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Past Lifetime and Inactivity Time: from Entropy to Coherent Systems

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Contents

Introduction

Information Theory was originally proposed by Claude Shannon in 1948 in the landmark paper entitled "A Mathematical Theory of Communication"[1]. In this paper the concept of entropy was adopted for the first time in a field other than thermodynamics and statistical mechanics. Since then, the interest in entropy has grown more and more and the current literature now focuses mainly on the analysis of residual lifetime, because "in survival analysis and in life testing one has information about the current age of the component under consideration"[2]. In recent years the interest has 'changed direction'. New notions of entropy have been introduced and are used to describe the past lifetime and the inactivity time of a given system or of a component that is found not to be working at the current time. Moreover inferences about the history of a system may be of interest in real life situations. So, the past lifetime and the inactivity time can also be analysed in the context of the theory of coherent systems.

The present work deals with the concepts of past lifetime and inactivity time in different contexts. I try to retrace my own course of the research during the three years of my Phd study. So the discussion unfolds as follows.

Chapter 1 - a prelude - presents a brief history of the concept of entropy, from the beginning to recent developments; the chapter also contains an introduction of basic concepts of the theory of reliability.

Chapter 2 discusses the state of art about past entropies. In particular the past entropy [10] and the cumulative past entropy (CPE) [12] are analysed in detail. But the question that arises may be:"Why past entropy?". It is reasonable to presume that in many realistic situations uncertainty is not necessarily related to the future but can also be referred to the past. For instance, if at time t, a system is found

[1] Shannon CE, A mathematical theory of communication, *Bell System Technical Journal* 27 (1948), after in Shannon CE and Weaver W, *The mathematical theory of communication.* The University of Illinois Press. Urbana (1964).

[2] N. Ebrahimi, How to measure uncertainty in the residual life time distribution. *Sankhya A* 58 (1996), p. 50.

to be broken, then the uncertainty of the system life relies on the past, in particular on which instant in $(0, t)$ it has failed. These measures are particularly suitable to describe the information in problems related to aging properties of reliability theory based on the past lifetime and on the inactivity time. For this reason it is interesting to introduce other past measure of information.

Chapter 3 builds on the discussion in the preceding chapter and proposes a new measure of past entropy, the cumulative Tsallis entropy (CTE) and its dynamic version (DCTE), based on past lifetime. These two measures are based on Tsallis entropy [54], presented in 1988 as possible generalization of Boltzmann-Gibbs statistics and plays an important role in the measuring uncertainty of random variables. CTE and DCTE also refer to other important sources, realiablility measures and some features related to stochastic orders. Moreover, in this chapter, is also presented a new aging class based on DCTE. Most of the results reported in this chapter has been published in the paper entitled *Some properties of cumulative Tsallis entropy* by Camilla Calì, Maria Longobardi and Jafar Ahmadi, published in 2017 in Physica A: Statistical Mechanics And Its Applications, vol. 486, pp. 1012-1021.

Chapter 4 - a brief interlude - discusses some foundational topics related to coherent systems. Structural properties of coherent systems are described also through the introduction of the concepts of distortion function and copula. In this chapter the main references are two fundamental books: *Statistical Theory of Reliability and Life Testing* by Richard E. Barlow and Frank Proshan published in 1975 and *An Introduction to Copulas* by Roger B. Nelsen published for the first time in 1998. These topics will be used in the following chapters. In particular, in the sequel the study of a special kind of entropy is strictly tied to the analysis of coherent systems.

Chapter 5 is devoted to the study of some properties for another measure of past entropy, the generalized cumulative past entropy (GCPE). For example, GCPE determines the underlying distribution. This measure is also analysed when it is referred to the lifetime of a coherent system with identically distributed components. Moreover a new generalized inaccuracy measure, the generalized cumulative Kerridge inaccuracy of order n is defined . Most of the results reported in this chapter has been published in the paper *Properties for generalized cumulative past measure of information* by Camilla Calì, Maria Longobardi and Jorge Navarro, published in 2018 (now is available only the online version) in Probability in the Engineering and Informational Sciences, doi:10.1017/S0269964818000360.

Chapter 6 analyses coherent systems under periodical inspections. This study was conducted during my visiting at the University of Murcia under the supervision of

Prof. Jorge Navarro. In real life situation the monitoring of a system can be scheduled at different times. Under these periodical inspections, the information about the system can be different and can be affected by the condition of the components of the system at two inspection times t_1 and t_2. Under this assumption the interest is on the inactivity time of the system T, $(t_2 - T|t_1 < T < t_2)$. Representations through distortion functions are obtained for the reliability functions of such inactivity times, considering coherent systems formed by possibly dependent components. Similar representations are obtained under other assumptions with partial information about component failures at times t_1 and t_2. The representations obtained are used to compare stochastically the inactivity times under different assumptions. In the last part some illustrative examples are provided. Most of the results reported in this chapter will be published in the paper *Inactivity times of coherent systems with dependent components under periodical inspections* by Jorge Navarro and Camilla Calì, now accepted for publication by the journal Applied Stochastic Models in Business and Industry.

Appendices contain extra material used in the thesis. Appendix A contains the list of the main stochastical orders. Appendix B contains the list of the main aging classes. Appendix C contains the tables Chapter 5 refers to.

Finally, I would like to thank my Phd advisor, Dr. Maria Longobardi, for her guidance and her continuous support throughout these three years and in writing the present work.

Chapter 1

Prelude - Entropy and Related Topics

1.1 "Choice, Uncertainty and Entropy": an Historical Overview

> "My greatest concern was what to call it. I thought of calling it *information*, but the word was overly used, so I decided to call it *uncertainty*. When I discussed it with John von Neumann, he had a better idea. Von Neumann told me «You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy really is, so in a debate you will always have the advantage.»".[1]

With this sentence, Claude Shannon gives us a curious starting point to retrace the history of information theory and its central concept, the information entropy.

The central event that established the birth of information theory was the publishing of the groundbreaking paper by Claude Shannon, A Mathematical Theory of Communication, in The Bell System Technical Journal between July and October 1948 [51]. In the introduction we can read:

> "The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another

[1] The sentence was pronounced by Claude Shannon as quoted in [53]. In spite of the precise bibliographic reference, this anecdote has been retold so many times that it has been classified by some authors as an urban legend in science. For further information and references on the matter, is availeble an intersting historical background at <http://www.eoht.info/page/Neumann-Shannon+anecdote>.

point. Frequently the messages have meaning [...] These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one selected from a set of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design. If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded as a measure of the information produced when one message is chosen from the set, all choices being equally likely"[2].

From a mathematical point of view, the main question that Claude Shannon pointed out in this paper was: "Can we define a quantity which will measure, in some sense, how much information is *produced* by such a process, or better, at what rate information is produced?"[3]. He supposed to have a set of possible events whose probabilities of occurrence are $p_1, p_2, \dots, p_n$. These probabilities are known, but that is all we know concerning which event will occur. Then "can we find a measure of how much *choice* is involved in the selection of the event or of how uncertain we are of the outcome?"[4]. Clearly, if there is such a measure, say $H(p_1, p_2, \dots, p_n)$, Shannon listed some basic properties that it is reasonable to require of H:

- Continuity: H should be continuous in the p_i;
- Monotonicity: if all the p_i are equal, $p_i = \frac{1}{n}$, then H should be a monotonic increasing function of n;
- Recursion: if a choice be broken down into two successive choices, the original H should be the weighted sum of the individual values of H.

Theorem 1.1. *The only H satisfying the three above assumptions is of the form:*

$$H = -K \sum_{i=1}^{n} p_i \log_2 p_i \tag{1.1}$$

where K is a positive constant.

[2]Shannon CE, A mathematical theory of communication, *Bell System Technical Journal* 27 (1948), after in Shannon CE and Weaver W, *The mathematical theory of communication.* The University of Illinois Press. Urbana (1964), p. 31.

[3]Ibid., p. 48.

[4]Ibid., p. 49.

The base of the logarithm is 2 because in Shannon's entropy the units are bits (binary information digits).The *bit* is also called *shannon* (in symbol, *Sh*): one shannon is defined as the the entropy of a system with two equiprobable states, so 1 $Sh = 1$ *bit*.

It is clear that there are close parallels between the mathematical expressions for the thermodynamic entropy of a physical system in the statistical thermodynamics and the new definition of information entropy, as Shannon himself highlighted:

> "Quantities of the form $H = -K\sum_{i=1}^{n} p_i \log_2 p_i$ (the constant K merely amounts to a choice of a unit of measure) play a central role in information theory as measures of information, choice and uncertainty. The form of H will be recognized as that of entropy as defined in certain formulations of statistical mechanics where p_i is the probability of a system being in cell i of its phase space. H is then, for example, the H in Boltzmann's famous H theorem"[5].

A suitable extension of the Shannon entropy to the absolutely continuous case is the so-called *differential entropy*. Let X be an absolutely continuous random variable with probability density function (pdf) $f(\cdot)$, then its differential entropy is given by

$$H(X) = -\mathbb{E}[\log f(X)] = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx, \tag{1.2}$$

where log is the natural logarithm.

Starting from this definition, other entropies have also been introduced and studied from the mathematical point of view.

One of the first generalization of (1.2) was proposed by Khinchin [24] in 1957, by choosing a convex function $\phi(x)$ such that $\phi(1) = 0$. He defined this new measure for an absolutely continuous random variable X as:

$$H_\phi(X) = \int_{-\infty}^{+\infty} f(x)\phi\left(f(x)\right)dx. \tag{1.3}$$

$H(X)$ can be derived from (1.3) by choosing $\phi(x) = -\log x$.

In 1961, Renyi [45] proposed a generalized version of entropy of order α, whose version for absolutely continuous random variable X is:

$$H_\alpha(X) = \frac{1}{\alpha-1}\log\int_{-\infty}^{+\infty}\left(f(x)\right)^\alpha dx \quad \alpha\neq 1, \alpha>0.$$

[5] Ibid., p. 50.

When α tends to 1, $H^\alpha(X)$ tends to $H(X)$.

Tsallis entropy was introduced by Tsallis [54] in 1988 and it is a generalization of Boltzmann-Gibbs statistics. For a continuous random variable X with pdf $f(x)$, Tsallis entropy of order α is defined by

$$T_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_{-\infty}^{+\infty} f^\alpha(x)dx\right); \qquad \alpha \neq 1, \quad \alpha > 0. \tag{1.4}$$

Clearly as $\alpha \to 1$ then $T_\alpha(X)$ reduces to $H(X)$, given in (1.2).

1.2 Relative Information and Inaccuracy

During the same years, other information measures, which examined two probability distributions associated with the same random experiment, were also investigated.

In 1951, Kullback and Leibler [25] were concerned with

> "the statistical problem of discrimination, by considering a measure of the 'distance' or 'divergence' between statistical populations in terms of our measure of information. For the statistician two populations differ more or less according as to how difficult it is to discriminate between them with the best test"[6].

Given X and Y two absolutely continuous random variables with probability density function $f(\cdot)$ and $g(\cdot)$, respectively,

$$\log \frac{f(x)}{g(x)}$$

is called the *information in x* for discrimination between X and Y.
The Kullback-Leibler divergence is defined by

$$H_{KL}(X|Y) = \int_{-\infty}^{-\infty} f(x) \log \frac{f(x)}{g(x)} dx. \tag{1.5}$$

Another step forward in this direction was the definition of inaccuracy introduced by Kerridge [23] in 1961. Suppose that the experimenter asserts that the probability of the i-th eventuality is p_i, when the true probability is q_i, with $\sum p_i = \sum q_i = 1$.

[6]S. Kullback, R.A. Leibler, On information and sufficiency. *Annals of Mathematics Statistics* 22 (1951), p. 79.

In the same way of Shannon, Kerridge demonstrated that the only definition which satisfies a set of intuitively reasonable assumptions listed by him was:

$$H_K(P;Q) = -\sum_{i=1}^{n} p_i \log q_i. \tag{1.6}$$

Kerridge inaccuracy can be considered also a generalization of Shannon entropy because, obviously, when $p_i = q_i$ for all i, then (1.6) reduces to (1.1).

In 1968, Nath [29] extended this concept of inaccuracy to the case of continuous distributions. If $f(x)$ is the actual probability density function related to an absolutely continuous random variable X and $g(x)$ is the true probability density function assigned by the experimenter, related to another absolutely continuous random variable Y, then the Kerridge inaccuracy is defined as

$$H_K(f;g) = -\int_{-\infty}^{+\infty} f(x) \log g(x) dx. \tag{1.7}$$

Also in this case, when $f(x) = g(x)$ for all x, then (1.7) reduces to (1.2).

1.3 Some Concepts of Mathematical Theory of Reliability

> "What is mathematical reliability theory? Generally speaking, it is a body of ideas, mathematical models, and methods directed toward the solution of problems in predicting, estimating, or optimizing the probability of survival, mean life, or, more generally, life distribution of components or systems; other problems considered in reliability theory are those involving the probability of proper functioning of the system at either a specified or an arbitrary time, or the proportion of time the system is functioning properly. In a large class of reliability situations, maintenance, such as replacement, repair, or inspection, may be performed, so that the solution of the reliability problem may influence decisions concerning maintenance policies to be followed"[7].

Let X be a nonnegative absolutely continuous random variable with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$. The *reliability function*

[7] R.E. Barlow, F. Proschan, *Mathematical Theory of Reliability.* Wiley, New York (1965), p. xiii.

(or *survival function*) is defined as:

$$\bar{F}(x) = P(X > x) = 1 - F(x) = \int_x^{+\infty} f(x)dx. \tag{1.8}$$

By the definition in (1.8), $\bar{F}$ is monotonically decreasing and is a right-continuous function such that $\bar{F}(0) = 1$ and $lim_{x\to+\infty}\bar{F}(x) = 0$.
In the context of theory of reliability, $\bar{F}(x)$ gives the probability that a system of interest - in this case X represent the lifetime of such system - will survive beyond any given specified time, in this case represented by x.

The *hazard rate function*, also known as the *failure rate function*, at time t is defined as:

$$\lambda(t) = \lim_{x\to 0} \frac{P(t < X < t + x | X \geq t)}{x} = \lim_{x\to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{\bar{F}(t)}$$

so that

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}, \tag{1.9}$$

when $f(t)$ exists and $\bar{F}(t) > 0$. This function has a useful probabilistic interpretation: $\lambda(t)$ represents the probability that an object or a system of age t will fail in the interval $[t, t + x]$. A strictly relation between reliability function and hazard rate function can be obtained by integrating both sides of (1.9),

$$\int_0^x \lambda(t)dt = -log\bar{F}(x),$$

and then $\lambda(t)$ uniquely determines the underlying reliability function via the relation

$$\bar{F}(x) = \exp\left[-\int_0^x \lambda(t)dt\right].$$

Keilson and Sumita [22] were among the first to define *reversed hazard rate function* (or *reversed failure rate function*), calling it the "dual failure function":

$$\tau(t) = \lim_{x\to 0} \frac{P(t - x < X < t | X \leq t)}{x} = \lim_{x\to 0} \frac{1}{x} \frac{F(t) - F(t-x)}{F(t)}$$

so that

$$\tau(t) = \frac{f(t)}{F(t)}, \tag{1.10}$$

when $f(t)$ exists and $F(t) > 0$. Also this function has a useful probabilistic interpretation: $\tau(t)$ represents the probability that an object or a system will fail in the interval $[t-x,t]$, since it has been found failed at time t. A strictly relation between cumulative distribution function and reversed hazard rate function can be obtained by integrating both sides of (1.10),

$$\int_{x}^{+\infty} \tau(t)dt = logF(x),$$

and then $\tau(t)$ uniquely determines the underlying distribution function via the relation

$$F(x) = \exp\left[-\int_{x}^{+\infty} \tau(t)dt\right]. \tag{1.11}$$

If X describes the random lifetime of a biological system, such as an organism or a cell, then the random variable

$$X_t = [X - t \,|\, X > t] \tag{1.12}$$

describes the *residual lifetime* of the system at age t, that is the additional lifetime given that the system has survived up to time t. Hence, if the system has survived up to time t, the uncertainty about the remaining lifetime is measured by means of X_t. Let us denote the *mean residual life*(MRL) by $m(t)$:

$$m(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[X - t \,|\, X > t\right] = \frac{1}{\bar{F}(t)} \int_{0}^{+\infty} (x-t)dF(x),$$

where $F(\cdot)$ and $\bar{F}(\cdot)$ are the distribution function and the survival function of X, respectively. Writing $(x-t) = \int_t^x du$ and employing Fubini-Tonelli's theorem, yields the equivalent formula

$$m(t) = \frac{1}{\bar{F}(t)} \int_{t}^{+\infty} \bar{F}(x)dx, \qquad \forall t \geq 0 : \bar{F}(t) > 0. \tag{1.13}$$

It is known (see, for istance, [15]) that each of the functions $\bar{F}$, λ and m uniquely determines the other two. More specifically

$$\bar{F}(t) = \exp\left(-\int_{0}^{t} \lambda(x)dx\right),$$

and

$$\lambda(t) = \frac{m'(t)+1}{m(t)}.$$

1.4 The Residual Entropy

> "Frequently, in survival analysis and in life testing one has information about the current age of the component under consideration. In such cases, the age must be taken into account when measuring uncertainty"[8].

The differential entropy in (1.2) is not appropriate in the situation described above because there is no connection with the age. For this reason, in 1996 Ebrahimi [15] proposed a more realistic approach, defining the so called *residual entropy.*
Let X be random variable that represents the lifetime of a component and X_t the residual lifetime of the component at age t defined in (1.12). So the residual entropy is defined as a dynamic measure of entropy based on Shannon differential entropy, given by:

$$H(f;t) = -\int_0^{+\infty} f_{X_t}(x) \log f_{X_t}(x) dx \tag{1.14}$$

where $f_{X_t}(x)$ denotes the probability density function of the random variable X_t:

$$f_{X_t}(x) = \begin{cases} \frac{f(x+t)}{\bar{F}(t)} \,, & \text{if} \quad x > t \\ 0 \,, & \text{otherwise.} \end{cases}$$

$H(f;t)$ basically measures the expected uncertainty contained in the conditional density of the random variable $(X-t)$ given $X>t$ about the predictability of remaining lifetime of the unit. Then $H(f;t)$ can be expressed as

$$\begin{aligned} H(f;t) &= -\int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \\ &= \log \bar{F}(t) - \frac{1}{\bar{F}(t)} \int_t^{+\infty} f(x) \log f(x) dx \\ &= 1 - \frac{1}{\bar{F}(t)} \int_t^{+\infty} f(x) \log \lambda(x) dx. \end{aligned}$$

In compliance with the definition, $H(f;t)$ can be negative and also it can be infinite. However Ebrahimi (see [15], Theorem 2.2) showed that when $m(t) < \infty$ then $H(f;t)$ is finite. Moreover in 2004, Belzunce *et al.* demonstrated that if $H(f;t)$ is increasing in t, then it uniquely determines the underlying reliability function (see [4], Theorem 1). It is interesting to note that $H(f;0)$ is the differential Shannon entropy in (1.2) for nonnegative random variables.

[8] N. Ebrahimi, How to measure uncertainty in the residual life time distribution. *Sankhya A* 58 (1996), p. 50.

1.5 The Cumulative Residual Entropy

All the definitions of entropy presented in the previous sections are related in some way to the definition of differential entropy of Shannon in (1.2). However, although the analogy between definitions (1.1) and (1.2) (and so with the other definitions of entropy), the differential entropy is an inaccurate extension of the Shannon discrete entropy, as stated by Reza [46] already in 1994. He summarizes in particular three basic points to be discussed:

- the differential entropy may be negative;
- the differential entropy may become infinitely large or the integral may not exist;
- the differential entropy does not remain invariant under the transformation of the coordinate systems.

So in the last 25 years various attempts have been made in order to define possible alternative information measures that have analogy with Shannon entropy, but have also reasonable mathematical properties.

In 2004 Rao *et al.* [44] introduced the concept of *cumulative residual entropy* (CRE) defined as

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}(x) \log \bar{F}(x) dx, \tag{1.15}$$

where X is a nonnegative absolutely continuous random variable and $\bar{F}(x)$ is the reliability function of X. The idea was to replace the density function with the reliability function in differential Shannon entropy in (1.2). This new definition overcomes the problems mentioned above, while retaining many of the important properties of Shannon entropy. In particular CRE has the following important properties:

- CRE is always non-negative;
- CRE has consistent definitions in both the continuous and discrete domains;
- CRE can be easily computed from sample data and these computations asymptotically converge to the true values.

Rao states that

> "The distribution function is more regular than the density function, because the density is computed as the derivative of the distribution. Moreover, in practice what is of interest and/or measurable is the distribution

function. For example, if the random variable is the life span of a machine, then the event of interest is not whether the life span equals t, but rather whether the life span exceeds t. Our definition also preserves the well established principle that the logarithm of the probability of an event should represent the information content in the event"[9].

[9]M. Rao, Y. Chen, B.C. Vemuri, F. Wang, Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory* 50 (2004), p.1221.

Chapter 2

Past Entropies

2.1 Why Past Entropy?

In many real life situations uncertainty is not necessarily related to the future but it can be also referred to the past. For instance, we consider a system that can be observed only at certain preassigned inspection times. We assume that at time t the system is inspected for the first time and it is found in a not working condition. In this situation, the uncertainty of the system cannot be referred to the future but to the past lifetime and, as a conseguence, also to the inactivity time. The inactivity time, in particular, describes the time elapsing between the failure of a system and the time when it is found to be broken. So it seems natural to introduce a notion of uncertainty that is suitable to measure information when the uncertainty is related to the past, a dual concept of the cumulative residual entropy related to uncertainty on the future lifetime of a system. So, one of the random variables that is used in these new definitions have to represent the *past lifetime* at time t, that is

$$ {}_tX = [X \,|\, X \le t], \qquad t > 0. $$

Moreover, in reliability theory, the duration of the time between an inspection time t and the failure time X, given that at time t the system has been found failed, is called *inactivity time* - it was introduced in 1996 by Ruiz and Navarro [47] - and it is represented by the random variable

$$ X^{(t)} = [t - X \,|\, X \le t], \qquad t > 0. $$

In this context in 2002 Di Crescenzo and Longobardi [10] introduced the concept of *past entropy* and in 2009 [12] the concept of *cumulative entropy* (in the rest of the thesis we refer to this definition called it *cumulative past entropy*, CPE).

2.2 Past Entropy

Let X be a nonnegative continuous random variable that represents the lifetime of a system or of a component of a system. In analogy with the definition of Ebrahimi in (1.14), Di Crescenzo and Longobardi [10] defined the *past entropy* at time t as

$$H^*(f;t) = -\int_0^{+\infty} f_{X^{(t)}}(x) \log f_{X^{(t)}}(x) dx$$

where $f_{X^{(t)}}(x)$ denotes the probability density function of the inactivity time $X^{(t)}$, given by

$$f_{X^{(t)}} = \begin{cases} \frac{f(t-x)}{F(t)} \ , & \text{if} \quad x < t \\ 0 \ , & \text{otherwise.} \end{cases} \tag{2.1}$$

For example, given that at time t a component of a system is found to be broken, $H^*(f;t)$ basically measures the uncertainty about its past lifetime. From (2.1), $H^*(f;t)$ can be expressed as

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx. \tag{2.2}$$

Note that (2.2) can be obtained also as differential entropy of the random variable ${}_tX$ that represents the past lifetime at time t.

Making use of definition of reversed hazard rate function in (1.10), $H^*(f;t)$ can be rewritten as

$$\begin{aligned} H^*(f;t) &= \log F(t) - \frac{1}{F(t)} \int_0^t f(x) \log f(x) dx \\ &= 1 - \frac{1}{F(t)} \int_0^t f(x) \log \tau(x) dx. \end{aligned} \tag{2.3}$$

As an immediate conseguence of (2.3), also the derivative of $H^*(f;t)$ can be written in terms of the reversed hazard rate function $\tau(t)$:

$$\frac{d}{dt} H^*(f;t) = \tau(t) \left[1 - H^*(f;t) - \log \tau(t)\right].$$

Note that if $f(t)$ is decreasing in $t > 0$, then $H^*(f;t)$ is increasing in $t > 0$. However, this property can be proved under the weaker assumption that the reversed hazard rate function of X is decreasing, i.e. X is DRHR, in compliance with the definition given in (B.4).

2.3 Cumulative Past Entropy

2.3.1 Definition and Basic Properties

In analogy with the definition of cumulative residual entropy by Rao in (1.15), Di Crescenzo and Longobardi [12] defined the *cumulative past entropy* (CPE) of a non-negative random variable X as

$$\mathcal{CE}(X) = -\int_0^{+\infty} F(x)\,\log F(x)dx, \tag{2.4}$$

where $F(x) = P(X \leq x)$ is the distribution function of X. The cumulative past entropy has some basic properties:

(i) $\mathcal{CE}(X)$ takes values in $[0, +\infty]$ and is equal to 0 if and only if X is a constant;

(ii) $\mathcal{CE}(X) = \mathcal{E}(X)$ if the distribution of X is symmetric with respect to μ (where $\mu = \mathbb{E}(X)$ is finite), i.e. if $F(\mu + x) = 1 - F(\mu - x)$ for all $x \in \mathbb{R}$;

(iii) $\mathcal{CE}(Y) = a\mathcal{CE}(X)$ where $Y = aX + b$ with $a > 0$ and $b \geq 0$; so $\mathcal{CE}(X)$ is a shift-independent measure.

Similarly to the normalized cumulative residual entropy introduced by Rao [43], Di Crescenzo and Longobardi [12] introduced also a normalized version of (2.4). For a nonnegative random varible X with $0 < \mathbb{E}(X) < +\infty$, they defined the *normalized cumulative past entropy* as

$$\mathcal{NCE}(X) = \frac{\mathcal{CE}(X)}{\mathbb{E}(X)} = -\frac{1}{\mathbb{E}(X)}\int_0^{+\infty} F(x)\,\log F(x)dx.$$

2.3.2 More on Reliability Theory

The inactivity time $X^{(t)}$ is well described also through its mean value, called *mean inactivity time* (MIT) given by

$$\tilde{\mu}(t) = \mathbb{E}\left[X^{(t)}\right] = \mathbb{E}\left[t - X \,|\, X \leq t\right] = \frac{1}{F(t)}\int_0^t (t-x)dF(x),$$

where $F(\cdot)$ is the distribution function of X. Writing $(t - x) = \int_x^t du$ and employing Fubini-Tonelli's theorem, yields the equivalent formula

$$\tilde{\mu}(t) = \frac{1}{F(t)}\int_0^t F(x)dx. \qquad \forall t \geq 0 : F(t) > 0. \tag{2.5}$$

The mean inactivity time has a stricly relation with the *mean past lifetime* of X defined as

$$\mu(t) = \mathbb{E}\left[{}_tX\right] = \mathbb{E}\left[X \mid X \leq t\right] = \int_0^t 1 - \frac{F(x)}{F(t)} dx. \tag{2.6}$$

So, from (2.5) and (2.6), we can state that

$$\tilde{\mu}(t) = t - \mu(t). \tag{2.7}$$

The derivative of the mean inactivity time of X can be expressed in term of the reversed hazard rate function, defined in (1.10) (if existing), as states the following theorem.

Theorem 2.1. *Let X be an absolutely continuous random variable with distribution function F, reversed hazard rate function τ and the mean inactivity time $\tilde{\mu}$. Then*

$$\tilde{\mu}'(t) = 1 - \tau(t)\tilde{\mu}(t), \tag{2.8}$$

and

$$F(t) = \exp\left(-\int_t^{+\infty} \frac{1 - \tilde{\mu}'(x)}{\tilde{\mu}(x)} dx\right) \tag{2.9}$$

for all t such that $F(t) > 0$.

Note that the relation in (2.9) is the same relation that we have found in (1.11).

The cumulative past entropy of a random lifetime X is strictly related to the concept of mean inactivity time, $\tilde{\mu}(t)$. In particular, $\mathcal{CE}(X)$ can be expressed as the expectation of its mean inactivity time evaluated at X, as it is shown in the following theorem.

Theorem 2.2. *Let X be a nonnegative random variable X with mean inactivity time $\tilde{\mu}(t)$ and cumulative entropy $\mathcal{CE}(X) < +\infty$. Then*

$$\mathcal{CE}(X) = \mathbb{E}\left[\tilde{\mu}(X)\right].$$

The cumulative past entropy of a random lifetime X is also strictly related to the concept of reversed hazard rate of X, $\tau(t)$, as it is shown in the following theorem.

Theorem 2.3. *Let X be a nonnegative random variable X with cumulative entropy $\mathcal{CE}(X) < +\infty$. Then*

$$\mathcal{CE}(X) = \mathbb{E}\left[T^{(2)}(X)\right],$$

where

$$T^{(2)}(x) := -\int_x^{+\infty} \log F(z) dz = \int_x^{+\infty} \left[\int_z^{+\infty} \tau(u) du\right] dz, \quad x \geq 0.$$

The normalized cumulative past entropy has an alternative expression in terms of Bonferroni curve $B_F[\cdot]$, as it is stated in the following proposition.

Proposition 2.1. *Let X be a nonnegative random variable with finite non-vanishing mean, then the normalized cumulative past entropy can be expressed as:*

$$\mathcal{NCE}(X) = 1 - \mathbb{E}\{B_F\left[F(X)\right]\}.$$

2.3.3 Proportional Reversed Hazards Model

In 1998, Gupta *et al.* [18] proposed a model called *proportional reversed hazards model* (PRHM) expressed by a nonnegative absolutely continous random variable X_θ^* whose distribution function is the θ-th power of the distribution function of X:

$$F_\theta^*(x) = P(X_\theta^* \leq x) = \left[F(x)\right]^\theta, \qquad x \in S_X, \tag{2.10}$$

where θ is a positive real number and $F(x)$ is the distribution function of X and S_X is the support of X.
From this definition is interesting comparing the cumulative past entropy of θX and X_θ^*.

Proposition 2.2. *Let X be a nonnegative absolutely continuous random variable, then*

$$\mathcal{CE}(\theta X) \geq (\leq) \mathcal{CE}(X_\theta^*)$$

if $\theta \geq 1$ ($0 < \theta \leq 1$).

The proportional reversed hazards model is largely used for the analysis of data on parallel systems (a parallel structure with n components works if at least one of the components works). Note that the right-hand side of (2.10) is the distribution function of the maximum of indipendent and identically distributed (IID) random variables (and then it rapresents also the distribution function of the lifetime of a parallel system), as it is shown in the following corollary.

Corollary 2.1. *Let $X_1, X_2, \ldots, X_n$ be IID random variables. Then*

$$\mathcal{CE}(nX_1) \geq \mathcal{CE}(\max\{X_1, X_2, \ldots, X_n\}).$$

2.3.4 Dynamic Cumulative Past Entropy

A dynamic version of (2.4) was also introduced and studied by Di Crescenzo and Longobardi [12] to describe the uncertainty related to the past. For istance, let consider a system thet begins to work at time 0 and, after an inspection at time t, it is found to be broken. In this situation, the random variable ${}_tX = [X|X \leq t]$ describes the past lifetime of the system at age t.
So the *dynamic cumulative past entropy* is defined as

$$\mathcal{CE}(X;t) = -\int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx, \qquad t > 0 : F(t) > 0. \tag{2.11}$$

Note that $\mathcal{CE}(X;t)$ is exactly the cumulative past entropy of the random variable ${}_tX$.

For the dynamic cumulative past entropy two results hold, that are similar to properties that hold for the cumulative past entropy (see Property (iii) and Theorem 2.2).

Proposition 2.3. *Let $Y = aX + b$ with $a > 0$ and $b \geq 0$. Then,*

$$\mathcal{CE}(Y;t) = a\mathcal{CE}\left(X; \frac{t-b}{a}\right), \qquad t \geq b.$$

Theorem 2.4. *Let X be a nonnegative random variable with mean inactivity time $\tilde{\mu}(\cdot)$ and dynamic cumulative past entropy $\mathcal{CE}(X;t) < +\infty$. Then*

$$\mathcal{CE}(X;t) = \mathbb{E}\left[\tilde{\mu}(X)|X \leq t\right], \qquad t > 0.$$

Note that from Theorems 2.2 and 2.4 it follows that $\mathcal{CE}(X;t)$ is nonnegative for all t with

$$\lim_{t \to 0^+} \mathcal{CE}(X;t) = 0, \qquad \lim_{t \to b^-} \mathcal{CE}(X;t) = \mathcal{CE}(X),$$

for any random variable X with support $(0, b)$, with b finite or infinite. For this reason $\mathcal{CE}(X;t)$ cannot be decresing in t and, moreover, the following theorem gives a condition in order for the dynamic cumulative past entropy to be increasing.

Theorem 2.5. *Let X be a nonnegative absolutely continuous random variable and let $t \geq 0$ be such that $F(t) > 0$. Then $\mathcal{CE}(X;t)$ is increasing in t if and only if $\mathcal{CE}(X;t) \leq \tilde{\mu}(t)$.*

In the previous sections are described the most important results about past entropies, for further details refer to Di Crescenzo and Longobardi [13] and Longobardi [27].

2.4 Past Relative Information and Inaccuracy

After the definition of the new measures of entropy based on the past lifetime, also the measures of "'distance' or 'divergence' between statistical populations in terms of our measure of information"[1] - as Kullback and Leibler wrote in 1951 - have to be redefined.

As in Section 1.2, based on (1.5), Di Crescenzo and Longobardi [11] studied a measure of divergence which constitutes a distance between two past lifetimes distributions. Given two nonnegative random lifetimes X and Y having distribution functions $F(\cdot)$ and $G(\cdot)$ and density function $f(\cdot)$ and $g(\cdot)$, respectively, the discrimination measure between past lifetimes is

$$H^*_{KL}(X|Y,t) = H_{KL}({}_tX|{}_tY) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)/F(t)}{g(x)/G(t)} dx.$$

where ${}_tX$ and ${}_tY$ are the random variable that describe the past lifetime at time t. If we have a system with true distribution function $F(\cdot)$ and reference distribution $G(\cdot)$, then $H^*_{KL}(X|Y,t)$ can also be interpreted as a measure of distance between $G_{X^{(t)}}(x)$ and the true distribution $F_{X^{(t)}}(x)$, where $X^{(t)}$ is the inactivity time $X^{(t)}$. It is also proved that $H^*_{KL}(X|Y,t)$ is constant if and only if X and Y satisfy the proportional reversed hazard model (PRHM), defined in (2.10).

In 2013, Di Crescenzo and Longobardi [14] defined also the *cumulative Kerridge inaccuracy*, the cumulative version of the measure of innaccuracy due to Kerridge, defined in (1.7). Given two random lifetimes X and Y having distribution functions $F(\cdot)$ and $G(\cdot)$, the cumulative Kerridge inaccuracy is defined as

$$K[F,G] = -\int_0^{+\infty} F(x) \log G(x)\, dx. \tag{2.12}$$

[1] S. Kullback, R.A. Leibler, On information and sufficiency. *Annals of Mathematics Statistics* 22 (1951), p. 79.

Chapter 3

New Measures of Past Lifetime - Part I

3.1 Cumulative Tsallis Entropy

In 1988 Tsallis [54] introduced a new measure of information, the *Tsallis entropy*, as a generalization of Boltzmann-Gibbs statistics. For a nonnegative continuous random variable X with pdf $f(x)$, recall, from (1.4), that *Tsallis entropy of order* α is defined as

$$T_\alpha(X) = \frac{1}{\alpha - 1}\left[1 - \int_0^{+\infty} f^\alpha(x)dx\right]; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{3.1}$$

Clearly as $\alpha \to 1$ then $T_\alpha(X)$ reduces to differential Shannon entropy $H(X)$, given in (1.2). Several researchers used Tsallis entropy in many physical applications, such as developing the statistical mechanics of large scale astrophysical systems, image processing and signal processing. Recently, Kumar [26] studied Tsallis entropy for k-record statistics from some continuous probability models, Baratpour and Khammar [1] proposed some applications of this entropy to order statistics and provided relations with some stochastic orders, Zhang [56] obtained some quantitative characterizations of the uniform continuity and stability properties of the Tsallis entropies.

Based on (3.1), Sati and Gupta [49] proposed a *cumulative residual Tsallis entropy of order* α (CRTE), which is given by

$$\eta_\alpha(X) = \frac{1}{\alpha - 1}\left[1 - \int_0^{+\infty} (\bar{F}(x))^\alpha dx\right]; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{3.2}$$

The Tsallis entropy in (3.1) can also be expressed as

$$T_\alpha(X) = \frac{1}{\alpha - 1}\int_0^{+\infty} \left[f(x) - f^\alpha(x)\right] dx; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{3.3}$$

By (3.3), Rajesh and Sunoj [42] introduced an alternative measure of CRTE of order α as

$$\xi_\alpha(X) = \frac{1}{\alpha - 1}\int_0^{+\infty} \left[\bar{F}(x) - (\bar{F}(x))^\alpha\right] dx; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{3.4}$$

Tsallis entropy is suitable, for example, to give more information about the intrinsic stucture (in particular the intrinsic fluctuations) of a physical systems through the parameter α that characterizes this entropy (see, for instance, Wilk and Wlodarczyk [55]).

Then, motivated by (3.1)-(3.4), we propose the *cumulative Tsallis entropy* (CTE) based on definition of $\mathcal{CE}(X)$ as

$$\mathcal{C}\xi_\alpha(X) = \frac{1}{\alpha - 1}\int_0^{+\infty} \left[F(x) - F^\alpha(x)\right] dx; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{3.5}$$

It is easy to show that, when $\alpha \to 1$, $\mathcal{C}\xi_\alpha(X)$ reduces to $\mathcal{CE}(X)$.

There is a strict relation between the proposed CTE in (3.5) and mean inactivity time in (2.5), as shown by the next result.

Theorem 3.1. *Let X be a nonnegative continuous random variable with cumulative distribution function $F(x)$ and density function $f(x)$, then*

$$\mathcal{C}\xi_\alpha(X) = \mathbb{E}\left[\tilde{\mu}(X)F^{\alpha-1}(X)\right],$$

where $\tilde{\mu}(x)$ is the mean inactivity time of X.

Proof. First of all, note that from (2.5), we have

$$\tilde{\mu}(x)F(x) = \int_0^x F(u)du.$$

and, differentiating both side of the identity with respect to x, we have

$$\frac{d}{dx}\left(\tilde{\mu}(x)F(x)\right) = F(x). \tag{3.6}$$

The cumulative Tsallis entropy in (3.5) can be written as

$$\begin{aligned}\mathcal{C}\xi_\alpha(X) &= \frac{1}{\alpha - 1}\left[\int_0^{+\infty} F(x)dx - \int_0^{+\infty} F^\alpha(x)dx\right] \\ &= \frac{1}{\alpha - 1}\left[\int_0^{+\infty} F(x)dx - \int_0^{+\infty} \frac{d}{dx}\left(\tilde{\mu}(x)F(x)\right) F^{\alpha-1}(x)dx\right].\end{aligned}$$

Then, using the integration by part for the second integral in the right-hand side we have

$$\mathcal{C}\xi_\alpha(X) = \int_0^{+\infty} \left[\tilde{\mu}(x)F^{\alpha-1}(x)f(x)\right] dx,$$

which completes the proof. □

Corollary 3.1. *Let X be a nonnegative continuous random variable with cumulative distribution function $F(x)$. If $\alpha > 1 (0 < \alpha < 1)$, then $\mathcal{C}\xi_\alpha(X) < (>) \mathbb{E}\left[\tilde{\mu}(X)\right]$.*

Example 3.1. If X is uniformly distributed on $(0, c)$, then

$$\mathcal{C}\xi_\alpha(X) = \frac{c}{2(\alpha+1)}, \qquad \tilde{\mu}(x) = \frac{x}{2} \qquad \text{and} \qquad \mathbb{E}\left[\tilde{\mu}(X)\right] = \frac{c}{4}.$$

So, for $\alpha > 1$ then $\mathcal{C}\xi_\alpha(X) < \mathbb{E}\left[\tilde{\mu}(X)\right]$, instead for $0 < \alpha < 1$ then $\mathcal{C}\xi_\alpha(X) \geq \mathbb{E}\left[\tilde{\mu}(X)\right]$, which confirm Corollary 3.1. Note that we have defined $\mathcal{C}\xi_\alpha(X)$ only for $\alpha \neq 1$, but in this particular case for $\alpha = 1$, $\mathcal{C}\xi_\alpha(X) = \mathbb{E}\left[\tilde{\mu}(X)\right]$.

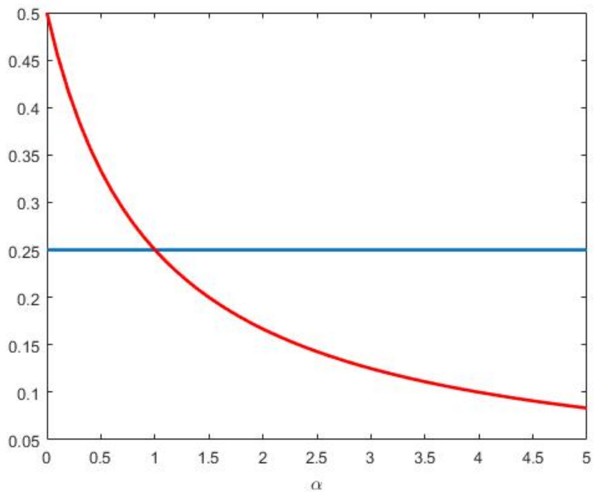

Figure 3.1: The figure refers to Example 3.1 in the case that X is uniformly distributed on $(0, 1)$, then $\mathbb{E}\left[\tilde{\mu}_X(X)\right] = \frac{1}{4}$ (blue line) and $\mathcal{C}\xi_\alpha(X) = \frac{1}{2(\alpha+1)}$ (red line) are plotted. Note that we have defined $\mathcal{C}\xi_\alpha(X)$ only for $\alpha \neq 1$, but in this particular case for $\alpha = 1$, $\mathcal{C}\xi_\alpha(X) = \mathbb{E}\left[\tilde{\mu}(X)\right]$.

Theorem 3.2. *$\mathcal{C}\xi_\alpha(X) = 0$ if, and only if, X is degenerate, while $\mathcal{C}\xi_\alpha(X) > 0$ for any non-negative and absolutely continuous random variable X.*

Proof. Let $0 < \alpha < 1$. In this case $F(x) \leq F^{\alpha}(x)$, thus $\mathcal{C}\xi_{\alpha}(X) \geq 0$. If $\alpha > 1$, then from $F(x) \geq F^{\alpha}(x)$ we have $\mathcal{C}\xi_{\alpha}(X) \geq 0$. If X is degenerate, then $\mathcal{C}\xi_{\alpha}(X) = 0$. Conversely, if $\mathcal{C}\xi_{\alpha}(X) = 0$, then

$$\int_0^{+\infty} [F(x) - F^{\alpha}(x)]\, dx = 0,$$

because $\alpha \neq 1$. The integrand function is non-negative for all x or is non-positive for all x, according to the value of α. Thus we can state that $F(x)(1 - F^{\alpha-1}(x)) = 0$. For this reason, $F(x) = 0$ or $F(x) = 1$, that is, X is degenerate. □

In the next result, we discuss the effect of increasing transformation on CTE.

Lemma 3.1. *Let X be a nonnegative continuous random variable with cdf F and take $Y = \phi(X)$, where $\phi(.)$ is a strictly increasing differentiable function. Then*

$$\mathcal{C}\xi_{\alpha}(Y) = \frac{1}{\alpha - 1} \int_{\max\{0,\phi^{-1}(0)\}}^{+\infty} [F(x) - F^{\alpha}(x)]\, \phi'(x) dx.$$

Remark 3.1. If $\phi(u) = au + b$, $a > 0$ and $b \geq 0$, then

$$\mathcal{C}\xi_{\alpha}(Y) = a\mathcal{C}\xi_{\alpha}(X).$$

Theorem 3.3. *Let X be a nonnegative absolutely continuous random variable with density function $f(x)$, if $\alpha \geq 1 (0 < \alpha \leq 1)$ then $\mathcal{C}\xi_{\alpha}(X) \leq (\geq) \mathcal{CE}(X)$.*

Proof. If $\alpha > 1$ $(0 < \alpha < 1)$ we can write

$$\begin{aligned}
\mathcal{C}\xi_{\alpha}(X) &= \frac{1}{\alpha - 1} \left[\int_0^{+\infty} (F(x) - F^{\alpha}(x))\, dx \right] \\
&= \frac{1}{\alpha - 1} \left[\int_0^{+\infty} F(x) \left(1 - F^{\alpha-1}(x)\right) dx \right] \\
&\leq (\geq) \frac{1}{\alpha - 1} \left[- \int_0^{+\infty} F(x) \left(\log F^{\alpha-1}(x)\right) dx \right] \\
&= \mathcal{CE}(X),
\end{aligned}$$

where the inequality is obtained using the fact that for $u > 0$, $1 - u \leq -\log u$. □

It should be mentioned that Theorem 3.3 also follows by Lemma 3.1, thanks to the Theorem 2.2 in which we proved that $\mathcal{CE}(X) = \mathbb{E}(\tilde{\mu}(X))$.

Example 3.2. If X is uniformly distributed on $(0,1)$, then $\mathcal{CE}(X) = \mathbb{E}(\tilde{\mu}(X)) = \frac{1}{4}$ and $\mathcal{C}\xi_\alpha(X) = \frac{1}{2(\alpha+1)}$. So we can refer again to the Figure 3.1: for $\alpha > 1$, $\mathcal{C}\xi_\alpha(X) \leq \mathcal{CE}(X)$ and for $0 < \alpha < 1$, $\mathcal{C}\xi_\alpha(X) \geq \mathcal{CE}(X)$ which confirms Theorem 3.3.

In the next result we refer to the definition of usual stochastic order, that can be found in the Appendix A in (A.1) and (A.2).

Lemma 3.2. *Let X and Y be two nonnegative continuous random variables with distribution functions F and G and finite mean $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively.*
If $X \leq_{ST} Y$, then

$$|\mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y)| \leq \mathbb{E}(Y) - \mathbb{E}(X), \quad \text{for } 1 < \alpha \in \mathbb{N}.$$

Proof. Suppose $\alpha \in \mathbb{N}$ and $\alpha > 1$, then from (3.5) we can write

$$\begin{aligned} \mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) &= \frac{1}{\alpha-1}\int_0^{+\infty} \{[F(t) - F^\alpha(t)] - [G(t) - G^\alpha(t)]\}dt \\ &= \frac{1}{\alpha-1}\int_0^{+\infty} [F(t) - G(t)]\left[1 - \sum_{i=1}^{\alpha} F^{i-1}(t)G^{\alpha-i}(t)\right]dt. \end{aligned}$$

By assumption $X \leq_{ST} Y$, then we reach the following inequality

$$\begin{aligned} -\int_0^{\infty} [F(t) - G(t)]dt &\leq \mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) \\ &\leq \frac{1}{\alpha-1}\int_0^{+\infty} [F(t) - G(t)]dt. \end{aligned} \tag{3.7}$$

Since X and Y are nonnegative random variables, then (3.7) completes the proof. □

In the next example, we show that $X \leq_{ST} Y$ does not imply $\mathcal{C}\xi_\alpha(X) < \mathcal{C}\xi_\alpha(Y)$, in general.

Example 3.3. Let X and Y be two random variables with cdfs $F(x) = x, 0 < x < 1$ and $G(y) = y^2, 0 < y < 1$, respectively. From (3.5), we have

$$\begin{aligned} \mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) &= \frac{1}{2(\alpha+1)} - \frac{2}{3(2\alpha+1)} \\ &= \frac{2\alpha-1}{6(\alpha+1)(2\alpha+1)}. \end{aligned}$$

So, $\mathcal{C}\xi_\alpha(X) < (>) \mathcal{C}\xi_\alpha(Y)$ whenever $\alpha < (>) \frac{1}{2}$ while clearly, $X \leq_{ST} Y$, as it is shown in Figure 3.2.

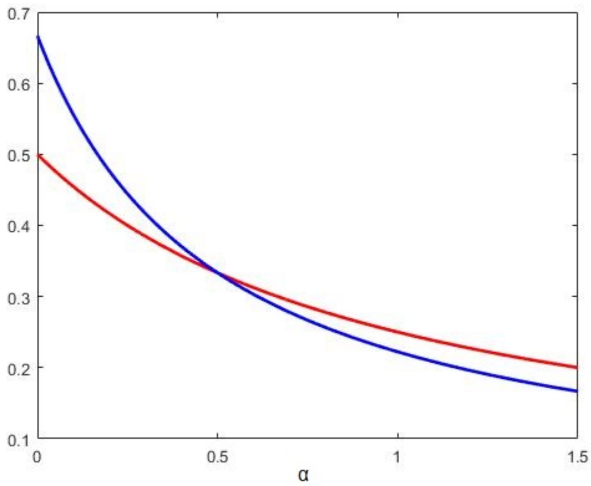

Figure 3.2: The figure refers to Example 3.3: $\mathcal{C}\xi_\alpha(X)$ is plotted in red line and $\mathcal{C}\xi_\alpha(Y)$ is plotted in blue line

.

Lemma 3.3. *Let $X_1, ..., X_n$ be IID nonnegative continuous random variables with common cdf F. Then, for $1 < \alpha \in \mathbb{N}$:*

(i) $\mathcal{C}\xi_\alpha(X_{n:n}) \leq n\mathbb{E}(X)$;

(ii) $\mathcal{C}\xi_\alpha(X_{n:n}) \leq n\mathcal{C}\xi_\alpha(X)$;

(iii) $\mathcal{C}\xi_\alpha(X_{1:n}) \leq \mathbb{E}(X)$,

where $X_{1:n} = \min\{X_1, ..., X_n\}$ *and* $X_{n:n} = \max\{X_1, ..., X_n\}$.

Proof. (i) From (3.5) we have

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X_{n:n}) &= \frac{1}{\alpha-1}\int_0^{+\infty}[F^n(x) - F^{n\alpha}(x)]dx \\
&= \frac{1}{\alpha-1}\int_0^{+\infty}[F(x) - F^\alpha(x)][F^{n-1}(x) + F^\alpha(x)F^{n-2}(x) + \ldots + F^\alpha(x)]dx \\
&\leq \frac{n}{\alpha-1}\int_0^{+\infty}[F(x) - F^\alpha(x)]dx \\
&\leq \frac{n}{\alpha-1}\int_0^{+\infty}[F(x) - (1 - \alpha\bar{F}(x))]dx \\
&= \frac{n}{\alpha-1}\int_0^{+\infty}[(\alpha-1)\bar{F}(x)]dx \\
&\leq n\int_0^{+\infty}\bar{F}(x)dx \\
&= n\mathbb{E}(X),
\end{aligned}$$

for nonnegative random variable X. In the fourth line, we use Bernoulli's inequality, that is $(1+u)^n \geq 1+nu$ for $x > -1$ and for $n \in \mathbb{N}$, so in this case we can write

$$F^{\alpha}(x) = \left(1-\bar{F}\right)^{\alpha} \geq 1 - \alpha\bar{F}(x).$$

(ii) From first inequality, it is deduced that $\mathcal{C}\xi_{\alpha}(X_{n:n}) \leq n\mathcal{C}\xi_{\alpha}(X)$ for $1 < \alpha \in \mathbb{N}$.
(ii) Using Bernoulli's inequality, for non-negative random variable X,

$$\begin{aligned}
\mathcal{C}\xi_{\alpha}(X_{1:n}) &= \frac{1}{\alpha-1}\int_0^{+\infty}\{[1-\bar{F}^n(x)] - [1-\bar{F}^n(x)]^{\alpha}\}dx \\
&\leq \frac{1}{\alpha-1}\int_0^{+\infty}[1-\bar{F}^n(x) - (1-\alpha\bar{F}^n(x))]dx \\
&= \int_0^{+\infty}\bar{F}^n(x)dx \\
&\leq \mathbb{E}(X).
\end{aligned}$$

□

Consider a system consisting of n components with IID lifetimes $X_1, ..., X_n$. Then, it is known that for series and parallel systems, the lifetimes of the system are $X_{1:n}$ and $X_{n:n}$, respectively. Thus, Lemma 3.3 provides upper bounds for cumulative Tsallis entropies of series and parallel systems based on the mean lifetime of their components.

Example 3.4. Let $X_1, ..., X_n$ be IID nonnegative continuous random variables uniformly distributed on $(0,1)$. In this case we have $\mathbb{E}(X) = \frac{1}{2}$ and

$$\mathcal{C}\xi_{\alpha}(X_{n:n}) = \frac{1}{\alpha-1}\int_0^1 [x^n - x^{n\alpha}]dx = \frac{n}{(n+1)(n\alpha+1)}.$$

So, for $1 < \alpha \in \mathbb{N}$ and for $n \geq 1$, $\mathcal{C}\xi_{\alpha}(X_{n:n}) \leq \frac{n}{2}$, which confirm (i).
As seen in Example 1, $\mathcal{C}\xi_{\alpha}(X) = \frac{1}{2(\alpha-1)}$, so it easy to prove that (ii) holds.
With an easy computation also (iii) holds. For simplicity, if we have a system consisting of only $n = 2$ components, we obtain:

$$\mathcal{C}\xi_{\alpha}(X_{1:2}) = \frac{1}{\alpha-1}\left[\frac{2}{3} - \frac{1}{2}B\left(1+\alpha, \frac{1}{2}\right)\right],$$

where $B\left(1+\alpha, \frac{1}{2}\right)$ is the Euler beta function (for $\alpha > 0$). So, for $1 < \alpha \in \mathbb{N}$, $\mathcal{C}\xi_{\alpha}(X_{1:2}) \leq \frac{1}{2}$, which confirm (iii) for $n = 2$.

3.1.1 Proportional Reversed Hazards Model

Let X and Y be two non-negative continuous random variables with distribution functions F and G, such that satisfy the proportional reversed hazards model in (2.10), that is

$$G(t) = (F(t))^{\theta}, \quad \text{for all } t \in S_X,$$

where S_X is the support of X and $\theta > 0$.

Lemma 3.4. *Let X and Y be two nonnegative continuous random variables with cumulative distribution functions F and G, respectively. If F and G satisfy the PRHR model, then*

$$(\alpha - 1)\mathcal{C}\xi_{\alpha}(Y) = (\alpha\theta - 1)\mathcal{C}\xi_{\alpha\theta}(X) - (\theta - 1)\mathcal{C}\xi_{\theta}(X). \tag{3.8}$$

Proof.

$$\begin{aligned}
\mathcal{C}\xi_{\alpha}(Y) &= \frac{1}{\alpha - 1}\int_0^{+\infty}[G(x) - G^{\alpha}(x)]dx \\
&= \frac{1}{\alpha - 1}\int_0^{+\infty}[F^{\theta}(x) - F^{\alpha\theta}(x)]dx \\
&= \frac{1}{\alpha - 1}\left\{\int_0^{+\infty}[F(x) - F^{\alpha\theta}(x)]dx - \int_0^{+\infty}[F(x) - F^{\theta}(x)]dx\right\},
\end{aligned}$$

from which we obtain (3.8). □

It is obvious that if X and Y have the same distribution then $\mathcal{C}\xi_{\alpha}(X) = \mathcal{C}\xi_{\alpha}(Y)$, but the converse doesn't hold. Using the proportional reversed hazards model we show a counterexample.
Suppose X has uniform distribution in $(0, b)$ with $b > 0$, i.e., $F(x) = x/b, 0 < x < b$ and X and Y satisfy the PRH model, then $\mathcal{C}\xi_{\alpha}(X) = \dfrac{b}{2(\alpha+1)}$ and from Lemma 3.4, we have

$$(\alpha - 1)\mathcal{C}\xi_{\alpha}(Y) = (\alpha\theta - 1)\frac{b}{2(\alpha\theta + 1)} - (\theta - 1)\frac{b}{2(\theta + 1)}.$$

If $\theta = \dfrac{1}{\alpha}$, then $\mathcal{C}\xi_{\alpha}(X) = \mathcal{C}\xi_{\alpha}(Y)$.
This means that $\mathcal{C}\xi_{\alpha}(X)$ does not uniquely characterize the distribution of X.

3.1.2 Computation of Cumulative Tsallis Entropy

In this section we provide an explicit expression for CTE for some continuous distributions. Moreover we note that, for a particular choice of α, CTE has an interesting relation with Gini index, $G(X)$, as we show in the following lemma.

Lemma 3.5. *Let X be nonnegative continuous random variable, with mean $\mathbb{E}(X)$, distribution function $F(x)$ and Gini index $G(X)$. Then, for $\alpha = 2$*

$$\mathcal{C}\xi_2(X) = \mathbb{E}\,(X)G(X).$$

Proof. For a nonnegative continuous random variable X, the Gini index is given by (see, for example, Hanada [20]):

$$G(X) := 1 - \frac{1}{\mathbb{E}(X)} \int_0^{+\infty} [1 - F(x)]^2 \, dx.$$

From this expression, we have:

$$\begin{aligned}
\mathbb{E}(X)\,G(X) &= \mathbb{E}(X) - \int_0^{+\infty} \bar{F}^2(x)dx \\
&= \int_0^{+\infty} \left[\bar{F}(x) - \bar{F}^2(x)\right] dx \\
&= \int_0^{+\infty} \left[\bar{F}(x)(1 - \bar{F}(x))\right] dx \\
&= \int_0^{+\infty} \left[F(x)(1 - F(x))\right] dx \\
&= \mathcal{C}\xi_2(X)
\end{aligned}$$

□

Note that, from the second equality, for $\alpha = 2$, then $\mathcal{C}\xi_2(X) = \xi_2(X)$.

In the following estimations, the expressions for Gini index are found in [17].

- Exponential distribution: $F(x) = 1 - e^{-\lambda x}$ for $x > 0$ and $\lambda > 0$.

$$\mathcal{C}\xi_\alpha(X) = \frac{H_\alpha - 1}{(\alpha - 1)\lambda},$$

where H_α is the harmonic number (α must be a positive integer) given by

$$H_\alpha = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\alpha} = \sum_{k=1}^{\alpha} \frac{1}{k}.$$

For $\alpha = 2$,

$$\mathcal{C}\xi_2(X) = \frac{1 + \frac{1}{2} - 1}{\lambda} = \frac{1}{2\lambda} = \mathbb{E}(X)\,G(X),$$

where $\mathbb{E}(X) = 1/\lambda$ and $G(X) = 1/2$.

- Uniform distribution on (a,b): $F(x) = (x-a)/(b-a)$ for $0 < a < x < b$.

$$\mathcal{C}\xi_\alpha(X) = \frac{b-a}{2(\alpha+1)}.$$

For $\alpha = 2$,

$$\mathcal{C}\xi_2(X) = \frac{b-a}{6} = \mathbb{E}(X)\, G(X),$$

where $\mathbb{E}(X) = \frac{a+b}{2}$ and $G(X) = \frac{b-a}{3(a+b)}$.

- Power function I distribution on (0,1): $F(x) = x^k$ for $0 < x < 1$ and $k > 0$.

$$\mathcal{C}\xi_\alpha(X) = \frac{k}{(k+1)(\alpha k+1)}.$$

For $\alpha = 2$,

$$\mathcal{C}\xi_2(X) = \frac{k}{(k+1)(2k+1)} = \mathbb{E}(X)\, G(X),$$

where $\mathbb{E}(X) = \frac{k}{k+1}$ and $G(X) = \frac{1}{2k+1}$.

- Fréchet distribution: $F(x) = exp\left[-\left(\frac{x}{\lambda}\right)^{-k}\right]$ for $x > 0$, $k > 0$ and $\lambda > 0$.

$$\mathcal{C}\xi_\alpha(X) = \frac{\lambda\left(1-\alpha^{\frac{1}{k}}\right)\Gamma\left[\frac{k-1}{k}\right]}{(\alpha-1)},$$

where $\Gamma[\cdot]$ is the Gamma function.
For $\alpha = 2$,

$$\mathcal{C}\xi_2(X) = \lambda\left(1-2^{\frac{1}{k}}\right)\Gamma\left[\frac{k-1}{k}\right] = \mathbb{E}(X)\, G(X),$$

where $\mathbb{E}(X) = \lambda\,\Gamma\left[\frac{k-1}{k}\right]$ and $G(X) = 2^{1/k} - 1$.

3.2 Dynamic Cumulative Tsallis Entropy

Rajesh and Sunoj [42] proposed the *dynamic cumulative residual Tsallis entropy* as

$$\begin{aligned}\psi_\alpha(X;t) = \xi_\alpha(X_t) &= \frac{1}{\alpha-1}\int_0^{+\infty}\left[\bar{F}_{X_t}(x) - \bar{F}^{\alpha}_{X_t}(x)\right]dx \\ &= \frac{1}{\alpha-1}\left(m(t) - \int_t^{+\infty}\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\alpha}dx\right),\end{aligned}$$

for $\alpha \neq 1$ and $\alpha > 0$, where $\bar{F}_{X_t}(x)$ is the survival function of the residual lifetime X_t and $m(t)$ is the mean residual lifetime defined in (1.13).

We propose the *dynamic cumulative Tsallis entropy* (DCTE) of a nonnegative absolutely continuous random variable X as

$$\mathcal{C}\psi_\alpha(X;t) = \mathcal{C}\xi_\alpha({}_tX) = \frac{1}{\alpha-1}\int_0^{+\infty}\left[F_{{}_tX}(x) - F^\alpha_{{}_tX}(x)\right]dx$$

for $\alpha \neq 1$ and $\alpha > 0$, where $F_{{}_tX}$ is the distribution function of the past lifetime ${}_tX = [X \mid X \leq t]$.
Dynamic cumulative Tsallis entropy, $\mathcal{C}\psi_\alpha(X;t)$, can be rewritten as

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\int_0^t\left[\frac{F(x)}{F(t)} - \frac{F^\alpha(x)}{F^\alpha(t)}\right]dx \\ &= \frac{1}{\alpha-1}\left(\tilde{\mu}(t) - \int_0^t\left(\frac{F(x)}{F(t)}\right)^\alpha dx\right),\end{aligned} \tag{3.9}$$

where $\tilde{\mu}(t)$ is the mean inactivity time of X defined in (2.5).

There is an identity for the dynamic cumulative residual Tsallis entropy and the dynamic cumulative Tsallis entropy.

Theorem 3.4. *Let X be a random variable with support in $[0,b]$ and symmetric with respect to $b/2$, that is $F(x) = \bar{F}(b-x)$ for $0 \leq x \leq b$. Then*

$$\mathcal{C}\psi_\alpha(X;t) = \psi_\alpha(X;b-t), \quad 0 \leq t \leq b.$$

Proof. We have

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\int_0^t\left[\frac{F(x)}{F(t)} - \frac{F^\alpha(x)}{F^\alpha(t)}\right]dx \\ &= \frac{1}{\alpha-1}\int_0^t\left[\frac{\bar{F}(b-x)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(b-x)}{\bar{F}^\alpha(b-t)}\right]dx \\ &= -\frac{1}{\alpha-1}\int_b^{b-t}\left[\frac{\bar{F}(y)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(y)}{\bar{F}^\alpha(b-t)}\right]dy \\ &= \frac{1}{\alpha-1}\int_{b-t}^b\left[\frac{\bar{F}(y)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(y)}{\bar{F}^\alpha(b-t)}\right]dy.\end{aligned}$$

□

Example 3.5. If X is uniformly distributed in $[0,b]$, for $0 \leq t \leq b$ we have

$$\mathcal{C}\psi_\alpha(X;t) = \frac{t}{2(\alpha+1)} \quad \text{and} \quad \psi_\alpha(X;t) = \frac{b-t}{2(\alpha+1)},$$

which is in agreement with Theorem 3.4.

As in Lemma 3.1, we now discuss the effect of increasing transformation on the DCTE.

Lemma 3.6. *Let $Y = \phi(X)$ an increasing differentiable function, then dynamic cumulative Tsallis entropy for the random variable Y is given by*

$$\begin{aligned}\mathcal{C}\psi_\alpha(Y;t) &= \frac{1}{(\alpha-1)F(\phi^{-1}(t))}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} F(x)\phi'(x)dx \\ &- \frac{1}{(\alpha-1)F^\alpha(\phi^{-1}(t))}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} F(x)^\alpha\phi'(x)dx.\end{aligned}$$

Remark 3.2. If $Y = aX + b$, with $a > 0$ and $b \geq 0$, then

$$\mathcal{C}\psi_\alpha(Y;t) = a\mathcal{C}\psi_\alpha\left(X;\frac{t-b}{a}\right), \quad t \geq b.$$

In the next theorem we use the definition of dispersive order, that can be found in the Appendix A in (A.10) and (A.12).

Theorem 3.5. *If $Y \geq_{DISP} (\leq_{DISP}) X$, then*

$$\mathcal{C}\psi_\alpha(Y;t) \geq (\leq)\mathcal{C}\psi_\alpha(X;\phi^{-1}(t)),$$

where ϕ is an increasing differentiable function which satisfies

$$\phi(x) - \phi(x^*) \geq x - x^* \quad \textit{whenever} \quad x \geq x^*. \tag{3.10}$$

Proof. The condition (3.10) implies that

$$\frac{d}{dx}\phi(x) \geq 1.$$

Note that from the hypotesis that $Y \geq_{DISP} X$, using th definition (A.12), we know that $Y =_{ST} \phi(X)$ for some ϕ which satisfies the condition (3.10). So we can apply Lemma 3.6, and we have

$$\begin{aligned}\mathcal{C}\psi_\alpha(Y;t) &= \frac{\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} \phi'(x)\left[F(x)F^{\alpha-1}(\phi^{-1}(t)) - F^\alpha(x)\right]dx}{(\alpha-1)F^\alpha(\phi^{-1}(t))} \\ &\geq \frac{1}{(\alpha-1)}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)}\left[\frac{F(x)}{F(\phi^{-1}(t))} - \frac{F^\alpha(x)}{F^\alpha(\phi^{-1}(t))}\right]dx \\ &= \mathcal{C}\psi_\alpha(X;\phi^{-1}(t)).\end{aligned}$$

For $Y \leq_{DISP} X$ the proof is similar. □

3.2.1 Monotonicy Properties of Dynamic Cumulative Tsallis Entropy

In this section we study some properties of $\mathcal{C}\psi_\alpha(X;t)$, in particular its monotonicy. Note that all the next results highlight the strict relation between the dynamic cumulative Tsallis entropy and the mean inactivity time.

Theorem 3.6. *Let X be a nonnegative absolutely continuous random variable with mean inactivity time $\tilde{\mu}(t)$ and reversed hazard rate $\tau(t)$; then $\mathcal{C}\psi_\alpha(X;t)$ is increasing (decreasing), if and only if,*

$$\mathcal{C}\psi_\alpha(X;t) \leq (\geq) \frac{\tilde{\mu}(t)}{\alpha}, \quad t > 0.$$

Proof. From the identity (3.9), we can write

$$(\alpha - 1)\mathcal{C}\psi_\alpha(X;t) = \tilde{\mu}(t) - \frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}. \tag{3.11}$$

Differentiating both side of (3.11) with respect to t, using (2.8), we have

$$(\alpha - 1)\mathcal{C}\psi'_\alpha(X;t) = \tau(t)\left[-\tilde{\mu}(t) + \alpha\frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}\right]. \tag{3.12}$$

Substituting (3.11) in (3.12), we obtain

$$\mathcal{C}\psi'_\alpha(X;t) = \tau(t)\left[\tilde{\mu}(t) - \alpha\mathcal{C}\psi_\alpha(X;t)\right]. \tag{3.13}$$

By the definition in in (1.10), $\tau(t) \geq 0$ for all t, and this complete the proof. $\square$

Example 3.6. Let X be a random variable with probability density:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,1] \cup [2,3] \\ 0 & \text{otherwise.} \end{cases}$$

Its mean inactivity time is given by:

$$\tilde{\mu}(t) = \begin{cases} \frac{t}{2} & \text{if} \quad 0 \leq t \leq 1 \\ t - \frac{1}{2} & \text{if} \quad 1 \leq t \leq 2 \\ \frac{t^2-2t+3}{2(t-1)} & \text{if} \quad 2 \leq t \leq 3, \end{cases}$$

and its dynamic cumulative Tsallis entropy is given by:

$$\mathcal{C}\psi_\alpha(X;t) = \begin{cases} \frac{t}{2(\alpha+1)} & \text{if} \quad 0 \le t \le 1 \\ \frac{1}{2(\alpha+1)} & \text{if} \quad 1 \le t \le 2 \\ \frac{(t-1)^{\alpha-1}-1}{(\alpha-1)(t-1)^\alpha} + \frac{t-1}{2(\alpha+1)} & \text{if} \quad 2 \le t \le 3. \end{cases}$$

$\mathcal{C}\psi_\alpha(X;t)$ and $\frac{\tilde{\mu}(t)}{\alpha}$ are plotted in Figure 3.3, where it is shown that $\mathcal{C}\psi_\alpha(X;t) \le \frac{\tilde{\mu}(t)}{\alpha}$ for all $t \ge 0$, with $\mathcal{C}\psi_\alpha(X;t)$ increasing for all t, as Theorem 3.6 states, while $\tilde{\mu}(t)$ does not exhibit the same behavior.

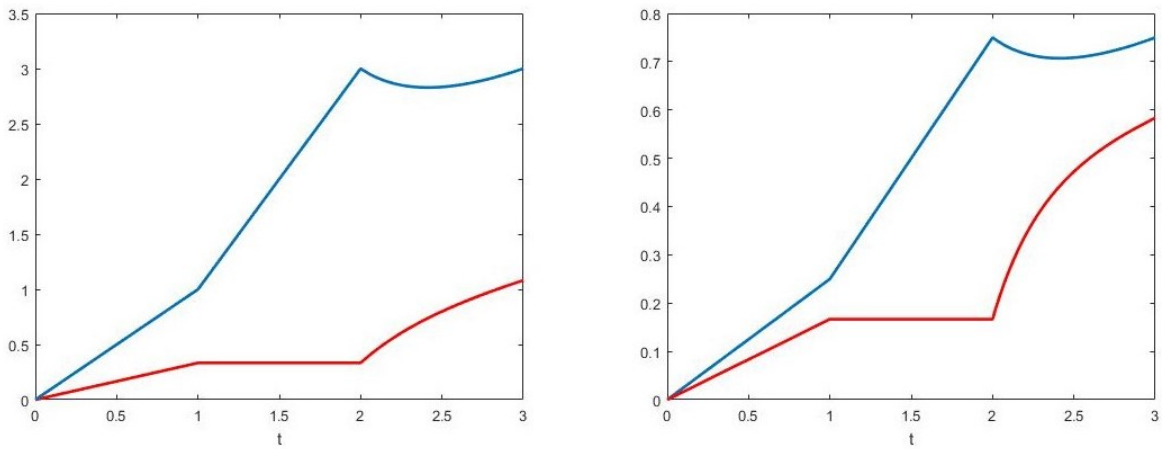

Figure 3.3: The figure refers to Example 3.6. The DCTE $\mathcal{C}\psi_\alpha(X;t)$ (red line) and $\frac{\tilde{\mu}(t)}{\alpha}$ (blue line) are plotted, on the left for $\alpha = \frac{1}{2}$ and on the right for $\alpha = 2$.

Theorem 3.7. *Let X be a nonnegative absolutely continuous random variable, with mean inactivity time $\tilde{\mu}(t)$ then*

$$\mathcal{C}\psi_\alpha(X;t) = \frac{\mathbb{E}(\tilde{\mu}(X)F^{\alpha-1}(X)|X \le t)}{F^{\alpha-1}(t)}.$$

Proof. From the identity (3.9) we have

$$\begin{aligned} \mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\left[\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\int_0^t F^\alpha(x)dx\right] \\ &= \frac{1}{\alpha-1}\left[\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\int_0^t \frac{d}{dx}\left(\tilde{\mu}(x)F(x)\right)F^{\alpha-1}(x)dx\right] \\ &= \frac{\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\left(\tilde{\mu}(t)F^\alpha(t) - (\alpha-1)\int_0^t[\tilde{\mu}(x)F^{\alpha-1}(x)f(x)]dx\right)}{\alpha-1} \\ &= \frac{\int_0^t[\tilde{\mu}(x)F^{\alpha-1}(x)f(x)]dx}{F^\alpha(t)}, \end{aligned} \tag{3.14}$$

where in the second equality we use the relation in (3.6) and in the third equality we use the integration by parts. The theorem is proved noting that

$$\mathbb{E}(\tilde{\mu}(X)F^{\alpha-1}(X)|X \leq t) = \frac{1}{F(t)}\int_0^t [\tilde{\mu}(x)F^{\alpha-1}(x)f(x)]dx.$$

□

A nonnegative random variable X is said to be increasing in mean inactivity time (IMIT) if $\tilde{\mu}(\cdot)$ is increasing on $(0, +\infty)$ (see Appendix B, definition (B.1)). Using this definition we obtain the following result.

Corollary 3.2. *Let X be a nonnegative random variable with increasing in mean inactivity time function $\tilde{\mu}(t)$, that is IMIT. Then*

$$\mathcal{C}\psi_\alpha(X;t) \leq \frac{\tilde{\mu}(t)}{\alpha}$$

Proof. If X is IMIT then $\tilde{\mu}(x) \leq \tilde{\mu}(t)$ for $x \leq t$. From (3.14), we have

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &\leq \frac{\int_0^t[\tilde{\mu}_X(t)F^{\alpha-1}(x)f(x)]dx}{F^\alpha(t)} \\ &= \frac{\tilde{\mu}_X(t)}{F^\alpha(t)}\int_0^t F^{\alpha-1}(x)f(x)dx = \frac{\tilde{\mu}(t)}{\alpha}.\end{aligned}$$

□

Let us give a new definition for a random variable X.

Definition 3.1. *X is said to have increasing (decreasing) dynamic cumulative Tsallis entropy (IDCTE (DDCTE)) if $\mathcal{C}\psi_\alpha(X;t)$ is increasing (decreasing) in $t \geq 0$.*

Remark 3.3. Combining Corollary 3.2 and Theorem 3.6 we obtain that:

$$X \in IMIT \Rightarrow \mathcal{C}\psi_\alpha(X;t) \leq \frac{\tilde{\mu}(t)}{\alpha} \Rightarrow X \in IDCTE.$$

In the following theorem we study the relations between $\mathcal{C}\psi_\alpha(X;t)$ and both the mean inactivity time $\tilde{\mu}(t)$, defined in (2.5) and the mean past lifetime $\mu(t)$, defined in (2.6).

Theorem 3.8. *Let X be a random variable with support in $[0,b]$, with b finite. For all $t \in [0,b]$ and for $\alpha > 1$ we have*

(i) $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$ *if and only if* $F(t) = \left(\frac{t}{b}\right)^{\frac{k}{1-k}}$, $k = \frac{c}{\alpha - c(\alpha-1)}$,

(ii) $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\mu(t)$ *if and only if* $F(t) = \left(\frac{t}{b}\right)^{\frac{1-k}{k}}$, $k = \frac{\alpha c}{\alpha + c(\alpha-1)}$,

where c is a constant such that $0 < c < 1$.

Proof. (i) Let $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$ for all $t \in [0,b]$. Differentiating both side with respect to t we have:

$$\mathcal{C}\psi'_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}'(t).$$

On the other hand, from (3.13) and from (2.8)

$$\tau(t)\left[\tilde{\mu}(t) - \alpha\mathcal{C}\psi_\alpha(X;t)\right] = \frac{c}{\alpha}\left[1 - \tau(t)\tilde{\mu}(t)\right].$$

Then using the assumption $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$, we obtain

$$\tau(t)\tilde{\mu}(t) = k,$$

where $k = \frac{c}{\alpha - c(\alpha-1)}$ is a constant such that $0 < k < 1$ for $0 < c < 1$ and $\alpha > 1$. Note that (2.8) gives

$$\tau(t) = \frac{1 - \tilde{\mu}'(t)}{\tilde{\mu}(t)}, \tag{3.15}$$

then we have

$$\tilde{\mu}'(t) = 1 - k.$$

This differential equation yields $\tilde{\mu}(t) - \tilde{\mu}(0) = (1-k)t$, but from definition we note that $\tilde{\mu}(0) = 0$, so $\tilde{\mu}(t) = (1-k)t$. Finally, we obtain

$$\tau(t) = \frac{k}{1-k}\frac{1}{t}.$$

By the relation in (1.11), this implies that

$$F(t) = \left(\frac{t}{b}\right)^{\frac{k}{(1-k)}}, \qquad 0 \le t \le b.$$

(ii) Let $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\mu(t)$ for all $t \in [0,b]$. By differentiating with respect to t we have

$$\mathcal{C}\psi'_\alpha(X;t) = \frac{c}{\alpha}\mu'(t).$$

By the relation in (2.7), the derivative of mean past lifetime with respect to t can be expressed as

$$\mu'(t) = 1 - \tilde{\mu}(t),$$

so that $\tilde{\mu}(t) - \alpha \mathcal{C}\psi_\alpha(X;t) = \dfrac{c}{\alpha}\tilde{\mu}(t)$. Using the assumption $\mathcal{C}\psi_\alpha(X;t) = \dfrac{c}{\alpha}\mu(t)$ and (2.7) we obtain

$$\tilde{\mu}(t) = kt,$$

where $k = \dfrac{\alpha c}{\alpha + c(\alpha - 1)}$ is a constant such that $0 < k < 1$ for $0 < c < 1$ and $\alpha > 1$. So, by the relation in (3.15), we have

$$\tau(t) = \frac{1-k}{tk}.$$

Again by the relation in (1.11), this implies that

$$F(t) = \left(\frac{t}{b}\right)^{\frac{1-k}{k}}, \qquad 0 \le t \le b.$$

The converse for both (i) and (ii) is quite straightforward. $\square$

Chapter 4

Interlude - Coherent Systems and Related Topics

4.1 Structural Properties of Coherent Systems

A system with n components is a Boolean function $\psi : \{0,1\}^n \to \{0,1\}$ which has two possible states:

- $\psi = 1$, if the system is working,
- $\psi = 0$, if the system has failed.

Generally, it is assumed that the state of the system is completely determined by the states of the components, so that

$$\psi = \psi(x_1, x_2, \dots, x_n)$$

where

$$x_i = \begin{cases} 1 & \text{if the } i\text{-th component is working} \\ 0 & \text{if the } i\text{-th component has failed.} \end{cases}$$

The function ψ is called the structure function of the system.
A system ψ is a *coherent system* if:

1. ψ is increasing in every component;
2. for every $i = 1, 2, \dots, n$, ψ is strictly increasing in x_i for some specific values of $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

The first property means that if a system is working and we replace a failed component by a functioning component, then the system must be working. The second property says that every component is relevant for the system in some situations, that is, for each i there exists $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ such that the system works, that is $\psi(x_1, x_2, \ldots, x_n) = 1$ if and only if $x_i = 1$. Hence $\psi(0,0,\ldots,0) = 0$ and $\psi(1,1,\ldots,1) = 1$.

Series and parallel systems are the two most used coherent systems. A series system functions if and only if each components works, so it is defined by the structure function

$$\psi(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n),$$

or, equivalently,

$$\psi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$$

for $x_i \in \{0,1\}$, for each i.
Analogously, a parallel system functions if and only if at least one component works, so it is defined by the structure function

$$\psi(x_1, x_2, \ldots, x_n) = \max(x_1, x_2, \ldots, x_n).$$

or, equivalently,

$$\psi(x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i) = \coprod_{i=1}^{n} x_i$$

for $x_i \in \{0,1\}$, for each i.
In general, the k-out-of-n system, is the system which works if at least k of its n components work. The structure function of the k-out-of-n system is

$$\psi(x_1, x_2, \ldots, x_n) = x_{n-k+1:n},$$

where $(x_{1:n}, x_{2:n}, \ldots, x_{n:n})$ is the increasing ordered vector obtained from the vector $(x_1, x_2, \ldots, x_n)$. In particular, $k = 1$ represents the parallel system and $k = n$ represents series systems.

For the analysis of the coherent systems is useful to introduce two important subsets of the set of indices that represent the components of the system.

Definition 4.1 (Path set). *A path set for a given system ψ with n components is a set of indices $P \subseteq \{1, 2, \ldots, n\}$ such that if the components in P work, then the system works.*

Definition 4.2 (Minimal path set). *A path set P is a minimal path set if it does not contain other path sets.*

Note that a system works if and only if at least one of the series systems obtained from its minimal path sets works. Then the structure function of a system can be written in terms of its minimal path sets as

$$\psi(x_1, x_2, \ldots, x_n) = \max_{1 \le j \le r} \min_{i \in P_j} x_i, \tag{4.1}$$

where the sets $P_1, P_2, \ldots, P_r$ are all the minimal path sets of the system. It is easy to prove that the minimal path sets of a coherent system satisfy

$$P_1 \cup P_2 \cup \cdots \cup P_r = \{1, 2, \ldots, n\}.$$

Definition 4.3 (Cut set). *A cut set for a given system ψ with n components is a set of indices $K \subseteq \{1, 2, \ldots, n\}$ such that if the components in K fail, then the system fails.*

Definition 4.4 (Minimal cut set). *A cut set K is a minimal cut set if it does not contain other cut sets.*

Note that a system fails if and only if at least one of the parallel systems obtained from its minimal cut sets fails. Then the structure function can be written in terms of its minimal cut sets as

$$\psi(x_1 x_2 \cdots x_n) = \min_{1 \le j \le s} \max_{i \in K_j} x_i, \tag{4.2}$$

where the sets $K_1, K_2, \ldots, K_s$ are all the minimal cut sets of the system. It is easy to prove that the minimal cut sets of a coherent system satisfy

$$K_1 \cup K_2 \cup \cdots \cup K_r = \{1, 2, \ldots, n\}.$$

A system is completely determined by its minimal cut sets and its minimal path sets.

Let $X_1, \ldots, X_n$ are the lifetimes of the components in a coherent system. In the general case, from (4.1) and (4.2), the lifetime T of the coherent system is given by

$$T = \min_{i=1,\ldots r} \max_{j \in K_i} X_j = \max_{i=1,\ldots s} \min_{j \in P_i} X_j. \tag{4.3}$$

So the lifetime of the system can be represented by $T = \psi(X_1, \ldots, X_n)$. Moreover note that the lifetime of a k-out-of-n system coincides with the order statistic $X_{n-k+1:n}$ from $X_1, \ldots, X_n$.

The distribution function F_T of the system T is

$$F_T(t) = P(T \leq t) = P\left(\min_{i=1,\dots r} \max_{j \in K_i} X_j \leq t\right) = P\left(\cup_{i=1}^{r} \{\max_{j \in K_i} X_j \leq t\}\right)$$

and, by using the inclusion-exclusion formula,

$$F_T(t) = \sum_{i=1}^{r} P\left(\max_{j \in K_i} X_j \leq t\right) - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} P\left(\max_{\ell \in K_i \cup K_j} X_\ell \leq t\right) + \dots + (-1)^{r+1} P\left(\max_{j \in K_1 \cup \dots \cup K_r} X_j \leq t\right)$$

for all t.

Let $X_1, \dots, X_n$ are the lifetimes of the components in a coherent system. If they are independent and identically distributed (IID), with the common distribution of the components represented by F, it holds a theorem due to Samaniego [48].

Theorem 4.1. *If T is the lifetime of a coherent system with IID component lifetimes $X_1, \dots, X_n$, having a common continuous distribution F, then*

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t),$$

where $s_1, \dots, s_n$ are coefficients such that $\sum_{i=1}^{n} s_i = 1$ and that don't depend on F and where $\bar{F}_{i:n}$ is the reliability function of the order statistic $X_{i:n}$. Moreover, these coefficients satisfy $s_i = P(T = X_{i:n})$ for $i = 1, \dots, n$.

The vector $s = (s_1, \dots, s_n)$ is called the *signature* of the system.

Let $X_1, \dots, X_n$ be the lifetimes of the components in a coherent system. They are exchangeable (EXC), if

$$(X_1, \dots, X_n) =_{ST} (X_{\sigma(1)}, \dots, X_{\sigma(n)}),$$

for any permutation σ. Note that EXC implies ID (that is identically distributed, dependent or independent).
So, if $X_1, \dots, X_n$ are EXC, there are two definitions due to Navarro, Ruiz and Sandoval [37].

Definition 4.5 (Minimal signature)**.** *If T is a coherent system with exchangeable components, the minimal signature is the vector $a = (a_1, \dots, a_n) \in \mathbb{R}$, with $\sum_{i=1}^{n} a_i = 1$, such that*

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t), \tag{4.4}$$

where $\bar{F}_{1:i}(t)$ is the reliability function of $X_{1:i} = \min\{X_1, \dots, X_n\}$.

Definition 4.6 (Maximal signature). *If T is a coherent system with exchangeable components, the maximal signature is the vector $b = (b_1, \dots, b_n) \in \mathbb{R}$, with $\sum_{i=1}^n b_i = 1$, such that*

$$\bar{F}_T(t) = \sum_{i=1}^n b_i \bar{F}_{i:i}(t), \tag{4.5}$$

where $\bar{F}_{i:i}(t)$ is the reliability function of $X_{i:i} = \max\{X_1, \dots, X_n\}$.

Note that minimal and maximal signatures do not depend on the joint distribution and that they can have negative components. Also note that it is possible to compute the system reliability from the minimal (maximal) signatures and series (parallel) reliability functions.
Table 4.1 contains the minimal and maximal signatures for all the possible coherent systems with three exchangeable components. Note that the minimal signature of a system is equal to the maximal signature of its dual system (that is the system $\psi^D(x) = 1 - \psi(1 - x)$, given a structure ψ).

System	$\phi(X)$	Minimal signature (a_1, a_2, a_3)	Maximal signature (b_1, b_2, b_3)
Series	$\min(X_1, X_2, X_3) = X_{(1:3)}$	$(0, 0, 1)$	$(3, -3, 1)$
Series-parallel	$\min(X_1, \max(X_2, X_3))$	$(0, 2, -1)$	$(1, 1, -1)$
2-out-of-3	$\max_{1 \le i < j \le 3} \min(X_i, X_j) = X_{(2:3)}$	$(0, 3, -2)$	$(0, 3, -2)$
Parallel-series	$\max(X_1, \min(X_2, X_3))$	$(1, 1, -1)$	$(0, 2, -1)$
Parallel	$\max(X_1, X_2, X_3) = X_{(3:3)}$	$(3, -3, 1)$	$(0, 0, 1)$

Table 4.1: Minimal and maximal signatures for coherent system with three exchangeable components.

4.2 Distortion Functions

The *distorted distribution* associated to a distribution function F and to an increasing continuous function, called *distortion function* $q : [0, 1] \to [0, 1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)).$$

Also for the reliability functions $\bar{F}$ there is a similar expression

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t))$$

where $\bar{F}_q = 1 - F_q$ and $\bar{q}(u) = 1 - q(1-u)$ is called the *dual distortion function*. Note that $\bar{q}$ is also a distortion function, that is, it is continuous, increasing and satisfies $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$.

In 2014, Navarro *et al.* [32] extended the concept of the distorted distributions to the concept of *generalized distorted distributions* which are univariate distribution functions obtained by distorting n distribution functions.
The generalized distorted distribution associated to n distribution functions $F_1, \ldots, F_n$ and to an increasing continuous multivariate distortion function $Q : [0,1]^n \to [0,1]$ such that $Q(0, \ldots, 0) = 0$ and $Q(1, \ldots, 1) = 1$ is defined by

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{4.6}$$

There is a similar expression for the respective reliability functions

$$\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \tag{4.7}$$

where $\bar{F}_i = 1 - \bar{F}_i$, for $i = 1, \ldots, n$, $\bar{F}_Q = 1 - \bar{F}_Q$ and

$$\bar{Q}(u) = 1 - \bar{Q}(1-u_1, 1-u_2, \ldots, 1-u_n)$$

is called the *dual distortion function*. Note that $\bar{Q}$ is also a distortion function, that is, it is continuous, increasing and satisfies $\bar{Q}(0, 0, \ldots, 0) = 0$ and $\bar{Q}(1, 1, \ldots, 1) = 1$.

Let T is a random variable that represents the lifetime of a coherent system, $X_1, \ldots, X_n$ are the lifetimes of the units and $F_i(x_i) = P(X_i \leq x_i)$ the marginal distribution function of the i-th component for $i = 1, \ldots, n$. It is clear that also the distribution function and the reliability function, F_T and $\bar{F}_T$, of the system can be computed from $F_1, \ldots, F_n$ using the generalized distorted distributions given in (4.6) and (4.7) as

$$F_T(t) = Q(F_1(t), \ldots, F_n(t)) \tag{4.8}$$

and

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).$$

The distorted distribution model is a very flexible model, which is useful to study different concepts in a unified way. For example, it is possible to obtain ordering properties for distorted distributions as shown in the following theorems given in [31, 33, 35].

Theorem 4.2. *Let $F_{q_1} = q_1(F)$ and $F_{q_2} = q_2(F)$ be two distorted distributions of two random variables X_1 and X_2 based on the same distribution function F and on the distortion functions q_1 and q_2, respectively. Let $\bar{q}_1$ and $\bar{q}_2$ be the respective dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ *for all F if and only if* $\bar{q}_1 \leq \bar{q}_2$ *in* $[0,1]$.

(ii) $X_1 \leq_{HR} X_2$ *for all F if and only if* $\bar{q}_2/\bar{q}_1$ *is decreasing in* $[0,1]$.

(iii) $X_1 \leq_{RHR} X_2$ *for all F if and only if* q_2/q_1 *is increasing in* $[0,1]$.

(iv) $X_1 \leq_{LR} X_2$ *for all F if and only if* $\bar{q}_2'/\bar{q}_1'$ *is decreasing in* $[0,1]$.

The analogous results hold for generalized distorted distributions.

Theorem 4.3. *Let $F_{Q_1} = Q_1(F_1, \ldots, F_n)$ and $F_{Q_2} = Q_2(F_1, \ldots, F_n)$ be two generalized distorted distributions of two random variables X_1 and X_2 based on the same distribution functions $F_1, \ldots, F_n$ and on the generalized distortion functions Q_1 and Q_2, respectively. Let $\bar{Q}_1$ and $\bar{Q}_2$ be the respective generalized dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ *for all $F_1, \ldots, F_n$ if and only if* $\bar{Q}_1 \leq \bar{Q}_2$ *in* $[0,1]^n$.

(ii) $X_1 \leq_{HR} X_2$ *for all $F_1, \ldots, F_n$ if and only if* $\bar{Q}_2/\bar{Q}_1$ *is decreasing in* $[0,1]^n$.

(iii) $X_1 \leq_{RHR} X_2$ *for all $F_1, \ldots, F_n$ if and only if* Q_2/Q_1 *is increasing in* $[0,1]^n$.

Note that the comparisons in the theorems 4.2 and 4.3 are distribution-free with respect to the common component reliability.

4.3 Copulas

The definition of *copula* originated in the paper by Sklar [52].

Definition 4.7. *A function $C : [0,1]^d \to [0,1]$ is a d-copula if, and only if, the following conditions hold:*

1. $C(u_1, \ldots, u_d) = 0$ *if* $u_j = 0$ *for at least one index* $j \in \{1, \ldots, d\}$;
2. *when all the arguments of C are equal to 1, but possibly for the j-th one, then*

$$C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j;$$

3. C is d-increasing.

Properties *1.* and *2.* together are called the boundary conditions of a d-copula. In particular, by property *2.* the univariate marginals of C are uniform on $[0, 1]$. Note that from the previous conditions C is continuous.

A particular case is when $d = 2$.

Definition 4.8. *A function $C : [0, 1]^2 \to [0, 1]$ is a 2-copula if, and only if, the following conditions hold:*

1. *$C(0, u) = C(u, 0) = 0$, for every $u \in [0, 1]$;*
2. *$C(1, u) = C(u, 1) = 1$, for every $u \in [0, 1]$*
3. *C is 2-increasing, that is for all a_1, a_2, b_1, b_2 in $[0, 1]$, with $a_1 \leq b_1$ and $a_2 \leq b_2$,*

$$V_C(]a, b]) = C(a_1, a_2) - C(a_1, b_2) - C(b_1, a_2) + C(b_1, b_2) \geq 0,$$

where $V_C(]a, b])$ expresses the probability that a random variable U distributed according to the copula C takes values in $]a, b]$.

Suppose that T is the random variable representing the lifetime of a system and $X_1, \ldots, X_n$ are the lifetime of the units. If the component lifetimes are dependent, then this dependancy can be represented by the joint distribution of the random vector $(X_1, \ldots, X_n)$

$$F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n). \tag{4.9}$$

Every joint distribution function for a random vector implicitly contains the description of both the marginal behaviour and of their dependence structure. The copula approach provides a way of highlighting the description of the dependence structure.

Theorem 4.4. *Let $(F_1, \ldots, F_d)$ be univariate distribution functions and let C be any d-copula. Then the function $H : \mathbb{R}^d \to [0, 1]$ defined, for every point $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, by*

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

is a d-dimensional distribution function with marginal distribution given by $(F_1, \ldots, F_d)$.

So the joint distribution in (4.9) can be written in terms of copula function, as it is shown in the following theorem.

Theorem 4.5 (Sklar's theorem). *Let X be a random vector $X = (X_1, \dots, X_n)$, let $P(X_1 \leq x_1, \dots, X_n \leq x_n)$ be the joint distribution function of X and let $F_i(x_i) = P(X_i \leq x_i)$ for $i = 1, \dots, n$ be its marginals. Then there exists a n-copula C such that, for every point $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^n$,*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \tag{4.10}$$

If the marginals $F_1, \dots, F_n$ are continuous, then the copula C is uniquely defined.

All the properties described above can be also referred to a joint reliability function

$$\bar{F}(x_1, \dots, x_n) = P(X_1 \geq x_1, \dots, X_n \geq x_n),$$

that can be expressed in terms of copula function thanks to Sklar's theorem as

$$\bar{F}(x_1, \dots, x_n) = K(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)),$$

where K is the so called *survival copula*. Note that also the survival copula K is a copula, that is, it is a distribution function and it is not a reliability function. Actually, the survival copula of the random vector $(X_1, \dots, X_n)$ is the distribution function of the random vector $(\bar{F}_1(X_1), \dots, \bar{F}_n(X_n))$, where F_i is the marginal reliability function of X_i. Moreover K can be computed from C and vice versa.

Proposition 4.1. *If $T = \psi(X_1, \dots, X_n)$ is the lifetime of a coherent system, then*

$$\bar{F}_T(t) = \bar{Q}_\psi^K(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{4.11}$$

where $\bar{Q}_\psi^K$ is a distortion function that depends on the structure function ψ and on the survival copula K of $(X_1, \dots, X_n)$. Moreover,

$$\bar{Q}_\psi^K(u_1, \dots, u_n) = \sum_{I \subseteq \{1, \dots, r\}} (-1)^{|I|+1} K_I(u_1, \dots, u_n),$$

where

$$K_I(u_1, \dots, u_n) = K(u_1^I, \dots, u_n^I)$$

where $u_j^I = u_j$ whenever $j \in \cup_{i \in I} P_i$ or $u_j^I = 1$ whenever $j \notin \cup_{i \in I} P_i$ with $P_1, \dots, P_r$ the minimal path set of the system.

The function $\bar{Q}_\psi^K$ is a multivariate (dual) distortion function defined in (4.7) and it is important to highlight that it depends both on the structure function ψ and on the dependance between the component lifetimes, that is well expressed by survival copula K.

For example, if T is the series system $X_{1:n}$, then $\bar{Q}_\psi^K = K$. Hence, the survival copulas are valid distortion functions, but, in general, $\bar{Q}_\psi^K$ needs not to be a copula function.

In the same way, the distribution function of the system can also be written as

$$F_T(t) = Q_\psi^C(F_1(t), \ldots, F_n(t)), \tag{4.12}$$

where Q_ψ^K is a multivariate distortion function defined in (4.6) that depends on the structure function ψ and on the copula C of $(X_1, \ldots, X_n)$.

For example, if T is the parallel system $X_{n:n}$, then $Q_\psi^C = C$. Hence, the copulas are valid distortion functions, but, in general, Q_ψ^K needs not to be a copula function.

There are some particular cases of interest in which the preceding expression can be simplified.

If the component are identically distributed (ID), that is $F_1 = \ldots = F_n = F$, the equation (4.11) and (4.12) can be written, respectively as

$$\bar{F}_T(t) = \bar{q}_\psi^K(\bar{F}(t))$$

where $\bar{q}_\psi^K(x) = \bar{Q}_\psi^K(x, \ldots, x)$ and

$$F_T(t) = q_\psi^C(F(t)), \tag{4.13}$$

where $q_\psi^C(x) = Q_\psi^C(x, \ldots, x)$.

In the IID case (4.4) and (4.5) can be written, respectively, as

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}^i(t),$$

where $a = (a_1, \ldots, a_n)$ is the minimal signature and $\bar{F}(t)$ is the common relability function of the components and

$$F_T(t) = \sum_{i=1}^{n} b_i F^i(t), \tag{4.14}$$

where $b = (b_1, \ldots, b_n)$ is the maximal signature and $F(t)$ is the common distribution function of the components.

Chapter 5

New Measures of Past Entropy - Part II

5.1 Generalized Cumulative Past Entropy

As we have seen in the Chapter 1, starting from Shannon entropy and, later on, from Rao's cumulative residual entropy, a number of alternative measure of information have been proposed in the literature.
In particular, recently, some extensions of the last one have been proposed.
The *generalized cumulative residual entropy* (GCRE) was defined by Psarrakon and Navarro [41] for a nonnegative random variable X with reliability function $\bar{F}(\cdot)$ as

$$\mathcal{E}_n(X) = \int_0^{+\infty} \bar{F}(x) \frac{[\Lambda_X(x)]^n}{n!} dx, \qquad \text{for } n = 0, 1, 2, \dots \tag{5.1}$$

where $\Lambda_X(x) = -\log \bar{F}(x)$ and where, by convention, $0(\log 0)^n = 0$ for $n = 1, 2, \dots$. Note that $\mathcal{E}_0(X) = E(X)$ and $\mathcal{E}_1(X) = \mathcal{E}(X)$, where $\mathcal{E}(X)$ is the cumulative residual entropy defined in (1.15).

In analogy with (5.1), Kayal [21] defined the *generalized cumulative past entropy* (GCPE) of a nonnegative random variable X with distribution function $F(\cdot)$ as

$$\mathcal{CE}_n(X) = \int_0^{+\infty} F(x) \frac{[T_X(x)]^n}{n!} dx, \qquad \text{for } n = 0, 1, 2, \dots$$

where $T_X(t) = -\log F(t)$. Note that $\mathcal{CE}_1(X) = \mathcal{CE}(X)$. However $\mathcal{CE}_0(X) = +\infty$. Kayal [21] also introduced the *dynamic generalized cumulative past entropy* (DGCPE), as

$$\mathcal{CE}_n(X;t) = \mathcal{CE}_n({}_tX) = \frac{1}{n!} \int_0^t \frac{F(x)}{F(t)} \left[-\log \frac{F(x)}{F(t)} \right]^n dx, \qquad t > 0: \; F(t) > 0,$$

for $n = 0, 1, 2, \ldots$ where ${}_tX = (X|X \leq t)$ is the random variable that describes the past lifetime of a system at age t. Note that $\mathcal{CE}_1(X;t) = \mathcal{CE}(X;t)$, where $\mathcal{CE}(X;t)$ is the dynamic version of cumulative past entropy defined in (2.11), and

$$\mathcal{CE}_0(X;t) = \int_0^t \frac{F(x)}{F(t)} dx = \tilde{\mu}(t), \tag{5.2}$$

where $\tilde{\mu}(\cdot)$ is the mean inactivity time of X.

5.2 Some Properties of DGCPE Functions

An important property to require for DGCPE $\mathcal{CE}_n(X;t)$ is that, under some assumptions and for a fixed n, it uniquely determines the distribution function F_X. To prove that this property holds, we need some preliminary results. The first one is a property of systems of differential equations (see, e.g., [19]).

Theorem 5.1. *Let U be an open set in $\mathbb{R}^{n+1}$ and let $f_i : U \to \mathbb{R}$ be continuous functions such that $\frac{\partial f_i}{\partial y_j}$ is continuous on U for all $i,j = 1, \ldots, n$. Then, for every $(x_0, z_1, \ldots, z_n) \in U$, there exist a unique solution of the following system of differential equations*

$$\begin{cases} y_1' &= f_1(x, y_1, \ldots, y_n) \\ & \cdots \\ y_n' &= f_n(x, y_1, \ldots, y_n) \end{cases}$$

such that $y_i(x_0) = z_i$ for $i = 1, \ldots, n$ which can be continued up to the boundary of U.

The second result is a property of DGCPE functions stated in Theorem 4.9 of [21].

Theorem 5.2. *Let X be a nonnegative absolutely continuous random variable, then*

$$\mathcal{CE}_n'(X;t) = \tau(t) \left[\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)\right] \tag{5.3}$$

for $n = 1, 2, \ldots,$ where $\tau(\cdot)$ is the reversed hazard rate function of X.

Now we can state the following result.

Theorem 5.3. *Let X be an absolutely continuous random variable with distribution function F and differentiable DGCPE functions $\mathcal{CE}_i(X;t)$ for $i = 0, 1, \ldots, n$ and $t \geq 0$. Let us also assume that*

$$\mathcal{CE}_n(X;t) \neq \mathcal{CE}_{n-1}(X;t) \tag{5.4}$$

for all $t \geq 0$ such that $F(t) > 0$ and let $t_0 \geq 0$. Then the values $\mathcal{CE}_i(X;t_0)$ for $i = 0, 1, \ldots, n-1$ and the function $\mathcal{CE}_n(X;t)$ uniquely determine F.

Proof. From (5.3), for all $t \geq 0$ such that $F_X(t) < 1$, we have

$$\begin{cases} \mathcal{CE}'_1(X;t) &= \tau(t)\left[\mathcal{CE}_0(X;t) - \mathcal{CE}_1(X;t)\right] \\ \mathcal{CE}'_2(X;t) &= \tau(t)\left[\mathcal{CE}_1(X;t) - \mathcal{CE}_2(X;t)\right] \\ & \dots \\ \mathcal{CE}'_n(X;t) &= \tau(t)\left[\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)\right]. \end{cases}$$

From the last equality and from the hypothesis in (5.4), we can write

$$\tau(t) = \frac{\mathcal{CE}'_n(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)}.$$

Substituting in the previous equalities, we obtain

$$\mathcal{CE}'_j(X;t) = \frac{\mathcal{CE}_{j-1}(X;t) - \mathcal{CE}_j(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)}\mathcal{CE}'_n(X;t), \qquad j = 1, 2, \dots, n-1.$$

Recall that $\mathcal{CE}_0(X;t) = \tilde{\mu}(t)$ as it is shown in (5.2). From the relation (2.8) we have

$$\mathcal{CE}'_0(X;t) = 1 - \tau(t)\mathcal{CE}_0(X;t)$$

and then we get the expression:

$$\mathcal{CE}'_0(X;t) = 1 - \frac{\mathcal{CE}'_n(X;t)\mathcal{CE}_0(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)}.$$

If $\mathcal{CE}_n(X;t)$ is a known function, we have to solve the following system of differential equations:

$$\begin{cases} y'_0 = f_0(t, y_0, \dots, y_{n-1}) \\ y'_1 = f_1(t, y_0, \dots, y_{n-1}) \\ \dots \\ y'_{n-1} = f_{n-1}(t, y_0, \dots, y_{n-1}), \end{cases}$$

where $y_j = \mathcal{CE}_j(X;t)$ for $j = 0, 1, 2, \dots, n-1$,

$$f_0(t, y_0, \dots, y_{n-1}) = \frac{\mathcal{CE}'_n(X;t)y_0}{y_{n-1} - \mathcal{CE}_n(X;t)} + 1$$

and

$$f_j(t, y_0, \dots, y_{n-1}) = \frac{y_{j-1} - y_j}{y_{n-1} - \mathcal{CE}_n(X;t)}\mathcal{CE}'_n(X;t), \qquad j = 1, 2, \dots, n-1.$$

From Theorem 5.1, this system of differential equations has a unique solution when we fix the initial values $\mathcal{CE}_j(X;t_0)$, with $j = 0, 1, \dots, n-1$. So $\mathcal{CE}_n(X;t)$ uniquely determines $\mathcal{CE}_j(X;t)$ for $j = 0, 1, \dots, n-1$ and in particular $\mathcal{CE}_0(X;t) = \tilde{\mu}(t)$. Then, from(2.9), we can obtain $F(t)$. $\square$

The following example shows how the preceding theorem can be used to characterize particular models.

Example 5.1. Let us consider a random variable X with the following Power distribution $F(t) = t^c/b^c$ for $t \in [0, b]$, where $b, c > 0$. Then, a straightforward calculation shows that

$$\mathcal{CE}_0(X;t) = \frac{t}{c+1} \qquad \text{and} \qquad \mathcal{CE}_1(X;t) = \frac{c\,t}{(c+1)^2}$$

for $t \in [0, b]$. Note that

$$\mathcal{CE}_1(X;t) = \frac{c\,t}{(c+1)^2} = \frac{c}{c+1}\,\mathcal{CE}_0(X;t) < \mathcal{CE}_0(X;t)$$

for all $t \in [0, b]$. Therefore, this model can be characterized by using Theorem 5.3 from the preceding expression for $\mathcal{CE}_1(X;t)$ and the initial value $\mathcal{CE}_0(X;0) = 0$.

5.3 GCPE of Coherent Systems

Let X be a random variable absolutely continuous with distribution function F and density function f. Suppose that F has support (α, β) with $0 \leq \alpha < \beta \leq +\infty$, then the GCPE can be written as

$$\begin{aligned}
\mathcal{CE}_n(X) &= \frac{1}{n!}\int_0^{+\infty} F(x)[-\log(F(x))]^n dx \\
&= \frac{1}{n!}\int_\alpha^\beta \phi_n(F(x))dx \\
&= \frac{1}{n!}\int_0^1 \frac{\phi_n(u)}{f(F^{-1}(u))}du \qquad (5.5)
\end{aligned}$$

for $n = 1, 2, \ldots$, where F^{-1} is the inverse function of F in (α, β) and where

$$\phi_n(u) := u[-\log(u)]^n \geq 0, \qquad 0 < u < 1.$$

Note that $\phi(0) = \phi(1) = 0$.

Example 5.2. Let X have a uniform distribution in $(0, b)$, then

$$\begin{aligned}
\mathcal{CE}_n(X) &= \frac{1}{n!}\int_0^{+\infty} F(x)[-\log(F(x))]^n dx \\
&= \frac{1}{n!}\int_0^{b} \frac{x}{b}\left[-\log\frac{x}{b}\right]^n dx \\
&= \frac{b}{n!2^{n+1}}\left|\Gamma\left(n+1, -2\log\frac{x}{b}\right)\right|_0^b \\
&= \frac{b}{n!2^{n+1}}\Gamma(n+1) = \frac{b}{2^{n+1}}, \qquad n = 1, 2, \ldots. \qquad (5.6)
\end{aligned}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function and $\Gamma(\cdot) = \Gamma(\cdot, 0)$ is the Gamma function. Note that (5.6) is the same result that we have evaluating

$$\frac{c}{n!}\int_0^1 \phi_n(u)du, \qquad n = 1, 2, \ldots,$$

considering that in this case $f(F^{-1}(u)) = f(u)$.

Let T be the lifetime of a coherent system with n identically distributed (ID) components. Recalling the identity (4.13) then its distribution function F_T can be written as

$$F_T(t) = q(F(t))$$

where F is the common distribution of the component lifetimes and where q is a distortion function.
In particular, if the components are indipendent and identically distributed (IID), we can use the result in (4.14), then

$$q(u) = \sum_{i=1}^{m} b_i u^i,$$

where the vector $(b_1, b_2, \ldots, b_m)$ is the maximal signature of the system.

Example 5.3. Referring to Table 4.1, for a 2-out-of-3 system with IID components, we have $q(u) = 3u^2 - 2u^3$.

Hence, proceeding as above, the GCPE of T can be written as

$$\begin{aligned} \mathcal{CE}_n(T) &= \frac{1}{n!}\int_0^{+\infty} F_T(x)[-\log(F_T(x))]^n dx \\ &= \frac{1}{n!}\int_0^{+\infty} \phi_n(F_T(x))dx \\ &= \frac{1}{n!}\int_0^{+\infty} \phi_n(q(F(x)))dx \\ &= \frac{1}{n!}\int_0^{1} \frac{\phi_n(q(u))}{f(F^{-1}(u))}du \end{aligned} \tag{5.7}$$

for $n = 1, 2, \ldots$.

Example 5.4. For a 2-out-of-3 system with IID components with a uniform distribution in $(0, 1)$, we have

$$\mathcal{CE}_1(T) = -\int_0^1 (3u^2 - 2u^3)\log(3u^2 - 2u^3)du = \frac{9}{8} - \frac{27}{32}\ln 3 \cong 0.198 < \mathcal{CE}_1(X) = 0.25$$

and

$$\mathcal{CE}_2(T) = \frac{1}{2}\int_0^1 (3u^2 - 2u^3)\left(-\log(3u^2 - 2u^3)\right)^2 du \cong 0.111 < \mathcal{CE}_2(X) = 0.125.$$

Starting from the inequalities in the previous example, we can obtain the following general result.

Proposition 5.1. *Let T be the lifetime of a coherent system with m ID components and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. If $\phi_n(q(u)) \geq \phi_n(u)$ (resp. $\leq$), then $\mathcal{CE}_n(T) \geq \mathcal{CE}_n(X_1)$ (resp. $\leq$).*

Proof. The proof is immediate comparing the expressions in (5.6) and in (5.7). □

The condition $\phi_n(q(u)) \geq \phi_n(u)$ is not a necessary condition for $\mathcal{CE}_n(T) \geq \mathcal{CE}_n(X_1)$ to hold: consider, for example, a 2-out-of-3 system with IID components having a uniform distribution as in the Example 5.4.

Using the dispersive ordering, defined in (A.10), (A.11) and (A.12), we obtain the following result.

Proposition 5.2. *Let T_1 and T_2 be the lifetimes of two coherent systems with the same structure and with ID components having common distributions F_1 and F_2, respectively, and having a common copula function C. If $T_1 \leq_{DISP} T_2$, then $\mathcal{CE}_n(T_1) \leq \mathcal{CE}_n(T_2)$ for $n = 1, 2, \ldots$.*

Proof. As the systems have the same structure and the components a common copula, then both systems have a common distortion function q. If we use the equivalent definition of dispersive order given in (A.11), from (5.7) we have

$$\mathcal{CE}_n(T_1) = \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{f_1(F_1^{-1}(u))}du \leq \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{f_2(F_2^{-1}(u))}du = \mathcal{CE}_n(T_2),$$

for $n = 1, 2, \ldots$. □

Remark 5.1. In particular, from (5.7) we obtain that, if $X \leq_{DISP} Y$, then $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$ for $n = 1, 2, \ldots$. Therefore these conditions are necessary conditions for the dispersive order to hold. Hence $\mathcal{CE}_n(X)$ can also be seen as a dispersion measure.
In particular, $\mathcal{CE}_1(X)$ can also be connected with a dispersion measure, the Gini mean difference (or mean absolute difference) defined as

$$D_G(X) := 2\int_0^{+\infty} F(t)(1 - F(t))dt,$$

where F is the distribution function of X. As $0 \leq x(1-x) \leq -x \log x$ for all $x \in (0, 1)$, we have

$$D_G(X) = 2\int_0^{+\infty} F(t)(1 - F(t))dt \leq 2\int_0^{+\infty} F(t)(-\log(F(t)))dt = 2\mathcal{CE}_1(X).$$

Therefore, $\mathcal{CE}_1(X)$ can also be seen as a dispersion measure closely connected with the Gini mean difference.

We can obtain bounds for $\mathcal{CE}_n(T)$ in terms of $\mathcal{CE}_n(X_1)$ in the case of identically distributed components as follows.

Proposition 5.3. *Let T be the lifetime of a coherent system with ID components with common distribution function F (and common probability density function f) and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. Then*

$$B_{1,n}\mathcal{CE}_n(X_1) \leq \mathcal{CE}_n(T) \leq B_{2,n}\mathcal{CE}_n(X_1)$$

for $n = 1, 2, \ldots$, where

$$B_{1,n} = \inf_{u\in(0,1)} \frac{\phi_n(q(u))}{\phi_n(u)} \quad \textit{and} \quad B_{2,n} = \sup_{u\in(0,1)} \frac{\phi_n(q(u))}{\phi_n(u)}.$$

Proof. From (5.7), the upper bound can be obtained as follows

$$\begin{aligned}
\mathcal{CE}_n(T) &= \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{f(F^{-1}(u))}du \\
&= \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{\phi_n(u)}\frac{\phi_n(u)}{f(F^{-1}(u))}du \\
&\leq \frac{1}{n!}\sup_{u\in(0,1)}\frac{\phi_n(q(u))}{\phi_n(u)}\int_0^1 \frac{\phi_n(u)}{f(F^{-1}(u))}du \\
&= \left(\sup_{u\in(0,1)}\frac{\phi_n(q(u))}{\phi_n(u)}\right)\mathcal{CE}_n(X_1)
\end{aligned}$$

for $n = 1, 2, \ldots$.
The lower bound can be obtained in a similar way. □

Example 5.5. As we see in the Example 5.3, for a 2-out-of-3 system with IID components, the distortion function is $q(u) = 3u^2 - 2u^3$ and, from the preceding proposition, we obtain

$$0 \leq \mathcal{CE}_1(T) \leq 1.01201\ \mathcal{CE}_1(X_1)$$

for any F. The results given by the precending proposition improve that given in Proposition 5.1. In this example we can't apply Proposition 5.1 because the upper bound is greater than 1 and the lower bound is smaller than 1.
Analogously, for $n = 2, 3$, we obtain

$$\mathcal{CE}_2(T) \leq 1.17433\ \mathcal{CE}_2(X_1)$$

and

$$\mathcal{CE}_3(T) \leq 1.50256\ \mathcal{CE}_3(X_1).$$

The preceding proposition can be extended also to compare the GCPE of two systems as follows.

Proposition 5.4. *Let T_1 and T_2 be the lifetimes of two coherent systems with ID components and with distortion functions q_1 and q_2, respectively. Let $\phi_n(u) = u[-\log(u)]^n$. Then*

$$\left(\inf_{u\in(0,1)}\frac{\phi_n(q_2(u))}{\phi_n(q_1(u))}\right)\mathcal{CE}_n(T_1) \leq \mathcal{CE}_n(T_2) \leq \left(\sup_{u\in(0,1)}\frac{\phi_n(q_2(u))}{\phi_n(q_1(u))}\right)\mathcal{CE}_n(T_1)$$

for $n = 1, 2, \ldots$.

Proof. The proof goes similarly to the proof of Proposition 5.3, but in the second equalities we multiply and divide by $\phi_n(q_1(u))$. □

Note that, if

$$\sup_{u\in(0,1)} \frac{\phi_n(q_2(u))}{\phi_n(q_1(u))} \leq 1,$$

then $\mathcal{CE}_n(T_2) \leq \mathcal{CE}_n(T_1)$.

We can obtain additional bounds for $\mathcal{CE}_n(T)$ when the ID components have a bounded density.

Proposition 5.5. *Let T be the lifetime of a coherent system with ID components and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. Let us assume that the components have an absolutely continuous distribution F with probability density function f and support S.*

(i) If $f(x) \leq M$ for all $x \in S$, then

$$\mathcal{CE}_n(T) \geq \frac{1}{M(n!)} \int_0^1 \phi_n(q(u))du$$

for $n = 1, 2, \ldots$.

(ii) If $f(x) \geq L > 0$ for all $x \in S$, then

$$\mathcal{CE}_n(T) \leq \frac{1}{L(n!)} \int_0^1 \phi_n(q(u))du$$

for $n = 1, 2, \ldots$.

Proof. If $f(x) \leq M$ for all $x \in S$, then

$$\begin{aligned} \mathcal{CE}_n(T) &= \frac{1}{n!} \int_0^1 \frac{\phi_n(q(u))}{f(F^{-1}(u))} du \\ &\geq \frac{1}{n!} \int_0^1 \frac{\phi_n(q(u))}{M} du \\ &= \frac{1}{M(n!)} \int_0^1 \phi_n(q(u))du \end{aligned}$$

for $n = 1, 2, \ldots$.

The other bound is obtained in a similar way. □

A particular case of the previous proposition, is when $T = X$. In this case we obtain

$$\mathcal{CE}_n(X) \geq \frac{1}{M(n!)} \int_0^1 \phi_n(u)du$$

whenever $f(x) \leq M$ for all $x \in S$, and

$$\mathcal{CE}_n(X) \leq \frac{1}{L(n!)} \int_0^1 \phi_n(u)du$$

whenever $f(x) \geq L > 0$ for all $x \in S$, for $n = 1, 2, \ldots$.

Example 5.6. The bounds are attained when X has a uniform distribution in $(0, b)$. In this case

$$\begin{aligned}
\mathcal{CE}_n(X) &= \frac{1}{n!} \int_0^b \frac{x}{b} \left[-\log \frac{x}{b} \right]^n dx \\
&= \frac{(-1)^n}{n!b} \left[-\int_0^b \frac{x}{2} n \left(\log \frac{x}{b} \right)^{n-1} dx \right] \\
&= \frac{1}{2} \left[\int_0^b \frac{(-1)^{n-1}}{(n-1)!} \frac{x}{b} \left(\log \frac{x}{b} \right)^{n-1} dx \right] \\
&= \frac{1}{2} \mathcal{CE}_{n-1}(X) \quad \text{for} \quad n = 1, 2, \ldots.
\end{aligned}$$

where the second equality is obtained by integration by part.
Hence

$$\mathcal{CE}_n(X) = \frac{1}{2} \mathcal{CE}_{n-1}(X) = \ldots = \frac{1}{2^n} \mathcal{CE}_0(X) = \frac{1}{2^n} \int_0^b \left(\frac{x}{b} \right) dx = \frac{1}{2^n} \frac{b}{2}.$$

Therefore $\mathcal{CE}_n(X) = b/2^{n+1}$ (as we see in the Example 5.2) for $n = 0, 1, \ldots$. Using the identity (5.5) and noting that $f(F^{-1}(u)) = 1/b$ for all $u \in (0, b)$,

$$\mathcal{CE}_n(X) = \frac{1}{n!} \int_0^1 \phi_n(u)du = \frac{b}{2^{n+1}}$$

from which

$$\mathcal{CE}_n(X) = \frac{1}{M 2^{n+1}},$$

because $f(x) \leq 1/b$ for all $x \in (0, b)$.

Thanks to the previous example we can state that the bounds for $\mathcal{CE}_n(X)$ and a general distribution F can be written as

$$\mathcal{CE}_n(X) \geq \frac{1}{M 2^{n+1}}$$

whenever $f(x) \leq M$ for all $x \in S$, and

$$\mathcal{CE}_n(X) \leq \frac{1}{L2^{n+1}}$$

whenever $f(x) \geq L > 0$ for all $x \in S$, for $n = 1, 2, \ldots$.

The following example shows how to obtain the bounds for a system with dependent and identically distributed (DID) components.

Example 5.7. Let us consider the parallel system $T = X_{2:2} = \max(X_1, X_2)$ with DID components having a common distribution F. Then the joint distribution function of the random vector (X_1, X_2) can be written as in the equation (4.10):

$$P(X_1 \leq x_1, X_2 \leq x_2) = C(F(x_1), F(x_2)),$$

where C is a given copula. Then the system distribution can be written as

$$F_{2:2}(t) = P(X_{2:2} \leq t) = P(X_1 \leq t, X_2 \leq t) = C(F(t), F(t)) = q(F(t)),$$

where $q(u) = C(u, u)$.
For example, let us consider the following Clayton copula with $\theta = 1$, that is

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\theta} = \frac{uv}{u + v - uv}, \quad 0 \leq u, v \leq 1. \tag{5.8}$$

Then $q(u) = u/(2 - u)$. Hence, from Proposition 5.3, we obtain

$$B_{1,1} = \inf_{u \in (0,1)} \left(\frac{1}{(2-u)\log u} \log\left(\frac{u}{2-u}\right)\right)$$

and

$$B_{2,1} = \sup_{u \in (0,1)} \left(\frac{1}{(2-u)\log u} \log\left(\frac{u}{2-u}\right)\right)$$

and then

$$\frac{1}{2}\, \mathcal{CE}_1(X_1) \leq \mathcal{CE}_1(T) \leq 2\, \mathcal{CE}_1(X_1).$$

As

$$\int_0^1 \phi_1(q(u))du = -2\log 2 + \frac{\pi^2}{6} = 0.2586397,$$

from Proposition 5.5, we also obtain

$$\frac{0.2586397}{M} \leq \mathcal{CE}_1(T) \leq \frac{0.2586397}{L}$$

whenever $L \leq f(x) \leq M$ for all $x \in S$.

In Appendix C, in Table C.1 we give the distortion functions for all the coherent systems with 1-4 IID components and in Table C.2 we give $\mathcal{CE}_n(T)$ for these systems when the components have a standard exponential distribution ($\mu = 1$). We also give the respective bounds obtained from Proposition 5.3 and Proposition 5.5. In this case the upper bound cannot be obtained from Proposition 5.5 since we have $L = 0$, so the upper bound is $+\infty$. In particular,

$$\mathcal{CE}_n(X_1) \geq \frac{1}{n!}\int_0^1 \phi_n(u)du = \frac{1}{2^{n+1}}.$$

Instead the value of M, defined in Proposition 5.5, is $M = 1$ and hence

$$D_n := \frac{1}{n!}\int_0^1 \phi_n(q(u))du$$

is a lower bound for $\mathcal{CE}_n(T)$. Note that $B_{1,n}$, $B_{2,n}$ and D_n do not depend on F and so these values can be used for other models (with possible different values for $\mathcal{CE}_n(T)$ and M). Note that $B_{1,n} = 0$ for all the coherent systems with 2–4 components. So the lower bound is D_n and the upper bound is $B_{2,n}\mathcal{CE}_n(X_1)$.

5.4 Generalized Cumulative Kerridge Inaccuracy

Let X and Y be two random lifetimes having distribution function F and G, respectively. In analogy with (2.12), let us now introduce the *generalized cumulative Kerridge inaccuracy of order* n defined as

$$K_n[F,G] = \frac{1}{n!}\int_0^{+\infty} F(x)[-\log G(x)]^n dx. \tag{5.9}$$

We now give a probabilistic meaning of the generalized cumulative inaccuracy in terms of (5.10) and (5.13). First we note that Kayal [21] introduces the following decreasing convex function:

$$T_{n,X}^{(2)}(x) = \frac{1}{n!}\int_x^{+\infty} [-\log F(z)]^n dz. \tag{5.10}$$

Moreover, Lemma 3.12 in [21] provides the following expression of the GCPE:

$$\mathcal{CE}_n(X) = \mathbb{E}[T_{n,X}^{(2)}(X)]. \tag{5.11}$$

In the following proposition we give a similar result for the measure defined in (5.9).

Proposition 5.6. *Let X and Y two nonnegative absolutely continuous random variables having distribution functions F and G, respectively. Then we have*

$$K_n[F,G] = \mathbb{E}\left[T_{n,Y}^{(2)}(X)\right], \tag{5.12}$$

where

$$T^{(2)}_{n,Y}(x) = \frac{1}{n!}\int_x^{+\infty}[-\log G(z)]^n dz. \tag{5.13}$$

Proof. From (5.9), by Fubini-Tonelli's theorem, it follows

$$\begin{aligned}
K_n[F,G] &= \frac{1}{n!}\int_0^{+\infty} F(x)[-\log G(x)]^n dx \\
&= \frac{1}{n!}\int_0^{+\infty}\left[\int_0^x dF(t)\right][-\log G(x)]^n dx \\
&= \int_0^{+\infty}\left[\int_t^{+\infty}\frac{[-\log G(x)]^n}{n!}dx\right] dF(t) \\
&= \int_0^{+\infty} T^{(2)}_{n,Y}(t)dF(t),
\end{aligned}$$

which immediately yields the result by virtue of (5.13). □

In the following propositions, we obtain a connection between our measure of discrimination and some stochastic orders.

Here we use the usual stochastic order, defined in (A.1) and in (A.2).

Proposition 5.7. *Let X and Y be nonnegative random variables having distribution functions F and G, respectively. If $X \le_{ST} Y$, then*

$$K_n[G,F] \le \mathcal{CE}_n(X) \le K_n[F,G]$$

for $n = 1, 2, \ldots$.

Proof. By assumption, from the equivalent definition in (A.2), $F(t) \ge G(t)$ for all $t \ge 0$. Then, for $n = 1, 2, \ldots$,

$$K_n[F,G] = \frac{1}{n!}\int_0^{+\infty} F(x)[-\log G(x)]^n dx \ge \frac{1}{n!}\int_0^{+\infty} F(x)[-\log F(x)]^n dx$$

and

$$K_n[G,F] = \frac{1}{n!}\int_0^{+\infty} G(x)[-\log F(x)]^n dx \le \frac{1}{n!}\int_0^{+\infty} F(x)[-\log F(x)]^n dx.$$

□

We remark that $X \le_{ST} Y$ does not imply in general $\mathcal{CE}_n(X) \le \mathcal{CE}_n(Y)$.

Now we use the decreasing convex order, defined in (A.9).

Proposition 5.8. *Let X and Y be nonnegative random variables with distribution functions F and G, respectively. If $X \leq_{DCX} Y$, then*

$$\mathcal{CE}_n(X) \leq K_n[G, F]$$

for $n = 1, 2, \ldots$.

Proof. By assumption $E[\phi(X)] \leq E[\phi(Y)]$ for all decreasing convex function $\phi : \mathbb{R} \to \mathbb{R}$. Note that from (5.11) and (5.12) both GCPE and $K_n[G, F]$ can be expressed as mean value of $T^{(2)}_{n,X}(\cdot)$. So, the proof follows by noting that $T^{(2)}_{n,X}$ is a decreasing convex function. □

5.5 Empirical Generalized Cumulative Kerridge Inaccuracy

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from a lifetime distribution with absolutely continuous cumulative distribution function $F(x)$. Kayal [21] defined the empirical GCPE as

$$\mathcal{CE}_n(\hat{F}_m) = \frac{1}{n!} \int_0^{+\infty} \hat{F}_m(x)[-\log \hat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots, \tag{5.14}$$

where

$$\hat{F}_m(x) = \frac{1}{m} \sum_{i=1}^{m} I_{(X_i \leq x)}, \quad x \in \mathbb{R}$$

is the empirical distribution of the sample and I is the indicator function. Denoting $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ as the order statistics of the sample, (5.14) can be written as

$$\mathcal{CE}_n(\hat{F}_m) = \sum_{j=1}^{m-1} \frac{1}{n!} \int_{X_{j:m}}^{X_{j+1:m}} \hat{F}_m(x)[-\log \hat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots. \tag{5.15}$$

Moreover, if F is continuous, then

$$\hat{F}_m(x) = \begin{cases} 0, & x < X_{1:m} \\ \dfrac{j}{m}, & X_{j:m} \leq x < X_{j+1:m}, \quad j = 1, 2, \ldots, m-1 \\ 1, & x \geq X_{m:m} \end{cases}$$

Hence, (5.15) can be rewritten as

$$\mathcal{CE}_n(\hat{F}_m) = \frac{1}{n!} \sum_{j=1}^{m-1} U_{j+1} \frac{j}{m} \left(-\log \frac{j}{m} \right)^n, \quad n = 1, 2, \ldots,$$

where

$$U_1 = X_{1:m}, \quad U_i = X_{i:m} - X_{i-1:m}, \qquad i = 1, 2, \ldots, m$$

are the sample spacings corresponding to the sample order statistics.

Let us now consider another random sample $Y_1, Y_2, \ldots, Y_m$ of nonnegative, absolutely continuous IID random variables and denote its empirical GCPE by

$$\mathcal{CE}_n(\hat{G}_m) = \frac{1}{n!} \int_0^{+\infty} \hat{G}_m(y)[-\log \hat{G}_m(y)]^n dy,$$

where $\hat{G}_m$ is the empirical distribution of the second sample.

According to (5.9) we define the *empirical generalized cumulative inaccuracy* as

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{n!} \int_0^{+\infty} \hat{F}_m(u)[-\log \hat{G}_m(u)]^n du.$$

It can be expressed as

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{n!} \sum_{j=1}^{m-1} \left(-\log \frac{j}{m}\right)^n \int_{Y_{j:m}}^{Y_{j+1:m}} \hat{F}_m(u) du,$$

where $Y_{1:m}, Y_{2:m}, \ldots, Y_{m:m}$ are the order statistics of the new sample. Let us denote by

$$N_j = \sum_{i=1}^{m} I_{(X_i \leq Y_{j:m})}, \quad j = 1, 2, \ldots, m$$

the number of random variables of the first sample that are less than or equal to the j-th order statistic of the second sample. We rename by $X_{j,1} < X_{j,2} < \ldots$ the random variables of the first sample belonging to the interval $(Y_{j:m}, Y_{j+1:m}]$. From the above position we have:

$$\int_{Y_{j:m}}^{Y_{j+1:m}} \hat{F}_m(u) du = \frac{N_j}{m}[Y_{j+1:m} - Y_{j:m}] + \frac{1}{m} \sum_{r=1}^{N_{j+1}-N_j} [Y_{j+1:m} - X_{j,r}].$$

Then

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{mn!} \sum_{j=1}^{m-1} \left[N_{j+1} Y_{j+1:m} - N_j Y_{j:m} - \sum_{r=1}^{N_{j+1}-N_j} X_{j,r} \right] \left(-\log \frac{j}{m}\right)^n.$$

Let us remember that the well known Glivenko-Cantelli's theorem states that:

$$\sup_x \mid \hat{F}_m(x) - F(x) \mid \to 0 \quad \text{a.s. as} \quad m \to +\infty.$$

Using this result we prove the following theorem, which discloses an asymptotic property of the empirical generalized cumulative inaccuracy.

Theorem 5.4. *Let X and Y be nonnegative random variables in L^p for some $p > 1$; the empirical generalized cumulative inaccuracy converges to the generalized cumulative inaccuracy of X and Y, that is*

$$K_n[\hat{F}_m, \hat{G}_m] \to K_n[F, G] \quad a.s. \ as \quad m \to \infty.$$

Proof. By the dominated convergence theorem, the integral of $\hat{F}_m(u)[-\log \hat{G}_m(u)]^n$ converges to that of $F(u)[-\log G(u)]^n$ on any finite interval. Hence, we only have to show that

$$\left| \int_1^{+\infty} \hat{F}_m(u)[-\log \hat{G}_m(u)]^n du - \int_1^{+\infty} F(u)[-\log G(u)]^n du \right| \to 0 \quad \text{a.s. as} \quad m \to \infty.$$

Note that, for a fixed n, the function $u[-\log(v)]^n$ is continuous in $[a, 1]^2$ for any $a > 0$ and so bounded. Hence, if $K = \min(\hat{F}_m(1), \hat{G}_m(1)) > 0$, we get

$$\left| \hat{F}_m(u)[-\log \hat{G}_m(u)]^n \right| \leq K.$$

Now, by applying the dominated convergence theorem and by virtue of the Glivenko-Cantelli's theorem, we obtain the thesis. □

Chapter 6

Inactivity Time of Coherent Systems under Periodical Inspections

The study of representations and comparisons of coherent systems is one of the most relevant topic in reliability theory. Several studies have been devoted to understand the behaviour of a system composed by different kind of components in order to evaluate its reliability. The reliability of a system depends on several factors such as the structure of the system, the behaviour of each component and the way the components are correlated with each other. In particular, recently, some authors have obtained representations for reliability of coherent system formed by components with possibly dependent lifetimes in terms of distorted distributions and with the help of copulas (for more details about a copula approach to reliability, see, e.g., Eryilmaz [16] and Navarro and Spizzichino [38]). For example, Navarro and Durante [34] studied the copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components. Using the same tools, Navarro *et al.* [36] obtained comparison results for inactivity times of k-out-of-n and general coherent systems with dependent components.

6.1 Why Inactivity Times?

In a real life situation, inference about the past lifetime of the system may be of interest. Suppose that a coherent system with lifetime T has failed some time before the inspection time $t > 0$. As in Section 2.1, we can consider the conditional random variable $X^{(t)} = (t - T|T \leq t)$, the inactivity time, which usually has a close connection with the so-called autopsy data, that are information obtained by examining the

component states of a failed system.
The monitoring of a system can be scheduled at different times, so it is believable that at a certain inspection time the system can be found broken. In our work, we consider only two inspwction time, t_1 and t_2, but more complex inspection plans can be analysed as well, starting from the results in the following sections. Under this double monitoring, we assume that the system is working at the first inspection time t_1 and it is broken at the second inspection time t_2, with $0 \leq t_1 < t_2$. So, in this case, the interest is on the history of such system, in particular on the inactivity time $(t_2 - T|t_1 < T < t_2)$. We study inactivity times in various cases, considering different informations about the states of the components available at the different inspection times.

6.2 Representations of Inactivity Times of Coherent Systems under Periodical Inspections

Let X be a nonnegative random variable representing the lifetime of a unit or a system with distribution function F. At time $t > 0$, we know that the unit is broken, then the inactivity time for that unit is

$$X^{(t)} = (t - X|X < t),$$

and the reliability function of $X^{(t)}$ is

$$\bar{F}^{(t)}(x) = P(X^{(t)} > x) = P(t - X > x|X < t) = \frac{P(X < t - x)}{P(X < t)} = \frac{F(t-x)}{F(t)}$$

for all $x \in [0, t]$, whenever $F(t) > 0$.

Under periodical inspections, we know that the unit is working at a time t_1 and that it is broken at a time t_2 with $0 \leq t_1 < t_2$, then the inactivity time in the interval (t_1, t_2) is given by

$$X^{(t_1,t_2)} = (t_2 - X|t_1 < X < t_2)$$

and its reliability function by

$$\begin{aligned}
\bar{F}^{(t_1,t_2)}(x) &= P(X^{(t_1,t_2)} > x) \\
&= P(t_2 - X > x|t_1 < X < t_2) \\
&= \frac{P(\{t_1 < X < t_2 - x\} \cap \{t_1 < X < t_2\})}{P(t_1 < X < t_2)} \\
&= \frac{F(t_2 - x) - F(t_1)}{F(t_2) - F(t_1)} \qquad (6.1)
\end{aligned}$$

for all $x \in [0, t_2 - t_1]$, whenever $F(t_2) - F(t_1) > 0$. Hence, the expected inactivity time is

$$\mathbb{E}(t_2 - X | t_1 < X < t_2) = \int_0^{t_2 - t_1} \frac{F(t_2 - x) - F(t_1)}{F(t_2) - F(t_1)} dx.$$

Under periodical inspections, we might have different information about the states of the system and of its component at the inspection times t_1 and t_2. We can consider different kinds of conditional distributions: we analyse three cases that are, in our opinion, the most realistic ones.

6.2.1 Case I

The simplest case is just to know that the system was working at a time t_1 and that it is broken at second inspection time t_2 with $0 \leq t_1 < t_2$. Then, the inactivity time of the system is given by

$$T^{(t_1,t_2)} = (t_2 - T | t_1 < T < t_2),$$

whenever $F_T(t_2) - F_T(t_1) > 0$. Recall that F_T can be calculated through generalized distortion function as in (4.8). Thus we obtain the following result.

Proposition 6.1. *If the component distribution functions satisfy $F_i(t_2) - F_i(t_1) > 0$ for $i = 1, \ldots, n$, then the reliability function of $T^{(t_1,t_2)} = (t_2 - T | t_1 < T < t_2)$ can be written as*

$$\bar{F}_T^{(t_1,t_2)}(x) = \bar{Q}^{(t_1,t_2)}(\bar{F}_1^{(t_1,t_2)}(x), \ldots, \bar{F}_n^{(t_1,t_2)}(x)),$$

where $\bar{F}_1^{(t_1,t_2)}, \ldots, \bar{F}_n^{(t_1,t_2)}$ are the reliability functions of the inactivity times of the components in the interval (t_1, t_2) and $\bar{Q}^{(t_1,t_2)}$ is a distortion function given by

$$\bar{Q}^{(t_1,t_2)}(\mathbf{u}) = \frac{Q(F_1(t_1)+u_1(F_1(t_2)-F_1(t_1)),\ldots,F_n(t_1)+u_1(F_n(t_2)-F_n(t_1)))-Q(F_1(t_1),\ldots,F_n(t_1))}{Q(F_1(t_2),\ldots,F_n(t_2))-Q(F_1(t_1),\ldots,F_n(t_1))} \tag{6.2}$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in [0,1]^n$.

Proof. From (6.1) the reliability function of $T^{(t_1,t_2)} = (t_2 - T | t_1 < T < t_2)$ is given by

$$\bar{F}_T^{(t_1,t_2)}(x) = \Pr(T^{(t_1,t_2)} > x) = \Pr(t_2 - T > x | t_1 < T < t_2) = \frac{F_T(t_2 - x) - F_T(t_1)}{F_T(t_2) - F_T(t_1)}$$

for all $x \in [0, t_2 - t_1]$ such that $F_T(t_2) - F_T(t_1) > 0$. Hence, from (4.8), we have

$$\bar{F}_T^{(t_1,t_2)}(x) = \frac{Q(F_1(t_2 - x), \ldots, F_n(t_2 - x)) - Q(F_1(t_1), \ldots, F_n(t_1))}{Q(F_1(t_2), \ldots, F_n(t_2)) - Q(F_1(t_1), \ldots, F_n(t_1))}. \tag{6.3}$$

Finally, we note that from (6.1), the reliability functions of the inactivity times of the components can be written as

$$\bar{F}_i^{(t_1,t_2)}(x) = \frac{F_i(t_2 - x) - F_i(t_1)}{F_i(t_2) - F_i(t_1)}$$

for $i = 1, \ldots, n$. Therefore

$$F_i(t_2 - x) = \bar{F}_i^{(t_1,t_2)}(x)(F_i(t_2) - F_i(t_1)) + F_i(t_1) \tag{6.4}$$

for $i = 1, \ldots, n$.
Now, substituting (6.4) in (6.3), we obtain and

$$\bar{F}_T^{(t_1,t_2)}(x) = \bar{Q}^{(t_1,t_2)}(\bar{F}_1^{(t_1,t_2)}(x), \ldots, \bar{F}_n^{(t_1,t_2)}(x))$$

where $\bar{Q}^{(t_1,t_2)}(u_1, \ldots, u_n)$ is the distortion function given in (6.2). □

Note that, in the preceding proof, we obtian an explicit expression for the distortion function $\bar{Q}^{(t_1,t_2)}$ and it depends on Q, $F_1(t_1), \ldots, F_n(t_1)$ and $F_1(t_2), \ldots, F_n(t_2)$. If $t_1 = 0$, then we obtain the representation given in [36].

Remark 6.1. If the components are identically distributed (ID), that is, $F_1 = \cdots = F_n = F$, then we get the following expression

$$\bar{F}_T^{(t_1,t_2)}(x) = \bar{q}^{(t_1,t_2)}(\bar{F}^{(t_1,t_2)}(x)),$$

where $\bar{q}^{(t_1,t_2)}(u) := \bar{Q}^{(t_1,t_2)}(u, \ldots, u)$ is a (univariate) distortion function and $\bar{F}^{(t_1,t_2)}$ is the common reliability function of the component inactivity times $(t_2 - X_i | t_1 < X_i < t_2)$ in the interval (t_1, t_2).

6.2.2 Case II

Now, we assume that we know which components are working and which have failed at the inspection times t_1 and t_2 with $0 \leq t_1 < t_2$. These informations about the components implies that the system has failed in the interval (t_1, t_2). This can be represented defining three subsets, W_1, W_2 and W_3, of the set of the indexes $I = \{1, \ldots, n\}$ as follows:

- W_1 represents the components that have failed in the interval $(0, t_1)$;
- W_2 represents the components that have failed in the interval (t_1, t_2);

- $W_3 = \{1,\ldots,n\} - W_1 - W_2$ represents the components that are working at time t_2 even if the system has failed in the interval (t_1, t_2).

Note that W_1 and/or W_3 can be empty sets and that we assume that the event $\{X_i > t_1, i \notin W_1\}$ implies $\{T > t_1\}$ and the event $\{X_i < t_2, i \in W_1 \cup W_2\}$ implies $\{T < t_2\}$.

Then, the inactivity time of the system is given by

$$T_{W_1,W_2}^{(t_1,t_2)} = (t_2 - T | E_{W_1,W_2})$$

where

$$E_{W_1,W_2} := \{X_i < t_1, i \in W_1; t_1 < X_j < t_2, j \in W_2; X_k > t_2, k \in W_3\}$$

We assume $P(E_{W_1,W_2}) > 0$ and that E_{W_1,W_2} implies $t_1 < T < t_2$. By using again (4.8) we obtain the following proposition.

Proposition 6.2. *If the component distribution functions satisfy $F_i(t_2) - F_i(t_1) > 0$ for $i = 1,\ldots,n$, then the reliability function of $T_{W_1,W_2}^{(t_1,t_2)}$ can be written as*

$$\bar{F}_{W_1,W_2}^{(t_1,t_2)}(x) = \bar{Q}_{W_1,W_2}^{(t_1,t_2)}(\bar{F}_1^{(t_1,t_2)}(x), \ldots, \bar{F}_n^{(t_1,t_2)}(x)), \tag{6.5}$$

where $\bar{Q}_{W_1,W_2}^{(t_1,t_2)}$ is a distortion function and $\bar{F}_1^{(t_1,t_2)}, \ldots, \bar{F}_n^{(t_1,t_2)}$ are the reliability functions of the inactivity times of the components in the interval (t_1, t_2).

Proof. The reliability function of $T_{W_1,W_2}^{(t_1,t_2)}$ is

$$\bar{F}_{W_1,W_2}^{(t_1,t_2)}(x) = P(T_{W_1,W_2}^{(t_1,t_2)} > x) = P(t_2 - T > x | E_{W_1,W_2}) = \frac{P(T < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})}$$

for $x \in [0, t_2 - t_1]$. The components in W_3 cannot fail before t_2. So they cannot cause the failure of the system before $t_2 - x$. Let $K_1, \ldots, K_s$ be the minimal cut sets of T which do not contain elements of W_3 (i.e. $K_i \subseteq W_1 \cup W_2$ for $i = 1, \ldots, s$) and let $T^* = \min_{i=1,\ldots,s} \max_{j \in K_i} X_j$. The lifetime of a system can be written of its minimal cut sets, then

$$\bar{F}_{W_1,W_2}^{(t_1,t_2)}(x) = \frac{P(T^* < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})} = \frac{P(\min_{i=1,\ldots,s} \max_{j \in K_i} X_j < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})}.$$

Now we can use the inclusion-exclusion formula obtaining

$$
\begin{aligned}
\bar{F}^{(t_1,t_2)}_{W_1,W_2}(x) &= \frac{P(\cup_{i=1}^{s}\{\max_{j\in K_i} X_j < t_2 - x\}, E_{W_1,W_2})}{P(E_{W_1,W_2})} \\
&= \sum_{i=1}^{s} \frac{P(\max_{j\in K_i} X_j < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})} + \\
&\quad - \sum_{i<j} \frac{P(\max_{z\in K_i\cup K_j} X_z < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})} + \dots \\
&\quad + (-1)^{s+1} \frac{P(\max_{j\in K_1\cup\dots\cup K_s} X_j < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})}. \qquad (6.6)
\end{aligned}
$$

Finally note that for $K \subseteq W_1 \cup W_2$, we have

$$
\begin{aligned}
P\left(\max_{j\in K} X_j < t_2 - x, E_{W_1,W_2}\right) &= P(t_1 < X_j < t_2 - x, j \in K \cup W_2; X_j < t_1, j \in W_1; X_j > t_2, j \in W_3) \\
&= \bar{Q}^{(t_1,t_2)}_{W_1,W_2,K}(\bar{F}^{(t_1,t_2)}_1(x), \dots, \bar{F}^{(t_1,t_2)}_n(x)), \qquad (6.7)
\end{aligned}
$$

where the last equality is obtained by using a procedure similar to that used in the proof of Proposition 6.1. Finally, using the preceding expression in (6.6) we prove (6.5). □

Note that (6.7) shows that $\bar{Q}^{(t_1,t_2)}_{W_1,W_2,C}(u_1,\dots,u_n)$ is constant in u_j for $j \notin W_2$.

Remark 6.2. If the components are identically distributed (ID) with common distribution F and common reliability function of the component inactivity times $\bar{F}^{(t_1,t_2)}$, then we get the following expression

$$\bar{F}^{(t_1,t_2)}_{W_1,W_2}(x) = \bar{q}^{(t_1,t_2)}_{W_1,W_2}(\bar{F}^{(t_1,t_2)}(x)),$$

where $\bar{q}^{(t_1,t_2)}_{W_1,W_2}(u) = \bar{Q}^{(t_1,t_2)}_{W_1,W_2}(u,\dots,u)$.

Another interesting particular case is when the components are independent. In this case we can state the following result that is a particular case of Proposition 6.2.

Proposition 6.3. *Let $K_1,\dots,K_s$ be all the minimal cut sets of a system T contained in $W_1 \cup W_2$. If the components are independent and their distribution functions satisfy $F_i(t_2) - F_i(t_1) > 0$ for $i = 1,\dots,n$, then the reliability function of $T^{(t_1,t_2)}_{W_1,W_2}$ can be written as*

$$\bar{F}^{(t_1,t_2)}_{W_1,W_2}(x) = \bar{Q}_{W_1,W_2}(\bar{F}^{(t_1,t_2)}_1(x), \dots, \bar{F}^{(t_1,t_2)}_n(x)),$$

where

$$\bar{Q}_{W_1,W_2}(u_1,\dots,u_n) = \sum_{i=1}^{s} \prod_{j\in K_i\cap W_2} u_j - \sum_{i<j} \prod_{z\in(K_i\cup K_j)\cap W_2} u_z + \dots + (-1)^{s+1} \prod_{j\in(K_1\cup\dots\cup K_s)\cap W_2} \bar{u}_j \quad (6.8)$$

is a distortion function.

Proof. In the components are independent, for $K \subseteq W_1 \cup W_2$, then the summations in (6.6) can be written as

$$
\begin{aligned}
S_K &= \frac{P(\max_{j\in K} X_j < t_2 - x, E_{W_1,W_2})}{P(E_{W_1,W_2})} \\
&= \frac{\prod_{j\in K\cap W_2} P(t_1 < X_j < t_2 - x) \prod_{j\in W_2 - K} P(t_1 < X_j < t_2) \prod_{j\in W_1} P(X_j < t_1) \prod_{j\in W_3} P(X_j > t_2)}{\prod_{j\in W_2} P(t_1 < X_j < t_2) \prod_{j\in W_1} P(X_j < t_1) \prod_{j\in W_3} P(X_j > t_2)} \\
&= \frac{\prod_{j\in K\cap W_2} P(t_1 < X_j < t_2 - x)}{\prod_{j\in K\cap W_2} P(t_1 < X_j < t_2)} \\
&= \prod_{j\in K\cap W_2} \bar{F}_j^{(t_1,t_2)}(x),
\end{aligned}
$$

where the last equality is obtained from (6.1). Then from (6.6) we get

$$
\begin{aligned}
\bar{F}^{(t_1,t_2)}_{W_1,W_2}(x) = \sum_{i=1}^{s} \prod_{j\in K_i \cap W_2} \bar{F}_j^{(t_1,t_2)}(x) - \sum_{i<j} \prod_{z\in (K_i\cup K_j)\cap W_2} \bar{F}_z^{(t_1,t_2)}(x) + \dots \\
+ (-1)^{s+1} \prod_{j\in (K_1\cup\dots\cup K_s)\cap W_2} \bar{F}_j^{(t_1,t_2)}(x)
\end{aligned}
$$

which completes the proof. □

Remark 6.3. An important fact here is that we have proved that, if the components are independent, then the distortion function $\bar{Q}_{W_1,W_2}$ does not depend on t_1 and t_2 (so we have deleted the superscript (t_1,t_2) used in the general case). Note that $\bar{Q}_{W_1,W_2}$ is constant in u_j for $j \notin W_2$. The sequence $K_1 \cap W_2, \dots, K_s \cap W_2$ cannot contain empty sets, because, if $K_i \cap W_2 = \emptyset$ for an $i \in \{1,\dots,s\}$, then $K_i \subseteq W_1$ so $T < t_1$. Moreover this sequence is not always a sequence of minimal cut sets since one of these sets can be included in another one.

6.2.3 Case III

We consider again the inspection times t_1 and t_2 with $0 \le t_1 < t_2$ but, in this case, we assume that t_1 is the first component failure time. We also assume that we know which component has failed at time t_1 and that this failure does not imply the system failure. These assumptions may seem unrealistic for some systems but, for others, they could be reasonable, for example if some continuous monitoring is made over some components. At the second inspection time we know that the components in the set W have failed causing the system failure, even if other components are working. Then, the inactivity time of the system is given by

$$
T^{(t_1,t_2)}_{i,W} = (t_2 - T | X_i = t_1; t_1 < X_j < t_2, j\in W; X_j > t_2, j\notin W),
$$

where we assume that the conditioning event has a positive probability and implies $t_1 < T < t_2$. More complex assumptions, for example when we know the exact failure times of more components, can be treated in a similar way by using the techniques developed below (see also [34]).
Without loss of generality, by choosing an appropriate structure function, we can assume that $i = n$ and that $W = \{1, \ldots, m\}$ for $m < n$. So the inactivity time of the system is given by

$$T_{n,W}^{(t_1,t_2)} = (t_2 - T | X_n = t_1; t_1 < X_j < t_2, j = 1, \ldots, m; X_j > t_2, j = m+1, \ldots, n-1).$$

Thus we can state the following proposition for this inactivity time with a representation similar (based on distortions) to that obtained in the preceding cases.

Proposition 6.4. *If $W = \{1, \ldots, m\}$ and the component distribution functions satisfy $F_i(t_2) - F_i(t_1) > 0$ for $i = 1, \ldots, n$, then the reliability function of $T_{n,W}^{(t_1,t_2)}$ can be written as*

$$\bar{F}_{n,W}^{(t_1,t_2)}(x) = \bar{Q}_{n,W}^{(t_1,t_2)}(\bar{F}_1^{(t_1,t_2)}(x), \ldots, \bar{F}_n^{(t_1,t_2)}(x)),$$

where $\bar{Q}_{n,W}^{(t_1,t_2)}$ is a distortion function and $\bar{F}_1^{(t_1,t_2)}, \ldots, \bar{F}_n^{(t_1,t_2)}$ are the reliability functions of the inactivity times of the components in the interval (t_1, t_2). Moreover, $\bar{Q}_{n,W}^{(t_1,t_2)}(u_1, \ldots, u_n)$ does not depend on $u_{m+1}, \ldots, u_n$.

Proof. First, we note that, from the copula representation of the joint distribution given in (4.10), the probability density function (pdf) $\mathbf{f}$ of $X_1, \ldots, X_n$ can be written as

$$\mathbf{f}(x_1, \ldots, x_n) = f_1(x_1) \ldots f_n(x_n) \partial_1 \ldots \partial_n C(F_1(x_1), \ldots, F_n(x_n)),$$

where $f_1, \ldots, f_n$ are the marginal probability density functions and $\partial_i C$ represents the partial derivative of C with respect to its i-th variable. The function $c = \partial_1 \ldots \partial_n C$ is the pdf of the copula C. Hence the conditional pdf of $(X_1, \ldots, X_{n-1} | X_n = x_1)$ is

$$\begin{aligned} \mathbf{f}_{1,\ldots,n-1|n}(x_1, \ldots, x_{n-1}|x_n) &= \frac{\mathbf{f}(x_1, \ldots, x_n)}{f_n(x_n)} \\ &= f_1(x_1) \ldots f_{n-1}(x_{n-1}) c(F_1(x_1), \ldots, F_n(x_n)) \end{aligned}$$

for x_n such that $f_n(x_n) > 0$. Therefore, its distribution function is

$$\begin{aligned} \mathbf{F}_{1,\ldots,n-1|n}(t_1, \ldots, t_{n-1}|x_n) &= \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_1(x_1) \ldots f_{n-1}(x_{n-1}) c(F_1(x_1), \ldots, F_n(x_n)) dx_{n-1} \ldots dx_1 \\ &= \int_0^{F_1(t_1)} \cdots \int_0^{F_{n-1}(t_{n-1})} c(v_1, \ldots, v_{n-1}, F(x_n)) dv_{n-1} \ldots dv_1 \\ &= \mathbf{Q}_{1,\ldots,n-1|n}(F_1(t_1), \ldots, F_{n-1}(t_{n-1})), \end{aligned}$$

where

$$\mathbf{Q}_{1,\ldots,n-1|n}(u_1,\ldots,u_{n-1}) = \int_0^{u_1}\cdots\int_0^{u_{n-1}} c(v_1,\ldots,v_{n-1},F(x_n))dv_{n-1}\ldots dv_1 \qquad (6.9)$$

depends on $F(x_n)$.

Hence, the reliability function of the inactivity time $T_{n,W}^{(t_1,t_2)}$ can be calculated as

$$\begin{aligned}\bar{F}_{n,W}^{(t_1,t_2)}(x) &= P(t_2 - T > x|X_n = t_1; t_1 < X_j < t_2, j\in I_m; X_j > t_2, j \in J_m)\\ &= \frac{P(T < t_2 - x, t_1 < X_j < t_2, j \in I_m; X_k > t_2, j \in J_m|X_n = t_1)}{P(t_1 < X_j < t_2, j\in I_m; X_j > t_2, j \in J_m|X_n = t_1)}, \qquad (6.10)\end{aligned}$$

where $I_m := \{1,\ldots,m\}$ and $J_m := \{m+1,\ldots,n-1\}$.
Recall that, from (4.3), the lifetime of the system can be written as

$$T = \min_{i=1,\ldots,r}\max_{k\in K_i} X_k$$

where $K_1,\ldots,K_r$ are the minimal cut sets of the system. As $X_j > t_2$ for $j = m+1,\ldots,n-1$, only the minimal cut sets included in $\{1,\ldots,m,n\}$ can cause the failure of the system in the interval (t_1,t_2). We can assume, without loss of generality, that those minimal cut sets are $K_1,\ldots,K_s$ for $1\le s\le r$. Let $K_i^* := K_i - \{n\}$ for $i = 1,\ldots s$ and let $T^* := \min_{i=1,\ldots,s}\max_{k\in K_i^*} X_k$. Using the inclusion-exclusion formula, the numerator in the expression (6.10) can be obtained as

$$\begin{aligned}N &= P(T < t_2 - x, t_1 < X_j < t_2, j\in I_m; X_j > t_2, j\in J_m|X_n = t_1)\\ &= P(T^* < t_2 - x, t_1 < X_j < t_2, j\in I_m; X_j > t_2, j\in J_m|X_n = t_1)\\ &= \sum_{i=1}^{s} P(t_1 < X_j < t_2 - x, j\in K_i^*; t_1 < X_j < t_2, j \in I_m - K_i^*; X_j > t_2, j \in J_m|X_n = t_1)+\\ &-\sum_{i=1}^{s-1}\sum_{j=1+1}^{s} P(t_1 < X_l < t_2 - x, l \in K_{i,j}^*; t_1 < X_l < t_2, l\in I_m - K_{i,j}^*; X_l > t_2, l\in J_m|X_n = t_1)+\ldots\\ &+(-1)^{s+1}P(t_1 < X_j < t_2 - x, j\in K_{1,\ldots,s}^*; t_1 < X_j < t_2, j\in I_m - K_{1,\ldots,s}^*; X_j > t_2, j\in J_m|X_n = t_1),\end{aligned}$$

where $K_{i,j}^* := K_i^* \cup K_j^*,\ldots,K_{1,\ldots,s}^* := K_1^* \cup \ldots K_s^*$.
Finally from (6.1) and (6.9), the probabilities in the previous expression can be written as functions of

$$\bar{F}_k^{(t_1,t_2)}(x) = \Pr(X_k^{(t_1,t_2)} > x) = \frac{F_k(t_2 - x) - F_k(t_1)}{F_k(t_2) - F_k(t_1)}$$

for $k \in I_m$. This concludes the proof. □

6.3 Comparison results

The distorted representations obtained in the preceding section can be used to compare inactivity times of systems under different assumptions on the sets W_1, W_2 or W. We use the usual stochastic order ($\leq_{ST}$), the hazard rate order ($\leq_{HR}$), the reversed hazard rate order ($\leq_{RHR}$) and the likelihood order ($\leq_{LR}$), all defined in Appendix A.

To this end recall that Theorem 4.2 and Theorem 4.3 provide necessary and sufficient conditions to obtain distribution-free orderings.
For example, all the inactivity times of case II have distorted distributions based on the same baseline reliability functions, so we can compare them for different choices of W_1 and W_2 or with the inactivity time considered in case I. Then from Theorem 4.3 and from Propositions 6.1 and 6.2, we obtain the following result.

Proposition 6.5. *Let T be the lifetime of a system with component lifetimes $X_1,\ldots,X_n$ having distribution functions $F_1,\ldots,F_n$, respectively. Let us assume that $F_i(t_2) > F_i(t_1)$ for all $i = 1,\ldots,n$ and some $0 \leq t_1 < t_2$. Let $T^{(t_1,t_2)} = (t_2 - T|t_1 < T < t_2)$ and $T^{(t_1,t_2)}_{W_1,W_2} = (t_2 - T|X_i < t_1, i \in W_1; t_1 < X_j < t_2, j \in W_2; X_k > t_2, k \in W_3)$. Then:*

- *(i) $T^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{ST}$) for all $F_1,\ldots,F_n$ if and only if $\bar{Q}^{(t_1,t_2)} \leq \bar{Q}^{(t_1,t_2)}_{W_1,W_2}$ ($\geq$) in $(0,1)^n$.*
- *(ii) $T^{(t_1,t_2)} \leq_{HR} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{HR}$) for all $F_1,\ldots,F_n$ if and only if $\bar{Q}^{(t_1,t_2)}_{W_1,W_2}/\bar{Q}^{(t_1,t_2)}$ is decreasing (increasing) in $(0,1)^n$.*
- *(iii) $T^{(t_1,t_2)} \leq_{RHR} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{RHR}$) for all $F_1,\ldots,F_n$ if and only if $Q^{(t_1,t_2)}_{W_1,W_2}/Q^{(t_1,t_2)}$ is increasing (decreasing) in $(0,1)^n$.*

In particular, if the components are identically distributed (ID), in compliance with Theorem 4.2, we have the following proposition.

Proposition 6.6. *Let T be the lifetime of a system with component lifetimes $X_1,\ldots,X_n$ having a common distribution function F. Let us assume that $F(t_2) > F(t_1)$ for some $0 \leq t_1 < t_2$. Let $T^{(t_1,t_2)} = (t_2 - T|t_1 < T < t_2)$ and $T^{(t_1,t_2)}_{W_1,W_2} = (t_2 - T|X_i < t_1, i \in W_1; t_1 < X_j < t_2, j \in W_2; X_k > t_2, k \in W_3)$. Then:*

- *(i) $T^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{ST}$) for all F if and only if $\bar{q}^{(t_1,t_2)} \leq \bar{q}^{(t_1,t_2)}_{W_1,W_2}$ ($\geq$) in $(0,1)$.*
- *(ii) $T^{(t_1,t_2)} \leq_{HR} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{HR}$) for all F if and only if $\bar{q}^{(t_1,t_2)}_{W_1,W_2}/\bar{q}^{(t_1,t_2)}$ is decreasing (increasing) in $(0,1)$.*
- *(iii) $T^{(t_1,t_2)} \leq_{RHR} T^{(t_1,t_2)}_{W_1,W_2}$ ($\geq_{RHR}$) for all F if and only if $q^{(t_1,t_2)}_{W_1,W_2}/q^{(t_1,t_2)}$ is increasing (decreasing) in $(0,1)$.*

(iv) $T^{(t_1,t_2)} \leq_{LR} T^{(t_1,t_2)}_{W_1,W_2}$ $(\geq_{LR})$ *for all* F *if and only if* $(\bar{q}^{(t_1,t_2)}_{W_1,W_2})'/(\bar{q}^{(t_1,t_2)})'$ *is decreasing (increasing) in* $(0,1)$.

Similar propositions can be stated to compare cases I and II with case III or to compare the different options in cases II and III.

Other interesting results can be obtained comparing inactivity time $X^{(t_1,t_2)} = (t_2 - X|t_1 < X < t_2)$ at different inspection times t_1 and t_2. In this direction we have the following Lemma where we need some stochastic orders, defined in Appendix A, and some aging classes, defined in Appendix B.

Lemma 6.1. *Let* X *be a nonnegative random variable with an absolutely continuous distribution function* F *and let* $X^{(t_1,t_2)} = (t_2 - X|t_1 < X < t_2)$*. Then:*

(i) $X^{(t_1,t_2)} \geq_{LR} X^{(t_1^*,t_2)}$ *for all* $0 \leq t_1 < t_1^* < t_2$.

(ii) If $X^{(t_1,z)}$ *is NBUHR for* $z \in [t_2, t_2^*]$ *and* $0 \leq t_1 < t_2 < t_2^*$*, then* $X^{(t_1,t_2)} \leq_{ST} X^{(t_1,t_2^*)}$.

(iii) If $X^{(t_1,t_2)}$ *is IHR for* $0 \leq t_1 < t_2 < t_2^*$*, then* $X^{(t_1,t_2)} \leq_{HR} X^{(t_1,t_2^*)}$.

(iv) If $X^{(t_1,t_2)}$ *is ILR for* $0 \leq t_1 < t_2 < t_2^*$*, then* $X^{(t_1,t_2)} \leq_{LR} X^{(t_1,t_2^*)}$.

Proof. (i) The pdf of the reliability function in (6.1) is

$$f^{(t_1,t_2)}(x) = \frac{f(t_2 - x)}{F(t_2) - F(t_1)} \tag{6.11}$$

for all $x \in [0, t_2 - t_1]$ and 0 elsewhere. If $t_1 < t_1^* < t_2$, then $t_2 - t_1^* < t_2 - t_1$ and we get

$$\frac{f^{(t_1,t_2)}(x)}{f^{(t_1^*,t_2)}(x)} = \frac{F(t_2) - F(t_1^*)}{F(t_2) - F(t_1)}$$

for all $x \in [0, t_2 - t_1^*]$ and $f^{(t_1,t_2)}(x)/f^{(t_1^*,t_2)}(x) = \infty$ for all $x \in (t_2 - t_1^*, t_2 - t_1]$. Therefore the ratio is increasing in x (in the union of their supports) and so, from definition (A.7), $X^{(t_1,t_2)} \geq_{LR} X^{(t_1^*,t_2)}$.

(ii) From (6.1) and (6.11), the hazard rate function of $X^{(t_1,t_2)}$ is

$$\lambda^{(t_1,t_2)}(x) = \frac{f^{(t_1,t_2)}(x)}{\bar{F}^{(t_1,t_2)}(x)} = \frac{f(t_2 - x)}{F(t_2 - x) - F(t_1)}$$

Then we define the function

$$g(z) := \frac{F(z - x) - F(t_1)}{F(z) - F(t_1)} = \bar{F}^{(t_1,z)}(x)$$

for $z \in [t_2, t_2^*]$. We have

$$g'(z) = \frac{f(z-x)\,[F(z)-F(t_1)] - f(z)\,[F(z-x)-F(t_1)]}{[F(z)-F(t_1)]^2}.$$

Hence, if $F(t_2 - x) > F(t_1)$, we get

$$g'(z) =_{sign} \frac{f(z-x)}{F(z-x)-F(t_1)} - \frac{f(z)}{F(z)-F(t_1)} = \lambda^{(t_1,z)}(x) - \lambda^{(t_1,z)}(0)$$

for $z \in [t_2, t_2^*]$ with $0 \le t_1 < t_2 < t_2^*$, where $a =_{sign} b$ means that a and b have the same sign. We assume that $X^{(t_1,z)}$ is NBUHR for $z \in [t_2, t_2^*]$, so by the definition in (B.3) we have $\lambda^{(t_1,z)}(x) \ge \lambda^{(t_1,z)}(0)$ for $z > 0$ and hence g is increasing. Then $\bar{F}^{(t_1,z)}(x)$ is increasing, that is $\bar{F}^{(t_1,t_2)}(x) \le \bar{F}^{(t_1,t_2^*)}(x)$ for $t_2 < t_2^*$ so, by the definition in (A.1), we have $X^{(t_1,t_2)} \le_{ST} X^{(t_1,t_2^*)}$.

(iii) If $X^{(t_1,t_2)}$ is IHR for $0 \le t_1 < t_2 < t_2^*$, by the definition in (B.2), $\lambda^{(t_1,t_2)}$ is increasing. We note that if $0 \le x \le t_2 - t_1$, then $x + t_2 - t_2^* \le x \le t_2 - t_1$, so

$$\lambda^{(t_1,t_2)}(x) = \frac{f(t_2-x)}{F(t_2-x)-F(t_1)} \ge \lambda^{(t_1,t_2)}(x+t_2-t_2^*) = \frac{f(t_2^*-x)}{F(t_2^*-x)-F(t_1)} = \lambda^{(t_1,t_2^*)}(x).$$

Therefore, using definition (A.4), $X^{(t_1,t_2)} \le_{HR} X^{(t_1,t_2^*)}$.

(iv) We use that $f^{(t_1,t_2^*)}/f^{(t_1,t_2)}$ is increasing if and only if

$$g(x) := \frac{f(t_2^*-x)}{f(t_2-x)}$$

is increasing in x in the interval $[0, t_2 - t_1]$. Differentiating we get

$$g'(x) =_{sign} -\frac{f'(t_2^*-x)}{f(t_2^*-x)} + \frac{f'(t_2-x)}{f(t_2-x)} = \eta^{(t_1,t_2)}(x) - \eta^{(t_1,t_2)}(x+t_2-t_2^*)$$

where

$$\eta^{(t_1,t_2)}(x) = -\frac{(f^{(t_1,t_2)})'(x)}{f^{(t_1,t_2)}} = \frac{f'(t_2-x)}{f(t_2-x)}.$$

We assume that $X^{(t_1,t_2)}$ is ILR for $0 \le t_1 < t_2 < t_2^*$, so, by the definition in (B.6), $\eta^{(t_1,t_2)}$ is increasing. Then we have $g' \ge 0$ and the LR order holds by the definition in (A.7).

□

Note that the hazard rate of $X^{(t_1,t_2)}$ can be written as

$$\lambda^{(t_1,t_2)}(x) = \frac{f(t_2-x)/F(t_2-x)}{1-F(t_1)/F(t_2-x)} = \frac{\tau(t_2-x)}{1-F(t_1)/F(t_2-x)}$$

for all $x \in [0, t_2 - t_1]$, where τ is the reversed hazard rate of X. If X is DRHR, by the definition in (B.4), τ is decreasing, so $\lambda^{(t_1,t_2)}(x)$ is increasing and hence $X^{(t_1,t_2)}$ is IHR (and so NBUHR) for all $t_1 < t_2$. Thus $X^{(t_1,t_2)} \leq_{HR} X^{(t_1,t_2^*)}$ (and also $X^{(t_1,t_2)} \leq_{ST} X^{(t_1,t_2^*)}$) holds for all for $t_2 < t_2^*$ when X is DRHR (ILR or DHR). So this ordering holds when X has an exponential or a Pareto distribution (since they are DHR). However, we do not know if this property is true (as one can expect) for other (non DRHR) models. Analogously, it is easy to see that $X^{(t_1,t_2)}$ is ILR if and only if $\eta =:= -f'/f$ is increasing in $[t_1, t_2]$. Hence, if X is ILR, by the definition in (B.6), we know that η is increasing in the support of X, then we have $X^{(t_1,t_2)} \leq_{LR} X^{(t_1,t_2^*)}$.

As an immediate consequence, from the preceding results and Proposition 6.3, we obtain the following proposition for the inactivity time of Case II.

Proposition 6.7. *If the components are independent and their distribution functions satisfy $F_i(t_2) - F_i(t_1) > 0$ for $i = 1, \ldots, n$, then the reliability function of $T_{W_1,W_2}^{(t_1,t_2)}$ is ST-decreasing in t_1. Moreover, if the components are DRHR (ILR or DHR), then it is ST-increasing in t_2.*

6.4 Illustrations

6.4.1 Series Systems

Consider a series system with two possibly dependent components and with lifetime $T = X_{1:2} = \min(X_1, X_2)$. As in Section 6.2, we consider different types of conditioning events for the system inactivity time.

Case I We assume that the system was working at a time t_1 and that it is broken at another time t_2 (that is $t_1 < T < t_2$) with $0 \leq t_1 < t_2$. In this case we do not have information about the components. However, note that $T > t_1$, implies $X_i > t_1$ for $i = 1, 2$. So we consider the inactivity time of the system $T^{(t_1,t_2)} = (t_2 - T \mid t_1 < T < t_2)$ whose reliability function is given by

$$\begin{aligned}\bar{F}_T^{(t_1,t_2)}(x) &= P\left(\min(X_1, X_2) < t_2 - x \mid t_1 < T < t_2\right) \\ &= \bar{Q}^{(t_1,t_2)}\left(\bar{F}_1^{(t_1,t_2)}(x), \bar{F}_2^{(t_1,t_2)}(x)\right),\end{aligned}$$

where $\bar{F}_1^{(t_1,t_2)}$ and $\bar{F}_2^{(t_1,t_2)}$ are the reliability functions of the inactivity times of the components and where, from Proposition 6.1, $\bar{Q}^{(t_1,t_2)}$ is a distortion function given by

$$\bar{Q}^{(t_1,t_2)}(u,v) = \frac{Q(F_1(t_1) + u(F_1(t_2) - F_1(t_1)), F_2(t_1) + v(F_2(t_2) - F_2(t_1))) - Q(F_1(t_1), F_2(t_1))}{Q(F_1(t_2), F_2(t_2)) - Q(F_1(t_1), F_2(t_1))}.$$

The distribution function of a series system is

$$\begin{aligned} F_{1:2}(t) &= P(X_{1:2} \leq t) \\ &= P(X_1 \leq t) + P(X_2 \leq t) - P(X_1 \leq t, X_2 \leq t) \\ &= Q(F_1(t), F_2(t)), \end{aligned}$$

where, as we have seen in (4.12), $Q(u,v) = u + v - C(u,v)$ and C is the copula of (X_1, X_2), because Q depends on the structure function and on the copula C of (X_1, X_2). Therefore

$$\bar{Q}^{(t_1,t_2)}(u,v) = \frac{z_u + z_v - C(z_u, z_v) - F_1(t_1) - F_2(t_1) + C(F_1(t_1), F_2(t_1)}{F_1(t_2) + F_2(t_2) - C(F_1(t_2), F_2(t_2)) - F_1(t_1) - F_2(t_1) + C(F_1(t_1), F_2(t_1))}$$

where $z_u := F_1(t_1) + u(F_1(t_2) - F_1(t_1))$ and $z_v := F_2(t_1) + v(F_2(t_2) - F_2(t_1))$.

If the two components are indipendents, that is, $C(u,v) = uv$ for every $(u,v) \in [0,1]^2$, then we have

$$\bar{Q}^{(t_1,t_2)}(u,v) = \frac{z_u + z_v - z_u z_v - F_1(t_1) - F_2(t_1) + F_1(t_1)F_2(t_1)}{F_1(t_2) + F_2(t_2) - F_1(t_2)F_2(t_2) - F_1(t_1) - F_2(t_1) + F_1(t_1), F_2(t_1)}$$

Case II We analyse the inactivity times obtained for different choices of W_1 (components that fail before t_1) and W_2 (components that fail between t_1 and t_2).

- $W_1 = \emptyset, W_2 = \{1,2\}$.

First we assume that, at a time t_1, both components were working and that they are broken at the second inspection time t_2, with $0 \leq t_1 < t_2$. In this case W_1 is empty, $W_2 = \{1,2\}$ and $W_3 = \{1,2\} - W_1 - W_2 = \emptyset$. The inactivity time of the system is

$$T^{(t_1,t_2)}_{\emptyset,\{1,2\}} = (t_2 - T \,|\, t_1 < X_1 < t_2, t_1 < X_2 < t_2)$$

whose reliability function is given by

$$\begin{aligned} \bar{F}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(x) &= P\left(\min(X_1, X_2) < t_2 - x \,|\, t_1 < X_1 < t_2, t_1 < X_2 < t_2\right) \\ &= \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}\left(\bar{F}_1^{(t_1,t_2)}(x), \bar{F}_2^{(t_1,t_2)}(x)\right), \end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ is a distortion function given by

$$\frac{C(z_u, F_2(t_2)) + C(F_1(t_1), F_2(t_1)) + C(F_1(t_2), z_v) - C(z_u, z_v) - C(F_1(t_1), F_2(t_2)) - C(F_1(t_2), F_2(t_1))}{C(F_1(t_2), F_2(t_2)) + C(F_1(t_1), F_2(t_1)) - C(F_1(t_2), F_2(t_1)) - C(F_1(t_1), F_2(t_2))},$$

where, as above, $z_u := F_1(t_1) + u(F_1(t_2) - F_1(t_1))$ and $z_v := F_2(t_1) + v(F_2(t_2) - F_2(t_1))$.

- $W_1 = \emptyset, W_2 = \{1\}$.

We assume that the first component was working at a time t_1 and that it is broken at the second inspection time t_2. We also know that the second component is working at time t_2 even if the system has clearly failed in the interval (t_1, t_2). In this case W_1 is empty, $W_2 = \{1\}$ and $W_3 = \{1,2\} - W_1 - W_2 = \{2\}$. Then the system inactivity time is

$$T_{\emptyset,\{1\}}^{(t_1,t_2)} = (t_2 - T \,|\, t_1 < X_1 < t_2, X_2 > t_2)$$

whose reliability function is

$$\begin{aligned}
\bar{F}_{\emptyset,\{1\}}^{(t_1,t_2)}(x) &= P\left(t_2 - T > x \,|\, t_1 < X_1 < t_2, X_2 > t_2\right) \\
&= \frac{P\left(t_1 < X_1 < t_2 - x, X_2 > t_2\right)}{P\left(t_1 < X_1 < t_2, X_2 > t_2\right)} \\
&= \bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}\left(\bar{F}_1^{(t_1,t_2)}(x), \bar{F}_2^{(t_1,t_2)}(x)\right),
\end{aligned}$$

where $\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}$ is a distortion function given by

$$\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}(u,v) = \frac{z_u + C(F_1(t_1), F_2(t_2)) - F_1(t_1) - C(z_u, F_2(t_2))}{F_1(t_2) + C(F_1(t_1), F_2(t_2)) - F_1(t_1) - C(F_1(t_2), F_2(t_2))}$$

where $z_u := F_1(t_1) + u(F_1(t_2) - F_1(t_1))$. Note that $\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}(u,v)$ does not depend (as expected) on v, that is $\bar{F}_2^{(t_1,t_2)}$.

If the components are independent, that is, $C(u,v) = uv$ for every $(u,v) \in [0,1]^2$, then we have

$$\bar{Q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}(u,v) = u + v - uv,$$

and

$$\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}(u,v) = u.$$

Note that they do not depend on t_1 and t_2, as we note in the Remark 6.3.
We have

$$\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}(u,v) = u \leq u + v - uv = \bar{Q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}(u,v),$$

so $T_{\emptyset,\{1\}}^{(t_1,t_2)} \leq_{ST} T_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ for all $t_1 < t_2$ and for all F_1, F_2.

If we want to study the HR ordering, from Proposition 4.3, (ii), we should analyse

$$\frac{\bar{Q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}(u,v)}{\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}(u,v)} = 1 - v + \frac{v}{u}.$$

As it is decreasing in u and increasing in v for all $(u,v) \in [0,1]^2$, $T^{(t_1,t_2)}_{\emptyset,\{1\}}$ and $T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ are not HR-ordered for all F_1, F_2.
However, if the components of such system are independent and identically distributed (IID) with a common distribution F, we have

$$\bar{q}^{(t_1,t_2)}_{\emptyset,\{1\}}(u) = \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1\}}(u,u) = u$$

and

$$\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u) = \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u,u) = 2u - u^2.$$

Then, to apply Proposition 4.2, (ii), we study the function

$$\frac{\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u)}{\bar{q}^{(t_1,t_2)}_{\emptyset,\{1\}}(u)} = 2 - u.$$

As it is decreasing for $u \in (0,1)$, we can state that $T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{HR} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ for all F and all $t_1 < t_2$. Even more, as

$$\frac{\left(\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}\right)'(u)}{\left(\bar{q}^{(t_1,t_2)}_{\emptyset,\{1\}}\right)'(u)} = 2 - 2u$$

is also decreasing, from Proposition 4.2, (iv), we obtain $T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{LR} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ for all F and all $t_1 < t_2$.

The following example shows that the HR order may not hold for inactivity time of systems with independent non-identically distributed components.

Example 6.1. Consider a series system with two independent components whose lifetimes X_1 and X_2 have exponential distribution with means $\mu_1 = 2$ and $\mu_2 = 1$, respectively. Set $t_1 = 0.2$ and $t_2 = 0.6$. In Figure 6.1 (left) we plot the hazard rate functions of $T^{(t_1,t_2)}_{\emptyset,\{1\}}$ (blue) and $T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ (red). Note that the hazard rate functions are not ordered. In Figure 6.1 (right) we plot the hazard rate functions when the components are IID with a common exponential distribution with mean 2. In this case, they are HR-ordered as stated above. This is true for all F and all $t_1 < t_2$.

In the following example we assume that the component lifetimes are ID and that they have a specific dependence structure, that is a specific copula.

Example 6.2. Consider a series system whose component lifetimes have a common distribution F and a Clayton copula with $\theta = 1$ defined in (5.8).

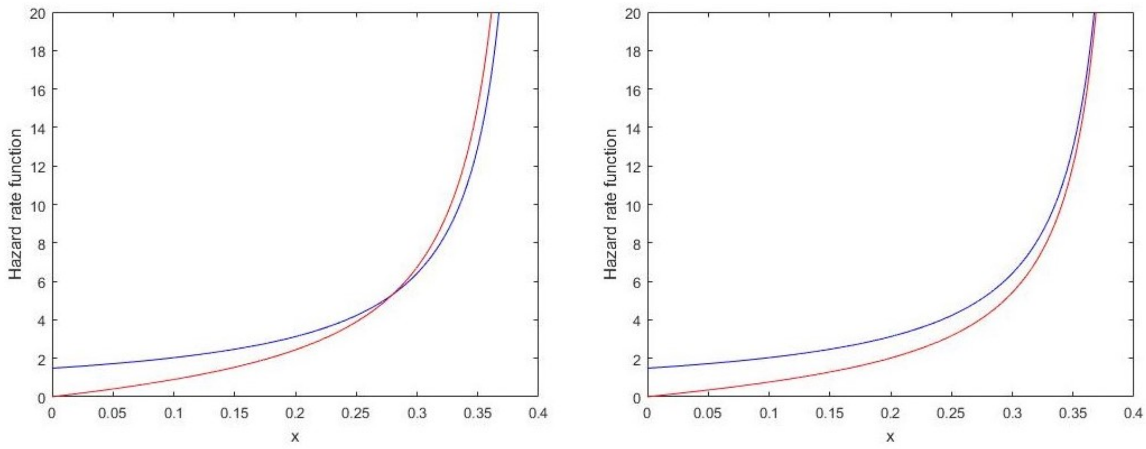

Figure 6.1: Hazard rate functions of $T_{\emptyset,\{1\}}^{(t_1,t_2)}$ (blue) and $T_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ (red) for inactivity time of the series system considered in Example 6.1 with independent components (left) or IID components (right).

Then, in Figure 6.2 we plot the dual distortion functions $\bar{q}_{\emptyset,\{1\}}^{(t_1,t_2)}$, $\bar{q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ and $\bar{q}^{(t_1,t_2)}$ when we set t_1 and t_2 such that $F(t_1) = 0.1$ and $F(t_2) = 0.5$. As we can see in Figure 6.2, these functions are ordered, and so

$$T_{\emptyset,\{1\}}^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)} \leq_{ST} T_{\emptyset,\{1,2\}}^{(t_1,t_2)}$$

for any F and for $0 < t_1 < t_2$ such that $F(t_1) = 0.1$ and $F(t_2) = 0.5$.
By analysing the ratio of the dual distortion functions $\bar{q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}/\bar{q}^{(t_1,t_2)}$ we obtain that it is decreasing and so, from Proposition 6.6, (ii), we also have

$$T^{(t_1,t_2)} \leq_{HR} T_{\emptyset,\{1,2\}}^{(t_1,t_2)}$$

for any F and for $0 < t_1 < t_2$ such that $F(t_1) = 0.1$ and $F(t_2) = 0.5$.

We can study, in a similar way, the performance of the dual distortion functions of the systems when the components are dependent and non identically distributed. In the following example we fix first u and then v to analyse the respective plots.

Example 6.3. Consider a series system whose components are not identically distributed. We use again the Clayton copula with $\theta = 1$ given above. Consider two distribution functions and the two inspection times such that $F_1(t_1) = 0.3$, $F_1(t_2) = 0.7$, $F_2(t_1) = 0.5$ and $F_2(t_2) = 0.9$. Then we fix $v = 0.2$ and, in Figure 6.3 (left), we plot the dual distortion functions $\bar{Q}_{\emptyset,\{1\}}^{(t_1,t_2)}$ (blue), $\bar{Q}^{(t_1,t_2)}$ (green) and $\bar{Q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ (red). In a

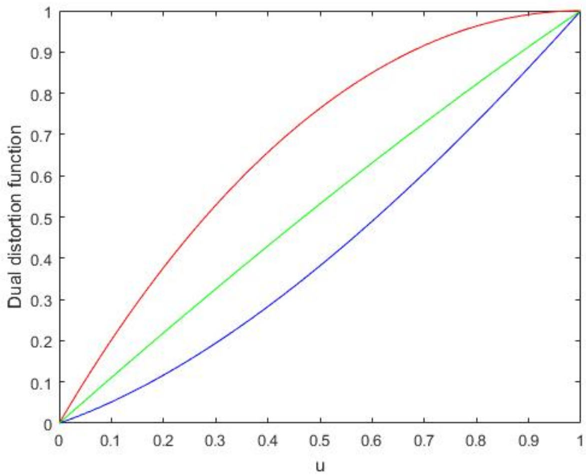

Figure 6.2: Dual distortion functions of $T^{(t_1,t_2)}_{\emptyset,\{1\}}$ (blue), $T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ (red) and $T^{(t_1,t_2)}$ (green) for the system in Example 6.2.

similar way, in Figure 6.3 (right), we plot the same functions for a fix $u = 0.2$. As we can see in Figure 6.3, the ordering

$$\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1\}} \leq \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}} \qquad \text{and} \qquad \bar{Q}^{(t_1,t_2)} \leq \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}},$$

may hold but $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1\}}$ and $\bar{Q}^{(t_1,t_2)}$ cross each other. So $T^{(t_1,t_2)}_{\emptyset,\{1\}}$ and $T^{(t_1,t_2)}$ are not ST-ordered for any F_1, F_2.

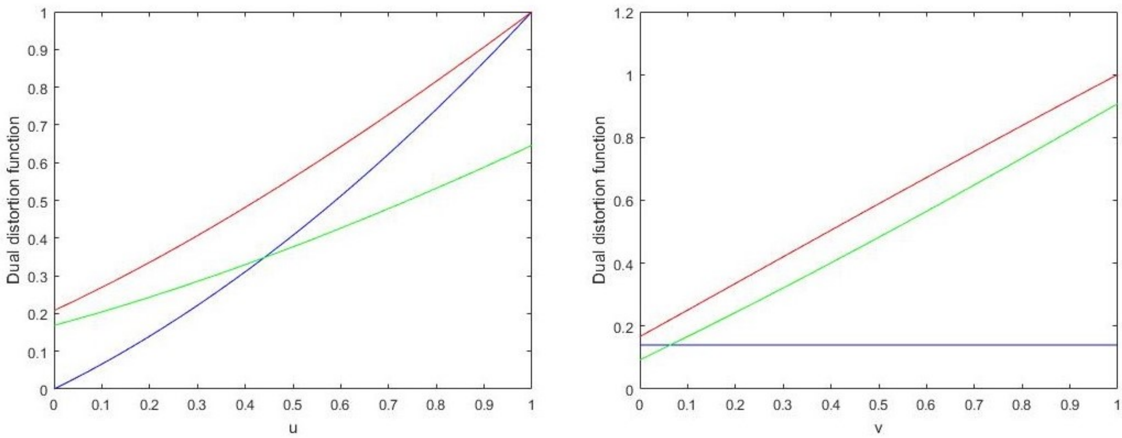

Figure 6.3: Dual distortion functions $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1\}}$ (blue), $\bar{Q}^{(t_1,t_2)}$ (green) and $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ (red) for the inactivity time of the series system considered in the Example 6.3 when we fix $v = 0.2$ (left) and $u = 0.2$ (right).

In the following example we show that the inactivity times considered in the previous example, can be ordered for some specific distributions.

Example 6.4. Consider again a series system whose components are not identically distributed and have a Clayton survival copula (with $\theta = 1$) and let us fix $0 \leq t_1 \leq t_2$ such that $F(t_1) = 0.2$ and $F(t_2) = 0.6$.
We assume first that the component lifetimes have both exponential distributions with means 2 and 1, respectively. Then in Figure 6.4 (left) we plot the reliability functions of the inactivity times $T^{(t_1,t_2)}_{\emptyset,\{1\}}$ (blue), $T^{(t_1,t_2)}$ (green) and $T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ (red). As we can see, we obtain:

$$T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{ST} T^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}.$$

By analysing the ratios $\bar{F}^{(t_1,t_2)}/\bar{F}^{(t_1,t_2)}_{\emptyset,\{1\}}$ and $\bar{F}^{(t_1,t_2)}_{\emptyset,\{1,2\}}/\bar{F}^{(t_1,t_2)}$, we find that both are increasing and so, from Proposition 6.5, (ii),

$$T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{HR} T^{(t_1,t_2)} \leq_{HR} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}.$$

However, if the components have Weibull distributions (i.e., $\bar{F}(x) = e^{-(\lambda x)^k}$) with parameters $(\lambda = 1, k = 0.5)$ and $(\lambda = 1, k = 1.5)$, respectively, we obtain the plots in Figure 6.4 (right) and so

$$T^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{ST} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}.$$

By analysing the ratios we see that, in this case, again from Proposition 6.6, (ii), we have

$$T^{(t_1,t_2)} \leq_{HR} T^{(t_1,t_2)}_{\emptyset,\{1\}} \leq_{HR} T^{(t_1,t_2)}_{\emptyset,\{1,2\}}.$$

Note that in all the examples $T^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ seems to be the greatest, and so the worst, one.

6.4.2 Parallel Systems

Consider now a parallel system with two possibly dependent components and with lifetime $T = X_{2:2} = \max(X_1, X_2)$. As in Section 6.2, we consider three possible types of conditioning events for the system inactivity time.

Case I Assume that the system was working at a time t_1 and that it is broken at the second inspection time t_2 with $0 \leq t_1 < t_2$ and that we do not have information about

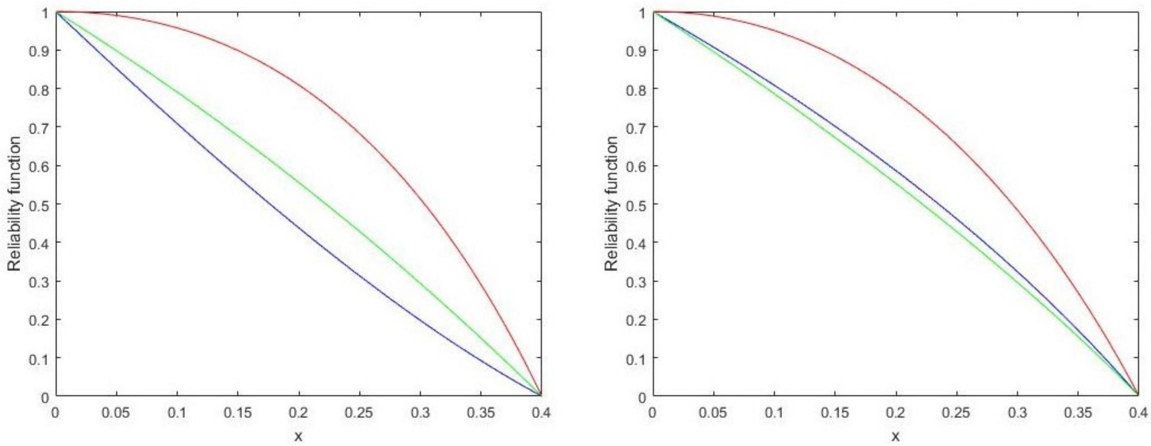

Figure 6.4: Reliability functions of $T_{\emptyset,\{1\}}^{(t_1,t_2)}$ (blue), $T^{(t_1,t_2)}$ (green) and $T_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ (red) for the series system considered in the Example 6.4 for components having both exponential distributions with means 2 and 1, respectively (left) and both Weibull distributions with parameters $\lambda = 1, k = 0.5$ and $\lambda = 1, k = 1.5$, respectively (right).

the components. Then we can consider the inactivity time of the system $T^{(t_1,t_2)} = (t_2 - T \mid t_1 < T < t_2)$ whose reliability function is given by

$$\begin{aligned} \bar{F}_T^{(t_1,t_2)}(x) &= P\left(\max(X_1, X_2) < t_2 - x \mid t_1 < T < t_2\right) \\ &= \bar{Q}^{(t_1,t_2)}\left(\bar{F}_1^{(t_1,t_2)}(x), \bar{F}_2^{(t_1,t_2)}(x)\right), \end{aligned}$$

where, from Proposition 6.1, $\bar{Q}^{(t_1,t_2)}(u,v)$ is a distortion function given by

$$\frac{Q(F_1(t_1) + u(F_1(t_2) - F_1(t_1)), F_2(t_1) + v(F_2(t_2) - F_2(t_1))) - Q(F_1(t_1), F_2(t_1))}{Q(F_1(t_2), F_2(t_2)) - Q(F_1(t_1), F_2(t_1))}.$$

For a parallel system we have

$$\begin{aligned} F_{2:2}(t) &= P(F_{2:2} \le t) \\ &= P(\max(X_1, X_2) \le t) \\ &= P(X_1 \le t, X_2 \le t) = C(F_1(t), F_2(t)), \end{aligned}$$

that is, $Q(u,v) = C(u,v)$. So

$$\bar{Q}^{(t_1,t_2)}(u,v) = \frac{C(z_u, z_v) - C(F_1(t_1), F_2(t_1))}{C(F_1(t_2), F_2(t_2)) - C(F_1(t_1), F_2(t_1))}$$

where $z_u := F_1(t_1) + u(F_1(t_2) - F_1(t_1))$ and $z_v := F_2(t_1) + v(F_2(t_2) - F_2(t_1))$.

Case II We analyse the inactivity times obtained for different choices of W_1 and W_2.

- $W_1 = \emptyset, W_2 = \{1, 2\}$.

Here we assume that at a time t_1 both components were working and that they are broken at the second inspection time t_2 with $0 \leq t_1 < t_2$. In this case $W_1 = \emptyset$, $W_2 = \{1, 2\}$ and $W_3 = \{1, 2\} - W_1 - W_2 = \emptyset$. Then we can consider the inactivity time

$$T^{(t_1,t_2)}_{\emptyset,\{1,2\}} = (t_2 - T \mid t_1 < X_1 < t_2, t_1 < X_2 < t_2),$$

whose reliability function is given by

$$\begin{aligned} \bar{F}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(x) &= P\left(\max(X_1, X_2) < t_2 - x \mid t_1 < X_1 < t_2, t_1 < X_2 < t_2\right) \\ &= \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}\left(\bar{F}^{(t_1,t_2)}_1(x), \bar{F}^{(t_1,t_2)}_2(x)\right), \end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u, v)$ is a distortion function given by

$$\frac{C(z_u, z_v) + C(F_1(t_1), F_2(t_1)) - C(F_1(t_1), z_v) - C(z_u, F_2(t_1))}{C(F_1(t_2), F_2(t_2)) + C(F_1(t_1), F_2(t_1)) - C(F_1(t_2), F_2(t_1)) - C(F_1(t_1), F_2(t_2))},$$

where $z_u := F_1(t_1) + u(F_1(t_2) - F_1(t_1))$ and $z_v := F_2(t_1) + v(F_2(t_2) - F_2(t_1))$.

- $W_1 = \{1\}, W_2 = \{2\}$

Assume that the system has failed in the interval (t_1, t_2), even if the first component stopped to work before the time t_1, instead the second component was working at a time t_1 and it is broken at the second inspection time t_2. In this case $T = X_2$, $W_1 = \{1\}$, $W_2 = \{2\}$ and $W_3 = \{1, 2\} - W_1 - W_2 = \emptyset$. Then we can consider the inactivity time

$$T^{(t_1,t_2)}_{\{1\},\{2\}} = (t_2 - T \mid X_1 < t_1, t_1 < X_2 < t_2),$$

whose reliability function is

$$\begin{aligned} \bar{F}^{(t_1,t_2)}_{\{1\},\{2\}}(x) &= P\left(\max(X_1, X_2) < t_2 - x | X_1 < t_1, t_1 < X_2 < t_2\right) \\ &= \bar{Q}^{(t_1,t_2)}_{\{1\},\{2\}}\left(\bar{F}^{(t_1,t_2)}_1(x), \bar{F}^{(t_1,t_2)}_2(x)\right), \end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\{1\},\{2\}}$ is a distortion function given by

$$\bar{Q}^{(t_1,t_2)}_{\{1\},\{2\}}(u, v) = \frac{C(F_1(t_1), z_v) - C(F_1(t_1), F_2(t_1))}{C(F_1(t_1), F_2(t_2)) - C(F_1(t_1), F_2(t_1))},$$

where $z_v := F_2(t_1) + v(F_2(t_2) - F_2(t_1))$. Note that $\bar{Q}^{(t_1,t_2)}_{\{1\},\{2\}}$ does not depend (as expected) on u, that is $\bar{F}^{(t_1,t_2)}_1$.

Case III Assume that the second component fails at time t_1 and that the first component fails in the interval (t_1, t_2) for $0 < t_1 < t_2$. Then the system fails in the interval (t_1, t_2) and we want to calculate the reliability function of the inactivity time

$$T_{2,\{1\}}^{(t_1,t_2)} = (t_2 - T | t_1 < X_1 < t_2, X_2 = t_1).$$

The pdf function of $(X_1 | X_2 = x_2)$ is

$$\mathbf{f}_{1|2}(x_1|x_2) = \frac{\mathbf{f}(x_1, x_2)}{f_2(x_2)} = f_1(x_1)\partial_1\partial_2 C(F_1(x_1), F_2(x_2))$$

for x_2 such that $f_2(x_2) > 0$. Therefore, its distribution function is

$$\begin{aligned}\mathbf{F}_{1|2}(x|x_2) &= \int_0^x f_1(x_1)\partial_1\partial_2 C(F_1(x_1), F_2(x_2))dx_1 \\ &= \partial_2 C(F_1(x), F_2(x_2))\end{aligned}$$

since $\lim_{u\to 0^+} \partial_2(u, v) = 0$ for all $v \in [0, 1]$. In particular, the distribution function of $(X_1 | X_2 = t_1)$ can be written (see expression (6.9)) as

$$\mathbf{F}_{1|2}(x|t_1) = P(X_1 \leq x | X_2 = t_1) = \partial_2 C(F_1(x), F_2(t_1)) = \mathbf{Q}_{1|2}(F_1(x)),$$

where $\mathbf{Q}_{1|2}(u) = \partial_2 C(u, F_2(t_1))$ is a distortion function.

Hence the reliability function of the inactivity time $T_{2,\{1\}}^{(t_1,t_2)}$ can be calculated as

$$\begin{aligned}P(T_{2,\{1\}}^{(t_1,t_2)} > x) &= P(t_2 - T > x | t_1 < X_1 < t_2, X_2 = t_1) \\ &= \frac{P(t_2 - T > x, t_1 < X_1 < t_2 | X_2 = t_1)}{P(t_1 < X_1 < t_2 | X_2 = t_1)},\end{aligned}$$

where

$$P(t_1 < X_1 < t_2 | X_2 = t_1) = \partial_2 C(F_1(t_2), F_2(t_1)) - \partial_2 C(F_1(t_1), F_2(t_1))$$

and

$$\begin{aligned}P(t_2 - T > x, t_1 < X_1 < t_2 | X_2 = t_1) &= P(t_1 < X_1 < t_2 - x | X_2 = t_1) \\ &= \partial_2 C(F_1(t_2 - x), F_2(t_1)) - \partial_2 C(F_1(t_1), F_2(t_1)).\end{aligned}$$

Finally, from (6.1), we have

$$F_1(t_2 - x) = \bar{F}_1^{(t_1,t_2)}(x)(F_1(t_2) - F_1(t_1)) + F_1(t_1)$$

and so

$$\bar{F}_{2,\{1\}}^{(t_1,t_2)}(x) = P(T_{2,\{1\}}^{(t_1,t_2)} > x) = \bar{Q}_{2,\{1\}}^{(t_1,t_2)}(\bar{F}_1^{(t_1,t_2)}(x), \bar{F}_2^{(t_1,t_2)}(x)),$$

where

$$\bar{Q}_{2,\{1\}}^{(t_1,t_2)}(u,v) = \frac{\partial_2 C(u(F_1(t_2)-F_1(t_1))+F_1(t_1),F_2(t_1)) - \partial_2 C(F_1(t_1),F_2(t_1))}{\partial_2 C(F_1(t_2),F_2(t_1)) - \partial_2 C(F_1(t_1),F_2(t_1))}.$$

Note that $\bar{Q}_{2,\{1\}}^{(t_1,t_2)}$ does not depends on v (as expected).

If the component are indipendent and so C is the product copula, then $\partial_2 C(u,v) = u$ and we obtain

$$\bar{Q}_{2,\{1\}}^{(t_1,t_2)}(u,v) = \frac{u(F_1(t_2)-F_1(t_1))+F_1(t_1)-F_1(t_1)}{F_1(t_2)-F_1(t_1)} = u,$$

that is,

$$\begin{aligned} \bar{F}_{2,\{1\}}^{(t_1,t_2)}(x) &= P(t_2 - T > x | t_1 < X_1 < t_2, X_2 = t_1) \\ &= \bar{F}_1^{(t_1,t_2)}(x) \\ &= P(t_2 - X_1 > x | t_1 < X_1 < t_2). \end{aligned}$$

However, if, as in the preceding examples, we chose the following Clayton copula

$$C(u,v) = \frac{uv}{u+v-uv}$$

then

$$\partial_2 C(u,v) = \frac{u^2}{(u+v-uv)^2}$$

and

$$\bar{Q}_{2,\{1\}}^{(t_1,t_2)}(u,v) = \frac{[F_1(t_1)+F_2(t_1)-F_1(t_1)F_2(t_1)]^2 A(z_u) - F_1^2(t_1)}{[F_1(t_1)+F_2(t_1)-F_1(t_1)F_2(t_1)]^2 A(F_1(t_2)) - F_1^2(t_1)}$$

where $z_u := u(F_1(t_2)-F_1(t_1))+F_1(t_1)$ and

$$A(x) := \left(\frac{x}{x+F_2(t_1)-xF_2(t_1)}\right)^2.$$

Example 6.5. Consider a parallel system whose component lifetimes have a common distribution F and the above Clayton copula. Then in Figure 6.5 (left) we plot $\bar{q}^{(t_1,t_2)}$ (blue), $\bar{q}_{\emptyset,\{1,2\}}^{(t_1,t_2)}$ (red), $\bar{q}_{2,\{1\}}^{(t_1,t_2)}$ (green) and $\bar{q}_{\{1\},\{2\}}^{(t_1,t_2)}$ (black) for $0 \le t_1 \le t_2$ such that $F(t_1) = 0.3$ and $F(t_2) = 0.5$. As we can see, we obtain

$$T_{\emptyset,\{1,2\}}^{(t_1,t_2)} \le_{ST} T^{(t_1,t_2)} \le_{ST} T_{2,\{1\}}^{(t_1,t_2)} \le_{ST} T_{\{1\},\{2\}}^{(t_1,t_2)}.$$

for any F and for $0 \le t_1 \le t_2$ such that $F(t_1) = 0.3$ and $F(t_2) = 0.5$. These ordering can be reinforced by plotting the ratios of the dual distortion functions. They are give in Figure 6.5 (right). As all of them are decreasing, so, from Proposition 6.6, (ii), they are also HR-ordered in the same way of the ST-order.

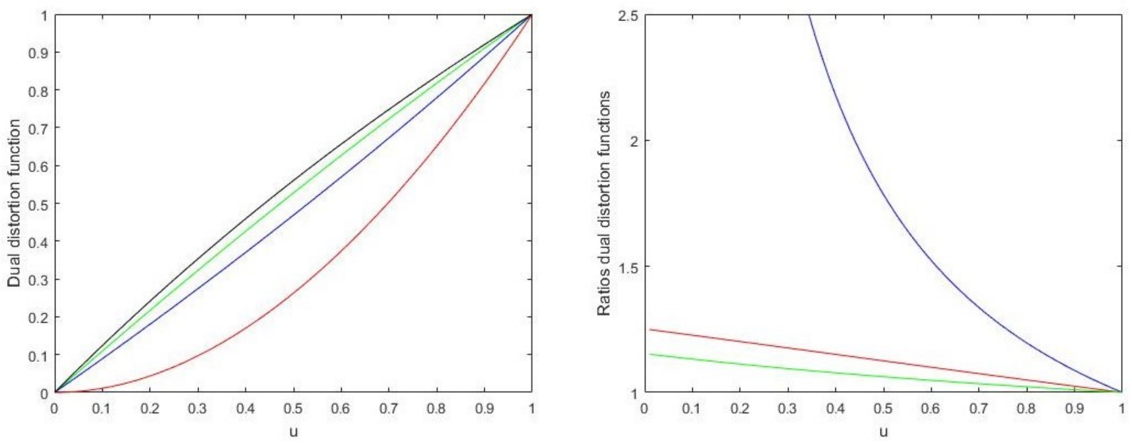

Figure 6.5: Dual distortion functions (left) $\bar{q}^{(t_1,t_2)}$ (blue), $\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ (red), $\bar{q}^{(t_1,t_2)}_{2,\{1\}}$ (green) and $\bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}$ (black) for the inactivity time of the parallel system considered in Example 6.5. The ratios of the dual distortion functions (right) are all decreasing, so the corresponding inactivity times they are also HR-ordered.

Some of the orders obtained in Example 6.5 hold also in a more general case, as we state in the following proposition.
First we need to recall that the diagonal section of a copula C is the function $\delta_C : [0,1] \to [0,1]$, defined by $\delta_C(u) = C(u,u)$. A theorem (see Theorem 3.2.12 in [39]) proves that, if δ is any function from $[0,1]$ to $[0,1]$ such that:

a) $\delta(1) = 1$;

b) $0 \leq \delta(t_2) - \delta(t-1) \leq 2(t_2 - t_1)$ for all $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$;

c) $\delta(t) \leq t$ for all $t \in [0,1]$,

then the function

$$C(u,v) = \min\left(u, v, \frac{\delta(u)+\delta(v)}{2}\right) \tag{6.12}$$

is a copula whose diagonal section is δ. This copula is called the *diagonal copula*.

Proposition 6.8. *Consider a parallel system with two dependent components with a common distribution F and the diagonal copula* (6.12)*. Then*

$$T^{(t_1,t_2)}_{\emptyset,\{1,2\}} \leq_{ST} T^{(t_1,t_2)} \leq_{ST} T^{(t_1,t_2)}_{\{1\},\{2\}}$$

for all F and all $0 \leq t_1 < t_2$.

Proof. From the expressions given above, $\bar{q}^{(t_1,t_2)}$, $\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ and $\bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}$ in terms of the diagonal copula as:

$$\bar{q}^{(t_1,t_2)}(u) = \frac{\delta(z_u)-\delta(a)}{\delta(b)-\delta(a)},$$

$$\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u) = \frac{\delta(z_u)+\delta(a)-2\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right)}{\delta(b)+\delta(a)-2\min\left(a, b, \frac{\delta(a)+\delta(b)}{2}\right)},$$

$$\bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}(u) = \frac{\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right)-\delta(a)}{\min\left(a, b, \frac{\delta(a)+\delta(z_u)}{2}\right)-\delta(a)},$$

where $a = F(t_1) < b = F(t_2)$ and $z_u = u(b-a)+a$. Note that $a < z_u < b$ and so $\delta(a) \le \delta(z_u) \le \delta(b)$ because $\delta(t)$ is a nondecreasing function on $[0,1]$. Hence $\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right) = a$, implies $\min\left(a, b, \frac{\delta(a)+\delta(b)}{2}\right) = a$. We also know that $\delta(u) \le u$ for all u. Thus we have to consider three possible cases:

- $\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right) = a$ and $\min\left(a, b, \frac{\delta(a)+\delta(b)}{2}\right) = a$.

$$\bar{q}^{(t_1,t_2)}(u) = \frac{\delta(z_u)-\delta(a)}{\delta(b)-\delta(a)},$$

$$\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u) = \frac{\delta(z_u)+\delta(a)-2a}{\delta(b)+\delta(a)-2a},$$

and

$$\bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}(u) = \frac{a-\delta(a)}{a-\delta(a)} = 1.$$

Then we have $\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}} \le \bar{q}^{(t_1,t_2)} \le \bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}$, because $2a \le \delta(a)+\delta(z_u) \le \delta(a)+\delta(b)$.

- $\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right) = \frac{\delta(a)+\delta(z_u)}{2}$ and $\min\left(a, b, \frac{\delta(a)+\delta(b)}{2}\right) = a$.

$$\bar{q}^{(t_1,t_2)}(u) = \frac{\delta(z_u)-\delta(a)}{\delta(b)-\delta(a)},$$

$$\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}(u) = 0,$$

and

$$\bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}(u) = \frac{\delta(z_u)-\delta(a)}{2a-2\delta(a)}.$$

Then we have $\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}} \le \bar{q}^{(t_1,t_2)} \le \bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}$ because, in this case, $2a \le \delta(a)+\delta(b)$.

- $\min\left(a, z_u, \frac{\delta(a)+\delta(z_u)}{2}\right) = \frac{\delta(a)+\delta(z_u)}{2}$ and $\min\left(a, b, \frac{\delta(a)+\delta(b)}{2}\right) = \frac{\delta(a)+\delta(b)}{2}$.

In this case the dual distortion function $\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2\}}$ does not exist and we have

$$\bar{q}^{(t_1,t_2)}(u) = \frac{\delta(z_u) - \delta(a)}{\delta(b) - \delta(a)} = \bar{q}^{(t_1,t_2)}_{\{1\},\{2\}}(u).$$

$\square$

6.4.3 Other Systems

In the following examples we analyse different cases studied in Section 6.2 for general coherent systems, different from series and parallel structures.

- Coherent system with lifetime $T = \max(X_1, \min(X_2, X_3))$

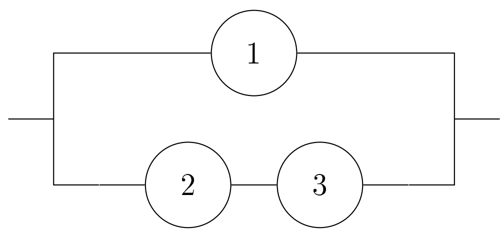

Figure 6.6: Coherent structure with lifetime $T = \max(X_1, \min(X_2, X_3))$

The minimal cut sets are $K_1 = \{1, 2\}$ and $K_2 = \{1, 3\}$. Then the system distribution can be obtained as follows

$$\begin{aligned}
F_T(t) &= P(\max(X_1, \min(X_2, X_3)) \le t)\\
&= P(X_1 \le t) + P(\min(X_2, X_3) \le t) - P(X_1 \le t, \min(X_2, X_3) \le t)\\
&= \mathbf{F}(t, \infty, \infty) + \mathbf{F}(\infty, t, t) - \mathbf{F}(t, t, t)\\
&= F_1(t) + C(1, F_2(t), F_3(t)) - C(F_1(t), F_2(t), F_3(t))\\
&= Q(F_1(t), F_2(t), F_3(t)),
\end{aligned}$$

where $Q(u_1, u_2, u_3) = u_1 + C(1, u_2, u_3) - C(u_1, u_2, u_3)$.

Assume now that for $0 < t_1 < t_2$, we know that component three failed before t_1, and that the first and the second components failed in the interval (t_1, t_2). This is case II of Section 6.2.2 with $W_1 = \{3\}$ and $W_2 = \{1, 2\}$. Note that these assumptions imply that the system fails in this interval (t_1, t_2). The system inactivity time considered is

$$T^{(t_1,t_2)}_{\{3\},\{1,2\}} = (t_2 - T | X_3 < t_1, t_1 < X_1 < t_2, t_1 < X_2 < t_2).$$

These conditions imply $T = X_1$ and so, for $x \in [0, t_2 - t_1]$, its reliability function is given by

$$\begin{aligned}
\bar{F}^{(t_1,t_2)}_{\{3\},\{1,2\}}(x) &= P(t_2 - T > x | X_3 < t_1, t_1 < X_1 < t_2, t_1 < X_2 < t_2) \\
&= P(X_1 < t_2 - x | X_3 < t_1, t_1 < X_1 < t_2, t_1 < X_2 < t_2) \\
&= \frac{P(t_1 < X_1 < t_2 - x, t_1 < X_2 < t_2, X_3 < t_1)}{P(t_1 < X_1 < t_2, t_1 < X_2 < t_2, X_3 < t_1)} \\
&= \frac{P(X_1 < t_2 - x, t_1 < X_2 < t_2, X_3 < t_1)}{P(t_1 < X_1 < t_2, t_1 < X_2 < t_2, X_3 < t_1)} - \frac{P(X_1 < t_1, t_1 < X_2 < t_2, X_3 < t_1)}{P(t_1 < X_1 < t_2, t_1 < X_2 < t_2, X_3 < t_1)} \\
&= \bar{Q}^{(t_1,t_2)}_{\{3\},\{1,2\}}(\bar{F}^{(t_1,t_2)}_1(x), \bar{F}^{(t_1,t_2)}_2(x), \bar{F}^{(t_1,t_3)}_3(x)),
\end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\{3\},\{1,2\}}(u_1, u_2, u_3)$ does not depend, as expected, on u_2, u_3.

In particular, if the components are independent, that is $C(u_1, u_2, u_3) = u_1u_2u_3$, then

$$\begin{aligned}
\bar{F}^{(t_1,t_2)}_{\{3\},\{1,2\}}(x) &= \frac{P(t_1 < X_1 < t_2 - x)P(t_1 < X_2 < t_2)P(X_3 < t_1)}{P(t_1 < X_1 < t_2)P(t_1 < X_2 < t_2)P(X_3 < t_1)} \\
&= \frac{P(t_1 < X_1 < t_2 - x)}{P(t_1 < X_1 < t_2)} \\
&= \bar{F}^{(t_1,t_2)}_1(x) \\
&= \bar{Q}_{\{3\},\{1,2\}}(\bar{F}^{(t_1,t_2)}_1(x), \bar{F}^{(t_1,t_2)}_2(x), \bar{F}^{(t_1,t_3)}_3(x))
\end{aligned}$$

with $\bar{Q}_{\{3\},\{1,2\}}(u_1, u_2, u_3) = u_1$.

This distortion function can also be computed from Proposition 6.3 as follows. The minimal cut sets included in $W_1 \cup W_2 = \{1, 2, 3\}$ are two, then $s = 2$. We have $K_1 \cap W_2 = \{1, 2\}$, $k_2 \cap W_2 = \{1\}$ and $(K_1 \cup K_2) \cap W_2 = \{1, 2, 3\}$ and from (6.8), $\bar{Q}_{\{3\},\{1,2\}}$ is given by

$$\bar{Q}_{\{3\},\{1,2\}}(u_1, u_2, u_3) = u_1u_2 + u_1 - u_1u_2 = u_1.$$

Note that the sets in the preceding sequence cannot be the minimal cut sets of a system since $\{1\} \subset \{1, 2\}$, as we state in Remark 6.3.

- Coherent system with lifetime $T = \min(X_1, \max(X_2, X_3))$

  The minimal cut sets are $K_1 = \{1\}$ and $K_2 = \{2, 3\}$.

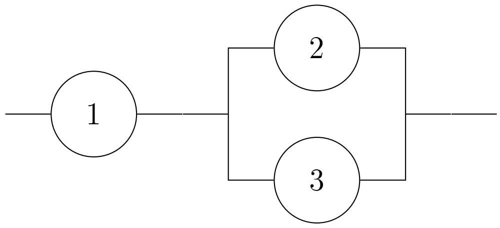

Figure 6.7: Coherent structure with lifetime $T = \min(X_1, \max(X_2, X_3))$

Assume that the three components fail in the interval (t_1, t_2) for $0 < t_1 < t_2$. This is case II of Section 6.2.2, with $W_1 = \emptyset$ and $W_2 = \{1, 2, 3\}$. Note that this assumption implies that the system fails in this interval (t_1, t_2). The system inactivity time is

$$T^{(t_1,t_2)}_{\emptyset,\{1,2,3\}} = (t_2 - T | t_1 < X_i < t_2,\ i = 1, 2, 3).$$

Note that $K_1 \cap W_2 = \{1\}$, $K_2 \cap W_2 = \{2, 3\}$ and $(K_1 \cup K_2) \cap W_2 = \{1, 2, 3\}$. So if the components are independent, from Proposition 6.3, its reliability function is

$$\begin{aligned}\bar{F}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}(x) &= \bar{F}^{(t_1,t_2)}_1(x) + \bar{F}^{(t_1,t_2)}_2(x)\bar{F}^{(t_1,t_2)}_3(x) - \bar{F}^{(t_1,t_2)}_1(x)\bar{F}^{(t_1,t_2)}_2(x)\bar{F}^{(t_1,t_2)}_3(x)\\ &= \bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}(\bar{F}^{(t_1,t_2)}_1(x)\bar{F}^{(t_1,t_2)}_2(x)\bar{F}^{(t_1,t_2)}_3(x))\end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3$.

Now we assume that for $0 < t_1 < t_2$, the second component failed before t_1, and that the other components fail in the interval (t_1, t_2). This is again case II with $W_1 = \{2\}$ and $W_2 = \{1, 3\}$. Note that these assumptions imply that the system fails in this interval (t_1, t_2). Noting that $K_1 \cap W_2 = \{1\}$, $K_2 \cap W_2 = \{3\}$ and $(K_1 \cup K_2) \cap W_2 = \{1, 3\}$ we can calculate as before

$$\begin{aligned}\bar{F}^{(t_1,t_2)}_{\{2\},\{1,3\}} &= \bar{F}^{(t_1,t_2)}_1(x) + \bar{F}^{(t_1,t_2)}_3(x) - \bar{F}^{(t_1,t_2)}_1(x)\bar{F}^{(t_1,t_2)}_3(x)\\ &= \bar{Q}^{(t_1,t_2)}_{\{2\},\{1,3\}}(\bar{F}^{(t_1,t_2)}_1(x)\bar{F}^{(t_1,t_2)}_2(x)\bar{F}^{(t_1,t_2)}_3(x))\end{aligned}$$

where $\bar{Q}^{(t_1,t_2)}_{\{2\},\{1,3\}}(u_1, u_2, u_3) = u_1 + u_3 - u_1u_3$. Note that we have obtained the dual distortion function of the series system formed by components one and three when they are independent.

A straightforward calculation shows that $\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}} \le \bar{Q}^{(t_1,t_2)}_{\{2\},\{1,3\}}$, so

$$T^{(t_1,t_2)}_{\emptyset,\{1,2,3\}} \le_{ST} T^{(t_1,t_2)}_{\{2\},\{1,3\}}$$

for all F_1, F_2, F_3 and all $0 \le t_1 < t_2$. If we analyse the ratio

$$\frac{\bar{Q}^{(t_1,t_2)}_{\{2\},\{1,3\}}(u_1, u_2, u_3)}{\bar{Q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}(u_1, u_2, u_3)} = \frac{u_1 + u_3 - u_1u_3}{u_1 + u_2u_3 - u_1u_2u_3}$$

we note that it is decreasing in u_1 and u_2 and increasing in u_3. Therefore $T^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}$ and $T^{(t_1,t_2)}_{\{2\},\{1,3\}}$ are not ordered in the HR sense for all F_1, F_2, F_3.

However, if we consider the case of IID components with a common distribution F, we obtain the ratio

$$\frac{\bar{q}^{(t_1,t_2)}_{\{2\},\{1,3\}}(u)}{\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}}(u)} = \frac{2u - u^2}{u + u^2 - u^3}$$

which is decreasing for $u \in [0,1]$. So, from Proposition 4.2, (ii),

$$T^{(t_1,t_2)}_{\emptyset,\{1,2,3\}} \leq_{HR} T^{(t_1,t_2)}_{\{2\},\{1,3\}}$$

for all F and all $0 \leq t_1 < t_2$. In this case we can also consider the ratio

$$\frac{(\bar{q}^{(t_1,t_2)}_{\{2\},\{1,3\}})'(u)}{(\bar{q}^{(t_1,t_2)}_{\emptyset,\{1,2,3\}})'(u)} = \frac{2 - 2u}{1 + 2u - 3u^2}$$

which is also decreasing for $u \in [0,1]$. So, from Proposition 4.2, (iv), we obtain

$$T^{(t_1,t_2)}_{\emptyset,\{1,2,3\}} \leq_{LR} T^{(t_1,t_2)}_{\{2\},\{1,3\}}$$

for all F and all $0 \leq t_1 < t_2$.

Appendix A

Stochastic Orders

The simplest way of comparing two distribution functions is by the associated means. However, such comparison is based on only two single numbers, therefore it is often not very informative. Moreover, the means sometimes do not exist. So, in the last 40 years, several stochastic orders, that take into account various forms of possible knowledge about the two underlying distribution functions, have been introduced. Stochastic orders are applicable to a lot of fields, such as reliability theory and survival analysis, biology (see, e.g., Calì and Longobardi [5]), economics, insurance, actuarial science (see, e.g., Pellerey [40]) and operations research.

Here we give formal definitions of some basic stochastic orders that we use in the text and their interrelationships. The main reference used is Shaked and Shanthikumar [50].

Let X and Y be two continuous random variables with density function f_X and f_Y, distribution functions F_X and F_Y and survival functions $\bar{F}_X$ and $\bar{F}_Y$, respectively.

- $X \leq_{ST} Y$ (in the usual stochastic order) if and only if

$$\bar{F}_X(t) \leq \bar{F}_Y(t) \quad \text{for all} \quad t \in \mathbb{R}, \tag{A.1}$$

  or, equivalently, if and only if

$$F_X(t) \geq F_Y(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{A.2}$$

- $X \leq_{HR} Y$ (in the hazard rate order) if and only if

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \quad \text{is increasing in} \quad t \in (-\infty, \max(u_X, u_Y)), \tag{A.3}$$

where u_X and u_Y denote the right endpoints of the supports of X and of Y, respectively.

Let $\lambda_X(t)$ and $\lambda_Y(t)$ be the hazard rate functions of X and Y, respectively. Then, equivalently, $X \leq_{HR} Y$ if and only if

$$\lambda_X(t) \geq \lambda_Y(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{A.4}$$

- $X \leq_{RHR} Y$ (in the reversed hazard rate order) if and only if

$$\frac{F_Y(t)}{F_X(t)} \quad \text{is increasing in} \quad t \in (\min(l_X, l_Y), +\infty), \tag{A.5}$$

where l_X and l_Y denote the left endpoints of the supports of X and of Y, respectively.

Let $\tau_X(t)$ and $\tau_Y(t)$ be the reversed hazard rate functions of X and Y, respectively. Then, equivalently, $X \leq_{RHR} Y$ if and only if

$$\tau_X(t) \leq \tau_Y(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{A.6}$$

- $X \leq_{LR} Y$ (in the likelihood ratio order) if and only if

$$\frac{f_Y(t)}{f_X(t)} \quad \text{is increasing in } t \text{ over the union of the supports of } X \text{ and } Y, \tag{A.7}$$

or, equivalently, if and only if

$$f_X(x) f_Y(y) \geq f_X(y) f_Y(x) \quad \text{for all} \quad x \leq y. \tag{A.8}$$

- $X \leq_{DCX} Y$ (in the decreasing convex order) if and only if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \quad \text{for all dencreasing convex functions} \quad \phi : \mathbb{R} \to \mathbb{R}. \tag{A.9}$$

- $X \leq_{DISP} Y$ (in the dispersive order) if and only if

$$F_X^{-1}(y) - F_X^{-1}(x) \leq F_Y^{-1}(y) - F_Y^{-1}(x) \quad \text{whenever} \quad 0 < x \leq y < 1, \tag{A.10}$$

where F_X^{-1} and F_Y^{-1} are the right continuous inverses of F_X and F_Y, defined as $F^{-1}(x) = \sup\{x : F(x) \leq x\}$.

Equivalently, $X \leq_{DISP} Y$ if and only if

$$f_Y(F_Y^{-1}(x)) \leq f_X(F_X^{-1}(x)) \quad \text{whenever} \quad x \in [0, 1]. \tag{A.11}$$

The dispersive order can be characterized also by comparing transformations of the random variables X and Y . So $X \leq_{DISP} Y$ if and only if

$$\begin{aligned} Y =_{ST} \phi(X) \quad \text{for some} \quad \phi \qquad & \text{which satisfies} \\ \phi(x) - \phi(x^*) \quad \geq \quad x - x^* \quad & \text{whenever} \quad x \geq x^*. \end{aligned} \tag{A.12}$$

The following relationships hold:

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & & \\ \Downarrow & & \Downarrow & & \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & \mathbb{E}(X) \leq \mathbb{E}(Y). \end{array}$$

Appendix B

Aging classes

The concept of aging is very important in reliability analysis: it describes how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in the literature according to their aging properties.

Here we give formal definitions of some basic aging notions that we use in the text and their interrelationships. The main references used are Barlow and Proshan [3], Chandra and Roy [8], Deshpande *et al.* [9].

Let X be a nonnegative random variable with distribution function F, survival functions $\bar{F}$ and density function f. Let $\lambda = f/\bar{F}$ be the hazard rate of X, $\tau = f/F$ the reversed hazard rate of X and $\tilde{\mu}$ the mean inactivity time of X. Then X is

- Increasing in Mean Inactivity Time, shortly written as IMIT, if and only if

$$\tilde{\mu}(t) \quad \text{is increasing in} \quad t > 0. \tag{B.1}$$

  Note that we don't give the definition of Decreasing in Mean Inactivity Time (DMIT) because we suppose that X is a nonnegative random variable (see Theorem 2.2 in [28]).

- Increasing (Decreasing) in Hazard Rate, shortly written as IHR (DHR), if and only if

$$\lambda(t) \leq (\geq) \lambda(s) \quad \text{for all} \quad 0 < t < s. \tag{B.2}$$

- New Better (Worse) than Used in Hazard Rate, shortly written as NBUHR (NWUHR), if and only if

$$\lambda(0) \leq (\geq) \lambda(t) \quad \text{for all} \quad t > 0. \tag{B.3}$$

- Increasing (Decreasing) Reversed Hazard Rate, shortly written as IRHR (DRHR), if and only if
$$\tau(t) \leq (\geq) \tau(s) \quad \text{for all} \quad 0 < t < s. \tag{B.4}$$

- Increasing (Decreasing) Likelihood Ratio, shortly written as ILR (DLR), if and only if
$$f \quad \text{is log-concave (log-convex),} \tag{B.5}$$
or, equivalently if and only if
$$\eta(t) := -\frac{f'(t)}{f(t)} \quad \text{is increasing for} \quad t > 0. \tag{B.6}$$

The following relationships hold:

$$\begin{array}{ccccccc} & & ILR & \Rightarrow & IHR & \Rightarrow & NBUHR \\ & & \Downarrow & & & & \\ DHR & \Rightarrow & DRHR & & & & \\ & & \Downarrow & & & & \\ & & IMIT & & & & \end{array}$$

Appendix C

Tables

Table C.1: Distortion functions q for all the coherent systems with 1-4 i.i.d. components.

N	T	$q(u)$
1	X_1	u
2	$X_{1:2} = \min(X_1, X_2)$	$2u - u^2$
3	$X_{2:2} = \max(X_1, X_2)$	u^2
4	$X_{1:3} = \min(X_1, X_2, X_3)$	$3u - 3u^2 + u^3$
5	$\min(X_1, \max(X_2, X_3))$	$u + u^2 - u^3$
6	$X_{2:3}$	$3u^2 - 2u^3$
7	$\max(X_1, \min(X_2, X_3))$	$2u^2 - u^3$
8	$X_{3:3} = \max(X_1, X_2, X_3)$	u^3
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	$4u - 6u^2 + 4u^3 - u^4$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u - 2u^3 + u^4$
11	$\min(X_{2:3}, X_4))$	$u + 3u^2 - 5u^3 + 2u^4$
12	$\min(X_1, \max(X_2, X_3), \max(X_2, X_4))$	$u + 2u^2 - 3u^3 + u^4$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$u + u^3 - u^4$
14	$X_{2:4}$	$6u^2 - 8u^3 + 3u^4$
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$	$5u^2 - 6u^3 + 2u^4$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$4u^2 - 4u^3 + u^4$
17	$\max(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$4u^2 - 4u^3 + u^4$
18	$\max(\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$3u^2 - 2u^3$
19	$\min(\max(X_1, X_2), \max(X_2, X_3), \max(X_3, X_4))$	$3u^2 - 2u^3$
20	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3, X_4))$	$2u^2 - u^4$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$2u^2 - u^4$
22	$\min(\max(X_1, X_2), \max(X_1, X_3, X_4), \max(X_2, X_3, X_4))$	$u^2 + 2u^3 - 2u^4$

23	$X_{3:4}$	$4u^3 - 3u^4$
24	$\max(X_1, \min(X_2, X_3, X_4))$	$3u^2 - 3u^3 + u^4$
25	$\max(X_1, \min(X_2, X_3), \min(X_2, X_4))$	$u^2 + u^3 - u^4$
26	$\max(X_{2:3}, X_4)$	$3u^3 - 2u^4$
27	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u^3 - u^4$
28	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$	u^4

Table C.2: $\mathcal{CE}_n(T)$ and bounds for $\mathcal{CE}_n(T)$ obtained from Proposition 5.3 and Proposition 5.5 for all the coherent systems given in Table C.1 with i.i.d. components having a standard exponential distribution ($\mu = 1$) and for $n = 1, 2, 3$.

N	n	$\mathcal{CE}_n(T)$	$B_{1,n}$	$B_{2,n}$	D_n	$B_{2,n}\mathcal{CE}_n(X_1)$
1	1	0.644934	1	1	0.25	0.644934
	2	0.202057	1	1	0.125	0.202057
	3	0.082323	1	1	0.0625	0.082323
2	1	0.322467	0	2	0.186915	1.289868
	2	0.101028	0	2	0.077728	0.404114
	3	0.041162	0	2	0.035552	0.164646
3	1	0.789868	0	2	0.222222	1.289868
	2	0.308228	0	4	0.148148	0.808228
	3	0.158586	0	8	0.098765	0.658586
4	1	0.214978	0	3	0.146386	1.934802
	2	0.067352	0	3	0.056225	0.606171
	3	0.027441	0	3	0.024833	0.246970
5	1	0.399153	0	1.056342	0.207481	0.681271
	2	0.142395	0	1.015601	0.101309	0.205209
	3	0.063844	0	1.004519	0.052122	0.082695
6	1	0.431768	0	1.012007	0.198046	0.652678
	2	0.172516	0	1.174332	0.111139	0.237282
	3	0.089776	0	1.502561	0.067755	0.123696
7	1	0.641597	0	1.150827	0.221637	0.742207
	2	0.233557	0	1.516010	0.130927	0.306320
	3	0.116690	0	2.132897	0.081244	0.175587
8	1	0.851469	0	3	0.1875	1.934802
	2	0.360179	0	9	0.140625	0.606171
	3	0.201894	0	27	0.105469	0.246970
9	1	0.161234	0	4	0.119811	2.579736
	2	0.050514	0	4	0.044013	0.808228
	3	0.020581	0	4	0.019079	0.329293
10	1	0.258196	0	2	0.165315	1.289868
	2	0.087430	0	2	0.070027	0.404114
	3	0.037390	0	2	0.032899	0.164647
11	1	0.288437	0	1.136607	0.174634	0.733036
	2	0.108046	0	1.041939	0.083079	0.210531

	3	0.050790	0	1.012924	0.043346	0.083387
12	1	0.346483	0	1.098163	0.192391	0.7082429
	2	0.123801	0	1.029188	0.091655	0.2079546
	3	0.056551	0	1.008789	0.047299	0.0830468
13	1	0.445965	0	1.012468	0.219642	0.652975
	2	0.163535	0	1.001239	0.111855	0.202307
	3	0.073025	0	1.000127	0.057992	0.082334
14	1	0.300888	0	1.020121	0.170339	0.657911
	2	0.121017	0	1.021857	0.088125	0.206473
	3	0.063164	0	1.15848	0.051613	0.095370
15	1	0.351058	0	1.005136	0.183012	0.648247
	2	0.136686	0	1.046841	0.095728	0.211521
	3	0.070459	0	1.22319	0.056271	0.100697
16,17	1	0.394934	0	1.000151	0.192397	0.645032
	2	0.154114	0	1.092515	0.103585	0.220750
	3	0.079293	0	1.326676	0.061669	0.109216
18,19	1	0.431768	0	1.012007	0.198046	0.652678
	2	0.172516	0	1.174332	0.111139	0.237282
	3	0.089776	0	1.502561	0.067755	0.123696
20,21	1	0.460559	0	1.045234	0.199342	0.674107
	2	0.190620	0	1.310963	0.117482	0.264889
	3	0.101742	0	1.807005	0.074229	0.148759
22	1	0.479865	0	1.09667	0.195363	0.707280
	2	0.206149	0	1.510379	0.120958	0.305183
	3	0.114162	0	2.300398	0.080054	0.189376
23	1	0.487245	0	1.159029	0.184466	0.747497
	2	0.213967	0	1.765755	0.11768	0.356783
	3	0.122231	0	3.02405	0.080962	0.248950
24	1	0.623248	0	1.057191	0.224525	0.681818
	2	0.212041	0	1.258253	0.124132	0.254239
	3	0.101572	0	1.618131	0.073492	0.133210
25	1	0.649105	0	1.24926	0.212759	0.805690
	2	0.250910	0	1.858282	0.133867	0.375479
	3	0.132327	0	2.973261	0.088163	0.244768
26	1	0.642545	0	1.343604	0.195741	0.866536
	2	0.257031	0	2.245712	0.127710	0.453762

	3	0.141765	0	4.118911	0.088438	0.339082
27	1	0.766283	0	2	0.197631	1.289868
	2	0.308478	0	4	0.136703	0.808228
	3	0.168417	0	8.197383	0.097174	0.674835
28	1	0.885292	0	4	0.16	2.579736
	2	0.390318	0	16	0.128	3.23291
	3	0.228564	0	64	0.1024	5.268687

Bibliography

[1] S. Baratpour, A. Khammar, Tsallis entropy properties of order statistics and some stochastic comparisons, *Journal of Statistical Research of Iran* 13 (2016), pp. 25-41.

[2] R.E. Barlow, F. Proschan, *Mathematical Theory of Reliability*. Wiley, New York (1965).

[3] R.E. Barlow, F. Proschan, *Statistical Theory of Reliability and Life Testing*. International Series in Decision Processes. Holt, Rinehart and Winston, New York (1975).

[4] F. Belzunce, J. Navarro, J. M. Ruiz, Y. del Águila, Some results on residual entropy function. *Metrika* 59 (2004), pp. 147-161.

[5] C. Calì, M. Longobardi, Some mathematical properties of the ROC curve and their applications. *Ricerche di Matematica* 64 (2015), pp. 391-402.

[6] C. Calì, M. Longobardi, J. Ahmadi, Some properties of cumulative Tsallis entropy. *Physica A: Statistical Mechanics And Its Applications* 486C (2017), pp. 1012-1021.

[7] C. Calì, M. Longobardi, J. Navarro, Properties for Generalized Cumulative Past Measures of Information. *Probability in the Engineering and Informational Sciences* (2018). doi:10.1017/S0269964818000360

[8] N.K. Chandra, D. Roy, Some results on reversed hazard rate. *Probability in the Engineering and Informational Sciences* 15 (2001), pp. 95-102.

[9] J.V. Deshpande, S.C. Kochar, H. Singh, Aspects of Positive Ageing. *Journal of Applied Probability* 23 (1986), pp. 748-758.

[10] A. Di Crescenzo, M. Longobardi, Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability* 39 (2002), pp. 434-440.

[11] A. Di Crescenzo, M. Longobardi, A measure of discrimination between past lifetime distributions. *Statistics and Probability Letters* 67 (2004), pp. 173-182.

[12] A. Di Crescenzo, M. Longobardi, On cumulative entropies. *Journal of Statistical Planning and Inference* 139 (2009), pp. 4072-4087.

[13] A. Di Crescenzo, M. Longobardi, More on cumulative entropy. *Cybernetics and Systems 2010* (Trappl R. ed.), pp. 181-186. Austrian Society for Cybernetic Studies, Vienna (2010).

[14] A. Di Crescenzo, M. Longobardi, Stochastic comparisons of cumulative entropies. *Stochastic Orders in Reliability and Risk, In Honor of Professor Moshe Shaked* (H. Li and X. Li, eds.) 208, pp. 167-182. Lecture Notes in Statistics, Springer, New York (2013).

[15] N. Ebrahimi, How to measure uncertainty in the residual life time distribution. *Sankhya A* 58 (1996), pp. 48-56.

[16] S. Eryilmaz, Modeling dependence between two multi-state components via copulas. *IEEE Transactions on Reliability* 63 (2014), pp. 715-720.

[17] G.M. Giorgi, S. Nadarajah, Bonferroni and Gini indices for various parametric families of distributions. *Metron* 68 (2010), pp. 23-46.

[18] R.C. Gupta, P.L. Gupta, R.D. Gupta, Modeling failure time data by lehman alternatives. *Communications in Statistics - Theory and Methods* 27 (1998), pp. 887-904.

[19] E. Hairer, P. Norsett, G. Wanner, *Solving ordinary differential equations I*, Springer-Verlag, Berlin, Heildelberg (2008).

[20] K. Hanada, A formula of Gini's concentration ratio and its applications to life tables. *Journal of Japanese Statistical Society* 19 (1983), pp. 293-325.

[21] S. Kayal, On generalized cumulative entropies. *Probability in the Engineering and Informational Sciences* 30 (2016), pp. 640-662.

[22] J. Keilson, U. Sunita, Uniform stochastic ordering and related inequalities. *Canadian Journal of Statistics* 10 (1982), pp. 181-198.

[23] D.F. Kerridge, Inaccuracy and Inference. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 23 (1961), pp. 184-194.

[24] A.J. Khinchin, *Mathematical Foundation of Information Theory.* Dover, New York (1957).

[25] S. Kullback, R.A. Leibler, On information and sufficiency. *Annals of Mathematical Statistics* 22 (1951), pp. 79-86.

[26] V. Kumar, Some results on Tsallis entropy measure and k-record values. *Physica A: Statistical Mechanics and its Applications* 462 (2016), pp. 667-673.

[27] M. Longobardi, Cumulative measures of information and stochastic orders. *Ricerche di Matematica* 63 (2014), pp. 209-223.

[28] A.K. Nanda, H. Singh, N. Misra, P. Paul, Reliability Properties of Reversed Residual Lifetime *Communications in Statistics - Theory and Methods* 32 (2003), pp. 2031-2042.

[29] P. Nath, Inaccuracy and coding theory *Metrika* 13 (1968), pp. 123-135.

[30] J. Navarro, C. Calì, Inactivity times of coherent systems with dependent components under periodical inspections. *Applied Stochastic Models in Business and Industry* (2018). In press.

[31] J. Navarro, Y. del Águila, M.A. Sordo, A. Suarez-Llorens, Stochastic ordering properties for systems with dependent identically distributed components. *Applied Stochastic Models in Business and Industry* 29 (2013), pp. 264-278.

[32] J. Navarro, Y. del Águila, M.A. Sordo, A. Suarez-Llorens, Preservation of reliability classes under the formation of coherent systems. *Applied Stochastic Models in Business and Industry* 30 (2014), pp. 444-454.

[33] J. Navarro, Y. del Águila, M.A. Sordo, A. Suarez-Llorens, Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. *Methodology and Computing in Applied Probability* 18 (2016), pp. 529-545.

[34] J. Navarro, F. Durante, Copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components. *Journal of Multivariate Analysis* 158 (2017), pp. 87-102.

[35] J. Navarro, M.C. Gomis, Comparisons in the mean residual life order of coherent systems with identically distributed components. *Applied Stochastic Models in Business and Industry* 32 (2015), pp. 33-47.

[36] J. Navarro, F. Pellerey, M. Longobardi, Comparison results for inactivity times of k-out-of-n and general coherent systems with dependent components. *Test* 26 (2017), pp. 822-846.

[37] J. Navarro, J.M. Ruiz, C.J. Sandoval, Properties of Coherent Systems with Dependent Components. *Communications in Statistics - Theory and Methods* 36 (2007), pp. 175-191.

[38] J. Navarro, F. Spizzichino, Comparisons of series and parallel systems with components sharing the same copula. *Applied Stochastic Models in Business and Industry* 26 (2010), pp. 775-791.

[39] R.B. Nelsen, *An introduction to copulas.* Second edition. Springer Series in Statistics. Springer, New York (2006).

[40] F. Pellerey, On the preservation of some orderings of risks under convolution. *Insurance: Mathematics and Economics* 16 (1995), pp. 23-30.

[41] G. Psarrakos, J. Navarro, Generalized cumulative residual entropy and record values. *Metrika* 76 (2013), pp.623-640.

[42] G. Rajesh, S.M. Sunoj, Some properties of cumulative Tsallis entropy of order α. *Statistical Papers* (2016), doi:10.1007/s00362-016-0855-7.

[43] M. Rao, More on a new concept of entropy and information. *Journal of Thoeretical Probability* 18 (2005), pp. 967-981.

[44] M. Rao, Y. Chen, B.C. Vemuri, F. Wang, Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory* 50 (2004), pp. 1220-1228.

[45] A. Renyi, On measures of entropy and information. *Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (1961), pp. 547-561.

[46] F.M. Reza, *An introduction to Information Theory.* Dover, New York (1994).

[47] J.M. Ruiz, J. Navarro, Characterizations based on conditional expectations of the double truncated distribution. *Annals of the Institute of Statistical Mathematics* 48 (1996), pp. 563-572.

[48] F.J. Samaniego, On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability* R-34 (1985), pp. 69-72.

[49] M.M. Sati, N. Gupta, Some characterization results on dynamic cumulative residual Tsallis entropy. *Journal of Probability and Statistics* (2015), doi:10.1155/2015/694203.

[50] M. Shaked, J.G. Shanthikumar, *Stochastic Orders*. Springer Series in Statistics. Springer, New York (2007).

[51] C.E. Shannon, A mathematical theory of communication. *Bell System Technical Journal* 27 (1948), pp. 379-423,623-656.

[52] A. Sklar, Fonctions de répartition à n dimensions et leurs marges. *Institut de Statistique de l'Université de Paris* 8 (1959), pp. 229-231.

[53] M. Tribus, E.C. McIrvine, Energy and information. *Scientific American* 224 (1971), pp. 178-184.

[54] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistic. *Journal of Statistical Physics* 52 (1988), pp. 479-487.

[55] G. Will, Z. Woldarczyk, Examples of a possible interpretation of Tsallis Entropy. *Physica A: Statistical Mechanics and its Applications* 341 (2008), pp. 4809-4813.

[56] Z. Zhang, Uniform Estimates on the Tsallis Entropies, *Letters in Mathematical Physics* 80 (2007), pp. 171-181.