Lefschetz Properties in Algebra, Geometry and Combinatorics

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Introduction and Historical Note

The Lefschetz properties are algebraic abstractions inspired by the so called Hard Lefschetz Theorem about the cohomology rings of smooth projective algebraic varieties over the complex number field indowed with the Euclidean topology, or, more generally, compact complex Khäler manifolds (see [49, 62]). The Hard Lefschetz Theorem was first stated by S. Lefschetz in [50], but his proof was not entirely rigorous. The first complete proof was given by Hodge in [43], using his theory of harmonic integrals. Today the standard proof uses the representation theory of the Lie algebra $sl_2(\mathbb{C})$ and it is due to Chern [11]. Lefschetz's original proof was made rigorous by Deligne (see [53]), who extended it to characteristic p.

For one hand, the Lefschetz theory for projective manifolds began by S. Lefschetz, and it was well established by the late 1950s, but, for the other hand, the investigation of the Lefschetz properties of Artinian algebras was started in the mid 1980s. Although there were limited developments on this topic in the 20-th century, in the last years this topic has attracted the attention of mathematicians from different areas. In fact, nowadays the Lefschetz properties are considered in a number of distinct contexts, such as Khäler manifolds, solvmanifolds (see [46]), arithmetic hyperbolic manifolds (see [4]), Shimura varieties (see [40]), convex polytopes (see [48]), Coxeter groups (see [59]), matroids, simplicial complexes (see [2, 34, 48, 68, 69]) among others. In these new contexts the Lefschetz properties showed to have intersections with the algebra itself, the geometry and the combinatorics.

The cornerstone of the algebraic theory of Lefschetz properties were the original papers of Stanley (see [68, 69, 71]) and the works of Watanabe, summarized in (see [39]). Watanabe's book is the first book on this subject, it combines techniques from algebraic geometry and from combinatorics to give a detailed account about Lefschetz properties.

A very important construction that appears many times in these works is the so called Nagata idealization also called trivial extension. In general, Nagata idealization is a useful tool, developed by Nagata, to convert any R-module M in a ideal of another ring, $A \ltimes M$. In our perspective the starting point is a very interesting isomorphism between the Nagata idealization of an ideal $I = (g_0, \dots, g_n) \subset \mathbb{K}[u_1, \dots, u_m]$ and the level algebra given by the annihilator of the I, in such way that the new ring is an Artinian Gorenstein algebra and we get an explicit formula for the Macaulay generator f (see [39, Proposition 2.77])

$$f = x_0 g_0 + \cdots + x_n g_n \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(1,d-1)}. \tag{0.1}$$

This bigraded polynomial is closely related with Gordan-Noether and Perazzo constructions of forms with vanishing Hessian (see [12, 31, 33, 35, 61]). It is not a coincidence since in [51] the authors present a Hessian criterion for the SLP saying that the vanishing of a (higher) Hessian implies the failure of SLP. This criterion was generalized in [32] also for the WLP using mixed Hessians. Following the original ideas of Gordan-Noether and Perazzo, the authors in [31] constructed families of polynomials whose k-th Hessian is zero.

A natural generalization of (0.1) should be the so called Nagata polynomial of order

d_1 (see Definition (1.4.1)). We study the Lefschetz properties for the algebras associated to Nagata polynomials of order d_1, the geometry of the Nagata hypersurfaces of order d_1 and the interaction between the combinatorics of f and the algebraic structure of A in the case that the g_i are square free monomials. We use a simplicial complex to study this case. Our first result is that the geometry of Nagata hypersurfaces is very similar to the geometry of the known hypersurfaces with vanishing Hessian. Hence these are hypersurfaces, satisfying at least a Laplace equation (see [13, 14]) and they are scroll hypersurfaces (see Theorem (1.3.9) and Corollary (1.3.10)). From the algebraic viewpoint, we studied the Lefschetz properties for higher order Nagata idealizations. In fact, our second main result is about the Lefschetz properties. We split the result in two cases:

$d_1 < d_2$: in this case we give examples with small numbers of summands where the SLP holds and we recall a result proved in [31] (see Proposition (1.3.5));

$d_1 \geq d_2$: in this case A has the WLP as proved in Proposition (1.3.7).

Finally our third main result is the Theorem (1.4.5) that gives a complete description of the structure of the algebra A, including the Hilbert vector and its ideal of presentation, in the case which the g_i are square free monomials.

For a standard graded Artinian Gorenstein algebra in general it is natural to try to understand its Hilbert function. When the codimension of the algebra is less than or equal to 3, all its Hilbert vector have been characterized (see [68] and [76]); in particular, they are unimodal, i.e. they never strictly increase after a strict decrease. While it is known that non unimodal Gorenstein h-vectors exist in every codimension greater than or equal to 5 (see [5, 6, 7]), it is open whether non unimodal Gorenstein h-vectors of codimension 4 exist. Historically, the first such example of a non unimodal Gorenstein h-vector was given by Stanley (see [68, Example 4.3]). He showed that the h-vector $(1, 13, 12, 13, 1)$ is indeed a Gorenstein h-vector and the non unimodality occurs here in degree 2. In [73] the authors showed that Stanley's example is optimal and for our purposes we call it minimal. Our main result is a generalization of this result. We study special Gorenstein h-vectors of type $(1, r, h_2, r, 1)$. Fixing the codimension r and denoting the least possible value that h_2 may assume by $f(r)$, we study the asymptotic behavior of $f(r)$. Stanley in [70] conjectured the existence of the following limit

$$\lim_{r\to\infty} \frac{f(r)}{r^{\frac{2}{3}}}$$

and he gives a precise value that is $6^{\frac{2}{3}}$. Bounds were given by Stanley in [72] and by Kleinschmidt in [47], but the precise limit was only proved in 2006 (see [58]). We construct a family of Gorenstein algebras called Full Perazzo algebras, related to the classical work of U. Perazzo about hypersurfaces with vanishing Hessian that are not cones (see [33, 61]). Our main result is that the Hilbert vectors of Full Perazzo algebras are always minimal (see Theorem (2.3.5)). Proving two Lemmas (Lemma (2.3.2) and Lemma (2.3.3)) about the monotonicity of the functions $\mu(r) = f(r)$ and $\delta(r) = r - f(r)$, we are able to give a simple proof of Stanley's conjecture (see Corollary (2.3.6)). We pointed out that the h-vector of the Stanley's example is a special case of a Full Perazzo algebra. Finally we introduce another family of Artinian Gorenstein algebras having non unimodal Gorenstein h-vectors: the Turan algebras that are Artinian Gorestein algebra presented by quadrics (see [34]). We have a conjecture about the asymptotic behaviour of Artinian Gorenstein algebra presented by quadric, (see Conjecture (2.4.5)).

Instead the focus of the last part of the thesis is the analysis of a particular class of curves: m-syzygy curves, i.e. reduced complex projective plane curves, whose Jacobian syzygy module has 3 generators (see Definition (3.2.2)). Some Dimca's papers have been of great inspiration, since they give us a detailed overview of this topic, classifying them carefully and highlighting their properties. Two topics, related to m-syzygy curves, have been argued: the Milnor algebras (see Definition (3.2.1)) and the Jacobian module (see Definition (3.3.1)). It is natural to wonder if the properties of Lefschetz type can hold for these algebras. In [45] the author proves that the Milnor algebra of a singular hypersurface of degree d in $\mathbb{P}^n$, with its singular locus of dimension at most $n-3$, has the WLP in degree $d-2$ (see [45, Proposition 3.1]), the SLP in degree $d-k-1$ at range k, for $k < d-1$ (see [45, Proposition 3.3]) and more generally, for a general hypersurface, SLP holds (see [45, Theorem 3.5]). About the Jacobian module, the speech is more complex; the authors in [19] have proved that the Jacobian module has some property of Lefschetz type. The minimal resolutions are the important tool for describe the Hilbert vector of the Milnor algebras and the Jacobian module of a m-syzygy curves. In [42], the authors describe the minimal resolution of an ideal, generated by 3 generators, and of its saturation, hence it allows to determine the Hilbert vector of a Milnor algebra and the Jacobian module of a m-syzygy curve respectively. Our main result has been to determine the Hilbert vector of the Jacobian module of a 3-syzygy curve (see Theorem (3.4.3)). Moreover we find the Hilbert vector of the maximal Tjurina curves (see Proposition (3.5.1)), and of the nodal curves, having only rational irreducible components (see Theorem (3.5.2)). A result due to Hartshorne, on the cohomology of some rank 2 vector bundles on $\mathbb{P}^2$, is used to get a sharp lower bound for the initial degree of the Jacobian module, under a semistability condition.

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Contents

Chapter 1

Lefschetz Properties for higher order Nagata idealization

1.1 Artinian Gorenstein algebras

We recall some basic facts about Artinian Gorenstein algebras. For a more detailed account, see [31, 39, 51, 56, 62].

In all the chapter $\mathbb{K}$ denotes a field of characteristic zero, unless any clarifications.

Definition 1.1.1. Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables and $I \subset R$ be a homogeneous Artinian ideal such that $I_1 = 0$. A graded Artinian $\mathbb{K}$-algebra $A = R/I = \oplus_{i=0}^{d} A_i$ is a *standard* graded Artinian $\mathbb{K}$-algebra if it is generated in degree 1 as an algebra.

Setting $h_i = \dim A_i$, the **Hilbert vector** is the vector $\mathrm{Hilb}(A) = (1, h_1, \dots, h_d)$. If $I_1 = 0$, then h_1 is called the codimension of A. The Hilbert vector is said to be **unimodal** if there exixts an integer $t \geq 1$ such that:

$$1 \leq \ldots \leq h_{t-1} \leq h_t \geq h_{t+1} \geq \ldots \geq h_d.$$

Moreover the Hilbert vector is said to be **symmetric** if

$$h_i = h_{d-i} \quad \forall i = 0, \dots, \lfloor \frac{d}{2} \rfloor.$$

Definition 1.1.2. A standard graded Artinian $\mathbb{K}$-algebra, $A = \oplus_{i=0}^{d} A_i$, is Gorenstein if and only if $\dim A_d = 1$ and the restriction of the multiplication of the algebra in complementary degree, that is, $A_k \times A_{d-k} \to A_d$ is a perfect paring for $k = 0, 1, \dots, d$ (see [51]).

If $A_j = 0$ for $j > d$, then d is called the **socle degree** of A.

Remark 1.1.3. Since $A_k \times A_{d-k} \to A_d$ is a perfect paring for $k = 0, 1, \dots, d$, it induces two $\mathbb{K}-$linear maps, $A_{d-k} \to A_k^*$, with $A_k^* := \operatorname{Hom}(A_k, A_d)$ and $A_k \to A_{d-k}^*$, with $A_{d-k}^* := \operatorname{Hom}(A_{d-k}, A_d)$, that are two isomorphisms.

Definition 1.1.4. Let $\mathbb{K}$ be a field and let $A = \bigoplus_{i=0}^{d} A_i$ be a graded Artinian $\mathbb{K}$-algebra with $A_0 = \mathbb{K}$ and $A_d \neq 0$. Let

$$\bullet : A_i \times A_{d-i} \to A_d$$

such that $\bullet(\alpha, \beta) := \alpha\beta$, be the restriction of the multiplication in A. The Artinian graded $\mathbb{K}$-algebra A satisfies the *Poincaré duality property* if:

1. $\dim_{\mathbb{K}} A_d = 1$;
2. $\bullet : A_i \times A_{d-i} \to A_d$ is no degenerate for every $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$.

It is clear that, by definition, a standard graded Artinian Gorenstein $\mathbb{K}$-algebra, $A = \oplus_{i=0}^{d} A_i$, satisfies the Poicaré duality property. In fact we have the following Proposition:

Proposition 1.1.5. *([30],[51, Proposition 1.4], [54, Proposition 2.1]) Let $A = \oplus_{i=0}^{d} A_i$ be a graded Artinian $\mathbb{K}$-algebra. Then A satisfies the Poincaré duality property if and only if it is Gorenstein.*

All graded Artinian $\mathbb{K}$-algebras satisfying the Poincaré duality property have the symmetric Hilbert vector, hence, by Proposition (1.1.5), the Hilbert vector of a standard graded Artinian Gorenstein $\mathbb{K}$-algebra is symmetric. Moreover let's specify that, given a standard graded Artinian Gorenstein $\mathbb{K}$-algebra, its Hilbert vector is not always unimodal; infact in [5] we can find examples of standard graded Artinian Gorenstein $\mathbb{K}$-algebras which the Hilbert vector is not unimodal (see also the next sections).

Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables. we denote by $R_d = \mathbb{K}[x_0, \dots, x_n]_d$ the $\mathbb{K}$-vector space of homogeneous polynomials of degree d. We denote by $Q = \mathbb{K}[X_0, \dots, X_n]$ the ring of differential operators of R, where $X_i = \frac{\partial}{\partial x_i}$ for $i = 0, \dots, n$. We denote by $Q_k = \mathbb{K}[X_0, \dots, X_n]_k$ the $\mathbb{K}$-vector space of homogeneous differential operators of R of degree k.
For each $d \geq k \geq 0$ there exist natural $\mathbb{K}$-bilinear maps $R_d \times Q_k \to R_{d-k}$ defined by differentiation:

$$(f, \alpha) \to f_\alpha := \alpha(f).$$

Let $f \in R$ be a homogeneous polynomial of degree $\deg f = d \geq 1$, we define:

$$\operatorname{Ann}(f) := \{\alpha \in Q | \alpha(f) = 0\} \subset Q.$$

This is called the **annihilator** of f. Since $\operatorname{Ann}(f)$ is a homogeneous ideal of Q, we can define $A = Q/\operatorname{Ann}(f)$.

Remark 1.1.6. $V : f = 0 \subset \mathbb{P}^n$ is a cone if and only if $\operatorname{Ann}(f)_1 \neq 0$. We know that $V : f = 0$ in $\mathbb{P}^n$ is a cone if and only if the partial derivatives of the first order are linearly dependent. In fact the $\operatorname{Ann}(f)_1 \neq 0$ if and only if exists a linear differential operator $F \in Q_1$ such that $F(f) = 0$. The annihilator of f is contained in the the maximal ideal, $(X_0, \dots, X_n)$, of Q. Thence we get:

$$F = \sum_{i=0}^{n} \alpha_i X_i \Leftrightarrow F(f) = \sum_{i=0}^{n} \alpha_i X_i(f) \Leftrightarrow 0 = \sum_{i=0}^{n} \alpha_i \frac{\partial f}{\partial x_i}$$

with coefficients $\alpha_0, \dots, \alpha_n$ are not all zero. We have a linear combination of partial derivatives of the first order, with coefficients not all zero, hence they are linearly dependent.

From now on, we assume that $\operatorname{Ann}(f)_1 = 0$, so that the hypersurface $V : f = 0 \subset \mathbb{P}^n$ is not a cone. Moreover, by Definition (1.1.1) and Definition (1.1.2), $A = Q/\operatorname{Ann}(f)$ is a standard graded Artinian Gorenstein $\mathbb{K}$-algebra. Conversely, by theory of inverse systems, we get the following characterization of standard graded Artinian Gorenstein $\mathbb{K}$-algebras:

Theorem 1.1.7 (Double annihilator theorem of Macaulay)**.** *Let $R = \mathbb{K}[x_0, \dots, x_n]$ and let $Q = \mathbb{K}[X_0, \dots, X_n]$ be the ring of differential operators. Let $A = \oplus_{i=0}^{d} A_i = Q/I$, with $I \subset Q$ homogeneous ideal, be a standard graded Artinian $\mathbb{K}$-algebra. Then A is Gorenstein if and only if there exists $f \in R_d$ such that $A \simeq Q/\operatorname{Ann}(f)$.*

A proof of this result can be found in [51, Theorem 2.1].

Remark 1.1.8. With the previous notation, let $A = \bigoplus_{i=0}^{d} A_i = Q/I$ be a standard graded Artinian $\mathbb{K}$-algebra. The socle degree of A coincides with the degree of the form f.

Now we can deal with standard bigraded Artinian Gorenstein algebras, i.e. Artinian Gorenstein algebras, $A = \oplus_{i=0}^{d} A_i$, such that

$$\begin{cases} A_d \neq 0 \\ A_k = \displaystyle\bigoplus_{i=0}^{k} A_{(i,k-i)} \text{ for } k < d. \end{cases}$$

The pair (d_1, d_2), such that $A_{(d_1,d_2)} \neq 0$ and $d_1 + d_2 = d$, is said the socle bidegree of A.

Remark 1.1.9. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A_{(i,j)}^* \simeq A_{(d_1-1,d_2-j)}$.

By abuse notation, we denote the polynomial ring viewed as standard bigraded ring in the set of variables $\{x_0, \dots, x_n\}$ and $\{u_1, \dots, u_m\}$ by $R = \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$.

Definition 1.1.10. A homogeneous polynomial $f \in R_{(d_1,d_2)}$ is said to be a bihomogeneous polynomial of total degree $\deg f = d = d_1 + d_2$, if f can be written in the following way:

$$f = \sum_{i=1}^{s} f_i g_i, \tag{1.1}$$

where $f_i \in \mathbb{K}[x_0, \dots, x_n]_{d_1}$ and $g_i \in \mathbb{K}[u_1, \dots, u_m]_{d_2}$, $\forall i \leq s$

Definition 1.1.11. A homogeneous ideal $I \subset R$ is a bihomogeneous ideal if

$$I = \bigoplus_{i,j=0}^{\infty} I_{(i,j)}$$

where $I_{(i,j)} = I \cap R_{(i,j)}$ $\forall i, j$.

Let $Q = \mathbb{K}[X_0, \dots, X_n, U_1, \dots, U_m]$ be the associated ring of differential operators and let $f \in R_{(d_1,d_2)}$ be a bihomogeneous polynomial of total degree $d = d_1 + d_2$, then $I = \operatorname{Ann}(f) \subset Q$ is a bihomogeneous ideal and $A = Q/I$ is a standard bigraded Artinian Gorenstein algebra of socle bidegree (d_1, d_2) and codimension $N = n + m + 1$.

Remark 1.1.12. Let $f \in R_{(d_1,d_2)}$ be a bihomogeneous polynomial of degree (d_1, d_2) and let A be the associated bigraded algebra of socle bidegree (d_1, d_2), then for $i > d_1$ or $i > d_2$:

$$I_{(i,j)} = Q_{(i,j)}.$$

In fact for all $\alpha \in Q_{(i,j)}$ with $i > d_1$ or $j > d_2$ we get $\alpha(f) = 0$, so $Q_{(i,j)} = I_{(i,j)}$. As consequence, we have the following decomposition for all A_k:

$$A_k = \bigoplus_{i \leq d_1, j \leq d_2, i+j=k} A_{(i,j)}.$$

Furthermore for $i < d_1$ and $j < d_2$, the evaluation map $Q_{(i,j)} \to A_{(d_1-i,d_2-j)}$ given by $\alpha \to \alpha(f)$ provides the following short exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow Q_{(i,j)} \longrightarrow A_{(d_1-i,d_2-j)} \longrightarrow 0.$$

1.2 Weak and Strong Lesfchetz Property

Definition 1.2.1. Let $A = \oplus_{i=0}^{d} A_i$, $A_d \neq 0$, be a graded Artinian algebra. A has the weak Lefschetz property (for short WLP) if there exists an element $L \in A_1$ such that the multiplication map

$$\times L \colon A_i \to A_{i+1}$$

has full rank for all $0 \leq i \leq d - 1$.

The element $L \in A_1$ with this property, is said to be **weak Lefschetz element**.

Proposition 1.2.2. *[39, Proposition 3.2] Let $A = \oplus_{i=0}^{d} A_i$, $A_d \neq 0$ be a standard graded Artinian $\mathbb{K}-$algebra. If A has the WLP, then A has a unimodal Hilbert vector.*

Definition 1.2.3. Let $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, be a graded Artinian algebra. A has the strong Lefschetz property (for short SLP), if there exists an element $L \in A_1$ such that the multiplication map

$$\bullet L^k : A_i \to A_{i+k}$$

has full rank for all $0 \leq i \leq d$ and $1 \leq k \leq d - i$.

The element $L \in A_1$, with this property, is said to be **strong Lefschetz element.**

Let $A = \oplus_{i=0}^{d} A_i, A_d \neq 0$ be a graded Artinian $\mathbb{K}$-algebra. If A has the SLP, then, for $k = 1$, A has the WLP. Hence, by Proposition (1.2.2), A has a unimodal Hilbert vector.

Definition 1.2.4. Let $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, be a graded Artinian $\mathbb{K}-$algebra. A has the strong Lefschetz property in the narrow sense if there exists an element $L \in A_1$ such that the multiplication map $\bullet L^{d-2i} : A_i \to A_{d-i}$ is bijective for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$.

If a graded Artinian $\mathbb{K}-$algebra A has the strong Lefschetz property in the narrow sense, then the Hilbert vector of A is unimodal and symmetric as the following Theorem shows:

Theorem 1.2.5. *[39] Let $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, be a graded Artinian $\mathbb{K}-$algebra. A has the SLP and the Hilbert vector is symmetric if and only if A has the SLP in the narrow sense.*

Corollary 1.2.6. *Let $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$ be a standard graded Artinian Gorenstein algebra the two condition SLP and SLP in the narrow sense are equivalent.*

Proof. Since A is a standard graded Artinian Gorenstein algebra, hence the Hilbert vector is symmetric. Thence A has the SLP if and only if A has the SLP in the narrow sense, by Theorem (1.2.5). □

The Lefschtez properties are linked to the concept of the complete intersection ideals and almost complete intersection ideals. Let $\mathbb{K}$ be a field of characteristic zero and let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring.

Definition 1.2.7. An artinian ideal $I \subset R$ is called monomial complete intersection ideal if

$$I = (x_0^{\alpha_0}, \ldots, x_n^{\alpha_n}).$$

The graded, Artinian $\mathbb{K}$-algebra, R/I, is said monomial complete intersection.

Theorem 1.2.8. *Every monomial complete intersection* $R/(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$ *has the SLP with* $\sum_{i=0}^n x_i$ *as strong Lefschetz element.*

Proof. See [56, Theorem 1.1]. □

Remark 1.2.9. By [39, Theorem 2.23], a graded, Artinian, complete intersection $\mathbb{K}$-algebra is a Gorenstein $\mathbb{K}$-algebra.

Definition 1.2.10. An ideal $I \subset R$ is said to be almost complete intersection ideal, if it is generated by one more homogeneous form than codimension of R, i.e.

$$I = (f_1, \ldots, f_{n+1})$$

where $\deg f_i = d_i$, $d_i \leq d_{i+1}$ and $d_{n+1} \leq \sum_{i=1}^{n} d_i - n$.

In recent years, many authors have worked with the goal to find conditions for which a graded Artinian $\mathbb{K}$-algebra can satisfy the Lefschetz properties (WLP or SLP). Certainly a very important criterion is due to J. Watanabe, called *criterion Hessian*, linking the study of the Lefschetz properties to the higher Hessians.

Definition 1.2.11. Let $f \in R_d$ be a homogeneous polynomial, let $A = \bigoplus_{i=0}^{d} A_i = \dfrac{Q}{\operatorname{Ann}(f)}$ be the associated Artinian Gorenstein algebra and let $\mathcal{B} = \{\alpha_j | j = 1, \ldots, \sigma_k\} \subset A_k$ be an ordered $\mathbb{K}$-basis of A_k. The *k-th Hessian matrix* of f with respect to is

$$\operatorname{Hess}_f^k := (\alpha_i \alpha_j(f))_{i,j=1}^{\sigma_k}.$$

The *k-th Hessian* of f with respect to is

$$\operatorname{hess}_f^k := \det(\operatorname{Hess}_f^k).$$

Remark 1.2.12. If $k = 0$, the Hessian is just $\operatorname{hess}_f^0 = f$. Instead the Hessian of order $k = 1$ with respect to the standard basis is just the classical Hessian.

Theorem 1.2.13. *[51, 75] Let notation be as above. An element* $L = a_1X_1 + \ldots + a_nX_n \in A_1$ *is a strong Lefschetz element of* $A = Q/\operatorname{Ann}(f)$ *if and only if*

$$\operatorname{hess}_f^k(a_1, \ldots, a_n) \neq 0$$

for all $k = 0, \ldots, \lfloor d/2 \rfloor$*. In particular, if for some* $k \leq \lfloor \frac{d}{2} \rfloor$ *we have* $\operatorname{hess}_f^k = 0$*, then* A *does not have the SLP.*

1.3 Nagata polynomial of order d_1

Definition 1.3.1. Let A be a ring and M be a A-module. The idealization of M, $A \ltimes M$, is the product set $A \times M$ in which addition and multiplication are defined as follows:

$$(a, m) + (b, n) = (a + b, m + n) \text{ and } (a, m).(b, n) = (ab, bm + an).$$

The following is a known result whose proof can be found in [39, Theorem 2.77].

Theorem 1.3.2. *Let $R = \mathbb{K}[u_1, \ldots, u_n]$ and $R' = \mathbb{K}[u_1, \ldots, u_n, x_0, \ldots, x_n]$ be polynomial rings and let $Q = \mathbb{K}[\partial_1, \ldots, \partial_n]$ and $Q' = \mathbb{K}[\partial_1, \ldots, \partial_n, \delta_0, \ldots, \delta_n]$ the associated ring of differential operators. Let $I = (g_1, \ldots, g_m) \subset Q$ be an ideal generated by forms of degree d and let $A = Q/Ann(g_1, \ldots, g_m)$ be the associated level algebra. Let $f = x_0 g_0 + \ldots + x_m g_m \in R'$ be a bihomogeneous polynomial and let $A' = Q'/Ann(f)$ be the associated algebra. Considering I as an A-module, we have*

$$A \ltimes I \simeq A'$$

Definition 1.3.3. A bihomogeneous polynomial $f = \in \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]_{(d_1,d_2)}$ such that:

$$f = \sum_{i=0}^{s} x_i^{d_1} g_i \tag{1.2}$$

where g_i polynomials in the $u_1, \ldots, u_m$ variables, of degree d_1, is called a *Nagata polynomial* of order d_1, if the polynomials g_i are linearly independent and they depend on all variables.

By Theorem (1.3.2), the algebra $A = Q/\operatorname{Ann}(f)$ can be realized as a trivial extension and it is said Nagata idealization of order d_1, socle degree $d_1 + d_2$ and codimension $n + m + 1$.

Let $R = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring and $f \in R_{(d_1,d_2)}$, with $d_1 \geq 1$, be a polynomial of type $f = \sum_{i=0}^{n} x_i^{d_1} g_i$, where g_i is a polynomial in $u_1, \ldots, u_m$ variables, for all $i = 0, \ldots, m$. We denote by $Q = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$ the ring of differential operators of R, where $X_i = \frac{\partial}{\partial x_i}$, for $i = 0, \ldots, n$ and $U_j = \frac{\partial}{\partial u_j}$, for $j = 1, \ldots, m$. Let $A = \frac{Q}{\operatorname{Ann}(f)}$ the associated algebra.

In the case $d_1 < d_2$, we have an example such that A has the SLP, hence A has the WLP:

Example 1.3.4. Let $f = x^2 u^3 + y^2 v^3$ be a bihomogeneous polynomial. Hence A has bidegree $(2, 3)$, Hilbert vector $(1, 4, 6, 6, 4, 1)$ and A has the SLP. By the Hessian criterion, Theorem 1.2.13, there are two Hessians to control, $\operatorname{hess}_f^1 \neq 0$ and $\operatorname{hess}_f^2 \neq 0$.

If the number of summands in f is great enough, we get the following proposition:

Proposition 1.3.5. *[31, Proposition 2.5] Let $x_0, \dots, x_n$ and $u_1, \dots, u_m$ be independent sets of indeterminates with $n \geq m \geq 2$. For $j = 1, \dots, s$, let $f_j \in \mathbb{K}[x_0, \dots, x_n]_{d_1}$ and $g_j \in \mathbb{K}[u_1, \dots, u_m]_{d_2}$ be linearly independent forms with $1 \leq d_1 < d_2$. If $s > \binom{m-1+d_1}{d_1}$, then the form of degree $d_1 + d_2$ given by*

$$f = f_1 g_1 + \cdots + f_s g_s$$

satisfies

$$\mathrm{hess}_f^k = 0$$

Corollary 1.3.6. *Let A be a Nagata idealization of order $d_1 < d_2$, then A fails SLP.*

If we consider $d_1 \geq d_2$, we have the following Proposition:

Proposition 1.3.7. *With the same notations, if $d_1 \geq d_2$, then A has the WLP and $L = \sum_{i=0}^{n} X_i$ is a weak Lefschetz element.*

Proof. (*The idea of this result was shared by the work group in Banff*).
We denote by $k = \lfloor \frac{d_1+d_2}{2} \rfloor$. We note that $d_1 \geq k$. In fact, by hypothesis $d_1 \geq d_2$, hence:

$$d_1 + d_1 \geq d_1 + d_2 \Rightarrow \frac{2d_1}{2} \geq \frac{d_1 + d_2}{2} \Rightarrow d_1 \geq \frac{d_1 + d_2}{2} \geq \lfloor \frac{d_1 + d_2}{2} \rfloor = k.$$

We have:

$$A_k = A_{(k,0)} \oplus A_{(k-1,1)} \oplus \cdots \oplus A_{(k-d_2,d_2)}.$$

We want to prove that for $L = X_0 + \ldots + X_n \in Q[X_0, \dots, X_n]_1$

$$\bullet L \colon A_{(k-i,i)} \to A_{(k-i+1,i)}$$

has maximal rank for all $i = 0, \dots, d_2$. Since A is a standard graded Artinian Gorenstein algebra it is enough to check it in the middle (see [55, Proposition 2.1]).
We denote $\omega_j = X_j^{k-i} \alpha_j$, where $\alpha_j \in Q[U_1, \dots, U_m]_i$, for $j = 0, \dots, n$ and we suppose that $\{\omega_j\}$ is a basis for $A_{(k-i,i)}$. Hence we get

$$\sum_j b_j \omega_j = 0 \Rightarrow b_j = 0.$$

It implies that the $\alpha_j(g_j)$ are linear independent in $\mathbb{K}(x_1, \dots, x_n)$.
Let $\Omega_j = X_j^{k-i+1} \alpha_j = \bullet L(\omega_j)$, we want to prove that $\{\Omega_0, \dots, \Omega_n\}$ is a linear independent system for $A_{(k-i+1,i)}$. We consider the following linear combination $\sum_j c_j \Omega_j = 0$. By definition, we get:

$$0 = \sum_j c_j \Omega_j(f) = \sum_j c_j \Omega_j \left(\sum_i x_i^{d_1} g_i \right) = \sum_j c_j x_j^{d_1-k+i-1} \alpha_j(g_j).$$

Since $\alpha_j(g_j)$ are linear independent in $\mathbb{K}(x_1,\ldots,x_n)$, for all $j=0,\ldots,n$, we have

$$c_j x_j^{d_1-k+i-1}=0 \Rightarrow c_j=0.$$

The result follows. $\square$

For this case, there is nothing we are able to say about the SLP.

In relation to Nagata polynomial of order d_1, from the point of view of the geometry, we can introduce a particular class of hypersurface in $\mathbb{P}^N$ determinated by a polinomial of the type (1.2) and decribe their geometry.

Definition 1.3.8. Let $R=\mathbb{K}[x_0,\ldots,x_n,u_1,\ldots,u_m]$ be the polynomial ring, with $\mathbb{K}$ an algebraically closed field. Let $f\in R$ be a Nagata polynomial of order d_1 and degree $\deg f=d=d_1+d_2$. The hypersurface $X=V(f)\subset\mathbb{P}^N$ is called a Nagata hypersurface of order d_1.

Let $X=V(f)\subset\mathbb{P}^N$ be a Nagata hypersurface of order d_1. We can consider two linear space respectively $\mathbb{P}^{m-1}$ with coordinates $u_1,\ldots,u_m$ and $\mathbb{P}^n$ with coordinates $x_0,x_1,\ldots,x_n$. Let $p_\alpha\in\mathbb{P}^{m-1}$ be a point and we consider the following linear space of dimension $n+1$:

$$\mathcal{L}_\alpha:=\langle p_\alpha,\mathbb{P}^n\rangle=\{\langle p_\alpha,q\rangle : q\in\mathbb{P}^n\}.$$

If we consider the intersection $\mathcal{L}_\alpha$ with X, we obtain a variety Y_α. Y_α is reducible whose irreducible components are the linear space $\mathbb{P}^n$ and a variety, called *residue* and denoted by $\tilde{Y}_\alpha$. $\tilde{Y}_\alpha$ is a cone of vertex p_α over a $(n-1)-$dimensional basis.

Theorem 1.3.9. *A Nagata hypersurface $X=V(f)\subset\mathbb{P}^N$ of order e consists of the union of the residue parts $\tilde{Y}_\alpha$, i.e.*

$$X=\cup_\alpha\tilde{Y}_\alpha.$$

Proof. Fixed a point $p_\alpha=(0:\ldots:0:a_1:\ldots:a_m)\in\mathbb{P}^{m-1}$ and let $\overline{p}=(\overline{x_0}:\ldots:\overline{x_n}:0:\ldots:0)$ be a point in $\mathbb{P}^n$. We consider the line that joins the points p_α and $\overline{p}$:

$$\mathscr{L}_\alpha:\begin{cases} x_0=\lambda\overline{x_0}\\ \ldots\ldots\ldots\\ x_n=\lambda\overline{x_n}\\ u_1=\mu a_1\\ \ldots\ldots\ldots\\ u_m=\mu a_m\end{cases}$$

with $\lambda,\mu\in\mathbb{K}$.
Since $X=V(f)$ is a Nagata hypersurface of order d_1, we have:

$$f=x_0^{d_1}g_0+\ldots+x_n^{d_1}g_n.$$

If we consider the intersection between the line $\mathscr{L}_\alpha$ and the Nagata hypersurface X, we get:

$$f_{\mathscr{L}_\alpha} = \lambda^{d_1}\overline{x_0}^{d_1} g_0(\mu a_1, \dots, \mu a_m) + \dots + \lambda^{d_1}\overline{x_n}^{d_1} g_n(\mu a_1, \dots, \mu a_m) = \lambda^{d_1}\mu^{d_2}\sum_{i=0}^{n}\overline{x_i}^{d_1} g_i(\underline{a})$$

where $\underline{a}$ is the vector $(a_1, \dots, a_m)$.

Since p_α and $\overline{p}$ are points of X, then $\sum_{i=0}^{n}\overline{x_i}^{d_1} g_i(\underline{a}) = 0$. Therefore

$$\tilde{Y}_\alpha = V\left(\sum_{i=0}^{n}\overline{x_i}^{d_1} g_i(\underline{a})\right)$$

and, by arbitrariness of the points $p_\alpha \in \mathbb{P}^{m-1}$ and $\overline{p} \in \mathbb{P}^n$, we have $\cup_\alpha \tilde{Y}_\alpha = X$. $\square$

As consequence of the above theorem, we can say how many linear spaces there are in a Nagata hypersurface of order e. We note that $\mathbb{P}^{m-1}$ and $\mathbb{P}^n$ are linear spaces on X. Thus we have:

Corollary 1.3.10. *Let $X = V(f) \subset \mathbb{P}^N$ be a Nagata hypersurface of order d_1. There is a family of lines of dimension $m+n-1$ on X.*

Proof. Let $p_\alpha \in \mathbb{P}^{m-1}$ be a point, then there is a family of lines of dimension n that joins p_α and the linear space $\mathbb{P}^n$, for all $p_\alpha \in \mathbb{P}^{m-1}$. This family covers $\tilde{Y}_\alpha$. Then we have a family of lines of dimension $(n) + (m-1) = n+m-1$ on X. The singular locus of X contains $\mathbb{P}^{m-1}$.

Conversely, let $\overline{p} \in \mathbb{P}^n$ be a point, then there is a family of lines of dimension $m-1$ that joins $\overline{p}$ and all points q in the linear space $\mathbb{P}^{m-1}$. So the proof follows. $\square$

1.4 Simplicial Nagata idealization of order k

Definition 1.4.1. A bihomogeneous polynomial

$$f = \sum_{i=0}^{n} x_i^k g_i \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(k,d-k)} \tag{1.3}$$

is called a simplicial Nagata polynomial of order k if all g_i are square free monomials.

The following combinatorial constructions were inspired by [34].

Definition 1.4.2. Let $V = \{u_1, \dots, u_m\}$ be a finite set. A *simplicial complex* Δ with vertex set V is a collection of subsets of V, i.e. a subset of the power set 2^V, such that for all $A \in \Delta$ and for all subset $B \subset A$, we have $B \in \Delta$.

We say that Δ is a simplex if $\Delta = 2^V$.
The members of Δ are referred as *faces* and the maximal faces (respect to the inclusion) are the *facets*. The vertex set of Δ is also called $0-$*skeleton*. If $A \in \Delta$ and $|A| = k$, it is called a $(k-1)-$face, or a face of dimension $k-1$: the $0-$faces are the vertices and the $1-$faces are called *edges*.

Definition 1.4.3. If all the facets have the same dimension $d > 0$, the complex is said to be *pure*.

Let Δ be a pure simplicial complex of dimension $d > 0$ with vertex set $V = \{u_1, \dots, u_m\}$, we denote by f_k the number of $(k-1)$-faces, hence $f_0 = 1$, $f_1 = m$, f_{d+1} is the number of facets of Δ and $f_j = 0$, for $j > d+1$.

Remark 1.4.4. There is a natural bijection between the square free monomials, of degree r, in the variables $u_1, \dots, u_m$, and the $(r-1)$-faces of the simplex 2^V, with vertex set $V = \{u_1, \dots, u_m\}$. In fact, a square free monomial $g = u_{i_1} \cdots u_{i_r}$, in the variables $u_1, \dots, u_m$, corresponds to the finite subset of 2^V given by $\{u_{i_1}, \dots, u_{i_r}\}$. To any finite subset F of 2^V, we associate the monomial $m_F = \prod_{u_i \in F} u_i$ of square free type.

An important result about simplicial Nagata idealization can be found in [34, Theorem 3.2].

Let $f \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ be a simplicial Nagata polynomial of order k:

$$f = \sum_{r=0}^{n} x_r^k g_r \tag{1.4}$$

with g_r monomials in variables $u_1, \dots, u_m$ of degree $k+1$.

We want to characterize the Hilbert vector of the algebras associated to the Nagata polynomial of type (1.4).
Let Δ be a pure simplicial complex of dimension k, with vertex set $V = \{u_1, \dots, u_m\}$. We denote by f_k the number of $(k-1)$-faces, hence $f_0 = 1, f_1 = m, f_{k+1}$ is the number of the facets of Δ and $f_j = 0$ for $j > k+1$.

The facets of Δ, associated to f, corresponding to the monomials g_i, will be labeled by g_i. The associated algebra is $A_\Delta = Q/\operatorname{Ann}(f_\Delta)$. By abuse of notation, we will always denote f_Δ with f and A_Δ with A.

If $p \in \mathbb{K}[u_1, \dots, u_m]$ is a square free monomial, we denote by $P \in \mathbb{K}[U_1, \dots, U_m]$ the dual differential operator $P = p(U_1, \dots, U_m)$.

Theorem 1.4.5. *[10, Theorem 3.5] Let $f \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(k,k+1)}$ be a simplicial Nagata polynomial of order k:*

$$f = \sum_{r=1}^{n} x_r^k g_r$$

with g_r monomials in variables $u_1,\dots,u_m$ of degree $k+1$. Let Δ be a pure simplicial complex of dimension k and let $A = Q/\operatorname{Ann}(f)$. Then

$$A = \bigoplus_{i=0}^{d=2k+1} A_i \text{ where } A_i = A_{(i,0)} \oplus A_{(i-1,1)} \oplus \cdots \oplus A_{(0,i)}, \quad A_d = A_{(k,k+1)}$$

1. *for all $j = 1,\dots,k+1$:*

$$\dim A_{(i,j)} = \begin{cases} f_j & \text{for} \quad i = 0 \\ (n+1)\cdot \overline{f_j} & \text{for} \quad 1 \le i < k \\ f_{k+1-j} & \text{for} \quad i = k \end{cases}$$

   where $\overline{f_j}$ is the number of the subfaces, of dimension $j-1$, of the facet, g_i, of Δ.

2. *$I = \operatorname{Ann}(f)$ is generated by*
   (a) *$\langle X_0,\dots,X_n\rangle^{k+1}$ and $U_1^2,\dots,U_m^2$;*
   (b) *the monomials in I representing the minimal faces of the complement of Δ, Δ^c;*
   (c) *the monomials $X_r^i P_r$, for $i = 1,\dots,k$, such that, fixed the facet M_r of Δ, corresponding to the monomial g_r, P_r is the dual differential operator of p_r; p_r is a monomial in the variables $u_1,\dots,u_m$, corresponding to a face M' of Δ s.t. $M' \cap M_r = \emptyset$;*
   (d) *the binomials $X_r^k\tilde{G}_r - X_s^k\tilde{G}_s$ where $g_r = \tilde{g}_r g_{rs}$ and $g_s = \tilde{g}_s g_{rs}$ and g_{rs} represents a common subface of g_r, g_s.*

Proof. 1. Let f be of type (1.4) associated to the pure simplicial complex Δ of dimension k. The variables $u_1,\dots,u_m$ represent the vertices of Δ.

We consider the following cases:

- for $i = 0$ and $j = 1,\dots,k+1$, $A_{(0,j)}$ is generated by the only monomials of degree j, in the variables $U_1,\dots,U_{k+1}$, that do not annihilate f. These monomials represent $(j-1)-$ faces of Δ. We need to show that they are linearly independent over $\mathbb{K}$.
  Consider $\{\Omega_1,\dots,\Omega_\nu\}$ a system of monomials of $Q_{(0,j)}$, where Ω_s, for $s = 1,\dots,\nu$, is associated to any $(j-1)$-face ω. We take any linear combination:

$$0 = \sum_{r=0}^{\nu} c_r\Omega_r(f) = \sum_{r=0}^{\nu} c_r \sum_{s=0}^{n} x_s^k \Omega_r(g_s) = \sum_{s=0}^{n} x_s \sum_{r=0}^{\nu} \Omega_r(g_s).$$

  Therefore we get $\sum_{r=0}^{\nu} c_r\Omega_r(g_s) = 0$, for all $s = 0,\dots,n$. For each $r = 0,\dots,\nu$, there is a $s = 0,\dots,n$, such that if $\Omega_r(g_s) \neq 0$, then $c_r = 0$ for all r. Hence $\dim A_{(0,j)} = f_j$, where f_j is the number of $(j-1)-$faces of Δ.

- for $1 \le i < k$ and $j = 1, \dots, k+1$, the generators of $A_{(i,j)}$ are the monomials of type $X_s^i U_{r_1} U_{r_2} \cdots U_{r_j}$ for $s = 0, \dots, n$, for all j. Fix $s = 0, \dots, n$, and let M_s be the facet of Δ, corresponding to the monomial g_s, the monomial $U_{r_1} U_{r_2} \cdots U_{r_j}$ of $Q_{(0,j)}$ is the dual differential operator of the monomial $u_{r_1} u_{r_2} \cdots u_{r_j}$, that gives the $(j-1)-$dimensional subfaces of M_s. The monomials $X_s^i U_{r_1} U_{r_2} \cdots U_{r_j}$ for $s = 0, \dots, n$, for all j are linearly independent. In fact, denoting by Ω_s^i the monomial $X_s^i U_{r_1} U_{r_2} \cdots U_{r_j}$, for $s = 0, \dots, n$, we note that:

$$\Omega_s^i(f) = c x_s^{k-i} (U_{r_1} \cdots U_{r_j})(g_s) \neq 0$$

since $(U_{r_1} \cdots U_{r_j})(g_s)$ identifies the vertices of the $(j-1)-$dimensional face. We get:

$$\sum_{s=0}^{n} c_s \Omega_s^i(f) = 0 \Leftrightarrow c_s = 0 \quad \forall s.$$

For $s = 0, \dots, n$, in correspondence of $\Omega_s^i(f)$, we can get a number of $(j-1)-$dimensional faces of Δ. Denoting such number by $\overline{f}_j$, we have $\dim A_{(i,j)} = (n+1) \cdot \overline{f_j}$.

- for $i = k$ and $j = 1, \dots, k$, by duality $A^*_{(0,k+1-j)} \simeq A_{(k,j)}$, thence we have:

$$\dim A_{(k,j)} = \dim A^*_{(0,k+1-j)} = f_{k+1-j}.$$

2. Let $I = \mathrm{Ann}(f)$ be the annihilator. We consider the following exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow Q_{(i,j)} \longrightarrow A_{(k-i,k+1-j)} \longrightarrow 0. \tag{1.5}$$

we have the following cases:

- for $i = 0$ and $1 \le j \le k+1$, we have by (1.5)

$$\dim A_{(0,j)} = f_j \Rightarrow \dim I_{(0,j)} = \dim Q_{(0,j)} - f_j.$$

Since $A_{(0,j)}$ has a basis given by the $(j-1)-$faces of Δ, then $I_{(0,j)}$ is generated by monomials representing all the $(j-1)-$faces of the complement of Δ. In I, it is enough to consider the minimal faces of Δ^c, by definition of ideal.
We note that in $I_{(0,2)}$ there are also the monomials $U_1^2, \dots, U_m^2$, since the monomials g_i, in the variables $u_1, \dots, u_m$ are square free.

- for $1 \le i < k$ and $1 \le j \le k+1$, fix the facet M_r of Δ, corresponding to g_r, since $A_{(i,j)}$ has a basis given by the $(j-1)-$dimensional subfaces of M_r, then $I_{(i,j)}$ is generated by monomials $X_r^i P_r$ where P_r is the dual differential operator of p_r; p_r is a monomial in the variables $u_1, \dots, u_m$, corresponding to a $(j-1)-$dimensional face $\overline{M}_r$ s.t. $\overline{M}_r \in \Delta^c$ or $\overline{M}_r \in \Delta$ and $\overline{M}_r \cap M_r = \emptyset$.
- for $i = k$ and $1 \le j \le k+1$, we fix two facets of Δ, M_r and M_s, corresponding to the monomials g_r and g_s, and such that $M_r \cap M_s \neq \emptyset$. Let $M_{rs} = M_r \cap$

M_s; we denote the monomial corresponding to it by g_{rs}. We consider $\tilde{M}_r = M_r \backslash M_{rs}$ and $\tilde{M}_s = M_s \backslash M_{rs}$. Let $\tilde{g}_r$ and $\tilde{g}_s$ be the monomials corresponding to $\tilde{M}_r$ and $\tilde{M}_s$. We note that $\tilde{M}_r \cap \tilde{M}_s = \emptyset$. Hence the binomials, $X_r^k \tilde{G}_r - X_s^k \tilde{G}_s$, are in $I_{(k,j)}$, where $\tilde{G}_r$ and $\tilde{G}_s$ are the dual differential operators of $\tilde{g}_r$ and $\tilde{g}_s$ respectively.
Let us consider the following exact sequence:

$$0 \longrightarrow I_{(k,j)} \longrightarrow Q_{(k,j)} \longrightarrow A_{(0,k+1-j)} \longrightarrow 0,$$

we get $\dim I_{(k,j)} = \dim Q_{(k,j)} - f_{k+1-j}$. Let $\tilde{Q}_{(k,j)}$ be the $\mathbb{K}-$space spanned by all the monomials $X_r^k \tilde{G}_r$, where $\tilde{G}_r$ is the dual differential operator of g_r that is a monomial in the variables $u_1, \ldots, u_m$, corresponding to a subface of M_r. Let $\overline{I}_{(k,j)} \subset I_{(k,j)}$ be the $\mathbb{K}$-vector space spanned by the monomials $X_r^k P_r$, where P_r is the dual differential operator of the monomial, in the variables $u_1, \ldots, u_m, p_r$, not corresponding to a subface of M_r. They are two $\mathbb{K}-$vector spaces s.t. $Q_{(k,j)} = \tilde{Q}_{(k,j)} \oplus \overline{I}_{(k,j)}$. We consider the ideal $\tilde{I}_{(k,j)} \subset \tilde{Q}_{(k,j)}$. The exact sequence given by evaluation restricted to $\tilde{Q}_{(k,j)}$ becomes:

$$0 \longrightarrow \tilde{I}_{(k,j)} \longrightarrow \tilde{Q}_{(k,j)} \longrightarrow A_{(0,k+1-j)} \longrightarrow 0.$$

We note:

$$\begin{aligned}\dim I_{(k,j)} &= \dim Q_{(k,j)} - f_{k+1-j} = \dim \tilde{Q}_{(k,j)} + \dim \overline{I}_{(k,j)} - f_{k+1-j} = \\ &= \dim \tilde{I}_{(k,j)} + f_{k+1-j} + \dim \overline{I}_{(k,j)} - f_{k+1-j} = \dim \tilde{I}_{(k,j)} + \dim \overline{I}_{(k,j)}.\end{aligned}$$

Hence $I_{(k,j)} = \tilde{I}_{(k,j)} \oplus \overline{I}_{(k,j)}$. The generators of $\tilde{I}_{(k,j)}$ are the binomial $X_r^k \tilde{G}_r - X_s^k \tilde{G}_s$ precisely. The result follows.

Moreover for $i = k+1$ and $j = 0$, it is clear that $I_{(k+1,0)} = (X_0, \ldots, X_n)^{k+1}$. In fact $X_i^{k+1}(f) = 0$, for $i = 0, \ldots, n$, since the monomials in $x_0, \ldots, x_n$ of f have degree k, by Remark (1.1.12).

□

We discuss the following example:

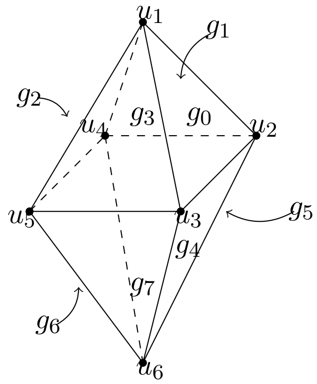
Example 1.4.6. Let $V = \{u_1, \ldots, u_6\}$ be a finite set. We have:

$$2^V = \{\emptyset, \{u_1\}, \ldots, \{u_6\}, \ldots, \{u_1, \ldots, u_6\}\}$$

Let Δ be the following simplicial complex:

$$\Delta = \{\emptyset, \underbrace{\{u_1\}, \ldots, \{u_6\}}_{\text{vertices}}, \underbrace{\{u_1, u_2\}, \ldots, \{u_5, u_6\}}_{\text{edges}}, \underbrace{\{u_1, u_2, u_3\}, \ldots, \{u_2, u_3, u_6\}}_{2-\text{faces}}\}$$

It is given by two pyramids, with the common basis, of vertices $u_1, \ldots u_5$ and u_6 and faces labeled by $g_0, \ldots, g_6$ and g_7:



The 2−faces are the facets of Δ, then Δ is pure of dimension 2.
Let

$$f = f_\Delta = x_0^2u_1u_2u_3 + x_1^2u_1u_2u_4 + x_2^2u_1u_4u_5 + x_3^2u_1u_3u_5+ \\ + x_4^2u_2u_3u_6 + x_5^2u_2u_4u_6 + x_6^2u_4u_5u_6 + x_7^2u_3u_5u_6$$

be the bihomogeneous polynomial of degree 5. It is a Nagata polynomial of order 2 and the monomials $g_0 = u_1u_2u_3$, $g_1 = u_1u_2u_4$, $g_2 = u_1u_4u_5$, $g_3 = u_1u_3u_5$, $g_4 = u_2u_3u_6$, $g_5 = u_2u_4u_6$, $g_6 = u_4u_5u_6$ and $g_7 = u_3u_5u_6$ are of square free type.

We have

$$A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$$

and the Hilbert vector is given by:

$$h_0 = 1 = h_5 \text{ and } h_1 = 14 = h_4.$$

We calculate $h_2 = \dim A_2$ and $h_3 = \dim A_3$.
By Theorem (1.4.5), we have

$$h_2 = \dim A_2 = \dim A_{(2,0)} + \dim A_{(1,1)} + \dim A_{(0,2)} = \\ = f_3 + 8 \cdot \overline{f_1} + f_2 = 8 + 8 \cdot 3 + 12 = 44$$

$$h_3 = \dim A_3 = \dim A_{(3,0)} + \dim A_{(2,1)} + \dim A_{(1,2)} + \dim A_{(0,3)} = \\ = 0 + f_2 + 8 \cdot \overline{f_2} + f_3 = 12 + 8 \cdot 3 + 8 = 44$$

Hence the Hilbert vector is $(1, 14, 44, 44, 14, 1)$.

By Theorem (1.4.5), $I = \operatorname{Ann}(f)$ is generated by:

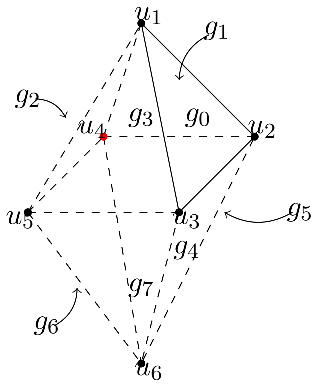
- $\langle X_0, \dots, X_7\rangle^3$ and $U_1^2, \dots, U_m^2$, by the part (2a);

- since the complement of Δ is:

$$\Delta^c = \{\underbrace{\{u_1,u_6\},\ldots,\{u_2,u_5\}}_{\text{diagonals}}, \underbrace{\{u_1,u_2,u_6\},\ldots,\{u_1,u_5,u_6\}}_{2-\text{faces}},$$
$$\underbrace{\{u_1,u_2,u_3,u_5\},\ldots,\{u_2,u_3,u_5,u_6\}}_{3-\text{faces}},\ldots,\{u_1,\ldots,u_6\}\}$$

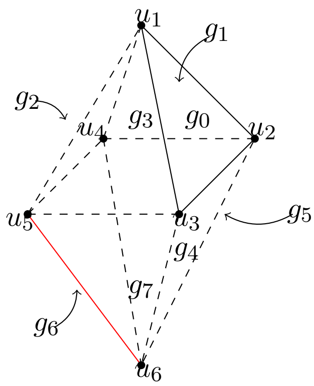
and since diagonals are the minimal faces of Δ^c, then the monomials U_1U_6, U_3U_4 and U_2U_5 are in $I = \mathrm{Ann}(f)$, by the part (2b).

- For $i = 1, 2$, fix the facet $M_0 = \{u_1, u_2, u_3\} \in \Delta$, corresponding to the monomial g_0, we have that the monomial p_0 represents:
  - one of the remaining vertices, for example u_4:



  and finally we get: $P_0 = p_0(U_1, \ldots, U_8) = U_4$. Thence the monomial of degree $i+1$, $X_0^iU_4$ is in $I = \mathrm{Ann}(f)$. The other monomials of this type are obtained with the same procedure.

  - one of the remaining edges, for example the edge that joins the vertices u_5 and u_6:

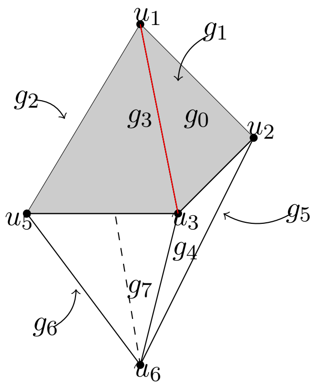


  We get:

$$P_0 = p_0(U_1, \ldots, U_8) = U_5U_6$$

and the monomial of degree $i+2$, $X_0^i U_5 U_6$, is in $I = \mathrm{Ann}(f)$, by part (2c). The other monomials of this type are obtained by the same way.

- The faces g_0 and g_3 have the common edge that joins the vertices $u_1 u_3$:



$g_{1,3}$ represents the edge that joins the vertices u_1 and u_3. $\tilde{g_0}$ and $\tilde{g_3}$ represent the vertices u_2 and u_5 respectively. We have:

$$\tilde{G_0} = \tilde{g_0}(U_1, \ldots, U_6) = U_2 \text{ and } \tilde{G_3} = \tilde{g_3}(U_1, \ldots, U_6) = U_5.$$

The binomial, of degree 3, $X_0^2 U_2 - X_3^2 U_5$, is in $I = \mathrm{Ann}(f)$ by the part (2d). The other binomials of degree 3 of this type are obtained by the same procedure. We note that the faces g_0 and g_2 have the common vertex u_1, hence, in the ideal $I = \mathrm{Ann}(f)$ there is the binomial, of degree 4, $X_0^2 U_2 U_3 - X_2^2 U_4 U_5$; the other binomials, of degree 4, are obtained by the same procedure.

Chapter 2

Asymptotic behaviour of lenght five Gorenstein Hilbert function

2.1 Classical Bounds of Hilbert function

We recall some classical bounds for the growth of the Hilbert function of Artinian $\mathbb{K}$-algebras. For a more detailed account, see [73]. The following three basic results are due to Macaulay, Gotzmann and Green; before stating them, we need to recall the following definition:

Definition 2.1.1. Let n and i be positive integers. The i-binomial expansion of n, denoted by $n_{(i)}$, is

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j} \tag{2.1}$$

where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$.

An expansion of type (2.1) always exists and is unique (see, e.g., [8, Lemma 4.2.6]). Following [8], we define for any integers a and b,

$$(n_{(i)})^b_a = \binom{n_i + b}{i + a} + \binom{n_{i-1} + b}{i - 1 + a} + \cdots + \binom{n_j + b}{j + a}$$

where we set $\binom{m}{c} = 0$ whenever $m < c$ or $c < 0$.

Theorem 2.1.2. *Let $A = R/I$ be a standard graded $\mathbb{K}$-algebra, and $L \in A$ a general linear form (according to the Zariski topology). Denote by h_d the degree d entry of the Hilbert function of A and by h'_d the degree d entry of the Hilbert function of $A/(L)$. Then:*

(Macaulay)

$$h_{d+1} \leq ((h_d)_{(d)})^{+1}_{+1}.$$

(Gotzmann) *If* $h_{d+1} = ((h_d)_{(d)})^{+1}_{+1}$ *and* I *is generated in degrees* $\leq d+1$, *then*

$$h_{d+s} = ((h_d)_{(d)})^s_s \text{ for all } s \geq 1$$

.

(Green)

$$h'_d \leq ((h_d)_{(d)})^{-1}_0.$$

Proof. For *Macaulay*, see [8, Theorem 4.2.10]. For *Gotzmann*, see [8, Theorem 4.3.3] or [36]. For *Green*, see [37, Theorem 1]. □

Definition 2.1.3. A sequence of nonnegative integers $h = (1, h_1, h_2, \ldots, h_i, \ldots)$ is said to be an $\mathcal{O}$-sequence if it satisfies Macaulay's Theorem (2.1.2) for all i.

Recall that when A is artinian and Gorenstein, then its Hilbert function is a finite, symmetric $\mathcal{O}$-sequence. We recall a useful theorem proved in [77, Theorem 3.5].

Theorem 2.1.4. *Let* h_{d-1}, h_d *and* h_{d+1} *be three integers such that* $((h_d)_{(d)})^{-1}_{-1} = h_{d-1}$ *and* $((h_d)_{(d)})^{+1}_{+1} = h_{d+1}$. *Suppose that* $h_{d-1} + \alpha, h_d$ *and* h_{d+1}, *for some integer* $\alpha > 0$, *are the entries of degree* $d-1, d$ *and* $d+1$ *of the h-vector of an algebra* A. *Then* A *has depth zero and an* α*-dimensional socle in degree* $d-1$.

2.2 Construction of non unimodal Hilbert vectors

We recall that the Artinian Gorenstein $\mathbb{K}$-algebras of codimension 3 have the Hilbert vector always unimodal (see [68] and [76]), for codimension 4 it is an open question if there is a non unimodal Gorenstein Hilbert vector. The first example of a non unimodal Gorenstein Hilbert vector in codimension 5 was given in [5]. A way to construct non unimodal Hilbert vector in codimension ≥ 5 is to consider the idealization of a generic level algebra of type 2, (see [44]). Prof. *Ricardo Machado*, from UFRPE, created a routine using Macaulay2 to choose two polynomials in random way and find the Hilbert function of the associated Nagata idealization. By Macaulay duality we consider a bihomogeneous polynomial of degree d in $\mathbb{K}[x_1, x_2, u_1, u_2, u_3]$ of this type

$$f = x_1 g_1 + x_2 g_2$$

with $g_i \in \mathbb{K}[u_1, u_2, u_3]_{d-1}$ for $i = 1, 2$. Using this construction in codimension 5 the first socle degree where we find a non unimodal Hilbert vector is $d = 16$, moreover, in this case we get:

$$(1, 5, 12, 22, 35, 51, 70, 91, 90, 91, 70, 51, 35, 22, 12, 5, 1)$$

that is exactly the Iarrobino's Example. Moreover in socle degree 18 we have the following non unimodal Hilbert vector:

$$(1, 5, 12, 22, 35, 51, 70, 92, 111, 110, 111, 92, 70, 51, 35, 22, 12, 5, 1).$$

In socle degree 20, we have:

$$(1, 5, 12, 22, 35, 51, 70, 92, 117, 133, 132, 133, 117, 92, 70, 51, 35, 22, 12, 5, 1)$$

and finally in socle degree 21 we have:

$$(1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 144, 144, 145, 117, 92, 70, 51, 35, 22, 12, 5, 1).$$

From a historic viewpoint the first example of a non unimodal Gorenstein Hilbert vector was given by Stanley, it is $(1, 13, 12, 13, 1)$. The following construction generalizes Stanley example. This construction will be very useful in the sequel.

Definition 2.2.1. Let $\mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring in the $n+1$ variables $x_0, \ldots, x_n$ and in the m variables $u_1, \ldots, u_m$. A **Perazzo polynomial** is a reduced bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$, of degree d, of type

$$f = \sum_{i=1}^{n} x_i g_i \tag{2.2}$$

with $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$, for $i = 0, \ldots, n$, linearly independent and algebraically dependent polynomials in the variables $u_1, \ldots, u_m$.

Remark 2.2.2. By Definition (1.4.1), a Perazzo polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$ of degree d is a Nagata polynomial, hence, by Theorem (1.3.2), the algebra $A = Q/\operatorname{Ann}(f)$, associated to f, where $Q = \mathbb{K}[X_1, \ldots, X_n, U_1, \ldots, U_m]$ is the ring of the differential operators, can be realized as a Nagata idealization of order 1, socle degree d and codimension $n+m$.

By above Remark (2.2.2), we can give the following definition:

Definition 2.2.3. Let $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$ be a Perazzo polynomial of degree d. The algebra $A = Q/\operatorname{Ann}(f)$ associated to f is called **Perazzo algebra** and it is a begraded algebra of socle degree d and codimension $n+m$.

Now we fix $m \geq 2$ and we consider the m variables $u_1, \ldots, u_m$. Let us consider

$$M_j = u_{j_1}^{\alpha_{j_1}} \cdots u_{j_m}^{\alpha_{j_m}} \text{ for } j = 1, \ldots, \tau_m$$

where $\tau_m := \binom{m+d-2}{d-1}$ and $\alpha_{j_1} + \cdots + \alpha_{j_m} = d-1$.

Definition 2.2.4. A bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_{\tau m}, u_1, \ldots, u_m]_{(1,d-1)}$ of degree d of type:

$$f = \sum_{j=1}^{\tau_m} x_j M_j \tag{2.3}$$

is called **Full Perazzo polynomial** of type m.

Remark 2.2.5. As in Remark (2.2.2), let $f \in \mathbb{K}[x_1, \dots, x_{\tau_m}, u_1, \dots, u_m]_{(1,d-1)}$ be a Full Perazzo polynomial of type m and of degree d, the algebra $A = Q/\operatorname{Ann}(f)$, associated to f, where $Q = \mathbb{K}[X_1, \dots, X_{\tau_m}, U_1, \dots, U_m]$ is the ring of the differential operators, can be realized as a Nagata idealization of order 1, socle degree d and codimension $m + \tau_m$.

By above Remark (2.2.5), we can give the following definition:

Definition 2.2.6. Let $f \in \mathbb{K}[x_1, \dots, x_\tau, u_1, \dots, u_m]_{(1,d-1)}$ be a Full Perazzo polynomial of degree d. The algebra $A = Q/\operatorname{Ann}(f)$ associated to f is called **Full Perazzo algebra** and it is a begraded algebra of socle degree d and codimension $m + \tau_m$.

Proposition 2.2.7. *Let A be a Full Perazzo algebra of type $m \geq 2$ an socle degree d. Then for $k = 0, \dots, \lfloor \frac{d}{2} \rfloor$*

$$h_k = \dim A_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}$$

Proof. Using the bigrading of A and considering that the polynomial f has degree 1 in tha all variables $x_1, \dots, x_{\tau_m}$, fixed $k = 0, \dots, \lfloor \frac{d}{2} \rfloor$, we have the following decomposition:

$$A_k = A_{(0,k)} \oplus A_{(1,k-1)}.$$

for $A_{(0,k)}$; it is clear that $A_{(0,k)} = Q_{(0,k)}$, hence $\dim A_{(0,k)} = \dim Q_{(0,k)} = \binom{m+k-1}{k}$;

for $A_{(1,k-1)}$; by Remark (1.1.9), we have $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$ and $A_{(0,d-k)} = Q_{(0,d-k)}$, hence $\dim A_{(1,k-1)} = \dim Q_{(0,d-k)} = \binom{m+d-k-1}{d-k}$.

□

2.3 A conjecture of R. Stanley

Let $\mathbb{K}$ be a field of characteristic zero. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the graded polynomial ring and let $Q = \mathbb{K}[X_1, \dots, X_n]$ be the differenzial ring, with $X_i = \frac{\partial}{\partial x_i}$, for $i = 1, \dots, n$. Moreover let $\operatorname{Ann}(f) \subset Q$ be the annihilator of a homogeneous polinomial $f \in R$. Let n and d be two integers, we consider the following set:

$$\mathcal{G}(n,d) := \left\{ A : \ A \simeq \frac{Q}{\operatorname{Ann}(f)} \right\}$$

with some $f \in \mathbb{K}[x_1, \dots, x_n]_d$ homogeneous polynomial of degree d, i.e. it is the family of standard graded artinian Gorenstein $\mathbb{K}$-algebras of codimension n and socle degree d.

Moreover, let $0 < k < d$ be a integer, we can define the following functions:

$$\mu_k(n,d) = \min_{A \in \mathcal{G}} \{\dim A_k\} \quad \delta_k(n,d) = n - \mu_k(n,d).$$

Fixed d, we can consider the above functions without dependence by d, hence $\mu_k(n,d) = \mu_k(n)$ and $\delta_k(n,d) = \delta_k(d)$.

Stanley in [70] conjectured that

$$\lim_n \frac{f(n)}{n^{\frac{2}{3}}} = 6^{\frac{2}{3}}$$

where in our notation $f(r) = \mu_2(r)$. He guessed the precise value of the limit even if he was not able to prove the existence of the limit. Bounds were given by Stanley in [72] and by Kleinschmidt in [47], but the precise limit was only proved in 2006 (see [58]). We will give a new proof of this result studing a family of algebras that actually reach the limit, asymptotically.

First of all we prove some introductory lemmas about the monotonicity of the functions μ_k and δ_k.

Lemma 2.3.1.

$$\mu_k(n+1) \leq \mu_k(n) + 1 \quad \forall k = 0, 1, \ldots, d$$

Proof. Let $f \in R_d$ be a homogeneous polynomial of degree d such that $A = R/\operatorname{Ann}(f)$ is a standard graded Artinian Gorenstein $\mathbb{K}-$algebra, for which $\mu_k(n) = \dim A_k$. We denote by R' the polynomials ring in $n+1$ variables $\mathbb{K}[x_1, \ldots, x_n, x_{n+1}]$. We take $f' \in R'_d$, s.t. $f' = f + x_{n+1}^d$, and we denote by $A' = R'/\operatorname{Ann}(f')$ the standard graded Artinian Gorenstein $\mathbb{K}-$algebra. Hence $A'_k = A_k \oplus < x_{n+1}^k >$. Therefore

$$\mu_k(n+1) \leq \dim A'_k = \dim A_k + 1 = \mu_k(n) + 1.$$

□

As consequence of the Lemma (2.3.1), we have the following:

Lemma 2.3.2. *The function $\delta_k(n)$ is non-decreasing in n.*

Proof. It's enought to show that $\delta_k(n+1) \geq \delta_k(n)$. Since By Lemma (2.3.1), we have:

$$\delta_k(n+1) = n + 1 - \mu_k(n+1) \geq n + 1 - \mu_k(n) - 1 = \delta_k(n).$$

□

Lemma 2.3.3. *The function $\mu_k(n)$ is non-decreasing in n.*

Proof. Let $A \in \mathcal{G}(n+1, d)$ be an algebra such that $\dim A_k = \mu_k(n+1)$. Hence $A \simeq Q'/\operatorname{Ann}(f)$ with $f \in \mathbb{K}[x_1, \ldots, x_n, x_{n+1}]_d$ and $Q' = \mathbb{K}[X_1, \ldots, X_{n+1}]$. Since $A \simeq Q/\operatorname{Ann}(f)$ is a standard, graded, Artinian and Gorenstein algebra, hence $\operatorname{Ann}(f)_1 = 0$ and the hypersurface $X = V(f) \subset \mathbb{P}^n$ is not a cone, by Remark (1.1.6). Up to a projective transformation, we can take a homogeneous polynomial of this type:

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0) \in \mathbb{K}[x_1, \ldots, x_n]$$

and it defines a hypersurpace $Y = V(g) \subset H = P^{n-1}$ where $H = V(x_{n+1})$. In particular we have $V(g) = V(f) \cap V(x_{n+1})$, i.e. $H = X \cap H$ and it is not a cone. By Proposition (1.1.6), $\operatorname{Ann}(g)_1 = 0$ and $B \simeq Q/\operatorname{Ann}(g)$ is a standard graded Artinian Gorenstein algebra, i.e $B \in \mathcal{G}(n, d)$. Considering that $\dim B_k \leq \dim A_k$, by definition of the function $\mu_k(n)$, we have:

$$\mu_k(n) \leq \dim B_k \leq \dim A_k = \mu_k(n+1).$$

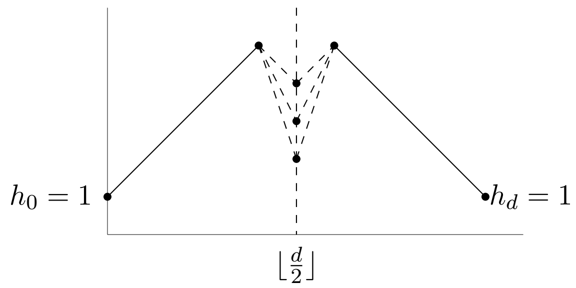
□

Remark 2.3.4. Notice that from the previows lemmas it is easy to see that both functions μ_k and δ_k satisfy the following monotic behaviour for each n.

$$\mu_k(n+1) = \mu_k(n) \ \text{ or } \ \mu_k(n+1) = \mu_k(n) + 1.$$

$$\delta_k(n+1) = \delta_k(n) \ \text{ or } \ \delta_k(n+1) = \delta_k(n) + 1.$$

The study of the function $\mu_k(n)$ is very important for our purposes; fixed $k = \lfloor \frac{d}{2} \rfloor$, for every $A \in \mathcal{G}(n, d)$, we can have different Hilbert vectors that are non unimodal, and the study of the asymptotic behavior of the function $\mu_k(n)$ gives us informations about the “depth" of the Hibert function in the middle part:



The first problem we have is to find a way to construct Hilbert vectors that are non unimodal.

Now we want to analyze the case $d = 4$, i.e. we consider $\mathcal{G}(n, 4)$ the family of all Gorenstein algebras in codimension n and socle degree 4. By abuse of notation, we denote $\mathcal{G}(n, 4)$ by $\mathcal{G}(n)$. For every $A \in \mathcal{G}(n)$, the Hilbert vector will always be of type

$$(1, n, \mu(n), n, 1)$$

denoting $\mu_2(n)$ by $\mu(n)$. The following result is a generalization of the main result of [73].

Theorem 2.3.5. *Let $A \in \mathcal{G}(r)$ be a Gorenstein $\mathbb{K}$-algebra of codimension r, with $r = m + \binom{m+2}{3}$. Then $\mu(r) = m(m+1)$.*

Proof. We want to show that the following Gorenstein Hilbert vector

$$\left(1, m + \binom{m+2}{3}, m(m+1), m + \binom{m+2}{3}, 1\right)$$

is minimal for $r = m + \binom{m+2}{3}$. It is enough to prove that the following Hilbert vector

$$H = \left(1, m + \binom{m+2}{3} - 1, m(m+1) - 1, m + \binom{m+2}{3} - 1, 1\right)$$

is not a Gorenstein Hilbert vector. By contradiction, we assume that the Hilbert vector H is Gorenstein, i.e. exists $A \simeq Q/I$, with f a homogeneous polynomial of degree 4 and $I = \mathrm{Ann}(f)$, such that its Hilbert vector is exactly the Hilbert vector H. Let $\ell \in Q$ be a generic linear form and $S = Q/(\ell)$. We get the following exact sequence:

$$0 \longrightarrow Q/(I:\ell)(-1) \longrightarrow Q/I \longrightarrow S/\overline{I} \longrightarrow 0$$

with $\overline{I} = \frac{(I,\ell)}{\ell}$. Notice that the conductor $(I:\ell)$ is also a Gorenstein ideal since $(I:\ell) = \{\alpha \in Q | \alpha.L(f) = 0\} = \{\alpha \in Q | \alpha(\frac{\partial f}{\partial \ell})\} = \mathrm{Ann}(\frac{\partial f}{\partial \ell})$.

The following diagram represents the only possible values for the Hilbert functions of Q/I, $Q/(I:\ell)(-1)$ and $S/\overline{I}$ respectively:

$$\begin{array}{ccccc}
1 & m + \binom{m+2}{3} - 1 & m(m+1) - 1 & m + \binom{m+2}{3} - 1 & 1 \\
 & 1 & a & a & 1 \\
\hline
1 & m + \binom{m+2}{3} - 2 & b & c &
\end{array}$$

with

$$b := m(m+1) - 1 - a \qquad c := m + \binom{m+2}{3} - 1 - a.$$

We note that $c - b = \binom{m}{3}$. By Green's Theorem (2.1.2) and Macaulay's Theorem (2.1.2), we get:

$$b = \binom{m}{2} = \binom{m}{m-2} \qquad c = \binom{m+1}{3} = \binom{m+1}{m-2}$$

$$a = \binom{m}{2} + \binom{m-1}{1}.$$

Let J be the ideal generated by the components of I in degrees 2 and 3 and $\overline{J} = \frac{(J,\ell)}{(\ell)}$. In degree ≤ 3, if we replace I by J we get the same table of Hilbert functions. Since $S/\overline{J}$ has maximal growth from degree 2 to degree 3 and $\overline{J}$ has no new generators in degree ≥ 4, then by Gotzmann's Theorem (2.1.2) we get

$$h_{S/\overline{J}}(t) = \binom{m-2+t}{m-2}$$

for $t \geq 2$. In $\mathbb{P}^{m+\binom{m+2}{3}-3}$, we consider $\tau = m + \binom{m+2}{3} - 2$, then $\overline{J}$ is the saturated ideal of $\overline{X} \subset \mathbb{P}^{\tau-1}$ for all $t \geq 2$. Therefore, up to saturation, J is the ideal of a union of $X = \mathbb{P}^{m-1} \subset \mathbb{P}^{\tau}$ and a finite number, say s, of points, $\{P_1, \dots, P_s\}$ in $\mathbb{P}^{\tau}$. We want to determine the upper and lower bound for $h_{Q/J}(4)$.

Upper bound: we use Macaulay's Theorem (2.1.2), since $h_{Q/J}(3) = \binom{m+2}{3} + \binom{m-1}{1}$, then

$$h_{Q/J}(4) \leq \left((h_{Q/J}(3))_{(3)}\right)^{+1}_{+1} \Rightarrow h_{Q/J}(4) \leq \binom{m+3}{4} + \binom{m}{2}.$$

Lower bound: we use the fact $h_{Q/J}(4) \geq h_{Q/J^{sat}}(4) = s + \binom{m+3}{4}$

Therefore

$$s + \binom{m+3}{4} \leq h_{Q/J}(4) \leq \binom{m+3}{4} + \binom{m}{2}$$

in particular $0 \leq s \leq \binom{m}{2}$. Hence all the various cases remain to be discussed as in the proof of [73, Proposition 3.1]. □

The following is a new short proof of the Theorem in [58] solving Stanley's conjecture.

Corollary 2.3.6. *Let $A \in \mathcal{G}(n)$ be a Gorenstein algebra of codimension r. Then*

$$\lim_{r \to \infty} \frac{[\mu(r)]}{r^{2/3}} = 6^{2/3}.$$

Proof. Fixed the integer $\mathcal{P}_k = k + \binom{k+2}{3}$, it is clear that:

$$\mathcal{P}_k \leq r \leq \mathcal{P}_{k+1}. \tag{2.4}$$

Applying the function $\mu(r)$, we have by Theorem (2.3.3):

$$\mu(\mathcal{P}_k) \leq \mu(r) \leq \mu(\mathcal{P}_{k+1}).$$

By Theorem (2.3.5), we get:

$$k(k+1) \leq \mu(r) \leq (k+1)(k+2).$$

In particular:

$$k^2 + o(k) \leq \mu(r) \leq k^2 + o(k)$$

with $o(k)$ all terms of lower degree in k. Hence we get:

$$k^6 + o(k) \leq [\mu(r)]^3 \leq k^6 + o(k). \tag{2.5}$$

Since it holds (2.4), then:

$$\frac{k^3}{6} + o(k) \leq r \leq \frac{k^3}{6} + o(k)$$
$$\frac{k^6}{6^2} + o(k) \leq r^2 \leq \frac{k^6}{6^2} + o(k)$$
$$\frac{1}{\frac{k^6}{6^2} + o(k)} \leq \frac{1}{r^2} \leq \frac{1}{\frac{k^6}{6^2} + o(k)}$$

Multiplying, we get:

$$\frac{k^6+o(k)}{\frac{k^6}{6^2}+o(k)} \leq \frac{[\mu(r)]^3}{r^2} \leq \frac{k^6+o(k)}{\frac{k^6}{6^2}+o(k)}.$$

Since in both sides the limit exists and are the same, therefore $\lim\limits_{r\to\infty} \frac{[\mu(r)]^3}{r^2} = 6$ and the result follows. $\square$

2.4 A new conjecture

A standard graded $\mathbb{K}$-algebra $A = R/I$ with $R = \mathbb{K}[x_1, \dots, x_n]$ is called presented by quadrics if the homogeneous ideal I is generated by quadratic forms. These quadrics are important in many areas, see [34, 57].

In [57] the authors conjectured that all Artinian Gorenstein algebras presented by quadrics shoud have the Weak Lefschetz property, but it is false. In [34] the authors found a family of Artinian Gorenstein algebras presented by quadrics whose Hilbert vectors were non unimodal.

A family of Artinian Gorestein algebra, presented by quadrics, having non unimodal Hilbert function, is the Turan algebra, introduced in [34] and inspired by the famous Turan's Graph Theorem.

Definition 2.4.1. Let $2 \leq a_1 \leq \dots \leq a_{d-1}$ be integers. The **Turan complex** of order $a_1, \dots, a_{d-1}$, $\mathcal{K} = \mathcal{TK}(a_1, \dots, a_{d-1})$ is the homogeneous simplicial complex whose facets set is the cartesian product

$$\pi = \prod_{i=1}^{d-1} \{1, 2, \dots, a_i\}.$$

The associated algebra is called the **Turan algebra** of order $(a_1, \dots, a_{d-1})$ and denoted by $TA(a_1, \dots, a_{d-1})$.

The following Theorem characterizes the Hilbert function of a Turan algebra:

Theorem 2.4.2. *[34, Theorem 3.7] Every Turan algebra $TA(a_1, \dots, a_{d-1})$ is presented by quadrics. Its Hilbert vector is given by $h_k = s_k + s_{d-k}$ where $s_k = s_k(a_1, \dots, a_{d-1})$ is the elementary symmetric polynomial of order k.*

The following corollary is very important, since it ensure the non unimodality of the Hilbert function of these algebras:

Corollary 2.4.3. *[34, Corollary 3.8] Let $A = TA(a_1, \dots, a_{d-1})$ be the Turan algebra of order $(a_1, \dots, a_{d-1})$ with $a_1 \approx \dots \approx a_{d-1}$ large enough. Then* $\mathrm{Hilb}(A)$ *is totally non unimodal, that is*

$$\dim A_1 > \dim A_2 > \dots > \dim A_{\lfloor \frac{d}{2} \rfloor}.$$

Remark 2.4.4. Notice that if we take $a \leq b \leq c$ in such way that $|x - y| \leq 1$ for all $x, y \in \{a, b, c\}$, then we can define $r = 3a + s$ with $s \in \{0, 1, 2\}$ and the Turan algebra $TA(r)$ associated to the Turan complex with respect to a, b, c. For this family of algebras it is easy to verify that

$$\lim_{r\to\infty} \frac{[\dim TA_k(r)]}{r^{2/3}} = 6.$$

Let $\mathcal{QG}(r)$ be the family of Artinian Gorenstein algebras presented by quadrics with socle degree 4 and codimension r. Let us call $\nu(r) = \min_{A\in\mathcal{QG}(r)} \{\dim A_2\}$.

Conjecture 2.4.5. Let $A \in \mathcal{QG}(r)$ be a Gorenstein algebra of codimension r. Then

$$\lim_{r\to\infty} \frac{[\nu(r)]}{r^{2/3}} = 6.$$

Chapter 3

The Hilbert vector of the Jacobian module of a plane curve

3.1 The minimal resolutions

Let A be a ring and let M be a finitely generated A-module. Let $f_1, \dots, f_n$ be elements of M, a syzygy between the f_i, $i = 1, \dots, n$, is a n-uple, $a_1, \dots, a_n \in A$, such that

$$\sum_{i=1}^{n} a_i f_i = 0.$$

The set of the syzygies of $f_1, \dots, f_n$ is a A-module. Moreover if A is a notherian ring then every A-module is finitely generated, in particular the A-module of the syzygies of $f_1, \dots, f_n$, denoted by $\mathrm{Syz}(f_1, \dots, f_n)$, is finitely generated.

Proposition 3.1.1. *For any A-module M exists a exact sequence*

$$A^s \xrightarrow{\ \psi\ } A^t \xrightarrow{\ \varphi\ } M \longrightarrow 0 \tag{3.1}$$

with s and t positive integers.

A sequence of type (3.1) is called *presentation* of M. If $\{f_1, \dots, f_t\}$ is the set of generators of M and $\{g_1, \dots, g_s\}$ os the set of generators of $\mathrm{Syz}(f_1, \dots, f_t)$, we can get a presentation of M setting $\varphi(e_i) = f_i$, for $i = 1, \dots, t$ with e_i element of the canonical basis of A^t and $\psi(\underline{e}_j) = g_j$, for $j = 1, \dots, s$ with $\underline{e}_j$ element of the canonical basis of A^s. Hence it is possible to extend to left the exact sequence of type (3.1) to a exact sequence having the module of the second syzygies, the module of the third syzygies and so on. By above Proposition, we can give the following definition:

Definition 3.1.2. A free resolution of a A-module M is a exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where $F_i \cong A^{d_i}$ for every i. If exists a positive integer l such that $F_{l+1} = F_{l+2} = \cdots = 0$, but $F_l \neq 0$, then the resolution is said to be finite of length l and it is written in the following way:

$$0 \longrightarrow F_l \longrightarrow F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Theorem 3.1.3 (Hilbert's syzygy theorem)**.** *Every finitely generated module over a polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$, in $n+1$ indeterminates, over a field $\mathbb{K}$ has a free resolution of length at most $n+1$.*

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring. A graded $A-$module is an A-module M with a decomposition

$$M = \bigoplus_{-\infty}^{+\infty} M_i$$

as abelian group such that $A_i M_j \subset M_{i+j}$ for all $i \geq 0$ and $j \in \mathbb{Z}$.

Remark 3.1.4. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a finitely generated graded A-module. Then M_i, for all i, is a $\mathbb{K}$-vector subspace of M of finite dimension.

By above Remark, given a finitely generated graded A-module M, we can give the following definitions of Hilbert function of M, n-shift of M and graded morphism of any degree between two graded A-modules:

Definition 3.1.5. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a finitely generated graded $A-$module, the Hilbert function of M is a function:

$$H_M \colon \mathbb{Z} \to \mathbb{N}$$

such that $H_M(t) := \dim_{\mathbb{K}} M_t$, for all $t \in \mathbb{Z}$.

Definition 3.1.6. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a graded A-module. The n-shift of M, denoted by $M(n)$, where $n \in \mathbb{Z}$, is a graded A-module such that $M(n)_d := M_{n+d}$.

The module $(A^m)(n)$ is the same module $(A(n))^m$ and they are graded free A-modules called **twisted**. The elements e_i are a basis of $A(n)^m$ amd they are homogeneous elements of degree $-n$.

Definition 3.1.7. Let M and N be graded A-modules. A morphism $\varphi \colon M \to N$ is said to be a graded morphism of degree d if $\varphi(M_t) \subset N_{t+d}$ for any $t \in \mathbb{Z}$.

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a finitely generated graded A-module. Given $\{r_1, \dots, r_m\}$ a set of homogeneous elements of M, of degree $d_1, \dots, d_m$ respectively, that generate M as A-module, we can define the following graded morphism of degree 0:

$$F_0 = \bigoplus_{i=1}^{m} A(-d_i) \xrightarrow{\varphi_0} M$$

such that it associates the i-th element, e_i, of the basis of F_0, to r_i, for $i = 1, \dots, m$. We note that it is necessary the twist in F_0, since the graded morphism preserves the degree. Now we can consider the $\ker(\varphi_0)$, denoted by $M_1 \subseteq F_0$, that is finitely generated and it is the set of syzygies of M. Choosing a finite number of homogeneous generators for M_1, we can define a morphism

$$F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M$$

such that $\mathrm{im}(\varphi_1) = M_1 = \ker(\varphi_0)$. So we can construct a resolution of M. In particular we can give the following definition:

Definition 3.1.8. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a graded A-module. A graded resolution of M is a resolution

$$\cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

where every F_i is a free, graded, twisted A-module of type $A(-d_1) \oplus \cdots \oplus A(-d_m)$ and every morphism φ_i is graded of degree 0.
Moreover a graded resolution is minimal if for all $i \geq 1$ the elements no zero of the matrix of φ_i have positive degree.

Moreover the Theorem (3.1.3) holds in the graded case:

Theorem 3.1.9. *Every finitely generated graded module over a polynomial ring $\mathbb{K}[x_0, \dots, x_n]$, in $n+1$ indeterminates, over a field $\mathbb{K}$ has a free graded resolution of length at most $n+1$.*

Proposition 3.1.10. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and let M be a finitely generated graded $A-$module. For all finite graded resolution of M*

$$0 \longrightarrow F_l \longrightarrow F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

we have

$$H_M(t) = \dim_{\mathbb{K}} M_t = \sum_{j=0}^{l} (-1)^j \dim_{\mathbb{K}} (F_j)_t.$$

3.2 m-syzygy curves

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables x, y, z with complex coefficients and let $f \in S$ be a homogeneous polynomial of degree $d > 0$. We denote by J_f the **Jacobian ideal** of f, i.e. the homogeneous ideal in S spanned by partial derivatives f_x, f_y and f_z of f. The graded S-module $AR(f) \subset S^3$ of all Jacobian relations for f is defined by

$$AR(f) := \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}$$

and it called **Jacobian syzygies** of f. The **minimal degree** of a Jacobian relation for the polynomial f is the positive integer $mdr(f)$ defined to be the smallest integer $m \geq 0$ such that $AR(f)_m \neq (0)$. Let $C : f = 0$ be a reduce curve of degree d in the complex projective plane $\mathbb{P}^2$, then $AR(f) = AR(C)$ and we assume $mdr(f) \geq 1$, since, when $mdr(f) = 0$, C is a pencil of lines.

Definition 3.2.1. The Milnor (or Jacobian) algebra of the curve $C : f = 0$ is the corresponding graded quotient ring, denoted by $M(f)$, S/J_f.

By Hilbert Syzygy Theorem (3.1.9), the graded Jacobian algebra $M(f)$ has a free graded resolution of length 3; in fact we can consider the following graded morphism of degree 0:

$$F_1 = S^3(1-d) \xrightarrow{(f_x, f_y, f_z)} F_0 = S$$

given by $(a, b, c) \to af_x + bf_y + cf_z$. It is clear that $AR(f)$ is the kernel of the above morphism and it has the following minimal resolution

$$0 \longrightarrow F_3(d-1) \longrightarrow F_2(d-1).$$

Definition 3.2.2. The curve $C : f = 0$ is said to be a m-syzygy curve if the module F_2 has rank m, i.e. the module $AR(f)$ is generated by m homogeneous syzygies, $\{r_1, \dots, r_m\}$, of degrees $d_i = \deg r_i$ ordered such that

$$1 \leq d_1 \leq \ldots \leq d_m.$$

The multiset $(d_1, \dots, d_m)$ is called **exponent** of the curve C and the set $\{r_1, \dots, r_m\}$ is called **minimal set of generators** for the module $AR(f)$.

It is clear that $d_1 = mdr(f)$. Moreover when $AR(f)$ is a free module of rank 2, the curve $C : f = 0$ is said to be a **free** 2-syzygy curve, see [3, 17, 22, 65, 66, 67].

Example 3.2.3. All curves $C : f = 0$ in $\mathbb{P}^2$ of type:

$$f = y^{d-1}z + x^d + ax^2y^{d-2} + bxy^{d-1} + cy^d$$

of degree $d \geq 5$ and $a \neq 0$ are a family of free curves. They are given by *A. Simis* and *A. Tohăneanu* and they are cuspidal rational curves.

The m-syzygy curves have been carefully classified and studied in these last years. In particular our study is about the m-syzygy curves with $m \geq 3$.

Definition 3.2.4. A 3-syzygy curve $C : f = 0$ is said to be **nearly free** if the graded S-module $AR(f)$ has a minimal generator system of syzygies r_1, r_2, r_3, such that their degree, d_1, d_2 and d_3 respectively, satisfy $d_3 = d_2$ and $d_1 + d_2 = d$.

Example 3.2.5. The sextic $C : f = (x^3+y^3+z^3)(x+y)(x+z)(y+z)$ in $\mathbb{P}^2$, that is union of 3 lines and an elliptic curve, is a nearly free curve, with exponents $(d_1, d_2, d_3) = (2, 4, 4)$. Moreover it is a smooth curve.

The nearly free 3-syzygy curves have been introduced in [23] and studied in [1, 3, 17, 18, 52].

Definition 3.2.6. A 3-syzygy line arrangement $C : f = 0$ is said to be **plus-one generated line arrangement** of level d_3 when $d_1+d_2 = d$ and $d_3 > d_2$. By extension, a $3-$syzygy curve C is said to be a **plus-one generated curve** of level d_3 when $d_1+d_2 = d$ and $d_3 > d_2$.

Example 3.2.7. The curve $C : f = (x^2+y^2-2xz)^2 - (x^2+y^2)z^2$ in $\mathbb{P}^2$, called *limaçon*, is a plus one generated curve with exponents $(d_1, d_2, d_3) = (2, 2, 3)$. Moreover it has 3 singularities: one of type A_1 located at $p_1 = (0 : 0 : 1)$ and two of type A_2 located at $(1 : \pm i : 0)$.

The exponents of the curves of the Examples (3.2.5) and (3.2.7) are determined by a computer algebra software, called *Singular*, see [15].

Remark 3.2.8. The nearly free curves and the plus-one generated curves satisfy $mdr(f) = d_1 \leq \frac{d}{2}$.

Mainly the interest of the free, nearly free and one-plus generated curves comes from the analysis the following problem:

Problem. It it true that a reduced plane curve $C : f = 0$ which is rational cuspidal is either free or nearly free?

This problem is known to be true when the degree of C is even and in particular for all odd degree $d \leq 33$. In the other cases the problem is assumed as conjecture, i.e. a reduced plane curve $C : f = 0$ which is rational cuspidal is either free or nearly free.

Moreover it is possible to consider a weak condition respect to be a reduced, rational and cuspidal plane and we have a particular class of rational curves called *nearly cuspidal*:

Definition 3.2.9. A nearly cuspidal curve is a reduced plane curve $C : f = 0$, having only cusps (i.e. unibranch singularities) except for one singular point which has two branches.

Example 3.2.10. The curve C of the Example (3.2.7) is a rational and nearly cuspidal curve, with the point $p_1 = (0 : 0 : 1)$ having two branches.

About to the reduced plane and nearly cuspidal curves, we can show the following theorem:

Theorem 3.2.11. *Let $C : f = 0$ be a plane, rational and nearly cuspidal curve of even degree $d \geq 2$. Then C is a free curve or a nearly free or a plus-one generated curve.*

The following result is important, since we can know the minimal resolution for the Milnor algebra $M(f)$, where $C : f = 0$ is a m-syzygy curve:

Lemma 3.2.12. *[42, Lemma 1.1] Let $(R, \mathfrak{m})$ be a positively graded Noetherian local ring and let $I \subset \mathfrak{m}$ be a 3-generated homogeneous ideal, the graded minimal free resolution*

$$0 \longrightarrow \bigoplus_{m=1}^{r-2} R(-D_m) \xrightarrow{\varphi_3} \bigoplus_{i=1}^{r} R(-d_i) \xrightarrow{\varphi_2} \bigoplus_{i=1}^{3} R(-a_i) \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0.$$

Assume $D_1 \geq \cdots \geq D_{r-2}$ and $d_1 \geq \cdots \geq d_r$. Then $D_m \geq d_m + 1$ for all $1 \leq m \leq r-2$.

By Lemma (3.2.12), since the Jacobian ideal J_f is generated by three elements, we can consider the general form of the minimal resolution for the Milnor algebra $M(f)$ of a m-syzygy curve $C : f = 0$ that is assumed to be not free, namely

$$0 \longrightarrow \bigoplus_{i=1}^{m-2} S(-e_i) \longrightarrow \bigoplus_{i=1}^{m} S(1-d-d_i) \longrightarrow S^3(1-d) \longrightarrow S \tag{3.2}$$

with $e_1 \leq \ldots \leq e_{m-2}$, $d_1 \leq \ldots \leq d_m$ and

$$e_i = d + d_{i+2} - 1 + \epsilon_i$$

for $i = 1, \ldots, m-2$ and some integers $\epsilon_i \geq 1$.

Remark 3.2.13. When the curve $C : f = 0$ in $\mathbb{P}^2$ is a free curve, hence $m = 2$, the minimal resolution of $M(f)$ is:

$$0 \longrightarrow S(1-d-d_1) \oplus S(1-d-d_2) \longrightarrow S^3(1-d) \longrightarrow S$$

With the above notation, for a 3-syzygy curve $C : f = 0$, the minimal resolution become

$$0 \longrightarrow S(-e) \longrightarrow \bigoplus_{i=1}^{3} S(1-d-d_i) \longrightarrow S^3(1-d) \longrightarrow S$$

and, by lemma (3.2.12), we have $e \geq d + d_3$. Moreover, for $k \geq e-2$, one has the obvious formula:

$$\dim M(f)_k = \binom{k+2}{2} - 3\binom{k-d+3}{2} + \sum_{i=1}^{3} \binom{k-d+3+d_i}{2} - \binom{k-e+2}{2}$$

that give us a formula for calculation of the Hilbert vector of $M(f)$, when $C : f = 0$ is a 3-syzygy curve, assumed to be not free. In particular, given a curve $C : f = 0$ in $\mathbb{P}^2$, it is possible to determine its corresponding Hilbert vector of $M(f)$, considering various cases.

It follows the definition of some invariants associated with a Milnor algebra $M(f)$ that will be useful in next sections.

Definition 3.2.14. For a reduce plane curve $C : f = 0$ of degree d, two integers are defined as follows:

- the **coincidence threshold**

$$ct(f) := \max\{q : \dim M(f)_k = \dim M(f_s)_k \quad \forall k \leq q\}$$

  with f_s a homogeneous polynomial in S of degree d such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

- the **stability threshold**

$$st(f) = \min\{q : \dim M(f)_k = \tau(C) \quad \forall k \geq q\}$$

  with $\tau(C)$ Tjiurina number of the curve C.

3.3 Minimal resolution of the Jacobian module

Let $C : f = 0$ be a reduced complex plane curve of $\mathbb{P}^2$ and consider the jacobian ideal $J_f = (f_x, f_y, f_z)$. Let $\hat{J}_f$ be the saturation of the ideal J_f with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in $S = \mathbb{C}[x, y, z]$.

Definition 3.3.1. The Jacobian module is the quotient module $N(f) = \hat{J}_f/J_f$.

The graded S-module $N(f)$ of a plane curve $C : f = 0$ of degree d, is an artinian algebra $N(f) = \bigoplus_{i=0}^{3(d-2)} N(f)_i$. Denoting $n(f)_i = \dim N(f)_i$, for all i, our interest is to study the Hilbert vector of $N(f)$. We can define the following invariants for a curve $C : f = 0$:

$$\sigma(C) := \min\{j : n(f)_j \neq 0\} = \operatorname{indeg}(N(f)), \quad \nu(C) := \max\{n(f)_j\}_j .$$

Setting $T = 3(d-2)$, the Jacobian module $N(f)$ enjoys a weak Lefschetz type property, see [19] for this result and [38, 39, 45] for Lefschetz properties of Artinian algebras. In fact it is important the following theorem:

Theorem 3.3.2. *[19, Theorem 4.1] Let $J = (f_0, f_1, f_2)$ be a dimension one almost complete intersection and let $N = H^0_{\mathbf{m}}(S/J) = I/J_f$ be the 0-degree local cohomology of the corresponding algebra $M = S/J$. Then there exists a Lefschetz element for N. More precisely, for a generic linear form $l \in S_1$, the multiplication by l induces injective morphisms $N_i \to N_{i+1}$ for $i < (d_0 + d_1 + d_2 - 3)/2$ and surjective morphisms $N_i \to N_{i+1}$ for $i \geq i_0 = (d_0 + d_1 + d_2 - 3)/2$.*

Since the Jacobian ideal $J_f = (f_x, f_y, f_z)$ of a smooth complex plane curve $C : f = 0 \subset \mathbb{P}^2$ is a complete intersection, and $\deg f_x = \deg f_y = \deg f_z = d-1$, then, by Theorem (3.3.2), we have that for a generic linear form $\ell \in S_1$ the morphism $N(f)_i \to N(f)_{i+1}$ is injective for $i < \lfloor \frac{T}{2} \rfloor$ and $N_i \to N_{i+1}$ surjective for $i \geq \lfloor \frac{T}{2} \rfloor$.

Hence, by Proposition (1.2.2), the Hilbert vector of $N(f)$ is unimodal:

$$n(f)_0 \leq n(f)_1 \leq \ldots \leq n(f)_{\lfloor \frac{T}{2} \rfloor -1} \leq n(f)_{\lfloor \frac{T}{2} \rfloor} \geq n(f)_{\lfloor \frac{T}{2} \rfloor +1} \geq \ldots \geq n(f)_T \tag{3.3}$$

and $\nu(C) = n(f)_{\lfloor \frac{T}{2} \rfloor}$. Moreover the self duality of the graded S-module $N(f)$, see [42, 64, 74], implies that

$$n(f)_j = n(f)_{T-j}, \tag{3.4}$$

for any integer j, in particular $n(f)_k \neq 0$ exactly for $k = \sigma(C), \ldots, T - \sigma(C)$.

We recall an important result:

Proposition 3.3.3. *[42, Proposition 1.3] Let $S = \mathbb{K}[x_0, x_1, x_2]$ and let $I \subset S$ be an ideal of codimension 2 generated by 3 linearly independent forms of degree $d \geq 1$ with minimal graded free resolution*

$$\begin{array}{l} 0 \longrightarrow \bigoplus_{i=1}^{r-2} S(-D_i) \longrightarrow \bigoplus_{i=1}^{r} S(-d_i) \longrightarrow \\ S^3(-d) \longrightarrow R \longrightarrow R/I \longrightarrow 0 \end{array} \tag{3.5}$$

with $r \geq 3$. Then:

- *the minimal free resolution of $\hat{I}/I$ as an S-module has the form*

$$\begin{array}{l} 0 \longrightarrow \bigoplus_{i=1}^{r-2} S(-D_i) \longrightarrow \bigoplus_{i=1}^{r} S(-d_i) \longrightarrow \bigoplus_{i=1}^{r} S(d_i - 3d) \longrightarrow \\ \bigoplus_{i=1}^{r-2} S(D_i - 3d) \longrightarrow \hat{I}/I \longrightarrow 0 \end{array}$$

where the leftmost map is the same as that of (3.5);

- *if in addition $\hat{I}_d = I_d$ and $\hat{I}_i = 0$ for $i < d$, then the resolution of $\hat{I}$ is*

$$0 \longrightarrow \bigoplus_{i=1}^{r} S(-(3d - d_i)) \longrightarrow S^3(-d) \bigoplus_{i=1}^{r-2} S(-(3d - D_i)) \longrightarrow \hat{I} \longrightarrow 0$$

Thence the minimal resolution of $N(f)$, obtained from (3.2) and by Proposition (3.3.3), is

$$0 \to \oplus_{i=1}^{m-2} S(-e_i) \to \oplus_{i=1}^{m} S(1-d-d_i) \to \oplus_{i=1}^{m} S(d_i - 2(d-1)) \to \oplus_{i=1}^{m-2} S(e_i - 3(d-1)).$$

It follows that

$$\sigma(C) = 3(d-1) - e_{m-2} = 2(d-1) - d_m - \epsilon_{m-2}. \tag{3.6}$$

since $e_{m-2} = d + d_m - 1 + \epsilon_{m-2}$, with some integer $\epsilon_{m-2} \geq 1$.
Let $C : f = 0$ be a reduced curve of degree d in $\mathbb{P}^2$. We recall that the sheafification of $AR(f)$, denoted by $E_C := \widetilde{AR(f)}$, is a rank two vector bundle on $\mathbb{P}^2$, see [1, 63, 64]. We set:

$$ar(f)_m = \dim AR(f)_m = \dim H^0(\mathbb{P}^2, E_C(m))$$

for any integer m. Associated to the vector bundle E_C there is the normalized vector bundle $\mathcal{E}_C$, which is the twist of E_C such that $c_1(\mathcal{E}_C) \in \{-1, 0\}$. More precisely,

when $d = 2d' + 1$ is odd

$$\mathcal{E}_C = E_C(d') \quad c_1(\mathcal{E}_C) = 0 \quad c_2(\mathcal{E}_C) = 3(d')^2 - \tau(C); \tag{3.7}$$

and

when $d = 2d'$ is even

$$\mathcal{E}_C = E_C(d'-1) \quad c_1(\mathcal{E}_C) = -1 \quad c_2(\mathcal{E}_C) = 3(d')^2 - 3d' + 1 - \tau(C), \tag{3.8}$$

see [26, Section 2].

Remark 3.3.4. The vector bundle E_C is stable if and only if $\mathcal{E}_C$ has no sections, see [60, Lemma 1.2.5]. This is equivalent to $r = mdr(f) \geq \frac{d}{2}$, see [64, Proposition 2.4]. Moreover by [26, Theorem 2.2] and using the formulas (3.7) and (3.8), we have that for a stable vector E_C, $c_2(\mathcal{E}_C) = \nu(C)$. Moreover, the vector bundle E_C is semistable if and only if $r = mdr(f) \geq (d-1)/2$, see again [60, Lemma 1.2.5], a condition that occurs in our Theorem (3.6.1) below.

The important key point is the identification

$$H^1(C, E_C(k)) = N(f)_{k+d-1}$$

for any integer k, see [64, Proposition 2.1]. Hence the study of the dimension of the Jacobian module $N(f)$ is equivalent to the study of the dimension of $H^1(C, E_C(k))$.

Theorem 3.3.5. *Let $C : f = 0$ be a reduced, non free curve of degree d and set $r = mdr(f)$. Then one has the following.*

- *if $r \geq \frac{d}{2}$, then, for $2d - 4 - r \leq k \leq d - 2 + r$*

$$n(f)_k = \begin{cases} 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) \text{ for } d = 2d' + 1 \\ 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) \text{ for } d = 2d' \end{cases} \tag{3.9}$$

- *if $r < \frac{d}{2}$, then $n(f)_k = \nu(C)$, for $d + r - 3 \leq k \leq 2d - r - 3$. Moreover $n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) - 1$.*

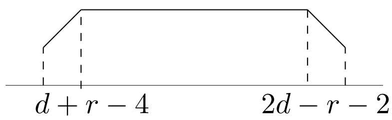
Proof. See [26, Theorem 3.1 and Theorem 3.2]. □

By Theorem 3.3.5, in the stable situation of the vector bundle E_C, the points $(k, n(f)_k)$ lie on an upward pointing parabola. Moreover, using the formulas (3.7) and (3.8), the claim (3.9) can be written:

$$n(f)_k = \begin{cases} \nu(C) - (j - \lfloor \frac{T}{2} \rfloor)(j - \lceil \frac{T}{2} \rceil) \text{ for } d = 2d' + 1 \\ \nu(C) - (j - \frac{T}{2}) \text{ for } d = 2d' \end{cases}$$

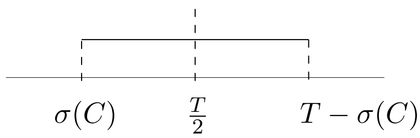
with $T = 3(d-2)$.

On the other hand, in the unstable situation of the vector bundle E_C, assuming C is not free, the points $(k, n(f)_k)$ lie on a horizontal line segment, with a one-unit drop at the extremities:



3.4 Results on the Hilbert vector of $N(f)$ for 3-syzygy curves

It is known that $C : f = 0$ is a free curve if and only if $\nu(C) = 0$, hence $N(f) = 0$. It follows that the Jacobian ideal is satured, i.e. $\hat{J}_f = J_f$, (see [20, 67]). The first nontrivial case is that of nearly free curves; indeed, by [23, Corollary 2.17], we have that for a nearly free curve $C : f = 0$ one has $N(f) \neq 0$ and $\nu(C) = 1$. Moreover $\sigma(C) = d + d_1 - 3$ and this describes completely the Hilbert vector of the Jacobian module of a nearly free curve. In particular it has the following shape.



For the case of the 3−syzygy curves, we recall the following two results.

Theorem 3.4.1. *[27, Theorem 3.9] Let $C : f = 0$ be a 3−syzygy curve with exponents (d_1, d_2, d_3) and set $e = d_1 + d_2 + d_3$. Then the minimal free resolution of $N(f)$ as a graded S−module has the form*

$$0 \to S(-e) \to \oplus_{i=1}^{3} S(1 - d - d_i) \to \oplus_{i=1}^{3} S(d_i + 2 - 2d) \to S(e + 3 - 3d), \qquad (3.10)$$

where the leftmost map is the same as in the resolution (3.2), when $m = 3$. In particular,

$$\sigma(C) = 3(d-1) - (d_1 + d_2 + d_3).$$

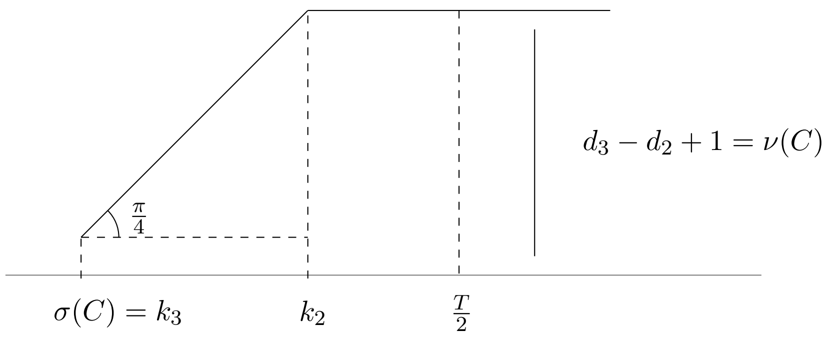
Corollary 3.4.2. *[27, Corollary 3.10] Let $C : f = 0$ be a plus-one generated curve of degree $d \geq 3$ with (d_1, d_2, d_3), which is not nearly free, i.e. $d_2 < d_3$. Set $k_j = 2d - d_j - 3$ for $j = 1, 2, 3$. Then one has the following minimal free resolution of $N(f)$ as a graded $S-$module:*

$$0 \to S(-d - d_3) \to S(-d - d_3 + 1) \oplus S(-k_1 - 2) \oplus S(-k_2 - 2) \to$$
$$\to S(-k_1 - 1) \oplus S(-k_2 - 1) \oplus S(-k_3 - 1) \to S(-k_3).$$

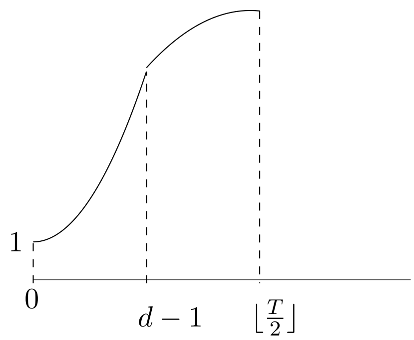
In particular $\sigma(C) = k_3$ and the Hilbert vector of $N(f)$ is given by following formulas:

1. $n(f)_j = 0$ *for* $j < k_3$ *and* $k_3 < \frac{T}{2}$;
2. $n(f)_j = j - k_3 + 1$ *for* $k_3 \leq j \leq k_2$, *and* $k_2 = d + d_1 - 3 \leq \frac{T}{2}$;
3. $n(f)_j = d_3 - d_2 + 1 = \nu(C)$ *for* $k_2 \leq j \leq \frac{T}{2}$.

By above corollary, the Hilbert vector of the Jacobian module of a plus-one generated curve of degree d and level d_3 has the following shape, *where we have drawn only the part corresponding to $j \leq T/2$, due to the symmetry* (3.4).



As an example, let $C : f = 0$ be a smooth curve of degree d, another class of 3-syzygy curves, where $d_1 = d_2 = d_3 = d - 1$. It is known that the Hilbert function of the Milnor algebra $M(f)$ is in this case $(\frac{1-t^{d-1}}{1-t})^3$. For a smooth curve we have $N(f) = M(f)$, hence $n(f)_j = \dim M(f)_j$ and the Hilbert vector of the Jacobian module $N(f)$ has the following shape. It is interesting to notice the change in convexity when we pass through the value $j = d - 1$.



For a general 3−syzygy curve, we have the following result (see [9]).

Theorem 3.4.3. *Let $C : f = 0$ be a 3−syzygy curve of degree d, not plus-one generated, and exponents $d_1 \leq d_2 \leq d_3$. Setting $e = d_1 + d_2 + d_3$ and $k_i = 2(d-1) - d_i$, $i = 1, 2, 3$, then the following hold:*

$$n(f)_k = \begin{cases} 0 \text{ for } k < \sigma \\ \binom{k-\sigma+2}{2} \text{ for } \sigma \leq k < k_3 \\ \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} \text{ for } k_3 \leq k < k_2 \\ \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} - \binom{k-k_2+2}{2} \text{ for } k_2 \leq k < T_0, \end{cases}$$

where $\sigma = \sigma(C) = 3(d-1) - e$ and

$$T_0 = \begin{cases} k_1 - 2 \text{ if } d_1 \geq \frac{d}{2} \\ d + d_1 - 4 \text{ if } d_1 < \frac{d}{2}. \end{cases}$$

Note that $n(f)_k$ is known for $T_0 \leq k \leq \frac{T}{2}$ in view of Theorem (3.3.5), hence the information on the Hilbert vector is complete.

Proof. Let $\sigma \geq 0$, since, by theorem [27, Theorem 2.3], $d_1 + d_2 > d > d - 1$, we have:

$$\sigma = 3(d-1) - (d_1 + d_2 + d_3) < 3(d-1) - (d-1) - d_3 = 2(d-1) - d_3 = k_3.$$

Moreover if $2d_2 \leq d$, one has

$$d_1 + d_2 \leq 2d_2 \leq d,$$

but it is not true by theorem [27, Theorem 2.3], hence it holds $2d_2 \geq d + 1$. Thence

$$2d_2 \geq d + 1 \Rightarrow 2d_2 > d + 2 \Rightarrow -2d_2 < -d - 2$$
$$\Rightarrow 4(d-1) - 2d_2 < 4(d-1) - d - 2 \Rightarrow 2k_2 < 3(d-2) \Rightarrow k_2 < \frac{T}{2}.$$

By Theorem (3.4.1), the minimal resolution of $N(f)$ is

$$0 \to S(-e) \to \oplus_{i=1}^{3} S(k_i - 3(d-1)) \to \oplus_{i=1}^{3} S(-k_i) \to S(e - 3(d-1)).$$

We note that $k_3 \leq k_2 \leq k_1$. Moreover fixed $\sigma \leq k < k_3$, we have $-k_i + k < 0$ for any i. Hence we get:

$$n(f)_k = \dim S(e - 3(d-1))_k = \dim S_{e-3(d-1)+k} = \dim S_{k-\sigma} = \binom{k-\sigma+2}{2}.$$

Now we consider when $k_3 \leq k < k_2 \leq k_1$. We have $-k_i + k < 0$, for $i = 1, 2$. We note that $k_2 < d_1 + d - 1$, in fact, since C is a 3−syzygy curve not plus-one generated, it holds $d_1 + d_2 > d$, by Theorem [27, Theorem 2.3]. Thence:

$$d_1 > d - d_2 \Rightarrow d_1 + d > 2d - d_2 \Rightarrow d_1 + d - 1 > 2(d-1) + 1 - d_2 > k_2.$$

We can say that $1 - d - d_i + k < 0$ for any $i = 1, 2, 3$. Hence

$$\begin{aligned} n(f)_k &= \dim S(e+3-3d)_k - \dim S(-k_3)_k = \\ &= \dim S_{e+3-3d+k} - \dim S_{-k_3+k} = \\ &= \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} = \end{aligned}$$

By calculations, we get:

$$\frac{4(d-1) - 2d_3 - 3 - (2\sigma - 3)}{2} k + \cdots = (k_3 - \sigma)k + \cdots$$

The coefficient $k_3 - \sigma = 2(d-1) - d_3 - 3(d-1) + e = d_1 + d_2 - (d-1) \geq 2$. In fact, if we consider $\delta = (d + d_1 - 4) - k_2$, we get:

$$(d_1 + d_2 - d) - 2 \geq -1 \Rightarrow d_1 + d_2 - d \geq 1$$

since $d_1 + d_2 > d$. Hence $d_1 + d_2 - (d-1) = d_1 + d_2 - d + 1 \geq 2$.
Now we consider two cases:

$k_1 \leq \frac{T}{2}$: we note that

$$k_1 \leq \frac{T}{2} \Leftrightarrow d_1 > \frac{d}{2}.$$

By Theorem (3.3.5), $n(f)_k$ is known for $k \in [2d - 4 - d_1, d - 2 + d_1]$. Setting $T_0 = 2d-4-d_1 = 2(d-1)-d_1-2 = k_1-2$, we can say that, for $k_2 \leq k < T_0 = k_1-2$,

$$n(f)_k = \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} - \binom{k-k_2+2}{2}$$

considering that $-k_1 + k < 0$.

$k_1 > \frac{T}{2}$: we note that

$$k_1 > \frac{T}{2} \Leftrightarrow d_1 \leq \frac{d}{2}.$$

By Theorem (3.3.5), $n(f)_k = \nu(C)$ is known for $k \in [d+d_1-4, 2d-2-d_2]$. setting $T_0 = d + d_1 - 4$, we can say that for $k_2 \leq k < T_0$,

$$n(f)_k = \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} - \binom{k-k_2+2}{2}$$

considering that $-k_1 + k < 0$.

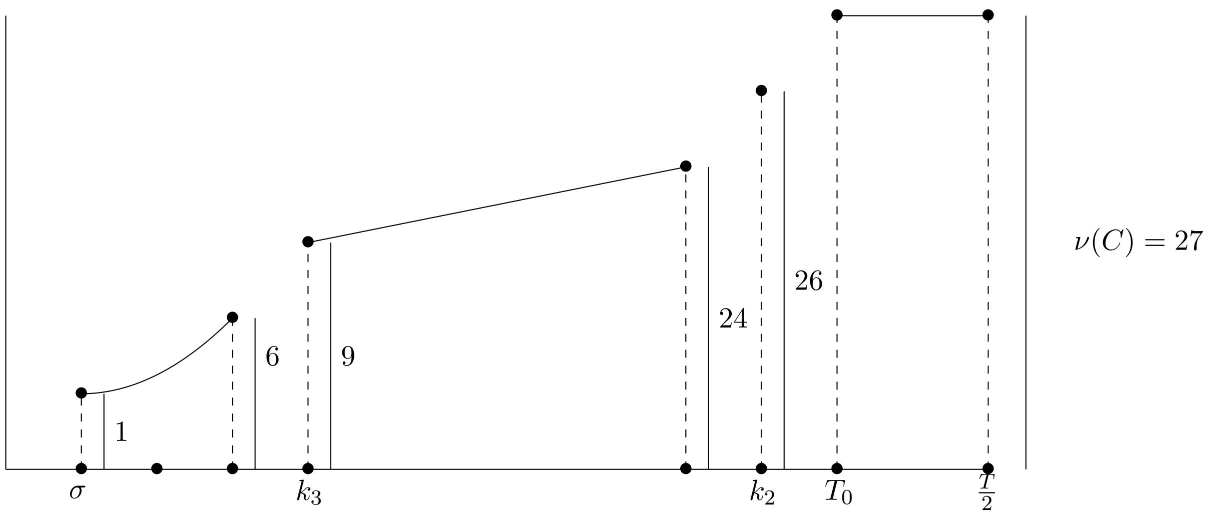
□

Example 3.4.4. Let $C : f = (x^9 + y^4z^5)^7 + xz^{62}$ be a singular curve of degree $d = 63$. It is a 3−syzygy curve not plus-one generated; in fact $d_1 = 9 < d_2 = 56 < d_3 = 62$ and $d_1 + d_2 = 65 > 63$. We have:

$$e = \sum_{i=1}^{3} d_i = 127 \quad \sigma = 59 \quad k_3 = 62 \quad k_2 = 68 \ .$$

Since $d_1 < \frac{d}{2}$, $T_0 = d + d_1 - 4 = 68 = k_2$. Then we get:

$$
\begin{array}{cccccccccccccc}
n(f) = (0\dots0 & 1 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 26 & 27 & 27 & 27 \quad \dots) \\
 & \uparrow & & & \uparrow & & & & & & \uparrow & & & \\
 & \sigma = 59 & & & k_3 = 62 & & & & & & k_2 = 68 & & &
\end{array}
$$

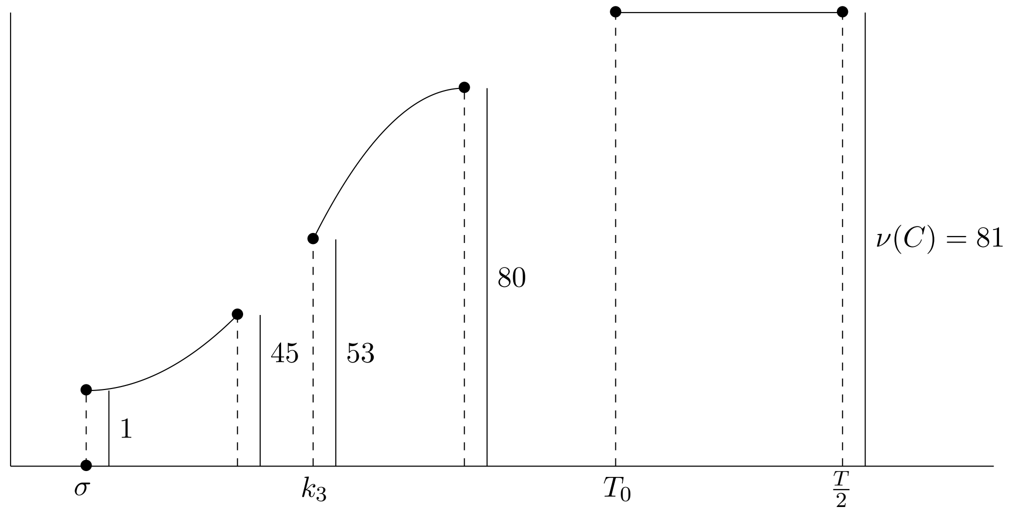


Example 3.4.5. Let $C : f = (x+y)^2(x-y)^2(x+2y)^2(x-2y)^2(x+3y)^2(x-3y)^2(x+4y)^2(x-4y)^2(x+5y)^2(x-5y)^2+z^{20}$ be a singular curve of degree $d = 20$. It is a 3−syzygy curve not plus-one generated; in fact $d_1 = 9 < d_2 = d_3 = 19$ and $d_1 + d_2 = 28 > 20$. We have:

$$e = \sum_{i=1}^{3} d_i = 47 \quad \sigma = 10 \quad k_3 = k_2 = 19.$$

Since $d_1 < \frac{d}{2}$, $T_0 = d + d_1 - 4 = 25$. Then we get:

$$
\begin{array}{cccccccccccccccc}
n(f) = (0\dots0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 53 & 60 & 66 & 71 & 75 & 78 & 80\dots) \\
 & \uparrow & & & & & & & & & \uparrow & & & & & & \\
 & \sigma = 10 & & & & & & & & & k_3 = 19 & & & & & &
\end{array}
$$



We note that in this example the linear part in the middle is missing.

3.5 Maximal Tjurina curves and nodal curves

We assume in this section that $r = d_1 \geq d/2$.
A reduced plane curve $C : f = 0$ of degree d is called a maximal Tjurina curve if the global Tjurina number $\tau(C)$ equals the du Plessis-Wall upper bound, namely if

$$\tau(C) = (d-1)(d-r-1) + r^2 - \binom{2r-d+2}{2}, \tag{3.11}$$

(see [24, 28, 29]). We know that a reduced plane curve $C : f = 0$ of degree d is a maximal Tjurina curve if and only if one has $d_1 = d_2 = \cdots = d_m = r$, $e_1 = e_2 = \cdots = e_{m-2} = d+r$ and $m = 2r - d + 3$, see [24, Theorem 3.1]. Using now the equality (3.6), it follows that in this case

$$\sigma(C) = 2d - r - 3. \tag{3.12}$$

Theorem (3.3.5) yields then the following result.

Proposition 3.5.1. *Let $C : f = 0$ be a maximal Tjurina curve of degree d with $r = d_1 \geq \frac{d}{2}$. Then the Hilbert vector of the Jacobian module $N(f)$ is given by the following*

$$n(f)_k = \begin{cases} 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) \text{ for } d = 2d' + 1 \\ 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) \text{ for } d = 2d' \end{cases}$$

for $2d - 3 - r \leq k \leq d - 3 + r$ and $n(f)_k = 0$ otherwise.

Consider now an arbitrary reduced curve $C : f = 0$ of degree d, satisfying $r = d_1 \geq \frac{d}{2}$. Then we clearly have

$$n(f)_k = m(f)_k - d(f)_k$$

where $m(f)_k = \dim M(f)_k$ and $d(f)_k = \dim S_k/(\hat{J}_f)_k$. Since we have to determine $n(f)_k$ only for $k \leq \frac{T}{2}$ by symmetry, and since $ct(f) \geq d-2+r > \frac{T}{2}$, it follows that

$$n(f)_k = m(f_s)_k - d(f)_k,$$

with f_s a homogeneous polynomial in S of degree d such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$ and $k \leq \frac{T}{2}$. In particular, we have to determine only the values $d(f)_k$ for $k \leq \frac{T}{2}$. On the other hand, we know that

$$d(f)_k = \tau(C),$$

for $k \geq T - ct(C)$, see [16, Proposition 2]. In particular, this equality holds for $k \geq 3(d-2)-(d-2+r) = 2d-4-r$, see also the proof of [26, Theorem 3.1]. Assume now that $C : f = 0$ is a nodal curve in $\mathbb{P}^2$. Then $r = d_1 \geq d-2 \geq \frac{d}{2}$ for $d \geq 4$, (see [20, Example 2.2 (i)]). Let $\mathcal{N}$ denote the set of nodes of the curve C and let $\operatorname{def} S_k(\mathcal{N})$ denote the defect of the set $\mathcal{N}$ with respect to the linear system S_k. Then it is known that

$$d(f)_k = |\mathcal{N}| - \operatorname{def} S_k(\mathcal{N}),$$

(see [16]). Then [21, Corollary 1.6] implies that $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k > d-3$ and $\operatorname{def} S_k(\mathcal{N}) = n(C) - 1$ for $k = d-3$, where $n(C)$ denotes the number of irreducible components of C. If all the irreducible components of C are rational, then [25, Theorem 2.7] shows that $n(f)_k = 0$ for $k \leq d-3$. These facts imply the following.

Theorem 3.5.2. *Let $C : f = 0$ is a nodal curve in $\mathbb{P}^2$ of degree $d \geq 4$. Then one has the following, with f_s a homogeneous polynomial in S of degree d such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$..*

$$n(f)_k = \begin{cases} m(f_s)_k - |\mathcal{N}| \text{ for } d-3 < k \leq T/2 \\ m(f_s)_k - |\mathcal{N}| + n(C) - 1 \text{ for } k = d-3. \end{cases}$$

Moreover, when all the irreducible components of C are rational, one has in addition $n(f)_k = 0$ for $k \leq d-3$.

3.6 Relation to a result by Hartshorne

Let $C : f = 0$ be a curve of degree d, and let $r = mdr(f)$ be the minimal degree of a Jacobian syzygy for f, we can give some informations about the invariant $\sigma(C)$, using a result by Hartshorne, namely [41, Theorem 7.4].

Theorem 3.6.1. *Let $C : f = 0$ be a curve of degree d, and let $r = mdr(f)$ be the minimal degree of a Jacobian syzygy for f. Assume that $r \geq (d-1)/2$, in other words that the rank 2 vector bundle E_C is semistable. Then we have the following.*

1. *If $d = 2d'+1$ is odd, then*

$$\sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1.$$

2. If $d = 2d'$ is even, then

$$\sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 5d' + r - 3.$$

The above inequalities are sharp, in particular they are equalities when C is a maximal Tjurina curve with $r \geq \frac{d}{2}$.

Proof. We discuss only the case $d = 2d' + 1$, the other case being completely similar. One has

$$n(f)_k = h^1(\mathbb{P}^2, \mathcal{E}_C(k - 3d')).$$

Moreover $h^0(\mathbb{P}^2, \mathcal{E}_C(t)) = h^0(\mathbb{P}^2, E_C(t + d')) \neq 0$ if and only if $t + d' \geq r$. Hence the minimal t satisfying this condition is $t_m = r - d' \geq 0$. Then [41, Theorem 7.4] implies that $n(f)_k = 0$ when

$$k - 3d' \leq -c_2(\mathcal{E}) + t_m^2 - 2.$$

Using the formula for t_m above, and the formula for $c_2(\mathcal{E})$ given in the equations (3.7), we get that $n(f)_k = 0$ when

$$k \leq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 2,$$

which clearly implies our claim (1). The fact that the inequality in (1) is in fact an equality when C is a maximal Tjurina curve with $r \geq \frac{d}{2}$ follows by a direct computation. Indeed, using the above definition of a maximal Tjurina curve of degree $d = 2d' + 1$, namely the equality (3.11), we see that

$$\tau(C) = 2(d')^2 + 2rd' - r^2 - r + d'.$$

Hence

$$\tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1 = 2d - r - 3 = \sigma(C),$$

where the lasy equality follows from (3.12). □

Example 3.6.2. Let $C : f = 0$ be a curve of degree $d = 2d' + 1$, having a unique node as singularities. Then it is known that $r = d - 1 = 2d'$, and $\tau(C) = \sigma(C) = 1$. The inequality in Theorem (3.6.1 (1)) is in this case

$$1 \geq d'(3 - 2d'),$$

hence the two terms in this inequality can be far apart in some cases.

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