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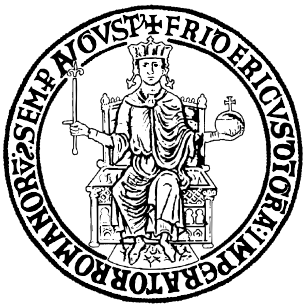

PhD Program in 'Scienze Matematiche ed Informatiche'
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Landscapes of Codes

rank distance codes and intersection problems in finite projective spaces

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to the Man I will be ...

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Contents

Preface

„Εὐομολόγητον, ἔφη· τοῦ γὰρ ἀεὶ ὄντος
ἡ γεωμετρικὴ γνῶσίς ἐστιν.“

Πλάτων, Πολιτεία, VII, 527[1].

The aim of this thesis is to highlight once again how Geometry, and in particular Combinatorics, is visual knowledge.
I feel very close to the Plato's idea of Geometry, which is something that has always been there, that lives in its own immutable and eternal rules.
This is even more alive in *Combinatorics*: *'the art of putting things in nice order'*.
In the term Combinatorics many areas of mathematics are enclosed. Combinatorics is related to many applications ranging from Logic to Statistical Physics, *Galois geometries*, *Cryptography*, *Information* and *Coding Theory*.
In particular Galois geometries, that is, geometric structures defined over a finite field, are well known to be rich in combinatorial properties and appear to be a good setting in which many problems on codes theory can be translated. However, the geometric and algebraic methods used to address these problems often intertwine and the form of algebraic expressions becomes itself a geometric object to be observed.
Also, the title *'Landscapes of Codes'* tries to summarize how coding theory, finite fields and in particular Galois geometries lend themselves as visual art, as landscape; the use of geometrical configurations and of the mutual position among objects are vision of a mental landscape.
It is precisely from here that this work starts: putting the arguments that I have met in my PhD in order. In this dissertation, we will go deeper into analyzing particular classes of codes and some topics in the so-called *extremal Combinatorics*.

[1] *„The knowledge at which geometry aims is the knowledge of the eternal.“*

Briefly, this thesis is divided into three blocks. In the first, we investigate the theory of maximum rank distance (MRD) codes whose codewords are symmetric, alternating or Hermitian matrices. In the linearized polynomials setting, we explore how the already known classes of such codes can be seen as the intersection of an appropriate code with the restricted ambient in which 'they live'. We solve the equivalence issues and we compute their automorphisms group. Moreover, we characterize these latters and present a new class of maximum symmetric codes.
In the second part, we recall some notions about finite projective spaces and in this context we introduce the q-analogue of the Erdős-Ko-Rado problem originally stated in set theory. After having retraced the known results in this topic, we study maximal families of k-dimensional subspaces in $\mathrm{PG}(n,q)$, $n \geq k+2$ and $k \geq 3$, pairwise intersecting in at least a $(k-2)$-space. We also give some upper bounds on the size of relevant families, exploring the largest examples.
In the last part, we introduce the subspace codes theory as the geometrical counterpart of the intersection problems with assigned size arisen from the set theory. Finally, we generalize the concept of equidistant constant-dimension codes with the notion of SPID (*Subspace Pre-assigned Intersection Dimensions*). The *junta code*, i.e. a highly regular structure that extends the notion of *sunflower*, is defined. In a vector setting, we analyze the space spanned by the elements of a SPID with two intersection dimensions and determinine a *geometrical junta bound*. In particular for two consecutive assigned values of the intersection, we show that this threshold is sharp.

More precisely, the dissertation is organized as follows.

In **Chapter 1**, we provide the basis for the rank metric codes. A *rank metric code* $\mathcal{C}$ consists of a non-empty set of matrices of $\mathbb{F}_q^{m\times n}$, where $\mathbb{F}_q$ is the Galois field of q elements, endowed with the rank metric, i.e.

$$d(A,B) = \mathrm{rk}(A-B),$$

for each $A, B \in \mathbb{F}_q^{m\times n}$. Delsarte introduced this metric in 1978, [31]. Here, we will call *d-code*, a rank metric code $\mathcal{C}$ such that the rank distance between any pair of its codewords is at least d, with $1 \leq d \leq \min\{m,n\}$. Furthermore, since the main goal in coding theory is to find codes of maximum possible size for a given minimum distance, he proved that the rank metric codes must obey to a simple analogue of the Singleton bound in classical coding theory. The codes achieving this bound are called *maximum rank distance* (or shortly MRD) *codes*. Delsarte also constructed the first family of linear MRD-codes for each possible set of parameters. Few years later, Gabidulin in [47, Section 4] presented the same class of MRD-codes defined as evaluations of polynomials

over a finite field belonging to a particular class. Nowadays, these codes are called *Delsarte-Gabidulin codes*.
In Section 1.1, we introduce the natural notion of *isometry* in the metric space $(\mathbb{F}_q^{m\times n}, d)$, i.e. a bijective map $\Phi : \mathbb{F}_q^{m\times n} \to \mathbb{F}_q^{m\times n}$ such that

$$d(A, B) = d(\Phi(A), \Phi(B))$$

for each $A, B \in \mathbb{F}_q^{m\times n}$ and the *equivalence* between two codes is recalled.
There are many ways of representing rank-metric codes. Basically, they are subsets of the space of homomorphisms from one vector space to another, or the space of bilinear forms from the product of two vector spaces to the underlying field. Often, however, one chooses to work with a particular class of poynomials: *linearized polynomials*. Such polynomials were introduced for the first time by Ore in 1933, [97]. More precisely, if σ is a generator of the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, then a linearized polynomial is an expression of the form

$$a_0x + a_1x^{\sigma} + \ldots + a_kx^{\sigma^k}$$

with $a_0, \ldots, a_k \in \mathbb{F}_q$. They correspond to the $\mathbb{F}_q$-endomorphisms of $\mathbb{F}_{q^n}$ seen as a vector space over $\mathbb{F}_q$ and, hence, they can be used to describe related objects such as $\mathbb{F}_q$-subspaces, MRD-codes, etc.
In Section 1.3, we look at the relations between the maximum rank distance codes whose codewords are invertible and some algebraic structures, the *quasifields* and *semifields*.
A finite semifield is a division algebra with a finite number of elements in which multiplication is not necessarily associative and in which a multiplicative identity element exists. If we only assume left-distributivity of multiplication over addition, we have a (left) quasifield. Semifields were first studied by Dickson [36], [37] and then by Albert [2] and Knuth [80]. We summarize the known examples of MRD-codes in Section 1.5. More precisely, we recall the family found by Gabidulin in [47] and generalized by him and Kshevetskiy in [50] called *generalized Gabidulin codes*. We describe the new family found by Sheekey in [109] known as *twisted Gabidulin codes* and the Trombetti-Zhou codes investigated in [112] .

Rank metric codes of symmetric, alternating and Hermitian matrices have been studied respectively in [30], [105] [106] and [107], by constructing q-polynomials associated with symmetric, alternating and Hermitian matrices of order n. In all such settings a heavy use of the theory of *association schemes* leads to the determination of bounds, in general different from the analogue of the Singleton one, on the size of maximum linear codes, and to the construction of examples attaining these bounds. The first part of **Chapter 2** deals with translating in the linearized polynomials setting isometries groups of matrix spaces with relevant restrictions. We use results by Wan contained in [117].

With *Guglielmo Lunardon, Rocco Trombetti* and *Yue Zhou*, we elaborate on these examples [**85**]. In Section 2.4, we show that they are the intersection of their ambient space, with a suitable unique MRD-code beloging to the equivalence class of a generalized Gabidulin code. Consequently, the automorphism groups of such linear d-codes are determined.
Moreover, in Theorem 2.5.1, we give a characterization of the d-codes equivalent to the known ones in symmetric, alternating and Hermitian context. Finally, we will show a set of q-linear polynomials over $\mathbb{F}_{q^{2m}}$ such that it forms a symmetric linear 2-code not equivalent to the code constructed by Kai-Uwe Schimdt in [106].

The aim of **Chapter 3** is to give an overview of so-called *Erdős-Ko-Rado problems*. Here, some results on intersection problems, in the set theory first and in the finite projective spaces after, are collected.
More precisely, a crucial result which gave rise to a lot of research in the area of '*extremal Combinatorics*' was published in 1961: Erdős, Rado and Ko studied the size of the largest sets (contained in a finite set) intersecting pairwise non-trivially, [41]. In [118], Wilson generalized this result to the families of k-subsets (i.e. with size k) sharing at least $t \geq 1$ elements. In honour of the three mathematicians mentioned above, these families were called Erdős-Ko-Rado sets (hereafter EKR sets). Moreover, finding the largest sets of pairwise non-trivially intersecting elements has been known as EKR *problem*.
The beauty of these topics is that often they can be defined very easily, based on the elementary notion of intersection among sets, but they require very difficult arguments to solve them or high-level mathematicians as Erdős was.
Here, after having recalled some basic notions on the finite projective spaces and some remarkable substructures in them, we introduce the q-analogue of EKR-problem. More precisely, in a projective context, the EKR problem translates into studying families of k-dimensional projective subspaces that have at least a fixed intersection dimension. Moreover, the research is focused on studying the maximal families. These are sets of k-spaces mutually sharing at least a t-space, not extendable to larger families with the same property. The main goal is to obtain the size of these maximal families which, clearly, may differ from the largest example.
Very important for the next chapter will be the huge work contained in M. De Boeck's PhD thesis, where he investigated the maximal EKR sets of planes in projective spaces and in polar spaces, simultaneously. Also, he characterized the maximal ones with sufficiently large size, [27].

In **Chapter 4**, some results obtained *with Jozefien D'haeseleer, Leo Storme* and *Ago-Erik Riet* during my research stay at Gent University are collected.
Starting from the work of Eisfeld, [39], and from the classification of maximal

EKR sets of projective planes due to De Boeck, [27], in Section 4.1, we analyze the sets of *solids* (i.e. 3-dimensional projective spaces) in $\mathrm{PG}(n,q)$, $n \geq 5$, such that every two solids intersect in at least a line, dividing the discussion into whether or not a particular configuration of solids in such family exists. So, let $\mathcal{S}$ be a maximal set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq k+2$ with the properties described above. We say that in $\mathcal{S}$ there is a *configuration*, if $\mathcal{S}$ contains three k-spaces A, B and C such that they have no $(k-3)$-space in common.
When there is a configuration in $\mathcal{S}$, Lemma 4.1.1 is crucial: the solids not contained in $\langle A, B\rangle$ meet this latter in a plane. Then, we consider the space α generated by the planes arising from such intersections, and we discuss properties of the set $\mathcal{S}$ of solids depending on the dimension of α. In Section 4.2, we generalize these results for sets of k-spaces, $k > 3$, pairwise intersecting in at least a $(k-2)$-dimensional subspace in $\mathrm{PG}(n,q)$ with $n \geq k+2$. Again, we discuss the largest examples giving some upper bounds on the size of these relevant families.
In both cases, we assume that all the elements in such family do not have a point or a $(k-3)$-space in common, respectively, otherwise we can investigate the quotient space with respect to the common space and we can refer to [27].

Chapter 5 shows how the theory of subspace codes can be reread as the analogue of subsets families of a given set mutually meeting in sets of assigned sizes. It opens with the notion of *I-intersecting family*, [4]. Let I be a set of non-negative integers, a family $\mathcal{F}$ of subsets of a set Ω is *I-intersecting* if $|X \cap Y| \in I$ for every distinct $X, Y \in \mathcal{F}$ and the easiest example is surely the sunflower: a family of k-sets (hereafter k-uniform family) such that all elements go through the same space. The main result in this theory is the *Deza Theorem*, [34].
In Section 5.2, the theory of subspace codes is introduced. A *subspace code* $\mathcal{C}$ is a non-empty collection of subspaces of $\mathbb{F}_q^n$. We may equipped the set of all the vector subspaces of $\mathbb{F}_q^n$ with two metrics in order to measure the distance between two codewords in $\mathcal{C}$. These distances and this mathematical approach were proposed by Kötter and Ksichishang in [77]. Such metrics are known as the *subspace distance* and the *injection* one.
The subspace codes arise from the context of correcting adversarial packet insertions in linear network coding, [98]. The link between the subspace codes theory and the rank distance codes is certainly the *lifting map* described in Section 5.2.1. It provides some simple examples of *constant-dimension codes*, i.e. codes whose codewords are vector subspaces with the same dimension. We recall some results in the setting of equidistant constant-dimension codes: the codewords of such code are subspaces with the same dimension intersecting mutually in a space with fixed dimension. In literature they are known as SCID (*Subspaces with Constant Intersection Dimension*). Some simple bounds

on the size of SCIDs in $\mathbb{F}_q^n$ are obtained *with Ferdinand Ihringer*, hoping it will lead to a fruitful collaboration.

Finally, in **Chapter 6**, *with Rocco Trombetti* and *Leo Storme*, we give a generalization of equidistant constant-dimension subspace codes: the SPID*s* (*Subspaces with Pre-assigned Intersection dimension*). Based on the work of Barrolleta *et al.* in [7], we analyze their structure. More precisely, we look at the subspace spanned by constant-dimension subspace codes whose codewords have subspace distance in an assigned set of integers. However, providing some resctrictions hold, together with Ferdinand Ihringer, we give a result on the size of a SPID.
In Section 6.1, we introduce the notion of *junta* and, similarly to what was done for SCIDs in [7], we determine a *geometrical junta bound*. This is the dimension of the space generated by a $(k; k-t_1, k-t_2)-$SPID after which it is definitely a junta. In the case of $(k; k-t, k-t+1)$-SPID, we show that this bound is sharp and we classify the examples attaining the largest dimension as one of the four infinite families properly described in Section 6.3.

It should be noted that the word 'restriction' or the attribute 'restricted' will often appear. This will represent the *fil rouge* of the whole work. In this case the restrictions do not restrict (sorry for the pun) the properties of mathematical objects described. On the contrary, they bring richness.

1 Codes with rank distance

„Apri la mente a quel ch'io ti paleso
e fermalvi entro; ché non fa scïenza,
sanza lo ritenere, avere inteso.“

Dante Alighieri, Divina Commedia, Purgatorio, V, 40-42.

In 1978, Delsarte introduced rank-distance codes as the *q-analogues* of linear error correcting codes endowed with the Hamming distance, [31]. He showed that the parameters of these codes must obey a *Singleton-like bound* on the size. The codes achieving this bound are called *maximum rank distance* or shortly MRD-*codes*.
Delsarte constructed the first family of linear MRD-codes although from the perspective of bilinear forms and such examples exist for each possible set of parameters.
Few years later, Gabidulin presented the same class of MRD-codes defined as evaluation of polynomials belonging to subspaces of linearized polynomials, [47, Section 4]. These codes are essentially the counterpart of Reed-Solomon codes in the rank metric setting, and nowdays they are known as *Delsarte-Gabidulin codes*.

In 1991, Roth studied the *crisscross* errors pattern since it comes out in different applications such as in memory chips, magnetic tapes, distributed and cloud storage systems. By trying to correct this kind of errors, rank metric codes reappeared again and a new interest arose, [101].

In this chapter we will outline some basic notions about matrix codes with the rank metric. In particular, we shall explore their very close connection with linearized polynomials, their algebraic invariants and, finally, we will present, in a linearized polynomials setting, the classes of the most known and studied MRD-codes. All this will be done to better understand the next chapter.

1.1 Introduction and Preliminaries

Let $\mathbb{F}_q$ be the finite field of q elements, with q a prime power. Denoted by $\mathbb{F}_q^{m\times n}$ the set of order $m \times n$ matrices with entries in $\mathbb{F}_q$, we consider the map d defined by

$$d(A, B) = \mathrm{rk}(A - B), \tag{1.1.1}$$

for $A, B \in \mathbb{F}_q^{m\times n}$.
The map d is called *rank distance* or *rank metric* on $\mathbb{F}_q^{m\times n}$.
For the sake of completeness, we shall show that d bears the name *metric* not unjustly.

Lemma 1.1.1. *The map d is a metric on $\mathbb{F}_q^{m\times n}$.*

Proof. Clearly, $\mathrm{rk}(A-B) \geq 0$ for all $A, B \in \mathbb{F}_q^{m\times n}$ and it is null if and only if A and B coincide. It is trivial that $\mathrm{rk}(A-B) = \mathrm{rk}(B-A)$ for each $A, B \in \mathbb{F}_q^{m\times n}$ as well.
To show that the triangular inequality holds, we prove that

$$\mathrm{rk}(A + B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$$

for all $A, B \in \mathbb{F}_q^{m\times n}$.
Then, let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ be matrices in $\mathbb{F}_q^{m\times n}$, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are column vectors of A and B, respectively.
The rank of the matrix A is the dimension over $\mathbb{F}_q$ of the vector space spanned by its columns, in formula

$$\mathrm{rk}(A) = \dim_{\mathbb{F}_q}\langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle.$$

Similarly,

$$\mathrm{rk}(B) = \dim_{\mathbb{F}_q}\langle \mathbf{b}_1, \ldots, \mathbf{b}_n\rangle \quad \text{and} \quad \mathrm{rk}(A + B) = \dim_{\mathbb{F}_q}\langle \mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n\rangle.$$

We claim that

$$\langle \mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n\rangle \subset \langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle + \langle \mathbf{b}_1, \ldots, \mathbf{b}_n\rangle.$$

Indeed, any vector $\mathbf{x} \in \langle \mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n\rangle$ can be written as

$$\mathbf{x} = \lambda_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + \lambda_n(\mathbf{a}_n + \mathbf{b}_n)$$

for some scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}_q$. Hence,

$$\begin{aligned}\mathbf{x} &= \lambda_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + \lambda_n(\mathbf{a}_n + \mathbf{b}_n) = \\ &(\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n) + (\lambda_1\mathbf{b}_1 + \cdots + \lambda_n\mathbf{b}_n) \\ &\in \langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle + \langle \mathbf{b}_1, \ldots, \mathbf{b}_n\rangle,\end{aligned}$$

and hence the claim is proved. Then, we have

$$\mathrm{rk}(A+B) = \dim_{\mathbb{F}_q}\langle \mathbf{a}_1+\mathbf{b}_1,\dots,\mathbf{a}_n+\mathbf{b}_n\rangle \le \dim_{\mathbb{F}_q}(\langle \mathbf{a}_1,\dots,\mathbf{a}_n\rangle + \langle \mathbf{b}_1,\dots,\mathbf{b}_n\rangle) \le \dim_{\mathbb{F}_q}\langle \mathbf{a}_1,\dots,\mathbf{a}_n\rangle + \dim_{\mathbb{F}_q}\langle \mathbf{b}_1,\dots,\mathbf{b}_n\rangle = \mathrm{rk}(A)+\mathrm{rk}(B).$$

So, for every $A, B, C \in \mathbb{F}_q^{m\times n}$

$$d(A,C) = \mathrm{rk}(A-C) = \mathrm{rk}((A-B)+(B-C)) \le \mathrm{rk}(A-B)+\mathrm{rk}(B-C) = d(A,B)+d(B,C).$$

□

A non-empty subset $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ is called *rank distance code*, RD-*code* for short, and the *minimum distance* of $\mathcal{C}$ is defined as

$$d(\mathcal{C}) = \min_{\substack{M,N\in\mathcal{C}\\ M\neq N}} d(M,N)\,.$$

Often, an RD-code $\mathcal{C}$ with minimum distance d will be called d-code.
When $\mathcal{C}$ is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{m\times n}$, we say that it is an $\mathbb{F}_q$-*linear* RD-code and its dimension $\dim_{\mathbb{F}_q}\mathcal{C}$ is defined to be the dimension of $\mathcal{C}$ as subspace over $\mathbb{F}_q$.
Also, we say that a d-code $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ is *additive* if $\mathcal{C}$ is a subgroup of $(\mathbb{F}_q^{m\times n}, +)$. Clearly, an $\mathbb{F}_q$-linear d-code is additive, while it is straightforward to see that each additive d-code is $\mathbb{K}$-linear for some subfield $\mathbb{K}$ of $\mathbb{F}_q$.

For the applications in classical coding theory [48, 77, 78], given the positive integers m, n and $1 \le d \le \min\{m,n\}$, it is desirable to have d-codes which are as large as possible in size.
In [31], Delsarte proved that the size of each RD-code must satisfy an upper bound, the so-called *Singleton-like bound*. Here, we show a proof of such a threshold due to Gorla and Ravagnani.

Theorem 1.1.2 ([54], Theorem 21)**.** *Let $\mathcal{C}$ be an* RD*-code of $\mathbb{F}_q^{m\times n}$ with minimum distance d, then*

$$|\mathcal{C}| \le q^{\max\{m,n\}(\min\{m,n\}-d+1)}$$

Proof. Consider the application $\pi : \mathcal{C} \to \mathbb{F}_q^{(m-d+1)\times n}$, where $\pi(\mathcal{C})$ is the matrix obtained by $\mathcal{C}$ by deleting its first $d-1$ rows, i.e. π is the projection on the last $m-d+1$ rows. Since d is the minimum distance of $\mathcal{C}$, it follows that π is injective and hence

$$|\mathcal{C}| = |\pi(\mathcal{C})| \le q^{n(m-d+1)}.$$

Hence, consider $\pi' : \mathcal{C} \to \mathbb{F}_q^{m\times(n-d+1)}$ the projection on the last $n-d+1$ columns. As before, π' is injective and so

$$|\mathcal{C}| = |\pi'(\mathcal{C})| \le q^{m(n-d+1)}$$

Hence the result. □

If the cardinality of the code $\mathcal{C}$ meets this bound, we say that $\mathcal{C}$ is a *Maximum Rank Distance code*, MRD-*code* for short, or *maximum d-code*. The first examples of MRD-codes were found independently by Gabidulin and Delsarte in [47] and [31], then rediscovered and generalized by Gabidulin and Kshevetskiy in [50]. We will introduce these examples in the Section 1.5

Since $(\mathbb{F}_q^{m\times n}, d)$ is a metric space, the notion of *isometry* arises naturally. More precisely, let $m, n \geq 2$, a bijective map $\Phi : \mathbb{F}_q^{m\times n} \to \mathbb{F}_q^{m\times n}$, is called *isometry*, if it preserves the rank metric distance, i.e.

$$d(A, B) = d(\Phi(A), \Phi(B)) \tag{1.1.2}$$

for each $A, B \in \mathbb{F}_q^{m\times n}$. Then, we say that two RD-codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if there exists an isometry Φ such that $\Phi(\mathcal{C}) = \mathcal{C}'$. These isometries have been classified by Hua in odd characteristic, under suitable restrictions, in [68]. Moreover, this topic is studied in even characteristic case by Wan in [116]. In the following, we summerize these results as reported in [117]. In particular we will focus on the finite fields case.

Theorem 1.1.3 ([117], Theorem 3.4)**.** *Let $\mathbb{F}_q$ be a finite field and let m, n be integers greater than one. Then a bijective map $\Phi : \mathbb{F}_q^{m\times n} \to \mathbb{F}_q^{m\times n}$ is an isometry if and only if there exist matrices $P \in \mathrm{GL}(m, q), Q \in \mathrm{GL}(n, q)$ and $R \in \mathbb{F}_q^{m\times n}$ such that*

$$\Phi(X) = PX^\sigma Q + R$$

for all $X \in \mathbb{F}_q^{m\times n}$, where σ is a field automorphism of $\mathbb{F}_q$ acting on the entries of X, or, but only in the case $m = n$,

$$\Phi(X) = P(X^t)^\sigma Q + R$$

for all $X \in \mathbb{F}_q^{m\times n}$, where X^t is the transpose of the matrix X.

According to Theorem 1.1.3, if $m \neq n$, two d-codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if and only if there exist $P \in \mathrm{GL}(m, q)$, $Q \in \mathrm{GL}(n, q)$, $R \in \mathbb{F}_q^{m\times n}$ and a field automorphism σ of $\mathbb{F}_q$ such that

$$\mathcal{C}' = \{PC^\sigma Q + R \mid C \in \mathcal{C}\}. \tag{1.1.3}$$

If $m = n$, in addition to the possibility described above, we say that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent even if there are two non-singular matrices P and $Q \in \mathbb{F}_q^{n\times n}$ and σ, a field automorphism of $\mathbb{F}_q$, such that

$$\mathcal{C}' = \{P(C^t)^\sigma Q + R \mid C \in \mathcal{C}\}.$$

In the following, when we will consider the equivalence between two RD-codes, we will refer only to (1.1.3). This relation is often called *strong equivalence*, see

[6] and [94]. We indicate the equivalence between $\mathcal{C}$ and $\mathcal{C}'$ by the symbol $\mathcal{C} \simeq \mathcal{C}'$ and denote by $[\mathcal{C}]_{\simeq}$ the equivalence class of $\mathcal{C}$ regarding relevant equivalence relation.
Moreover, $\mathrm{Aut}(\mathcal{C})$ will indicate the *automorphism group* of $\mathcal{C}$, which is the group of all isometries of $\mathbb{F}_q^{m\times n}$ fixing $\mathcal{C}$.
When $\mathcal{C}$ and $\mathcal{C}'$ are additive, we may assume that R to be the null map. Indeed, since $C = 0$ is in $\mathcal{C}$, then $R \in \mathcal{C}'$ and

$$\mathcal{C}' - R = \{\mathcal{C}' - R \mid C' \in \mathcal{C}'\} = \mathcal{C}'$$

For further details on the equivalence of RD-codes, see also [109].

Let $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ be an RD-code, the *weight* of its codeword C is the rank of C. The *rank weight distribution* of $\mathcal{C}$ is a sequence of numbers

$$A_j = |\{C \in \mathcal{C} \,|\, \mathrm{rk}(C) = j\}|$$

for $j = 0, 1, \ldots, \min\{m, n\}$.
In general, it is difficult to determine the rank weight distribution of a given code. However, MRD-codes with the same parameters have the same rank weight distribution which is completely determined, see [31], [47]. Moreover in [89], the authors showed that the spectrum of an MRD-code is *complete*,

Proposition 1.1.4 ([89], Lemma 2.1)**.** *Let $\mathcal{C}$ be an* MRD*-code in $\mathbb{F}_q^{m\times n}$ with minimum distance d and suppose $m \leq n$. Assume that the null matrix is in $\mathcal{C}$. Then, for any $0 \leq \ell \leq m - d$, we have $A_{d+\ell} > 0$, i.e. there exists at least one matrix $C \in \mathcal{C}$ such that $\mathrm{rk}(C) = d + \ell$.*

Let $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ a rank distance code, the *adjoint* code of $\mathcal{C}$ is the set

$$\mathcal{C}^{\top} = \{C^t \mid C \in \mathcal{C}\}, \tag{1.1.4}$$

while the *Delsarte-dual code* of an additive RD-code $\mathcal{C}$ is

$$\mathcal{C}^{\perp} = \{N \in \mathbb{F}_q^{m\times n} \mid \mathrm{Tr}(MN^t) = 0 \ \forall M \in \mathcal{C}\} \subset \mathbb{F}_q^{m\times n}, \tag{1.1.5}$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a square matrix of order $m \times m$.
It is straightforward to see that the map

$$(M, N) \in \mathbb{F}_q^{m\times n} \times \mathbb{F}_q^{m\times n} \longmapsto Tr(MN^t) \in \mathbb{F}_q$$

is a non-degenerate, symmetric bilinear form on $\mathbb{F}_q^{m\times n}$. We will recall some basic notions about bilinear forms in Chapter 2.
Delsarte proved a relation between an $\mathbb{F}_q$-linear MRD-code and its Delsarte-dual code, more precisely

Theorem 1.1.5 ([31], Theorem 5.5)**.** *Let $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ be an $\mathbb{F}_q$-linear* MRD*-code with minimum distance $d = d(\mathcal{C}) \geq 1$. If $m \leq n$, then the Delsarte-dual code $\mathcal{C}^\perp$ is an $\mathbb{F}_q$-linear* MRD*-code with*

$$\dim_{\mathbb{F}_q}(\mathcal{C}^\perp) = mn - \dim_{\mathbb{F}_q}(\mathcal{C})$$

and with minimum distance $m - d + 2$.

In general, it is difficult to tell whether two rank metric codes with the same parameters are equivalent or not. In [84], useful tools to face with this problem were introduced by Liebhold and Nebe. Let $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ be an RD-code, the *left* and *right idealisers* $L(\mathcal{C})$ and $R(\mathcal{C})$ are defined as the sets

$$L(\mathcal{C}) = \{X \in \mathbb{F}_q^{m\times m} \mid XC \in \mathcal{C} \text{ for all } C \in \mathcal{C}\},$$
$$R(\mathcal{C}) = \{Y \in \mathbb{F}_q^{n\times n} \mid CY \in \mathcal{C} \text{ for all } C \in \mathcal{C}\},$$

respectively.
In [89], they appear with the name of *middle* and *right nucleus*, respectively.

Proposition 1.1.6 ([89], Proposition 4.1)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive* RD*-codes of $\mathbb{F}_q^{m\times n}$. If $\mathcal{C}_1 \simeq \mathcal{C}_2$, then their left (resp. right) idealisers are equivalent.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent, then there exist a field automorphism σ, $P \in GL(m, q)$ and $Q \in GL(n, q)$ such that

$$\mathcal{C}_2 = \{PC^\sigma Q \mid C \in \mathcal{C}_1\}$$

Then, we have

$$L(\mathcal{C}_2) = \{PT^\sigma P^{-1} \mid T \in L(\mathcal{C}_1)\}$$

and

$$R(\mathcal{C}_2) = \{Q^{-1}T^\sigma Q \mid T \in R(\mathcal{C}_1)\}$$

□

Both the idealisers of the adjoint code and Delsarte-dual code of a rank metric code $\mathcal{C}$ are linked. Indeed, we have the following

Proposition 1.1.7 ([89], Proposition 4.2)**.** *Let $\mathcal{C}$ be an $\mathbb{F}_q$-linear* RD*-code in $\mathbb{F}_q^{m\times n}$. The following statements hold:*

i) $L(\mathcal{C}^\top) = R(\mathcal{C})^\top$ *and* $R(\mathcal{C}^\top) = L(\mathcal{C})^\top$*;*

ii) $L(\mathcal{C}^\perp) = L(\mathcal{C})^\perp$ *and* $R(\mathcal{C}^\perp) = R(\mathcal{C})^\perp$*.*

Theorem 1.1.8 ([89], Theorem 5.4 and Corollary 5.6)**.** *Let $\mathcal{C}$ be an $\mathbb{F}_q$-linear* MRD*-code in $\mathbb{F}_q^{m\times n}$ with minimum distance $d > 1$. If $m \leq n$, then $L(\mathcal{C})$ is a field with $|L(\mathcal{C})| \leq q^m$. If $m \geq n$, then $R(\mathcal{C})$ is a finite field with $|R(\mathcal{C})| \leq q^n$. In particular, when $m = n$ $L(\mathcal{C})$ and $R(\mathcal{C})$, are both finite fields.*

1.2 Linearized polynomials

In this section, we will recall a different representation of RD-codes as a particular class of polynomials over finite fields, the so-called *linearized polynomials*. They fit into a broader theory started by Ore in 1933: the theory of *non-commutative polynomial rings*, see [96] and [97].

Let $\mathbb{F}_{q^n}$ be a finite field of order q^n with q a prime power. A *linearized polynomial* or a *q-polynomial* over $\mathbb{F}_{q^n}$ is a polynomial of the form

$$f(x) = \sum_{i=0}^{k} c_i x^{q^i}$$

where $c_i \in \mathbb{F}_{q^n}$ and k is a positive integer.
We will denote the set of these polynomials by $\mathscr{L}_{n,q}[x]$. If k is the largest integer such that $c_k \neq 0$, we say that k is the *q-degree* of f, in short $\deg_q(f)$. It is straightforward to show that a linearized polynomial f defines an $\mathbb{F}_q$-linear map of $\mathbb{F}_{q^n}$, when $\mathbb{F}_{q^n}$ is viewed as an $\mathbb{F}_q$-vector space.
Now, consider two $\mathbb{F}_q$-vector spaces V_n and V_m with dimension n and m, respectively. Fixed an $\mathbb{F}_q$-basis in both vector spaces, any (additive, $\mathbb{F}_q$-linear, resp.) RD-code $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ can be seen as a (subgroup, $\mathbb{F}_q$-subspace, resp.) subset of maps in $\mathrm{Hom}_{\mathbb{F}_q}(V_n, V_m)$, the set of $\mathbb{F}_q$-linear map from V_n to V_m. Morever if $n \geq m$, we can always regard V_m as a subspace of V_n and identify $\mathrm{Hom}_{\mathbb{F}_q}(V_n, V_m)$ with the subspace of those $\varphi \in \mathrm{Hom}_{\mathbb{F}_q}(V_n, V_n)$ such that $\mathrm{Im}\,\varphi \subset V_m$.
Clearly, up to isomorphism, we can suppose that V_n is $\mathbb{F}_{q^n}$ seen as a $\mathbb{F}_q$-vector space of dimension n. Hence, let $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) = \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^n}, \mathbb{F}_{q^n})$ be the set of all $\mathbb{F}_q$-linear maps of $\mathbb{F}_{q^n}$ in itself. It is well known that each element of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$ can be represented in a unique way as a linerized polynomial over $\mathbb{F}_{q^n}$ with q-degree at most $n-1$, see [83].
So, denoted by $\tilde{\mathscr{L}}_{n,q}[x]$ the quotient $\mathscr{L}_{n,q}[x]/(x^{q^n} - x)$, the algebraic structure $(\tilde{\mathscr{L}}_{n,q}[x], +, \circ, \cdot)$, where $+$ is addition of polynomials, $\circ$ is the composition of polynomials modulo $x^{q^n} - x$ and $\cdot$ is the scalar multiplication by elements of $\mathbb{F}_q$, is isomorphic to the $\mathbb{F}_q$-algebra $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$.
Let f be a linearized polynomial in $\tilde{\mathscr{L}}_{n,q}[x]$, the *kernel* of f is the set of its roots and the values assumed by f form the *image subspace*, see [83]. Clearly, if $\deg_q(f) = k$, the kernel and the image of f has dimension at most k and at least $n-k$, respectively, as $\mathbb{F}_q$-vector spaces. This follows from the fact that such a polynomial may have at most q^k roots, and hence its kernel (when viewed as a linear transformation) has dimension at most k, implying that its rank is at least $n-k$.
Actually, this turns out to be a special case of the following more general result

Theorem 1.2.1 ([56],Theorem 5). *Let $\mathbb{L}$ be a cyclic Galois extension of a field $\mathbb{F}$ of degree n, and suppose that σ generates the group* $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$. *Let k be an integer satisfying $1 \leq k < n$, and let $c_0, c_1, \ldots, c_{k-1}$ be elements of $\mathbb{L}$, not all zero. Then the $\mathbb{F}$-linear transformation defined as*

$$f(x) = c_0 x + c_1 x^{\sigma} + \ldots + c_{k-1} x^{\sigma^{k-1}}$$

has rank at least $n - k + 1$.

Taking $\mathbb{L} = \mathbb{F}_{q^n}$, $\mathbb{F} = \mathbb{F}_q$, and $x^\sigma = x^q$ returns the above statement about linearized polynomials; if we take $x^\sigma = x^{q^s}$ for some s relatively prime to n, then we get the so-called *q^s-polynomials* of the form

$$c_0 x + c_1 x^{q^s} + \ldots + c_{k-1} x^{q^{s(k-1)}}, \tag{1.2.1}$$

and their rank is at least $n - k + 1$.

1.2.1 RD-codes in linearized polynomials setting

Let $\mathbb{F}_{q^n}$ be the finite field of order q^n, where q is a prime power and consider the set

$$\tilde{\mathscr{L}}_{n,q}[x] = \left\{ \sum_{i=0}^{n-1} c_i x^{q^i} \ : \ c_0, c_1, \ldots, c_{n-1} \in \mathbb{F}_{q^n} \right\}.$$

In light of the above, $\tilde{\mathscr{L}}_{n,q}[x]$ is isomorphic to the $\mathbb{F}_q$-algebra of $\mathbb{F}_q$-linear maps of the n-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_{q^n}$ in itself and in this setting the rank distance in (1.1.1) translates simply as

$$d(f_1, f_2) = \mathrm{rk}(f_1 - f_2)$$

with $f_1, f_2 \in \tilde{\mathscr{L}}_{n,q}[x]$.
Since any RD-code $\mathcal{C} \subset \mathbb{F}_q^{m \times n}$, with $m \leq n$, can be considered as an appropriate subset of $\tilde{\mathscr{L}}_{n,q}[x]$, we will reformulate some of the notions recalled in Section 1 in terms of q -polynomials over $\mathbb{F}_{q^n}$, and in particular if no further requests are made on the kernel or the image space of such maps, we will suppose to refer to square matrices of order n with entries on $\mathbb{F}_q$.
A linearized polynomial in $\tilde{\mathscr{L}}_{n,q}[x]$ is called *invertible* or *permutation* q-polynomial, if it admits an inverse with respect to $\circ$.
Let $\mathrm{Tr}_{q^n/q}$ be the trace function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$[1], the map

$$T : (x, y) \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \to \mathrm{Tr}_{q^n/q}(xy) \in \mathbb{F}_q, \tag{1.2.2}$$

[1]For $x \in \mathbb{F}_{q^n}$, the trace $\mathrm{Tr}_{q^n/q}(x)$ of x on $\mathbb{F}_q$ is defined by

$$\mathrm{Tr}_{q^n/q}(x) = x + x^q + \ldots + x^{q^{n-1}}.$$

It is a surjective $\mathbb{F}_q$-linear map from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$, see [83].

is a non-degenerate $\mathbb{F}_q$-bilinear form on $\mathbb{F}_{q^n}$. If $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ is an $\mathbb{F}_q$-linear map of $\mathbb{F}_{q^n}$, then the *adjoint* map of f with respect to $T(\cdot,\cdot)$, i.e. the map such that

$$T(x, f(y)) = T(y, f^\top(x))$$

for each $x, y \in \mathbb{F}_{q^n}$, is the q-polynomial

$$f^\top(x) = \sum_{i=0}^{n-1} a_{n-i}^{q^i} x^{q^i}.$$

In fact, the adjoint of f is equivalent to the transpose of the matrix in $\mathbb{F}_q^{n\times n}$ derived from f. If $f = f^\top$, we say that f is *self-adjoint* with respect to $T(\cdot,\cdot)$. If $\mathcal{C} \subset \tilde{\mathscr{L}}_{n,q}[x]$ is an RD-code, then the adjoint code of $\mathcal{C}$ defined in (1.1.4) turns to be

$$\mathcal{C}^\top = \{f^\top \,:\, f \in \mathcal{C}\}.$$

Also the notion of Delsarte-dual code can be written in terms of q-polynomials, see for example [90]. Indeed, let $B : \tilde{\mathscr{L}}_{n,q}[x] \times \tilde{\mathscr{L}}_{n,q}[x] \to \mathbb{F}_q$ be the bilinear form given by

$$B(f,g) = \mathrm{Tr}_{q^n/q}\left(\sum_{i=0}^{n-1} f_i g_i\right)$$

where $f(x) = \sum_{i=0}^{n-1} f_i x^{q^i}$ and $g(x) = \sum_{i=0}^{n-1} g_i x^{q^i}$. In this setting, the Delsarte-dual code $\mathcal{C}^\perp$ of an RD-code $\mathcal{C}$ is the set of q-polynomials

$$\mathcal{C}^\perp = \{f \in \tilde{\mathscr{L}}_{n,q}[x] \mid B(f,g) = 0 \ \forall g \in \mathcal{C}\}.$$

Similarly, we may express the equivalence in this structure by repeating what we have done before with matrices. More precisely, two sets of q-polynomials in $\tilde{\mathscr{L}}_{n,q}[x]$, say $\mathcal{C}$ and $\mathcal{C}'$, are equivalent if there exist two permutation q-polynomials g_1, g_2 in $\tilde{\mathscr{L}}_{n,q}[x]$ and $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ such that

$$\mathcal{C}' = \{g_1 \circ f^\rho \circ g_2(x) + r(x) : f \in \mathcal{C}\}, \tag{1.2.3}$$

where $r(x) \in \tilde{\mathscr{L}}_{n,q}[x]$, and $f^\rho(x) = \sum a_i^\rho x^{q^i}$, if $f(x) = \sum a_i x^{q^i}$. Note that if $q = p^e$ for p a prime, and ρ is the automorphism such that $x \mapsto x^{p^i}$, then $f^\rho(x) = x^{p^i} \circ f \circ x^{p^{ne-i}}$.

Although, as seen before, an *isometry* may cover the possibility

$$\mathcal{C}' = \{g_1 \circ f^{\top\rho} \circ g_2(x) + r(x) : f \in \mathcal{C}\}.$$

Let g_1, ρ, g_2, r be as above. In the following, we will use the symbol $\Phi_{g_1,\rho,g_2,r}$ to denote the map of $\tilde{\mathscr{L}}_{n,q}[x]$ defined by

$$f(x) \mapsto g_1 \circ f^\rho \circ g_2(x) + r(x) \qquad \mod x^{q^n} - x$$

Since for the remaining part of this chapter we will work with additive RD-codes, we will assume $r(x)$ is the null map.
Furthermore, the left and right idealisers of a code $\mathcal{C} \subset \tilde{\mathscr{L}}_{n,q}[x]$ can be written as

$$L(\mathcal{C}) = \{\varphi(x) \in \tilde{\mathscr{L}}_{n,q}[x] \mid \varphi \circ f \in \mathcal{C} \ \forall f \in \mathcal{C}\},$$
$$R(\mathcal{C}) = \{\varphi(x) \in \tilde{\mathscr{L}}_{n,q}[x] \mid f \circ \varphi \in \mathcal{C} \ \forall f \in \mathcal{C}\},$$

where we consider the code $\mathcal{C}$ as a set matrices acting on vectors $\mathbf{x}$ of $\mathbb{F}_q^n$ as $\mathbf{x} \mapsto \mathbf{x}M$, with $M \in \mathcal{C}$.
Finally, by Theorem 1.2.1, all the considerations and results above can be stated for q^s-polynomials where s is an integer relative prime to n.

1.3 (Pre)semifields and quasifields

Let $\mathbb{F}_{q^n}$ be the finite field of order q^n, q a prime power, and suppose that $\mathcal{C} \subset \mathscr{L}_{n,q}[x]$ is a maximum d-code. If $d = n$, then because of the Singleton-like bound, $|\mathcal{C}| = q^n$ and $\mathcal{C}$ consists of q^n invertible endomorphisms of $\mathbb{F}_{q^n}$. Such a set is called *spread set*, [33]. In particular, if $\mathcal{C}$ is an additive spread set, it is called *semifield spread set*, [88].
In this section, we shall explore the link between the MRD-codes with minimum distance n and some algebraic structures: the *quasifields*, see [28]. We briefly recall the definition and some properties of quasifields.

Definition 1.3.1. A set $\mathcal{Q}$ with at least two elements and with two operations $+, \star : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ is called (right) *quasifield*, if the operations $+$ and $\star$ satisfy the following axioms:

i) $(\mathcal{Q}, +)$ is an abelian group with identity element 0.

ii) There is an identity e in $(\mathcal{Q}, \star)$, i.e.

$$e \star a = a \star e$$

for each $a \in \mathcal{Q}$.

iii) For each $a, b \in \mathcal{Q}$ with $a \neq 0$, there exists exactly one element x in $\mathcal{Q}$ such that

$$a \star x = b$$

iv) For each distinct $a, b, c \in \mathcal{Q}$, there exists exactly one element x in $\mathcal{Q}$ such that

$$x \star a = x \star b + c$$

v) Right distributivity property holds, i.e.

$$(a+b)\star c = a\star c + b\star c$$

for each $a, b, c \in \mathcal{Q}$.

A quasifield satisfying also the left distributivity law is called *semifield*. A *presemifield* satisfies the axioms $i), iii), iv), v)$ and the left distributivity law. Clearly, we refer to a finite quasifield or a finite semifield, if the underlying set is finite.
The quasifields were first studied by Veblen and Wedderburn in [115], while the semifields were first studied by Dickson in the early 1900s, see [36] and [37]. This topic was further developed by Albert [1] and Knuth [80], in the 1960s. This research was motivated by the fact that semifields give rise to a certain class of projective planes: the *translation planes*, [81].
A very recent research on semifields has been performed due to surprising connections in finite geometry, see for istance [82].

One easily shows that the additive group of a finite quasifield $\mathcal{Q}$ is elementary abelian, and the additive order of its elements is called the *characteristic* of $\mathcal{Q}$. Let $\mathcal{Q}$ be a right quasifield, the *kernel* of $\mathcal{Q}$ is the set

$$\mathrm{Ker}\,\mathcal{Q} = \{c \in \mathcal{Q} \mid c\star(a+b) = c\star a + c\star b \text{ and } c\star(a\star b) = (c\star a)\star b \ \ \forall a, b \in \mathcal{Q}\}.$$

If $\mathcal{Q}$ is a finite quasifield, $\mathrm{Ker}\,\mathcal{Q}$ is isomorphic to a finite field and, moreover $\mathcal{Q}$ can be structured as a finite dimensional left vector space over its kernel.
Contained in a semifield $\mathcal{S}$ there are the following important substructures, each of which is isomorphic to a finite field, if $\mathcal{S}$ is finite:

$$\begin{aligned}
\mathcal{N}_l(\mathcal{S}) &= \{x \in \mathcal{S} \mid (x\star y)\star z = x\star(y\star z),\ \forall y, z \in \mathcal{S}\}\\
\mathcal{N}_m(\mathcal{S}) &= \{y \in \mathcal{S} \mid (x\star y)\star z = x\star(y\star z), \forall x, z \in \mathcal{S}\}\\
\mathcal{N}_r(\mathcal{S}) &= \{z \in \mathcal{S} \mid (x\star y)\star z = x\star(y\star z),\ \forall x, y \in \mathcal{S}\}
\end{aligned}$$

These are known as the *left nucleus*, the *middle nucleus*, the *right nucleus*. The intersection of the three nuclei is called *associative center* $\mathcal{N}(\mathcal{S})$ and the *commutative center*, i.e. the elements of $\mathcal{N}(\mathcal{S})$ commuting with all the elements of $\mathcal{S}$, is called *center* of $\mathcal{S}$ and is denoted by $\mathcal{Z}(\mathcal{S})$.
A finite semifield $\mathcal{S}$ can be seen as a division algebra over its center (i.e. an algebra such that for any element $a \in \mathcal{S}$ and any non-zero element $b \in \mathcal{S}$ there exists precisely one element $x \in \mathcal{S}$ with $a = b\star x$ and precisely one element $y \in \mathcal{S}$ such that $a = y\star b$), as a left vector space over its left nucleus, as a left vector space and right vector space over its middle nucleus, and, finally, as a right vector space over its right nucleus.

Now, let $\mathcal{C} \subset \tilde{\mathscr{L}}_{n,q}[x]$ be a spread set. We may assume that the null polynomial belongs to $\mathcal{C}$. Indeed, if the latter is not in $\mathcal{C}$, we can choose an element $g \in \mathcal{C}$ and we can replace $\mathcal{C}$ by

$$\mathcal{C} - g = \{f - g \mid f \in \mathcal{C}\}.$$

Since in $\mathcal{C}$ all elements are invertible, then we may further assume that the polynomial x belongs to $\mathcal{C}$ replacing $\mathcal{C}$ by $h^{-1} \circ \mathcal{C}$ with $h \in \mathcal{C}$.
Then, it is straightforward to show that the non-null polynomials in $\mathcal{C}$ act regularly on the non-zero elements of $\mathbb{F}_{q^n}$, i.e. for any element $b \in \mathbb{F}_{q^n}$ there is a unique element $f(x) \in \mathcal{C}$ such that $f(1) = b$. We denote such element by the symbol $f_b(x)$. Then, we can assume that

$$\mathcal{C} = \{f_b(x) \mid b \in \mathbb{F}_{q^n}\},$$

where the null-map and the the polynomial x are determined by 0 and 1, respectively.
Defined the multiplication by $a \star b = f_b(a)$, the algebraic structure $\mathcal{Q} = (\mathbb{F}_{q^n}, +, \star)$ is a right quasifield and $\mathbb{F}_q$ is a subfield of its kernel.
Conversely, if $\mathcal{Q} = (\mathbb{F}_{q^n}, +, \star)$ is a finite quasifield with $\mathbb{F}_q \leq \operatorname{Ker} \mathcal{Q}$, the set

$$\{f_b : x \in \mathbb{F}_{q^n} \mapsto x \star b \in \mathbb{F}_{q^n} \mid b \in \mathbb{F}_{q^n}\}$$

defines a spread set $\mathcal{C} \subset \tilde{\mathscr{L}}_{n,q}[x]$. We summerize the discussion in the following

Theorem 1.3.2 ([28], Theorems 2, 3 and 4)**.** *Let $\mathcal{C} \subset \tilde{\mathscr{L}}_{n,q}[x]$ be a spread set, then there exists a finite quasifield $\mathcal{Q} = (\mathbb{F}_{q^n}, +, \star)$ with $\mathbb{F}_q \leq \operatorname{Ker} \mathcal{Q}$ and $\dim_{\mathbb{F}_q} \mathcal{Q} = n$ and vice versa.*
If $\mathcal{C}$ is an additive code, $\mathcal{Q}$ is a finite semifield and $\mathbb{F}_q$ is contained in $\mathcal{N}(\mathcal{Q})$.
If $\mathcal{C}$ is an $\mathbb{F}_q$-linear code with minimum distance n, $\mathcal{Q}$ is division algebra over $\mathbb{F}_q$, with $\mathbb{F}_q \leq \mathcal{Z}(\mathcal{Q})$ and $\dim_{\mathbb{F}_q} \mathcal{Q} = n$.

Actually, the theorem above was originally stated for a MRD-code of non-singular matrices of order n with entries over a field $\mathbb{K}$. Here, we preferred to translate it in the setting of q-polynomials over $\mathbb{F}_{q^n}$.

Hence, quasifields and semifields give examples of MRD-codes with the maximum possible value of the minimum distance. Now, the natural notion of equivalence for semifields is the so-called *isotopism*. Presisely, let $(\mathcal{S}, +, \star)$ and $(\mathcal{S}', +, \star')$ be two semifields, an isotopism between $\mathcal{S}$ and $\mathcal{S}'$ is a triple of non-singular additive maps F, G and H from $\mathcal{S}$ to $\mathcal{S}'$ such that

$$F(x) \star' G(y) = H(x \star y)$$

for all $x, y \in \mathcal{S}$. If such a triple (F, G, H) exists $\mathcal{S}$ and $\mathcal{S}'$ are called *isotopic*. The set of semifields isotopic to a semifield $\mathcal{S}$ is called the *isotopism class* of $\mathcal{S}$. In [28], the authors showed that isotopic semifields give equivalent MRD-codes and viceversa.

Theorem 1.3.3 ([28], Remark 2)**.** *Let $(\mathcal{S}, +, \star)$ and $(\mathcal{S}', +, \star')$ be finite semifields and suppose that $\mathbb{F}_q$ be is contained in both of $\mathcal{Z}(\mathcal{S})$ and $\mathcal{Z}(\mathcal{S}')$. Denote by $\mathcal{C}$ and $\mathcal{C}'$ the corresponding semifield spread sets of $\mathcal{S}$ and $\mathcal{S}'$, respectively. Then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if and only if $\mathcal{S}$ and $\mathcal{S}'$ are isotopic over the prime field of $\mathbb{F}_q$.*

1.4 Puncturing of an RD-code

Puncturing is a well-known operation on codes in the Hamming metric, see for example [53, Section 14.4].
In [10], the authors introduced a natural rank metric analogue. As we observed in Section 1.2, a matrix may be seen as a map between vector spaces, then by restricting the domain and/or range to a subspace, we can obtain a matrix of smaller size.
More precisely, let $\mathbb{F}_q$ be the finite field with q elements and consider $\mathcal{C}$, a rank distance code in $\mathbb{F}_q^{n\times n}$. Given an $m \times n$ matrix A of rank m, $m \leq n$, with entries over $\mathbb{F}_q$, it is clear that the set

$$A\mathcal{C} = \{AC : C \in \mathcal{C}\}$$

is a rank distance code in $\mathbb{F}_q^{m\times n}$. We say that the code $A\mathcal{C}$, which we will denote by $\mathcal{P}_A(\mathcal{C})$, is obtained by *puncturing* $\mathcal{C}$ with A and $\mathcal{P}_A(\mathcal{C})$ is called *punctured code* of $\mathcal{C}$ by A. In [10, Corollary 35], Byrne and Ravagnani proved that a punctured code obtained by an MRD-code is still an MRD-code. For the sake of completeness, we shall briefly retrace this result. First of all, we shall recall a classic rank inequality.

Lemma 1.4.1 (**Sylvester's rank inequality**, [52])**.** *Let A be an $m\times n$ matrix and M an $n \times \ell$ matrix with entries over a field $\mathbb{K}$. Then*

$$\mathrm{rk}(AM) \geq \mathrm{rk}(A) + \mathrm{rk}(M) - n.$$

Theorem 1.4.2 ([24], Theorem 3.2)**.** *Let $\mathcal{C} \subset \mathbb{F}_q^{n\times n}$ be an* MRD*-code with minimum distance d. Let A be an $m \times n$ matrix over $\mathbb{F}_q$ with rank m, with $n-d < m \leq n$. Then the punctured code $\mathcal{P}_A(\mathcal{C})$ is an* MRD*-code of $\mathbb{F}_q^{m\times n}$ with minimum distance $d' = d + m - n$.*

Proof. We first show that the map

$$C \in \mathcal{C} \mapsto AC \in \mathcal{P}_A(\mathcal{C})$$

is injective. Assume $AC_1 = AC_2$ for some distinct matrices $C_1, C_2 \in \mathcal{C}$. Then $A(C_1 - C_2) = 0$, giving

$$\dim_{\mathbb{F}_q}(\mathrm{Ker}\, A) \geq \mathrm{rk}(C_1 - C_2) \geq d > 0,$$

thus $\mathrm{rk}(A) = m - \dim(\mathrm{Ker}\, A) < m$, a contradiction. Therefore, $|A\mathcal{C}| = |\mathcal{C}| = q^{n(n-d+1)} = q^{n(m-d'+1)}$, where $d' = d+m-n$. By the Sylvester's rank inequality, we have

$$\mathrm{rk}(AC_1 - AC_2) \geq \mathrm{rk}(A) + \mathrm{rk}(C_1 - C_2) - n \geq m + d - n = d' > 0.$$

Hence the claim. □

In [109, Remark 10] Sheekey posed the problem to understand whenever different codes obtained by puncturing an MRD-code are equivalent or not, or whether there should exist examples of MRD-codes which cannot be obtained by puncturing. In [24], Csajbók and Siciliano, investigating punctured codes in the framework of bilinear forms, proved that generalized twisted Gabidulin codes contain many MRD-codes which are inequivalent to the MRD-codes obtained by puncturing generalized Gabidulin codes (see next the section for twisted and generalized Gabidulin codes).

1.5 Known examples of MRD-codes

In this section, we list some known examples of MRD-codes, necessary for understanding the results of the next chapter. In the following, we are going to present the known maximum rank distance codes by using their representation as sets of linearized polynomials of $\tilde{\mathscr{L}}_{n,q}[x]$. In particular, we will highlight some useful properties of the first discovered family of linear MRD-codes. Most of these results are contained in [109].

In [31], Delsarte found this family using the bilinear form theory and he referred to it as the *Singleton systems*. In [47], Gabidulin presented the same class of MRD-codes as evaluation of polynomials belonging to subspaces of linearized polynomials.
Let k, n be positive integers with $k \leq n$, a *Gabidulin code* with stated parameters is the set of linearized polynomials

$$\mathcal{G}_{n,k} = \left\{ \sum_{i=0}^{k-1} a_i x^{q^i} \ : \ a_0, a_1, \ldots, a_{k-1} \in \mathbb{F}_{q^n} \right\}.$$

Later, in [50], Kshevetskiy and Gabidulin generalized the previous construction obtaining the so-called *generalized Gabidulin codes* and they can be written as follows

$$\mathcal{G}_{n,k,s} = \left\{ \sum_{i=0}^{k-1} a_i x^{q^{si}} \ : \ a_0, a_1, \ldots, a_{k-1} \in \mathbb{F}_{q^n} \right\}. \tag{1.5.1}$$

with $\gcd(s, n) = 1$ and $k \leq n$. With these parameters, the code $\mathcal{G}_{n,k,s}$ is an $\mathbb{F}_q$-subspace of $\tilde{\mathscr{L}}_{n,q}[x]$ of dimension kn and any non-zero element in $\mathcal{G}_{n,k,s}$ has

rank greater than or equal to $d = n-k+1$. Hence, $\mathcal{G}_{n,k,s}$ is a linear MRD-code with minimum distance d. Moreover, it is easy to show that

$$L(\mathcal{G}_{n,k,s}) = R(\mathcal{G}_{n,k,s}) \simeq \mathbb{F}_{q^n},$$

see for istance [84] and [94].
Each code $\mathcal{G}_{n,1,s}$ is a semifield spread set, and all are equivalent and correspond to the field $\mathbb{F}_{q^n}$.

We note that, in general, the map

$$a_0x + a_1x^q + \ldots + a_{n-1}x^{q^{k-1}} \longmapsto a_0x + a_1x^{q^s} + \ldots + a_{n-1}x^{q^{s(k-1)}}$$

does not preserve the rank distance. Indeed, in [50] it was shown that there exist codes in $\mathcal{G}_{n,k,s}$ inequivalent to any in $\mathcal{G}_{n,k}$ for particular values of k, s and q. However, it is easy to check that $\mathcal{G}^{\top}_{n,k,s} = x^{q^{sk}} \circ \mathcal{G}_{n,k,s}$, and hence we have the following

Lemma 1.5.1 ([109], Lemmas 1 and 2)**.** *Each generalized Gabidulin code $\mathcal{G}_{n,k,s}$ is equivalent to its adjoint $\mathcal{G}^{\top}_{n,k,s}$ and the Delsarte-dual code $\mathcal{G}^{\perp}_{n,k,s}$ of a generalized Gabidulin code $\mathcal{G}_{n,k,s}$ is equivalent to $\mathcal{G}_{n,n-k,s}$.*

Note also that the generalized Gabidulin codes form a chain:

$$\mathcal{G}_{n,1,s} \leq \mathcal{G}_{n,2,s} \leq \ldots \leq \mathcal{G}_{n,n,s} = \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}).$$

Here, we premise a useful result to calculate the group of automorphisms of a generalized Gabidulin code $\mathcal{G}_{n,k,s}$. We will try to figure out which subspaces of a generalized Gabidulin code $\mathcal{G}_{n,k,s}$ are equivalent to another generalized Gabidulin code $\mathcal{G}_{n,r,s}$, with $r \leq k$.

Proposition 1.5.2 ([109], Theorem 3)**.** *A subspace U of $\mathcal{G}_{n,k,s}$, $k \leq n-1$, is equivalent to $\mathcal{G}_{n,r,s}$ if and only if there exist invertible linearized polynomials g, h such that*

$$U = \Phi_{g,id,h}(\mathcal{G}_{n,r,s}) = \{g \circ f \circ h : f \in \mathcal{G}_{n,r,s}\},$$

where $g_0 = 1$ and $\deg_{q^s}(g) + \deg_{q^s}(h) \leq k - r$.

Proof. Clearly if g and h are invertible linearized polynomials satisfying the condition on degrees, then U is contained in $\mathcal{G}_{n,k,s}$, and it is equivalent to $\mathcal{G}_{n,r,s}$. Note that for any $\beta \in \mathbb{F}^*_{q^n}$ and any $j \in \{0, \ldots, n-1\}$, we have that

$$\{g \circ f \circ h : f \in \mathcal{G}_{n,r,s}\} = \{g \circ (\beta x^{q^{sj}}) \circ f \circ (\beta^{-1}x^{q^{s(n-j)}}) \circ h : f \in \mathcal{G}_{n,r,s}\},$$

and hence we may assume without loss of generality that $g_0 = 1$.
Consider $g \circ \alpha x^{q^{sj}} \circ h$, where $\alpha \in \mathbb{F}_{q^n}$. Then the coefficient of $x^{q^{sm}}$ is

$$c_{m,j}(\alpha) = \sum_{i=0}^{n-1} g_i h_{m-i-j}^{q^{si}} \alpha^{q^{si}}. \tag{1.5.2}$$

where indices are taken modulo n. If U is contained in $\mathcal{G}_{n,k,s}$, we must have for each $m \geq k, j \leq r-1$, $c_{m,j}(\alpha) = 0$ for every $\alpha \in \mathbb{F}_{q^n}$. Hence for all $m \geq k, j \leq r-1$ and $i \in \{0, \ldots, n-1\}$ we have that

$$g_i h_{m-i-j} = 0.$$

As $g_0 \neq 0$, we get that $g_m = 0$ for all $m \geq k$. Let $\deg_{q^s}(g) = \ell$, $\deg_{q^s}(h) = t$, and so $g_\ell h_t \neq 0$. Then $h_{m-\ell-r+1} = 0$ for all $m \in \{k, \ldots, n-1\} \neq \emptyset$, and hence $t \leq k - \ell - r$, proving the claim. □

Thanks to the result above, we obtain a complete description of the automorphism group of the Gabidulin codes.

Theorem 1.5.3 ([109], Theorem 4)**.** *The automorphism group of the generalized Gabidulin code $\mathcal{G}_{n,k,s}$ is given by*

$$\{\Phi_{\alpha x, \rho, \beta x} \mid \alpha, \beta \in \mathbb{F}_{q^n}^*, \rho \in \mathrm{Aut}(\mathbb{F}_q)\} \tag{1.5.3}$$

Proof. Clearly the set in 1.5.3 is a subgroup of $\mathrm{Aut}(\mathcal{G}_{n,k,s})$.
Suppose $\Phi_{g,\rho,h}(\mathcal{G}_{n,k,s}) = \mathcal{G}_{n,k,s}$ for some invertible linearized polynomias g, h and some $\rho \in \mathrm{Aut}(\mathbb{F}_q)$. As $\mathcal{G}_{n,k,s}^{\rho} = \mathcal{G}_{n,k,s}$ for all $\rho \in \mathrm{Aut}(\mathbb{F}_q)$, we may assume that ρ is the identity. Then $g = g' \circ (\alpha x^{q^{si}})$ for some $\alpha \in \mathbb{F}_{q^n}^*$, $i \in \{0, \ldots, n-1\}$, where $g'_0 = 1$. Let $h' = x^{q^{si}} \circ h$. Then $\Phi_{g',id,h'}(\mathcal{G}_{n,k,s}) = \mathcal{G}_{n,k,s}$, and by the proof of Proposition 1.5.2 , we must have $\deg_{q^s}(g') + \deg_{q^s}(h') = 0$. Hence $g' = x$ and $h' = \beta^{q^{si}} x$ for some $\beta \in \mathbb{F}_{q^n}^*$, and so $\Phi_{g,id,h} = \Phi_{\alpha x^{q^{si}}, id, \beta x^{q^{s(n-i)}}}$. □

More recently, in [109], Sheekey constructed a new family of linear MRD-codes for all parameters. Let $\mathbb{F}_{q^n}$ be the finite field of order q^n, q a prime power, $\eta \in \mathbb{F}_{q^n}$ such that $\mathrm{N}_{q^n/q}(\eta) \neq (-1)^{nk}$ [2] and let s be an integer relatively prime to n. The set of q^s-polynomials

$$\mathcal{H}_{n,k}(\eta, h) = \{a_0 x + a_1 x^{q^s} + \ldots + a_{k-1} x^{q^{s(k-1)}} + a_0^{q^h} \eta x^{q^{sk}} \mid a_0, a_1, \ldots, a_{k-1} \in \mathbb{F}_{q^n}\}$$

with $k \leq n-1$, is an $\mathbb{F}_q$-linear MRD-code of dimension nk. This code is known as *generalized twisted Gabidulin code*. In [90], Lunardon, Trombetti

[2] For $x \in \mathbb{F}_{q^n}$, the norm $\mathrm{N}_{q^n/q}(x)$ of x over $\mathbb{F}_q$ is defined by

$$\mathrm{N}_{q^n/q}(x) = x^{1+q+\ldots+q^{n-1}} = x^{(q^n-1)/(q-1)}.$$

It is a group homomorphism from $\mathbb{F}_{q^n}^*$ onto $\mathbb{F}_q^*$, see [83].

and Zhou determined the automorphism group of the generalized twisted Gabidulin codes. However, note that the generalized Gabidulin codes and twisted Gabidulin codes are both proper subsets of this class. Indeed, when $s = 1$, $\mathcal{H}_{n,k,s}(\eta, h)$ is the twisted Gabidulin code, when $\eta = 0$, $\mathcal{H}_{n,k,s}(\eta, h)$ is exactly the generalized Gabidulin code $\mathcal{G}_{n,k,s}$. In particular, when $k = 1$, all elements in $\mathcal{H}_{n,1,s}(\eta, h)$ are of the type

$$a_0 x + \eta a_0^{q^h} x^{q^s},$$

for $a_0 \in \mathbb{F}_{q^n}$. They define the multiplication of a generalized twisted field, a presemifield found by Albert [2]. Also, if $\eta \neq 0$, the authors in [90] determined its left and right idealisers

$$L(\mathcal{H}_{n,k,s}(\eta, h)) \simeq \mathbb{F}_{q^{gcd(n,h)}} \quad \text{and} \quad R(\mathcal{H}_{n,k,s}(\eta, h)) \simeq \mathbb{F}_{q^{gcd(n,sk-h)}}$$

As for generalized Gabidulin codes, the class of generalized twisted Gabidulin codes is closed by the adjoint operation and by Delsarte duality described in (1.1.4) and (1.1.5), more precisely

Proposition 1.5.4 ([90], Proposition 4.2 and 4.3)**.** *The Delsarte dual code and the adjoint code of $\mathcal{H}_{n,k,s}(\eta, h)$ is equivalent to the code $\mathcal{H}_{n,n-k,s}(-\eta, n-h)$ and to the code $\mathcal{H}_{n,k,s}(1/\eta, sk - h)$, respectively.*

In [109], the author proved that $\mathcal{G}_{n,k,s}$ is equivalent to $\mathcal{H}_{n,k,1}(\eta, h)$ if and only if $k \in \{1, n-1\}$ and $h \in \{0, 1\}$, while the equivalence issue between $\mathcal{H}_{n,k,s}(\eta, h)$ and $\mathcal{H}_{n,k,t}(\theta, g)$ has been completely solved in [90]. In particular, as a consequence we obtain the following

Corollary 1.5.5. *Let n, k, s, t be integers such that $1 \leq k \leq n$ and s, t relatively prime to n. The generalized Gabidulin codes $\mathcal{G}_{n,k,s}$ and $\mathcal{G}_{n,k,t}$ are equivalent if and only if $s \equiv \pm t \pmod{n}$.*

By similar techniques used by J. Sheekey in [109], Trombetti and Zhou found a new example of MRD-code of $\tilde{\mathscr{L}}_{n,q}$, with n even and q odd, [112]. More precisely, the set

$$\mathcal{D}_{k,s}(\gamma) = \left\{ ax + \sum_{j=1}^{k-1} c_i x^{q^{sj}} + \gamma b x^{q^{sk}} \ : \ c_1, \ldots, c_{k-1} \in \mathbb{F}_{q^n}, \ a, b \in \mathbb{F}_{q^{n/2}} \right\} \tag{1.5.4}$$

with $\gcd(s, n) = 1$ and $\gamma \in \mathbb{F}_{q^n}$ such that $N_{q^n/q}(\gamma)$ is a non-square in $\mathbb{F}_q$, defines a maximum rank distance code with minimum distance $d = n - k + 1$. Both its idealisers are isomorphic to $\mathbb{F}_{q^{n/2}}$.

2 Rank distance codes with restrictions

„Wir haben eine ältere Offenbarung
als jede geschriebene, die Natur.“

Friedrich W.J. Schelling, System des transzendentalen Idealismus.

It should be noted that the theory of RD-codes recalled in the previous chapter does not require particular restrictions for the codewords. Then, we could refer to these codes with the name of *unrestricted* RD-codes.
In [30], [105], [106] and [107], the authors studied RD-codes with prescribed restrictions for their elements. More precisely, rank codes whose codewords are symmetric, alternating and Hermitian matrices with entries over a finite field.
In [51], Gabidulin *et al.* investigated rank codes containing a linear subcode of symmetric matrices. They showed that these codes have a good behavior from the point of view of errors correction. More details on applications of restricted codes to the coding theory can be found in [29].
In [30], Delsarte and Goethals and later Kai-Uwe Schimdt, in [105], [106], [107], explored restricted RD-codes, leading to the determination of bounds on the size of such additive or non-additive d-codes in general different from the Singleton-like one. Some bounds are proven to be tight by exhibiting examples. Moreover, under certain conditions, the rank weight distribution of these codes is determined by their parameters, see [106, Section 3.1].

In this chapter after recalling the known literature on rank metric codes with the aforementioned restrictions, we shall focus on such examples. More precisely, we determine their automorphism groups and solve the equivalence issue for them. Finally, in the last section, we shall exhibit a maximum symmetric 2-code which is not equivalent to the one with same parameters constructed in [106].

2.1 Sesquilinear, bilinear and Hermitian forms

Before introducing other concepts from the theory of RD-codes, in this section we will recall the well-known notion of sesquilinear form on a vector space from which the notion of symmetric, alternating bilinear form and Hermitian form follows. So, let $\mathbb{V} = \mathbb{V}(n, q)$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$ and let σ be an automorphism of $\mathbb{F}_q$. A *σ-sesquilinear form* on $\mathbb{V}$ with *accompanying automorphism* σ is a map

$$S : \mathbb{V} \times \mathbb{V} \to \mathbb{F}_q,$$

satisfying the following conditions

i) $S(u + v, w + z) = S(u, w) + S(u, z) + S(v, w) + S(v, z)$

ii) $S(\lambda u, \mu v) = \lambda \, \mu^\sigma \, S(u, v)$,

for all $u, v, w, z \in \mathbb{V}$ and for all $\lambda, \mu \in \mathbb{F}_q$.
Hence, fixed an $\mathbb{F}_q$-basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{V}$, a σ-sesquilinear form on $\mathbb{V}$ is uniquely determined by its matrix $S = (s_{ij}) \subset \mathbb{F}_q^{n \times n}$ with

$$s_{ij} = S(e_i, e_j),$$

for $i, j = 1, \ldots, n$. The rank of this matrix doesn't depend on the choice of the basis and is defined to be the *rank* of $S(\cdot, \cdot)$.

Now, if σ is the identity automorphism of $\mathbb{F}_q$, a σ-sesquilinear form $B(\cdot, \cdot)$ on $\mathbb{V}$ is properly called *$\mathbb{F}_q$-bilinear form*.
In particular, an $\mathbb{F}_q$-bilinear form $B(\cdot, \cdot)$ on $\mathbb{V}$ is *symmetric* if

$$B(u, v) = B(v, u) \qquad \forall u, v \in \mathbb{V}. \tag{2.1.1}$$

An *alternating* $\mathbb{F}_q$-bilinear form B on $\mathbb{V}$, instead, is a bilinear form such that for all $u \in \mathbb{V}$

$$B(u, u) = 0 \tag{2.1.2}$$

from which the additional property

$$B(u, v) + B(v, u) = 0 \tag{2.1.3}$$

follows. By (2.1.1), (2.1.2) and (2.1.3), we may conclude that the matrix of a symmetric (resp. alternating) bilinear form is symmetric (resp. skew-symmetric). Conversely, to any symmetric (resp. skew-symmetric) matrix of

order n with elements in $\mathbb{F}_q$ corresponds a symmetric (resp. alternating) $\mathbb{F}_q$-bilinear form.

Now, let $\mathbb{V}(n, q^2)$ be n-dimensional vector space over the finite field of order q^2 and let $\sigma : a \mapsto a^q$ be the involuntory automorphism of $\mathbb{F}_{q^2}$.
A *Hermitian form* $H : \mathbb{V} \times \mathbb{V} \to \mathbb{F}_{q^2}$ is a σ-sesquilinear form on $\mathbb{V}$ satisfying the following property

$$H(v, u) = H(u, v)^\sigma, \tag{2.1.4}$$

for all $u, v \in \mathbb{V}$. By considerations similar to those above, fixed an $\mathbb{F}_{q^2}$-basis of $\mathbb{V}$, a Hermitian form on $\mathbb{V}$ is uniquely determined by a matrix A of order n with entries in $\mathbb{F}_{q^2}$ such that

$$A = \bar{A}^t, \tag{2.1.5}$$

where we indicate $\bar{A}$, the matrix obtained from A applying to each entry the involutory automorphism of $\mathbb{F}_{q^2}$. Conversely, given a matrix in $\mathbb{F}_{q^2}^{n\times n}$ as in (2.1.5), a Hermitian form on $\mathbb{V}$ is uniquely determined. We will still call *rank* of the Hermitian form $H(\cdot, \cdot)$, the rank of A.

First of all, we prove the following slight generalization to any σ-sesquilinear form of [105, Lemma 13].

Proposition 2.1.1. *Let ℓ be an arbitrary integer and let σ be an automorphism of $\mathbb{F}_q$. For each m-dimensional $\mathbb{F}_q$-subspace U of $\mathbb{F}_{q^n}$, every σ-sesquilinear form*

$$S : U \times \mathbb{F}_{q^n} \to \mathbb{F}_q$$

can be written in the following form

$$S(x, y) = \mathrm{Tr}_{q^n/q}\left(\sum_{j=0}^{m-1} a_j y^\sigma x^{q^{s(j-\ell)}}\right),$$

for uniquely determined $a_0, a_1, \ldots, a_{m-1} \in \mathbb{F}_{q^n}$ and s an integer relatively prime to n.

Proof. Since there are q^{mn} σ-sesquilinear forms from $U \times \mathbb{F}_{q^n}$ to $\mathbb{F}_q$ and the trace is linear, it is enough to show that, if $S(x, y)$ is identically zero, then $a_0 = a_1 = \ldots = a_{m-1} = 0$. If $S(x, y)$ is identically zero, then

$$\sum_{j=0}^{m-1} a_j x^{q^{s(j-\ell)}} \tag{2.1.6}$$

equals zero for all $x \in U$. If the a_j's are not all zero, then (2.1.6) has at most q^{m-1} zeros in $\mathbb{F}_{q^n}$. Since $|U| = q^m$, this completes the proof.

□

In particular, choosen $\ell = 0$ and σ the identity automorphism, by Proposition 2.1.1, each bilinear form, say $B(\cdot,\cdot)$, defined over $\mathbb{F}_{q^n}$ seen as a vector space over $\mathbb{F}_q$, can be written as

$$B(x,y) = \mathrm{Tr}_{q^n/q}(f(x)y),$$

where $f(x) \in \tilde{\mathscr{L}}_{n,q}[x]$ and the rank of the bilinear form $B(\cdot,\cdot)$ equals the rank of the $\mathbb{F}_q$-linearized polynomial $f(x)$.
Note explicitly that we get (1.2.2) by putting $f(x) = x$.

2.2 Symmetric and alternating RD-codes

In [105], by exploiting close relationships between symmetric and alternating bilinear forms as described in [30], the author derived an upper bound on the size of symmetric RD-codes in even characteristic. Later in [106], he generalized the previous results in each characteristic. He obtained in certain cases their rank weight distribution and provided constructions of maximum symmetric additive codes for all possible parameters.

Before recalling these results, we note that if $B(\cdot,\cdot)$ is a symmetric $\mathbb{F}_q$-bilinear form of $\mathbb{F}_{q^n}$, by Proposition 2.1.1, there exists a q-polynomial $f(x)$ such that $B(x,y) = \mathrm{Tr}_{q^n/q}(f(x)y)$, and by (2.1.1) we must have for all $x, y \in \mathbb{F}_{q^n}$,

$$\mathrm{Tr}_{q^n/q}(f(y)x) = \mathrm{Tr}_{q^n/q}(f(x)y).$$

It is customary to verify that, $\forall x, y \in \mathbb{F}_{q^n}$,

$$\mathrm{Tr}_{q^n/q}(f(y)x) = \mathrm{Tr}_{q^n/q}(f(x)y) = \mathrm{Tr}_{q^n/q}(xf^{\top}(y)),$$

which means that f is a self-adjoint map with respect to $T(\cdot,\cdot)$ given in (1.2.2). Therefore, by suitably choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$, we can identify the set of symmetric bilinear forms over $\mathbb{F}_{q^n}$, with the $\frac{n(n+1)}{2}$-dimensional subspace $S_n(q) \subset \tilde{\mathscr{L}}_{n,q}[x] \simeq \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$ of self-adjoint $\mathbb{F}_q$-linear maps of $\mathbb{F}_{q^n}$. More precisely,

$$S_n(q) = \left\{ \sum_{i=0}^{n-1} c_i x^{q^i} \ : \ c_{n-i} = c_i^{q^{(n-i)}} \ \text{ for } i \in \{0,1,\ldots,n-1\} \right\}. \tag{2.2.1}$$

Now, we give some results and a description of the known examples of maximum additive d-codes presented in [31], [106] in terms of q-polynomials.
Regarding upper bounds for such d-codes, parts of the following results can be found in [106, Theorem 3.3, Lemma 3.5 and 3.6] and [105, Corollary 7, Remark 8]. The last open case, i. e. when q and d both even was proved in [108].

Theorem 2.2.1. *Let $\mathcal{C}$ be a d-code in $S_n(q)$. If $\mathcal{C}$ is additive*

$$|\mathcal{C}| \leq \begin{cases} q^{\frac{n(n-d+2)}{2}}, & \text{if } n-d \text{ is even} \\ q^{\frac{(n+1)(n-d+1)}{2}}, & \text{if } n-d \text{ is odd.} \end{cases} \tag{2.2.2}$$

If d is odd and $\mathcal{C}$ is not necessarily additive, then

$$|\mathcal{C}| \leq \begin{cases} q^{\frac{n(n-d+2)}{2}}, & \text{if } n \text{ is odd} \\ q^{\frac{(n+1)(n-d+1)}{2}}, & \text{if } n \text{ is even.} \end{cases} \tag{2.2.3}$$

If d is even and $\mathcal{C}$ is not necessarily additive, then

$$|\mathcal{C}| \leq \begin{cases} q^{\frac{n(n-d+3)}{2}}\left(\frac{1+q^{-n+1}}{q+1}\right), & \text{if } n \text{ is odd} \\ q^{\frac{(n+1)(n-d+2)}{2}}\left(\frac{1+q^{-n+d-1}}{q+1}\right), & \text{if } n \text{ is even.} \end{cases} \tag{2.2.4}$$

We note explicitly that when d is even, the upper bound for non-additive codes is greater than that for additive codes. However, it is not known whether there exists a non-additive code exceeding the bound for additive ones.
In [106], Kai-Uwe Schmidt showed that for additive codes the bounds are tight presenting the following class of additive (actually $\mathbb{F}_q$-linear) codes in $S_n(q)$: for any integer $1 \leq d \leq n$ such that $n-d$ is even and s coprime with n, consider the following subset of $S_n(q)$

$$\mathcal{S}_{n,d,s} = \left\{ b_0 x + \sum_{i=1}^{\frac{n-d}{2}} \left(b_i x^{q^{si}} + (b_i x)^{q^{s(n-i)}} \right) \ : \ b_0, b_1, \ldots, b_{\frac{n-d}{2}} \in \mathbb{F}_{q^n} \right\}. \tag{2.2.5}$$

We shall show that the set in (2.2.5) attains the bound in (2.2.2).

Theorem 2.2.2 ([106],Theorem 4.4)**.** *Let n, d and s be integers such that $1 \leq d \leq n$, $n-d$ even and s coprime with n. The set $\mathcal{S}_{n,d,s}$ is a maximum d-code in $S_n(q)$.*

Proof. Since $\mathcal{S}_{n,d,s}$ is $\mathbb{F}_q$-subspace of $S_n(q)$, it is sufficient to show that, if $f \in \tilde{\mathscr{L}}_{n,q}[x]$ is not null, then the $\mathbb{F}_q$-bilinear form

$$B_f(x,y) = Tr_{q^n/q}(f(x)y) \tag{2.2.6}$$

with

$$f(x) = b_0 x + \sum_{i=1}^{\frac{n-d}{2}} \left(b_i x^{q^{si}} + (b_i x)^{q^{s(n-i)}} \right)$$

has rank at least d.
Then, let $f(x)$ be a nonzero linearized polynomial. Observe that $B_f(x,y) = 0$ for each $y \in \mathbb{F}_{q^n}$ if and only if $f(x) = 0$ and we note that $f\left(x^{q^s \frac{n-d}{2}}\right)$ has at most

q^{n-d} zeros in $\mathbb{F}_{q^n}$. Since $x \mapsto x^{q^s \frac{n-d}{2}}$ is an automorphism of $\mathbb{F}_{q^n}$, the dimension of the kernel of the $\mathbb{F}_q$-linear map f is at most $n-d$, so that B_f has rank at least d, as required. □

In [106], in order to obtain maximum symmetric d-codes with $n-d$ odd, the author showed that it is sufficient to construct $(d+2)$-codes in $S_{n+1}(q)$. So, let $\mathcal{C}$ be a subset of $S_n(q)$ and for every $f \in \mathcal{C}$, let B_f be the $\mathbb{F}_q$-bilinear form on $\mathbb{F}_{q^n}$ defined by f. As described in 1.4, if $\mathbb{W}$ is an $(n-1)$-dimesional subspace of $\mathbb{F}_{q^n}$, the *punctured* set (with respect to $\mathbb{W}$) of $\mathcal{C}$ is the set

$$\mathcal{C}^\star = \{B_{f|_{\mathbb{W}}} \mid f \in \mathcal{C}\}$$

where

$$B_{f|_{\mathbb{W}}} : (x, y) \in \mathbb{W} \times \mathbb{W} \to \mathrm{Tr}_{q^n/q}(f(x)y)$$

is the restriction of $B_f(\cdot,\cdot)$ onto $\mathbb{W}$. So, by simply *puncturing* the $(d+2)$-code $\mathcal{S}_{n+1,d+2,s}$ of $S_{n+1}(q)$, we obtain a maximum d-code in $S_n(q)$, in fact

Theorem 2.2.3 ([106], Theorem 4.1)**.** *Suppose that $\mathcal{C}$ is a maximum additive $(d+2)$-code in $S_{n+1}(q)$ for some $d \geq 1$ such that $n-d-1$ is even. Then every punctured set $\mathcal{C}^\star$ is a maximum additive d-code in $S_n(q)$.*

Observe that in the case of minimum distance $d = n$, the upper bound for a maximum symmetric rank code is identical to the Singleton-like bound, and thus, according to Theorem 1.3.2, it corresponds to a quasifield. In particular, in the additive case, it corresponds to a semifield. It is known that semifields defined by a subspace of symmetric matrices are obtained from semifields in which the multiplication is *commutative*.

Theorem 2.2.4 ([75])**.** *Equivalence classes of additive maximum symmetric additive rank distance codes in $\tilde{\mathscr{L}}_{n,q}[x]$ with minimum distance n are in one-to-one correspondence with isotopy classes defined by commutative semifields.*

However, we note that this correspondence is not direct. Indeed the semifield spread set of a commutative semifield does not necessarily consist of symmetric matrices. One has to perform the semifield operation known as *transposition*, part of the *Knuth orbit* of a semifield described in [80], in order to obtain a set of symmetric matrices. For this reason, it is common in the literature on semifields to refer to *symplectic semifields* rather than commutative semifields.

Now, let $B(\cdot,\cdot)$ be an *alternating* $\mathbb{F}_q$-bilinear form on $\mathbb{F}_{q^n}$. By Proposition 2.1.1, Equations (2.1.2) and (2.1.3), and again properly choosing an $\mathbb{F}_q$-basis

of $\mathbb{F}_{q^n}$, the set of alternating bilinear forms with entries running over $\mathbb{F}_q$ can be seen as the following subset of q-polynomials:

$$A_n(q) = \left\{ \sum_{i=1}^{n-1} c_i x^{q^i} \ : \ c_{n-i} = -c_i^{q^{(n-i)}} \ \text{ for } i \in \{1, 2, \ldots, n-1\} \right\}. \tag{2.2.7}$$

Clearly, $A_n(q)$ is an $\frac{n(n-1)}{2}$-dimensional subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$ and it is well known that the *rank* of each element of $A_n(q)$ is necessarily even, [117].
Recall, in the alternating setting, the result of the same sort of Theorem 2.2.1 due to Delsarte and Goethals

Theorem 2.2.5 ([30],Theorem 4)**.** *Let $m = \lfloor \frac{n}{2} \rfloor$ and let $\mathcal{C}$ be a $2e$-code in $A_n(q)$, then*

$$|\mathcal{C}| \leq q^{\frac{n(n-1)}{2m}(m-e+1)}.$$

The authors exhibited a class of $\mathbb{F}_q$-linear maximal codes in $A_n(q)$ for any characteristic, and any odd value of n and a class of non-linear maximum alternating codes for n and q even, generalizing the result obtained, for $q = 2$, by Kerdock [76].
For the purposes of this thesis, we recall only the class of linear ones. Let $2 \leq d = 2e \leq n-1$, and let s be an integer coprime with n. Then the set of q-polynomials

$$\mathcal{A}_{n,d,s} = \left\{ \sum_{i=e}^{\frac{n-1}{2}} \left(b_i x^{q^{si}} - (b_i x)^{q^{s(n-i)}} \right) \ : \ b_e, \ldots, b_{\frac{n-1}{2}} \in \mathbb{F}_{q^n} \right\} \tag{2.2.8}$$

turns to be a maximum alternating d-code [30, Theorem 7].

With an abuse of notation, we will use the symbols $S_n(q)$ and $A_n(q)$ to indicate the set of symmetric or skew-symmetric matrices of order n with entries over $\mathbb{F}_q$, respectively. Cleary, both sets are metric spaces with respect to the rank distance d. Therefore, it is natural to study the isometries of such spaces in the sense of (1.1.2). But first, we recall the following result

Theorem 2.2.6 ([117], Theorem 5.4)**.** *Let $\mathbb{F}_q$ be a finite field, q a prime power, and let n be an integer greater than one. Then a bijective map $\Psi : S_n(q) \to S_n(q)$ preserves the rank distance one with its inverse if and only if there exist $P \in GL(n,q)$, $S \in S_q(n)$, $a \in \mathbb{F}_q^*$ and σ a field automorphism of $\mathbb{F}_q$ such that*

$$\Psi(X) = aPX^\sigma P^t + S \tag{2.2.9}$$

for all $X \in S_n(q)$, unless $n = 3$ and $q = 2$.
In this case there exists an extra bijective map $\tilde{\Psi}$ of the form

$$\tilde{\Psi} : S_3(2) \longrightarrow S_3(2) \ : \begin{cases} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \\ \\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 1 \\ x_{13} & 1 & x_{33} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11}+1 & x_{12}+1 & x_{13}+1 \\ x_{12}+1 & x_{22} & 1 \\ x_{13}+1 & 1 & x_{33} \end{pmatrix} \end{cases} \tag{2.2.10}$$

for all $x_{11}, x_{22}, x_{33}, x_{12}, x_{13} \in \mathbb{F}_2$, and each composition between a bijective map with the form as in (2.2.11) and the extra bijective map above.

The extra map $\tilde{\Psi} : S_3(2) \longrightarrow S_3(2)$ in (2.2.10) preserves the rank distance one with its inverse, but it is not an isometry in terms of the rank distance as one can see in the following example, see [117, Section 5.7].

Consider the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ in $S_3(2)$. Its rank distance from zero matrix is 2, while the distance of its image under $\tilde{\Psi}$ from the zero matrix is 3. Hence, as consequence we have the following

Corollary 2.2.7. *Let $\mathbb{F}_q$ be a finite field, q a prime power, and let n be an integer greater than one. Then a bijective map $\Psi : S_n(q) \to S_n(q)$ is an isometry with respect to the rank distance if and only if there exist $P \in GL(n, q)$, $S \in S_n(q)$, $a \in \mathbb{F}_q^*$ and σ a field automorphism of $\mathbb{F}_q$ such that*

$$\Psi(X) = aPX^{\sigma}P^t + S \tag{2.2.11}$$

for all $X \in S_n(q)$.

The case of the isometries of the skew-symmetric matrices space is slightly different,

Theorem 2.2.8 ([117], Theorem 4.4 and Corollary 4.6)**.** *Let $\mathbb{F}_q$ be a finite field, q a prime power, and let $n \geq 4$. Then a bijective map $\Psi : A_n(q) \to A_n(q)$ is an isometry if and only if there exist $P \in GL(n, q)$, $S \in A_q(n)$, $a \in \mathbb{F}_q^*$ and σ a field automorphism of $\mathbb{F}_q$ such that*

$$\Psi(X) = aP(X^{\circ})^{\sigma}P^t + S$$

where $X \longmapsto X^{\circ}$ is

a) if $n > 4$, the identity map

b) if $n = 4$, either the identity map or the map

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & x_{12} & x_{13} & x_{23} \\ -x_{12} & 0 & x_{14} & x_{24} \\ -x_{13} & -x_{14} & 0 & x_{34} \\ -x_{23} & -x_{24} & -x_{34} & 0 \end{pmatrix}.$$

Denote by the symbol $X_n(q)$ either the subspace $S_n(q)$ or $A_n(q)$, as consequence of Corollary 2.2.7 and Theorem 2.2.8, it is readily verified that for given $a \in \mathbb{F}_q^*$, $\rho \in \mathrm{Aut}(\mathbb{F}_q)$, g a permutation q-polynomial over $\mathbb{F}_{q^n}$, and $r_0 \in X_n(q)$, the map $\Psi : X_n(q) \to X_n(q)$ defined by

$$\Psi_{a,g,\rho,r_0}(f) = ag \circ f^\rho \circ g^\top(x) + r_0(x), \tag{2.2.12}$$

preserves the rank distance on $X_n(q)$. Moreover, the vice versa is also true if $X_n(q) = S_n(q)$, except when $n \leq 4$ if $X_n(q) = A_n(q)$.
For two subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of X_n, if there exists a map Ψ_{a,g,ρ,r_0} defined as in (2.2.12) for certain a, g, ρ and r_0 such that

$$\mathcal{C}_2 = \{\Psi_{a,g,\rho,r_0}(f) : f \in \mathcal{C}_1\},$$

then we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* in $X_n(q)$, and to distinguish this relation from the one defined in Section 1, we write $\mathcal{C}_1 \cong \mathcal{C}_2$.

2.3 Hermitian RD-codes

Let $\mathbb{F}_{q^{2n}}$ be the finite field of order q^{2n} equipped with the involutory automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^2}$. Let Tr_{q^{2n}/q^2} be the trace function of $\mathbb{F}_{q^{2n}}$ over $\mathbb{F}_{q^2}$, it is easy to check that for all $x \in \mathbb{F}_{q^{2n}}$, $\mathrm{Tr}_{q^{2n}/q^2}(x)^q = \mathrm{Tr}_{q^{2n}/q^2}(x^q)$, and the map

$$S : (x, y) \in \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}} \to \mathrm{Tr}_{q^{2n}/q^2}(xy^q) \tag{2.3.1}$$

is a non-degenerate sesquilinear form of $\mathbb{F}_{q^{2n}}$ with accompanying automorphism $a \mapsto a^q$.
Again by Proposition 2.1.1, every such a sesquilinear form can be written in the following fashion:

$$S(f(x), y) = \mathrm{Tr}_{q^{2n}/q^2}(f(x)\, y^q),$$

where $f(x) \in \tilde{\mathscr{L}}_{n,q^2}[x]$ is a q^2-polynomial with coefficients in $\mathbb{F}_{q^{2n}}$ with $\deg_{q^2}(f) \leq n-1$.
Now, let $f(x) = \sum_{i=0}^{n-1} a_i x^{q^{2i}}$ be an element of $\tilde{\mathscr{L}}_{n,q^2}[x]$; it can be viewed as an element of $\mathrm{End}_{\mathbb{F}_{q^2}}(\mathbb{F}_{q^{2n}})$. It is easy to show that $S(f(x), y)^q = S(\tilde{f}(y), x)$ for all $x, y \in \mathbb{F}_{q^{2n}}$ where

$$\tilde{f}(x) = f^{\top q}(x^{q^2}) = \sum_{i=0}^{n-1} a_i^{q^{2n-2i+1}} x^{q^{2(n-i+1)}}. \tag{2.3.2}$$

Here $f^\top$ denotes the adjoint map of f as an $\mathbb{F}_{q^2}$-linear map, i.e.,

$$f^\top = \sum_{i=0}^{n-1} a_{n-i}^{q^{2i}} x^{q^{2i}}.$$

and $f^{\top q}(x)$ means taking the q-th power of each coefficients of $f^\top(x)$. It is straightforward to verify that $\tilde{(\cdot)}$ in (2.3.2) is involutionary on each $\mathbb{F}_{q^2}$-linear map. Note explicitly that

$$\tilde{f}(x) = x^q \circ f^\top \circ x^q \tag{2.3.3}$$

for every $f \in \tilde{\mathscr{L}}_{n,q^2}[x]$. Then, let $H(\cdot,\cdot)$ be a Hermitian form on $\mathbb{F}_{q^{2n}}$. By (2.1.4) and Proposition 2.1.1, we obtain

$$S(f(y),x) = H(y,x) = H(x,y)^q = S(f(x),y)^q = S(\tilde{f}(y),x)$$

for all $x, y \in \mathbb{F}_{q^{2n}}$.
Hence, we may identify the set of Hermitian forms defined on $\mathbb{F}_{q^{2n}}$ with the set of q^2-polynomials $f(x) \in \tilde{\mathscr{L}}_{n,q^2}[x]$ such that $\tilde{f}(x) = f^\top(x)$ for every $x \in \mathbb{F}_{q^{2n}}$ or equivalently as the set

$$H_n(q^2) = \left\{ \sum_{i=0}^{n-1} c_i x^{q^{2i}} \ : \ c_{n-i+1} = c_i^{q^{2n-2i+1}}, \quad i \in \{0,1,2,\ldots,n-1\} \right\}, \tag{2.3.4}$$

where the indices of the c_i's are taken modulo n. The set $H_n(q^2)$ is an n^2-dimensional $\mathbb{F}_q$-vector subspace of $\mathrm{End}_{\mathbb{F}_{q^2}}(\mathbb{F}_{q^{2n}})$.
We explicitly note that if $f(x) = \sum_{i=1}^{n-1} c_i x^{q^{2i}} \in H_n(q^2)$ with n odd, then $c_{(n+1)/2} \in \mathbb{F}_{q^n}$.

Now, still using the symbol $H_n(q^2)$ to indicate the set of Hermitian matrices of order n over $\mathbb{F}_{q^2}$, we recall the result about the isometries of this space.

Theorem 2.3.1 ([117],Theorem 6.4)**.** *Let $H_n(q^2)$ be the set of Hermitian matrices of order n with entries over $\mathbb{F}_{q^2}$, $n > 1$. Then a bijective map $\Theta : H_n(q^2) \to H_n(q^2)$ is an isometry with respect to the rank distance if and only if there exist $a \in \mathbb{F}_q^*$, $\rho \in \mathrm{Aut}(\mathbb{F}_{q^2})$, $P \in GL(n,q^2)$, and $H \in H_n(q^2)$ such that*

$$\Theta(X) = a\, P X^\rho \bar{P}^t + H \tag{2.3.5}$$

for all $X \in H_n(q^2)$.

In the following result, we shall show how the maps in (2.3.5) turn to be in the linearized polynomials setting.

Theorem 2.3.2. *Let $H_n(q^2)$ be the set of $\mathbb{F}_{q^2}$-Hermitian form on $\mathbb{F}_{q^{2n}}$, $n > 1$ in (2.3.4). Then a bijective map $\Theta : H_n(q^2) \to H_n(q^2)$ is an isometry with respect to the rank distance if and only if there exist $a \in \mathbb{F}_q^*$, $\rho \in \mathrm{Aut}(\mathbb{F}_{q^2})$, g a permutation q^2-polynomial over $\mathbb{F}_{q^{2n}}$, and $r_0 \in H_n(q^2)$ such that*

$$\Theta = \Theta_{a,g,\rho,r_0}(f) = a\, g \circ f^{\rho} \circ g^{\top q^{2n-1}}(x) + r_0(x), \tag{2.3.6}$$

Proof. First of all, since the set of permutation q^2-polynomials is isomorphic to the group $\mathrm{GL}(n, q^2)$, we note that the size of the set

$$\{\Theta_{a,g,\rho,r_0} : a \in \mathbb{F}_q^*, \rho \in \mathrm{Aut}(\mathbb{F}_{q^2}), r_0 \in H_n(q^2),\ g \in \tilde{\mathscr{L}}_{n,q^2}[x] \text{ with } \mathrm{rk}(g) = n\,\}$$

equals the size of the set of maps with the shape as in (2.3.5) and it is straightforward to see that these maps preseve the rank distance. So, to show the claim, it is enough to see that such a map fixes the polynomials in the set $H_n(q^2)$. Then, consider Θ_{a,g,ρ,r_0} as in (2.3.6) and let $f \in H_n(q^2)$. Since

$$H_n(q^2) = \{f(x) \in \tilde{\mathscr{L}}_{n,q^2}[x] : \tilde{f}(x) = f(x)\}$$

and by (2.3.2), we have that

$$\begin{aligned} x^q \circ \Big((a\, g \circ f^{\rho} \circ g^{\top q^{2n-1}})(x) + r_0(x)\Big)^{\top} \circ x^q = \\ a\, x^q \circ (g^{q^{2n-1}} \circ f^{\top\rho} \circ g^{\top}) \circ x^q + x^q \circ r_0^{\top} \circ x^q = \\ a\, x^q \circ (g^{q^{2n-1}} \circ x^{q^{2n-1}} \circ x^q \circ f^{\top\rho} \circ x^q \circ x^{q^{2n-1}} \circ g^{\top} \circ x^q) + \tilde{r}_0(x) = \\ a\, (x^q \circ g^{q^{2n-1}} \circ x^{q^{2n-1}}) \circ (x^q \circ f^{\top\rho} \circ x^q) \circ (x^{q^{2n-1}} \circ g^{\top} \circ x^q) + r_0(x) \end{aligned} \tag{2.3.7}$$

Then, since $x^q \circ g^{q^{2n-1}} \circ x^{q^{2n-1}} = g(x)$, $x^{q^{2n-1}} \circ g^{\top} \circ x^q$ is equivalent to take the q^{2n-1}-th power of the $g^{\top}$'s coefficients and $f(x) \in H_n(q^2)$, we obtain

$$a\,(g \circ \tilde{f}^{\rho} \circ g^{\top q^{2n-1}})(x) + r_0(x) = (a\, g \circ f^{\rho} \circ g^{\top q^{2n-1}})(x) + r_0(x). \tag{2.3.8}$$

□

So, as in Subsection 2.2, if for $\mathcal{C}_1$, and $\mathcal{C}_2 \in H_n(q^2)$, there exists a map Θ_{a,g,ρ,r_0} defined as in (2.3.6) for certain a, g, ρ and r_0 such that

$$\mathcal{C}_2 = \{\Theta_{a,g,\rho,r_0}(f) : f \in \mathcal{C}_1\},$$

then we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* in $H_n(q^2)$, and write $\mathcal{C}_1 \cong \mathcal{C}_2$.

Regarding upper bounds for codes in this context, in [107], Kai-Uwe Schimdt proved the following

Theorem 2.3.3 ([107], Theorem 1 and 2)**.** *Assume that $\mathcal{C}$ is a d-code in $H_n(q^2)$, then*

$$|\mathcal{C}| \leq \begin{cases} q^{n(n-d+1)}, & \text{for } d \text{ odd or } \mathcal{C} \text{ additive} \\ (-1)^{n+1} q^{n(n-d+1)} \frac{((-q)^{n-d+2}-1)+(-q)^n((-q)^{n-d+1}-1)}{(-q)^{n-d+2}-(-q)^{n-d+1}}, & \text{for } d \text{ even} \end{cases} \tag{2.3.9}$$

In [107], he also provided constructions of Hermitian additive d-codes that attain the first bound in (2.3.9) for all possible n and d, except if n and d are both even and $3 < d < n$. Now, we will recall these examples.
Let s be an odd integer coprime with n. The following two classes of $\mathbb{F}_q$-linear codes in $H_n(q^2)$, are presented only for $s = 1$. However, by Theorem 1.2.1 the case with $s \in \mathbb{Z}$, $s \neq 1$ follows.

Proposition 2.3.4 ([107], Theorem 4)**.** *Let n, d be integers with opposite parity such that $1 \leq d \leq n-1$. Then, the set*

$$\mathcal{H}_{n,d,s} = \left\{ \sum_{j=1}^{\frac{n-d+1}{2}} \left((b_j x)^{q^{2s(n-j+1)}} + b_j^{q^s} x^{q^{2sj}} \right) : \ b_1, b_2, \ldots, b_{\frac{n-d+1}{2}} \in \mathbb{F}_{q^{2n}} \right\}, \tag{2.3.10}$$

is a maximum $\mathbb{F}_q$-linear Hermitian d-code .

Proof. It is readily verified that the maps

$$H(x, y) = \mathrm{Tr}_{q^{2n}/q^2}(f(x)y^q)$$

with $f(x) \in \mathcal{H}_{n,d,s}$, are Hermitian forms and that the linearity of the trace function implies that the set in (2.3.10) is $\mathbb{F}_q$-linear. Therefore, it is enough to show that $H(\cdot, \cdot)$ has rank at least d unless $f(x)$ is null polynomial. Then, consider the $\mathbb{F}_q$-linear polynomial

$$f(x) = \sum_{j=1}^{\frac{n-d+1}{2}} \left((b_j x)^{q^{2s(n-j+1)}} + b_j^{q^s} x^{q^{2sj}} \right).$$

We have

$$f(x^{q^{s(n-d-1)}}) = \sum_{j=1}^{\frac{n-d+1}{2}} \left((b_j)^{q^{2s(n-j+1)}} x^{q^{s(n-d-2j+1)}} + b_j^{q^s} x^{q^{s(n-d+2j-1)}} \right).$$

This is a polynomial of degree at most $q^{2s(n-d)}$ and so has at most $q^{2(n-d)}$ zeros. Since the sesquilinear form in (2.3.1) is non-degenerate and the kernel of a nonzero $f(x) \in \mathcal{H}_{n,d,s}$ has dimension at most $n-d$ over $\mathbb{F}_{q^2}$, the set

$$\{x \in \mathbb{F}_{q^{2n}} : H(x, y) = 0 \text{ for all } y \in \mathbb{F}_{q^{2n}}\}$$

is an $(n-d)$-dimensional vector space over $\mathbb{F}_{q^2}$. Therefore, $f(x)$ has rank at least d unless $b_1 = \ldots = b_{(n-d+1)/2} = 0$, as required. □

Now, suppose that n and d are both odd integers such that $1 \leq d \leq n$ and s as above. By a similar proof to that of Theorem 2.3.4, the author showed that the set

$$\mathcal{E}_{n,d,s} = \left\{ (b_0\, x)^{q^{s(n+1)}} + \sum_{j=1}^{\frac{n-d}{2}} \left((b_j x)^{q^{s(n+2j+1)}} + b_j^{q^s}\, x^{q^{s(n-2j+1)}} \right) : \; b_0 \in \mathbb{F}_{q^n} \right. \\ \left. \text{and } b_1, \ldots, b_{\frac{n-d}{2}} \in \mathbb{F}_{q^{2n}} \right\} \tag{2.3.11}$$

turns to be a maximum $\mathbb{F}_q$-linear Hermitian d-code [107, Theorem 5].
The research for maximum codes in $H_n(q^2)$ with minimum rank distance $d = n$ is closely related to the problem of finding maximal *partial spread sets* in $H_n(q^2)$ see Subsection 3.2.1, [57], [113] and [114].
The sets in (2.3.10) and (2.3.11) are examples of maximum additive d-codes in $H_n(q^2)$ for every possible n and d except when both n and d are even. Constructions of maximum Hermitian additive d-codes can be obtained for $d = 2$ and for $d = n$, independently from whether n is even or odd. However, the existence and the construction of maximum additive d-codes in $H_n(q^2)$ when n and d are even integers satisfying $4 \leq d \leq n-2$ still remain an open problem. Finally, we have already seen that the bound for additive codes in Theorem 2.3.3 can be surpassed by non-additive codes whenever n is even and $d = n$, a direct construction is shown in [107, Theorem 6].
To conlude this section and since in what follows, different equivalence relations will be used for relevant codes depending on their ambient space, we collect them in the following table

Ambient	Symbol	Definition	Notation
$\tilde{\mathscr{L}}_{n,q}[x]$	$\mathcal{C}_1 \simeq \mathcal{C}_2$	$\mathcal{C}_2 = \{\Phi_{g_1,\rho,g_2,h}(f) : f \in \mathcal{C}_1\}$	g_1, g_2 are two permutation q-polynomials, $h(x) \in \tilde{\mathscr{L}}_{n,q}[x]$, $\rho \in \mathrm{Aut}(\mathbb{F}_q)$
$S_n(q)$, $A_n(q)$	$\mathcal{C}_1 \cong \mathcal{C}_2$	$\mathcal{C}_2 = \{\Psi_{a,g,\rho,r_0}(f) : f \in \mathcal{C}_1\}$	g is a permutation q-polynomial, $r_0 \in S_n(q)$ (resp. $r_0 \in A_n(q)$), $\rho \in \mathrm{Aut}(\mathbb{F}_q)$, $a \in \mathbb{F}_q$
$H_n(q^2)$	$\mathcal{C}_1 \cong \mathcal{C}_2$	$\mathcal{C}_2 = \{\Theta_{a,g,\rho,r_0}(f) : f \in \mathcal{C}_1\}$	g is a permutation q^2-polynomial, $r_0 \in H_n(q^2)$, $\rho \in \mathrm{Aut}(\mathbb{F}_{q^2})$, $a \in \mathbb{F}_q$

2.4 Automorphism groups of known restricted maximum additive codes

In this section, we will give a different description of the restricted d-codes introduced before. They will be obtained as the intersection of the ambient space in which they 'live' with a suitable code which is equivalent to a generalized Gabidulin code with minimum distance d. Later, we shall determine their automorphism group. As we have seen in Chapter 1, the computation of the automorphism group of these codes may serve to determine some algebraic invariants which are useful when facing with the equivalence issue, as described in [90] and [109]. Moreover, this problem has never been dealt with the known restricted maximum codes.

In order to do that, we recall that the symbol $X_n(q)$ denotes here one of the subspaces $S_n(q)$ or $A_n(q)$ of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. On the other hand the symbol $H_n(q^2)$ will be used to denote the n^2-dimensional $\mathbb{F}_q$-subspace of $\mathrm{End}_{\mathbb{F}_{q^2}}(\mathbb{F}_{q^{2n}})$ associated to the Hermitian forms defined on $\mathbb{F}_{q^{2n}}$, with accompanying automorphism $a \mapsto a^q$.
So, we start by giving an alternative description of such d-codes in terms of the intersection of their ambient space with suitable subspaces of $\tilde{\mathscr{L}}_{n,q}[x]$ or $\tilde{\mathscr{L}}_{n,q^2}[x]$, when dealing with the Hermitian setting. More precisely,

Proposition 2.4.1. *Let n, s and d be integers such that $1 \leq d \leq n$ and $\gcd(s,n) = 1$. Let $\mathcal{G} = \mathcal{G}_{n,n-d+1,s} \subset \tilde{\mathscr{L}}_{n,q}[x]$ be the generalized Gabidulin code with minimum distance d, then we have the following*

(i) $\mathcal{S}_{n,d,s} = \mathcal{G}' \cap S_n(q)$, *where* $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{n+d}{2})}}$,

(ii) $\mathcal{A}_{n,d,s} = \mathcal{G}' \cap A_n(q)$, *where* $\mathcal{G}' = \mathcal{G} \circ x^{q^{s\frac{d}{2}}}$.

Moreover, let $\mathcal{G} = \mathcal{G}_{n,n-d+1,s} \subset \tilde{\mathscr{L}}_{n,q^2}[x]$ be the generalized Gabidulin code with minimum distance d, then we have the following

(iii) $\mathcal{H}_{n,d,s} = \mathcal{G}' \cap H_n(q^2)$, *where* $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(n+d+1)}}$,

(iv) $\mathcal{E}_{n,d,s} = \mathcal{G}' \cap H_n(q^2)$, *where* $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(d+1)}}$.

Proof. Let $f(x) = \sum_{i=0}^{n-d} a_i x^{q^{si}}$ be an element of $\mathcal{G}_{n,n-d+1,s}$. Each element in $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(\frac{n+d}{2})}}$ has the following form:

$$\sum_{i=0}^{n-d} a_i x^{q^{s(\frac{n+d}{2}+i)}} = \sum_{i=0}^{\frac{n-d}{2}-1} a_i x^{q^{s(\frac{n+d}{2}+i)}} + \sum_{i=\frac{n-d}{2}}^{n-d} a_i x^{q^{s(\frac{n+d}{2}+i)}} =$$

$$\sum_{j=0}^{\frac{n-d}{2}} a_{j+\frac{n-d}{2}} x^{q^{sj}} + \sum_{j=\frac{n+d}{2}}^{n-1} a_{j-\frac{n+d}{2}} x^{q^{sj}} =$$

$$a_{\frac{n-d}{2}} x + \sum_{j=1}^{\frac{n-d}{2}} \Big(a_{\frac{n-d}{2}+j} x^{q^{sj}} + a_{\frac{n-d}{2}-j} x^{q^{s(n-j)}} \Big).$$

It is clear that $\mathcal{G}'^{\top} = \mathcal{G}'$, and by intersecting $\mathcal{G}'$ with $S_n(q)$, we get the following conditions

$$a_{\frac{n-d}{2}-i} = a_{\frac{n-d}{2}+i}^{q^{s(n-i)}}, \qquad i = 1, 2, \ldots, \frac{n-d}{2}.$$

Hence, each element in $\mathcal{G}' \cap S_n(q)$ has the following shape:

$$a_{\frac{n-d}{2}} x + \sum_{i=1}^{\frac{n-d}{2}} \Big(a_{\frac{n-d}{2}+i} x^{q^{si}} + (a_{\frac{n-d}{2}+i} x)^{q^{s(n-i)}} \Big),$$

this proves (i).
Let be $f(x) = \sum_{i=0}^{n-d} a_i x^{q^{si}}$ an element of $\mathcal{G}_{n,n-d+1,s}$. Suppose $d = 2e$ and we compose $f(x)$ on the right with the monomial $x^{q^{se}}$, we obtain

$$\sum_{i=0}^{n-d} a_i x^{q^{s(i+e)}} = \sum_{i=e}^{n-e} a_{i-e}\, x^{q^{si}} =$$

$$\sum_{i=e}^{\frac{n-1}{2}} a_{i-e}\, x^{q^{si}} + \sum_{i=\frac{n+1}{2}}^{n-e} a_{i-e}\, x^{q^{si}} =$$

$$\sum_{i=e}^{\frac{n-1}{2}} \Big(a_{i-e}\, x^{q^{si}} + a_{n-e-i}\, x^{q^{s(n-i)}} \Big).$$

Let $\mathcal{G}' = \{ f(x^{q^{se}}) : f(x) \in \mathcal{G}_{n,n-d+1,s} \}$, by intersecting $\mathcal{G}'$ with $A_n(q)$ the statement in (ii) follows.
Regarding (iii) and (iv), let $f(x) = \sum_{i=0}^{n-d} a_i^{q^s} x^{q^{2si}}$ be any element in $\mathcal{G}_{n,n-d+1,s} \subset \tilde{\mathscr{L}}_{n,q^2}[x]$. Composing $f(x)$ on the right with the monomial $x^{q^{s(n+d+1)}}$, we obtain

$$a_0^{q^s} x^{q^{s(n+d+1)}} + \sum_{i=1}^{\frac{n-d-1}{2}} a_i^{q^s} x^{q^{2s\left(\frac{n+d+1}{2}+i\right)}} + \sum_{i=\frac{n-d+1}{2}}^{n-d} a_i^{q^s} x^{q^{2s\left(\frac{n+d+1}{2}+i\right)}} =$$

$$\sum_{j=0}^{\frac{n-d+1}{2}} a_{j+\frac{n-d-1}{2}}^{q^s} x^{q^{2sj}} + \sum_{j=\frac{n+d+1}{2}}^{n-1} a_{j-\frac{n+d+1}{2}}^{q^s} x^{q^{2sj}} =$$

$$\sum_{j=1}^{\frac{n-d+1}{2}} \left(a_{\frac{n-d+1}{2}-j}^{q^s} x^{q^{s(2n-2j+2)}} + a_{\frac{n-d-1}{2}+j}^{q^s} x^{q^{2sj}} \right).$$

By intersecting $\mathcal{G}'$ with $H_n(q^2)$, we get the following conditions

$$c_j^{q^{s(2n-2j+1)}} = a_{\frac{n-d-1}{2}+j}^{q^{s(2n-2j+2)}} = c_{n-j+1} = a_{\frac{n-d+1}{2}-j}^{q^s}, \text{ for } j = 1, 2, \ldots, \frac{n-d+1}{2}.$$

Hence,

$$\mathcal{G}' \cap H_n(q^2) = \mathcal{H}_{n,d,s}.$$

In a similar way, by composing an element $f(x) \in \mathcal{G}_{n,n-d+1,s}$ with $x \mapsto x^{q^{s(d+1)}}$, we obtain

$$\sum_{i=0}^{n-d} a_i^{q^s} x^{q^{2s\left(\frac{d+1}{2}+i\right)}} = a_{\frac{n-d}{2}}^{q^s} x^{q^{s(n+1)}} + \sum_{i=0}^{\frac{n-d}{2}-1} a_i^{q^s} x^{q^{s(2i+d+1)}} + \sum_{i=\frac{n-d}{2}+1}^{n-d} a_i^{q^s} x^{q^{s(2i+d+1)}} =$$

$$a_{\frac{n-d}{2}}^{q^s} x^{q^{s(n+1)}} + \sum_{i=1}^{\frac{n-d}{2}} a_{i-1}^{q^s} x^{q^{s(2i+d-1)}} + \sum_{j=\frac{n-d}{2}+1}^{n-d} a_j^{q^s} x^{q^{s(2j+d+1)}}.$$

Setting $i = \frac{n-d}{2} - \ell + 1$ and $j = \frac{n-d}{2} + m$, we have

$$a_{\frac{n-d}{2}}^{q^s} x^{q^{s(n+1)}} + \sum_{\ell=1}^{\frac{n-d}{2}} a_{\frac{n-d}{2}-\ell}^{q^s} x^{q^{s(n-2\ell+1)}} + \sum_{m=1}^{\frac{n-d}{2}} a_{\frac{n-d}{2}+m}^{q^s} x^{q^{s(n+2m+1)}} =$$

$$a_{\frac{n-d}{2}}^{q^s} x^{q^{s(n+1)}} + \sum_{j=1}^{\frac{n-d}{2}} \left(a_{\frac{n-d}{2}-j}^{q^s} x^{q^{n-2j+1}} + a_{\frac{n-d}{2}+j}^{q^s} x^{q^{s(n+2j+1)}} \right).$$

Again by intersecting $\mathcal{G}'$ with the Hermitian space $H_n(q^2)$, we get:

$$\begin{cases} a_{\frac{n-d}{2}}^{q^s} \in \mathbb{F}_{q^n} \\ \\ c_{\frac{n+1}{2}+j}^{q^{s(2n-2j)}} = a_{\frac{n-d}{2}+j}^{q^{s(2n-2j+1)}} = c_{\frac{n+1}{2}-j} = a_{\frac{n-d}{2}-j}^{q^s}, \end{cases}$$

which finally gives the result. □

Regarding the punctured set obtained from $\mathcal{S}_{n+1,d+2,s}$, we can consider $\mathbb{F}_{q^{n+1}} = \mathbb{W} \oplus \mathbb{K}$, where $\mathbb{K} = \langle \eta \rangle_q$ with $\eta \in \mathbb{F}_{q^{n+1}}^*$ and $\mathbb{W}$ is an n-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^{n+1}}$.
Let s be a positive integer coprime with $n+1$, let $1 \leq d \leq n-1$ such that $n-d$ is odd, and consider the $\mathbb{F}_q$-vector space $\mathcal{U}'_\eta$ of $\mathcal{G}' = \mathcal{G}_{n+1,n-d+2,s} \circ x^{q^{s\left(\frac{n+d+1}{2}\right)}}$ defined as follows

$$\mathcal{U}'_\eta = \left\{ \sum_{i=1}^{\frac{n-d+1}{2}} \left(c_i (x^{q^{si}} - x\eta^{q^{si}-1}) + c_{n+1-i} (x^{q^{s(n+1-i)}} - x\eta^{q^{s(n+1-i)}-1}) \right) \right. \\ \left. : c_i, c_{n+1-i} \in \mathbb{F}_{q^{n+1}}, \ i \in \left\{ 1, 2, \ldots, \frac{n-d+1}{2} \right\} \right\}. \tag{2.4.1}$$

We notice that $\mathcal{U}'_\eta$ has dimension $(n+1)(n-d+1)$, and it is made up of all maps $f \in \mathcal{G}'$ such that $\mathbb{K} \subset \mathrm{Ker} f$. Let

$$S_{n+1}(q) \cap \mathcal{U}'_\eta = \Bigg\{ \sum_{i=1}^{\frac{n-d+1}{2}} \Big(b_i (x^{q^{si}} - x\eta^{q^{si}-1}) + b_i^{q^{s(n+1-i)}} (x^{q^{s(n+1-i)}} - x\eta^{q^{s(n+1-i)}-1}) \Big) \ : \ b_1, \ldots, b_{\frac{n-d+1}{2}} \in \mathbb{F}_{q^{n+1}} \Bigg\}.$$

Clearly each polynomial f in this set has at most q^{n-d+1} roots in $\mathbb{F}_{q^{n+1}}$. Furthermore, since f is a linearized polynomial, we can write $f(x+u) = f(x)+f(u)$ for all $x, u \in \mathbb{F}_{q^{n+1}}$. But $\mathbb{K} \subset \mathrm{Ker} f$ which implies that, if $f(x) = 0$, then $f(x+u) = 0$ for all $u \in \mathbb{K}$. For each $x \in \mathbb{W}$ and each $u \in \mathbb{K}^*$, we have $x + u \notin \mathbb{V}$, so the number of roots of the polynomial f in $\mathbb{W}$ is at most q^{n-d}, i.e.

$$\dim(\mathrm{Ker}\, f \cap \mathbb{W}) \leq n - d.$$

Hence, for each $f \in S_{n+1}(q) \cap \mathcal{U}'_\eta$, the rank of the symmetric bilinear form on $\mathbb{W}$

$$B_{f|_{\mathbb{W}}} : (x, y) \in \mathbb{W} \times \mathbb{W} \to \mathrm{Tr}_{q^n/q}(f(x)y)$$

is at least d and the set

$$\mathcal{T}_{n,d,s}(\eta) = (\mathcal{S}_{n+1,d,s} \cap \mathcal{U}'_\eta)_{|\mathbb{W}} = \left\{ B_{f|_{\mathbb{W}}} \ : \ f \in S_{n+1}(q) \cap \mathcal{U}'_\eta \right\}$$

is a symmetric $\mathbb{F}_q$-linear maximum d-code of size $q^{(n+1)\frac{n-d+1}{2}}$.

By Proposition 2.4.1 (i), we have the following.

Corollary 2.4.2. *Let $(n+1, s) = 1$, and $1 \leq d \leq n-1$. Let $\eta \in \mathbb{F}^*_{q^{n+1}}$ and let $\mathbb{W}$ be an n-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^{n+1}}$ such that $\mathbb{F}_{q^{n+1}} = \mathbb{W} \oplus \langle \eta \rangle_q$. Then the d-code*

$$\mathcal{T}_{n,d,s}(\eta) = (\mathcal{U}'_\eta \cap S_{n+1}(q))_{|\mathbb{W}}, \tag{2.4.2}$$

is maximum, where $\mathcal{U}'_\eta$ is the $\mathbb{F}_q$-subspace in (2.4.1).

Clearly, if η_1 and η_2 are linearly dependent over $\mathbb{F}_q$, then $\mathcal{T}_{n,d,s}(\eta_1) = \mathcal{T}_{n,d,s}(\eta_2)$. Furthermore, we notice that $\mathcal{U}'_\eta \cap S_{n+1}(q) \subset \mathcal{S}_{n+1,d,s}$, while

$$\mathcal{T}_{n,d,s}(\eta) = (\mathcal{U}'_\eta \cap S_{n+1}(q))_{|\mathbb{W}} = (\mathcal{S}_{n+1,d+2,s})_{|\mathbb{W}}. \tag{2.4.3}$$

In the remaing part of this section we prove that the subspaces $\mathcal{G}' \subset \tilde{\mathscr{L}}_{n,q}[x]$ ($\mathcal{G}' \subset \tilde{\mathscr{L}}_{n,q^2}[x]$) defined in Proposition 2.4.1, are the unique elements in $[\mathcal{G}_{n,n-d+1,s}]_\simeq$ satisfying the properties (i) and (ii) ((iii) and (iv)) of Proposition 2.4.1. More precisely, we have the following

Theorem 2.4.3. *Let n, s and d be integers such that $d \geq 1$ and $\gcd(s, n) = 1$.*

(*i*) *Let* $W \subset \tilde{\mathscr{L}}_{n,q}[x]$ *be an* $(n-d+1)n$*-dimensional subspace of* $\tilde{\mathscr{L}}_{n,q}[x]$ *such that* $W \in [\mathcal{G}_{n,n-d+1,s}]_{\simeq}$*, and* $W \cap S_n(q) = \mathcal{S}_{n,d,s}$ *(respectively,* $W \cap A_n(q) = \mathcal{A}_{n,d,s}$*).*

Then $W = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s\frac{n+d}{2}}}$ *(respectively,* $W = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s\frac{d}{2}}}$*).*

(*ii*) *Let* $W \subset \tilde{\mathscr{L}}_{n,q^2}[x]$ *be an* $(n-d+1)n$*-dimensional* $\mathbb{F}_{q^2}$*-subspace such that* $W \in [\mathcal{G}_{n,n-d+1,s}]_{\simeq}$ *and* $W \cap H_n(q^2) = \mathcal{H}_{n,d,s}$ *(respectively,* $W \cap H_n(q^2) = \mathcal{E}_{n,d,s}$*).*

Then, $W = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(n+d+1)}}$ *(respectively,* $W = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(d+1)}}$*).*

Proof. (*i*) Since W is equivalent to $\mathcal{G}_{n,n-d+1,s}$, there exists a rank-preserving map $\Phi_{g,\rho,h}$ such that

$$\Phi_{g,\rho,h}(\mathcal{G}_{n,n-d+1,s}) = W.$$

As $\mathcal{G}^{\rho}_{n,n-d+1,s} = \mathcal{G}_{n,n-d+1,s}$ for all $\rho \in \mathrm{Aut}(\mathbb{F}_q)$, we may assume that ρ is the identity. Hence, the elements of W are

$$g \circ \left(\sum_{j=0}^{n-d} \alpha_j x^{q^{sj}} \right) \circ h = \sum_{j=0}^{n-d} (g \circ \alpha_j x^{q^{sj}} \circ h) = \sum_{j=0}^{n-d} \left(\sum_{m=0}^{n-1} c_{m,j}(\alpha_j) x^{q^{sm}} \right) =$$

$$\sum_{m=0}^{n-1} \left(\sum_{j=0}^{n-d} c_{m,j}(\alpha_j) \right) x^{q^{sm}},$$

with $\alpha_j \in \mathbb{F}_{q^n}$ for all $j \in J = \{0, 1, \ldots, n-d\}$ and

$$c_{m,j}(\alpha_j) = \sum_{i=0}^{n-1} g_i h^{q^{si}}_{m-i-j} \alpha_j^{q^{si}}.$$

The indices here are taken modulo n.
Suppose that

$$W \cap S_n(q) = \mathcal{S}_{n,d,s}.$$

By (2.2.1) and (2.2.5), we have that $L_m(\underline{\alpha}) = \sum_{j=0}^{n-d} c_{m,j}(\alpha_j)$ is equal to zero for each $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-d})$, $m \in M = \{\frac{n-d}{2}+1, \frac{n-d}{2}+2, \ldots, n-(\frac{n-d}{2}+1)\}$. In particular $L_m(\underline{\alpha}) = 0$ when $\underline{\alpha} = (0, \ldots, 0, \alpha_j, 0, \ldots, 0)$, with $\alpha_j \in \mathbb{F}_{q^n}$, $m \in M$ and $j \in J$. Then

$$c_{m,j}(\alpha) = 0 \qquad \text{for all } \alpha \in \mathbb{F}_{q^n} \text{ and } m \in M, j \in J.$$

Hence, we obtain the following conditions:

$$\begin{cases} g_i h^{q^{si}}_{m-i-j} = 0 \\ i \in I := \{0, 1, ..., n-1\},\ j \in J,\ m \in M. \end{cases} \tag{2.4.4}$$

As g is an invertible q-polynomial, there exists at least an integer $i_0 \in I$ such that $g_{i_0} \neq 0$. It is straightforward to verify that

$$\left\{ m - j + \frac{n-d}{2} \ : \ j \in J \ \text{ and } \ m \in M \right\} = \{1, 2, ..., n-1\}\,.$$

Hence, we get that for each given $i \in I$, by letting j varying in J and $m \in M$, integers $m - i - j$ equal, modulo n, all elements in I with the only exception of $\frac{n+d}{2} - i$. By inspecting the equations in System (2.4.4), this easily implies that there exists a unique index i_0 between 0 and $n-1$, such that $g_{i_0} \neq 0$ and $h_{\frac{n+d}{2}-i_0} \neq 0$, and all others g_i and h_i are zero.

Hence, $g(x) = \gamma x^{q^{si_0}}$ and $h(x) = \delta x^{q^{s(\frac{n+d}{2}-i_0)}}$ with $\gamma, \delta \in \mathbb{F}_{q^n}$.
On the other hand if

$$W \cap A_n(q) = \mathcal{A}_{n,d,s},$$

by (2.2.7) and taking into account (2.2.8), we may conclude that $L_m(\underline{\alpha})$ is equal to zero for each $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-d}) \in \mathbb{F}_{q^n}^{n-d+1}$, $m \in M = M_1 \cup M_2 = \{0, 1, \ldots, \frac{d}{2} - 1\} \cup \{n - (\frac{d}{2} - 1), \ldots, n-1\}$. In particular we have,

$$c_{m,j}(\alpha) = 0 \qquad \text{for all } \ \alpha \in \mathbb{F}_{q^n} \text{ and } \ m \in M, j \in J.$$

Hence, we obtain a similar set of conditions as in System (2.4.4). Also in this case, there exists $i_0 \in I$ such that $g_{i_0} \neq 0$. Again, one easily verifies that

$$\left\{ m - j - \frac{d}{2} \ : \ j \in J \ \text{ and } \ m \in M_1 \cup M_2 \right\} = \{1, 2, ..., n-1\}.$$

Also, for each given $i \in I$, by letting j varying in J and $m \in M$, integers $m - i - j$ equal, modulo n, elements of I except $\frac{d}{2} - i$. Discussing as in the previous part, this leads to prove that there exists a unique index i_0 between 0 and $n-1$, such that $g_{i_0} \neq 0$ and $h_{\frac{d}{2}-i_0} \neq 0$, and all others g_i and h_i are zero.

Hence we have that $g(x) = \gamma x^{q^{si_0}}$ and $h(x) = \delta x^{q^{s(\frac{d}{2}-i_0)}}$ with $\gamma, \delta \in \mathbb{F}_{q^n}^*$. This concludes the proof.

(*ii*) It is similar to that of point (*i*). For this reason here we omit the computations. □

As a direct consequence of Theorems 2.4.3 and 1.5.3, we may state the following result.

Corollary 2.4.4. *Let d and s be integers such that $1 < d < n$ and $\gcd(n, s) = 1$. Let $\mathcal{C} \in X_n(q)$ be a d-code.*

(i) If either $\mathcal{C} = \mathcal{S}_{n,d,s}$ or $\mathcal{C} = \mathcal{A}_{n,d,s}$, then we have

$$\mathrm{Aut}(\mathcal{C}) = \left\{ \Psi_{a,\gamma x^{q^r},id} \ : \ a \in \mathbb{F}_q^*,\ \gamma \in \mathbb{F}_{q^n}^*,\ r \in \{0, ..., n-1\} \right\}.$$

(ii) If $\mathcal{C} \in H_n(q^2)$ *and either* $\mathcal{C} = \mathcal{H}_{n,d,s}$ *or* $\mathcal{C} = \mathcal{E}_{n,d,s}$*, then we have*

$$\mathrm{Aut}(\mathcal{C}) = \left\{\Theta_{a,\gamma^q x^{q^{2r}},id} : \ a \in \mathbb{F}_q^*,\ \gamma \in \mathbb{F}_{q^{2n}}^*,\ r \in \{0,...,n-1\}\right\}.$$

Proof. (i) Let $\mathcal{G} = \mathcal{G}_{n,n-d+1,s}$. We first observe that $\mathrm{Aut}(\mathcal{G}') = \mathrm{Aut}(\mathcal{G}_{n,n-d+1,s})$, whenever $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{n+d}{2})}}$ or $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{d}{2})}}$. Nonetheless, in [109] it was proven that if $0 \leq r \leq n-1$, then

$$\mathrm{Aut}(\mathcal{G}_{n,n-d+1,s}) = \left\{\Phi_{\alpha x^{q^r},id,\beta x^{q^{n-r}}} \mid \alpha, \beta \in \mathbb{F}_{q^n}^*\right\}.$$

Now, assume that either $\mathcal{C} = \mathcal{S}_{n,d,s}$ or $\mathcal{C} = \mathcal{A}_{n,d,s}$.
Since each element in the set

$$A = \{\Phi_{a\gamma x^{q^r},id,\gamma^{q^{n-r}} x^{q^{n-r}}} \mid \ a \in \mathbb{F}_q^*,\ \gamma \in \mathbb{F}_{q^n}^*\}, \tag{2.4.5}$$

fixes both $S_n(q)$ and $A_n(q)$; we get, as a consequence of Proposition 2.4.1, that A is a subgroup of $\mathrm{Aut}(\mathcal{C})$. Conversely, let $\Phi \in \mathrm{Aut}(\mathcal{C})$. Of course, by points (i) and (ii) of Proposition (2.4.1), we get

$$\Phi(\mathcal{G}') \cap \Phi(X_n) = \mathcal{C},$$

whenever $X_n = S_n(q)$ and $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{n+d}{2})}}$, or $X_n(q) = A_n(q)$ and $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{d}{2})}}$, respectively. This also means that $\mathcal{D} = \Phi(\mathcal{G}') \cap X_n(q) \supseteq \mathcal{C}$.
Now, assume that $\mathcal{D} \supset \mathcal{C}$. Then $\mathcal{D}$ would be a d-code in $X_n(q)$. If $X_n(q) = S_n(q)$ then $|\mathcal{D}| > q^{n(n-d+2)/2}$, while if $X_n(q) = A_n(q)$ then $|\mathcal{D}| > q^{n(n-d+1)/2}$. By Theorems 2.2.1 and 2.2.5 this is clearly not possible. Hence,

$$\Phi(\mathcal{G}') \cap X_n(q) = \mathcal{C}. \tag{2.4.6}$$

However, Equation (2.4.6) contradicts Theorem 2.4.3, unless we have $\Phi(\mathcal{G}') = \mathcal{G}'$, which implies that Φ is an element of A. This conclude the proof of point (i).

(ii) Assume now that $\mathcal{C} \subset H_n(q^2)$ is either $\mathcal{H}_{n,d,s}$ or $\mathcal{E}_{n,d,s}$. Again, it is trivial to see that, in both cases, each element in the set

$$A = \{\Phi_{a\gamma^q x^{q^{2r}},id,\gamma^{q^{2n-2r+1}} x^{q^{2n-2r}}} : \ a \in \mathbb{F}_q^*,\ \gamma \in \mathbb{F}_{q^{2n}}^*\},$$

fixes $\mathcal{C}$. Moreover, an easy computation also shows that if either $\mathcal{C} = \mathcal{H}_{n,d,s}$ and $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(n+d+1)}}$, or $\mathcal{C} = \mathcal{E}_{n,d,s}$ and $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(d+1)}}$; we have

$$\mathrm{Aut}(\mathcal{G}') \cap \mathrm{Aut}(\mathcal{C}) = A. \tag{2.4.7}$$

Now, let $\Phi = \Phi_{f,\rho,g}$ where f and g are two invertible q^2-polynomials in $\tilde{\mathscr{L}}_{n,q^2}[x]$, and $\rho \in \mathrm{Aut}(\mathbb{F}_{q^2})$, be an element of $\mathrm{Aut}(\mathcal{C})$, and suppose that Φ does not belong to A. Then by (2.4.7), $\Phi(\mathcal{G}') \neq \mathcal{G}'$ and this leads again to a contradiction by Theorem 2.4.3. □

We end this section by proving the following result about the equivalence of codes.

Theorem 2.4.5. *Let $d \geq 1$. Two maximum d-codes $\mathcal{S}_{n,d,s}$ and $\mathcal{S}_{n,d,s'}$ (respectively, $\mathcal{A}_{n,d,s}$ and $\mathcal{A}_{n,d,s'}$), where s and s' are integers satisfying $\gcd(s,n) = \gcd(s',n) = 1$, or, two maximal d-codes $\mathcal{H}_{n,d,s}$ and $\mathcal{H}_{n,d,s'}$ (respectively, $\mathcal{E}_{n,d,s}$ and $\mathcal{E}_{n,d,s'}$), where s and s' are integers satisfying $\gcd(s,2n) = \gcd(s',2n) = 1$, are equivalent if and only if $s \equiv \pm s' \pmod{n}$.*

Proof. We give the proof in symmetric and alternating setting only. Similar arguments lead to the result for the two known constructions in the Hermitian setting. For this reason, we omit the details here.
Suppose that $s \equiv \pm s' \pmod{n}$. Let $\mathcal{G}'_s = \mathcal{G}_s \circ x^{q^{s(\frac{n+d}{2})}}$ and $\mathcal{G}'_{s'} = \mathcal{G}_{s'} \circ x^{q^{s'(\frac{n+d}{2})}}$ (respectively, $\mathcal{G}'_s = \mathcal{G}_s \circ x^{q^{s(\frac{d}{2})}}$ and $\mathcal{G}'_{s'} = \mathcal{G}_{s'} \circ x^{q^{s'(\frac{d}{2})}}$).
By Proposition 2.4.1 points (i) and (ii),

$$\mathcal{S}_{n,d,s} = \mathcal{G}'_s \cap S_n(q) \text{ and } \mathcal{S}_{n,d,s'} = \mathcal{G}'_{s'} \cap S_n(q),$$

(respectively, $\mathcal{A}_{n,d,s} = \mathcal{G}'_s \cap A_n(q)$ and $\mathcal{A}_{n,d,s'} = \mathcal{G}'_{s'} \cap A_n(q)$).
Since $s \equiv \pm s' \pmod{n}$, by [90, Theorem 4.4 and 4.8, (a)], we have that

$$\mathcal{G}_{s'} = \Phi_{ux^{q^r}, id, vx^{q^{n-r}}}(\mathcal{G}_s) = ux^{q^r} \circ \mathcal{G}_s \circ vx^{q^{n-r}},$$

for two given elements $u, v \in \mathbb{F}_{q^n}$. Hence,

$$\mathcal{G}'_{s'} = (ux^{q^r} \circ \mathcal{G}_s \circ vx^{q^{n-r}}) \circ x^{q^{(\pm s+kn)(\frac{n+d}{2})}},$$

(respectively, $\mathcal{G}'_{s'} = (ux^{q^r} \circ \mathcal{G}_s \circ vx^{q^{n-r}}) \circ x^{q^{(\pm s+kn)(\frac{d}{2})}}$).
If $s' \equiv s \pmod{n}$, from equation above we get

$$\mathcal{G}'_{s'} = ux^{q^r} \circ \mathcal{G}'_s \circ v^{q^{s\frac{n-d}{2}}} x^{q^{n-r}}, \tag{2.4.8}$$

(respectively, $\mathcal{G}'_{s'} = ux^{q^r} \circ \mathcal{G}'_s \circ v^{q^{-s(\frac{d}{2})}} x^{q^{n-r}}$).
If otherwise $s' \equiv -s \pmod{n}$, we have

$$\mathcal{G}'_{s'} = ux^{q^r} \circ (\mathcal{G}_s \circ x^{q^{-s(\frac{n+d}{2})}}) \circ v^{q^{s\frac{n+d}{2}}} x^{n-r}, \tag{2.4.9}$$

(respectively, $\mathcal{G}'_{s'} = ux^{q^r} \circ (\mathcal{G}_s \circ x^{q^{-s(\frac{d}{2})}}) \circ v^{q^{s(\frac{d}{2})}} x^{q^{n-r}}$).
Since $\mathcal{G}'^{\top}_{s'} = \mathcal{G}'_{s'}$, by comparing coefficients in Equation (2.4.8) we get that it must necessarily be $u = av'$ with $a \in \mathbb{F}_q^*$ and $v' = v^{q^{s\frac{n-d}{2}}}$ (respectively, $u = av'$ with $a \in \mathbb{F}_q^*$ and $v' = v^{q^{-s(\frac{d}{2})}}$).

In a similar way, by comparing coefficients in Equation (2.4.9), we find $u = av'$ with $a \in \mathbb{F}_q^*$, where $v' = v^{q^{s\frac{n+d}{2}}}$ (respectively, $u = av'$ with $a \in \mathbb{F}_q^*$ and $v' = v^{q^{-s(\frac{d}{2})}}$).
Hence,

$$\Phi_{av'x^{q^r},id,v'x^{q^{n-r}}}(\mathcal{S}_{n,d,s}) = \Psi_{a,v'x^{q^r}}(\mathcal{S}_{n,d,s}) = \mathcal{S}_{n,d,s'},$$

(respectively, $\Phi_{av'x,\rho,v'x}(\mathcal{A}_{n,d,s}) = \mathcal{A}_{n,d,s'}$), where $s' \equiv \pm s \pmod n$.
Conversely, suppose that $\mathcal{S}_{n,d,s}$ and $\mathcal{S}_{n,d,s'}$ (respectively, $\mathcal{A}_{n,d,s}$ and $\mathcal{A}_{n,d,s'}$) are equivalent. Denote by $\Psi = \Psi_{a,g,\rho}$ the map such that $\Psi(\mathcal{S}_{n,d,s}) = \mathcal{S}_{n,d,s'}$ (respectively, $\Psi(\mathcal{A}_{n,d,s}) = \mathcal{A}_{n,d,s'}$).
As $\gcd(s,n) = \gcd(s',n) = 1$, we may assume that $s' \equiv es \pmod n$. In the remaining part of the proof we will write down the computations only in the symmetric context. Similar arguments can be applied in the alternating case leading to the same achievement.
Each element $f \in \mathcal{S}_{n,d,s}$ has the following shape:

$$f(x) = b_0 x + \sum_{i=1}^{\frac{n-d}{2}} \left(b_i x^{q^{si}} + (b_i x)^{q^{s(n-i)}} \right).$$

Let $g = \sum_{i=0}^{n-1} a_i x^{q^{si}} \in \mathbb{F}_{q^n}[x]$.
Discussing as in the proof of Theorem 2.4.3 we have that each element in $\Psi(\mathcal{S}_{n,d,s})$ can be written as follows

$$\sum_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} \left(b_0^{q^{s(n-i)}} a_i^{q^{s(n-i)}} a_{k+i}^{q^{s(n-i)}} + \sum_{r=1}^{\frac{n-d}{2}} \left(b_r^{q^{s(n-i-r)}} a_i^{q^{s(n-i)}} a_{k+i+r}^{q^{s(n-i-r)}} + b_r^{q^{s(n-i)}} a_i^{q^{s(n-i)}} a_{k+i-r}^{q^{s(r-i)}} \right) \right) \right) x^{q^{sk}}. \tag{2.4.10}$$

By comparing the coefficients of the term $x^{q^{ks}}$ in $\Psi(\mathcal{S}_{n,d,s})$ and in $\mathcal{S}_{n,d,s'}$ we get

$$\sum_{i=0}^{n-1} \left(b_0^{q^{s(n-i)}} a_i^{q^{s(n-i)}} a_{k+i}^{q^{s(n-i)}} + \sum_{r=1}^{\frac{n-d}{2}} \left(b_r^{q^{s(n-i-r)}} a_i^{q^{s(n-i)}} a_{k+i+r}^{q^{s(n-i-r)}} + b_r^{q^{s(n-i)}} a_i^{q^{s(n-i)}} a_{k+i-r}^{q^{s(r-i)}} \right) \right) = 0, \tag{2.4.11}$$

for each $k \in \{je : \frac{n-d}{2} < j < \frac{n+d}{2}\}$ and all $\lambda = (b_0, \cdots, b_{\frac{n-d}{2}}) \in \mathbb{F}_{q^n}^{\frac{n-d}{2}+1}$.
By taking $b_0 \neq 0$ and $b_j = 0$ for $j \neq 0$, from above Equation (2.4.11) we have

$$a_i a_{k+i} = 0 \tag{2.4.12}$$

for $i = 0, 1, \ldots, n-1$. Similarly, for each $r \in \{1, \ldots, \frac{n-d}{2}\}$, letting b_r be the unique nonzero elements among all b_j, from (2.4.11) we can derive

$$\sum_{i=0}^{n-1} \left(a_{i-r}^{q^{s(r-i)}} a_{k+i}^{q^{s(n-i)}} + a_i^{q^{s(n-i)}} a_{k+i-r}^{q^{s(r-i)}} \right) b_r^{q^{s(n-i)}} = 0.$$

As the above equation holds for any $b_r \in \mathbb{F}_{q^n}$, it implies

$$a_{i-r}^{q^{s(r-i)}} a_{k+i}^{q^{s(n-i)}} + a_i^{q^{s(n-i)}} a_{k+i-r}^{q^{s(r-i)}} = 0$$

for every i, which means

$$a_{i-r}^{q^{sr}} a_{k+i} + a_i a_{k+i-r}^{q^{sr}} = 0. \tag{2.4.13}$$

Since g is a permutation q-polynomial, there must be at least one coefficient a_i, $i \in \{0, \ldots, n-1\}$ which is different from zero. Let a_{i_0} denote a non-zero coefficient.
By letting $i = i_0$ in (2.4.12), we get

$$a_{je+i_0} = 0 \text{ for } \frac{n-d}{2} < j < \frac{n+d}{2}.$$

By taking $i = i_0$ and $i = r + i_0$ in (2.4.13) respectively, together with the equation above, we can derive

$$a_{i_0+je-r} = a_{i_0+je+r} = 0 \text{ for } \frac{n-d}{2} < j < \frac{n+d}{2} \text{ and } 1 \le r \le \frac{n-d}{2}.$$

Hence,

$$a_{je+i+i_0} = 0 \text{ for } \frac{n-d}{2} < j < \frac{n+d}{2} \text{ and } -\frac{n-d}{2} \le i \le \frac{n-d}{2}.$$

As $a_{i_0} \neq 0$, the equation

$$je + i \equiv 0 \pmod{n}$$

should have no solution for $\frac{n-d}{2} < j < \frac{n+d}{2}$ and $-\frac{n-d}{2} \le i \le \frac{n-d}{2}$. As there are $d-1$ elements in $\{je \pmod{n} : \frac{n-d}{2} < j < \frac{n+d}{2}\}$ and $n-d+1$ elements in $\{i : -\frac{n-d}{2} \le i \le \frac{n-d}{2}\}$, $a_{je+i+i_0} = 0$ implies all $a_j = 0$ for $j \neq i_0$.
Thus $g(x) = a_{i_0} x^{q^{i_0}}$. However, if $e \not\equiv \pm 1 \pmod{n}$, i.e. $s \not\equiv \pm s' \pmod{n}$, by Corollary 2.4.4. it is obvious that $\Psi_{a, a_{i_0} x^{q^{i_0}}, \rho}(\mathcal{S}_{n,d,s})$ is not in $\mathcal{S}_{n,d,s'}$. Therefore, we must have $s \equiv \pm s' \pmod{n}$. □

2.5 A characterization of known additive constructions

In this section we will show that the properties stated in Proposition 2.4.1 characterize the known examples of maximum d-codes in restricted setting. More precisely, we prove the following

Theorem 2.5.1. *Let n, s be two integers such that $n \geq 4$ and $\gcd(s, n) = 1$, let d be an integer such that $1 \leq d \leq n-1$. Let $\mathcal{D} \subset X_n(q)$ be a maximum d-code.*

(i) If $X_n(q) = S_n(q)$, then $\mathcal{D} \in [\mathcal{S}_{n,d,s}]_{\cong}$ if and only if there is a unique subspace V of $\tilde{\mathscr{L}}_{n,q}[x]$, such that

(a) $V \in [\mathcal{G}']_{\simeq}$ where $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s\frac{n+d}{2}}}$;

(b) $V = V^{\top}$, where $V^{\top} = \{f^{\top} : f \in V\}$;

(c) $V \cap S_n(q) = \mathcal{D}$.

(ii) If $X_n(q) = A_n(q)$, then $\mathcal{D} \in [\mathcal{A}_{n,d,s}]_{\cong}$ if and only if there is a unique subspace V of $\tilde{\mathscr{L}}_{n,q}[x]$, such that

(a) $V \in [\mathcal{G}']_{\simeq}$ where $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s\frac{d}{2}}}$;

(b) $V = V^{\top}$, where $V^{\top} = \{f^{\top} : f \in V\}$;

(c) $V \cap A_n(q) = \mathcal{D}$.

Proof. Let us prove the sufficiency first. Assume $\mathcal{D} \in [\mathcal{C}]_{\cong}$ where either $\mathcal{C}$ is $\mathcal{S}_{n,d,s}$ or $\mathcal{C}$ is $\mathcal{A}_{n,d,s}$. Hence, there exists a rank-preserving map of type $\Psi = \Psi_{a,g,\rho}$, with $a \in \mathbb{F}_q^*$, $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ and g a permutation q-polynomial, such that $\Psi(\mathcal{C}) = \mathcal{D}$.
Let $V = \Phi_{ag,\rho,g^{\top}}(\mathcal{G}')$, where $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{n+d}{2})}}$ if $X_n(q) = S_n(q)$ and $\mathcal{C} = \mathcal{S}_{n,d,s}$, $\mathcal{G}' = \mathcal{G} \circ x^{q^{s(\frac{d}{2})}}$ if $X_n(q) = A_n(q)$ and $\mathcal{C} = \mathcal{A}_{n,d,s}$.
In both cases it is easy to see that ${\mathcal{G}'}^{\top} = \mathcal{G}'$. Hence, V satisfies the properties (a) and (b). Moreover, as $\Phi_{ag,\rho,g^{\top}}$ fixes $X_n(q)$, applying (i) of Proposition 2.4.1, we obtain that

$$V \cap X_n(q) = \Phi_{ag,\rho,g^{\top}}(\mathcal{G}') \cap X_n(q) = \Psi(\mathcal{G}' \cap X_n(q)) = \Psi(\mathcal{C}) = \mathcal{D}.$$

Hence V satisfies (c).
Next, let us show the uniqueness. To this aim suppose that V and V' are two subspaces of $\tilde{\mathscr{L}}_{n,q}[x]$ both satisfying conditions (a), (b) and (c).

In particular we have that

$$V \cap X_n(q) = \mathcal{D} = V' \cap X_n(q).$$

By hypothesis $\mathcal{D} = \Psi(\mathcal{C})$ and Ψ fixes $X_n(q)$. This means that there is an element in $[\mathcal{G}_{n,n-d+1,s}]_{\simeq}$ different from $\mathcal{G}'$, intersecting X_n in $\mathcal{C}$. Indeed, $\Phi^{-1}_{ag,\rho,g^\top}(V')$. This, by Theorem 2.4.3 (i), is a contradiction.
Now, let us prove the necessity. By (a), $\mathcal{G}'$ and V are equivalent, then there exists a map $\Phi = \Phi_{g,\rho,h}$ such that $V = \Phi(\mathcal{G}')$. Since again ${\mathcal{G}'}^\top = \mathcal{G}'$, by using condition (b), we have

$$\Phi^\top(\mathcal{G}') = \Phi_{h^\top,\rho,g^\top}(\mathcal{G}') = V. \tag{2.5.1}$$

Now, from (2.5.1) and taking into account that $V = \Phi(\mathcal{G}')$, we get

$$\Phi_{g^{-1}\circ h^\top,\mathrm{id},g^\top\circ h^{-1}}(\mathcal{G}') = \mathcal{G}'.$$

Hence, by Theorem 1.5.3, we get

$$g^{-1}\circ h^\top = \alpha x^{q^r} \text{ and } g^\top \circ h^{-1} = \beta x^{q^{n-r}},$$

with $\alpha, \beta \in \mathbb{F}^*_{q^n}$.
In particular, $r \equiv 0 \pmod n$ and consequently $\beta = \alpha^{-1}$, $g = h^\top \circ \beta x$, and $\Phi = \Phi_{h^\top\circ\beta x,\rho,h}$.
We show that $\Phi(\mathcal{C}) \cap \mathcal{D}$ contains at least one element which is different from the null map. In fact, by (c), we have

$$\dim_{\mathbb{F}_q}(\Phi(\mathcal{C}) \cap \mathcal{D}) \geq \dim_{\mathbb{F}_q}\Phi(\mathcal{C}) + \dim_{\mathbb{F}_q}\mathcal{D} - \dim_{\mathbb{F}_q} V = n.$$

Hence, let f be an element of $\mathcal{C}$ such that $\Phi(f) \in \mathcal{D}$. Since $\Phi(f) \in \mathcal{D} \subset S_n(q)$, we have that $\Phi^\top(f) = \Phi(f)$. Consequently,

$$f^\rho(\beta x) = \beta f^\rho(x) \quad \text{for each } x \in \mathbb{F}_{q^n}.$$

Hence $\beta \in \mathbb{F}_q$ and

$$\mathcal{D} = \Phi(\mathcal{G}') \cap X_n(q) = \Phi(\mathcal{G}' \cap X_n(q)) = \Psi_{\beta,h^\top,\rho}(\mathcal{C}).$$

Hence $\mathcal{D} \in [\mathcal{C}]_{\cong}$. □

A similar result can be stated also for the two known constructions of maximum d-codes in $H_n(q^2)$.

Theorem 2.5.2. *Let n, s be two integers such that $\gcd(s, 2n) = 1$, and let d be an integer such that $d > 1$. Then we have the following*

- (i) *$\mathcal{C} \in [\mathcal{H}_{n,d,s}]_{\cong}$ if and only if there is a unique subspace V of $\tilde{\mathscr{L}}_{n,q^2}[x]$, such that*

(a) $V \in [\mathcal{G}']_{\simeq}$ *where* $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(n+d+1)}}$;
(b) $V = \tilde{V}$, *where* $\tilde{V} = \{\tilde{f} \,:\, f \in V\}$;
(c) $V \cap H_n(q^2) = \mathcal{C}$.

(*ii*) $\mathcal{C} \in [\mathcal{E}_{n,d,s}]_{\cong}$ *if and only if there is a unique subspace* V *of* $\tilde{\mathscr{L}}_{n,q^2}[x]$, *such that*

(a) $V \in [\mathcal{G}']_{\simeq}$ *where* $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(d+1)}}$;
(b) $V = \tilde{V}$, *where* $\tilde{V} = \{\tilde{f} \,:\, f \in V\}$;
(c) $V \cap H_n(q^2) = \mathcal{C}$.

Proof. The proof is similar to that of the previous theorem; taking into account that, in this case we have $\tilde{\mathcal{G}'} = \mathcal{G}'$, whenever $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(n+d+1)}}$ or $\mathcal{G}' = \mathcal{G}_{n,n-d+1,s} \circ x^{q^{s(d+1)}}$. □

2.6 A new additive symmetric 2-code

In this section we exhibit a symmetric 2-code which is not equivalent to the one with the same parameters shown in Theorem 2.2.2. So, let q be an odd prime power, m and s two integers such that $m \geq 2$ and $\gcd(s, 2m) = 1$. Let $\mathrm{N}_{q^{2m}/q}$ be the norm function of $\mathbb{F}_{q^{2m}}$ over $\mathbb{F}_q$ and let η be an element of $\mathbb{F}_{q^{2m}}$ such that $\mathrm{N}_{q^{2m}/q}(\eta)$ is not a square.
As we described in Section 1.5, the set

$$\mathcal{D}_{k,s}(\eta) = \left\{ ax + \sum_{j=1}^{k-1} c_i x^{q^{js}} + \eta b x^{q^{ks}} \;:\; c_1, \cdots, c_{k-1} \in \mathbb{F}_{q^{2m}},\ a, b \in \mathbb{F}_{q^m} \right\}$$

is a maximum rank distance code with minimum distance $d = 2m - k + 1$, [112].
Now, consider the following set of q-polynomials

$$\mathcal{S} = \left\{ a_0 x + \sum_{j=1}^{m-2} a_j x^{q^{sj}} + \eta b x^{q^{s(m-1)}} + a x^{q^{sm}} + \eta^{q^{s(m+1)}} b^{q^s} x^{q^{s(m+1)}} \right.$$
$$\left. + \sum_{j=1}^{m-2} (a_j x)^{q^{s(2m-j)}} \;:\; a_0, a_1, \ldots, a_{m-2} \in \mathbb{F}_{q^{2m}} \text{ and } a, b \in \mathbb{F}_{q^m} \right\}.$$

It is straightforward to see that, if we set $\mathcal{D}' = \mathcal{D}_{2m-1,s}(\eta) \circ x^{q^{sm}}$ then

$$\mathcal{S} = \mathcal{D}' \cap S_{2m}(q) \quad \text{and} \quad \mathcal{S} = \mathcal{D}'^{\top} \cap S_{2m}(q).$$

Clearly, since $\mathcal{D}'$ is a 2-code and $|\mathcal{S}| = q^{2m^2}$, by Theorem 2.2.1, it is a maximum symmetric $\mathbb{F}_q$-linear code with minimum distance $d = 2$. We will show this in an algebraic way as well, but before we recall the following

Lemma 2.6.1 ([109], Lemma 3)**.** *Let* $f(x) = \sum_{i=1}^{k-1} f_i x^{q^i} \in \tilde{\mathscr{L}}_{n,q}[x]$ *with* q*-degree* k*. If* f *has rank* $n-k$*, then* $\mathrm{N}_{q^n/q}(f_0) = (-1)^{kn}\mathrm{N}_{q^n/q}(f_k)$*.*

Proof. For any k-dimensional $\mathbb{F}_q$-subspace U of $\mathbb{F}_{q^n}$, there is a unique monic linearized polynomial with $\deg_q(f) = k$ that annihilates U, i.e. the set of its roots contains U. It is the polynomial

$$m_U(x) = \prod_{u \in U} (x - u),$$

see [83, Theorem 3.52]. Clearly, every linearized polynomial of q-degree k annihilating U is a multiple of $m_U(x)$ by an element of $\mathbb{F}_{q^n}$, and hence it suffices to prove the result for any particular linearized polynomial of degree k annihilating U. Choose an $\mathbb{F}_q$-basis $\{u_0, u_1, \ldots, u_{k-1}\}$ of U, and define a linearized polynomial f as the determinant of the following $(k+1) \times (k+1)$ matrix

$$f(x) = \begin{pmatrix} x & x^q & \cdots & x^{q^k} \\ u_0 & u_0^q & \cdots & u_0^{q^k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k-1} & u_{k-1}^q & \cdots & u_{k-1}^{q^k} \end{pmatrix} = f_0 x + f_1 x^q + \ldots + f_k x^{q^k}. \tag{2.6.1}$$

Then it is straightforward to see that f annihilates U, because plugging in any $u \in U$ for x, we get that the firt row is an $\mathbb{F}_q$-linear combination of the remaining rows. Furthermore, expanding along the top row we see that $f_0 = (-1)^k f_k^q$, and so $\mathrm{N}_{q^n/q}(f_0) = (-1)^{kn}\mathrm{N}_{q^n/q}(f_k)$, proving the claim. □

Now, we will show that any map in $\mathcal{S}$ has rank strictly greater than one. In fact, let $f_m = f \circ x^{q^{sm}}$, where $f \in \mathcal{S}$. Then the coefficients of terms x and $x^{q^{s(2m-1)}}$, of f_m are c and ηb, respectively. As a consequence of the lemma above, the rank of $f(x)$ is then at least two. Hence, $\mathcal{S}$ is a maximum 2-code of $S_{2m}(q)$.

Theorem 2.6.2. *The* 2*-code* $\mathcal{S} \in S_{2m}(q)$ *is not equivalent to* $\mathcal{S}_{2m,2,s}$*.*

Proof. Assume by way of contradiction that $\mathcal{S}$ is equivalent to $\mathcal{S}_{2m,2,s}$. Then there must be a map $\Psi = \Psi_{a,g^\top,\rho}$ such that $\Psi(\mathcal{S}) = \mathcal{S}_{2m,2,s}$, where $a \in \mathbb{F}_q$, $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ and $g(x) = \sum_{i=0}^{2m-1} g_i x^{q^{is}}$ is a permutation q-polynomial with coefficients in $\mathbb{F}_{q^{2m}}$.
Consider $g^\top \circ \alpha^\rho x \circ g$, where $\alpha \in \mathbb{F}_{q^{2m}}$. By computation the coefficient of $x^{q^{ms}}$ is

$$a_m(\alpha) = \sum_{i=0}^{2m-1} g_{2m-i}^{q^{si}} \, g_{m-i}^{q^{si}} \alpha^{\rho q^{si}} \tag{2.6.2}$$

where indices are taken modulo $2m$. Since the coefficient of the term with q-degree ms of $\mathcal{S}_{2m,2,s}$ is zero, we obtain

$$g_{2m-i}\ g_{m-i} = 0 \quad \text{for each} \quad i = 1, 2, \ldots, m.$$

Define $\mathrm{supp}(g) = \{i \in \{0, \ldots, m-1\} : g_i \neq 0\}$ and let $c \in \mathbb{F}_{q^m}$. Similarly, the coefficient of degree q^{ms} of the composition $g^{\top} \circ c^{\rho} x^{q^{ms}} \circ g$ is equal to

$$a_m(c) = \sum_{i=0}^{2m-1} g_i^{q^{s(2m-i)}} g_i^{q^{s(m-i)}} c^{\rho q^{s(2m-i)}} = \sum_{i \in \mathrm{supp}(g)} g_i^{q^{s(2m-i)}} g_i^{q^{s(m-i)}} c^{\rho q^{s(m-i)}}.$$

Obviously, since

$$(g_i^{q^{s(2m-i)}} g_i^{q^{s(m-i)}})^{q^m} = g_i^{q^{s(2m-i)}} g_i^{q^{s(m-i)}}$$

for all $i \in \mathrm{supp}(g)$, the polynomial above has coefficients in $\mathbb{F}_{q^m}$. On the other hand, as the coefficient of the term with q-degree ms in $\mathcal{S}_{2m,2,s}$ is zero, $a_m(c) = 0$ for all $c \in \mathbb{F}_{q^m}$. This implies that $g_i = 0$ for $i \in \mathrm{supp}(g)$. Therefore g is the null polynomial which contradicts the permutation property of g. $\square$

3 Intersection problems in finite projective spaces

„Sono
molto
irrequieta
quando
mi legano
allo spazio.“

Alda Merini, Aforismi e Magie.

One of the classical problems in extremal combinatorics is to determine the size of the largest families of pairwise non-trivially intersecting subsets of a finite set.
In 1961, the mathematicians Pál Erdős, Chao Ko and Richard Rado published an influential paper in which they solved this problem, [41].
Their result became a milestone that inspired many mathematicians on this topic. In addition to set theory, similar problems, known as *intersection problems*, were and are still studied in many different structures including multisets, groups and projective and polar geometries, [26]. In 2013, M. De Boeck and L. Storme drew up a survey paper in which the current status of this topic is collected in the various settings.
In honour of the three authors of the original paper, these problems are called *Erdős-Ko-Rado problems*, briefly EKR problems, and the generalisations of this theorem in the various settings are called *Erdős-Ko-Rado theorems*.

In this chapter, after the classical results on EKR families in set theory are traced, we shall recall some notions on finite projective spaces and we will state some intersection problems in them, called the *q-analogues* of the EKR problems.

3.1 The original Erdös-Ko-Rado problem

In this section we will retrace the original Erdos-Ko-Rado problem in set theory and we will give the background for the next sections where we will focus on the Erdos-Ko-Rado problems in the finite projective spaces.
As mentioned before, the problem of finding the largest sets of pairwise non-trivially intersecting elements was solved by Erdős *et al.* in 1961,

Theorem 3.1.1 ([41]). *If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of $\mathcal{S}$ are pairwise not disjoint, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$.*

Note that in case $n = 2k$, there are many examples attaining this upper bound. Indeed, for every subset with size k, in short a *k-subset*, there is precisely one disjoint k-subset in the set, so any family of k-subsets constructed by picking one k-subset from each such pair has the size in the theorem above. In case of $n < 2k$, the problem is trivial: two subsets of size k cannot be disjoint.
Note that the upper bound in Theorem 3.1.1 is attained if $\mathcal{S}$ is the set of all subsets of size k containing a fixed element of Ω, such a family is called a *point-pencil*. The orginal Erdős-Ko-Rado result was generalised and improved by Wilson in 1984,

Theorem 3.1.2 ([118]). *Let k be a positive integer and $1 \leq t \leq k$. If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geq (t+1)(k-t+1)$, such that the elements of $\mathcal{S}$ pairwise intersect in at least t elements, then $|\mathcal{S}| \leq \binom{n-t}{k-t}$.*
Moreover, if $n \geq (t+1)(k-t+1)+1$, then equality holds if and only if $\mathcal{S}$ is the set of all the subsets of size k through a fixed subset of Ω of size t.

Actually, for general $t \geq 1$, a similar result was already obtained in [41], but the bound $n \geq t + (k-t)\binom{k}{t}^3$ was required.
In [118], Wilson also showed that the bound in Theorem 3.1.2 is tight. More precisely, let $\mathcal{F}$ be the set of all subsets of size k meeting a fixed subset of Ω of size $t+2$ in at least $t+1$ elements, $k \geq t+1$. Clearly, the elements of $\mathcal{F}$ meet pairwise in at least t elements. If $n = (t+1)(k-t+1)$, then

$$|\mathcal{F}| = (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}$$

equals the size of the set described in Theorem 3.1.2. If $n \leq (t+1)(k-t+1)-1$, then $\mathcal{F}$ is larger than the size of the set described in this theorem.

Generalizing the above mentioned, we will call the set of all the subsets of size k containing a fixed subset of size t is called a *t-pencil*, clearly an 1-pencil is a

point-pencil.

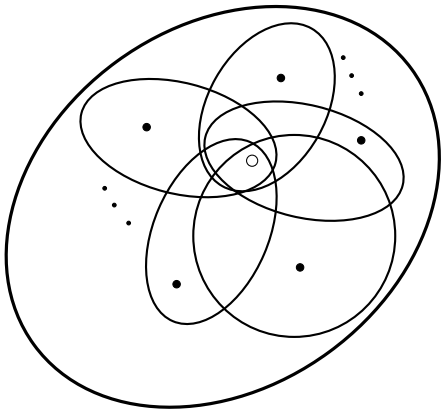

Figure 3.1: A point-pencil.

Note that if the parameter $t \geq 1$, then $\mathcal{S}$ is a collection of subsets of size k of an arbitrary set, which are pairwise not disjoint.
In literature this family is called an *Erdős-Ko-Rado set* and classification of the largest Erdős-Ko-Rado sets is called the *Erdős-Ko-Rado problem*, in short EKR *problem*. Related results are obtained by Hilton and Milner, [63]. They described the largest Erdős-Ko-Rado sets which are not contained in a point-pencil.

Theorem 3.1.3 ([63])**.** *Let Ω be a set of size n and let $\mathcal{S}$ be an Erdős-Ko-Rado set of k-subsets in Ω, $k \geq 3$ and $n \geq 2k+1$. If there is no element in Ω which is contained in all subsets in $\mathcal{S}$, then*

$$|\mathcal{S}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

Moreover, equality holds if and only if

i) either $\mathcal{S}$ is the union of F, for some fixed k-subset F, and the set of all k-subsets G of Ω containing a fixed element $x \notin F$, such that $G \cap F \neq \emptyset$,

ii) or else $k = 3$ and $\mathcal{S}$ is the set of all subsets of size 3 having an intersection of size at least 2 with a fixed subset F of size 3.

Studying the original Erdős-Ko-Rado problem led to investigations in graph theory. More precisely, this issue can be translated into properties of the *Johnson graph* and the *Kneser graph*, see [15, Chapter 9]. Finally, an interesting generalisation of the Erdős-Ko-Rado problem can be found in [44], where Frankl explored the largest *r-wise intersecting families* of k-subsets of a given finite set.

3.2 Incidence geometry and projective spaces

In this section we will introduce the basic concepts that will be useful to understand the remainder part of this thesis. The aim is to avoid the ambiguity that would arise by not stating some definitions or theorems. For this purpose, we will refer to standard references by Buekenhout ([18], [19]), Hirschfeld and Thas ([64], [66]) and Mazzocca ([92]).

Let $\mathcal{P}$ be a non-empty set, whose elements are called *points* and let $\mathcal{L}$ be a set, whose elements are called either *lines* or *blocks*. Denote by $\mathcal{I}$ an *incidence relation* between an element of $\mathcal{P}$ and an element of $\mathcal{L}$ that we will consider symmetric. The triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ will be called an *incidence geometry*. Often the incidence will be either $\subseteq$ or $\supseteq$. In these cases we will omit $\mathcal{I}$ and denote the incidence geometry by $(\mathcal{P}, \mathcal{L})$ and either the set $\mathcal{L}$ of lines will be identified with a set of subsets of $\mathcal{P}$ or vice versa. Moreover, when a point P is in relation with a line ℓ, we will use the usual terminology 'the point P is incident with line ℓ', 'the point P is 'on the line ℓ, 'the line ℓ passes through the point P' etc. .
A *homomorphism* $\alpha : (\mathcal{P}, \mathcal{L}) \to (\mathcal{P}', \mathcal{L}')$ of incidence geometries is a map $\alpha : \mathcal{P} \to \mathcal{P}'$ such that the image under α of every line in $\mathcal{L}$ is contained in a line of $\mathcal{L}'$. If it is injective and the image of every line in $\mathcal{L}$ is a line in $\mathcal{L}'$ then the homomorphism is also called an *embedding*. The notions *isomorphism* and *automorphism* are defined in the obvious way.

Now, we recall the notion of *projective spaces over fields*. Let $\mathbb{V}$ be a $(n+1)$-dimensional vector space over a field $\mathbb{F}$. We will denote by $\mathrm{PG}(\mathbb{V})$ the set of 1-dimensional subspaces of $\mathbb{V}$, that will be called *points* of $\mathrm{PG}(\mathbb{V})$. We will call *lines*, *planes*, *solids*, *k-dimensional projective subspaces*, *hyperplanes* respectively the 2-dimensional, 3-dimensional, $(k+1)$-dimensional, n-dimensional subspaces of $\mathbb{V}$ seen as set of points of $\mathrm{PG}(\mathbb{V})$.
If $\mathbb{W}$ is an $(k+1)$-dimensional subspace of $\mathbb{V}$ we will denote by W the corresponding k-dimensional subspace of $\mathrm{PG}(\mathbb{V})$ while a point of the projective space will be indicated by $\langle \mathbf{x} \rangle$. If X is any subset of points of $\mathrm{PG}(\mathbb{V})$, we will denote by $\langle X \rangle$ the smallest projective subspace that contains X and we will refer to it as the projective space *spanned* by X. In particular, we will indicate

by $\langle U_1, \ldots, U_s \rangle$ the projective space spanned by the set of projective subspaces $\{U_1, \ldots, U_s\}$ of $\mathrm{PG}(\mathbb{V})$.
For two subspaces U and W of $\mathrm{PG}(\mathbb{V})$, the *intersection* $U \cap V$ is the largest subspace which is contained in both U and V. We can immediately generalise this notion to $U_1 \cap \ldots \cap U_s$ for subspaces $U_1, \ldots, U_s$ of $\mathrm{PG}(\mathbb{V})$. Finally, we consider the empty set as a projective subspace with dimension -1.

Definition 3.2.1. Let S_j be the set of all projective subspaces of $\mathrm{PG}(\mathbb{V})$ with dimension j, for every $j = -1, 0, 1, \ldots, n$. The pair $(\mathrm{PG}(\mathbb{V}), (\mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n))$ is the *n-dimensional projective space* associated to $\mathbb{V}$. We will refer to it just by $\mathrm{PG}(\mathbb{V})$. The integer $n = \dim(\mathbb{V}) - 1$ is called the *dimension* of $\mathrm{PG}(\mathbb{V})$

We will denote by $\mathrm{PG}(n, \mathbb{F})$ the n-dimensional projective space associated to $\mathbb{V} = \mathbb{F}^{n+1}$. If $n = 1$, then $\mathrm{PG}(1, \mathbb{F})$ is also called the *projective line* over $\mathbb{F}$, while, if $n = 2$, then $\mathrm{PG}(2, \mathbb{F})$ is also called the *projective plane* over $\mathbb{F}$.
Clearly, any subspace of a projective space can also be seen as a projective space and each projective space $\mathrm{PG}(\mathbb{V})$ over a vector space $\mathbb{V}$ induces an incidence geometry of its points and lines.
The dimension theorem for vector subspaces implies the *Grassmann identity* for subspaces of a projective space:

$$\dim(U) + \dim(W) = \dim\langle U, W \rangle + \dim(U \cap W) \tag{3.2.1}$$

for all subspaces U and W of $\mathrm{PG}(\mathbb{V})$.

Since each point in $\mathrm{PG}(\mathbb{V})$ corresponds to a 1-dimensional vector space in $\mathbb{V}$, if we consider a non-zero vector $\mathbf{x}$, then it is possible to define the *coordinates* of the corresponding projective point $P = \langle \mathbf{x} \rangle$: they are the components of the vector $\mathbf{x}$ in a fixed $\mathbb{F}$-basis of $\mathbb{V}$, up to a non-zero scalar multiple. They are called *homogeneous coordinates* and we will denote them by $(x_0, x_1, \ldots, x_n)$.
A hyperplane W in $\mathrm{PG}(n, \mathbb{F})$ corresponds to a vector hyperplane $\mathbb{W}$ and then it is the set of points whose homogeneus coordinates $(x_0, \ldots, x_n)$ satisfy a linear equation $a_0x_0 + a_1x_1 + \ldots + a_nx_n = 0$. In the same way, a k-space S_k is the set of points whose homogeneous coordinates $(x_0, x_1, \ldots, x_n)$ satisfy the equations $A(x_0, x_1, \ldots, x_n)^t = 0$, where A is an $(n-k)\times(n+1)$ matrix of rank $n-k$ over $\mathbb{F}$.

Let $\mathrm{PG}(\mathbb{V})$ be the projective space underlying the vector space $\mathbb{V}$, an *automorphism* of its incidence geometry is called more properly *collineation*.
It is straightforward to see that the map

$$(x_0, \ldots, x_n) \in \mathbb{F}^{n+1} \mapsto A(x_0^\sigma, \ldots, x_n^\sigma)^t \in \mathbb{F}^{n+1},$$

with A a non-singular $(n+1) \times (n+1)$-matrix and σ a field automorphism of $\mathbb{F}$, induces a collineation of $\mathrm{PG}(n, \mathbb{F})$. We can denote this collineation by

(A, σ). The set of these maps is denoted by $\mathrm{P\Gamma L}(n+1, \mathbb{F})$. The *fundamental theorem of projective geometry* states that every collineation of $\mathrm{PG}(n, \mathbb{F})$, $n \geq 2$, arises from a non-singular matrix and a field automorphism as before. The collineations (A, id), with $A \in \mathrm{GL}(n+1, \mathbb{F})$ are called *projectivities*. The group of all projectivities of $\mathrm{PG}(n, \mathbb{F})$ is denoted by $\mathrm{PGL}(n+1, \mathbb{F})$. Cleary, in $\mathrm{PG}(1, \mathbb{F})$ every bijection of the points gives rise to a collineation. Hence, the group $\mathrm{P\Gamma L}(2, \mathbb{F})$ in general is not the full collineation group.
Two subsets S_1 and S_2 of $\mathrm{PG}(n, \mathbb{F})$ are called $\mathrm{P\Gamma L}$-*equivalent* (resp. PGL-*equivalent*) if and only if there exists a collineation (resp. a projectivity) such that $\alpha(S_1) = S_2$.

A projective space is finite if the underlying field is finite. So, if $\mathbb{F}_q$ is the finite field of order q, with q a prime power, we can consider the finite projective space $\mathrm{PG}(n, \mathbb{F}_q)$ that will generally be denoted by $\mathrm{PG}(n, q)$. The integer q is called *order* of the projective space.
Probably, the best known finite projective space is the projective plane $\mathrm{PG}(2, 2)$, it is called the *Fano plane*.

Since a finite projective space is linked to a finite vector space, we can easily count the number of subspaces of a certain dimension in $\mathrm{PG}(n, q)$ using the *Gaussian coefficient*

Definition 3.2.2. Let q be a prime power, let a be a non-negative integer and let $0 \leq b \leq a$. The *q-ary Gaussian coefficient* of a and b is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \frac{(q^a-1)(q^{a-1}-1)\cdots(q^{a-b+1}-1)}{(q^b-1)(q^{b-1}-1)\cdots(q-1)} & \text{if } a, b > 0 \\ 1 & \text{otherwise.} \end{cases} \tag{3.2.2}$$

So, the number of k-dimensional subspaces in $\mathrm{PG}(n, q)$ equals $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$, i.e. the number of subspaces with vector dimension $k+1$ in the vector space $\mathbb{V}(n+1, q)$, while the number of a k-dimensional spaces through a fixed t-dimensional space, $0 \leq t \leq k$ in $\mathrm{PG}(n, q)$ is $\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Moreover, we will indicate the number $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q$ by the symbol $\theta_{n,q}$.

Finally, let S_m be a subspace of dimension m, $m \leq n-2$; of $\mathrm{PG}(n, q)$ and consider the incidence geometry whose points are the $(m+1)$-dimensional subspaces containing S_m and whose lines are the $(m+2)$-dimensional subspaces containing S_m. The incidence relation is induced by the incidence of $\mathrm{PG}(n, q)$. This incidence geometry is called the *quotient geometry* of $\mathrm{PG}(n, q)$ w.r.t. S_m. It is denoted by $\mathrm{PG}(n, q)/S_m$ and it is easy to show that it is isomorphic to $\mathrm{PG}(n-m-1, q)$, [9].

3.2.1 Reguli and (partial) spreads

In this subsection we will introduce two remarkable substructures in finite projective spaces: *reguli* and (*partial*) *spreads*. So, let $\mathrm{PG}(n,q)$ be a projective space over the finite field $\mathbb{F}_q$. We call a set $\mathcal{S}$ of subspaces of $\mathrm{PG}(n,q)$ *skew* if no two distinct subspaces of $\mathcal{S}$ have a point in common. We also speak of *skew* subspaces.
So, let $\mathcal{S}$ be a set of skew subspaces, a line is called a *transversal* of $\mathcal{S}$ if it meets each subspace of $\mathcal{S}$ exactly in one point.

Lemma 3.2.3. *Let $\mathbb{P} = \mathrm{PG}(\mathbb{V})$ be an n-dimensional projective space. Let l_1 and l_2 be two skew lines, and denote by P a point outside l_1 and l_2. Then there is at most one transversal of l_1 and l_2 through P. If $n = 3$ then there is exactly one transversal of l_1 and l_2 through P.*

Proof. Assume that there are two transversals g_1 and g_2 through P. Then each of these transversals meets the lines l_1 and l_2 in different points. So g_1 and g_2 span a plane, which contains the two skew lines l_1 and l_2, a contradiction. Now suppose that $\mathbb{P}$ is a 3-dimensional. Then, by (3.2.1), the plane $\langle P, l_1\rangle$ must intersect the line l_2 in some point Q. Thereforem the line PQ intersects l_1 and l_2. Hence it is a transversal of l_1 and l_2. □

Now, let $\mathbb{P} = \mathrm{PG}(3,q)$ be the 3-dimensional projective space of order q. A nonempty skew set $\mathcal{R}$ of lines of $\mathbb{P}$ is called *regulus* if the following are true:

a) through each point of each line of $\mathcal{R}$ there is a transversal of $\mathcal{R}$,

b) through each point of a transversal of $\mathcal{R}$ there is a line of $\mathcal{R}$.

Note that if we consider the set $\mathcal{R}'$ of all transversals of a regulus $\mathcal{R}$ again form a regulus, we call it *opposite regulus* of $\mathcal{R}$.

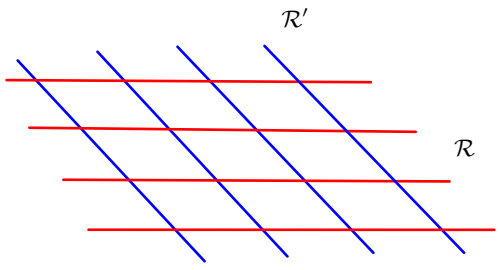

Figure 3.2: A regulus $\mathcal{R}$ and its opposite regulus $\mathcal{R}'$.

Clearly, since $\mathbb{P}$ is finite then any regulus consists of exactly $q+1$ lines. In particular, it is possible to show that if l_1, l_2 and l_3 are three skew lines of $\mathbb{P}$, then there is exactly one regulus through l_1, l_2 and l_3 in $\mathbb{P}$, [9, Section 2.4]. Now, consider a skew set of t-spaces in $\mathrm{PG}(n,q)$. It is called *partial t-spread*. If a partial t-spread cannot be extended to a larger one, then it is called *maximal*. A partial t-spread is called *t-spread* if it covers all points of $\mathrm{PG}(n,q)$.

For the sake of completeness, we will report a result of Segre about the existence of a spread in a finite projective space, [103]. The proof that we propose here is in [65, Section 4].

Theorem 3.2.4. *Let $\mathrm{PG}(n,q)$ be the n-dimensional projetive space over $\mathbb{F}_q$. There exists a t-spread $\mathcal{S}$ of $\mathrm{PG}(n,q)$ if and only if $t+1$ divides $n+1$.*

Proof. If there exists a spread $\mathcal{S}$, then the number of points in $\mathrm{PG}(t,q)$ divides the number of points in $\mathrm{PG}(n,q)$ that is

$$\frac{\theta_{n,q}}{\theta_{t,q}} = \frac{q^{n+1}-1}{q^{t+1}-1}.$$

It is an integer if and only if $t+1$ divides $n+1$. Now, let s be an integer such that

$$n+1 = (t+1)(s+1). \tag{3.2.3}$$

Consider the finite field $\mathbb{F}_{q^{t+1}}$ and let $f(x)$ be an irreducible polynomial of degree $t+1$ over $\mathbb{F}_q$ and let α be a root of $f(x)$ in $\mathbb{F}_{q^{t+1}}$, then every element $\beta \in \mathbb{F}_{q^{t+1}}$ can be written as

$$\beta = x_0 + x_1\alpha + \ldots + x_t\alpha^t$$

where $x_j \in \mathbb{F}_q$, for all $j \in \{0,1,\ldots,t\}$. So, if we take the $s+1$ elements $\beta_0,\ldots,\beta_s \in \mathbb{F}_{q^{t+1}}$, they can be written as

$$\beta_i = x_{i0} + x_{i1}\alpha + \ldots + x_{it}\alpha^t$$

where $i \in \{0,1,\ldots,s\}$. The $n+1$ elements x_{ij} in $\mathbb{F}_q$, say in lexicographical order, can be interpreted as homogeneous coordinates of a point in $\mathrm{PG}(n,q)$. Thus each point of $\mathrm{PG}(n,q)$ is given by an $(s+1)$-ple $(\beta_0,\ldots,\beta_s)$ of elements of $\mathbb{F}^{q^{t+1}}$. Let $\gamma_0,\ldots,\gamma_s$ be any elements, not all zero, of $\mathbb{F}_{q^{t+1}}$. Then the equations

$$\beta_i\gamma_j = \beta_j\gamma_i \tag{3.2.4}$$

for $i \in \{0,1,\ldots,s\}$ and $j \in \{0,1,\ldots,t\}$, define a t-dimensional space S_t in $\mathrm{PG}(n,q)$. In fact, the equations (3.2.4) give $s(t+1)$ linearly independent equations in the x_{ij} and so define a subspace of dimension $n-s(t+1)=t$. Each $(s+1)$-ple $\gamma=(\gamma_0,\ldots,\gamma_s)$ corresponds to a point $P(\gamma)$ in $\mathrm{PG}(s,q^{t+1})$.

As $P(\gamma)$ varies in $\mathrm{PG}(s, q^{t+1})$, so S_t varies through a partition of $\mathrm{PG}(n, q)$. Indeed, since $\mathrm{PG}(s, q^{t+1})$ contains $\begin{bmatrix} s+1 \\ 1 \end{bmatrix}_{q^{t+1}}$ points, by 3.2.3, we obtain this number of t-dimensional spaces. Thus the number of points in all these spaces is

$$\theta_{s,q^{t+1}} \cdot \theta_{t,q} = \theta_{n,q}$$

exactly the number of points in $\mathrm{PG}(n, q)$ and every point is in some S_t, so two spaces S_t cannot intersect. Hence, the set $\mathcal{S}$ of these t-dimensional spaces form a partition of $\mathrm{PG}(n, q)$. □

Maximal (partial) t-spreads have been the subject of much research. Especially, the maximal partial line spreads in $\mathrm{PG}(3, q)$ have received a lot of attention, see for example [60] and [61]. In [3] and [103], it is shown that the study of t-spreads in $\mathrm{PG}(2t+1, q)$ is equivalent to the study of finite *translation planes* and their strict relation to the finite quasifields is explored. In [42], some bounds on the size of (partial) t-spreads are summarized. Moreover, relevant survey papers are [82] and [88].

3.3 The Erdős-Ko-Rado problem in finite projective spaces

As we mentioned before, the EKR problem, originated in set theory, can be generalized in a natural way to many other structures.
Here, we will focalise on its current state-of-the-art in the finite projective spaces to better understand the work that we will present in Chapter 4. In the latter setting and in the vector space setting, this problem is known as the *q-analogue* of the Erdős-Ko-Rado problem.
In general, by the term *q-analogue*, one refers to a mathematical expression parameterized by a quantity q that generalizes a known expression and may reduce to the original one in the limit for $q \to 1$. For istance, the formula in (3.2.2) is often called the *q-analogue binomial coefficient*, but also in other contexts, expressions as factorial, Fibonacci numbers and others have an equivalent q-analogue. Often, this term goes on to designate problems in many different combinatorial structures as we shall see.

Now, let q be a prime power and let $\mathrm{PG}(n, q)$ be the n-dimensional projective space of order q. Clearly, results on families of vector spaces pairwise intersecting in at least a vector space with fixed dimension can be interpreted in projective spaces, and vice versa, so we prefer to state all the results in a geometric setting.
In this context, in $\mathrm{PG}(n, q)$ an Erdős-Ko-Rado set is a family of k-dimensional subspaces, in short *k-spaces*, such that they are pairwise no skew. We will

denote it by EKR(k,n). In 1975, Hsieh proved the q-analogue for Theorem 3.1.1, [67]. We will report a slight improvement that combines the results due to Frankl and Wilson with Tanaka's.

Theorem 3.3.1 ([45], Theorem 1 and [110], Theorem 3)**.** *Let k and t be integers, with $0 \leq t \leq k$. Let $\mathcal{S}$ be a set of k-spaces in* PG(n,q)*, pairwise intersecting in at least a t-space.*

(i) If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed t-space of PG(n,q)*, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.*

(ii) If $2k-t \leq n \leq 2k$, then $|\mathcal{S}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.

In the theorem above, if we set $t = 0$, we obtain

Corollary 3.3.2. *Let $\mathcal{S}$ be an* EKR(k,n) *set in* PG(n,q)*. If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed point of* PG(n,q)*, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed hyperplane.*

Note that in Theorem 3.3.1 the condition $n \geq 2k-t$ is not a restriction, since any pair of k-dimensional subspaces in PG(n,q), with $n \leq 2k-t$, meets in at least a t-dimensional subspace.
Furthermore, it is clear that new families of any size below the size of the largest example can be found by deleting elements from it and so we are focused on *maximal* families, these are sets of k-spaces pairwise intersecting in at least a t-space, not extendable to larger families with the same property. Clearly, maximal families of this sort may have different sizes and so that we can refer to the first, the second, etc. largest example.
Related to this question, we report the q-analogue of the Hilton-Milner result on the second-largest maximal Erdős-Ko-Rado sets of subspaces in a finite projective space, due to Blokhuis *et al.* Also here, in the context of projective spaces, a set of subspaces through a fixed t-space will be called a *t-pencil*, a *point-pencil* if $t = 0$ and a *line-pencil* if $t = 1$.

Theorem 3.3.3 ([11] Theorem 1.3, Proposition 3.4)**.** *Let $\mathcal{S}$ be a maximal* EKR(k,n) *set in* PG(n,q)*, with $n \geq 2k+2$, $k \geq 2$ and $q \geq 3$ (or $n \geq 2k+4$, $k \geq 2$ and $q = 2$). If $\mathcal{S}$ is not a point-pencil, then*

$$|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{k(k+1)} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q + q^{k+1}.$$

Moreover, if equality holds, then

(i) either $\mathcal{S}$ consists of all the k-spaces through a fixed point P, meeting a fixed $(k+1)$-space τ, with $P \in \tau$, in a j-space, $j \geq 1$, and all the k-spaces in τ,

(ii) or else $k = 2$ and $\mathcal{S}$ is the set of all the planes meeting a fixed plane π in at least a line.

Regarding the EKR problem for $k = 1$ has been solved completely. Indeed, in $\mathrm{PG}(n, q)$ with $n \geq 3$, a maximal $\mathrm{EKR}(1, n)$ set is either the set of all the lines through a fixed point or the set of all the lines contained in a fixed plane. It is possible to generalize this result for a maximal family $\mathcal{S}$ of k-spaces, pairwise intersecting in a $(k-1)$-space, in a projective space $\mathrm{PG}(n, q)$, $n \geq k+2$.

Theorem 3.3.4 ([15], Section 9.3)**.** *Let $\mathcal{S}$ be a set of projective k-spaces, pairwise intersecting in a $(k-1)$-space in $\mathrm{PG}(n, q)$, $n \geq k+2$, then all the k-spaces of $\mathcal{S}$ go through a fixed $(k-1)$-space or they are contained in a fixed $(k+1)$-space.*

Not only the largest examples of EKR sets are studied. Indeed, Mussche considered small Erdős-Ko-Rado sets and obtained the following

Theorem 3.3.5 ([95], Theorem 2.45)**.** *If k is a prime power, a maximal $\mathrm{EKR}(k)$ set of size k^2+k+1 exists in $\mathrm{PG}(n, q), n \geq k^2+k$.*

Note that the size of the EKR set is independent by q and, even if not mentioned in the original paper, the hypothesis about n is necessary for the proof.

Also the EKR problem for sets of projective planes was analysed. By Grassman identity, it is trivial if $n \leq 4$. For $n = 5$, Blokhuis, Brouwer and Szönyi classified the six largest examples, finding the following

Theorem 3.3.6 ([12], Section 6)**.** *Let $\mathcal{S}$ be a maximal* EKR *set of planes in the projective space $\mathrm{PG}(5, q)$, with $|\mathcal{S}| \geq 3q^4+3q^3+2q^2+q+1$. Then one of the following cases occurs.*

a) $|\mathcal{S}| = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ and $\mathcal{S}$ is the set of planes through a fixed point P or the set of planes in a 4-space τ.

b) $|\mathcal{S}| = 1 + q(q^2+q+1)^2$ and $\mathcal{S}$ is one of the following: the set of planes intersecting a fixed plane π in at least a line, the set of planes that either are contained in a 3-space σ or else intersect σ in a line through a fixed point $P \in \sigma$, or the set of planes such that either pass through a fixed line l or else are in a 4-space $l \subset \tau$ and intersect l in a point.

c) $|\mathcal{S}| = 3q^4+3q^3+2q^2+q+1$ and $\mathcal{S}$ is the set of all planes that intersect π in a line through P, all planes in τ that intersect π in a line, and all

planes through P in τ, with P a point, π a plane and τ a 4-space such that $P \in \pi \subset \tau$.

In [27], M. De Boeck investigated the maximal EKR$(2, n)$ sets in PG(n, q) with $n \geq 5$. He characterized maximal EKR$(2, n)$ sets with sufficiently large size and showed that they belong to one of the 11 known examples, explicitly described in his work. More precisely,

Theorem 3.3.7 ([27], Theorem 3.1)**.** *Let $\mathcal{S}$ be a maximal* EKR *set of planes in a projective space* PG$(n, q), n \geq 5$*. Let ρ be the subspace of* PG(n, q) *generated by the elements of $\mathcal{S}$. If* $\dim(\rho) \geq 6$*, then $\mathcal{S}$ belongs to one of 11 known types of maximal* EKR *sets.*

The largest among these known types of maximal EKR sets, the point-pencil, contains $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$ elements, while the smallest one contains 7 elements. A classification result for large maximal Erdős-Ko-Rado sets of planes follows.

Theorem 3.3.8 ([12] Section 6 and [27] Theorem 4.6)**.** *Let $\mathcal{S}$ be a maximal* EKR *set of planes in a projective space $PG(n, q)$, $n \geq 5$, such that*

$$|\mathcal{S}| \geq 3q^4 + 3q^3 + 2q^2 + q + 1.$$

i) If $5 \leq n \leq 6$, then $\mathcal{S}$ belongs to one of 6 known types of maximal EKR *sets.*

ii) If $n \geq 7$, then $\mathcal{S}$ belongs to one of 10 known types of maximal EKR *sets.*

Again, each of the types mentioned in the theorem above is explicitly described in [27].

The tecniques used in all these papers besides being purely combinatory, involve the matrix calculation, as in [12] and in [26], or the properties of *Grassmann graph*. It is the graph with vertices the k-spaces in PG(n, q) and such that two vertices are adjacent if the corresponding k-spaces meet in a $(k-1)$-space.
Finally in [21], as in the set case, Chowdhury and Patkós studied the q-analogue of EKR problem for r-wise intersecting k-spaces in the projective spaces setting.

4 Maximal sets of k-spaces pairwise intersecting in at least a $(k-2)$-space

„Beauty is truth, truth beauty, - that is all
Ye know on earth, and all ye need to know.“

JOHN KEATS, Ode on a Grecian Urn, 49-50.

In this chapter, as natural step further with respect to the work of Eisfeld [39] and to the classification of EKR$(2,n)$ due to M. Boeck [27], we investigate sets of k-spaces in PG(n,q) pairwise intersecting in at least a $(k-2)$-space.
First, in Section 4.1, we analyze the sets of solids in PG(n,q), $n \geq 5$, such that every two solids intersect in at least a line, dividing the discussion on the existence or otherwise of some particular configurations of solids in those family and we give an overview of the largest examples of these sets.
Then, in Section 4.2, we generalize these results for sets of k-spaces, $k > 3$, pairwise intersecting in at least a $(k-2)$-dimensional subspace in PG(n,q) with $n \geq k+2$. Again, we discuss the largest examples giving some upper bounds on the size of these relevant families.
In both cases, we assume that all the elements in such a family do not have a point or a $(k-3)$-space in common, respectively, otherwise we can investigate the quotient space with respect to the common space and refer to [27]. Furthermore, we will suppose that these sets of subspaces are maximal.
Finally, since we will give upper bounds on the size of the largest examples, we will indicate the order of such families of k-spaces in PG(n,q), for q very large, using the *big O notation* and we will write $\begin{bmatrix} n \\ k \end{bmatrix}$, for $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and θ_n for $\theta_{n,q}$ if the field size q is clear from the context.

So, let $\mathcal{S}$ be a set of k-spaces with the properties described above. As we mentioned before, in our arguments we distinguish between two cases, depending on whether there exists a *configuration* or not in $\mathcal{S}$.
More precisely, three k-spaces A, B and C in PG(n,q), $n \geq k+2$, form a

configuration if they have no $(k-3)$-space in common.

Note that every two k-spaces of $\mathcal{S}$ in a configuration meet in a $(k-2)$-space and that for $k=3$, three solids A, B, C form a configuration if and only if $A \cap B \cap C = \emptyset$.

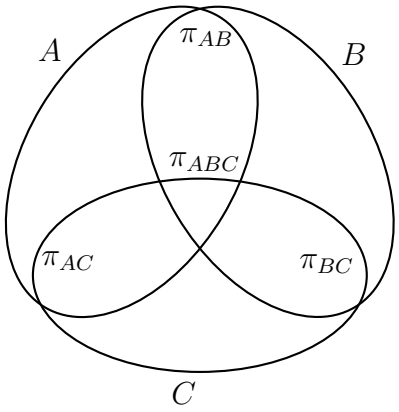

Figure 4.1: A three k-spaces configuration in $\mathcal{S}$.

4.1 Solids pairwise intersecting in at least a line

Let $\mathcal{S}$ be a maximal set of solids pairwise intersecting in at least a line in the projective space $\mathrm{PG}(n,q)$, with $n \geq 5$, and we suppose that there is no point contained in all the solids of $\mathcal{S}$. Note that the set of all the solids in a fixed 5-space is an example of a maximal set of solids pairwise intersecting in at least a line, with size $\begin{bmatrix} 6 \\ 4 \end{bmatrix} = O(q^8)$. Therefore we assume that the set $\mathcal{S}$ of solids spans at least a 6-space.
Now we will split our investigation: in the following subsection we suppose that the set $\mathcal{S}$ of solids contains a configuration formed by the solids A, B, C and we show in Lemma 4.1.1 that the solids not contaied in $\langle A, B \rangle$ meet the latter space in a plane. Then we consider the space α generated by the planes arising as such an intersection, and we discuss properties of the set $\mathcal{S}$ of solids depending on the dimension of α.

4.1.1 There is a configuration

Suppose that there exist three solids A, B and C in $\mathcal{S}$ that form a configuration, with $A \cap B = l_{AB}$, $A \cap C = l_{AC}$, $B \cap C = l_{BC}$. Note that $\langle A, B\rangle = \langle B, C\rangle = \langle A, C\rangle$. Since $\mathcal{S}$ spans at least a 6-space, let $\{D_i \mid i \in I\}$ be the family of the solids of $\mathcal{S}$ not contained in $\langle A, B\rangle$, where I is a certain set of indices. We start with the following

Lemma 4.1.1. *Let $\mathcal{S}$ be a maximal set of solids pairwise intersecting in at least a line. If there exists a configuration of solids A, B, C in $\mathcal{S}$, then a solid of $\mathcal{S}$ not in $\langle A, B\rangle$ intersects l_{AB}, l_{AC} and l_{BC} in a point.*

Proof. Consider a solid $E \in \mathcal{S}$ not in $\langle A, B\rangle$. We show that E intersects l_{AB} exactly in a point. The arguments for l_{AC} and l_{BC} are similar.
Suppose first that E contains the line l_{AB}. The solid E also has at least a line l in common with C. As $l_{AB} \cap l = \emptyset$, and l and l_{AB} span E, we have that E lies in $\langle A, C\rangle = \langle A, B\rangle$. This contradicts the hypothesis. Suppose now that E is disjoint from l_{AB}. Then E contains a line of A and a line of B which are disjoint. Again, this implies that E is spanned by these two lines and lies in $\langle A, B\rangle$. Hence, a solid of $\mathcal{S}$ not in $\langle A, B\rangle$, intersects l_{AB} in a point. □

Note that there is no transversal line to the lines l_{AB}, l_{BC} and l_{AC} as these lines span a 5-space. This implies, by Lemma 4.1.1, that all the solids D_i not in $\langle A, B\rangle$, meet $\langle A, B\rangle$ in a plane. To make our discussion easier, for the remainder of this section we will work under the following assumption:

($\diamond$) There are three solids A, B and C in $\mathcal{S}$ such that they form a configuration and α is the span of all the planes $D_i' = D_i \cap \langle A, B\rangle$, with $i \in I$.

Now, we distinguish between several cases depending on the dimension of α.

4.1.1.1 α is a plane

In this case we note that $\forall i \in I$, $D_i' = \alpha$, so all the solids in $\mathcal{S}$, not in $\langle A, B\rangle$, meet $\langle A, B\rangle$ in the plane α.
A solid in $\langle A, B\rangle \cap \mathcal{S}$ needs to have at least a line in common with every D_i not in $\langle A, B\rangle$. This implies that every solid of $\mathcal{S}$ in $\langle A, B\rangle$ meets α in at least a line. By the Grassmann identity, every two solids in $\langle A, B\rangle$ meet in at least a line.
If n is the dimension of the ambient projective space, we have at most $\theta_{n-3} - \theta_2 = O(q^{n-3})$ solids in $\mathcal{S}$ outside of $\langle A, B\rangle$; this is the number of solids through α in $\mathrm{PG}(n, q)$ excluded those in $\langle A, B\rangle$, and at most $\theta_2 \cdot \left(\begin{bmatrix}4\\2\end{bmatrix} - \theta_2\right) + \theta_2 = O(q^6)$ solids of $\mathcal{S}$ in $\langle A, B\rangle$, where the first term is the number of solids that have exactly a line in common with α and the second one, the number of solids through α in $\langle A, B\rangle$.

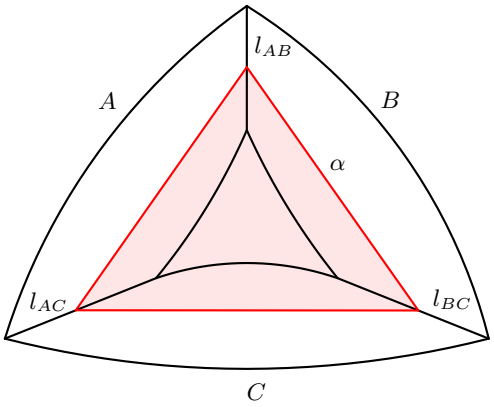

Figure 4.2: There is a configuration in $\mathcal{S}$ and $\dim(\alpha) = 2$

4.1.1.2 α is a solid

For every $i \in I$, the plane D'_i is spanned by three points of the three lines l_{AB}, l_{AC} and l_{BC} respectively. Hence we can suppose that α is spanned by l_{AB} and two points P_{AC}, P_{BC} of l_{AC}, l_{BC} respectively. Note that all the solids D_i have a plane in common with α and contain the line $P_{AC}P_{BC}$, so all the solids not in $\langle A, B\rangle$ already intersect in a line inside $\langle A, B\rangle$. We will show that all the solids of $\mathcal{S}$ in $\langle A, B\rangle$ have a plane in common with α or contain $P_{AC}P_{BC}$. To remark that $\mathcal{S}$ can contain the solid α, as $P_{AC}P_{BC} \subset \alpha$.

Proposition 4.1.2. *Under the assumption* $(\diamond)$*, and if* $\dim(\alpha) = 3$*, all the solids of* $\mathcal{S}$ *in* $\langle A, B\rangle$ *have a plane in common with* α *or contain the line* $P_{AC}P_{BC}$*.*

Proof. Consider a solid E of $\mathcal{S}$ in $\langle A, B\rangle$, not having a plane in common with α. As E needs to contain a line of every plane D'_i, E needs to contain the line $P_{AC}P_{BC}$. □

In this case, the number of solids of $\mathcal{S}$ outside of $\langle A, B\rangle$ is at most $\theta_1\Big(\theta_{n-3} - \theta_2\Big) = O(q^{n-2})$, where θ_1 is the number of the planes D'_i through the line $P_{AC}P_{BC}$ in α, times the number of solids, through a plane in α, not contained in $\langle A, B\rangle$.
While, there are at most $\begin{bmatrix}4\\2\end{bmatrix} + \Big(\theta_3 - \theta_1\Big)\Big(\theta_2 - 1\Big) = O(q^5)$ solids of $\mathcal{S}$ in $\langle A, B\rangle$, where the first term $\begin{bmatrix}4\\2\end{bmatrix}$ is the number of solids (the solid α included) through the line $P_{AC}P_{BC}$ in $\langle A, B\rangle$ and the second one, the number of solids in $\langle A, B\rangle$ that meet α in a plane not through the line $P_{AC}P_{BC}$.

4.1.1.3 α is a 4-space

By Lemma 4.1.1, we can suppose that $\alpha = \langle P_{AB}, l_{AC}, l_{BC} \rangle$ with P_{AB} a point on l_{AB}. Note that there exist solids D_i, D_j, with $i, j \in I$, such that their intersections D_i' and D_j' with α, meet in a point. Indeed, by Theorem 3.3.4, if all the planes D_i' would pairwise intersect in a line, then these planes lie in a fixed solid or contain a fixed line. Both possibilities contradict this case where α is a 4-space. Moreover, if $D_i' \cap D_j'$ is a point, then the corresponding solids D_i and D_j also need to intersect outside of $\langle A, B \rangle$.
Now, let $\mathcal{L}$ be the set of lines $D_i \cap C$, $i \in I$, and we discuss the construction of the solids in $\langle A, B \rangle$.

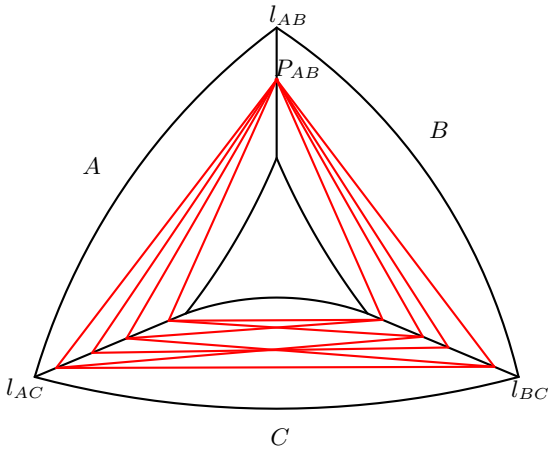

Figure 4.3: There is a configuration in $\mathcal{S}$ and $\dim(\alpha) = 4$

Proposition 4.1.3. *Under the assumption* $(\diamond)$ *and if* $\dim(\alpha) = 4$*, a solid of* $\mathcal{S}$ *in* $\langle A, B \rangle$ *either*

i) is contained in α*, or*

ii) contains P_{AB} *and a line* r *of* C*, intersecting all the lines of* $\mathcal{L}$*.*

Proof. Case 1. Suppose that E is a solid of $\mathcal{S}$ in $\langle A, B \rangle$, not containing P_{AB}. As E needs to contain at least a line of every plane D_i', E contains at least a point of every line in the planes D_i'. This implies that E contains lines l_A, l_B in the planes $\langle P_{AB}, l_{AC} \rangle$ and $\langle P_{AB}, l_{BC} \rangle$ respectively. Remark that l_A and l_B lie in α, are disjoint and span E. This implies that $E \subset \alpha$, so this is the first possibility in the statement. Remark that if $E \subset \alpha$, then E intersects all the planes D_i' in at least a line.
Case 2. So, now we can suppose that E contains the point P_{AB} and intersects

α in a plane γ. The plane γ is the span of P_{AB} and the line $r = \gamma \cap C$. As $E \cap D_i$ is at least a line of the plane D_i' for every $i \in I$, and since every two lines in a plane meet, we have that r has to intersect all the lines of $\mathcal{L}$. Hence we find the second possibility. □

Then, we distinguish two cases to analyze, given a solid of $\mathcal{S}$ in $\langle A, B\rangle$, not in α, the possibilities for the line r of the point *ii*) in the previous proposition, depending on the structure of $\mathcal{L}$.

Case 4.1. There is a line $l \in \mathcal{L}$ that intersects all the lines of $\mathcal{L}$

Note that there cannot be two lines in $\mathcal{L}$ intersecting all the lines of $\mathcal{L}$, as otherwise all the lines of $\mathcal{L}$ lie in a plane or go through a fixed point in C. This gives a contradiction as all the lines of $\mathcal{L}$ span C and at least two points of both l_{AB} and l_{BC} are covered by the lines of $\mathcal{L}$.

Let $P_A = l \cap l_{AC}$ and $P_B = l \cap l_{BC}$. Since every line $m \neq l$ of $\mathcal{L}$ intersects the lines l_{AC}, l_{BC} and l, then follows that m contains the point P_A or the point P_B. As a consequence of the Proposition 4.1.3, we have that a solid of $\mathcal{S}$ in $\langle A, B\rangle$, not contained in α, goes through P_{AB} and it meets C in a line r of the plane $\langle P_B, l_{AC}\rangle$ through P_A or in a line of $\langle P_A, l_{BC}\rangle$ through P_B.

Now, for every plane D_i' different from $\langle l, P_{AB}\rangle$, there are at most $\theta_2 - \theta_1 = q^2$ ways to extend this plane to a solid D_i as this solid also has to meet several solids of $\mathcal{S}$ outside of $\langle A, B\rangle$. As the plane $\langle l, P_{AB}\rangle$ already meets all the planes D_i' in a line in $\langle A, B\rangle$, there are $\theta_{n-3} - \theta_2 = O(q^{n-3})$ ways to extend $\langle l, P_{AB}\rangle$ to a solid not in $\langle A, B\rangle$. For the solids inside $\langle A, B\rangle$, there are θ_4 solids in α, and at most $(2q+1)\Big(\theta_2 - \theta_1\Big)$ solids of the second type in the statement of Proposition 4.1.3, not contained in α. The first factor of this last counting is the number of planes $\langle P_{AB}, m\rangle$ in $\langle A, B\rangle$, with m a line in $\langle P_B, l_{AC}\rangle$ through P_A or a line in $\langle P_A, l_{BC}\rangle$ through P_B. The second factor is the number of ways to extend this plane to a solid not contained in α.
In total, we have at most $2q \cdot q^2 + 1 \cdot q^3\theta_{n-6} + \theta_4 + (2q+1)q^2 = O(q^{\max\{n-3,4\}})$ solids in $\mathcal{S}$.

Case 4.2. For every line in $\mathcal{L}$, there exists another line in $\mathcal{L}$ disjoint to the given line

Depending on the structure of the set $\mathcal{L}$ of lines, as in the subsection before, we discuss the construction of the solids in $\langle A, B\rangle$ not contained in α. We have different possibilities for the line r of C from Proposition 4.1.3:

i) Suppose there are three pairwise disjoint lines in $\mathcal{L}$, then these lines are part of a unique regulus $\mathcal{R}$.

a) If $\mathcal{L}$ is contained in $\mathcal{R}$, then $|\mathcal{L}| \leq q+1$ and r is a line of the opposite regulus $\mathcal{R}^c$. Hence there are $q+1$ possibilities for r.

b) If $\mathcal{L}$ is not contained in $\mathcal{R}$, then there are at most two lines, namely l_{AC} and l_{BC}, in $\mathcal{R}^c$, intersecting all the lines of $\mathcal{L}$. Let $l \in \mathcal{L} \setminus \mathcal{R}$. If there was a third line r meeting all lines of $\mathcal{L}$, then $r \in \mathcal{R}^c$. But then there would be three lines, namely r, l_{AC} and l_{BC}, in $\mathcal{R}^c$, all of them intersecting l. Hence l also has to lie in $\mathcal{R}$, a contradiction. In this case there are at most 2 possibilities for r and $|\mathcal{L}| \leq (q+1)^2$.

ii) Suppose there are no three pairwise disjoint lines in $\mathcal{L}$. In this case we can prove the following lemma.

Lemma 4.1.4. *The set $\mathcal{L}$ is contained in the union of two point-pencils such that their vertices are contained either in l_{BC} or in l_{AC}.*

Proof. we can suppose that $\mathcal{L}$ contains at least two disjoint lines l_1, l_2, since the lines of $\mathcal{L}$ span the solid C. Let $P_i = l_{AC} \cap l_i$ and $Q_i = l_{BC} \cap l_i$ for $i = 1, 2$. Now, we shall distinguish various cases depending on the size of $\mathcal{L}$:

- If $|\mathcal{L}| = 2$, clearly $\mathcal{L}$ is contained in the union of the point-pencils through P_1 and P_2 or equivalently through Q_1 and Q_2.
- If $|\mathcal{L}| = 3$, let l_3 be the line of $\mathcal{L} \setminus \{l_1, l_2\}$. Then l_3 contains at least one of the points P_1, P_2, Q_1, Q_2 (w.l.o.g. P_1). Again we find that $\mathcal{L}$ is contained in the union of the point-pencils through P_1 and P_2. Note that since for every line in $\mathcal{L}$, there exists another line in $\mathcal{L}$ disjoint from it, l_3 cannot be the line P_1Q_2.
- If $|\mathcal{L}| = 4$, let $l_3 \neq l_4$ be the lines of $\mathcal{L} \setminus \{l_1, l_2\}$. Then l_3 and l_4 both contain at least one of the points P_1, P_2, Q_1, Q_2. W.l.o.g. we can suppose that l_3 contains the point P_1. Now, if $l_3 = P_1Q_2$, l_4 must contain P_2 or Q_1 as otherwise either l_1, l_2 and l_4 are three pairwise disjoint lines or there exists a line (l_3) that meets all the other ones. So, we find that $\mathcal{L}$ is contained in the union of the point-pencils through P_1 and P_2 or through Q_1 and Q_2. If $l_3 \neq P_1Q_2$, then l_4 must contain P_1,P_2 or Q_2 as otherwise either l_2, l_3 and l_4 are three pairwise disjoint lines. So, we obtain the lemma statement. Note explicitly that $l_4 \neq P_1Q_2$. As otherwise l_4 meets all lines l_i, $i < 4$.
- If $|\mathcal{L}| \geq 5$, let l_3 be a line of $\mathcal{L} \setminus \{l_1, l_2, P_1Q_2, P_2Q_1\}$. Then l_3 contains one of the points P_1, P_2, Q_1, Q_2. W.l.o.g. we can suppose that l_3 contains the point P_1. If $\mathcal{L} = \{l_1, l_2, l_3, P_1Q_2, P_2Q_1\}$ or if $\mathcal{L}$ only contains lines through P_1 and the lines l_2, P_2Q_1 then $\mathcal{L}$ is contained in the point-pencils through P_1 and P_2. Now, we can suppose that

$\mathcal{L} \setminus \{l_1, l_2, P_1Q_2, P_2Q_1\}$ also contains a line l_4 not through P_1. As $\mathcal{L}$ contains no three pairwise disjoint lines, l_4 contains the point P_2 or Q_2 (w.l.o.g. P_2). Every other line of $\mathcal{L}$ must contain P_1 or P_2 and hence we find again that $\mathcal{L}$ is contained in the union of the point-pencils through P_1 and P_2.

□

By using the notations in the lemma above, since $\mathcal{L}$ contains no 3 pairwise disjoint line, every line $l_0 \in \mathcal{L} \setminus \{l_1, l_2\}$ contains at least one of the points P_1, P_2, Q_1, Q_2 . From Lemma 4.1.4 we find the following possibilities for the set $\mathcal{L}$:

a) if $\mathcal{L}$ only contains two lines l_1, l_2, then l_1 and l_2 are disjoint and we find $(q+1)^2$ possibilities for r, as every such line is defined by a point of l_1 and a point of l_2.

b) if $\mathcal{L}$ contains at least 3 elements and is contained in the union of a line l_0 and a point-pencil through a point P then $|\mathcal{L}| \leq q+2$. Let $P_0 = l_0 \cap l_{AC}$, $Q_0 = l_0 \cap l_{BC}$ and suppose w.l.o.g. that $P \in l_{AC}$. A line r that meets all lines of $\mathcal{L}$ is a line that contains P and a point of l_0 or is a line that contains Q_0 and lies in the plane $\langle P, l_{BC}\rangle$. Hence there are at most $2q+1$ possibilities for the line r.

c) if $\mathcal{L}$ contains at least 3 elements and is contained in the union of two point-pencils through the points P and Q respectively such that $\mathcal{L}$ contains at least two lines through P and at least two lines through Q. Then $|\mathcal{L}| \leq 2(q+1)$. A line r that meets all lines of $\mathcal{L}$ is the line l_{AC}, the line l_{BC} or the line PQ if $PQ \neq l_{AC}$, l_{BC} and $PQ \notin \mathcal{L}$. Hence there are at most 3 possibilities for the line r.

For every plane D_i', with $i \in I$, there are at most $\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = q^2$ ways to extend the plane to a solid D_i, as this solid also has to meet several solids of $\mathcal{S}$ outside of $\langle A, B\rangle$. And since the number of planes D_i' equals the number of lines in $\mathcal{L}$, there are at most $(q+1)\cdot q^2, (q+1)^2 \cdot q^2, 2\cdot q^2, (q+2)\cdot q^2, 2(q+1)\cdot q^2$ solids outside of $\langle A, B\rangle$, respectively, dependent on the four cases above.
For the solids inside $\langle A, B\rangle$, there are θ_4 solids in α and $(q+1)\cdot q^2, 2\cdot q^2, (q+1)^2 \cdot q^2, (2q+1)\cdot q^2, 3\cdot q^2$ solids of the second type of Proposition 4.1.3, respectively. We find these numbers by multiplying the number of possibilities for the line r and the number q^2 of 3-spaces through a plane in $\langle A, B\rangle$, not contained in α. So in total we have at most $\theta_4 + (q^2+2q+3)\cdot q^2 = O(2q^4)$ solids, using case *ib*) or *iia*).

4.1.1.4 α is a 5-space

We start with a lemma that will often be used in this subsection.

Lemma 4.1.5. *Under the assumption* ($\diamond$)*, every two planes D'_i and D'_j share a point on l_{AB}, l_{AC} or l_{BC}.*

Proof. Consider two solids D_i and D_j in $\mathcal{S}$, with corresponding planes D'_i and D'_j in $\langle A, B\rangle$. Since D_i and D_j meet in at least a line, D'_i and D'_j have to meet in at least a point. If D'_i and D'_j do not meet in a point of l_{AB}, l_{AC} or l_{BC}, then these planes define 6 different intersection points $P_1, \ldots, P_6$ on the lines l_{AB}, l_{AC} and l_{BC}. As $\langle D'_i, D'_j\rangle = \langle P_1, \ldots, P_6\rangle = \langle l_{AB}, l_{AC}, l_{BC}\rangle$, we find that D'_i and D'_j span a 5-space, so these planes are disjoint, a contradiction. □

If α is a 5-space, we distinguish two cases, depending on the planes $D'_i = D_i \cap \langle A, B\rangle$, $i \in I$.

Lemma 4.1.6. *Under the assumption* ($\diamond$)*, if* $\dim(\alpha) = 5$*, we have one of the following possibilities for the planes D'_i:*

i) There are four possibilities for the planes D'_i: $\langle P_1, P_3, P_6\rangle$, $\langle P_1, P_4, P_5\rangle$, $\langle P_2, P_4, P_6\rangle$ and $\langle P_2, P_3, P_5\rangle$, where $P_1, P_2 \in l_{AB}, P_3, P_4 \in l_{BC}$ and $P_5, P_6 \in l_{AC}$.

ii) There are three points $P \in l_{AB}, Q \in l_{BC}$ and $R \in l_{AC}$ so that every plane D'_i contains at least two of the three points in the set $\{P, Q, R\}$.

Proof. We prove the Lemma by construction and we start with a plane, we say D'_1, intersecting l_{AB}, l_{BC} and l_{AC} in the points P, Q and R' respectively. *Case (a): There exists a plane D'_2 such that $D'_1 \cap D'_2$ is a point* (w.l.o.g. P, see Lemma 4.1.5) *and let $D'_2 \cap l_{BC}$ be Q' and $D'_2 \cap l_{AC}$ be R.* In this case we know that there exists a third plane D'_3 intersecting l_{AB} in a point P' different from P (as $\dim(\alpha) = 5$). Then D'_3 needs at least a point of D'_2 and D'_1. This implies that D'_3 contains Q and R or Q' and R' (w.l.o.g. Q and R) by Lemma 4.1.5. Now there are two possibilities:

i) There exists a plane $D'_4 = \langle P', Q', R'\rangle$, and then, by construction, we cannot add another plane D'_i. (In the formulation of the lemma $P = P_1, P' = P_2, Q = P_3, Q' = P_4, R = P_5, R' = P_6$.)

ii) There exists no plane $D'_4 = \langle P', Q', R'\rangle$, then, by construction, we see that all the planes need to contain at least two of the three points P, Q, R by Lemma 4.1.5.

Case (b): all the planes D'_i intersect pairwise in a line. Then all these planes have to lie in a solid (contradiction since they span a five-space) or they go through a fixed line l. In this last case, l cannot be one of the lines l_{AB}, l_{AC}, l_{BC}

and also, l cannot intersect one of these lines, as otherwise all the planes D'_i would contain the intersection point of this line and l (which gives a contradiction since $\dim(\alpha) = 5$). Consider now the disjoint lines l and l_{AB}. Then all the planes D'_i would contain l and a point of l_{AB}, but this implies that $\dim(\alpha) = 3$ which also gives a contradiction. We conclude that this case cannot happen. □

Case 5.1. There are four intersections D'_i.

In this situation, we can show that the only solids of $\mathcal{S}$ in $\langle A, B\rangle$ are A, B and C.

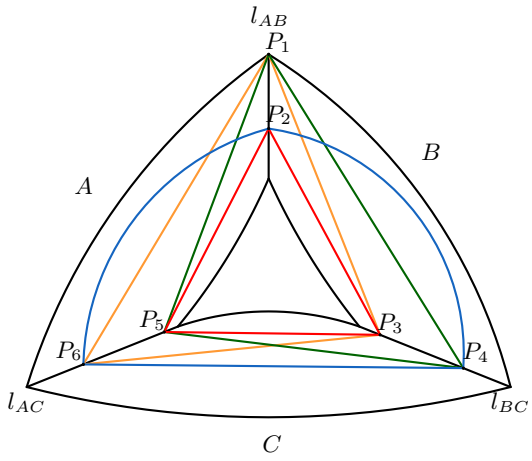

Figure 4.4: There is a configuration in $\mathcal{S}$ and $\dim(\alpha) = 5$

Proposition 4.1.7. *Under the assumption* ($\diamond$)*, and if* $\dim(\alpha) = 5$*, the only solids of* $\mathcal{S}$ *in* $\langle A, B\rangle$ *are* A, B *and* C*.*

Proof. Let $P_1, \ldots, P_6$ be as in the first case of Lemma 4.1.6 (the intersection points of $D_i \cap \langle A, B\rangle$ with the lines l_{AB}, l_{AC}, l_{BC}), and let E be a solid in $\langle A, B\rangle$ different from A, B, C. The solid E cannot contain all the points $P_1, \ldots, P_6$, by its dimension, so we can suppose that $P_1 \notin E$. As E has a line in common with every plane D'_i, E has at least a point in common with every line of these planes D'_i. This implies that E has at least a point in common with $P_1P_3, P_1P_4, P_1P_5, P_1P_6$ or equivalently, a line l_A in common with $\langle P_1, l_{AC}\rangle$ and a line l_B in common with $\langle P_1, l_{BC}\rangle$. If $l_A = l_{AC}$ and $l_B = l_{BC}$, then $E = C$, so we can suppose that $l_A \neq l_{AC}$. We show that in this case $P_2 \in E$.
If $P_2 \notin E$, then E also contains a point of $P_2P_3, P_2P_4, P_2P_5, P_2P_6$. As $l_A \neq l_{AC}$, E contains at least a plane of A. But then E contains precisely a line l_B of B, since E is three-dimensional. This line l_B is disjoint to A as it has to

intersect the lines P_1P_3, P_1P_4, and P_2P_3, P_2P_4 in points different from P_1 and P_2 respectively, and hence lies in the plane $\langle P_1, l_{BC}\rangle$ and in the plane $\langle P_2, l_{BC}\rangle$. This gives that $l_B = l_{BC}$, and so, this line in E would be disjoint from A.
As E contains a plane and a line that are skew, E has to be four-dimensional, a contradiction, so $P_2 \in E$.
Since E cannot contain $P_2, P_3, \ldots, P_6$ (by the dimension), we can suppose that $P_1, P_6 \notin E$. Then, by the previous arguments and symmetry, we know that P_2 and P_5 lie in E. In A, the solid E needs an extra point P of P_1P_6. This gives that E contains the plane $\gamma = \langle P, P_2, P_5\rangle$ of A. As E also needs at least a point of each line $P_1P_3, P_1P_4, P_6P_3, P_6P_4$, E needs at least one extra line, disjoint to γ. Again, this results a contradiction due to its dimension. □

There are $4 \cdot \left(\theta_2 - \theta_1\right)$ solids not in $\langle A, B\rangle$. This number follows since there are 4 planes D_i' and two solids, intersecting $\langle A, B\rangle$ in different planes D_i, have to intersect outside of $\langle A, B\rangle$. There are only 3 solids, A, B, C in $\langle A, B\rangle$.

Case 5.2. Every plane D_i' contains at least two of the points P, Q, R.

Remark that in this situation we have at least the red, green and blue plane (see Figure 4.5) as planes D_i'. In the following Proposition, we prove how the solids in $\langle A, B\rangle$ lie with respect to the points P, Q, R.

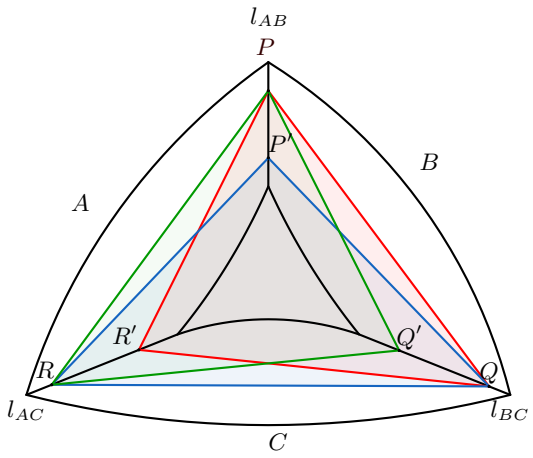

Figure 4.5: There is a configuration in $\mathcal{S}$ and $\dim(\alpha) = 5$

Proposition 4.1.8. *Under the assumption* $(\diamond)$*, and if* $\dim(\alpha) = 5$*, all the solids of* $\mathcal{S}$ *in* $\langle A, B\rangle$*, contain at least two of the points* P, Q, R*.*

Proof. Let E be a solid of $\mathcal{S}$ in $\langle A, B\rangle$, different from A, B and C. Suppose $P \notin E$, then we have to prove that E contains the points R and Q. We find that E contains subspaces in A and B, intersecting the lines PR, PR', $P'R$ and $PQ, PQ', P'Q$, respectively (see Figure 4.5). Hence E meets A in a line l_{AE} through R and a point of PR', or E has a plane π_{AE} in common with A. By symmetry E meets B in a line l_{BE} through Q and a point of PQ', or E has a plane π_{BE} in common with B.

a) If $\dim(A \cap E) = \dim(B \cap E) = 2$ then the planes π_{AE}, π_{BE} meet in a point of l_{AB} as they cannot contain the line l_{AB} since $P \notin E$. Hence E contains two planes meeting in a point, which gives a contradiction since $\dim(E) = 3$.

b) If $\dim(A \cap E) = 2$ and $\dim(B \cap E) = 1$ then $\pi_{AE} \cap l_{AB} = l_{BE} \cap l_{AB}$. First note that $l_{BE} \cap l_{AB}$ is not empty by dimension of E. Now, if $\pi_{AE} \cap l_{AB} \neq l_{BE} \cap l_{AB}$, then $l_{AB} \subset E$, which gives a contradiction as $P \notin E$. Since l_{BE} can only meet l_{AB} in the point P we find a contradiction, again as $P \notin E$.

Hence we know that E contains a line $l_A \subset A$ through R and a line $l_B \subset B$ through Q, which proves the proposition. □

There are $\big(3 \cdot \theta_1 - 2\big)\big(\theta_2 - \theta_1\big)$ solids not in $\langle A, B\rangle$ as two solids, intersecting $\langle A, B\rangle$ in different planes, have to intersect outside of $\langle A, B\rangle$, and there are at most $3 \cdot \theta_1 - 2$ planes D_i'. There are at most $\theta_2 + 3q^2\theta_2$ solids in $\langle A, B\rangle$, namely all the solids through the plane $\langle P, Q, R\rangle$ and all solids precisely through two of the three points P, Q, R in $\langle A, B\rangle$.

We conclude this section by enunciating a result about the upper bound on the size of the largest examples.

Proposition 4.1.9. *In the projective space* $\mathrm{PG}(n, q)$*, with* $n \geq 5$*, let* $\mathcal{S}$ *be a maximal set of solids pairwise intersecting in at least a line such that there is no point contained in all the elements of* $\mathcal{S}$*. Under the assumption* $(\diamond)$*,*

i) if there are no solids outside of $\langle A, B\rangle$*, then* $\mathcal{S}$ *is the set of solids in a 5-space and* $|\mathcal{S}| = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

ii) if $\dim(\alpha) = 2$*, then* $|\mathcal{S}| \leq \theta_3 + \theta_2\Big(\begin{bmatrix} 4 \\ 2 \end{bmatrix} - \theta_2\Big) = O(q^{\max\{n-3,6\}})$.

iii) if $\dim(\alpha) = 3$*, then* $|\mathcal{S}| \leq q^3\theta_1\theta_{n-6} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + q^3\theta_1^2 = O(q^{\max\{n-2,5\}})$.

iv) Let $\mathcal{L} = \{D_i \cap C \mid i \in I\}$ *and* $\dim(\alpha) = 4$

a) if there is a line $l \in \mathcal{L}$ *that meets all the lines of* $\mathcal{L}$*, then*

$$|\mathcal{S}| \leq \theta_{n-3} + q^2(q^2+5q+1) = O(q^{\max\{n-3,4\}}).$$

b) if for every line in $\mathcal{L}$, there exists another line in $\mathcal{L}$ disjoint from it, then

$$|\mathcal{S}| \leq \theta_4 + q^2(q^2+2q+3) = O(2q^4).$$

v) if $\dim(\alpha) = 5$, *the size of* $\mathcal{S}$ *is at most*

$$4q^2 + 3 = O(4q^2)$$

or at most

$$(3q+1)q^2 + \theta_2(3q^2+1) = O(3q^4)$$

depending on the fact that either there are four intersections for the planes D_i' or there exist three points $P \in l_{AB}, Q \in l_{BC}, R \in l_{AC}$ such that every plane D_i' contains at least two of these points P, Q and R.

	dim(α)	**Order of solids not in** $\langle A,B\rangle$	**Order of solids in** $\langle A,B\rangle$
PG$(5,q)$		/	q^8
PG$(n,q), n>5$	2	q^{n-3}	q^6
	3	q^{n-2}	q^5
	4	*Case 4.1.*	
		q^{n-3}	q^4
		Case 4.2.	
		q^4	$2q^4$
	5	*Case 5.1.*	
		$4q^2$	3
		Case 5.2.	
		$3q^3$	$3q^4$

Table 4.1: Upper bound on the order of the largest examples size in the case there is a configuration.

4.1.2 There is no configuration

Now, in this subsection, we will work under the assumption:

($\diamond\diamond$) There is no configuration of solids in $\mathcal{S}$.

As a direct consequence of ($\diamond\diamond$), we obtain the following result.

Corollary 4.1.10. *Under the assumption ($\diamond\diamond$), every three solids in $\mathcal{S}$ have at least a point in common.*

We may assume that there exist two solids, A and B, meeting each other in a line l_{AB}. Otherwise all the solids would pairwise meet each other in a plane in which case the classification is known. We can also suppose that we have no point or line contained in all the solids of $\mathcal{S}$.

To investigate the structure of $\mathcal{S}$ under the assumption ($\diamond\diamond$), we divide the discussion into two cases:

Case 1. *There is a solid C intersecting l_{AB} in a point P_C and intersecting A and B in the lines l_{AC} and l_{BC}* (see Figure 4.6). Suppose there is another solid D intersecting l_{AB} in a point P_D different from P_C and intersecting A in the line l_{AD} and B in at least a line. We know there is such a solid since not all the solids go through P_C. As A, C, D have a point in common, l_{AC} and l_{AD} have to intersect. Similarly in B, l_{BC} and $B \cap D$ have to intersect. Let Z be the solid spanned by l_{AB}, l_{AC}, l_{BC}.

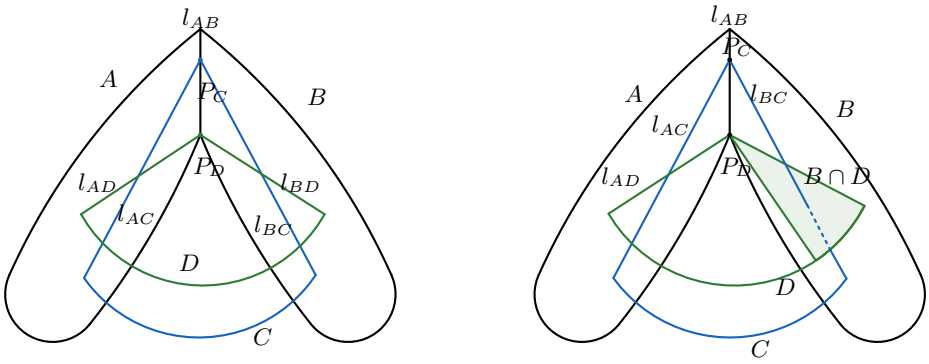

Figure 4.6: There is no configuration in $\mathcal{S}$, Case 1. .

Proposition 4.1.11. *If the assumption ($\diamond\diamond$) holds, all the solids in $\mathcal{S}$ have a plane in common with Z.*

Proof. It is easy to see that A, B, C, D have a plane in common with Z. Consider now a solid E not through l_{AB}, and through a point $P_E \neq P_C$ of l_{AB}. Then E contains a point P_1 of $A \cap C = l_{AC}$, and a point P_2 of l_{BC}, see Corollary 4.1.10. This implies that E contains a plane (spanned by P_E, P_1, P_2) of Z. Before we continue with a solid through the point P_C, we distinguish two cases.
Suppose first that $B \cap D$ is the line l_{BD}. Then the argument also works for a solid through the point P_C and not containing l_{AB}, as here we can work with the lines l_{AD} and l_{BD} instead of l_{AC} and l_{BC} respectively.

If $B \cap D$ is a plane, then we can remark that $C \cap D$ is a line and not a plane. Otherwise, C would contain a line of $B \cap D$ which would imply that C contains a plane of B, which contradicts the assumption in Case 1. So here we can follow the same argument by replacing B by C: then D intersects A and C in a line, not through P_C, to prove that the solid E has a plane in common with Z.
Consider now a solid F through l_{AB}. If F contains l_{AC}, then F contains the plane spanned by l_{AB}, l_{AC} in Z, so suppose that F does not contain l_{AC}. Then we can use the same arguments as above where we replace B by C. □

Remark 4.1.12. Note that Proposition 4.1.11 also works when there are four solids X, Y, Z, T, such that

a) X, Y both intersect A in a line and these lines have different points in common with l_{AB},

b) Z, T both intersect B in a line and these lines have different points in common with l_{AB}.

Remark that the pairs X, Y and Z, T of solids are not necessarily disjoint.

There are θ_3 possibilities for the planes in Z, and through a plane, there are θ_{n-3} solids, so there are at most $\theta_3\Big(\theta_{n-3} - 1\Big) + 1 = O(q^n)$ solids in this case.

Case 2. *Every solid, not through the line l_{AB}, has a plane in common with A or B.* Remark that a solid, intersecting l_{AB} in a point, cannot have a plane in common with both A and B due to the dimension. By taking into account Remark 4.1.12, there are only two subcases left.

Case 2.1. *All the solids not through the line l_{AB} have a plane in common with A and a line in common with B.* Remark that all these lines in B pairwise intersect (Corollary 4.1.10), and so, lie in a plane λ in B through l_{AB}. This implies that all these solids, not through l_{AB}, lie in the 4-space Y spanned by λ and A. Consider now a solid C through l_{AB}. Then, again by Corollary 4.1.10, C has to contain at least a plane through l_{AB} of Y. This follows since all the solids not through l_{AB} lie in Y and not all these solids meet l_{AB} in the same point.

So, we obtain that all the solids not through l_{AB} lie in a fixed 4-space Y and all the solids through l_{AB} intersect Y in at least a plane.
There are θ_4 solids in Y and at most $\theta_2 \cdot \Big(\theta_{n-3} - \theta_1\Big) = O(q^{n-1})$ solids through l_{AB}, not in Y.

Case 2.2. *There is one solid C, not through l_{AB}, intersecting B in a plane and A in a line, and all other solids not through l_{AB} intersect A in a plane*

and B in a line. All these lines in B lie in a plane λ through l_{AB} (by Corollary 4.1.10). We show that all the solids have a plane in common with the solid Z spanned by λ and the line $C \cap A = l_{AC}$.

Proposition 4.1.13. *Under the assumption $(\diamond\diamond)$, all the solids in $\mathcal{S}$ have a plane in common with Z.*

Proof. Since the solid Z is spanned by $\lambda \subset B$ and the line l_{AC}, we find that the proposition holds for the solids A and B, and also for C since the plane $C \cap B$ intersects λ in at least a line. Consider a solid $D \neq C$, intersecting l_{AB} in a point different from $P_C = C \cap l_{AB}$. Remark that there exists such a solid, as otherwise all the solids would contain the point P_C. Note that $D \cap A$ is a plane in A, intersecting l_{AC} in a point not contained in l_{AB}, by Corollary 4.1.10, and $B \cap D$ is a line in λ, by the definition of λ. This implies that D has at least a plane in common with Z.
Consider a solid $E \neq C, D$, containing P_C, but not through l_{AB}. By Corollary 4.1.10, E contains a point of $D \cap B$ and since $E \cap A$ is a plane, $E \cap A$ has to meet the plane $Z \cap A$ in a line. This implies that E also has a plane in common with Z.
Consider now a solid D through l_{AB}. We will show that D also intersects Z in a plane. Suppose E is a solid not through l_{AB} and intersecting l_{AB} in a point different from P_C. Remark that the solid E exists as otherwise all solids of $\mathcal{S}$ would contain the point P_C. Since the line $E \cap C$, disjoint from l_{AB}, lies in Z and D needs to have a point Q in common with $E \cap C$ by Corollary 4.1.10, we see that D contains the plane $\langle Q, l_{AB} \rangle$ of Z. □

Remark 4.1.14. Note that if there exists another solid S, not through l_{AB}, that meets l_{AB} in P_C and A in a line not in $\langle l_{AC}, l_{AB} \rangle$, then, by Proposition 4.1.13, we obtain two different solids Z_C and Z_S that have the plane λ in common and both meet all the solids of $\mathcal{S}$ in a plane. Hence the solids of $\mathcal{S}$ have to go through λ, or have to lie in the 4-space spanned by Z_C and Z_S.

Again, there are θ_3 possibilities for the planes in Z, and through a plane, there are θ_{n-3} solids, so there are at most $\theta_3\big(\theta_{n-3}-1\big)+1 = O(q^n)$ solids in this case.

To conclude this section, we summarize the results obtained and we give an overview of the largest examples of $\mathcal{S}$.

Theorem 4.1.15. *Let $\mathcal{S}$ be a maximal set of solids pairwise intersecting in at least a line such that there is no point contained in all the solids of $\mathcal{S}$ and let $A, B \in \mathcal{S}$, intersecting in a line l_{AB}. Under the assumption $(\diamond\diamond)$, either*

(i) there exists a solid Z such that all the solids of $\mathcal{S}$ have a plane in common with Z, or

(ii) all the solids of $\mathcal{S}$ not through l_{AB} lie in a fixed 4-space Y and all the solids through l_{AB} intersect Y in at least a plane.

In particular, the size of $\mathcal{S}$ is at most either $q\theta_3\theta_{n-4}+1 = O(q^n)$ or at most $\theta_4 + q^2\theta_2\theta_{n-5} = O(q^{n-1})$, respectively.

4.2 Generalization to k-spaces pairwise intersecting in at least a $(k-2)$-space

From now on, we generalize the previous results to $k > 3$. As in Section 4.1, let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in the projective space $\mathrm{PG}(n,q)$ with $n \geq k+2$ and we suppose that there is no point contained in all the k-spaces of $\mathcal{S}$.
Note, again, that the set of all k-spaces in a fixed $(k+2)$-space is an example of a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space, with size $\begin{bmatrix} k+3 \\ 2 \end{bmatrix} = O(q^{2k+2})$. Therefore we assume that $\mathcal{S}$ spans at least a $(k+3)$-space and we distinguish two cases depending on whether there is a configuration or not in $\mathcal{S}$.

4.2.1 There is a configuration

Suppose there exist three k-spaces A, B and C in $\mathcal{S}$ that form a configuration. Let $A \cap B = \pi_{AB}$, $A \cap C = \pi_{AC}$, $B \cap C = \pi_{BC}$. Note that $\langle A,B\rangle = \langle B,C\rangle = \langle A,C\rangle$ and, by Grassmann's formula, $A \cap B \cap C = \pi_{ABC}$ is a $(k-4)$-space.
Since $\mathcal{S}$ is not contained in a $(k+2)$-space, let $\{D_i \mid i \in I\}$ be the family of the k-spaces of $\mathcal{S}$ not contained in $\langle A,B\rangle$, where I is a certain set of indices.
In this section, by investigating the quotient space of π_{ABC} and using the results for $k=3$, it will become clear that we find several results for general k. We first present a lemma that follows from the existence of a configuration in $\mathcal{S}$.

Lemma 4.2.1. *If there exists a configuration of k-spaces A, B and C in $\mathcal{S}$, then a k-space of $\mathcal{S}$ not in $\langle A,B\rangle$ contains π_{ABC} and intersects π_{AB}, π_{AC} and π_{BC} in a $(k-3)$-space through π_{ABC}.*

Proof. Consider a k-space E of $\mathcal{S}$ not in $\langle A,B\rangle$. Clearly

$$k-2 \leq \dim(E \cap \langle A,B\rangle) \leq k-1.$$

If $\dim(E\cap\langle A,B\rangle) = k-2$, then this $(k-2)$-space has to lie in the $(k-4)$-space π_{ABC}, which gives a contradiction. So we can suppose that $\dim(E\cap\langle A,B\rangle) = k-1$. Moreover, E has to meet at least one of the three k-spaces, A, B or C in a $(k-2)$-space, as otherwise E lies in $\langle A,B\rangle$, again a contradiction. Then,

without loss of generality, we can suppose that $\dim(E \cap C) = k-2$. Since $E \cap A$ and $E \cap B$ are both contained in the $(k-1)$-space $E \cap \langle A, B\rangle$ and they have at least dimension $k-2$, $E \cap \pi_{AB}$ has dimension at least $k-3$. Similarly, we have also that $\dim(E \cap \pi_{AC}) \geq k-3$ and $\dim(E \cap \pi_{BC}) \geq k-3$. These last two inequalities imply that $\dim(E \cap \pi_{AC} \cap \pi_{BC}) = \dim(E \cap \pi_{ABC}) \geq k-4$ as $\dim(E \cap \langle \pi_{AC}, \pi_{BC}\rangle) = \dim(E \cap C) = k-2$. Since $\dim(\pi_{ABC}) = k-4$, E contains π_{ABC}. By the investigation of the quotient space with respect to π_{ABC} and Lemma 4.1.1, we obtain the result. □

Again, by Lemma 4.2.1, all the k-spaces D_i not contained in $\langle A, B\rangle$, intersect $\langle A, B\rangle$ in a $(k-1)$-space $D'_i = D_i \cap \langle A, B\rangle$. In the following discussion we will work under the assumption:

$(\diamond)_G$ There is a configuration of k-spaces A, B and C in $\mathcal{S}$, and α is the span of all the $(k-1)$-spaces $D'_i = D_i \cap \langle A, B\rangle$, with $i \in I$.

We distinguish between several cases depending on the dimension of α.

4.2.1.1 α is a $(k-1)$-space

In this case we can remark that $\forall i \in I, D_i \cap \langle A, B\rangle = \alpha$, so all the k-spaces not in $\langle A, B\rangle$ meet $\langle A, B\rangle$ in α.
A k-space of $\mathcal{S}$ in $\langle A, B\rangle$ needs to have at least a $(k-2)$-space in common with every D_i not in $\langle A, B\rangle$. This implies that every k-space in $\langle A, B\rangle$ meets α in at least a $(k-2)$-space. The condition that every two k-spaces in $\langle A, B\rangle$ meet in at least a $(k-2)$-space is fulfilled by the dimension.

Let n be the dimension of the ambient projective space. Here, we have at most $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = O(q^{n-k})$ k-spaces of $\mathcal{S}$ outside of $\langle A, B\rangle$; this is the number of k-spaces through a fixed $(k-1)$-space α in $\mathrm{PG}(n, q)$ excluded those in $\langle A, B\rangle$. There are at most $\begin{bmatrix} k \\ 1 \end{bmatrix} q^2(q^2+q+1)$ k-spaces in $\langle A, B\rangle$ meeting in α in a $(k-2)$-space, and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ k-spaces through α in $\langle A, B\rangle$. This implies that $|\mathcal{S}| \leq \theta_{n-k} + q^2\theta_{k-1}\theta_2$.

4.2.1.2 α is a k-space

As every D'_i $(i \in I)$, contains π_{ABC} and a $(k-3)$-space through π_{ABC} in π_{AB}, π_{AC} and π_{BC} respectively, we can suppose that α is spanned by π_{AB} and two points P_{AC}, P_{BC} of π_{AC}, π_{BC} outside of π_{ABC} respectively. Remark that all the k-spaces D_i have a $(k-1)$-space D'_i in common with α and contain $\langle \pi_{ABC}, P_{AC}P_{BC}\rangle$, so all the k-spaces not in $\langle A, B\rangle$ already intersect in a $(k-2)$-space inside $\langle A, B\rangle$. We will show that all the k-spaces of $\mathcal{S}$ in $\langle A, B\rangle$ have a $(k-1)$-space in common with α or contain $\langle \pi_{ABC}, P_{AC}P_{BC}\rangle$.

Proposition 4.2.2. *Under assumption $(\diamond)_G$, all the k-spaces of $\mathcal{S}$ in $\langle A, B\rangle$ have a $(k-1)$-space in common with α or contain $\alpha \cap C = \langle \pi_{ABC}, P_{AC}P_{BC}\rangle$.*

Proof. Consider a k-space E of $\mathcal{S}$ in $\langle A, B\rangle$, not having a $(k-1)$-space in common with α. As E needs to contain a $(k-2)$-space of every D'_i, E needs to contain $\langle \pi_{ABC}, P_{AC}P_{BC}\rangle$. □

Here we have at most $\begin{bmatrix}2\\1\end{bmatrix}\left(\begin{bmatrix}n-k+1\\1\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right) = O(q^{n-k+1})$ k-spaces of $\mathcal{S}$ outside of $\langle A, B\rangle$. This is the number of $(k-1)$-spaces D'_i times the number of k-spaces through a $(k-1)$-space in $\mathrm{PG}(n,q)$ not contained in $\langle A, B\rangle$; and $\begin{bmatrix}4\\2\end{bmatrix} + \left(\begin{bmatrix}k+1\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)\left(\begin{bmatrix}3\\1\end{bmatrix} - 1\right)$ k-spaces of $\mathcal{S}$ in $\langle A, B\rangle$ where the first term is the number of k-spaces through $\langle \pi_{ABC}, P_{AC}P_{BC}\rangle$ whereas the second one, the number of k-spaces that meet α in a $(k-1)$-space not through $\langle \pi_{ABC}, P_{AC}P_{BC}\rangle$.

4.2.1.3 α is a $(k+1)$-space

Again since every D'_i contains π_{ABC} and a $(k-3)$-space through π_{ABC} in π_{AB}, π_{AC} and π_{BC} respectively, by Lemma 4.2.1 we can suppose that α is spanned by π_{AC}, π_{BC} and a point P_{AB} of π_{AB} outside of π_{ABC}.

Proposition 4.2.3. *Under assumption $(\diamond)_G$, a k-space of $\mathcal{S}$ in $\langle A, B\rangle$ is contained in α or contains π_{ABC}, $\alpha \cap \pi_{AB}$ and a line in $C \setminus \pi_{ABC}$ that intersects with all the $(k-2)$-spaces $D_i \cap C$.*

Proof. For the k-spaces through π_{ABC} we can investigate the quotient space of π_{ABC} and refer to Subsection 4.1 and 4.2. These results imply that a k-space in $\langle A, B\rangle$ through π_{ABC} is contained in α or contains $D_i \cap \pi_{AB}$ and a line in $C \setminus \pi_{ABC}$ that intersects with all the $(k-2)$-spaces $D_i \cap C$.
Now we suppose that E is a k-space in $\langle A, B\rangle$ and not through π_{ABC}. As E contains at least a $(k-2)$-space of all the $(k-1)$-spaces D'_i, we find that E contains at least a hyperplane of π_{ABC}, a point of $\alpha \cap (\pi_{AB} \setminus \pi_{ABC})$ and a line of $\alpha \cap (\pi_{AC} \setminus \pi_{ABC})$ and $\alpha \cap (\pi_{BC} \setminus \pi_{ABC})$. So here again we see that $E \subset \alpha$. □

The upper bound $\begin{bmatrix}n-k+1\\1\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}$ on the number of k-spaces of $\mathcal{S}$ outside of $\langle A, B\rangle$ and the upper bound $(q^2+5q+1)\left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)$ on the number of k-spaces inside $\langle A, B\rangle$ through π_{ABC}, follows from Section 4.1.1.3. Instead the number of k-spaces inside $\langle A, B\rangle$ not through π_{ABC} are at most $\begin{bmatrix}k+2\\1\end{bmatrix} - \begin{bmatrix}5\\1\end{bmatrix} = O(q^{k+1})$.

4.2.1.4 α is a $(k+2)$-space

In this case we prove that all the k-spaces contain π_{ABC}. This implies the possibility to investigate such case by analyzing the quotient space with respect to π_{ABC} and using Section 4.1.1.4.

Proposition 4.2.4. *Under assumption $(\diamond)_G$, every k-space in $\mathcal{S}$ contains π_{ABC}.*

Proof. By Lemma 4.2.1, we know that all the k-spaces outside of $\langle A, B\rangle$ contain π_{ABC}. It is also clear that A, B and C contain π_{ABC}.
Suppose that there is a k-space E in $\langle A, B\rangle$, not through π_{ABC}. As E contains a hyperplane of all the $(k-1)$-spaces D'_i, E has to contain a hyperplane of π_{ABC}, a line of $\pi_{AB} \setminus \pi_{ABC}$, a line of $\pi_{BC} \setminus \pi_{ABC}$ and a line of $\pi_{AC} \setminus \pi_{ABC}$. This would imply that $\dim(E) = k+1$, which gives the contradiction. □

Clearly, in order to have an estimate of the number of k-spaces in and outside of $\langle A, B\rangle$, by the previous proposition, we can use the results for $k = 3$ in Section 4.1.1.4: $|\mathcal{S}| \leq 4 \cdot \left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right) + 3$ or $|\mathcal{S}| \leq \left(3 \cdot \begin{bmatrix}2\\1\end{bmatrix} - 2\right)\left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right) + \begin{bmatrix}3\\1\end{bmatrix}\left(3q^2+1\right)$.

Proposition 4.2.5. *In the projective space $\mathrm{PG}(n, q)$, with $n \geq k+2$ and $k > 3$, let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space such that there is no point contained in all the elements of $\mathcal{S}$. Under the assumption $(\diamond)_G$,*

i) if there are no k-spaces outside of $\langle A, B\rangle$, then $\mathcal{S}$ is the set of k-spaces in a $(k+2)-$space and $|\mathcal{S}| = \begin{bmatrix}k+3\\k+1\end{bmatrix} = O(q^{2k+2})$.

ii) if $\dim(\alpha) = k-1$, then $|\mathcal{S}| \leq \theta_{n-k} + q^2\theta_{k-1}\theta_2 = O(q^{\max\{n-k,k+3\}})$.

iii) if $\dim(\alpha) = k$, then

$$|\mathcal{S}| \leq q^3\theta_{n-k-3}\theta_1 + \begin{bmatrix}4\\2\end{bmatrix} + q^3\theta_{k-2}\theta_1 = O(q^{\max\{n-k+1,k+2\}})$$

iv) If $\dim(\alpha) = k+1$, then

$$|\mathcal{S}| \leq q^3\theta_{n-k-3} + q^2(q^2+5q+1) + q^5\theta_{k-4} = O(q^{\max\{n-k,k+1\}})$$

v) if $\dim(\alpha) = k+2$, every k-space goes through π_{ABC}, and the size of $\mathcal{S}$ is at most

$$4q^2 + 3 = O(4q^2)$$

or at most

$$(3q+1)q^2 + \theta_2(3q^2+1) = O(3q^4).$$

4.2.2 There is no configuration

Here again, for the remainder of this section, we will assume that

$(\diamond\diamond)_G$ There is no configuration in $\mathcal{S}$.

Hence, we start with a result that follows immediately by $(\diamond\diamond)_G$,

Corollary 4.2.6. If the assumption $(\diamond\diamond)_G$ holds, then every three k-spaces of $\mathcal{S}$ have at least a $(k-3)$-space in common.

Consider two k-spaces A and B, intersecting in a $(k-2)$-space π_{AB}. We can find those two k-spaces, as otherwise all subspaces would intersect pairwise in a $(k-1)$-space. Again, the k-spaces go through a fixed $(k-1)$-space or all the k-spaces lie in a $(k+1)$-dimensional space. We also suppose that we have no $(k-3)$- or $(k-2)$-pencil as in this case we can investigate the quotient space and use the known EKR results.

Case 1. *There is a k-space C intersecting π_{AB} in a $(k-3)$-space π_C and intersecting A and B in the $(k-2)$-spaces π_{AC} and π_{BC} through π_C.*
Suppose there is another solid D intersecting π_{AB} in a $(k-3)$-space π_D different from π_C and intersecting A in the $(k-2)$-space π_{AD} and B at least in a $(k-2)$-space. We know there is such a k-space since not all the k-spaces go through π_C and since π_D cannot contain a $(k-1)$-space of both A and B by the dimension. By symmetry, here, we can suppose that $\dim(D \cap A) = \dim(\pi_{AD}) = k-2$. As A, C, D have a $(k-3)$-space in common, π_{AC} and π_{AD} have at least a point in common outside of π_{AB}. Analogously in B, π_{BC} and $B \cap D$ have a point in common outside of π_{AB}.
Let Z be the k-space spanned by $\pi_{AB}, \pi_{AC}, \pi_{BC}$.

Proposition 4.2.7. *If assumption $(\diamond\diamond)_G$ holds, then all the k-spaces of $\mathcal{S}$ have a $(k-1)$-space in common with Z.*

Proof. It is clear that A, B, C, D have a $(k-1)$-space in common with Z.
For a k-space through the $(k-4)$-space $A \cap B \cap C \cap D$, we investigate the quotient space of $A \cap B \cap C \cap D$ and refer to Lemma 4.1.11. Consider now a k-space E not through $A \cap B \cap C \cap D$. Then E contains a $(k-3)$-space of π_{AB} that intersects $A \cap B \cap C$ in a $(k-4)$-space. As $E \cap \pi_{AC}$ and $E \cap \pi_{BC}$ are at least $(k-3)$-dimensional, we see that E contains an extra point of $\pi_{AC} \setminus \pi_{AB}$ and of $\pi_{BC} \setminus \pi_{AB}$ respectively. This implies that E contains at least a $(k-1)$-space of Z. □

Remark 4.2.8. Note that Proposition 4.2.7 is true also if there are two k-spaces X and Y intersecting π_{AB} in different $(k-3)$-spaces and both intersecting A in different $(k-2)$-spaces, and two k-spaces Z and T, intersecting

π_{AB} also in different $(k-3)$-spaces and intersecting B both in a $(k-2)$-space. Note that it is not necessary that $\{X,Y\} \cap \{Z,T\} = \emptyset$.

There are θ_k possibilities for a $(k-1)$-space in Z, and there are θ_{n-k} k-spaces through a $(k-1)$-space , so there are at most $q\theta_k\theta_{n-k-1}+1$ k-spaces in this case.

Case 2. *Every k-space, not through π_{AB}, has a $(k-1)$-space in common with A or B.* Remark that a k-space, intersecting π_{AB} in a $(k-3)$-space, cannot have a $(k-1)$-space in common with both A and B due to the dimension of π_{AB}. By taking into account Remark 4.2.8, there are only two subcases left.

Case 2.1. *All the k-spaces not through π_{AB} have a $(k-1)$-space in common with A and a $(k-2)$-space in common with B.* Remark that all these $(k-2)$-spaces in B pairwise intersect in a $(k-3)$-space (Corollary 4.2.6). Since there is no $(k-3)$-space contained in all these $(k-2)$-spaces, they lie in a $(k-1)$-space β in B through π_{AB}. This implies that all these k-spaces, not through π_{AB}, lie in the $(k+1)$-space Y spanned by β and A. Consider now a k-space C through π_{AB}. Then, again by Corollary 4.2.6, C has to contain at least a $(k-1)$-space of Y, as all the k-spaces not through π_{AB} lie in Y.

Remark 4.2.9. All the k-spaces not through π_{AB} lie in a fixed $(k+1)$-space Y and all the k-spaces through π_{AB} intersect Y in at least a $(k-1)$-space.

There are θ_{k+1} k-spaces in Y and at most $\begin{bmatrix}3\\1\end{bmatrix}\cdot\left(\begin{bmatrix}n-k+1\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right) = O(q^{n-k+2})$ through π_{AB}, not in Y.

Case 2.2 *From Remark 4.2.8 and Case 2., we can suppose there is precisely one k-space C not through π_{AB}, intersecting B in a $(k-1)$-space and A in a $(k-2)$-space. All other k-spaces not through π_{AB} intersect A in a $(k-1)$-space and B in a $(k-2)$-space.* by Corollary 4.2.6, all these $(k-2)$-spaces in B lie in a $(k-1)$-space β through π_{AB} . We show that all the k-spaces have a hyperplane in common with the k-space Z spanned by β and the $(k-2)$-space $C \cap A = \pi_{AC}$.

Proposition 4.2.10. *If the assumption $(\diamond\diamond)_G$ holds, all the k-spaces have a $(k-1)$-space in common with Z.*

Proof. We see that this holds for all the k-spaces not through π_{AB} and for A and B. Consider now a k-space D through π_{AB}. We will show that D also intersects Z in a $(k-1)$-space. Suppose E is a k-space not through π_{AB} and intersecting π_{AB} in a $(k-3)$-space different from $C \cap \pi_{AB}$. As $E \cap C$ is a $(k-2)$-space that lies in Z and D needs to have a $(k-3)$-space in common with $E \cap C$ by Corollary 4.2.6, we see that D contains a $(k-1)$-space of Z. □

Remark 4.2.11. If there exists another k-space S, not through π_{AB}, that meets π_{AB} in $C \cap \pi_{AB}$ and A in a $(k-2)$-space not in $\langle \pi_{AC}, \pi_{AB} \rangle$, then by the previous proposition we obtain two different k-spaces Z_C and Z_S. They have the $(k-1)$-space β in common and they both meet all the elements of $\mathcal{S}$ in a $(k-1)$-space. Hence the elements of $\mathcal{S}$ have to go through β, or has to lie in the $(k+1)$-space spanned by Z_C and Z_S.

Again, there are θ_k possibilities for a $(k-1)$-space in Z, and there are θ_{n-k} k-spaces through a $(k-1)$-space , so there are at most $q\theta_k\theta_{n-k-1}+1$ k-spaces in this case.

We summarize the results obtained in this section in the following proposition.

Proposition 4.2.12. *Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space such that there is no point contained in all the k-spaces of $\mathcal{S}$ and let $A, B \in \mathcal{S}$, intersecting in a $(k-2)$-space. Under the assumption $(\diamond\diamond)_G$, either*

(i) there exists a k-space Z such that all the spaces in $\mathcal{S}$ have a $(k-1)$-space in common with Z, or

(ii) all the elements of $\mathcal{S}$ not through π_{AB} lie in a fixed $(k+1)$-space Y and all the k-spaces through π_{AB} intersect Y in at least a $(k-1)$-space.

In particular the size of $\mathcal{S}$ is at most either $q\theta_k\theta_{n-k-1}+1 = O(q^n)$ or at most $\theta_{k+1} + q^2\theta_2\theta_{n-k-2} = O(q^{n-k+2})$, respectively.

5 Subspace codes as q-analogues of set systems with restricted intersections

„Y si no es la vid, será
aquel girasol que está
viendo cara a cara al sol,
tras cuyo hermoso arrebol
siempre moviéndose va.“

Pedro Calderón de la Barca, El Mágico Prodigioso, Scene 3, 203-207.

Recently, there has been a new interest in codes whose codewords are vector subspaces of a given vector space over a finite field $\mathbb{F}_q$. These codes have many applications in random network coding and they were introduced by Ho *et al.* in [59]. A mathematical approach was proposed by Kötter, Kschischang in [77] and, later with Silva, in [78]. They are called *subspace codes* and considered to be the q-analogues of classical codes with the Hamming distance or, as we will see in this chapter, the analogues of some results in the extremal set theory originally developed by Fisher in 1940, [43].

5.1 Sets systems with restricted intersections

In Section 3.1, we focused on families of subsets of a given set such that they have the same size and meet pairwise in a subset with *at least* a certain number of elements. A variation of this topic was explored by Fisher in 1940, [43]. More precisely, he took into account this question:

How many subsets of a set of cardinality n can pairwise share the same number of elements?

Later, Bose recognized a few years later that the validity of Fisher's results extends to far more general circumstances: he showed that if in a set, every pair of subsets with the same size has equal intersection size, then the number of these subsets does not exceed the number of the set elements, [13].
An answer to the above question was given by Majumdar in 1953, [91], and rediscovered by Isbell in 1959, [72], when the size of subsets in the family may vary

Theorem 5.1.1 (**Nonuniform Fisher Inequality**). *Let $X_i, \dots, X_m$ be distinct subsets of an n-set such that for every $i \neq j$, $|X_i \cap X_j| = t$ where $1 \leq t < n$. Then $m \leq n$.*

Now, we can state a generalization both above mentioned topics and those ones covered in the original EKR problem. Precisely,

Definition 5.1.2. Let Ω be a set with size n and let I be a set of non negative integers. A family $\mathcal{F}$ of subsets of Ω is *I-intersecting* if $|X \cap Y| \in I$ for every distinct $X, Y \in \mathcal{F}$. Moreover, if the element of $\mathcal{F}$ have the same size k, $\mathcal{F}$ is called a *k-uniform I-intersecting* family.

Clearly, as usual in extremal combinatorics, such a definition gives rise to the problem of determining the size of the largest I-intersecting family both in the uniform and in the non-uniform case. A partial result in this direction was obtained by Ray-Chaudhuri and Wilson

Theorem 5.1.3 (**Ray-Chaudhuri-Wilson Theorem**, [99]). *Let Ω be an n-set and let $\mathcal{F}$ be a k-uniform I-intersecting family of subsets of Ω where $|I| \leq k$. Then*

$$|\mathcal{F}| \leq \binom{n}{|I|}.$$

This is the best possible as long as the answer is to be a function of the parameters n and $|I|$ only. Indeed, a k-uniform $\{0, 1, \dots, k-1\}$-intersecting family of an n-set has cardinality $\binom{n}{k}$.
As we have seen in Theorem 3.1.1 and 3.1.2, stronger results are to be expected if more information on the set I is taken into account.

The rate of growth for fixed t and k (with $t \leq k$) and for very large n was considered as well. Note that for the Ray-Chaudhuri-Wilson bound with $I = \{0, 1, \ldots, t-1\}$, it is $O(n^t)$ and this rate of growth can actually be achieved for any given t and k, in fact

Theorem 5.1.4. *For every integers $k \geq t \geq 1$ and $n \geq 2k^2$, there exists a k-uniform family $\mathcal{F}$ of size strictly greater than $(n/2k)^t$ of subsets of an n-set such that $|X \cap Y| \leq t-1$ for any two distinct sets $X, Y \in \mathcal{F}$.*

In Section 3.1, we recalled the concept of t-pencil of an assigned set. Also here, we have a highly regular configuration, called *sunflower*,

Definition 5.1.5. A family $\mathcal{F} = \{X_1, \ldots, X_m\}$ is a *sunflower* with m *petals* if

$$X_i \cap X_j = \bigcap_{k=1}^{m} X_k$$

for every distinct $i, j \in \{1, \ldots, m\}$. The common intersection of the members of a sunflower is called *kernel* or *center*.

Note that a family of disjoint sets may be considered as a sunflower with empty center. An interesting result due to Erdős and Rado shows that 'large' k-uniform families are sunflowers. More precisely,

Theorem 5.1.6 (**Sunflower Theorem**, [40])**.** *If $\mathcal{F}$ is a k-uniform family of subsets of an n-set with more than $k!(m-1)^k$ elements, then $\mathcal{F}$ contains a sunflower with m petals.*

Proof. We proceed by induction on k. For $k = 1$, we have more than $(m-1)$ elements (disjoint 1-sets), so any m of them form a sunflower with m petals. Now let $k > 2$ and let $\mathcal{T} = \{X_1, \ldots, X_r\}$ be a maximal family of pairwise disjoint members of $\mathcal{F}$. If $r > m$, these sets form a sunflower with $r > m$ petals, and hence the statement. Assume that $r < m$, and let $Y = \bigcup_{i=1}^{m} X_i$, Then $|Y| \leq k(m-1)$. By the maximality of the family $\mathcal{T}$, every member of $\mathcal{F}$ intersects Y. Therefore, there exists an element $x \in Y$, contained in at least

$$\frac{|\mathcal{F}|}{|Y|} > \frac{k!(m-1)^k}{k(m-1)} = (k-1)!(m-1)^{k-1}$$

members of $\mathcal{F}$. Then x can be deleted from these sets and consider the $(k-1)$-uniform family

$$\mathcal{F}_x = \{X \setminus \{x\} \,:\, X \in \mathcal{F},\, x \in X\}.$$

By the induction hypothesis, this family contains a sunflower $\mathcal{S}$ with m petals. Adding x to each member of $\mathcal{S}$ we obtain a subfamily of $\mathcal{T}$ which forms a sunflower with m petals. □

Even today, it is a major open problem whether or not there exists a positive integer C such that every k-uniform family with C^k members necessarily contains a sunflower with three petals.
Finally, a useful result in this context is the following Deza's theorem

Theorem 5.1.7 ([34]). *If every pair of members in a k-uniform family $\mathcal{F}$ shares t elements, then either $|\mathcal{F}| \leq k^2 - k + 1$ or $\mathcal{F}$ is a sunflower, i.e. all the pairwise intersections are the same t-set.*

As we will see, the sunflower is one of the most important structures in the q-analogue of these topics.

5.2 Subspace codes

Let $\mathcal{G}_q(n,k)$ be the set of all the k-dimensional spaces of the vector space $\mathbb{F}_q^n$, with $0 \leq k \leq n$. It is called *k-Grassmannian* where $0 \leq k \leq n$. A *subspace code* $\mathcal{C}$ is a non-empty subset of $\mathcal{G}_q(n) = \bigcup_{k=0}^{n} \mathcal{G}_q(n,k)$, i.e. $\mathcal{C}$ is a collection of subspaces of $\mathbb{F}_q^n$.
Unlike classical coding theory or rank-metric codes where each codeword is a vector or a matrix respectively, in this context each codeword of $\mathcal{C}$ is itself an entire space of vectors.
Indeed, for understanding better the 'q-analogy', we recall that, in the classical coding theory, the *Hamming distance* between two codewords in the vector space $\mathbb{F}_q^n$ is the number of position in which they differ, while the *weight* of a codeword is the number of non-zero entries in it. The *minimum Hamming distance* of a code $\mathcal{A} \subseteq \mathbb{F}_q^n$ is the smallest distance between two distinct codewords while a code is called *constant weight code* if all its codewords have the same weight. Now, in the theory of *subspace codes*, the subsets are replaced by subspaces of a vector space over a finite field, their sizes by the dimensions of the related subspaces and the minimum Hamming distance in the minimum distance of the code in some metric.
Even though this theory is born in a vector setting, as we will retrace in this chapter, in literature some results are obtained in a projective setting to highlight the geometric aspects involved.
A code in which each codeword has the same dimension, i.e. a code contained in a k-Grassmannian, is called a *constant-dimension* code.
Also in this context, it is possible to define a metric in the set $\mathcal{G}_q(n)$. One possible distance between two spaces U and V in $\mathbb{F}_q^n$ is the so-called *subspace metric*

$$d_S(U,V) = \dim(U) + \dim(V) - 2\dim(U \cap V),$$

introduced in [77]. Another way to measure the distance in a subspace code is the *injection distance*. It is introduced in [79] and given by

$$d(U,V) = \max\{\dim(U), \dim(V)\} - \dim(U \cap V).$$

It is easy to see that they are effectively two metrics. Moreover, they are closely related indeed

$$d(U,V) = \frac{1}{2}(d_{\mathrm{S}}(U,V) + |\dim(U) - \dim(V)|) \quad \forall\, U, V \in \mathbb{F}_q^n. \tag{5.2.1}$$

In particular, if U and V have the same dimension, $d_{\mathrm{S}}(U,V) = 2d(U,V)$. The Grassman identity in (3.2.1) gives the following alternative expressions

$$d(U,V) = \dim\langle U, V\rangle - \min\{\dim(U), \dim(V)\}$$

and

$$d_{\mathrm{S}}(U,V) = \dim\langle U, V\rangle - \dim(U \cap V).$$

Also in this context, we can define in the natural way the *minimum distance* of a subspace code $\mathcal{C}$ but it depends on the metric used. In fact, it is defined as

$$d(\mathcal{C}) = \min_{\substack{U,V\in\mathcal{C}\\ U\neq V}} d(U,V) \quad \text{or} \quad d_{\mathrm{S}}(\mathcal{C}) = \min_{\substack{U,V\in\mathcal{C}\\ U\neq V}}, d_{\mathrm{S}}(U,V),$$

if we refer to the injection metric or the subspace metric, respectively. Clearly, by (5.2.1)

$$d(\mathcal{C}) \geq \frac{1}{2} d_{\mathrm{S}}(\mathcal{C}),$$

Moreover, by (5.2.1), for the constant-dimension codes we can only study their properties with one of the two distances introduced before.
A subspace code $\mathcal{C}$ is called an $(n,d)_q$-*code* if $d(\mathcal{C}) = d$, and it is called an $(n,d,k)_q$-*code* if, additionally, $\mathcal{C} \subseteq \mathcal{G}_q(n,k)$. Similarly, $\mathcal{C}$ is called an $(n,d)_q^{\mathrm{S}}$-*code* if $d_{\mathrm{S}}(\mathcal{C}) = d$. The latter notation follows the convention that if a concept is defined for the injection metric, then the analogous concept for the subspace metric is denoted by a superscript $'\mathrm{S}'$.

As we have seen for rank-distance codes, also here we are interested in constructing subspace codes as large as possible in size. Hence, we denote by $A_q(n,d)$, $A_q^{\mathrm{S}}(n,d)$ the sizes of a largest subspace code $\mathcal{C}$ in $\mathbb{F}_q^n$ with mininum distance d in the injection or subspace metric, respectively. While, we will indicate by $A_q(n,d,k)$ the size of a largest subspace code $\mathcal{C} \subseteq \mathcal{G}_q(n,k)$ with mininum distance d in the injection metric. In general, it is not easy to evaluate these quantities, this has given rise to extensive research. In [73], the authors collected general results on lower and upper bounds of (constant-)subspace

codes.

Clearly, as we have told before, the investigation and the construction of such objects are closely linked to the geometric properties of the finite projective spaces. Indeed, a constant-subspace code can be seen as a family of projective subspaces such that they pairwise meet in subspaces with dimension in a certain set of integers and these objects are exactly the q-analogue of the intersecting families recalled in the section before.
Indeed, studying the structure of an $(n+1, d, k+1)_q$-code is equivalent to explore the largest maximal examples of k-dimensional projective spaces pairwise intersecting in at most a $(k-d)$-space in $\mathrm{PG}(n, q)$ and vice versa.

5.2.1 Lifted RD-Codes

In this section we shall describe as the rank-metric codes, recalled in the first chapter of this thesis, produce the simplest construction of subspace codes. This link between these classes of codes was first proposed in [102] and then rediscovered in [77] for the special case where the rank-metric code is a Gabidulin code. This construction was later explained in the context of both subspace and injection distance. To avoid confusion with the other distances introduced before, we will indicate the rank-distance defined in Chapter 1 by d_{R}.

Let $X \in \mathbb{F}_q^{k\times m}$ and consider the subspace

$$\Lambda(X) = \langle\, [I_k \;\; X] \,\rangle \in \mathcal{G}_q(k+m, k)$$

where we indicate $\langle [I_k \;\; X] \rangle$ the vector subspace spanned by the rows of the matrix $[I_k \;\; X]$ of order $k \times (k+m)$. The k-space $\Lambda(X)$ is called the *lifting* of X. Similarly, for a matrix code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$, the subspace code

$$\Lambda(\mathcal{C}) = \{\Lambda(X) \,|\, X \in \mathcal{C}\}$$

is called the *lifting* of $\mathcal{C}$ and the map $X \mapsto \Lambda(X)$ is called the *lifting map*.
Since every subspace corresponds to a unique matrix in RREF (*reduced row echelon form*), i.e. a matrix such that

i) all nonzero rows are above any rows of all zeros,

ii) the *pivot element*, this is the first nonzero element from the left of nonzero row, is 1 and it is always strictly to the right of the pivot element of the row above it,

iii) each column containing a pivot has zeros everywhere else,

the lifting map is injective, and therefore $|\Lambda(\mathcal{C})| = |\mathcal{C}|$. Note that $\Lambda(\mathcal{C})$ is a constant-dimension code. Now, we shall show that a subspace code constructed by lifting inherits the distance properties of its underlying rank-metric code, more precisely

Lemma 5.2.1 (**Lifting Lemma**, [74]). *For all* $X, X' \in \mathbb{F}_q^{k\times m}$ *and* $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$,

$$d(\Lambda(X), \Lambda(X')) = d_{\mathrm{R}}(X, X'),$$

$$d(\Lambda(\mathcal{C})) = d_{\mathrm{R}}(\mathcal{C})$$

Proof. We have

$$\begin{aligned} d(\Lambda(X), \Lambda(X')) &= \dim\langle \Lambda(X), \Lambda(X')\rangle - \min\{\dim \Lambda(X), \dim \Lambda(X')\} \\ &= \mathrm{rk}\begin{pmatrix} I_k & X \\ I_k & X' \end{pmatrix} - k \\ &= \mathrm{rk}\begin{pmatrix} I_k & X \\ 0 & X' - X \end{pmatrix} - k \\ &= \mathrm{rk}(X - X') = d_{\mathrm{R}}(X, X') \end{aligned}$$

The second statement immediately follows from the first one. □

In particular, let $\mathcal{C} \subseteq \mathbb{F}_q^{k\times(n-k)}$ be an MRD code with minimum distance d, and without loss of generality let $k \leq n - k$. Then its lifting $\Lambda(\mathcal{C})$ is an $(n, d, k)_q$-code with size

$$|\Lambda(X)| = q^{(n-k)(k-d+1)} \tag{5.2.2}$$

Clearly, this cardinality gives a lower bound on $A_q(n, d)$, in fact optimazing k in (5.2.2), we have

$$A_q(n, d) \geq q^{(n-\lceil \frac{n}{2} \rceil)(\lceil \frac{n}{2} \rceil d+1)}.$$

5.3 Equidistant constant-dimension codes

In this section, we will recall some results about a particular class of constant-dimension subspace codes where the codewords have all the same distance. An *equidistant constant-dimension subspace code* or *ℓ-intersecting code* is a set of k-subspaces of $\mathbb{F}_q^n$ mutually intersecting in an ℓ-space, where $\ell < k$ and $n \geq 2k - \ell$. These subspace codes are often called in the literature $(k; \ell)$-SCIDs, i.e. *Subspaces with Constant Intersection Dimension*. This term was coined in [39].
The largest equidistant constant-dimension subspace code in $\mathbb{F}_q^n$ is said to be *optimal*.
Some optimal binary (i.e. over $\mathbb{F}_2$) equidistant codes form a structure called

partial projective plane. It was defined and studied by Hall in [58].
In this context, the q-analogue of Definition 5.1.5 follows in a very natural way: in a vector setting, an *ℓ-sunflower* is a $(k;\ell)$-SCID $\mathcal{S}$ in which all codewords intersect in the same ℓ-subspace Z. Again, the ℓ-subspace Z is called the *center* of $\mathcal{S}$. Of course, not all ℓ-sunflowers in $\mathbb{F}_q^n$ with the same number of elements (or *petals*) span a subspace of the same dimension. A sunflower $\mathcal{S}$ is said to be of *maximal dimension* if its petals span a subspace of $\mathbb{F}_q^n$ of largest dimension. It is clear that a sunflower is of *maximal dimension* if any element meets the subspace generated by all other elements precisely in the center Z. However, as we have discussed in Section 5.1, from a random network coding point of view the vector (partial) spreads, introduced in a projective setting in Subsection 3.2.1, are 0-sunflowers. Instead, as consequence of Theorem 3.3.4, all the $(k;k-1)$-SCIDs are sunflowers or *balls*, these are sets of k-spaces in a fixed $(k+1)$-space.

In [35], by using a slight modification of Theorem 5.1.7, a crucial result was proved. Here below, we will state it as a result on subspace codes.

Theorem 5.3.1. *If a $(k;\ell)$-SCID in the vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ has more than*

$$\left(\frac{q^k-q^\ell}{q-1}\right)^2+\frac{q^k-q^\ell}{q-1}+1$$

elements, then it is an ℓ-sunflower.

The lower bound of the previous theorem is called the *sunfower bound*. So, we obtain that the largest SCIDs are sunflowers. In other terms, Theorem 5.3.1 sets an upper bound on the size of *non-trivial* subspace codes.
It is believed that the sunflower bound is too large and that already for smaller sizes a $(k;\ell)$-SCID is already a sunflower.
On the other hand, in [21], Chowdhury *et. al.* stated a conjecture, attributed to Deza: if a $(k;\ell)$-SCID in $\mathbb{F}_q^n$ has more than $\theta_{k,q}$ codewords, then the code is a sunflower.
In [42], the authors presented a construction of non-trivial 1-intersecting code in $\mathcal{G}_q(n,k)$, $n \geq \binom{k+1}{2}$, whose size is $\theta_{k,q}$, but in general these families are not optimal. By a computer search, they exhibited a non-sunflower code in $\mathcal{G}_2(6,3)$ with sixteen elements. Clearly, it is a counterexample to Deza's conjecture.
In [5], Bartoli and Pavese showed, by using projective geometry tecniques, that the size of an optimal $(3,1)$-SCID in $\mathbb{F}_2^6$ is twenty. They also provided an example and proved that such a family is unique up to collineations. Their paper concludes the problem started by Beutelspacher *et al.* . Indeed, in [8], a classification of optimal 1-intersecting codes in $\mathcal{G}_q(6,3)$, $q>2$, was obtained. Again, they explored these families in the projective setting, so their results

are described as sets of planes pairwise intersecting in a point.

In equidistant constant-dimension codes theory, some particular SCIDs play an important role, they are called *primitive*. A $(k;\ell)$-SCID $\mathcal{S}$ in the vector space $\mathbb{F}_q^n$ is called *primitive*, if it satisfies the following properties:

i) $\mathcal{S}$ spans the entire space,

ii) there is no nonzero vector contained in all elements of $\mathcal{S}$, i.e. $\bigcap_{\pi\in\mathcal{S}}\pi = \{\mathbf{0}\}$,

iii) each element π of $\mathcal{S}$ is spanned by $\{\pi\cap\sigma:\ \sigma\in\mathcal{S}\setminus\{\pi\}\}$,

iv) $n\geq 2k$.

In [39], primitive $(k;k-2)$-SCID of $\mathbb{F}_q^n$ are extensively studied, following the results in [8]. It turns out that there is essentially only one new example of primitive $(k;k-2)$-SCID for $k\geq 4$. More specifically, an example of primitive $(4;2)$-SCID is given, which is shown to be unique up to collineation. The primitive $(k;k-2)$-SCIDs were fully covered since the nonexistence of primitive $(k;k-2)$-SCIDs for $k\geq 5$ is also proved. In [7], constructions of primitive $(k;k-t)$-SCIDs with $t\geq 3$ are exhibited.

Note that in the vector setting, a q-analogue of Fisher inequality holds as well. A proof of following result can be found in [86], by using hypergraph tecniques

Theorem 5.3.2 (**Fisher Inequality for vector spaces**). *Let $\mathcal{S}$ be a $(k;\ell)$-*SCID *in the vector space $\mathbb{F}_q^n$ with $1\leq\ell<k<n$. Then $|\mathcal{S}|\leq\begin{bmatrix}n\\1\end{bmatrix}_q$.*

Moreover, with a simple arguments, we may show the following

Proposition 5.3.3. *Let $\mathcal{S}$ be a $(k;\ell)$-*SCID *in $\mathbb{F}_q^n$. Then $|\mathcal{S}|\leq\begin{bmatrix}k\\\ell\end{bmatrix}\cdot\theta_{k-1}$ or $\mathcal{S}$ is an ℓ-sunflower.*

Proof. Choose a subspace T which meets all elements of $\mathcal{S}$ in at least an ℓ-space such that $t=\dim(T)$ is minimal. Clearly, $\ell\leq t\leq k$.
Now, if $t=\ell$, then $\mathcal{S}$ is an ℓ-sunflower. If $\ell+1\leq t\leq k$, we show that there exists a ℓ-dimensional subspace L in T such that lies in at least $|\mathcal{S}|/\begin{bmatrix}t\\\ell\end{bmatrix}$ elements of $\mathcal{S}$. By contradiction, suppose that each ℓ-spaces M in T is contained in s_M elements of $\mathcal{S}$, with $s_M<|\mathcal{S}|/\begin{bmatrix}t\\\ell\end{bmatrix}$. Then,

$$|\mathcal{S}|\leq\sum_{\substack{\dim M=\ell\\ M\subseteq T}}s_M<|\mathcal{S}|,$$

obtaining a contradiction. Hence, there is an ℓ-space L in T such that lies in at least $|\mathcal{S}|/\begin{bmatrix}t\\\ell\end{bmatrix}$ elements of $\mathcal{S}$. As $t\leq k$, L is contained in at least $|\mathcal{S}|/\begin{bmatrix}k\\\ell\end{bmatrix}$

elements of $\mathcal{S}$ as well. Now, as L is not contained in all elements of $\mathcal{S}$, there exists $\pi \in \mathcal{S}$ such that meets L in the null space. Now, each subspace in $\mathcal{S}$ through L must meet π in a different 1-space, otherwise two subspaces through π would intersect in an $(\ell+1)$-space. Hence, L lies in at most θ_{k-1} elements of $\mathcal{S}$. Let s_L be the number of elements of $\mathcal{S}$ that contain L, then

$$\frac{|\mathcal{S}|}{\begin{bmatrix} k \\ \ell \end{bmatrix}} \leq s_L \leq \theta_{k-1},$$

and this shows the claim. □

We note explicitly that the bound in proposition above is independent of n. Since for some values of k,ℓ and n the estimates in the Fisher inequality and in Proposition 5.3.3 have the same orders, we have the following

Corollary 5.3.4. *Let $\mathcal{S}$ be a $(k;\ell)$-*SCID *in $\mathbb{F}_q^n$ with $1 \leq \ell < k < n$. Then $|\mathcal{S}| \leq \min\{\theta_{n-1}, \begin{bmatrix} k \\ \ell \end{bmatrix} \cdot \theta_{k-1}\}$ or $\mathcal{S}$ is an ℓ-sunflower.*

and from the last part of the proof above, we obtain

Corollary 5.3.5. *Let $\mathcal{S}$ be a $(k;\ell)$-*SCID *in $\mathbb{F}_q^n$. If $\mathcal{S}$ is not an ℓ-sunflower, then no ℓ-space lies in more than θ_{k-1} elements of $\mathcal{S}$.*

Finally, we note explictly that most of the definitions in this and the section before can be stated for subspaces families of a finite dimensional vector space on any field.

6 Geometrical junta bound for sets of subspaces with two intersection dimensions

„Portami il girasole ch'io lo trapianti
nel mio terreno bruciato dal salino,
e mostri tutto il giorno agli azzurri specchianti
del cielo l'ansietà del suo volto giallino.“

Eugenio Montale, Ossi di Seppia.

In [7] and [62], the authors used a different approach with respect to the one explained in the previous chapter to explore the properties of an equidistant constant-dimension code $\mathcal{C}$. More precisely, they looked at the vector subspace spanned by $\mathcal{C}$ or by an appropriate subsets of its codewords.
For istance, let $\mathbb{V} = \mathbb{V}(\mathbb{F})$ be a vector space over a (possibly finite) field $\mathbb{F}$ and let $\mathcal{S}$ be a $(k; k-t)$-SCID. In [62], defined the subspace

$$S = \langle \pi_1, \dots, \pi_n \rangle \text{ and } I = \langle \pi_i \cap \pi_j \,|\, 1 \leq i < j \leq n \rangle,$$

L. Hernandez Lucas established several upper bounds for $\dim S + \dim I$ in different situations. In particular, she showed that $\dim S + \dim I \leq nk$ and if $(n-1)(k-t) \leq k$, this bound is tight.
In [7], Barrolleta *et al.* investigated SCIDs that span a large subspace and they proved that again sunflowers are the 'largest' SCIDs. More precisely,

Theorem 6.0.1 (Theorem 2, [7])**.** *Let $\mathcal{S}$ be a $(k; k-t)$-*SCID *in a vector space $\mathbb{V}$, with $\mathcal{S} \geq 3$ and $3 \leq t \leq k-1$. If $\dim\langle \mathcal{S} \rangle \geq k + (t-1)(n-1) + 2$ then $\mathcal{S}$ is a $(k-t)$-sunflower.*

The threshold integer in the theorem statement is called the *geometrical sunflower bound*. The authors showed that is sharp by presenting two families of SCIDs that are not sunflowers, but $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$ is attained.

In this chapter, we will take consider the q-analogue of problem described in Section 5.1. Some properties of the constant-dimension subspace codes whose codewords have distance in an assigned set of integers will be investigated and the concept of *junta* will be introduced.
Later, we will focus on the case when only two intersection values for the codewords are assigned. In the same vein of [7], we will generalize Theorem 6.0.1 by determining an upper bound for the dimension of the vector space spanned by the elements of a non-junta code. In addition, if the intersection values are consecutive, we prove that such a bound is tight, and classify the examples attaining the largest dimension as one of four infinite families properly described.

6.1 SPIDs and juntas

In this section we introduce a natural generalization of the concept of $(k;\ell)$-SCID and sunflower. So, let $k, \ell_1, \ell_2, ..., \ell_v$ be non-negative integers such that $\ell_1, \ell_2, ..., \ell_v < k$. We give the following

Definition 6.1.1. A family $\mathcal{S} = \{\pi_1, \pi_2, \dots, \pi_n\}$ of k-spaces of $\mathbb{V}$, is a $(k; \ell_1, \ell_2, ..., \ell_v)$-SPID (*Subspaces with Pre-assigned Intersection Dimensions*) if for each pair of distinct subspaces $\pi_i, \pi_j \in \mathcal{S}$, we have $\dim(\pi_i \cap \pi_j) \in \{\ell_1, \ell_2, ..., \ell_v\}$.

Clearly, when $v = 1$, we get back the definition of a $(k;\ell)$-SCID in $\mathbb{V}$. Also the notion of ℓ-sunflower in $\mathbb{V}$ can be naturally generalized as follows: we say that a $(k; \ell_1, \ell_2, ..., \ell_v)$-SPID $\mathcal{S} = \{\pi_1, \pi_2, \dots, \pi_n\}$ is an ℓ-*junta* in $\mathbb{V}$, if all elements of $\mathcal{S}$ pass through a common ℓ-space of $\mathbb{V}$.
Note that this structure has already been defined with the name of pencil in Chapter 3. Here, we will use the term 'junta' borrowing it by the paper of Dinur and Friedgut in the set theory context, [38].

Before determining a lower bound after which every $(k; \ell_1, \ell_2)$-SPID in $\mathbb{V}$ is a junta code, we give an estimate on the size of a $(k; \ell, \ell+1)$-SPID

Proposition 6.1.2. *Let $\mathcal{S}$ be a $(k; \ell, \ell+1)$-SPID in $\mathbb{F}_q^n$ such that $\mathcal{S}$ is not an 1-junta. Then $|\mathcal{S}| \leq \theta_{k-1}^2 \begin{bmatrix} k \\ \ell \end{bmatrix} / \theta_{\ell-1}$ or there exists an $(\ell+1)$-space meeting all elements of $\mathcal{S}$.*

Proof. Choose a subspace T which meets all elements of $\mathcal{S}$ in at least an ℓ-space such that $t = \dim(T)$ is minimal. Clearly, $\ell < t \leq k$. First consider the case $t > \ell + 1$. By the same tecnique used in Proposition 5.3.3, we find a 1-space $p \in T$ which lies in at least $|\mathcal{S}|\theta_{\ell-1}/\theta_{k-1}$ elements of $\mathcal{S}$. Since p does not lie in all elements of $\mathcal{S}$, there is a k-space $\pi \in \mathcal{S}$ with $p \notin \pi$. Each element of $\mathcal{S}$ through p has to meet π in at least an ℓ-space L. We have at most $\begin{bmatrix} k \\ \ell \end{bmatrix}$

choices for L. Since $t > \ell + 1$ and $\dim\langle L, p\rangle = \ell + 1$, we find a $\pi' \in \mathcal{S}$ which meets $\langle L, p\rangle$ in at most a $(\ell - 1)$-space. Each member of $\mathcal{S}$ through $\langle L, p\rangle$ still has to meet π', but these meet in the trivial space outside of $\langle L, p\rangle$. Hence, we have at most θ_{k-1} members of $\mathcal{S}$ through $\langle L, p\rangle$. This completes the proof. □

Clearly if $\mathcal{S}$ is a $(k; \ell, \ell + 1) -$ SPID and there is a 1-space p contained in all the elements of $\mathcal{S}$, we can pass to the quotient vector space with respect to p and we can apply the proposition above to obtain an upper bound of $\mathcal{S}$.

Now, let $\mathcal{S} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ be a $(k; \ell_1, \ell_2, \ldots, \ell_v)$-SPID. As in [7], for each $j \in \{1, ..., n\}$, the differences

$$\delta_j = \dim\langle \pi_1, \ldots, \pi_j\rangle - \dim\langle \pi_1, \ldots, \pi_{j-1}\rangle,$$

where we put π_0 the null space (i.e., $\delta_1 = k$), will be an important tool during our discussion. Clearly, the δ_j's depend on the order in which subspaces in $\mathcal{S}$ are labeled. Now, we give a proof of a well-known fact which will play a crucial role in the reamining part of this chapter.

Proposition 6.1.3. *Let $k, t_1, t_2, ..., t_v \in \mathbb{N}$ be integers such that $k > t_1 > t_2 > \cdots > t_v \geq 1$. Let $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ be a $(k; k - t_1, k - t_2, ..., k - t_v)$-SPID in a vector space $\mathbb{V}$, with $n \geq 3$. Then there exists a permutation σ of the indices in the set $I_n = \{1, 2, \ldots, n\}$ such that*

$$t_1 = \delta_2(\mathcal{S}^\sigma) \geq \delta_3(\mathcal{S}^\sigma) \geq \ldots \geq \delta_n(\mathcal{S}^\sigma)$$

with

$$\delta_j(\mathcal{S}^\sigma) = \dim\langle \pi_{\sigma(1)}, \ldots, \pi_{\sigma(j)}\rangle - \dim\langle \pi_{\sigma(1)}, \ldots, \pi_{\sigma(j-1)}\rangle$$

Proof. Let $m \in \mathbb{N}$ be the maximum integer for which there exist m k-spaces, $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}$ of $\mathcal{S}$, forming a $(k - t_1)$-sunflower of maximal dimension; obviously $m \geq 2$.
Consider

$$\max_{\substack{1\leq i\leq n \\ i\neq i_1,\ldots,i_m}} \dim(\pi_i \cap \langle \pi_h \,|\, h \neq i\rangle),$$

then there exists an integer, say i_n, in $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$ such that

$$\dim(\pi_{i_n} \cap \langle \pi_h \,|\, h \neq i_n\rangle) = \max_{\substack{1\leq i\leq n \\ i\neq i_1,\ldots,i_m}} \dim(\pi_i \cap \langle \pi_h \,|\, h \neq i\rangle).$$

Similarly, let

$$\max_{\substack{1\leq i\leq n \\ i\neq i_1,\ldots,i_m,i_n}} \dim(\pi_i \cap \langle \pi_h \,|\, h \neq i, i_n\rangle),$$

then there exists an integer, say $i_{n-1} \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m, i_n\}$, such that

$$\dim(\pi_{i_{n-1}} \cap \langle \pi_h \,|\, h \neq i_{n-1}, i_n\rangle) = \max_{\substack{1\leq i\leq n \\ i\neq i_1,\ldots,i_m,i_n}} \dim(\pi_i \cap \langle \pi_h \,|\, h \neq i, i_n\rangle).$$

After $n-m$ steps, we obtain a sequence of indices $(i_{m+1}, \dots, i_n)$. Let σ be a permutation of the indices $\{1, ..., n\}$, fixing the set $\{i_1, i_2, \dots, i_m\}$ and such that $\sigma(j) = i_j$ for every $j = m+1, \dots, n$. Now, consider $\mathcal{S}^\sigma = \{\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)}\}$. We will show that

$$\delta_{j+1}(\mathcal{S}^\sigma) \leq \delta_j(\mathcal{S}^\sigma) \quad \text{for all} \quad j = 2, \dots, n-1.$$

First of all, we have that $\delta_j(\mathcal{S}^\sigma) \leq t_1$, for each $j = 2, \dots, n$; indeed we have

$$\delta_j(\mathcal{S}^\sigma) = k - \dim(\pi_{\sigma(j)} \cap \langle \pi_{\sigma(1)}, \dots, \pi_{\sigma(j-1)} \rangle) \leq k - \dim(\pi_{\sigma(j)} \cap \pi_{\sigma(1)}) \leq t_1.$$

Also, since $\pi_{\sigma(1)}, \dots, \pi_{\sigma(m)}$ form a $(k-t_1)$-sunflower of maximal dimension $\delta_j(\mathcal{S}^\sigma) = t_1$, with $2 \leq j \leq m$.
Note that

$$\dim(\pi_{i_{j+1}} \cap \langle \pi_h \,|\, h \neq i_{j+1}, \dots, i_n \rangle) \geq \dim(\pi_{i_j} \cap \langle \pi_h \,|\, h \neq i_j, \dots, i_n \rangle)$$

for all $m+1 \leq j \leq n-1$, because otherwise we would have

$$\dim(\pi_{i_{j+1}} \cap \langle \pi_h \,|\, h \neq i_{j+1}, \dots, i_n \rangle) < \dim(\pi_{i_j} \cap \langle \pi_h \,|\, h \neq i_j, \dots, i_n \rangle) \leq \dim(\pi_{i_j} \cap \langle \pi_h \,|\, h \neq i_j, i_{j+2}, \dots, i_n \rangle),$$

a contradiction by the definition of i_{j+1}. Then

$$\delta_{j+1}(\mathcal{S}^\sigma) = k - \dim(\pi_{i_{j+1}} \cap \langle \pi_h \,|\, h \neq i_{j+1}, \dots, i_n \rangle) \leq k - \dim(\pi_{i_j} \cap \langle \pi_h \,|\, h \neq i_j, \dots, i_n \rangle) = \delta_j(\mathcal{S}^\sigma).$$

This concludes the proof. □

Remark 6.1.4. We note explicitly that a $(k; k-t)$-SCID $\mathcal{S} = \{\pi_1, \dots, \pi_n\}$ in $\mathbb{V}$, have parameters

$$(\delta_1, \delta_2, \dots, \delta_n) = (k, t, \dots, t) \tag{6.1.1}$$

if and only if $\mathcal{S}$ is a $(k-t)$-sunflower of maximal dimension. Clearly, sequence in (6.1.1) is independent of the indices labelling.

6.2 Large $(k; k-t_1, k-t_2)$-SPIDs are juntas

In this section we focus on the case where only two values for the intersection dimensions are possible. Moreover, we will focus on $(k; k-t_1, k-t_2)$-SPIDs spanning a large subspace of the ambient vector space $\mathbb{V}$. The next result is a generalization of [7, Theorem 2] to $(k; k-t_1, k-t_2)$-SPID. More precisely:

Theorem 6.2.1. *Let $k, t_1, t_2 \in \mathbb{N}$ such that $k > t_1 > t_2 \geq 1$. Let $\mathcal{S}$ be a $(k; k-t_1, k-t_2)$-SPID in a vector space $\mathbb{V}$, with $|\mathcal{S}| \geq 3$. If $\dim\langle \mathcal{S} \rangle \geq k + (t_1 - 1)(n-1) + 2$, then $\mathcal{S}$ is a $(k-t_1)$-junta.*

Proof. By Proposition 6.1.3, without loss of generality we can sort the spaces in $\mathcal{S}$ in such a way that the components in $\delta = (\delta_1, \dots, \delta_n)$ are non-increasing. In particular, we can choose as first m spaces, $m \geq 2$, those forming a $(k-t_1)$-sunflower of maximal dimension. Also, we note that such an ordering is not necessarily unique. Clearly, $\delta_1 = k$ and by Remark 6.1.4 the integer m is the largest index for which $\delta_m = t_1$. Let V' be the center of the sunflower formed by $\pi_1, \dots, \pi_m$; hence we get $\dim V' = k - t_1$.
Assume that $\mathcal{S}$ is not a $(k-t_1)$-junta, so we can find a subspace $\pi_r \in \mathcal{S}$ not containing V'. We denote $k - t_1 - \dim(\pi_r \cap V')$ by ε; hence, $\varepsilon \geq 1$. Also, in the quotient vector space $\Pi = \langle \mathcal{S} \rangle / (V' \cap \pi_r)$, we have that $\dim_\Pi \pi_r = t_1 + \varepsilon$, and that $\dim_\Pi(\pi_r \cap \pi_i) \in \{\varepsilon, \varepsilon + t_1 - t_2\}$, for each $1 \leq i \leq m$. Moreover, the subspaces $(\pi_r \cap \pi_i)/(V' \cap \pi_r)$, are linearly independent[1], with $i = 1, \dots, m$. Hence,

$$\begin{aligned}\delta_r = \dim \pi_r - \dim(\langle \pi_1, \dots, \pi_{r-1} \rangle \cap \pi_r) &\leq \dim_\Pi \pi_r - \dim_\Pi \langle \pi_1 \cap \pi_r, \dots, \pi_m \cap \pi_r \rangle \\ &= t_1 + \varepsilon - \sum_{i=1}^{m} \dim_\Pi(\pi_r \cap \pi_i) \leq t_1 + \varepsilon - m \cdot \varepsilon \leq t_1 - m + 1\end{aligned}$$

Since $(\delta_1, \dots, \delta_n)$ is nonincreasing, we find that

$$\begin{aligned}\dim\langle \mathcal{S} \rangle = \sum_{i=1}^{n} \delta_i &= k + \sum_{i=2}^{m} \delta_i + \sum_{i=m+1}^{r-1} \delta_i + \sum_{i=r}^{n} \delta_i \\ &\leq k + (m-1)t_1 + (r-m-1)(t_1 - 1) + (n-r+1)(t_1 - m + 1) \\ &= k + (n-1)(t_1 - 1) - (n-r)(m-2) + 1 \\ &\leq k + (n-1)(t_1 - 1) + 1,\end{aligned} \tag{6.2.1}$$

which proves the theorem. □

Remark 6.2.2. We point out here that contrary to what happens for SCIDs in general the bound stated above is not tight. For instance, with same notation used in Theorem 6.2.1; if $t_1 > t_2$ and there exists an integer s such that $r > s > m$ and $\delta_s \leq t_2$, we can slightly improve on the lower bound stated in

[1]It is enough to show that $\pi_j \cap \pi_r \cap \langle \pi_i \cap \pi_r \,|\, i \neq j \rangle = \pi_r \cap V'$, with $j = 1, \dots, m$. So

$$V' \cap \pi_r \subseteq \pi_j \cap \pi_r \cap \langle \pi_i \cap \pi_r \,|\, i \neq j \rangle \subseteq V' \cap \pi_r.$$

Then we obtain that $\dim_\Pi(\pi_j \cap \pi_r \cap \langle \pi_i \cap \pi_r \,|\, i \neq j \rangle) = 0$

Theorem 6.2.1. In fact, if this is the case by re-writing (6.2.1), we get

$$\begin{aligned}
&\dim\langle\mathcal{S}\rangle = \sum_{i=1}^{n}\delta_i = k + \sum_{i=2}^{m}\delta_i + \sum_{i=m+1}^{s-1}\delta_i + \sum_{i=s}^{r-1}\delta_i + \sum_{i=r}^{n}\delta_i \\
&\leq k + (m-1)t_1 + (s-m-1)(t_1-1) + (r-s)t_2 + (n-r+1)(t_1-m+1) \\
&= k + (n-1)(t_1-1) - (n-r)(m-2) - (r-s)(t_1-t_2-1) + 1 \\
&\leq k + (n-1)(t_1-1) - (t_1-t_2) + 2.
\end{aligned} \tag{6.2.2}$$

This possibility is verified if the first $r-1$ spaces form a $(k-t_1)$-junta with $\dim(\pi_s \cap \pi_j) = k - t_2$ for some $j \in \{1, \ldots, s-1\}$. In what follows we exhibit an example.
Let $k, t_1, t_2 \in \mathbb{N}$ such that $k > t_1 > t_2 + 1 > 1$ and consider $t_1 - t_2 + 1 \leq m \leq \min\{t_1 + 1, n-1\}$. Let $V', X, N_1, \ldots, N_m$, $M_{m+1}, \ldots, M_{s-1}$ and $P_s, \ldots, P_{n-1}$ be linearly independent subspaces of $\mathbb{V}$ such that

a) $\dim V' = k - t_1$,

b) $\dim X = t_1 - m + 1$,

c) $\dim N_i = t_1$ for $i = 1, \ldots, m$,

d) $\dim M_j = t_1 - 1$ for $j = m+1, \ldots, s-1$,

e) $\dim P_\ell = t_2$ for $\ell = s, \ldots, n-1$

Let $L_i = \{n_{i1}, \ldots, n_{i\,t_1-t_2}\}$ be a set of linearly independent 1-spaces in N_i, for $i = 1, \ldots, m$, $|L_i| = t_1 - t_2$, and we choose in L_i a 1-space, for example n_{i1}. Now, let $p_{m+1}, \ldots, p_{s-1}$ be 1-spaces in $\langle n_{11}, \ldots, n_{m1}\rangle \setminus \{n_{11}, \ldots, n_{m1}\}$ and let W be a $(k-t-1)$-space in V'. Then we define the sets $\pi_1, \ldots, \pi_n$ as follows.

- $\pi_1 = \langle V', N_1\rangle$, $\pi_2 = \langle V', N_2\rangle$, …, $\pi_m = \langle V', N_m\rangle$,
- $\pi_{m+1} = \langle V', p_{m+1}, M_{m+1}\rangle$, …, $\pi_{s-1} = \langle V', p_{s-1}, M_{s-1}\rangle$,
- $\pi_s = \langle V', Q_s, P_s\rangle$,…,$\pi_{n-1} = \langle V', Q_{n-1}, P_{n-1}\rangle$
- $\pi_n = \langle W, n_{11}, \ldots, n_{m1}, X\rangle$.

where $Q_s, \ldots, Q_{n-1}$ are $(t_1 - t_2)$-spaces equal to $\langle L_i\rangle$ for some $i \in \{1, \ldots, m\}$. It easy to verify that $\dim(\pi_i \cap \pi_j) \in \{k - t_1, k - t_2\}$ for $i, j = 1, \ldots, n$.
Hence, the set $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting in a space of dimension $k - t_1$ or $k - t_2$, i.e. a $(k; k-t_1, k-t_2)$-SPID. As not all pairwise intersections equal the same $(k-t_1)$-space, $\mathcal{S}$ is not a $(k-t_1)$-junta. Now, we have that

$$\langle\mathcal{S}\rangle = \langle\pi_1, \ldots, \pi_n\rangle = \langle V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{s-1}, P_s, \ldots, P_{n-1}, X\rangle.$$

So, by hypothesis

$$\begin{aligned}\dim\langle\mathcal{S}\rangle &= k + (m-1)t_1 + (s-m-1)(t_1-1) + (n-s)t_2 + (t_1-m+1)\\ &= k + (n-1)(t_1-1) - (n-s)(t_1-t_2-1) + 1\\ &\leq k + (n-1)(t_1-1) - (t_1-t_2) + 2.\end{aligned}$$

As in Theorem 6.2.1, we find that the array δ corresponding to $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is as follows:

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t_1, \ldots, t_1}_{m-1 \text{ times}}, \underbrace{t_1-1, \ldots, t_1-1}_{s-m-1 \text{ times}}, \underbrace{t_2, \ldots, t_2}_{n-s \text{ times}}, t_1-m+1).$$

However, in the following we will show that if in addition we ask that the two possible values for the dimensions of the intersection between elements of the SPID are consecutive integers, then the bound in Theorem 6.2.1 is sharp. Toward this aim we briefest the following

Proposition 6.2.3. *Let $\mathcal{S}$ be a $(k; k-t_1, k-t_2)$-*SPID *in a vector space $\mathbb{V}$, with $|\mathcal{S}| \geq 3$, such that $\dim\langle\mathcal{S}\rangle = k + (n-1)(t_1-1) + 1$.*
Then, there is no $(k-t_1)$-sunflower of maximal dimension with at least three petals in $\mathcal{S}$, if and only if any non-increasing sequence

$$(\delta_1, \delta_2, \ldots, \delta_n) = (k, t_1, t_1-1, \ldots, t_1-1). \tag{6.2.3}$$

Moreover, fixed a labelling of indices such that the sequence is as in (6.2.3), any permutation σ that fixes the first two spaces of such an ordering, does not change the sequence.

Proof. By Proposition 6.1.3, without loss of the generality, we can suppose that the spaces $\pi_1, \ldots, \pi_n$ of $\mathcal{S}$ are labelled in such a way that the sequence $(\delta_1, \delta_2, \ldots, \delta_n)$ is non-increasing.
The necessity is obvious because if any such a sequence $(\delta_1, \delta_2, \ldots, \delta_n)$ is like in (6.2.3), then, by Proposition 6.1.3 and by Remark 6.1.4, $\mathcal{S}$ can not contain a $(k-t_1)$-sunflower of maximal dimension with at least three petals.
Clearly $\dim(\pi_1 \cap \pi_2) = k - t_1$ and, by hypothesis, for any non-increasing sequence the largest index m for which $\delta_m = t_1$ is 2. Now, if $\delta_n \leq t_1 - 2$,

$$\dim\langle\mathcal{S}\rangle = \sum_{i=1}^{n} \delta_i = k + t_1 + \sum_{i=3}^{n-1} \delta_i + \delta_n \leq k + (n-1)(t_1-1),$$

a contradiction. Then $(\delta_1, \delta_2, \ldots, \delta_n) = (k, t_1, t_1-1, \ldots, t_1-1)$.
Now, we shall show that any permutation of the indices fixing the first two spaces does not change the sequence (6.2.3). First of all, we notice that

$$\dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t_1 + 1 \qquad \text{for all } j = 3, \ldots, n$$

Indeed, for $3 \leq j \leq n$,

$$k-t_1 \leq \dim(\pi_j \cap \pi_1) \leq \dim(\pi_j \cap \langle \pi_1, \pi_2 \rangle) \leq \dim(\pi_j \cap \langle \pi_1, \pi_2, \ldots, \pi_{j-1} \rangle) = k-t_1+1$$

and if $\dim(\pi_j \cap \langle \pi_1, \pi_2 \rangle) = k - t_1$, then

$$\pi_j \cap \pi_1 = \pi_j \cap \langle \pi_1, \pi_2 \rangle = \pi_j \cap \pi_2 = \pi_1 \cap \pi_2$$

this is a contradiction because π_1, π_2, π_j form a $(k-t)$-sunflower of maximal dimension.
Since

$$k - t_1 + 1 = \max_{3 \leq i \leq n} \dim(\pi_i \cap \langle \pi_h \,|\, h \neq i \rangle) \geq \dim(\pi_j \cap \langle \pi_h \,|\, h \neq j \rangle) \geq \dim(\pi_i \cap \langle \pi_1, \pi_2 \rangle) = k - t_1 + 1,$$

we obtain that

$$\dim(\pi_j \cap \langle \pi_h \,|\, h \in I \rangle) = k-t_1+1, \qquad \text{for all } j = 3, \ldots, n$$

for any $I \subseteq I_n$ with $1, 2 \in I$ and $j \notin I$.
Now, let σ be a permutation of I_n that fixes the set I_2, then

$$\delta_j(\mathcal{S}^\sigma) = k - \dim(\pi_{\sigma(j)} \cap \langle \pi_{\sigma(1)}, \pi_{\sigma(2)}, \ldots, \pi_{\sigma(j-1)} \rangle) = k - \dim(\pi_{\sigma(j)} \cap \langle \pi_1, \pi_2, \pi_{\sigma(3)}, \ldots, \pi_{\sigma(j-1)} \rangle) = t_1 - 1,$$

for all $j = 3, \ldots, n$. □

6.3 Constructions of $(k; k-t, k-t+1)$-SPIDs

Let $t \in \mathbb{N}$ such that $2 \leq t \leq k-1$. In this section we will construct $(k; k-t, k-t+1)$-SPIDs $\mathcal{S}$ which are not $(k-t)$-juntas, but where $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$.
Let $m \in \mathbb{N}$ be a positive integer such that $m > 2$. The first costruction provides one such a $(k; k-t, k-t+1)$-SPID with parameters

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m),$$

containing a $(k-t)$-sunflower of maximal dimension.

- **Class I**

Let $2 < m \leq \min\{t+1, n-1\}$. Let $V', X, N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t$, $\dim X = t-m+1$, $\dim N_i = t$ for $i = 1, \ldots, m$ and $\dim M_j = t-1$ for $j = m+1, \ldots, n-1$ (Figure 6.1).
Let $n_1, \ldots, n_m$ be 1-spaces in $N_1, \ldots, N_m$ respectively. Also, let $p_{m+1}, \ldots, p_{n-1}$ be 1-spaces in $\langle n_1, \ldots, n_m \rangle$ such that either

(a) at least two of them are the same 1-space, or

(b) at least one of them is equal to n_i, with $i \in \{1, \ldots, m\}$.

Let W be a $(k-t-1)$-space in V'. Then we define the sets $\pi_1, \ldots, \pi_n$ as follows.

- $\pi_1 = \langle V', N_1 \rangle$, $\pi_2 = \langle V', N_2 \rangle$, ..., $\pi_m = \langle V', N_m \rangle$,
- $\pi_{m+1} = \langle V', M_{m+1}, p_{m+1} \rangle$, ..., $\pi_{n-1} = \langle V', M_{n-1}, p_{n-1} \rangle$,
- $\pi_n = \langle W, n_1, \ldots, n_m, X \rangle$.

By (a) and (b), it is clear that the pairwise intersection of distinct spaces π_i and π_j, $i, j = 1, \ldots n-1$ is or the $(k-t)$-space V' or a $(k-t+1)$-space containing V'. Moreover, since each of the spaces $\pi_1, \ldots, \pi_{n-1}$ contains an unique 1-space from the set $\{n_1, \ldots, n_m, p_{m+1}, \ldots, p_{n-1}\}$ (note that by the property (a) and (b) in this set some 1-spaces could be equal), also $\dim(\pi_n \cap \pi_i) = k-t$ for all $i = 1, \ldots, n-1$.
Hence, the set $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting in a space of dimension $k-t$ or $k-t+1$. As not all pairwise intersections equal the same $(k-t)$-space, $\mathcal{S}$ is not a $(k-t)$-junta.
The set $\{n_1, \ldots, n_m, p_{m+1}, \ldots, p_{n-1}\}$ is contained in $\langle N_1, \ldots, N_m \rangle$ and also $W \subset V'$. Then

$$\langle \mathcal{S} \rangle = \langle \pi_1, \ldots, \pi_n \rangle = \langle V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{n-1}, X \rangle.$$

Since $V', X, N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ are linearly independent spaces of $\mathbb{V}$, we find that

$$\begin{aligned} \dim\langle \mathcal{S} \rangle &= k - t + m \cdot t + (n-1-m) \cdot (t-1) + t - m + 1 \\ &= k + (n-1)(t-1) + 1. \end{aligned}$$

As in Theorem 6.2.1, we find that the array δ corresponding to $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is as follows:

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m).$$

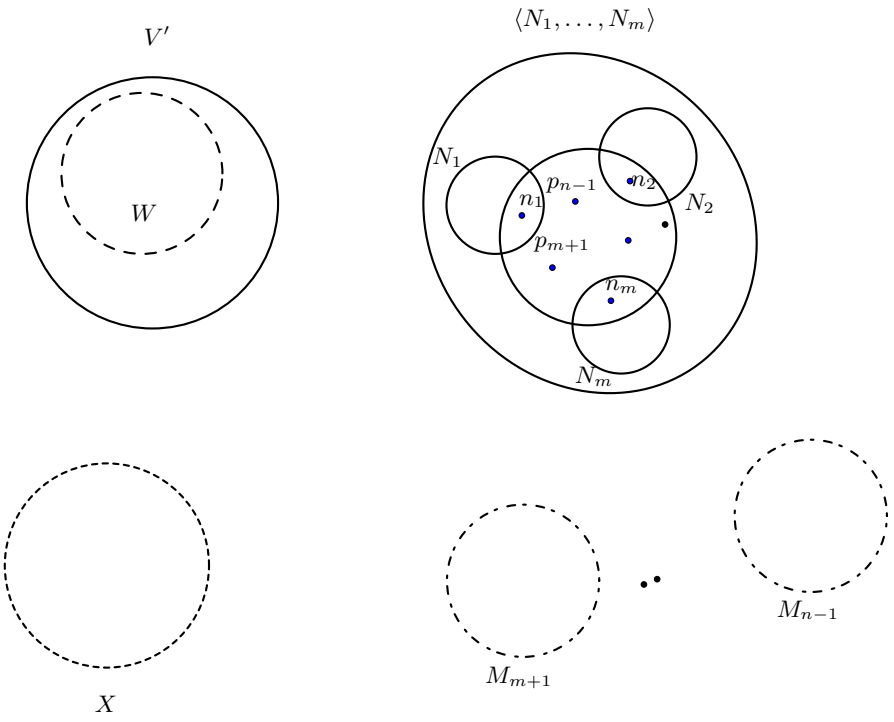

Figure 6.1: The $(k; k-t, k-t+1)$-SPID described in Class I.

Lemma 6.3.1. *Let $\mathcal{S}$ be a $(k; k-t, k-t+1)$-SPID of $\mathbb{V}$ where $2 \leq t \leq k-1$, such that $\mathcal{S}$ is not a $(k-t)$-junta, with $|\mathcal{S}| \geq 3$.*
If $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$ and there exists a $(k-t)$-sunflower of maximal dimension with at least three petals in $\mathcal{S}$, then $\mathcal{S}$ is equivalent to the SPID exhibited in Class I.

Proof. As in Theorem 6.2.1, we can sort the spaces in $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ in such a way that the sequence $(\delta_1, \ldots, \delta_n)$ is non-increasing. In particular, we can choose as first m spaces, $m \geq 3$, those that form a $(k-t)$-sunflower of maximal dimension with largest size.
Since $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$, we have that (6.2.1) of Theorem 6.2.1 holdes with equality. Then,

$$(\delta_1, \delta_2, \ldots, \delta_n) = (k, \underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m), \tag{6.3.1}$$

Consider $\mathcal{S}' = \{\pi_1, \ldots, \pi_{n-1}\}$. Since $\dim\langle \mathcal{S}' \rangle = k + (n-2)(t-1) + m - 1 \geq k + (n-2)(t-1) + 2$, then $\mathcal{S}'$ is a $(k-t)$-junta.
Let V' be the common $(k-t)$-space through which the k-spaces $\pi_1, \ldots, \pi_{n-1}$ pass, and denote $k - t - \dim(\pi_n \cap V')$ by ε as in Theorem 6.2.1. Since $\mathcal{S}$ is not

a junta, $\varepsilon \geq 1$; indeed, by Theorem 6.2.1 we have $\varepsilon = 1$. Then let W be the $(k-t-1)$-subspace $\pi_n \cap V'$.
Furthermore, we note that the first m k-subspaces in $\mathcal{S}$ meets pairwise in V'. Hence, there exist t-subspaces $N_1, \ldots, N_m$ with $i = 1, \ldots, m$ such that $N_1, ..., N_m, V'$ are linearly indipendent, and $\pi_i = \langle V', N_i \rangle$.
By hypothesis, there exist at least two k-spaces in $\mathcal{S}$, say π_i and π_j, such that $\dim(\pi_i \cap \pi_j) = k - t + 1$.
We shall show that

$$\dim(\pi_n \cap \pi_j) = k - t \qquad \text{for all } j \in \{1, \ldots, n-1\}.$$

For this purpose, suppose by contradiction that there exists $j \in \{1, \ldots, n-1\}$ such that $\dim(\pi_n \cap \pi_j) = k - t + 1$; we may distinguish two cases:

(*a*) $j \in \{1, \ldots, m\}$. Then there are two 1-spaces n_{j_1} and n_{j_2} in $\pi_n \cap \pi_j$ not in V', and there is at least another 1-space $n_i \in \pi_n \cap \pi_i$, for all $i \in \{1, \ldots, m\} \setminus \{j\}$ not in V'. Without losing any generality we may choose, up to equivalence, the N_i's, in such a way that $n_1 \in N_1, \ldots, n_m \in N_m$ and $\langle n_{j_1}, n_{j_2} \rangle \subseteq N_j$. Hence,

$$\pi_n \cap \langle \pi_1, \ldots, \pi_{n-1} \rangle \supseteq \langle W, n_1, \ldots, n_{j-1}, n_{j_1}, n_{j_2}, n_{j+1}, \ldots, n_m \rangle,$$

obtaining that

$$t - m + 1 = \delta_n \leq k - \dim\langle W, n_1, \ldots, n_{j-1}, n_{j_1}, n_{j_2}, n_{j+1}, \ldots, n_m \rangle = t - m,$$

a contradiction.

(*b*) $j \in \{m+1, \ldots, n-1\}$. Since from the point (*a*) $\dim(\pi_n \cap \pi_i) = k - t$ for every $i = 1, \ldots, m$, π_n contains the 1-spaces $n_1 \in \pi_1, \ldots, n_m \in \pi_m$, meeting V' trivially. Furthermore, since $\dim(\pi_n \cap \pi_j) = k - t + 1$, there are two 1-spaces $n', n'' \in \pi_n \cap \pi_j$ not in V'. Suppose that $\langle n', n'' \rangle \cap V'$ is not the trivial space, then it belongs to $V' \setminus W$, but then V' would be completely in π_n that is a contradiction.
Now, we assume that $\langle n', n'' \rangle \subseteq \langle V', n_1, \ldots, n_m \rangle$, then

$$\pi_j \cap \langle V', n_1, \ldots, n_m \rangle \supseteq \langle V', n', n'' \rangle$$

and

$$t - 1 = \delta_j \leq k - \dim\langle V', n', n'' \rangle = t - 2.$$

So, the space $\langle n', n'' \rangle$ is not contained in $\langle V', n_1, \ldots, n_m \rangle$, but it meets $\langle V', n_1, \ldots, n_m \rangle$ in a 1-space otherwise

$$\pi_n \cap \langle \pi_1, \ldots, \pi_{n-1} \rangle \supseteq \langle W, n_1, \ldots, n_m, n', n'' \rangle$$

obtaining $t - 1 = \delta_n \leq t - 2$. Let $p \in \langle n', n'' \rangle \setminus \langle V', n_1, \ldots, n_m \rangle$, then

$$t - m + 1 = \delta_n \leq k - \dim\langle W, n_1 \ldots, n_m, p \rangle = t - m$$

again a contradiction.

Hence, since $\delta_n = t - m + 1$ and π_n must intersect V' in the $(k-t-1)$-dimensional subspace W, we get that $\pi_n = \langle W, n_1, \ldots, n_m, X\rangle$ for suitable points $n_1 \in N_1, \ldots, n_m \in N_m$ and a $(t-m+1)$-dimensional subspace X such that $V', N_1, \ldots, N_m, X$ are linearly independent.
Since $\pi_n \cap \pi_j$ is a $(k-t)$-space contained in $\langle W, n_1, \ldots, n_m\rangle$ there exists a 1-space p_j in $\langle n_1, \ldots, n_m\rangle$. Moreover, since $\delta_j = t-1$, it is immediate that each $\pi_j = \langle V', M_j, p_j\rangle$, $j = m+1, \ldots, n-1$, with M_j is a $(t-1)$-space and such that $V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{n-1}$ and X are linearly independent.
Note explicitly that if $\dim(\pi_i \cap \pi_j) = k-t+1$, with $i, j \in \{m+1, \ldots, n-1\}$, then $p_i = p_j$, where $\pi_i = \langle V', p_i, M_i\rangle$ and $\pi_j = \langle V', p_j, M_j\rangle$.
Indeed let $\pi_i \cap \pi_j = \langle V', n'\rangle$. This space is contained in $\langle \pi_1, \pi_2, \ldots, \pi_m\rangle$, since if $n' \notin \langle \pi_1, \pi_2, \ldots, \pi_m\rangle$, assuming $j > i$,

$$\pi_j \cap \langle \pi_1, \ldots, \pi_i\rangle \supseteq \langle V', p_j, n'\rangle,$$

obtaining $\delta_j \leq t-2$. So, $\langle V', n'\rangle \subseteq \langle \pi_1, \ldots, \pi_m\rangle$.
Now, since

$$\pi_i \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle = \langle V', p_i\rangle$$
$$\pi_j \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle = \langle V', p_j\rangle$$

have dimension $k-t+1$ and

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle,$$

$\pi_i \cap \pi_j$ is equal to $\langle V', p_i\rangle$ and $\langle V', p_j\rangle$. Now, assumed $j > i$, if $p_i \neq p_j$ then $t-1 = \delta_j \leq t-2$, a contradiction.
Suppose that there exist $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, n-1\}$ such that $\dim(\pi_i \cap \pi_j) = k-t+1$, then there exists another 1-space $n' \in N_i$ and

$$\langle V', n'\rangle = \pi_i \cap \pi_j \subseteq \pi_j \cap \langle \pi_1, \ldots, \pi_m\rangle = \langle V', p_j\rangle.$$

Then $p_j \in \langle V', n'\rangle$ and since $\delta_j = t-1$, $p_j \in \langle n_1, \ldots, n_m\rangle \cap N_i$. This implies that $p_j = n_i$.
Note explicitly that a k-space π_j in $\mathcal{S}$ with $j \in \{m+1, \ldots, n-1\}$ can meet at most one π_i with $i \in \{1, \ldots, m\}$ in a $(k-t+1)$-space.
Finally, we can suppose that in $\mathcal{S}$ there exists a k-space π_j with $j \in \{m+1, \ldots, n-1\}$ that intersects π_i with $i \in \{1, \ldots, m\}$ and π_h with $h \in \{m+1, \ldots, n-1\}$ in two $(k-t+1)$-spaces. For previous results $p_h = p_j = n_i$. So $\mathcal{S}$ is isomorphic to one of the examples presented in Class I. □

6.3.1 SPIDs with $\delta = (k, t, t-1, t-1, ..., t-1)$

Next we exhibit three classes of $(k; k-t, k-t+1)$-SPIDs which are not $(k-t)$-juntas.

- **Class II**

Choose integers $n \geq 3$ and k, t such that $2 \leq t \leq k-1$. Let W be a $(k-t+1)$-subspaces of $\mathbb{V}$, and X_1, X_2 t-spaces such that $\dim\langle X_1, X_2\rangle = 2t-1$. Moreover, consider $M_3, \dots M_n$ $(t-1)$-subspaces of $\mathbb{V}$ such that W, $\langle X_1, X_2\rangle$, $M_3, \dots, M_n$ are linearly independent. Let W_1 and W_2 be a $(k-t)$-space in W, (Figure 6.2).
Then we define the sets $\pi_1, \dots, \pi_n$ as follows:

- $\pi_1 = \langle W_1, X_1\rangle$, $\pi_2 = \langle W_2, X_2\rangle$,
- $\pi_3 = \langle W, M_3\rangle$, $\dots$, $\pi_n = \langle W, M_n\rangle$.

Now, since $\dim(X_1 \cap X_2) = 1$, $\pi_1 \cap \pi_2$ is a $(k-t)$-space and these spaces meet the other ones in W_1 or W_2, while $\{\pi_3, \dots, \pi_n\}$ is a $(k-t+1)$-sunflower with center W. Clearly,

$$\langle \mathcal{S}\rangle = \langle \pi_1, \dots, \pi_n\rangle = \langle W, X_1, X_2, M_3, \dots, M_n\rangle.$$

Since W, $\langle X_1, X_2\rangle$, $M_3, \dots, M_n$ are linearly independent, we find that

$$\begin{aligned}\dim\langle \mathcal{S}\rangle &= k-t+1+2t-1+(n-2)\cdot(t-1)\\ &= k+(n-1)(t-1)+1.\end{aligned}$$

Again, as in Theorem 6.2.1, using the ordering $\pi_1, \dots, \pi_n$, we find that

$$(\delta_2, \dots, \delta_n) = (t, t-1, \dots, t-1).$$

In particular, we observe that the examples in this class contain $(k-t+1)$-sunflowers of maximal dimension, but does not contain $(k-t)$-sunflowers of maximal dimension. Nonetheless, they are $(k-t-1)$-juntas.

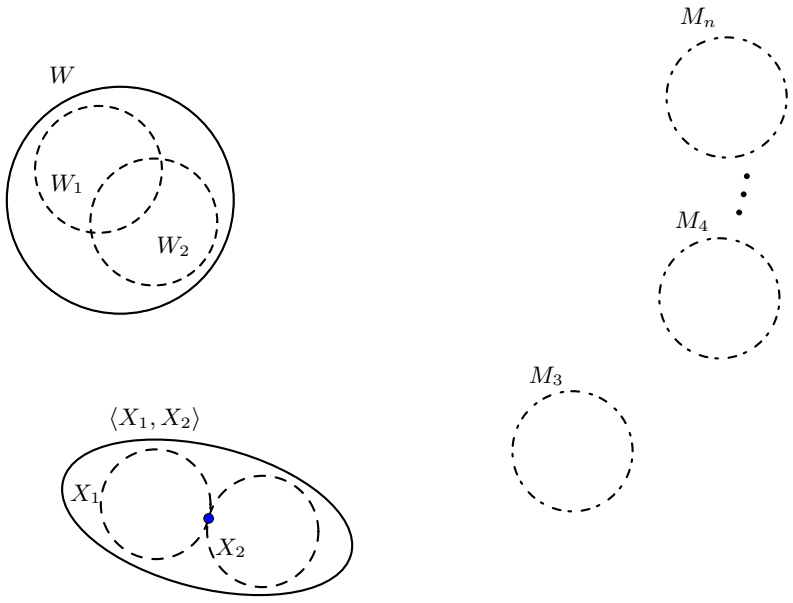

Figure 6.2: The $(k; k-t.k-t+1)$-SPIDs described in Class II.

- **Class III**

Choose integers $n \geq 3$, $2 \leq s < n$ and k, t such that $2 \leq t \leq k-1$. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}$ with q a prime power such that $\frac{q^{k-t+2}-1}{q-1} \geq s+1$. Let V', $\langle X_1, X_2\rangle, M_3, \ldots, M_n$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t+2$, $\dim X_1 = t$, $\dim X_2 = t-1$ and $\dim M_i = t-1$ for $i = 3, \ldots, n$.
Let $W_0, W_1, \ldots, W_s$ be distinct $(k-t+1)$-spaces in V' such that $W_1, \ldots, W_s$ go through a $(k-t)$-space W (Figure 6.3). We define the sets

$$\pi_1 = \langle W, X_1\rangle,\ \pi_2 = \langle W_0, X_2\rangle,$$
$$\pi_3 = \langle W_1, M_3\rangle, \ldots,\ \pi_{m_1} = \langle W_1, M_{m_1}\rangle,$$
$$\pi_{m_1+1} = \langle W_2, M_{m_1+1}\rangle, \ldots,\ \pi_{m_2} = \langle W_2, M_{m_2}\rangle$$
$$\ldots$$
$$\pi_{m_{s-1}+1} = \langle W_s, M_{m_{s-1}+1}\rangle, \ldots,\ \pi_n = \langle W_s, M_n\rangle.$$

Clearly the set $\mathcal{S}$ is a $(k; k-t, k-t+1)$-SPID such that it is not a $(k-t)$-junta and

$$\langle \mathcal{S}\rangle = \langle \pi_1, \ldots, \pi_n\rangle = \langle V', X_1, X_2, M_3, \ldots, M_n\rangle.$$

Since V', $X_1, X_2, M_3, \ldots, M_n$ are linearly independent, we find that

$$\begin{aligned}\dim\langle\mathcal{S}\rangle &= k-t+2+2t-2+(n-2)(t-1)\\ &= k+(n-1)(t-1)+1.\end{aligned}$$

As in Theorem 6.2.1, using the ordering $\pi_1, \ldots, \pi_n$, we find that

$$(\delta_2, \ldots, \delta_n) = (t, t-1, \ldots, t-1).$$

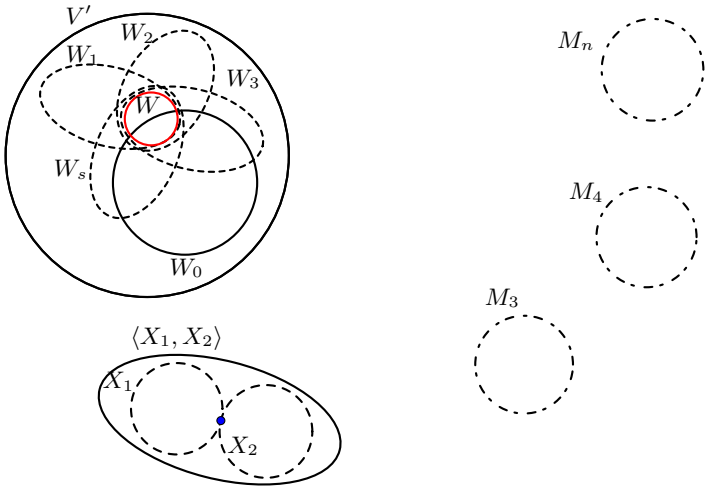

Figure 6.3: The $(k; k-t.k-t+1)$-SPIDs described in Class III.

Examples in this class may contain $(k-t)$-sunflowers not of maximal dimension and $(k-t+1)$-sunflowers of maximal dimension.

- **Class IV**

Choose integers $n \geq 3$, $2 \leq s < n$ and k, t such that $2 \leq t \leq k-1$. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}$ with q a prime power such that $\frac{q^{k-t+2}-1}{q-1} \geq s+2$. Let V', $M_1, \ldots, M_n$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t+2$. and $\dim M_i = t-1$, for $i = 1, \ldots, n$.
Let $V_0, W_0, W_1, \ldots, W_s$ be $s+2$ $(k-t+1)$-spaces in V' such that they not go through the same $(k-t)$-space, with $W_1, \ldots, W_s$ distinct (Figure 6.4). We define the sets

$$\pi_1 = \langle V_0, M_1\rangle,\ \pi_2 = \langle W_0, M_2\rangle,$$

$$\pi_3 = \langle W_1, M_3\rangle, \ldots, \pi_{m_1} = \langle W_1, M_{m_1}\rangle,$$
$$\pi_{m_1+1} = \langle W_2, M_{m_1+1}\rangle, \ldots, \pi_{m_2} = \langle W_2, M_{m_2}\rangle$$
$$\ldots$$
$$\pi_{m_{s-1}+1} = \langle W_s, M_{m_{s-1}+1}\rangle, \ldots, \pi_n = \langle W_s, M_n\rangle$$

Clearly the set $\mathcal{S}$ is a $(k; k-t, k-t+1)$-SPID such that it is not a $(k-t)$-junta and

$$\langle \mathcal{S}\rangle = \langle \pi_1, \ldots, \pi_n\rangle = \langle V', M_1, M_2, M_3, \ldots, M_n\rangle.$$

Since V', $M_1, M_2, M_3, \ldots, M_n$ are linearly independent, we find that

$$\begin{aligned}\dim\langle \mathcal{S}\rangle &= k - t + 2 + n \cdot (t-1)\\ &= k + (n-1)(t-1) + 1.\end{aligned}$$

As in Theorem 6.2.1, using the ordering $\pi_1, \ldots, \pi_n$, we find that

$$(\delta_2, \ldots, \delta_n) = (t, t-1, \ldots, t-1).$$

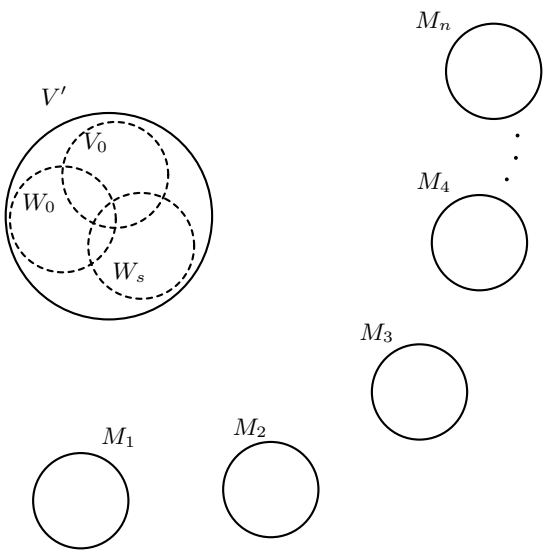

Figure 6.4: The $(k; k-t.k-t+1)$-SPIDs described in Class IV.

The examples in this last class may contain $(k-t+1)$-sunflowers of maximal dimension and $(k-t)$-sunflowers not of maximal dimension.

Lemma 6.3.2. *Let $\mathcal{S}$ be a $(k; k-t, k-t+1)$-SPID $(2 \leq t \leq k-1)$ in a vector space $\mathbb{V}$ such that $|\mathcal{S}| \geq 3$ and $\mathcal{S}$ is not a $(k-t)$-junta.*
If $\dim\langle\mathcal{S}\rangle = k+(n-1)(t-1)+1$ and there is no a $(k-t)$-sunflower of maximal dimension with at least three petals then $\mathcal{S}$ is equivalent to one of the examples described in Class II, III, or IV.

Proof. As in Theorem 6.2.1, we can sort $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ in such a way that δ is non-increasing. By Proposition 6.1.3 and 6.2.3, we have that

$$(\delta_1, \delta_2, \ldots, \delta_n) = (k, t, t-1, \ldots, t-1). \tag{6.3.2}$$

Moreover, as consequence of Proposition 6.2.3, $\dim(\pi_4 \cap \langle\pi_1, \pi_2, \pi_3\rangle) = \dim(\pi_4 \cap \langle\pi_1, \pi_2\rangle) = k-t+1$, then we obtain that

$$\pi_4 \cap \pi_3 \subseteq \pi_4 \cap \langle\pi_1, \pi_2, \pi_3\rangle = \pi_4 \cap \langle\pi_1, \pi_2\rangle \subseteq \langle\pi_1, \pi_2\rangle.$$

By Proposition 6.2.3, eventually rearranging the spaces $\pi_3, \ldots, \pi_n$ in $\mathcal{S}$, we can repeat the previous argument, getting

$$\pi_i \cap \pi_j \subseteq \langle\pi_1, \pi_2\rangle,$$

for each π_i and π_j with $i, j \in \{3, \ldots, n\}$.
Now, in $\mathcal{S}' = \{\pi_3, \ldots, \pi_n\}$, we can define the following binary relation

$$\pi_i \sim \pi_j \iff \pi_i \cap \langle\pi_1, \pi_2\rangle = \pi_j \cap \langle\pi_1, \pi_2\rangle,$$

for $i, j = 3, \ldots, n$.
Clearly, $\sim$ is an equivalence relation on $\mathcal{S}'$. The k-spaces of an equivalence class meet $\langle\pi_1, \pi_2\rangle$ in the same $(k-t+1)$-space. In this way, we have that $W_1, \ldots, W_s$, where $1 \leq s \leq n-3$, are $(k-t+1)$-dimensional spaces in $\langle\pi_1, \pi_2\rangle$, pairwise intersecting in a $(k-t)$-space[2].

[2]Indeed, let π_i and π_j be k-spaces of $\mathcal{S}'$ in different equivalence classes. Then, by Proposition 6.2.3, $\dim(\pi_i \cap \langle\pi_1, \pi_2\rangle) = \dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k-t+1$,

$$\pi_i \cap \langle\pi_1, \pi_2\rangle = W_\ell \text{ and } \pi_j \cap \langle\pi_1, \pi_2\rangle = W_m,$$

with for some $\ell, m \in \{1, \ldots, s\}$ distinct. Hence

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle\pi_1, \pi_2\rangle = W_\ell \cap W_m.$$

Since $k-t \leq \dim(\pi_i \cap \pi_j) = \dim(W_\ell \cap W_m)$ and W_ℓ, W_m are distinct $(k-t+1)$-subspaces,

$$\dim(W_\ell \cap W_m) = k-t.$$

Since the relation $\sim$ induces a partition $J_1, J_2, \ldots, J_s$ on the index set $\{3, 4, \ldots, n\}$, by Proposition 6.2.3, we can label appropriately the elements of $\mathcal{S}$, obtaining

$$\begin{aligned} \pi_{j_1} &= \langle W_1, M_{j_1} \rangle && \text{with } j_1 \in J_1 \\ \pi_{j_2} &= \langle W_2, M_{j_2} \rangle && j_2 \in J_2, \\ &\ldots \\ \pi_{j_s} &= \langle W_s, M_{j_s} \rangle && j_s \in J_s. \end{aligned}$$

where the elements in the set $\{M_{j_h} : j_h \in J_h,\ h \in \{1, 2, \ldots, s\}\}$, are linearly independent $(t-1)$-spaces. In fact, if two of them intersect not trivially, this would immediately contradict (6.3.2).
We divide the rest of the proof in two steps:

1) First of all, we look at the case in which all spaces of $\mathcal{S}'$ meet $\langle \pi_1, \pi_2 \rangle$ in the same space, say W. Since for all $3 \leq j \leq n$ and $i = 1, 2$ $\pi_i \cap \pi_j = \pi_i \cap W$[3] we shall show that

$$\dim(\pi_i \cap \pi_j) = k - t.$$

So, suppose that either the space π_1 or π_2 contains W ($W \not\subseteq \pi_1 \cap \pi_2$, $\dim(\pi_1 \cap \pi_2) = k - t$). We can consider, without loss of generality, that π_1 contains W. Then $\pi_2 \cap W = \pi_1 \cap \pi_2$; in fact, we have that $\pi_1 \cap \pi_2 \supseteq \pi_2 \cap W = \pi_2 \cap \pi_j$ with $j \in \{3, ..., n\}$. But then $\mathcal{S}$ is a $(k-t)$-junta; a contradiction.

Hence, $\pi_1 \cap W$ and $\pi_2 \cap W$ are $(k-t)$-spaces, they are distinct otherwise $\mathcal{S}$ is again a $(k-t)$-junta[4]. More precisely, they are two hyperplanes of W, and then they have to meet in a $(k-t-1)$-space W' in W. Hence, we may always choose a basis of $\mathbb{V}$ in such a way that the following happens

$$\pi_1 \cap W = \langle W', n_1 \rangle \text{ and } \pi_2 \cap W = \langle W', n_2 \rangle$$

with n_1, n_2 distinct 1-spaces in $W_1 \setminus W'$ with W', n_1, n_2 linearly independent. Then, there exist X_1 and X_2, t-spaces such that they have a 1-space in common and

$$\pi_1 = \langle W', n_1, X_1 \rangle, \quad \pi_2 = \langle W', n_2, X_2 \rangle.$$

[3]Recall that $W = \pi_j \cap \langle \pi_1, \pi_2 \rangle$. Then

$$\pi_i \cap \pi_j \subseteq \pi_j \cap \langle \pi_1, \pi_2 \rangle = W$$

and clearly $\pi_i \cap \pi_j \subseteq W \cap \pi_i$. On the other hand, since

$$W = \pi_j \cap \langle \pi_i, \pi_2 \rangle \subseteq \pi_j,$$

then $W \cap \pi_i \subseteq \pi_j \cap \pi_i$.

[4]Otherwise, $\pi_1 \cap W = \pi_1 \cap \pi_2 = \pi_2 \cap W \subseteq W$, and hence $\pi_1 \cap \pi_2$ is contained in every k-space of $\mathcal{S}$.

This means that $\mathcal{S}$ is isomorphic to one of the examples in Class II.

2) Now, we suppose that $s \geq 2$. In this case $W_1, \ldots, W_s$ are $(k-t+1)$-spaces pairwise intersecting in a $(k-t)$-space. Hence, by Theorem 3.3.4, either

   (a) they have a $(k-t)$-space in common, or

   (b) they lay in a $(k-t+2)$-space V'.

   Note explicitly that for $s = 2$, (a) and (b) are equivalent. If $s \geq 3$, we shall show that

$$\dim\langle W_1, W_2, \ldots, W_s\rangle = k-t+2 \tag{6.3.3}$$

   which is equivalent to prove that, for all $1 \leq h \leq s$,

$$W_h \subseteq \langle W_1, W_2\rangle. \tag{6.3.4}$$

   Suppose that $W_1, W_2, \ldots, W_s$ go through a $(k-t)$-space in $\langle \pi_1, \pi_2\rangle$ and let $\pi_{j_1}, \pi_{j_2}, \pi_{j_h}$ be k-spaces belonging to different equivalence classes with respect to $\sim$, where

$$\pi_{j_1} = \langle W_1, M_{j_1}\rangle \quad \pi_{j_2} = \langle W_2, M_{j_2}\rangle \quad \pi_{j_h} = \langle W_h, M_{j_h}\rangle.$$

   Since there is no a sunflower of maximal dimension with at least three petals, we have

$$\dim(\pi_{j_h} \cap \langle \pi_{j_1}, \pi_{j_2}\rangle) \geq k-t+1.$$

   Then, by applying Grassmann Formula, we obtain

$$\begin{aligned} k-t+1 &\leq \dim(\pi_{j_h} \cap \langle \pi_{j_1}, \pi_{j_2}\rangle) = 2k+t-\dim\langle W_1, W_2, W_h, M_{j_1}, M_{j_2}, M_{j_h}\rangle \\ &= 2k+t-3(t-1)-(\dim W_h + \dim\langle W_1, W_2\rangle - \dim(W_h \cap \langle W_1, W_2\rangle), \end{aligned}$$

   so $\dim(W_h \cap \langle W_1, W_2\rangle \geq k-t+1$, then by W_h dimension we have the property (6.3.4).
   Hence, we suppose $s \geq 2$ and, by (6.3.4), all $(k-t+1)$-spaces $W_1, \ldots, W_s$ lie in a $(k-t+2)$-space, say V'.
   Obviously V' is contained in $\langle \pi_1, \pi_2\rangle$ and

$$k-t \leq \dim(\pi_i \cap V') \leq k-t+1 \qquad \text{for } i = 1, 2. \tag{6.3.5}$$

   Indeed, since for any $j \in \{3, \ldots, n\}$,

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle \pi_1, \pi_2\rangle = \pi_i \cap W_h \subseteq \pi_i \cap V'$$

for $i = 1, 2$ and for some $h \in \{1, \dots, s\}$, then the left inequality in (6.3.5) follows.
On the other hand, if $\dim(\pi_i \cap V') \geq k - t + 2$, for $i = 1$ or 2, then V' is contained either in π_1 or in π_2 (not in both, $\dim(\pi_1 \cap \pi_2) = k - t$). Without loss of generality, we can suppose that V' is contained in π_1. Then

$$\pi_2 \cap \pi_{j_h} = \pi_2 \cap W_h \subseteq \pi_2 \cap V' \subseteq \pi_1 \cap \pi_2,$$

for $h \in \{1, \dots, s\}$. This implies that $\pi_1 \cap \pi_2$ is contained in all elements of $\mathcal{S}$ and it is a $(k-t)$-junta.

Furthermore, $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are distinct subspaces. Indeed,

($\diamond$) if $\pi_1 \cap V' = \pi_2 \cap V'$ and it is a $(k-t+1)$-space, then $\pi_1 \cap \pi_2$ is a $(k-t+1)$-space, a contradiciton;

($\diamond\diamond$) if $\pi_1 \cap V' = \pi_2 \cap V'$ is a $(k-t)$-space, since for $i = 1, 2$ and $h = 1, \dots, s$, $\pi_i \cap W_h$ has dimension at least $k - t$ and $W_h \subseteq V'$, we have that

$$\pi_1 \cap W_h = \pi_1 \cap V' = \pi_2 \cap V' = \pi_2 \cap W_h.$$

This implies that $\pi_1 \cap \pi_2$ is contained in all element of $\mathcal{S}$, a contradiction.

Now, let W_h be a $(k-t+1)$-space with $1 \leq h \leq s$, then

$$\begin{aligned} k - t = &\dim(\pi_1 \cap \pi_2) \geq \dim(\pi_1 \cap \pi_2 \cap V') \geq \dim(\pi_1 \cap \pi_2 \cap W_h) \geq \\ &\dim(\pi_1 \cap W_h) + \dim(\pi_2 \cap W_h) - \dim W_h \geq \\ &2(k-t) - k + t - 1 = k - t - 1. \end{aligned} \tag{6.3.6}$$

By the inequalities (6.3.5) and (6.3.6), the discussion is reduced only to the following three cases:

(i) $\dim(\pi_1 \cap V') = \dim(\pi_2 \cap V') = k - t$ (and $\dim(\pi_1 \cap \pi_2 \cap V') = k - t - 1$).

(ii) π_1 and π_2 meet V' in subspaces with different dimension.

(iii) $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are two hyperplanes of V'.

Case (i): We shall show that for all $3 \leq j \leq n$,

$$\pi_j \cap \langle \pi_1, \pi_2 \rangle = \langle \pi_1 \cap V', \pi_2 \cap V' \rangle. \tag{6.3.7}$$

Since $\pi_j \cap \langle \pi_1, \pi_2 \rangle \subseteq V'$ and π_1 and π_2 meet V' in a $(k-t)$-space,

$$\pi_j \cap \pi_1 = \pi_1 \cap V' \text{ and } \pi_j \cap \pi_2 = \pi_2 \cap V' \ ,$$

obtaining that

$$\pi_j \cap \langle \pi_1, \pi_2 \rangle \supseteq \langle \pi_j \cap \pi_1, \pi_j \cap \pi_2 \rangle = \langle \pi_1 \cap V', \pi_2 \cap V' \rangle.$$

Since they are both $(k-t)$-spaces in V', we obtain the claim in (6.3.7). Hence, every π_j, $j = 3, \ldots, n$, meets $\langle \pi_1, \pi_2 \rangle$ always in the same $(k-t+1)$-subspace, then $s = 1$, a contradiction.

Case (ii): We can suppose, without loss of generality, that

$$\dim(\pi_1 \cap V') = k - t \quad \text{and} \quad \dim(\pi_2 \cap V') = k - t + 1.$$

Clearly, $\pi_1 \cap V' \not\subseteq \pi_2 \cap V'$ otherwise

$$\pi_1 \cap W_h = \pi_1 \cap V' \subseteq \pi_2 \cap V'.$$

This implies that $\pi_1 \cap \pi_2 \subseteq W_h$ for $h = 1, \ldots, s$, then $\mathcal{S}$ is a $(k-t)$-junta. Since $W = \pi_1 \cap \pi_2 \cap V'$ is a $(k-t-1)$-space, there exist a t-space X_1 contained in π_1 and a $(t-1)$-space X_2 contained in π_2 both disjoint from V', for $i = 1, 2$ and such that $\langle X_1, X_2 \rangle = 2t - 2$. Then

$$\pi_1 = \langle W, X_1 \rangle \text{ and } \pi_2 = \langle W, X_2 \rangle.$$

We note explicitly that

$$\pi_1 \cap V' = \pi_1 \cap W_h \subseteq W_h, \tag{6.3.8}$$

for $h \in \{1, \ldots, s\}$.

Case (iii): Now, we suppose that $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are hyperplanes of V', say V_0 and W_0, respectively. Then, there exists X_i, $i = 1, 2$, a $(t-1)$-space in π_i disjoint from V' such that

$$\pi_1 = \langle V_0, X_1 \rangle \text{ and } \pi_2 = \langle W_0, X_2 \rangle.$$

Again, by Grassmann Formula, we obtain that X_1, X_2, V' are linearly independent and
$\dim\langle X_1, X_2 \rangle = 2t - 2$.

So, the discussion in case (ii) provide us with an example which isomorphic to one of those described in Class III, while (ii) gives an example isomorphic to one described in Class IV. □

Remark 6.3.3. Let $\mathcal{W} = \{W_1, \ldots, W_s, \pi_2 \cap V'\}$ be the set of $(k-t+1)$-spaces in V' with $2 \leq s \leq n-3$.
In the Case (ii), if $s \geq 3$ and by formula (6.3.8), the first s subspaces in $\mathcal{W}$ form a sunflower with center $\pi_1 \cap V'$, with $\pi_2 \cap V'$ not through $\pi_1 \cap V'$.
In the Case (iii), considered $\pi_1 \cap V'$ and $\pi_2 \cap V'$, one of them or both could be in $\{W_1, \ldots, W_s\}$.
If $s = 2$, at most one between $\pi_1 \cap V'$ and $\pi_2 \cap V'$ can coincide with W_1 or W_2. Otherwise, $W_1 \cap W_2 = \pi_1 \cap \pi_2$ and it is contained in all elements of $\mathcal{S}$.
In particular, if $s = 2$ and $n = 4$, it is straightforward to see that exactly one between $\pi_1 \cap V'$ and $\pi_2 \cap V'$ must necessarily be equal to W_1 or W_2.

Theorem 6.3.4. *Let $\mathcal{S}$ be a $(k; k-t; k-t+1)$-*SPID *in a vector space $\mathbb{V}$, with $|\mathcal{S}| \geq 3$ and $2 \leq t \leq k-1$. If the dimension of $\langle \mathcal{S} \rangle$ is $k+(n-1)(t-1)+1$, then $\mathcal{S}$ is either a $(k-t)$-junta or $\mathcal{S}$ is isomorphic to one of the examples described in Class I, II, III or IV.*

Proof. We assume that $\mathcal{S}$ is not a $(k-t)$-junta. We denote the elements of $\mathcal{S}$ by $\pi_1, \pi_2, \ldots, \pi_n$. We will consider all possible orderings of the spaces in $\mathcal{S}$ such that the parameters $(\delta_2, \ldots, \delta_n)$ are non-increasing.
Since $\dim\langle \mathcal{S} \rangle = k+(n-1)(t-1)+1$, we have the equality in (6.2.1) of Theorem 6.2.1 . Hence, if $m \geq 3$ we have

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m), \tag{6.3.9}$$

otherwise $m = 2$ and we have

$$(\delta_2, \ldots, \delta_n) = (t, t-1, \ldots, t-1). \tag{6.3.10}$$

- Suppose that we can find a permutation of $\mathcal{S}$ such that $(\delta_2, \ldots, \delta_n)$ is as in (6.3.9), for $m \geq 3$, then by Lemma 6.3.1 it follows that $\mathcal{S}$ is isomorphic to one of the examples described in Class I.

- If there is permutation of $\mathcal{S}$ such that $\delta_n \leq t-2$, then in $\mathcal{S}$ there is a $(k-t)$-sunflower of maximal dimension with at least three petals and then there is a permutation of $\mathcal{S}$ such that $(\delta_2, \ldots, \delta_n)$ is as in (6.3.9). This case has been covered in the preceding point.
  So, we can assume that for any permutation of $\mathcal{S}$ the tuple $(\delta_2, \ldots, \delta_n)$ is as in (6.3.10). By Proposition 6.2.3, there is no $(k-t)$-sunflower of maximal dimension with at least three petals and the result follows by Lemma 6.3.2.

□

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„Felix qui potuit rerum cognoscere causas“

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