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Isoperimetric problems for Robin and Steklov eigenvalues: linear and nonlinear cases

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Introduction

In this thesis we mainly address some isoperimetric problems and our interest is focused on the ones which involve the spectrum of some boundary value problems for second order elliptic operators. The study of them needs different fields of mathematics as spectral theory, partial differential equations, calculus of variations and shape optimization.
An important question on the optimization of eigenvalues was asked by Lord Rayleigh in his book "The theory of Sound" (1894). He conjectured that the disk minimizes the first Dirichlet Laplacian eigenvalue (the first frequency of the fixed membrane) among all planar sets with given area. In the 20s of the XX Century, Faber in [47] and Krahn in [62] gave a positive answer to the above conjecture, proving that, if Ω is a bounded open subset of $\mathbb{R}^n$, the following inequality holds

$$\lambda_1^D(\Omega)\,|\Omega|^{\frac{2}{n}} \geq \lambda_1^D(B)\,|B|^{\frac{2}{n}}$$

where λ_1^D is the first Dirichlet eigenvalue of the Laplace operator and B is any ball of $\mathbb{R}^n$. When we consider the eigenvalues of the Laplacian with Neumann boundary conditions, we know that the first eigenvalue is equal to zero for every open bounded set with Lipschitz boundary. For this reason we have to consider the first nontrivial eigenvalue, that we denote by μ_2^N. Contrary to the Dirichlet case, the relevant inequality for μ_2^N is a maximization result, proved by Szegö in [76] in the two dimensional case and by Weinberger in higher dimension in [81]. In particular, they proved that, if Ω is a bounded open set with Lipschitz boundary, it holds the following inequality

$$\mu_2^N(\Omega)\,|\Omega|^{\frac{2}{n}} \leq \mu_2^N(B)\,|B|^{\frac{2}{n}}\,.$$

In other words this inequality states that, among all bounded open set with Lipschitz boundary and fixed measure, the ball maximizes the first nontrivial Neumann-Laplacian eigenvalue.
Yet another important boundary condition is the Robin one. Let Ω be a bounded open set with Lipschitz boundary, the Robin-Laplacian eigenvalue problem is the following

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\frac{\partial u}{\partial \nu}$ is the outer normal derivative and α is a real number. The spectrum of this problem is discrete and it forms a sequence

$$\lambda_1(\alpha,\Omega) \leq \lambda_2(\alpha,\Omega) \leq \ldots \leq \lambda_k(\alpha,\Omega) \leq \ldots \nearrow +\infty.$$

Moreover, the first eigenvalue has the following variational characterization

$$\lambda_1(\alpha,\Omega) = \inf_{\substack{v\in H^1(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx + \alpha\int_{\partial\Omega} v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\,dx}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. It is clear that Dirichlet and Neumann boundary conditions are special cases of the Robin boundary conditions. Indeed, if $\alpha = 0$ the first Robin eigenvalue coincides with the first trivial Neumann eigenvalue, while in the case $\alpha = +\infty$ the first Robin eigenvalue coincides with the first Dirichlet eigenvalue. When α is positive, for any bounded open set Ω with Lipschitz boundary, it holds

$$\lambda_1(\alpha, \Omega) \geq \lambda_1(\alpha, B_r), \tag{2}$$

where B_r is a ball of radius r such that $|B_r| = |\Omega|$. The inequality (2) was proved by Bossel in [16] in the two dimensional case and by Daners in [35] in higher dimension and by Bucur and Giacomini in [26] for non smooth domains.
In the case $\alpha < 0$ the framework completely changes. In 1977 Bareket in [11] proved that the ball is the maximum within a class of nearly circular domains and for a range of boundary parameter α. This result suggested to her to conjecture that the ball maximizes the first Robin-Laplacian eigenvalue among all the bounded smooth domains of given measure for any negative value of α.
After the appearance of the Bareket's paper, in 2015 Ferone, Nitsch and Trombetti in [48] proved that the ball is a local maximizer among all the bounded open Lipschitz set with fixed volume which are "*close*" in L^∞ sense to the ball. In 2015 Freitas and Krejčiřík in [50] disproved the Bareket's conjecture showing that the first Robin-Laplacian eigenvalue on a spherical shell is greater than the one on a ball with the same measure for a suitable large negative α. This is quite surprising because, to the best of our knowledge, the first eigenvalue of the problem (1) with a suitable large negative α is the first one for which the ball is not a maximum or minimum with fixed measure. However, in the same paper, they proved that, among all the bounded planar domains of class C^2 and fixed area, the ball is a maximum for the first Robin-Laplacian eigenvalue for α negative sufficiently small. Moreover, the problem of maximizing the first Robin-Laplacian eigenvalue for α negative and for $n \geq 3$ is still open.
If, instead of the measure, the perimeter of sets is fixed, in 2017 Antunes, Freitas and Krejčiřík in [7] proved that among all bounded planar domains of class C^2, a ball is a maximum for the first Robin-Laplacian eigenvalue for any negative value of α. Moreover, in 2018 Bucur, Ferone, Nitsch and Trombetti in [24] have shown that, among all bounded, open and convex sets with given perimeter the ball is still a maximizer for the first Robin-Laplacian eigenvalue for any negative value of α and for all dimensions.
Other interesting questions arise in the case of Steklov eigenvalue problem. Let Ω be a bounded, open and connected subset of $\mathbb{R}^n$ with Lipschitz boundary and let us consider the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

It is well-known (see, for instance [8, 19, 58]) that the spectrum is discrete and that there exists a sequence of eigenvalues

$$0 = \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \ldots \leq \sigma_k(\Omega) \leq \ldots \leq\nearrow +\infty$$

called Steklov eigenvalues of Ω. The first Steklov eigenvalue is zero,while the first non-trivial has the following variational characterization

$$\sigma_2(\Omega) = \inf_{\substack{v \in H^1(\Omega)\setminus\{0\} \\ \int_{\partial\Omega} v \, d\mathcal{H}^{n-1} = 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}}.$$

If we take $\Omega = B_r(0)$, where $B_r(0)$ is the ball of radius r centered at the origin, then we have

$$\sigma_2\left(B_r(0)\right) = \frac{1}{r}.$$

Moreover, we know that $\sigma_2(B_r(0))$ has multiplicity n and the corresponding eigenfunctions are $u_{i+1}(x) = x_i$, with $i = 1, \dots, n$. In 1954 Weinstock in [82, 83] considered the problem of maximizing $\sigma_2(\Omega)$ in the plane, keeping fixed the perimeter of Ω. More precisely, he proved that, if Ω is a bounded, open and simply connected subset of $\mathbb{R}^2$ with Lipschitz boundary, the following inequality holds

$$\sigma_2(\Omega)P(\Omega) \leq \sigma_2(B)P(B), \tag{3}$$

where $P(\Omega)$ is the Euclidean perimeter of the set Ω. The inequality (3) states that, among all planar bounded, open, Lipschitz and simply connected sets with prescribed perimeter, $\sigma_2(\Omega)$ is maximum for the disk. In 2017 Bucur, Ferone, Nitsch and Trombetti in [25] generalized the Weinstock inequality (3) in any dimension, when the set Ω is in the class of the convex sets. More precisely, they proved that, if Ω is a bounded open convex subset of $\mathbb{R}^n$ then

$$\sigma_2(\Omega)P(\Omega)^{\frac{1}{n-1}} \leq \sigma_2(B)P(B)^{\frac{1}{n-1}} \tag{4}$$

and the equality holds if and only if Ω is a ball. We observe that (4) and the classical isoperimetric inequality implies the following result for convex sets

$$\sigma_2(\Omega)\left|\Omega\right|^{\frac{1}{n}} \leq \sigma_2(B)\left|B\right|^{\frac{1}{n}}. \tag{5}$$

Actually, in 2001 Brock in [22] proved that (5) holds for any bounded open set with Lipschitz boundary. More precisely he proved the following inequality

$$\sum_{i=2}^{n+1} \frac{1}{\sigma_i(\Omega)} \geq nr, \tag{6}$$

where $\sigma_i(\Omega)$ is the i-th Steklov eigenvalue of the Laplacian and r is the radius of a ball with the same measure as Ω. We stress that the inequality (6) is weaker than (4) because it contains the measure but it is stronger because it holds without geometric restrictions and it concerns the sum of first nontrivial Steklov eigenvalues of Ω. Recently, Brasco, De Philippis and Ruffini in [19] have proved the following quantitative version of inequality (6)

$$\frac{1}{|\Omega|^{\frac{1}{n}}} \sum_{i=2}^{n+1} \frac{1}{\sigma_i(\Omega)} \geq \frac{n}{\omega_n^{\frac{1}{n}}} \left[1 + c_n \mathcal{A}_{\mathcal{F}}(\Omega)^2\right],$$

where ω_n is the measure of the n-dimensional unit ball, $\mathcal{A}_{\mathcal{F}}(\Omega)$ is the Fraenkel asymmetry of the set Ω and c_n is an explicit constant which depends only on the dimension.

All the results listed before are the background of this work of thesis, that is mainly focused on the study of some isoperimetric problems related to Robin and Steklov eigenvalues.

The Chapter 1 is devoted to recall some definitions and to state some useful propositions for this thesis. We introduce the definitions of Finsler norm, Wulff shape and anisotropic perimeter, we recall the first variation of Euclidean and anisotropic perimeter [2, 5, 10, 12, 38, 65, 75]. Moreover we recall some definitions and results concerning the quermassintegrals [75]. Finally we introduce the Hausdorff distance and the concept of nearly spherical sets [4, 18, 46, 52, 53, 75].

In Chapter 2 we consider the Robin eigenvalue problem with negative boundary parameter for the Laplacian and for its anisotropic version, which is called Finsler-Laplacian.

Section 2.1 is devoted to the study of problem (1), when $\alpha < 0$, and contains all the results obtained in [78]. Using a shape optimization technique we obtain a monotonicity property for the first Robin-Laplacian eigenvalue for spherical shells in $\mathbb{R}^2$: more precisely we get that, if $r_2 < \tilde{r_2}$ then

$$\lambda_1(\alpha, A_{r_1,r_2}) < \lambda_1(\alpha, A_{r_1,\tilde{r_2}})$$

where $A_{r_1,r_2} = B_{r_2} \setminus \overline{B}_{r_1}$. Moreover, if Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, we observe that, when the parameter $\alpha = -\sqrt[n]{\omega_n/|\Omega|} = -1/r$, problem (1) on a ball B_r is equivalent to the following problem

$$\begin{cases} \Delta u = 0 & \text{in } B_r \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{r} u & \text{in } \partial B_r \end{cases}$$

where B_r is a ball such that $|B_r| = |\Omega|$. In this setting, we obtain that

$$\lambda_2\left(-\frac{1}{r}, \Omega\right) \leq \lambda_2\left(-\frac{1}{r}, B_r\right) = 0$$

where $\lambda_2(\alpha, \Omega)$ is the second Robin-Laplacian eigenvalue. This last result is generalized by Freitas and Laugesen in [51] for any $\alpha \in \left[-\frac{n+1}{nr}, 0\right]$, where r is the radius of a ball with the same measure as Ω.

Section 2.2 collects the results contained in [70]. We generalize what is contained in [7, 50] to the anisotropic case, using a method which is inspired by the parallel coordinates technique of Payne and Weinberger explained in [71] (this method was introduced by Makai in [66] and Pólya in [72]). Let F be a Finsler norm (see Section 1.2) and let Ω be an open bounded connected set of $\mathbb{R}^2$ with C^2 boundary. We consider the anisotropic version of problem (1), that is

$$\begin{cases} -\operatorname{div}\left(F(\nabla u)\nabla F(\nabla u)\right) = \lambda_{1,F}(\alpha, \Omega) u & \text{in } \Omega \\ \langle F(\nabla u)\nabla F(\nabla u), \nu\rangle + \alpha F(\nu) u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where at the left hand side of the first equation there is the so-called Finsler-Laplacian and ν is the usual outward unit normal to $\partial\Omega$. The first eigenvalue of problem (7) has the following variational characterization

$$\lambda_{1,F}(\alpha, \Omega) = \inf_{\substack{v \in H^1(\Omega) \\ v \not\equiv 0}} \frac{\displaystyle\int_\Omega F^2(\nabla v)\, dx + \int_{\partial\Omega} v^2 F(\nu)\, d\mathcal{H}^1}{\displaystyle\int_\Omega v^2\, dx}.$$

Recalling that, in the plane, a Wulff shape of radius r centered at x_0 is defined as

$$\mathcal{W}_r(x_0) = \left\{\xi \in \mathbb{R}^2 \ : \ F^o(\xi - x_0) < r\right\}$$

we prove the following two results. In the first one we state that, for all bounded open connected planar set Ω of class C^2, there exists a negative constant $\alpha_* = \alpha_*(|\Omega|)$ such that for all $\alpha \in [\alpha_*, 0]$

$$\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, \mathcal{W}_\Omega^*),$$

where $\mathcal{W}_\Omega^*$ is a Wulff shape with the same are as Ω. In the second one we have that, for all bounded open connected planar set Ω of class C^2 and for any $\alpha \leq 0$

$$\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, \widetilde{\mathcal{W}_\Omega}),$$

where $\widetilde{\mathcal{W}_\Omega}$ is a Wulff shape with the same anisotropic perimeter as Ω.
In Chapter 3 we study some geometric properties of the eigenvalues associated to the p-Laplace operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$$

and to the anisotropic p-Laplace operator

$$\mathcal{Q}_p u = \operatorname{div}\left(\frac{1}{p}\nabla[F^p](\nabla u)\right),$$

where F is a Finsler norm and $1 < p < +\infty$, with Robin boundary conditions.
In Section 3.1, the results obtained in [56] are discussed. We consider the following eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \ell\,|u|^{p-2}\,u & \text{in } \Omega \\ F^{p-1}(\nabla u)\,\langle \nabla F(\nabla u), \nu\rangle + \beta(x)F(\nu)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where Ω is a bounded open connected set with $C^{1,\alpha}$ boundary, $\alpha \in]0,1[$ and $\beta : \partial\Omega \to [0,+\infty[$ such that $\beta \in L^1(\partial\Omega)$ and verifies

$$\int_{\partial\Omega} \beta(x)F(\nu)\;d\mathcal{H}^{n-1} > 0.$$

The first eigenvalue of $\mathcal{Q}_p$ has the following variational characterization

$$\ell_1(\beta,\Omega) = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega F^p(\nabla v)\;dx + \int_{\partial\Omega} \beta(x)\,|v|^p\,F(\nu)\;d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v|^p\;\;dx} \tag{9}$$

and the minimizers of (9) are weak solution to problem (8). When $\beta = \bar\beta$ is a positive constant and F is the Euclidean norm, this problem is studied in [23, 32] and it is addressed to a generic Finsler norm in [37]. A first result that we obtain in this section is a monotonicity property for $\ell_1(\bar\beta,\cdot)$: if Ω_1 and Ω_2 are two bounded open connected sets with $C^{1,\alpha}$ boundary, with Ω_2 convex, for which there exists a Wulff shape $\mathcal{W}_r$ such that $\Omega_1 \subset \mathcal{W}_r \subset \Omega_2$ then

$$\ell_1(\bar\beta,\Omega_2) \le \ell_1(\bar\beta,\Omega_1).$$

This result is proved for $F(\xi) = |\xi|$ and $p = 2$ in [57]. Then we prove a representation formula for $\ell_1(\beta,\Omega)$ and finally we prove a Faber-Krahn type inequality and a Cheeger type inequality. Precisely we get

$$\ell_1(\beta,\Omega) \ge \ell_1(\beta,\mathcal{W}_r),$$

where Ω is a bounded open connected set with $C^{1,\alpha}$ boundary, $\mathcal{W}_r$ is a Wulff shape such that $|\mathcal{W}_r| = |\Omega|$ and $\beta(x) = w(F^o(x))$, with w a suitable function such that

$$w(t) \ge C(r)t$$

for some constant $C(r)$. On the other hand, we obtain the following anisotropic weighted Cheeger inequality

$$\ell_1(\beta,\Omega) \ge h_{\beta,F}(\Omega) - (p-1)\left\|\beta_\Omega^{\frac{p}{p-1}}\right\|_{L^\infty(\overline{\Omega})},$$

where β_Ω is a function defined in the whole Ω whose trace on $\partial\Omega$ is the function β and $h_{\beta,F}(\Omega)$ is the anisotropic weighted Cheeger constant defined in the Paragraph 3.1.5. The same inequality is proved in the Euclidean case in [61] for $p = 2$ and $\beta = \bar{\beta}$ positive constant.
In Section 3.2 we collect the results described in [68] and we study some properties of the first eigenvalue of the p-Laplacian on a convex set Ω of $\mathbb{R}^n$, that contains holes, with Robin conditions on the external boundary and Neumann conditions on the internal boundary. If we denote by Γ_0 the external boundary and by Γ_1 the internal boundary, we deal with the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda_p^{RN} |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases} \tag{10}$$

where $\beta \in \mathbb{R} \setminus \{0\}$. The case $\beta = 0$, which coincides with the Neumann case, is trivial, since the first eigenvalue is identically zero and the relative eigenfunctions are constant.
The first eigenvalue of problem (10), i.e. the lowest eigenvalue, is variationally characterized by

$$\lambda_p^{RN}(\beta,\Omega) = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^p \, dx + \beta \int_{\Gamma_0} |v|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v|^p \, dx}. \tag{11}$$

As we have stress before, Makai in [66] and Pólya in [72] introduced the method of interior parallels, used by Payne and Weinberger in [71], to study the Laplacian eigenvalue problem with external Robin boundary condition and with Neumann internal boundary condition in the plane. Here, we generalize these tools to show that the annulus maximizes the first p-Laplacian eigenvalue (11) among convex sets Ω with holes, with fixed measure and fixed external perimeter. Precisely, we get

$$\lambda_p^{RN}(\beta,\Omega) \le \lambda_p^{RN}(\beta, A_{r_1,r_2}),$$

where A_{r_1,r_2} is an annulus with the same measure and external perimeter as Ω.
When $\beta = +\infty$, this gives an answer to the open problem 5 in [58, Chap. 3], restricted to convex sets with holes. More precisely, our proof is based on the use of particular test functions, called web functions, used e.g. in [17, 24, 30], and on the study of their level sets. Similarly, but only for positive value of β, we also study the p-torsional rigidity type problem:

$$\frac{1}{T_p^{RN}(\beta,\Omega)} = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^p \, dx + \beta \int_{\Gamma_0} |v|^p \, d\mathcal{H}^{n-1}}{\left|\displaystyle\int_\Omega v \, dx\right|^p};$$

in particular, this problem leads to, up to a suitable normalization,

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

It is known that the ball maximizes the torsional rigidity with Robin boundary conditions (see, for instance [27]) among all bounded open set with Lipschitz boundary and given measure. Here we show that the annulus minimizes the torsional rigidity $T_p^{RN}(\beta,\Omega)$ among convex sets having holes, where the measure and the external perimeter are fixed. Precisely, we obtain

$$T_p^{RN}(\beta,\Omega) \geq T_p^{RN}(\beta, A_{r_1,r_2}),$$

where A_{r_1,r_2} is an annulus with the same measure and external perimeter as Ω.
In Chapter 4 we extend the results obtained in [25] where the authors prove that, if Ω is a bounded open convex subset of $\mathbb{R}^n$

$$\frac{\displaystyle\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}}{P(\Omega)\ |\Omega|^{\frac{2}{n}}} \geq \omega_n^{\frac{-2}{n}}, \tag{12}$$

where the equality holds if and only if Ω is a ball centered at the origin.In order to prove (12) the authors use the notion of shape derivative and the inverse mean curvature flow. In Section 4.1, which contains the results obtained in [69], we prove an anisotropic version of (12). More precisely, we consider the following scale invariant functional

$$\mathcal{F}(\Omega) = \frac{\displaystyle\int_{\partial\Omega} [F^o(x)]^p F(\nu)\, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_{\partial\Omega} F(\nu)\, d\mathcal{H}^{n-1}\right) |\Omega|^{\frac{p}{n}}},$$

where $p > 1$, ν is the outward unit normal to $\partial\Omega$ and F is a Finsler norm with its dual norm F^o. We show that

$$\mathcal{F}(\Omega) \geq \kappa_n^{-\frac{p}{n}},$$

where κ_n is the measure a Wulff shape of unitary radius and the equality holds if and only if Ω is a Wulff shape centered at the origin. To prove the above inequality, we adapt the arguments of proof in [25]. We investigate the first variation of $\mathcal{F}(\Omega)$ and thanks to an approximation argument, we can compute it assuming the smoothness of the boundary of the sets. A fundamental tool is the inverse anisotropic mean curvature flow, which is studied in [44, 84]. Roughly speaking, the smooth boundary $\partial\Omega$ of an open set $\Omega = \Omega(0)$ flows by anisotropic inverse mean curvature if there exists a time dependent family $(\partial\Omega(t))_{t\in[0,T)}$ of smooth boundaries such that the anisotropic normal velocity at any point $x \in \partial\Omega(t)$ is equal to the inverse of the anisotropic mean curvature of $\partial\Omega(t)$ at x. We give the exact definition of anisotropic mean curvature, that we denote by H_F and anisotropic normal in Paragraph 1.2.2. We make also use of the following anisotropic version of the Heintze-Karcher inequality

$$\int_{\partial\Omega} \frac{F(\nu)}{H_F}\, d\mathcal{H}^{n-1} \geq \frac{n}{n-1}\,|\Omega|$$

which is proved for the Euclidean case in [73] and for the anisotropic case in [85].
The aim of Section 4.2, which presents the results obtained in [55], is to get a quantitative version of the inequality (4), that holds for convex sets. More precisely, denoting by $B_r(x)$ the ball of radius r with center at the origin, considering the following asymmetry functional

$$\mathcal{A}_{\mathcal{H}}(\Omega) = \min_{x\in\mathbb{R}^n} \left\{ \left(\frac{d_{\mathcal{H}}(\Omega, B_r(x))}{r} \right) \ :\ P(\Omega) = P(B_r(x)) \right\},$$

we obtain that there exists $\bar{\delta} > 0$ such that for any bounded, open and convex set Ω of $\mathbb{R}^n$ with $\sigma_2(B_r) \leq (1+\bar{\delta})\sigma_2(\Omega)$, where B_r is a ball with same perimeter as Ω, it holds

$$\frac{\sigma_2(B_r)-\sigma_2(\Omega)}{\sigma_2(\Omega)} \geq \begin{cases} \frac{16}{9\pi}\left(\mathcal{A}_{\mathcal{H}}(\Omega)\right)^{\frac{5}{2}} & \text{if } n=2 \\ \frac{2}{3}\sqrt{\pi}\, g\left(\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta}\right)^2\right) & \text{if } n=3 \\ \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}} & \text{if } n\geq 4, \end{cases} \tag{13}$$

where β and β_n are suitable constants and g is the inverse function of $f(t) = t\log\left(\frac{1}{t}\right)$, for $0 < t < e^{-1}$. The key role in the proof of (13) is played by a quantitative version of the following weighted inequality for convex sets

$$\frac{\displaystyle\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}}{|\Omega|\, P(\Omega)^{\frac{1}{n-1}}} \geq \frac{n}{(n\omega_n)^{\frac{1}{n-1}}}.$$

For $n \geq 3$ it is obtained by means of a Fuglede's approach (see [52]). However, the planar case is treated in a different way, indeed we use the representation of a two dimensional bounded, open and convex set via support function.

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Chapter 1

Preliminaries

1.1 Notations

In this thesis we denote by B_r a ball of $\mathbb{R}^n$ of radius r and by $B_r(x_0)$ the ball of radius r centered at x_0. Moreover, we denote by ω_n the Lebesgue measure of a ball B_1 and we define the annulus $A_{r_1,r_2} = B_{r_2} \setminus \overline{B}_{r_1}$, where the balls are centered at the same point.

1.2 The Finsler norm and the anisotropic perimeter

Let F be a convex, even, 1-homogeneous and non negative function defined in $\mathbb{R}^n$. Then F is a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R},\, \xi \in \mathbb{R}^n, \tag{1.1}$$

and such that

$$a|\xi| \leq F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^n, \tag{1.2}$$

for some constants $0 < a \leq b$. Moreover, throughout this thesis we will assume that $F \in C^2(\mathbb{R}^n \setminus \{0\})$, and

$$\nabla^2[F^p] \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

with $1 < p < +\infty$.

The polar function $F^o \colon \mathbb{R}^n \to [0, +\infty[$ of F is defined as

$$F^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{F(\xi)}.$$

It is easy to verify that also F^o is a convex function which satisfies properties (1.1) and (1.2). Furthermore,

$$F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{F^o(\xi)}.$$

The above equality implies the following anisotropic version of the Cauchy Schwarz inequality

$$|\langle \xi, \eta\rangle| \leq F(\xi)F^o(\eta), \qquad \forall \xi, \eta \in \mathbb{R}^n.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^n \colon F^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. We put $\kappa_n = |\mathcal{W}|$, where $|\mathcal{W}|$ denotes the Lebesgue measure of $\mathcal{W}$. More generally, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape of radius r centered at x_0 with measure $\kappa_n r^n$, and $\mathcal{W}_r(0) = \mathcal{W}_r$. Moreover, in Section 4.1, we assume that $\mathcal{W}$ is uniformly convex, i.e. there exists a constant $c > 0$ such that the principal curvatures $k_i(\mathcal{W}) > c$, for every $i = 1, \ldots, n-1$.

The following properties of F and F^o hold true:

$$\begin{gathered} \langle \nabla F(\xi), \xi \rangle = F(\xi), \quad \langle \nabla F^o(\xi), \xi \rangle = F^o(\xi), \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \\ F(\nabla F^o(\xi)) = F^o(\nabla F(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \\ F^o(\xi) \nabla F\left(\nabla F^o(\xi)\right) = F(\xi) \nabla F^o\left(\nabla F(\xi)\right) = \xi \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{gathered}$$

We recall the definition of anisotropic perimeter for a bounded, Lipschitz open set:

Definition 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. The anisotropic perimeter of Ω is*

$$P_F(\Omega) = \int_{\partial\Omega} F(\nu) \, d\mathcal{H}^{n-1}$$

where ν denotes the Euclidean unit outer normal to $\partial\Omega$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

We can also define the anisotropic perimeter in a more general way as in [2, 3]. Let Ω be a bounded open set and let E be a measurable subset of $\mathbb{R}^n$: the anisotropic perimeter of E in Ω is

$$P_F(E;\Omega) = \sup \left\{ \int_E \operatorname{div} \varphi dx \ : \ \varphi \in C_c^\infty(\Omega;\mathbb{R}^n), \ F^o(\varphi) \leq 1 \right\}.$$

It is clear that the anisotropic perimeter of E in Ω is finite if and only if the Euclidean perimeter of E in Ω

$$P(E;\Omega) = \sup \left\{ \int_E \operatorname{div} \varphi dx \ : \ \varphi \in C_c^\infty(\Omega;\mathbb{R}^n), \ |\varphi| \leq 1 \right\}.$$

and, by the aforementioned properties of F we obtain that

$$aP(E;\Omega) \leq P_F(E;\Omega) \leq bP(E;\Omega).$$

Furthermore, the anisotropic perimeter of a measurable subset Ω of $\mathbb{R}^n$ is $P_F(\Omega) = P_F(\Omega;\mathbb{R}^n)$ and it holds an isoperimetric inequality for the anisotropic perimeter (see for instance [2, 28, 33, 40, 49]).

Theorem 1.2. *Let Ω be a subset of $\mathbb{R}^n$ with finite perimeter. Then, denoting with $|\Omega|$ the n-dimensional Lebesgue measure of Ω,*

$$P_F(\Omega) \geq n\kappa_n^{\frac{1}{n}} \left|\Omega\right|^{1-\frac{1}{n}}$$

and equality holds if and only if Ω is homothetic to a Wulff shape.

Let Ω be a bounded and open set of $\mathbb{R}^n$, the anisotropic distance of a point $x \in \Omega$ to the boundary $\partial\Omega$ is defined as

$$d_F(x, \partial\Omega) = \inf_{y \in \partial\Omega} F^o(x - y).$$

By the properties of the Finsler norm F, the distance function satisfies

$$F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega$$

For the properties of the anisotropic distance function we refer, for instance, to [31].
We can define also the anisotropic inradius of Ω as

$$r_F(\Omega) = \sup\{d_F(x, \partial\Omega),\ x \in \Omega\}. \tag{1.4}$$

We recall the following so-called weighted anisotropic isoperimetric inequality (see for instance [15] and [20])

$$\int_{\partial\Omega} f(F^o(x))F(\nu)\, d\mathcal{H}^{n-1} \geq \int_{\partial\mathcal{W}_R} f(F^o(x))F(\nu)\, d\mathcal{H}^{n-1} = f(R)P_F(\mathcal{W}_R), \tag{1.5}$$

where $\mathcal{W}_R$ is a Wulff shape such that $|\Omega| = |\mathcal{W}_R|$ and $f\colon [0,+\infty[\to [0,+\infty[$ is a nondecreasing function such that

$$g(z) = f(z^{\frac{1}{n}})z^{1-\frac{1}{n}}, \qquad 0 \leq z \leq R^n,$$

is convex with respect to z.
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, the anisotropic Cheeger constant of Ω is defined as follows

$$h_F(\Omega) = \inf_{U\subset\Omega} \frac{P_F(U)}{|U|}. \tag{1.6}$$

In [36] the authors prove that

$$\frac{1}{r_F(\Omega)} \leq h_F(\Omega) \leq \frac{n}{r_F(\Omega)}, \tag{1.7}$$

where $r_F(\Omega)$ is the anisotropic inradius defined in (1.4).

1.2.1 The first variation of euclidean perimeter

For the content of this paragraph we refer, for instance, to Chapter 2 in [10] and Section 17.3 in [65]. We start from recalling the definition of tangential gradient.

Definition 1.3. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ with C^∞ boundary and let $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. We can define the tangential gradient of u for almost every $x \in \partial\Omega$ as follows:*

$$\nabla_\tau u(x) = \nabla u(x) - \langle \nabla u(x), \nu(x)\rangle \nu(x),$$

whenever ∇u exists at x, where $\nu(x)$ is the Euclidean unit outer normal vector to $\partial\Omega$.

If we consider a vector field $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, we can also define the tangential divergence of T on $\partial\Omega$ by the formula

$$\mathrm{div}_\tau T = \mathrm{div} T - \langle \nabla T\, \nu, \nu\rangle.$$

The following theorem is an extension to hypersurfaces in $\mathbb{R}^n$ of Gauss-Green theorem (see in [65] Theorem 11.8 combined with Remark 17.6).

Theorem 1.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Then there exists a continuous scalar function $H_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ such that for every $\varphi \in C^1_c(\mathbb{R}^n)$*

$$\int_{\partial\Omega} \nabla_\tau \varphi(x)\ d\mathcal{H}^{n-1} = \int_{\partial\Omega} \varphi(x) H_{\partial\Omega}(x)\nu(x)\ d\mathcal{H}^{n-1}.$$

The scalar function $H_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ is the so-called mean curvature.

Remark 1.5. Using the definition of tangential divergence, the Gauss-Green theorem can be reformulated in the following way:

$$\int_{\partial\Omega} \operatorname{div}_\tau T(x)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} H_{\partial\Omega}(x)\langle T(x), \nu(x)\rangle\, d\mathcal{H}^{n-1},$$

for every $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$.

A $1-$parameter family of diffeomorphisms of $\mathbb{R}^n$ is a smooth function

$$(x,t) \in \mathbb{R}^n \times (-\epsilon, \epsilon) \mapsto \phi(x,t),$$

for $\epsilon > 0$ such that, for each fixed $|t| < \epsilon$, $\phi(\cdot, t)$ is a diffeomorphism. We consider here a particular class of $1-$parameter family of diffeomorphisms such that $\phi(x,t) = x + tT(x) + O(t^2)$, with $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$. In [65, Theorem 17.5] the following theorem is proved.

Theorem 1.6. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary and let $\{\phi(\cdot,t)\}_{|t|<\epsilon}$ be a $1-$parameter family of diffeomorphisms as previously defined. We denote by $\Omega(t)$ the image of Ω through $\phi(\cdot,t)$. Then,*

$$P(\Omega(t)) = P(\Omega) + t \int_{\partial\Omega} \operatorname{div}_\tau T(x)\, d\mathcal{H}^{n-1} + o(t).$$

Using now the Gauss-Green theorem and this last theorem, we obtain the following expression for the first variation of the perimeter of an open set with C^∞ boundary:

$$\frac{d}{dt} P(\Omega(t))|_{t=0} = \int_{\partial\Omega} H_{\partial\Omega}(x)\langle T(x), \nu(x)\rangle\, d\mathcal{H}^{n-1}.$$

1.2.2 The first variation of anisotropic perimeter

We give now the following definitions.

Definition 1.7. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. At each point of $\partial\Omega$ we define the F-normal vector*

$$\nu^F(x) = \nabla F(\nu(x)),$$

sometimes called the Cahn-Hoffman field.

Definition 1.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. For every $x \in \partial\Omega$, we define the F-mean curvature*

$$H^F_{\partial\Omega}(x) = \operatorname{div}_\tau \left(\nu^F(x)\right).$$

In [12, Theorem 3.6] we find the computation of the first variation of the anisotropic perimeter. We report its statement; in the proof is used the first variation of the euclidean perimeter.

Theorem 1.9. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. For $t \in \mathbb{R}$, let $\phi(\cdot,t) : \mathbb{R}^n \to \mathbb{R}^n$ be a family of diffeomorphisms such that $\phi(\cdot,0) = Id$ and $\phi(\cdot,t) - Id$ has compact support in $\mathbb{R}^n$, for t in a neighborhood of 0. Set $\Omega(t)$ the image of Ω through $\phi(\cdot,t)$. Then*

$$\frac{d}{dt} P_F(\Omega(t))|_{t=0} = \int_{\partial\Omega} H^F_{\partial\Omega}(x)\langle \nu(x), g(x)\rangle d\mathcal{H}^{n-1}, \tag{1.8}$$

where $g(x) := \dfrac{\partial \phi(x,t)}{\partial t}|_{t=0}$.

For more details on this part the reader is referred to [84] and [12].

1.3 Quermassintegrals: definition and some properties

For the content of this section we refer, for instance, to [75]. Let $\emptyset \neq \Omega_0 \subseteq \mathbb{R}^n$ be a compact and convex set. We define the outer parallel body of Ω_0 at distance ρ as the Minkowski sum

$$\Omega_0 + \rho B_1(0) = \{x + \rho y \in \mathbb{R}^n \mid x \in \Omega_0,\ y \in B_1(0)\}.$$

The Steiner formula asserts that

$$|\Omega_0 + \rho B_1(0)| = \sum_{i=0}^{n} \binom{n}{i} W_i(\Omega_0)\rho^i. \tag{1.9}$$

The coefficients $W_i(\Omega_0)$ are known as quermassintegrals and some of them have an easy interpretation:

- $W_0(\Omega_0) = |\Omega_0|$;
- $nW_1(\Omega_0) = P(\Omega_0)$;
- $W_n(\Omega_0) = \omega_n$.

Let us assume now that Ω_0 is also of class C^2_+, i.e. Ω_0 has boundary of class C^2 and has non-vanishing Gaussian curvature.

We give now some definitions and recall some basic properties that we will use in the following. We introduce, for $j = 1, \cdots, (n-1)$, H_j the j-th normalized elementary symmetric function of the principal curvatures $k_1, \cdots, k_{n-1}$ of $\partial\Omega_0$:

$$H_j := \binom{n-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le n-1} k_{i_1} \cdots k_{i_j}$$

and we put $H_0 = 1$. We have that

$$W_i(\Omega_0) = \frac{1}{n} \int_{\partial\Omega_0} H_{i-1}\, d\mathcal{H}^{n-1}, \quad i = 1, \cdots, n,$$

and a Steiner formula for the quermassintegrals holds:

$$W_m(\Omega_0 + \rho B_1(0)) = \sum_{i=0}^{n-m} W_{m+i}(\Omega_0)\rho^i, \quad m = 0, \cdots, n-1,$$

that gives back (1.9) in the case $m = 0$. Moreover, we have that

$$\lim_{\rho \to 0^+} \frac{P(\Omega_0 + \rho B_1(0)) - P(\Omega_0)}{\rho} = n(n-1)W_2(\Omega_0)$$

and, in the case Ω_0 of class C^2_+, from the last equality, we obtain

$$\lim_{\rho \to 0^+} \frac{P(\Omega_0 + \rho B_1(0)) - P(\Omega_0)}{\rho} = (n-1) \int_{\partial\Omega_0} H_1\, d\mathcal{H}^{n-1}.$$

We recall also the Aleksandrov-Fenchel inequalities

$$\left(\frac{W_j(\Omega_0)}{\omega_n}\right)^{\frac{1}{n-j}} \ge \left(\frac{W_i(\Omega_0)}{\omega_n}\right)^{\frac{1}{n-i}}, \tag{1.10}$$

for $0 \leq i < j < n$, with equality if and only if Ω_0 is a ball. If we put in the last inequality $i = 0$ and $j = 1$ we obtain the classical isoperimetric inequality, that is

$$P(\Omega_0)^{\frac{n}{n-1}} \geq n^{\frac{n}{n-1}} \omega_n^{\frac{1}{n-1}} |\Omega_0| \,.$$

We will also need the case in (1.10) when $i = 1$ and $j = 2$:

$$W_2(\Omega_0) \geq n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} P(\Omega_0)^{\frac{n-2}{n-1}}. \tag{1.11}$$

We denote by $d_e(x)$ the distance function from the boundary of Ω_0. We use the following notations:

$$\Omega_{0,t} = \{x \in \Omega_0 \ : \ d_e(x) > t\}, \qquad t \in [0, r(\Omega_0)],$$

where by $r(\Omega_0)$ we denote the Euclidean inradius of Ω_0. We state now the following two lemmas, whose proofs can be found in [17] and [24].

Lemma 1.10. *Let Ω_0 be a bounded, convex, open set in $\mathbb{R}^n$. Then, for almost every $t \in (0, r_{\Omega_0})$, we have*

$$-\frac{d}{dt} P(\Omega_{0,t}) \geq n(n-1) W_2(\Omega_{0,t})$$

and equality holds if Ω_0 is a ball.

By simply applying the chain rule formula and recalling that $|\nabla d_e(x)| = 1$ almost everywhere, it remains proved the following.

Lemma 1.11. *Let $f : [0, +\infty) \to [0, +\infty)$ be a non decreasing C^1 function and let $\tilde{f} : [0, +\infty) \to [0, +\infty)$ a non increasing C^1 function. We define $u(x) := f(d_e(x))$, $\tilde{u}(x) := \tilde{f}(d_e(x))$ and*

$$E_{0,t} := \{x \in \Omega_0 \ : \ u(x) > t\},$$
$$\tilde{E}_{0,t} := \{x \in \Omega_0 \ : \ \tilde{u}(x) < t\}.$$

Then,

$$-\frac{d}{dt} P(E_{0,t}) \geq n(n-1) \frac{W_2(E_{0,t})}{|\nabla u|_{u=t}}, \tag{1.12}$$

and

$$\frac{d}{dt} P(\tilde{E}_{0,t}) \geq n(n-1) \frac{W_2(\tilde{E}_{0,t})}{|\nabla \tilde{u}|_{\tilde{u}=t}}. \tag{1.13}$$

1.4 Hausdorff distance and nearly spherical sets

Let E be a bounded open set with Lipschitz boundary, we define the boundary momentum of E as

$$W(E) = \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1}. \tag{1.14}$$

Moreover, we recall the definition of Hausdorff distance between two non-empty compact sets $E, G \subset \mathbb{R}^n$, that is (see for instance [75]):

$$d_{\mathcal{H}}(E, G) = \inf \left\{ \varepsilon > 0 \ : \ E \subset G + B_\varepsilon(0), \ G \subset E + B_\varepsilon(0) \right\}.$$

Note that, in the case E and G are convex sets, we have $d_{\mathcal{H}}(E, G) = d_{\mathcal{H}}(\partial E, \partial G)$ and the following rescaling property holds

$$d_{\mathcal{H}}(tE, tG) = t \, d_{\mathcal{H}}(E, G), \quad t > 0.$$

Let $E \subset \mathbb{R}^n$ be a bounded, open, convex set, we need to consider the following asymmetry functional related to E:

$$\mathcal{A}_{\mathcal{H}}(E) = \min_{x \in \mathbb{R}^n} \left\{ \left(\frac{d_{\mathcal{H}}\left(E, B_R(x)\right)}{R} \right), \, P(\Omega) = P(B_R(x)) \right\}. \tag{1.15}$$

We introduce the definition of convergence in measure.

Definition 1.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, let (E_j) be a sequence of measurable subsets of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be a measurable set. We say that (E_j) converges in measure in Ω to E, and we write $E_j \to E$, if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$, or in other words, if $\lim_{j\to\infty} |(E_j \Delta E) \cap \Omega| = 0$.*

We recall also that the perimeter is lower semicontinuous with respect to the local convergence in measure, that means, if the sequence of sets (E_j) converges in measure in Ω to E, then

$$P(E;\Omega) \le \liminf_{j\to\infty} P(E_j;\Omega).$$

As a consequence of the Rellich-Kondrachov theorem, the following compactness result holds; for a reference see for instance [4, Theorem 3.39].

Proposition 1.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and let (E_j) be a sequence of measurable sets of $\mathbb{R}^n$, such that $\sup_j P(E_j;\Omega) < \infty$. Then, there exists a subsequence (E_{j_k}) converging in measure in Ω to a set E, such that*

$$P(E;\Omega) \le \liminf_{k\to\infty} P(E_{j_k};\Omega).$$

Another useful property concerning the sets of finite perimeter is stated in the next approximation result, see [4, Theorem 3.42].

Proposition 1.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and let E be a set of finite perimeter in Ω. Then, there exists a sequence of smooth, bounded open sets (E_j) converging in measure in Ω and such that $\lim_{j\to\infty} P(E_j;\Omega) = P(E;\Omega)$.*

In the particular case of convex sets, the following lemma holds.

Lemma 1.15. *Let (E_j) be a sequence of convex subsets of $\mathbb{R}^n$ such that $E_j \to B_1$ in measure, then $\lim_{j\to\infty} P(E_j) = P(B_1)$.*

Proof. Since, in the case of convex sets, the convergence in measure implies the Hausdorff convergence, we have that $\lim_{j\to\infty} d_{\mathcal{H}}(E_j, B_1) = 0$ (see for instance [46]). Thus, for j large enough, there exists ε_j, such that

$$(1-\varepsilon_j)B_1 \subset E_j \subset (1+\varepsilon_j)B_1.$$

Being the perimeter monotone with respect to the inclusion of convex sets then

$$(1-\varepsilon_j)^{n-1}P(B_1) \le P(E_j) \le (1+\varepsilon_j)^{n-1}P(B_1).$$

When j goes to infinity, we have the claim. □

Moreover, it holds this lemma, which states a bound for the diameter of a convex set (see [46]).

Lemma 1.16. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open, convex set. There exists a positive constant $C(n)$ such that*

$$\mathrm{diam}(E) \le C(n)\frac{P(E)^{n-1}}{|E|^{n-2}}. \tag{1.16}$$

In this section we give also the definition of nearly spherical sets and we recall some of their basic properties (see for instance [18, 48, 52, 53]). In the following we denote by $\mathbb{S}^{n-1}$ the boundary of the unit ball centered at the origin.

Definition 1.17. *Let $n \geq 2$. An open, bounded set $E \subset \mathbb{R}^n$ is said a nearly spherical set parameterized by v, if there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that*

$$\partial E = \left\{ y \in \mathbb{R}^n \colon y = x(1+v(x)),\, x \in \mathbb{S}^{n-1} \right\},$$

with $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \frac{1}{2}$.

Note also that $||v||_{L^\infty(\mathbb{S}^{n-1})} = d_{\mathcal{H}}(E, B_1(0))$. The perimeter, the volume and the boundary momentum of a nearly spherical set are given by

$$P(E) = \int_{\mathbb{S}^{n-1}} (1+v(x))^{n-2} \sqrt{(1+v(x))^2 + |\nabla_\tau v(x)|^2}\, d\mathcal{H}^{n-1}, \tag{1.17}$$

$$|E| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (1+v(x))^n \ d\mathcal{H}^{n-1}, \tag{1.18}$$

$$W(E) = \int_{\mathbb{S}^{n-1}} (1+v(x))^n \sqrt{(1+v(x))^2 + |\nabla_\tau v(x)|^2}\, d\mathcal{H}^{n-1}. \tag{1.19}$$

Finally, we recall two lemmas that we will use later. The first one is an interpolation result; for its proof we refer for instance to [52, 53].

Lemma 1.18. *If $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}} v\, d\mathcal{H}^{n-1} = 0$, then*

$$||v||^{n-1}_{L^\infty(\mathbb{S}^{n-1})} \leq \begin{cases} \pi ||\nabla_\tau v||_{L^2(\mathbb{S}^1)} & n = 2 \\ 4||\nabla_\tau v||^2_{L^2(\mathbb{S}^2)} \log \dfrac{8e||\nabla_\tau v||^2_{L^\infty(\mathbb{S}^2)}}{||\nabla_\tau v||^2_{L^2(\mathbb{S}^2)}} & n = 3 \\ C_n ||\nabla_\tau v||^2_{L^2(\mathbb{S}^{n-1})} ||\nabla_\tau v||^{n-3}_{L^\infty(\mathbb{S}^{n-1})} & n \geq 4 \end{cases}$$

For this second lemma see for instance [53].

Lemma 1.19. *Let $n \geq 2$. There exists an universal $\varepsilon_0 < \frac{1}{8}$ such that, if E is a convex, nearly spherical set with $|E| = |B_1|$ and $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$, then*

$$||\nabla_\tau v||^2_{L^\infty(\mathbb{S}^{n-1})} \leq 8||v||_{L^\infty(\mathbb{S}^{n-1})}. \tag{1.20}$$

Finally, we prove the following

Lemma 1.20. *Let $n \geq 2$ and let $E \subseteq \mathbb{R}^n$ be a bounded, convex, nearly spherical set with $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$, then*

$$d_{\mathcal{H}}(E, E^*) \leq \left(16 \left(\frac{9}{8} \right)^n + n + 1 \right) d_{\mathcal{H}}(E, E^\sharp), \tag{1.21}$$

where E^* and $E^\sharp$ are the balls centered at the origin having the same perimeter and the same volume as E respectively.

Proof. By the properties of the Hausdorff distance, we get

$$\begin{aligned} d_{\mathcal{H}}(E, E^*) &\leq d_{\mathcal{H}}(E, E^\sharp) + d_{\mathcal{H}}(E^*, E^\sharp) = d_{\mathcal{H}}(E, E^\sharp) + \left(\frac{P(E)}{n\omega_n} \right)^{\frac{1}{n-1}} - \left(\frac{|E|}{\omega_n} \right)^{\frac{1}{n}} \\ &= d_{\mathcal{H}}(E, E^\sharp) + \left(\frac{|E|}{\omega_n} \right)^{\frac{1}{n}} \left[\left(\frac{P(E)}{n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}} \right)^{\frac{1}{n-1}} - 1 \right]. \end{aligned} \tag{1.22}$$

We stress that, in the square brackets, we have the isoperimetric deficit of E, which is scaling invariant. Let $G \subset \mathbb{R}^n$ be a convex, nearly spherical set parameterized by v_G, with $||v_G||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$ and $|G| = |B_1|$. Being G nearly spherical and $||v_G||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$, from the isoperimetric inequality, (1.17), Lemma 1.19, and recalling that $\varepsilon_0 < \frac{1}{8}$ we get

$$\begin{aligned}
\left(\frac{P(G)}{n\omega_n^{\frac{1}{n}}\,|G|^{\frac{n-1}{n}}}\right)^{\frac{1}{n-1}} - 1 \leq \frac{P(G)}{n\omega_n} - 1 \\
&= \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} \left((1+v_G(x))^{n-2}\sqrt{(1+v_G(x))^2 + |\nabla_\tau v_G(x)|^2} - 1\right) \\
&\leq \left(n + 8\left(\frac{9}{8}\right)^n\right)||v_G||_{L^\infty(\mathbb{S}^{n-1})} + \left(\frac{9}{8}\right)^{n-2}\|\nabla_\tau v_G\|^2_{L^\infty(\mathbb{S}^{n-1})} \\
&\leq \left(16\left(\frac{9}{8}\right)^n + n\right)||v_G||_{L^\infty(\mathbb{S}^{n-1})}.
\end{aligned}$$

As a consequence, recalling that $||v_G||_{L^\infty(\mathbb{S}^{n-1})} = d_{\mathcal{H}}(G, B_1(0))$,

$$\left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}\left[\left(\frac{P(E)}{n\omega_n^{\frac{1}{n}}\,|E|^{\frac{n-1}{n}}}\right)^{\frac{1}{n-1}} - 1\right] \leq \left(16\left(\frac{9}{8}\right)^n + n\right) d_{\mathcal{H}}(E, E^\sharp).$$

Using this inequality in (1.22), we get the claim. □

Chapter 2

Euclidean and anisotropic eigenvalue problems involving Robin boundary conditions with negative parameter

In this chapter we deal with Robin eigenvalue problems with negative parameter of the Laplacian and of its anisotropic version, the so-called Finsler Laplacian, which is defined as

$$\mathcal{Q}_2 u = \operatorname{div}\left(F(\nabla u)\nabla F(\nabla u)\right),$$

where F is a Finsler norm of $\mathbb{R}^n$ as is defined in the Section 1.2.

2.1 Some remarks on the Robin-Laplacian eigenvalues

In this section we consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary, $\frac{\partial u}{\partial \nu}$ is the normal derivative and $\alpha < 0$.
We provide, in dimension $n = 2$, a monotonicity result for the first eigenvalue of the problem (2.1) among all the annuli when we fix the inner radius. Moreover, in any dimension, we get an isoperimetric inequality for the second eigenvalue of the problem (2.1) for a particular value of the parameter α.

2.1.1 Preliminary Results

We recall some properties of the eigenvalues of problem (2.1). They form a sequence $\lambda_1(\alpha,\Omega) \leq \lambda_2(\alpha,\Omega) \leq \ldots \leq \lambda_m(\alpha,\Omega) \leq \ldots$ such that $\lambda_m(\alpha,\Omega) \to \infty$, and they can be characterized with *min-max* formulation, that is

$$\lambda_m(\alpha,\Omega) = \inf_{\substack{E_m \subset H^1(\Omega) \\ \dim E_m = m}} \left(\max_{\substack{v \in E_m \\ v \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2 dx + \alpha \int_{\partial\Omega} v^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 dx} \right). \tag{2.2}$$

In particular, the first one is given by

$$\lambda_1(\alpha,\Omega)=\inf_{\substack{v\in H^1(\Omega)\\ v\neq 0}}\frac{\displaystyle\int_\Omega|\nabla v|^2dx+\alpha\int_{\partial\Omega}v^2d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2dx}. \tag{2.3}$$

Using the constant as test function in the Rayleigh quotient (2.3), we obtain the following inequality, which allows to see that $\lambda_1(\alpha,\Omega)<0$:

$$\lambda_1(\alpha,\Omega)\le\alpha\frac{\mathcal{H}^{n-1}(\partial\Omega)}{|\Omega|},$$

where $\mathcal{H}^{n-1}(\Omega)$ stands for the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ and $|\Omega|$ stands for the Lebesgue measure of Ω. The above inequality implies that the first eigenvalue is not bounded from below when the volume is fixed. If Ω is connected, as in [63] , one can see that the first eigenvalue is simple and has a positive associated eigenfunction.
Having in mind this fact, we obtain that the associated eigenfunction to problem (2.1) on the annulus, defined as $A_{r_1,r_2}=B_{r_2}\setminus\overline{B}_{r_1}$, is radial, and then we can write problem (2.1) as follows

$$\begin{cases}-\frac{1}{r^{n-1}}\left[r^{n-1}\phi'(r)\right]' & =\lambda_1(\alpha,A_{r_1,r_2})\phi(r),\ r_1<r<r_2\\ -\phi'(r_1)+\alpha\phi(r_1) & =0\\ \phi'(r_2)+\alpha\phi(r_2) & =0\end{cases} \tag{2.4}$$

where $u_1(x)=\phi(|x|)$ is the first eigenfunction in A_{r_1,r_2}. The solutions of (2.4) are given by

$$\phi(r)=r^{-p}\left[C_1K_p\left(\sqrt{\lambda_1(\alpha,A_{r_1,r_2})}\,r\right)+C_2I_p\left(\sqrt{\lambda_1(\alpha,A_{r_1,r_2})}\,r\right)\right], \tag{2.5}$$

where C_1 and C_2 are implicitly defined by the boundary conditions as in [50], and where the functions I_p and K_p are modified Bessel functions of order p, see for instance [1], and

$$p=\frac{n-2}{2}.$$

For a long time, it was conjectured that balls maximize λ_1 among bounded open Lipschitz sets with given volume. In [50], the authors disprove such conjecture by showing that there exists an annulus, for which $|A_{r_1,r_2}|=|B_r|$ such that

$$\lambda_1(\alpha,A_{r_1,r_2})>\lambda_1(\alpha,B_r)$$

for α negative big enough. More precisely, they prove the following asymptotics for λ_1 as $\alpha\to-\infty$

$$\begin{aligned}\lambda_1(\alpha,A_{r_1,r_2})&=-\alpha^2+\frac{(n-1)\alpha}{r_2}+o(\alpha)\\ \lambda_1(\alpha,B_r)&=-\alpha^2+\frac{(n-1)\alpha}{r}+o(\alpha).\end{aligned} \tag{2.6}$$

In order to prove Theorem 2.3 in the next subsection, we use the classical Hadamard formula, see for instance [9] , and in order to compute $d\lambda_1(\alpha,\Omega;V)$ we have

$$d\lambda_1(\alpha,\Omega;V)=\int_{\partial\Omega}\left(|\nabla u_1|^2-\lambda_1(\Omega)u_1^2-2\alpha^2u_1^2+\alpha Hu_1^2\right)\langle V,\nu\rangle d\mathcal{H}^1 \tag{2.7}$$

where $\Omega\subset\mathbb{R}^2$ is smooth, H is the mean curvature at a point x of $\partial\Omega$, ν is the unit outward normal vector of boundary $\partial\Omega$ and V is a smooth vector field defined on $\partial\Omega$.

Before proceeding, we recall that an area preserving vector field is a smooth vector field $V : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\int_{\partial\Omega} \langle V, \nu \rangle \, d\mathcal{H}^1 = 0,$$

and applying (2.7) to A_{r_1,r_2}, we obtain the following stationary condition:

Proposition 2.1. *Let A_{r_1,r_2} be an annulus of $\mathbb{R}^2$ and let V be an area preserving vector field in A_{r_1,r_2}, if*

$$\phi^2(r_2)\left(k^2 - \alpha^2 + \frac{\alpha}{r_2}\right) - \phi^2(r_1)\left(k^2 - \alpha^2 - \frac{\alpha}{r_1}\right) = 0 \text{ then } d\lambda_1(\alpha, A_{r_1,r_2}; V) = 0. \quad (2.8)$$

Here ϕ is the function given in (2.5), $k^2 = -\lambda_1(\alpha, A_{r_1,r_2})$ and α is the negative parameter in the Robin boundary conditions.

Proof. By (2.7), we get

$$\begin{aligned} d\lambda_1(\alpha, A_{r_1,r_2}; V) &= \int_{\partial A_{r_1,r_2}} \left(|\nabla u|^2 + k^2u^2 - 2\alpha^2u^2 + \alpha H u^2\right) \langle V, \nu \rangle \, d\mathcal{H}^1 \\ &= \left(k^2 - \alpha^2 + \frac{\alpha}{r_2}\right) \phi^2(r_2) \int_{\partial B_{r_2}} \langle V, \nu \rangle \, d\mathcal{H}^1 \\ &+ \left(k^2 - \alpha^2 - \frac{\alpha}{r_1}\right) \phi^2(r_1) \int_{\partial B r_1} \langle V, \nu \rangle \, d\mathcal{H}^1, \end{aligned}$$

and, having in mind that the vectorial field V is volume preserving, or equivalently

$$\int_{\partial A_{r_1,r_2}} \langle V, \nu \rangle \, d\mathcal{H}^1 = 0 \Rightarrow \int_{\partial B_{r_1}} \langle V, \nu \rangle \, d\mathcal{H}^1 = -\int_{\partial B_{r_2}} \langle V, \nu \rangle \, d\mathcal{H}^1$$

and then

$$\begin{aligned} d\lambda_1(\alpha, A_{r_1,r_2}; V) &= \Bigg[\phi^2(r_2)\left(k^2 - \alpha^2 + \frac{\alpha}{r_2}\right) \\ &- \phi^2(r_1)\left(k^2 - \alpha^2 - \frac{\alpha}{r_1}\right)\Bigg] \int_{\partial B_{r_2}} \langle V, \nu \rangle \, d\mathcal{H}^1, \end{aligned}$$

which implies (2.8). ☐

Let

$$G(r_2) = \phi^2(r_2)\left(k^2 - \alpha^2 + \frac{\alpha}{r_2}\right) - \phi^2(r_1)\left(k^2 - \alpha^2 - \frac{\alpha}{r_1}\right),$$

using the area constraint ${r_2}^2 - {r_1}^2 = C$, the boundary conditions in (2.4) and assuming that r_1 and r_2 are as in (2.8), we obtain

$$\begin{aligned} \frac{dG}{dr_2}(r_2) &= -2\alpha\phi^2(r_2)\left(k^2 - \alpha^2 + \frac{\alpha}{r_2} + \frac{1}{2{r_2}^2}\right) \\ &- \frac{2\alpha\phi^2(r_1)\, r_2}{r_1}\left(k^2 - \alpha^2 - \frac{\alpha}{r_1} + \frac{1}{2{r_1}^2}\right). \end{aligned}$$

Using the asymptotics (2.6), we have

$$\frac{dG}{dr_2}(r_2) = -2\alpha\phi^2(r_2)\left(\frac{1}{2r_2^2} + o(\alpha)\right) - \frac{2\alpha\phi^2(r_1)r_2}{r_1}\left(-\frac{\alpha}{r_2} - \frac{\alpha}{r_1} + \frac{1}{2r_1^2} + o(\alpha)\right)$$

and $\frac{dG}{dr_2}(r_2)$ is positive for α smaller than a negative critical value, said α_c.
In order to prove the Theorem 2.4, we need the following weighted isoperimetric inequality from [15] :

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, B_r a ball of radius r centered at the origin, such that $|\Omega| = |B_r|$, and $\psi : [0, +\infty) \to [0, +\infty)$ a non-decreasing function such that*

$$\left(\psi(t^{\frac{1}{n}}) - \psi(0)\right) t^{1-\frac{1}{n}}$$

is convex for every $t \geq 0$

$$\int_{\partial\Omega} \psi(|x|) d\mathcal{H}^{n-1} \geq \int_{\partial B_r} \psi(|x|) d\mathcal{H}^{n-1}. \tag{2.9}$$

Another important remark, in order to prove the Theorem 2.4 in the next subsection, concerns the eigenvalues of the Steklov-Laplacian problem,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

where Ω is a bounded open connected set with Lipschitz boundary. The eigenvalues of (2.10) form a sequence $0 = \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \ldots \leq \sigma_m(\Omega) \leq \ldots$ and they can be characterized, like in [58], with the variational formulation

$$\sigma_m(\Omega) = \min_{\substack{v \in H^1(\Omega) \\ v \not\equiv 0}} \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 dx}{\displaystyle\int_{\partial\Omega} v^2 d\sigma} \ : \ \int_{\partial\Omega} v\zeta_i d\sigma = 0, \ i = 1, \ldots, m-1 \right\},$$

where ζ_i is the eigenfunction associated to the eigenvalue $\sigma_i(\Omega)$.
It is known that $\sigma_2(B_r) = \sigma_3(B_r) = \ldots = \sigma_{n+1}(B_r) = \frac{1}{r}$ and the associated eigenfunctions are $\zeta_i(x) = x_{i-1}$ with $i = 2, \ldots, n+1$. For that reason, choosing in problem (2.1) $\alpha = \sigma_2(B_r) = \frac{1}{r}$, we obtain $\lambda_2(B_r) = 0$

2.1.2 Main Results

First, we investigate a monotonicity property for $\lambda_1(\alpha, A_{r_1,r_2})$ with respect to r_2, when $A_{r_1,r_2} \subset \mathbb{R}^2$, using (2.7) as in [9].

Theorem 2.3. *Let V_1 be the following vectorial field in $\mathbb{R}^2$*

$$V_1(x) = \begin{cases} \nu & \text{if } |x| = r_2 \\ 0 & \text{otherwise} \end{cases}$$

where ν is the unit outward normal vector of ∂B_{r_2}, then

$$d\lambda_1(\alpha, A_{r_1,r_2}; V_1) > 0.$$

where $d\lambda_1(\alpha, A_{r_1,r_2}; V_1)$ is the shape derivative of $\lambda_1(\alpha, \cdot)$ given in (2.7). In particular, if $r_2 < \tilde{r_2}$ than

$$\lambda_1(\alpha, A_{r_1,r_2}) < \lambda_1(\alpha, A_{r_1,\tilde{r_2}}).$$

Proof. When $n = 2$ (2.4) becomes

$$\begin{cases} \phi''(r) + \dfrac{\phi'(r)}{r} + \lambda\phi(r) = 0 \\ \phi'(r_1) = \alpha\phi(r_1) \\ \phi'(r_2) = -\alpha\phi(r_2) \end{cases} \tag{2.11}$$

where $\lambda = \lambda_1(\alpha, A_{r_1,r_2})$.
From (2.7) we obtain

$$d\lambda\left(\alpha, A_{r_1,r_2}; V_1\right) = 2\pi r_2 \phi^2\left(r_2\right)\left(-\lambda - \alpha^2 + \frac{\alpha}{r_2}\right) \tag{2.12}$$

and using (2.12) we can prove the statement by proving that

$$\left(\lambda + \alpha^2 - \frac{\alpha}{r_2}\right) < 0.$$

Setting $z = \dfrac{\phi'}{\phi}$ (having in mind that $\phi > 0$), using (2.11), we obtain that z satisfies

$$\frac{dz}{dr} + z^2 + \frac{z}{r} + \lambda = 0 \text{ in } (r_1, r_2) \tag{2.13}$$

and then

$$\frac{dz}{dr}(r_2) = -\left(\lambda + \alpha^2 - \frac{\alpha}{r_2}\right).$$

From the boundary conditions in (2.11) we have $z(r_1) = \alpha$ and $z(r_2) = -\alpha$. Then defining

$$\xi = \sup\left\{\rho \in (r_1, r_2) : z(\rho) < 0\right\}, \tag{2.14}$$

we have that $\xi < r_2$ and $z(\xi) = 0$, and using (2.13) we obtain that

$$\frac{dz}{dr}(\xi) = -\lambda > 0.$$

Our aim is to prove that $\frac{dz}{dr}(r_2) > 0$. Let ξ_1 be defined by

$$\xi_1 = \sup\left\{\rho \in (\xi, r_2) : \frac{dz}{dr}(\rho) > 0\right\}, \tag{2.15}$$

by (2.14), we have $z(\xi_1) > 0$, moreover, if $\xi_1 < r_2$, by (2.15) we have

$$\frac{dz}{dr}(\xi_1) = 0.$$

Differentiating (2.13) we get

$$\frac{d^2z}{dr^2}(\xi_1) > 0,$$

which gives a contradiction. Then necessarily $\xi_1 = r_2$ and by continuity $\frac{dz}{dr}(r_2) \geq 0$. If $\frac{dz}{dr}(r_2) = 0$, differentiating (2.13), we obtain again

$$\frac{d^2z}{dr^2}(r_2) > 0,$$

but this is a contradiction to $r_2 = \xi_1$. This implies $\frac{dz}{dr}(r_2) > 0$ and hence the theorem is proved. □

The second result that we want to prove is an isoperimetric inequality for the second eigenvalue of the problem (2.1), which is defined in the equation (2.2), for a particular value of the parameter α. First of all we observe that, when the parameter $\alpha = -\sqrt[n]{\frac{\omega_n}{|\Omega|}}$, where

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)},$$

is the measure of the unit ball, the problem (2.1) on the ball is equivalent to the following Steklov-Laplacian problem

$$\begin{cases} -\Delta u = 0 & \text{in } B_r \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{r} u & \text{on } \partial B_r \end{cases}$$

where B_r is the ball such that $|B_r| = |\Omega|$.

Theorem 2.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let B_r be the ball with the same measure as Ω, that is $r = \sqrt[n]{\frac{|\Omega|}{\omega_n}}$. When $\alpha = -\frac{1}{r}$ the following inequality holds*

$$\lambda_2\left(-\frac{1}{r}, \Omega\right) \leq \lambda_2\left(-\frac{1}{r}, B_r\right) = 0.$$

Proof. The min-max formulation (2.2) for the second eigenvalue of problem (2.1) allows to write

$$\lambda_2\left(-\frac{1}{r}, \Omega\right) \leq \max_{v \in E_2} \frac{\displaystyle\int_\Omega |\nabla v|^2 dx - \frac{1}{r}\int_{\partial\Omega} v^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 dx} \tag{2.16}$$

where E_2 is a 2-dimensional space of the $H^1(\Omega)$. We choose E_2 as the subspace spanned by the coordinate function x_i and a constant function, and then, denoting by $a_i \in \mathbb{R}$ the constant achieving the maximum in (2.16), we have

$$\begin{aligned} \lambda_2\left(-\frac{1}{r}, \Omega\right) &\leq \frac{\displaystyle\int_\Omega |\nabla (x_i + a_i)|^2 dx - \frac{1}{r}\int_{\partial\Omega} (x_i + a_i)^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega (x_i + a_i)^2 dx} \\ &= \frac{\displaystyle |\Omega| - \frac{1}{r}\int_{\partial\Omega} (x_i + a_i)^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega (x_i + a_i)^2 dx}. \end{aligned} \tag{2.17}$$

From (2.17), adding for every index, from 1 to n, we obtain the following inequality

$$\lambda_2\left(-\frac{1}{r}, \Omega\right) \leq \frac{\displaystyle n|\Omega| - \frac{1}{r}\int_{\partial\Omega} |x + a|^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |x + a|^2 dx},$$

and from that, by means of inequality (2.9), using a simple change of variables, we have

$$\lambda_2\left(-\frac{1}{r}, \Omega\right) \leq \frac{\displaystyle n|\Omega| - \frac{1}{r}\int_{\partial B_r(-a)} |x + a|^2 d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |x + a|^2 dx} = 0 = \lambda_2\left(-\frac{1}{r}, B_r\right),$$

and this completes the proof. □

2.1.3 What happens to λ_1 when we pinch the ball?

We know that, if u_1 is the eigenfunction of problem associated to $\lambda_1(\alpha, B_r)$, we have

$$\lambda_1(\alpha, B_r) = \frac{\displaystyle\int_{B_r} |\nabla u_1|^2 dx + \alpha \int_{\partial B_r} u_1^2 d\sigma}{\displaystyle\int_{B_r} u_1^2 dx} = \frac{\displaystyle\int_{B_r} |\nabla u_1|^2 dx + n\alpha\omega_n u_1^2(r) r^{n-1}}{\displaystyle\int_{B_r} u_1^2 dx}.$$

Let $\epsilon > 0$, we consider the annulus $A_{\epsilon,r'}$, with $r' > r$ such that $|A_{\epsilon,r'}| = |B_r|$ and let u_1 be the function in $H^1(B_{r'})$ defined by the following statement

$$w(x) = \begin{cases} u_1(x) & \text{if } x \in B_r \\ u_1(r) & \text{if } x \in B_{r'} \setminus B_r. \end{cases}$$

We have

$$\lambda_1(\alpha, A_{\epsilon,r'}) \leq \frac{\displaystyle\int_{A_{\epsilon,r'}} |\nabla w|^2 dx + \alpha \int_{\partial A_{\epsilon,r'}} w^2 d\mathcal{H}^{n-1}}{\displaystyle\int_{A_{\epsilon,r'}} w^2 dx} \tag{2.18}$$

$$= \frac{\displaystyle\int_{B_r'} |\nabla w|^2 dx + \int_{B_\epsilon} |\nabla w|^2 dx - \alpha \left(\int_{\partial B_{r'}} w^2 d\mathcal{H}^{n-1} + \int_{\partial B_\epsilon} w^2 d\mathcal{H}^{n-1} \right)}{\displaystyle\int_{B_{r'}} w^2 dx - \int_{B_\epsilon} w^2 dx}.$$

We have

$$\int_{B_{r'}} |\nabla w|^2 dx = \int_{B_r} |\nabla u_1|^2 dx,$$

$$\int_{B_\epsilon} |\nabla w|^2 dx = o(\epsilon^n),$$

$$\alpha \left(\int_{\partial B_{r'}} w^2 d\mathcal{H}^{n-1} + \int_{\partial B_\epsilon} w^2 d\mathcal{H}^{n-1} \right) = n\alpha\omega_n r^{n-1} u_1^2(r) - \mathcal{O}(\epsilon^{n-1}),$$

$$\int_{B_{r'}} w^2 dx - \int_{B_\epsilon} w^2 dx = \int_{B_r} u_1^2 dx + \mathcal{O}(\epsilon^n).$$

From (2.18) and the above equations, we have

$$\lambda_1(\alpha, B_r) - \lambda_1(\alpha, A_{\epsilon,r'}) \geq \frac{\mathcal{O}(\epsilon^{n-1})}{\displaystyle\int_{B_r} u_1^2 dx + \mathcal{O}(\epsilon^n)}$$

then, for ϵ small enough, we have $\lambda_1(\alpha, B_r) > \lambda_1(\alpha, A_{\epsilon,r'})$.

2.2 Two estimates for the first Robin eigenvalue of the Finsler Laplacian with negative boundary parameter

Let Ω be a bounded open connected subset of $\mathbb{R}^2$ of class C^2. In this section we are interested to the following variational problem

$$\lambda_{1,F}(\alpha,\Omega) = \min_{\substack{v\in H^1(\Omega)\\ v\neq 0}} \frac{\displaystyle\int_\Omega (F(\nabla v))^2\ dx + \alpha \int_{\partial\Omega} |v|^2 F(\nu)\ d\mathcal{H}^1}{\displaystyle\int_\Omega |v|^2\ dx}, \tag{2.19}$$

where F is a Finsler norm as defined in Section 1.2, α is a negative constant and ν is the Euclidean outward unit normal vector to $\partial\Omega$.
Using a constant as test function, we obtain the following inequality

$$\lambda_{1,F}(\alpha,\Omega) \leq \alpha\frac{P_F(\Omega)}{|\Omega|} < 0. \tag{2.20}$$

The minimizers u of problem (2.19) satisfy the following problem

$$\begin{cases} -\mathrm{div}\,(F(\nabla u)\nabla F(\nabla u)) = \lambda_{1,F}(\alpha,\Omega)u & \text{in } \Omega \\ \langle F(\nabla u)\nabla F(\nabla u), \nu\rangle + \alpha F(\nu)u = 0 & \text{on } \partial\Omega \end{cases}$$

that is, in the weak sense

$$\int_\Omega F(\nabla u)\,\langle\nabla F(\nabla u), \nabla\varphi\rangle\ dx + \alpha\int_{\partial\Omega} u\varphi F(\nu)\ d\mathcal{H}^1 = \lambda_{1,F}(\alpha,\Omega)\int_\Omega u\varphi\ dx, \tag{2.21}$$

for all $\varphi \in H^1(\Omega)$.
Here we prove two isoperimetric inequalities for $\lambda_{1,F}(\alpha,\Omega)$: in the first one we prove that the Wulff shape maximizes $\lambda_{1,F}(\alpha,\Omega)$ when we fix the volume for certain values of α and in the second one we show that the Wulff Shape maximizes $\lambda_{1,F}(\alpha,\Omega)$ when we fix the anisotropic perimeter for all negative parameter α.

2.2.1 Isoperimetric Estimates with an Area Constraint

In this part of the chapter we are interested to find an estimate for $\lambda_{1,F}(\alpha,\Omega)$ when is given a volume constraint

Theorem 2.5. *For bounded planar domains of class C^2 and fixed area, there exists a negative number α_*, depending only on the area, such that the following inequality holds $\forall\alpha \in [\alpha_*, 0]$:*

$$\lambda_{1,F}(\alpha,\Omega) \leq \lambda_{1,F}(\alpha,\mathcal{W}^*_\Omega),$$

where $\mathcal{W}^*_\Omega$ is the Wulff shape of the same area as Ω.

In order to prove Theorem 2.5, we adapt in the anisotropic case the proof of Freitas and Krejčiřík contained in [50]. This proof makes use of the classical method of parallel coordinates, developed for the Euclidean case in [71] and for the Riemannian case in [74]. We assume that $\partial\Omega$ is composed by a finite union of C^2 Jordan curves $\Gamma_0,\dots,\Gamma_N$, where Γ_0 is the outer boundary of Ω, i.e. Ω lies in the interior Ω_0 of Γ_0. We observe that, if $N = 0$, then Ω is simply connected and $\Omega = \Omega_0$. We denote by

$$L_0^F = P_F(\Omega_0)$$

the outer anisotropic perimeter. Therefore, by the anisotropic isoperimetric inequality in Theorem 1.2, denoting by κ the measure of the unit Wulff shape in dimension 2 we have

$$(L_0^F)^2 \geq 4\kappa A_0,$$

where $A_0 = |\Omega|$ denotes the area of Ω (not of Ω_0).
We now introduce the **anisotropic parallel coordinate method** based at the outer boundary Γ_0. Let $\rho_F : \Omega_0 \to]0, \infty[$ be the anisotropic distance function from the outer boundary Γ_0:

$$\rho_F(x) = d_F(x, \Gamma_0).$$

Let

$$A_F(t) = |\{x \in \Omega \ : \ 0 < \rho_F(x) < t\}|$$

denote the area of $\tilde{\Omega}_t = \{x \in \Omega \ : \ 0 < \rho_F(x) < t\}$ and let

$$L_F(t) = \int_{\rho_F^{-1}(t) \cap \Omega} F(\nu) \ d\mathcal{H}^1.$$

Remark 2.6. We obtain that, as in [50], using [39, Lemma 3.1], for almost every $t \in [0, r_F(\Omega_0)]$,

$$A'_F(t) = L_F(t). \tag{2.22}$$

Step 1: Use of the Anisotropic Parallel Coordinates

Let $\phi : [0, |\Omega|] \to \mathbb{R}$ be a smooth function and consider the test function

$$u = \phi \circ A_F \circ \rho_F,$$

which is Lipschitz in Ω. Using the anisotropic parallel coordinates, the coarea formula and the fact that $F(\nabla \rho_F) = 1$, we obtain the following relations:

$$\begin{aligned}
||u||^2_{L^2(\Omega)} &= \int_\Omega u^2(x) \, dx = \int_\Omega (\phi \circ A_F \circ \rho_F(x))^2 \, dx = \\
&= \int_0^{r_F(\Omega)} \left(\int_{\{\rho_F(x)=t\}} (\phi \circ A_F \circ \rho_F(x))^2 \frac{1}{|\nabla \rho_F(x)|} \, d\mathcal{H}^1 \right) \, dt \\
&= \int_0^{r_F(\Omega)} \phi(A_F(t))^2 \, P_F(\{\rho_F(x) < t\}) \, dt = \\
&= \int_0^{r_F(\Omega)} \phi(A_F(t))^2 \, A'_F(t) \, dt;
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \left(F^2 \left(\nabla u(x)\right)\right) dx &= \int_\Omega F^2 \left(\phi' \left(A_F \circ \rho_F \left(x\right)\right) A'_F \left(\rho_F \left(x\right)\right) \nabla \rho_F \left(x\right)\right) dx = \\
&= \int_\Omega \left(\phi' \left(A_F \circ \rho_F \left(x\right)\right)\right)^2 \left(A'_F \left(\rho_F \left(x\right)\right)\right)^2 dx \\
&= \int_0^{r_F(\Omega)} \left(\phi' \left(A_F \left(t\right)\right)\right)^2 \left(A'_F \left(t\right)\right)^3 dt;
\end{aligned}$$

$$\begin{aligned}
\int_{\partial\Omega} |u(x)|^2 F(\nu) \, d\mathcal{H}^1 &= \int_{\partial\Omega} \left(\phi \circ A_F \circ \rho_F \left(x\right)\right)^2 F(\nu) \, d\mathcal{H}^1 = \\
&= \left(\phi \circ A_F \left(0\right)\right)^2 P_F(\Omega) \geq \phi^2(0) \, L_0^F.
\end{aligned}$$

Therefore we have that

$$\lambda_{1,F}(\alpha,\Omega) \leq \frac{\displaystyle\int_0^{r_F(\Omega)} \left(\phi'\left(A_F\left(t\right)\right)\right)^2 \left(A_F'\left(t\right)\right)^3 dt + \alpha\, \phi^2(0)\, L_0^F}{\displaystyle\int_0^{r_F(\Omega)} \phi(A_F(t))^2\, A_F'(t)\, dt}.$$

Step 2: from Domains to Annuli

We adapt in the anisotropic case the idea contained in [71]. We consider the following change of variables:

$$R(t) := \frac{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_F(t)}}{2\kappa} \tag{2.23}$$

on the interval $[r_1, r_2]$, where

$$r_1 := R\left(r_F\left(\Omega\right)\right) = \frac{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_0}}{2\kappa}, \qquad r_2 := R(0) = \frac{L_0^F}{2\kappa}. \tag{2.24}$$

Remark 2.7. Thanks to Theorem 1.2, the transformation (2.23) is well defined on the set $[0, r_F(\Omega)]$.

We introduce now the function

$$\psi(r) := \phi\left(\frac{\left(L_0^F\right)^2}{4\kappa} - \kappa r^2\right)$$

and we obtain the following expressions:

$$\int_\Omega u^2(x)\, dx = 2\kappa \int_{r_1}^{r_2} \left(\psi(r)\right)^2 r\, dr;$$

$$\int_\Omega \left(F^2\left(\nabla u(x)\right)\right) dx = 2\kappa \int_{r_1}^{r_2} \left(\psi'(r)\right)^2 \left(R'(t)\right)^2 r\, dr;$$

$$\int_{\partial\Omega} |u(x)|^2 F(\nu)\, d\mathcal{H}^1 \geq L_0^F\, \psi(r_2)^2.$$

Remark 2.8. The radii in (2.24) are such that the F-annulus $A^F_{r_1,r_2} := \mathcal{W}_{r_2} \setminus \overline{\mathcal{W}}_{r_1}$ has the same area A_0 as the original domain Ω. We observe that the transformation (2.23) maps the internal part of $\partial\tilde{\Omega}_t$ into the Wulff shape of radius $R(t)$; so Γ_0 is mapped into the boundary of a Wulff shape of equal anisotropic perimeter. Moreover, $\tilde{\Omega}_t$ is mapped in the anisotropic annulus of area $A_F(t)$.

Proposition 2.9. *Let Ω be a bounded planar domain of class C^2, then*

$$|R'(t)| \leq 1,$$

where R is defined in (2.23)*.*

Proof. From (2.22) follows that, for almost every $t \in [0, r_F(\Omega)]$ we have

$$R'(t) = -\frac{L_F(t)}{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_F(t)}}. \tag{2.25}$$

In the convex case, using the Steiner formulas

$$|\Omega + t\mathcal{W}| = |\Omega| + P_F(\Omega)t + \kappa t^2;$$
$$P_F(\Omega + t\mathcal{W}) = P_F(\Omega) + 2\kappa t$$

we obtain for almost every $t \in [0, r_F(\Omega)]$

$$L_F(t) \leq L_0^F - 2\kappa t;$$
$$A_F(t) = \int_0^t L_F(v)\, dv \leq L_0^F t - \kappa t^2.$$

Nevertheless, these inequalities hold for bounded domains with C^2 boundary. Therefore,

$$L_F(t)^2 \leq \left(L_0^F\right)^2 - 4\kappa A_F(t),$$

and putting this in (2.25) the thesis follows. □

We obtain this upper bound

$$\lambda_{1,F}(\alpha, \Omega) \leq \inf_{\psi \neq 0} \frac{\displaystyle\int_{r_1}^{r_2} \psi'(r)^2 r\, dr + \alpha\, r_2\, \psi(r_2)^2}{\displaystyle\int_{r_1}^{r_2} \psi(r)^2 r\, dr} =: \mu(\alpha, A^F_{r_1,r_2}),$$

so the infimum is attained for the first eigenfunction of the Finsler Laplacian in $A^F_{r_1,r_2}$, with anisotropic Robin boundary condition on $\partial\mathcal{W}_2$ and anisotropic Neumann boundary conditions on $\partial\mathcal{W}_1$. Therefore we have proved the following proposition.

Proposition 2.10. *Let $\alpha \leq 0$. For any bounded planar domain Ω of class C^2,*

$$\lambda_{1,F}(\alpha; \Omega) \leq \mu(\alpha, A^F_{r_1,r_2}),$$

where $A^F_{r_1,r_2}$ is the anisotropic annulus of the same area as Ω with radii (2.24).

Step 3: from Annuli to Disks

Let $\mathcal{W}_{r_1,r_2}$ be the Wulff shape of the same area as the anisotropic annulus $A^F_{r_1,r_2}$, which has the same area A_0 as Ω. So, we have that

$$r_3 = \sqrt{\frac{A_0}{\kappa}},$$

where r_3 is the radius of $\mathcal{W}_{r_1,r_2}$. In [50] we find the following asymptotics as $\alpha \to 0$:

$$\lambda_{1,F}(\alpha, \mathcal{W}_{r_1,r_2}) = 2\alpha \frac{r_3}{r_3^2} + O(\alpha^2) \quad \text{(Robin Wulff);} \tag{2.26}$$

$$\mu(\alpha, A^F_{r_1,r_2}) = 2\alpha \frac{r_2}{r_3^2} + O(\alpha^2) \quad \text{(Neumann-Robin annulus).} \tag{2.27}$$

Using them we can prove that, for $\alpha < 0$ small enough,

$$\mu(\alpha, A^F_{r_1,r_2}) \leq \lambda_{1,F}(\alpha, \mathcal{W}_{r_1,r_2}), \tag{2.28}$$

where $\mathcal{W}_{r_1,r_2}$ is the Wulff shape of the same area as the anisotropic annulus $A^F_{r_1,r_2}$. Thus, we have proved the following theorem.

Proposition 2.11. *For any bounded domain Ω of class C^2, there exists a negative number $\alpha_0 = \alpha_0(A_0, L_0^F)$ such that*

$$\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, \mathcal{W}^*_\Omega)$$

holds $\forall \alpha \in [\alpha_0, 0]$, where $\mathcal{W}^*_\Omega$ is the Wulff shape of the same area as Ω.

Remark 2.12. Using the above asymptotics we can show that

$$\frac{d}{d\alpha}\lambda_{1,F}(\alpha, \Omega)|_{\alpha=0} = \frac{P_F(\Omega)}{|\Omega|}.$$

Step 4: Uniform Behavior and Conclusion

In order to complete the proof of the Theorem 2.5, it remains only to show the following fact.

Proposition 2.13. *The constant α_0 of Proposition 2.11 is independent of L_0^F.*

Following [50], we need to show that the neighbourhood of zero in which (2.28) does not degenerate in both cases when $r_1 \to 0$ and $r_2 \to +\infty$. So, we are going to prove that α_0 remains bounded away from 0 uniformly in this two instances. We fix $\epsilon > 0$ and we consider

$$r_1 = \sqrt{(2\epsilon r_3 + \epsilon^2)}, \qquad r_2 = r_3 + \epsilon,$$

where r_3 is fixed and equal to $\sqrt{A_0/\kappa}$. In an analogous way to the one reported in [50], it can be proved that there exists $\alpha^* < 0$ such that the curve $\Gamma_A : \alpha \longmapsto \mu(\alpha, A^F_{r_1,r_2})$ stays below the curve $\Gamma_B : \alpha \longmapsto \lambda_{1,F}(\alpha, \mathcal{W}_{r_3})$ for all $\epsilon > 0$ and $\forall \alpha \in]\alpha^*, 0[$. Because of the simplicity of the eigenvalues, both the curves are analytic. Moreover, taking into account the asymptotics (2.26) and (2.27) we have that

$$\frac{d}{d\alpha}\mu(\alpha, \mathcal{W}_{r_1,r_2}) \leq \frac{d}{d\alpha}\lambda_{1,F}(\alpha, A^F_{r_1,r_2}).$$

Remark 2.14. We prove that the curves Γ_A are concave in α. Let $\epsilon > 0$ and let ψ be the first eigenfunction $\mu(\alpha + \epsilon, A^F_{r_1,r_2})$ of the Laplacian in the anisotropic annulus: we can choose ψ such that $\|\psi\|_{L^2(A^F_{r_1,r_2})} = 1$, so we have

$$\mu(\alpha + \epsilon, A^F_{r_1,r_2}) = \int_{r_1}^{r_2} \psi'(r)^2 r\, dr + (\alpha + \epsilon)\, r_2\, \psi(r_2)^2.$$

Let φ be the first eigenfunction $\mu(\alpha, A^F_{r_1,r_2})$ with $\|\varphi\|_{L^2(A^F_{r_1,r_2})} = 1$:

$$\mu(\alpha, A^F_{r_1,r_2}) = \int_{r_1}^{r_2} \phi'(r)^2 r\, dr + \alpha\, r_2\, \phi(r_2)^2.$$

Now, putting ϕ as a test function in the variational formula of $\mu(\alpha + \epsilon, A^F_{r_1,r_2})$ we obtain

$$\mu(\alpha + \epsilon, A^F_{r_1,r_2}) \leq \int_{r_1}^{r_2} \phi'(r)^2 r\, dr + (\alpha + \epsilon)\, r_2\, \phi(r_2)^2 = \mu(\alpha, A^F_{r_1,r_2}) + \epsilon\, r_2\, \phi(r_2)^2.$$

In order to prove our claim, we need only to show that

$$\frac{d}{d\alpha}\mu(\alpha, A^F_{r_1,r_2}) = r_2\, \phi(r_2)^2.$$

We prove the following more general result.

Lemma 2.15. *Let Ω be a bounded domain of $\mathbb{R}^2$ with C^2 boundary and let u_α be an eigenfunction associated to the eigenvalue $\lambda_{1,F}(\alpha, \Omega)$, solution to* (2.21)*, such that $\|u_\alpha\|_{L^2(\Omega)} = 1$. Then,*

$$\lambda'_{1,F}(\alpha, \Omega) := \frac{d\lambda_{1,F}(\alpha, \Omega)}{d\alpha} = \int_{\partial\Omega} u_\alpha^2 F(\nu) d\mathcal{H}^1. \tag{2.29}$$

Proof. From the variational characterization (2.19) and using the fact that $\|u_\alpha\|_{L^2(\Omega)} = 1$ we have

$$\lambda_{1,F}(\alpha, \Omega) = \int_\Omega F^2(\nabla u_\alpha)\, dx + \alpha \int_{\partial\Omega} u_\alpha^2 F(\nu)\, d\mathcal{H}^1. \tag{2.30}$$

Deriving both sides of (2.30) with respect to α, we obtain

$$\lambda'_{1,F}(\alpha,\Omega) = 2\int_\Omega F(\nabla u_\alpha)\left\langle \nabla F(\nabla u_\alpha), \nabla u'_\alpha\right\rangle \, dx + \int_{\partial\Omega} u_\alpha^2 F(\nu)\, d\mathcal{H}^1 \\ + 2\alpha\int_{\partial\Omega} u_\alpha u'_\alpha F(\nu)\, d\mathcal{H}^1. \quad (2.31)$$

Using the weak formulation (2.21) of the problem in the equation (2.31), remembering that u'_α is the derivative with respect to α and it is in the set of the test functions by standard elliptic regularity theory, we obtain

$$\lambda'_{1,F}(\alpha,\Omega) = 2\lambda_{1,F}(\alpha,\Omega)\int_\Omega u_\alpha u'_\alpha \, dx + \int_{\partial\Omega} u_\alpha^2 F(\nu)\, d\mathcal{H}^1, \quad (2.32)$$

and, having in mind that, from the condition $\|u_\alpha\|_{L^2(\Omega)} = 1$,

$$\int_\Omega u_\alpha u'_\alpha \, dx = 0$$

we get, from (2.32), the equation (2.29). □

Therefore, since the Γ_A are concave in α and their derivative with respect to α are increasing with ϵ, we have that the tangent to the curve corresponding to a specific anisotropic annulus intersects Γ_B at one and only one point, α_1, to the left of zero. Thanks to the concavity we can say that, for larger value of ϵ, any Γ_A that intersects Γ_B must do so to the left of α_1.

As far as the case when ϵ is small, we follow closely the proof presented in [50]. We study the intersection points of the two curves Γ_A and Γ_B, comparing the following two equations; the first equation is the equation of the Wulff shape

$$kI_1(kr_3) + \alpha I_0(kr_3) = 0;$$

the second equation is the one of the Neumann-Robin anisotropic annulus

$$K_1(k\sqrt{2\epsilon r_3+\epsilon^2})\left[kI_1\left(k\left(r_3+\epsilon\right)\right) + \alpha I_0\left(k\left(r_3+\epsilon\right)\right)\right] - \\ I_1(k\sqrt{2\epsilon r_3+\epsilon^2})\left[kK_1\left(k\left(r_3+\epsilon\right)\right) - \alpha K_0\left(k\left(r_3+\epsilon\right)\right)\right] = 0.$$

We denote here with I_ν and K_ν the modified Bessel functions (for their properties we refer to [1]). The solution in α of the intersection is given by

$$\alpha = -k\frac{I_1(kr_3)}{I_0(kr_3)}.$$

The proof that there are no intersections between Γ_A and Γ_B for α close to zero is the same as the one presented in [50]. In this way we have proved Proposition 2.13.

2.2.2 Isoperimetric Estimates with a Perimeter Constraint

Now we deal with problem of maximizing $\lambda_{1,F}(\alpha,\Omega)$ under anisotropic perimeter constraint.
Using the method of parallel coordinates we are able to prove also the following theorem.

Theorem 2.16. *Let $\alpha \le 0$ and let $\Omega \subseteq \mathbb{R}^2$ a bounded domain of class C^2. Then*

$$\lambda_{1,F}(\alpha,\Omega) \le \lambda_{1,F}(\alpha,\widetilde{\mathcal{W}_\Omega}),$$

where $\widetilde{\mathcal{W}_\Omega}$ is the Wulff shape with the same anisotropic perimeter as Ω.

The crucial step in order to prove this theorem is given by the following proposition.

Proposition 2.17. *Let* $\alpha < 0$*. For any* $0 < r_1 < r_2$ *we have*

$$\mu(\alpha, A_{r_1,r_2}) \leq \lambda_{1,F}(\alpha, \mathcal{W}_{r_2}).$$

Proof. By symmetry, $\lambda_{1,F}(\alpha, \mathcal{W}_{r_2})$ is the smallest eigenvalue of the following one-dimensional problem

$$\begin{cases} -r^{-(d-1)} & [r^{d-1}\phi'(r)]' = \lambda_{1,F}(\alpha, \mathcal{W}_{r_2})\ \phi(r), r \in [0, r_2] \\ & \phi'(0) = 0 \\ & \phi'(r_2) + \alpha\phi(r_2) = 0. \end{cases} \tag{2.33}$$

We can choose the associated function ϕ_1 to be positive and normalized to 1 and this eigenfunction can be used as a test function. Integrating by parts, we obtain

$$\mu(\alpha, A_{r_1,r_2}) \leq \lambda_{1,F}(\alpha, \mathcal{W}_{r_2}) - r_1\phi(r_1)\phi'(r_1). \tag{2.34}$$

Since ϕ_1 satisfies (2.33), we have for all $r \in [0, r_2]$

$$\left[r\phi_1(r)\phi_1'(r)\right]' = -\lambda_{1,F}(\alpha, \mathcal{W}_{r_2}) r\phi_1(r)^2 + r\phi_1'(r)^2 \geq 0.$$

and the inequality is due to (2.20). From the above inequality the function $g(r) := r\phi(r)\phi'(r)$ is non-decreasing and using (2.34), we obtain the desired result. □

Remark 2.18. The following monotonicity result holds true. Let be $\mathcal{W}_R$ be a Wulff shape of radius R. If $\alpha < 0$, then

$$R \mapsto \lambda_{1,F}(\alpha, \mathcal{W}_R)$$

is strictly increasing. The above result is proven for the disks in [7].

Proof of Theorem 2.16. Firstly, we observe that the measure of $\mathcal{W}_{r_2}$ is greater than the measure of $A^F_{r_1,r_2}$ and the anisotropic perimeter of $\mathcal{W}_{r_2}$, which is equal to L^F_0, is less than the anisotropic perimeter of $A^F_{r_1,r_2}$. Using Theorem 2.10 and Proposition 2.17 we obtain the thesis for simply connected domains, i.e. when $L^F_0 = P_F(\Omega)$. Concerning the general case, when there are multiple connected domains, thanks to Remark 2.18, we have that

$$\lambda_{1,F}(\alpha, \mathcal{W}_{r_2}) \leq \lambda_{1,F}(\alpha, \mathcal{W}_{r_3}),$$

where $r_3 = P_F(\Omega)/2\kappa$ for all $\alpha \leq 0$. □

Chapter 3

Eigenvalue problems for p-Laplacian type operators with Robin boundary conditions

In this chapter we study some Robin eigenvalue problems for the classical p-Laplacian

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

and its anisotropic version, defined a (3.2).

3.1 On the first Robin eigenvalue of a class of anisotropic operators

3.1.1 The Robin eigenvalue problem of $\mathcal{Q}_p$

In this section we study the following eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \ell |u|^{p-2} u & \text{in } \Omega \\ F^{p-1}(\nabla u)\left\langle \nabla F(\nabla u), \nu \right\rangle + \beta(x) F(\nu)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$\mathcal{Q}_p u := \operatorname{div}\left(\frac{1}{p}\nabla[F^p](\nabla u)\right) \tag{3.2}$$

is the anisotropic p-Laplacian, Ω is a bounded open connected subset of $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary, F is a Finsler as in Section 1.2, ν is the Euclidean unit outward normal to $\partial\Omega$ and the function $\beta : \partial\Omega \to [0, +\infty[$ belongs to $L^1(\partial\Omega)$ and verifies

$$\int_{\partial\Omega} \beta(x) F(\nu)\, d\mathcal{H}^{n-1} = m > 0. \tag{3.3}$$

We get a mononicity result for the first eigenvalue of the problem (3.1), that we denote by $\ell_1(\beta, \cdot)$, when β is a constant, among bounded open connected sets Ω_1 and Ω_2, with convex Ω_2 such that $\Omega_1 \subset \mathcal{W}_R \subset \Omega_2$, where $\mathcal{W}_R$ is a Wulff shape. Furthermore, we prove a representation formula for $\ell_1(\beta, \Omega)$ and from that we obtain a Faber-Krahn type inequality and a Cheeger type inequality.
Firstly, we stress that, the assumption (1.3) on F ensures that the operator $\mathcal{Q}_p$ is elliptic, hence there exists a positive constant γ such that

$$\frac{1}{p}\sum_{i,j=1}^{n} \nabla^2_{\xi_i\xi_j}[F^p](\eta)\xi_i\xi_j \geq \gamma |\eta|^{p-2}|\xi|^2,$$

for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$. For $p = 2$, $\mathcal{Q}_2$ is the so-called Finsler Laplacian, and when $F(\xi) = |\xi| = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean norm, $\mathcal{Q}_p$ reduces to the well known p-Laplace operator (see, for instance [64]).
From now on in this section we will write $\bar{\beta}$ instead of β when β is a positive constant.

Definition 3.1. *A function $u \in W^{1,p}(\Omega)$, $u \not\equiv 0$ is an eigenfunction to* (3.1) *if $\beta(\cdot)|u|^p \in L^1(\partial\Omega)$ and*

$$\int_\Omega F^{p-1}(\nabla v)\,\langle \nabla F(\nabla v), \nabla\varphi\rangle\, dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}u\varphi F(\nu) d\mathcal{H}^{n-1} = \ell \int_\Omega |u|^{p-2}u\varphi dx \quad (3.4)$$

for any test function $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\partial\Omega)$. The corresponding number ℓ, is called Robin eigenvalue.

The smallest eigenvalue of (3.1), $\ell_1(\beta,\Omega)$ has the following variational characterization

$$\begin{aligned}\ell_1(\beta,\Omega) &= \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} J[\beta, v] \\ &= \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega F^p(\nabla v)dx + \int_{\partial\Omega} \beta(x)|v|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v|^p dx}. \quad (3.5)\end{aligned}$$

By definition we have

$$\ell_1(\beta,\Omega) \le \lambda_D(\Omega),$$

where $\lambda_D(\Omega)$ is the first Dirichlet eigenvalue of $\mathcal{Q}_p$. Indeed choosing as test function in (3.5), the first Dirichlet eigenfunction u_D of $\lambda_D(\Omega)$ in the Rayleigh quotient, we get

$$\begin{aligned}\ell_1(\beta,\Omega) &= \min_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_\Omega [F(\nabla v)]^p dx + \int_{\partial\Omega} \beta|v|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v|^p dx} \\ &\le \frac{\displaystyle\int_\Omega [F(\nabla u_D)]^p dx + \int_{\partial\Omega} \beta|u_D|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u_D|^p dx} = \frac{\displaystyle\int_\Omega [F(\nabla u_D)]^p dx}{\displaystyle\int_\Omega |u_D|^p\, dx} = \lambda_D(\Omega).\end{aligned}$$

The following existence result holds.

Proposition 3.2. *Let $\beta \in L^1(\partial\Omega)$, $\beta \ge 0$ be such that* (3.3) *holds. Then there exists a positive minimizer $u \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ of* (3.5) *which is a weak solution to* (3.1) *in Ω with $\ell = \ell_1(\beta,\Omega)$. Moreover $\ell_1(\beta,\Omega)$ is positive and it is simple, that is the relative eigenfunction u is unique up to a multiplicative constant.*

Proof. Let $u_k \in W^{1,p}(\Omega)$ be a minimizing sequence of (3.5) such that $\|u_k\|_{L^p(\Omega)} = 1$. Then, being u_k bounded in $W^{1,p}(\Omega)$ there exists a subsequence, still denoted by u_k and a function $u \in W^{1,p}(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$, such that $u_k \to u$ strongly in $L^p(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. Then u_k converges to u in $L^p(\partial\Omega)$ and then almost everywhere on $\partial\Omega$ to u up to subsequences. Then by weak lower semicontinuity and Fatou's lemma we get

$$\ell_1(\beta,\Omega) = \lim_{k\to+\infty} J[\beta, u_k] \ge J[\beta, u],$$

then $\beta(\cdot)|u|^p \in L^1(\partial\Omega)$ and u is an eigenfunction related to $\ell_1(\beta,\Omega)$ by definition. Moreover $u \in L^\infty(\Omega)$. To see that, we can argue exactly as in [37] to get that $u \in L^\infty(\Omega)$. Now the L^∞-estimate, the assumption (1.3) and the properties of F allow to apply standard regularity results (see [45], [77]), and obtain that $u \in C^{1,\alpha}(\Omega)$. In order to prove that $\ell_1(\beta,\Omega) > 0$, we proceed by contradiction supposing that there exists β_o which verifies (3.3) and such that $\ell_1(\beta_o,\Omega) = 0$. Then there exists $u_{\beta_o} \in C^{1,\alpha}(\Omega) \cap L^\infty(\overline{\Omega})$ such that $u_{\beta_o} \geq 0$, $\|u_{\beta_o}\|_{L^p(\Omega)} = 1$ and

$$0 = \ell_1(\beta_o,\Omega) = \int_\Omega F^p(\nabla u_{\beta_o})dx + \int_{\partial\Omega} \beta_o\, u^p_{\beta_o}\, F(\nu) d\mathcal{H}^{n-1}.$$

Then u_{β_o} has to be constant in $\overline{\Omega}$ and then $u^p_{\beta_o} \displaystyle\int_{\partial\Omega} \beta_o F(\nu) = u^p_{\beta_o} m = 0$. Being $m > 0$, then $u_{\beta_o} = 0$ in $\overline{\Omega}$, and this is not true. Hence $\ell_1(\beta_o,\Omega) > 0$.

Finally to prove the simplicity of the eigenfunctions we can proceed exactly as in [37]. For completeness we recall the main steps. Let u, w be positive minimizers of the functional $J[\beta,\cdot]$ defined in (3.5) such that $\|u\|_{L^p(\Omega)} = \|w\|_{L^p(\Omega)} = 1$, and let us consider the function $\eta_t = (tu^p + (1-t)w^p)^{1/p}$, with $t \in [0,1]$. Obviously, $\|\eta_t\|_{L^p(\Omega)} = 1$. Clearly it holds:

$$J[\beta,u] = \ell_1(\beta,\Omega) = J[\beta,w]. \tag{3.6}$$

In order to compute $J[\beta,\eta_t]$ we observe that by using the homogeneity and the convexity of F it is not hard to prove that (see for instance [14, 21, 37] for the precise computation)

$$F^p(\nabla\eta_t) \leq tF^p(\nabla v) + (1-t)F^p(\nabla w). \tag{3.7}$$

Hence recalling (3.6), we obtain

$$J[\beta,\eta_t] \leq tJ[\beta,u] + (1-t)J[\beta,w] = \ell_1(\beta,\Omega),$$

and then η_t is a minimizer for J. This implies that the equality holds in (3.7). Thence, uniqueness follows, arguing e.g. as in [37]. □

The following result characterizes the first eigenfunctions.

Proposition 3.3. *Let $\beta \in L^1(\partial\Omega)$, $\beta \geq 0$ be such that* (3.3) *holds. Let $\eta > 0$ and $v \in W^{1,p}(\Omega)$, $v \not\equiv 0$ and $v \geq 0$ in Ω such that*

$$\begin{cases} -\mathcal{Q}_p v = \eta v^{p-1} & \text{in } \Omega \\ F^{p-1}(\nabla v)\,\langle \nabla F(\nabla v), \nu\rangle + \beta F(\nu) v^{p-1} = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Definition 3.1. Then v is a first eigenfunction of (3.1), *and $\eta = \ell_1(\beta,\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$ be a positive eigenfunction related to $\ell_1(\beta,\Omega)$. Choosing $u^p/(v+\varepsilon)^{p-1}$, with $\varepsilon > 0$, as test function in the Definition 3.1 for the solution v, and arguing exactly as in [37], we get the claim. □

Remark 3.4. We observe that Propositions 3.2 and 3.3 generalize the results proved respectively in [42] for the Euclidean norm and in [37] when $\beta(x) = \beta$ is a positive constant.

Theorem 3.5. *Let $\beta \in L^1(\partial\Omega)$, $\beta \geq 0$ and such that* (3.3) *holds. The following properties hold for $\ell_1(\beta,\Omega)$*

(i) $\forall t > 0,\ \ell_1(\beta(\frac{x}{t}), t\Omega) = t^{-p}\ell_1(t^{p-1}\beta(y),\Omega), \quad x \in \partial(t\Omega), y \in \partial\Omega\ ;$

(ii) $\ell_1(\beta,\Omega) \le \frac{m}{|\Omega|}$;

(iii) $a^p \ell_{\mathcal{E}}(a^{1-p}\beta,\Omega) \le \ell_1(\beta,\Omega) \le b^p \ell_{\mathcal{E}}(b^{1-p}\beta,\Omega)$,
where a, b are defined in (1.2) *and* $\ell_{\mathcal{E}}(a^{1-p}\beta,\Omega)$, $\ell_{\mathcal{E}}(b^{1-p}\beta,\Omega)$ *are the first Robin eigenvalue for the Euclidean p-Laplacian corresponding respectively to the function* $a^{1-p}\beta$ *and* $b^{1-p}\beta$*;*

(iv) *If* $\beta(x) \ge \bar\beta > 0$, *for almost* $x \in \partial\Omega$, *then*

$$\sup_{|\Omega|=k} \ell_1(\beta,\Omega) = +\infty$$

Proof. By the homogeneity of F, we have:

$$\ell_1\left(\beta\left(\frac{x}{t}\right), t\Omega\right) =$$

$$\min_{\substack{\varphi\in W^{1,p}(t\Omega)\\ \varphi\neq 0}} \frac{\displaystyle\int_{t\Omega} F^p(\nabla\varphi(x))dx + \int_{\partial(t\Omega)} \beta\left(\frac{x}{t}\right)|\varphi(x)|^p F(\nu(x))d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{t\Omega}|\varphi(x)|^p dx} =$$

$$\min_{\substack{v\in W^{1,p}(\Omega)\\ v\neq 0}} \frac{\displaystyle t^{-p}\int_{t\Omega} F^p\left(\nabla v\left(\frac{x}{t}\right)\right)dy + \int_{\partial(t\Omega)} \beta\left(\frac{x}{t}\right)|v\left(\frac{x}{t}\right)|^p F\left(\nu\left(\frac{x}{t}\right)\right)d\mathcal{H}^{n-1}(y)}{\displaystyle t^n\int_{\Omega}|v(y)|^p dy}$$

$$= \min_{\substack{v\in W^{1,p}(\Omega)\\ v\neq 0}} \frac{\displaystyle t^{n-p}\int_{\Omega} F^p(\nabla v(y))dy + t^{n-1}\int_{\partial\Omega} \beta(y)|v(y)|^p F(\nu(y))d\mathcal{H}^{n-1}(y)}{\displaystyle t^n\int_{\Omega}|v(y)|^p dy} =$$

$$t^{-p}\ell_1(t^{p-1}\beta(y),\Omega).$$

In order to obtain the second property, it is sufficient to consider a non-zero constant as test function in (3.5).

Now we prove the inequality in the right-hand side in *(iii)*. The proof of the other inequality is similar. By using (3.5) and (1.2), we obtain that

$$\ell_1(\beta,\Omega) = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_{\Omega} F^p(\nabla v)dx + \int_{\partial\Omega}\beta(x)|v|^p F(\nu)d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega}|v|^p dx} \le$$

$$\inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} b^p \frac{\displaystyle\int_{\Omega}|\nabla v|^p dx + b^{1-p}\int_{\partial\Omega}\beta(x)|v|^p d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega}|v|^p dx} =$$

$$b^p \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} \frac{\displaystyle\int_{\Omega}|\nabla v|^p dx + \int_{\partial\Omega} b^{1-p}\beta(x)|v|^p d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega}|v|^p dx} = b^p\ell_{\mathcal{E}}(b^{1-p}\beta,\Omega),$$

where last equality follows, by definition of $\ell_{\mathcal{E}}(b^{1-p}\beta,\Omega)$.

Finally we give the proof of (iv). Clearly $\ell_1(\beta,\Omega)\geq\ell_1(\bar\beta,\Omega)$, then by [37, Proposition 3.1], we know that

$$\ell_1(\bar\beta,\Omega)\geq\left(\frac{p-1}{p}\right)^p\frac{\bar\beta}{r_F(\Omega)\left(1+\bar\beta^{\frac{1}{p-1}}r_F(\Omega)\right)} \tag{3.8}$$

where $r_F(\Omega)$ is the anisotropic inradius of the subset Ω . The claim follows constructing a sequence of convex sets Ω_k with $|\Omega_k|=1$ and such that $r_F(\Omega_k)\to 0$, for $k\to\infty$. Let $k>0$, proceeding as in [36, 41], it is possible to consider the $n-$rectangles $\Omega_k=\left]-\frac{1}{2k},\frac{1}{2k}\right[\times\left]-\frac{k^{\frac{1}{n-1}}}{2},\frac{k^{\frac{1}{n-1}}}{2}\right[^{n-1}$ and suppose that $r_F(\Omega_k)=\dfrac{1}{2k}F^o(e_1)$. Then we obtain

$$\ell_1(\bar\beta,\Omega_k)\geq\left(\frac{p-1}{p}\right)^p\frac{4k^2\bar\beta}{F^o(e_1)\left(2k+\bar\beta^{\frac{1}{p-1}}F^o(e_1)\right)}\to+\infty \text{ for } k\to\infty.$$

□

3.1.2 The anisotropic radial case

In this paragraph we recall some properties of the first eigenvalue of $\mathcal{Q}_p$ with Robin boundary condition when Ω is a Wulff shape. We suppose that $\beta=\bar\beta$ is a positive constant then we consider

$$\ell_1\left(\bar\beta,\mathcal{W}_R\right)=\min_{\substack{v\in W^{1,p}(\mathcal{W}_R)\\ v\not\equiv 0}}J\left[\bar\beta,v\right]=\min_{\substack{v\in W^{1,p}(\mathcal{W}_R)\\ v\not\equiv 0}}\frac{\displaystyle\int_{\mathcal{W}_R}[F(\nabla v)]^p dx+\bar\beta\int_{\partial\mathcal{W}_R}|v|^pF(\nu)d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathcal{W}_R}|v|^p dx}, \tag{3.9}$$

where $\mathcal{W}_R=R\,\mathcal{W}=\{x\colon F^o(x)<R\}$, with $R>0$, and $\mathcal{W}$ is the Wulff shape centered at the origin.

By Proposition 3.2, the minimizers of (3.9) solve the following problem:

$$\begin{cases}-\mathcal{Q}_p v=\ell_1\left(\bar\beta,\mathcal{W}_R\right)|v|^{p-2}v & \text{in } \mathcal{W}_R,\\ F^{p-1}(\nabla v)\left\langle\nabla F(\nabla v),\nu\right\rangle+\bar\beta F(\nu)|v|^{p-2}v=0 & \text{on } \partial\mathcal{W}_R.\end{cases} \tag{3.10}$$

In [23, 35, 37] the authors prove the following result

Theorem 3.6. *Let $v_p\in C^{1,\alpha}(\mathcal{W}_R)\cap C(\overline{\mathcal{W}_R})$ be a positive solution to problem* (3.10)*. Then $v_p(x)=\rho_p(F^o(x))$, with $x\in\overline{\mathcal{W}}_R$, where $\rho_p(r)$, $r\in[0,R]$, is a decreasing function such that $\rho_p\in C^\infty(0,R)\cap C^1([0,R])$ and it verifies*

$$\begin{cases}-(p-1)(-\rho_p'(r))^{p-2}\rho_p''(r)+\dfrac{n-1}{r}(-\rho_p'(r))^{p-1}=\ell_1(\bar\beta,\mathcal{W}_R)\rho_p(r)^{p-1},\ r\in]0,R[,\\ \rho_p'(0)=0,\\ -(-\rho_p'(R))^{p-1}+\bar\beta(\rho_p(R))^{p-1}=0.\end{cases} \tag{3.11}$$

Remark 3.7. We observe that the first eigenvalue in the Wulff $\mathcal{W}_R=\{F^o(x)<R\}$ is the same for any norm F. In particular it coincides with the first Robin eigenvalue in the Euclidean ball B_R for the p-Laplace operator. Finally we emphasize that in this case the eigenfunctions have more regularity because $\bar\beta$ is a positive constant.

Theorem 3.6, as in [23, 37], suggests to consider , for every $x \in \mathcal{W}_R$, the following function

$$f(r_x) = \frac{(-\rho_p'(r_x))^{p-1}}{(\rho_p(r_x))^{p-1}} = \frac{\big[F\big(\nabla v_p(x)\big)\big]^{p-1}}{v_p(x)^{p-1}} = \frac{\big[F\big(\nabla v_p(x)\big)\big]^{p-1}\big\langle \nabla F\big(\nabla v_p(x)\big), \nu\big\rangle}{v_p(x)^{p-1}F(\nu)}, \quad (3.12)$$

where

$$r_x = F^o(x), \quad 0 \le r_x \le R.$$

Let us observe that f is nonnegative, $f(0) = 0$ and

$$f(R) = \frac{(-\rho_p'(R))^{p-1}}{(\rho_p(R))^{p-1}} = \frac{\big[F\big(\nabla v_p(x)\big)\big]^{p-1}}{v_p(x)^{p-1}} = \frac{\big[F\big(\nabla v_p(x)\big)\big]^{p-1}\big\langle \nabla F\big(\nabla v_p(x)\big), \nu\big\rangle}{v_p(x)^{p-1}F(\nu)} = \bar{\beta}$$

where $x \in \partial\mathcal{W}_R$.
The following result proved in the Euclidean case in [23] and in [37] in the anisotropic case, states that the first Robin eigenvalue is monotone decreasing with respect the set inclusion in the class of Wulff shapes.

Lemma 3.8. *The function $r \to \ell_1\left(\bar{\beta}, \mathcal{W}_r\right)$ is strictly decreasing in $]0,\infty[$.*

In [23] and [37] the authors prove also the following monotonicity property for the function f defined in (3.12).

Lemma 3.9. *Let f be the function defined in* (3.12)*. Then $f(r)$ is strictly increasing in $[0, R]$.*

In the next result we prove a convex property for the function f.

Theorem 3.10. *Let f be the function defined in* (3.12)*. Then the function*

$$g(z) = f(z^{\frac{1}{n}})z^{1-\frac{1}{n}}, \qquad 0 \le z \le R^n,$$

is convex with respect to z.

Proof. We first observe that by (3.11) it holds

$$\begin{aligned} f'(r) = \frac{d}{dr}\left(\frac{-\rho_p'(r)}{\rho_p(r)}\right)^{p-1} &= (p-1)f^{\frac{p-2}{p-1}}\left(\frac{-\rho_p''}{\rho_p} + \left(\frac{\rho_p'}{\rho_p}\right)^2\right) \\ &= \ell_1(\bar{\beta}, \mathcal{W}_R) - \frac{(n-1)}{r}f + (p-1)f^{\frac{p}{p-1}} \quad \forall r \in]0, R[. \quad (3.13) \end{aligned}$$

Then

$$\begin{aligned} g'(z) &= \frac{1}{n}f'(z^{\frac{1}{n}}) + \frac{(n-1)}{n}\frac{f(z^{\frac{1}{n}})}{z^{\frac{1}{n}}} \\ &= \frac{1}{n}\left(\ell_1(\bar{\beta}, \mathcal{W}_R) - \frac{(n-1)}{z^{\frac{1}{n}}}f(z^{\frac{1}{n}}) + (p-1)f^{\frac{p}{p-1}}(z^{\frac{1}{n}})\right) + \frac{(n-1)}{n}\frac{f(z^{\frac{1}{n}})}{z^{\frac{1}{n}}} \\ &= \frac{\ell_1(\bar{\beta}, \mathcal{W}_R)}{n} + \frac{(p-1)}{n}f^{\frac{p}{p-1}}(z^{\frac{1}{n}}), \end{aligned}$$

which is increasing and this implies the thesis. $\square$

Finally the following comparison result for f holds

Theorem 3.11. *Let f be the function defined in* (3.12). *Then there exists a positive constant $C = C(R)$ such that*

$$f(r) \leq r\, C(R), \quad \textit{for} \;\; 0 \leq r \leq R$$

Proof. By (3.13) and by Lemma 3.9 we obtain that f verifies the following equation

$$f'(r) = \ell_1(\bar{\beta}, \mathcal{W}_R) - \frac{(n-1)}{r} f(r) + (p-1) f^{\frac{p}{p-1}}(r) \leq C(R) - \frac{(n-1)}{r} f(r) \tag{3.14}$$

where

$$C(R) = \ell_1(\bar{\beta}, \mathcal{W}_R) + (p-1) f(R)^{\frac{p}{p-1}} = \ell_1(\bar{\beta}, \mathcal{W}_R) + (p-1)\bar{\beta}^{\frac{p}{p-1}} \tag{3.15}$$

Then by (3.14) multiplying both sides by r^{n-1} we get

$$f'(r) r^{n-1} + (n-1) r^{n-2} f(r) \leq C(R) r^{n-1},$$

and

$$\frac{d}{dr}\left(r^{n-1} f(r)\right) \leq C(R) r^{n-1}$$

Then the claim follows integrating both sides between 0 and r. □

Remark 3.12. The results contained in Lemma 3.9 and Theorem 3.10 ensures that $f(r)$ is an admissible weight for the weighted anisotropic isoperimetric inequality quoted in (1.5)

3.1.3 A monotonicity property for $\ell_1\left(\bar{\beta}; \Omega\right)$

In this paragraph we assume that $\beta = \bar{\beta}$ is a positive constant. The first Robin eigenvalue $\ell_1(\bar{\beta}, \Omega)$ has not, in general, a monotonicity property with respect the set inclusion. For instance in [34] in the Euclidean case, for the Laplace operator, the authors give a counterexample. More precisely, they construct a suitable sequence of sets Ω_k such that $P(\Omega_k) \to \infty$, $B_1(0) \subset \Omega_k \subset B_{1+\varepsilon}(0)$ which verify

$$\ell_{\mathcal{E}}(\bar{\beta}, \Omega_k) > \ell_{\mathcal{E}}(\bar{\beta}, B_1(0)) > \ell_{\mathcal{E}}(\bar{\beta}, B_2(0)).$$

Here $B_r(x_0)$ denotes the Euclidean ball with radius r and centered at the point x_0 and $\ell_{\mathcal{E}}(\bar{\beta}, B_r(x_0))$ is the Euclidean first Robin eigenvalue of the Laplacian of the ball $B_r(x_0)$. In what follows we prove a monotonicity type property for the first Robin eigenvalue of the operator $\mathcal{Q}_p$ with respect the set inclusion. In the Euclidean case for the Laplace operator we refer the reader for instance to [57].

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with $C^{1,\alpha}$ boundary, $\alpha \in]0,1[$. Let $\mathcal{W}_R$ be a Wulff shape such that $\Omega \subset \mathcal{W}_R$ and $\bar{\beta}$ a positive constant. Then*

$$\ell_1(\bar{\beta}, \mathcal{W}_R) \leq \ell_1(\bar{\beta}, \Omega).$$

Proof. Let v_p be the positive eigenfunction associated to $\ell_1(\bar{\beta}, \mathcal{W}_R)$ and let Ω be a subset of $\mathcal{W}_R$.

Then for every $x \in \partial\Omega$, we can consider $f(r_x)$ as in (3.12) in order to get that the following Robin boundary condition on $\partial\Omega$ holds

$$\left[F\left(\nabla v_p(x)\right)\right]^{p-1} \left\langle \nabla F\left(\nabla v_p(x)\right), \nu \right\rangle + f(r_x) v_p(x)^{p-1} F(\nu) = 0. \tag{3.16}$$

Having in mind that $\Omega \subset \mathcal{W}_R$ and using (3.16), we have that v_p solves the following problem

$$\begin{cases} -\mathcal{Q}_p v_p = \ell_1(\bar{\beta}, \mathcal{W}_R) v_p^{p-1} & \text{in } \Omega \\ \left[F\left(\nabla v_p\right)\right]^{p-1} \left\langle \nabla F\left(\nabla v_p\right), \nu \right\rangle + f(r_x) {v_p}^{p-1} F(\nu) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.17}$$

Using (3.17) and Lemma 3.9

$$
\begin{aligned}
\ell_1(\bar\beta,\mathcal{W}_R) &= \frac{\displaystyle\int_\Omega [F(\nabla v_p)]^p dx + \int_{\partial\Omega} f(r_x)|v_p|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v_p|^p dx} \\
&= \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [F(\nabla u)]^p dx + \int_{\partial\Omega} f(r_x)|u|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx} \\
&\le \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [F(\nabla u)]^p dx + \int_{\partial\Omega} \bar\beta|u|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx} \\
&= \ell_1(\bar\beta,\Omega)
\end{aligned}
$$

□

When Ω contains a Wulff shape we have the following result

Theorem 3.14. *Let $\Omega\subset\mathbb{R}^n$ be a bounded and convex open set with $C^{1,\alpha}$ boundary, $\alpha\in]0,1[$. Let $\mathcal{W}_R$ be a Wulff shape such that $\mathcal{W}_R\subset\Omega$, then*

$$\ell_1(\bar\beta,\Omega)\le\ell_1(\bar\beta,\mathcal{W}_R).$$

Proof. First of all, we take the positive eigenfunction v_p associated to $\ell_1(\bar\beta,\mathcal{W}_R)$. By Theorem 3.6 $v_p(x)=\varrho_p(F^o(x))$, and by (3.11) we can extend ϱ_p up to $+\infty$ and then v_p in $\mathbb{R}^n$. Let us consider the super-level set

$$\mathcal{W}_+=\{x\in\mathbb{R}^n\colon v_p(x)>0\}.$$

By the property of v_p, $\mathcal{W}_+$ is a Wulff shape and clearly $\mathcal{W}_R\subset\mathcal{W}_+$.

Moreover, v_p solves the following equation

$$-\mathcal{Q}_p v_p=\ell_1(\bar\beta,\mathcal{W}_R)v_p^{p-1} \text{ in } \mathcal{W}_+.$$

To prove the theorem we consider the set $\tilde\Omega=\Omega\cap\mathcal{W}_+$. Being Ω convex and due to the radially decreasing of the eigenfunction, three possible cases can occur.

Case 1. $\partial\tilde\Omega=\partial\Omega$. Then in this case $\mathcal{W}_R\subset\Omega\subset\mathcal{W}_+$ and $\tilde\Omega=\Omega$. Then for $x\in\partial\Omega$ we put $r_x=F^o(x)$ and we can compute

$$f(r_x)=\frac{(-\rho_p'(r_x))^{p-1}}{(\rho_p(r_x))^{p-1}}.$$

Then arguing as in the proof of Theorem 3.13 and recalling that by Lemma 3.9, $f(r_x)\ge\bar\beta$, for any $x \in \partial\Omega$ we get

$$\begin{aligned}
\ell_1(\bar\beta, \mathcal{W}_R) &= \frac{\displaystyle\int_\Omega [F(\nabla v_p)]^p dx + \int_{\partial\Omega} f(r_x)|v_p|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |v_p|^p dx} \\
&= \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [F(\nabla u)]^p dx + \int_{\partial\Omega} f(r_x)|u|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx} \\
&\geq \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [F(\nabla u)]^p dx + \int_{\partial\Omega} \bar\beta|u|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx} \\
&= \ell_1(\bar\beta, \Omega)
\end{aligned}$$

and the first case is proved.

Case 2. $\partial\tilde\Omega \cap \partial\Omega \neq \emptyset$ and $\partial\tilde\Omega \cap \partial\Omega \neq \partial\Omega$. Then $\partial\tilde\Omega \cap \mathcal{W}_+ \neq \emptyset$. Moreover, on $\partial\tilde\Omega \cap \partial\Omega$ the eigenfunction v_p is positive, while on $\partial\tilde\Omega \cap \partial\mathcal{W}_+$ it is equal to zero. In particular, for every $x \in \partial\tilde\Omega \cap \partial\Omega$ we still have that $f(r_x) \geq \bar\beta$ as in the *Case 1*. We define the following test function $\varphi \in W^{1,p}(\Omega)$

$$\varphi(x) = \begin{cases} v_p(x) & \text{in } \tilde\Omega \\ 0 & \text{in } \Omega \setminus \tilde\Omega. \end{cases}$$

Then

$$\begin{aligned}
\ell_1(\bar\beta, \mathcal{W}_R) &= \frac{\displaystyle\int_{\tilde\Omega} [F(\nabla v_p)]^p dx + \int_{\partial\tilde\Omega\cap\partial\Omega} f(r_x) v_p^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_{\tilde\Omega} v_p^p dx} \\
&= \frac{\displaystyle\int_\Omega [F(\nabla\varphi)]^p dx + \int_{\partial\tilde\Omega\cap\partial\Omega} f(r_x)\varphi^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^p dx} \\
&\geq \frac{\displaystyle\int_\Omega [F(\nabla\varphi)]^p dx + \int_{\partial\tilde\Omega\cap\partial\Omega} \bar\beta\varphi^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^p dx} \\
&= \frac{\displaystyle\int_\Omega [F(\nabla v)]^p dx + \int_{\partial\Omega} \bar\beta\varphi^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^p dx} \\
&\geq \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [F(\nabla u)]^p dx + \int_{\partial\Omega} \bar\beta|u|^p F(\nu) d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx} \\
&= \ell_1(\bar\beta, \Omega)
\end{aligned}$$

and the second case is proved.

Case 3. $\partial\tilde{\Omega} \cap \partial\Omega = \emptyset$. Then $\tilde{\Omega} = \mathcal{W}_+ \subset \Omega$. Using the monotonicity result in Lemma 3.8 we obtain that $\ell_1(\bar{\beta}, \mathcal{W}_R) \geq \ell_1(\bar{\beta}, \mathcal{W}_+)$. Denoting with $v_p^{(1)}$ the eigenfunction associated to $\ell_1(\bar{\beta}, \mathcal{W}_+)$ and defining $\tilde{\Omega}^{(1)} = \Omega \cap \{v_p^{(1)}(x) > 0\}$ and repeating the division in three possible cases, after a finite number of steps we could be either in *Case 1* or in *Case 2*. □

By Theorems 3.13 and 3.14 we get the following monotonicity property for ℓ_1 for constant β.

Corollary 3.15. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be two bounded open connected set with $C^{1,\alpha}$ boundary and let Ω_2 be a convex set. Let $\mathcal{W}_R$ be a Wulff shape such that $\Omega_1 \subset \mathcal{W}_R \subset \Omega_2$. Then $\ell_1(\bar{\beta}, \Omega_2) \leq \ell_1(\bar{\beta}, \Omega_1)$.*

3.1.4 A representation formula for $\ell_1(\beta, \Omega)$

We are interested to prove a level set representation formula for the first eigenvalue $\ell_1(\beta, \Omega)$ of the following problem

$$\begin{cases} -\mathcal{Q}_p v = \ell |v|^{p-2} v & \text{in } \Omega \\ F^{p-1}(\nabla v)\, \langle \nabla F(\nabla v), \nu \rangle + \beta F(\nu) |v|^{p-2} v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

When $\beta = \bar{\beta}$ is a nonnegative constant, a similar result can be found in [23] in the Euclidean case and in [37] for the anisotropic case. Our aim is to extend the known results assuming that β is in general a nonnegative function defined on $\partial\Omega$. In the next we will use the following notation. Let $\tilde{u}_p$ be the first positive eigenfunction such that $\max \tilde{u}_p = 1$. Then, for $t \in [0, 1]$,

$$\begin{aligned} U_t &= \{x \in \Omega \colon \tilde{u}_p > t\}, \\ S_t &= \{x \in \Omega \colon \tilde{u}_p = t\}, \\ \Gamma_t &= \{x \in \partial\Omega \colon \tilde{u}_p > t\}. \end{aligned}$$

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with $C^{1,\alpha}$ boundary and let $\alpha \in]0,1[$. Let β be a function belonging to $L^1(\partial\Omega)$, $\beta \geq 0$ and such that* (3.3) *holds. Let $\tilde{u}_p \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ be a positive minimizer of* (3.5) *with $\|\tilde{u}_p\|_\infty = 1$. Then for a.e. $t \in]0,1[$ the following representation formula holds*

$$\ell_1(\beta, \Omega) = \mathcal{F}_\Omega \left(U_t, \frac{[F(\nabla \tilde{u}_p)]^{p-1}}{\tilde{u}_p^{p-1}} \right), \tag{3.19}$$

where $\mathcal{F}_\Omega$ is defined as

$$\mathcal{F}_\Omega(U_t, \varphi) = \frac{1}{|U_t|} \left(-(p-1) \int_{U_t} \varphi^{p'}\, dx + \int_{S_t} \varphi F(\nu)\, d\mathcal{H}^{n-1} + \int_{\Gamma_t} \beta F(\nu)\, d\mathcal{H}^{n-1} \right). \tag{3.20}$$

Proof. Let $0 < \varepsilon < t < 1$ and we define

$$\psi_\varepsilon = \begin{cases} 0 & \text{if } \tilde{u}_p \leq t \\ \dfrac{\tilde{u}_p - t}{\varepsilon} \dfrac{1}{\tilde{u}_p^{p-1}} & \text{if } t < \tilde{u}_p < t + \varepsilon \\ \dfrac{1}{\tilde{u}_p^{p-1}} & \text{if } \tilde{u}_p \geq t + \varepsilon. \end{cases}$$

The functions ψ_ε belong to $W^{1,p}(\Omega)$ and their pointwise limit as $\varepsilon \to 0$ is $\tilde{u}_p^{-(p-1)}\chi_{U_t}$. Moreover, it can be proved that

$$\nabla\psi_\varepsilon = \begin{cases} 0 & \text{if } \tilde{u}_p < t \\ \dfrac{1}{\varepsilon}\left((p-1)\dfrac{t}{\tilde{u}_p} + 2 - p\right)\dfrac{\nabla\tilde{u}_p}{\tilde{u}_p^{p-1}} & \text{if } t < \tilde{u}_p < t+\varepsilon \\ -(p-1)\dfrac{\nabla\tilde{u}_p}{\tilde{u}_p^p} & \text{if } \tilde{u}_p > t+\varepsilon. \end{cases}$$

Then choosing ψ_ε as test function in (3.4), we get that the first integral is

$$\begin{aligned} -(p-1)\int_{U_{t+\varepsilon}} \frac{[F(\nabla\tilde{u}_p)]^p}{\tilde{u}_p^p}dx + \frac{1}{\varepsilon}\int_{U_t\setminus U_{t+\varepsilon}} \frac{[F(\nabla\tilde{u}_p)]^p}{\tilde{u}_p^{p-1}}\left((p-1)\frac{t}{\tilde{u}_p} + 2 - p\right)dx \\ = -(p-1)\int_{U_{t+\varepsilon}} \frac{[F(\nabla\tilde{u}_p)]^p}{\tilde{u}_p^p}dx + \\ + \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left((p-1)\frac{t}{\tau} + 2 - p\right)\int_{S_\tau}\frac{[F(\nabla\tilde{u}_p)]^{p-1}}{\tilde{u}_p^{p-1}}F(\nu)d\mathcal{H}^{n-1}, \end{aligned}$$

where last equality follows by the coarea formula. Then, reasoning as in [23] and [37] we get that

$$\begin{aligned} \int_\Omega [F(\nabla\tilde{u}_p)]^{p-1}\langle\nabla F(\nabla\tilde{u}_p), \nabla\psi_\varepsilon\rangle\,dx \xrightarrow{\varepsilon\to 0} -(p-1)\int_{U_t}\frac{[F(\nabla\tilde{u}_p)]^p}{\tilde{u}_p^p}dx + \\ \int_{S_t}\frac{[F(\nabla\tilde{u}_p)]^{p-1}}{\tilde{u}_p^{p-1}}F(\nu)d\mathcal{H}^{n-1}. \end{aligned}$$

As regards the other two integrals in (3.4), we have

$$\begin{aligned} \int_{\partial\Omega}\beta\tilde{u}_p^{p-1}\psi_\varepsilon F(\nu)\,d\mathcal{H}^{n-1} = \\ \int_{\Gamma_{t+\varepsilon}}\beta F(\nu)\,d\mathcal{H}^{n-1} + \int_{\Gamma_t\setminus\Gamma_{t+\varepsilon}}\beta\frac{\tilde{u}_p - t}{\varepsilon}F(\nu)\,d\mathcal{H}^{n-1} \\ \xrightarrow{\varepsilon\to 0} \int_{\Gamma_t}\beta F(\nu)\,d\mathcal{H}^{n-1}, \end{aligned}$$

by dominated convergence theorem and by monotone convergence theorem and the definition of ψ_ε,

$$\ell_1(\beta,\Omega)\int_\Omega \tilde{u}_p^{p-1}\psi_\varepsilon dx \quad \xrightarrow{\varepsilon\to 0} \quad \ell_1(\beta,\Omega)|U_t|.$$

Summing the three limits, we get (3.19). □

When we consider a generic test function we have

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with $C^{1,\alpha}$ boundary and let $\alpha \in]0,1[$. Let φ be a nonnegative function in Ω such that $\varphi \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$. If $\varphi \not\equiv [F(\nabla\tilde{u}_p)]^{p-1}/\tilde{u}_p^{p-1}$, where $\tilde{u}_p$ is the eigenfunction given in Theorem 3.16, and $\mathcal{F}_\Omega$ is the functional defined in* (3.20)*, then there exists a set $S \subset]0,1[$ with positive measure such that for every $t \in S$ it holds that*

$$\ell_1(\beta,\Omega) > \mathcal{F}_\Omega(U_t,\varphi). \tag{3.21}$$

The proof is similar to that obtained in [23] and [37], and we only sketch it here. It can be divided in two main steps. First, we claim that, if

$$w(x) := \varphi - \frac{[F(\nabla \tilde{u}_p)]^{p-1}}{\tilde{u}_p^{p-1}}, \qquad I(t) := \int_{U_t} w \frac{F(\nabla \tilde{u}_p)}{\tilde{u}_p}\, dx,$$

then the function $I :]0,1[\to \mathbb{R}$ is locally absolutely continuous and

$$\mathcal{F}_\Omega(U_t, \varphi) \le \ell_1(\beta, \Omega) - \frac{1}{|U_t|\, t^{p-1}} \left(\frac{d}{dt} \left(t^p I(t)\right)\right) \tag{3.22}$$

for almost every $t \in]0,1[$. Second, we show that the derivative $\frac{d}{dt}\left(t^p I(t)\right)$ is positive in a subset of $]0,1[$ with nonzero measure.
In order to prove (3.22), using the representation formula (3.19) we obtain that, for a.e. $t \in]0,1[$,

$$\begin{aligned}
\mathcal{F}_\Omega(U_t, \varphi) &= \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left(\int_{S_t} w F(\nu)\, d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \left(\varphi^{p'} - \frac{[F(\nabla \tilde{u}_p)]^p}{\tilde{u}_p^p} \right) dx \right) \\
&\le \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left(\int_{S_t} w F(\nu)\, d\mathcal{H}^{n-1} - p \int_{U_t} w \frac{F(\nabla \tilde{u}_p)}{\tilde{u}_p} dx \right) \\
&= \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left(\int_{S_t} w F(\nu)\, d\mathcal{H}^{n-1} - p\, I(t) \right)
\end{aligned} \tag{3.23}$$

where the inequality in (3.23) follows from the inequality
$\varphi^{p'} \ge v^{p'} + p' v^{p'-1}(\varphi - v)$, with $\varphi,\ v \ge 0$. Proceeding as in [37] and using the coarea formula we, obtain for a.e. $t \in]0,1[$

$$-\frac{d}{dt}(t^p I(t)) = t^{p-1} \left(\int_{S_t} w F(\nu)\, d\mathcal{H}^{n-1} - p\, I(t) \right). \tag{3.24}$$

Substituting (3.24) in (3.23) we obtain (3.22). We can conclude the proof, arguing by contradiction exactly as in [23, Theorem 3.2], indeed is possible to see that the function $t^p I(t)$ has positive derivative in a set of positive measure. This fact with (3.22) give us the inequality (3.21).

3.1.5 Applications

In this section we use the representation formula given in Theorem 3.16 in order to get some estimates for $\ell_1(\beta, \Omega)$.

A Faber-Krahn type inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with $C^{1,\alpha}$ boundary, $\alpha \in]0,1[$ and let $\mathcal{W}_R$ be the Wulff shape centered at the origin with radius R such that $|\Omega| = |\mathcal{W}_R|$. Let $\bar{\beta}$ be a positive constant and let us consider the following Robin eigenvalue problem in $\mathcal{W}_R$ for $\mathcal{Q}_p$ (3.10). Let $w(t)$, $t \in [0, +\infty[$, be a non negative continuous function such that

$$w(t) \ge C(R)\, t, \tag{3.25}$$

where $C(R) = \ell_1(\bar{\beta}, \mathcal{W}_R) + (p-1)\bar{\beta}^{\frac{p}{p-1}}$ is the constant appearing in (3.15). Let us consider the following Robin eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \ell_1(\beta, \Omega) |u|^{p-2} u & \text{in } \Omega, \\ F^{p-1}(\nabla u) \left\langle \nabla F(\nabla u), \nu \right\rangle + \beta(x) F(\nu) |u|^{p-2} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.26}$$

where

$$\beta(x) = w(F^o(x)), \quad x \in \partial\Omega. \tag{3.27}$$

As a consequence of the representation formula (3.16) for $\ell_1(\beta, \Omega)$ we get the following Faber-Krahn inequality.

Theorem 3.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with $C^{1,\alpha}$ boundary, let $\alpha \in]0,1[$ and let $\mathcal{W}_R$ be the Wulff shape such that $|\Omega| = |\mathcal{W}_R|$. Let $w(t)$, $t \in [0, +\infty[$, be a non negative continuous function which verifies* (3.25) *and let $\beta(x)$ be the function defined in* (3.27)*. Then,*

$$\ell_1(\bar{\beta}, \mathcal{W}_R) \leq \ell_1(\beta, \Omega). \tag{3.28}$$

where $\bar{\beta} = w(R)$.

Proof. We construct a suitable test function in Ω for (3.21). Let v_p be a positive eigenfunction of the radial problem (3.10) in $\mathcal{W}_R$. By Theorem 3.6, v_p is a function depending only by $F^o(x)$, $v_p = \rho_p(F^o(x))$, and then we can argue as in Paragraph 3.1.2 defining the function

$$f(r_x) = \varphi_\star(x) = \frac{\left[F\left(\nabla v_p(x)\right)\right]^{p-1} \left\langle \nabla F\left(\nabla v_p(x)\right), \nu \right\rangle}{v_p(x)^{p-1} F(\nu)} = \frac{(-\rho_p'(r_x))^{p-1}}{(\rho_p(r_x))^{p-1}},$$

where $r_x = F^o(x) \in [0, R]$

Denoted by $\mathcal{W}_s = \{x \in \mathcal{W}_R \colon v_p(x) > s\}$, $0 < s < R$, clearly $\mathcal{W}_s$ is a Wulff shape centered at the origin and by Theorem 3.16 we get

$$\begin{aligned} &\ell_1(\beta(R), \mathcal{W}_R) = \\ &\qquad F_{\mathcal{W}_R}(\mathcal{W}_s, \varphi_\star) = \frac{1}{|\mathcal{W}_s|}\left(-(p-1)\int_{\mathcal{W}_s} \varphi_\star^{p'}\, dx + \int_{\partial\mathcal{W}_s} \varphi_\star F(\nu) d\mathcal{H}^{n-1}\right) \end{aligned} \tag{3.29}$$

Let $\tilde{u}_p$ be the first eigenfunction of (3.26) in Ω such that $\|\tilde{u}_p\|_\infty = 1$. For $x \in \Omega$ we set $\tilde{u}_p(x) = t$, $0 < t < 1$. Then we consider the Wulff shape $\mathcal{W}_{r(t)}$, centered at the origin, where $r(t)$ is the positive number such that $|U_t| = |\mathcal{W}_{r(t)}|$. Then, we define the following test function

$$\varphi(x) := f(r(t)) = f(F^o(x)).$$

We stress that clearly $r(t) < R$. Our aim is to compare $\mathcal{F}_\Omega(U_t, \varphi)$ with $\mathcal{F}_{\mathcal{W}_R}(\mathcal{W}_{r(t)}, \varphi_\star)$. Then by (3.29) with $s = r(t)$ we have to show that

$$\begin{aligned} \mathcal{F}_\Omega(U_t, \varphi) &\geq \frac{1}{|\mathcal{W}_{r(t)}|}\left(-(p-1)\int_{\mathcal{W}_{r(t)}} \varphi_\star^{p'}\, dx + \int_{\partial\mathcal{W}_{r(t)}} \varphi_\star F(\nu) d\mathcal{H}^{n-1}\right) \\ &= \mathcal{F}_{\mathcal{W}_R}(\mathcal{W}_{r(t)}, \varphi_\star). \end{aligned}$$

We first observe that by [67, Section 1.2.3], being $|U_t| = |\mathcal{W}_{r(t)}|$ for all $t \in]0,1[$

$$\int_{U_t} \varphi^{p'}\, dx = \int_{\mathcal{W}_{r(t)}} \varphi_\star^{p'}\, dx.$$

Moreover, from the weighted isoperimetric inequality quoted in Remark 3.12, Theorem 3.11 and the assumption (3.27) on β we get

$$
\begin{aligned}
\int_{\partial \mathcal{W}_{r(t)}} \varphi_\star F(\nu) d\sigma = \int_{\partial \mathcal{W}_{r(t)}} f(r(t)) F(\nu) d\mathcal{H}^{n-1} &\leq \int_{\partial U_t} f(F^o(x)) F(\nu) d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} f(F^o(x)) F(\nu) d\mathcal{H}^{n-1} + \int_{\Gamma_t} f(F^o(x)) F(\nu) d\mathcal{H}^{n-1} \\
&= \int_{S_t} \varphi F(\nu) d\mathcal{H}^{n-1} + \int_{\Gamma_t} f(F^o(x)) F(\nu) d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} \varphi F(\nu) d\mathcal{H}^{n-1} + C(R) \int_{\Gamma_t} F^o(x) F(\nu) d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} \varphi F(\nu) d\mathcal{H}^{n-1} + \int_{\Gamma_t} w(F^o(x)) F(\nu) d\mathcal{H}^{n-1}
\end{aligned}
$$

and this concludes the proof. □

Remark 3.19. When $\beta = \bar{\beta}$ is a nonnegative constant (3.28) is proved in [37] in the anisotropic case and in [23, 32] in the Euclidean case.

A Cheeger type inequality for $\ell_1(\beta, \Omega)$

In this part we introduce the anisotropic weighted Cheeger constant and, using the representation formula, we prove an anisotropic weighted Cheeger inequality for $\ell_1(\beta, \Omega)$. Following [29] we give

Definition 3.20. *Let $g \colon \overline{\Omega} \to]0, \infty[$ be a continuous function the weighted anisotropic Cheeger constant is defined as follows*

$$
h_{g,F}(\Omega) = \inf_{U \subset \Omega} \frac{\displaystyle\int_{\partial U} g F(\nu) d\mathcal{H}^{n-1}}{|U|} = \inf_{U \subset \Omega} \frac{P_F(g, U)}{|U|}.
$$

We observe that when $g(x) = c$ is a constant then

$$
h_{g,F}(\Omega) = c \inf_{U \subset \Omega} \frac{P_F(U)}{|U|} = c\, h_F(\Omega),
$$

where $h_F(\Omega)$ is the anisotropic Cheeger constant defined in (1.6) . In [29] it is proved that actually $h_{g,F}(\Omega)$ is a minimum that is there exists a set $C \subset \Omega$ such that

$$
h_{g,F}(\Omega) = \frac{P_F(g, C)}{|C|},
$$

and we refer to C as a weighted Cheeger set.

We observe that for suitable weight g the constant $h_{g,F}(\Omega)$ verifies an anisotropic isoperimetric inequality

Theorem 3.21. *Let $g(x) = w(F^o(x)) = w(r)$, $r \geq 0$ with w a non negative and nondecreasing function such that*

$$
w(r^{\frac{1}{n}}) r^{1 - \frac{1}{n}}, \qquad 0 \leq r \leq R^n,
$$

is convex with respect to r. Then

$$
h_{g,F}(\Omega) \geq h_{g,F}(\mathcal{W}_R) = \frac{n w(R)}{R},
$$

where $\mathcal{W}_R$ is a Wulff shape with the same measure as Ω.

Proof. The proof follows immediately from Remark 3.12. □

When $\beta = \bar{\beta}$ is a nonnegative constant and $p = 2$ in [61] the following Cheeger inequality is proved in the Robin eigenvalue Euclidean case

$$\ell_{\mathcal{E}}(\bar{\beta}, \Omega) \geq \begin{cases} h(\Omega)\bar{\beta} - \bar{\beta}^2 & \text{always} \\ \frac{1}{4}\left[h(\Omega)\right]^2 & \text{if } \bar{\beta} \geq \frac{1}{2}h(\Omega) \end{cases} \tag{3.30}$$

In the next result we extend (3.30) to the anisotropic case for any $1 < p < \infty$ considering β not in general constant.

Theorem 3.22. *Let us consider problem* (3.18) *with* $\beta \in C\left(\overline{\Omega}\right)$ *such that* $\beta \geq 0$. *Then the following weighted anisotropic Cheeger inequality holds*

$$\ell_1(\beta, \Omega) \geq h_{\beta,F}(\Omega) - (p-1)\|\beta^{p'}\|_{L^\infty(\overline{\Omega})}, \tag{3.31}$$

where $p' = \frac{p}{p-1}$.

Proof. Using β as test function in (3.21) we obtain

$$\begin{aligned} \ell_1(\beta, \Omega) \geq \mathcal{F}(U_t, \beta) = \\ \frac{1}{|U_t|}\left(-(p-1)\int_{U_t}\beta^{p'}dx + \int_{S_t}\beta F(\nu)d\mathcal{H}^{n-1} + \int_{\Gamma_t}\beta F(\nu)d\mathcal{H}^{n-1}\right) \\ = \frac{1}{|U_t|}\left(-(p-1)\int_{U_t}\beta^{p'}dx + \int_{\partial U_t}\beta F(\nu)d\mathcal{H}^{n-1}\right) \geq -(p-1)\|\beta^{p'}\|_\infty + h_{\beta,F}(\Omega). \end{aligned}$$

□

Remark 3.23. We observe that the previous result continues to hold if we take $\beta \in C(\partial\Omega)$. Indeed in this case from a classical result, see for instance [54, Theorem 4.I], we know that the function β is the trace of a nonnegative function $\beta_\Omega \in C\left(\overline{\Omega}\right)$. Then inequality (3.31) holds with $\beta = \beta_\Omega$.

We emphasize the inequality (3.31) in the particular case of $\beta = \bar{\beta}$ is a nonnegative constant.

Corollary 3.24. *The first eigenvalue* $\ell_1(\bar{\beta}, \Omega)$ *of* (3.18) *on a fixed bounded open connected set* $\Omega \subset \mathbb{R}^n$ *with* $C^{1,\alpha}$ *boundary, with* $\alpha \in]0, 1[$, *satisfies*

$$\ell_1(\bar{\beta}, \Omega) \geq \begin{cases} h_F(\Omega)\bar{\beta} - (p-1)\bar{\beta}^{\frac{p}{p-1}} & \textit{always} \\ \frac{1}{p^p}\left[h_F(\Omega)\right]^p & \textit{if } \bar{\beta} \geq \frac{1}{p^{p-1}}\left[h_F(\Omega)\right]^{p-1} \end{cases} \tag{3.32}$$

Proof. From the Theorem 3.22 we obtain, using the constant function $\bar{\beta}$ as test, we obtain the first part of the inequality. For the second part, is suitable using as test function in the functional $\mathcal{F}_\Omega(U_t, \cdot)$, the constant $\frac{1}{p^{p-1}}\left[h_F(\Omega)\right]^{p-1}$ under the assumption that the constant $\bar{\beta} \geq \frac{1}{p^{p-1}}\left[h_F(\Omega)\right]^{p-1}$. □

Remark 3.25. From the anisotropic Cheeger inequality for constant $\bar{\beta}$ we obtain immediately a lower bound for $\ell_1(\bar{\beta}, \Omega)$ in terms of the anisotropic inradius of Ω different from (3.8) by using (1.7).

Remark 3.26. By *(ii)* Theorem 3.5 and Corollary 3.24 we obtain for $\bar{\beta} \geq \frac{1}{p^{p-1}}\left[h_F(\Omega)\right]^{p-1}$ the anisotropic Cheeger inequality (3.32) for the first Dirichlet eigenvalue of $\mathcal{Q}_p$

$$\lambda_D(\Omega) \geq \ell_1(\bar{\beta}, \Omega) \geq \frac{1}{p^p}\left[h_F(\Omega)\right]^p.$$

3.2 Sharp estimates for the first p-Laplacian eigenvalue and for the p-torsional rigidity on convex sets with holes

Throughout this section, we denote by Ω a set such that $\Omega = \Omega_0 \setminus \Theta$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. We define $\Gamma_0 := \partial\Omega_0$ and $\Gamma_1 := \partial\Theta$.

3.2.1 Eigenvalue problems

Let $1 < p < +\infty$, we deal with the following p-Laplacian eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda_p^{RN}(\beta,\Omega)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{3.33}$$

We denote by $\partial u/\partial\nu$ the outer normal derivative of u on the boundary and by $\beta \in \mathbb{R}\setminus\{0\}$ the Robin boundary parameter, observing that the case $\beta = +\infty$ gives asimptotically the Dirichlet boundary condition. Now we give the definition of eigenvalue and eigenfunction of problem (3.33).

Definition 3.27. *The real number λ is an eigenvalue of* (3.33) *if and only if there exists a function $u \in W^{1,p}(\Omega)$, not identically zero, such that*

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle \, dx + \beta\int_{\Gamma_0} |u|^{p-2}u\varphi \, d\mathcal{H}^{n-1} = \lambda\int_\Omega |u|^{p-2}u\varphi \, dx$$

for every $\varphi \in W^{1,p}(\Omega)$. The function u is called eigenfunction associated to λ.

In order to compute the first eigenvalue we use the variational characterization, that is

$$\lambda_p^{RN}(\beta,\Omega) = \min_{\substack{w\in W^{1,p}(\Omega)\\ w\not\equiv 0}} J_0[\beta, w] \tag{3.34}$$

where

$$J_0[\beta,w] := \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \beta\int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p \, dx}$$

We observe that Ω_0 is convex and hence it has Lipschitz boundary; this ensures the existence of minimizers of the analyzed problems.

Proposition 3.28. *Let $\beta \in \mathbb{R}\setminus\{0\}$. There exists a minimizer $u \in W^{1,p}(\Omega)$ of* (3.34), *which is a weak solution to* (3.33).

Proof. First we consider the case $\beta > 0$. Let $u_k \in W^{1,p}(\Omega)$ be a minimizing sequence of (3.34) such that $||u_k||_{L^p(\Omega)} = 1$. Then, being u_k bounded in $W^{1,p}(\Omega)$, there exist a subsequence, still denoted by u_k, and a function $u \in W^{1,p}(\Omega)$ with $||u||_{L^p(\Omega)} = 1$, such that $u_k \to u$ strongly in L^p and almost everywhere and $\nabla u_k \rightharpoonup \nabla u$ weakly in L^p. As a consequence, u_k converges strongly to u in $L^p(\partial\Omega)$ and so almost everywhere on $\partial\Omega$ to u. Then, by weak lower semicontinuity:

$$\lim_{k\to+\infty} J_0[\beta,u_k] \geq J_0[\beta,u].$$

We consider now the case $\beta < 0$. Let $u_k \in W^{1,p}(\Omega)$ be a minimizing sequence of (3.34) such that $||u_k||_{L^p(\partial\Omega)} = 1$. Now, since β is negative, we have the equi-boundness of the functional $J_0[\beta, \cdot]$, i.e. there exists a constant $C < 0$ such that $J_0[\beta, u_k] \leq C$ for every $k \in \mathbb{N}$. As a consequence

$$||\nabla u_k||^p_{L^p(\Omega)} - C||u_k||^p_{L^p(\Omega)} \leq -\beta,$$

and so

$$||u||^p_{W^{1,p}(\Omega)} \leq L,$$

where $L := -\beta/\min\{1, -C\}$. Then, there exist a subsequence, still denoted by u_k, and a function $u \in W^{1,p}(\Omega)$ such that $u_k \to u$ strongly in L^p and $\nabla u_k \rightharpoonup \nabla u$ weakly in L^p. So u_k converges strongly to u in $L^p(\partial\Omega)$, and so

$$J_0[\beta, u] \leq \liminf_{k\to\infty} J_0[\beta, u_k] = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} J_0[\beta, v].$$

Finally, u is strictly positive in Ω by the Harnack inequality (see [79]).

□

Now we state some basic properties on the sign and the monotonicity of the first eigenvalue.

Proposition 3.29. *If $\beta > 0$, then $\lambda_p^{RN}(\beta, \Omega)$ is positive and if $\beta < 0$, then $\lambda_p^{RN}(\beta, \Omega)$ is negative. Moreover, for all $\beta \in \mathbb{R} \setminus \{0\}$, $\lambda_p^{RN}(\beta, \Omega)$ is simple, that is all the associated eigenfunctions are scalar multiple of each other and can be taken to be positive.*

Proof. Let $\beta > 0$, then trivially $\lambda_p^{RN}(\Omega) \geq 0$. We prove that $\lambda_p^{RN}(\Omega) > 0$ by contradiction, assuming that $\lambda_p^{RN}(\Omega) = 0$. Thus, we consider a non-negative minimizer u such that $||u||_{L^p(\Omega)} = 1$ and

$$0 = \lambda_p^{RN}(\Omega, \beta) = \int_\Omega |\nabla u|^p \, dx + \beta \int_{\Gamma_0} |u|^p \, d\mathcal{H}^{n-1}.$$

So, u has to be constant in Ω and consequently u is 0 in Ω, which contradicts the fact that the norm of u is unitary.

If $\beta < 0$, choosing the constant as test function in (3.34), we obtain

$$\lambda_p^{RN}(\beta, \Omega) \leq \beta \frac{P(\Omega_0)}{|\Omega|} < 0.$$

Let $u \in W^{1,p}(\Omega)$ be a function that achieves the infimum in (3.34). First of all we observe that

$$J_0[\beta, u] = J_0[\beta, |u|],$$

and this fact implies that any eigenfunction must have constant sign on Ω and so we can assume that $u \geq 0$. In order to prove the simplicity of the eigenvalue, we proceed as in [14, 37]. We give here a sketch of the proof. Let u, w be positive minimizers of the functional $J_0[\beta, \cdot]$, such that $||u||_{L^p(\Omega)} = ||w||_{L^p(\Omega)} = 1$. We define $\eta_t = (tu^p + (1-t)w^p)^{1/p}$, with $t \in [0, 1]$ and we have that $||\eta_t||_{L^p(\Omega)} = 1$. It holds that

$$J_0[\beta, u] = \lambda_p^{RN}(\beta, \Omega) = J_0[\beta, w]. \tag{3.35}$$

Moreover by convexity the following inequality holds true:

$$|\nabla \eta_t|^p \leq t|\nabla u|^p + (1-t)|\nabla w|^p. \tag{3.36}$$

Using now (3.35), we obtain

$$\lambda_p^{RN}(\beta,\Omega) \leq J_0[\beta,\eta_t] \leq tJ_0[\beta,u] + (1-t)J_0[\beta,w] = \lambda_p^{RN}(\beta,\Omega),$$

and then η_t is a minimizer for $J_0[\beta,\cdot]$; so we have equality in (3.36), and the uniqueness follows. $\square$

Proposition 3.30. *The map $\beta \to \lambda_p^{RN}(\beta,\Omega)$ is Lipschitz continuous and non-decreasing with respect to $\beta \in \mathbb{R}$. Moreover $\lambda_p^{RN}(\beta,\Omega)$ is concave in β.*

Proof. Let $\beta_1, \beta_2 \in \mathbb{R}$ such that $\beta_1 < \beta_2$ and let $w \in W^{1,p}(\Omega)$ be not identically 0. We observe that

$$\int_\Omega |\nabla w|^p \, dx + \beta_1 \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1} \leq \int_\Omega |\nabla w|^p \, dx + \beta_2 \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}.$$

Now, passing to the infimum on w and taking into account the variational characterization, we obtain $\lambda_p^{RN}(\beta_1,\Omega) \leq \lambda_p^{RN}(\beta_2,\Omega)$.

We prove that $\lambda_p^{RN}(\beta,\Omega)$ is concave in β. Indeed, for fixed $\beta_0 \in \mathbb{R}$, we have to show that

$$\lambda_p^{RN}(\beta,\Omega) \leq \lambda_p^{RN}(\beta_0,\Omega) + \left(\lambda_p^{RN}\right)'(\beta_0,\Omega)\left(\beta - \beta_0\right), \tag{3.37}$$

for every $\beta \in \mathbb{R}$. Let u_0 the eigenfunction associated to $\lambda_p^{RN}(\beta_0,\Omega)$ and normalized such that $\int_\Omega u_0^p \, dx = 1$. Hence, we have

$$\lambda_p^{RN}(\beta,\Omega) \leq \int_\Omega |\nabla u_0|^p \, dx + \beta \int_{\Gamma_0} |u_0|^p \, d\mathcal{H}^{n-1}. \tag{3.38}$$

Now, summing and subtracting to the left hand side of (3.38) the quantity $\beta_0 \int_{\Gamma_0} |u_0|^p d\mathcal{H}^{n-1}$, taking into account that

$$\lambda_p^{RN}(\beta_0,\Omega) = \int_\Omega |\nabla u_0|^p \, dx + \beta_0 \int_{\Gamma_0} |u_0|^p \, d\mathcal{H}^{n-1},$$

and the fact that

$$\left(\lambda_p^{RN}\right)'(\beta_0,\Omega) = \int_{\Gamma_0} |u_0|^p \, d\mathcal{H}^{n-1},$$

we obtain the desired result (3.37). $\square$

Now we state a result relative to the eigenfunctions of problem (3.33) on the annulus.

Proposition 3.31. *Let $r_1, r_2 \in \mathbb{R}$ such that $r_2 > r_1 \geq 0$, and let u be the minimizer of problem (3.34) on the annulus A_{r_1,r_2}. Then u is strictly positive and radially symmetric, in the sense that $u(x) =: \psi(|x|)$. Moreover, if $\beta > 0$, then $\psi'(r) < 0$ and if $\beta < 0$, then $\psi'(r) > 0$.*

Proof. The first claim follows from the simplicity of $\lambda_p^{RN}(\beta, A_{r_1,r_2})$ and from the rotational invariance of problem (3.33). For the second claim, we consider the problem (3.33) with the boundary parameter $\beta > 0$. The associated radial problem is:

$$\begin{cases} -\dfrac{1}{r^{n-1}}\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' = \lambda_p^{RN}(\beta, A_{r_1,r_2})\psi^{p-1}(r) & \text{if } r \in (r_1,r_2), \\ \psi'(r_1)|\psi'(r_1)|^{p-2} = 0, \\ |\psi'(r_2)|^{p-2}\psi'(r_2) + \beta\psi^{p-1}(r_2) = 0. \end{cases}$$

We observe that for every $r \in (r_1, r_2)$

$$-\frac{1}{r^{n-1}}\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' = \lambda_p^{RN}(\beta,\Omega)\psi^{p-1}(r) > 0, \tag{3.39}$$

and, as a consequence,

$$\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' < 0.$$

Taking into account the boundary conditions $\psi'(r_1) = 0$, it follows that $\psi'(r) < 0$, since

$$|\psi'(r)|^{p-2}\psi'(r)r^{n-1} < 0.$$

If $\beta < 0$, by Remark 3.29, $\lambda_p^{RN}(\beta, A_{r_1,r_2}) < 0$ and consequently the left side of the equation (3.39) is negative, and hence $\psi'(r) > 0$.

□

3.2.2 Torsional rigidity

Let $\beta > 0$, we consider the torsion problem for the $p-$Laplacian. More precisely we are interested in

$$\frac{1}{T_p^{RN}(\beta,\Omega)} = \min_{\substack{w\in W^{1,p}(\Omega)\\ w\not\equiv 0}} K_0[\beta, w], \tag{3.40}$$

where

$$K_0[\beta, w] := \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \beta \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\left|\displaystyle\int_\Omega w \, dx\right|^p}.$$

Problem (3.40), up to a suitable normalization, leads to

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ |Du|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |Du|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{3.41}$$

In the following, we state some results for the torsion rigidity problems analogous to the ones stated in the previous paragraph for the eigenvalue problems. The proofs can be easily adapted.

Proposition 3.32. *Let $\beta > 0$, then the following properties hold.*

- *There exists a positive minimizer $u \in W^{1,p}(\Omega)$ of* (3.40) *which is a weak solution to* (3.41) *in Ω.*
- *Let $r_1, r_2 \in \mathbb{R}$ such that $r_2 > r_1 \geq 0$, and ψ be the relative solution to* (3.40) *on the annulus A_{r_1,r_2}. Then ψ is strictly positive, radially symmetric and strictly decreasing.*
- *The map $\beta \mapsto \dfrac{1}{T_p^{RN}(\beta,\Omega)}$ is positive, Lipschitz continuous, non-increasing and concave with respect to β.*

3.2.3 Main results

In this paragraph we state and prove the main results. In the first theorem, we study the problem (3.34), in the second one the problem (3.40). We consider a set Ω as defined at the beginning of this section.

Theorem 3.33. *Let $\beta \in \mathbb{R} \setminus \{0\}$ and let Ω be such that $\Omega = \Omega_0 \setminus \Theta$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. Let $A = A_{r_1,r_2}$ be the annulus having the same measure of Ω and such that $P(B_{r_2}) = P(\Omega_0)$. Then,*

$$\lambda_p^{RN}(\beta, \Omega) \leq \lambda_p^{RN}(\beta, A).$$

Proof. We divide the proof in two cases, distinguishing the sign of the Robin boundary parameter.
Case 1: $\beta > 0$**.** We start by considering problem (3.34) with positive value of the Robin parameter. The solution v to (3.34) is a radial function by Proposition 3.31 and we denote by v_m and v_M the minimum and the maximum of v on A. We construct the following test function defined in Ω_0:

$$u(x) := \begin{cases} G(d_e(x)) & \text{if } d_e(x) < r_2 - r_1 \\ v_M & \text{if } d_e(x) \geq r_2 - r_1, \end{cases} \tag{3.42}$$

where G is defined as

$$G^{-1}(t) = \int_{v_m}^{t} \frac{1}{g(\tau)}\, d\tau,$$

with $g(t) = |\nabla v|_{v=t}$, defined for $v_m \leq t < v_M$, and $d_e(\cdot)$ denotes the distance from $\partial\Omega_0$. We observe that $v(x) = G(r_2 - |x|)$ and u satisfy the following properties: $u \in W^{1,p}(\Omega_0)$,

$$|\nabla u|_{u=t} = |\nabla v|_{v=t}$$
$$u_m := \min_{\Omega_0} u = v_m = G(0)$$
$$u_M := \max_{\Omega_0} u \leq v_M.$$

We need now to define the following sets:

$$\begin{aligned} E_{0,t} :=& \{x \in \Omega_0 \ : \ u(x) > t\}, \\ A_t :=& \{x \in A \ : \ v(x) > t\}, \\ A_{0,t} :=& A_t \cup \overline{B}_{r_1}. \end{aligned} \tag{3.43}$$

For simplicity of notation, we will denote by A_0 the set $A_{0,0}$, i.e. the ball B_{r_2}. Since $E_{0,t}$ and $A_{0,t}$ are convex sets, inequalities (1.12) and (1.11) imply

$$-\frac{d}{dt}P(E_{0,t}) \geq n(n-1)\frac{W_2(E_{0,t})}{g(t)} \geq n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(E_{0,t}))^{\frac{n-2}{n-1}}}{g(t)},$$

for $u_m < t < u_M$. Moreover, it holds

$$-\frac{d}{dt}P(A_{0,t}) = n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(A_{0,t}))^{\frac{n-2}{n-1}}}{g(t)},$$

for $v_m < t < v_M$. Since, by hypothesis, $P(\Omega_0) = P(B_{r_2})$, using a comparison type theorem, we obtain

$$P(E_{0,t}) \leq P(A_{0,t}),$$

for $v_m \le t < u_M$. Let us also observe that

$$\mathcal{H}^{n-1}(\partial E_{0,t} \cap \Omega) \le P(E_{0,t}) \le P(A_{0,t}). \tag{3.44}$$

Using now the coarea formula and (3.44):

$$\begin{aligned}\int_\Omega |\nabla u|^p \, dx &= \int_{u_m}^{u_M} g(t)^{p-1} \, \mathcal{H}^{n-1}(\partial E_{0,t} \cap \Omega) \, dt \\ &\le \int_{u_m}^{u_M} g(t)^{p-1} P(E_{0,t}) \, dt \le \int_{v_m}^{v_M} g(t)^{p-1} P(A_{0,t}) \, dt = \int_A |\nabla v|^p \, dx. \end{aligned} \tag{3.45}$$

Since, by construction, $u(x) = u_m = v_m$ on Γ_0, then

$$\int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1} = u_m^p P(\Omega_0) = v_m^p P(A_0) = \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}. \tag{3.46}$$

Now, we define $\mu(t) = |E_{0,t} \cap \Omega|$ and $\eta(t) = |A_t|$ and using again coarea formula, we obtain, for $v_m \le t < u_M$,

$$\begin{aligned}\mu'(t) = -\int_{\{u=t\}\cap\Omega} \frac{1}{|\nabla u(x)|} \, d\mathcal{H}^{n-1} &= -\frac{\mathcal{H}^{n-1}(\partial E_{0,t} \cap \Omega)}{g(t)} \ge -\frac{P(E_{0,t})}{g(t)} \\ &\ge -\frac{P(A_{0,t})}{g(t)} = -\int_{\{v=t\}} \frac{1}{|\nabla v(x)|} \, d\mathcal{H}^{n-1} = \eta'(t).\end{aligned}$$

This inequality holds true also if $0 < t < v_M$. Since $\mu(0) = \eta(0)$ (indeed $|\Omega| = |A|$), by integrating from 0 to t, we have:

$$\mu(t) \ge \eta(t), \tag{3.47}$$

for $0 \le t < v_M$. If we consider the eigenvalue problem (3.34), we have

$$\int_\Omega u^p \, dx = \int_0^{v_M} pt^{p-1}\mu(t)dt \ge \int_0^{v_M} pt^{p-1}\eta(t) \, dt = \int_A v^p \, dx. \tag{3.48}$$

Using (3.45)-(3.46)-(3.48), we achieve

$$\begin{aligned}\lambda_p^{RN}(\beta, \Omega) &\le \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \beta \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p \, dx} \\ &\le \frac{\displaystyle\int_A |\nabla v|^p \, dx + \beta \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_A v^p \, dx} = \lambda_p^{RN}(\beta, A).\end{aligned}$$

Case 2: $\beta < 0$. We consider now the problem (3.34) with negative Robin external boundary parameter. By Proposition 3.29 the first p-Laplacian eigenvalue is negative. We observe that v is a radial function. We construct now the following test function defined in Ω_0:

$$u(x) := \begin{cases} G(d_e(x)) & \text{if } d_e(x) < r_2 - r_1 \\ v_m & \text{if } d_e(x) \ge r_2 - r_1, \end{cases}$$

where G is defined as

$$G^{-1}(t) = \int_t^{v_M} \frac{1}{g(\tau)} \, d\tau,$$

with $g(t) = |\nabla v|_{v=t}$, defined for $v_m < t \le v_M$ with $v_m := \min_A v$ and $v_M := \max_A v$. We observe that u satisfies the following properties: $u \in W^{1,p}(\Omega_0)$, $|\nabla u|_{u=t} = |\nabla v|_{v=t}$ and

$$
\begin{aligned}
u_m &:= \min_{\Omega} u \ge v_m;\\
u_M &:= \max_{\Omega} u = v_M.
\end{aligned}
$$

We need now to define the following sets:

$$
\begin{aligned}
\tilde{E}_{0,t} =& \{x \in \Omega_0 \ : \ u(x) < t\},\\
\tilde{A}_t =& \{x \in A \ : \ v(x) < t\};\\
\tilde{A}_{0,t} =& \tilde{A}_t \cup \overline{B}_{r_1}.
\end{aligned}
$$

For simplicity of notation, we will denote by $\tilde{A}_0$ the set $\tilde{A}_{0,0}$, i.e. the ball B_{r_2}. Since $\tilde{E}_{0,t}$ and $\tilde{A}_{0,t}$ are now convex sets, by inequalities (1.13) and (1.11), we obtain

$$
\frac{d}{dt} P(\tilde{E}_{0,t}) \ge n(n-1)\frac{W_2(\tilde{E}_{0,t})}{g(t)} \ge n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}} \frac{\left(P(\tilde{E}_{0,t})\right)^{\frac{n-2}{n-1}}}{g(t)}.
$$

Moreover, it holds

$$
\frac{d}{dt} P(\tilde{A}_{0,t}) = n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}} \frac{\left(P(\tilde{A}_{0,t})\right)^{\frac{n-2}{n-1}}}{g(t)}.
$$

Since, by hypothesis, $P(\Omega_0) = P(B_{r_2})$, using a comparison type theorem, we obtain

$$
P(\tilde{E}_{0,t}) \le P(\tilde{A}_{0,t}),
$$

for $u_m \le t < v_M$. Moreover, we have

$$
\mathcal{H}^{n-1}(\partial\tilde{E}_{0,t} \cap \Omega) \le P(\tilde{E}_{0,t}) \le P(\tilde{A}_{0,t}). \tag{3.49}
$$

Using the coarea formula and (3.49),

$$
\begin{aligned}
\int_{\Omega} |\nabla u|^p \, dx &= \int_{u_m}^{u_M} g(t)^{p-1} \, \mathcal{H}^{n-1}(\partial\tilde{E}_{0,t} \cap \Omega) \, dt\\
&\le \int_{u_m}^{u_M} g(t)^{p-1} P(\tilde{E}_{0,t}) \, dt \le \int_{v_m}^{v_M} g(t)^{p-1} P(\tilde{A}_{0,t}) \, dt = \int_A |\nabla v|^p \, dx.
\end{aligned} \tag{3.50}
$$

Since, by construction, $u(x) = u_M = v_M$ on Γ_0, it holds

$$
\int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1} = u_M^p P(\Omega_0) = v_M^p P(A_0) = \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}. \tag{3.51}
$$

We define now $\tilde{\mu}(t) = |\tilde{E}_{0,t} \cap \Omega|$ and $\tilde{\eta}(t) = |\tilde{A}_t|$ and using coarea formula, we obtain, for $u_m \le t < v_M$,

$$
\begin{aligned}
\tilde{\mu}'(t) = \int_{\{u=t\}\cap\Omega} \frac{1}{|\nabla u(x)|} \, d\mathcal{H}^{n-1} &= \frac{\mathcal{H}^{n-1}(\partial\tilde{E}_{0,t} \cap \Omega)}{g(t)} \le \frac{P(\tilde{E}_{0,t})}{g(t)}\\
&\le \frac{P(\tilde{A}_{0,t})}{g(t)} = \int_{\{v=t\}} \frac{1}{|\nabla v(x)|} \, d\mathcal{H}^{n-1} = \tilde{\eta}'(t).
\end{aligned}
$$

Hence $\mu'(t) \leq \eta'(t)$ for $v_m \leq t \leq v_M$. Then, by integrating from t and v_M:

$$|\Omega| - \tilde{\mu}(t) \leq |A| - \tilde{\eta}(t),$$

for $v_m \leq t < v_M$ and consequently $\tilde{\mu}(t) \geq \tilde{\eta}(t)$.

Let us consider the eigenvalue problem (3.34). We have that

$$\int_\Omega u^p \, dx = u_M^p |\Omega| - \int_{u_m}^{u_M} pt^{p-1} \tilde{\mu}(t) dt \leq v_M^p |A| - \int_{v_m}^{v_M} pt^{p-1} \tilde{\eta}(t) \, dt = \int_A v^p \, dx. \quad (3.52)$$

By (3.50)-(3.51)-(3.52), we have

$$\lambda_p^{RN}(\beta, \Omega) \leq \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \beta \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p \, dx} \leq$$

$$\leq \frac{\displaystyle\int_A |\nabla v|^p \, dx + \beta \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_A v^p \, dx} = \lambda_p^{RN}(\beta, A).$$

□

Theorem 3.34. *Let $\beta > 0$ and let Ω be such that $\Omega = \Omega_0 \setminus \Theta$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. Let $A = A_{r_1,r_2}$ be the annulus having the same measure of Ω and such that $P(B_{r_2}) = P(\Omega_0)$. Then,*

$$T_p^{RN}(\beta, \Omega) \geq T_p^{RN}(\beta, A).$$

Proof. Let v be the function that achieves the minimum in (3.40) on the annulus A. We consider the test function as in (3.42) and the superlevel sets as in (3.43). By (3.47) we have

$$\int_\Omega u \, dx = \int_0^{v_M} \mu(t) dt \geq \int_0^{v_M} \eta(t) \, dt = \int_A v \, dx \quad (3.53)$$

In this way, using (3.45)-(3.46)-(3.53), we conclude

$$\frac{1}{T_p^{RN}(\beta, \Omega)} \leq \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \beta \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\left| \int_\Omega u \, dx \right|^p}$$

$$\leq \frac{\displaystyle\int_A |\nabla v|^p \, dx + \beta \int_{\Gamma_0} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\left| \int_A v \, dx \right|^p} = \frac{1}{T_p^{RN}(\beta, A)}.$$

□

We conclude with some remarks.

Remark 3.35. In [6] the authors prove that the annulus maximezes the first eigenvalue of the p-Laplacian with Neumann condition on internal boundary and Dirichlet condition on external boundary, among sets of $\mathbb{R}^n$ with holes and having a sphere as outer boundary. We explicitly observe that our result include this case, since

$$\lim_{\beta\to+\infty}\lambda_p^{RN}(\beta,\Omega)=\lambda_p^{DN}(\Omega),$$

where with $\lambda_p^{DN}(\Omega)$ we denote the first eigenvalue of the p-Laplacian endowed with Dirichlet condition on external boundary and Neumann condition on internal boundary.

Remark 3.36. Let us remark that in the case $p=2$, we know explicitly the expression of the solution of the problems described in this section on the annulus $A=A_{r_1,r_2}$. The function that achieves the minimum in $\lambda=\lambda_p^{RN}(\beta,A)$ is

$$v(r)=Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)r^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(\sqrt{\lambda}r)-J_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)r^{1-\frac{n}{2}}Y_{\frac{n}{2}-1}(\sqrt{\lambda}r),$$

where Y_σ and J_σ are the Bessel functions of order σ (for their properties we refer to [1, 80]), with the condition

$$\begin{aligned}&Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_1)[r_2^{1-\frac{n}{2}}J_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)\sqrt{\lambda}+\beta r_2^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(\sqrt{\lambda}r_2)]-\\&\quad J_{\frac{n}{2}-2}(\sqrt{\lambda}r_1)[r_2^{1-\frac{n}{2}}Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)\sqrt{\lambda}+\beta r_2^{1-\frac{n}{2}}Y_{\frac{n}{2}-1}(\sqrt{\lambda}r_2)]=0.\end{aligned}$$

The function that achieves the minimum $1/T=1/T_p^{RN}(\beta,A)$ is

$$v(r)=\frac{1}{2Tn}r^2+c_1\frac{(1-n)}{r^n}+c_2,$$

with

$$\begin{cases}c_1=\frac{1}{\beta T}\left(\frac{r_2}{n}-\frac{r_1^n}{nr_2^{n-1}}+\frac{\beta r_2^2}{2n}+\frac{(n-1)\beta}{n}\left(\frac{r_1}{r_2}\right)^n\right)\\c_2=-\frac{1}{nT}r_1^n.\end{cases}$$

Chapter 4

Isoperimetric inequalities and stability issue of the Weinstock inequality for convex sets

In this chapter we extend the results obtained [25] in two ways. Let Ω be a bounded open convex set: in the Section 4.1 we generalize the isoperimetric inequality

$$\frac{\displaystyle\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}}{P(\Omega)\ |\Omega|^{\frac{2}{n}}} \geq \omega_n^{\frac{-2}{n}} \tag{4.1}$$

to a functional involving the anisotropic p-momentum, the anisotropic perimeter and the volume; in the Section 4.2 we get a stability result for the inequality

$$\frac{\displaystyle\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}}{|\Omega|\, P(\Omega)^{\frac{1}{n-1}}} \geq \frac{n}{(n\omega_n)^{\frac{1}{n-1}}},$$

that is obtained easily from the (4.1), and then we obtain a stability result for the Weinstock inequality for convex sets

$$\sigma_2(\Omega) P(\Omega)^{\frac{1}{n-1}} \leq \sigma_2(B_R(x)) P(B_R(x))^{\frac{1}{n-1}} \tag{4.2}$$

where σ_2 is the first nontrivial eigenvalue of the Steklov-Laplacian problem (2.10).

4.1 Anisotropic isoperimetric inequalities involving boundary momentum, perimeter and volume

Let Ω be a bounded, open set of $\mathbb{R}^n$ with Lipschitz boundary. Let $1 < p < +\infty$, we consider the following scale invariant functional:

$$\mathcal{F}(\Omega) = \frac{\displaystyle\int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}}{\left[\displaystyle\int_{\partial\Omega} F(\nu(x)) \, d\mathcal{H}^{n-1}\right] |\Omega|^{\frac{p}{n}}},$$

where $\nu_{\partial\Omega}(x)$ is the Euclidean unit outer normal at $x \in \partial\Omega$. We define the anisotropic p-boundary momentum of Ω as

$$M_F(\Omega) = \int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}.$$

The main result of this section is the following.

Theorem 4.1. *Let Ω be a bounded, open, convex set of $\mathbb{R}^n$. The following inequality holds true:*

$$\mathcal{F}(\Omega) \geq \kappa_n^{-\frac{p}{n}}$$

and equality holds only for Wulff shapes centered at the origin.

Remark 4.2. We observe that from this last theorem follows a particular case of (2.9). If we take $F(x) = |x|$, we obtain

$$\left(\int_{\partial\Omega} |x|^p \, d\mathcal{H}^{n-1} \right)^n \geq n^n \omega_n^{1-p} \, |\Omega|^{n+p-1} \, .$$

In what follows we will need the following definitions:

- $r_{\max}^F(\Omega) := \max \left\{ F^o(x) \mid x \in \bar{\Omega} \right\}$.
- $x_{\max}^F(\Omega) \in \partial\Omega$ is such that $F^o(x_{\max}^F(\Omega)) = r_{\max}^F(\Omega)$;
- the anisotropic p-excess function $E_F(\Omega) := (r_{\max}^F(\Omega))^{p-1} - \dfrac{M_F(\Omega)}{n\,|\Omega|}$.

In order to prove our main theorem, we need some intermediate results that we are now going to illustrate. The general way of proceeding is analogous to the one presented in [25].

4.1.1 The first variation of the p-momentum in the smooth case

Let Ω be a subset of $\mathbb{R}^n$ with C^∞ boundary. We consider the following transformations:

$$\phi(x,t) = x + t\varphi(x)\nu^F(x), \tag{4.3}$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\nu^F(x) = \nabla F(\nu(x))$ is the anisotropic normal. We recall that

$$\Omega(t) := \{x + t\varphi(x) \, \nu^F(x) \mid x \in \Omega\}.$$

From (1.8), we have that

$$\begin{aligned}
&\frac{d}{dt} P_F(\Omega(t))|_{t=0} = \int_{\partial\Omega} H_{\partial\Omega}^F(x) \langle \nu(x), \varphi(x)\nu^F(x) \rangle \, d\mathcal{H}^{n-1}(x) = \\
&= \int_{\partial\Omega} H_{\partial\Omega}^F(x)\varphi(x)\langle \nu(x), \nabla F(\nu(x)) \rangle \, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} H_{\partial\Omega}^F(x)\varphi(x) F(\nu(x)) \, d\mathcal{H}^{n-1}(x),
\end{aligned}$$

where the last equality holds true because of the properties of a Finsler norm. We recall also the variation of the volume of a set:

$$\frac{d}{dt} \left|\Omega(t)\right| |_{t=0} = \int_{\partial\Omega} \varphi(x) F(\nu(x)) \, d\mathcal{H}^{n-1}(x).$$

Proposition 4.3. *Let Ω and $\Omega(t)$ be the subsets of $\mathbb{R}^n$ previously defined. Then*

$$\begin{aligned}
&\frac{d}{dt} M_F(\Omega(t))|_{t=0} = \\
&= p \int_{\partial\Omega} (F^o(x))^{p-1} \, \langle \nabla F^o(x), \varphi(x) \, \nu^F(x) \rangle F(\nu(x)) \, d\mathcal{H}^{n-1}(x) + \\
&+ \int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, H_{\partial\Omega}^F(x)\varphi(x) \, d\mathcal{H}^{n-1}(x).
\end{aligned}$$

Proof. Considering the change of variables given by (4.3), i.e. $y = \phi(x,t)$, we have that

$$
\begin{aligned}
&\frac{d}{dt} M_F(\Omega(t))|_{t=0} = \\
&= \int_{\partial\Omega} \frac{d}{dt} \left([F^o(\phi(x,t))]^p\right) F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0} + \\
&+ \int_{\partial\Omega} (F^o(\phi(x,t)))^p \frac{d}{dt} \left[F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))\right]|_{t=0}.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\int_{\partial\Omega} \frac{d}{dt} \left([F^o(\phi(x,t))]^p\right) F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0} \\
&= \int_{\partial\Omega} p\,(F^o(\phi(x,t)))^{p-1} \langle \nabla F^o(\phi(x,t)), \varphi(x)\nu^F(x)\rangle F(\nu(\phi(x,t))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0}.
\end{aligned}
$$

Moreover, from the first variation of the perimeter (1.8), we can say that

$$
\frac{d}{dt}\left[F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))\right]|_{t=0} = H^F_{\partial\Omega}(x)\varphi(x)F(\nu(x))d\mathcal{H}^{n-1}(x).
$$

The thesis follows. $\square$

Considering now the derivative of the quotient, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}((\Omega(t))|_{t=0} = \\
&= \frac{1}{P_F(\Omega)^2\,|\Omega|^{\frac{p}{n}}} \left\{ p\int_{\partial\Omega} \left[(F^o(x))^{p-1}\langle\nabla F^o(x), \nu^F(x)\rangle F(\nu(x)) \right.\right. \\
&\qquad \left. -\frac{M_F(\Omega)}{n\,|\Omega|} F(\nu(x))\right] \varphi(x)\, d\mathcal{H}^{n-1}(x) + \\
&\qquad \left. + \int_{\partial\Omega}\left[(F^o(x))^p - \frac{M_F(\Omega)}{P_F(\Omega)}\right] H^F_{\partial\Omega}(x)\, F(\nu(x))\varphi(x)\, d\mathcal{H}^{n-1}(x)\right\}.
\end{aligned}
$$

Let be $T > 0$; we choose, as in [85],

$$
\varphi(x) = \frac{1}{H^F_{\partial\Omega}(x)},
$$

and we have that

$$
\frac{\partial}{\partial t}\phi(x,t) = \frac{\nu^F(x)}{H^F_{\partial\Omega}(x)},
$$

for every $t \in [0,T]$. This one parameter family of diffeomorphisms gives rise to the so called inverse anisotropic mean curvature flow (IAMCF). Concerning this family of flows, local and global existence and uniqueness have been studied in [85, 60, 73].

Remark 4.4. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex set of class C^∞. Ω is called F-mean convex if its anisotropic mean curvature is strictly positive and, in this case, we say that $\Omega \in C^{\infty,+}_F$. In [85] is proved that, if $\Omega(0) = \Omega \in C^{\infty,+}_F$, then there exists an unique smooth solution $\phi(\cdot,t)$ of the inverse mean curvature flow in $[0,+\infty]$. Moreover the surface $\partial\Omega(t)$, for every $t > 0$, is the boundary of a smooth convex set in $C^{\infty,+}_F$ that asymptotically converges to a Wulff shape as $t \to +\infty$.

Substituting this φ in the derivative of the quotient and taking into account the fact that

$$\int_{\partial\Omega}\left[(F^o(x))^p-\frac{M_F(\Omega)}{P_F(\Omega)}\right]F(\nu(x))\,d\mathcal{H}^{n-1}(x)=0,$$

we obtain

$$\frac{d}{dt}\mathcal{F}((\Omega(t))|_{t=0}= \tag{4.4}$$

$$\frac{p}{P_F(\Omega)^2\,|\Omega|^{\frac{p}{n}}}\int_{\partial\Omega}\left[(F^o(x))^{p-1}\langle\nabla F^o(x),\nu^F(x)\rangle F(\nu(x))-\frac{M_F(\Omega)}{n\,|\Omega|}F(\nu(x))\right]\frac{d\mathcal{H}^{n-1}(x)}{H^F_{\partial\Omega}(x)}=$$

$$=\frac{p}{P_F(\Omega)^2\,|\Omega|^{\frac{p}{n}}}\int_{\partial\Omega}\left[(F^o(x))^{p-1}\langle\nabla F^o(x),\nu^F(x)\rangle-\frac{M_F(\Omega)}{n\,|\Omega|}\right]\frac{F(\nu(x))}{H^F_{\partial\Omega}(x)}\,d\mathcal{H}^{n-1}(x).$$

4.1.2 Existence of a minimizer

Proposition 4.5. *There exists a convex set minimizing $\mathcal{F}(\cdot)$.*

Proof. Given a convex set Ω, we can take a minimizing sequence $(\Omega_i)_i$, having the same volume of Ω. By Blaschke selection Theorem in [75, Theorem 1.8.7], it is enough to show that the Ω_i's are all contained in the same Wulff. For the sake of simplicity, we suppose that $|\Omega_i|=\kappa_n$ and, since any Wulff $\mathcal{W}$ with centered at the origin is such that $\mathcal{F}(\mathcal{W})=\kappa_n^{-\frac{p}{n}}$, we have that

$$\lim_{i\to+\infty}\mathcal{F}(\Omega_i)\le\kappa_n^{-\frac{p}{n}},$$

and consequently

$$\lim_{i\to+\infty}\frac{M_F(\Omega_i)}{P_F(\Omega_i)}\le 1.$$

Arguing by contradiction, if we assume that $\lim_{i\to+\infty}\operatorname{diam}_F(\Omega_i)=+\infty$, from convexity follows easily that $\lim_{i\to+\infty}P_F(\Omega_i)=+\infty$. Thereafter, if $\mathcal{W}_2$ is the Wulff of anisotropic radius 2 centered at the origin, it is enough to observe that

$$\lim_{i\to+\infty}\frac{\int_{\partial\Omega_i\cap\mathcal{W}_2}F(\nu(x))\,d\mathcal{H}^{n-1}(x)}{\int_{\partial\Omega_i\setminus\mathcal{W}_2}F(\nu(x))\,d\mathcal{H}^{n-1}(x)}=0$$

and

$$\lim_{i\to+\infty}\frac{M_F(\Omega_i)}{P_F(\Omega_i)}\ge\lim_{i\to+\infty}\frac{2^p}{1+\frac{\int_{\partial\Omega_i\cap\mathcal{W}_2}F(\nu(x))\,d\mathcal{H}^{n-1}(x)}{\int_{\partial\Omega_i\setminus\mathcal{W}_2}F(\nu(x))\,d\mathcal{H}^{n-1}(x)}}=2^p,$$

which gives a contradiction. So the diameters of the Ω_i's are equibounded. Moreover, arguing as before, we can show that $\Omega_i\cap\mathcal{W}_2\neq\emptyset$ definitely. Therefore we have the claim. □

4.1.3 A minimizer cannot have negative Excess

Remark 4.6. There exist sets with negative anisotropic p-Excess. We prove this fact in dimension 2 and for $p=2$. We consider the elliptic metric

$$F(x,y)=\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}};$$

we know that its polar is this elliptic norm

$$F^o(x,y)=\sqrt{a^2x^2+b^2y^2}.$$

We consider now the following convex domain:

$$R_\epsilon = \left\{ (x,y) \in \mathbb{R}^2 \ : \ |x| \le \frac{1}{\epsilon}, \ \ |y| \le \epsilon \right\}.$$

From the computations we obtain that $|R_\epsilon| = 4$, $r^F_{\max}(R_\epsilon) = a/\epsilon + O(\epsilon^3)$ and $M_F(R_\epsilon) = (4a^2/3b)(1/\epsilon^3) + 4a/\epsilon + O(\epsilon)$.

Lemma 4.7. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then*

$$(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F(x) \rangle - \frac{M_F(\Omega)}{n\,|\Omega|} \le E_F(\Omega). \tag{4.5}$$

Proof. We observe that

$$\langle \nabla F^o(x), \nu^F(x) \rangle = \langle \nabla F^o(x), \nabla F(\nu(x)) \rangle \le F(\nabla F^o(x)) F^o(\nabla F(\nu(x))) = 1,$$

for the properties of the Finsler norm F. □

We prove now a fact, that is an analogous of a property holding in the Euclidean case (see [25, Remark 2]).

Remark 4.8. Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then

$$\int_{\partial\Omega} \left[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F(x) \rangle - \frac{M_F(\Omega)}{n\,|\Omega|} \right] F(\nu(x))\, d\mathcal{H}^{n-1}(x) \le 0. \tag{4.6}$$

Proof. In order to prove (4.6), we observe that

$$\begin{aligned}
&\int_{\partial\Omega} \left[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F(x) \rangle F(\nu(x)) - \frac{M_F(\Omega)}{n\,|\Omega|} F(\nu(x)) \right] d\mathcal{H}^{n-1}(x) = \\
&\int_{\partial\Omega} \left[(F^o(x))^{p-1} \langle \nabla F^o(x), \nabla F(\nu(x)) \rangle F(\nu(x)) - \frac{M_F(\Omega)}{n\,|\Omega|} F(\nu(x)) \right] d\mathcal{H}^{n-1}(x) \\
&\le \int_{\partial\Omega} \left[(F^o(x))^{p-1} F(\nu(x)) \right] \, d\mathcal{H}^{n-1}(x) - \frac{M_F(\Omega)}{n\,|\Omega|} P_F(\Omega) \\
&\le \int_{\partial\Omega} [(F^o(x))^{p-1} F(\nu(x))] \, d\mathcal{H}^{n-1}(x) - \frac{M_F(\Omega) P_F(\Omega)}{\int_{\partial\Omega} F^o(x) F(\nu(x))\, d\mathcal{H}^{n-1}(x)}
\end{aligned}$$

and the last inequality holds since

$$n\,|\Omega| = \int_{\partial\Omega} \langle x, \nu(x) \rangle \, d\mathcal{H}^{n-1}(x) \le \int_{\partial\Omega} F^o(x) F(\nu(x))\, d\mathcal{H}^{n-1}(x),$$

for the properties of the Finsler norms. Using now Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\partial\Omega} (F^o(x))^{p-1} F(\nu(x))\, d\mathcal{H}^{n-1}(x) \\
&\le \left[\int_{\partial\Omega} \left[(F^o(x))^{p-1} \right]^{\frac{p}{p-1}} F(\nu(x))\, d\mathcal{H}^{n-1}(x) \right]^{\frac{p-1}{p}} \ (P_F(\Omega))^{\frac{1}{p}} \\
&= \left[\int_{\partial\Omega} (F^o(x))^{p} F(\nu(x))\, d\mathcal{H}^{n-1} \right]^{\frac{p-1}{p}} \ (P_F(\Omega))^{\frac{1}{p}}
\end{aligned}$$

and

$$\int_{\partial\Omega} F^o(x) F(\nu(x))\, d\mathcal{H}^{n-1}(x) \le \left[\int_{\partial\Omega} (F^o(x))^{p} F(\nu(x))\, d\mathcal{H}^{n-1}(x) \right]^{\frac{1}{p}} \ (P_F(\Omega))^{\frac{p-1}{p}}.$$

Finally, from these last two inequalities follows that

$$\left(\int_{\partial\Omega}\left[(F^o(x))^{p-1}\, F(\nu(x))\right]\, d\mathcal{H}^{n-1}(x)\right)\left(\int_{\partial\Omega} F^o(x)F(\nu(x))\, d\mathcal{H}^{n-1}(x)\right) \le M_F(\Omega)P_F(\Omega).$$

□

We recall now this lemma (see [85]), which will be used in the next proofs. This is the anisotropic version of the Heintze-Karcher inequality, whose proof in the Euclidean case can be found in [73].

Lemma 4.9. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ with C^2 boundary, then*

$$\int_{\partial\Omega} \frac{F(\nu(x))}{H^F_{\partial\Omega}(x)}\, d\mathcal{H}^{n-1}(x) \ge \int_{\partial\mathcal{W}_R} \frac{F(\nu(x))}{H^F_{\partial\mathcal{W}_R}(x)}\, d\mathcal{H}^{n-1}(x)$$

where $\mathcal{W}_R$ is a Wulff such that $|\mathcal{W}_R| = |\Omega|$.

Proposition 4.10. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such that*

$$E_F(\Omega) < 0,$$

then Ω is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. We firstly assume that $\Omega \in C^{\infty,+}_F$. Since $E_F(\Omega) \neq 0$, Ω is not a Wulff shape centered at the origin. Then, from (4.5) and (4.4), we have

$$\mathcal{F}'(\Omega) \le \frac{p}{P_F(\Omega)^2\, |\Omega|^{\frac{p}{n}}} E_F(\Omega) \int_{\partial\Omega} \frac{d\mathcal{H}^{n-1}(x)}{H^F_{\partial\Omega}(x)} < 0.$$

We suppose now that $\Omega \notin C^{\infty,+}_F$ and we assume by contradiction that Ω minimizes the functional $\mathcal{F}(\cdot)$. We can find a decreasing (in the sense of inclusion) sequence of sets $(\Omega_k)_{k\in\mathbb{N}} \subset C^{\infty,+}_F$ that converges to Ω in the Hausdorff sense. We have that

$$\lim_{k\to+\infty} |\Omega_k| = |\Omega|\,; \qquad \lim_{k\to+\infty} P_F(\Omega_k) = P_F(\Omega);$$

$$\lim_{k\to+\infty} M_F(\Omega_k) = M_F(\Omega); \qquad \lim_{k\to+\infty} r^F_{\max}(\Omega_k) = r^F_{\max}(\Omega).$$

We now consider the IAMCF for every Ω_k and we denote by $\Omega_k(t)$, for $t \ge 0$, the family generated in this way. We let $\Omega_k(0) = \Omega_k$. Using Hadamard formula (see [59]), we obtain:

$$\frac{d}{dt}\, |\Omega_k(t)| = \int_{\partial\Omega_k(t)} \frac{F(\nu(x))}{H^F_{\partial\Omega_k(t)}} d\mathcal{H}^{n-1}(x);$$

$$\frac{d}{dt} P_F(\Omega_k(t)) = P_F(\Omega_k(t)).$$

We have also that

$$\frac{d}{dt} r^F_{\max}(\Omega_k(t)) \le \frac{r^F_{\max}(\Omega_k(t))}{n-1}. \tag{4.7}$$

We prove now this last inequality. From definition of $x^F_{\max}(\Omega(t))$ and (4.3) in the IAMCF case, we have that

$$r^F_{\max}(\Omega(t)) = F^o(x^F_{\max}(\Omega(t)));$$

$$x^F_{\max}(\Omega(t)) = x^F_{\max}(\Omega) + \frac{t\nu^F}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))}.$$

Then

$$\frac{d}{dt}r^F_{\max}(\Omega(t)) = \frac{d}{dt}F^o(x^F_{\max}(\Omega(t))) = \left\langle \nabla F^o(x^F_{\max}(\Omega(t)), \frac{\nu^F(x^F_{\max}(\Omega))}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} \right\rangle \leq$$
$$\leq F(\nabla F^o(x^F_{\max}(\Omega(t))))F^o(\nu^F(x^F_{\max}(\Omega)))\frac{1}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} =$$
$$= F(\nabla F^o(x^F_{\max}(\Omega(t))))F^o(\nabla F(\nu(x^F_{\max}(\Omega))))\frac{1}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} =$$
$$= \frac{1}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} \leq \frac{r^F_{\max}(\Omega)}{n-1},$$

since F is a Finsler norm and therefore it is true that $F(\nabla F^o(x)) = F^o(\nabla F(x)) = 1$. We can then repeat this last inequality for every Ω_k. From (4.7) follows that

$$r^F_{\max}(\Omega_k(t)) \leq r^F_{\max}(\Omega_k)e^{\frac{t}{(n-1)}}, \text{ for } t > 0.$$

Analogous computations to the ones reported in [25, Proposition 2.4] lead to a contradiction with the minimality of Ω and therefore to the claim. □

4.1.4 A minimizer cannot have positive Excess

We start observing that there exist sets with positive excess.

Remark 4.11. We consider the case $n = 2$ and $p = 2$. The norm that we take into consideration is

$$F(x, y) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}};$$

and its polar is:

$$F^o(x, y) = \sqrt{a^2x^2 + b^2y^2}.$$

We define

$$\mathcal{E}_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid a^2(1-\epsilon)^2x^2 + b^2(1+\epsilon)^2y^2\}.$$

We have that

$$r^F_{\max}(\mathcal{E}_\epsilon) = 1 + \epsilon + o(\epsilon)$$

and

$$|\mathcal{E}_\epsilon| = \frac{\pi}{ab}(1 + \epsilon^2 + o(\epsilon)).$$

Computing the momentum, we find that

$$M_F(\mathcal{E}_\epsilon) = \frac{2}{ab(1-\epsilon)^2(1+\epsilon)^2}\left(\pi + \epsilon\int_0^\pi \cos(2t)\, dt\right) + o(\epsilon) = \frac{2}{ab(1-\epsilon)^2(1+\epsilon)^2}(\pi + o(\epsilon))$$

and so it results that $E_F(\mathcal{E}_\epsilon) = \epsilon + o(\epsilon)$.

In the following, we will use the notations: $\underline{0} \in \mathbb{R}^{n-1}$ and $x' = (x_1, \dots, x_{n-1})$.
We consider the halfspace T_ϵ that has outer Euclidean normal pointing in the direction given by the outer Euclidean normal to Ω in the point $x^F_{\max}(\Omega)$ and intersecting Ω at a distance ϵ from $x^F_{\max}(\Omega)$. We define the sets:

$$\Omega_\epsilon := \Omega \cap T_\epsilon,$$

$$A_\epsilon := \partial\Omega_\epsilon \cap \partial T_\epsilon,$$

$$C_\epsilon := \partial\Omega \cap T^c_\epsilon,$$

where T_ϵ^c is the complement of T_ϵ in $\mathbb{R}^n$, and we define the following quantities, that vanish as ϵ goes to 0:

$$\Delta M_F = M_F(\Omega_\epsilon) - M_F(\Omega);$$

$$\Delta V = |\Omega_\epsilon| - |\Omega|\,;$$

$$\Delta P_F = P_F(\Omega_\epsilon) - P_F(\Omega).$$

Considering Remark 2.2 in [43], we can choose the coordinate in such a way that the x_n axis lies in the direction of the outer normal to T_ϵ and we denote the coordinates of $x^F_{\max}(\Omega)$ by $x^F_{\max}(\Omega) =: (x'_0, y_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Moreover, we call $A'_\epsilon \subseteq \mathbb{R}^{n-1}$ the projection of A_ϵ onto $\{x_n = 0\}$.

Let $g : A'_\epsilon \to \mathbb{R}$ the concave function describing C_ϵ. Since the class of open and bounded convex set with positive mean curvature is dense in the class of open and bounded convex set, we can assume, in particular, that Ω is strictly convex and, consequently, that g is a function of class $C^1(A'_\epsilon)$, for $\epsilon > 0$ small enough. Let $h : A'_\epsilon \to \mathbb{R}$ defined by $h(x') = g(x') - (y_0 - \epsilon)$, so h is equal to 0 on $\partial A'_\epsilon$.

We observe that $g : A'_\epsilon \to \mathbb{R}$ is such that for any $x = (x', x_n) \in C_\epsilon$ we have $x_n = g(x')$. We call $G(x) := x_n - g(x')$ and, as a consequence, C_ϵ is the level set $G(x) = 0$; the Euclidean outer unit normal to C_ϵ in a point $x = (x', x_n) \in C_\epsilon$ is given by $\nabla G(x')/||\nabla G(x')||$, i.e.

$$\nu_{C_\epsilon}(x) = \frac{(-\nabla g(x'), 1)}{\sqrt{1 + \nabla g(x')^2}}$$

Since $\nabla g(x'_0) = \underline{0}$, we have that

$$-\Delta P_F = \int_{A'_\epsilon} \left[F(-\nabla g(x'), 1) - F(\underline{0}, 1)\right] dx'.$$

Lemma 4.12. *We claim that*

$$\int_{A'_\epsilon} \langle \nabla_{x'} F(\underline{0}, 1), -\nabla g(x') \rangle dx' = 0$$

Proof. Since

$$\int_{A'_\epsilon} \langle \nabla_{x'} F(\underline{0}, 1), -\nabla g(x') \rangle dx' = -\sum_{i=1}^{n-1} \int_{A'_\epsilon} \frac{\partial F}{\partial x_i}(\underline{0}, 1) \frac{\partial g}{\partial x_i}(x')\, dx',$$

it is enough to prove that, for every $i = 1, \dots (n-1)$,

$$\int_{A'_\epsilon} \frac{\partial F}{\partial x_i}(\underline{0}, 1) \frac{\partial g}{\partial x_i}(x')\, dx' = \frac{\partial F}{\partial x_i}(\underline{0}, 1) \int_{A'_\epsilon} \frac{\partial g}{\partial x_i}(x')\, dx' = 0.$$

Using the divergence theorem and the fact that h is equal to 0 on $\partial A'_\epsilon$,

$$\int_{A'_\epsilon} \frac{\partial g}{\partial x_i}(x')\, dx' = \int_{A'_\epsilon} \frac{\partial h}{\partial x_i}(x')\, dx' = \int_{A'_\epsilon} \operatorname{div}\left(h(x') e_i\right)\, dx' = \int_{\partial A'_\epsilon} \langle h(x') e_i, \nu_{\partial A'_\epsilon}(x') \rangle d\mathcal{H}^{n-2}(x') = 0,$$

where e_i is the vector having all zero coordinates, except the i-coordinate equal to 1. $\square$

Lemma 4.13. *There exists a positive constant $C(\Omega)$ such that for all $\epsilon > 0$ small enough, we have that*

$$|\Delta V| \leq C(\Omega) |\Delta P_F|. \tag{4.8}$$

Proof. There exists a Wulff shape centered in the origin, that we denote with $\mathcal{W}_{\max}$, that contains Ω and that it is tangent to Ω in the point $x^F_{\max} = (x'_0, y_0)$, with $x'_0 \in \mathbb{R}^{n-1}$ and $y_0 \in \mathbb{R}$. Moreover, since $\mathcal{W}$ is uniformly convex, there exists a ball $\bar{B}$ that contains $\mathcal{W}_{\max}$ and that is tangent to $\mathcal{W}_{\max}$ in $x^F_{\max}(\Omega)$. Let $c > 0$ be the positive constant such that , for all $i = 1, \cdots, n-1$, $k_i(\mathcal{W}) > c$, with $k_i(\mathcal{W})$ principal curvature of $\mathcal{W}$. If we denote by $\bar{R}$ the radius of $\bar{B}$, that is centered at a point $(x'_0, y_c) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have that $\bar{R} = r^F_{\max}(\Omega)/c$.

We have that $A_\epsilon \subseteq \bar{B} \cap \partial T_\epsilon$ and we denote by $\tilde{R}$ the radius of the $(n-1)$-dimensional ball $\bar{B} \cap \partial T_\epsilon$. Now, we have that

$$\operatorname{diam}(A_\epsilon) \le \operatorname{diam}(\bar{B} \cap \partial T_\epsilon) = 2\tilde{R} \le 2\sqrt{2\epsilon \bar{R}}. \tag{4.9}$$

We observe that

$$-\Delta V = \int_{A'_\epsilon} h(x')dx' \ge \epsilon \frac{\mathcal{L}^{n-1}(A'_\epsilon)}{n}. \tag{4.10}$$

Using (4.10), (4.9) and the Sobolev Poincaré inequality

$$\begin{aligned}
-\Delta V &= \int_{A'_\epsilon} h(x')\,dx' \le \left(\int_{A'_\epsilon} h(x')\,dx' \right)^2 \frac{n}{\epsilon \mathcal{L}^{n-1}(A'_\epsilon)} \le \\
&\le C(n) \frac{\left(\mathcal{L}^{n-1}(A'_\epsilon)\right)^{2/(n-1)}}{\epsilon} \int_{A'_\epsilon} ||\nabla h||^2 dx' \le C(n) 2\bar{R} \left(\omega_{n-1}\right)^{2/(n-1)} \int_{A'_\epsilon} ||\nabla h||^2 dx'
\end{aligned}$$

We now consider the function, $x' \in \mathbb{R}^{n-1} \to F(x', 1)$. Using the Taylor expansion with the Lagrange reminder:

$$\begin{aligned}
&F(-\nabla g(x'), 1) - F(\underline{0}, 1) = \langle \nabla_{x'} F(\underline{0}, 1), -\nabla g(x') \rangle + \frac{1}{2} (-\nabla g(x'))^T D^2 F(\tilde{x_y}, 1)(-\nabla g(x')) \ge \\
&\ge \langle \nabla_{x'} F(\underline{0}, 1), -\nabla g(x') \rangle + c ||\nabla g(x')||^2.
\end{aligned}$$

Integrating the last chain of inequalities and using the result in Lemma 4.12, we can conclude

$$-\Delta P_F \ge C(\Omega) \int_{A'_\epsilon} ||\nabla g(x')||^2 \, dx'.$$

We point out that, with the last inequality, we have also proved that $-\Delta P_F \ge 0$. □

Lemma 4.14. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$, then*

$$\Delta M_F \le p \left(r^F_{\max}(\Omega)\right)^{p-1} \Delta V + \left(r^F_{\max}(\Omega)\right)^p \Delta P_F + o(\Delta P_F) + o(\Delta V). \tag{4.11}$$

Proof.

$$\begin{aligned}
-\Delta M_F &= \int_{C_\epsilon} (F^o(x))^p \, F(\nu_{\partial\Omega}(x)) \, d\mathcal{H}^{n-1}(x) - \int_{A_\epsilon} (F^o(x))^p \, F(\underline{0}, 1) \, d\mathcal{H}^{n-1}(x) = \\
&= \int_{A'_\epsilon} \left(F^o(x', g(x'))\right)^p F(-\nabla g(x'), 1) \, dx' - F(\underline{0}, 1) \int_{A'_\epsilon} \left(F^o\left(x', y_0 - \epsilon\right)\right)^p \, dx' = \\
&= \int_{A'_\epsilon} \left[\left(F^o(x', g(x'))\right)^p - \left(F^o(x', y_0 - \epsilon)\right)^p \right] F(-\nabla g(x'), 1) \, dx' + \\
&+ \int_{A'_\epsilon} \left[F(-\nabla g(x'), 1) - F(\underline{0}, 1) \right] \left(F^o(x', y_0 - \epsilon)\right)^p \, dx' = I_1 + I_2.
\end{aligned}$$

Firstly, we take into consideration I_2.

Claim 1:

$$F^o(x', y_0 - \epsilon) = r^F_{\max}(\Omega) + o(1),$$

where we use the following notation: $q(\epsilon) =: o(\epsilon^n)$ if $\lim_{\epsilon\to 0} q(\epsilon)/\epsilon^n = 0$.

Using Taylor

$$\begin{aligned} F^o(x', y_0 - \epsilon) &= F^o(x'_0, y_0) + \langle \nabla F^o(x'_0, y_0), (x' - x'_0, -\epsilon)\rangle + o(||(x' - x'_0, -\epsilon)||) = \\ &= r^F_{\max}(\Omega) + \langle \nabla F^o(x'_0, y_0), (x' - x'_0, -\epsilon)\rangle + o(||(x' - x'_0, -\epsilon)||). \end{aligned}$$

For the Cauchy-Schwarz inequality:

$$\begin{aligned} |\langle \nabla F^o(x'_0, y_0), (x' - x_0, -\epsilon)\rangle| &\le ||\nabla F^o(x'_0, y_0)|| \sqrt{||x' - x_0||^2 + \epsilon^2} \le \\ &\le ||\nabla F^o(x'_0, y_0)|| \sqrt{\max_{x'\in A'_\epsilon}\{||x' - x'_0||\} + \epsilon^2} = o(1). \end{aligned}$$

So we have the claim.

Using Claim 1, we have that

$$\begin{aligned} I_2 &= \int_{A'_\epsilon} \left[F(-\nabla g(x'), 1) - F(\underline{0}, 1)\right] \left(r^F_{\max}(\Omega) + o(1)\right)^p \, dx' = \\ &= \int_{A'_\epsilon} \left[F(-\nabla g(x'), 1) - F(\underline{0}, 1)\right] \left(\left(r^F_{\max}(\Omega)\right)^p + o(1)\right) \, dx' = \left(r^F_{\max}(\Omega)\right)^p \Delta P_F(\Omega) + o(1)\Delta P_F = \\ &= \left(r^F_{\max}(\Omega)\right)^p \Delta P_F + o(\Delta P_F). \end{aligned}$$

We study now I_1.

From the convexity inequality we have

$$\left(F^o(x', g(x'))\right)^p - \left(F^o(x', y_0 - \epsilon)\right)^p \ge p\left(F^o(x', y_0 - \epsilon)\right)^{p-1} \langle \nabla F^o(x', y_0 - \epsilon), \left(\underline{0}, h(x')\right)\rangle.$$

Using the last convexity inequality we have

$$\begin{aligned} I_1 &= \int_{A'_\epsilon} \left[\left(F^o(x', g(x'))\right)^p - \left(F^o(x', y_0 - \epsilon)\right)^p\right] F(-\nabla g(x'), 1) \, dx' \ge \\ &\ge \int_{A'_\epsilon} p\left(F^o(x', y_0 - \epsilon)\right)^{p-1} \langle \nabla F^o(x', y_0 - \epsilon), (0, h(x'))\rangle F(-\nabla g(x'), 1) \, dx' = \\ &= \int_{A'_\epsilon} p\left(F^o(x', y_0 - \epsilon)\right)^{p-1} \frac{\partial F^o}{\partial x_n}(x', y_0 - \epsilon) h(x') F(-\nabla g(x'), 1) \, dx'. \end{aligned}$$

Claim 2

$$\frac{\partial F^o}{\partial x_n}(x', y_0 - \epsilon) = \frac{F^o(x'_0, y_0)}{(y_0 - \epsilon)} + o(1).$$

Using Taylor and the property $\langle \nabla F^o(\xi), \xi\rangle = F^o(\xi)$, we have that

$$\begin{aligned} F^o(x'_0, y_0 - \epsilon) &= F^o(x', y_0 - \epsilon) + o(1) = \langle \nabla F^o(x', y_0 - \epsilon), (x', y_0 - \epsilon)\rangle + o(1) = \\ &= \langle \nabla_{x'} F^o(x', y_0 - \epsilon), x'\rangle + (y_0 - \epsilon)\frac{\partial F^o}{\partial x_n}(x', y_0 - \epsilon) + o(1), \end{aligned}$$

and consequently

$$\frac{\partial F^o}{\partial x_n}(y, y_0 - \epsilon) = \frac{F^o(x'_0, y_0 - \epsilon)}{(y_0 - \epsilon)} - \frac{1}{(y_0 - \epsilon)}\langle \nabla_{x'} F^o(x', y_0 - \epsilon), x'\rangle + o(1). \tag{4.12}$$

Considering the fact that $\nabla F^o(x'_0, y_0) = (\underline{0}, 1)$, we have that

$$\langle \nabla_{x'} F^o(x', y_0 - \epsilon), x' \rangle = \sum_{i=1}^{n-1} x_i \frac{\partial F^o}{\partial x_i}(x', y_0 - \epsilon) = \sum_{i=1}^{n-1} x_i \left(\frac{\partial F^o}{\partial x_i}(x'_0, y_0) + o(1) \right) = o(1).$$

So, from (4.12) and Claim 1, we obtain the claim

$$\frac{\partial F^o}{\partial x_n}(x', y_0 - \epsilon) = \frac{F^o(x'_0, y_0)}{(y_0 - \epsilon)} + o(1).$$

Claim 3

$$F(-\nabla g(x'), 1) = F(\underline{0}, 1) + o(1).$$

Using Taylor and the facts that $\nabla g = \nabla h$ is continuous and $\nabla h(x'_0) = 0$, we obtain

$$F(-\nabla h(x'), 1) = F(\underline{0}, 1) + \langle \nabla F(\underline{0}, 1), (-\nabla h(x'), 0) \rangle + o(|| - \nabla h(x')||) = F(\underline{0}, 1) + o(1),$$

Using Claim 1, Claim 2 and Claim 3:

$$\begin{aligned}
I_1 &\geq \int_{A'_\epsilon} p \left(r^F_{\max}(\Omega) + o(1) \right)^{p-1} \left(\frac{F^o(x'_0, y_0)}{(y_0 - \epsilon)} + o(1) \right) h(x') \left(F(\underline{0}, 1) + o(1) \right) \, dx' \\
&= \int_{A'_\epsilon} p \left(r^F_{\max}(\Omega) \right)^{p-1} \frac{F^o(x'_0, y_0)}{(y_0 - \epsilon)} h(y) F(\underline{0}, 1) \, dy + o(-\Delta V) \geq \\
&\geq \int_{A'_\epsilon} p \left(r^F_{\max}(\Omega) \right)^{p-1} \frac{y_0}{y_0 - \epsilon} h(x') \, dx' + o(-\Delta V) \geq p \left(r^F_{\max}(\Omega) \right)^{p-1} (-\Delta V) + o(-\Delta V),
\end{aligned}$$

where we have used the fact that

$$F(\underline{0}, 1) F^o(x'_0, y_0) \geq |\langle (\underline{0}, 1), (x'_0, y_0) \rangle| = y_0.$$

□

Lemma 4.15. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then,*

$$\frac{M_F(\Omega)}{P_F(\Omega)} \leq \left(r^F_{\max}(\Omega) \right)^p$$

and equality holds if and only if Ω is a Wulff shape centered at the origin.

Proof. If Ω is a Wulff shape, then

$$\frac{M_F(\Omega)}{P_F(\Omega)} = \left(r^F_{\max}(\Omega) \right)^p \frac{P_F(\Omega)}{P_F(\Omega)} = \left(r^F_{\max}(\Omega) \right)^p.$$

If Ω is not a Wulff shape, consider the set

$$S := \{ x \in \partial\Omega \ : \ F^o(x) < r^F_{\max}(\Omega) \}.$$

Since F^o is a continuous function, $\mathcal{H}^{n-1}(S) > 0$ and, by definition of $r^F_{\max}(\Omega)$, we have that

$$\partial\Omega \setminus S = \{ x \in \partial\Omega \ : \ F^o(x) = r^F_{\max}(\Omega) \}.$$

Thus, we obtain

$$\begin{aligned}
\frac{M_F(\Omega)}{P_F(\Omega)} &= \frac{\displaystyle\int_S [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega \setminus S} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x)}{P_F(\Omega)} \\
&< \frac{\displaystyle\int_S \left(r^F_{\max}(\Omega) \right)^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega \setminus S} \left(r^F_{\max}(\Omega) \right)^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x)}{P_F(\Omega)} = \left(r^F_{\max}(\Omega) \right)^p
\end{aligned}$$

□

Proposition 4.16. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such that*

$$E_F(\Omega) > 0, \tag{4.13}$$

then Ω is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. Using (4.11), we have that

$$\Delta\mathcal{F} = \frac{1}{|\Omega|^{\frac{p}{n}} P_F(\Omega)} \left(\Delta M_F - \frac{\Delta P_F}{P_F(\Omega)} M_F(\Omega) - \frac{p}{n} \frac{\Delta V}{|\Omega|} M_F(\Omega) \right) + o(\Delta P_F) + o(\Delta V) = \tag{4.14}$$

$$\leq \frac{1}{|\Omega|^{\frac{p}{n}} P_F(\Omega)} \left[p \left(\left(r^F_{\max}(\Omega)\right)^{p-1} - \frac{M_F(\Omega)}{n\,|\Omega|} \right) \Delta V + \right.$$

$$\left. \left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} \right) \Delta P_F \right] + o(\Delta P_F) + o(\Delta V) =$$

$$= \frac{1}{|\Omega|^{\frac{p}{n}} P_F(\Omega)} \left[pE_F(\Omega)\Delta V + \left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} \right) \Delta P_F \right] + o(\Delta P_F) + o(\Delta V)$$

Since (4.13) holds, Ω cannot be a ball centered at the origin. From Lemma 4.15, follows that

$$(r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} > 0.$$

Considering also that $\Delta V < 0$ and $\Delta P_F < 0$, we can conclude that

$$\Delta\mathcal{F} < 0.$$

□

4.1.5 Wulff shapes are the unique minimizers having vanishing Excess

Proposition 4.17. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such that*

$$E_F(\Omega) = 0, \tag{4.15}$$

then either Ω is the Wulff shape centered at the origin or it is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. From (4.8), (4.15), (4.14), we obtain the following expression

$$\Delta\mathcal{F} = \frac{1}{|\Omega|^{\frac{p}{n}} P_F(\Omega)} \left[\left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} \right) \Delta P_F \right] + o(\Delta P_F).$$

If

$$(r^F_{\max}(\Omega))^p = \frac{M_F(\Omega)}{P_F(\Omega)},$$

then Ω is a Wulff shape centered at the origin. If $\Delta\mathcal{F} < 0$, then Ω is not a minimizer. Thus, we have proved the desired claim. □

4.2 A quantitative Weinstock inequality for convex sets

4.2.1 An isoperimetric inequality

In [15] the authors proved a weighted isoperimetric inequality, which has as a consequence the following inequality involving the boundary momentum $W(E)$ defined in (1.14). More precisely, it is proved that, if $E \subseteq \mathbb{R}^n$ is a bounded open with Lipschitz boundary, then

$$\frac{W(E)}{|E|^{\frac{n+1}{n}}} \geq n\omega_n^{-1/n}, \tag{4.16}$$

and equality holds for any ball centered at the origin. The inequality (4.16) implies that, among sets with fixed volume, the boundary momentum is minimal on balls centered at the origin.
An isoperimetric inequality for a functional involving the quantities $P(E)$, $W(E)$ and $|E|$ is proved in [82] in the planar case and then in [25] in any dimension, restricting to the class of convex sets. More precisely, if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set, it is proved that

$$\mathcal{J}(E) = \frac{W(E)}{P(E)\ |E|^{\frac{2}{n}}} \geq \omega_n^{\frac{-2}{n}} = \mathcal{J}(B_1(0)) \tag{4.17}$$

where equality holds only on balls centered at the origin.
In the same spirit, we define the following functional

$$I(E) = \frac{W(E)}{|E|\, P(E)^{\frac{1}{n-1}}}. \tag{4.18}$$

The following isoperimetric inequality holds.

Proposition 4.18. *Let $n \geq 2$. For every bounded, open, convex set $E \subset \mathbb{R}^n$, it holds*

$$I(E) \geq \frac{n}{(n\omega_n)^{\frac{1}{n-1}}} = I(B_1(0)). \tag{4.19}$$

Equality holds only for balls centered at the origin.

Proof. The proof follows easily by using inequality (4.17), the standard isoperimetric inequality and observing that

$$I(E) = \mathcal{J}(E) \left(\frac{P(E)}{|E|^{1-\frac{1}{n}}} \right)^{\frac{n-2}{n-1}}.$$

□

Our aim is to prove a quantitative version of (4.19). From now on, we will use the following notation

$$\mathcal{D}(E) = I(E) - \frac{n}{(n\omega_n)^{\frac{1}{n-1}}} = I(E) - I(B_1(0)). \tag{4.20}$$

4.2.2 Stability for nearly spherical sets

Following Fuglede's approach (see [52]), we first prove a quantitative version of (4.19) for nearly spherical sets as in Definition 1.17, when $n \geq 3$.

Theorem 4.19. *Let $n \geq 3$. There exists $\varepsilon = \varepsilon(n) > 0$, such that if $E \subseteq \mathbb{R}^n$ is a nearly spherical set with $P(E) = P(B_1)$ and $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon$, then*

$$\frac{3^n}{n\omega_n}||v||_{W^{1,1}(\mathbb{S}^{n-1})} \geq \mathcal{D}(E) \geq \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{4.21}$$

Proof. Setting $v = tu$, with $||u||_{W^{1,\infty}} = 1/2$, we have $||v||_{W^{1,\infty}} = t||u||_{W^{1,\infty}} = t/2$. Thus, using the expressions of $P(E)$ and $W(E)$ given in (1.17) and (1.19), we get

$$\mathcal{D}(E) = \frac{n}{P(B_1)^{\frac{1}{n-1}}}\left(\frac{\displaystyle\int_{\mathbb{S}^{n-1}} (1+tu(x))^n \sqrt{(1+tu(x))^2 + t^2|\nabla_\tau u(x)|^2}\, d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{S}^{n-1}} (1+tu(x))^n\, d\mathcal{H}^{n-1}} - 1\right) \tag{4.22}$$

$$= \frac{n}{P(B_1)^{\frac{1}{n-1}}}\left(\frac{\displaystyle\int_{\mathbb{S}^{n-1}} (1+tu(x))^n \left(\sqrt{(1+tu(x))^2 + t^2|\nabla_\tau u(x)|^2} - 1\right) d\mathcal{H}^{n-1}}{n\,|E|}\right).$$

Now we prove the lower bound in (4.21). Firstly we take into account the numerator in (4.22). Let $f_k(t) = (1+tu)^k\sqrt{(1+tu)^2 + t^2|\nabla_\tau u|^2}$. An elementary calculation shows that

$$f_k(0) = 1, \qquad f_k'(0) = (k+1)u, \qquad f_k''(0) = (k+1)ku^2 + |\nabla_\tau u|^2$$
$$f_k'''(\tau) \leq 2(k+2)(k+1)k\left(|u|^3 + |u||\nabla_\tau u|^2\right) \tag{4.23}$$

for any $\tau \in (0,t)$. Thus, since the numerator of (4.22) is given by $f_n(t) - (1+tu)^n$, using the Lagrange expression of the remainder term, we can Taylor expand up to the third order, obtaining

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} &(1+tu(x))^n \left(\sqrt{(1+tu(x))^2 + t^2|\nabla_\tau u(x)|^2} - 1\right) d\mathcal{H}^{n-1}\\ &\geq t\int_{\mathbb{S}^{n-1}} u\,d\mathcal{H}^{n-1} + nt^2\int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} + \frac{1}{2}t^2\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}\\ &\qquad - C_1(n)\varepsilon t^2\int_{\mathbb{S}^{n-1}} \left(u^2 + |\nabla_\tau u|^2\right) d\mathcal{H}^{n-1}.\end{aligned} \tag{4.24}$$

Since $P(E) = P(B_1)$, we have

$$\int_{\mathbb{S}^{n-1}} (1+tu(x))^{n-2}\sqrt{(1+tu(x))^2 + t^2|\nabla_\tau u(x)|^2}\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} 1\,d\mathcal{H}^{n-1}.$$

Using (4.23) for f_{n-2}, we infer

$$\begin{aligned}t\int_{\mathbb{S}^{n-1}} u\,d\mathcal{H}^{n-1} &\geq -\frac{n-2}{2}t^2\int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - \frac{t^2}{2(n-1)}\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}\\ &\qquad - C_2(n)\varepsilon t^2\int_{\mathbb{S}^{n-1}} \left(u^2 + |\nabla_\tau u|^2\right) d\mathcal{H}^{n-1}.\end{aligned} \tag{4.25}$$

Since $n \geq 3$, using inequality (4.25) in (4.24), we get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} &(1+tu(x))^n \left(\sqrt{(1+tu(x))^2 + t^2|\nabla_\tau u(x)|^2} - 1\right) d\mathcal{H}^{n-1}\\ \geq &\left(\frac{n+2}{2} - C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} + \left(\frac{n-2}{2(n-1)} - C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}\\ &\geq \left(\frac{n-2}{2(n-1)} - C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} u^2 + |\nabla_\tau u|^2 d\mathcal{H}^{n-1},\end{aligned} \tag{4.26}$$

where $C_3(n) = C_1(n) + C_2(n)$. Choosing $\varepsilon = \frac{n-2}{4C_3(n-1)}$, we obtain

$$\mathcal{D}(E) \geq \frac{n-2}{4(n-1)} ||tu||^2_{W^{1,2}(\mathbb{S}^{n-1})} = \frac{n-2}{4(n-1)} ||v||^2_{W^{1,2}(\mathbb{S}^{n-1})},$$

which is the lower bound in (4.21). Then, recalling that $\|v\|_\infty \leq \frac{1}{2}$ we have

$$\begin{aligned}
\frac{W(E)}{n\,|E|} - 1 &= \frac{\int_{\mathbb{S}^{n-1}} (1+v(x))^n \left(\sqrt{(1+v(x))^2 + |\nabla_\tau v(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}}{n\,|E|} \\
&\leq \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} \left(\sqrt{(1+|v(x)|)^2 + |\nabla_\tau v(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}}{n\,|E|} \\
&\leq \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} \left(\sqrt{(1+|v(x)| + |\nabla_\tau v(x)|)^2} - 1 \right) \, d\mathcal{H}^{n-1}}{n\,|E|} \\
&\leq \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} (|v(x)| + |\nabla_\tau v(x)|) \, d\mathcal{H}^{n-1}}{n\,|E|} \leq \frac{3^n}{n\omega_n} ||v||_{W^{1,1}(\mathbb{S}^{n-1})},
\end{aligned}$$

where last inequality follows from the following estimate

$$n\,|E| = \int_{\mathbb{S}^{n-1}} (1+v(x))^n \, d\mathcal{H}^{n-1} \geq n\omega_n \left(\frac{1}{2}\right)^n.$$

□

Remark 4.20. Observe that the proof of the lower bound in (4.21) does not seem to work in the planar case. The reason is that for $n = 2$ the coefficient of $\|\nabla_\tau u\|^2_{L^2(\mathbb{S}^{n-1})}$ in (4.26) could be negative.

4.2.3 Stability for convex sets

Before completing the proof of the quantitative version of the inequality (4.19), we need the following useful technical lemmas.

Lemma 4.21. *Let $n \geq 2$. There exists $M > 0$ such that, if $K \subseteq \mathbb{R}^n$ is an open, convex set with finite perimeter and $I(K) \leq \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, then $K \subset Q_M$, where Q_M is the hypercube centered at the origin with edge M.*

Proof. Since the functional is scale invariant, we can assume $|K| = 1$. Let $L > 1$, we have

$$\begin{aligned}
W(K) = \int_{\partial K} |x|^2 d\mathcal{H}^{n-1} &= \int_{(\partial K)\cap Q_L} |x|^2 d\mathcal{H}^{n-1} + \int_{\partial K \setminus Q_L} |x|^2 d\mathcal{H}^{n-1} \\
&\geq \int_{\partial K \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(K; C(Q_L)),
\end{aligned}$$

where by $C(Q_L)$ we denote the complementary set of Q_L in $\mathbb{R}^n$. Since K is convex, also $K \cap Q_L$ is convex and then

$$P(K) \leq P(K; C(Q_L)) + P(K; Q_L) \leq P(K; C(Q_L)) + 2nL^{n-1}, \tag{4.27}$$

by the monotonicity of the perimeter. Suppose $P(K) > L^n$; then, equation (4.27) gives $P(K; C(Q_L)) \geq L^n - 2nL^{n-1}$ and, as a consequence,

$$I(K) \geq \frac{\displaystyle\int_{\partial K \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(K; C(Q_L))}{(P(K; C(Q_L)) + 2nL^{n-1})^{\frac{1}{n-1}}} > \frac{L^{n+2} - L^{n+1}}{L^{\frac{n}{n-1}}}.$$

The previous inequality leads to a contradiction for L large enough, since we are assuming $I(K) < \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, while the last term of the above inequality diverges when $L \to \infty$. Thus, there exists L_0 such that, for every convex set K with $I(K) \leq \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, we have $P(K) < L_0^n$. Since $|K| = 1$ and $P(K) \leq L_0^n$, using (1.16), we get

$$\operatorname{diam}(K) \leq C(n) L_0^{n(n-1)}.$$

The last inequality proves (4.27), if we choose $M = C(n) L_0^{n(n-1)}$. □

Lemma 4.22. *Let $(K_j) \subseteq \mathbb{R}^n$, $n \geq 2$, be a sequence of convex sets such that $I(K_j) \leq \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$ and $P(K_j) = P(B_1)$. Then, there exists a convex set $K \subseteq \mathbb{R}^n$ with $P(K) = P(B_1)$ and such that, up to a subsequence,*

$$|K_j \Delta K| \to 0 \quad \textit{and} \quad I(K) \leq \liminf I(K_j). \tag{4.28}$$

Proof. The existence of the limit set K comes from the proof of Lemma 4.21: since $I(K_j) < \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, there exists $M > 0$ such that $K_j \subset Q_M$ and $P(K_j) = P(B_1)$ for every $i \in \mathbb{N}$. Thus, the sequence $\{\chi_{K_j}\}_{j\in\mathbb{N}}$ is precompact in $BV(Q_M)$ and so there exists a subsequence and a set K such that $|K \Delta K_j| \to 0$. Moreover, from Lemma 1.15, we have that $P(K) = P(B_1)$. Note that we can write

$$W(K) = \sup\left\{ \int_K \operatorname{div}\left(|x|^2 \phi(x)\right) dx, \quad \phi \in C_c^1(Q_M, \mathbb{R}^n), \quad ||\phi||_\infty \leq 1 \right\}.$$

Observing that

$$\int_K |\operatorname{div}\left(|x|^2 \phi(x)\right)| dx \leq M ||\operatorname{div}\phi||_\infty + M^2,$$

using the dominate convergence theorem, we have that the functional

$$K \to \int_K \operatorname{div}\left(|x|^2 \phi(x)\right) dx$$

is continuous with respect to the L^1 convergence. Hence, since $W(K)$ is obtained by taking the supremum of continuous functionals, it is lower semicontinuous. As a consequence, we obtain the inequality (4.28). □

The next result allows us to reduce the study of the stability issue to nearly spherical sets.

Lemma 4.23. *Let $n \geq 2$. For every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set with $P(E) = P(B_1)$ and $\mathcal{D}(E) < \delta_\varepsilon$, with $\mathcal{D}(E)$ defined as in (4.20), then there exists a Lipschitz function $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that E is a nearly spherical set parameterized by v and $\|v\|_{W^{1,\infty}} \leq \varepsilon$.*

Proof. Firstly, we prove that $d_{\mathcal{H}}(E, B_1(0)) < \varepsilon$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that, for every $j \in \mathbb{N}$, there exists a convex set E_j with $I(E_j) - \dfrac{n}{(n\omega_n)^{\frac{1}{n-1}}} < \dfrac{1}{j}$, $d_{\mathcal{H}}(E_j, B_1(0)) \geq \varepsilon_0$ and $P(E_j) = P(B_1)$. By Lemma 4.22, we have that there exists a convex set E such that E_j converges to E in measure and $P(E) = P(B_1)$. From the semicontinuity of $W(E)$, we have that $I(E) \leq \liminf I(E_j) \leq \dfrac{n}{(n\omega_n)^{\frac{1}{n-1}}}$. Since $B_1(0)$ is the only minimizer of the functional I, we obtain the contradiction. Then, since E is convex and $d_{\mathcal{H}}(E, B_1(0)) \leq \varepsilon$, E contains the origin and so there exists a Lipschitz function $v \in L^\infty(\mathbb{S}^{n-1})$, with $||v||_\infty < \varepsilon$, such that

$$\partial E = \{x(1 + v(x)),\, x \in \mathbb{S}^{n-1}\}.$$

Now, in order to complete the proof, we have only to show that $\|v\|_{W^{1,\infty}(\mathbb{S}^{n-1})}$ is small when $\mathcal{D}(E)$ is small. This is a consequence of Lemma 1.19. $\square$

Now we can prove the stability result for the inequality (4.19). We first consider the case $n \geq 3$. The two dimensional case will be discussed separately in the next section.

Theorem 4.24. *Let $n \geq 3$. There exists $\delta > 0$ such that if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set with $\mathcal{D}(E) \leq \delta$, then*

$$\left(\frac{n\omega_n}{P(E)}\right)^{1/(n-1)} d_{\mathcal{H}}(E, E^*) \leq \begin{cases} \beta\sqrt{\mathcal{D}(E)\log\frac{1}{\mathcal{D}(E)}} & n = 3 \\ \beta_n\,(\mathcal{D}(E))^{\frac{2}{n+1}} & n \geq 4, \end{cases} \tag{4.29}$$

where $\mathcal{D}(E)$ is defined in (4.20) *and E^* is the ball centered at the origin with $P(E^*) = P(E)$ and*

$$\beta = \frac{2^{-\frac{29}{6}}}{9}, \quad \beta_n = \left(\frac{n-2}{4(n-1)C_n^{\frac{1}{n-1}}}2^{-\frac{5n-7}{2}}\right)^{\frac{2}{n+1}}\left(16\left(\frac{9}{8}\right)^n + n + 1\right)^{-1} \tag{4.30}$$

Remark 4.25. We observe that inequality (4.29) implies the following

$$\mathcal{A}_{\mathcal{H}}(E) \leq \begin{cases} \beta\sqrt{\mathcal{D}(E)\log\frac{1}{\mathcal{D}(E)}} & n = 3 \\ \beta_n\,(\mathcal{D}(E))^{\frac{2}{n+1}} & n \geq 4, \end{cases} \tag{4.31}$$

where $\mathcal{A}_{\mathcal{H}}(E)$ is the asymmetry defined in (1.15). We emphasize that (4.29) and (4.31) are not equivalent, because $\mathcal{A}_{\mathcal{H}}(E)$ is in general different from $d_{\mathcal{H}}(E, E^*)$, since one does not know where is centered the optimal ball for (1.15). For instance, if E is a ball not centered at the origin, we have that $\mathcal{A}_{\mathcal{H}}(E) = 0$, but $d_{\mathcal{H}}(E, E^*) > 0$. On the other hand, since the functional $I(\cdot)$ is not translational invariant, it admits a very unique minimizer once a value of the perimeter is fixed, that is the ball centered at the origin and with the right radius. Thus, it seems more reasonable to use $d_{\mathcal{H}}(E, E^*)$ in (4.29), since it measures how different is the set E from *the* minimizer of $I(\cdot)$.

Proof. Since the functional I is scaling invariant, we can suppose that E is a convex set with $P(E) = P(B_1)$. We fix now $\varepsilon > 0$. Using Lemma 4.23, we can suppose that there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ with $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$ such that

$$\partial E = \{x(1 + v(x)),\, x \in \mathbb{S}^{n-1}\}.$$

Then, if we take ε small enough, by Theorem 4.19, we obtain

$$\mathcal{D}(E) \geq \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})}.$$

Let $K = \lambda E$, with λ such that $|K| = |B_1|$. From the isoperimetric inequality, it follows that $\lambda > 1$. Since the quantity $I(E)$ is scaling invariant, we have that $I(K) = I(E)$ and, from the definition of K, that

$$\partial K = \{\lambda x(1+v(x)), x \in \mathbb{S}^{n-1}\} = \{x(1+(\lambda - 1 + \lambda v(x))), x \in \mathbb{S}^{n-1}\}.$$

Using the definition of λ , we obtain

$$\lambda^n - 1 = \frac{|B_1|}{|E|} - 1 = \frac{\sum_{k=1}^n \binom{n}{k} \int_{\mathbb{S}^{n-1}} v^k d\mathcal{H}^{k-1}}{|E|}$$

and, as a consequence,

$$\lambda - 1 = \frac{\sum_{k=1}^n \binom{n}{k} \int_{\mathbb{S}^{n-1}} v^k d\mathcal{H}^{k-1}}{|E| \sum_0^{n-1} \lambda^k}.$$

Let now $h(x) = \lambda - 1 + \lambda v(x)$. Note that $||h||_{W^{1,\infty}(\mathbb{S}^{n-1})} < 2^n ||v||_{W^{1,\infty}(\mathbb{S}^{n-1})}$ and that $\lambda^n \in (1,2)$. Moreover, using Hölder inequality, it is easy to check that

$$||h||^2_{L^2(\mathbb{S}^{n-1})} \leq 2^{n+2}||v||^2_{L^2(\mathbb{S}^{n-1})} \quad \text{and} \quad ||\nabla_\tau h||^2_{L^2(\mathbb{S}^{n-1})} \leq 2^{1/n}||\nabla_\tau v||^2_{L^2(\mathbb{S}^{n-1})}.$$

Thus,

$$\mathcal{D}(K) = \mathcal{D}(E) \geq \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})} \geq 2^{-n-1}\frac{n-2}{4(n-1)}||h||^2_{W^{1,2}(\mathbb{S}^{n-1})}.$$

Let $g = (1+h)^n - 1$. Then, since $|K| = |B_1|$, we have $\int_{\mathbb{S}^{n-1}} g d\mathcal{H}^{n-1} = 0$ and, from the smallness assumption on u, we immediately have $\frac{1}{2}|h| \leq |g| \leq 2|h|$ and $\frac{1}{2}|\nabla_\tau h| \leq |\nabla_\tau g| \leq 2|\nabla_\tau h|$. Now we have to distinguish the cases $n = 3$ and $n \geq 4$, since we are going to apply the interpolation Lemma 1.18 to g. In the case $n \geq 4$, recalling that C_n is the constant given by the Sobolev embedding in Lemma 1.18, we get

$$\begin{aligned}
||h||_{L^\infty(\mathbb{S}^{n-1})} &\leq 2||g||_{L^\infty(\mathbb{S}^{n-1})} \leq 2C_n ||\nabla_\tau g||^{\frac{2}{n-1}}_{L^2(\mathbb{S}^{n-1})} ||\nabla_\tau g||^{\frac{n-3}{n-1}}_{L^\infty(\mathbb{S}^{n-1})} \\
&\leq C_n ||\nabla_\tau h||^{\frac{2}{n-1}}_{L^2(\mathbb{S}^{n-1})} ||\nabla_\tau h||^{\frac{n-3}{n-1}}_{L^\infty(\mathbb{S}^{n-1})} \leq 8^{\frac{n-3}{2(n-1)}} C_n ||\nabla_\tau h||^{\frac{2}{n-1}}_{L^2(\mathbb{S}^{n-1})} ||h||^{\frac{n-3}{2(n-1)}}_{L^\infty(\mathbb{S}^{n-1})},
\end{aligned}$$

where in the last inequality we use (1.20). From the above chain of inequalities we deduce

$$||h||^{\frac{n+1}{2}}_{L^\infty(\mathbb{S}^{n-1})} \leq 8^{\frac{n-3}{2}} C_n^{\frac{1}{n-1}} ||\nabla_\tau h||^2_{L^2(\mathbb{S}^{n-1})}$$

and finally, recalling that $K = \lambda E$ and $|K| = |B_1|$, we get

$$\begin{aligned}
\mathcal{D}(E) &\geq 2^{-n-1}\frac{n-2}{4(n-1)}||\nabla_\tau h||^2_{L^2(\mathbb{S}^{n-1})} \geq \gamma_n ||h||^{\frac{n+1}{2}}_{L^\infty(\mathbb{S}^{n-1})} \\
&= \gamma_n d_{\mathcal{H}}(K, B_1(0))^{\frac{n+1}{2}} = \gamma_n \left(\frac{d_{\mathcal{H}}(E, E^\sharp)}{|E|^{\frac{1}{n}}}\right)^{\frac{n+1}{2}},
\end{aligned}$$

where $\gamma_n = \dfrac{n-2}{4(n-1)C_n^{\frac{1}{n-1}}} 2^{-\frac{5n-7}{2}}$.

So, using (1.21) and the isoperimetric inequality, we obtain the desired result (4.29) in the case $n \geq 4$. We proceed in an analogous way in the case $n = 3$. Firstly we observe that, by definition of h it is quickly checked that $||v||_{W^{1,1}(\mathbb{S}^2)} \leq ||h||_{W^{1,1}(\mathbb{S}^2)}$. Then, the upper bound in (4.29) in terms of h, can be written as follows

$$\mathcal{D}(E) = \mathcal{D}(K) \leq \bar{C}||h||_{W^{1,1}(\mathbb{S}^2)}, \tag{4.32}$$

with $\bar{C}$ positive constant depending on the dimension. Applying Lemma 1.18 to g and using Lemma 1.19, we obtain:

$$\begin{aligned}
||h||^2_{L^\infty(\mathbb{S}^2)} &\leq 4||g||^2_{L^\infty(\mathbb{S}^2)} \leq 16||\nabla_\tau g||^2_{L^2(\mathbb{S}^2)} \log\left[\frac{8e||\nabla_\tau g||^2_{L^\infty(\mathbb{S}^2)}}{||\nabla_\tau g||^2_{L^2(\mathbb{S}^2)}}\right] \\
&\leq 64||\nabla_\tau h||^2_{L^2(\mathbb{S}^2)} \log\left[\frac{2^7e||\nabla_\tau h||^2_{L^\infty(\mathbb{S}^2)}}{||\nabla_\tau h||^2_{L^2(\mathbb{S}^2)}}\right] \leq 64||\nabla_\tau h||^2_{L^2(\mathbb{S}^2)} \log\left[\frac{2^{10}e\ ||v||_{L^\infty(\mathbb{S}^2)}}{||\nabla_\tau v||^2_{L^2(\mathbb{S}^2)}}\right].
\end{aligned}$$

Choosing now $||h||_{L^\infty(\mathbb{S}^2)}$ small enough, from the upper bound in (4.21), we have

$$||h||^2_{L^\infty(\mathbb{S}^2)} \leq 64||\nabla_\tau h||^2_{L^2(\mathbb{S}^2)} \log\left[\frac{1}{\mathcal{D}(E)}\right], \tag{4.33}$$

and, as a consquence, using (4.21) and (4.33),

$$\begin{aligned}
\mathcal{D}(E) \log\left(\frac{1}{\mathcal{D}(E)}\right) &\geq \frac{1}{8}||\nabla_\tau v||^2_{L^2(\mathbb{S}^2)} \log\left(\frac{1}{\mathcal{D}(E)}\right) \\
&\geq 2^{-\frac{29}{3}}||h||^2_{L^\infty(\mathbb{S}^2)} \frac{\log\left(\frac{1}{\mathcal{D}(E)}\right)}{\log\left(\frac{1}{\mathcal{D}(E)}\right)} = 2^{-\frac{29}{3}}||h||^2_{L^\infty(\mathbb{S}^2)}.
\end{aligned}$$

□

4.2.4 Optimality issue

In this section we will show the sharpness of inequality (4.29) and, as a consequence, the sharpness for the exponent in inequality (4.29). We start by taking into exam the case $n = 3$.

Theorem 4.26. *Let $n = 3$. There exists a family of convex sets $\{E_\alpha\}_{\alpha>0}$ such that for every α*

$$\mathcal{D}(E_\alpha) \to 0, \quad \text{when } \alpha \to 0$$

and

$$d_{\mathcal{H}}(E_\alpha, E^*_\alpha) = C\sqrt{\mathcal{D}(E_\alpha) \log\frac{1}{\mathcal{D}(E_\alpha)}} \tag{4.34}$$

where C is a suitable positive constant independent of α.

Proof. We follow the idea contained in [52, Example 3.1] and recall it here for the convenience of the reader. Let $\alpha \in (0, \pi/2)$ and consider the following function $\omega = \omega(\varphi)$ defined over $\mathbb{S}^2$ and depending only on the spherical distance φ, with $\varphi \in [0, \pi]$, from a prescribed north pole $\xi^* \in \mathbb{S}^2$:

$$\omega = \omega(\varphi) = \begin{cases} -\sin^2\alpha \log(\sin\alpha) + \sin\alpha\,(\sin\alpha - \sin\varphi) & \text{for } \sin\varphi \leq \sin\alpha \\ -\sin^2(\alpha) \log(\sin\varphi) & \text{for } \sin\varphi \geq \sin\alpha. \end{cases}$$

Let $g := \omega - \bar{\omega}$, with $\bar{\omega}$ the mean value of ω, i.e.

$$\bar{\omega} = \int_0^{\pi/2} \omega(\varphi) \sin\varphi \, d\varphi = (1 - \log 2)\, \alpha^2 + O(\alpha^3),$$

when α goes to 0, and let

$$R := (1 + 3g)^{1/3} = 1 + h.$$

The C^1 function $R = R(\varphi)$ determines in polar coordinates (R, φ) a planar curve. We rotate this curve about the line $\xi^*\mathbb{R}$, determining in this way the boundary of a convex and bounded set, that we call E_α. We can observe that h and g are the same functions contained in the proof of Theorem 4.24. The set E_α is indeed a nearly spherical set, which has h as a representative function and with $|E_\alpha| = |B_1|$. Therefore, taking into account the computations contained in the proof of Theorem 4.24 relative to the functions h and g and the ones contained in [52] combined with (4.21), we have

$$||g||_{L^\infty(\mathbb{S}^2)} = \alpha^2 \log \frac{1}{\alpha} + O(\alpha^2),$$

$$||h||_{L^\infty(\mathbb{S}^2)} \geq \frac{1}{2} ||g||_{L^\infty(\mathbb{S}^2)} = \frac{1}{2}\alpha^2 \log \frac{1}{\alpha} + O(\alpha^2), \tag{4.35}$$

and

$$||\nabla_\tau h||^2_{L^2(\mathbb{S}^2)} = ||\nabla_\tau g||^2_{L^2(\mathbb{S}^2)} = \alpha^4 \log\left(\frac{1}{\alpha}\right) + O(\alpha^4).$$

Using (4.32), we obtain:

$$\mathcal{D}(E_\alpha) = O\left(\alpha^4 \log \frac{1}{\alpha}\right)$$

Consequently

$$\mathcal{D}(E_\alpha) \log\left(\frac{1}{\mathcal{D}(E_\alpha)}\right) = O\left(\alpha^2 \log \frac{1}{\alpha}\right)^2. \tag{4.36}$$

So, we have that $\mathcal{D}(E_\alpha) \to 0$ as α goes to 0 and, combining (4.35) with (4.36), we have the validity of (4.34). $\square$

We show now the sharpness of the quantitative Weinstock inequality in dimension $n \geq 4$.

Theorem 4.27. *Let $n \geq 4$. There exists a family of convex sets $\{P_\alpha\}_{\alpha>0}$ such that*

$$\mathcal{D}(P_\alpha) \to 0, \quad \textit{when} \quad \alpha \to 0$$

and

$$d_{\mathcal{H}}(P_\alpha, P_\alpha^*) \geq C(n)\, (\mathcal{D}(P_\alpha))^{2/(n+1)},$$

where $C(n)$ is a suitable positive constant.

Proof. In this proof we follow the construction given in [52, Example 3.2]. Let $\alpha \in]0, \pi/2[$ and let P_α be the convex hull of $B_1(0) \cup \{-p, p\}$, where $p \in \mathbb{R}^n$ is given by

$$|p| = \frac{1}{\cos\alpha}.$$

We have that

$$|P_\alpha| = \omega_n + \frac{2}{n(n+1)} \omega_{n-1} \alpha^{n+1} + O(\alpha^{n+3})$$

and

$$P(P_\alpha) = nV(P_\alpha).$$

We provide here the computation of the boundary momentum, that is

$$W(P_\alpha) = \frac{2\omega_{n-1}}{n(n+1)} \frac{(\sin(\alpha))^{(n-1)}}{\cos(\alpha)} \left(n^2 + n + 2\tan^2(\alpha)\right) \\ + 2(n-1)\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} - \int_0^\alpha \sin^{n-2}(\theta)\, d\theta\right].$$

Since $n > 2$, we have

$$(n\omega_n)^{\frac{1}{n-1}} V(P_\alpha) P(P_\alpha)^{\frac{1}{n-1}} \mathcal{D}(P_\alpha) = (n\omega_n)^{\frac{1}{n-1}} \frac{2\omega_{n-1}}{n+1} \frac{(n-2)}{n(n-1)} \alpha^{n+1} + o(\alpha^{n+3}).$$

Since $d_{\mathcal{H}}(P_\alpha, P_\alpha^*)$ behaves asimptotically as α^2, we have proved the desired claim. $\square$

4.2.5 The planar case

In this section we discuss the stability of the isoperimetric inequality (4.19) in the plane. This case is treated in a different way since the proof given in Section 3 does not seem to be adapted to the planar case, as explained in Remark 4.20. Moreover, we observe that, in two dimension, the inequality (4.17) contained in [25] and the inequality (4.19) are proved by Weinstock in [82], using the representation of a two dimensional convex set via its support function. We recall here the definition of support function (see for instance [75]).

Definition 4.28. *Let $E \subset \mathbb{R}^2$ be a closed convex set with $\emptyset \neq E \neq \mathbb{R}^2$. The support function of E is defined by*

$$h(\theta) := \sup\{\langle x, \theta\rangle \ : \ x \in E\} \quad \text{for} \ \ \theta \in \mathbb{S}^1,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$.

Let $E \subset \mathbb{R}^2$ be an open, convex set in the plane containing the origin and let $h(\theta)$ be the support function of E with $\theta \in [0, 2\pi]$. Weinstock proved in [82] the following inequality (see also [25] for details)

$$\pi W(E) - P(E)\,|E| \geq \frac{P(E)}{2} \int_0^{2\pi} p^2(\theta)\, d\theta \geq 0, \tag{4.37}$$

where, for every $\theta \in [0, 2\pi]$, $p(x)$ is defined by

$$h(\theta) = \frac{P(E)}{2\pi} + p(\theta).$$

By the definition of support function, it holds

$$\int_0^{2\pi} h(\theta)\, d\theta = P(E).$$

Moreover, since E is convex, we have

$$h(\theta) + h''(\theta) \geq 0,$$

where h'' has to be understood in the distributional sense. Then, the function p verifies

$$\int_0^{2\pi} p(\theta)\, d\theta = 0,$$

and

$$\frac{P(E)}{2\pi} + p(\theta) + p''(\theta) \geq 0. \tag{4.38}$$

We observe that

$$\|p\|_{L^\infty([0,2\pi])} = d_{\mathcal{H}}(E, E^*), \tag{4.39}$$

where E^* is the disc centered at the origin having the same perimeter as E. Consider $\theta_0 \in [0, 2\pi]$ such that $\|p\|_{L^\infty([0,2\pi])} = p(\theta_0)$. By using property (4.38), it is not difficult to prove the following result.

Proposition 4.29. *Let p be as above, then*

$$p(\theta) \geq \gamma(\theta),$$

where $\gamma(\theta) := p(\theta_0) - \frac{1}{2}\left(\frac{P(E)}{2\pi} + p(\theta_0)\right)(\theta - \theta_0)^2$ is a parabola which vanishes at the following points

$$\theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}}.$$

Proof. By property (4.38), we obtain

$$\begin{aligned} p(\theta) = p(\theta_0) + \int_{\theta_0}^{\theta} p'(t)\, dt = p(\theta_0) + \int_{\theta_0}^{\theta}\int_{\theta_0}^{t} p''(s)\, ds\, dt \\ \geq p(\theta_0) + \int_{\theta_0}^{\theta}\int_{\theta_0}^{t} -\left(\frac{P(E)}{2\pi} + p(s)\right) ds\, dt \\ \geq p(\theta_0) - \left(\frac{P(E)}{2\pi} + p(\theta_0)\right)\frac{(\theta - \theta_0)^2}{2}, \end{aligned}$$

which is the claim. Then, p is above the parabola γ, that attains its zeros at the following points:

$$\theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}}.$$

This concludes the proof. □

Inequality (4.37) implies Weinstock inequality but it hides also a stability result. Indeed, by using the previous Proposition, we get the following quantitative Weinstock inequality in the plane.

Theorem 4.30. *There exists $\delta > 0$ such that, if $E \subset \mathbb{R}^2$ is a bounded, open, convex set with $\mathcal{D}(E) \leq \delta$, then*

$$\frac{16}{9\pi^2}\left(2\pi\frac{d_{\mathcal{H}}(E, E^*)}{P(E)}\right)^{\frac{5}{2}} \leq \mathcal{D}(E),$$

where $\mathcal{D}(E)$ is defined in (4.20). Moreover, the exponent $\frac{5}{2}$ is sharp.

Proof. Since the functional $\mathcal{D}$ is scaling invariant, we can assume that E is a convex set of finite measure with $P(E) = 2\pi$. From Lemma 4.23, if we take a sufficiently small ε, there exists $\delta > 0$ such that, if $\mathcal{D}(E) \leq \delta$, then E contains the origin, its boundary can be parametrized as above by means of the support function and, by (4.39),

$$d := \|p\|_{L^\infty([0,2\pi])} \leq \varepsilon.$$

Under these assumptions, since in particular $|d| < \frac{1}{2}$, Proposition 4.29 gives

$$p(\theta) \geq d - \left(\frac{1+d}{2}\right)(\theta-\theta_0)^2 \geq d - \frac{(\theta-\theta_0)^2}{4}. \tag{4.40}$$

Denoting by $\theta_{1,2}$ the zeros of the parabola $d - \frac{(\theta-\theta_0)^2}{4}$, that are

$$\theta_{1,2} = \theta_0 \pm 2\sqrt{d},$$

by using (4.37), the isoperimetric inequality, Hölder inequality and (4.40), we get

$$\begin{aligned}\mathcal{D}(E) = \frac{W(E)}{P(E)\,|E|} - \frac{1}{\pi} = \frac{\pi W(E) - P(E)\,|E|}{\pi P(E)\,|E|} &\geq \frac{1}{2\pi^2}\int_0^{2\pi} p^2(\theta)\,d\theta \\ &> \frac{1}{2\pi^2}\int_{\theta_1}^{\theta_2} p^2(\theta)\,d\theta \geq \frac{1}{2\pi^2(\theta_2-\theta_1)}\left(\int_{\theta_1}^{\theta_2} p(\theta)\,d\theta\right)^2 > \frac{16}{9\pi^2}d^{\frac{5}{2}}.\end{aligned}$$

By (4.39) and (1.15), being $P(E) = 2\pi$, we get the claim. In order to conclude the proof, we have to show the sharpness of the exponent. We construct a family of convex sets E_ε, with $P(E_\varepsilon) = 2\pi$, such that

$$\mathcal{D}(E_\varepsilon) \to 0 \text{ for } \varepsilon \to 0,$$

and

$$\|p\|_{L^\infty([0,2\pi])} = \varepsilon + o\left(\varepsilon^{\frac{3}{2}}\right)$$

Let us consider the convex set E having the following support function:

$$h(\theta) = 1 + p(\theta), \quad \theta \in [0, 2\pi],$$

where the function p is the following

$$p(\theta) = \begin{cases} b & \text{if } \theta \in [0, \pi-\alpha] \\ c - \frac{(\theta-\pi)^2}{4} & \text{if } \theta \in [\pi-\alpha, \pi+\alpha] \\ b & \text{if } \theta \in [\pi+\alpha, 2\pi]. \end{cases}$$

Here the parameters α, b and c are

$$\alpha = 2\sqrt{\varepsilon}, \quad b = -\frac{4}{3\pi}\varepsilon^{\frac{3}{2}}, \quad c = \varepsilon - \frac{4}{3\pi}\varepsilon^{\frac{3}{2}}.$$

By construction, we have that

$$P(E_\varepsilon) = 2\pi \quad \text{and} \quad \int_0^{2\pi} p(\theta)\,d\theta = 0.$$

We recall that (see for instance [82, 83, 25])

$$\begin{cases} |E_\varepsilon| = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \left(h^2(\theta) + h(\theta)h''(\theta)\right)\,d\theta \\ W(E_\varepsilon) = \displaystyle\int_0^{2\pi}\left(h^3(\theta) + \frac{1}{2}h^2(\theta)h''(\theta)\right)\,d\theta. \end{cases}$$

Arguing as in the proof of Weinstock inequality, a simple calculation gives

$$\begin{aligned}\pi W(E_\varepsilon) - P(E_\varepsilon)\,|E_\varepsilon| &= \pi\int_0^{2\pi} p^2(\theta)\left(2 + p(\theta) + \frac{1}{2}p''(\theta)\right)\,d\theta \\ &= 2\pi\int_0^{2\pi} p^2(\theta)\,d\theta + \pi\int_0^{2\pi} p^3(\theta)\,d\theta + \frac{\pi}{2}\int_0^{2\pi} p^2(\theta)p''(\theta)\,d\theta = C\varepsilon^{\frac{5}{2}} + O(\varepsilon^3),\end{aligned}$$

where C is a positive constant. This concludes the proof. □

4.2.6 Main Theorem

In this paragraph we state and prove the main theorem of this section, which is a stability result for the Weinstock inequality (4.2) in the convex sets case.

Theorem 4.31. *Let $n \geq 2$. There exists $\bar{\delta} > 0$ such that for every $\Omega \subset \mathbb{R}^n$ bounded, convex open set with $\sigma_2(B_R) \leq (1+\bar{\delta})\,\sigma_2(\Omega)$, where B_R is a ball with $P(B_R) = P(\Omega)$, then*

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} \geq \begin{cases} \frac{16}{9\pi}\left(\mathcal{A}_{\mathcal{H}}(\Omega)\right)^{\frac{5}{2}} & \text{if } n=2 \\ \frac{2}{3}\sqrt{\pi}\, g\left(\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta}\right)^2\right) & \text{if } n=3 \\ \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}} & \text{if } n \geq 4, \end{cases}$$

where β and β_n are defined in (4.30) *and g is the inverse function of $f(t) = t\log\left(\frac{1}{t}\right)$, for $0 < t < e^{-1}$.*

Proof. The proof is a consequence of Theorems 4.24 and 4.30. Since all the quantities involved are invariant under translations, we can assume that $\partial\Omega$ has the origin as barycenter. Under this assumption in [25] it is proved that

$$\sigma_2(\Omega) \leq \frac{n\,|\Omega|}{W(\Omega)}.$$

It holds

$$\sigma_2(B_R) = \frac{1}{R} = \left[\frac{n\omega_n}{P(\Omega)}\right]^{1/(n-1)}$$

then, using the previous inequality and (4.18), we have

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} = \frac{\sigma_2(B_R)}{\sigma_2(\Omega)} - 1 \geq \frac{W(\Omega)}{n\,|\Omega|}\left(\frac{n\omega_n}{P(\Omega)}\right)^{1/(n-1)} - 1 = \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\mathcal{D}(\Omega).$$

Let δ be as in Theorem 4.24. Then if Ω is such that $\sigma_2(B_R) \leq (1+\bar{\delta})\sigma_2(\Omega)$, with $\bar{\delta} = \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\delta$ then $\mathcal{D}(\Omega) \leq \delta$ and, for $n \geq 4$ from (4.29) in Theorem 4.24, we get

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} \geq \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\frac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}}.$$

If $n = 3$, we can conclude a similar way, observing that $f(t) = t\log\left(\frac{1}{t}\right)$ is invertible for $0 < t < e^{-1}$. Thus, being $D(\Omega)$ small, we can explicit it in (4.29), obtaining the thesis. The result in two dimension follows from Theorem 4.30. □

Bibliography

[1] M. S. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover, New York, USA (1965)

[2] A. Alvino, V. Ferone, P. - L. Lions, G. Trombetti, *Convex symmetrization and applications.* Ann. Inst. H. Poincaré Anal. Non Linéaire 14. 2 (1997) 275-293

[3] M. Amar, G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients.* Ann. Inst. H. Poincaré. Anal. Non Linéaire 11. 1 (1994) 91-133

[4] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford University Press, Oxford (2000)

[5] B. Andrews, *Volume-preserving anisotropic mean curvature flow.* Indiana Univ. Math. J. 50. 2 (2001) 783-827

[6] T. V. Anoop, K. Ashok Kumar, *On the reverse Faber-Krahn inequalities.* Preprint arXiv:1806.11491

[7] P. R. S. Antunes, P. Freitas, D. Krejčiřík, *Bounds and extremal domains for Robin eigenvalues with negative boundary parameter.* Adv. Calc. Var. 10. 4 (2017) 357-380

[8] C. Bandle, *Isoperimetric inequalities and applications.* Pitman, Boston (1980)

[9] C. Bandle, A. Wagner, *Second domain variation for problems with Robin Boundary conditions.* J. Optim. Theory Appl. 167. 2 (2015) 430-436

[10] G. Bellettini, *Lecture notes on mean curvature flow: barriers and singular perturbations.* Edizioni della Normale, Pisa (2013)

[11] M. Bareket, *On an isoperimetric inequality for the first eigenvalue of a boundary value problem.* SIAM J. Math. Anal. 8. 2 (1977) 280-287

[12] G. Bellettini, M. Novaga, G. Riey, *First variation of anisotropic energies and crystalline mean curvature for partitions.* Interfaces and Free Bound. 5. 3 (2003) 331-356.

[13] M. Belloni, V. Ferone, and B. Kawohl. *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators.* Z. Angew. Math. Phys. 54. 5 (2003) 771-783.

[14] M. Belloni, B. Kawohl, *A direct uniqueness proof for equations involving the p-Laplace operator.* Manuscripta Math. 190. 2 (2002) 229-231

[15] M. F. Betta, F. Brock, A. Mercaldo, M. R. Posteraro *A weighted isoperimetric inequality and applications to symmetrization.* J. Inequal. Appl. 3. 4 (1999) 215-240

[16] M. H. Bossel, *Membranes élastiquement liées: extension du theoreme de Rayleigh-Faber-Krahn et de l'inegalité de Cheeger*. C. R. Acad. Sci. Paris Sér. I Math. 302. 1 (1986) 47-50

[17] B. Brandolini, C. Nitsch, C. Trombetti, *An upper bound for nonlinear eigenvalues on convex domains by means of the isoperimetric deficit*. Arch. Math. (Basel) 94. 4 (2010) 391-400.

[18] L. Brasco, G. De Philippis, *Spectral inequalities in quantitative form*. Shape optimization and spectral theory. De Gruyter Open, Poland, Warsaw (2017) 201-281.

[19] L. Brasco, G. De Philippis, B. Ruffini, *Spectral oprimization for the Stekloff-Laplacian: the stability issue*. J. Funct. Anal. 262. 11 (2012) 4675-4710

[20] L. Brasco, G. Franzina, *An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities*. NoDEA Nonlinear Differential Equations Appl. 20. 6 (2013) 1795-1830

[21] L. Brasco, G. Franzina, *A note on positive eigenfunctions and hidden convexity*. Arch. Math. (Basel) 99. 4 (2012) 367-374.

[22] F. Brock, *An isoperimetric inequality for eigenvalues of the Stekloff problem*. ZAMM Z. Angew. Math. Mech. 81. 1 (2001) 69-71

[23] D. Bucur, D. Daners, *An alternative approach to the Faber-Krahn inequality for Robin problems*. Calc. Var. Partial Differential Equations 37. 1-2 (2010) 75-86

[24] D. Bucur, V. Ferone, C. Nitsch, C. Trombetti, *A sharp estimate for the first Robin-Laplacian eigenvalue with negative boundary parameter*. Preprint arXiv:1810.06108 to appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.

[25] D. Bucur, V. Ferone, C. Nitsch, C. Trombetti, *Weinstock inequality in higher dimensions*. to appear in J. Differential Geom.

[26] D. Bucur, A. Giacomini, *A variational approach to the isoperimetric inequality for the Robin eigenvalue problem*. Arch. Ration. Mech. Anal. 198. 3 (2010) 927-961

[27] D. Bucur, A, Giacomini, *The Saint-Venant inequality for the Laplace operator with Robin boundary conditions*. Milan J. Math. 83. 2 (2015) 327-343

[28] H. Busemann, *The isoperimetric problem for Minkowski area*. Amer. J. Math. 71 (1949) 743-762

[29] V. Caselles, G. Facciolo, E. Meinhardt, *Anisotropic Cheeger sets and applications*. SIAM J. Imaging Sci. 2. 4 (2009) 1211-1254

[30] G. Crasta, I. Fragalà, F. Gazzola, *A sharp upper bound for the torsional rigidity of rods by means of web functions*. Arch. Ration. Mech. Anal. 164. 3 (2002) 189-211

[31] G. Crasta, A. Malusa, *The distance function from the boundary in a Minkowski space*. Trans. Amer. Math. Soc. 359. 12 (2007) 5725-5759

[32] Q.-y. Dai, Y.-x. Fu, *Faber-Krahn inequality for Robin problems involving p-Laplacian*. Acta Math. Appl. Sin. Engl. Ser. 27. 1 (2011) 13-28

[33] B. Dacorogna, C. E. Pfister, *Wulff theorem and best constant in Sobolev inequality*. J. Math. Pures Appl. (9) 71. 2 (1992) 97-118

[34] E. N. Dancer, D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions.* J. Differential Equations 138. 1 (1997) 86-132

[35] D. Daners, *A Faber-Krahn inequality for Robin problems in any space dimension.* Math. Ann. 335. 4 (2006) 767-785

[36] F. Della Pietra, G. di Blasio, N. Gavitone, *Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle.* Adv. Nonlinear Anal. 9. 1 (2020) 278-291

[37] F. Della Pietra, N. Gavitone, *Faber-Krahn inequality for anisotropic eigenvalue problems with Robin boundary conditions.* Potential Anal. 41. 4 (2014) 1147-1166

[38] F. Della Pietra, N. Gavitone, *Relative isoperimetric inequality in the plane: the anisotropic case.* J. Convex Anal. 20. 1 (2013) 157-180

[39] F. Della Pietra, N. Gavitone, *Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators.* Math. Nachr. 287. 2-3 (2014) 194-209

[40] F. Della Pietra, N. Gavitone, *Symmetrization with respect to the anisotropic perimeter and applications.* Math. Ann. 363. 3-4 (2015) 953-971.

[41] F. Della Pietra, N. Gavitone, S. Guarino Lo Bianco, *On functionals involving the torsional rigidity related to some classes of nonlinear operators.* J. Differential Equations 265. 12 (2018) 6424-6442

[42] F. Della Pietra, N. Gavitone, H. Kovařík, *Optimizing the first eigenvalue of some quasilinear operators with respect to boundary conditions.* ESAIM Control Optim. Calc. Var. 23. 4 (2017) 1381-1395

[43] F. Della Pietra, N. Gavitone, G. Piscitelli, *A sharp weighted anisotropic Poincaré inequality for convex domains.* C. R. Math. Acad. Sci. Paris 355. 7 (2017) 748-752

[44] F. Della Pietra, N. Gavitone, C. Xia, *Motion of level sets by inverse anisotropic mean curvature.* Preprint arXiv:1804.06639

[45] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal. 7. 8 (1983) 827-850

[46] L. Esposito, N. Fusco, C. Trombetti, *A quantitative version of the isoperimetric inequality: the anisotropic case.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4. 4 (2005) 619-651.

[47] G. Faber, *Beweis dass unter allen homogenen Membranen von gleicher Flache und gleichen Spannungndie kreiformiger den tiefsten Grundton gibt.* Sitzungber, Acad. Wiss. (1923) 169-172

[48] V. Ferone, C. Nitsch, C. Trombetti, *On a conjectured reverse Faber-Krahn inequality for a Stecklov-type Laplacian eigenvalue.* Commun. Pure Appl. Anal. 14. 1 (2015) 63-82

[49] I. Fonseca, S. Müller, *A uniqueness proof for the Wulff theorem.* Proc. Roy. Soc. Edinburgh Sect. A 119. 1-2 (1991) 125-136

[50] P. Freitas, D. Krejčiřík, *The first Robin eigenvalue with negative boundary parameter.* Adv. Math. 280 (2015) 322-339

[51] P. Freitas, R. Laugesen, *From Neumann to Steklov and beyond, via Robin: the Weinberger way.* Preprint arxiv:1810.07461

[52] B. Fuglede, *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbb{R}^n$. Trans. Amer. Math. Soc. 314. 2 (1989) 619-638.

[53] N. Fusco, *The quantitative isoperimetric inequality and related topics.* Bull. Math. Sci. 5. 3 (2015) 517–607.

[54] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.* Rend. Sem. Mat. Univ. Padova 27 (1957) 284-305

[55] N. Gavitone, D. A. La Manna, G. Paoli, L. Trani *A quantitative Weinstock inequality for convex sets.* to appear in Calc. Var. Partial Differential Equations

[56] N. Gavitone, L. Trani, *On the first robin eigenvalue of a class of anisotropic operators.* Milan J. Math. 86. 2 (2018) 201–223

[57] T. Giorgi, R. G. Smits, *Monotonicity results for the principal eigenvalue of the generalized Robin Problem.* Illinois J. Math. 49. 4 (2005) 1133-1143

[58] A. Henrot, *Extremum problems for eigenvalues of elliptic operators.* Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006)

[59] A. Henrot, M. Pierre, *Variation et optimisation de formes. Une analyse géométrique.* Mathématiques & Applications. Birkhäuser Verlag, Basel (2006)

[60] G. Huisken, T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality.* J. Differential Geom. 59. 3 (2001) 353-437

[61] J. B. Kennedy, *On the isoperimetric problem for the Laplacian with Robin and Wentzell boundary conditions.* University of Sydney, School of Mathematics and Statistics, Ph.D. Thesis (2010)

[62] E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenshaft des Kreises* Math. Ann. 94. 1 (1925) 97-100

[63] A. A. Lacey, J. R. Ockendon, J. Sabina, *Multidimensional reaction diffusion equations with nonlinear boundary conditions.* SIAM J. Appl. Math. 58. 5 (1998) 1622-1647

[64] P. Lindqvist, *Notes on the p-Laplace equation.* Report. University of Jyväskylä Department of Mathematics and Statistics, 102 (2006)

[65] F. Maggi, *Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory.* Cambridge Studies in Advanced Mathematics (2012)

[66] E. Makai, *On the principal frequency of a convex membrane and related problems.* Czechoslovak Math. J. 9. 1 (1959) 66-70

[67] V. G. Maz'ja, *Sobolev Spaces.* Springer Verlag, Berlin (1985)

[68] G. Paoli, G. Piscitelli, L.Trani, *Sharp estimates for the first p-Laplacian eigenvalue and for the p-torsional rigidity on convex sets with holes.* Preprint arXiv:1908.00362

[69] G. Paoli, L. Trani, *Anisotropic isoperimetric inequalities involving boundary momentum, perimeter and volume.* Nonlinear Anal. 187 (2019) 229-246

[70] G. Paoli, L. Trani, *Two estimates for the first Robin eigenvalue of the finsler Laplacian with negative boundary parameter*. J. Optim. Theory Appl. 181. 3 (2019) 743–757

[71] L. E. Payne, H. F. Weinberger, *Some isoperimetric inequalities for membrane frequencies and torsional rigidity*. J. Math. Anal. Appl. 2 (1961) 210-216

[72] G. Pólya, *Two more inequalities between physical and geometrical quantities*. J. Indian Math. Soc. (N.S.) 24 (1960) 413-419

[73] A. Ros, *Compact hypersurfaces with costant higher order mean curvature*. Rev. Mat. Iberoamericana 3. 3-4 (1987) 447-453.

[74] A. Savo, *Lower bounds for the nodal length of eigenfunctions of the Laplacian*. Ann. Global Anal. Geom. 19. 2 (2001) 133-151

[75] R. Schneider, *Convex bodies: the Brunn-Minkowski theory. Second expanded edition.* Cambridge University Press (2014)

[76] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*. J. Ration. Mech. Anal. 3 (1954) 343-356

[77] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*. J. Differential Equations 51. 1 (1984) 126-150

[78] L. Trani, *Some Remarks on Robin-Laplacian Eigenvalues*. Rend. Acc. Sc. Fis. Mat. Napoli 84 (2017) 87-96

[79] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*. Comm. Pure Appl. Math. 20 (1967) 721-747

[80] G. N. Watson, *A treatise on the theory of Bessel functions*. Cambridge University Press (1995)

[81] H. F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*. J. Rational Mech. Anal. 5 (1956) 633-636

[82] R. Weinstock, *Inequalities for a classical eigenvalue problem*. Department of Math., Stanford Univ., Tech. Rep. 37 (1954)

[83] R. Weinstock, *Inequalities for a classical eigenvalue problem*. J. Rational Mech. Anal. 3 (1954) 745-753

[84] C. Xia, *Inverse anisotropic mean curvature flow and a Minkowski type inequality*. Adv. Math. 315 (2017) 102-129

[85] C. Xia, X. Zhang, *ABP estimate and geometric inequalities*. Comm. Anal. Geom. 25. 3 (2016) 685-708