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Collusion in Auctions:

Secret vs Public Rings

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Abstract

This thesis analyzes whether and how an auctioneer who is aware that a ring of colluding bidders is present at the auction should reveal this information to other nonring bidders. We compare auction outcomes in four scenarios: (i) when bidders bid noncooperatively, (ii) when the presence of the ring is unknown to other bidders, (iii) when the ring presence is common knowledge, and (iv) when the ring is secretly known to nonring bidders.

The thesis consists of four chapters.

The first chapter motivates the analysis and lays the argument that conditional on ring presence, an auctioneer may improve his position depending on how he informs bidders regarding his suspicion/knowledge that a ring is present at the auction.

The Second and Third chapters are devoted to the formal analysis in private and common value auctions respectively.

In Chapter 2, we consider single-object first-price and second-price private value auctions with colluding bidders, and show that with uniformly distributed signals and a ring containing all-but-one of the bidders, it is best for the auctioneer to announce publicly if he knows that a ring is present. In this context, the first price auction is preferred to the second price auction. However, when the ring presence is known secretly to the nonring bidder, then the auctioneer is better-off under the second price auction.

In Chapter 3 we consider pure common value second price auctions. Assuming the underlying signals are uniform, we show for a family of valuation functions that publicly announcing the ring presence is a dominant strategy for the auctioneer, due to the fact that it leads bidders to bid higher than when bidding purely competitively. This in addition incentivizes an auctioneer to always announce publicly that a ring is present even if it is not, essentially making bidders bid higher without using a shill.

In Chapter 4, we present a brief overview of future research. We describe a *Bid Coordination Mechanism* a la Marshall and Marx (2007) for the first price auction based on repeated interactions.

To my parents Aminata Tanja Ceesay and Abdou Ceesay, and the departed soul of my grandmother Alhaja Aisha Tanja
(We ask Allaah to grant her a lofty position in Jannah)

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Contents

Chapter 1

Introduction

Auctions have been used since time immemorial as an allocation mechanism. From the sale of brides in Ancient Greece to putting the whole Empire of Rome on sale in 193 A.D Cassady (1967), Tulipmania in The Netherlands History Today (2018), the *"Great Auction of Slaves at Savannah Georgia"* Southern Spaces (2010), Privatization in Russia Boycko et al. (1993), the spectrum auctions of the 1990's and early 2000's Klemperer (2002), the Bank of England's Product Mix auction in 2007 Klemperer (2010) to The Gambia's licensing of blocks for oil exploration World Oil (2018). The ubiquity of auctions across spheres ranging from government procurement and solicitation of tenders for large-scale projects to online auctions on *ebay* (although there has been a sharp decline in the latter Einav et al. (2018)) gives stimulus to the study and analysis of auctions.

The main benefit to allocating an item via auctions comes from the competitive process. The likelihood of *Bidder Collusion* defined by Wang and Chen (2016) as

> *"an arrangement among a group of bidders (a bidding ring) that is designed to limit competition between the participants and thus capture some of the rents that would be transferred to the auctioneer in the case of competitive bidding"*

is a major impediment to this process. The reduced competition as a result of ring presence usually leads to loss of revenue for the auctioneer, and in some cases renders the auction inefficient.[1] This makes the consideration that bidders may cheat of paramount concern for the

[1] The inefficiency that may be induced by collusion is a major cause of concern for developing economies when auctions are used as allocation mechanisms for privatization, as the main objective is to get the objects to those who value them most Maskin (2000).

auctioneer Klemperer (2002). Bid rigging is a criminal offence in most countries.[2] Individuals or firms who have been found to engage in anticompetitive practices are prosecuted and incur high fines, and in some cases prison sentences. For this reason bidding rings take measures to conceal their presence at auctions.

Early research on the issue of collusion in auctions focused on analysing which auctions were susceptible to manipulation by bidding rings and were mainly geared towards independent private values. Abstracting away from certain internal workings of the ring (incentive to reveal true value to other members), Robinson (1985) showed that if bidders were ex-ante symmetric, the second price auction was more susceptible to collusion. Mailath and Zemsky (1991) considered ex-ante heterogenous bidders in second price auctions and showed that it was possible for rings to design incentive compatible mechanisms that deter ring members from shirking on agreed terms at the main auction. A similar approach was taken by McAfee and McMillan (1992) for first price auctions with ex-ante homogenous bidders, analysing cases for which the ring was able to make side payments and otherwise. They showed that provided the ring could enforce members' bidding strategies at the main auction,[3] it was possible to design efficient incentive-compatible mechanisms. Graham and Marshall (1987) examined bidding rings in Second price and English auctions. Incorporating a paradigm detailing the internal workings of rings, they showed the negative effects of ring presence on sellers revenue and showed that the auctioneer can offset the loss in revenue by imposing a higher reserve price.

Another strand in the literature focuses on measures to deter bidding rings from auctions. One such measure proposed by Marshall and Marx (2009) involves restricting information about bidder identity at auctions. This serves to make it difficult for rings to monitor members' bids at the auction, and gives incentives to ring members to renege on agreed ring terms. Other measures involve the construction of collusive-proof mechanisms for which bidding as a ring is made infeasible Pavlov (2008), Che and Kim (2006), Che and Kim (2009). These collusion-proof measures involve complexities that make them difficult to implement in practice Marshall and Marx (2012).

Given the proliferation of bidding rings, most bidders will go into an auction wary of the fact that they might be competing against bidding rings Marshall and Marx (2012).[4] For instance, in England, the Office of Fair Trading (OFT) conducted investigations into allegations of bid rigging by 103 construction firms, reported in the decision OFT (2009), in which fines amounting

[2]In the United States, it is a violation of the Sherman Act of 1890.

[3]Basically employing a *Bid Submission Mechanism* as in Marshall and Marx (2007)

[4]Baldwin et al. (1997) examined forest service timber sales in the Pacific Northwest using a dataset spanning the periods 1975 - 1981 and found that "*winning bids are best explained by collusion*" Other empirical studies documenting collusion include Asker (2010), Porter (2005) and Pesendorfer (2000).

to £192.2 million were imposed on the guilty parties. The decision noted that *Cover Pricing*[5] was so pervasive a phenomenon that even textbooks on bidding[6] referenced it and some went as far as providing guidance as to its implementation. Among the testimonies by the guilty parties was

> *"Cover pricing is a universally accepted practice within the construction industry and has been so for as far back as most people can remember ... cover pricing is so common as to be regarded as endemic within the construction industry, and has been practiced by all companies regardless of their size'."*[7]

This means auctioneers will also be wary that bidding at the main auction might be compromised. This raises the following concerns

- Compared to the case where the auction is purely competitive, how do bidders alter their bidding strategies when they suspect that a ring is present? What impact does this have on auctioneer's expected revenue?
- Will an auctioneer who believes that a ring is present at the auction want to make his belief known, or conceal it? If the decision is to reveal his suspicion, should the auctioneer reveal it publicly or secretly to the presumed nonring bidders?
- Will the auctioneer's choice to reveal or conceal depend on the valuations structure of bidders - That is whether we are under an independent private values framework or a common values framework?
- Based on the answers to the aforementioned concerns, is it possible that an auctioneer can cheat at the auction by convincing bidders that they are facing a ring when they are not?
- How does the conclusion to the aforementioned queries change with respect to the auction format?

Chapter 2 is devoted to analyzing these concerns in the independent private values paradigm. For the first price auction, the analysis proceeds in three stages. The first stage involves analysis based on arbitrary value distributions. We show that bidding secretly on behalf of the ring leads the ring representative to bid lower than he would have under noncooperation. The second

[5] A form of Bid rigging in reverse auctions involving a selected bidder who is chosen by the bidding ring to be the winner of the item at an agreed price (usually higher than the fair price) while other ring members submit significantly higher bids, to give a false impression of competition. This leads the principal to pay a higher price than he would have if the bidding process was uncompromised.

[6] OFT (2009) pp. 400 - 401.

[7] OFT (2009) pp. 398.

stage of the analysis follows from complexities with deriving explicit bidding functions in certain environments, necessitating simplification along two dimensions. First we restrict the size of the ring, by assuming that the ring is composed of all-but-one of the bidders. We refer to such a ring composition as an *almost all-inclusive* ring. Secondly, we assume that bidders values are ex-ante uniformly distributed. Under these assumptions, our analysis shows that a nonring bidder who is privately informed about ring presence bids (weakly) lower than when he (the nonring bidder) is unaware of ring presence. This results from the fact that forming the ring means eliminating competition which increases the probability of winning. Thus the ring representative can afford to bid lower compared to when the auction is noncooperative. The nonring bidder who is aware of this also is incentivised to bid lower than he would have if he believed that the auction was purely competitive, as he is facing less competition and a potentially lower competing price. However, when the presence of the ring is common knowledge, the auction becomes a standard two-bidder asymmetric auction, for which closed form solutions are generally not feasible. This leads to the third stage of the analysis where we adopt numerical estimates from Marshall et al. (1994).

We provide the following example to illustrate the results.

An Example with Independent Private Values

Suppose we have a first price independent private values auction with 3 bidders A, B and C whose values x_A, x_B and x_C are each drawn uniformly from $[0, 1]$. Let their realized values be

Bidder	A	B	C
Value	0.25	0.5	0.75

When the auction is purely noncooperative, Riley and Samuelson (1981) show that each bidder bids $\frac{2}{3}$ of their value. That is, they submit

Bidder	A	B	C
Bid	0.1667	0.3333	0.5

Bidder C wins, and the auctioneer receives 0.5.

Suppose now that bidders B and C collude without the knowledge of bidder A. Most ring mechanisms involve a designated bidder representing the group at the main auction (usually the ring member with the highest value) Graham and Marshall (1987), while other members remain absent or submit phony bids (usually very close to the seller's reserve price). This means that bidder C is chosen to bid competitively on behalf of the ring, while bidder B bids zero. How does bidder C choose his bid?

First of all, bidder C knows that bidder A bids $\frac{2}{3}$ of his value. When bidder C bids an amount b, he wins if $\frac{2}{3}x_A < b \implies x_A < \frac{3}{2}b$ and this happens with probability $\frac{3}{2}b$. So b is chosen to maximize

$$(0.75 - b) \cdot \frac{3}{2}b$$

which yields $b = \frac{0.75}{2} = 0.375$. That is, the best response for the ring representative is to bid half of his value.

Bidder	A	B	C
Bid	0.1667	0	0.375

Therefore, bidder C wins since $0.1667 < 0.375$, but the auctioneer receives 0.375, which is lower than 0.5, the amount he receives under noncooperation.

Hence, the formation of the ring hurts the auctioneer.

We purposely choose bidders B and C as the ring members to have a case under which ring formation is profitable, as the ring comprises of the two members with the highest values. In this sense, forming the ring eliminates bidder C's closest rival, and makes the auction less competitive. So the auctioneer loses revenue as a result.

In the thesis, we show in general that the ring representative bids lower than he would have under noncooperation, leading to a lower expected revenue for the auctioneer.

Given that the auctioneer can anticipate this, can he do better by revealing his knowledge of ring presence? If so, should the revelation be done publicly or in secret?

Suppose the auctioneer secretly announces his knowledge to bidder A. In principle, the nonring bidder infers that he is competing against a bidder whose value distribution is the highest of two random uniform draws from $[0, 1]$. That is, he infers that the distribution function for bidder C is x_C^2, and bidder C bids half of his value. Therefore, with a bid $\hat{b}$, the nonring bidder wins if $\frac{1}{2}x_C < \hat{b} \implies x_C < 2\hat{b}$ and this happens with probability $(2\hat{b})^2$. So he chooses this b to maximize

$$(0.25 - \hat{b}) \cdot (2\hat{b})^2$$

yielding $\hat{b} = 0.1667$, the same amount he submits under the noncooperative case.

Bidder	A	B	C
Bid	0.1667	0	0.375

Bidder C wins, since $0.1667 < 0.375$ and the auctioneer receives 0.375 which is the same amount as when the ring is operating in secret.

Essentially, secretly revealing his knowledge of ring presence to the nonring bidder does not help the auctioneer.

In the thesis, we show that when the value of the nonring bidder is sufficiently high, he bids even lower than under noncooperation, due to the anticipated lower bid by the ring representative.

Suppose the auctioneer publicly announces his knowledge of ring presence. The auction effectively becomes a two bidder asymmetric auction between bidder A and bidder C. Using the approximate bidding functions of Fibich and Gavious (2003) the bids submitted by bidder A and bidder C are 0.2425 and 0.5460 respectively.[8]

Bidder	A	B	C
Bid	0.2425	0	0.5460

Hence, bidder C wins, and the auctioneer receives 0.5460, which is higher than the amount he would have received if he remained quiet about ring presence.

Hence, the auctioneer benefits from publicly revealing his suspicion.

In essence, the example suggests that when an auctioneer suspects that a ring might be present at the auction, he is better-off making his suspicion publicly known.

In the thesis, we show that this conclusion holds more generally.

In Chapter 3, we make similar analyses under a pure common values second price auction. We specify a family of value functions under which we show that publicly revealing the presence

[8]The approximate bidding functions are

$$b_A(\theta) = \theta - \left(\frac{2}{\theta+\theta^2}\right)^2 \int_0^\theta \left(\frac{\psi+\psi^2}{2}\right)^2 d\psi + \frac{1}{\theta+\theta^2}\left[\int_0^\theta \left(\frac{\psi+\psi^2}{2}\right)^2\right]^3 \int_0^1 \frac{1}{\left[\int_0^\psi \left(\frac{u+u^2}{2}\right)^2\right]^2} \frac{16}{(1+\psi)^2} d\psi$$

$$b_C(\theta) = \theta - \left(\frac{2}{\theta+\theta^2}\right)^2 \int_0^\theta \left(\frac{\psi+\psi^2}{2}\right)^2 d\psi - \frac{1}{\theta+\theta^2}\left[\int_0^\theta \left(\frac{\psi+\psi^2}{2}\right)^2\right]^3 \int_0^1 \frac{1}{\left[\int_0^\psi \left(\frac{u+u^2}{2}\right)^2\right]^2} \frac{16}{(1+\psi)^2} d\psi$$

of the ring is the best option for the auctioneer. In addition, under this family of valuation structures, the ring presence being common knowledge leads other nonring bidders to bid higher than they would have under noncooperation. In this sense, the auctioneer does not necessarily need shills to artificially raise prices. The fact that bidders change their bidding strategies based on the auctioneer's revelation of information stems from the fact that under noncooperation, conditional on winning, a bidder can only infer the signal of the second highest bidder from the price he pays. So estimates the value of the item based on these two signals. However, when a bidder is informed that he is bidding against a ring, he knows that the ring representative bids based on ring member signals. In this sense, when the bidder wins against the ring representative, he can in principle infer the signals of the ring members via the price he pays, and can make an updated estimate of the value of the item upon winning, which is more informative than the estimated value under noncooperation. In this sense, the bidder suffers less winner's curse upon winning. We further demonstrate that whether this additional information leads him to bid higher or lower depends on the valuation structure.

We illustrate our results with the following example

An Example with Pure Common Values

Assume a second price pure common values auction with three bidders A, B and C with independent private signals x_A, x_B and x_C drawn uniformly from $[0, 1]$ with common value defined by

$$v(x_A, x_B, x_C) = x_A \cdot x_B \cdot x_C$$

Suppose the realization of their signals is

Bidder	A	B	C
Signal	0.25	0.5	0.75

When the auction is noncooperative and bidders use symmetric bidding strategies, Milgrom and Weber (1982) have shown that each bidder bids using the strategy

$$\beta(x_i) = \underbrace{\mathbb{E}[V|x_i, \max\{x_j\}_{j\neq i} = x_i]}_{x_i \cdot x_i \cdot \frac{1}{2}x_i} = \frac{x_i^3}{2}, \; i = \{A, B, C\}$$

Therefore, the bids submitted are

Bidder	A	B	C
Bid	0.0078125	0.0625	0.2109375

Bidder C wins and the auctioneer receives 0.0625.

Suppose now, that bidders B and C decide to bid as a group and bidder C is chosen to bid competitively on behalf of the ring while bidder B submits zero. In this sense, they have a joint signal $x_B \cdot x_C$. Given their realized values, their valuation structure going into the auction is

$$x_A \cdot x_B \cdot x_C = x_A \cdot 0.375$$

How does bidder C choose how much to bid?
If bidder A is unaware of the ring presence, he bids $\beta(x_A) = \frac{x_A^3}{2}$. Therefore, if bidder C wins the auction, he pays $\rho = \frac{x_A^3}{2}$ from which he infers that $x_A = (2\rho)^{\frac{1}{3}}$. So bidder C is happy to win as long as he makes a profit. That is, as long as the price he pays is less than his inferred value for the item upon winning. This is equivalent to

$$\rho \leq 0.375 \cdot (2\rho)^{\frac{1}{3}}$$

Therefore, the highest price at which bidder C be willing to win the auction is a price such that he breakseven. That is $\bar{\rho}$ such that

$$\bar{\rho} = 0.375 \cdot (2\bar{\rho})^{\frac{1}{3}} \implies \underbrace{\bar{\rho} = 2^{\frac{1}{2}} \cdot (0.375)^{\frac{3}{2}}}_{\substack{\text{That is,} \\ \bar{\rho}=2^{\frac{1}{2}}\cdot(x_B \cdot x_C)^{\frac{3}{2}}}} \approx 0.32475$$

Therefore, the bids submitted are

Bidder	A	B	C
Bid	0.0078125	0	0.32475

Bidder C wins and the auctioneer receives 0.0078125 which is less than 0.0625, the amount he receives under noncooperation.

Hence we have a case where the formation of the ring hurts the auctioneer.

Again, we purposely choose the ring to comprise of the bidders with the two highest signals, as this eliminates competition between them. This is the reason behind the loss in revenue.

We show in the thesis that this holds generally in expected value, provided the underlying *distribution of signals is uniform.*

Hence, if the auctioneer is silent about his suspicion/knowledge of ring presence, he gets 0.0078125.

Suppose the auctioneer secretly announces his suspicion to bidder A. The first inference that bidder A makes is that he is bidding against bidder C whose signal is $x_B \cdot x_C$, and who uses the bidding strategy $2^{\frac{1}{2}} \cdot (x_B \cdot x_C)^{\frac{3}{2}}$. Therefore, if bidder A wins the auction, he pays a price $\kappa = 2^{\frac{1}{2}} \cdot (x_B \cdot x_C)^{\frac{3}{2}}$ and infers that $x_B \cdot x_C = \frac{\kappa^{\frac{2}{3}}}{2^{\frac{1}{3}}}$. So he is only happy when he makes a profit. That is, when

$$\kappa \leq 0.25 \cdot \frac{\kappa^{\frac{2}{3}}}{2^{\frac{1}{3}}}$$

Therefore, he bids $\bar{\kappa}$ such that

$$\bar{\kappa} = 0.25 \cdot \frac{\bar{\kappa}^{\frac{2}{3}}}{2^{\frac{1}{3}}} \implies \bar{\kappa} = \underbrace{\frac{(0.25)^3}{2}}_{\frac{x_A^3}{2}} = 0.0078125$$

Essentially, bidding the same amount as in the noncooperative case.

We show in the thesis that this holds under a specific family of valuation structures (for which the multiplicative structure is a subset), and under arbitrary value distributions.

The bids submitted are

Bidder	A	B	C
Bid	0.0078125	0	0.32475

Bidder C wins, and the auctioneer receives 0.0078125.

Hence, secretly revealing the ring presence does not help the auctioneer.

Suppose however that the auctioneer publicly announces ring presence. It becomes common knowledge that bidder C has signal $x_B \cdot x_C$. Bidder A and bidder C choose bids to best-respond to each other. It is easy to see that the symmetric equilibrium strategy involves both bidders bidding square their signals. That is, using the strategy $\tilde{\beta}(x_i) = x_i^2$.[9] Therefore, the bids submitted are

[9] We show that x_A^2 and $(x_B \cdot x_C)^2$ are mutual best responses.
Suppose bidder C uses the strategy $(x_B \cdot x_C)^2$. If bidder A wins the auction, he pays a price $\rho = (x_B \cdot x_C)^2$, and infers $x_B \cdot x_C = \rho^{\frac{1}{2}}$. Therefore he is happy to win as long as he makes a profit. That is, as long as $\rho \leq x_A \cdot \rho^{\frac{1}{2}}$. So he bids $\hat{\rho}$ such that

$$\hat{\rho} = x_A \cdot \hat{\rho}^{\frac{1}{2}} \implies \hat{\rho} = x_A^2$$

Hence x_A^2 is a best response to $(x_B \cdot x_C)^2$. A similar argument shows that the converse is true.

Bidder	A	B	C
Bid	0.0625	0	0.140625

First, observe that Bidder C wins and the auctioneer receives 0.0625 which is more than 0.0078125, the amount he would have received had he been silent about ring presence or revealed it secretly to bidder A.

Hence, publicly announcing the presence of the ring benefits the auctioneer compared to when he is silent about ring presence or announces it privately to the nonring bidder.

We show in the thesis that this holds generally in expected value.

Also note that a bidder with signal x_i uses the strategy $\frac{x_i^3}{2}$ under noncooperation, while he uses the strategy x_i^2 when he is convinced that ring presence is common knowledge. For $x_i \in [0,1]$, $x_i^2 > \frac{x_i^3}{2}$. This presents the auctioneer with a novel way to artificially raise bidding even if no ring is present, as announcing ring presence publicly leads bidders to bid higher.

We show in the thesis that this result is contingent on the valuation structure.

Chapter 2

Secret vs Public Rings in Private Value Auctions

2.1 Introduction

In this chapter we analyse the consequences for the auctioneer of revealing vs concealing his knowledge of ring presence in the private values framework.

As a motivation, consider a bidding ring of size K from N symmetric bidders in a first price auction with independent private values. Equilibrium bidding in first price auctions is based on "*balancing the risk of losing against profitability of winning*" Rothkopf and Harstad (1994). A higher bid increases the probability of winning, while simultaneously reducing the profit upon winning. More bidders in the auction implies more competition, hence reducing the probability of winning, and this in turn leads participants to bid higher. If a group of bidders agree among themselves not to compete against each other, the immediate consequence is that the number of active bidders is $N-K+1<N$. This means that the designated ring bidder only competes with only $N-K$ bidders. This implies a lower risk of losing, and the ring representative can afford to bid lower. If this fact (the presence of the ring) remains unknown to the non-ring bidders, they will perceive that they are competing against $N-1$ bidders. If however they realize that there are only $N-K$ bidders, they might also be incentivised to bid lower (despite the fact that they know they are effectively bidding against a ring representative whose value is the highest among the ring members). So knowledge of ring presence may be of benefit to the non-ring bidders. On the contrary however, this has the opposite effect for the seller as it means he makes a lower revenue as all parties submit lower bids. If it is the case that knowing that a ring is present

causes other participants to submit lower bids, the ring representative might be induced to bid even lower, and other non-ring members being aware of this possibility might also be induced to further reduce their bids and so on. In second price auctions however, such a scenario is unlikely to occur as bidding truthfully is always a dominant strategy regardless of the identity of other participants. An auctioneer who anticipates the possibility of the scenario above will want to weigh the two auction formats to see which is more robust in terms of expected revenue.

We can phrase our main questions as follows:

- If an auctioneer knows that a ring is present at the auction, should she reveal her knowledge of ring presence to the nonring bidders? If yes, how? Should she publicly announce it or reveal it secretly to the bidders she believes are not part of the ring? What would be bidders' responses to these announcements compared to the purely competitive case?
- Conditional on ring presence, under what announcement policy is the first price auction more desirable in terms of auctioneer expected revenue than the second price auction?

It is in this regard that we consider single-object private - value second price and first price auctions with *"almost all-inclusive rings"* under the following informational assumptions:

- The Presence of the ring is unknown to the non-ring bidder: This covers the case where the ring is able to operate without the knowledge of outsiders. In this case, other bidders believe that the auction is purely competitive and they bid as if they are facing $N-1$ other similar bidders.
- The presence of the ring is known privately by the non-ring bidder: This case covers the case under which the ring believes it has perfectly concealed its presence, but nonring bidders are aware (or at least anticipate) that the ring is present.
- The presence of the ring is common knowledge: This covers the case for which nonring bidders know they are facing a ring and vice versa. This scenario is likely to occur when the auction involves bidding by industrial cartels, where affiliations are likely to be an open secret among participants in the industry. This may also be the case where a new entrant, say a foreign company competes with locals for government projects. It is reasonable to see that if the foreign company is relatively *Strong*, locals have an incentive to bid as a group, and the foreign company is likely to anticipate this, and the group knows the

foreign company will anticipate, and so on.

Under these scenarios, we specify the strategies employed by the ring representative and the non-ring bidder in both auctions. Given the difficulty associated with explicit derivation of strategies in the first-price auction, we aid our analysis further by assuming that bidders draw their valuations from a uniform distribution with support $[0, 1]$. We find that in general, conditional on a ring being present, the auctioneer prefers that the presence of the ring be publicly known. When the ring presence is unknown to the nonring bidders, the auctioneer prefers the first price auction. This stems from the fact that in the first price auction, the presence of the ring (organized such that the ring representative is the member with the highest value) exerts a positive externality on nonring bidders in the form of a higher probability of winning, as bidding on behalf of the ring involves bidding lower than in the non-cooperative case, and this increases the chance that non-ring bidders win the auction. In essence, there is a "*right kind of bias*" Milgrom (2004) towards the nonring bidder in the first price auction.

In the event that the ring presence is known secretly to the non-ring bidder, we find that the auctioneer is better off under the second price auction. This differs from the conclusion under when ring presence is publicly known, as employing results from Kirkegaard (2012), the first price auction yields higher expected revenue in this context. We illustrate that a possible explanation for the opposition in revenue rankings is the fact that equilibrium strategies when the ring presence is private knowledge involves the non-ring bidder submitting a constant amount across a range of values. This implies that when deciding between first price and second price auctions, practitioners should take into consideration that the superiority of the first price auction only holds when ring presence is assumed to be common knowledge.

The scope of the chapter is at the intersection of the literature on asymmetric auctions and auctions with colluding bidders. In particular we contribute to the literature on asymmetric first price auctions with independent private values. Furthermore, we complement on the discussion regarding when auctioneers prefer first price auctions to second price auctions arguing based on the response of other non-ring members when they become aware of the presence of a ring at the auction. Our analysis also provides another dimension along which off-equilibrium bids observed at first price auctions may be explained: Bidders may expect a ring at the auction, causing them to depress their bids to amounts lower than would have been expected in the non-cooperative equilibrium.

The rest of the chapter is organized as follows; In Section 2.2, we develop the framework for our analysis. Sections 2.3 and 2.4 are devoted to the analysis of bidding strategies in the second price auction and first price auction respectively. Section 2.5 demonstrates under the underlying uniform $[0,1]$ assumption, the implications of the assumed informational structures on seller's revenue, and other comparative static analyses. Section 2.6 concludes.

2.2 Framework

There is a single item for sale to $N \geq 3$ risk-neutral bidders $\mathcal{N} = \{1, 2, \ldots, N\}$, each $i \in \mathcal{N}$ assigning independent private values $X_i \in [0,1]$ drawn from an atomless distribution $F(\cdot)$ with corresponding density $f(\cdot)$. Denote $Y^1_{\mathcal{N}/\{i\}}, Y^2_{\mathcal{N}/\{i\}}, \ldots, Y^{N-1}_{\mathcal{N}/\{i\}}$ as the largest, second-largest, ..., smallest of X_j for $j \in \mathcal{N}/\{i\}$. $\mathbb{P}(A)$ is shorthand for "*probability that A*". The ring in question is assumed to be an *almost all-inclusive ring* - all but one of the bidders are members. We call bidders submitting competitive bids *active bidders*. The objective of the auctioneer is to maximize revenue. We also adopt the convention of referring to the non-ring bidder with male pronouns, and the ring representative with female pronouns.

We use the term *auctioneer* and *seller* interchangeably.

2.2.1 Informational Structures

The following are the informational structures[1] we will consider

Purely Non-Cooperative Case

This is the scenario when there is no ring at the auction, and everyone is bidding independently. Everyone is an active bidder in this case.

Concealed Case

This is the case for which the ring is present at the auction, but is able to conceal it's presence from the Non-Ring Bidder. This corresponds to the auctioneer being silent about his knowledge of ring presence.

Public Knowledge

This is the case for which the ring is present at the auction, and it's presence is commonly known to all participants at the auction. This corresponds to the auctioneer publicly revealing his knowledge of ring presence.

[1] The term *Informational Structures* has been used by Cheng and Tan (2010) in a similar context.

Private Knowledge

This is the case for which the ring is present at the auction, and they think they are operating in secret, but the presence of the ring is privately known to the Non-ring bidder. This corresponds to the auctioneer revealing his knowledge of ring presence privately to the nonring bidder.

2.2.2 Terminology

The Ring Representative

The ring member who is chosen to bid competitively on behalf of the ring, while others submit phony bids - which we normalize to zero.

Secret Ring

The ring which believes it operates concealed. Encompasses the ring under the Concealed and Private Knowledge cases.

Public Ring

The ring under Public Knowledge.

2.2.3 Timing of the Game

- Nature assigns bidders' values
- The ring mechanism is implemented, and
  a ring representative is chosen
- Bidders proceed to the main auction

2.2.4 Ring Mechanism

We generalise the *Strong Cartel* mechanism of McAfee and McMillan (1992) for first price auctions to cover arbitrary standard auctions. This mechanism involves a reporting stage - which determines the ring representative and payoffs to other ring members - and bidding in the main auction. The ring has to decide how to coordinate bidding at the main auction, and ensure that ring members are truthful and stick to ring agreements. The full description is as follows:

– Stage 1: Ring members report their values to a ring center, and the member with the highest reported value is designated by the center as the ring representative and is required to pay an amount $b(x)$ based on her reported value. This amount is divided among the other ring members. That is, other ring members receive $\frac{b(x)}{N-m-1}$, where m denotes the number of nonring bidders.

– Stage 2: Bidders proceed to the main auction. Losing bidders from Stage 1 are forced to bid zero, while the ring representative bids competitively against other nonring bidders, pocketing the profit in the event that he wins.[2] In this sense, this mechanism is a *Bid Submission Mechanism* in the language of Marshall and Marx (2007).[3]

In this context, incentive compatibility involves reporting the true value to the ring center in Stage 1.[4]

Lemma 2.2.1. McAfee and McMillan (1992) *Suppose that there are m nonring members. The mechanism given above is incentive compatible given*

$$b(x) = \frac{1}{F(x)^{N-m}} \int_0^x (\theta - \rho) M(\theta)(N - m - 1)F(\theta)^{N-m-1} f(\theta) d\theta$$

where ρ is the price paid by the ring representative in the event that he wins at the main auction, and $M(x)$ is the probability that the ring representative wins at the main auction.

Furthermore, we show that ring members' ex-ante expected payoff from this mechanism is the same as designating the member that carries the highest value for the good to bid on behalf of the ring at the main auction, other ring members submitting zero, and the ring members sharing the profit equally in the event that the ring wins.

Consider an arbitrary ring member j with value x, and suppose in Stage 1, he reports a

[2] One way to think about this is that bids are submitted by the center on behalf of each ring member, and they just have to be present at the auction

[3] Given that the absence of an enforcement mechanism for bid submission in the first price auction gives ring members an incentive to deviate, an alternative approach to enforcement will be to describe a repeated games scenario where bidders use grim-trigger strategies to punish defectors. The difficulty with this approach is that defection is only spotted if it affects the outcome of the main auction. That is, only if the defector wins at the main auction. One may defect and still lose if his bid is in-between the ring representative's bid and the nonring bidder's bid. So deviation is not perfectly observable, and the ring mechanism might involve public strategies, or under the current circumstances, a very high discount factor.

[4] Similar mechanisms are considered by Eso and Schummer (2004) and Balzer (2019) for two-bidder auctions. However, by construction, these mechanisms involve all bidders being part of the ring, in contrast to the case of an almost-all-inclusive ring that we analyze.

value $\hat{x}$. His expected payoff based on this report is

$$\pi(x,\hat{x}) = \overbrace{F(\hat{x})^{N-m-1}}^{\substack{\text{Probability} \\ \hat{x}\text{ is} \\ \text{highest} \\ \text{reported} \\ \text{value}}}[\overbrace{(x-\rho)M(x)}^{\substack{\text{Expected} \\ \text{payoff} \\ \text{from} \\ \text{main} \\ \text{auction}}} - \overbrace{b(\hat{x})}^{\substack{\text{Amount} \\ \text{paid to} \\ \text{other} \\ \text{ring} \\ \text{members}}}]+ \\ \underbrace{(1-F(\hat{x})^{N-m-1})}_{\substack{\text{Probability} \\ \hat{x}\text{ is not} \\ \text{highest} \\ \text{reported} \\ \text{value}}} \underbrace{\int_{\hat{x}}^{1} \frac{b(\theta)}{N-m-1}(N-m-1)\frac{f(\theta)F(\theta)^{N-m-2}}{1-F(\hat{x})^{N-m-1}}d\theta}_{\substack{\text{Expected} \\ \text{amount} \\ \text{received} \\ \text{from} \\ \text{ring} \\ \text{representative}}} \tag{2.1}$$

This simplifies to

$$\pi(x,\hat{x}) = F(\hat{x})^{N-m-1}[(x-\rho)M(x) - b(\hat{x})] + \int_{\hat{x}}^{1} b(\theta)f(\theta)F(\theta)^{N-m-2}d\theta \tag{2.2}$$

In essence we can show

Lemma 2.2.2.

$$\int_0^1 \pi(x,x)f(x)dx = \int_0^1 (x-\rho)M(x)F(x)^{N-m-1}f(x)dx$$

Proof. From (2.2),

$$\int_0^1 \pi(x,x)f(x)dx = \int_0^1 (x-\rho)M(x)F(x)^{N-m-1}f(x)dx \\ \underbrace{-\int_0^1 b(x)F(x)^{N-m-1}f(x)dx + \int_0^1\int_x^1 b(\theta)f(\theta)F(\theta)^{N-m-2}d\theta f(x)dx}_{=0} \tag{2.3}$$

Since

$$\int_0^1\int_x^1 b(\theta)f(\theta)F(\theta)^{N-m-2}d\theta f(x)dx = \int_0^1 b(\theta)f(\theta)F(\theta)^{N-m-2}\Big(\int_0^\theta f(x)dx\Big)d\theta \\ = \int_0^1 b(\theta)f(\theta)F(\theta)^{N-m-1}d\theta = \int_0^1 b(x)F(x)^{N-m-1}f(x)dx$$

□

In this regard, we lose no generality by proceeding directly to the main auction, where the ring representative bids competitively against nonring bidders, other ring members bidding zero, and proceeds being shared equally afterwards.

2.3 The Second-Price Auction

We start by analysing the bidding strategies employed under each informational structure and the effect on seller's expected revenue with the noncooperative case as the benchmark.

The Purely Non-Cooperative Equilibrium

Theorem 2.3.1. Vickrey (1961) *In a Single-Object second price auction, the symmetric increasing equilibrium strategy involves active bidders bidding truthfully, regardless of the number of bidders or the asymmetry of their distributions. That is ,* $\forall i$

$$\beta(x_i) = x_i$$

An immediate consequence is that for the second price auction, conditional on ring presence, whether or not the presence of the ring is known to the non-ring member is irrelevant for the auctioneer, as the behavior of bidders do not change. Hence it makes no difference to the auctioneer whether or not he reveals the presence of the ring to nonring bidders as bidding strategies remain the same. However, it is easy to see that the auctioneer's expected revenue is lower when a ring is present. This is due to the fact that the expected revenue to the seller is the expected second highest bid. We can demonstrate this fact for an arbitrary number $m \leq 1$ of non-ring bidders.

In the absence of the ring, the expected second highest bid has distribution described by

$$\mathbb{G}_N(x) = F(x)^N + NF(x)^{N-1}(1 - F(x))$$

while in the presence of the ring, the distribution of second highest bids is given by

$$\mathbb{G}_R(x) = F(x)^N + mF(x)^{N-1}(1 - F(x)) + F(x)^m(1 - F(x)^{N-m})$$

$$\mathbb{G}_R(x) - \mathbb{G}_N(x) = F(x)^m - (F(x)^N + (N - m)F(x)^{N-1}(1 - F(x)))$$

Lemma 2.3.1.

$$\mathbb{G}_R(x) \geq \mathbb{G}_N(x)$$

Proof. We compare

$$F(x)^N + (N - m)F(x)^{N-1}(1 - F(x)) \text{ with } F(x)^m$$

This is straightforward. Denote $\theta = F(x)$. It is equivalent to comparing θ^m and $\kappa(\theta) = \theta^N + (N-m)\theta^{N-1}(1-\theta)$, for $\theta \in [0,1]$

- $\theta = 0 \implies \theta^m = 0,\ \kappa(\theta) = 0$
- $\theta = 1 \implies \theta^m = 1,\ \kappa(\theta) = 1$
- For $\theta \in (0,1)$, comparing θ and $\kappa(\theta)$ is equivalent to comparing 1 and $s(\theta) = \theta^{N-m} + (N-m)\theta^{N-m-1}(1-\theta)$. $s(1) = 1$, and $s^{'}(\theta) = (N-m)(N-m-1)(1-\theta)\theta^{N-m-2} > 0$ for $N > m+1$.[5] Therefore, for $\theta \in (0,1)$, $s(\theta) < 1$ which implies $\theta > \theta^N + (N-1)\theta^{N-1}(1-\theta)$

Therefore,

$$F(x)^m \geq F(x)^N + (N-m)F(x)^{N-1}(1-F(x)) \iff \mathbb{G}_R(x) \geq \mathbb{G}_N(x)$$

$\square$

Therefore, $\mathbb{G}_R(x)$ first order stochastically dominates $\mathbb{G}_N(x)$,implying that the revenue is lower in the presence of the ring.[6] Furthermore,

$$\mathbb{G}_R(x) - \mathbb{G}_N(x) = F(x)^m - (F(x)^N + (N-m)F(x)^{N-1}(1-F(x)))$$

is clearly decreasing in m, implying that the lower the ring size, the better.

In summary,

Proposition 1. *In the second price auction, all active bidders bid truthfully regardless of the informational structure. This implies that the presence of the ring has no impact on the probability of winning of nonring bidders as they need their values to be the highest among all active bidders. Furthermore by stifling competition among themselves, the ring induces a loss in expected revenue for the seller, regardless of whether they operate in secret or not, and more ring members implies more revenue loss.*

Hence in a second price auction with colluding bidders, the auctioneer cannot improve his position by revealing the presence of the ring to other nonring members.

Furthermore, in the second price auction, ring presence does not impact efficiency. This is because the ring representative has the highest value from ring members and since active bidders

[5] The case $m = N-1$ corresponds to the Purely Non-Cooperative Case.

[6] This case corresponds to comparing the asymmetric auction with the *Symmetric Benchmark* as in Cantillon (2008), who proves a more general result via the inequality of arithmetic and geometric means.

at the main auction bid their values, the ring representative only wins if her value is higher than that of the other nonring members. In essence, the winner at the main auction is the bidder with the highest value. So the auction remains efficient.

2.4 The First-Price Auction

2.4.1 Equilibrium Behavior under Noncooperation

Let $H(\cdot) = F(\cdot)^{N-1}$ denote the distribution of $Y^1_{\mathcal{N}/\{i\}}$, with density $h(\cdot)$.

Theorem 2.4.1. Riley and Samuelson (1981) *The symmetric equilibrium increasing bidding strategy $\beta(\cdot)$ is given by*

$$\beta(x_i) = x_i - \frac{1}{H(x_i)} \int_0^{x_i} H(\psi) d\psi$$

The idea behind this result is that a bidder i wins with bid b if $\beta(Y^1_{\mathcal{N}/\{i\}}) < b \iff Y^1_{\mathcal{N}/\{i\}} < \beta^{-1}(b)$, and this happens with probability $H(\beta^{-1}(b))$. Therefore, given value x_l, the best response for bidder l is $\hat{b}$ such that

$$\hat{b} \equiv \arg\max (x_i - \hat{b}) H(\beta^{-1}(\hat{b}))$$

The first order condition yields

$$-H(\beta^{-1}(\hat{b})) + \frac{(x_l - \hat{b})}{\beta'(\beta^{-1}(\hat{b}))} h(\beta^{-1}(\hat{b})) = 0 \tag{2.4}$$

coupled with the fact that $\beta(\cdot)$ is a symmetric equilibrium strategy.

2.4.2 Equilibrium Behavior in the Concealed Case

Consider an arbitrary number m of non-ring bidders. Since the presence of the ring is unknown to the nonring bidders, they maintain the bidding strategy $\beta(\cdot)$. Denote $\beta^{-1}(\cdot) \equiv \phi(\cdot)$. The strategy of the ring representative in this case is to best respond to $\beta(\cdot)$. Denote the realized value of the ring representative and the non-ring bidder by x_R.

Theorem 2.4.2. *When the ring operates undetected, given that the ring representative has value $x_R \in (0, 1]$, she submits a $\tilde{b}$ which satisfies*

$$-F(\phi(\tilde{b})) + m f(\phi(\tilde{b}))\phi'(\tilde{b})(x_R - \tilde{b}) = 0$$

and the additional condition that $\tilde{b} = 0$ *for* $x_R = 0$.

Proof. The proof is straight-forward. Let $\mathcal{M}$ denote the set of nonring bidders. When the ring representative submits a bid b, she wins if $\beta(x_l) < b \iff x_l < \beta^{-1}(b) \equiv \phi(b) \forall l \in \mathcal{M}$, and this happens with probability $F(\phi(b))^m$. The expected profit of the ring representative based on this bid is

$$(x_R - b)F(\phi(b))^m$$

So she chooses a $\tilde{b}$ such that

$$\tilde{b} \equiv \arg\max(x_R - b)F(\phi(b))^m$$

The first order condition yields upon simplification

$$-F(\phi(\tilde{b})) + mf(\phi(\tilde{b}))\phi'(\tilde{b})(x_R - \tilde{b}) = 0 \tag{2.5}$$

Furthermore, bidding higher than zero for $x_R = 0$ is weakly dominated by $\tilde{b} = 0$ since it results in a loss when the ring representative wins. $\square$

The almost all-inclusive ring case corresponds to $m = 1$, and the noncooperative case corresponds to $m = N - 1$. As a next step, we can compare the bid submitted in this case with the bid submitted in the noncooperative case. That is we compare the ring representative's bid when $m < N - 1$ with her bid when $m = N - 1$. Maintaining the convention $\beta^{-1}(\cdot) \equiv \phi(\cdot)$ note that with a generic value x, the first order condition (2.4) from Theorem 2.4.1 can be rewritten as $-H(\phi(\hat{b})) + (x - \hat{b})\phi'(\hat{b})h(\phi(\hat{b})) = 0, \equiv -F^{N-1}(\phi(\hat{b})) + (x - \hat{b})\phi'(\hat{b})(N-1)f(\phi(\hat{b}))F^{N-2}(\phi(\hat{b})) = 0$

$$\iff -F(\phi(\hat{b})) + f(\phi(\hat{b}))\phi'(\hat{b})(x - \hat{b})(N - 1) = 0 \tag{2.6}$$

(2.6) characterizes the bid of the ring representative under non-cooperation. Define

$$\Omega(b) = -F(\phi(b)) + mf(\phi'(b))\phi'(b)(x - b)$$

Then (2.6) can be rewritten as

$$\Omega(\hat{b}) + (N - m - 1)f(\phi(\hat{b}))\phi'(\hat{b})(x - \hat{b}) = 0 \tag{2.7}$$

and

$$\Omega(\tilde{b}) = 0 \tag{2.8}$$

Theorem 2.4.3. *When the ring representative is bidding on behalf of the ring with the belief that the ring presence is concealed, the bid $\tilde{b}$ that she submits is lower in comparison to her bid $\hat{b}$ when she is bidding non-cooperatively. That is*

$$\hat{b} \geq \tilde{b}$$

Proof. (2.7) and (2.8) implies

$$\Omega(\tilde{b}) = \Omega(\hat{b}) + \overbrace{(N-m-1)}^{>0}\overbrace{f(\phi(\hat{b}))}^{\geq 0} \cdot \underbrace{\overbrace{\phi'(\hat{b})}^{\geq 0}}_{\substack{\text{since } \beta(\cdot) \\ \text{is} \\ \text{increasing}}} \cdot \overbrace{(x-\hat{b})}^{>0} \implies \Omega(\tilde{b}) \geq \Omega(\hat{b})$$

Optimality of $\tilde{b}$ requires that $\Omega'(\tilde{b}) \leq 0$. Therefore,

$$\Omega(\tilde{b}) \geq \Omega(\hat{b}) \implies \hat{b} \geq \tilde{b}$$

□

The intuition for this result is straightforward. Forming the ring reduces the number of active bidders in the auction, hence it is less competitive. So there is a lower chance that the ring representative faces a bidder with a value at least as high as hers. This implies that the ring representative can afford to shade her bid further than in the non-cooperative case. Formally when she bids $\hat{b}$,

$$\underbrace{F(\phi(\hat{b}))^{N-1}}_{\substack{\text{Probability} \\ \text{of winning} \\ \text{under} \\ \text{non-cooperation}}} \leq \underbrace{F(\phi(\hat{b}))^{m}}_{\substack{\text{Probability} \\ \text{of winning} \\ \text{under} \\ \text{ring}}}$$

Therefore, when bidding on behalf of the ring, she can achieve the same probability of winning as she would have in the non-cooperative case with a lower bid $\mathfrak{b}$ and earn a higher expected profit. That is there is a $\mathfrak{b} < \hat{b}$ such that

$$F(\phi(\hat{b}))^{N-1} = F(\phi(\mathfrak{b}))^{m} \implies (x_R - \hat{b})F(\phi(\hat{b}))^{N-1} < (x_R - \mathfrak{b})F(\phi(\mathfrak{b}))^{m}$$

Therefore she has more room to make the tradeoff between amount paid and the probability of winning. This is because on one hand, any bid in the range $(\mathfrak{b}, \hat{b})$ ensures a higher probability of winning than in the non-cooperative case. However the payment upon winning is higher than $\mathfrak{b}$. Therefore the objective will be to find a $\tilde{b} \in (\mathfrak{b}, \hat{b})$ that balances the *payment - probability of* *winning* tradeoff.

So far the discussion has been quite general with regards to the size of the ring. In the next subsection, the objective is to examine the case for which the presence of the ring is publicly known. In this case, the auction becomes equivalent to an asymmetric $m+1$-bidder first price auction,where m symmetric bidders - the nonring bidders - have values drawn from $F(\cdot)$ and the remaining bidder -the ring representative - has value drawn from $F(\cdot)^{N-m}$ where the $F(\cdot)$ is common knowledge. Bidding strategies in this case are characterized as solutions to a system of differential equations as in Lebrun (1999) and Maskin and Riley (2000) for whom explicit derivations are intractable generally.[7] Most of the qualitative results in this context center around the case with two bidders; particularly the case with *Strong* and *Weak* bidders. This necessitates a further modification of the assumption regarding the composition of the ring. Subsequently, we further assume that the ring is "*almost all inclusive*" (that is $m = 1$). This enables the characterization of the ring representative and the nonring bidder as the *Strong* and *Weak* bidders respectively.

2.4.3 Equilibrium Behavior under Public Knowledge

Here, we consider the case where it is common knowledge among the bidders that the non-ring bidder is effectively competing with only one serious bidder with valuation drawn from $F(\cdot)^{N-1}$. This scenario can be interpreted as a standard asymmetric first-price auction, with a *strong* and a *weak* bidder. The *strong* bidder being the ring representative, and the *weak* bidder being the non-ring bidder.

First I review the definitions of *weak* and *strong* and then I illustrate the connection with our case.

2.4.3.1 Weak and Strong Bidders

Given two bidders S and W, with values distributed according to cdf $F_S(\cdot)$ and $F_W(\cdot)$ respectively, with densities $f_S(\cdot)$ and $f_W(\cdot)$. Denote the *Reverse Hazard Function* by

$$\mathbb{Q}_i(x) = \frac{f_i(x)}{F_i(x)}$$

[7] Kirkegaard (2012) uses a mechanism design approach, albeit without explicit derivations of the bidding functions.

$i \in \{S, W\}$. Bidder S is said to be *Stronger* than Bidder W if

$$\mathbb{Q}_S(x) > \mathbb{Q}_W(x) \equiv \frac{f_S(x)}{F_S(x)} > \frac{f_W(x)}{F_W(x)}$$

Essentially, *Strength* is synonymous with *Reverse Hazard Rate Dominance.*

In relation to our case, we can interpret the current bidding environment as a two bidder asymmetric second price auction for which

- The ring representative draws values from $H(\cdot) = F(\cdot)^{N-1}$
- The non-ring bidder draws values from $F(\cdot)$

$$\frac{h(x)}{H(x)} = \frac{(N-1)f(x)F(x)^{N-2}}{F(x)^{N-1}} = \frac{(N-1)f(x)}{F(x)} > \frac{f(x)}{F(x)}$$

implying that the ring representative is the *stronger* bidder. Existence of continuous increasing equilibria in this setting has been characterized by Maskin and Riley (2000), Athey (2001) and in an earlier paper by Lebrun (1999). In general, these strategies are characterized as solutions to the system of differential equations specified below

Theorem 2.4.4. Maskin and Riley (2000), Lebrun (1999) *Suppose the ring representative and the non-ring bidder use the continuous increasing strategies $\beta_R(\cdot)$ and $\beta_N(\cdot)$ respectively, with $\beta_R^{-1}(\cdot) \equiv \phi_R(\cdot)$ and $\beta_N^{-1}(\cdot) \equiv \phi_N(\cdot)$ and denote the highest possible bid as $\bar{b}$. These strategies can be implicitly characterised by the system of differential equations*

$$-F(\phi_R(b)) + f(\phi_R(b))\phi_R'(b)(\phi_N(b) - b)(N-1) = 0$$
$$-F(\phi_N(b)) + f(\phi_N(b))\phi_N'(b)(\phi_R(b) - b) = 0$$

with boundary conditions $\beta_N(0) = \beta_R(0) = 0$, $\beta_N(1) = \beta_R(1) = \bar{b}$

The properties of this kind of auction has been studied more generally by Maskin and Riley (2000) and Kirkegaard (2012). A central result in this theme is the fact that the *Weak* bidder bids more aggressively than the strong bidder. A summary of the major findings given by Kaplan and Zamir (2015) are the following

- The *Strong* bidder is less aggressive than the *Weak* bidder. That is, for any realisation of values x, $\beta_R(x) \leq \beta_N(x)$.
- When strength is manifested in the form of a *Stretch* ($F_S(x) = \theta F_W(x)$, $\theta \in (0, 1)$) or a *shift* ($F_S(x + \theta) = F_W(x)$, $\theta > 0$), the auctioneer prefers the first price auction to the second price auction.

– The *Strong* bidder obtains higher expected profit from the second price auction than the first price auction. The opposite is true for the *Weak* bidder.

Our current notion of the ring representative being the *Strong* bidder and the nonring bidder being the *Weak* bidder does not correspond to strength by *Shifting* or *Stretching*. This makes revenue comparisons generally intractable without explicitly formulating the respective bidding strategies. Furthermore, the same difficulty is encountered when analysing the strategies adopted in the case of private knowledge.

However, with an underlying uniform distribution, defining a *Symmetric Benchmark* against which to compare outcomes from an asymmetric auction, Cantillon (2008) analyzed a two-bidder asymmetric first price auction and showed that with respect to a *Symmetric Benchmark*, the auctioneer's revenue is lower under the asymmetric case. Furthermore, she showed that greater asymmetry induced greater revenue loss.

Therefore, we proceed by simplification in the following direction:

– We fix the underlying distribution $F(\cdot)$ to be the uniform $[0, 1]$. This is a standard assumption in several papers on asymmetric first price auctions. In particular, it is the distribution mostly assumed when applying numerical methods.

This simplification also allows us to use the results from Cantillon (2008) regarding the effect of asymmetries on expected revenue. Some terminology from the paper is required

– *Configuration*: The n-tuple of value distributions $(F_1, F_2, \ldots, F_N)$.

– *The Symmetric Benchmark*: $F = \left(\prod_i^N F_i\right)^{\frac{1}{N}}$.

– Denote by $\mathbb{R}_{U,[V]}$ the expected revenue to the seller from Uth price auction with configuration $[V]$.

Lemma 2.4.1. (Cantillon (2008), Theorem 3)*: Consider two configurations of bidders* $[A] = (F^{N-1}, F)$ *and* $[B] = (F^{N-\alpha}, F^{\alpha})$ *with* $1 < \alpha \leq \frac{N}{2}$ *and* F *being a uniform distribution function. The expected revenue from configuration* $[A]$ *is strictly lower than that from configuration* $[B]$*.*

In particular, the lemma implies that the revenue from a first price asymmetric two-bidder auction with value distributions F^{N-1} and F is less than that from the symmetric auction with distributions $F^{\frac{N}{2}}$ and $F^{\frac{N}{2}}$.

Furthermore, by the revenue equivalence theorem, under configuration $[B'] = (F^{\frac{N}{2}}, F^{\frac{N}{2}})$, the first price and second price auctions yield the same revenue. That is $\mathbb{R}_{1,[B']} = \mathbb{R}_{2,[B']}$.

Additionally, in the second-price auction, the expected revenue for the seller is the same under the two bidder auction with configuration $[B'] = (F^{\frac{N}{2}}, F^{\frac{N}{2}})$ and the N bidder auction with configuration $[B''] = (F^{\frac{N}{2}}, F^{\frac{N}{2}}, 1, \dots, 1)$.[8] That is $\mathbb{R}_{2,[B']} = \mathbb{R}_{2,[B'']}$.

A second theorem from Cantillon (2008) relates the revenue from the N bidder second price auction with configuration $[B''] = (F^{\frac{N}{2}}, F^{\frac{N}{2}}, 1, \dots, 1)$ with the revenue from the configuration characterised by the symmetric benchmark $[B^s] = (F, F, \dots, F)$.[9] Note that the configuration $[B^s] = (F, F, \dots, F)$ characterizes the distribution of values under Noncooperation, while $[A] = (F^{N-1}, F)$ characterizes the value distributions under Public Knowledge.

Lemma 2.4.2. (Cantillon (2008), Theorem 1): *In the second price auction, the revenue from the configuration $[B'']$ is lower than the revenue from the symmetric benchmark $[B^s]$. That is*

$$\mathbb{R}_{2,[B'']} < \mathbb{R}_{2,[B^s]}$$

Therefore,

Proposition 2. *When the underlying distribution is uniform, the auctioneer gets a lower expected revenue under public knowledge of ring presence compared to the noncooperative case. That is*

$$\mathbb{R}_{1,[A]} < \mathbb{R}_{1,[B^s]}$$

Proof.

$$\underbrace{\mathbb{R}_{1,[A]} < \mathbb{R}_{1,[B']}}_{\text{By Lemma 2.4.1}} \overbrace{= \mathbb{R}_{2,[B']}}^{\substack{\text{Revenue}\\ \text{Equivalence}}} = \underbrace{\mathbb{R}_{2,[B'']} < \mathbb{R}_{2,[B^s]}}_{\text{By Lemma 2.4.2}} \overbrace{= \mathbb{R}_{1,[B^s]}}^{\substack{\text{Revenue}\\ \text{Equivalence}}} \tag{2.9}$$

□

Since we have no way of explicitly representing seller's expected revenue in the public knowledge case, it is unclear whether the revenue in the public knowledge case is lower or higher compared with the revenues in the private knowledge and the concealed case. In the section 2.5, we explicitly characterise the seller's expected revenue in the other cases, and compare with simulated values from Marshall et al. (1994) for the public knowledge case.

First, we describe the behavior of the non-ring bidder in the private knowledge case.

2.4.4 Equilibrium Behavior under Private Knowledge

Here, the scenario is that the ring operates with the belief that its presence is concealed. However, the non-ring bidder is aware of the presence and the belief of the ring. In that regard, the strategy

[8]The expected revenue in both cases is the expected second highest value, which has distribution $F^N + 2F^{\frac{N}{2}}(1 - F^{\frac{N}{2}})$

[9]$\mathbb{R}_{1,[B^s]}$ is the auctioneer's expected revenue in the first price auction under non-cooperation, and $\mathbb{R}_{1,[A]}$ is the auctioneer's expected revenue in the first price auction under public knowledge of the ring.

of the ring representative is the same as in Theorem 2.4.2 with $m = 1$. That is for $x_R \in (0, 1]$, he submits a $\tilde{b}$ such that

$$-F(\phi(\tilde{b})) + f(\phi(\tilde{b}))\phi'(\tilde{b})(x_R - \tilde{b}) = 0$$

and he bids $\tilde{b} = 0$ when $x_R = 0$.

The non-ring bidder has to best-respond to this strategy. However, the difficulty with expressing $\tilde{b}$ explicitly makes this intractable. Again, this necessitates the simplification that bidders' values are uniformly distributed under which we obtain closed-form solutions.

2.5 Characterization under the Uniform Distribution

In this section, we adopt the convention that underlying distributions are uniform and proceed to characterise explicitly where possible - using the results from previous sections - the bidding strategies and expected revenues, and make comparative static analyses.

Formally, each X_i is identically and independently uniformly distributed on $[0, 1]$.
Here,

$$F(x) = x,\ f(x) = 1,\ H(x) = x^{N-1} \text{ and } h(x) = (N-1)x^{N-2}$$

2.5.1 Purely Non-Cooperative Equilibrium

With this specification, the equilibrium bidding strategy adopted by bidders under noncooperation, described in Theorem 2.4.1 becomes

$$\beta(x_i) = x_i - \frac{1}{H(x_i)} \int_0^{x_i} H(\psi) d\psi = x_i - \frac{1}{x_i^{N-1}} \int_0^{x_i} \psi^{N-1} d\psi = \frac{N-1}{N} x_i$$

With inverse

$$\phi(b) = \frac{N}{N-1} b$$

The equilibrium bidding strategy under noncooperation is

$$\beta(x_i) = \frac{N-1}{N} x_i$$

2.5.2 Equilibrium Behavior under the Concealed Case

When the ring operates undetected, the ring representative submits $\tilde{b}$ as specified in Theorem 2.4.2. Therefore for $x_R \in (0, 1]$, $\tilde{b}$ is implicitly defined by

$$-\phi(b) + \phi'(b)(x_R - b) = 0$$

which becomes

$$-\frac{N}{N-1}\tilde{b} + \frac{N}{N-1}(x_R - \tilde{b}) = 0 \implies \tilde{b} \equiv \tilde{b}(x_R) = \frac{x_R}{2}$$

Hence[10]

The bidding strategies under the concealed case are

$$\bar{\beta}_N(x_N) = \frac{N-1}{N}x_N$$

and

$$\bar{\beta}_R(x_R) = \frac{1}{2}x_R$$

2.5.3 Equilibrium Behavior under Private Knowledge

Given that the ring representative uses a strategy $\tilde{\beta}_R(x_R) = \frac{1}{2}x_R$ believing that the non-ring bidder is unaware of the ring's presence, the strategy of the non-ring bidder is to best-respond to $\tilde{\beta}_R(x_R)$.

Proposition 3. *When the presence of the ring is privately known to the non-ring bidder, the equilibrium bidding strategies for the ring representative and the non-ring bidder are $\tilde{\beta}_R(x_R)$ and $\tilde{\beta}_N(x_N)$ respectively, characterised by*

$$\tilde{\beta}_R(x_R) = \frac{1}{2}x_R \text{ and } \tilde{\beta}_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{1}{2}\right\}$$

Proof. First of all, the non-ring bidder bids no higher than $\frac{1}{2}$, as any bid $b = \frac{1}{2} + \epsilon$, $\epsilon > 0$ is strictly dominated by $b' = \frac{1}{2} + \frac{\epsilon}{k}$, $k > 1$. Therefore, $\tilde{\beta}_N(x_N) \leq \frac{1}{2}$ $\forall x_N \in [0, 1]$. Secondly upon submitting b , the non-ring bidder wins if $\tilde{\beta}_R(x_R) < b \iff \frac{x_R}{2} < b \iff x_R < 2b$, and this

[10] In fact, $\tilde{b}(x_R) = \frac{x_R}{2}$ holds true for any strategy of the form $\beta(x_N) = Cx_N$ by the nonring bidder (where C is a positive constant). For in this case, $\phi(b) = \frac{1}{C}b$, and the best response of the ring representative is $\tilde{b}$ such that

$$-\frac{1}{C}\tilde{b} + \frac{1}{C}(x_R - \tilde{b}) = 0 \implies \tilde{b} = \frac{x_R}{2}$$

with the additional condition that $\tilde{b} = 0$ for $x_R = 0$.

happens with probability $(2b)^{N-1}$. Therefore b is chosen such that

$$b \equiv \arg\max_{\hat{b}}(x_N - \hat{b})(2\hat{b})^{N-1} \iff b = \frac{N-1}{N}x_N$$

Therefore, the strategy of the non-ring bidder is

$$\tilde{\beta}_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{1}{2}\right\}$$

This can be rewritten as

$$\begin{cases} \tilde{\beta}_N(x_N) = \frac{N-1}{N}x_N, & x_N \in \left[0, \frac{N}{2(N-1)}\right] \\ \tilde{\beta}_N(x_N) = \frac{1}{2}, & x_N \in \left(\frac{N}{2(N-1)}, 1\right] \end{cases}$$

□

The equilibrium bidding strategies under Private Knowledge are

$$\hat{\beta}_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{1}{2}\right\}$$

and

$$\hat{\beta}_R(x_R) = \frac{1}{2}x_R$$

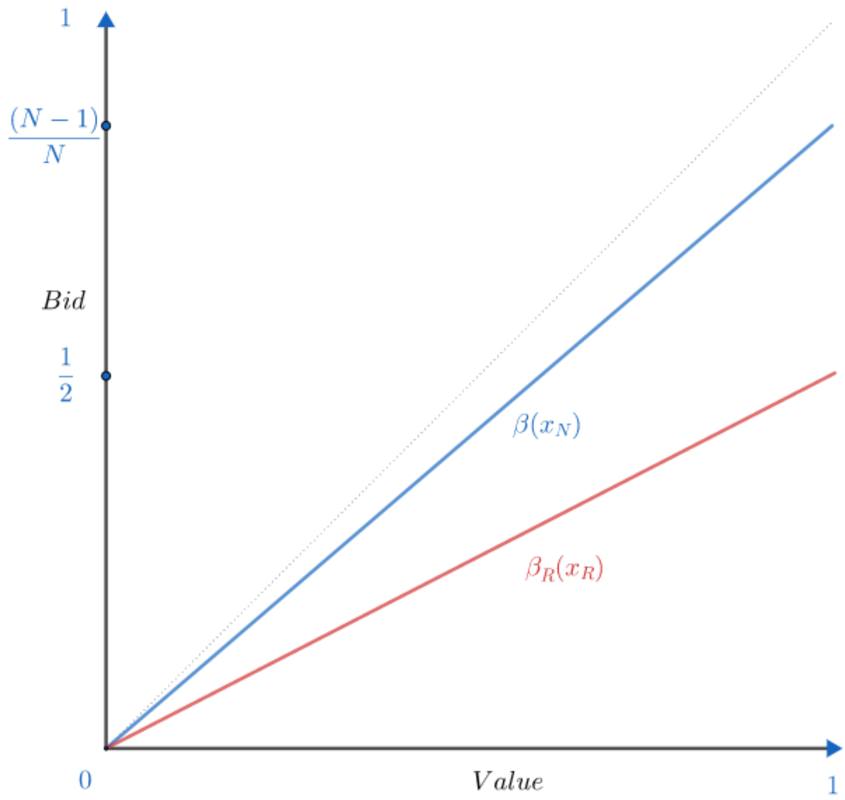

Figure 2.1: Equilibrium bidding strategies when the ring is concealed.

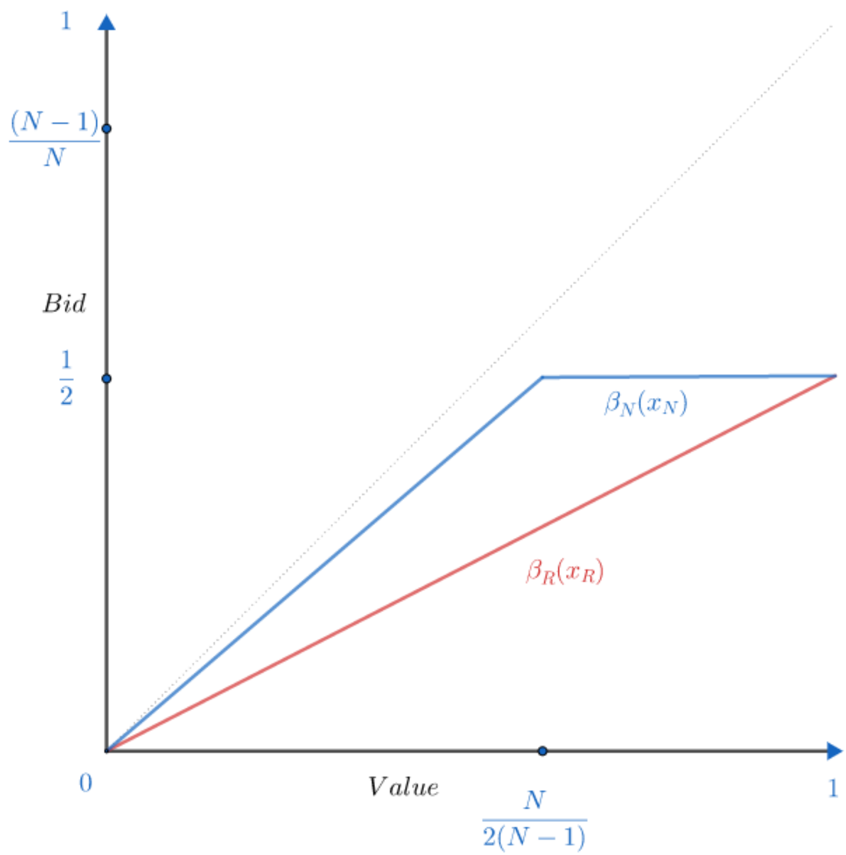

Figure 2.2: Equilibrium Bidding Strategies under Private Knowledge.

2.5.4 Equilibrium Strategies under Public Knowledge

This case corresponds to a two-bidder asymmetric first price auction, where strictly increasing and differentiable strategies are characterized by the system of differential equations given in Theorem 2.4.4. With $F(x) = x$,

The equilibrium bidding strategies under public knowledge $\tilde{\beta}_N(x_N)$ and $\tilde{\beta}_R(x_R)$ with inverses $\phi_N(\cdot)$ and $\phi_R(\cdot)$ are characterised by the system

$$-\phi_R(b) + \phi_R'(b)(\phi_N(b) - b)(N-1) = 0$$

$$-\phi_N(b) + \phi_N'(b)(\phi_R(b) - b) = 0$$

with boundary conditions $\beta_N(0) = \beta_R(0) = 0$, $\beta_N(1) = \beta_R(1) = \bar{b}$

Closed form solutions for general cases are rare. Kaplan and Zamir (2012) derive closed form solutions for two bidder asymmetric auctions where bidders values are drawn from uniform distributions with different supports. Cheng (2006) derives for values drawn from distributions $F(x) = x^{\alpha}$ and $H(x) = (\frac{x}{\delta})^{\lambda}$, where $\delta = \frac{\lambda(\alpha+1)}{\alpha(\lambda+1)}$, improving on Plum (1992) who analyzed the case for which $\lambda = \alpha$ and δ arbitrary. Our case involves two bidders with values drawn from $F(x) = x$ and $H(x) = x^{N-1}$, for which there is no closed form solution as yet. For the most part, the most popular approach to this case are numerical methods as in Marshall et al. (1994), Gayle and Richard (2008), Fibich and Gavish (2011) and Hubbard et al. (2015).

For subsequent analyses relating to the Public Knowledge case, we use estimates from Marshall et al. (1994) with $N = 101$.

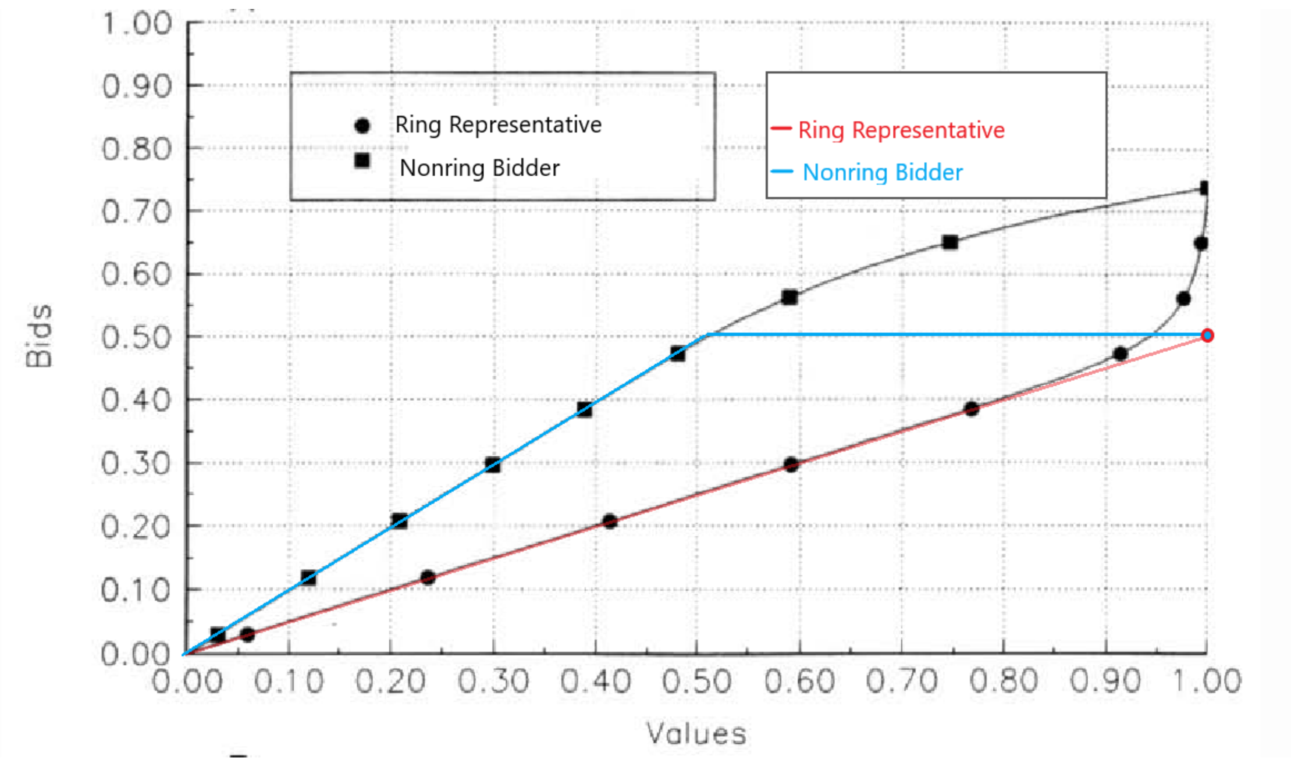

Figure 2.3: Equilibrium Bidding Strategies under private and public knowledge of ring presence.

2.5.5 Auctioneer Expected Revenue Comparison

In this subsection, we analyze the preferences of the auctioneer (based on expected revenue) over informational structures within auctions, and his preferences over auctions across informational structures.

2.5.5.1 Preferred Informational Structure under the First Price Auction

We derive the auctioneer's expected revenue under Non-Cooperation, the Concealed case, and the Private Knowledge case. Then we compare with the numerical estimates of the Public Knowledge case.

In the absence of the ring, the expected revenue to the seller is

$$\boxed{\mathbb{E}_1[\mathbb{R}_{\text{No Ring}}] = \int_0^1 \frac{N-1}{N}\psi \cdot N\psi^{N-1} d\psi = \frac{N-1}{N+1}}$$

Under Private Knowledge, the seller's expected revenue is the expected maximum of $\beta_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{1}{2}\right\}$ and $\beta_R(x_R) = \frac{1}{2}$. That is

$$\mathbb{E}_1[\mathbb{R}_{\text{Private}}] = \mathbb{E}[\max\{\beta_N(x_N), \beta_R(x_R)\}]$$

In the case where the presence of the ring is completely concealed, the bidding strategies are $\beta_R(x_R) = \frac{1}{2}x_R$ for the ring representative, and $\beta(x_N) = \frac{N-1}{N}x_N$ for the non-ring bidder. The expected revenue to the auctioneer is

$$\mathbb{E}_1[\mathbb{R}_{\text{Concealed}}] = \mathbb{E}[\max\{\beta(x_N), \beta_R(x_R)\}]$$

Note that $\beta_N(x_N) \leq \beta(x_N)$, since $\beta_N(x_N)$ involves bidding equal to $\frac{1}{2}$ ($< \frac{N-1}{N}x_N$) for $x_N \in [\frac{N}{2(N-1)}, 1]$. This implies

$$\underbrace{\mathbb{E}[\max\{\beta_N(x_N), \beta_R(x_R)\}]}_{\mathbb{E}_1[\mathbb{R}_{\text{Private}}]} \leq \underbrace{\mathbb{E}[\max\{\beta(x_N), \beta_R(x_R)\}]}_{\mathbb{E}_1[\mathbb{R}_{\text{Concealed}}]}$$

Implying that when a ring operates at the auction, the auctioneer prefers that their presence is discreet compared to being privately known by the nonring bidder. This is reasonable as bidding on behalf of the ring involves bid shading by the ring representative. If this is known to the non-ring bidder, then he has the incentive to also shade his bid as well.

The expected revenue for the seller when the ring is concealed is given by (2.13) as

$$\boxed{\mathbb{E}_1[\mathbb{R}_{\text{Concealed}}] = \frac{5N^2-4}{8N(N+1)}}$$

While the expected revenue under private knowledge is given by (2.16) as[11]

[11] The derivation of the expected revenue is shown in the next section.

$$\mathbb{E}_1[\mathbb{R}_{\text{Private}}] = \frac{2N^2 - N - 2}{4(N^2 - 1)}$$

We compare these quantities with estimates of the expected revenue under public knowledge for $N = 101$.

The estimates show that the auctioneer in general prefers that the auction be noncooperative. However, when the ring is present at the first price auction, the auctioneer prefers that their presence be publicly known.

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.9804	0.6188	0.4975	0.6578

Table 2.1: Revenue Comparisons in the First Price Auction

Figure 2.3 represents the estimated bidding strategies under public knowledge (from Marshall et al. (1994)) with the bidding strategies under private knowledge superimposed in colour. We see that

- The maximum possible bid is higher under public knowledge than under private knowledge. The maximum possible bid under private knowledge is $\frac{1}{2}$, while the maximum possible bid under private knowledge is given by Marshall et al. (1994) as

$$\bar{b} = 1 - \left(\frac{N}{2^{N-1}} \cdot \left(\frac{N}{2(N-1)}\right)^{N-1}\right)^{\frac{1}{N-2}}$$

  which for $N = 101$ is $\bar{b} = 0.7391$.

- Overall, the ring representative bids higher under public knowledge than under private knowledge.

- For realizations of values roughly in the range $[0.505, 1]$, the nonring bidder bids significantly higher under public knowledge.

Together, these three observations may help explain why the expected revenue of the auctioneer is higher under public knowledge than under private knowledge. Note that the strategy of the nonring bidder under private knowledge involves a flat bidding schedule over the range $\left[\frac{N}{2(N-1)}, 1\right]$. In this range, the nonring bidder wins with probability equal to 1, and the seller receives a payment of $\frac{1}{2}$. In essence, the seller's payment does not change over a long range of bidder values, in contrast to the public knowledge case for which bidding strategies are strictly increasing.

We summarise in the following proposition

Proposition 4. *In the first price auction, the best case for the auctioneer is when all bidders bid noncooperatively. However, in the presence of a ring, the auctioneer prefers that the ring presence be public knowledge. Otherwise, it is better that the ring remains concealed.*

$$\mathbb{E}_1[\mathbb{R}_{\text{No Ring}}] > \mathbb{E}_1[\mathbb{R}_{\text{Public}}] > \mathbb{E}_1[\mathbb{R}_{\text{Concealed}}] \geq \mathbb{E}_1[\mathbb{R}_{\text{Private}}]$$

2.5.5.2 Preferred Informational Structure under the Second Price Auction

We have already stated in Proposition 1 that the seller is hurt by ring presence. Furthermore, since active bidders employ the dominant strategy of bidding their values, conditional on ring presence, the auctioneer is indifferent with regards to whether or not their presence is known as his revenue remains the same. We characterise the expected revenues in each of the informational settings below.

In the absence of the ring, the revenue equivalence theorem implies that the revenue in the second price auction is the same as in the first price auction. That is

$$\mathbb{E}_2[\mathbb{R}_{\text{No Ring}}] = \frac{N-1}{N+1}$$

In the presence of the ring, since the bidding strategies of both the ring representative and the non-ring bidder remains unchanged (each bidder submitting a bid equivalent to their private value), we note that

$$\mathbb{E}_2[\mathbb{R}_{\text{Public}}] = \mathbb{E}_2[\mathbb{R}_{\text{Private}}] = \mathbb{E}_2[\mathbb{R}_{\text{Concealed}}] \tag{2.10}$$

The expected revenue in these cases can be computed as the expected second-highest of x_R and x_N. It has distribution described by

$$Q(x) = x^N + x(1 - x^{N-1}) + x^{N-1}(1 - x) = x + x^{N-1} - x^N$$

with density

$$q(x) = 1 + (N-1)x^{N-2} - Nx^{N-1}$$

Therefore,

$$\mathbb{E}_2[\mathbb{R}_{\text{Public}}] = \int_0^1 xq(x)dx = \int_0^1 x + (N-1)x^{N-1} - Nx^N dx = \frac{(N+2)(N-1)}{2N(N+1)} < \frac{N-1}{N+1} = \mathbb{E}_2[\mathbb{R}_{\text{No Ring}}]$$

So we can conclude the following

Proposition 5. *In the second price auction, the auctioneer is better off without a ring. In the presence of the ring however, the auctioneer is indifferent with regards to whether their presence is known or not.*

$$\mathbb{E}_2[\mathbb{R}_{\text{No Ring}}] > \mathbb{E}_2[\mathbb{R}_{\text{Concealed}}] = \mathbb{E}_2[\mathbb{R}_{\text{Public}}] = \mathbb{E}_2[\mathbb{R}_{\text{Private}}]$$

2.5.5.3 Preferred Auction Format across Informational Structures

Next, we compare for each informational structure, the auctioneer's preferred auction format between the first price and the second price auction. We proceed by ranking expected revenues under Non-Cooperation, the Concealed case, the Private Knowledge case, and - via numerical estimates - the Public Knowledge case.

Preferred Auction Format Under Noncooperation

In the absence of a ring, the *Revenue Equivalence Theorem* implies that the auctioneer is indifferent between the first and second price auctions.

$$\mathbb{E}_1[\mathbb{R}_{\text{No Ring}}] = \mathbb{E}_2[\mathbb{R}_{\text{No Ring}}]$$

Proposition 6. *In the absence of a ring, the auctioneer is indifferent between the first price and the second price auction.*

Preferred Auction Format when the Ring is Concealed

In the first price auction, the bids submitted are $\beta_N(x_N) = \frac{N-1}{N}x_N$ and $\beta_R(x_R) = \frac{1}{2}x_R$. The expected revenue of the seller can be expressed as the sum of the ex-ante expected payments of both bidders.

The ring representative wins when $\frac{N-1}{N}x_N < \frac{1}{2}x_R \implies x_N < \frac{N}{2(N-1)}x_R$, and this occurs with probability $\frac{N}{2(N-1)}x_R$. Therefore ex-ante, the expected payment for the ring representative is

$$\int_0^1 \frac{x_R}{2} \cdot \frac{N}{2(N-1)}x_R \cdot (N-1)x_R^{N-2}dx_R = \frac{N}{4(N+1)} \tag{2.11}$$

For the nonring bidder, he wins when $\frac{1}{2}x_R < \frac{N-1}{N}x_N \implies x_R < \frac{2(N-1)}{N}x_N$. This happens with probability

$$\mathbb{P}\Big(x_R < \frac{2(N-1)}{N}x_N\Big) = \begin{cases} \Big(\frac{2(N-1)}{N}x_N\Big)^{N-1} & \text{for } x_N \in \Big[0, \frac{N}{2(N-1)}\Big) \\ 1 & \text{for } x_N \in \Big[\frac{N}{2(N-1)}, 1\Big] \end{cases}$$

So his ex-ante expected payment is

$$\int_0^{\frac{N}{2(N-1)}} \frac{N-1}{N}x_N \cdot \Big(\frac{2(N-1)}{N}x_N\Big)^{N-1} dx_N + \int_{\frac{N}{2(N-1)}}^1 \frac{N-1}{N}x_N dx_N = \frac{3N^2-4}{8N(N+1)} \tag{2.12}$$

the auctioneer's expected revenue is thus

$$\mathbb{E}_1[\mathbb{R}_{\text{Concealed}}] = (2.11) + (2.12) = \frac{5N^2-4}{8N(N+1)} \tag{2.13}$$

in the second price auction, the revenue is given in (2.5.5.2) as

$$\mathbb{E}_2[\mathbb{R}_{\text{Concealed}}] = \frac{N+2}{2N} \cdot \frac{N-1}{N+1} < \frac{5N^2-4}{8N(N+1)} = \mathbb{E}_1[\mathbb{R}_{\text{Concealed}}]$$

Therefore, the auctioneer prefers the first price auction. The intuition for this result stems from the fact that for the second price auction, the presence of the ring does not change the probability of winning of the nonring bidder, as he only wins when his value is the highest. However, in the first price auction, since bidding on behalf of the ring involves the ring representative bidding lower than he would have in the noncooperative case, the nonring bidder has a higher chance of winning when the ring is in operation. Formally, in the second price auction, the nonring bidder wins if his value is higher than x_R. That is, with probability

$$\mathbb{P}\Big(x_R < x_N\Big)$$

However, in the first price auction, he wins with probability

$$\mathbb{P}\Big(\frac{1}{2}x_R < \frac{N-1}{N}x_N\Big) \equiv \underbrace{\mathbb{P}\Big(x_R < \frac{2(N-1)}{N}x_N\Big)}_{\text{Probability of winning under first price}} > \underbrace{\mathbb{P}\Big(x_R < x_N\Big)}_{\text{Probability of winning under second price}}$$

since $\frac{2(N-1)}{N} > 1$. Basically, the first price auction increases the probability that the object goes to the nonring bidder. In the words of Milgrom (2004), it induces a "*right kind of bias*" in favor of the nonring bidder. So

Proposition 7. *When the auctioneer knows that a ring will operate without the knowledge of the nonring bidder, he should employ a first price auction instead of a second price auction.*

$$\mathbb{E}_1[\mathbb{R}_{\text{Concealed}}] > \mathbb{E}_2[\mathbb{R}_{\text{Concealed}}]$$

Preferred Auction Format Under Private Knowledge

In the first price auction, the bids submitted are $\beta_N(x_N) = \min\{\frac{N-1}{N}x_N, \frac{1}{2}\}$ and $\beta_R(x_R) = \frac{1}{2}x_R$ and we express the seller's expected revenue as the sum of ex-ante expected payments of both bidders. Again, the ring representative wins when

$$\frac{N-1}{N}x_N < \frac{1}{2}x_R \implies x_N < \frac{N}{2(N-1)}x_R$$

and this occurs with probability $\frac{N}{2(N-1)}x_R$. Therefore ex-ante, the expected payment for the ring representative is

$$\int_0^1 \frac{x_R}{2} \cdot \frac{N}{2(N-1)}x_R \cdot (N-1)x_R^{N-2}dx_R = \frac{N}{4(N+1)} \tag{2.14}$$

In the same vein, for the nonring bidder, he wins when

$$\frac{1}{2}x_R < \frac{N-1}{N}x_N \implies x_R < \frac{2(N-1)}{N}x_N$$

This happens with probability

$$\mathbb{P}\Big(x_R < \frac{2(N-1)}{N}x_N\Big) = \begin{cases} \left(\frac{2(N-1)}{N}x_N\right)^{N-1} & \text{for } x_N \in \Big[0, \frac{N}{2(N-1)}\Big) \\ 1 & \text{for } x_N \in \Big[\frac{N}{2(N-1)}, 1\Big] \end{cases}$$

So that his ex-ante expected payment is

$$\int_0^{\frac{N}{2(N-1)}} \frac{N-1}{N}x_N \cdot \Big(\frac{2(N-1)}{N}x_N\Big)^{N-1}dx_N + \int_{\frac{N}{2(N-1)}}^1 \frac{1}{2}dx_N = \frac{N^2-2}{4(N^2-1)} \tag{2.15}$$

the auctioneer's expected revenue is thus

$$\mathbb{E}_1[\mathbb{R}_{\text{Private}}] = (2.14) + (2.15) = \frac{2N^2 - N - 2}{4(N^2 - 1)} \tag{2.16}$$

However,

$$\mathbb{E}_1[\mathbb{R}_{\text{Private}}] = \frac{2N^2 - N - 2}{4(N^2 - 1)} < \frac{N-1}{N+1} = \mathbb{E}_2[\mathbb{R}_{\text{Private}}]$$

so the auctioneer prefers the second price auction in this case.

We summarize the findings in the following proposition.

Proposition 8. *When the presence of the ring is privately known to the nonring bidder, the auctioneer is better off under the second price auction compared to the first price auction.*

$$\boxed{\mathbb{E}_1[\mathbb{R}_{\text{Private}}] < \mathbb{E}_2[\mathbb{R}_{\text{Private}}]}$$

The result here is in contrast to that for the concealed case. This stems from the fact that despite increasing the probability that the nonring bidder wins the item, the nonring bidder bids lower than in the concealed case.

Preferred Auction Format Under Public Knowledge

The remainder of the analysis concerns the case for which ring presence is public knowledge. As indicated previously, this corresponds to a two bidder asymmetric auction, with a strong (the ring representative) and a weak (the nonring) bidder. Numerical simulations by Marshall et al. (1994) suggest that the revenue to the seller is higher in the first price auction. Indeed Kirkegaard (2012) provides a sufficient condition for this ranking in the following lemma

Lemma 2.5.1. Kirkegaard (2012) *Let F_S and F_W denote the distributions of the* Strong *and* Weak *bidder respectively such that $\forall\ x < y$ in their common support, $\frac{F_S(x)}{F_S(y)} < \frac{F_W(x)}{F_W(y)}$. If*

$$\int_x^{F_S^{-1}(F_W(x))} (f_W(x) - f_S(\psi))d\psi \geq 0$$

$\forall x$, the seller's revenue from the first price auction will be higher than the revenue from the second price auction.

In our case, $F_S(x) = x^{N-1}$, and $F_W(x) = x$. So that

$$\int_x^{F_S^{-1}(F_W(x))} (f_W(x) - f_S(\psi))d\psi = \int_x^{x^{\frac{1}{N-1}}} (1 - (N-1)\psi^{N-2})d\psi = x^{\frac{1}{N-1}} - x - [x - x^{\frac{1}{N-1}}] = 0$$

implying that seller's revenue is higher under the first price auction. Again as in the previous subsection, a possible explanation could be the fact that under private knowledge equilibrium strategies in the first price auction for the nonring bidder entails a flat bidding schedule over the range $\left[\frac{N}{2(N-1)}, 1\right]$. In this range, the nonring bider wins with probability equal to 1, and the seller receives a payment of $\frac{1}{2}$. In essence, the seller's payment does not change over a long range of bidder values, in contrast to the public knowledge case for which bidding strategies are strictly increasing.

Proposition 9. *When the presence of the ring is public knowledge, the auctioneer prefers the first price auction.*

$$\mathbb{E}_1[\mathbb{R}_{\text{Public}}] > \mathbb{E}_2[\mathbb{R}_{\text{Public}}]$$

Estimated revenues for $N = 101$ are shown in Table 2.2.

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.9804	0.6188	0.4975	0.6578
Second Price	0.9804	0.4999	0.4999	0.4999

Table 2.2: Revenue Comparisons across Auctions

In summary,

Proposition 10. *The second price auction should be the preferred format only if the ring presence is private knowledge. Otherwise the seller is at least better off under the first price auction.*

2.5.5.4 Summary of Auctioneer Expected Revenue Comparisons

Overall, the auctioneer prefers that the auction be noncooperative, as he earns the maximum profit when all bidders compete. However, subject to ring presence, the auctioneer prefers that the ring be made publicly known. Otherwise, it is better that the ring remains concealed rather than be privately known by the nonring bidder. Furthermore, across auctions, the auctioneer is weakly better-off under the first price auction under Noncooperation, the Concealed and the Public Knowledge Cases. However, under Private Knowledge, the auctioneer prefers the second price auction.

2.5.6 Optimal Environment for Bidders

In this subsection, we examine the preferred informational structure (based on expected profit) for bidders within auctions and preferred auction format across informational structures. Starting with the first price auction, we derive for the nonring bidder and the ring members; expected profits under Noncooperation, the Concealed and Private Knowledge cases, and compare with numerical estimates of the public knowledge case. We repeat the same sequence for the second price auction. Then we compare for each informational structure, the nonring bidder and ring members' preferences between the first price and second price auction.

2.5.6.1 Preferred Informational Structure under the First Price Auction

2.5.6.1.1 The Non-Ring Bidder

Expected Profit under Noncooperation

The expected profit under noncooperation can be computed as

$$\pi^N_{1,\text{No Ring}} = \int_0^1 \left(x_N - \frac{N-1}{N}x_N\right) \underbrace{\mathbb{P}(x_j < x_N, \forall j \neq N)}_{\mathbb{P}\left(x_R < x_N\right)} dx_N = \frac{1}{N(N+1)}$$

Expected Profit under the Concealed Case

The characterization of expected profits in this case is two-fold. On the one hand, there is a notion of *Presumed Expected Profit* - $\pi^N_{1,\text{Presumed}}$, which stems from the fact that the non-ring bidder believes that the auction is purely non-cooperative, and on the other hand there is the notion of *Actual Expected Profit* - $\pi^N_{1,\text{Actual}}$ - due to ring presence.

$$\pi^N_{1,\text{Presumed}} = \pi^N_{1,\text{No Ring}}$$

$$\pi^N_{1,\text{Actual}} = \int_0^1 \left(x_N - \frac{N-1}{N}x_N\right)\mathbb{P}\left(x_R < \frac{2(N-1)}{N}x_N\right)dx_N$$

Proposition 11. *In the first price auction, the Non-ring bidder benefits from the presence of a concealed ring. That is*

$$\pi^N_{1,Actual} \geq \pi^N_{1,Presumed}$$

Proof. This follows from the fact that

$$\mathbb{P}\Big(x_R < \frac{2(N-1)}{N}x_N\Big) \geq \mathbb{P}\Big(x_R < x_N\Big)$$

□

We can decompose the benefit to the non-ring bidder of the presence of the ring into two parts. The first part involves a reduction in competition in the auction, due to the fact that there is only one other serious bidder in the auction. The second part has to do with the lower bid by the ring representative, which in turn increases the chance of winning of the non-ring bidder.

$$\pi^N_{1,\text{Actual}} \equiv \pi^N_{1,\text{Concealed}} = \int_0^{\frac{N}{2(N-1)}} \big(x_N - \frac{N-1}{N}x_N\big)\big(\frac{2(N-1)}{N}x_N\big)^{N-1}dx_N + \int_{\frac{N}{2(N-1)}}^1 \big(x_N - \frac{N-1}{N}x_N\big)dx_N$$

$$\boxed{\implies \pi^N_{1,\text{Concealed}} = \frac{N}{4(N+1)(N-1)^2} + \frac{3N^2-8N+4}{8N(N-1)^2}}$$

<u>Expected Profit under Private Knowledge</u>

The ring representative uses $\hat{\beta}_R(x_R) = \frac{1}{2}x_R$. Furthermore, she has a chance of winning only if $x_N \in [0, \frac{N}{2(N-1)}]$, and for this range of x_N, we have $\beta(x_N) = \hat{\beta}_N(x_N)$. Therefore, the non-ring bidder's probability of winning

$$1 - \mathbb{P}(\text{Ring Representative Wins})$$

is the same when the ring is concealed and under private knowledge. However, for $x_N \in [\frac{N}{2(N-1)}, 1]$, $\hat{\beta}_N(x_N) \leq \beta(x_N)$ meaning that the non-ring bidder pays more when under the concealed case compared to the private knowledge case. We can explicitly formulate the expected profit as

$$\pi^N_{1,\text{Private}} = \int_0^{\frac{N}{2(N-1)}} \big(x_N - \frac{N-1}{N}x_N\big)\big(\frac{2(N-1)}{N}x_N\big)^{N-1}dx_N + \int_{\frac{N}{2(N-1)}}^1 \big(x_N - \frac{1}{2}\big)dx_N$$

$$\boxed{\implies \pi^N_{\text{Private}} = \frac{N}{4(N+1)(N-1)^2} + \frac{N(N-2)}{8(N-1)^2}}$$

<u>Expected Profit under Public Knowledge</u>

We compare with Public knowledge estimates for $N = 101$

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.000097	0.003712	0.1249	0.0412

Table 2.3: Nonring Bidder Profit Across Informational Structures

We see that

$$\pi^N_{1,\text{Private}} > \pi^N_{1,\text{Public}} > \pi^N_{1,\text{Concealed}} > \pi^N_{1,\text{No Ring}}$$

Proposition 12. *In the first price auction, the nonring bidder benefits in general from the stifling of competition. Furthermore, he is most happy when the presence of the ring is privately known to him. This follows directly from the fact that the ring representative bids higher under public knowledge compared to under private knowledge.*

2.5.6.1.2 Ring Members

Expected Profit under Noncooperation

$$\pi^R_{1,\text{No Ring}} = \frac{1}{N+1}$$

Expected Profit when Ring Presence is Concealed / Private Knowledge

We have already mentioned above that the ring representative's chance of winning is the same under the Concealed and Private Knowledge cases. She earns the same expected profit since she uses the same strategy in both cases. Furthermore, we can explicitly derive the expected ring profit as

$$\Pi^R_{1,\text{Concealed}} = \Pi^R_{1,\text{Private}} = \int_0^1 (x_R - \frac{1}{2}x_R)\frac{N}{2(N-1)}x_R \cdot (N-1)x_R^{N-2}dx_R = \frac{N}{4(N+1)}$$

Given that the ring shares proceeds equally among members, each member of the ring earns an expected profit of

$$\pi^R_{1,\text{Concealed}} = \pi^R_{1,\text{Private}} = \frac{N}{4(N-1)(N+1)}$$

<u>Expected Profit under Public Knowledge</u>

We compare with Public Knowledge estimates

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.000097	0.002475	0.002475	0.0025

Table 2.4: Ring Member's Profit Across Informational Structures

We see that

$$\pi^R_{1,\text{Public}} > \pi^R_{1,\text{Concealed}} = \pi^R_{1,\text{Private}} > \pi^R_{1,\text{No Ring}}$$

Proposition 13. *In the first price auction, ring members profit the most when ring presence is public knowledge.*

2.5.6.2 Preferred Informational Structure under the Second Price Auction

2.5.6.2.1 The Non-Ring Bidder

We have already seen from section 2.3 that the equilibrium bidding strategy for each agent regardless of the informational structure (whether or not a ring is present and their presence is publicly or privately known) involves bidding truthfully - i.e equal to his/her value. For the non-ring bidder, conditional on winning, he pays the highest competing value. That is $\max_{j \neq N} x_j$. Since the ring is organized such that the ring representative is the member with the highest value, $\max_{j \neq N} x_j = x_R$. Therefore, conditional on winning, the non-ring bidder pays the same amount.

Proposition 14. *In the second price auction, the presence of the ring exerts no positive externality on the non-ring bidder. Hence, he is unaffected by the ring presence and he is indifferent between the informational structures.*

$$\pi^N_{2,\text{No Ring}} = \pi^N_{2,\text{Concealed}} = \pi^N_{2,\text{Private}} = \pi^N_{2,\text{Public}}$$

2.5.6.2.2 The Ring Representative

Since the strategy of the non-ring bidder remains the same for every informational structure, the ring profit remains the same. Furthermore, ring members benefit from ring formation as each ring member earns expected profit

$$\pi^R_{2,} = \frac{1}{N-1}\int_0^1 (x_R - \frac{1}{2}x_R)x_R \cdot (N-1)\cdot x_R^{N-2}dx_R = \frac{1}{2(N+1)}$$

which is higher than $\frac{1}{N+1}$, the expected profit under noncooperation.

$$\pi^R_{2,\text{Concealed}} = \pi^R_{2,\text{Private}} = \pi^R_{2,\text{Public}} > \pi^R_{2,\text{No Ring}}$$

Proposition 15. *In the second price auction, ring members benefit from ring formation, and they are indifferent as to whether or not their presence is hidden or known to the non-ring bidder*

2.5.6.3 Preferred Auction Format across Informational Structures

We compare bidders' preference of auction formats across informational structures.

2.5.6.3.1 The Non-Ring Bidder

In the second price auction, Proposition 1 implies that the expected profit for the non-ring bidder is the same across all informational structures. The revenue equivalence theorem implies that the expected profit for the non-ring bidder under the various informational structures in the second price auction is the same as his expected profit in the first price auction under noncooperation. However, the non-ring bidder benefits from the presence of the ring in the first price case. Therefore, the non-ring bidder is weakly better off under the first price auction.

Proposition 16. *The nonring bidder prefers the first price auction to the second price auction.*

2.5.6.3.2 Ring Members

For $N = 101$, we compare ring members' profits from the first price auction and the second price auction in the table below

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.000097	0.002475	0.002475	0.0025
Second Price	0.000097	0.0049	0.0049	0.0049

Table 2.5: Ring Member's Profit Across Auctions

we find that

Proposition 17. *Ring members prefer the second price auction to the first price auction.*

An implication of the results so far is that there is no incentive for the ring to exert effort to keep their presence hidden from the non-ring bidder. Therefore, if the risk of prosecution is low, the ring does not have to worry about it's presence being known, at least from the point of view of the other bidders.

2.5.6.4 Summary of Expected Profit Comparisons

Conditional on ring presence, the nonring bidder benefits most from the suppressed competition in the first price auction. Furthermore, he is most happy when the ring presence is privately known to him.

The second price auction is the most favorable for the ring. As far as informational structures, ring members earn more (weakly) under the public knowledge case compared to all other cases.

Furthermore, an additional caveat is the fact that the private knowledge case aptly models the case for which the nonring bidder suspects that a ring is present. Imagine a scenario where a subset of bidders go into the auction suspecting that a ring might be present when in fact, there isn't. This group will bid using the strategy

$$\hat{\beta}_N(\theta) = \min\left\{\frac{N-1}{N}\theta, \frac{1}{2}\right\} \leq \beta(\theta) = \frac{N-1}{N}\theta$$

What this implies is the fact that despite the absence of a bidding ring, merely suspecting that a ring might be present at the auction leads bidders to suppress their bids compared to when they are convinced that there is no ring, and this depresses the auctioneer's expected revenue .

Proposition 18. *In the first price auction, an auctioneer who is sure that a ring will not be present at the auction should make sure that bidders do not harbour any suspicion of ring presence as it leads them to bid lower.*

Remark: We have already mentioned that the ring presence does not affect efficiency in the second price auction regardless of the informational structure, since in each case, active bidders bid their values and the winner is the bidder with the highest value.

However, in the first price auction, ring presence leads to inefficiency. This stems from the fact that for each informational structure, the equilibrium bidding strategies involve the ring representative bidding less-aggressively compared to the nonring bidder.[12]

To see this, fix any informational structure, and let the bidding strategies of the nonring bidder and the ring representative (as a function of their values) be $\tau_N(x_N)$ and $\tau_R(x_R)$ respectively. For any $\theta \in (0, 1)$, $\tau_N(\theta) > \tau_R(\theta)$. Since bidding strategies are continuous, $\exists \epsilon > 0$ such that $\tau_N(\theta - \epsilon) > \tau_R(\theta)$. However $\theta - \epsilon < \theta$, meaning that the nonring bidder could win even when his value is less than that of the ring representative, in which case the object goes to the active bidder with the lower value.

[12]See Figure 2.1, Figure 2.2 and Figure 2.3

2.6 Conclusion

In this chapter, we have analyzed bidder strategies in Second and First price Independent Private Value auctions with colluding bidders. Assuming an almost all-inclusive ring, we characterised equilibrium behavior under different informational structures corresponding to four cases: (i) when there is no ring at the auction, (ii) when there is a ring and their presence is concealed, (iii) when there is a ring an their presence is privately known to the nonring bidder, and (iv) when there is a ring and their presence is common knowledge. The case for which ring presence is common knowledge corresponds to a two - bidder asymmetric auction, whose equilibrium bidding strategies in the first price auction are almost impossible to explicitly derive in general. This necessitated the simplification that the underlying distribution of bidders be the uniform $[0, 1]$ for which we could use simulated estimates from Marshall et al. (1994).

A summary of the main results are as follows

- Rings are bad for the auctioneer in general. However, loss in expected revenue varies across informational structures.
- A common conclusion is the idea that the first price auction is more robust to collusion than the second price auction in terms of seller expected revenue. However, we show that this is contingent on the assumed informational structure. In particular, we find that the second price auction is more robust than the first price auction when the ring presence is privately known to the nonring bidder.
- In the first price auction, simply suspecting that a ring is present induces bidders to depress their bids, leading to a loss in revenue for the auctioneer, even without a ring being present.
- When a ring is present at the auction, the weakly dominant strategy for the auctioneer is to publicly reveal their presence.

A major limitation of the analyses relates to it's simplicity - the assumption of *almost all inclusive rings*[13] and uniformly distributed values. Nonetheless, the simplifying assumptions were convenient as they facilitated explicit characterizations of bidder strategies for some informational structures in the first price auction. Furthermore, it would be reasonable to think that the pattern of behavior by the ring representative and nonring bidders in the first price auction should extend beyond our simplistic case. This boils down to the fact that the major driver of price in the auction is competition. In the first price auction, suppressing competition

[13] We partially analyse the result for more than one bidder in Appendix 4.1.4.2

by forming a ring allows the ring representative to bid lower. Furthermore, nonring bidders who become aware of ring presence also have the incentive to shade their bids due to a lower bid by the ring representative and less competition due to ring formation. Hence the conclusion that the auctioneer is worst off when the ring presence is secretly known by nonring bidders may extend beyond the single nonring bidder and uniformly distributed values cases.

Chapter 3

Secret vs Public Rings in Common Value Auctions

3.1 Introduction

In this chapter, we extend the analysis of the Chapter 2 to cover a common values framework. The motivation for considering common values stems from the analysis of Pagnozzi (2011) in ascending auctions with affiliated values, where he demonstrated that in addition to reducing competition among each other, bidding rings could also influence other non-ring bidders to bid lower than usual. The ring mechanism employed in this scenario involves all ring members being present at the main auction, with the designated representative bidding competitively, while other ring members drop out early. This premature exit sends misleading signals of a low value to other non-ring bidders, causing them to depress their bids as well. This has the combined effect of further reducing the seller's revenue and further reducing the probability of winning of the other non-ring bidders. A nonring bidder who is aware of this tactic takes note and is wary when updating his estimate of the value of the item based on observed dropout prices. In such a case, it is in the interest of both the nonring bidder and the seller that ring presence is not concealed. This is because nonring bidders who are aware of the ring's tactic are not influenced to depress their bids. Hence the seller can as well expect higher profit compared to when nonring bidders are oblivious of ring presence.

We argue that the benefits to the knowledge of ring presence applies to sealed-bid common value auctions as well. Typically, in standard second price common value auctions, the only information that a bidder can elicit upon winning is the signal of the highest competing bidder.

Based on this inference, he makes an estimate of the value of the item. In the event that such a bidder is made aware that he is bidding against a ring, he knows that the ring representative bids based on the signals of the ring members. In essence, when the nonring bidder wins in such a context, there is the possibility of inferring more information with regards to the signals of his competitors, leading to a better estimate of the value of the item. We suspect that this may lead the nonring bidder to adjust his bidding strategy compared to when he is unaware of the ring's presence. Whether this will be in favor of or against the auctioneer will depend on the nature of the value function. Specifying a family of value functions, we describe a common value auction environment in which the participants determine the value of the item on sale based on their signals and an aggregate of their competitors' signals.

As in Chapter 2, we analyse the case of an *almost all-inclusive ring* and a valuation structure such that the common value is completely specified by the bidders' signals. Under this framework, it becomes easy to see how the nonring bidder changes his behavior with respect to knowing that a ring is present compared to bidding in the noncooperative case. This is due to the fact that with this specification, when the nonring bidder is oblivious of ring presence, the most he believes he can infer about the value of the item upon winning is the value conditional on his signal and the inferred signal of the highest competitor based on the winning price. However, when he is aware that he is bidding against the ring, given that the ring representative bids based on the signals of all ring members, upon winning the nonring bidder in principle can infer the aggregate of his competitors' signals from the price he pays, and hence can infer the actual value of the item. We show that based on the nature of the aggregate, whether or not the nonring bidder will alter his bidding strategy based on knowledge of ring presence depends on whether the ring presence is publicly known, or is privately known to him. In parallel with the rhetoric of the previous chapter, this corresponds to saying that if a seller who knows that a ring is present decides to reveal this information, whether the nonring bidder will alter his bidding strategy compared to his strategy in the noncooperative case depends on whether the seller reveals this information publicly or secretly to the nonring bidder.

In line with the previous chapter, we can phrase our main questions thus:

- If an auctioneer knows that a ring is present at the auction, should she reveal the ring presence to bidders?
- If yes, how? Should she publicly announce it or reveal it secretly to the nonring bidders?
- What would be bidders' responses to these announcements compared to the non-cooperative case?

Our work closely parallels that of Mares and Harstad (2003), who consider the role of privately revealed information by the seller on expected revenue. However, in contrast to their work, our specification asserts that the actual value of the item is completely determined by bidder signals, The role of auctioneer information in our context is to aid the winner of the auction in estimating the ex-post value via the price paid.

If the aggregate of competitors' signals is less than the average, and signals are uniformly distributed, conditional on ring presence, we show that publicly revealing the ring presence is the best policy for the auctioneer. This stems from the fact that when the ring presence is publicly known, the nonring bidder bids higher compared to when he is oblivious of ring presence. Furthermore, since publicly announcing that a ring is present leads nonring bidders to bid higher compared to when they believe that the auction is noncooperative, the auctioneer has an incentive to induce bidders to believe a ring is present even when the auction is noncooperative. This presents a novel way that an auctioneer may cheat that does not involve shill bidding.

It is worth noting that the conclusion that public revelation is the best strategy is not a consequence of the analysis of the impact of public information revelation in Milgrom and Weber (1982). This is because their analysis rests on the specification that the auctioneer's revealed information affects the valuation of bidders ex-ante. Our case concerns the scenario for which the exact specification of the value depends only on bidders signals. The impact of auctioneer information in this case only concerns the inferred value upon winning. That is, ex-post.

The rest of the chapter proceeds as follows: In Section 3.2, we specify the framework for our analysis. Section 3.3 is devoted to the characterisation of bidding strategies under the various informational structures. We introduce the additional specification that signals are uniformly distributed in Section 3.4, and we compare expected revenues and profits for the auctioneer and bidders across informational structures. Section 3.5 shows that the results of section 3.4 do not necessarily hold under a different valuation structure. Section 3.6 concludes.

3.2 Framework

3.2.1 Pure Common Values

Assume a pure common values framework with $N \geq 3$ risk-neutral bidders $\mathcal{N} = \{1, 2, \ldots, N\}$ each receiving an iid signal $X_i \in [0, 1]$ drawn from an atomless distribution $F(\cdot)$ with corresponding density $f(\cdot)$ strictly positive on $[0, 1]$. The common value is V defined by

$$V = \pi(X_1, \ldots, X_N)$$

Denote $Y^1_{\mathcal{N}/\{i\}}$,$Y^2_{\mathcal{N}/\{i\}}$,...,$Y^{N-1}_{\mathcal{N}/\{i\}}$ as the largest, second-largest, ..., smallest of X_j for $j \in \mathcal{N}/\{i\}$. Also denote $\mathbf{X}_{-i} = \{X_j\}_{j \neq i}$. Let a subset ($\mathcal{K} \subset \mathcal{N}$, $\mathcal{K} = \{1, 2, \ldots, k\}$) of bidders decide to form a ring. Suppose for the sake of analysis that the ring is able to design a stable incentive compatible mechanism such that the ring member with the highest signal bids on behalf of the ring, and the other members submit phony bids. When bidding for the ring, the ring representative knows the signals $x_1, x_2, \ldots, x_k$. A salient feature of second price common value auctions is the multiplicity of equilibria Bikhchandani and Riley (1991), Krishna (2009), Milgrom (1981). Criteria for equilibrium selection in this context usually involves choosing equilibria that survive some form of perturbation of the value function Larson (2009). Subsequently, where applicable, we impose that bidders use continuous symmetric strategies.

3.2.2 All-but-One of the Bidders are Part of the Ring

Suppose $|\mathcal{K}| = N - 1$. We refer to this scenario as a case of an “almost all-inclusive ring".

3.2.3 Special family of Valuation Functions

We consider a family of value functions of the form

$$V(X_i, \mathbf{X}_{-i}) = X_i \cdot g(\mathbf{X}_{-i}) = X_j \cdot g(\mathbf{X}_{-j})$$

where

$$g : [0, 1]^{N-1} \longrightarrow [0, 1]$$

is a Borel-Measurable function, increasing and symmetric in all its arguments, and $\mathbf{X}_{-i} \equiv \{X_j\}_{j \neq i}$. We refer to $g(\cdot)$ as the *Auxiliary Signal.*[1]

[1] Technically, $g(\cdot)$ is also a random variable since we have assumed that it is Borel-Measurable

Assume additionally that

$$g(\mathbf{X}_{-i}) \leq \left(\frac{1}{N-1}\right) \sum_{j \neq i} X_j \tag{3.1}$$

This structure ensures that when the ring is competing against the non-ring bidder, the resulting environment can be interpreted as a two-bidder auction, where the non-ring bidder has signal X_N, and the ring representative has signal $g(\mathbf{X}_{-N})$. This allows us to explore the dynamics in bidding behavior across informational structures, in particular for the nonring bidder, as in each case, he only has to make inference concerning the auxiliary signal $g(\mathbf{X}_{-N})$.

We give the following anecdotal motivation for the case in question.

The figure below shows some of the offshore blocks for exploration in The Gambia.

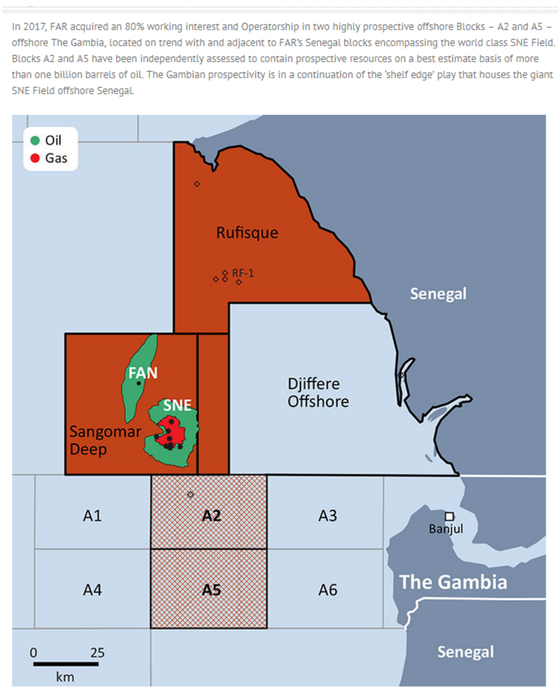

Figure 3.1: Gambia Oil Tract Adjacent Blocks

In March 2017, Australian exploration company *Far Limited* secured an 80 percent working interest in Blocks $A2$ and $A5$, originally licensed to Erin Energy, while licenses for blocks $A4$, $A3$ and $A6$ are yet to be assigned.[2] Blocks $A2$ and $A5$ were estimated to contain just over a million barrels of oil. In October 2018, drilling commenced on block $A2$, and the first well was found to contain "more water than oil" and was abandoned. This development is unfavorable not only for *Far Limited* (who might have lower revised estimates of the amount of oil reserve in the blocks), but also for potential investors interested in the remaining unlicensed blocks. If

[2]Block $A1$ was assigned to *British Petroleum*

we suppose that all potential investors - future bidders - use the same geoscientific techniques with regards to estimating the amount of oil, then going into the competition stages for the remaining blocks, they will be less confident with regards to estimating the actual amount of oil in the reserve based on their and other competitors' initial estimates as the remaining blocks are in close proximity with blocks $A2$ and $A5$. It is this kind of a scenario that we capture with the assumption $g(\mathbf{X}_{-i}) < \frac{1}{N-1}\sum_{j\neq i} X_j$ in the sense that the final aggregate is less than the average.

In this sense, bidding at the auction is *risky* for bidders, and a subset of the bidders may decide to pool information together and bid as a group as a means of risk sharing. Suppose a subset of bidders form the said ring . If the auctioneer suspects that the ring has formed, should he withhold this information or make it known?

3.2.4 Timing

- Nature assigns bidders' signals
- The ring mechanism is implemented, and a ring representative is chosen
- Bidders proceed to the main auction

3.3 The Second Price Auction

This section is devoted to the characterisation of bidders' equilibrium bidding strategies under the various informational structures. We start by analysing the equilibrium strategy adopted under the noncooperative case, using it as a benchmark. Then we proceed to strategies adopted under the Public Knowledge, Concealed and Private Knowledge Cases respectively. Recall that the Public Knowledge case refers to the case where the auctioneer publicly relays her knowledge of ring presence, the Private Knowledge case refers to the case where the revelation is made privately to the nonring bidder, and the Concealed case refers to the case where the auctioneer is silent about ring presence.

3.3.1 The Purely Non-Cooperative Equilibrium

When all bidders bid individually, there exists a symmetric increasing equilibrium strategy characterized below. Define

$$v(x, y) = \mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = y]$$

Theorem 3.3.1. (Theorem 6, Milgrom and Weber (1982)) *The unique symmetric equilibrium strategy $\beta(\cdot)$ is given by*

$$\beta(x_i) = v(x_i, x_i) = \mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = x_i]$$

$\forall i \in \mathcal{N}$

The intuition for the theorem is as follows. Since all bidders other than bidder i use $\beta(\cdot)$, suppose bidder i wins when he submits a bid b and he pays a price $\bar{\theta}$. He infers that $\beta(Y^1_{\mathcal{N}/\{i\}}) = \bar{\theta} \iff Y^1_{\mathcal{N}/\{i\}} = \beta^{-1}(\bar{\theta})$, and his expected value for the item conditional on winning is

$$\mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = \beta^{-1}(\bar{\theta})]$$

and he is happy to win so long as he makes a profit. That is, as long as

$$\bar{\theta} \leq \mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = \beta^{-1}(\bar{\theta})]$$

So the maximum amount he will bid is a θ such that he breaks even. That is θ satisfying

$$\theta = \mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = \beta^{-1}(\theta)] \tag{3.2}$$

$\beta(\cdot)$ being an equilibrium bidding strategy requires $\theta = \beta(x_i)$. Substituting $\theta = \beta(x_i)$ into (3.2), we get

$$\beta(x_i) = \mathbb{E}[V|X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = x_i] = v(x_i, x_i)$$

In our case,

Under Noncooperation, the symmetric equilibrium bidding strategy is

$$\beta(x_i) = x_i \cdot \mathbb{E}[g(\mathbf{X}_{-i})|Y^1_{N/\{i\}} = x_i] \tag{3.3}$$

And the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{No Ring}}] = \mathbb{E}_{\text{2nd Max}}\{\beta(x_i)\}$$

In the derivation of the strategy $\beta(\cdot)$, there was no need to impose symmetry of distribution of signals. In essence, provided the value function remains the same, the main determinant - other than his own signal - of the strategy of any player is the highest competing bid, from which he can infer the signal of the highest competing bidder. This suggests that the equilibrium bids when the presence of the ring is common knowledge will have a similar structure, as the signals in this case are X_N and $g(\mathbf{X}_{-N}) = \mathbf{X}_R$.

3.3.2 Equilibrium Behavior under Public Knowledge

Suppose that the presence of the ring is common knowledge among bidders. In this case, the auction is equivalent to a two-bidder auction with asymmetric bidders, with the non-ring bidder having signal x_N and the ring representative having signal $\mathbf{x}_R = g(\mathbf{x}_{-N})$. The equilibrium bidding behavior for the ring representative and *Bidder N* is as follows:

$$\text{Define } \tilde{v}(a,b) = \mathbb{E}[V|X_i = a, g(\mathbf{X}_{-i}) = b]$$

Theorem 3.3.2. *When ring presence is common knowledge, there exists a continuum of equilibria*

$$\tilde{\beta}_N(x_N) = \tilde{v}(x_N, \phi(x_N))$$
$$\tilde{\beta}_R(\mathbf{x}_R) = \tilde{v}(\phi^{-1}(\mathbf{x}_R), \mathbf{x}_R)$$

where

$$\phi : [0,1] \longrightarrow [0,1]$$

is increasing and surjective.

Proof. The approach is identical to Krishna (2009). First, since $g(.)$ is Borel-Measurable, $g(\mathbf{X}_{-N})$ is also a random variable. In that sense, $g(\mathbf{X}_{-N})$ can be interpreted as the signal of the ring bidder. And the rest of the proof follows as in Krishna (2009), pg 118-119.

We show that no bidder can do better.

Suppose the ring representative and bidder N use $\tilde{\beta}_N(x_N)$ and $\tilde{\beta}_R(\mathbf{x}_R)$ respectively, and also (without loss of generality) that bidder N wins the auction.

Then $\tilde{\beta}_N(x_N) > \tilde{\beta}_R(\mathbf{x}_R) = \tilde{\beta}_N(\phi^{-1}(\mathbf{x}_R))$,[3] which implies $x_N > \phi^{-1}(\mathbf{x}_R)$.[4] Therefore

$$\underbrace{\tilde{v}(x_N, \mathbf{x}_R)}_{\substack{\text{His inferred}\\ \text{value from}\\ \text{winning}}} > \underbrace{\tilde{v}(\phi^{-1}(\mathbf{x}_R), \mathbf{x}_R)}_{\text{The amount he pays}}$$

So he is happy to win at this price and cannot improve his situation as it does not hinge on his bid. It is straightforward to show that the ring representative is satisfied to lose at the current price. Since $\phi(x_N) > \mathbf{x}_R$. If the ring bidder were to win, he would pay $\tilde{v}(x_N, \phi(x_N)) > \tilde{v}(x_N, \mathbf{x}_R)$. So he overpays. Therefore, he has no problem losing.

The choice of ϕ is random, which implies a continuum (with respect to the space of equilibria).

□

The equilibrium strategies above are similar to that of Milgrom (1981) and Bikhchandani and Riley (1991), who prove the existence of a continuum of asymmetric equilibrium strategies in second-price auction with symmetric bidders.[5] The focus of the analysis will be on symmetric strategies. In this regard, we restrict to equilibria for which $\phi(x) = x$, so that

Corollary 3.3.1. *The symmetric equilibrium strategies are*

$$\tilde{\beta}_N(x_N) = \tilde{v}(x_N, x_N)$$
$$\tilde{\beta}_R(\mathbf{x}_R) = \tilde{v}(\mathbf{x}_R, \mathbf{x}_R)$$

So

[3] Since $\tilde{\beta}_N(x_N) = x_N \cdot \phi(x_N)$ and $\tilde{\beta}_R(\mathbf{x}_R) = \phi^{-1}(\mathbf{x}_R) \cdot \mathbf{x}_R = \phi^{-1}(\mathbf{x}_R) \cdot \phi(\phi^{-1}(\mathbf{x}_R)) = \tilde{\beta}_N(\phi^{-1}(\mathbf{x}_R))$

[4] Due to the fact that $\tilde{\beta}_N(\cdot)$ is increasing. To see this, let $a, b \in [0,1]$ be given, such that $\tilde{\beta}_N(a) > \tilde{\beta}_N(b)$. We need to show that $a > b$. Suppose the contrary, that $a < b$. Since $\phi(\cdot)$ is strictly increasing, this implies $\phi(a) < \phi(b) \implies \underbrace{a \cdot \phi(a)}_{\tilde{\beta}_N(a)} < \underbrace{b \cdot \phi(b)}_{\tilde{\beta}_N(b)}$, contradicting the assumption that $\tilde{\beta}_N(a) > \tilde{\beta}_N(b)$. So $\tilde{\beta}_N(a) > \tilde{\beta}_N(b) \implies a > b$. We also need to show that the converse $a > b \implies \tilde{\beta}_N(a) > \tilde{\beta}_N(b)$ is true. This is straightforward. $\phi(\cdot)$ strictly increasing implies that $\phi(a) > \phi(b) \implies \underbrace{a \cdot \phi(a)}_{\tilde{\beta}_N(a)} > \underbrace{b \cdot \phi(b)}_{\tilde{\beta}_N(b)}$.

[5] Milgrom (1981) considered the case for a two-bidder auction. Bikhchandani and Riley (1991) generalized the result to the N-bidder case

Under Public Knowledge, the equilibrium bidding strategies adopted by the ring representative and the nonring bidder are respectively

$$\tilde{\beta}_R(\mathbf{x}_R) = \mathbf{x}_R^2 \text{ and } \tilde{\beta}_N(x_N) = x_N^2$$

The auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Public}}] = \mathbb{E}_{\text{2nd Max}}\{\tilde{\beta}_N(x_N), \tilde{\beta}_R(\mathbf{x}_R)\}$$

Furthermore, we can compare the bidding strategies employed in this case to those used in the noncooperative case. In particular,

Proposition 19. *Compared to the noncooperative case, the nonring bidder submits a higher bid under common knowledge. That is,*

$$\beta(x_N) \leq \tilde{\beta}_N(x_N)$$

Proof. Since

$$\left(\frac{1}{N-1}\right)\sum_{j\neq i} X_j \geq g(\mathbf{X}_{-i})$$

By definition, $Y^1_{\mathcal{N}/\{N\}} \geq X_j$, $\forall j \neq N$. This implies $Y^1_{\mathcal{N}/\{N\}} \geq \left(\frac{1}{N-1}\right)\sum_{j\neq N} X_j \geq g(\mathbf{X}_{-N})$. So

$$Y^1_{\mathcal{N}/\{N\}} = x_N \implies g(\mathbf{X}_{-N}) \leq x_N$$

Therefore,

$$\mathbb{E}[g(\mathbf{X}_{-N})|Y^1_{\mathcal{N}/\{N\}}] \leq x_N \iff \underbrace{x_N \cdot \mathbb{E}[g(\mathbf{X}_{-N})|Y^1_{\mathcal{N}/\{N\}}]}_{\beta(x_N)} \leq \underbrace{x_N^2}_{\tilde{\beta}_N(x_N)} \tag{3.4}$$

$\square$

The intuition behind this result follows from the following observation: Under noncooperation, the only information that the nonring bidder can infer upon winning the item is the highest signal of his competitors. However, under public knowledge, upon winning, the nonring bidder is able to infer the entire aggregate. In this sense, the nonring bidder's inferred estimate of the value of the item is more informative under public knowledge than under noncooperation. Hence, the nonring bidder suffers less winner's curse compared to the noncooperative case, leading him to bid more aggressively.

3.3.3 Equilibrium Behavior under the Concealed Case

Consider the case under which the ring is able to conceal its presence from the nonring bidder. In this scenario, the nonring bidder still maintains the noncooperative bidding strategy $\beta(\cdot)$, as he believes he is bidding competitively against other $N-1$ identical bidders. The objective of the ring representative will be to best respond to $\beta(\cdot)$ conditional on the reported ring signals. We derive the ring representative's strategy for a very general specification of the value function.

3.3.3.1 The General Case

We derive the equilibrium bidding strategy for the ring representative in a very general context involving an arbitrary set of ring members $\mathcal{K}$, and a general value function $V(X_1, X_2, \cdots, X_N, \epsilon)$.

Theorem 3.3.3. *When the ring is able to operate undetected in the auction, the unique equilibrium bidding strategy for the ring representative is to bid a value ω, such that*

$$\omega = \mathbb{E}[V|X_1 = x_1, \ldots, X_k = x_k, Y^1_{\mathcal{N}/\mathcal{K}} = \beta^{-1}(\omega)]$$

Proof. In the case where the ring representative wins the auction at price ρ, he can infer the signal of the highest non-ring bidder, since this bidder uses the equilibrium strategy

$$\beta(Y^1_{\mathcal{N}/\mathcal{K}}) = v(Y^1_{\mathcal{N}/\mathcal{K}}, Y^1_{\mathcal{N}/\mathcal{K}}) = \rho \iff Y^1_{\mathcal{N}/\mathcal{K}} = \beta^{-1}(v(Y^1_{\mathcal{N}/\mathcal{K}}, Y^1_{\mathcal{N}/\mathcal{K}})) = \beta^{-1}(\rho)$$

Therefore, the ring bidder's estimated valuation conditional on winning is

$$\mathbb{E}[V|X_1 = x_1, \ldots, X_k = x_k, Y^1_{\mathcal{N}/\mathcal{K}} = \beta^{-1}(\rho)]$$

So he is happy to win so long as the price ρ is at most equal to his estimated value, otherwise he makes a loss. That is, as long as

$$\rho \leq \mathbb{E}[V|X_1 = x_1, \ldots, X_k = x_k, Y^1_{\mathcal{N}/\mathcal{K}} = \beta^{-1}(\rho)]$$

Therefore, the highest he will bid is a ρ such that he breakseven. That is, ρ such that

$$\rho = \mathbb{E}[V|X_1 = x_1, \ldots, X_k = x_k, Y^1_{\mathcal{N}/\mathcal{K}} = \beta^{-1}(\rho)]$$

□

Corollary 3.3.2. *For the family of valuation functions $V = u(X_i, g(\mathbf{X}_{-i}))$ and $|\mathcal{K}| = N - 1$,*

$$\bar{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)]$$

$$\omega = \mathbf{x}_R \cdot \beta^{-1}(\omega)$$

When the ring is concealed, the equilibrium bidding strategies for the ring representative and the nonring bidder are respectively

$$\bar{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)]$$

and

$$\bar{\beta}_N(x_N) = \mathbb{E}[V|X_N = x_N, Y^1_{N/\{N\}} = x_N]$$

and the auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Concealed}}] = \mathbb{E}_{\text{2nd Max}}\{\bar{\beta}_R(\mathbf{x}_R), \bar{\beta}_N(x_N)\}$$

3.3.4 Equilibrium Behavior under Private Knowledge

Now we consider the situation where the nonring bidder is aware of the ring, but the ring operates with the belief that their presence is concealed. That us, when the auctioneer reveals the presence of the ring secretly to the nonring bidder. Denote the bidding strategies adopted by the ring representative and the nonring bidder as $\hat{\beta}_R(\mathbf{x}_R)$ and $\hat{\beta}_N(x_N)$ respectively. Since the ring is unaware that their presence is known, the ring representative employs the same strategy as in the concealed case. That is, $\hat{\beta}_R(\mathbf{x}_R) = \bar{\beta}_R(\mathbf{x}_R)$. The objective of the nonring bidder will be to best-respond to $\hat{\beta}_R(\mathbf{x}_R)$.

Theorem 3.3.4. *Under private knowledge, the nonring bidder does not change his bid relative to the nonncooperative case. That is*

$$\hat{\beta}_N(x_N) = \beta(x_N)$$

Proof. If he wins at a price ω, the nonring bidder can infer the realised value of $g(\mathbf{X}_{-N})$ from the amount he pays. His inferred value conditional on winning is

$$\mathbb{E}[V|X_N = x_N, g(\mathbf{X}_{-N}) = \mathbf{x}_R]$$

He makes profit provided that

$$\underbrace{x_N \cdot \mathbf{x}_R}_{\substack{\text{My inferred} \\ \text{value} \\ \text{conditional} \\ \text{on winning}}} > \underbrace{\beta^{-1}(\omega) \cdot \mathbf{x}_R}_{\text{The amount I pay}} \tag{3.5}$$

and this is satisfied when my realised value of X_N, $x_N > \beta^{-1}(\omega) \implies \beta(x_N) > \omega$. Therefore,

- Gaining corresponds to the condition $\beta(x_N) > \omega$. This is equivalent to winning with bid $\beta(x_N)$.
- Making losses corresponds to the condition $\beta(x_N) < \omega$.
- Being indifferent to winning or losing corresponds to the condition $\beta(x_N) = \omega$. This is equivalent to earning zero surplus with bid $\beta(x_N)$.

I will show that the nonring bidder does not benefit from bidding higher or lower than $\beta(x_N)$.

Suppose he bids higher. That is, he submits a bid $\bar{b} > \beta(x_N)$. There are three possible cases that may arise:

- Case 1: $\omega < \beta(x_N) < \bar{b}$
  - In this case, the non ring bidder wins, and makes profit since $\omega < \beta(x_N)$.
- Case 2: $\beta(x_N) < \omega < \bar{b}$
  - In this case, the non ring bidder wins, but loses money as $\beta(x_N) < \omega$.
- Case 3: $\beta(x_N) < \bar{b} < \omega$
  - In this case, the non ring bidder loses, and is happy to lose, since $\beta(x_N) < \omega$.

$$\beta(x_N) \textit{ weakly dominates any } \bar{b} > \beta(x_N)$$

I follow a similar argument to show that he does not profit from bidding lower.

Suppose he bids lower. That is, he submits a bid $\underline{b} > \beta(x_N)$.

- Case 1: $\omega < \underline{b} < \beta(x_N)$
  - In this case, the non ring bidder wins, and makes profit since $\omega < \beta(x_N)$. In essence he is not hurt by depressing his bid.
- Case 2: $\underline{b} < \omega < \beta(x_N)$

  - In this case, the non ring bidder loses, but could have won by bidding $\beta(x_N)$, and made a positive profit. Hence in this case, depressing his bid lost him potential profit.

- Case 3: $\underline{b} < \beta(x_N) < \omega$
  - In this case, the non ring bidder loses, and is happy to lose, since $\beta(x_N) < \omega$.

$$\beta(x_N) \textit{ weakly dominates any } \underline{b} < \beta(x_N)$$

Altogether, we conclude that the best response for the nonring bidder is $\beta(x_N)$. $\square$

Under Private Knowledge, the equilibrium strategy profile for the ring representative and the nonring bidder are respectively:

$$\hat{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)]$$

and

$$\hat{\beta}_N(x_N) = \mathbb{E}[V|X_N = x_N, Y^1_{N/\{N\}} = x_N]$$

and the auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Private}}] = \mathbb{E}_{\text{2nd Max}}\{\hat{\beta}_R(\mathbf{x}_R), \hat{\beta}_N(x_N)\}$$

The auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{Private}}] = \mathbb{E}_{\text{2nd Max}}\{\omega, \beta(x_N)\}$$

We can hence state the following proposition

Proposition 20. *When the presence of the ring is privately revealed to the nonring bidder, he does not change his bidding strategy with respect to the noncooperative case.*

Remark 1: The result above indicates that the nonring bidder has the same best response to $N-1$ noncooperating bidders as he does against the ring representative under private knowledge.
Remark 2: This implies that the pair of private knowledge strategies correspond to an asymmetric equilibrium of the public knowledge case. Hence our analysis of the distinction in bidder behavior between the private and public knowledge cases is contingent on our restriction that equilibrium strategies are symmetric in the public knowledge case.

Furthermore, we can analyse the benefits (costs) to the auctioneer when bidders suspect the presence of a ring, when there is none. The results vary depending on whether they expect a public or private ring.

When they expect a private knowledge scenario, they use the strategy $\beta(x_i)$, and the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}^{\text{Private}}_{\text{Suspect}}] = \mathbb{E}_{\text{ 2nd Max}}\{\beta(x_i)\}$$

while when they expect a public knowledge scenario, they use the strategy $\tilde{\beta}_i(x_i)$, and the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}^{\text{Public}}_{\text{Suspect}}] = \mathbb{E}_{\text{ 2nd Max}}\{\tilde{\beta}_i(x_i)\}$$

Since $\beta(x_i) < \tilde{\beta}_i(x_i)$,

$$\mathbb{E}[\mathbb{R}^{\text{Private}}_{\text{Suspect}}] < \mathbb{E}[\mathbb{R}^{\text{Public}}_{\text{Suspect}}]$$

and we state the following proposition.

Proposition 21. *In the absence of a ring, the auctioneer can improve his position by convincing bidders that they are facing a public ring.*

3.3.4.1 A Brief Recap of The Main Ideas

The nonring bidder bids higher under Public Knowledge compared to the noncooperative case.
Privately revealing the ring presence to the nonring bidder does not cause him to change his behavior with respect to the noncooperative case.

This implies that the auctioneer can make bidders bid higher by convincing them that they are facing a public ring. Hence, even when there is no ring at the auction, an auctioneer can artificially raise prices in this way.

3.4 Characterisation under the Uniform Distribution

We introduce the additional specification that the underlying signals are uniformly distributed. This is motivated by the fact that the implicit nature of the equilibrium bidding strategy of the ring representative under private knowledge means it is not immediately clear how to express the auctioneer's expected revenue in this case. We further simplify by fixing a functional form for the auxiliary signal $g(\cdot)$. This is done by assuming a value function of the form

$$V(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} X_i$$

with each X_i independently and uniformly distributed on $[0, 1]$.

In the event of ring presence,

- The nonring bidder (Bidder N) knows only his signal x_N.
- The ring representative knows all signals but x_N.

We can rewrite the value function as

$$V(X_1, \ldots, X_N) = \prod_{i} X_i = X_i \cdot \Big(\prod_{j \neq i} X_j\Big) = X_i \cdot g(\mathbf{X}_{-i})$$

where $g(\mathbf{X}_{-i}) = \prod_{j \neq i} X_j$. Note that $g(\mathbf{X}_{-i})$ is indeed lower than $\frac{1}{N-1}\sum_{j \neq i} X_j$.[6]

With this specification, we derive the equilibrium bidding strategies under the different informational structures and make relevant comparative static analyses relating to auctioneer expected revenue and bidders' expected profit.

3.4.1 Equilibrium Strategies under Noncooperation

Under the purely Non-Cooperative case, each bidder uses the strategy[7]

$$\beta(x_i) = v(x_i, x_i) = \mathbb{E}[V | X_i = x_i, Y^1_{\mathcal{N}/\{i\}} = x_i] = x_i \cdot x_i \cdot \underbrace{\overbrace{\frac{x_i}{2}}^{\substack{\mathbb{E}[x_j | x_j < x_i] \\ j \neq i}} \cdots \frac{x_i}{2}}_{N-2 \text{ times}} \tag{3.6}$$

[6]This follows from

$$g(\mathbf{X}_{-i}) = \underbrace{\prod_{j \neq i} X_j \leq \big(\prod_{j \neq i} X_j\big)^{\frac{1}{N-1}}}_{\substack{\text{Since the} \\ X_j\text{'s are between} \\ 0 \text{ and } 1}} \overbrace{\leq \frac{1}{N-1}\sum_{æ \neq i} X_j}^{\substack{\text{Via the} \\ \text{Inequality of} \\ \text{Arithmetic} \\ \text{and Geometric} \\ \text{means}}}$$

[7]Note that in general, for $V = \prod_i X_i$ and any arbitrary distribution $F(\cdot)$ with density $f(\cdot)$, we have the following

$$v(x, x) = x^2 \int_0^x \int_0^{y^2_{\mathcal{N}/\{i\}}} \cdots \int_0^{y^{N-2}_{\mathcal{N}/\{i\}}} \big(\prod_{s=2}^{N-2} y^s_{\mathcal{N}/\{i\}}\big) \frac{(N-2)! \prod_{s=2}^{N-2} f(y^s_{\mathcal{N}/\{i\}})}{F(x)^{N-2}} dy^{N-2}_{\mathcal{N}\{i\}} \cdots dy^2_{\mathcal{N}/\{i\}}$$

The Symmetric Equilibrium Bidding Strategy under Noncooperation is

$$\beta(x_i) = \frac{x_i^N}{2^{N-2}}$$

3.4.2 Equilibrium Strategies under Public Knowledge

Here, we fix the idea behind Theorem 3.3.2 by deriving the equilibrium bidding strategies based on the fact that the farthest that bidders will go in terms of bidding is a price such that they breakeven. The argument proceeds as follows:

Suppose the ring representative uses a strategy

$$\tilde{\beta}_R(g(\mathbf{X}_{-N})) = \alpha\Big(\prod_{i\neq N} x_i\Big)^{\gamma}$$

Again, for simplicity, denote $g(\mathbf{X}_{-N})$ as $\mathbf{X}_R$, and $\Big(\prod_{i\neq N} x_i\Big)$ as $\mathbf{x}_R$, so that the ring representative's strategy can be rewritten as

$$\tilde{\beta}_R(\mathbf{X}_R) = \alpha\big(\mathbf{x}_R\big)^{\gamma}$$

When the nonring bidder wins, he does so at a price p^* such that

$$p^* = \alpha\Big(\mathbf{x}_R\Big)^{\gamma}$$

he infers that the auxiliary signal, $\mathbf{x}_R$ is

$$\mathbf{x}_R = \left(\frac{p^*}{\alpha}\right)^{\frac{1}{\gamma}}$$

Therefore, his estimate $\mathbb{E}[V|X_l = x_l, \mathbf{X}_R = \mathbf{x}_R]$ of the value of the item is

$$\mathbb{E}[V|X_N = x_N, \mathbf{X}_R = \left(\frac{p^*}{\alpha}\right)^{\frac{1}{\gamma}}] = x_N * \left(\frac{p^*}{\alpha}\right)^{\frac{1}{\gamma}}$$

He is happy as long as he makes a profit. That is, as long as $p^* \leq x_N\left(\frac{p^*}{\alpha}\right)^{\frac{1}{\gamma}}$. Therefore, he is happy to win until a price $\mathbf{P_{NR}}$ such that [8]

$$\mathbf{P_{NR}} = x_N\Big(\frac{\mathbf{P_{NR}}}{\alpha}\Big)^{\frac{1}{\gamma}}$$
$$\iff \mathbf{P_{NR}} = \frac{\big(x_N\big)^{\frac{\gamma}{\gamma-1}}}{\alpha^{\frac{1}{\gamma-1}}}$$

In the same vein, when the ring bidder wins at price $\hat{p}$, she infers that

$$x_N = \hat{p}^{\frac{\gamma-1}{\gamma}}\alpha^{\frac{1}{\gamma}}$$

So she stays in until a $\mathbf{P_R}$ such that

$$\mathbf{P_R} = \mathbf{x}_R\mathbf{P_R}^{\frac{\gamma-1}{\gamma}}\alpha^{\frac{1}{\gamma}}$$
$$\iff \mathbf{P_R}^{\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}}\mathbf{x}_R$$
$$\iff \mathbf{P_R} = \alpha\Big(\mathbf{x}_R\Big)^{\gamma}$$

In essence, both agents are best-responding to each other. Therefore, $\tilde{\beta}_R(\mathbf{x}_R) = \alpha\Big(\mathbf{x}_R\Big)^{\gamma}$, $\tilde{\beta}_N(x_N) = \Big(\frac{x_N^{\gamma}}{\alpha}\Big)^{\frac{1}{\gamma-1}}$ characterizes the equilibrium strategies for any $(\alpha, \gamma) \in \mathbb{R}^2$. For convenience, we restrict $\gamma \neq 1$.[9] So we have a continuum of equilibria.

$$\Bigg\{\tilde{\beta}_R(\mathbf{x}_R) = \alpha\Big(\mathbf{x}_R\Big)^{\gamma}, \tilde{\beta}_N(x_N) = \Big(\frac{x_N^{\gamma}}{\alpha}\Big)^{\frac{1}{\gamma-1}}, (\alpha, \gamma) \in \mathbb{R}^2, \gamma \neq 1\Bigg\} \tag{3.7}$$

In terms of Theorem 3.3.2 with $\tilde{v}(a, b) = \mathbb{E}[V|X_i = a, g(\mathbf{X}_{-i}) = b]$, the strategies above correspond to

$$\tilde{\beta}_N(x_N) = \tilde{v}(x_N, \phi(x_N)) = x_N * \Big(\frac{x_N}{\alpha}\Big)^{\frac{1}{\gamma-1}}$$
$$\tilde{\beta}_R(\mathbf{x}_R) = \tilde{v}(\phi^{-1}(\mathbf{x}_R), \mathbf{x}_R) = \Big(\mathbf{x}_R\Big) * \alpha\Big(\mathbf{x}_R\Big)^{\gamma-1}$$

where $\phi^{-1}(x) = \alpha x^{\gamma-1}$. Symmetry requires $\phi(x) = x \iff (\alpha, \gamma) = (1, 2)$

Therefore, as in Theorem 3.3.2,

[8]This is sort of a break-even price

[9]This is in order that the bidding strategies do not explode.

The symmetric equilibrium strategy profile under Public knowledge is

$$\tilde{\beta}_R(\mathbf{x}_R) = \left(\mathbf{x}_R\right)^2$$
$$\tilde{\beta}_N(x_N) = \left(x_N\right)^2$$

3.4.3 Equilibrium Strategies under the Concealed case

Under the concealed case, the nonring bidder is obliviuos to the presence of the ring. Hence, he believes that the auction is purely competitive. Hence, he bids as in the noncooperative case. That is, he uses the strategy $\bar{\beta}_N(x_N) = \frac{x_N^N}{2^{N-2}}$. The strategy for the ring representative is given by Corollary 3.3.2 as $\omega = \mathbf{x}_R \cdot \beta^{-1}(\omega)$.

$\beta^{-1}(\omega) = (2^{N-2} \cdot \omega)^{\frac{1}{N}}$, which implies

$$\omega = \mathbf{x}_R \cdot (2^{N-2} \cdot \omega)^{\frac{1}{N}} \implies \bar{\beta}_R(\mathbf{x}_R) = \omega = \left(\mathbf{x}_R\right)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$$

The equilibrium strategy profile under the Concealed case is

$$\bar{\beta}_R(\mathbf{x}_R) = \left(\mathbf{x}_R\right)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$$
$$\bar{\beta}_N(x_N) = \frac{x_N^N}{2^{N-2}}$$

3.4.4 Equilibrium Strategies under Private Knowledge

In the private knowledge case, the ring believes that it is concealed. Hence, the strategy for the ring representative is the same as in the Concealed case. The nonring bidder has to best-respond to this strategy. Theorem 3.3.4 implies that the nonring bidder does not bid differently from the concealed case. That is, the nonring bidder maintains the strategy $\frac{x_N^N}{2^{N-2}}$. The fact that the nonring bidder behaves the same seems counterintuitive. Indeed, to fix this idea, we can use a breakeven price argument like we have done in the public knowledge case.

Suppose the nonring bidder wins at a price ρ. This implies that

$$\rho = \left(\mathbf{x}_R\right)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}} \implies \mathbf{x}_R = 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

Therefore, his inferred value for the item upon winning is

$$x_N \cdot 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

and he is happy to win provided he makes a profit. That is, provided

$$\rho \leq x_N \cdot 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

Therefore, the maximum he will bid is an amount such that he breakseven. That is, a $\bar{\rho}$ such that

$$\bar{\rho} = x_N \cdot 2^{\frac{2-N}{N}} \bar{\rho}^{\frac{N-1}{N}} \implies \bar{\rho} = \frac{x_N^N}{2^{N-2}}$$

Essentially the result implied by Theorem 3.3.4.

The strategy profile under private knowledge is

$$\hat{\beta}_R(\mathbf{x}_R) = \left(\mathbf{x}_R\right)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$$
$$\hat{\beta}_N(x_N) = \frac{x_N^N}{2^{N-2}}$$

3.4.5 Auctioneer Revenue Analysis

In this subsection, I consider the implications of the aforementioned information structures on seller's revenue, using the formulation $V = \prod_i X_i$ and with 5 bidders (4 in the ring).

Revenue When Ring Presence is Common Knowledge

The bids submitted in this case are $\tilde{\beta}_N(x_N) = \left(x_N\right)^2$ and $\tilde{\beta}_R(\mathbf{x}_R) = \left(\mathbf{x}_R\right)^2$.

The ring representative wins when

$$\left(\mathbf{x}_R\right)^2 > (x_N)^2 \iff \mathbf{x}_R > x_N$$

and vice versa. $\mathbf{x}_R = \prod_{j \neq N} x_j$ is a product of $N-1$ independent uniform $[0,1]$ random variables. It's pdf is given in Dettmann and Georgiou (2009) as

$$s(\psi) = \frac{(-1)^{N-2} \log^{N-2}(\psi)}{(N-2)!}$$

then the probability that $\mathbf{X}_R < \theta$ for some θ

$$S(\theta) = \int_0^\theta \frac{(-1)^{N-2}\log^{N-2}(\psi)}{(N-2)!}d\psi$$

It is easy to show that[10]

$$S(\theta) = \sum_{i=1}^{N-1} \frac{(-1)^{i-1}\theta\log^{i-1}(\theta)}{(i-1)!}$$

In our case with $N = 5$, this implies that the distribution of the auxiliary signal can be expressed as

$$S(\theta) = \theta - \theta\log(\theta) + \frac{\theta}{2}\log^2(\theta) - \frac{\theta}{6}\log^3(\theta)$$

The expected revenues for both the noncooperative and the public knowledge cases can be computed by hand. In particular, for the public knowledge case, we can express the seller's expected revenue as[11]

$$\mathbb{E}[\mathbb{R}_{\text{Public}}] = \int_0^1 (1-F(\theta))(1-S(\theta))d\tilde{v}(\theta,\theta) =$$
$$\int_0^1 (1-\theta)\Big(1-\theta+\theta\log(\theta) - \frac{\theta}{2}\log^2(\theta) + \frac{\theta}{6}\log^3(\theta)\Big)\cdot 2\theta d\theta \approx 0.00974$$

In a similar fashion we can show that the expected revenue under noncooperation can be expressed as

$$\mathbb{E}[\mathbb{R}_{\text{No Ring}}] = \int_0^1 (1-F(\theta)^5 - 5F(\theta)^4(1-F(\theta))dv(\theta,\theta) =$$
$$\int_0^1 (1-\theta^5 - 5\theta^4(1-\theta))\cdot\frac{5}{8}\theta^4 = \frac{1}{36}$$

However, due to the relative difficulty with manually computing the expected revenue under private knowledge, we resort to numerical methods. We simulate the auctioneer's expected revenue and bidders' expected profit across the different informational structures. The results are presented in Table 3.1 below.

[10] We show this in Appendix 4.1.4.2

[11] This formulation is due to Proposition 2 in Cheng (2006). It follows from the fact that under public knowledge, both bidders use the symmetric equilibrium strategy $\eta(\theta) = \tilde{v}(\theta,\theta)$. Therefore, the expected revenue for the auctioneer is the expected second-highest bid, which can be expressed as

$$\mathbb{E}[\mathbb{R}_{\text{Public}}] = \int_0^1 \tilde{v}(\theta,\theta)d(\underbrace{F(\theta)S(\theta) + F(\theta)(1-S(\theta)) + S(\theta)(1-F(\theta))}_{1-(1-F(\theta))(1-S(\theta))})$$

Integration by parts with the condition that $\tilde{v}(0,0) = 0$ yields our result.

Informational Structure	Seller Expected Revenue	Ring Member Expected Profit	Nonring Bidder Expected Profit
NonCooperative	0.02782098	0.0006937737	0.0006937737
Secret / Private Ring	0.009461446	0.005287447	0.0006791732
Public Ring	0.009748861	0.0001632564	0.02088852
Convince of Public Ring	0.4763		

Table 3.1: Revenue and Profits Across Informational Structures

The numbers in blue are used to signify the best case scenario, while the ones in red indicate the worst case respectively. Secret Ring refers to the ring under the Concealed case. Private Ring refers to the ring under the Private Knowledge case, while Public Ring refers to the ring under Public Knowledge.

We find that

- Compared to the Secret, Public and Private Ring cases, the auctioneer is most happy when every party bids noncooperatively. However, in the event that a ring is present at the auction, he prefers that their presence be made public.
- The best case for the nonring bidder is under public knowledge. The rationale is pretty straightforward. By construction, the nonring bidder starts as the *Stronger* bidder, in the sense that the signal of the nonring bidder *First Order Stochastically Dominates* that of the ring representative.[12] Under public knowledge, both the ring representative and the nonring bidder use the same strategies. This implies that it is more likely that the nonring bidder wins the item.
- For the ring representative, his best case is under the Concealed and Private Knowledge cases.
- In the absence of a ring, convincing bidders that they are facing a public ring significantly raises the auctioneer's expected revenue compared to the Noncooperative case.

In addition, we compare the probability of winning across the informational structures

[12] We show this in Lemma .0.1

Informational Structure	Ring Representative	Nonring Bidder
NonCooperative	0.2	0.2
Secret / Private Ring	0.664689	0.335311
Public Ring	0.062264	0.937736

Table 3.2: Probability of Winning Across Informational Structures

In terms of chances of winning, the best case for the nonring bidder is the public knowledge case. However, for the ring representative, the best case is private knowledge case. In fact, ring members are worse off under the public ring case than the noncooperative case. In essence, we are presented with a scenario for which the ring only benefits from operating when they believe their presence is concealed. That is, when the ring operates at the auction, they endeavor to conceal their presence.

3.4.5.1 A Summary of the Revenue and Profit Comparisons

Conditional on ring presence, the auctioneer is better off with a public ring compared to a secret ring.

The ring prefers that their presence be concealed.

The nonring bidder is better off competing against a public ring than a secret ring.

Overall, the rankings across informational structures are

- Auctioneer's Expected Revenue
  - $\mathbb{E}[\mathbb{R}_{\text{No Ring}}] > \mathbb{E}[\mathbb{R}_{\text{Public}}] > \mathbb{E}[\mathbb{R}_{\text{Concealed / Private}}]$
- Nonring Bidder Expected Profit
  - $\mathbb{E}[\pi^N_{\text{Public}}] > \mathbb{E}[\pi^N_{\text{No Ring}}] > \mathbb{E}[\pi^N_{\text{Concealed / Private}}]$
- For the Ring Members
  - $\mathbb{E}[\pi^R_{\text{Concealed / Private}}] > \mathbb{E}[\pi^N_{\text{No Ring}}] > \mathbb{E}[\pi^N_{\text{Public}}]$

3.5 Analysis with a Different Valuation Structure

In this section, we highlight the fact that some of the results under the valuation family with $g(\mathbf{X}_{-i}) \leq \frac{1}{N-1}\sum_{j\neq i} X_j$ do not hold for certain other valuation functions. In particular, the result that the nonring bidder bids higher under public knowledge does not always hold.

Assume as in the previous section that bidders' signals are uniformly distributed, but the common value now takes the additive form

$$V(X_1, X_2, \ldots, X_N) = \sum_{i=1}^{N} X_i$$

We can rewrite $V(X_1, X_2, \ldots, X_N) = X_i + \underbrace{\sum_{j \neq i} X_j}_{g(\mathbf{X}_{-i})}$, where the auxiliary signal is not necessarily between 0 and 1. Denote the nonring bidder's signal as x_N. When the ring is formed, the ring representative bids using $g(\mathbf{X}_{-N})$, which we denote as $\mathbf{x}_R$. In this sense, $V = x_N + \mathbf{x}_R$. However, note that in this case, $g(\mathbf{X}_{-i}) \geq \frac{1}{N-1} \sum_{j \neq i} X_j$.

3.5.1 Noncooperative Equilibrium

Under noncooperation, the equilibrium bidding strategy according to Milgrom and Weber (1982) is

$$\beta(x_i) = \mathbb{E}[V | X_i = x_i, Y^1_{N/\{i\}} = x_i] = x_i + x_i + \underbrace{\frac{x_i}{2} + \cdots + \frac{x_i}{2}}_{N-2 \text{ times}} = \frac{N+2}{2} x_i$$

The noncooperative equilibrium strategy is

$$\beta(x_i) = \frac{N+2}{2} x_i$$

3.5.2 Equilibrium under the Concealed Case

When the ring believes they are operating in secret, the ring representative has to best-respond to $\beta(x_i)$.

When the ring representative wins at a price ρ. she infers that

$$\frac{N+2}{2} x_N = \rho \implies x_N = \frac{2}{N+2} \rho$$

Therefore, her inferred estimate of the value of the item upon winning is $\mathbf{x}_R + \frac{2}{N+2}\rho$, and he is happy to win as long as $\rho \leq \mathbf{x}_R + \frac{2}{N+2}\rho$. So he bids $\bar{\rho}$ such that

$$\bar{\rho} = \mathbf{x}_R + \frac{2}{N+2} \bar{\rho} \implies \bar{\rho} = \frac{N+2}{N} \mathbf{x}_R$$

Under the concealed case, the equilibriun bidding strategies for the nonring bidder and the ring representative are respectively

$$\bar{\beta}_N(x_N) = \frac{N+2}{2}x_N$$

and

$$\bar{\beta}_R(\mathbf{x}_R) = \frac{N+2}{N}\mathbf{x}_R$$

3.5.3 Private Knowledge Equilibrium

Given that the ring believes it is concealed, the ring representative bids $\bar{\rho}$. The nonring bidder best-responds to this strategy.

When the nonring bidder wins at a price κ, he infers that $\frac{N+2}{N}\mathbf{x}_R = \kappa \implies \mathbf{x}_R = \frac{N}{N+2}\kappa$. Therefore his inferred value is $x_N + \frac{N}{N+2}\kappa$, and he is hapy to win so long as $\kappa \leq x_N + \frac{N}{N+2}\kappa$. So he bids

$$\hat{\kappa} = x_N + \frac{N}{N+2}\hat{\kappa} \implies \hat{\kappa} = \frac{N+2}{2}x_N$$

The equilibrium strategies under private knowledge are

$$\hat{\beta}_N(x_N) = \frac{N+2}{2}x_N$$

and

$$\hat{\beta}_R(\mathbf{x}_R) = \frac{N+2}{N}\mathbf{x}_R$$

As in the case with $g(\mathbf{X}_{-i}) = \prod_{j \neq i} X_j$, under private knowledge, the nonring bidder does not change his bid with respect to the noncooperative case.

3.5.4 Symmetric Public Knowledge Equilibrium

It is easy to show that in the public knowledge case, the pair

$$\tilde{\beta}_R(\mathbf{x}_R) = \alpha \cdot \mathbf{x}_R$$

$$\tilde{\beta}_N(x_N) = \frac{\alpha}{\alpha - 1}x_N$$

are equilibrium strategies. We show that $\alpha \cdot \mathbf{x}_R$ and $\frac{\alpha}{\alpha-1}x_N$ are mutual best responses.

Suppose the ring representative uses $\alpha \cdot \mathbf{x}_R$.

When the nonring bidder wins at a price ρ, he infers that $\alpha \cdot \mathbf{x}_R = \rho \implies \mathbf{x}_R = \frac{\rho}{\alpha}$. Therefore, he is happy to win until a price $\bar{\rho}$ such that

$$\bar{\rho} = x_N + \frac{\bar{\rho}}{\alpha} \implies \bar{\rho} = \frac{\alpha}{\alpha - 1} x_N$$

Essentially, the nonring bidder uses the strategy $\frac{\alpha}{\alpha-1} x_N$ as a best response to $\alpha \cdot \mathbf{x}_R$.
Suppose now, that the nonring bidder uses $\frac{\alpha}{\alpha-1} x_N$.
When the ring representative wins at ρ, she infers that $\frac{\alpha}{\alpha-1} x_N = \rho \implies x_N = \frac{\alpha-1}{\alpha}\rho$, and he is happy to win up to a $\hat{\rho}$ such that

$$\hat{\rho} = \mathbf{x}_R + \frac{\alpha - 1}{\alpha}\hat{\rho} \implies \hat{\rho} = \alpha \cdot \mathbf{x}_R$$

So $\alpha \cdot \mathbf{x}_R$ and $\frac{\alpha}{\alpha-1} x_N$ are mutual best responses. Furthermore, symmetry requires
$\alpha = \frac{\alpha}{\alpha-1} \implies \alpha = 2$.

The symmetric equilibrium strategies under public knowledge are

$$\tilde{\beta}_N(x_N) = 2x_N$$

and

$$\tilde{\beta}_R(\mathbf{x}_R) = 2\mathbf{x}_R$$

Comparing the equilibrium strategy of the nonring bidder under private knowledge ($\frac{N+2}{2} x_N$) and under public knowledge ($2x_N$), since $N \geq 3$,

$$\frac{N+2}{2} > 2 \implies \frac{N+2}{2} x_N > 2x_N$$

implying that the nonring bidder bids higher under private knowledge.

This conclusion is in contrast to the result under the multiplicative valuation structure. A corollary of this observation is also the conclusion that the fact that the auctioneer can cheat by convincing bidders that they are facing a public ring is also contingent on the valuation structure, as bidders bid less when the valuation structure is additive. We summarise this fact in the following proposition.

Proposition 22. *When the valuation structure is additive, nonring bidders bid higher under* *public knowledge compared to the noncooperative, concealed and private knowledge cases.*

3.6 Conclusion

We have analyzed the question of whether in a pure common value second price auction, an auctioneer who suspects that a ring might be present at the auction should reveal this information, and if so, whether this revelation should be made publicly to all participants or privately to other nonring bidders. We showed that for a particular family of value functions with an underlying uniform distribution, publicly revealing this information is the best strategy for the auctioneer. This is because it leads nonring bidders to bid higher than they would have when the auction is purely noncooperative. Furthermore, we showed that a consequence of this result is that an auctioneer might always want to convince bidders publicly that a ring exists even if a ring is not present, as the higher bids by bidders upon receiving this information means the auctioneer earns higher expected revenue compared to the noncooperative case. We also illustrated that the results do not necessarily hold when the valuation structure is different.

As in the previous chapter, a limitation of this study has to do with the assumption of an *almost all-inclusive* ring and the assumption that the underlying signals are uniformly distributed. A further issue is the fact that the results also rest on the assumption that bidders use symmetric equilibrium strategies where possible, in particular for the Public Knowledge case. Future work will be geared towards relaxing these assumptions.

Chapter 4

Future Research Direction

In Chapter 2, we assumed that the ring members used the *Bid Submission Mechanism* of McAfee and McMillan (1992), where the ring center could enforce submission of recommended bids at the main auction. The aim for future work will be to explicitly characterise how such enforcement can be implemented. We adopt an infinitely-repeated games framework where the ring members enforce compliance by using grim-trigger strategies. In particular, ring members agree to bid according to the ring center's recommendation, and punish defectors by reverting to the noncooperative equilibrium bidding indefinitely.

4.1 Framework

4.1.1 How the Ring Works

The ring contains $N-1$ members. At the beginning of each period, nature assigns each participant a value drawn independently from a uniform $[0,1]$ distribution. Each member reports their value. The bidder with the highest value is chosen as the ring representative, and bids competitively on behalf of the ring at the main auction, while other bidders submit zero. Conditional on winning, proceeds are divided equally among ring members. Given that all ring members report their true values, the ring has to put measures in place that enforce bidders to bid as required. In essence, the ring has to make sure that

- Ring members do not want to break the ring. In particular, either
  * Make sure that ring members are patient enough to not want to deviate by bidding higher than the ring representative at the main auction and keeping the proceeds for themselves, or

- * Make sure that the bid recommended to the ring representative to submit at the main auction is high enough to make deviation unprofitable.

– The ring being *broken* means all participants bid noncooperatively in subsequent auctions.

– We assume that at the end of the main auction, all the submitted bids are observed, and so deviation can be immediately spotted after the auction. In this sense, we are dealing with an infinitely repeated game with perfect monitoring.

4.1.2 The Main Auction

The nonring bidder is unaware that the remaining bidders are operating as a ring, and so submits the noncooperative bid. That is, a nonring bidder with value x_N bids $\beta(x_N) = \frac{N-1}{N}x_N$. Suppose all values in the ring have been truthfully reported to the ring center, and the two highest reported values are θ and $\theta_{-\epsilon}$. The problem of how much the ring representative should bid at the main auction can be approached in two ways.

– *One way is to choose a bid that maximizes the stage profit provided no ring member deviates, and then specify a threshold for the discount factor $\bar{\delta}$ such that no ring member finds it profitable to deviate.*

This involves choosing a $\hat{b}$ such that

$$\hat{b} \equiv \arg\max \underbrace{(\theta - b)}_{\substack{\text{Ring}\\ \text{Profit}\\ \text{upon}\\ \text{winning}}} \cdot \underbrace{\mathbb{P}(\beta(x_N) < b)}_{\substack{\text{Probability}\\ \text{of}\\ \text{winning}}} = \arg\max(\theta - b) \cdot \overbrace{\frac{N}{N-1}b}^{\substack{\mathbb{P}\left(x_N < \frac{N}{N-1}b\right)\\ =\\ \mathbb{P}\left(\frac{N-1}{N}x_N < b\right)}} \implies \hat{b} = \frac{1}{2}\theta \qquad (4.1)$$

With expected stage profit

$$\Pi_{\text{Stage}} = \left(\theta - \frac{1}{2}\theta\right) \cdot \mathbb{P}\left(\beta(x_N) < \frac{1}{2}\theta\right) = \frac{1}{2}\theta \cdot \frac{N}{N-1} \cdot \frac{1}{2}\theta = \frac{N}{4(N-1)}\theta^2$$

Shared among $N-1$ bidders, the expected stage profit from being a member of the ring is

$$\pi_{\text{Stage}} = \frac{1}{N-1} \cdot \Pi_{\text{Stage}} = \frac{N}{4(N-1)^2}\theta^2 \qquad (4.2)$$

Ring members do not deviate if

$$\underbrace{(1-\delta)\frac{N}{4(N-1)^2}\theta^2+\delta\cdot\overbrace{\mathbb{E}_\theta\left[\frac{N}{4(N-1)^2}\theta^2\right]}^{\substack{\frac{N}{4(N-1)(N+1)}\\=\\\frac{N}{4(N-1)^2}\int_0^1\theta^2\cdot(N-1)\theta^{N-2}d\theta}}}_{\text{Expected Lifetime Profit from Sticking to Ring Agreement}}\geq\underbrace{(1-\delta)\frac{N}{4(N-1)}\theta^2+\frac{\delta}{N(N+1)}}_{\text{Maximum Expected Lifetime Profit from Deviation}} \tag{4.3}$$

where $\frac{1}{N(N+1)}$ is the expected profit when bidders revert to noncooperation.
This yields

$$\delta\geq\underbrace{\frac{\left(\frac{N}{N-1}\theta^2\right)}{\left(\frac{N}{N-1}\theta^2+\frac{N-2}{N(N+1)}\right)}}_{\bar{\delta}}$$

with threshold discount factor

$$\hat{\delta}=\frac{\left(\frac{N}{N-1}\theta^2\right)}{\left(\frac{N}{N-1}\theta^2+\frac{N-2}{N(N+1)}\right)}$$

$\bar{\delta}\to 1$ for θ close to 1 and large N. This implies in essence that to maintain such a scheme, ring members have to be extremely patient.

– *Another approach is to consider for any arbitrary discount factor δ, the ring representative's bid ρ such that no party wants to break the ring. This ρ is chosen in the following way.*

We want to chose a ρ such that the second-highest valuing ring member with value $\theta_{-\epsilon}$ very close to θ does not want to deviate from the ring agreement. In essence, ρ is such that

$$\begin{aligned}\max_{\rho}\quad & (\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)\\ \text{subject to}\quad & \end{aligned} \tag{4.4}$$

$$\underbrace{\left(\frac{1}{N-1}\right)(1-\delta)(\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\pi_C\geq(1-\delta)(\theta_{-\epsilon}-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\frac{1}{N(N+1)}}_{\text{Second-highest valuing ring member does not want to break the ring}} \tag{4.5}$$

where π_C is the expected continuation payoff when the ring does not break. Basically, we choose ρ to maximize per-period ring profits such that no ring member wants to deviate.

Therefore, for $\theta_{-\epsilon}\approx\theta$, our task is equivalent to solving the following programming problem

4.1.3 The Programming Problem

$$\max_{\rho} \quad (\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho) \qquad \text{subject to} \tag{4.6}$$

$$\Big(\frac{1}{N-1}\Big)(1-\delta)(\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\pi_C \geq (1-\delta)(\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\frac{1}{N(N+1)} \tag{4.7}$$

The major difficulty relates to how to characterise π_C. Hence we proceed by specifying different possible functional forms for ρ.

4.1.4 Attempting Different Forms

$$\max_{\rho} \quad (\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho) \qquad \text{subject to} \tag{4.8}$$

$$\Big(\frac{1}{N-1}\Big)(1-\delta)(\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\pi_C \geq (1-\delta)(\theta-\rho)\cdot\mathbb{P}(\beta(x_N)<\rho)+\delta\frac{1}{N(N+1)} \tag{4.9}$$

4.1.4.1 Attempt 1 - ($\rho = a\theta$) for some constant a

Using this specification,

$$\mathbb{P}(\beta(x_N)<\rho)=\mathbb{P}(\beta(x_N)<a\theta)=\mathbb{P}(\frac{N-1}{N}x_N<a\theta)=\mathbb{P}(x_N<\frac{N}{N-1}a\theta)=a\frac{N}{N-1}\theta$$

and

$$\pi_C=\frac{1}{N-1}\mathbb{E}_\theta[(\theta-a\theta)\mathbb{P}(\beta(x_N)<\rho)]=\frac{1}{N-1}\mathbb{E}_\theta[\theta(1-a)a\frac{N}{N-1}\theta]=$$
$$\frac{1}{N-1}a(1-a)\frac{N}{N-1}\mathbb{E}_\theta[\theta^2]=a(1-a)\frac{N}{(N-1)(N+1)}$$

the objective function is

$$(\theta-\rho)\mathbb{P}(\beta(x_N)<\rho)=(\theta-a\theta)\mathbb{P}(\frac{N-1}{N}x_N<a\theta)=a(1-a)\frac{N}{N-1}\theta^2$$

The constraint is

$$\Big(\frac{1}{N-1}\Big)(1-\delta)a(1-a)\frac{N}{N-1}\theta^2+\delta a(1-a)\frac{N}{(N-1)(N+1)} \geq (1-\delta)a(1-a)\frac{N}{N-1}\theta^2+\delta\frac{1}{N(N+1)}$$

$$\implies a(1-a)\Big[\underbrace{\frac{N(N-2)(1-\delta)}{(N-1)^2}}_{\psi}\theta^2 - \underbrace{\frac{\delta N}{(N-1)(N+1)}}_{\mu}\Big] + \delta\underbrace{\frac{1}{N(N+1)}}_{\gamma} \leq 0$$

And the programming problem becomes

$$\max_{a} \qquad a(1-a)\frac{N}{N-1}\theta^2 \tag{4.10}$$

subject to

$$a(1-a)(\psi\theta^2-\mu)+\gamma \leq 0 \tag{4.11}$$

The lagrangian is

$$\mathcal{L}(a) = a(1-a)\frac{N}{N-1}\theta^2 - \lambda\Big(a(1-a)[\psi\theta^2-\mu]+\gamma\Big)$$

The optimality conditions are

$$\mathcal{L}'(a) = (1-2a)\Big(\frac{N}{N-1}\theta^2 - \lambda(\psi\theta^2-\mu)\Big) = 0 \tag{4.12}$$

$$\lambda(a(1-a)(\psi\theta^2-\mu)+\gamma) = 0,\ \lambda \geq 0 \tag{4.13}$$

– Case 1: $\lambda = 0$

This implies

$$(1-2a)\frac{N}{N-1}\theta^2 = 0 \implies a = \frac{1}{2}$$

For this be a valid solution, it should satisfy (4.11), $\implies$

$$\frac{1}{4}(\psi\theta^2-\mu)+\gamma \leq 0 \implies (\psi\theta^2-\mu)+4\gamma \leq 0 \implies \theta \leq \Big(\frac{\mu-4\gamma}{\psi}\Big)^{\frac{1}{2}}$$

Substituting the values of μ, γ and δ,

$$\theta \leq \underbrace{\Big(\big(\frac{\delta}{1-\delta}\big)\frac{(N-1)(N-2)}{N^2(N+1)}\Big)^{\frac{1}{2}}}_{\bar{\theta}} \tag{4.14}$$

This solution holds for all valid values of θ provided $\bar{\theta} = 1$. However, this implies

$$\Big(\big(\frac{\delta}{1-\delta}\big)\frac{(N-1)(N-2)}{N^2(N+1)}\Big)^{\frac{1}{2}} = 1 \implies \delta = \underbrace{\frac{N^2(N+1)}{N^2(N+1)+(N-1)(N-2)}}_{\bar{\delta}}$$

This means $a = \frac{1}{2}$ can only be sustained for $\delta \geq \bar{\delta}$. However, we started with the condition that δ be arbitrary. Therefore $a = \frac{1}{2}$ is not valid for all δ, and hence is not a solution.

– Case 2: $\lambda > 0$

This implies

$$a(1-a)(\psi\theta^2 - \mu) + \gamma = 0 \implies a^2 - a - \frac{\gamma}{\psi\theta^2 - \mu} = 0 \implies$$
$$a = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\psi\theta^2 - \mu - 4\gamma}{\psi\theta^2 - \mu}}$$

Giving a as a function of θ. However, we started with the assumption that a was a constant with respect to θ. So the solution here is invalid.

Putting both cases together, we can conclude that the specification $\rho = a\theta$ yields no valid solution.

4.1.4.2 Attempt 2 - (ρ constant)

Under this specification,

$$\mathbb{P}(\beta(x_N) < \rho) = \mathbb{P}(\frac{N-1}{N}x_N < \rho) = \frac{N\rho}{N-1}$$

and

$$\pi_C = \frac{1}{N-1}\mathbb{E}_\theta[(\theta-\rho)\mathbb{P}(\beta(x_N) < \rho)] = \frac{1}{N-1}\mathbb{E}_\theta\Big[(\theta-\rho)\frac{N\rho}{N-1}\Big] =$$
$$= \frac{N\rho}{(N-1)^2}[\mathbb{E}_\theta(\theta) - \rho] = \frac{N\rho}{(N-1)^2}\Big[\frac{N-1}{N} - \rho\Big] = \frac{\rho}{(N-1)^2}[N(1-\rho)-1]$$

With constraint

$$\frac{1}{N-1}(1-\delta)(\theta-\rho)\frac{N\rho}{N-1} + \frac{\delta\rho(N(1-\rho)-1)}{(N-1)^2} \geq (1-\delta)(\theta-\rho)\frac{N\rho}{N-1} + \frac{\delta}{N(N+1)}$$
$$\implies \underbrace{\frac{N(N-2)(1-\delta)}{(N-1)^2}}_{\alpha}\rho(\theta-\rho) - \underbrace{\frac{\delta}{(N-1)^2}}_{\omega}\rho(N(1-\rho)-1) + \underbrace{\frac{\delta}{N(N+1)}}_{\gamma} \leq 0$$

$$\equiv \rho^2(\omega N - \alpha) + \rho(\alpha\theta - \omega N + \omega) + \gamma \leq 0$$

And the programming problem becomes

$$\max_{\rho} \quad (\theta-\rho)\frac{N\rho}{N-1} \tag{4.15}$$

subject to

$$\rho^2(\omega N-\alpha)+\rho(\alpha\theta-\omega N+\omega)+\gamma\leq 0 \tag{4.16}$$

$$\mathcal{L}(\rho)=(\theta-\rho)\frac{N\rho}{N-1}-\lambda(\rho^2(\omega N-\alpha)+\rho(\alpha\theta-\omega N+\omega)+\gamma)$$

Optimality requires

$$\mathcal{L}'(\rho)=(\theta-2\rho)\frac{N}{N-1}-\lambda(2\rho(\omega N-\alpha)+\alpha\theta-\omega N+\omega)=0 \tag{4.17}$$

$$\lambda(\rho^2(\omega N-\alpha)+\rho(\alpha\theta-\omega N+\omega)+\gamma)=0,\ \lambda\geq \tag{4.18}$$

– Case 1: $\lambda=0$

This implies

$$(\theta-2\rho)\frac{N}{N-1}=0 \implies \rho=\frac{\theta}{2}$$

which is not a constant, and hence is not a valid solution.

– Case 2: $\lambda>0$

This implies

$$\rho^2(\omega N-\alpha)+\rho(\alpha\theta-\omega N+\omega)+\gamma=0$$

$$\implies \rho=\frac{(\omega N-\omega-\alpha\theta)\pm\sqrt{(\alpha\theta-\omega N+\omega)^2-4\gamma(\omega N-\alpha)}}{2(\omega N-\alpha)}$$

which again is a function of θ, and not a valid solution.

We can conclude from both cases that the specification (ρ constant) yields no valid solution.

The objective for subsequent research will be to examine more functional forms in the search for a valid solution.

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Appendix

Appendix to Chapter 3

Extension of Equilibrium Strategies to $[0, \omega]$ for any $\omega > 0$

Modifying the framework in section 5, with the values $X_i \in [0, \omega]$ for any $\omega > 0$ we can make the following conclusions

Proposition 23. *In the first price auction, under private knowledge, the equilibrium bidding strategies are*

$$\beta_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{\omega}{2}\right\}$$
$$\beta_R(x_R) = \frac{1}{2}x_R$$

Proof. First, we note that this implies

$$F(x) = \frac{x}{\omega}, f(x) = \frac{1}{\omega}, H(x) = \left(\frac{x}{\omega}\right)^{N-1} \text{ and } h(x) = \frac{N-1}{\omega} \cdot \left(\frac{x}{\omega}\right)^{N-2}$$

The noncooperative equilibrium is therefore

$$\beta(x_i) = x_i - \left(\frac{\omega}{x_i}\right)^{N-1} \int_0^{x_i} \left(\frac{\psi}{\omega}\right)^{N-1} d\psi = \frac{N-1}{N}x_i$$

This implies $\phi(b) = \frac{N}{N-1}b$ and $\phi'(b) = \frac{N}{N-1}$. When the ring believes that they are operating covertly, given her value x_R, the strategy of the ring representative is to submit $\tilde{b}$ that satisfies the differential equation given by Theorem 2.4.2, which we have previously found to be $\tilde{b} \equiv \beta_R(x_R) = \frac{1}{2}x_R$.

Equilibrium Under Private Knowledge

When the ring is privately known to the non-ring bidder his strategy is to best-respond to

$\beta_R(x_R) = \frac{1}{2}x_R$. First of all, this implies that he bids no higher than $\frac{\omega}{2}$, as this is the maximum possible bid for the ring representative, and any $b = \frac{\omega}{2}+\epsilon$, $\epsilon > 0$ is strictly dominated by $b = \frac{\omega}{2}+\frac{\epsilon}{k}$ for any $k > 1$. Secondly, when he submits any $b < \frac{\omega}{2}$, he wins if $\frac{1}{2}x_R < b \iff x_R < 2b$, an this occurs with probability $\left(\frac{2b}{\omega}\right)^{N-1}$. Therefore, he chooses b to maximize

$$(x_N - b)\left(\frac{2b}{\omega}\right)^{N-1} \implies b = \frac{N-1}{N}x_N$$

Therefore, the non-ring bidder's best response is

$$\beta_N(x_N) = \min\left\{\frac{N-1}{N}x_N, \frac{\omega}{2}\right\}$$

□

Private Knowledge With More Than One Nonring Bidder

We relax the *almost all inclusive ring* assumption, and analyze equilibrium strategies as before, under the different informational structures when the underlying distribution is the Uniform $[0,1]$. We first derive the equilibrium bidding strategies under the Noncooperative, Concealed and Private Knowledge cases for an arbitrary number of ring and nonring bidders. Then for a case with 3 ring members and 2 nonring bidders, we analyse the consequences of the various informational structures for auctioneer's expected revenue and bidders' expected profit. We use the simulated values from Marshall et al. (1994) for the public knowledge case. The first difficulty with this case is the fact that compared to the *almost all inclusive* case, in the private knowledge case, it is not immediately clear if the strategy of the nonring bidders is unique. Nonetheless, we derive an equilibrium and show that in general, the conclusions from the one nonring bidder case hold.

Equilibrium Bidding Strategies

Proposition 24. *Assume a private knowledge case with $m > 1$ nonring bidders with values drawn from distribution $F(x) = x$ and the other $N-m$ bidders forming a ring, with the ring representative having value drawn from $F_R(x_R) = x_R^{N-m}$. We can show that one set of equilibrium bidding strategies adopted by the m bidders and bidder the ring representative (for $N > m+1$)*

are respectively

$$\beta_m(x) = \min\left\{\frac{N-1}{N}x, \frac{m}{m+1}\right\},\ \beta_m(0) = 0$$
$$\beta_R(x_R) = \frac{m}{m+1}x_R,\ \beta_R(0) = 0$$

Proof. No bidder who has realized value equal to zero bids higher than zero, as he makes a loss upon winning. Therefore, $\beta_m(0) = \beta_R(0) = 0$. Denote the set of nonring bidders as $\mathcal{M}$.

Ring Representative Strategy

By Theorem 2.4.2, for $x_R \in (0,1]$ the ring bidder's strategy is characterised by $\bar{b}$ that satisfies

$$-\frac{N}{N-1}\tilde{b} + m\frac{N}{N-1}(x_R - \tilde{b}) = 0 \implies \tilde{b} \equiv \tilde{b}(x_R) = \frac{m}{m+1}x_R$$

All Bidders $i \in \mathcal{M}$ Best Responding to $\beta_R(x_R)$

Given that the ring bidder uses $\beta_R(x_R) = \frac{m}{m+1}x_R$, first of all, no bidder $i \in \mathcal{M}$ bids higher than $\frac{m}{m+1}$, as it is the highest possible bid by bidder R, and any other bid $b' = \frac{m}{m+1} + \epsilon$ for $\epsilon > 0$ is strictly dominated by $b'' = \frac{m}{m+1} + \frac{\epsilon}{k}$, $k > 1$. Select an arbitrary bidder $l \in \mathcal{M}$ and suppose that for all bids less than $\frac{m}{m+1}$, all other bidders $j \in \mathcal{M}/\{l\}$ use a linear symmetric strategy $\mathfrak{b}_j(x_j) = \theta x_j$. For any bid $b < \frac{m}{m+1}$ submitted by bidder l, he wins if $\forall j \in \mathcal{M}/\{l\}$

$$\left(\theta x_j < b \text{ and } \frac{m}{m+1}x_R < b\right) \iff \left(x_j < \frac{b}{\theta} \text{ and } x_R < \frac{m+1}{m}b\right),$$

and this happens with probability $\left(\frac{b}{\theta}\right)^{m-1} \cdot \left(\frac{m+1}{m}b\right)^{N-m}$. Therefore he chooses b such that

$$b \equiv \arg\max_{\{b\}}(x_l - b) \cdot \left(\frac{b}{\theta}\right)^{m-1} \cdot \left(\frac{m+1}{m}b\right)^{N-m} = \arg\max_{\{b\}}(x_l - b) \cdot b^{N-1} \implies b = \frac{N-1}{N}x_l$$

The requirement that equilibrium strategies be symmetric implies that $\theta = \frac{N-1}{N}$. Therefore the best response of other bidders $i \in \mathcal{M}$ to $\beta_R(x_R)$ by bidder R is

$$\beta_m(x_i) = \min\left\{\frac{N-1}{N}x_i, \frac{m+1}{m}\right\}$$

□

Using these strategies together with public knowledge estimates from Marshall et al. (1994), I simulated via 1,000,000 draws, expected revenues for $N = 5$, 3 ring bidders and 2 individual nonring bidders. I report the results below

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.6667	0.6104	0.5894	0.6089
Second Price	0.6667	0.6000	0.6000	0.6000

Table 1: Revenue Comparison 3 Ring vs 2 individuals

As in the *almost all inclusive* case, the across all informational structures, the auctioneer is weakly better off under the first price auction than the second price auction, except under private knowledge. Furthermore, in the first price auction, conditional on ring presence, the private knowledge case remains the least-favored informational structure, followed by the public knowledge and the concealed case.

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.0333	0.0385	0.0385	0.0406
Second Price	0.0333	0.0556	0.0556	0.0556

Table 2: Ring Member Profit Profit Across Auctions and Informational Structures

Overall, the ring prefers the second price auction to the first price auction. Under the first price auction, the best informational structure for the ring is under public knowledge, while it is indifferent between the private knowledge and concealed case.

Auction Format	No Ring	Concealed	Private Knowledge	Public Knowledge
First Price	0.0333	0.0474	0.0541	0.0488
Second Price	0.0333	0.0333	0.0333	0.0333

Table 3: Nonring Bidder Profit Across Auctions and Informational Structures

The nonring bidder is weakly better off under the first price auction compared to the second price auction. Under the first price auction, the most preferred informational structure for the nonring bidder is under private knowledge.

In essence, we find the same results as under the all-inclusive ring case. This seems to suggest that our results hold for any arbitrary size of the ring.

On the Distribution of The Product of N iid Uniform $[0, 1]$ Random Variables

Lemma .0.1. *Define* $f_n = \frac{\partial}{\partial x} F_n$*, where*

$$f_n = \frac{(-1)^{n-1}(\log(x))^{n-1}}{(n-1)!}$$

Then it can be shown that

$$F_n = \sum_{j=1}^{n} \frac{(-1)^{j-1} x (\log(x))^{j-1}}{(j-1)!} \tag{19}$$

and

$$F_n \text{ is increasing in } n \tag{20}$$

Proof. We proceed by induction.

The assertion is true for $n = 1$, since $f_1 = 1$ and $F_1 = x$, and indeed $\frac{\partial}{\partial x} F_1 = f_1$. Suppose the conclusion holds for some $k \in \mathbb{N}$. That is, $\frac{\partial}{\partial x} F_k = f_k$. Then by definition,

$$F_{k+1} = \sum_{j=1}^{k+1} \frac{(-1)^{j-1} x (\log(x))^{j-1}}{(j-1)!} = \underbrace{\sum_{j=1}^{n} \frac{(-1)^{j-1} x (\log(x))^{j-1}}{(j-1)!}}_{F_k} + \frac{(-1)^k x (\log(x))^k}{k!}$$

and

$$f_{k+1} = \frac{(-1)^k (\log(x))^k}{k!} = \underbrace{\frac{(-1)^{k-1} (\log(x))^{k-1}}{(k-1)!}}_{f_k} \cdot \frac{(-1)\log(x)}{k}$$

Therefore,

$$F_{k+1} = F_k + \frac{(-1)^k x (\log(x))^k}{k!}$$

So that

$$\frac{\partial}{\partial x} F_{k+1} = f_k + \underbrace{\frac{(-1)^k (\log(x))^k}{k!}}_{f_{k+1}} + \underbrace{\frac{(-1)^k (\log(x))^{k-1}}{(k-1)!}}_{(-1) f_k} = f_{k+1}$$

Therefore

$$F_n = \sum_{j=1}^{n} \frac{(-1)^{j-1} x (\log(x))^{j-1}}{(j-1)!}$$

holds $\forall n$. $\square$

Also, note that

$$F_{k+1} = F_k + \underbrace{\frac{(-1)^k x(\log(x))^k}{k!}}_{\geq 0 \text{ for } x \in (0,1]} \implies F_{k+1} \geq F_k$$