of the Riemann and the Ricci tensor [128, 129]. The interest in those terms comes also from low-energy effective field theories of string theory and supergravity [130, 131].

These are the reasons we want to study the $f(R,\mathcal{G})$ theory in this chapter. Theories including the Gauss-Bonnet invariant have been extensively studied in the literature [132–136]. They are successful in describing the late-time acceleration of the Universe, quintessence and phantom behaviour as well as the transition from inflationary to the dark-energy epochs.

What we will do, is apply the Noether Symmetry Approach developed in chapter

6, in order to classify those models of $f(R,\mathcal{G})$ theory that accept Noether symmetries in the minisuperspace of cosmology. What is more, we will use these symmetries in order to find analytical cosmological solutions. In the last part of this chapter, we will consider the pure $f(\mathcal{G})$ theory of gravity and we will do the same in a spherically symmetric spacetime.

In this chapter, we will deal only with quantities calculated using the Levi-Civita connection; it is thus redundant to always use the $\circ$ everywhere and therefore we will omit it.

8.2 $f(R,\mathcal{G})$ gravity

The action of $f(R,\mathcal{G})$ gravity, where R is the Ricci scalar and $\mathcal{G}$ is the Gauss-Bonnet topological invariant given by

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\,, \tag{8.1}$$

reads

$$\mathcal{S}_{f(R,\mathcal{G})} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}f(R,\mathcal{G}) + \mathcal{S}_m\,. \tag{8.2}$$

We remind the reader that, a linear term in $\mathcal{G}$, appearing in the gravitational action (in four dimensions) does not contribute to the field equations being a topological invariant [56]. Varying the above action with respect to the metric, we get

$$\begin{aligned}
2f_{\mathcal{G}}RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R,\mathcal{G}) + 4f_{\mathcal{G}}R_{\mu}{}^{\alpha}R_{\alpha\nu} + 2f_{\mathcal{G}}R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} + 4f_{\mathcal{G}}R_{\mu\alpha\beta\nu}R^{\alpha\beta} - 2R\nabla_{\mu}\nabla_{\nu}f_{\mathcal{G}} + \\
+ 2g_{\mu\nu}R\Box f_{\mathcal{G}} + 4R_{\nu\alpha}\nabla^{\alpha}\nabla_{\mu}f_{\mathcal{G}} + 4R_{\mu\alpha}\nabla^{\alpha}\nabla_{\nu}f_{\mathcal{G}} - 4g_{\mu\nu}R_{\alpha\beta}\nabla^{\alpha}\nabla^{\beta}f_{\mathcal{G}} + R_{\mu\nu}f_{\mathcal{G}} + \\
+ 4R_{\mu\alpha\nu\beta}\nabla^{\alpha}\nabla^{\beta}f_{\mathcal{G}} + \nabla_{\mu}\nabla_{\nu}f_R + g_{\mu\nu}\Box f_R = 8\pi G_N T^M_{\mu\nu}\,,
\end{aligned} \tag{8.3}$$

where f_R and $f_{\mathcal{G}}$ are partial derivatives of $f(R,\mathcal{G})$ with respect to R and $\mathcal{G}$ respectively, and $T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}}$.

We want to seek for forms of the function $f\left(R,\mathcal{G}\right)$ compatible with the existence of Noether symmetries and then use these symmetries to find out solutions. We will develop our considerations in a cosmological minisuperspace considering a spatially flat Friedman-Robertson-Walker metric of the form

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right)\,. \tag{8.4}$$

In order to construct a point-like canonical Lagrangian, we introduce two Lagrange multipliers as

$$\mathcal{S} = \frac{1}{2\kappa}\int dt\sqrt{-g}\left[f\left(R,\mathcal{G}\right) - \lambda_1\left(R-\bar{R}\right) - \lambda_2\left(\mathcal{G}-\bar{\mathcal{G}}\right)\right]\,, \tag{8.5}$$

where $\bar{R}$ and $\bar{\mathcal{G}}$ are the Ricci scalar and the Gauss Bonnet invariant expressed in terms of the metric (8.4),

$$\bar{R} = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) \quad , \quad \bar{\mathcal{G}} = 24\frac{\dot{a}^2\ddot{a}}{a^3} . \tag{8.6}$$

The Lagrange multipliers λ_1 and λ_2 are given by varying the action with respect to R and $\mathcal{G}$ respectively and thus $\lambda_1 = \partial f/\partial R = f_R$ and $\lambda_2 = \partial f/\partial\mathcal{G} = f_{\mathcal{G}}$. After integrating out two total derivatives, we end up with the following point-like Lagrangian

$$L = a^3\left(f - \mathcal{G}f_{\mathcal{G}} - Rf_R\right) - 6af_R\dot{a}^2 - 6a^2 f_{R\mathcal{G}}\dot{a}\dot{\mathcal{G}} - 6a^2 f_{RR}\dot{a}\dot{R} - 8f_{\mathcal{G}\mathcal{G}}\dot{a}^3\dot{\mathcal{G}} - 8f_{R\mathcal{G}}\dot{a}^3\dot{R} . \tag{8.7}$$

The configuration space of (8.7) is $\mathcal{Q} = \{a, R, \mathcal{G}\}$ and its tangent space $T\mathcal{Q} = \{a, \dot{a}, R, \dot{R}, \mathcal{G}, \dot{\mathcal{G}}\}$. Hence the symmetry generating vector is

$$\mathbf{X} = \xi(t,a,R,\mathcal{G})\partial_t + \eta_a(t,a,R,\mathcal{G})\partial_a + \eta_R(t,a,R,\mathcal{G})\partial_R + \eta_{\mathcal{G}}(t,a,R,\mathcal{G})\partial_{\mathcal{G}} . \tag{8.8}$$

8.3 Classification using Noether Symmetries

Let us now apply the symmetry condition

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}\frac{d\xi}{dt} = \frac{dg}{dt} , \tag{8.9}$$

to the point-like Lagrangian (8.7). If symmetries exist, such a condition will fix the form of vector $\mathbf{X}$ as well as the form $f(R,\mathcal{G})$. By the above procedure, we obtain an overdetermined system of 27 partial differential equations. There are several different cases depending on the forms of $f(R,\mathcal{G})$ as well as on the values of integration constants. We summarize them in what follows; the function g on the right hand side of equation (8.9) is assumed constant, unless otherwise stated. All the c_i's mentioned below are integration constants.

There are two different classes of theories; those for which the derivative $f_{R\mathcal{G}}$ vanishes and those for which it does not.

1. For $f_{R\mathcal{G}} \neq 0$ we have the following cases:

   (a) If $f_{RR} \neq 0$, then

   - for $c_1 \neq c_2$, we get

$$f(R,\mathcal{G}) = R^{\frac{c_1+3c_2}{2c_1}}\tilde{f}\left(\frac{\mathcal{G}}{R^2}\right) + \frac{c_4}{3\left(c_2 - c_1\right)}\mathcal{G} , \tag{8.10}$$

where $\tilde{f}$ is an arbitrary function of $\mathcal{G}/R^2$. It admits the Noether symmetry vector

$$\mathbf{X} = (c_1 t + c_3)\partial_t + c_2 a\partial_a - 2c_1 R\partial_R - 4c_1 \mathcal{G}\partial_{\mathcal{G}} \,. \tag{8.11}$$

- For $c_1 = c_2$, the theory

$$f(R,\mathcal{G}) = R^2 \tilde{f}(\frac{\mathcal{G}}{R^2}) - \frac{c_4 \mathcal{G}\ln R}{2c_1}\,, \tag{8.12}$$

admits the Noether symmetry

$$\mathbf{X} = (c_1 t + c_3)\partial_t + c_1 a\partial_a - 2c_1 R\partial_R - 4c_1 \mathcal{G}\partial_{\mathcal{G}} \,. \tag{8.13}$$

(b) If $f_{RR} = 0$, the function f takes the form $f(R,G) = f_1(\mathcal{G}) + Rf_2(\mathcal{G})$ and the following cases are obtained:

- For $c_1 = 0$, the theory $f(R,G) = f_1(\mathcal{G}) + Rf_2(\mathcal{G})$ admits only the symmetry $\mathbf{X} = c_3\partial_t$ and the associated integral is the Hamiltonian E.
- For $c_1 \neq 0$ we have two cases

  i. If $c_2 \neq c_1$, then the theory

$$f(R,\mathcal{G}) = \frac{c_4}{3\,(c_2 - c_1)}\mathcal{G} + c_5 \mathcal{G}^{\frac{c_1+3c_2}{4c_1}} + c_6 R\mathcal{G}^{\frac{3c_2-c_1}{4c_1}}\,, \tag{8.14}$$

  admits the symmetry (8.11).

  ii. If $c_2 = c_1$, the theory

$$f(R,\mathcal{G}) = -\frac{c_4 \mathcal{G}\ln\mathcal{G}}{4c_1} + c_5\mathcal{G} + c_6\sqrt{\mathcal{G}}R\,, \tag{8.15}$$

  admits the symmetry (8.13).

2. The second class of theories are those for which $f_{R\mathcal{G}} = 0$, which means $f(R,\mathcal{G}) = f_1(R) + f_2(\mathcal{G})$. For these theories we find that

   (a) If $f_1''(R) \neq 0$ and

   i. $f_2''(\mathcal{G}) \neq 0$, there are three possible cases:

   - If $c_2 \neq c_1$ and $c_2 \neq -c_1/3$ the theory

$$f(R,\mathcal{G}) = \frac{c_4\mathcal{G}}{3c_2 - 3c_1} + c_5 R^{\frac{c_1+3c_2}{2c_1}} + c_6 \mathcal{G}^{\frac{c_1+3c_2}{4c_1}}\,, \tag{8.16}$$

   admits the symmetry vector given by Eq. (8.11).

- If $c_2 = c_1$ the theory

$$f(R,\mathcal{G}) = -\frac{c_4\mathcal{G}\ln\mathcal{G}}{4c_1} + c_5R^2 + c_6\mathcal{G}\,, \tag{8.17}$$

  admits the Noether vector given by Eq. (8.13).

- If $c_2 = -c_1/3$, the theory

$$f(R,\mathcal{G}) = -\frac{c_4}{4c_1}\mathcal{G} + \frac{c_5}{2c_1}\ln\left(\frac{\sqrt{\mathcal{G}}}{R}\right) + c_6\,, \tag{8.18}$$

  admits the symmetry with generator

$$\mathbf{X} = (c_1t + c_3)\partial_t - \frac{c_1}{3}a\partial_a - 2c_1R\partial_R - 4c_1\mathcal{G}\partial_\mathcal{G}\,. \tag{8.19}$$

ii. $f_2''(\mathcal{G}) = 0$, then f takes the form $f(R,\mathcal{G}) = f_1(R) + c_1\mathcal{G} + c_2$, and the system reduces to three equations, but it is not solvable for arbitrary $f_1(R)$.

(b) If $f_1''(R) = 0$ then we can write it as $f_1(R) = c_5R + c_6$. Thus we end up with the following cases

i. For theories which are General Relativity with $c_5 \neq 0$

A. and $f_2''(\mathcal{G}) \neq 0$, there are two different cases, one of which is non-trivial:

- when $c_1 \neq 0$, the theory

$$f(R,\mathcal{G}) = c_5R - \frac{c_4}{2c_1}\mathcal{G} + c_7\sqrt{\mathcal{G}}\,, \tag{8.20}$$

  admits the following symmetry

$$\mathbf{X} = (c_1t + c_3)\partial_t + \frac{c_1}{3}a\partial_a - 4c_1\mathcal{G}\partial_\mathcal{G}\,, \tag{8.21}$$

- while when $c_1 = 0$, the theory is $f(R,\mathcal{G}) = c_5R + c_6 + f_2(\mathcal{G})$ and admits only the symmetry $\mathbf{X} = c_3\partial_t$ with the Hamiltonian as integral of motion.

B. and $f_2''(\mathcal{G}) = 0 \Rightarrow f_2(\mathcal{G}) = c_7\mathcal{G}$ we obtain the following two non-trivial cases:

- For $c_6 \neq 0$ the theory has the form

$$f(R,\mathcal{G}) = c_5R + c_6 + c_7\mathcal{G}\,, \tag{8.22}$$

and the Noether vector becomes

$$\begin{aligned}\mathbf{X} =&\Bigg[\frac{a}{3}\left(c_3\sin\left(\frac{t}{c_0}\right)+c_2\cos\left(\frac{t}{c_0}\right)\right)+\\&+\frac{1}{\sqrt{a}}\left(c_4\sin\left(\frac{t}{2c_0}\right)+c_8\cos\left(\frac{t}{2c_0}\right)\right)\Bigg]\partial_a+\\&+\left[c_1-c_0\left(c_2\sin\left(\frac{t}{c_0}\right)+c_3\cos\left(\frac{t}{c_0}\right)\right)\right]\partial_t\,,\end{aligned}\tag{8.23}$$

where $c_0=\sqrt{2c_5/(3c_6)}$ is a redefinition of the integration constants. In addition, while in all the previous cases, the right hand side function g in Eq. (8.9) was constant, here it becomes non-trivial and takes the form

$$\begin{aligned}g(a,t) =&2\sqrt{c_5c_6}\Bigg[\sqrt{6}a^{3/2}\left(c_8\sin\left(\frac{t}{2c_0}\right)-c_4\cos\left(\frac{t}{2c_0}\right)\right)+\\&+\sqrt{\frac{2}{3}}a^3\left(c_2\sin\left(\frac{t}{c_0}\right)-c_3\cos\left(\frac{t}{c_0}\right)\right)\Bigg]+g_0\,.\end{aligned}\tag{8.24}$$

- For $c_6=0$ the theory is the Einstein-Hilbert action plus a topological invariant term (Gauss-Bonnet) $f(R,\mathcal{G})=c_5R+c_7\mathcal{G}$ which does not contribute to the dynamics in 4-dimensional spaces. In this case, the Noether vector takes the form

$$\mathbf{X}=\left(t\left(c_1t+c_2\right)+c_3\right)\partial_t+\left(\frac{1}{3}a\left(2c_1t+c_2\right)+\frac{c_4t+c_8}{\sqrt{a}}\right)\partial_a\,,\tag{8.25}$$

and the function g is again non-trivial

$$g(a)=-8a^{3/2}c_4c_5-\frac{8}{3}a^3c_1c_5+g_0\,.\tag{8.26}$$

ii. In the case where the theories do not contain any Ricci scalar in the action, i.e. $c_5=0$, the function f takes the form $f(R,\mathcal{G})=c_6+f_2(\mathcal{G})$. The only interesting theories in this case, are those for which $f_2''(\mathcal{G})\neq 0$, since if $f_2''(\mathcal{G})=0$, we will have only linear terms in $\mathcal{G}$ in the action and the theory is trivial for the above reasons. So, for $f_2''(\mathcal{G})\neq 0$ and

- $c_2\neq c_1$, we have the theory

$$f(R,\mathcal{G})=c_7\mathcal{G}^{\frac{c_1+3c_2}{4c_1}}+\frac{c_4}{3\left(c_2-c_1\right)}\mathcal{G}\,,\tag{8.27}$$

and its Noether symmetry is given by

$$\mathbf{X}=(tc_1+c_3)\partial_t+c_2a\partial_a-4c_1\mathcal{G}\partial_{\mathcal{G}}\,;\tag{8.28}$$

- for $c_2 = c_1$, the theory takes the form

$$f(R,\mathcal{G}) = c_7\mathcal{G} - \frac{c_4}{4c_1}\mathcal{G}\ln\mathcal{G}\,, \tag{8.29}$$

and its generator

$$\mathbf{X} = (tc_1 + c_3)\partial_t + c_1 a\partial_a - 4c_1\mathcal{G}\partial_{\mathcal{G}}\,. \tag{8.30}$$

It is obvious that, for each of these functions f and its generators, there exists an integral of motion, which we do not report here for the sake of simplicity. In conclusion, we have showed that, by Noether theorem, it possible to select specific models of a given theory of gravity, in this case $f(R,\mathcal{G})$ of the action (8.2). Models that admit Noether symmetries have generating vectors and associated integrals of motion. These symmetries can be used to find analytical solutions as we we see in the next section.

8.4 Cosmological Solutions

Finding out exact solutions is the main issue related to the search for symmetries. In other words, if the symmetries do not reduce dynamics, they are useless from the point of view of dynamical systems. Here we consider some specific forms of $f(R,\mathcal{G})$, selected above by the existence of the Noether symmetries, and search for cosmological solutions. From the generator of symmetries, we can calculate the zero-order invariants for each case. The Noether vector for several of above models, i.e. (8.10), (8.12), (8.14), (8.15), (8.16), (8.17), (8.18), (8.20), (8.27), (8.29), has the same form, with different constants of integration. From this, we get the Lagrange system

$$\frac{dt}{c_1 t} = \frac{da}{c_2 a} = -\frac{dR}{2c_1 R} = -\frac{d\mathcal{G}}{4c_1\mathcal{G}}\,, \tag{8.31}$$

and solving for $a(t)$, $R(t)$ and $\mathcal{G}(t)$, we get

$$a(t) = a_0 t^{c_2/c_1}\,,\ R(t) = \frac{r_0}{t^2}\,,\ \mathcal{G}(t) = \frac{g_0}{t^4}\,, \tag{8.32}$$

where a_0, r_0 and g_0 are constants. By substituting these into the Euler-Lagrange equations for a, R and $\mathcal{G}$ we can constrain the arbitrary functions $\tilde{f}(\mathcal{G}/R^2)$ in the (8.10) and (8.12), as well as the integration constants c_i's.

Let us study the different models one by one. Consider the model given by (8.10). We substitute it into the point-like Lagrangian (8.7) and write down the Euler-Lagrange equations for the variables of the configuration space. The equations for R and for $\mathcal{G}$, as expected from the Lagrange multipliers, give the expressions (8.6).

It is easy to see that the cosmological scale factor a has power law solutions of the form $a(t) = a_0 t^p$ for $c_2 = c_1(3p-1)/3$, and de-Sitter solutions $a(t) = a_0 e^{H_0 t}$, for $\tilde{f}(1/6) = 0$.

In the same way, the model given by (8.12) admits de Sitter solutions for $c_4 = 0$. Power-law solutions are obtained for some specific values of $\tilde{f}$. The model (8.14) admits power-law solutions for $c_2 = c_1(3p-1)/3$ and de-Sitter solutions for $c_6 = -c_5/\sqrt{6}$. The model (8.15) gives de-Sitter solutions as soon as $c_4 = 0$ and power-law solutions for

$$c_6 = \frac{c_4\sqrt{(p-1)p^3}(p+3)}{3\sqrt{6}c_1(p-1)p}\,.$$

For the model (8.16) power-law solutions exist for $c_2 = c_1(3p-1)/3$ and de-Sitter solutions for $c_6 = -6^{(3c_2+c_1)/4c_1}c_5$. The model (8.17) admits de-Sitter solutions only for $c_4 = 0$, but gives power-law solutions for

$$c_4 = \frac{18c_5(2p-1)c_1}{p(p+3)}\,.$$

The model (8.18) gives de-Sitter solutions for $c_6 = c_5 \ln 6/4c_1$ and power-law solutions for

$$c_6 = -\frac{c_5\left((4p^2-6p+2)\ln\left(-\frac{\sqrt{\frac{2}{3}}\sqrt{(p-1)p^3}}{p-2p^2}\right)+3p+1\right)}{4c_1\left(2p^2-3p+1\right)}\,.$$

The model (8.20) admits de-Sitter solutions for $c_7 = -\sqrt{6}c_5$ and power-law solutions for $c_7 = \sqrt{6}c_5\frac{p^3(p-1)}{p(p+1)}$. The model (8.27) admits only power-law solutions for $c_2 = -c_1/3$ or $c_2 = c_1(3p-1)/3$. Finally, the model (8.29) admits only power-law solutions for $p = -3$ and $p = 4/3$.

8.5 Pure $f(\mathcal{G})$ gravity

In the literature, an addition of an arbitrary function $f(\mathcal{G})$ to the Einstein-Hilbert action has been proposed [132]. Specifically, the theory given by the action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_N}R + f(\mathcal{G})\right]$$

has been extensively studied. In [361] they study cosmologically viable models, by studying the stability of a late-time de-Sitter solution and the existence of radiation and matter epochs. In [362] they study possible power-law scaling solutions by working on the scalar tensor equivalent of the above theory. In particular, in [363] they study cosmological perturbations and show that density perturbations cause instabilities. In [364], the author shows that the above theory is ruled out as a possible explanation of the late-time acceleration by solar system tests. Finally in [365] they study a mimetic version of the above theory and they find apart from accelerating solutions, solutions that unify the inflation era together with dark energy and in addition dark matter is described in the framework of this theory.

All of the above references deal with a theory that safely recovers GR in the background. This means that if one switches off the effect of the GB contribution, i.e. $f(\mathcal{G}) \to 0$, then the action reduces to the Einstein-Hilbert and we recover GR. This happened because of the great success that GR has with observations and because any new proposed theory should repeat the successes of GR together with more achievements. In this section, we propose a more radical scenario where GR does not exist in the background and gravity is given only by quadratic curvature invariants and specifically an arbitrary function of the GB term.

What we do is after writing down the equations of motion for this theory, we move to a spherically symmetric minisuperspace and we study possible Noether Symmetries. Specifically, we use the so called Noether Symmetry Approach [166, 359], which has been extensively used in the literature , [105, 110, 162, 170]as a geometric criterion to select forms of the arbitrary function $f(\mathcal{G})$ that are invariant under point transformations. Interestingly enough, we see that the only possible form that accepts symmetries is a power-law. By making use of these symmetries we find general exact spherically symmetric solutions, which for specific values of the power reduce to the known Schwarzschild and de-Sitter solutions.

8.5.1 $f(\mathcal{G})$ in spherical symmetry

The pure Gauss-Bonnet gravity is given by the action

$$S = \int \sqrt{-g} f(\mathcal{G})\, d^4x \,. \tag{8.33}$$

In four dimensions linear terms in $\mathcal{G}$ in the action, are trivial since $\mathcal{G}$ is an invariant. As we already mentioned in the intorduction, up to now, every author who studied $f(\mathcal{G})$ theories, considered also a Ricci scalar in the action, in order to recover General Relativity when $f \to 0$. In our case, we consider pure $f(\mathcal{G})$ theories and we claim that, it maybe the case that we can recover the success of GR without considering the Einstein-Hilbert action in the background. In this section, we will prove that this happens in spherically symmetric spacetimes, in vacuum and in [366] the authors prove it also in cosmology.

By varying this action with respect to the metric we get the equations of motion which are

$$2R\nabla_\mu\nabla_\nu f_\mathcal{G} - 2g_{\mu\nu}R\Box f_\mathcal{G} - 4R_\mu^\lambda\nabla_\lambda\nabla_\nu f_\mathcal{G} + 4R_{\mu\nu}\Box f_\mathcal{G} + 4g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma f_\mathcal{G} + \\ +4R_{\mu\nu\rho\sigma}\nabla^\rho\nabla^\sigma f_\mathcal{G} + \frac{1}{2}g_{\mu\nu}\left[f - \mathcal{G}f_\mathcal{G}\right] = 0\,, \quad (8.34)$$

where $f_\mathcal{G}$ is the derivative of $f(\mathcal{G})$ with respect to $\mathcal{G}$. In addition, we have one more equation, which is given by taking the trace of Eq. (8.34)

$$2R\Box f_\mathcal{G} - 8R^{\mu\nu}\nabla_\mu\nabla_\nu f_\mathcal{G} - 2\left(f - \mathcal{G}f_\mathcal{G}\right) = 0\,. \quad (8.35)$$

This can be seen as the equation of motion for the new scalar degree of freedom, in analogy with the scalaron in $f(R)$ theories.

We consider now the following spherically symmetric *Ansatz* for the metric

$$ds^2 = P(r)^2dt^2 - Q(r)^2dr^2 - r^2d\Omega^2\,, \quad (8.36)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the 2-sphere. Before we proceed, an important comment is necessary here; obviously, the metric (8.36) does not depend on time, which means that the Birkhoff's theorem should be valid in these theories. This is not proven and we take it for granted, as a simplification to the calculations. In future work we plan to extend the results to more generic spacetimes.

The Gauss-Bonnet term (8.1) takes now the form

$$\mathcal{G} = \frac{8P''(r)}{r^2P(r)Q(r)^2}\left(\frac{1}{Q(r)^2} - 1\right) + \frac{8P'(r)Q'(r)}{r^2P(r)Q(r)^3}\left(1 - \frac{3}{Q(r)^2}\right)\,. \quad (8.37)$$

In what follows, we calculate the point-like Lagrangian of our theory, in spherical symmetry. We introduce the Lagrange multiplier as

$$\mathcal{S} = \int d^4x\, r^2\sqrt{P(r)Q(r)}\left[f(\mathcal{G}) - \lambda\left(\mathcal{G} - \tilde{\mathcal{G}}\right)\right]\,, \quad (8.38)$$

with $\tilde{\mathcal{G}}$ being the Gauss-Bonnet term in spherical symmetry (8.37) and λ the Lagrange multiplier given by varying the action with respect to $\mathcal{G}$, i.e. $\lambda = f_\mathcal{G}$. We have set $\theta = \pi/2$. By substituting $\tilde{\mathcal{G}}$ and integrating out the second derivatives we obtain

$$\mathcal{L}(r,P,Q,\mathcal{G}) = \frac{8f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\mathcal{G}'(r)P'(r)}{Q(r)}\left(1 - \frac{1}{Q(r)^2}\right) + r^2P(r)Q(r)\left[f(\mathcal{G}) - \mathcal{G}(r)f_\mathcal{G}(\mathcal{G})\right]\,, \quad (8.39)$$

where $f_\mathcal{G}$ and $f_{\mathcal{G}\mathcal{G}}$ are the first and second derivatives of f with respect to $\mathcal{G}$. This is the point like Lagrangian of our theory in spherical symmetry. Apparently, even though its configuration space is $\mathcal{Q} = \{P, Q, \mathcal{G}\}$, its tangent space is 5-dimensional, i.e. $\mathcal{TQ} = \{P, P', Q, \mathcal{G}, \mathcal{G}'\}$, since no Q' appears in it. This means that Q is a cyclic variable and we could solve algebraically for Q, the associated Euler-Lagrange equation, $\partial_Q \mathcal{L} = 0$, substitute back in the Lagrangian (8.39), and obtain a 2-dimensional configuration space. However, the Noether theorem holds also for singular Lagrangians; since the calculations are much simpler in this case, we will proceed with (8.39).

8.5.2 Noether symmetries in Gauss-Bonnet gravity

The generator of the transformations in our case is given by

$$\mathbf{X} = \xi(r,\mathcal{G},P,Q)\partial_r + \eta^{\mathcal{G}}(r,\mathcal{G},P,Q)\partial_{\mathcal{G}} + \eta^{P}(r,\mathcal{G},P,Q)\partial_P + \eta^{Q}(r,\mathcal{G},P,Q)\partial_Q \,, \quad (8.40)$$

where ξ and η^i with $i = \{\mathcal{G}, P, Q\}$ are the coefficients of the vector and the i's are just indices to specify each coefficient. By applying Noether's theorem, i.e. Eq. (8.9), we obtain a system of 12 equations which are not all independent. It is straightforward to see that,

$$\xi(r,\mathcal{G},P,Q) = \xi(r)\,, \;\; \eta^{\mathcal{G}}(r,\mathcal{G},P,Q) = \eta^{\mathcal{G}}(\mathcal{G})\,,$$
$$\eta^{P}(r,\mathcal{G},P,Q) = \eta^{P}(P)\,, \;\; g(r,\mathcal{G},P,Q) = g(r)\,,$$

and the only non-trivial equations that survive are the following two

$$r^2 P\left[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})\right]\eta^{Q}(r,\mathcal{G},P,Q) + r\mathcal{G}PQ\Big[r\eta^{\mathcal{G}}(\mathcal{G})f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + rf_{\mathcal{G}}(\mathcal{G})\xi'(r) + 2f_{\mathcal{G}}(\mathcal{G})\xi(r)\Big] +$$
$$+ f(\mathcal{G})\left[r^2 Q\eta^{P}(P) + r^2 PQ\xi'(r) + 2rPQ\xi(r)\right] - \mathcal{G}Qr^2 f_{\mathcal{G}}(\mathcal{G})\eta^{P}(P) - g'(r) = 0\,, \quad (8.41)$$

$$Q\left(Q^2-1\right)\left\{f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\left[\eta^{\mathcal{G}}{}_{,\mathcal{G}}(\mathcal{G}) + \eta^{P}{}_{,P}(P) - \xi'(r)\right] + f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G})\eta^{\mathcal{G}}(\mathcal{G})\right\} -$$
$$- \left(Q^2-3\right) f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\eta^{Q}(r,\mathcal{G},P,Q) = 0\,, \quad (8.42)$$

with $f_{\mathcal{G}}$, $f_{\mathcal{G}\mathcal{G}}$ and $f_{\mathcal{G}\mathcal{G}\mathcal{G}}$ being the first, second and third derivative of $f(\mathcal{G})$ respectively. We solve Eq. (8.41) for η_Q and substitute in Eq. (8.42). After some non-trivial but straightforward calculations, we find that the only $f(\mathcal{G})$ which accepts symmetries has the form

$$f(\mathcal{G}) = \mathcal{G}^n\left(c_3 + c_4\mathcal{G}^m\right)\,. \quad (8.43)$$

where $n = \frac{7c_1 + c_2 - |c_1 - c_2|}{8c_1}$ and $m = \frac{|c_2 - c_1|}{4c_1}$. Its symmetry is given by the Noether vector

$$\mathbf{X} = c_1 r\partial_r + c_2 P\partial_P - 4c_1\mathcal{G}\,, \quad (8.44)$$

and the function on the right hand side of Noether's theorem is constant, i.e. $g(r) = c_5$. All the c_i's are constants of integration. Finally, the associated Noether integral, which is a conserved quantity, is given by

$$I = \xi \left(\dot{x}^i \frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \mathcal{L} \right) - \eta^i \frac{\partial \mathcal{L}}{\partial x^i} + g \,. \tag{8.45}$$

8.5.3 Spherically Symmetric solutions

It is now time to use the symmetries we found in order to find exact solutions. It is straightforward to construct the following Lagrange system, from the Noether vector

$$\frac{dt}{c_1 r} = \frac{dP}{c_2 P} = -\frac{d\mathcal{G}}{-4c_1 \mathcal{G}} \,, \tag{8.46}$$

and by solving it, we obtain the following solutions

$$P(r) = P_0 r^{c2/c1} \text{ and } \mathcal{G} = \mathcal{G}_0 / r^4 \,. \tag{8.47}$$

These are the zero-order invariants and satisfy the Euler-Lagrange equations for any $f(\mathcal{G})$ of the form (8.43).

From Eq. (8.43) we can use a simplified form of $f(\mathcal{G})$ to study possible spherically symmetric solutions. Specifically, we consider the form

$$f(\mathcal{G}) = f_0 \mathcal{G}^n \,, \tag{8.48}$$

which is obtained by setting $c_3 = f_0$ and $c_4 = 0$. In addition, we choose to study solutions for which the Birkhoff's theorem holds, i.e. $Q(r) = 1/P(r)$. It may not be true for every case and thus the solutions are not general enough, however, it is a good starting point to check if known spherically symmetric solutions can be recovered at specific limits of more general possible solutions.

The Lagrangian (8.39) becomes

$$\mathcal{L}(P, G) = -f_0(n-1)G(r)^{n-2} \left[4n(P(r)-1)G'(r)P'(r) + r^2 G(r)^2 \right] \,, \tag{8.49}$$

and the associated Euler-Lagrange equations $\frac{\partial \mathcal{L}}{\partial q^i} = \frac{d}{dr} \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ yield immediately

$$\mathcal{G}(r) = \frac{4}{r^2} \left[P(r)P''(r) - P''(r) + P'(r)^2 \right] \tag{8.50}$$

$$4f_0 n(n-1) \left[P(r) - 1 \right] G(r)^{n-3} \Big[G(r)G''(r) + (n-2)G'(r)^2 \Big] = 0 \,. \tag{8.51}$$

which is the Lagrange multiplier for the Gauss-Bonnet topological term and the E-L equation for $P(r)$ respectively. The Eq. (8.51) has three non-trivial solutions: the first one, which is the simplest, is the Minkowski solution, where $P(r) = 1$ and the associated Gauss-Bonnet term vanishes. Another solution with vanishing $\mathcal{G}$ is given by

$$P(r) = 1 \pm \sqrt{1 - 2k_1 r - 2k_1 k_2}\,, \tag{8.52}$$

where k_1 , k_2 are constants of integration. It is a general solution for any power n.

The most interesting solutions will be given by the last term. We solve it for $\mathcal{G}(r)$ to find

$$\mathcal{G}(r) = g_1\left[(n-1)r - g_2\right]^{1/(n-1)}\,, \tag{8.53}$$

where g_1 and g_2 are constants of integration. Now we substitute this into Eq. (8.50) and after solving for $P(r)$ we find

$$P(r) = 1 \pm \sqrt{1 - 2p_1 + p_2 r + \frac{g_1\left\{6g_2^2 + nr[4g_2 + (2n-1)r]\right\}\left[(n-1)r - g_2\right]^{\frac{1}{n-1}+2}}{2n(2n-1)(3n-2)(4n-3)}}\,, \tag{8.54}$$

with p_1, p_2 constants. This is a general spherically symmetric solution for any n. Apart from this one, we see that the E-L equations are satisfied for any n by the de-Sitter solution, $P(r) = 1 - \Lambda r^2/3$ where the Gauss-Bonnet term takes the constant value $\mathcal{G} = 8\Lambda^2/3$. Finally, by specifying the power n we can find more interesting spherically symmetric solutions. For example, for $n = 5/6$, $p_2 = 0$, $g_2 = 0$ and $p_1 = 1/2$, we can recover Schwarzschild solution and the GB term becomes $\mathcal{G} = 48M^2/r^6$.

8.6 Conclusions

In this chapter we study the $f(R,\mathcal{G})$ theory, where $\mathcal{G}$ is the Gauss-Bonnet topological term. Specifically, by using the Noether Symmetry Approach we classify those models that are invariant under point transformations, in a cosmological minisuperspace. Furthermore, we use the symmetries of these models to find exact cosmological solutions.

In addition, we propose a rather "radical" idea, which is a theory of gravity only dependent in the $f(\mathcal{G})$ function. Even though, theories in the form $R + f(\mathcal{G})$ have been studied in detail, such an idea is totally new, up to our knowledge.

In a spherically symmetric background, we study the Noether symmetries of such a theory and the only possible form that is invariant, is the power-law (8.43). Moreover, we use the symmetry that this action possesses to find exact spherically symmetric solutions. Specifically, for the simplified form $f(\mathcal{G}) = f_0\mathcal{G}^n$ we find analytical spherically symmetric solutions, which are much more general than those known in GR. Schwarzschild and de-Sitter solutions can also be recovered for specific values of the power.

As a final remark, we would like to stress that, even though in the literature most of the attempts try to safely recover GR, we see in this chapter that, if at the solution level we recover the successes of GR, it is not necessary to stick to the Einstein-Hilbert action as a background theory. Of course, this attempt is not enough to convince us that pure GB gravity could be the new theory of gravity, and a lot of ink has to still be spilt.

Chapter 9

Noether Symmetries in Gauss-Bonnet-teleparallel cosmology

As we saw in detail in the first part of this thesis, there is another, very interesting formulation of gravity, the teleparallel gravity. Specifically, the Teleparallel Equivalent of General Relativity uses the torsion scalar as an action. A natural extension has been proposed, in full analogy with $f(R)$ and Einstein-Hilbert action, which is called $f(T)$ theories of gravity. As a continuation of the previous chapter and inspired by the metric Gauss-Bonnet gravity, its teleparallel analog was constructed. The teleparallel Gauss-Bonnet term, $T_{\mathcal{G}}$, differs from the known one, $\mathcal{G}$, by a boundary term, $B_{\mathcal{G}}$, just like the torsion scalar differs from the Ricci scalar. In this chapter, we study the teleparallel alternative of $f(R, \mathcal{G})$ gravity theories, which is $f(T, T_{\mathcal{G}})$, where T is the torsion scalar and $T_{\mathcal{G}}$ is the teleparallel Gauss-Bonnet term. We use the Noether Symmetry Approach in the cosmological minisuperspace to classify the theories and find exact solutions.

9.1 Introduction

Extended theories of gravity are semi-classical approaches where the effective gravitational Lagrangian is modified, with respect to the Hilbert-Einstein one, by considering higher order terms of curvature invariants, torsion tensor, derivatives of curvature invariants and scalar fields (see for example [79, 99, 321, 349]). In particular, taking into account the Ricci, Riemann and Weyl invariants, one can construct terms like R^2, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\delta\sigma}R_{\mu\nu\delta\sigma}$, $W^{\mu\nu\delta\sigma}W_{\mu\nu\delta\sigma}$, that give rise to fourth-order theories in the metric formalism [234, 235]. Considering minimally or nonminimally coupled scalar fields to the geometry, we deal with scalar-tensor theories of gravity [236, 237]. Considering terms like $R\Box R$, $R\Box^k R$, we are dealing with higher-than fourth order theories [238, 239]. $f(R)$ gravity is the simplest class of these models where a generic function of the Ricci scalar R is considered. The interest for these extended models is related both to the problem of quantum gravity [79] and to the possibility to explain the accelerated expansion of the universe, as well as the structure formation, without invoking new particles in the matter/energy content of the universe [170, 174, 234–242, 349]. In other words, the attempt is to address the dark side of the universe by changing the geometric sector and remaining unaltered the matter sources with respect to the Standard Model of Particles. However, in the framework of this "geometric picture", the debate is very broad involving the fundamental structures of gravitational interaction. Just to summarize some points, gravity could be described only by metric (in this case we deal with a metric approach), or by metric and connections (in this case, we are considering a metric-affine approach [317]), or by a purely affine approach [243]. Furthermore, dynamics could be related to curvature tensor, as in the original Einstein theory, to both curvature and torsion [244], or to torsion only, as in the so called *teleparallel gravity* [98].

Starting from these original theories and motivations, one can build more complex Lagrangians, by using different combinations of curvature scalars and their derivatives, or topological invariants, such us the Gauss-Bonnet term, $\mathcal{G}$, as well as the torsion scalar T. Many theories have been proposed considering generic functions of such terms, like $f(\mathcal{G})$, $f(T)$, $f(R,\mathcal{G})$ and $f(R,T)$ [106, 131, 132, 134, 161, 166, 167, 245–256]. However, the problem is *how many* and *what kind* of geometric invariants can be used, and furthermore what kind of physical information one can derive from them. For example, it is well known that $f(R)$ gravity is the straightforward extension of the Hilbert-Einstein which is $f(R) = R$ and $f(T)$ is the extension of teleparallel gravity which is $f(T) = T$. However, if one wants to consider the whole information contained in curvature invariants, one has to take into account also combinations of Riemann, Ricci and Weyl tensors[1]. As discussed in [248], assuming a $f(R,\mathcal{G})$ theory means to consider the whole curvature budget and then all the degrees of freedom related to curvature.

Assuming the teleparallel formalism, a $f(T_{\mathcal{G}},T)$ theory, where $T_{\mathcal{G}}$ is the torsional counterpart of the Gauss-Bonnet topological invariant, means to exhaust all the degrees of freedom related to torsion and then completely extend $f(T)$ gravity. It is important to stress, as we will show below, that the Gauss-Bonnet invariant derived from curvature differs from the same topological invariant derived from torsion in less than a total derivative and then the dynamical information is the same in both representations. According to this result, the topological invariant allows a regularization of dynamics also in the teleparallel torsion picture (see [129, 248] for a discussion in the curvature representation).

[1]Clearly, this means that we are not considering higher-order derivative terms like $\Box R$, or derivative combinations of curvature invariants.

Notation reminder: in this chapter the only quantities we are going to use are related to teleparallel geometry and are calculated using the Weitzenböck connection. So the • is omitted.

9.2 $f(T, T_{\mathcal{G}})$ gravity

In order to incorporate spin in a geometric description, as well as to bring gravity closer to its gauge formulation, people started, some years ago, to study torsion in gravity [98, 244]. An extensive review of torsional theories (teleparallel, Einstein-Cartan, metric-affine, etc) is presented in [99]. If in the action of the teleparallel theory, i.e. in a curvature-free *vierbein* formulation, we replace the torsion scalar, T, with a generic function of it, we obtain the so called $f(T)$ gravity [113,164,257,340],

In this paper, we will study a theory whose Lagrangian is a generic function of the Gauss-Bonnet teleparallel term, $T_{\mathcal{G}}$ and the torsion scalar, T, i.e.

$$\mathcal{A} = \frac{1}{16\pi G_N} \int d^4xe \left[f(T_{\mathcal{G}}, T) + \mathcal{L}_m \right] , \tag{9.1}$$

which is a straightforward generalization of

$$\mathcal{A} = \frac{1}{16\pi G_N} \int d^4xe \left[f(T) + \mathcal{L}_m \right] , \tag{9.2}$$

where $\mathcal{L}_m$ is the standard matter that, in the following considerations, we will discard. It is important to note that the field equations of $f(T)$ gravity are of second order in the metric derivatives and thus simpler than those of $f(R)$ gravity, which are of fourth order [99].

The metric determinant $\sqrt{-g}$ can be derived from the determinant of the vierbeins h as follows. We have

$$E^{\mu}{}_{i} e^{i}{}_{\nu} = \delta^{\mu}{}_{\nu} \, , \; E^{\mu}{}_{i} e^{j}{}_{\mu} = \delta_{i}{}^{j} \, . \tag{9.3}$$

The relation between metric and vierbiens is given by

$$g_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu} \, , \tag{9.4}$$

where η_{ab} is the flat Minkowski metric. Finally, it is $|e| \equiv \det\left(e^{i}_{\mu}\right) = \sqrt{-g}$. More details on how the two formalisms are related can be found in [250].

The torsion scalar is given by the contraction

$$T = S^{\mu\nu}{}_{\rho} T^{\rho}{}_{\mu\nu} \tag{9.5}$$

where

$$S_\rho{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu{}_\rho T^{\sigma\nu}{}_\sigma - \delta^\nu{}_\rho T^{\sigma\mu}{}_\sigma\right) , \tag{9.6}$$

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right) , \tag{9.7}$$

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \tilde{\Gamma}^\alpha{}_{\mu\nu} , \tag{9.8}$$

are respectively the superpotential, the contorsion tensor, the torsion tensor and $\tilde{\Gamma}^\alpha{}_{\mu\nu}$ is the Weitzenböck connection.

We remind the reader that the relation between the torsion and the curvature scalar is given by

$$R = -T + 2T_\nu{}^{\nu\mu}{}_{,\mu} , \tag{9.9}$$

Following [161], the teleparallel equivalent of the Gauss-Bonnet topological invariant can be obtained as ,

$$\mathcal{G} = -T_\mathcal{G} + B_\mathcal{G} , \tag{9.10}$$

where the Gauss-Bonnet invariant, in terms of curvature, is

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} , \tag{9.11}$$

the teleparallel $T_\mathcal{G}$ invariant is given by

$$\begin{aligned} T_\mathcal{G} =&(K^{\alpha_1}{}_{ea}K^{e\alpha_2}{}_b K^{\alpha_3}{}_{fc}K^{f\alpha_4}{}_d - 2K^{\alpha_1\alpha_2}{}_a K^{\alpha_3}{}_{eb}K^e{}_{fc}K^{f\alpha_4}{}_d + \\ &+ 2K^{\alpha_1\alpha_2}{}_a K^{\alpha_3}{}_{eb}K^{e\alpha_4}{}_f K^f{}_{cd} + 2K^{\alpha_1\alpha_2}{}_a K^{\alpha_3}{}_{eb}K^{e\alpha_4}{}_{c,d})\delta^a{}_{\alpha_1}{}^b{}_{\alpha_2}{}^c{}_{\alpha_3}{}^d{}_{\alpha_4} , \end{aligned} \tag{9.12}$$

and the boundary term [106][2]

$$B_\mathcal{G} = \frac{1}{e}\partial_a\left[K_b{}^{ij}\left(K_b{}^{kl}{}_{,d} + K_d{}^m{}_c K_m{}^{kl}\right)\right]\delta^a{}_i\delta^b{}_j\delta^c{}_k\delta^d{}_l \tag{9.13}$$

In a four dimensional spacetime, the term $T_\mathcal{G}$ is a topological invariant, constructed out of torsion and contorsion tensor[3].

The field equations from the action (9.1) are then

$$\begin{aligned} &-4\left[e(\partial_\mu f_T)S_a{}^{\mu\beta} + \partial_\mu(eS_a{}^{\mu\beta})f_T - ef_T T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu}\right] + \partial_\mu\left(E^\mu_h E^\beta_b\left(Y^b{}_a{}^h - Y^h{}_a{}^b + Y_a{}^{[bh]}\right)\right) + \\ &+ T^i{}_{ab}E^\beta_h\left(Y^b{}_i{}^h - Y^h{}_i{}^b + Y_i{}^{[bh]}\right) - 2ef_{TT_\mathcal{G}}E^\beta_d K_m{}^{ij}K_b{}^k{}_e K_c{}^{el}{}_{,a}\delta^m{}_i\delta^b{}_j\delta^c{}_k\delta^d{}_l = 0 , \end{aligned} \tag{9.14}$$

[2]Strictly speaking, the difference between the Gauss-Bonnet term and its teleparallel equivalent is a total derivative. However, when integrated it behaves as a boundary term not contributing in the dynamics of the theory. That is where the name comes from.

[3]See Section 3 of [161] for the detailed derivation and discussion.

where

$$Y^b{}_{ij} = e f_{T_\mathcal{G}} X^b{}_{ij} - 2\partial_\mu \left(e f_{TT_\mathcal{G}} E^\mu_d K_c{}^{el} K_a{}^k{}_i \right) \delta^c{}_e \delta^a{}_l \delta^b{}_k \delta^d{}_e j \,, \tag{9.15}$$

$$X^a{}_{ij} = \frac{\partial T_\mathcal{G}}{\partial K_a{}^{ij}} \,, \tag{9.16}$$

and $f_A = \partial f / \partial A$ being $A = T, T_\mathcal{G}$.

9.3 $f(T, T_\mathcal{G})$ cosmology

Let us consider a a spatially flat FRW cosmology defined by the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \,, \tag{9.17}$$

from which we can express the teleparallel Gauss-Bonnet term as a function of the scale factor $a(t)$ [259]

$$T_\mathcal{G} = 24 \left[\frac{\dot{a}^2(t)\ddot{a}(t)}{a(t)^3} \right] \,. \tag{9.18}$$

As said above, we can discard the total derivative term (see also [106]) The torsion scalar is

$$T = -6 \left[\frac{\dot{a}^2(t)}{a^2(t)} \right] \,. \tag{9.19}$$

We can re-express (9.1) as a canonical point-like action by using the Lagrange multipliers as

$$\mathcal{A} = \frac{1}{16\pi G_N} \int dt \left[a^3 f(T, T_\mathcal{G}) - \lambda_1 \left(T_\mathcal{G} - \bar{T}_\mathcal{G} \right) - \lambda_2 \left(T - \bar{T} \right) \right] \,, \tag{9.20}$$

where $\bar{T}_\mathcal{G}$ and $\bar{T}$ are the Gauss-Bonnet term and the torsion scalar expressed by (9.18) and (9.19). The Lagrange multipliers are given by $\lambda_1 = a^3 \partial_{T_\mathcal{G}} f = a^3 f_{T_\mathcal{G}}$ and $\lambda_2 = a^3 \partial_T f = a^3 f_T$ and are obtained by varying the action with respect to $T_\mathcal{G}$ and T respectively. We can rewrite action (9.20) as

$$\mathcal{A} = \int \frac{dt a^3}{16\pi G_N} \left[f(T, T_\mathcal{G}) - f_{T_\mathcal{G}} \left(T_\mathcal{G} - \frac{24\dot{a}^2\ddot{a}}{a^3} \right) - f_T \left(T + 6\frac{\dot{a}^2}{a^2} \right) \right] \,, \tag{9.21}$$

and discarding total derivative terms, the final Lagrangian is

$$\mathcal{L} = a^3 \left(f - T_\mathcal{G} f_{T_\mathcal{G}} - T f_T \right) - 8\dot{a}^3 \left(\dot{T}_\mathcal{G} f_{T_\mathcal{G} T_\mathcal{G}} + \dot{T} f_{TT_\mathcal{G}} \right) - 6 f_T a \dot{a}^2 \,, \tag{9.22}$$

This is a point-like, canonical Lagrangian whose configuration space is $\mathbb{Q} = \{a, T, T_\mathcal{G}\}$ and tangent space is $\mathbb{TQ} = \{a, \dot{a}, T, \dot{T}, T_\mathcal{G}, \dot{T}_\mathcal{G}\}$. The Euler-Lagrange equations for

a, T and $T_{\mathcal{G}}$ are respectively

$$a^2\left(f - T_{\mathcal{G}} f_{T_{\mathcal{G}}} - T f_T\right) + 2 f_T \dot{a}^2 + 16 \dot{a}\ddot{a} \dot{f}_{T_{\mathcal{G}}} + 8 \dot{a}^2 \ddot{f}_{T_{\mathcal{G}}} + 4 \dot{f}_T a\dot{a} + 4 f_T a\ddot{a} = 0\,, \quad (9.23)$$

$$\left(a^2 T + 6\dot{a}^2\right) a f_{TT} - \left(a^3 T_{\mathcal{G}} - 24\dot{a}^2\ddot{a}\right) f_{TT_{\mathcal{G}}} = 0 \quad (9.24)$$

$$\left(a^3 T_{\mathcal{G}} - 24\dot{a}^2\ddot{a}\right) f_{T_{\mathcal{G}}T_{\mathcal{G}}} + \left(a^3 T + 6a\dot{a}^2\right) f_{TT_{\mathcal{G}}} = 0\,,\,. \quad (9.25)$$

As expected, for $f_{T_{\mathcal{G}}T_{\mathcal{G}}} \neq 0$ and $f_{TT_{\mathcal{G}}} \neq 0$, we obtain, from (9.25) and (9.24), the expressions (9.18) and (9.19) for the Gauss-Bonnet term and the torsion scalar. The energy condition $E_{\mathcal{L}} = 0$, associated with Lagrangian (9.22), is

$$E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}}\dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{T}}\dot{T} + \frac{\partial \mathcal{L}}{\partial \dot{T}_{\mathcal{G}}}\dot{T}_{\mathcal{G}} - \mathcal{L} = 0$$

corresponding to the 00-Einstein equation

$$24\dot{a}^3 \dot{f}_{T_{\mathcal{G}}} + 6 f_T a\dot{a}^2 + a^3\left(f - T_{\mathcal{G}} f_{T_{\mathcal{G}}} - T f_T\right) = 0\,. \quad (9.26)$$

Alternatively, the system (9.23)-(9.26) can be derived from the field equations (9.14).

9.4 Classification with Noether Symmetries

The Noether Symmetry Approach is explicitly given in chapter 6. In this case, the generator of the infinitesimal transformations takes the form

$$\mathbf{X} = \xi(t,a,T,T_{\mathcal{G}}) + \eta^a(t,a,T,T_{\mathcal{G}}) + \eta^T(t,a,T,T_{\mathcal{G}}) + \eta^{T_{\mathcal{G}}}(t,a,T,T_{\mathcal{G}})\,. \quad (9.27)$$

The Noether condition

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}\frac{d\xi}{dt} = \frac{dg}{dt}\,, \quad (9.28)$$

will give a system of 21 equations. The system is of course overdetermined, since our unknowns are the coefficients of the generator $\mathbf{X}$, $g(t,a,T,T_{\mathcal{G}})$ and $f(T,T_{\mathcal{G}})$.

The following cases are the only cases that are invariant under point-transformations and thus have Noether Symmetries:

1. If $f(T,T_{\mathcal{G}}) = f(T) + c\mathcal{G}$,

   the linear Gauss-Bonnet term in four dimensions will not contribute to the dynamics of the theory and thus, this case inherits all the symmetries of $f(T)$ gravity [170].

2. If $f(T,T_{\mathcal{G}}) = \tilde{f}(\frac{T}{T_{\mathcal{G}}^{1/2}})T_{\mathcal{G}}^{\frac{1}{4}+\frac{3c_2}{4c_1}} + \frac{c_3}{3(c_2-c_1)}T_{\mathcal{G}}$,

   the Noether vector has the form

$$\mathbf{X} = c_1 t\partial_t + c_2 a\partial_a - 2c_1 T\partial_T - 4c_1 T_{\mathcal{G}}\partial_{T_{\mathcal{G}}}\,. \quad (9.29)$$

3. For $f(T,T_{\mathcal{G}})=\tilde{f}(\frac{T}{T_{\mathcal{G}}^{1/2}})T_{\mathcal{G}}-\frac{c_3}{4c_1}T_{\mathcal{G}}\ln\left(T_{\mathcal{G}}\right)$,

   the Noether vector reads

$$\mathbf{X}=c_1t\partial_t+c_1a\partial_a-2c_1T\partial_T-4c_1T_{\mathcal{G}}\partial_{T_{\mathcal{G}}}\,. \tag{9.30}$$

4. If $f(T,T_{\mathcal{G}})=c_4T_{\mathcal{G}}(\frac{T}{T_{\mathcal{G}}^{2/3}})^{\frac{9}{2}-\frac{9c_2}{2c_1}}+\frac{c_3}{3(c_2-c_1)}T_{\mathcal{G}}$,

   the Noether vector reads

$$\mathbf{X}=c_1t\partial_t+\left(c_2a+c_5\right)\partial_a-2\frac{c_6+c_1a}{a}T\partial_T-\left(\frac{3c_5}{a}+4c_1\right)T_{\mathcal{G}}\partial_{T_{\mathcal{G}}}\,. \tag{9.31}$$

5. If $f(T,T_{\mathcal{G}})=c_4T_{\mathcal{G}}+\frac{3c_3}{2c_1}T_{\mathcal{G}}\ln\left(\frac{T}{T_{\mathcal{G}}^{2/3}}\right)$

   the Noether vector has the form

$$\mathbf{X}=c_1t\partial_t+\left(c_1a+c_5\right)\partial_a-2\frac{c_6+c_1a}{a}T\partial_T-\left(\frac{3c_5}{a}+4c_1\right)T_{\mathcal{G}}\partial_{T_{\mathcal{G}}}\,. \tag{9.32}$$

6. If $f(T,T_{\mathcal{G}})=\frac{c_3}{3(c_2-c_1)}T_{\mathcal{G}}+c_4T^{\frac{1}{2}+\frac{3c_2}{2c_1}}+c_5T_{\mathcal{G}}T^{\frac{3c_2}{2c_1}-\frac{3}{2}}$,

   the Noether vector reads

$$\mathbf{X}=c_1t\partial_t+\left(c_2a+c_6\right)\partial_a-2c_1T\partial_T-\left(\left(4ac_1+c_6\right)T_{\mathcal{G}}+\frac{c_4c_6\left(c_1+3c_2\right)}{3c_5\left(c_2-c_1\right)}\frac{T^2}{a}\right)\partial_{T_{\mathcal{G}}}\,. \tag{9.33}$$

7. and if $f(T,T_{\mathcal{G}})=c_5T_{\mathcal{G}}+c_4T^2-\frac{c_3}{2c_1}T_{\mathcal{G}}\ln(T)$,

   the Noether vector becomes

$$\mathbf{X}=c_1t\partial_t+\left(c_1a+c_6\right)\partial_a-2c_1T\partial_T+\left(\frac{4c_4c_6c_1}{c_3}\frac{T^2}{a}-\frac{4ac_1+c_6}{a}T_{\mathcal{G}}\right)\partial_{T_{\mathcal{G}}}\,. \tag{9.34}$$

In each case, the conserved quantity is calculated in a straightforward way from

$$I=\xi\left(\dot{q}^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}-\mathcal{L}\right)-\eta^i\frac{\partial\mathcal{L}}{\partial q^i}+g\,. \tag{9.35}$$

However, for simplicitly we omit it.

9.5 Cosmological solutions

We saw in the previous section that the Noether condition (9.28) gave us all the models of $f(T, T_{\mathcal{G}})$ theory, that are invariant under point-transformations. Here, we will use these symmetries to find exact cosmological solutions.

Specifically, as explained in the chapter 6, one can construct a Lagrange system (6.31) from the coefficients of the Noether vector $\mathbf{X}$ and after finding the zero-order (and sometimes the first-order) invariants to reduce the dynamics of the system and to find exact solutions.

In the list that follows, we show the necessary conditions that need to be satisfied in order for each theory to give de-Sitter and power-law solutions of the form

$$a(t) = a_0 e^{H_0 t} \quad \text{and} \quad a(t) = a_0 \left(\frac{t}{t_0}\right)^p . \tag{9.36}$$

Obviously, in every case the functional form of the torsion and the teleparallel Gauss-Bonnet scalar is given respectively by (9.19) and (9.18). We omit the first case, where the theory is identical to $f(T)$ since it has been extensively studied in the literature. The number in the list corresponds to the model in the classification of the previous section:

2. $\tilde{f}(x) = f_0 x^{\frac{3(c_2-c_1)}{2c_1}} \left(-3c_2 - 2c_1 x^2 + 2c_2 x^2\right)$,
3. $\tilde{f}(x) = f_0 + \frac{c_3}{18c_1}\left(4x^2 - 9\ln x\right)$,
4. only power law for $c_2 = \frac{2c_1 p}{3}$ and $c_2 = \frac{c_1}{3}(3p-1)$,
5. only power law for $p = 3/2$ and $p = 4/3$,
6. $c_5 = \frac{3c_2 c_4}{2(c_1 - c_2)}$,
7. $c_4 = \frac{2c_3}{9c_1}$.

9.6 Conclusions

In this chapter, we discussed a theory of gravity where the interaction Lagrangian consists of a generic function $f(T, T_{\mathcal{G}})$ of the teleparallel Gauss-Bonnet topological invariant, $T_{\mathcal{G}}$, and the torsion scalar T. The physical reason for this, is that in curvature gravity (GR and extensions) the Gauss-Bonnet term plays a significant role, especially when one consider the high energy limit of the theory of gravity, or the low energy limit of string theories and supergravity. In addition, theories of the form $f(R, \mathcal{G})$ can, among others, mimic the late-time acceleration of the Universe and be responsible for a smooth transition between inflation and dark energy epochs.

What we did is to present the essentials of this theory in cosmology and we to look for Noether symmetries. In particular, we classified all the models that are invariant under point-transformations and we found their symmetry vectors. Using these symmetries we calculated the necessary constraints in order for these models to have de-Sitter and power-law solutions.

Chapter 10

Noether Symmetries in Non-local teleparallel cosmology

Arising naturally as quantum loop effects, non-local corrections to general relativity have been studied extensively. In addition, if there would be any posibility to distinguish between General relativity and teleparallel gravity, this would be at quantum level, where such effects would take place. That is why in this chapter, we consider a non-local teleparallel theory of gravity. We will motivate it in the curvature formulation and then we will elaborate on the teleparallel description. Furthermore, we will use Noether Symmetries to classify invariant models under point-transformations and we will use their symmetries to find cosmological solutions.

10.1 Introduction

Motivated mostly from quantum loop corrections and in order to explain the late-time acceleration of the Universe, almost a decade ago, a non-local modification of the Einstein-Hilbert (EH) action has been proposed [137], and the new action has the following form[1]

$$\mathcal{S}_{\text{standard}-\text{NL}} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,\overset{\circ}{R}\left[1+f\Big((\Box^{-1}\overset{\circ}{R})\Big)\right] + \int d^4x\,\sqrt{-g}\,L_m\,, \quad (10.1)$$

with $\overset{\circ}{R}$ being the Ricci scalar, f is an arbitrary function which depends on the retarded Green function evaluated at the Ricci scalar, L_m is any matter Lagrangian and $\Box \equiv \partial_\rho(\sqrt{-g}g^{\sigma\rho}\partial_\sigma)/\sqrt{-g}$ is the scalar-wave operator, which can be written in

[1]In this chapter, we will deal with quantities calculated both with the Levi-Civita connection and with the Weitzenböck connection. For this reason, we restore back our notation introduced in chapter 1 and also used throughout the first part.

terms of the Green function $G(x,x')$ as

$$(\Box^{-1}F)(x) = \int d^4x' \sqrt{-g(x')}F(x')G(x,x') \,. \tag{10.2}$$

It is clear that by setting $f(\Box^{-1}\overset{\circ}{R}) = 0$, the above action is equivalent to the Einstein-Hilbert one plus the matter content. The non-locality is introduced by the inverse of the d'Alembert operator (see [137] for details). Corrections of this kind arise naturally as soon as quantum loop effects are studied and they are also considered as possible solution to the black hole information paradox [138,139]. Since then, a lot of studies of non-localities have been done in various contexts [140–146]. In [147–151], non local quantum gravity is fully discussed putting in evidence results and open issues. From the string theory point of view, in [152] they present some bouncing solutions, in [153] solutions of an expanding Universe with phantom dark energy and in [154] they generate non-Gaussianities during inflation. Emanating from infrared (IR) scales, a lot of progress has also been done. Unification of inflation with late-time acceleration, as well as, the dynamics of a local form of the theory have been studied in [155, 168]. In [156], they show that non-local gravity models do not alter the GR predictions for gravitationally bound systems, and also they are ghost-free and stable. Finally, in [157–159], they derived a technique to fix the functional form of the function f in the action, which is called nonlocal distortion function. The interested reader should see the detailed review by Barvinsky [160], which summarizes the non-local aspects both from the quantum-field theory point of view and from the cosmological one.

Along another track, teleparallel [98] and modified teleparallel theories of gravity [99, 161] have, in the last decade, gained a lot of attention trying not only to formulate gravity in a gauge invariant way, but also to interpret the late-time acceleration of the Universe, without invoking any cosmological constant. The idea is that gravity, instead of curvature, is mediated only through torsion. This means that, the theory is no more a geometrical theory, i.e. the trajectories of the particles are not described by geodesic equations, but just by some force equations, since torsion is seen as a force, similar to the Lorentz equation in electrodynamics. The Teleparallel Equivalent of General Relativity (TEGR) is a gauge description of the gravitational interactions and torsion defined through the Weitzenböck connection (instead of the Levi-Civita connection, used by GR, where the Equivalence Principle is strictly requested in order to make geodesic and metric structure to coincide). Hence, in this theory, the manifold is flat but endorsed with torsion. The dynamical fields of the theory are the vierbeins and their relation with the metric and the inverse of the metric is given by

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \,, \quad g^{\mu\nu} = \eta^{ab}E^\mu_a E^\nu_b \,, \tag{10.3}$$

where η_{ab} is the flat Minkowski metric and $E_a{}^{\mu}$ is the inverse of the tetrads. The action of TEGR is given by

$$\mathcal{S}_{\text{TEGR}} = -\frac{1}{16\pi G_N}\int d^4x e \overset{\bullet}{T} + \int d^4x\, e\, L_m\,, \tag{10.4}$$

with e being $e = \det(e^a{}_\mu) = \sqrt{-g}$ and T is the torsion scalar, which is given by the contraction

$$\overset{\bullet}{T} = \overset{\bullet}{S}{}^{\mu\nu}{}_\rho \overset{\bullet}{T}{}^\rho{}_{\mu\nu}\,, \tag{10.5}$$

where

$$\overset{\bullet}{S}_\rho{}^{\mu\nu} = \frac{1}{2}\left(\overset{\bullet}{K}{}^{\mu\nu}{}_\rho + \delta^\mu{}_\rho \overset{\bullet}{T}{}^{\sigma\nu}{}_\sigma - \delta^\nu{}_\rho \overset{\bullet}{T}{}^{\sigma\mu}{}_\sigma\right)\,, \tag{10.6}$$

$$\overset{\bullet}{K}{}^{\mu\nu}{}_\rho = -\frac{1}{2}\left(\overset{\bullet}{T}{}^{\mu\nu}{}_\rho - \overset{\bullet}{T}{}^{\nu\mu}{}_\rho - \overset{\bullet}{T}_\rho{}^{\mu\nu}\right)\,, \tag{10.7}$$

$$\overset{\bullet}{T}{}^\alpha{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\alpha{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^\alpha{}_{\mu\nu}\,, \tag{10.8}$$

are respectively the superpotential, the contorsion tensor, the torsion tensor and $\overset{\bullet}{\Gamma}{}^\alpha{}_{\mu\nu} = E^\alpha_a \partial_\mu e^a_\nu$ is the Weitzeböck connection. The teleparallelism condition gives the relation of the Ricci scalar with the torsion scalar, that is

$$\overset{\circ}{R} = -\overset{\bullet}{T} + \frac{2}{e}\partial_\mu(e\overset{\bullet}{T}{}^\mu) = -\overset{\bullet}{T} + \overset{\bullet}{B}\,. \tag{10.9}$$

Hence, we directly see that at the action level, the EH action with the TEGR action differ only by a boundary term and thus the descriptions are equivalent. This is easily generalized to a more complex action as soon as we substitute $\overset{\bullet}{T}$ with an arbitrary function of this, $f(\overset{\bullet}{T})$. This theory can present problems not being Lorentz invariant and because a covariant formulation of $f(\overset{\bullet}{T})$ gravity is still not very well accepted since the spin connection is a field without dynamics. Nevertheless, it is always possible to give rise to the correct field equations choosing suitable tetrads. For more details see the chapter 4.

The extra degrees of freedom introduced by f, do not allow us to find an exact relation between $f(\overset{\bullet}{T})$ and $f(\overset{\circ}{R})$, since now the boundary terms in (10.9), contribute to the field equations. These kind of theories and their extensions are of great interest [106, 162–165], since they provide theoretical interpretation of the accelerating expansion of the Universe and also accomodate the radiation and matter dominated phases of it. In specific cases, one can also find inflationary solutions and avoid the Big Bang singularity with bouncing solutions.

In the teleparallel framework, recently it was proposed a similar kind of non-local gravity based on the torsion scalar $\overset{\bullet}{T}$. In this theory, the action reads as follows [109]

$$\mathcal{S}_{\text{teleparalell}-\text{NL}} = -\frac{1}{16\pi G_N}\int d^4x\, e\, \overset{\bullet}{T} + \frac{1}{16\pi G_N}\int d^4x\, e\, \overset{\bullet}{T} f\Big((\Box^{-1}\overset{\bullet}{T})\Big) + \int d^4x\, e\, L_m\,, \tag{10.10}$$

where the function f depends on $\Box^{-1}\overset{\bullet}{T}$. The teleparallel equivalent of GR is recovered if $f(\Box^{-1}\overset{\bullet}{T}) = 0$. It is possible to show [109] that this theory is consistent with the cosmological data by SNe Ia + BAO + CC $+H_0$ observations. From (10.9), it is straightforward noticing that (10.1) and (10.10) correspond to different theories, where B is the term connecting them.

We now present a generalization of (10.1) and (10.10), which we call Generalized Non-local Teleparallel Gravity (GNTG). Its action is given by

$$\mathcal{S} = -\frac{1}{16\pi G_N}\int d^4x\, e\overset{\bullet}{T} + \frac{1}{16\pi G_N}\int d^4x\, e\,(\xi\overset{\bullet}{T}+\chi\overset{\bullet}{B}) f\Big((\Box^{-1}\overset{\bullet}{T}),(\Box^{-1}\overset{\bullet}{B})\Big) + \int d^4x\, e\, L_m\,. \tag{10.11}$$

Here, $f(\Box^{-1}T,\Box^{-1}B)$ is an arbitrary function of the nonlocal torsion and the nonlocal boundary terms. The greek letters ξ and χ denote coupling constants. It is easily seen, that by choosing $\xi = -\chi = -1$ one obtains the standard Ricci scalar. From (10.2), we directly see that the following relation is also true

$$\Box^{-1}\overset{\circ}{R} = -\Box^{-1}\overset{\bullet}{T} + \Box^{-1}\overset{\bullet}{B}\,, \tag{10.12}$$

and thus, if $f(\Box^{-1}\overset{\bullet}{T},\Box^{-1}\overset{\bullet}{B}) = f(-\Box^{-1}\overset{\bullet}{T}+\Box^{-1}\overset{\bullet}{B})$, the action takes the well known form $\overset{\circ}{R} f(\Box^{-1}\overset{\circ}{R})$ given by the action (10.1). Moreover, nonlocal teleparalell gravity given by the action (10.10) is recovered if $\chi = 0$ and $f(\Box^{-1}\overset{\bullet}{T},\Box^{-1}\overset{\bullet}{B}) = f(\Box^{-1}\overset{\bullet}{T})$. Starting from this theory, we can construct a scalar tensor analog by using Lagrange multipliers and we can constrain the distortion function f by the so-called Noether Symmetries Approach [166]. There is a huge amount of articles in the literature, which adopt the Noether Symmetry Approach to constrain the form of some classes theories (see for example [79, 105, 162] and references therein). In this way, one obtains models that, thanks to the existence of Noether Symmetries, present integrals of motion that allows to reduce dynamics and then, in principle, to find out exact solutions. Besides these technical points, the presence of symmetries fixes couplings and potentials with physical meaning [166]. In such a way, the approach can be considered a sort of criterion to "select" physically motivated theories [167].

Throughout the chapter we adopt the signature $(+,-,-,-)$, because it is the one usually used in Teleparallel gravity.

10.2 Generalized Non-local Cosmology

Since the field equations for the GNTG theory are very cumbersome, we will rerewrite the action (10.11) in a more suitable way using scalar fields, according to [168]. Specifically, the action can be rewritten introducing four scalar fields $\phi, \psi, \theta, \zeta$ as follows

$$\begin{aligned}\mathcal{S} &= -\frac{1}{16\pi G_N}\int d^4x\, e\overset{\bullet}{T} + \frac{1}{16\pi G_N}\int d^4x\, e\left[(\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})f(\phi,\varphi) + \theta(\Box\phi - \overset{\bullet}{T}) + \zeta(\Box\varphi - \overset{\bullet}{B})\right],\\ &= -\frac{1}{16\pi G_N}\int d^4x\, e\overset{\bullet}{T} + \frac{1}{16\pi G_N}\int d^4x\, e\left[(\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})f(\phi,\varphi) - \partial_\mu\theta\partial^\mu\phi - \theta\overset{\bullet}{T} - \partial_\mu\zeta\partial^\mu\varphi - \zeta\overset{\bullet}{B}\right],\end{aligned} \tag{10.13}$$

where we omitted the matter Lagrangian densities for simplicity. By varying this action with respect to θ and ζ we get $\phi = \Box^{-1}\overset{\bullet}{T}$ and $\varphi = \Box^{-1}\overset{\bullet}{B}$ respectively. In addition, by varying this action with respect to ϕ and φ we get

$$\Box\theta = (\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})\frac{\partial f(\phi,\varphi)}{\partial\phi}, \tag{10.14}$$

$$\Box\zeta = (\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})\frac{\partial f(\phi,\varphi)}{\partial\varphi}. \tag{10.15}$$

In the scalar representation it is not straightforward how to recover curvature or teleparallel nonlocal gravity. Let us explicitly recover these theories under the scalar-tensor formalism: by setting $\xi = -1 = -\chi$, $f(\phi,\varphi) = f(-\phi+\varphi)$, and $\theta = -\zeta$ we obtain standard non-local curvature gravity, namely

$$\mathcal{S}_{\mathrm{standard-NL}} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\left[\overset{\circ}{R} + \overset{\circ}{R}f(\psi) - \partial_\mu\zeta\partial^\mu\psi - \zeta\overset{\circ}{R}\right] + \int d^4x\, e\, L_m\,, \tag{10.16}$$

$$= \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\left[\overset{\circ}{R} + \overset{\circ}{R}f(\Box^{-1}\overset{\circ}{R})\right] + \int d^4x\, e\, L_m\,, \tag{10.17}$$

where $\psi = -\phi + \varphi$. On the other hand, the non-local TEGR is recovered if in the action (10.13) we choose $\xi = 1$, $\chi = 0$, $f(\phi,\varphi) = f(\phi)$ and $\zeta = 0$. We obtain

$$\mathcal{S}_{\mathrm{teleparallel-NL}} = \frac{1}{16\pi G_N}\int d^4x\, e\left[\overset{\bullet}{T}\,(f(\phi) - 1) - \partial_\mu\theta\partial^\mu\phi - \theta\overset{\bullet}{T}\right] + \int d^4x\, e\, L_m \tag{10.18}$$

$$= \frac{1}{16\pi G_N}\int d^4x\, e\left[\overset{\bullet}{T}\left(f(\Box^{-1}\overset{\bullet}{T}) - 1\right)\right] + \int d^4x\, e\, L_m\,. \tag{10.19}$$

A more general class of theories, like $-\overset{\bullet}{T} + (\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})f(\Box^{-1}\overset{\bullet}{T})$ or $-\overset{\bullet}{T} + (\xi\overset{\bullet}{T} + \chi\overset{\bullet}{B})f(\Box^{-1}\overset{\bullet}{B})$ can be obtained by setting $f(\phi,\varphi) = f(\phi)$ and $f(\phi,\varphi) = f(\varphi)$ respectively. Obviously, in these cases, one can change the values of ξ and χ to obtain other couplings like

$$S = \frac{1}{16\pi G_N}\int d^4x\, e\left[-\overset{\bullet}{T} + \overset{\bullet}{B}f(\Box^{-1}\overset{\bullet}{T})\right] + \int d^4x\, e\, L_m\,, \tag{10.20}$$

$$S = \frac{1}{16\pi G_N}\int d^4x\, e\left[-\overset{\bullet}{T} + \overset{\bullet}{T}f(\Box^{-1}\overset{\bullet}{B})\right] + \int d^4x\, e\, L_m\,, \tag{10.21}$$

$$S = \frac{1}{16\pi G_N}\int d^4x\, e\left[-\overset{\bullet}{T} + \overset{\bullet}{B}f(\Box^{-1}\overset{\bullet}{B})\right] + \int d^4x\, e\, L_m\,. \tag{10.22}$$

Fig. 10.1 is a comprehensive diagram representing all the theories that can be recovered from the action (10.13). Here, we have not considered unnatural couplings like $\overset{\circ}{R}f(\Box^{-1}\overset{\bullet}{T})$ or $\overset{\bullet}{T}f(\Box^{-1}\overset{\circ}{R})$ because $\overset{\circ}{R}$ and $\overset{\bullet}{T}$, $\overset{\bullet}{B}$ are quantities defined in different connections, so mixed terms like these are badly defined. The above half part of the figure represents different non-local teleparallel theories and the below part of it, the standard curvature counterpart. As it is easy to see, only TEGR and GR dynamically coincide while this is not the case for other theories defined by $\overset{\bullet}{T}$, $\overset{\circ}{R}$ and $\overset{\bullet}{B}$. From a fundamental point of view, this fact is extremely relevant because the various representations of gravity can have different dynamical contents. For example, it is well known that $f(\overset{\bullet}{T})$ gravity gives second order field equations while $f(\overset{\circ}{R})$ gravity, in metric representation, is fourth order. These facts are strictly related to the dynamical roles of torsion and curvature and their discrimination at fundamental level could constitute an important insight to really understand the nature of gravitational field (see [99] for a detailed discussion).

Varying the generalized non-local action (10.13) with respect to the tetrads, we get the following field equations

$$\begin{aligned} &2\left(1-\xi(f(\phi,\varphi)-\theta)\right)\left[\frac{1}{e}\partial_\mu(e\overset{\bullet}{S}_a{}^{\mu\beta}) - E_a^\lambda\overset{\bullet}{T}^\rho{}_{\mu\lambda}\overset{\bullet}{S}_\rho{}^{\beta\mu} - \frac{1}{4}E_a^\beta\overset{\bullet}{T}\right] - \frac{1}{2}\Big[(\partial^\lambda\theta)(\partial_\lambda\phi)E_a^\beta - \\ &-(\partial^\beta\theta)(\partial_a\phi) - (\partial_a\theta)(\partial^\beta\phi)\Big] - \frac{1}{2}\left[(\partial^\lambda\zeta)(\partial_\lambda\varphi)E_a^\beta - (\partial^\beta\zeta)(\partial_a\varphi) - (\partial_a\zeta)(\partial^\beta\varphi)\right] + \\ &+2\,\partial_\mu\left[f(\phi,\varphi)(\xi+\chi)-\theta-\zeta\right]E_a^\rho\overset{\bullet}{S}_\rho{}^{\mu\nu} + \left(E_a^\nu\Box - E_a^\mu\nabla^\nu\nabla_\mu\right)(\zeta-\chi f(\phi,\varphi)) = \kappa\Theta_a^\beta\,, \end{aligned} \tag{10.23}$$

where Θ_a^β is the general energy-momentum tensor.
Let us now take into account the tetrad $e_\beta^a = (1, a(t), a(t), a(t))$, which reproduces

the flat Friedmann-Robertson-Walker (FRW) metric $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$. For this geometry, the modified FRW equations are

$$3H^2(\theta - \xi f + 1) = \frac{1}{2}\dot{\zeta}\dot{\varphi} + \frac{1}{2}\dot{\theta}\dot{\phi} + 3H\big(\dot{\zeta} - \chi\dot{f}\big) + \kappa\rho_m\,, \tag{10.24}$$

$$\big(2\dot{H} + 3H^2\big)(\theta - \xi f + 1) = -\frac{1}{2}\dot{\zeta}\dot{\varphi} - \frac{1}{2}\dot{\theta}\dot{\phi} - \dot{f}(2H(\xi + 2\chi) + \chi) + 2H(2\dot{\zeta} + \dot{\theta}) + \ddot{\zeta} - \kappa p_m\,, \tag{10.25}$$

where ρ_m and p_m are the energy density and the pressure of the cosmic fluid respectively and dots denote differentiation with respect to the cosmic time. The equations for the scalar fields can be written as

$$6H^2 + 3H\dot{\phi} + \ddot{\phi} = 0\,, \tag{10.26}$$

$$6(\dot{H} + 3H^2) + 3H\dot{\varphi} + \ddot{\varphi} = 0\,, \tag{10.27}$$

$$-6H^2\left(\xi f_\varphi + 3\chi f_\varphi\right) - 6\dot{H}\chi f_\varphi + 3H\dot{\zeta} + \ddot{\zeta} = 0\,, \tag{10.28}$$

$$-6H^2\left(\xi f_\phi + 3\chi f_\phi\right) - 6\dot{H}\chi f_\phi + 3H\dot{\theta} + \ddot{\theta} = 0\,, \tag{10.29}$$

where the sub-indices represent the partial derivative $f_\phi = \partial f/\partial\phi$ and $f_\varphi = \partial f/\partial\varphi$. In the following section, we will use the Noether Symmetry Approach to seek for conserved quantities.

10.3 The Noether Symmetry Approach

Let us use the Noether Symmetry Approach [166, 169] in order to find symmetries and cosmological solutions for the generalized action (10.13). For simplicity, hereafter we will study the vacuum case, i.e., $\rho_m = p_m = 0$. It can be shown that the torsion scalar and the boundary term in a flat FRW are given by

$$\overset{\bullet}{T} = -6H^2\,, \quad \overset{\bullet}{B} = -18H^2 - 6\dot{H}\,, \tag{10.30}$$

so that the action (10.13) takes the following form

$$\mathcal{S} = \frac{1}{16\pi G_N}\int a^3 dt\left\{-6\frac{\dot{a}^2}{a^2}(\xi f(\phi,\varphi) - \theta - 1) - 6\Big(2\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\Big)(\chi f(\phi,\varphi) - \zeta) - \dot{\theta}\dot{\phi} - \dot{\zeta}\dot{\varphi}\right\}. \tag{10.31}$$

Considering the procedure in [166], we find that the point-like Lagrangian is given by

$$\mathcal{L} = 6a\dot{a}^2\big(\theta + 1 - \xi f(\phi,\varphi)\big) + 6a^2\dot{a}(\chi\dot{f}(\phi,\varphi) - \dot{\zeta}) - a^3\dot{\theta}\dot{\phi} - a^3\dot{\zeta}\dot{\varphi}\,. \tag{10.32}$$

The generator of infinitesimal transformations [169] is given by

$$\mathbf{X} = \lambda(t, x^\mu)\partial_t + \eta^i(t, x^\mu)\partial_i\,, \tag{10.33}$$

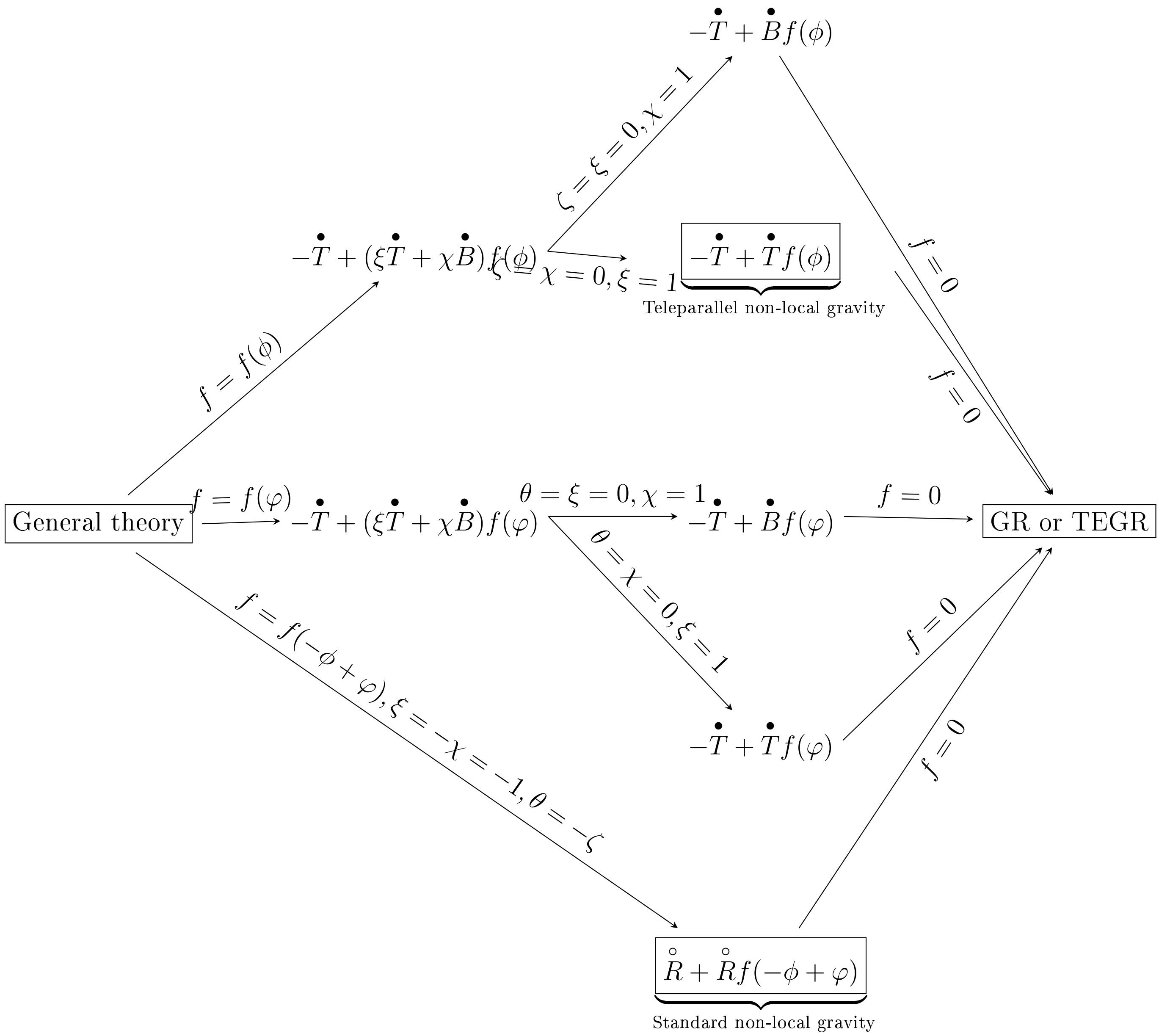


Figure 10.1: The diagram shows how to recover the different theories of gravity starting from the scalar-field representation of the general theory.

where $x^\mu=(a,\theta,\phi,\varphi,\zeta)$ and the vector η^i is

$$\eta^i(t,x^\mu)=\left(\eta^a,\eta^\theta,\eta^\phi,\eta^\varphi,\eta^\zeta\right). \tag{10.34}$$

In general, each function depends on t and x^μ. If there exists a function $h = h(t, x^\mu)$ such that

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}\frac{d\lambda}{dt} = \frac{dh}{dt}\,, \tag{10.35}$$

where $\mathcal{L} = \mathcal{L}(t, x^\mu, \dot{x}^\mu)$ is the Lagrangian of a system and $\mathbf{X}^{[1]}$ is the first prolongation of the vector $\mathbf{X}$ [169], then the Euler-Lagrange equations remain invariant under these transformations. The generator is a Noether symmetry of the system described by $\mathcal{L}$ and the relative integral of motion is given by

$$I = \lambda\left(\dot{x}^\mu\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \mathcal{L}\right) - \eta^i\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} + h\,. \tag{10.36}$$

In the next subsections, we will search for Noether symmetries in specific non-local Lagrangians, starting from the two cases ($\overset{\bullet}{T}f(\Box^{-1}\overset{\bullet}{T})$ and$\overset{\circ}{R}f(\Box^{-1}\overset{\circ}{R})$) and ending up to the general action (10.13). The set of generalized coordinates $x^\mu = \{t, a, \theta, \phi, \varphi, \zeta\}$ gives rise to the configuration space $\mathcal{Q} \equiv \{x^\mu\,, \mu = 1, ..., 6\}$ and the tangent space $\mathcal{TQ} \equiv \{x^\mu, \dot{x}^\mu\}$ of the Lagrangian $\mathcal{L} = \mathcal{L}(t, x^\mu, \dot{x}^\mu)$. Clearly, the procedure can be applied to many different models starting from Fig. 10.1.

10.4 Noether's symmetries in teleparallel non-local gravity with coupling $Tf(\Box^{-1}T)$

10.4.1 Finding Noether's symmetries

Let us first study the case where we recover the teleparalel non-local case studied in [109]. In this case, the torsion scalar $\overset{\bullet}{T}$ is coupled with a non-local function evaluated at the torsion scalar, that is $f(\Box^{-1}\overset{\bullet}{T}) = f(\phi)$. For Noether's symmetries, we need to consider,

$$f(\phi, \varphi) = f(\phi)\,, \quad \chi = 0\,, \quad \xi = 1 \quad \text{and} \quad \zeta = 0\,. \tag{10.37}$$

in the general action (10.13) and thus the Lagrangian becomes

$$\mathcal{L} = 6a\,(-f(\phi) + \theta + 1)\,\dot{a}^2 - a^3\dot{\theta}\dot{\phi}\,. \tag{10.38}$$

From Eq. (10.35), one derives a system of 16 equations for the coefficients of the Noether vector and the functions h, f. It can be immediately seen that the dependence on the coordinates of the Noether vector components is

$$\lambda(a,\theta,\phi,t) = \lambda(t)\,, \tag{10.39}$$
$$\eta_a(a,\theta,\phi,t) = \eta_a(a,\theta,\phi,t)\,, \tag{10.40}$$
$$\eta_\phi(a,\theta,\phi,t) = \eta_\phi(a,\phi,t)\,, \tag{10.41}$$
$$\eta_\theta(a,\theta,\phi,t) = \eta_\theta(a,\theta,t)\,, \tag{10.42}$$
$$h(a,\theta,\phi,t) = h(a,\theta,\phi)\,. \tag{10.43}$$

Note that we do not need to impose any ansantz to find out the symmetries. Hence, the equation for f reads

$$c_1 f'(\phi) - c_2 f(\phi) + c_2 - c_3 = 0\,, \tag{10.44}$$

where c_1, c_2 and c_3 are constants. There are two non trivial solutions ($f \neq$ constant) to (10.44) depending on the value of c_2, i.e.

$$f(\phi) = \begin{cases} c_7 e^{\frac{c_2\phi}{c_1}} - \frac{c_3}{c_2} + 1\,, & c_2 \neq 0\,, \\ c_7 + \frac{c_3}{c_1}\phi\,, & c_2 = 0\,, \end{cases} \tag{10.45}$$

where c_7 is another integration constant. From (10.19), we can notice that for having a TEGR (or GR) background we must have that $c_3 = c_2$ in the exponential form and $c_7 = 0$ in the linear form. The Noether vector has the following form

$$\mathbf{X} = (c_4 + c_5 t)\partial_t - \frac{1}{3}(c_2 - c_4)a\partial_a + (c_3 + c_2\theta)\partial_\theta + c_1\partial_\phi\,, \tag{10.46}$$

and the integral of motion is

$$\begin{aligned} I =& a^3 c_1\dot{\theta} + a^3 c_2(\theta+1)\dot{\phi} - a^3\left(c_4 t + c_5\right)\dot{\theta}\dot{\phi} + \\ &+ \left[4a^2\left(c_2 - c_4\right)\dot{a} + 6a\dot{a}^2\left(c_4 t + c_5\right)\right]\left(1 - f(\phi) + \theta\right) + c_6\,. \end{aligned} \tag{10.47}$$

10.4.2 Cosmological solutions

In the previous subsection we found that the form of the function f is constrained to be an exponential or a linear form of the non-local term (10.45). It can be shown that for the linear form, there are no power-law or de-Sitter solution. Here we will find solutions for the exponential form of the coupling function.
As we pointed out before, it is physically convenient to choose $c_2 = c_3$ in order to have a GR (or TEGR) background. Hence, in this section, we will assume this condition for the constants. For the exponential form of the function $f(\phi)$ given by (10.45), the Lagrangian (10.38) takes now the form

$$\mathcal{L} = -6a\dot{a}^2\left(c_7 e^{\frac{c_3\phi}{c_1}} - \theta - 1\right) - a^3\dot{\theta}\dot{\phi}\,, \tag{10.48}$$

so that the Euler-Lagrange equations are given by

$$c_1\left(4\dot{H}(c_7e^{\frac{c_2\phi}{c_1}}-\theta-1)-\dot{\theta}\dot{\phi}\right)+H\left(4c_2c_7\dot{\phi}e^{\frac{c_2\phi}{c_1}}-4c_1\dot{\theta}\right)+6c_1H^2\left(c_7e^{\frac{c_2\phi}{c_1}}-\theta-1\right)=0\,, \tag{10.49a}$$

$$6H^2+3H\dot{\phi}+\ddot{\phi}=0\,, \tag{10.49b}$$

$$-\frac{6c_2c_7}{c_1}H^2e^{\frac{c_2\phi}{c_1}}+\ddot{\theta}+3H\dot{\theta}=0\,, \tag{10.49c}$$

$$6H^2\left(-c_7e^{\frac{c_2\phi}{c_1}}+\theta+1\right)-\dot{\theta}\dot{\phi}+6\theta H^2=0\,, \tag{10.49d}$$

for a,θ,ϕ and the energy equation, respectively. If we consider de-Sitter solution for the scale factor,

$$a(t)=e^{H_0t}\Rightarrow H(t)=H_0\,,$$

we immediately find from (10.49b) that

$$\phi(t)=-2H_0t-\frac{\phi_1e^{-3H_0t}}{3H_0}+\phi_2\,. \tag{10.50}$$

For the sake of simplicity, we will choose $\phi_1=\phi_2=0$ otherwise Eq. (10.49c) cannot be integrated easily. By this assumption, we directly find that

$$\theta(t)=e^{-3H_0t}\left(-c_7(3H_0t+1)-\frac{\theta_1}{3H_0}\right)+\theta_2\,, \tag{10.51}$$

where θ_1 and θ_2 are integration constants and we needed to choose the branch $c_1=2c_2/3$, otherwise Eq. (10.49a) cannot be satisfied. Hence, from (10.49a) we directly see that $\theta_2=-1$, giving us the following cosmological solution,

$$a(t)=e^{H_0t}\,,\quad \phi(t)=-2H_0t\,,\quad \theta(t)=e^{-3H_0t}\left(-c_7(3H_0t+1)-\frac{\theta_1}{3H_0}\right)-1\,, \tag{10.52}$$

and

$$f(\phi)=c_7e^{-3H_0t}\,. \tag{10.53}$$

If we consider that the scale factor behaves as a power-law $a(t)=a_0t^p$, where p is a constant, from (10.49b) we directly find that

$$\phi(t)=\frac{6p^2\log(t-3pt)}{1-3p}+\frac{\phi_1}{1-3p}t^{1-3p}+\phi_0\,, \tag{10.54}$$

where ϕ_1 and ϕ_0 are integration constants that for simplicity (as we did before) we will assume that are zero, otherwise (10.49c) cannot be integrated directly. By doing this, we find

$$\theta(t)=\frac{c_1t^{1-3p}}{1-3p}+c_2+\frac{c_7(3p-1)(c_1-3c_1p)}{c_1(1-3p)^2-6c_2p^2}(t-3pt)^{\frac{6c_2p^2}{c_1-3c_1p}}\,, \tag{10.55}$$

where θ_0 and θ_1 are integration constants and we have assumed that $c_1 \neq \frac{6c_2p^2}{(3p-1)^2}$ and $p \neq 1/3$ since there are not solutions for these other two branches. By replacing this solution into (10.49a) we get that $c_2 = \frac{c_1(2-9p+9p^2)}{6p^2}$ and $\theta_1 = -1$ yielding the following solution

$$\phi(t) = \frac{6p^2 \log(t-3pt)}{1-3p}\,, \quad \theta(t) = c_7(1-3p)^{3-3p}t^{2-3p} + \frac{\theta_0 t^{1-3p}}{1-3p} - 1\,,$$
$$a(t) = a_0 t^p\,, \quad f(\phi) = c_7 e^{\frac{\left(9p^2-9p+2\right)\phi}{6p^2}}\,.$$

Note that the energy condition (10.49d) is satisfied and $p = 1/3$ is not a solution.

10.5 Noether's symmetries in curvature non-local gravity with coupling $Rf(\Box^{-1}R)$

10.5.1 Finding Noether's symmetries

Let us find now Noether's symmetries for the case where curvature non-local gravity is considered. We assume that the coupling $\mathring{R}f(\Box^{-1}\mathring{R})$ is present in the action. To recover this case, we must set

$$f(\phi,\varphi) = f(-\phi+\varphi) = f(\psi)\,, \quad \chi = 1\,, \quad \xi = -1\,, \quad \theta = -\zeta\,. \tag{10.56}$$

In this way, the Lagrangian (10.13) reads as follows

$$\mathcal{L} = 6a\dot{a}^2(f(\psi)+\theta+1) + 6a^2\dot{a}(f'(\psi)\dot{\psi}+\dot{\theta}) + a^3\dot{\theta}\dot{\psi}\,. \tag{10.57}$$

and Noether's condition equation (10.35), gives a system of 18 differential equations. The result is

$$\lambda(a,\theta,\psi,t) = \lambda(t) \quad \text{and} \quad h(a,\theta,\psi,t) = h(a,\theta,\psi)\,, \tag{10.58}$$

and the system reduces to 9 equations. However, the full system is still difficult to be solved without any assumption. A simple assumption is choosing $h(a,\theta,\psi) =$ constant. The last two equations of Noether condition for $f(\psi)$ are

$$2c_2f'(\psi) + c_1f(\psi) + c_1 - c_3 = 0\,, \tag{10.59}$$
$$2c_2f''(\psi) + c_1f'(\psi) = 0\,. \tag{10.60}$$

and the Noether vector results to be

$$\mathbf{X} = (c_5+c_4t)\partial_t + \frac{1}{3}a(c_4-c_1)\partial_a + (c_3+c_1\theta)\partial_\theta - 2c_2\partial_\psi\,. \tag{10.61}$$

Eqs. (10.59) and (10.60) are easily solved and the form of f is

$$f(\psi) = \begin{cases} -1 + \frac{c_3}{c_1} + c_6 e^{-\frac{c_1}{c_2}\psi} & c_1 \neq 0\,, \\ c_6 + \frac{c_3}{2c_2}\psi\,. & c_1 = 0 \end{cases}\,. \tag{10.62}$$

Again, the form of the function is either exponential or linear in $\psi = \Box^{-1}\overset{\circ}{R}$. This result is very interesting since, without further assumptions than $h = \text{const.}$, the symmetries give the same kind of couplings for both teleparallel and curvature non-local theories. These two couplings can be particularly relevant to get a renormalizable theory of gravity. As discussed in [147–149], the form of the coupling is extremely important to achieve a regular theory. In particular, the exponential coupling plays an important role in calculations. Here, the symmetry itself is imposing this kind of coupling. In other words, it is not put by hand but is related to a fundamental principle, i.e. the existence of the Noether symmetry.

10.5.2 Cosmological solutions

It is well known [168] that, non-local theories with exponential coupling, i.e. $\overset{\circ}{R}(1 + e^{\alpha\Box^{-1}\overset{\circ}{R}})$, have both de-Sitter and power-law solutions. In this section, we will verify that the Lagrangian (10.57) with the coupling (10.62), given by the symmetry, i.e.

$$\mathcal{L} = 6a\left(\frac{c_3}{c_1} + \theta\right)\dot{a}^2 + 3c_6 a e^{-\frac{c_1}{2c_2}\psi}\left(2\dot{a}^2 - \frac{c_1}{c_2}a\dot{a}\dot{\psi}\right) + 6a^2\dot{a}\dot{\theta} + a^3\dot{\theta}\dot{\psi}\,, \tag{10.63}$$

gives rise to these solutions. In order to recover the GR background, we will assume that $c_3 = c_1$.

Let us start from the de-Sitter case, where $a(t) = e^{H_0 t}$. The Euler-Lagrange equations for a, ψ, θ and the energy equation, read respectively

$$c_1^2 c_6\dot{\psi}^2 + 8c_2^2\dot{H}\left(\theta e^{\frac{c_1\psi}{2c_2}} + e^{\frac{c_1\psi}{2c_2}} + c_6\right) + 12c_2^2H^2\left(\theta e^{\frac{c_1\psi}{2c_2}} + e^{\frac{c_1\psi}{2c_2}} + c_6\right)$$
$$+4c_2^2\ddot{\theta}e^{\frac{c_1\psi}{2c_2}} - 2c_2^2\dot{\theta}\dot{\psi}e^{\frac{c_1\psi}{2c_2}} + 4c_2H\left(2c_2\dot{\theta}e^{\frac{c_1\psi}{2c_2}} - c_1c_6\dot{\psi}\right) - 2c_1c_2c_6\ddot{\psi} = 0\,, \tag{10.64a}$$

$$3c_1c_6\dot{H}e^{-\frac{c_1\psi}{2c_2}} + 6c_1c_6H^2e^{-\frac{c_1\psi}{2c_2}} - 3c_2H\dot{\theta} - c_2\ddot{\theta} = 0\,, \tag{10.64b}$$

$$6\dot{H} + 3H\dot{\psi} + 12H^2 + \ddot{\psi} = 0\,, \tag{10.64c}$$

$$H\left(6\dot{\theta} - \frac{3c_1c_6\dot{\psi}e^{-\frac{c_1\psi}{2c_2}}}{c_2}\right) + 6H^2\left(c_6e^{-\frac{c_1\psi}{2c_2}} + \theta + 1\right) + \dot{\theta}\dot{\psi} = 0\,. \tag{10.64d}$$

Eq. (10.64c) gives

$$\psi(t) = -4H_0t - \frac{\psi_1e^{-3H_0t}}{3H_0} + \psi_2\,, \tag{10.65}$$

where ψ_1 and ψ_0 are integration constants. For simplicity, to find analytical solutions, we set $\psi_1 = \psi_0 = 0$. Then, from Eq. (10.64b) we find

$$\theta(t) = \frac{3c_2c_6}{2c_1+3c_2}e^{\frac{4c_1H_0t-c_1\psi_2}{2c_2}} - \frac{\theta_1}{3H_0}e^{-3H_0t} + \theta_2\,,$$

and, in order to satisfy the other two Eqs. (10.64a) and (10.64d), we set $\theta_2 = -1$ and $c_2 = -c_1$. Finally, the following de-Sitter solution,

$$a(t) = e^{H_0t}\,,\quad \psi(t) = -4H_0t+\psi_2\,,\quad \theta(t) = 3c_6e^{\frac{\psi_2}{2}-2H_0t} - \frac{\theta_1}{3H_0}e^{-3H_0t} - 1\,,$$

is recovered and

$$f(\psi) = c_6e^{\psi/2}\,.$$

In the same spirit, if we assume that the scale factor with a power-law behavior as $a(t) = a_0t^p$, the system (10.64a)-(10.64d) yields the following solution,

$$a(t) = a_0t^p\,,\quad \psi(t) = \frac{6p(1-2p)}{3p-1}\ln(t)\,,\quad \theta(t) = \frac{c_6(3p-1)}{(p-1)}t^{-2p}-1\,,\quad f(\psi(t)) = c_6e^{\frac{\psi(1-3p)}{3(1-2p)}}\,.$$

This solution is valid for $p \neq 1/3$. Now, if one considers the linear form of $f(\psi) = c_6 + \frac{c_3}{2c_2}\psi$, it is also possible to find power-law solutions but only for $p = 1/2$, which corresponds to radiation. The non-trivial solution for this particular case is given by

$$\theta(t) = \theta_0\,,\quad a(t) = a_0t^{1/2}\,,\quad \psi(t) = -\frac{2c_2(\theta_0+2)}{c_3} - 2\psi_1t^{-1/2}\,,\quad f(\psi) = \frac{c_3\psi}{2c_2}+c_6\,,$$

where θ_0 and ψ_1 are constants.

10.6 Noether's symmetries in the general case

10.6.1 Finding Noether's symmetries

Let us consider now the generalized non-local action involving both teleparallel and curvature non-local contributions. The Lagrangian is

$$\mathcal{L} = 6\chi a^2\dot{a}\dot{\phi}f_\phi(\phi,\varphi)+6\chi a^2\dot{a}\dot{\varphi}f_\varphi(\phi,\varphi)-6\xi a\dot{a}^2f(\phi,\varphi)-6a^2\dot{a}\dot{\zeta}+6a\theta\dot{a}^2+6a\dot{a}^2-a^3\dot{\zeta}\dot{\varphi}-a^3\dot{\theta}\dot{\phi}\,, \tag{10.66}$$

from which we can derive several interesting theories as shown in the diagram, Fig. 10.1. The Noether condition (10.35) gives a system of 43 (non-independent) equations for the Noether vector components

$$\lambda(a,\theta,\phi,\varphi,\zeta,t)\,,\ \eta_a(a,\theta,\phi,\varphi,\zeta,t)\,,\ \eta_\phi(a,\theta,\phi,\varphi,\zeta,t)\,,$$
$$\eta_\varphi(a,\theta,\phi,\varphi,\zeta,t)\,,\ \eta_\theta(a,\theta,\phi,\varphi,\zeta,t)\,,\ \eta_\zeta(a,\theta,\phi,\varphi,\zeta,t)\,,$$

and the functions

$$h(a,\theta,\phi,\varphi,\zeta,t)\,,\ \ f(\phi,\varphi)\,.$$

We can see immediately, from the system, that

$$\lambda(a,\theta,\phi,\varphi,\zeta,t)=\lambda(t)\,,\ \ \eta_\phi(a,\theta,\phi,\varphi,\zeta,t)=\eta_\phi(a,\phi,\varphi,\zeta,t)\,,\ \ h(a,\theta,\phi,\varphi,\zeta,t)=h(a,\theta,\phi,\varphi,\zeta)\,.$$

The system now reduces to 19 equations that cannot be easily solved. Hence, as we did in the previous sections, we assume that $h(a,\theta,\phi,\varphi,\zeta)=\text{constant}=h$ and after some calculations we end up with the following three equations for $f(\phi,\varphi)$

$$\begin{aligned}-f_\varphi(\phi,\varphi)\Big(c_7\xi\varphi+c_6\xi+c_8\xi-6c_7\chi\Big)+f_\phi(\phi,\varphi)\left(-c_5\xi\varphi-c_4\xi+6c_5\chi\right)-6c_7\chi\phi f_{\varphi\phi}(\phi,\varphi)-\\ -6c_5\chi\phi f_{\phi\phi}(\phi,\varphi)+c_3\xi f(\phi,\varphi)-c_3+c_{10}-c_{12}=0\,,\end{aligned}\tag{10.67a}$$

$$\begin{aligned}6\left(c_7-c_3\right)\chi f_\varphi(\phi,\varphi)+6\chi\left(c_7\varphi+c_6+c_8\right)f_{\varphi\varphi}(\phi,\varphi)+6c_5\chi f_\phi(\phi,\varphi)+\\ +6\chi\left(c_5\varphi-c_7\phi+c_4\right)f_{\varphi\phi}(\phi,\varphi)-6c_5\chi\phi f_{\phi\phi}(\phi,\varphi)-c_{12}=0\,,\end{aligned}\tag{10.67b}$$

$$\begin{aligned}-\left(c_5\xi+c_3\chi\right)f_\phi(\phi,\varphi)-c_7\xi f_\varphi(\phi,\varphi)-6c_7\chi f_{\varphi\varphi}(\phi,\varphi)+\chi\left(c_5\varphi+c_4\right)f_{\phi\phi}(\phi,\varphi)+\\ +\chi\left(c_7\varphi-6c_5+c_6+c_8\right)f_{\varphi\phi}(\phi,\varphi)=0\,,\end{aligned}\tag{10.67c}$$

where all the c's are constants coming from the coefficients of the Noether vector. The system (10.67a)-(10.67c) can be easily integrated but, depending on the vanishing or not of some constants, different solutions can be derived. Specifically, we obtain seven different symmetries described below. The Noether vectors and the function f take the forms:

1. (a) For $c_7\neq 0$ and $c_3\neq 0\,, c_4\neq\frac{c_5}{c_7}(c_6+c_9)$, we have

$$\begin{aligned}\mathbf{X}=&(c_1t+c_2)\partial_t+\frac{1}{3}(c_1-c_3)a\partial_a+(c_4+c_5(6\ln a+\psi))\partial_\phi+\\ &+(c_6+c_7(6\ln a+\varphi)+c_9)\partial_\varphi+c_3\theta\partial_\theta+((c_3-c_7)\zeta-c_5\theta+c_8)\partial_\zeta\,.\end{aligned}$$

and

$$f(\phi,\varphi)=\frac{1}{\xi}+\frac{c_{11}\left(c_5c_6-c_4c_7+c_5c_9\right)}{c_3}\exp\left(\frac{c_3}{c_5c_6-c_4c_7+c_5c_9}\left(c_5\varphi-c_7\phi\right)\right)\,.\tag{10.68}$$

(b) For $c_7 \neq 0$ and $c_3 = 0\,, c_4 = \frac{c_5}{c_7}(c_6 + c_9)$, it is

$$\begin{aligned}\mathbf{X} =&(c_1 t + c_2)\partial_t + \frac{c_1}{3}a\partial_a + (c_4 + c_5(6\ln a + \varphi))\partial_\phi + \\ &+ (c_6 + c_7(6\ln a + \varphi) + c_9)\partial_\varphi + (c_8 - c_7\zeta - c_5\theta)\partial_\zeta\,.\end{aligned}$$

and

$$f(\phi,\varphi) = c_{11} + F(-c_7\phi + c_5\varphi)\,. \tag{10.69}$$

2. (a) i. For $c_7 = 0$ and $c_5 \neq 0$ and $c_3 \neq 0\,, c_5 \neq -c_6$, it is

$$\begin{aligned}\mathbf{X} =&(c_1 t + c_2)\partial_t + \frac{1}{3}(c_1 - c_3)a\partial_a + (c_4 + c_5(6\ln a + \varphi))\partial_\phi + \\ &+ (c_6 + c_9)\partial_\varphi + (c_{10} + c_3\theta)\partial_\theta + (c_3\zeta - c_5\theta + c_8)\partial_\zeta\,,\end{aligned}$$

and

$$f(\phi,\varphi) = \frac{c_3 - c_{10}}{\xi c_3} + c_{11}e^{\frac{c_3}{c_6 c_9}\varphi}\,. \tag{10.70}$$

ii. For $c_7 = 0$ and $c_5 \neq 0$ and $c_3 = 0\,, c_5 = -c_6$, it is

$$\mathbf{X} = (c_1 t + c_2)\partial_t + \frac{c_1}{3}a\partial_a + (c_4 + c_5(6\ln a + \varphi))\partial_\phi + (c_8 - c_5\theta)\partial_\zeta\,.$$

and

$$f(\phi,\varphi) = c_{11} + F(\varphi)\,. \tag{10.71}$$

(b) i. For $c_7 = 0$ and $c_5 = 0$ and $c_3 \neq 0\,, c_4 \neq 0$, it is

$$\mathbf{X} = (c_1 t + c_2)\partial_t + \frac{1}{3}(c_1 - c_3)a\partial_a + c_4\partial_\phi + (c_6 + c_9)\partial_\varphi + (c_{10} + c_3\theta)\partial_\theta + (c_8 + c_3\zeta)\partial_\zeta\,,$$

and

$$f(\phi,\varphi) = \frac{c_3 - c_{10}}{\xi c_3} + F(-\frac{c_6 + c_9}{c_4}\phi + \varphi)e^{\frac{c_3}{c_4}\phi}\,. \tag{10.72}$$

ii. A. For $c_7 = 0$ and $c_5 = 0$ and $c_3 = 0\,, c_4 = 0$ and $c_6 \neq -c_7$, it is

$$\mathbf{X} = (c_1 t + c_2)\partial_t + \frac{c_1}{3}a\partial_a + (c_6 + c_9)\partial_\varphi + c_{10}\partial_\theta + c_8\partial_\zeta\,,$$

and

$$f(\phi,\varphi) = \frac{c_{10}}{(c_6 + c_9)\xi}\varphi + F(\phi)\,. \tag{10.73}$$

B. For $c_7 = 0$ and $c_5 = 0$ and $c_3 = 0\,, c_4 = 0$ and $c_6 = -c_7$, it is

$$\mathbf{X} = (c_1 t + c_2)\partial_t + \frac{c_1}{3}a\partial_a + c_8\partial_\zeta\,,$$

and the equations are satisfied for any f.

Clearly, each of these symmetries specify a different Lagrangian and then a different dynamics. The fact that several symmetries exist for the same symmetry condition (10.35) is due to the fact that such a condition consists in a system of non-linear partial differential equations which have no unique general solution.

10.6.2 Cosmological Solutions

Let us now find cosmological solutions for the generalized Lagrangian (10.66). In principle, it is possible to find out cosmological solutions for each of the above cases depending on the coupling functions. Due to the physical importance of the exponential couplings, we will present cosmological solutions for the coupling function given by (10.68). However, the procedure for the other cases is the same.

In the case (10.68), we have the constraint given by the integration constants, that is $c_7 \neq 0\,, c_3 \neq 0\,, c_4 \neq \frac{c_5}{c_7}(c_6+c_9)$. Hence, the Euler-Lagrange equations obtained by (10.66), together with the energy condition, give a system of six differential equations for $a(t)\,, \phi(t)\,, \varphi(t)\,, \theta(t)$ and $\zeta(t)$.
Assuming that the scale factor of the universe behaves as de-Sitter $a(t) = e^{H_0 t}$, it is possible to find different kind of solutions depending on different cases for the constants. In all of these cases, the final cosmological solutions are almost the same. A general solution that one can easily find is

$$a(t) = e^{H_0 t}\,, \phi(t) = -2H_0 t\,,$$
$$\theta(t) = \frac{1}{3}e^{-3H_0 t}\left(-\frac{18c_{11}c_7\chi^2(c_7-3c_5)^2\exp\left(\frac{H_0 t(3c_5-2c_7)(\xi+3\chi)}{\chi(3c_5-c_7)}\right)}{(3c_5-2c_7)(3c_5\xi-2c_7\xi-3c_7\chi)}-\frac{\theta_1}{H_0}\right)\,,$$
$$\varphi(t) = -6H_0 t\,, \zeta(t) = \frac{1}{3}e^{-3H_0 t}\left(\frac{18c_{11}c_5\chi^2(c_7-3c_5)^2\exp\left(\frac{H_0 t(3c_5-2c_7)(\xi+3\chi)}{\chi(3c_5-c_7)}\right)}{(3c_5-2c_7)(3c_5\xi-2c_7\xi-3c_7\chi)}-\frac{\zeta_1}{H_0}\right)+\zeta_0\,,$$

and the coupling function f becomes

$$f(\phi,\varphi) = \frac{1}{\xi}-\frac{2c_{11}\chi(c_7-3c_5)^2}{3c_5\xi-2c_7\xi-3c_7\chi}\exp\left(-\frac{(3c_5\xi-2c_7\xi-3c_7\chi)(c_5\varphi-c_7\phi)}{2\chi(c_7-3c_5)^2}\right)\,, \tag{10.74}$$

where $\theta_1\,, \zeta_1$ and ζ_2 are integration constants and we need to set

$$c_3 = -\frac{(3c_5\xi-2c_7\xi-3c_7\chi)(-c_4c_7+c_5c_6+c_5c_9)}{2\chi(3c_5-c_7)^2}$$

. Apart from de-Sitter solutions, the system admits also power-law solutions. For example, by setting

$$c_3 = \frac{(9p^2-9p+2)\,(-c_4c_7+c_5c_6+c_5c_9)}{6p(3c_5p-c_5-c_7p)}$$

, we get the following solutions

$$
\begin{aligned}
&a(t) = t^p\,,\ \phi(t) = \frac{6p^2\ln(t-3pt)}{1-3p}\,,\ \varphi(t) = -6p\ln t\,,\\
\theta(t) &= \frac{6c_{11}c_7pt^{2-3p}(1-3p)^{\frac{c_7(2-3p)p}{-3c_5p+c_5+c_7p}}\left(p(\xi+3\chi)-\chi\right)}{3p-2} + \frac{\theta_1 t^{1-3p}}{1-3p}\,,\\
\zeta(t) &- \frac{6c_{11}c_5pt^{2-3p}(1-3p)^{-\frac{c_7p(3p-2)}{-3c_5p+c_5+c_7p}}\left(p(\xi+3\chi)-\chi\right)}{3p-2} + \zeta_0 + \frac{\zeta_1 t^{1-3p}}{1-3p}\,,
\end{aligned}
$$

and the coupling function f becomes

$$
f(\phi,\varphi) = \frac{1}{\xi} - \frac{6c_{11}p(-3c_5p+c_5+c_7p)}{9p^2-9p+2}\exp\left(-\frac{(9p^2-9p+2)\,(c_5\varphi-c_7\phi)}{6p(-3c_5p+c_5+c_7p)}\right)\,. \tag{10.75}
$$

The above procedure can be iterated for all the above couplings. We stress again the important fact that such couplings are not arbitrarily given but result from the existence of the symmetries.

10.7 Discussion and Conclusions

Motivated by an increasing amount of studies related to non-local theories, here we proposed a new generalized non-local theory of gravity including curvature and teleparallel terms. These kind of theories were introduced motivated by loop quantum effects and they have attracted a lot of interest since some of them are renormalizable [149]. Under suitable limits, the general action that we proposed can represent either curvature non-local theories with $\overset{\circ}{R}f(\Box^{-1}\overset{\circ}{R})$ based on [137] or teleparallel non-local theories $\overset{\bullet}{T}f(\Box^{-1}\overset{\bullet}{T})$ based on [109]. Since the theory is highly non-linear, it is possible to introduce four auxiliary scalar fields in order to rewrite the action in an easier way. Then, for a flat FRW cosmology, using the Noether Symmetry Approach, the coupling functions can be selected directly from the symmetries for the various models derived from the general theory. It is obvious that the theory (10.11) can give several models, depending on the values of the constants ξ and χ and on the form of the distortion function. We prove that, in most physically interesting cases, the only forms of the distortion function selected by the Noether Symmetries, are the exponential and the linear ones. According to the literature [155, 168], this is an important result, because, up to now, these kinds of couplings were chosen by hand in order to find cosmological solutions while, in our case, they come out from a first principle. In addition, there is a specific class of exponentials non-local gravity models which are renormalizable (see [148, 149]). This means that, the Noether

Symmetries dictate the form of the action and choose exponential form for the distortion function. As discussed in [167], the existence of Noether's symmetries is a selection criterion for physically motivated models. Finally, from models selected by symmetries, it is easy to find cosmological solutions like de-Sitter and power-law ones. The integrability of dynamics is guaranteed by the existence of first integrals. In forthcoming studies, the cosmological analysis will be improved in view of observational data.

Part III

Astrophysical Applications

Chapter 11

Maximum Size of Large Scale Structures

Apart from the cosmological scales, it is necessary to consider how the newly proposed theories of gravity behave at astrophysical scales. In the very end, one expects to have a unique theory at all scales and not a collection of many ones (of course, theories with screening mechanisms are still considered as unique). For this reason, the notion of the *maximum turnaround radius* has been used. Gravity is an attracting force and dark energy repelling; the point (or better surface) at which these two forces cancel each other is known as the *turnaround radius*. The maximum turnaround radius for a structure with a given mass, is the maximum size that it could possibly have in order to be stable. We use this stability criterion to test alternative theories of gravity as well as dark energy models. If a theory predicts a maximum possible size smaller than the actual observed size of structures of mass M, the latter are expected to be unstable in the framework of that theory.

11.1 Introduction

The Λ-cold dark matter (ΛCDM) model is widely considered to be the simplest and most successful theoretical description of our universe, and finds support from a wide range of cosmological observations. Despite its success, this model is unfortunately not without problems. While certain observational glitches have been reported from time to time [262–265], the biggest challenge the ΛCDM model has to face is the cosmological constant problem[1].

The tiny value of the observed cosmological constant, $\Lambda \sim \mathcal{O}(10^{-3}eV)^4$, that is

[1]The cosmological constant problem is not manifest only for ΛCDM but for any other theory with dynamical dark energy or alternative to GR.

needed for the model to be observationally viable, finds no compelling explanation from a quantum field theoretical point of view. There had been numerous attempts to explain the value of Λ by relating it to vacuum energy density of quantum fields, but all such attempts have either theoretical or observational inconsistencies [266, 267]. A related problem is that de Sitter space may also be unstable to quantum corrections [268, 269, 271, 272].

These conceptual and observational problems with the cosmological constant Λ have triggered in recent years vigorous research in alternatives to the ΛCDM model. The chief agenda of these alternative models is to generate the effect of the dark energy through additional matter fields (for instance quintessence [273]), or, by replacing the theory of gravity on which ΛCDM rests, i.e. General Relativity, by a different theory [265, 274](see also [275] for a recent critique of the current status of cosmology). In order to discriminate between such alternative theories of gravity and GR, it is necessary to test all their possible observable consequences with cosmological observations. The next generation of cosmological surveys will offer a huge boost in precision making such tests possible [276–280].

In this chapter, we are particularly interested in one possible test of ΛCDM and alternative theories of gravity, namely, the stability of the large scale cosmic structures [281] [2]. The maximum size of a large scale cosmic structure with a given mass M can be estimated using the maximum turnaround radius (or simply the turnaround radius $R_{\mathtt{TA}}$ for short). More precisely, $R_{\mathtt{TA}}$ is the point where for radially moving test particles the attraction due to normal matter is balanced by the repulsion due to the dark energy. Specific theories are expected to lead to estimates for the turnaround radius, which depend on the theory parameters. If a certain theory predicts a maximum possible size smaller than the actual observed size of structures of mass M, the latter are expected to be unstable in the framework of that theory. Thus, parameter ranges resulting to maximum possible sizes smaller than what we observe are ruled out.

The turnaround radius was calculated in [283, 284] in the wider context of geodesics of the Schwarzschild-de Sitter spacetime[3]. In a cosmological context, the turnaround radius for spherical structures was calculated for ΛCDM in [281], and in [285] for smooth dark energy. The turnaround radius as a cosmological observable was investigated in [286, 287]. In [288] it was proposed to look for the violation of the maximum upper bound of $R_{\mathtt{TA}}$ using the zero velocity surfaces of a large scale structure, by observing the peculiar velocity profiles of its members. It turns out that for structures as massive as $10^{15}M_\odot$ (e.g. the Virgo supercluster), the actual sizes lie very close and below the theoretical prediction of ΛCDM [281]. The struc-

[2] See also [282] for a different approach.

[3] The turnaround radius was called the “static radius” in [283, 284].

tures studied in the references above are at sufficiently low redshifts ($z \sim 10^{-2}$), and hence $R_{\mathtt{TA}}$ measurements could provide a local indication and check for dark energy. In other words, it does not require any data coming from the high redshift Supernovae or from the early universe.

Measuring the turnaround radius offers yet another way of putting constraints on alternative gravity models. For instance, the maximum turnaround radius has recently been calculated for a cubic galileon model [327]. A method to calculate the turnaround radius in generic gravitational theories was put foward in [289, 326] by considering timelike geodesics, in the framework of alternative gravity theories admitting McVittie-like [290–292] solutions. We agree with the general formula for the turnaround, eq. 21, of [289] but disagree in other results of that article which appear to be in conflict with ours. [4]

This chapter is organized as follows. In section 11.2 we derive a general formula for the calculation of the turnaround radius, valid in any metric theory of gravity obeying the Einstein Equivalence Principle (EEP), a necessary assumption as the geodesic equation is used in our derivation. We perform our derivation in steps: (i) we first calculate the maximum turnaround radius in the case of the ΛCDM model using static coordinates, (ii) we extend the static metric calculation to arbitrary theories (arbitrary static metrics), (iii) we re-calculate the turnaround radius using the McVittie metric and finally (iv) we relate the two types of calculation (static and McVittie) using cosmological perturbation theory in a general theory of gravity. Our result is (11.21). Our derivation makes it clear why the standard formula for the turnaround radius (which in fact agrees with eq. 21 of [289]) is valid for *any theory of gravity obeying the EEP*, once the solution for the potential Ψ is known (see eq. (11.15) and (11.23) below). In section 11.3 we consider a specific theory, the Brans-Dicke theory of gravity, as an example to demonstrate the use of our formula. Within the Brans-Dicke theory, we perform the calculation of the turnaround radius in two coordinate systems, arriving (as expected) at the same result. We firstly determine the solution to the field equations around a static spherically mass distribution and secondly around a spherical solution in an expanding universe and show that the two solutions are equivalent, related by a coordinate transformation.

[4] In [289] it is assumed that the potentials Ψ and Ψ have solutions $\sim \frac{Gm}{r}$ under the assumption of spherical symmetry (even in ΛCDM), where the constant m is the mass of the source. This however is incorrect as the correct solution (as can be verified by inspecting the McVittie solution) is $\sim \frac{Gm}{ar}$. Indeed, eq. 29 in [289] gives a time-dependent turnaround radius, which is in disagreement with the known result for ΛCDM . Our second source of disagreement is the recasting of the turnaround radius in terms of the areal radius. While we agree with the reasoning and with the relation between the comoving and areal radius, the contribution to the turnaround formula is of higher order in perturbation theory and should be neglected unless higher order in perturbation theory solutions are also used.

Our calculation yields $R_{\mathtt{TA}}^{(BD)} \approx R_{\mathtt{TA}}^{(\Lambda CDM)}\left(1+\frac{1}{3\omega}\right)$ for large Brans-Dicke parameter ω and hence it is always larger than ΛCDM, implying that the Brans-Dicke theory is also consistent with current data. Finally in section 11.4 we discuss the turnaround radius in the framework of $f(R)$ gravity. In this perspective, we assume $f(R)$ models which, at certain point, assume a constant curvature, $R = R_{\mathrm{dS}}$. We thus compute the $R_{\mathrm{TA,max}}$ in two different pictures. In the first approach, we assume spherically symmetric space-times. In the second, we discuss the cosmological case considering perturbations. As expected, curvature terms account for repulsive effects, mimicking dark energy even at the level of turnaround radius. In this way, we are able to put constraints on generic $f(R)$ models. We find that the observed bounds on $R_{\mathrm{TA,max}}$ are analogous if one considers spherically symmetric space-times or cosmological perturbations. We therefore derive limits over the form of $f(R)$ which should guarantee structure formation for any model even in the approximation of quasi-constant curvature.

Throughout this article we work with mostly positive signature of the metric, $(-,+,+,+)$ and use the greek alphabet for spacetime indices and latin alphabet for spatial indices. We use units where the speed of light is equal to unity.

11.2 The turnaround radius

11.2.1 The turnaround radius in GR with a cosmological constant

Let us first briefly present the case of GR with a cosmological constant, where the derivation of the turnaround radius is well known. This will be useful further below when we generalize the result to arbitrary metric theories of gravity.

In [285] the turnaround radius for a spherical mass M in the ΛCDM model was defined in the following way. Consider a stationary probe in a Schwarzschild-de Sitter (SdS) spacetime with metric

$$ds^2_{SdS} = -\left(1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2\right)dT^2 + \frac{dR^2}{1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2} + R^2 d\Omega \tag{11.1}$$

following a trajectory in spacetime with four-velocity

$$u^\mu = \left(\frac{1}{\sqrt{1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2}}, 0, 0, 0\right). \tag{11.2}$$

Here G_N is the measured Newtonian gravitational constant. The maximum turnaround radius is the point along a radial trajectory where the four-acceleration $a^\nu = u^\mu \nabla_\mu u^\nu$

of the probe vanishes. Using the SdS metric (11.1) yields $a^1 = \left(1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2\right)\left(\frac{G_N M}{R^2} - \frac{\Lambda}{3}R\right)$ and setting it to zero gives the turnaround radius,

$$R_{\mathtt{TA}} = \left(\frac{3G_N M}{\Lambda}\right)^{1/3} \tag{11.3}$$

in the case of GR with a cosmological constant.

In a different theory of gravity, the SdS metric (11.1) need not be a solution. However, assuming that a static solution exists of the form

$$ds^2 = -f(R)dT^2 + h(R)dR^2 + R^2 d\Omega \tag{11.4}$$

one can follow the same line of thought to define the turnaround radius by the vanishing of the four-acceleration for a stationary probe. This leads to the condition

$$f'(R) \equiv \frac{\partial f}{\partial R} \underset{\mathtt{at\ R=R_{TA}}}{\longrightarrow} 0 \tag{11.5}$$

supplying us with an algebraic equation for R, which must be solved in order to obtain the maximum turnaround radius $R_{\mathtt{TA}}$. The definition (11.5) is valid in any theory of gravity which obeys the weak equivalence principle and can be used to calculate the turnaround radius once the solution $f(R)$ is known. Let us also note that even if the spacetime is spherically symmetric but not static, the metric may still be brought into a diagonal form, in which case the condition (11.5) still holds, although the resulting turnaround radius will in general be time dependent.

The above definition (11.5) of the turnaround radius is not formulated in a covariant language, but can be made so. In particular, the turnaround radius corresponds to the locus where $u^\mu \nabla_\mu u^\nu = 0$ for a *stationary observer* in a spherically symmetric spacetime. With this definition one can calculate the turnaround radius in any coordinate system of choice, although, the definition depends on this particular choice of observer.

Our goal is to find a definition of the turnaround radius, suited for cosmology, equivalent to the definition above. Consider the McVittie metric [290–292]

$$ds^2_{McV} = -\left(\frac{1-\mu}{1+\mu}\right)^2 dt^2 + (1+\mu)^4 a^2 (dr^2 + r^2 d\Omega) \tag{11.6}$$

where $\mu = \frac{G_N M}{2ar}$, describing the exterior of a spherical mass in an expanding Universe evolving with scale factor $a(t)$. The field equations are

$$3H^2 = 8\pi G_N \rho \tag{11.7}$$

$$-2\frac{1+\mu}{1-\mu}\dot{H} - 3H^2 = 8\pi G_N P \tag{11.8}$$

where $H(t) \equiv \dot{a}/a$ is the Hubble parameter, $\rho = \rho(t)$ is the energy density and $P = P(t, r)$ the (inhomogeneous) pressure. [5] If $8\pi G_N \rho = \Lambda$ is a constant then this

[5] Having a homogeneous density, yet, inhomogeneous pressure seems somewhat unnatural.

spacetime reduces to the Schwarzschild-de Sitter spacetime in a different coordinate system to (11.1). To see this (and remembering always that H is a constant in Schwarzschild-de Sitter) define new coordinates $T(t,r)$ and $R(t,r)$ via

$$t = T - Q(R)\,, \tag{11.9}$$
$$R = (1+\mu)^2 ar\,, \tag{11.10}$$

with $Q(R)$ the solution to

$$\frac{\partial Q}{\partial R} = \frac{\sqrt{\frac{\Lambda}{3}}R}{\left(1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2\right)\sqrt{1 - \frac{2G_N M}{R}}} \tag{11.11}$$

so that one recovers (11.1).

How does the turnaround condition look-like from the McVittie's point of view? Since we already know the result in the case of the static Schwarzschild-de Sitter coordinate system, we can simply transform the conditions leading to that result, to the McVittie coordinate system. In particular, we need to transform the velocity vector field (11.2) of the stationary observer, into the new system [6] and apply the condition $u^\mu \nabla_\mu u^\nu = 0$. For this we need the inverse transformation of (11.10), i.e.

$$r(t,R) = \frac{R - G_N M + \sqrt{R^2 - 2G_N M R}}{2a}\,, \tag{11.12}$$

where we have chosen the positive sign of the square root. [7]

With the above transformation, the observer's velocity (11.2) becomes

$$u^\mu = \frac{1+\mu}{\sqrt{(1-\mu)^2 - H^2(1+\mu)^6 a^2 r^2}}(1, -rH, 0, 0). \tag{11.13}$$

Using the condition $u^\mu \nabla_\mu u^\nu = 0$ we find (remember H is constant)

$$2\mu = (1+\mu)^6 H^2 a^2 r^2 \tag{11.14}$$

which translates to (11.3) using (11.10).

11.2.2 New definition of the turnaround radius

We now present a new definition of the turnaround radius, valid in any theory of gravity obeying the EEP. In generic alternative theories of gravity that we deal with

[6] It is easy to show that using a stationary observer in the McVittie coordinate system fails. Indeed a stationary observer in one coordinate system is no longer stationary in the other.

[7] The negative sign also works, however, issues arise when one considers a perturbative analogue of the McVittie metric as we do further below.

in this chapter, the Schwarzschild-de Sitter metric will in general not be a solution. Neither will some general static spherically symmetric metric have an equivalent form, which resembles the McVittie metric. However, our interest is in cosmology, where a perturbed FRW metric always exists. Let us then consider the perturbed version of the McVittie construction of the previous subsection.

In the Newtonian gauge, the perturbed FRW metric takes the form

$$ds^2 = -\left(1+2\Psi\right)dt^2 + a^2\left(1-2\Phi\right)\gamma_{ij}dx^i dx^j \tag{11.15}$$

where Ψ and Φ are the two metric potentials and where we have assumed that γ_{ij} is flat, so that $\gamma_{ij}dx^i dx^j = dr^2 + r^2 d\Omega$ in spherical coordinates.

By inspection, when $\mu \ll 1$, the McVittie metric (11.6) may be interpreted as a perturbation on FRW sourced by a point-mass by identifying $\Psi = \Phi = -2\mu = -\frac{G_N M}{ar}$. We exploit this fact and re-cast the definition of the turnaround radius using cosmological perturbation theory. Starting from (11.13), we rotate into an arbitrary spatial direction, using $r^i = (x, y, z) = \frac{1}{2}\vec{\nabla}^i r^2$, where $\vec{\nabla}^i = \gamma^{ij}\vec{\nabla}_j$. The 3-vector r^i has components $(-rH, 0, 0)$ in the original coordinate system used in (11.13). We also use the Friedman equation, $\Lambda = 3H^2$, so that the equivalent version of (11.13) albeit in an arbitrary direction is

$$u^\mu = \frac{1+\mu}{\sqrt{(1-\mu)^2 - (Har)^2(1+\mu)^6}}(1, -\frac{1}{2}H\vec{\nabla}^i r^2). \tag{11.16}$$

This is the four-velocity of a test particle at rest in a coordinate system which is equivalent to (11.15). Taking the limit $\mu \ll 1$ and $aHr \ll 1$, corresponding to regions far away from both horizons, leads to

$$u^\mu = (1 - \Psi + Ha\Theta, -\frac{1}{a}\vec{\nabla}^i\Theta) \tag{11.17}$$

where we have defined the scalar function

$$\Theta = \frac{1}{2}aHr^2 \tag{11.18}$$

We have assigned the perturbation orders $\mathcal{O}(\Psi) \sim \mathcal{O}(H^2) \sim \mathcal{O}(\Theta^2) \sim \mathcal{O}(\dot{\Theta})$, which are remiscent of the Parametrized Post-Newtonian formalism. Indeed the vector field $\vec{\nabla}^i\Theta$ has all the properties of a spatial curl-less velocity field.

We have managed to create a covariant definition of the turnaround radius, which is adapted to cosmology. In particular one starts from the observer moving with velocity given by (11.17) and impose the EEP. The EEP implies the geodesic equation

$$u^\mu \nabla_\mu u^\nu = 0 \tag{11.19}$$

which in turn leads to

$$\vec{\nabla}_i \left[\dot{\Theta} - \frac{1}{a}\Psi + H\Theta \right] - \frac{1}{a}\vec{\nabla}^j \Theta \vec{\nabla}_j \vec{\nabla}_i \Theta = 0. \tag{11.20}$$

Since the last term can be written as $\vec{\nabla}^j\Theta\vec{\nabla}_j\vec{\nabla}_i\Theta = \frac{1}{2}\vec{\nabla}_i|\vec{\nabla}\Theta|^2$, we finally get the general *turnaround equation*

$$\vec{\nabla}_i \left[a\left(\dot{\Theta} + H\Theta\right) - \frac{1}{2}|\vec{\nabla}\Theta|^2 \right] = \vec{\nabla}_i \Psi \tag{11.21}$$

The above equation is valid in any theory of gravity obeying the EEP. Despite appearances the above equation is fully consistent in perturbation theory (remember the assignment of perturbation orders above). One should not treat (11.21) as a differential equation for Θ. Rather, one should assume a specific functional form for $\Theta(\vec{x}^i, t)$ and then given that functional form, as well as the solution for Ψ from the field equations of the theory, one should determine the 3-surface $\mathcal{F}(x^i) = \text{const}$ such that the equation holds. In the case of spherical symmetry Θ is given by (11.18), however, (11.21) may be used as a starting point for generalizing the turnaround radius calculation into a turnaround surface when the shape of the bound object is non-spherical. One possibility would be to consider a non-spherical function $\Theta(t, \vec{x})$ corresponding to some non-spherical surface.

Let us now return to our spherically-symmetric ansatz, i.e. $\Theta = \frac{1}{2}aHr^2$. In this case we have that $|\vec{\nabla}\Theta|^2 = a^2H^2r^2 = 2aH\Theta$, hence the LHS of (11.21) leads to

$$a\frac{\partial\dot{\Theta}}{\partial r} = a^2[H^2 + \dot{H}]r \tag{11.22}$$

and the turnaround equation simplifies to

$$a^2[H^2 + \dot{H}]r = \frac{\partial \Psi}{\partial r}. \tag{11.23}$$

The above equation which we name the *reduced turnaround equation* (due to spherical symmetry) can then be used to calculate the turnaround radius $R_{\mathtt{TA}} = ar$ given a Hubble parameter $H(t)$ and the solution to the potential Ψ, both of which are specified in a given theory, including a theory beyond GR.

From (11.23) a quick calculation gives the turnaround radius for the case of a cosmological constant as dark energy and for the case of a dark energy fluid with equation of state parameter w both within the GR framework. In both models the solution to the potential is $\Psi = -\frac{G_N M}{ar}$ [285]. What is different between the two models is the Hubble parameter. In the first case it is a constant given by

$H = \sqrt{\Lambda/3}$ so that (11.23) leads to (11.3), while in the second case it is given by $aH = H_0 a^{-(1+3w)/2}$ so that (since for dark energy $1 + 3w < 0$)

$$R_{\mathrm{TA}} = \left[-\frac{2G_N M}{(1+3w)H^2}\right]^{1/3} = \left[-\frac{2G_N M}{(1+3w)H_0^2}\right]^{1/3} a^{1+w} \tag{11.24}$$

We observe that when $w \neq -1$ the maximum turnaround radius is time-dependent. In the limit $w \to -1$, i.e. ΛCDM, the maximum turnaround radius agrees with the time-independent ΛCDM formula (11.3).

11.3 The turnaround radius of Brans-Dicke theory of gravity

The Brans-Dicke theory [77] can be thought of as a prototype alternative theory of gravity. Its action in the presence of a cosmological constant is given by

$$S = \frac{1}{16\pi G}\int \sqrt{-g}d^4x \left[\phi R - 2\Lambda - \frac{\omega}{\phi}(\nabla\phi)^2\right] + S_M, \tag{11.25}$$

where the scalar ϕ is the Brans-Dicke field, the constant ω is the Brans-Dicke parameter and S_M is the collective action for all matter fields present, which depends on the metric $g_{\mu\nu}$ but not on the scalar field. The shift of conceptual paradigm from GR in this theory is certainly the scalar field ϕ, whose non-minimal coupling with the Ricci scalar indicates a spacetime dependent gravitational coupling. In the limit $\omega \to \infty$ the scalar field ϕ must be a constant $\phi \to \phi_0$ in which case GR is recovered.

Solar system data severely constrain $\omega \gtrsim 40000$ [293, 294], thereby making it practically indistinguishable from General Relativity in our local neighbourhood. However, any test of gravity should be accompanied by a specification of the curvature and potential regime it is performed in [295]. In this sense cosmological constraints on Brans-Dicke theory should be treated independently from solar system tests as they lie in different regions of the gravitational parameter space.

Let us exemplify. As shown in [296], the Brans-Dicke theory arises as a specific limit of Horndeski theory [80, 297], the most general Lorentz-invariant scalar-tensor theory, having second order field equations in four dimensions. The Horndeski theory offers the possibility of realizing screening mechanisms such as the Vainshtein [298], the chameleon [299] and the symmetron [75] mechanisms. These mechanisms restore GR around the high-curvature/high-density environments of astrophysical bodies, such as the sun. Hence, it is possible that certain subsets of Horndeski theory which realize these mechanisms tend to Brans-Dicke theory in the low curvature environment of the cosmological regime but acquire corrections which send it back to GR in regions of high curvature. As such, cosmological constraints on Brans-Dicke theory give different information than solar system tests. In [296] the lower bound $\omega > 890$ at the 99% confidence level was placed (see also [270, 300]), using the latest Cosmic Microwave Background data from Planck. Future photometric and spectroscopic cosmological surveys are expected to increase this by a factor of $20 - 30$ [301, 302], making cosmological tests comparable to solar system tests.

In [303], the no hair theorems for the Brans-Dicke theory with $\Lambda > 0$ for stationary axisymmetric black holes and stars were discussed. It was shown there that no matter how large the Brans-Dicke parameter ω is, unless it is *infinite* (i.e., the theory coincides exactly with the General Relativity), there can exist no regular such solutions if asymptotic de Sitter boundary condition is imposed. The Brans-Dicke theory has also been investigated in the context of galactic dark matter in [304].

In order to pave the way for the calculation of the turnaround radius we construct solutions in Brans-Dicke theory with a cosmological constant. We consider two types of solutions, i.e. static spherically symmetric solutions and cosmological solutions, in order to apply both formulae (11.5) and (11.23) for the determination of the turnaround radius.

11.3.1 Stationary spherically symmetric point-mass solutions

Adopting a static spherically symmetric ansatz as in (11.4) and in addition that $\phi = \phi(R)$, the field equations are

$$\frac{1}{R}\left(\frac{h'}{h} + \frac{h-1}{R}\right) - \frac{1}{2}\omega\left(\frac{\phi'}{\phi}\right)^2 + \frac{f'}{2f}\frac{\phi'}{\phi} = \frac{8\pi G h}{\phi}\left[\rho + \frac{-\rho + 3P}{2\omega + 3}\right] \tag{11.26a}$$

$$\frac{1-h}{R^2} + \frac{1}{R}\frac{f'}{f} + \frac{2}{R}\frac{\phi'}{\phi} + \frac{f'}{2f}\frac{\phi'}{\phi} - \frac{1}{2}\omega\left(\frac{\phi'}{\phi}\right)^2 = \frac{8\pi G P}{\phi}h \tag{11.26b}$$

$$\frac{f''}{2f} - \frac{(f')^2}{2f^2} + \left(\frac{f'}{2f} - \frac{h'}{2h}\right)\left(\frac{f'}{2f} + \frac{1}{R}\right) + \frac{1}{2}\omega\left(\frac{\phi'}{\phi}\right)^2 - \frac{1}{R}\frac{\phi'}{\phi} = \frac{8\pi G h}{\phi}\left[P + \frac{\rho - 3P}{2\omega + 3}\right] \tag{11.26c}$$

and

$$\left[\frac{\sqrt{f}}{\sqrt{h}}R^2\phi'\right]' = \frac{8\pi G\,(\rho - 3P)}{2\omega + 3}\sqrt{fh}R^2 \tag{11.26d}$$

where ρ and P are the total density and pressure of matter respectively, including the cosmological constant. Consistency requires that the matter velocity has components $u^\mu = (\frac{1}{\sqrt{f}}, 0, 0, 0)$. In the Einstein equations above, we have used the scalar equation (11.26d) to eliminate the $\Box\phi$ terms.

A complete analytic solution of (11.26a)-(11.26d) is impossible. Indeed, as we discussed above, it has been shown that the Brans-Dicke theory with a cosmological constant does not admit stationary and spherically symmetric solutions, which are exterior solutions to a compact object and which have a cosmological horizon where the Brans-Dicke field is regular [303]. Clearly then, any spherically symmetric solution in this theory (in the presence of Λ) must be necessarily time-dependent. However, we expect this time-dependence to become more and more manifest only when we approach the cosmological horizon. As the turnaround radius is on much smaller scales, we take a different approach: perturbation theory.

Physical systems of interest are those where the Schwarzschild horizon R_s, the turnaround radius $R_{\mathtt{TA}}$ and the de Sitter horizon R_h are widely separated. To be more precise, in standard GR we have $R_s/R_{\mathtt{TA}} = 2G_N M(\frac{\Lambda}{3G_N M})^{1/3} \lesssim 10^{-8} - 10^{-4}$ for the most massive galaxy clusters in the range $M \sim 10^{11} - 10^{17} M_\odot$ while $R_{\mathtt{TA}}/R_h = (\frac{3G_N M}{\Lambda})^{1/3}\sqrt{\Lambda}/\sqrt{3} \lesssim 10^{-4} - 10^{-2}$. It thus seems like a good first approximation that $2G_N M/R \ll 1$ and $\Lambda R^2/3 \ll 1$, so that the Scharzschild-de Sitter spacetime may be considered as a perturbation around Minkowski for the scales of interest.[8]

We expand our variables as

$$f = 1 + U \tag{11.27}$$

$$h = 1 + V \tag{11.28}$$

$$\phi = \bar{\phi}_0(1 + \varphi) \tag{11.29}$$

so that U, V and φ are small compared to unity and $\bar{\phi}_0$ is a background value for ϕ. We consider a point-mass source in a spacetime filled with a cosmological constant so that the energy-density and pressure entering (11.26a)-(11.26d) take the form

$$8\pi G\rho = \frac{2GM}{R^2}\delta(R) + \Lambda \tag{11.30}$$

$$8\pi G P = -\Lambda \tag{11.31}$$

Consistently with our approximation both the point mass and Λ are treated as small perturbations. We start from (11.26a), linearize and then integrate to get

$$V = \frac{1}{\bar{\phi}_0(2\omega+3)}\left[2(\omega+1)\frac{2GM}{R} + \frac{2\omega-1}{3}\Lambda R^2\right]. \tag{11.32}$$

The above solution is then used in the linearized version of (11.26d), which when integrated gives

$$\varphi = -\frac{1}{\bar{\phi}_0(2\omega+3)}\left[\frac{2}{3}\Lambda R^2 - \frac{2GM}{R}\right]. \tag{11.33}$$

[8] One may instead perturb around a de Sitter, or even, a Schwarzschild-de Sitter spacetime. However, this introduces tremendous complication in solving the scalar equation and in the end, the Minkowski space approximation used here, where $2G_N M/R \ll 1$ and $\Lambda R^2/3 \ll 1$, is recovered.

Finally, the expressions for V and φ are used in the linearized version of (11.26b), leading after integration to

$$U = -\frac{1}{\bar{\phi}_0(2\omega+3)}\left[2(\omega+2)\frac{2GM}{R} + \frac{2\omega+1}{3}\Lambda R^2\right] \tag{11.34}$$

so that the metric is

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2(\omega+2)}{2\omega+3}\frac{2GM}{\bar{\phi}_0 R} - \frac{2\omega+1}{2\omega+3}\frac{\Lambda}{3\bar{\phi}_0}R^2\right]dT^2 \\ &+\left[1 + \frac{2(\omega+1)}{2\omega+3}\frac{2GM}{\bar{\phi}_0 R} + \frac{2\omega-1}{2\omega+3}\frac{\Lambda}{3\bar{\phi}_0}R^2\right]dR^2 + R^2 d\Omega \end{aligned} \tag{11.35}$$

11.3.2 Cosmological solutions with a point-mass source

Let us now construct cosmological solutions for the metric and the Brans-Dicke field with a point-mass source. By “cosmological” we mean that in the limit $M \to 0$, the metric becomes the Friedman-Robertson-Walker metric and so these solutions are the analogue of the McVittie solution in the case of GR. We construct our solution by first considering a background FRW solution and then adding the perturbation due to the mass (see also [305] for cosmological perturbation theory equations with an array of point masses).

FRW solutions

The FRW metric is

$$ds^2 = -dt^2 + a^2\gamma_{ij}^{(\kappa)}dx^i dx^j, \tag{11.36}$$

where $a(t)$ is the scale factor of cosmic time t, $\gamma_{ij}^{(\kappa)}$ is the 3-metric (used to raise and lower three-dimensional indices) of constant spatial curvature κ.

The Friedman equation in Brans-Dicke theory takes the form

$$3\left(H + \frac{1}{2}\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 + \frac{3\kappa}{a^2} = \frac{8\pi G}{\bar{\phi}}\bar{\rho} + \frac{2\omega+3}{4}\left(\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 \tag{11.37}$$

where $\bar{\rho}$ is the background energy-density of matter (including the cosmological constant), $H = \dot{a}/a$ is the time-dependent Hubble parameter and $\bar{\phi}$ is the homogeneous part of the scalar field adopted to the FRW symmetries. The scalar evolves according to

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} = \frac{8\pi G}{2\omega+3}(\bar{\rho} - 3\bar{P}) \tag{11.38}$$

where $\bar{P}$ is the background pressure of matter (including the cosmological constant).

It is straightforward to verify that an exact analytical solution is, in general, impossible, even if $8\pi G\bar{\rho} = \Lambda$ is a constant. Indeed, it can be shown that the de Sitter spacetime in no longer an exact solution of the field equations as it is in GR. [9]. This is equivalent to the non-existence of static-spherically symmetric solutions in the presence of a cosmological constant [303], as we have discussed in the previous subsection. Hence, we proceed using perturbation theory. Both the Friedman equation (11.37) and the scalar equation (11.38) suggest that the small parameter to use is

$$\epsilon = \frac{1}{2\omega + 3} \tag{11.39}$$

We are interested in the case of a flat universe filled with cosmological constant so that $3H_0^2\bar{\phi}_0 = 8\pi G\bar{\rho} = \Lambda = -8\pi G\bar{P}$. We construct the perturbative solution as a power series in ϵ which yields

$$\bar{\phi} = \bar{\phi}_0 \left[1 + 4\epsilon \ln a + \ldots\right] = \bar{\phi}_0 \left[1 + 4\epsilon H_0 t + \ldots\right] \tag{11.40}$$

$$H = H_0 \left[1 - \frac{4}{3}\epsilon - 2\epsilon \ln a + \ldots\right] = H_0 \left[1 - \frac{4}{3}\epsilon - 2\epsilon H_0 t + \ldots\right] \tag{11.41}$$

as can be verified by direct substitution. A more formal derivation which is valid for a generic matter field and curvature can be found in the appendix. The dependence of the scale factor on time t is found by integrating (11.41) so that to $\mathcal{O}(\epsilon)$ we find

$$a = \bar{a} \left[1 - \frac{4}{3}\epsilon H_0 t - \epsilon (H_0 t)^2 + \ldots\right] \tag{11.42}$$

where $\bar{a} = e^{H_0 t}$. The solutions found above are of course only valid close to $\ln a \sim 1$, i.e. for all times t such that $\epsilon H_0 t \ll 1$.

Perturbed FRW solutions

Including the point-mass in our system inevitably introduces spatial dependence in the solutions. Assuming that the point-mass is not too massive as to overclose the universe, we may treat its contribution as a perturbation on top of the FRW solution we have constructed. This requires perturbing the FRW metric to linear order as in (11.15) by adopting the Newtonian gauge. Likewise we perturb the scalar field

$$\phi = \bar{\phi}(1 + \varphi) \tag{11.43}$$

[9]It may be shown that an exact solution exists for $8\pi G\rho = \Lambda = \text{const}$ with $a = (t/t_0)^{\frac{2\omega+1}{2}}$ and $\phi = 4\Lambda t^2/(2\omega + 3)/(6\omega + 5)$. However, this requires that initially both the scalar and its first derivative vanish, i.e. $\bar{\phi}_0 = \bar{\phi}^{(in)} = 0$, and therefore this is a spurious solution of no physical significance and must be discarded.

where $\bar{\phi}$ is the background value and $\bar{\phi}\varphi$ the perturbation.

Before proceeding into solving the system, caution is warranted. Our background solution was arbitrarily close to de Sitter. We may then re-interpret the background FRW solution as being exact de Sitter plus small time-dependent perturbations. In other words set $a = \bar{a}(1+\delta_a)$ where from (11.42) we get $\delta_a = -\epsilon\left[\frac{4}{3}H_0 t + (H_0 t)^2\right]$. This also implies $H = H_0 + \dot{\delta}_a$, which may be checked for consistency with (11.41). Then we may define a new potential as $a^2(1-2\Phi) = \bar{a}^2(1-2\tilde{\Phi})$ so that $\tilde{\Phi} = \Phi - \delta_a$. The background field equations can only be satisfied under this transformation, if and only if a further transformation is also implemented: by observing that $\bar{\phi} = \bar{\phi}_0(1+\delta_\phi)$ with $\delta_\phi = 4\epsilon H_0 t$ from (11.40), we may transform δ_ϕ away via $\phi = \bar{\phi}(1+\varphi) = \bar{\phi}_0(1+\tilde{\varphi})$ so that $\tilde{\varphi} = \varphi + \delta_\phi$.

Consistency of this line of thought requires that $\mathcal{O}(\Phi) \sim \mathcal{O}(\tilde{\Phi}) \sim \mathcal{O}(\delta_a) \sim \mathcal{O}(\delta_\phi)$ so that when considering linearized perturbations we ignore terms like $\Phi\delta_a$ or δ_a^2, etc. This means that in the perturbation equations we may replace $a \to \bar{a}$, $H \to H_0$ and $\dot{\bar{\phi}} \to 0$ resulting in great simplification. A further consistency requirement is that since after the transformation the background scalar field is constant, the scalar field equation must be treated entirely perturbatively. With these considerations and letting $\vec{\nabla}_i$ to be the covariant derivative of γ_{ij}, the perturbed Einstein equations sourced by matter with density perturbation $\delta\rho = M\delta^{(3)}(a\vec{r})$ are as follows. Using the identity $\delta^{(3)}(a\vec{r}) = \delta(r)/(4\pi a^3 r^2)$, the $0-0$ perturbed Einstein equation is

$$-6H_0\left(\dot{\tilde{\Phi}} + H_0\Psi\right) + 3H_0\left(\dot{\tilde{\varphi}} + H_0\tilde{\varphi}\right) + \frac{2}{\bar{a}^2}\vec{\nabla}^2\left(\tilde{\Phi} - \frac{1}{2}\tilde{\varphi}\right) = \frac{2GM}{\bar{\phi}_0\bar{a}^3r^2}\delta(r) \tag{11.44}$$

and the $0-i$-Einstein equation is

$$2\vec{\nabla}_i\left(\dot{\tilde{\Phi}} + H_0\Psi\right) = \vec{\nabla}_i\left(\dot{\tilde{\varphi}} - H_0\tilde{\varphi}\right). \tag{11.45}$$

We combine (11.44) and (11.45), assume the quasistatic limit where $H_0^2\tilde{\varphi} \ll \vec{\nabla}^2\tilde{\varphi}$ and integrate to get

$$\tilde{\Phi} - \frac{1}{2}\tilde{\varphi} = -\frac{GM}{\bar{\phi}_0\bar{R}} \tag{11.46}$$

where we have defined

$$\bar{R} = \bar{a}r. \tag{11.47}$$

The perturbed scalar field equation is

$$\ddot{\tilde{\varphi}} + 3H_0\dot{\tilde{\varphi}} - \frac{1}{\bar{a}^2}\vec{\nabla}^2\tilde{\varphi} = \frac{\epsilon}{\bar{\phi}_0}\left[4\Lambda + \frac{2GM}{\bar{a}^3r^2}\delta(r) + 8\Lambda\Psi\right] \tag{11.48}$$

and after assuming the quasistatic limit and integrating gives

$$\tilde{\varphi} = 2\epsilon\left[\frac{GM}{\bar{\phi}_0\bar{R}} - H_0^2\bar{R}^2\right] \tag{11.49}$$

Hence, $\tilde{\Phi} = -\frac{GM(1-\epsilon)}{\bar{\phi}_0 \bar{R}} - \epsilon H_0^2 \bar{R}^2$ while $\Psi = -\frac{GM(1+\epsilon)}{\bar{\phi}_0 \bar{R}} + \epsilon H_0^2 \bar{R}^2$ after using the traceless-ij-Einstein equation $D_{ij}(\Phi - \Psi - \varphi) = 0$ and ignoring the kernel which results to pure gauge-solutions.

Therefore, the metric to $\mathcal{O}(\epsilon)$ is

$$ds^2 = -\left[1 - \frac{2GM}{\bar{\phi}_0 \bar{R}}(1+\epsilon) + 2\epsilon H_0^2 \bar{R}^2\right] dt^2 + \bar{a}^2 \left[1 + \frac{2GM}{\bar{\phi}_0 \bar{R}}(1-\epsilon) + 2\epsilon H_0^2 \bar{R}^2\right] \gamma_{ij} dx^i dx^j \tag{11.50}$$

Setting $\epsilon \to 0$ recovers the perturbed McVittie metric as expected, i.e. it recovers (11.6) in the limit $\mu \ll 1$.

11.3.3 The turnaround radius in Brans-Dicke theory

Having found the two types of solutions let us return to our original goal: the turnaround radius. A quick calculation using (11.5) along with the static spherically symmetric solution (11.35) yields

$$R_{\mathtt{TA}}^3 = \frac{3GM}{\Lambda}\frac{2\omega+4}{2\omega+1} \tag{11.51}$$

and taking the large ω (small ϵ) limit

$$R_{\mathtt{TA}} \approx \left(\frac{3GM}{\Lambda}\right)^{1/3}(1+\epsilon) \approx \left(\frac{3GM}{\Lambda}\right)^{1/3}\left(1 + \frac{1}{2\omega}\right) \tag{11.52}$$

to $\mathcal{O}(\epsilon) \sim \mathcal{O}(1/\omega)$.

Similarly, another quick calculation using (11.23) along with $H = H_0$ and the cosmological solution (11.50) yields once again (11.52). This should not come as a surprise. After all the two solutions (11.35) and (11.50) are in fact one and the same, after a coordinate transformation. This may be checked using the general form of such coordinate transformations between a static spherically symmetric space time and a perturbed FRW spacetime [306].

Note that we may also transform the cosmological solution back to the original FRW background given by (11.40), (11.41) and (11.42). In that case, the potential Φ acquires a pure time-dependence, which is in turn eliminated by a gauge-transformation. This introduces a time-dependence into Ψ and in order to use (11.23) we must determine the canonical form of Ψ as in [306]. This is found to be $\Psi = -\frac{GM(1+\epsilon)}{\bar{\phi}_0 R} - \epsilon H_0^2(\frac{4}{3} + 2H_0 t)R^2$ so that (11.23) along with (11.41) gives back (11.52).

In (11.52) we have found the turnaround radius in terms of the bare parameters of the theory, G and Λ. However, as is well known, the bare G in the Brans-Dicke action is not the actual measured Newtonian gravitational constant G_N. Indeed, the latter is defined as [7, 294, 307]

$$G_N = \frac{2(\omega+2)}{\bar{\phi}_0(2\omega+3)} G \approx \frac{1+\epsilon}{\bar{\phi}_0} G, \tag{11.53}$$

so that $g_{00} \approx -1 + 2G_N M/R$ as $R \to 0$. Hence, $3GM/\Lambda = (1-\epsilon)G_N M/H_0^2$. Furthermore, we should consider how we measure the cosmological constant. The Friedman equation (under the assumption that $\phi \approx$ const) is $3H^2 \approx \Lambda/\bar{\phi}_0 + 8\pi G_N(1-\epsilon)\rho_{\text{matter}}$. Hence, using cosmological observations one would measure $\Lambda_{\text{eff}} = \Lambda/\bar{\phi}_0$ rather than the bare Λ and we call this the effective cosmological constant. With these considerations the expression (11.52) should be adjusted accordingly to

$$R_{\mathtt{TA}} \approx \left(\frac{3G_N M}{\Lambda_{\text{eff}}}\right)^{1/3}\left(1+\frac{2}{3}\epsilon\right) \approx \left(\frac{3G_N M}{\Lambda_{\text{eff}}}\right)^{1/3}\left(1+\frac{1}{3\omega}\right) \tag{11.54}$$

which is our final result.

11.4 The turnaround radius in $f(R)$ gravity

In this section, we compute the turnaround radius in $f(R)$ gravity taking into account two cases:(i) static and spherically symmetric space-time, and (ii) cosmological perturbations.

Briefly, $f(R)$ gravity is obtained by substituting the Hilbert-Einstein action, linear in the Ricci scalar R, with an arbitrary function of R, i.e.

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N} f(R) + \mathcal{L}_m\right). \tag{11.55}$$

By varying the above action with respect to the metric, we get the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)f'(R) = 8\pi G_N T^M_{\mu\nu}\,, \tag{11.56}$$

where $T^M_{\mu\nu}$ is the energy-momentum tensor of matter. Another interesting equation, is the trace of Eq. (11.56),

$$f'(R)R - 2f(R) + 3\Box f'(R) = 8\pi G_N T^M\,, \tag{11.57}$$

which relates the Ricci scalar to $f(R)$ and its derivative in R, that is $f'(R)$.

Several efforts have been done to get constraints on viable $f(R)$ models [79, 274, 321, 336]. In particular, we can assume $f''(R) > 0$, in order to avoid tachyonic instabilities, and $f'(R) > 0$, to make the theory ghost-free.

11.4.1 The spherically symmetric case

As mentioned in the introduction 11.1, we can restrict our attention to $f(R)$ models with constant curvature solutions, i.e. $R = R_{\rm dS} = \text{const.}$. This $R_{\rm dS}$ is the solution of the trace Eq. (11.57), which, for constant curvature, takes the form

$$Rf'(R) - 2f(R) = 0\,. \tag{11.58}$$

Solutions[10] to this equation are known as de Sitter points [337, 338]. Moreover, following [339, 340], nonlinear models should have stable de Sitter points at late times. From the Eq. (11.56), using Eq. (11.58), we get

$$R_{\mu\nu} = \frac{f(R_{\rm dS})}{2f'(R_{\rm dS})} g_{\mu\nu} = \frac{R_{\rm dS}}{4} g_{\mu\nu}\,, \tag{11.59}$$

which gives rise to an effective cosmological constant of the form

$$\Lambda_{eff} = \frac{f(R_{\rm dS})}{2f'(R_{\rm dS})} = \frac{R_{\rm dS}}{4}\,. \tag{11.60}$$

The stability of these points is discussed in [337,341]. They asymptotically approach the (anti)-de Sitter space with $\Lambda = \Lambda_{\rm eff}$.

If we consider a static and spherically symmetric metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2\,, \tag{11.61}$$

the equations of motion (11.56), for both the 00 and 11 components, together with the trace equation, are respectively

$$\left(\frac{1}{2}Af(R) + \frac{A'f'(R)}{rB} - \frac{A'^2 f'(R)}{4AB} - \frac{A'B'f'(R)}{4B^2} + \frac{f'(R)A''}{2B}\right)\Bigg|_{R=R_{\rm dS}} = 0\,, \tag{11.62a}$$

$$\left(-\frac{1}{2}Bf(R) + \frac{A'^2 f'(R)}{4A^2} + \frac{B'f'(R)}{rB} + \frac{A'B'f'(R)}{4AB} - \frac{f'(R)A''}{2A}\right)\Bigg|_{R=R_{\rm dS}} = 0\,, \tag{11.62b}$$

$$\Big(Rf'(R) - 2f(R)\Big)\Big|_{R=R_{\rm dS}} = 0\,. \tag{11.62c}$$

We can easily see, from the first two equations, (11.62a) and (11.62b), that

$$B(r) = \frac{c}{A(r)}\,, \tag{11.63}$$

[10] Such solutions exist in the constant curvature $f(R)$ models and therefore, Birkhoff's theorem is valid [342, 343].

where c is an integration constant. The constant can be set equal to unity to recover the Minkowski space-time asymptotically. Hence, Eq. (11.62b) provides:

$$A(r) = a_1 - \frac{a_2}{r} - \frac{r^2}{6}\frac{f(R)}{f'(R)}\bigg|_{R_{\rm dS}} = a_1 - \frac{a_2}{r} - \frac{R_{\rm dS}}{12}r^2\,, \tag{11.64}$$

where, in the second line, we used Eq. (11.62c), and a_1, a_2 are constants. Without losing generality, we choose $a_1 = 1$ and $a_2 = 2G_{\rm eff}M/c^2$, where $G_{\rm eff}$ is the effective gravitational coupling. This allows to recover a Schwarzschild-like solution in the $R_{\rm dS} \to 0$ limit.

It is worth-mentioning that in the Einstein-Hilbert limit, i.e. $f(R) = R - 2\Lambda$, we can set $R_{\rm dS} = 4\Lambda > 0$, since Λ is positive-definite. As a consequence, we recover the known Schwarzschild-de Sitter solution (11.1).

Finally, the maximum turnaround radius for any $f(R)$ model with $R = R_{\rm dS}$ is given by Eq. (11.5)

$$A'(R_{\rm TA,max}) = 0 \Rightarrow R_{\rm TA,max} = \left(\frac{12G_{\rm eff}M}{R_{\rm dS}}\right)^{1/3}\,. \tag{11.65}$$

The maximum turnaround radius in any alternative theory of gravity can be, at most, 10% smaller than the corresponding one in GR. Thus, by comparing (11.65) with (11.3), we get the following constraint

$$\frac{G_{\rm eff}}{R_{\rm dS}} \geq \frac{0.18G_N}{\Lambda}\,. \tag{11.66}$$

At this point, a short comment on Eq. (11.58) is needed. Solutions $R_{\rm dS} = 0$, are not excluded *a priori*. However, these are trivial Minkowski solutions, instead of de Sitter ones, leading to neither expanding nor contracting universes. In this case, the maximum turnaround radius cannot be defined. In the next section, we will see that scalar cosmological perturbations give the same result as Eq. (11.66), together with a specific form for the gravitational coupling.

11.4.2 The cosmological case

In the previous section, we studied the turnaround radius derived from static and spherically symmetric space-times in $f(R)$ gravity. However, we want to find a more general formula for the turnaround radius and thus we turn to cosmology. Specifically, in this section, we are going to use Eq. (11.23) in order to see, whether we can extend the result (11.66).

To this end, let us consider a spherical cosmic structure described by a perfect fluid with non-relativistic matter, i.e. with pressure $P = 0$, and all its mass is assumed to be at the center, $r = 0$. We perturb this structure by a test fluid and we study its dynamics. The whole configuration can be described by a perturbed FRW metric, which in conformal Newtonian gauge, can be expressed in the form (11.15). For a detailed discussion about the relation between static and comoving coordinates, as well as the relation of the metric (11.15), with the Bardeen potentials, see [331].

The homogeneous background Eqs. (11.56) in a FRW flat space-time

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t))$$

, and with non-relativistic matter, i.e. $T_{\mu\nu} = \mathrm{diag}(-\rho, 0, 0, 0)$, are given by

$$3f'H^2 = 8\pi\rho + \frac{1}{2}(Rf' - f) - 3f''H\dot{R}\,, \tag{11.67a}$$

$$f'\left(2\dot{H} + 3H^2\right) = \frac{1}{2}(Rf' - f) - f'''\dot{R}^2 - f''\ddot{R} \tag{11.67b}$$

together with the continuity equation

$$\dot{\rho} + 3H\rho = 0\,. \tag{11.68}$$

The prime denotes differentiation with respect to R, the dot with respect to time t, and $H = \dot{a}/a$ is the Hubble parameter. In this case, the Ricci scalar is

$$R = 6\left(2H^2 + \dot{H}\right)\,, \tag{11.69}$$

and Eq. (11.68) gives $\rho = \rho_0/a^3$, where ρ_0 is the constant rest-mass density.

The perturbed energy momentum tensor is given by $T_{00} = -\rho - \delta\rho$, $T_{0i} = -\rho\delta\upsilon_i$ and $T_{ij} = 0$. Thus, the perturbed equations are [337, 344, 345]

$$-\frac{\Delta\Phi}{a^2} + 3H\left(H\Psi + \dot{\Phi}\right) = -\frac{1}{2f'}\left[8\pi G_N\delta\rho + \left(3H^2 + 3\dot{H} + \frac{\Delta}{a^2}\right)\delta f' - 3H\left(\delta\dot{f}' - 2\dot{f}'\Psi\right) + 3\dot{f}'\dot{\Phi}\right]\,, \tag{11.70a}$$

$$H\Psi + \dot{\Phi} = \frac{1}{2f'}\left[8\pi G_N\rho\delta\upsilon + \delta\dot{f}' - H\delta f' - \dot{f}'\Psi\right]\,, \tag{11.70b}$$

$$\Phi - \Psi = \frac{\delta f'}{f'}\,, \tag{11.70c}$$

$$\begin{aligned} 3\left(\dot{H}\Psi + H\dot{\Psi} + \ddot{\Phi}\right) + 6H\left(H\Psi + \dot{\Phi}\right) + \left(3H + \frac{\Delta}{a^2}\right)\Psi = \frac{1}{2f'}\Big[8\pi G_N\delta\rho + 3\delta\ddot{f}' + 3H\delta\dot{f}' - \\ - \left(6H^2 + \frac{\Delta}{a^2}\right)\delta f' - 3\dot{f}'\left(\dot{\Psi} + 2H\Psi + \dot{\Phi}\right) - 6\ddot{f}'\Psi\Big]\,, \end{aligned} \tag{11.70d}$$

$$\delta\ddot{f}' + 3H\delta\dot{f}' - \left(\frac{\Delta}{a^2} + \frac{R}{3}\right)\delta f' = \frac{8\pi G_N}{3}\delta\rho + \dot{f}'\left(\dot{\Psi} + 6H\Psi + 3\dot{\Phi}\right) + 2\ddot{f}'\Psi - \frac{1}{3}f'\delta R\,, \tag{11.70e}$$

where

$$\delta R = -2\left[3\left(\dot{H}\Psi + H\dot{\Psi} + \ddot{\Phi}\right) + 12H\left(H\Psi + \dot{\Phi}\right) + \frac{\Delta}{a^2}\left(\Psi - 2\Phi\right) + 3\dot{H}\Psi\right]. \tag{11.71}$$

As before, $f' = df/dR$, dot denotes differentiation with respect to cosmic time t and Δ is the Laplacian in comoving coordinates.

We can safely use the quasi-static approximation [346, 347] according to which the inhomogeneities Ψ and Φ are primarily produced by the spatial distribution of matter. This means that the spatial derivatives of the fields are dominant in the equations (see [337]). In this way, Eq. (11.70a), (11.70d), (11.70e) yield[11]

$$\frac{\Delta\Phi}{a^2} = \frac{1}{2f'}\left[8\pi G_N\delta\rho + \frac{\Delta\delta f'}{a^2}\right], \tag{11.72a}$$

$$\frac{\Delta\Psi}{a^2} = \frac{1}{2f'}\left[8\pi G_N\delta\rho - \frac{\Delta\delta f'}{a^2}\right], \tag{11.72b}$$

$$-\frac{\Delta\delta f'}{a^2} - \frac{R\delta f'}{3} = \frac{8\pi G_N\delta\rho}{3} - \frac{1}{3}f'\delta R, \tag{11.72c}$$

where $\delta R = -\frac{2\Delta}{a^2}\left(\Psi - 2\Phi\right)$. By using $\delta f' = f''\delta R$, (11.72a) and (11.72b) immediately give

$$\Phi = \frac{f''}{2f'}\delta R + \phi\,, \quad \Psi = -\frac{f''}{2f'}\delta R + \phi\,, \tag{11.73}$$

where ϕ satisfies the Poisson-like equation

$$\Delta\phi = \frac{4\pi G_N a^2}{f'}\delta\rho\,. \tag{11.74}$$

In addition, (11.72c) gives

$$\left(\Delta - a^2M^2\right)\delta R = -\frac{8\pi G_N a^2}{3f''}\delta\rho\,, \tag{11.75}$$

where we defined the mass term (of the scalar degree of freedom) as

$$M^2 = \frac{1}{3}\left[\frac{f'(R_{\rm dS})}{f''(R_{\rm dS})} - R_{\rm dS}\right]. \tag{11.76}$$

We notice that Eq.(11.75) is a modified-Helmholtz equation and, if we set $\delta\rho(r) \sim \mathcal{M}\delta(r)/a^3$, where $\mathcal{M}$ is the mass of the structure, Eq. (11.75) and Eq. (11.74) give respectively

$$\delta R = \frac{2G_N\mathcal{M}}{3f''ar}e^{-aMr}\,, \quad \phi = -\frac{G_N\mathcal{M}}{f'ar}\,, \tag{11.77}$$

[11] Eq. (11.70b) means that, in the considered order of approximation, the peculiar velocities of the test fluid are ignored and Eq. (11.70c) is in the correct order and it is satisfied, *a posteriori*, by Eq. (11.73).

where we assumed that $\delta R \to 0$ when r is large. Finally, the gravitational potentials from (11.73) become

$$\Psi = -\frac{G_N \mathcal{M}}{f' a r}\left(1 + \frac{e^{-aMr}}{3}\right), \; \Phi = -\frac{G_N \mathcal{M}}{f' a r}\left(1 - \frac{e^{-aMr}}{3}\right). \tag{11.78}$$

Before proceeding to the calculation of the turnaround radius through Eq.(11.23), let us comment on these results. First of all, Eq. (11.78) shows that the effective gravitational coupling is

$$G_{\text{eff}} = \frac{G_N}{f'(R_{\text{dS}})}\left(1 + \frac{e^{-Mr_{ph}}}{3}\right), \tag{11.79}$$

where $r_{ph} = ar$ is the physical distance. So, even in the weak field limit, deviations from GR, i.e. $f'(R) \neq 1$, become evident. Apart from this, we see that the inhomogeneities caused by the test fluid, together with the non-linearities of the theory, contribute as a Yukawa-like correction to the gravitational fields. This has been observed in different contexts [128, 348]. Such corrections can effectively explain the flat rotation curves of galaxies, without invoking any exotic form of matter [349]. Moreover, it is worth noticing that, with growing r, the Yukawa corrections vanish, while Ψ and Φ evolve towards a McVittie-like form, which is expected in scalar cosmological perturbations.

Let us proceed now with the calculation of the turnaround radius. Eq. (11.23) gives

$$r_{ph}^3 = \frac{12 G_N \mathcal{M}}{R_{\text{dS}} f'(R_{\text{dS}})}\left[1 + \frac{e^{-Mr_{ph}}}{3}\left(1 + \frac{1}{3} M r_{ph}\right)\right]. \tag{11.80}$$

As already mentioned, M is the effective mass related to the further degree of freedom of $f(R)$ gravity. Thus, $Mr \ll 1$ means that the mass of the scalar field is small and the related effective length is very large with respect to the the Solar System scale. On the other hand, if the mass M is large, it cannot have observable effects at late times[12]. Hence, Eq. (11.80) becomes of zeroth order in Mr

$$R_{\text{TA,max}} \simeq \left(\frac{12 G_N \mathcal{M}}{R_{\text{dS}} f'(R_{\text{dS}})}\right)^{1/3}. \tag{11.81}$$

Clearly, Eq. (11.81) is the same of Eq. (11.65) for $G_{\text{eff}} = G_N / f'(R_{\text{dS}})$, which is (11.79) for $Mr \ll 1$. Thus the constraint (11.66) becomes

$$R_{\text{dS}} f'(R_{\text{dS}}) \leq 5.48 \Lambda. \tag{11.82}$$

[12] One can claim that the mass M is a function of the curvature, and then of the energy density, and thus it is small at cosmological scale and large at Solar System scales like in the so-called Chameleon Mechanism [72, 350].

The maximum turnaround radius can be used to set a further criterion for the viability of $f(R)$ models, through the stability of cosmic structures. As we already mentioned in Sec. 11.4.1, for late times, when the matter density becomes negligible compared to Λ, the Ricci curvature scalar takes the value $R_{\rm dS} = 4\Lambda$; thus the upper bound for the first derivative of the model is $f'(R_{\rm dS}) \leq 1.37$ (which, of course, in GR is $f'(R_{\rm dS}) = 1$). Summarizing, viable $f(R)$ models are those which obey the following criteria

$$0 < f'(R) \leq 1.37\,, \quad f''(R) > 0 \text{ for } R \geq R_{\rm dS} \geq 0\,. \tag{11.83}$$

As an example we can consider a power-law model

$$f(R) = R + \alpha R^n\,, \quad n > 0\,, \tag{11.84}$$

where n is a positive real number and α is a dimensional constant. For $n = 0$, α plays the role of cosmological constant. In the literature, there are several constraints on n (e.g. [351] and references therein) and thus we will consider only $n > 2$. From the constraint (11.83), we find, for example, that for $n = 3$, it is $\alpha \lesssim 1/(20\Lambda^2)$, as shown in Fig.1.

11.5 Conclusions

In the second part of the thesis, we focused mainly on cosmological models and spacetimes. Indeed, most of the (infrared) modifications of gravity emanate from cosmology. However, it is equally important for a theory to be consistent also with astrophysical observations. That is why in this third part of the thesis, we focus on studying the stability of large scale structures in alternative theories of gravity.

Specifically, we have calculated the effect of generic alternative theories of gravity obeying the Einstein Equivalence Principle maximum size of a structure is given by the maximum turnaround radius $R_{\tt TA}$ – the point where the attraction due to the central mass gets balanced with the repulsion due to the dark energy, beyond which no compact mass distribution is possible. Thus any model predicting a maximum size of a structure with a given mass smaller than its actual observed size, gets ruled out on the basis of the stability of the structure. Conversely, if a given theory predicts a maximum size larger than the actual or observed size, the theory certainly persists. The theoretical prediction of ΛCDM on $R_{\tt TA}$ was shown to be absolutely consistent with the observed astrophysical data [281, 285], and it is only about 10% larger than the observed ones for large scale structures with $M \geq 10^{13} M_\odot$ which are yet to virialize and much larger for masses below that [286]. Thus, it is clear

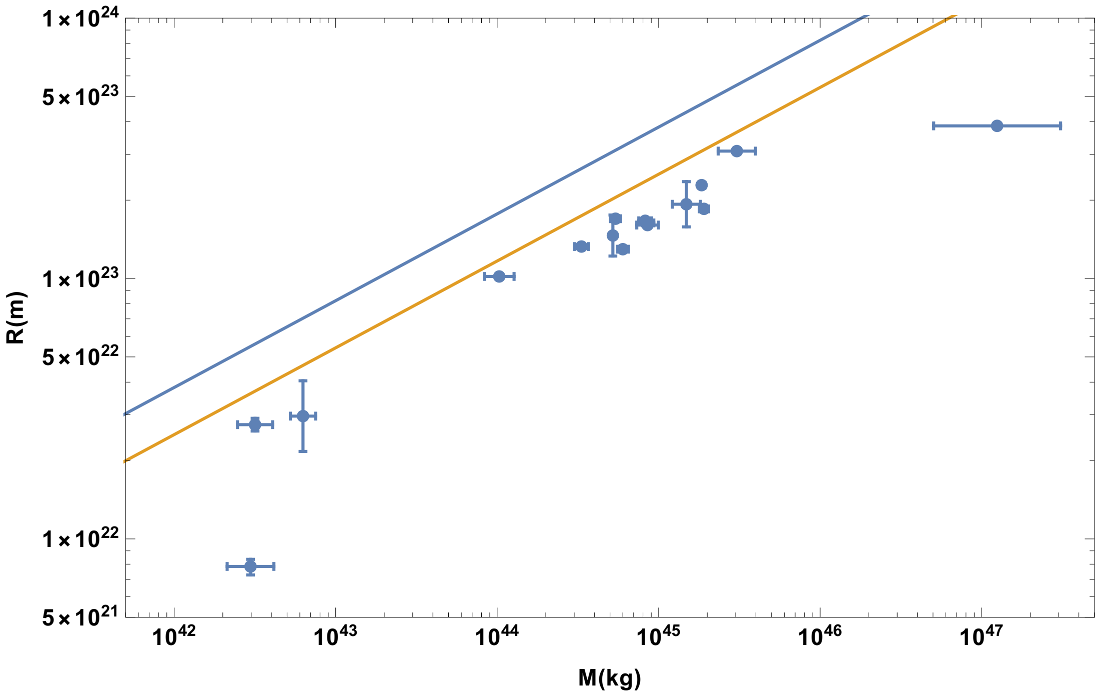


Figure 11.1: In the picture, the observational values of the turnaround radius for some astrophysical structures are reported. Details about the used data can be found in [323,324] and references therein. The theoretical bound of the maximum turnaround radius in ΛCDM model (blue line), as well as in the power-law $f(R)$model with $n = 3$ (orange line) are reported.

that in order to have a meaningful phenomenolo on the maximum size of large scale cosmic structures. Thegy with the maximum turnaround radius to constrain various models, we must consider large scale objects with $M \geq 10^{13} M_{\odot}$. In particular, such consideration completely rules out dark energy models with equation of state parameter $w < -2$ [285].

We have introduced a new definition of the maximum turnaround radius, given by the turnaround equation (11.21), valid in any theory of gravity obeying the EEP and for any non-spherical bound object. We have further adopted (11.21) under the simplified assumptions of a spherically symmetric setup and a time-dependent cosmological setup with spherically symmetric perturbations arriving at the same conclusions. In both cases we deal with spherical symmetry. As we discussed above, since the large scale structures we should apply the turnaround calculation to are yet to virialize, spherical symmetry seems to be a very good approximation for our current purpose. The members of such a structure would redistribute their kinetic energy in order to reach virialization and the structure would get smaller in size. Thus, non-sphericity would eventually be created, but at a later time. In particular, it was argued in [281] that even the maximum departure from non-sphericity is not very large for most of those structures, except that of the Corona-Borialis supercluster – which may not be a single structure at all. Nevertheless, it is quite instructive and interesting to extend the current formalism to include non-sphericity as well. One possibility is to start from the general turnaround equation (11.21) and consider a non-spherical function $\Theta(t,\vec{x})$, possibly corresponding to some non-spherical surface. Another possible way to do this without adhering to perturbation theory, would be to consider an axisymmetric generalization of the McVittie solution we investigated by putting in a rotation and also to consider the Sheth-Tormen statistical mass function instead of the Press-Schechter statistical mass function (see e.g. [308]) in the analysis of [286].

The most important point we have demonstrated is that the turnaround radii predicted by both spherically symmetric and cosmological spacetimes are the same – establishing it as a purely geometric, coordinate invariant quantity. Such equality was earlier established for ΛCDM in [281, 285]. As an application, we used the formalism in the context of the Brans-Dicke theory with a positive cosmological constant. Owing to the severe constraint of the Brans-Dicke parameter from the solar system data, $\omega \gtrsim 40000$ [294], we used a perturbative expansion in the Brans-Dicke parameter in terms of $\epsilon = 1/(2\omega+3)$ and showed that the maximum turnaround radius is always larger than that of the ΛCDM, Eq. (11.54) since our formula is only valid for $\omega \gg 1$. The increment of $R_{\mathtt{TA}}$ from the ΛCDM is apparent from Eq. (11.54) – depicting the increment of the term $G_N M$ for a finite and positive ω, keeping Λ fixed. The physical meaning behind this is related to the fact that since the gravitational attraction in Brans-Dicke is increased compared to GR (due to the additional scalar mediating gravity), we should move further radial distance away than ΛCDM in order to get it balanced by the repulsion of the dark energy whose value is being fixed. In other words, the maximum size of a structure with given mass should be regarded as the maximum length scale up to which it can hold itself against the repulsion due to the ambient dark energy. If we specify the latter, certainly $R_{\mathtt{TA}}$ would increase with increasing mass or gravitational coupling.

Furthermore, we studied whether it is possible to constrain $f(R)$ invoking the maximum turnaround radius. To do so, we considered two approaches: the first concerning a spherically symmetric metric and the second adopting cosmic perturbations. In both cases, we got analogous outcomes which allow the existence of stable structures according to a stability criterion which is $f'(R_{\rm dS}) \leq 1.37$.

Another important point to note here that we have used the definition of the mass and the cosmological constant as that of the General Relativity in Eq. (11.54). Certainly, this should not be the case in general and such parameters should be defined within the framework of the theory itself. However, as long as we are doing perturbation theory over ΛCDM, such notion seems practically reasonable. Similar considerations within the Brans-Dicke theory in the context of the Parameterized Post-Newtonian formalism can be found in [7, 307]. In any case, our result shows that the Brans-Dicke theory is perfectly consistent with the mass versus observed maximum sizes and hence the stability of structures.

It would be highly interesting to go beyond the first order perturbation theory considered here, in order to further investigate the stability issues. We hope to return to this in a future work.

Chapter 12

Epilogue

12.1 Summary

Let us now summarize in brief the work presented in this thesis. In chapter 1 we motivated the thesis, describing the current picture of gravity through its journey in history. We refered to what the theory implies at both astrophysical and cosmological scales and we discussed its riddles. Then, we splited the thesis into three parts: in the first one, we presented the geometric foundations of gravity and we formulated the Geometric Trinity of gravity, i.e. GR, TEGR and STEGR; in the second one, we used Noether's theorem to investigate and classify different theories of gravity in the cosmological minisuperspace and in the last one, we tried to set some astrophysical constraints on alternative theories through the notion of the turnaround radius.

In particular, in chapter 2 we presented the necessary criteria that a theory of gravity should satisfy and we discussed the affine structure of spacetime. Then, in the next three chapters, i.e. chapter 3, chapter 4 and chapter 5, we formulated the three alternative representations of gravitational interactions that are dynamically equivalent, General Relativity, Teleparallel and Symmetric Teleparallel gravity. The first one describes gravity as the effect of the curvature of spacetime, while the other two suggest that gravity is mediated through the torsion and non-metricity of spacetime respectively. We also discussed possible modifications of each of the above that have been considered in the literature.

In the first chapter of the second part, 6, we presented the so-called Noether Symmetry Approach (NSA). We know that symmetries play a significant role in field theories and for this reason we use the NSA as a geometric criterion to select theories of gravity. It turns out that the Lie and Noether symmetries of 2nd order differential equations can help us classify dynamical systems that are invariant under point transformations. Moreover, we can calculate the invariant of each symmetry and use them to reduce the dynamics of the system, in order to find exact solutions.

In the rest four chapters of the second part, we applied this method to several modified theories of gravity. In particular, in chapter 7, we considered the most general scalar-tensor theory (with a single scalar field) that leads to second order equations of motion, i.e. Horndeski's theory, and using the Noether Symmetry Approach we found those models that accept Noether Symmetries in cosmology. Furthermore, we mapped known scalar-tensor theories to the Horndeski action and found the necessary conditions that have to be satisfied to have cosmological solutions. In chapter 8 we classified those models of $f(R, \mathcal{G})$ gravity, where R is the Ricci scalar and $\mathcal{G}$ is the Gauss-Bonnet topological invariant, that have Noether symmetries in a cosmological minisuperspace. Those models of pure $f(\mathcal{G})$ gravity in a spherically symmetric spacetime were also considered. In both cases, we used the symmetries of the models to find exact solutions. In addition, in chapter 9, we studied the symmetries of the Teleparallel Gauss-Bonnet gravity, which is the teleparallel equivalent of $f(R, \mathcal{G})$ gravity and in chapter 10, we did the same for the non-local Teleparallel theories. All these theories are motivated by current observations.

A "good" theory of gravity, however, should behave correctly at astrophysical scales too. That is why, in the last part and specifically in chapter 11, we used the maximum turnaround radius of structures, which denotes the maximum size that a structure can have, as a stability criterion of the large scale cosmic structures, in order to test alternative theories of gravity. We derived a general formula for those theories that respect the Einstein Equivalence principle and we studied two examples: the Brans-Dicke theory and the $f(R)$ class of theories.

12.2 Conclusions and Future Aspects

One could reasonably argue, if the two alternatives of GR, i.e. TEGR and STEGR, are equivalent to it, then what is the motivation of studying them? And it turns out the advantages are a lot.

First of all, General Relativity is based on the *universality of free fall*, that is intrinsically prescribed in the weak equivalence principle. It is the only interaction in nature though, that exhibits it; no other fundamental interaction does. Teleparallel theories, being gauge theories can come along with universality, but can absolutely survive without it too. It is also argued [367] that since Newton's theory complies with non-universality, it is a more natural non-relativistic limit in Teleparallel theory than in GR. Furthermore, the strong equivalence principle of GR, establishes the local equivalence between gravity and inertia. This makes it impossible to reconcile with quantum mechanics, whose basic asset is the uncertainty principle that is fundamentally nonlocal, i.e. test particles do not follow a specific trajectory, but instead, infinitelly many trajectories, with different probabilities.

Subsequently, the physical description of gravity is totally different in Teleparallel gravity; curvature is replaced by torsion and geometry by a force. That is because teleparallelism is a gauge theory of the translations in Minkowski spacetime, i.e. the tangent space of each point of any spacetime. This gives two distinct characteristics to teleparallel theory: the first one is that it explains why gravitation has the Noether current of translations for source energy-momentum. The second one is that it differs from $U(1)$ or from Yang-Mills in the sense that the tangent bundle is soldered and not internal. However, it keeps every property that a gauge theory should have. Effectively, this means that it is more appropriate to be unified with the other three fundamental interactions, in contrast with GR, and in addition it is easier to be quantized.

Finally, a further advantage of teleparallel theories is that, it does not encapsulate the problems of the spin-2 theory constructed on the framework of GR. Specifically, being a gauge theory it does not geometrize the gravitational interaction and thus it is more natural to define a spin-2 field in its context.

That being said, the author hopes to have contributed a small piece to the huge structure of gravitation theories existing in the literature. It maybe proven in a couple of years that GR is the final theory of gravity (even though current observations do not converge to that), but even in that case we will have learned its limitations and we will have secured our faith in it. On the other hand, if the opposite is proven, then all the existing modifications of standard GR would be nothing more than toy-theories. In this sense, teleparallel theories could play a very important role. A trivial but realistic example is that, if for some reasons the equivalence principle will be disproved, then GR becomes useless. In that case, teleparallel theories could replace it.

All in all, it is very well known so far that GR together with the concordance model in cosmology, ΛCDM, have been really successful with experiments and observations. It is also true, that cosmology, astrophysics and high energy physics motivate the hunt for an alternative description of gravitational interactions. Up to now, a plethora of theories has been proposed and studied in this sense and of course it has been made significant progress in the field. However, we would connive if we would not notice that none of the proposed theories can be as successful as GR. The reason for this is even deeper; the common method to the day, to formulate a modified theory of gravity, is the so-called *trial and error* method. In practice, this is when one tries to violate one of the basic assumptions of GR and studies if the resulting theory is valid or not. Nevertheless, it may be the case that we should change our approach, chasing more the fundamental, underlying principles that a *good* theory of gravity should have. The future of gravity and cosmology is unknown; but it is in our hands to solve the current problems. Martin Amis (British novelist) recently claimed that “we’re five Einstein’s away from explaining the Universe’s existence”. We can either prove him right, by continuing to work wearing blinkers [369] or we can seize the day, following the hardest path, hoping that it leads us to the promised land. It is our decision.

Appendix A

Perturbative solution of the background FRW in Brans-Dicke theory

In this appendix we give a formal derivation of the perturbative background FRW solution presented in section 11.3.2. We give the derivation for a general matter source in the presence of curvature and specialize at the end to a constant-w component in a flat universe.

We eliminate the t-dependence in the background field equations by changing variables from t to $\ln a$ so that the Friedman equation (11.37) becomes

$$H^2 = \frac{\frac{8\pi G}{\bar{\phi}}\bar{\rho} - \frac{3\kappa}{a^2}}{3\left(1 + \frac{1}{2}\frac{d\ln\bar{\phi}}{d\ln a}\right)^2 - \frac{1}{4}\frac{1}{\epsilon}\left(\frac{d\ln\bar{\phi}}{d\ln a}\right)^2} \tag{A.1}$$

while the scalar equation (11.38) can be formally integrated to

$$\bar{\phi} = \bar{\phi}_0 + \epsilon\, 8\pi G \int d\ln a \frac{1}{a^3}\frac{1}{H}\int d\ln a(\bar{\rho} - 3\bar{P})\frac{a^3}{H} \tag{A.2}$$

We have set the initial condition $\dot{\bar{\phi}}^{(in)}$ to zero as it leads to a decaying solution.

The calculation now proceeds by expanding the fields as

$$\bar{\phi} = \bar{\phi}_0\left(1 + \sum_{n=1}^{\infty}\bar{\phi}_n\epsilon^n\right) \tag{A.3}$$

$$H = \bar{H}\left(1 + \sum_{n=1}^{\infty}h_n\epsilon^n\right) \tag{A.4}$$

where $\bar{H}$ is the time-dependent Hubble parameter in the limit $\epsilon \to 0$ (not to be confused with the Hubble constant H_0) and is given by $3\bar{H}^2 = \frac{8\pi G}{\bar{\phi}_0}\bar{\rho} - \frac{3\kappa}{a^2}$.

Let us define the operator $S[A,B]$ acting on functions A and B by

$$S[A,B] = \frac{8\pi G}{\bar{\phi}_0} \int d\ln a \frac{1}{a^3}\frac{1}{\bar{H}} A \int d\ln a (\bar{\rho} - 3\bar{P}) \frac{a^3}{\bar{H}} B. \tag{A.5}$$

This operator is then used to construct the perturbed variables $\bar{\phi}_n$ from the scalar integral (A.2). The first three expansion coefficients are

$$\bar{\phi}_1 = S[1,1] \tag{A.6a}$$
$$\bar{\phi}_2 = -S[1,h_1] - S[h_1,1] \tag{A.6b}$$
$$\bar{\phi}_3 = S[1,h_1^2 - h_2] + S[h_1,h_1] + S[h_1^2 - h_2, 1] \tag{A.6c}$$
$$\dots$$

The Friedman equation (A.1) is also perturbed to give

$$h_1 = -\frac{1}{2}\bar{\chi}_1\left(1 - \frac{1}{12}\bar{\chi}_1\right) - (1-\Omega_K)\frac{1}{2}\bar{\phi}_1 \tag{A.7a}$$

$$\begin{aligned} h_2 = &-\frac{1}{2}\left(\bar{\chi}_2 - \bar{\phi}_1\bar{\chi}_1 + \frac{1}{4}\bar{\chi}_1^2\right) + \frac{1}{12}\bar{\chi}_1\left(\bar{\chi}_2 - \bar{\phi}_1\bar{\chi}_1\right) + \frac{3}{8}\bar{\chi}_1^2\left(1 - \frac{1}{12}\bar{\chi}_1\right)^2 \\ &+ \frac{1}{2}(1-\Omega_K)\left[\frac{3}{4}\bar{\phi}_1^2 - \bar{\phi}_2 + \frac{1}{2}\bar{\phi}_1\bar{\chi}_1\left(1 - \frac{1}{12}\bar{\chi}_1\right) + \frac{1}{4}\Omega_K\bar{\phi}_1^2\right] \end{aligned} \tag{A.7b}$$

$$\dots$$

where $\bar{\chi}_n = d\bar{\phi}_n/d\ln a$ and $\Omega_K = \kappa a^{-2}/\bar{H}$. The final solution is constructed from (A.6) and (A.7) with the help of (A.5). In particular one proceeds as $\bar{\phi}_1 \to h_1 \to \bar{\phi}_2 \to h_2 \to \dots$ and so forth.

A particular case of interest is a flat universe with $\Omega_K = 0$ and matter with constant equation of state w. Then

$$\bar{\phi} = \bar{\phi}_0\left[1 + 2(\alpha + \alpha^2 + \alpha^3)\ln a + (2\alpha^2 + 4\alpha^3)\ln^2 a + \frac{4}{3}\alpha^3 \ln^3 a + \dots\right] \tag{A.8}$$

and

$$H = \bar{H}\left[1 - \frac{1}{6}\frac{5-3w}{1-w}\alpha - \frac{(1-3w)(3-w)}{24(1-w)^2}\alpha^2 - \left(\alpha + \frac{(1-3w)}{6(1-w)}\alpha^2\right)\ln a + \frac{1}{2}\alpha^2\ln^2 a + \dots\right] \tag{A.9}$$

where

$$\alpha = \frac{1-3w}{1-w}\epsilon \tag{A.10}$$

Clearly, in a radiation dominated Universe, the solution is $\bar{\phi} = constant$ and $H = \bar{H}$ as we would expect from the fact that the scalar couples to the trace of the energy-momentum tensor.

Imposing $w = -1$ in (A.8) and (A.9) and keeping terms to $\mathcal{O}(\epsilon)$ gives (11.40) and (11.41) after letting $\bar{H} = H_0 = \sqrt{\frac{\Lambda}{3\bar{\phi}_0}}$.

Appendix B

Basic variations

In this chapter we will provide all the necessary variations that were implicitly, or explicitly used throughout the thesis. Based on these variations one can calculate more complicated equations of motions, such as $f(R)$, $f(T)$, $f(\mathcal{G})$ theories, as well as theories with (non-)minimally coupled scalar fields.

B.1 Metric variations

Consider an $n-$dimensional (pseudo-)Riemaniann manifold described by a metric $g_{\mu\nu}$. Its inverse is defined through

$$g_{\mu\nu} g^{\alpha\nu} = \delta^{\alpha}_{\mu} \,, \tag{B.1}$$

where all the indices run from 0 to n. Varying that, and contracting with $g_{\alpha\beta}$ we get

$$\begin{aligned} (\delta g_{\mu\nu})\, g^{\alpha\nu} + g_{\mu\nu} \delta g^{\alpha\nu} &= 0 \Rightarrow \\ (\delta g_{\mu\nu})\, \delta^{\nu}_{\beta} + g_{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\nu} &= 0 \Rightarrow \\ \delta g_{\mu\beta} &= -g_{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\nu} \,. \end{aligned} \tag{B.2}$$

It is easy to see from (B.1) that, $g_{\mu\nu} g^{\mu\nu} = n$. Varying that, we get

$$g^{\mu\nu} \delta g_{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu} \,. \tag{B.3}$$

On the same track, the determinant of the metric is defined as

$$g = \det(g_{\mu\nu}) \,, \tag{B.4}$$

and from Jacobi's formula we easily see that

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu} \,. \tag{B.5}$$

Using this together with (B.3), we get

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\delta g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}\,(g_{\mu\nu}\delta g^{\mu\nu})\ . \tag{B.6}$$

The variations of the Christoffel symbols, Ricci and Riemann tensor can be found in any basic textbook of GR so their calculations will be omitted. We just write them down for completeness

$$\delta\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}\left[\nabla_{\beta}(\delta g_{\gamma\lambda}) + \nabla_{\gamma}(\delta g_{\beta\lambda}) - \nabla_{\lambda}(\delta g_{\beta\gamma})\right]\,, \tag{B.7}$$

$$\delta R^{\alpha}{}_{\beta\gamma\delta} = 2\nabla_{[\gamma}\delta\Gamma^{\alpha}{}_{\delta]\beta} = \nabla_{\gamma}\delta\Gamma^{\alpha}{}_{\delta\beta} - \nabla_{\delta}\delta\Gamma^{\alpha}{}_{\gamma\beta}\,, \tag{B.8}$$

$$\delta R_{\mu\nu} = \delta R^{\alpha}{}_{\mu\alpha\nu}\,. \tag{B.9}$$

In the Gauss-Bonnet scalar, contractions of the Ricci and Riemann tensor appear, so it would be interesting to obtain their variations. For the “squared” Riemann tensor we have

$$\begin{aligned}\delta(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) &= R_{\mu\nu\rho\sigma}\delta R^{\mu\nu\rho\sigma} + R^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma}\\ &= 2R^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma} + 4R_{\mu}{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\delta g^{\mu\nu}\\ &= 2R_{\alpha}{}^{\nu\rho\sigma}\delta R^{\alpha}{}_{\nu\rho\sigma} + 2R_{\mu}{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\delta g^{\mu\nu}\,.\end{aligned} \tag{B.10}$$

Finally, for contractions of the Ricci tensor we have similarly

$$\delta(R_{\mu\nu}R^{\mu\nu}) = R_{\mu\nu}\delta R^{\mu\nu} + R^{\mu\nu}\delta R_{\mu\nu} = 2R^{\mu\nu}\delta R_{\mu\nu} + 2R_{\mu\alpha}R^{\alpha}{}_{\nu}\delta g^{\mu\nu}\,. \tag{B.11}$$

B.2 Tetrad variations

Tetrad variations may not be so common in the literature. However, using the relations between metric and tetrad, one can obtain all the necessary results in a somewhat cumbersome but straightforward way.

Let us see some examples. The action of the Teleparallel Equivalent of General Relativity is given by

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4xeT + \int d^4xe\mathcal{L}_{\mathrm{mat}}\,, \tag{B.12}$$

where $\mathcal{L}_{\mathrm{mat}}$ is the matter Lagrangian. Varying (B.12) we get

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\,((\delta e)T + e\delta T) + \int d^4x\delta(e\mathcal{L}_{\mathrm{mat}})\,. \tag{B.13}$$

As we said before, using the relation $g^{\mu\nu} = \eta^{ab}E^{\mu}_{a}E^{\nu}_{b}$, it is easy to obtain

$$\delta g^{\mu\nu} = -\left(g^{\nu\beta}E^{\mu}_{a} + g^{\mu\beta}E^{\nu}_{a}\right)\delta e^{a}_{\beta}\,, \tag{B.14}$$

$$\delta e = eE^{\beta}_{a}\delta e^{a}_{\beta}\,. \tag{B.15}$$

In addition, in analogy with (B.1) we can also obtain from $E^\mu_m e^m_\nu = \delta^\mu_\nu$, the following

$$\delta E^\lambda_m = -E^\lambda_n E^\mu_m \delta e^n_\mu \quad \text{and} \quad \partial_\nu E^\lambda_m = -E^\lambda_n E^\mu_m \partial_\nu e^n_\mu \,. \tag{B.16}$$

Now, the second term in (B.13) gives

$$e\delta T = e\left(\frac{1}{4}\delta(T^{\mu\nu\beta}T_{\mu\nu\beta}) + \frac{1}{2}\delta(T^{\mu\nu\beta}T_{\nu\mu\beta}) - \delta(T^\mu T_\mu)\right) \,, \tag{B.17}$$

where by using the above we find that

$$\delta T^\beta{}_{\mu\nu} = -E^\beta_a T^\sigma{}_{\mu\nu}\delta e^a_\sigma + E^\beta_a(\partial_\mu \delta e^a_\nu - \partial_\nu \delta e^a_\mu) \,, \tag{B.18}$$

$$\delta T^\mu = -\left(E^\mu_a T^\sigma + g^{\mu\sigma}T_a + T^\sigma{}_a{}^\mu\right)\delta e^a_\sigma + g^{\mu\nu}E^\sigma_a\left(\partial_\sigma \delta e^a_\nu - \partial_\nu \delta e^a_\sigma\right) \,. \tag{B.19}$$

Coming back to (B.17), the three terms in the brackets are now expressed as

$$\delta(T^{\mu\nu\beta}T_{\mu\nu\beta}) = -4T^{\mu\nu\beta}T_{\mu\nu\sigma}E^\sigma_a \delta e^a_\beta + 4T_\mu{}^{\nu\beta}E^\mu_a \partial_\nu \delta e^a_\beta \,, \tag{B.20}$$

$$\delta(T^{\mu\nu\beta}T_{\nu\mu\beta}) = 2\left(T^{\lambda\nu\mu} - T^{\mu\nu\lambda}\right)T_{\nu\mu\beta}E^\beta_a \delta e^a_\lambda + \left(T^\mu{}_\nu{}^\lambda - T^\lambda{}_\nu{}^\mu\right)E^\nu_a \partial_\mu \delta e^a_\lambda \,, \tag{B.21}$$

$$\delta(T^\mu T_\mu) = -2\left(T^\lambda T^\beta{}_{\lambda\mu} + T^\beta T_\mu\right)E^\mu_a \delta e^a_\beta + 2\left(T^\beta E^\mu_a - T^\mu E^\beta_a\right)\partial_\beta \delta e^a_\mu \,. \tag{B.22}$$

If we integrate out all the boundary terms, finally we get

$$e\delta T = 4\left(-\partial_\mu(eS_a{}^{\mu\beta}) + eT^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu}\right)\delta e^a_\beta \,, \tag{B.23}$$

and the equations of motion of TEGR are given by

$$\frac{4}{e}\partial_\mu(eS_a{}^{\mu\beta}) - 4T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu} - TE^\beta_a = 16\pi G_N \mathcal{T}^\beta_a \,, \tag{B.24}$$

where we defined the energy-momentum tensor to be

$$\mathcal{T}^\beta_a = \frac{1}{e}\frac{\delta(e\mathcal{L}_{\text{mat}})}{\delta e^a_\beta} \,. \tag{B.25}$$

Appendix C

Notetion and Conventions

Here we will spell out the conventions and definitions used throughout the thesis. As already known, the three different representations of gravity were described. This means that different geometries and thus different connections were used to calculate all the necessary quantities. For this reason, we have to be careful and precise with the notation, in order to distinguish between the mathematical tools used and this is the reason which we include all the notations and conventions used throughout the thesis.

We use the letters/symbols $R_{\mu\nu}, \nabla_\alpha$, *etc* to denote the Ricci tensor, the covariant derivative and so on, that are defined through a general affine connection $\Gamma^\alpha{}_{\mu\nu}$. Moreover, symbols with an open circle on top, i.e. $\overset{\circ}{R}_{\mu\nu}, \overset{\circ}{\nabla}_\alpha$, *etc* are defined through the Levi-Civita connection $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = \{^\alpha{}_{\mu\nu}\}$; while others with a bullet on top, i.e. $\overset{\bullet}{R}_{\mu\nu}$, *etc* are defined through the Weitzenbock connection $\overset{\bullet}{\Gamma}{}^\alpha{}_{\mu\nu}$. However, in chapters like 8, 7, 11 we use only curvature based theories, so it is redundant to use the $\circ$ and therefore we omit it. In the same spirit, we do the same in chapter 9 where we use only quantities in teleparallel geometry.

Other symbols are denoted as follows:

$g^{\mu\nu}$:	Lorentzian metric,
g:	Determinant of $g^{\mu\nu}$,
$(\mu\nu)$:	Symmetrization over the indices μ and ν,
$[\mu\nu]$:	Anti-symmetrization over the indices μ and ν,
$e^i{}_\mu$:	Tetrad field,
$E^\mu{}_i$:	Inverse tetrad field,
e:	Determinant of tetrad field,
$\mathcal{S}_m$:	Matter action,
$\mathcal{L}_m$:	Matter lagrangian density,
$T^M_{\mu\nu}$:	Energy-momentum tensor of matter fields.

Finally, in the second and third parts whenever we study curvature based theories, we use the, usual in cosmology, signature $(-+++)$. However, when we discuss teleparallel theories, it is better to use the opposite one, i.e. $(+---)$. The speed of light c is set to unity throughout the thesis and Newton's constant is denoted by G_N.

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