Università degli Studi di Napoli *Federico II*

Dottorato di Ricerca in

FISICA

Ciclo XXXII

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Observers and Momenta in κ-Minkowski space-time

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In un attimo, prima ancora che la sua visione potesse formularsi in pensiero, riconobbe in ciò che vedeva il proprio occhio. [...]
Si era colto nell'atto del vedere: sfuggito alla banalità delle prospettive abituali, aveva guardato dappresso l'organo piccolo ed enorme, vicino benchè estraneo, vivo ma vulnerabile, dotato d'una potenza imperfetta seppur prodigiosa, da cui dipendeva per vedere l'universo. [...]
In un certo senso, l'occhio controbilanciava l'abisso.

Marguerite Yourcenar, L'Opera al Nero

Contents

Notation and Conventions

In this work we adopt the following notation and conventions.

Tensors and Indices

- We use **boldface** to indicate three dimensional vectors $\mathbf{v}$.
- Higher dimensional vectors are simply indicated by a letter v. When the nature of the index need to be specified we write v^a, which means that v^a is a vector whose coordinates are labelled by the index a.
- The above convention is extended to tensors. The position of an index denotes if it is covariant or contra variant in the usual way
- $3-$dimensional indices are denoted with lower case latin letters

$$i, j, k \in \{1, 2, 3\}$$

- $4-$dimensional indices are denoted with lower case greek letters

$$\mu, \nu, \rho \in \{0, 1, 2, 3\}$$

- Higer $d-$dimensional indices are denoted with UPPER case latin letters

$$A, B, C \in \{0, ..., d-1\}$$

Metric and signature

- The adopt the *mostly minus* convention for the metric of four dimensional Minkowski space-time $\eta_{\mu\nu} = \mathbf{diag}(+, -, -, -)$.
- The above rule is extended to the flat metric with any signature in arbitrary dimensions, unless otherwise stated. As an example, for the $5D$ Minkowski space we simply assume $\eta_{AB} = \mathbf{diag}(+, -, -, -, -)$.

Introduction

Beteween a rock and an hard place

Despite the idea of formulating a *theory of everything* is alluring, any physical theory has his own range of attainability. In other words, it is truly predictive only for systems whose physically observable quantities are in a range given by a certain typical scale. Usually one expects different theories, which are set at a different scales, to be compatible with each other in the overlapping range. As an example, Classical Mechanics is compatible with both Quantum Mechanics and Special Relativity in the limit of vanishing Planck constant $\hbar \to 0$ and infinite maximal speed $c \to \infty$ respectively. Quantum Mechanics is instead not compatible with Special Relativity because it lacks of homogeneity in the way time and space are concerned. In order to restore space-time homogeneity, physicists introduced the Quantum Field Theory which is again compatible with Special Relativity.

As a matter of fact, our comprehension of Physics is nowadays split between two extremely predictive, although incompatible, theories: namely the *Standard Model* of particle interactions and *General Relativity*. The Standard Model, developed in various steps along the second half of 20th century, is based on Quantum Filed theory, and gives an accurate description of (almost) any aspect of particle interactions. On the other hand, the theory of General Relativity, formulated by A. Einstein in 1916 [1], describes gravity in terms of the space-time geometry [2]. The incompatibility between these two theories arise from the following fact. The Standard Model is required to be a renormalizable field theory, this allows one to safely cast perturbation theory. On the other hand, the Einstein Hilbert Lagrangian [3] formulation of General Relativity turns out to be non-renormalizable. In other words, the Standard Model is able to trustfully describes all those processes where gravitational interaction between particles is negligible. This fixes the typical scale of the theory, which is expected to lose its predictability for processes with energy of the order of the Planck energy

$$E_P = \sqrt{\frac{\hbar c}{G_N}} \simeq 10^{16} TeV, \tag{1}$$

where G_N is the Newton constant of gravitation. At the energy scale (1) both gravitational and quantum effects must be considered [4, 5].

Presently, no laboratory is able to reproduce process with an energy of the order of the TeV, hence quantum-gravitational interactions are far from being directly observed. Important physical questions, like those related to the early universe and the intrinsic structure of space-time, are supposed to require an unified description of both Quantum Mechanics and General Relativity in order to be answered. The search for a Quantum Gravity theory is without doubt one of the greatest challenge of contemporary physics, and many efforts have been dedicated to the cause in the last decades. Despite many valid theories have been proposed, the problem remains open.

One of the main differences between Quantum Mechanics and General Relativity resides in the way observers are concerned. In the first case the observer is strictly related to the concept of measurement and observable. In the latter it is regarded in a more geometrical taste, since it is strictly related with the choice of a reference frame. An interesting point of view was proposed by S. Majid in 2000 [6] concerning the dualism between observer and observation. It is a mathematical fact that given a function f defined over a set X, to evaluate $f(x)$ on an element $x \in X$ is equivalent to evaluate $x(f)$ on the element $f \in \mathcal{F}(\mathcal{X})$ of the set of function over X. Then for any mathematical concept X can we can define maps (or representations) from X to a class of object representing the outcomes of measurements (usually real or complex numbers). Note that the above *self-dualism* is intrinsic in Quantum Mechanics. In this sense a Quantum Gravity theory would encode a duality between quantum matter and geometry [6]. In particular the Einstein's equations

$$G_{\mu\nu} \propto T_{\mu\nu} \tag{2}$$

should be regarded as a sort of self duality equation: the stress-energy $T_{\mu\nu}$ encodes how matter respond to geometry, just as the Einstein $G_{\mu\nu}$ tensor measures how the geometry responds to matter. However, this self duality makes sense only if one considers a theory of both quantum and gravity [6, 7]. From a mathematical point of view the self duality can be regarded as the unification between a group and his dual; this is properly defined by the concept of Hopf Algebra [8]. This provides the most general category containing both coordinate algebras $\mathbb{C}[\mathcal{G}]$ and the enveloping algebra $U[\mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$ generating $\mathcal{G}$. Although this mathematical structure has been introduced in 1940 [8], it found physical application as *Quantum Groups* only in recent times.

Despite the various proposed Quantum Gravity models are sometimes very different in both the conceptual assumption and technical implementations, they all seems to agree on at least one fact: there must be a minimum limit to the localization of events in the space-time [9, 10]. This can be understood at the level of Heisenberg's uncertainty principle using

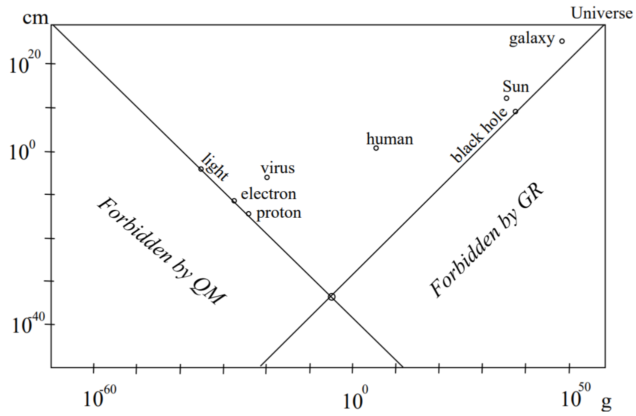


Figure 1: *The graph log plots typical mass-energy versus size of phenomena. The wedge region in the middle represents the range of physically allowed phenomena.* [6]

the following heuristic argument. Suppose to use a probe to measure a very short spatial distance. Due to Heisenberg principle, the uncertainty on the measured distance is lower the larger the energy of the probe is. Moreover, according to the theory of general relativity, the greater the probe mass-energy gets, the more the gravitational field (*i.e.* space-time curvature) is perturbed. This once again increases the measurement uncertainty. Due to these two counterbalancing effects, it is not possible to specify the position of the probe with precision lower then some minimum uncertainty.

In Fig. 1 it is sketched the relation between the typical dimension of an objects and its mass-energy. The region below the straight line on the right represent a regime which is forbidden by General Relativity. On its border there are the *black holes*, which are the objects with the maximal mass-energy density permitted by General Relativity with respect to their volume: increasing their mass will necessarily increase their volume. This can be regarded as a measurement of "how full" a region in space can be. On the other hand, the region below the other line is forbidden by quantum uncertainty. It can be regarded as a measurement of "how empty" a region in space can be [6]. The wedge region between the two forbidden ones represents the range of allowed physical phenomena: we, the humans, dwell there, far away from the two border lines. Notice how these two lines meet each other at the order of the *Planck length* $\lambda_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-35}m$. The above described scenario suggests that in a region of space-time with linear dimension of the order of the Planck length the geometry may appear quite different from the one we are familiar with in Special Relativity. As a consequence, also the way space-time is implemented as a differentiable manifold would change.

In particular, the non-commutative geometry turns out to play an fundamental role in some approaches to quantum gravity: in some of those a non-commutative space-time is directly assumed [11–13]; while in other it manifests itself only at an effective level; this second case is usually encountered in String Theory [9, 14, 15].

From an algebraic point of view, a non-commutative flat space-time can be regarded as a Minkowski space whose coordinates x_μ have a non trivial commutation relation. The most simple example of the above relations is given by homogeneous structure constant

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}, \tag{3}$$

were the $\theta_{\mu\nu}$ do not depend on the point. A space-time with this kind of commutation relation appears in the context of String Theory [15]. More in general, the commutator

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}(x) \tag{4}$$

may change from point to point in space-time.

Above we gave a general overview of some of the arguments motivating us to study non-commutative geometry and Hopf algebras as well as their possible applications in a context where both quantum and gravitational effects are relevant. However, our aim is not to provide a Quantum Gravity model, but it is to investigate the physical consequences that a non-commutative space may produce. The present work is focused on the $\kappa-$Poincaré $\mathcal{P}_\kappa$ quantum group and its homogeneous space-time $\kappa-$Minkowski $\mathcal{M}_\kappa$. This last is space-time whose coordinates close a Lie algebra with the following commutation relation

$$[x_0, x_i] = \frac{\mathrm{i}}{\kappa}x_j, \qquad [x_i, x_j] = 0 \tag{5}$$

where the deformation parameter κ as the dimension of the inverse of a length. The Hopf algebra of $\kappa-$Poincaré has been firstly studied by J. Lukierski and collaborators in 1991 [16, 17] as a deformation of the Poincaré group $ISO(3,1)$. In particular, the $\mathcal{P}_\kappa$ can be regarded as a "quantum symmetry" of the non-commutative $\kappa-$Minkowski space-time [18], in the sense that commutation relation (5) are preserved. A motivation to introduce such a deformation of $ISO(3,1)$ is the following. Since the Planck length λ_p is obtained as a combination of fundamental physical constants only, it is by itself a constant independent from the choice of the observer. It is clear that if λ_p is regarded as a measurable distance, it would be incompatible with the Lorentz transformations. The $\mathcal{P}_\kappa$ deformed symmetries "adjusts" the usual notion of Lorentz covariance in such a way that it is compatible with an invariant minimum length. This is the reason why in some context [12], the parameter κ is assumed to be of the order of the Planck length. Nevertheless, as it has been discussed in [19–21], the Planck length is also compatible with the usual notion of covariance.

Structure of the Thesis

This thesis is composed of three chapters which are organized in the following way.

First, we will propose a brief review of well known concepts in the field of non-commutative geometry and Hopf algebra. In particular we will focus our attention on the notion of duality and self duality, as well as on the most important deformation techniques; namely deformation quantization and twist deformation. These will be cast both for group deformation (Hopf algebras) ans their generators (Lie bialgebras).

In Chapter 2 we will discuss a way to look at the non-commutative coordinates x_μ of the $\kappa-$Minkowski space-time with the tools of the algebras of operators. We will briefly review the notion of states and pure states of a C^*-algebra and give some famous examples. In particular we will discuss how states and pure states represent probability densities and Dirac deltas in Classical Mechanics; while in Quantum Mechanics they represent density matrices and wave functions. In non-commutative geometry, whatever the algebra of coordinates one is dealing with is, states and pure states are strictly related to the notion of localization. The most famous example of this is in fact the Hesienberg uncertainty principle in quantum phase space.

Motivated by this, we associate to the algebra (5) a non-localizability principle of the form (2.34). This tells us that in $\kappa-$Minkowski it is not possible to know with absolute precision position and time of a given "event". We will proceed in analogy with what was done for quantum phase space and we will develop a theory of states and operators for the non-commutative space-time. We will introduce a representation of coordinates x_μ as operators on the Hilbert space $L^2(\mathbb{R}^3)$ whose elements $\psi(x)$ will be interpreted as *localizability functions*. In particular x_0 will be represented as a dilatation operator while x_i will acts multiplicatively. For seek of simplicity, we will consider polar basis (τ, r, θ, ϕ), and focus on the time $\tau = x_0$ and radial $r = \sqrt{x^i x_i}$ coordinates. As a result τ acts as a dilatation while r acts multiplicatively. By solving the eigenvalue problem for τ one gets a basis of improper eigenfunction T_τ for the Hilbert space as in (2.54). These functions are what plane-waves where in quantum mechanics.

However, one is allowed to perform a "turn of the table" and swap between the *time domain* and the *radial domain* and vice versa. In Sec 2.2.4 we will show how this can be performed via a Mellin transform and its anti-transformation as in (2.63) and (2.62). We can say that the Fourier transform in Quantum Mechanics is replaced by Mellin transform in quantum space-time. It is important to notice that the Mellin transform in (2.63) is an isometry, thus it does not change the localizability density $||\psi||^2$. In particular we will assume that the familiar interpretation of self adjoints operators as observables and eigenvalues as possible outcomes of a measurement still holds in this context; thus $||\psi||^2$ will be also regarded as a probability density. We stress that this is an assumption which is useful to discuss our physical results in a familiar way.

An interesting result we will obtain is that the *coherent* states of our models are *log-Gaussian* functions. As we will show in Sec 2.2.5, this allows to fully localize the origin of spatial coordinates at any given time. We will conclude that any observer is able to properly define at least one point in the *fuzzy* space-time which will be the origin of its coordinate system. This results may misleadingly seems to give a special role to a precise point in space, which would be a very unpleasant physical feature. Nevertheless, we will show in Sec 2.3 that this is not the case.

Sec 2.2.6 will be dedicated to develop a representation of the $\kappa-$Poincaré quantum group as operators on Hilbert space $L^2(\mathfrak{so}(3,1) \times \mathbb{R}^3)$. The states corresponds to the fuzzy transformations in the quantum symmetry group of $\kappa-$Minkowski.

In Sec. 2.3.2 we will explain our physical interpretation and the reason why we need the above mentioned representation. Up to Sec 2.3.2 we will never explicitly specify who the *observer* measuring $\hat{x}^0$ and $\hat{x}^i$ would be. Somehow, since the origin is a perfectly localized point at any time, we will implicitly assume that the observer was located there. Then, in order to change the observer one has to use elements of the κ-Poincaré quantum group. Accordingly, it will be impossible to locate the position of the transformed observer, because the translation sector is non-commutative. We will consider the algebra generated by the a's (translations) and Λ's (Lorentz matrices), and associate to a translated and Lorentz transformed observers a state of this algebra. As a result, transformation between different observers will also be *fuzzifyed*. In Sec 2.3.3 we will obtain some interesting properties which are true for every fuzzy-transformed observers. In particular we will obtain that:

- *The state obtained by transforming the origin state $|o\rangle$ via the $\mathcal{P}_\kappa$ state $|g\rangle$ in the representation of the κ-Poincaré algebra a^μ, $\Lambda^\mu{}_\nu$, sis the state which will assign, to any polynomials in the transformed coordinates $x'^\mu = a^\mu \otimes 1 + \Lambda^\mu{}_\nu \otimes x^\nu$, the same expectation value $|g\rangle$ alone would produce on the corresponding polynomial in a^μ.*

- *Whenever the state of the transformation is the identity $|o\rangle_{\mathcal{P}}$, the original observer and the transformed one, will agree on all measurements of time and position.*

- *By translating a state the uncertainty of the coordinates may only increase or remain unchanged; the latter case occurs for identity or pure temporal translations only.*

- *Despite the fact that a pure translation can only increase the variances of x^μ, under particular circumstances, it is still possible for the uncertainties on coordinates to* *decrease for a generic $\kappa-$Poincaré transformation.*

It has to be noted that we consider a regime which is not very natural in physics, namely we consider the effects of a quantum space-time for which the non-commutativity parameter of space, κ^{-1} is non zero, while $\hbar$ can be ignored. Thus, the whole model we will develop will be purely kinematic. A possible way to introduce a dynamics would be to restore the quanta of action. However, bringing $\hbar$ back into the picture would require us to consider momenta (either in the form of wave modes in a field-theoretical setting, or as quantity of motion of particles).

In Chapter 3 we will study the momentum space dual paired to $\kappa-$Minkowski coordinates. It has been widely discuss how in general the momentum space must be curved whenever the coordinate space is non-commutative and vice versa. In Sec 3.1.1 we will show this can be physically understood in terms of plane waves over $\kappa-$Minkowski.

We will use the following method to deduce the geometry of the curved momentum space. It is a fact that the five-dimensional Lorentz algebra $\mathfrak{so}(4,1)$ has a subalgebra which is isomorphic to the algebra (2.11). Thus the x_μ coordinates can be represented in terms of five dimensional matrices $\rho(x_\mu)$ as in (3.7); this also induce by exponentiation a representation of the group elements $G^*(p_\mu) = e^{\mathrm{i}p_i\rho(x^i)}e^{\mathrm{i}p_0\rho(x^0)}$ as in (3.9). The geometry of the momentum space can be deduced from the group orbits. If one considers the 5D Minkowski space $\mathcal{M}^5$ as ambient space, then given a fiducial vector $u \in \mathcal{M}^5$c the orbit coincide with the locus of points obtained by acting with $G^*(p_\mu)$ upon u^A for all choices of p^μ. The coordinates induced over such a manifold are $X^a = G^*(p_\mu)^A_B u^A$ and the corresponding induced metric is (3.13). This defines a submanifold in $\mathcal{M}^5$ which is diffeomorphic to the group manifold, which is the curved momentum space.

Since the orbits of the Lorentz group are disconnected, the choice of the fiducial vector u^A is not inconsequential. Indeed, we cannot transform a space-like fiducial vector into a time-like or light-like one with a Lorentz transformation. Hence, different values for the group Casimir $X^A(p)X^B(p)\eta_{AB}$ correspond to different orbits. As a consequence, we will classify the non-degenerate orbits in three families depending whether $X^A(p)X^B(p)\eta_{AB}$ is positive, negative, or null. With each one of these three possibilities we will associate a different (inequivalent) geometry of the momentum space.

In Sec 3.1.2 we will show that there are three classes of possible embedded submanifolds in $\mathcal{M}^5$ which are all diffeomorphic to group manifold of AN_3 [22, 23]. In addition to these three families of equivalent four-dimensional momentum spaces, there is also a family of degenerate cases. For the space-like choice , we will reproduce the result in [18] (the patch of de Sitter space that is covered by comoving coordinates). For a light-like fiducial vector, we will obtain a future-oriented light cone of the ambient Minkowski space (the limit of vanishing cosmological constant of the above case). For a time-like fiducial vector, we will have one of the two sheets of a Riemannian hyperbolic space, *i.e.* the positive-frequency mass-shell of a massive particle. These three manifolds are diffeomorphic to each other, and have the same topology as that of a plane. This is to be expected, because they are all diffemorphic to the group manifold of of AN_3. In Table 3.1 we will illustrate these results.

In Sec 3.1.3 we will repeat the above construction with a choosing representation (3.7) of $\mathfrak{an}_3$ in terms of 5×5 matrices. This corresponds to embed $\mathfrak{an}_3$ into $\mathfrak{so}(3,2)$ instead of $\mathfrak{so}(4,1)$. However, this choice will have some consequences on the corresponding momentum spaces. Again, we will search for the subgroups of $SO(3,2)$ that stabilize u^A. In the $u^A u^B \eta'_{AB} < 0$ case, we will have the Lorentz group $SO(3,1)$, as in [24]. For u^A light-like, the subgroup will be $ISO(2,1)$, *i.e.* the Poincaré group in 2+1 dimensions. Finally, in the $u^A u^B \eta'_{AB} > 0$ case, the group will be $SO(2,2)$. The obtained geometries are reported in Table 3.2

In Sec 3.2 we will study the symmetry group of the obtained momentum spaces. Since each one of those id associated to a families of orbits $(AN_3)\,u$, the symmetries of a momentum space coincide with the symmetries of the corresponding orbit. This can be constructed as the Inönü Wigner group contraction of the global symmetry group of the embedding space with respect to the subgroup which stabilizes the fiducial vector u (little group) as shown in Sec 3.2.1. In particular, starting from the $SO(4,1)$ group one finds that the symmetry group of the momentum in space-, light-, and time-like families will be generated by $\mathfrak{iso}(3,1)$, $\mathfrak{carr}(3,1)$ and $\mathfrak{iso}(4)$ Lie algebras respectively. On the other hands, for the space-, light-, and time-like families obtained from $SO(3,2)$ the symmetries will be generated by $\mathfrak{iso}(2,2)$, $\mathfrak{carr}(2,2)$ and $\mathfrak{iso}(3,1)$. This results will be reported in Table 3.2.2.

In Sec. 3.3 we will show that any of the above obtained symmetries can be made into a Lie bialgebra dual to the the algebra of $\kappa-$Minkowski in the sense of Sec. 1.4.3. We will consider $ISO(p,4-p)$: the group of isometries of a flat space with metric $g_{\mu\nu}$ of arbitrary signature; this also include degenerate metrics. In (3.71) we will find the most general r-matrix which both satisfies the *modified classical Yang Baxter equation* $(m-cYBE)$ and whose translation sector has a co-bracket as in (3.69). In particular, this last requirement will ensure that the resulting Lie bialgebra will be dual to $\kappa-$Minkowski. We will obtain that the introduced $r-$matrix is compatible with any signature of the metric. In other words, the algebra of isometries $\mathfrak{iso}(p,4-p)$ admits a quantum deformation which is dual the κ-Minkowski whatever the signature of the metric $g_{\mu\nu}$ would be.

This can be interpreted in the following way: each of the Quantum Group(s) (3.72) generated by the above described Lie bialgebras satisfy $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}$, and $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} = g_{\mu\nu}$ for any choice of matrices $g^{\mu\nu}$ and $g_{\rho\sigma}$. Furthermore the the left co-action $\Phi_{\Lambda,a}[x^\mu] =$

$\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes \mathbb{1}$ is a homomorphism for (2.11) *i.e.* it leaves $\kappa-$Minkowski space unchanged. In this sense, equations (3.74) and (3.72) are in fact a generalization of equations (2.82) and (2.84), which will be introduced in Sec. 2.2.6.

We conclude that there are momentum spaces associated to the $\kappa-$Minkowski non commutative space-time with all possible (degenerate or not) signatures. This is compatible with the results of Sec. 3.2.2.

The last chapter will be dedicated to our conclusions and outlook.

Chapter 1

Non Commutative Geometry

An Intuitive Picture

At the very beginning of our education we learned how to visualize abstract geometrical entities as concrete objects. As an example, we visualized points, lines and planes as "dots","straight lines" or "sheets" located somewhere. Later, we understood that such a realization is far from being exhaustive, and we developed a more precise mathematical formalism to describe geometrical concepts in a rigorous way. Nevertheless, the intuitive concept of "localization" survives trough such an update. Many physical evidences (like the quantization of phase space) suggests that the notion of perfectly localized geometrical entities has to be revised. Ironically, the same geometrical formalism turns out to suggests a possible generalization of itself.

Before developing such a generalization in a precise algebraic way, we prefer to take a step back and give a more intuitive description. Imagine to blur the above introduced "dots","lines" and "sheets". As a consequence one is not able to tell exactly any more where the original dot, line and surface was. One would ask then whether the information once encoded as geometrical entities has been completely erased or not. The answer to this question lies in the way the blurring procedure has been performed. A primitive way to think of *non-commutative geometry* (NCG) is that of a geometry where usual basic geometrical entities are replaced by blurred ones, thus instead of the exact coordinates of a point one has to think about some kind of localizability distribution. This is not so far from what one usually do when introducing probability density in quantum phase space.

Nevertheless, this fashionable pictures does not correspond to a general method to construct NCG. The concept of non-commutative geometry is in fact well defined on his own and does not necessarily need to relay on any pre-existing geometrical notions.

In this chapter we provide a general review about the tools provided by *non-commutative* *geometry* and their possible implementation.

1.1 Basic Concepts

Before commencing a precise treatment of non commutativity it is convenient to recap some of the basic concepts we will need later.

1.1.1 Covariance and Contravariance

Covariance and contravariance are key concept for almost any theoretical physics investigation. We start with a finite dimensional vector space V. It is well known that:

$$\{\dim(V) < \infty\} \;\Rightarrow\; \{\dim(V) = \dim(V^*)\}\,,$$

where V^* is the dual space of V. Thus V and V^* are isomorphic to each other. However, any isomorphism between V and V^* will depend on the choice of the basis. Although V and V^* are isomorphic to each other, there are no canonical isomorphism between them. On the other hand, there always exists a canonical isomorphism between $(V^*)^*$ and V. We have that $V^* = Hom(V, \mathbb{R})$ is the space of the homomorphisms from V to the real line $\mathbb{R}$.

Given a basis $\mathcal{B}_V = \{e_a\}$ of V it is always possible to define the canonical dual basis $\mathcal{B}_V^* = \{e^a\}$ with following rule

$$e^b(e_a) = \delta_a^b. \tag{1.1}$$

Thus, given a vector $x \in V$ and a covector $f \in V^*$ we have the following decomposition:

$$V \ni x = x^a e_a, \quad a \in \{1, ..., \dim(V)\}, \tag{1.2}$$

$$V^* \ni f = f_b e^b, \quad a \in \{1, ..., \dim(V^*))\}, \tag{1.3}$$

where we used

$$f_a := f(e_a) \tag{1.4}$$

Although we are able to find a canonical isomorphism between bases, the same is not true for components. Indeed, when one perform a transformation $S \in GL(\dim(V))$ on the basis $\{e_a\}$ one has

$$e_{a'} = S_{a'}{}^a e_a, \tag{1.5}$$

and using (1.1) one gets

$$e^{a'} = S^{a'}_{\ a} e^{a'}. \tag{1.6}$$

Those transformation are inverse one to each other

$$S_{a'}{}^b S^a_{\ b} = \delta^a_{a'}. \tag{1.7}$$

We say that $\{e_a\}$ is *covariant* and $\{e^a\}$ is *contravariant*. On the other hand, the components transforms in the opposite way

$$x^{a'} = S^{a'}_{\ a} x^a \quad (\textit{multiplication on the right}), \tag{1.8}$$

$$f_{a'} = S_{a'}{}^{a} f_a \quad (\textit{multiplication on the left}). \tag{1.9}$$

Let us introduce the following bilinear map

$$\langle \cdot, \cdot \rangle : V \times V^* \ni (x, f) \to \langle f, x \rangle = f(x) \in \mathcal{K}, \quad \mathcal{K} \in \{\mathbb{R}, \mathbb{C}\} \tag{1.10}$$

Notice that the above definition states that, given $x \in V$ is possible to associate to any $f \in V^*$ a value $x(f) := \langle f, x \rangle \in \mathcal{K}$. Thus, to any element $x \in V$ it is associated one in $(V^*)^*$, which is a functional over V^*.

The above construction can be generalized to the infinite dimensional case. However, one has to pay attention to some subtleties. Considers an infinite dimensional vector space V whose basis $\{e_\alpha\}$ is such that $\alpha \in \mathcal{I}$ with $\mathcal{I}$ an infinite set. Although we are still able to define the dual element of each e_α the corresponding e^α as in (1.1), what we get is not a basis of the dual space V^*. Indeed, the algebric dual V^* is in general larger than V. Nevertheless, if one introduces a topology $\mathcal{T}$ over V (*i.e.* V is a *topological space* then one has the notion of *topological continuous dual* space V' *i.e.* the space of continuous linear functional over V. It follows that V' is a linear subspace of V^*.

In particular, for any topological finite dimensional vector space the topological dual coincide with the algebric dual, the same is not true for infinite dimensional spaces. In the infinite dimensional case one usually requires that V is a *reflexive* space, *i.e.* that $(V')' = V$. However, the same vector space V can give rise to both reflexive and not reflexive topological spaces depending on the choice of the topology $\mathcal{T}$. In most of the concrete cases a suitable topological dual space is achieved by introducing on V the *strong* or the *weak* topology. For a more complete discussion about the topics of this section see [25–28].

1.1.2 Algebra

Consider a vector space *i.e.* a triple $(\mathcal{A}, +, \mathcal{K})$ such that $(\mathcal{A}, +)$ is an abelian group such that the action (*scalar multiplication*) of $\mathcal{K}$ over $\mathcal{A}$ is compatible with the internal operation $+$ of the group.

Definition 1.1.1. *A quintuple* $(\mathcal{A}, m, 1_{\mathcal{A}}, +, \mathcal{K})$ *is said to be an unital associative* ***algebra*** *if*

- $(A, +, \mathcal{K})$ *is a vector space*

- *$(A, m, +)$ is a ring*
- *multiplication m is compatible with scalar multiplication:*

$$\lambda\ m(\mathbf{a}, \mathbf{b}) = m(\lambda\mathbf{a}, \mathbf{b}) = m(\mathbf{a}, \lambda\mathbf{b}),\ \forall \mathbf{a}, \mathbf{b} \in \mathcal{A}, \qquad \forall \lambda \in \mathcal{K} \tag{1.11}$$

The multiplication m extends to a map $m(.,.) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ whose source is $\mathcal{A} \otimes \mathcal{A}$ and whose target is $\mathcal{A}$. Thus, its graph implementation is straightforward

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{\ m\ } \mathcal{A}. \tag{1.12}$$

Furthermore, the unit element $1_\mathcal{A}$ also defines the **identity** map from k to $\mathcal{A}$

$$\mathcal{K} \xrightarrow{\ 1_\mathcal{A}\ } \mathcal{A}. \tag{1.13}$$

such that given $\alpha \in \mathcal{K}$ the map returns $\alpha 1_\mathcal{A} \in \mathcal{A}$. The triple $(A, m, +)$ is also a **ring**, which means that $(\mathcal{A}, +)$ is an abelian group, and that the m is associative and distributive with respect to $+$. The associativity of m is expressed by the following equality

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m), \tag{1.14}$$

where the identification map **id** has not to be confused with the identity map $1_\mathcal{A}$ introduced in (1.13). The property (1.14) can be encoded in a graph as:

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A} \end{tikzcd} \quad , \tag{1.15}$$

which is required to **commute**.

Definition 1.1.2. *Consider two nodes $\mathcal{A}$, $\mathcal{B}$ in a diagram. A diagram is said to* ***commute*** *if for any element $a \in A$ the element $b \in B$ obtained following a path on the diagram is independent on the choice of the path.*

Remark. The associativity condition (1.14) is equivalent to require graph (1.15) to commute.

1.1.3 Algebra from a generic set

Consider a generic set X, *i.e.* a generic element of the *set category* $X \in$ "Sets". A *functor* is a map between different categories. Given X we consider the space $Lin(X)$ of all possible linear combination of the elements of $X = \{x_1, ..., x_i, ...\}$. Notice that $Lin(X)$ does not coincide with X itself. Indeed, the map which links X to $Lin(X)$ is a functor Φ from the category of sets to the category of linear spaces:

$$\Phi : \text{“Sets”} \ni X \longrightarrow \Phi(X) = Lin(X) \in \text{“Linear Spaces”} \tag{1.16}$$

We want to generate the dual space $Lin(X)^*$. This must be generated by $\mathcal{F}(X \to \mathcal{K})$ *i.e.* the functions from set X to $\mathcal{K}$. Indeed, given an element $\Sigma\alpha^i x_i \in Lin(X)$ and a function $f \in \mathcal{F}(X)$ we obtain a well defined linear combination

$$f(\Sigma\alpha^i x_i) = \Sigma\alpha^i f(x_i). \tag{1.17}$$

In this way, we are able to *dualize* vector spaces of any dimension.

Consider now $Lin(X)$ and $Lin(Y)$ generated by sets X and Y respectively. Suppose we have a map $\varphi : X \to Y$. We want to generalize this to a $\hat{\varphi} : Lin(X) \to Lin(Y)$, this is called the *extension* of φ

$$\hat{\varphi}\left(\Sigma\alpha^i x_i\right) = \Sigma\alpha^i \varphi(x_i). \tag{1.18}$$

To simplify notation the hat will be omitted. We want also to find a similar relation $\hat{\varphi}$ between duals. Thus we define:

$$\tilde{\varphi}(f_Y)[x] := \varphi\left(f_X(x)\right) \tag{1.19}$$

where $f_X \in \mathcal{F}(X)$ and $f_Y \in \mathcal{F}(Y)$. We say that $\tilde{\varphi}$ “reverses the arrows” of φ.

$$\begin{array}{ccc} Lin(X) & & \mathcal{F}(X) \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle\tilde{\varphi}} \\ Lin(Y) & & \mathcal{F}(Y) \end{array}$$

1.2 Hopf algebras and Quantum Groups

1.2.1 Bialgebras

In the last section we introduced the tool of dualization to build new structure from an existing one. In this way, the co-algebra is the structure dual to an algebra, *i.e.* the structure obtained by inverting any arrows in the graphs of an algebra. In this way one introduce the **co-product** just as the dual of (1.12):

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C}. \tag{1.20}$$

We also introduce a **co-unit** map **co-unit** $\varepsilon : \mathcal{C} \to \mathcal{K}$ map

$$\forall \mathbf{c} \in \mathcal{C} \quad \varepsilon(\mathbf{c}) = 1_{\mathcal{K}}, \in \mathcal{K} \tag{1.21}$$

which is dual to the identity unit map introduced in (1.13).

co-algebra

Definition 1.2.1. *Given a vector space $\mathcal{C}$, the quintuple $(\mathcal{C}, \Delta, \varepsilon, +, k)$ is said to be a co-unital coassociative* ***co-algebra*** *if*

- *Δ is compatible with scalar multiplication.*
- *Δ is co-associative*

$$(\Delta \otimes \mathrm{id}_{\mathcal{C}}) \circ \Delta = (\mathrm{id}_{\mathcal{C}} \otimes \Delta) \circ \Delta \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \mathcal{C} \otimes \mathcal{C} \arrow[d, "\Delta \otimes \mathrm{id}_{\mathcal{C}}"] \\ \mathcal{C} \otimes \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes \Delta"] & \mathcal{C} \end{tikzcd} \tag{1.22}$$

  which is to say that (1.22) has to commute.

- *ε is such that the following graph commute*

$$(\varepsilon \otimes \mathrm{id}_{\mathcal{C}}) \circ \Delta = \mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes \varepsilon) \circ \Delta \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \mathcal{C} \otimes \mathcal{C} \arrow[d, "\varepsilon \otimes \mathrm{id}_{\mathcal{C}}"] \\ \mathcal{C} \otimes \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes \varepsilon"] & \mathcal{K} \otimes \mathcal{C} = \mathcal{C} = \mathcal{C} \otimes \mathcal{K} \end{tikzcd} \tag{1.23}$$

  where we ientfy in the last step $k \otimes \mathcal{C}$ with $\mathcal{C}$ in the obvious way.

There are two notable types of co-products

Definition 1.2.2. *Given an element $c \in \mathcal{C}$, we say it to be a* ***group-like*** *element if*

$$\Delta c = c \otimes c, \tag{1.24}$$

Furthermore, if the above statement is true $\forall c \in \mathcal{C}$ we say that $\mathcal{C}$ ***group-like co-product****.*

Definition 1.2.3. *Given an element $c \in \mathcal{C}$, we say it to be a* ***primitive*** *element if*

$$\Delta c = c \otimes c, \tag{1.25}$$

Furthermore, if the above statement is true $\forall c \in \mathcal{C}$ we say that $\mathcal{C}$ has a ***primitive co-product****.*

The co-product Δ maps an element $c \in \mathcal{C}$ into $\Delta(c) \in \mathcal{C} \otimes \mathcal{C}$ which is not in general one single tensor product between two elements of $\mathcal{C}$ but a sum of many of those. The notation introduced by M. Sweedler [29–31] allows one to write the output of a co-product in a compact way

Sweedler notation

Definition 1.2.4. *We represent co-product $\Delta(c) \in \mathcal{C} \otimes \mathcal{C}$ as*

$$\Delta(c) = \sum_j c^j_{(1)} \otimes c^j_{(2)} \tag{1.26}$$

where j is the index of summation and the lower index (n) specify if the element $c^i_{(n)} \in \mathcal{C}$ has been taken from the first or the second leg of tensor product $\mathcal{C} \otimes \mathcal{C}$. Usually, this notation is further contracted as $\Delta(c) = c_{(1)} \otimes c_{(2)}$, this turns out to be extremely useful when one has to deal with multiple co-product.

Notice that even if the structure of algebra has been built by dualization this does not mean that any algebra is also a co-algebra and vice versa (usually they are not). However, objects which has both an algebra and a co-algebra compatible structures are of particular interest in physical applications.

Bialgebra

Definition 1.2.5. *A structure $(\mathcal{B}, m, 1_{\mathcal{B}}, \Delta, \varepsilon, \mathcal{K})$ is said to be a **bialgebra** if*

- $(\mathcal{B}, m, 1_{\mathcal{B}}, \mathcal{K})$ *is an algebra.*
- $(\mathcal{B}, \Delta, \varepsilon, \mathcal{K})$ *is a co-algebra.*
- *The morphisms $m, 1_{\mathcal{B},\Delta}$ and ε are compatible; which is to say:*

$$\begin{aligned} &\Delta \circ m = \\ &= m \otimes m \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ \Delta \otimes \Delta, \\ &\quad \text{where } \tau(a \otimes b) = b \otimes a, \quad \forall a, b \in \mathcal{B} \end{aligned} \qquad \begin{tikzcd} \mathcal{B} \otimes \mathcal{B} \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"] & \mathcal{B} \arrow[r, "\Delta"] & \mathcal{B} \otimes \mathcal{B} \\ \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[u, "m \otimes m"'] \end{tikzcd} \tag{1.27}$$

$$\varepsilon \circ m = \varepsilon \otimes \varepsilon \qquad \begin{tikzcd} \mathcal{B} \otimes \mathcal{B} \arrow[rr, "m"] \arrow[dr, "\varepsilon \otimes \varepsilon"] & & \mathcal{B} \arrow[dl, "\varepsilon"] \\ & \mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K} & \end{tikzcd} \tag{1.28}$$

$$\Delta \circ 1_{\mathcal{B}} = 1_{\mathcal{B}} \otimes 1_{\mathcal{B}} \qquad \begin{tikzcd} & \mathcal{K} \otimes \mathcal{K} = \mathcal{K} \arrow[dl, "1_{\mathcal{B}} \otimes 1_{\mathcal{B}}"'] \arrow[dr, "1_{\mathcal{B}}"] & \\ \mathcal{B} \otimes \mathcal{B} & & \mathcal{B} \arrow[ll, "\Delta"] \end{tikzcd} \tag{1.29}$$

$$\varepsilon \circ m = \varepsilon \otimes \varepsilon \qquad \begin{tikzcd} \mathcal{B} \otimes \mathcal{B} \arrow[rr, "m"] \arrow[dr, "\varepsilon \otimes \varepsilon"] & & \mathcal{B} \arrow[dl, "\varepsilon"] \\ & \mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K} & \end{tikzcd} \tag{1.30}$$

$$\mathrm{id}_{\mathcal{B}} = \varepsilon \circ 1_{\mathcal{B}} \qquad \begin{tikzcd} \mathcal{K} \arrow[dd, "\mathrm{id}_{\mathcal{B}}"] \arrow[dr, "1_{\mathcal{B}}"] & \\ & \mathcal{B} \arrow[dl, "\varepsilon"] \\ \mathcal{K} & \end{tikzcd} \tag{1.31}$$

where all the above graphs have to commute.

In other words, a bialgebra is both an algebra and a co-algebra whose product, unit, co-product and co-unit are compatible with each other.

1.2.2 Hopf algebras

It is quite easy to see that any a group $\mathcal{G}$ can be made into a bialgebra by setting $\Delta g = g \otimes g$ and $\varepsilon(g) = 1$. In particular, a bialgebra with such a co-product is said to be group-like. However, not all the group-like bialgebra are groups because in general a concept of inversion is not provided. Nevertheless, the concept of inversion itself needs to be generalized in order to achieve compatibility with both algebra and co-algebra structures.

Hopf Algebra

Definition 1.2.6. *An Hopf Algebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \varepsilon, S, \mathcal{K})$ is a bialgebra endowed with a linear* ***antipode*** *map $S : \mathcal{H} \to \mathcal{H}$ such that*

$$\begin{tikzcd} \mathcal{H} \otimes \mathcal{H} \arrow[d, "S\otimes\mathrm{id}"'] & \mathcal{H} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "1_{\mathcal{H}}\circ\varepsilon"] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "\mathrm{id}\otimes S"] \\ \mathcal{H} \otimes \mathcal{H} \arrow[r, "m"] & \mathcal{H} & \mathcal{H} \otimes \mathcal{H} \arrow[l, "m"'] \end{tikzcd} \tag{1.32}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = 1_{\mathcal{H}} \circ \varepsilon$$

where the above graph has to commute.

In Sweedler notation we have that

$$S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \varepsilon(a)1_{\mathcal{H}}, \quad \forall a \in \mathcal{H}. \tag{1.33}$$

It is possible to show that given a Hopf algebra the antipode is unique. Furthermore, we have:

$$S \circ m = m \circ (S \otimes S), \tag{1.34}$$

$$(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S, \tag{1.35}$$

$$S(1) = 1, \tag{1.36}$$

with $\tau(a \otimes b) = b \otimes a$. Not surprisingly, the usual group inversion $S(g) = g^{-1}$ satisfy all the above requirements, hence it is the antipode for any group viewed as a group-like Hopf algebra. In this sense one states that the antipode generalize the concept of inversion from groups to Hopf algebras.

It is of strong physical (and mathematical) interest to define the action of an algebra on some other structure.

Left action of an Algebra onto another algebra

Definition 1.2.7. *Given an algebra $\mathcal{H}$, we define its* ***left action*** *(or* ***representation)***[1] $\triangleright$ *on another algebra $\mathcal{A}$ any linear map $\alpha : \mathcal{H} \otimes \mathcal{A} \ni h \otimes a \to \alpha(h \otimes a) := h \triangleright a \in \mathcal{A}$ such that*

$$\alpha(m(h \otimes g) \otimes a) = \alpha(h \otimes \alpha(g \otimes a)) \qquad \begin{tikzcd} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{A} \arrow[dl, "m\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\alpha"] & \\ \mathcal{H} \otimes \mathcal{A} \arrow[dr, "\alpha"'] & & \mathcal{H} \otimes \mathcal{A} \arrow[dl, "\alpha"] \\ & \mathcal{A} & \end{tikzcd} \tag{1.37}$$

$$\alpha(h \otimes a) = \varepsilon(h)a \qquad \begin{tikzcd} & \mathcal{H} \otimes \mathcal{A} \arrow[dl, "\varepsilon\otimes\mathrm{id}"'] \arrow[dr, "\alpha"] & \\ \mathcal{K} \otimes \mathcal{A} \arrow[rr, phantom, "\cong"] & & \mathcal{A} \end{tikzcd} \tag{1.38}$$

hold $\forall h, g \in \mathcal{H}, \forall a \in \mathcal{A}$ and the above graphs are commutative.

$$\alpha(m(h \otimes g) \otimes a) = \alpha(h \otimes \alpha(g \otimes a)) \qquad \forall h, g \in \mathcal{H}, \forall a \in \mathcal{A} \tag{1.39}$$

and

$$\alpha(h \otimes a) = \varepsilon(h)a \qquad \forall h \in \mathcal{H}. \tag{1.40}$$

The above statement can be rewritten in terms of left action symbol $\triangleright$ as follows:

$$m(h \otimes g) \triangleright a = h \triangleright (g \triangleright a), \tag{1.41}$$

$$h \triangleright 1_{\mathcal{A}} = \varepsilon(h) 1_{\mathcal{A}}. \tag{1.42}$$

However, such an action does not usually preserve the structures of the object on which it is applied. Not surprisingly, this is something one would avoid in most of the physical application. It follows that one has usually to introduce the concept of *covariant action.*

[1]The terms **action** and **representation** are equivalent and will both be used in this thesis.

Definition 1.2.8. *An Hopf algebra $\mathcal{H}$ is said to* ***act covariantly*** *on an algebra $(\mathcal{A}, \cdot, 1, \mathcal{K})$ if*

$$\forall h \in \mathcal{H}, \qquad h \triangleright a \cdot b = (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright a), \qquad \forall a, b \in \mathcal{A}. \; . \tag{1.43}$$

In this case, we say that $\mathcal{A}$ is an H-module algebra.

Hopf algebras may also be represented on co-algebras. In this case, we say C to be a $\mathcal{H}$-module co-algebra if

$$\Delta(h \triangleright c) = (\Delta h) \triangleright \Delta c = \sum (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)}), \qquad \forall h \in \mathcal{H}, \; \forall c \in \mathcal{C}. \tag{1.44}$$

We also define the action of an Hopf algebra $\mathcal{H}$ on a generic structure. In this case, we replace $\mathcal{A}$ in any instance with the structure one wish to act on. In the same way, we generalize the concept of H-module and representation.

Now that the mathematical notion of Hopf algebra has been rigorously introduced it is a good time to look at it with some physical intuition. As already stated above, the co-product map defines a rule to link any object from a single copy of $\mathcal{H}$ to one in $\mathcal{H} \otimes \mathcal{H}$, which is made by two copies of the same algebra. These two copies can be represented independently *i.e.* each leg of the output of a co-product may act on a different structure. As an example, consider to have the Poincaré group $\mathcal{P}$ in 4 dimension. It is well known that irreducible representation of $\mathcal{P}$ can be classified by eigenvalues of its central elements. In physics, these representations are associated to particles with different mass and different spin [4, 32]. Of course, when a physical system undergoes a Poincaré transformation this does not affect the reference frame only, but also any internal degrees of freedom associated to particles (spin). In other words, we have one single element $g \in \mathcal{P}$ that has to be represented on different vector spaces in a compatible way. As an example, consider a system composed by a subsystem of spin 1/2 and another of spin 1, which undergoes a finite rotation $R_z(\theta) = \exp(J_z\theta)$ of an angle θ around the z axis. As we know from elementary courses in Quantum Mechanics, in order to correctly transform a state one needs to apply the tensor product $R_z^{s=1/2}(\theta) \otimes R_z^{s=1}(\theta)$ of two suitable representations of the same element of the group. This last procedure works because the rotation sector $SO(3)$ is a subgroup and its co-product has to be group-like: $\Delta R(\theta) = R(\theta) \otimes R(\theta) \in SO(3) \otimes SO(3)$. In other words, the above mentioned transformation rule on the state is obtained representing each one of the two $SO(3)$ on the correct inner space. On the other hand, if one consider an infinitesimal rotation then one has to deal with the Lie algebra $\mathfrak{so}(3)$. In this case, the above mentioned state transforms with $J_z \otimes 1 + 1 \otimes J_z$ (which is obtained by expanding $R(\theta) \otimes R(\theta)$ up to first order in θ). Not surprisingly, it turns out that a Lie algebra $\mathcal{L}$ can be made into co-algebra by introducing a primitive co-product.

1.2.3 Algebra, co-algebra and dualization

All the above discussion revolves around the same basic notion: *algebras and co-algebras are dual to each other*. Indeed we introduced co-algebras by dualizing all the structure that appeared in the definition of algebra. However, the correspondence between these two structures goes beyond just this, and can be precisely described with the notions introduced in Sec. 1.1.1 and Sec. 1.1.3.

Consider a co-algebra $(\mathcal{C}, \Delta, \varepsilon, \mathcal{K})$. Since $\mathcal{C}$ is a vector space over $\mathcal{K}$, it is possible to introduce its adjoint as $\mathcal{C}^* = Lin(\mathcal{C})$ as well as an inner product map $\langle \cdot, \cdot \rangle : \mathcal{C}^* \otimes \mathcal{C} \to \mathcal{K}$ as in (1.10). Then, we define the map $m_{\mathcal{C}^*} : \mathcal{C}^* \otimes \mathcal{C}^* \to \mathcal{C}^*$ such that

$$\langle m_{\mathcal{C}^*}(\phi, \psi), c \rangle := \langle \phi \otimes \psi, \Delta c \rangle, \qquad \forall \phi, \psi \in \mathcal{C}^*, \ \forall c \in \mathcal{C}, \tag{1.45}$$

$$\langle 1_{\mathcal{C}^*}, c \rangle := \varepsilon(c), \qquad \forall c \in \mathcal{C}. \tag{1.46}$$

We say that $m_{\mathcal{C}^*}$ is the adjoint of Δ. It is easy to check that, due to the co-algebra structure of $\mathcal{C}$, the above defined $m_{\mathcal{C}^*}$ and $1_{\mathcal{C}^*}$ satisfy all the axioms given in Sec. 1.1.2. Similarly, also given an algebra $(\mathcal{A}, m, 1, \mathcal{K})$ we have that $(\mathcal{A}^*, \Delta_{\mathcal{A}^*}, \varepsilon_{\mathcal{A}^*}, \mathcal{K})$ defines a co-algebra where $\Delta_{\mathcal{A}^*}$ is the adjoint[2] of m.

Given a **colagebra** $(\mathcal{C}, \Delta, \varepsilon, \mathcal{K})$ it is always possible to define by **adjunction** an **algebra** $(\mathcal{C}^*, m_{\mathcal{C}^*}, 1_{\mathcal{C}^*}, \mathcal{K})$, and vice versa.

This suggest that also for an Hopf algebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \varepsilon, \mathcal{K})$ and its dual $\mathcal{H}^*$ this duality should work. Indeed, it is quite obvious that if one introduces the adjoint maps $m_{\mathcal{H}^*}, 1_{\mathcal{H}^*}, \Delta_{\mathcal{H}^*}$ and $\varepsilon_{\mathcal{H}^*}$ as

$$\langle m_{\mathcal{H}^*}(\phi, \psi), c \rangle := \langle \phi \otimes \psi, \Delta(a) \rangle, \tag{1.47}$$

$$\langle \Delta_{\mathcal{H}^*}(\phi), a \otimes b \rangle := \langle \phi, m(a, b) \rangle, \qquad \forall \phi, \psi \in \mathcal{H}^*, \ \forall a, b \in \mathcal{H} \tag{1.48}$$

$$\langle 1_{\mathcal{H}^*}, a \rangle := \varepsilon(a), \tag{1.49}$$

$$\varepsilon_{\mathcal{H}^*}(\phi) := \langle \phi, 1_{\mathcal{H}} \rangle, \tag{1.50}$$

then, $(\mathcal{H}^*, m_{\mathcal{H}^*}, 1_{\mathcal{H}^*}, \Delta_{\mathcal{H}^*}, \varepsilon_{\mathcal{H}^*}, \mathcal{K})$ is a bialgebra. Furthermore, if we introduce the adjoint map $S_{\mathcal{H}^*}$ as

$$\langle S_{\mathcal{H}^*}(\phi), a \rangle := \langle \phi, S(a) \rangle, \qquad \forall \phi \in \mathcal{H}^*, \forall a \in \mathcal{H}, \tag{1.51}$$

then $(\mathcal{H}^*, m_{\mathcal{H}^*}, 1_{\mathcal{H}^*}, \Delta_{\mathcal{H}^*}, \varepsilon_{\mathcal{H}^*}, S_{\mathcal{H}^*}, \mathcal{K})$ is also an Hopf algebra.

Dually paired Hopf algebras

Definition 1.2.9. *Two Hopf algebras $\mathcal{H}$ and $\mathcal{H}^*$ are said to be* ***dually paired*** *if there exists an inner product $\langle \cdot, \cdot \rangle : \mathcal{H}^* \otimes \mathcal{H} \to \mathcal{K}$ such that the all of the equalities from* (1.47) *up to*

[2] In case of infinite dimensional algebra one may introduce a weaker notion of *nondegenerate dual pairing*.

(1.51) *are satisfied.*

Notice that in (1.47)-(1.51) we always used ":= " and not just "= ". Indeed, all those relation can be used constructively to produce an Hopf algebra $\mathcal{H}^*$ dual paired to a given one $\mathcal{H}$. However, the dual pair of an Hopf algebra is not unique. Nevertheless, this procedure turns out to be extremely useful when one want to give the Hopf algebra structure to an object which as an algebra is already well known to be dual to some Hopf algebra. As an example, the structure of space-time coordinate can be worked out from that of its conjugate momenta.

Commutativity and co-commutativity

Definition 1.2.10. *An Hopf algebra* $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \varepsilon, \mathcal{K})$ *is said to be*

$$\begin{array}{lll} \textbf{\textit{commutative}} & \textit{if} & m \circ \tau = m, \\ \textbf{\textit{co-commutative}} & \textit{if} & \tau \circ \Delta = \Delta. \end{array}$$

Furthermore, given a co-commutative Hopf algebra $\mathcal{H}$ which is dual paired with $\mathcal{H}^*$ one has

$$\begin{aligned} \langle \phi \otimes \psi, \Delta(a) \rangle &= \langle \phi, a_{(1)} \rangle \langle \psi, a_{(2)} \rangle = \langle \phi, a_{(2)} \rangle \langle \psi, a_{(1)} \rangle \\ &= \langle \psi, a_{(1)} \rangle \langle \phi, a_{(2)} \rangle = \langle \psi \otimes \phi, \Delta(a) \rangle \end{aligned} \tag{1.52}$$

and using (1.47) we have $m_{\mathcal{H}^*}(\psi \otimes \phi) = m_{\mathcal{H}^*}(\phi \otimes \psi)$, thus $\mathcal{H}^*$ is commutative. We have just shown that the dual of a **commutative** Hopf algebra is **co-commutative** and vice versa.

An example of Hopf algebra: finite groups

Perhaps, the most intuitive example of Hopf algebra is given by finite groups. Consider a group $\mathcal{G}$ which is finite, we can construct the vector space $\mathcal{KG}$ spanned by the elements of $\mathcal{G}$ over the field $\mathcal{K}$

$$\mathcal{KG} \ni \mathbf{v} := \sum_{g \in \mathcal{G}} \alpha(g) \mathbf{e}_g. \tag{1.53}$$

Such a vector space $\mathcal{KG}$ endowed with the following map

$$\Delta g = g \otimes g, \tag{1.54}$$

$$\varepsilon(g) = 1, \tag{1.55}$$

$$S(g) = g^{-1}, \tag{1.56}$$

$$\tag{1.57}$$

satisfy all the compatibility relation introduced in Sec. 1.2.2. In other words, $(\mathcal{KG}, \cdot, e, \Delta, \varepsilon, \mathcal{K})$ is an Hopf algebra, where e is the unit element of the group and $\cdot$ is the composition law.

An example of Hopf algebra: functions over a compact topological group

Consider a compact topological group $\mathcal{G}$, we already discussed that groups are Hopf algebra with group-like co-product and group inversion as antipode. Now, consider the space of continuous function $C(\mathcal{G})$ over the group. It can be shown that the relation $C(\mathcal{G}) \otimes C(\mathcal{G}) = C(\mathcal{G} \times \mathcal{G})$ is always true for finite dimensional groups while it is true for infinite groups endowed with a suitable completion of the tensor product. Then $(C(\mathcal{G}), \cdot, \eta, \Delta, \epsilon, \mathcal{K})$ is an Hopf algebra with

$$
\begin{aligned}
(f \cdot h)(g) &= f(g)h(g), \quad \forall f, h \in C(\mathcal{G}),\ \forall g_i \in \mathcal{G} && (1.58)\\
\Delta(f)(g_1 \otimes g_2) &= f(g_1 g_2), && (1.59)\\
\eta(x) &= x1, \qquad\qquad \forall x \in \mathcal{K} && (1.60)\\
\epsilon(f) &= f(e), && (1.61)\\
(S(f))(g) &= f(g^{-1}), && (1.62)
\end{aligned}
$$

where in (1.60) we used the identity function $1(.)$ such that $\mathcal{G} \ni g \to 1(g) = 1 \in \mathcal{K}$ and in(1.61) we used the neutral element $e \in \mathcal{G}$. In this way, the Hopf algebra $C(\mathcal{G})$ knows the structure of the group $\mathcal{G}$. Indeed the multiplication law of the group is implemented as the co-product while the identity corresponds to the co-unit map.

Furthermore, the *Gelfand-Naimark theorem* states that any **commutative** Hopf algebra $C(\mathcal{G})$ is equivalent to a compact topological group G. This correspondence is lost if one consider a non commutative Hopf algebra of function, and what is left is what we call a *quantum group*. We will discuss in more detail these structures in later sections.

An example of dual pair: Universal Enveloping $U(\mathfrak{g})$ and $C(\mathcal{G})$

Consider a Lie algebra $\mathfrak{g}$ and its **universal enveloping algebra** $U(\mathfrak{g})$. If one introduce the following maps

$$
\begin{aligned}
\Delta(x) &= x \otimes 1 + 1 \otimes x, \quad \forall x \in \mathfrak{g}, && (1.63)\\
\eta(\alpha) &= \alpha 1, \quad \forall \alpha \in \mathcal{K}, && (1.64)\\
\varepsilon(x) &= \begin{cases} 0 & \forall x \neq 1 \\ 1 & \text{if} x = 1 \end{cases}, && (1.65)\\
S(x) &= -x && (1.66)
\end{aligned}
$$

then $(\mathfrak{g}, \cdot, 1, \Delta, \varepsilon, \mathcal{K})$ is an Hopf algebra with primitive co-product, as we anticipated at the end of Sec. 1.2.2. Furthermore, the antipode of any element is just the opposite of that element. Notice that (1.63)-(1.66) have been defined only on the element of $\mathfrak{g}$. It is possible to extend these map to the whole $U(\mathfrak{g})$ and the Hopf algebra axioms would still hold. Thus, also $(U(\mathfrak{g}), \cdot, 1, \Delta, \varepsilon, \mathcal{K})$ is an Hopf algebra too. Notice that (1.63) is invariant under swap τ, thus the algebra is co-commutative. As we already stated at the end of Sec. 1.2.2, $\Delta(x)$ can be used to compute the action on the tensor product of two objects. In this case, the co-commutativity of (1.63) tells us that also the element x acts as a derivation. Furthermore, if $\mathcal{G}$ is the group obtained from $\mathfrak{g}$ via exponentiation, then, the Hopf algebras $C(\mathcal{G})$ and $U(\mathfrak{g})$ are dual paired.

1.2.4 Dual Action

In Sec. 1.2.2 we introduced the concept of left action. However, one can also define a right action of $\mathcal{H}$ on an algebra $\mathcal{A}$. A precise definition can be obtained from that of a left action replacing the map $\triangleright$ with

$$\triangleleft : \ \mathcal{A} \otimes \mathcal{H} \ni a \otimes h \to a \triangleleft h \in \mathcal{A} \tag{1.67}$$

in any instance of (1.41) and (1.42). In the same way, we also deduce covariance for a right action from (1.43) and (1.44). In particular, right and left action are interchanged under dualization, as we will soon clarify.

Consider an algebra $\mathcal{A}$ whose dual algebra is $\mathcal{A}^*$ and an Hopf algebra H. Suppose a left action $\triangleright$ of $\mathcal{H}$ on $\mathcal{A}$ is given, then $h \triangleright a$ is an element of $\mathcal{A}$. Then, the inner product can be used to define an action $\alpha^*_h(.)$ of $\mathcal{H}$ on $\mathcal{A}$

$$\langle \phi, h \triangleright a \rangle := \langle \alpha^*_h(\phi), a \rangle, \quad \forall a \in \mathcal{A},\ \forall \phi \in \mathcal{A}^*,\ \forall h \in \mathcal{H}. \tag{1.68}$$

Then we also have

$$\langle \phi, m(h \otimes g) \triangleright a \rangle = \langle \alpha^*_{m(h\otimes g)}(\phi), a \rangle \quad \forall a \in \mathcal{A},\ \forall \phi \in \mathcal{A}^*,\ \forall h, g \in \mathcal{H}, \tag{1.69}$$

$$\langle \phi, h \triangleright (g \triangleright a) \rangle = \langle \alpha^*_h(\phi), g \triangleright a \rangle = \langle \alpha^*_g(\alpha^*_h(\phi)), a \rangle. \tag{1.70}$$

Since $\triangleright$ is a left action we have $\langle \phi, m(h \otimes g) \triangleright a \rangle = \langle \phi, h \triangleright (g \triangleright a) \rangle$. It follows that $\alpha^*_{m(h\otimes g)}(\phi) = \alpha^*_g(\alpha^*_h(\phi))$, which is to say that $\triangleleft^* : \mathbb{A}^* \otimes \mathcal{H} \ni phi \otimes h \to \phi \triangleleft^* h := \alpha^*_h(a) \in \mathcal{A}^*$ is a right action.

Given a left action $\triangleright$ on an algebra $\mathcal{A}$ it also defines a right action $\triangleleft^*$ on the dual algebra $\mathcal{A}^*$ by dualization

$$\langle \phi, h \triangleright a \rangle := \langle \phi \triangleleft^* h, a \rangle, \quad \forall a \in \mathcal{A},\ \forall \phi \in \mathcal{A}^*,\ \forall h \in \mathcal{H}. \tag{1.71}$$

We give now some relevant example of representation of algebras.

Example: Left and right regular action

Consider an algebra $(\mathcal{A}, \cdot, 1, \mathcal{K})$. The product of two element $a \cdot b$ defines a representation of the algebra on itself. In this way, one defines the **left regular action** $\overset{\text{reg}}{\triangleright}$ and the **right regular action** $\overset{\text{reg}}{\triangleleft}$ as follows:

$$a \overset{\text{reg}}{\triangleright} b := a \cdot b =: a \overset{\text{reg}}{\triangleleft} b, \qquad a, b \in \mathcal{A} \tag{1.72}$$

Now, suppose $\mathcal{A}$ to be also a bialgebra, then both $\overset{\text{reg}}{\triangleright}$ and $\overset{\text{reg}}{\triangleleft}$ are in general not covariant.

Example: Left and right canonical action

Consider an algebra $\mathcal{A}$ together with its dual $\mathcal{A}^*$. The right $\overset{\text{can}}{\triangleright}$a and left $\overset{\text{can}}{\triangleleft}$ **canonical action** of $\mathcal{A}$ on $\mathcal{A}^*$ are define by dualization:

$$\langle \phi \overset{\text{can}}{\triangleleft} a, b \rangle := \langle \phi, a \overset{reg}{\triangleright} b \rangle, \qquad \forall a, b \in \mathcal{A}, \forall \phi \in \mathcal{A}^* \tag{1.73}$$

$$\langle b \overset{\text{can}}{\triangleright} \phi, a \rangle := \langle \phi, a \overset{reg}{\triangleleft} b \rangle. \tag{1.74}$$

We apply dualization on the first line and get the explicit form of the action

$$\langle \phi \overset{\text{can}}{\triangleleft} a, b \rangle = \langle \phi, a \cdot b \rangle = \langle \Delta\phi, a \otimes b \rangle = \langle \phi_{(1)}, a \rangle \langle \phi_{(2)}, b \rangle. \tag{1.75}$$

Similarly, we derive explicit form of $b \overset{\text{can}}{\triangleright} \phi$ from (1.74). The explicit form of canonical action are

$$\phi \overset{\text{can}}{\triangleleft} a = \langle \phi_{(1)}, a \rangle \phi_{(2)}, \qquad \forall a \in \mathcal{A},\ \forall \phi \in \mathcal{A}^*, \tag{1.76}$$

$$a \overset{\text{can}}{\triangleright} \phi = \phi_{(1)} \langle \phi_{(2)}, a \rangle. \tag{1.77}$$

Both left and right canonical representation are covariant.

Example: the adjoint action

We want to have an action of an Hopf algebra on itself. We define the the left $\overset{\text{ad}}{\triangleright}$ and right $\overset{\text{ad}}{\triangleleft}$ adjoint maps as

$$a \overset{\text{ad}}{\triangleright} b = a_{(1)} b\, S(a_{(2)}), \qquad \forall a, b \in \mathcal{H}, \tag{1.78}$$

$$b \overset{\text{ad}}{\triangleleft} a = S(a_{(1)}) b\, a_{(2)}. \tag{1.79}$$

Both the above representation are covariant. Furthermore, if $\mathcal{H}^*$ and $\mathcal{H}$ are dual paired Hopf algebras, then

$$\langle \phi \overset{\text{ad}}{\triangleleft} b, a \rangle = \langle \phi, b \overset{\text{ad}}{\triangleright} a \rangle, \phi \in \mathcal{H}^*, a \in \mathcal{H}. \tag{1.80}$$

1.3 Bicross product

In this section we study a new kind of structures firstly introduced by S. Majid and known as *bicross product*. This is a new operation between two objects which allows one two construct new Hopf algebra from two existing one. First, we introduce the concept co-action

Co-action

Definition 1.3.1. *Given a co-algebra $\mathcal{H}$ and a vector space V, a right co-action β is a linear map from $V \to V \otimes \mathcal{H}$ such that*

$$(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta, \tag{1.81}$$

$$id = (id \otimes \varepsilon) \circ \beta. \tag{1.82}$$

Equivalently, the following graph

$$(\beta \otimes \mathrm{id}) \circ \beta = (\mathrm{id} \otimes \Delta) \circ \beta, \quad \begin{tikzcd} & V \otimes \mathcal{H} \otimes \mathcal{H} & \\ V \otimes \mathcal{H} \arrow[ur, "\beta\otimes\mathrm{id}"] & & V \otimes \mathcal{H} \arrow[ul, "\mathrm{id}\otimes\Delta"'] \\ & V \arrow[ul, "\beta"] \arrow[ur, "\beta"'] & \end{tikzcd} \tag{1.83}$$

$$\mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \beta, \quad \begin{tikzcd} & V \otimes \mathcal{H} \arrow[dl, "\mathrm{id}\otimes\varepsilon"'] & \\ V \otimes \mathcal{K} \arrow[rr, phantom, "\cong"] & & V \arrow[ul, "\beta"'] \end{tikzcd} \tag{1.84}$$

where the above graph have to commute.

In Sweedler notation the (1.83) and (1.84) read

$$\sum v^{(1)(1)} \otimes v^{(1)(2)} \otimes v^{(2)} = \sum v^{(1)} \otimes v^{(2)} \otimes v^{(3)}, \tag{1.85}$$

$$\sum v^{(1)} \varepsilon(v^{(2)}) = v. \tag{1.86}$$

Moreover, a vector space V which satisfies (1.85) and (1.86) is said to be a $\mathcal{H}$-*right comodule*. Notice that the graph in (1.83) and (1.84), are just dual to those used in the definition of the action (1.37) and (1.38). Furthermore, consider two finite dimensional dually paired Hopf algebras $\mathcal{H}$ and $\mathcal{H}^*$, and suppose β to be a co-representation of $\mathcal{H}^*$ on some vector space $\mathcal{V}$. Then, we are able to define the following map $\alpha : \mathcal{H} \otimes V \to V$ as

$$\alpha_h = (\mathrm{id} \otimes \langle \cdot, \cdot \rangle) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \tag{1.87}$$

which turns out to be an action

$$h \triangleright v = \sum v^{(1)} \langle h, v^{(2)} \rangle. \tag{1.88}$$

Given a finite dimensional Hopf algebra $\mathcal{H}$, then a left action of $\mathcal{H}$ corresponds to a right co-action on$\mathcal{H}^*$. Moreover, if $\mathcal{A}$ is an left- $\mathcal{H}$ module algebra it is also a right-$\mathcal{H}^*$ co-module algebra. The same holds for any co-algebra $\mathcal{C}$ with the obvious replacements.

This is the reason why given a group G an action of the function $\mathcal{K}(G)$ on some vector space V correspond to a coaction of the vector field $\mathcal{KG}$ over the same vector space V.

Example of co-action: The co-product

Given a bialgebra H the co-product map $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ satisfy all the axioms of a co-action. Indeed, (1.85) and (1.86) are equivalent to (1.22) and (1.23). Furthermore, (1.27) ensures that $\mathcal{B}$ is a $\mathcal{B}$-comodule

Example: Right co-regular coaction

Given a dual pair $\mathcal{H}$ and $\mathcal{H}^*$, we define a right action of $\mathcal{H}^*$ on $\mathcal{H}$ as

$$h \triangleleft \phi = \sum h_{(1)} \langle h_{(2)}, \phi \rangle, \qquad \forall h \in \mathcal{H}, \forall \phi \in \mathcal{H}^*. \tag{1.89}$$

Then, given the map $\beta : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}$

$$\beta_h(\phi) = \sum \phi^{(1)} \otimes \phi^{(2)} := \begin{cases} \phi^{(1)} \text{ such that } \langle h, \phi^{(1)} \rangle = \langle h_{(2)}, \phi \rangle \\ \phi^{(2)} := h_{(1)} \end{cases} \tag{1.90}$$

we have $\sum h_{(1)} \langle h_{(2)}, \phi \rangle = \sum \langle h, \phi^{(1)} \rangle \phi^{(2)}$. This is the *right co-regular co-action* of $\mathcal{H}$ on $\mathcal{H}^*$ since it satisfy all the co-action axioms. Indeed, if we take $(\beta_h \otimes \mathrm{id}) \circ \beta_h$ we have

$$\langle h, \phi^{(1)(1)} \rangle \phi^{(1)(2)} \otimes \phi^{(2)} = h_{(1)} \langle h_{(2)}, \phi^{(1)} \rangle \otimes \phi^{(2)} = \tag{1.91}$$

$$= h_{(1)(1)} \otimes h_{(1)(2)} \langle h, \phi \rangle = \langle h, \phi \rangle \Delta \phi^{(2)} \tag{1.92}$$

where $\Delta h_{(1)} = h_{(1)(1)} \otimes h_{(1)(2)} = \Delta \phi^{(2)}$; thus the last step is just the outcome of $(\mathrm{id} \otimes \Delta) \circ \beta_h$. We also have $\langle h, \phi^{(1)} \varepsilon(\phi^{(2)}) = \langle h, \phi \rangle$ and $\langle hg, \phi \rangle = \langle h, \phi^{(1)}_{(1)} \rangle \langle h, \phi^{(1)}_{(2)} \rangle \phi^{(2)}_{(1)} \phi^{(2)}_{(2)}$. It follows that $\mathcal{H}^*$ is a $\mathcal{H}-$comodule algebra.

Consider two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$. Note that the graph representing the right action$\triangleleft$ of $\mathcal{B}$ on $\mathcal{A}$ is **not dual** to the one representing the left co-action β of $\mathcal{A}$ on $\mathcal{B}$

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{\triangleleft} \mathcal{A} \ , \qquad \mathcal{B} \xrightarrow{\beta} \mathcal{A} \otimes \mathcal{B} \tag{1.93}$$

Nevertheless, the above structures can be used to build up from $\mathcal{A}$ and $\mathcal{B}$ a new algebra

Bicross product algebra $\bowtie$

Definition 1.3.2. *Given two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$, a **bicross product algebra**$\mathcal{A} \bowtie \mathcal{B}$ is a tensor product Hopf algebra $(\mathcal{A} \otimes \mathcal{B},\ 1_{\mathcal{A}\bowtie\mathcal{B}},\ \Delta_{\mathcal{A}\bowtie\mathcal{B}},\ \varepsilon_{\mathcal{A}\bowtie\mathcal{B}},\ S_{\mathcal{A}\bowtie\mathcal{B}},\ \mathcal{K})$ whose maps are*

$$(x \otimes \phi) \cdot (y \otimes \psi) = xy_{(1)} \otimes (\phi \triangleleft y_{(2)})\psi, \qquad \forall x, y \in \mathcal{A}, \forall \phi, \psi \in \mathcal{B}, \tag{1.94}$$

$$1_{\mathcal{A}\bowtie\mathcal{B}} = 1_{\mathcal{A}} \otimes 1_{\mathcal{B}}, \tag{1.95}$$

$$\Delta_{\mathcal{A}\bowtie\mathcal{B}}(x \otimes \phi) = \left(x_{(1)} \otimes {x_{(2)}}^{(1)}\phi_{(1)}\right) \otimes \left({x_{(2)}}^{(2)}\phi_{(2)}\right), \tag{1.96}$$

$$\varepsilon_{\mathcal{A}\bowtie\mathcal{B}} = \varepsilon_{\mathcal{A}}(x)\varepsilon_{\mathcal{B}}(\phi), \tag{1.97}$$

$$S_{\mathcal{A}\bowtie\mathcal{B}}(x \otimes \phi) = \left(1_{\mathcal{A}} \otimes S_{\mathcal{B}}(x^{(1)}\phi)\right) \cdot \left(S_{\mathcal{A}}x^{(2)} \otimes 1_{\mathcal{A}}\right), \tag{1.98}$$

and whose action $x \triangleleft \phi$ and co-action $\beta_\phi(x) = \sum x^{(1)} \otimes x^{(2)}$ satisfy

$$\varepsilon_{\mathcal{A}}(x \triangleleft \phi) = \varepsilon_{\mathcal{A}}(x)\varepsilon_{\mathcal{B}}(\phi), \tag{1.99}$$

$$\Delta_{\mathcal{A}}(x \triangleleft \phi) = (x_{(1)} \triangleleft \phi_{(2)}){x_{(2)}}^{(1)} \otimes (\phi_{(2)} \triangleleft {x_{(2)}}^{(2)}), \tag{1.100}$$

$$\beta(\phi\psi) = (\phi^{(1)} \triangleleft \psi_{(1)}){\psi_{(2)}}^{(1)} \otimes \phi^{(2)}{\psi_{(2)}}^{(2)}, \tag{1.101}$$

$${\phi_{(1)}}^{(1)}(x \triangleleft \phi_{(2)}) \otimes {\phi_{(1)}}^{(2)} = (x \triangleleft \phi_{(1)}){\phi_{(2)}}^{(1)} \otimes {\phi_{(2)}}^{(2)}, \tag{1.102}$$

*which is to say that $x \triangleleft \phi$ and $\beta_\phi(x)$ are **compatible**.*

The single elements $x \in \mathcal{A}$ and $\phi \in \mathcal{B}$ can be mapped respectively into $X = x \otimes 1$ and $\Phi = 1 \otimes \phi$ in $\mathcal{A} \bowtie \mathcal{B}$; in particular, the X's and the Φ's are the generators of such an algebra. Any element in $\mathcal{A} \bowtie \mathcal{B}$ is just the product (1.94) of an X with a Φ

$$X \cdot \Phi = (x \otimes 1)(1 \otimes \phi) = x \otimes \phi. \tag{1.103}$$

Here, the ordering plays a role. If we swap X and Φ we obtain

$$\Phi X = (1 \otimes \phi)(x \otimes 1) = x_{(1)} \otimes (\phi \triangleleft x_{(2)}). \tag{1.104}$$

In other words, the generators of $\mathcal{A} \bowtie \mathcal{B}$ do note commute

$$[X, \Phi] = x \otimes \phi - x_{(1)} \otimes (\phi \triangleleft x_{(2)}), \tag{1.105}$$

it follows that $\mathcal{A} \bowtie \mathcal{B}$ can be seen as the universal enveloping algebra generated by $\mathcal{A}$ and $\mathcal{B}$ modulo relation (1.105). Notice that the above construction not only gives a completely general method to obtain a new Hopf algebra, but also gives the commutation rules of the new algebra

1.4 Deformed Symmetries

The symmetry group of a given theory plays a crucial role in its physical interpretation. Nevertheless, there are cases in which the usual notion of group turns out to be insufficient to properly describe the symmetries of a physical theory [33–38]. This lead to the introduction of new generalised structure to describe symmetries which took the name of Quantum Group. Ironically, a Quantum Group has not a group structure nor it has to be necessarily "quantum" in the usual physical meaning of the term. In this section we will give some of the most common deformation procedures used to obtain quantum group, together with some concrete realizations and examples.

1.4.1 Quasitriangular Quantum Group

This approach to deformation is based on the concept of quasitriangular Hopf algebras. In Sec. 1.2.2 we said that an Hopf algebra is co-commutative if $\Delta \circ \tau = \Delta$. We want to relax this property, but keep the non-commutativity under control. In particular, we want an Hopf algebra which is commutative up to conjugation of an element $R \in \mathcal{H} \otimes \mathcal{H}$.

Quasitriangular Hopf algebra

Definition 1.4.1. *Given an Hopf algebra $\mathcal{H}$ and an invertible element $R \in \mathcal{H} \otimes \mathcal{H}$, we say that the pair $(\mathcal{H}, R)$ is a* ***quasitriangular*** *Hopf algebra if R satisfies*

$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{1.106}$$

$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{1.107}$$

$$\tau \circ \Delta h = R(\Delta h)R^{-1}, \qquad h \in \mathcal{H}. \tag{1.108}$$

In this case, R is said to be the ***quasitriangular element*** *of $(\mathcal{H}, R)$.*

In the above definition the notation $R_{ij} \in \overset{n}{\otimes} \mathcal{H}$ is used to describe higher tensor elements build up using the "legs" of $R = \sum R^{(1)} \otimes R^{(2)}$ as follows:

$$R_{ij} := \sum 1 \otimes ...1 \otimes R^{(1)} \otimes 1 \otimes ... \otimes 1 \otimes R^{(2)} \otimes ... \otimes 1. \tag{1.109}$$

In other words, we replace the i-th and the j-th elements in $\mathbb{1} \in \overset{n}{\otimes} \mathcal{H}$ with $R^{(1)}$ and $R^{(2)}$ respectively. Notice that (1.108) states that to swap elements of a co-product is the same as properly apply R. In this sense we say that *the quasitriangular structure R keeps the non commutativity under control*. In other words, although the algebra is non commutative, the failure in commutativity is encoded by the action of R. Furthermore, given a quasitriangular $(\mathcal{H}, R)$ one has that

$$(\varepsilon \otimes \mathrm{id})R = (\mathbf{id} \otimes \Delta)R = 1, \tag{1.110}$$

$$(S \otimes \mathrm{id})R = R^{-1}, \tag{1.111}$$

$$(\mathrm{id} \otimes S)R^{-1} = R, \tag{1.112}$$

as can be easily checked using the above definition and the properties co-unit and antipode. In particular, from (1.111) and (1.112) follows that $(S \otimes S)R = R$. Moreover, using (1.106),(1.107) and (1.108), one has that the quasitriangular element R of any $(\mathcal{H}, R)$ satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.113}$$

which is known as the abstract *quantum Yang-Baxter equation* (QYBE) [39–41]. The quasitriangular structure R is also called the *universal* $R-$ *matrix* because given any representation ρ of H on some vector space, then also $(\rho \otimes \rho)(R)$ satisfy the QYBE (1.113). This is extremely important because just as the R-matrix tells us how to deal with non commutativity in an abstract sense, the same will do $(\rho \otimes \rho)(R)$ for any possible representations, which are what one concretely uses in physical applications. We stress that the universal $R-$matrix is far from being unique, and different universal R-matrices are compatible with the same Hopf algebra. As an example, given any co-commutative $\mathcal{H}$ it is easy to check that the element $R = 1 \otimes 1$ satisfies (1.106), (1.107) and (1.108), *i.e.* $1 \otimes 1$ is a quasitriangular structure.

1.4.2 Twist Deformation

In the previous section we stated that given an Hopf algebra $\mathcal{H}$, it is in general compatible with different $R-$matrix each one gives a different quantum group $(\mathcal{H}, R)$. In the present section we want to introduce a mechanism called **twist**, such that given a $(\mathcal{H}, R)$ produced a new (twisted) $(\mathcal{H}, R')$ with the same algebra but different R matrix. This turns out to be very useful when one already knows a trivial $R-$matrix and wants to get a new one. This is exactly what happens for any group G when seen as a quasitriangular quantum group $(G, 1 \otimes 1)$.

First, we need to introduce the concept of cochains, coboundary and cocycles. Consider an Hopf algebra $\mathcal{H}$ with co-product Δ. Any **invertible** element $\chi \in \mathcal{H}^{\otimes n}$ is said to be an $n-$**cochain**. We also introduce the map $\Delta_i : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n+1)}$ obtained by replacing in

$\mathrm{id} \otimes ... \otimes \mathrm{id} \in \mathcal{H}^{\otimes n}$ the $i-th$ element with Δ as follows:

$$\Delta_i := \overbrace{\mathrm{id} \otimes \cdots \otimes}^{i-1} \Delta \otimes \cdots \otimes \mathrm{id}, \tag{1.114}$$

where, $i \in 1, \ldots, n$ and

$$\Delta_0 := 1 \otimes (.), \qquad \Delta_{n+1} := (.) \otimes 1, \tag{1.115}$$

by convention.

coboundary

Definition 1.4.2. *Given a $n-$cochain, its* ***coboundary*** *is the $(n+1)-$cochain given by*

$$\partial\chi = (\partial_+\chi)(\partial_-\chi^{-1}) := \left(\sum_{i\in\text{“evens”}} \Delta_i\chi\right)\left(\sum_{i\in\text{“ odds”}} \Delta_i\chi^{-1}\right), \tag{1.116}$$

where the product are taken in increasing order in i.

An $n-$chain χ is said to be a $n-$cocycle if $\partial\chi = 1$. An $n-$cochain is said to be co-unital if $\varepsilon_i(\chi) = 1$ for all i, where ε_i is obtained by replacing Δ with ε in (1.114). As an example, given any group-like element γ is a $1-$co-cycle since we have $\Delta\gamma = \gamma\otimes\gamma$ and $\partial\gamma = (\gamma\otimes 1)(\gamma^{-1}\otimes\gamma^{-1})(1\otimes\gamma) = 1$, similarly it can be proved to be co-unital also. It follows that any element of a group $\mathcal{G}$ seen as an Hopf algebra in the usual way is automatically a co-unital $1-$co-cycle. Now, consider $n = 2$. A invertible $h \in \mathcal{H}\otimes\mathcal{H}$, then it is a $2-cocycle$ if $\partial h = 1$ which is equivalent to

$$(1\otimes h)(\mathrm{id}\otimes\Delta h) = (\Delta h\otimes\mathrm{id})(h\otimes 1), \tag{1.117}$$

which is also co-unital if $(\varepsilon\otimes\mathrm{id})(h) = 1$. Then, taken any invertible $\gamma\in\mathcal{H}$ with $\varepsilon\gamma = 1$ we have that $\partial(\partial\gamma) = 1$, so for $\partial\gamma$ is a co-unital $2-$cocyle[3].More in general, It can be proved that given any co-unital $2-$cocycle $h \in \mathcal{H}\otimes\mathcal{H}$ and any co-unital invertible element $\gamma\in\mathcal{H}$ **cohomologue** element h_γ of h defined by

$$h_\gamma := (\partial_+\gamma)h(\partial_-\gamma^{-1}) = (\gamma\otimes\gamma)h\Delta h^{-1} \tag{1.118}$$

is a co-unital $2-cocycle$. In other words, the space $Cohom(k, \mathcal{H})$ of non-Abelian cohomology over $\mathcal{H}$ is the set of co-unital $2-$cocycles modulo transformations of the (1.118) kind. In particular, the space non-Abelian cohomolog of an Hopf algebra plays a crucial role in the proof of a fundamental result due to V. Drinfel'd [42], of which we report just the statement:[4]

[3]This is trivial if γ is a $1-$cocycle. If γ is not a $1-cocycle$, then one has to write $\partial\gamma$ explicitly as in (1.116) and then check it to be equal to 1.

[4]For a complete proof of the Drinfel'd Theorem see [41].

Theorem 1.4.1. ***(Drinfel'd Theorem)*** *Given any quasitriangular Hopf algebra* $(\mathcal{H}, R)$ *and any co-unital* $2-$*cocycle* F*, there is a new quasitriangular Hopf algebra* $(\mathcal{H}_F, R_F)$ *defined by the same algebra and couint but different co-product* Δ_F*,* $R-$*matrix* R_F *and antipode* S_F*. These last are defined* $\forall h \in \mathcal{H}_{\mathcal{F}}$ *as follows:*

$$\Delta_F h = F(\Delta h)F^{-1} \quad R_F = F_{12}RF^{-1} \quad S_F h = U(S\ h)U^{-1}, \tag{1.119}$$

where $U = \sum F^{(1)}(SF^{(2)})$ *is invertible.*

We say that that the $2-$cocycle F *twists* the quasitriangular Hopf algebra (H, R) into a new one (H_F, R_F) which is said to be *twist deformed.* We stress that since $R \neq R_F$ these two structures lead to different non-commutativity *i.e.* different quasitriangular quantum groups. However two different twists , say F and E may produce isomorphic quasitriangular Hopf algebra, which is to say that there exist an inner automorphism between them. In particular, this happens any time F is **cohomologus** to E, *i.e.* there exists some $\gamma \in \mathcal{H}$ such that $E = F_\gamma$ in the sense of (1.118). In this case, the twist can be easily undone via inner automorphism and both $(\mathcal{H}_E, R_E)$ and $(\mathcal{H}_F, R_F)$ define the same quantum group. In other words, the only twists which effectively produce new quantum groups are those who are cohomologically non trivial. It is also possible, although very rare, that the whole $Cohom(\mathcal{K}, \mathcal{H})$, in which peculiar case all twists of $(\mathcal{H}, R)$ are isomorphic to itself and the twist deformation method fails. However, in the vast majority of cases twist deformation produces genuinely new quantum groups. The twist mechanism has also been used by Drinfel'd as a "deformation quantization procedure". Indeed, he showed that any triangular Poisson-Lie group can be deformed into a triangular Hopf algebra [43].

1.4.3 Lie bialgebras and deformation quantisation

In the previous section we introduced the concept of quantum group as an Hopf algebra obtained by deformation from some already known groups or the enveloping algebra of a Lie algebra. In this section instead we want to consider a deformation at the Lie algebra $\mathfrak{g}$ level itself [7, 44]. This is different from what we did in Sec. 1.2.2 because what we want to dualize here is not the product of $\mathfrak{g}$ of its algebra structure but the bracket biliniar map$[.,.] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ itself.

Lie bialgebra

Definition 1.4.3. *A triple* $(\mathfrak{g},\ [.,.]\ ,\ \delta)$ *is said to be a Lie bialgebra if* $(\mathfrak{g},\ [.,.])$ *is a Lie algebra and the* ***cobraket*** *map*

$$\delta :\ \mathfrak{g} \ni X \longrightarrow \delta X = \sum X_{[1]} \otimes X_{[2]} \in \mathfrak{g} \otimes \mathfrak{g} \tag{1.120}$$

satisfies the following properties

$$\delta = -\,\tau \circ \delta, \tag{1.121}$$

$$(\mathrm{id} \otimes \delta) \circ \delta X + \text{cyclic} = 0, \tag{1.122}$$

$$\delta \circ [,] = ([,] \otimes \mathrm{id} + \mathrm{id} \otimes [,]) \circ (\mathrm{id} \otimes \delta) - ([,] \otimes \mathrm{id} + \mathrm{id} \otimes [,]) \circ (\delta \otimes \mathrm{id}). \tag{1.123}$$

In particular, (1.122) is known as the co-*Jacobi identity* because it is obtained reversing arrows in the usual Jacobi identity of $\mathfrak{g}$. Furthermore, in terms of the adjoint representation ad of $\mathfrak{g}$ on itself

$$X \triangleright Y = \mathrm{ad}_X Y := [X, Y] \tag{1.124}$$

(1.123) reads

$$\delta([X,Y]) = (\mathrm{id} \otimes \mathrm{ad}_X + \mathrm{ad}_X \otimes \mathrm{id})\delta(Y) - (\mathrm{id} \otimes \mathrm{ad}_Y + \mathrm{ad}_Y \otimes \mathrm{id})\delta(X) = \mathrm{ad}_X(\delta Y) - \mathrm{ad}_Y(\delta X), \tag{1.125}$$

where define $ad_X(\delta Y) := (\mathrm{id} \otimes \mathrm{ad}_X + \mathrm{ad}_X \otimes \mathrm{id})\delta(Y)$ the adjoint action on co-brackets.

We also introduce another action. Note that $[,]$ and $d\delta$ are dual to each other. Furthermore, just like Hopf algebras, also Lie bialgebras can be *selfdual* [41]. Indeed, consider the dual space $\mathfrak{g}^*$, then

$$\langle [\phi, \psi], X \rangle = \langle \phi \otimes \psi, \delta X \rangle, \tag{1.126}$$

$$\langle \phi, [X, Y] \rangle = \langle \delta\phi, X \otimes Y \rangle, \tag{1.127}$$

where $X, Y \in \mathfrak{g}$ and $\phi, \psi \in \mathfrak{g}^*$, as it can be easily checked using Lie bialgebra definition. Furthermore, any pair of generic Lie bialgebras endowed with a bilinear map $\langle , \rangle$ such that their brackets and co-brackets satisfy (1.126) and (1.127) are said to be **dual paired**. Notice also that δ is a $1-$cocycle for a Lie algebra just like Δ was for Hopf algebra.

Coboundary Lie bialgebra

Definition 1.4.4. *Consider a Lie bialgebra* $(\mathfrak{g}, [,], \delta)$ *together with* $r = \sum r^{[1]} \otimes r^{[2]} \in \mathfrak{g} \otimes \mathfrak{g}$. *We say that* $(\mathfrak{g}, r)$ *is a* ***coboundary*** *Lie bialgebra if* $\delta = \partial r$, *which is to say*

$$\delta X = \mathrm{ad}_X(r) = \sum [X, r^{[1]}] \otimes r^{[2]} + r^{[1]} \otimes [X, r^{[2]}]. \tag{1.128}$$

Since δ is the coboundary of the $1-$cochain r, the co-braket is chomologically trivial in the sense of (1.118). However, this does not mean that the whole Lie bialgebra is, in particular, $[,]$ may not be. There is a method due to Drinfel'd [45, 46] to check if a given $r \in \mathfrak{g} \otimes \mathfrak{g}$ is compatible with (1.128). First, we introduce the *Schouten bracket* $[[,]] : \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ as

$$[[r, s]] = [r_{12}, s_{13}] + [r_{12}, s_{23}] + [r_{13}, s_{23}], \qquad r, s \in \mathfrak{g} \otimes \mathfrak{g} \tag{1.129}$$
$$= \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[2]} + r^{[1]}\forall \otimes [r^{[2]}, s^{[1]}] \otimes s^{[2]} + r^{[1]} \otimes s^{[1]} \otimes [r^{[2]}, s^{[2]}]. \tag{1.130}$$

It is possible to prove [46] that the triple $(\mathfrak{g}, [,], r)$ *is a coboundary Lie bialgebra if and only if*

$$\mathrm{ad}_X([[r, r]]) = 0, \quad \text{and} \quad \mathrm{ad}_X(r + r_{21}) = 0, \tag{1.131}$$

where $r_{21} = r^{[2]} \otimes r^{[1]}$. *Furthermore, it is said to be* ***quasi-triangular*** *if* $[[r, r]] = 0$. *Moreover, if also* $\tau(r) = -r$, *then it is said to be* ***triangular***

Above we introduced the is the *classical Yang Baxter equation*(CYBE)

$$[[r, r]] = 0, \tag{1.132}$$

which is for quasitriangular Lie bialgebras what (1.113) was for quasitriangular Hopf algebras.

When the Lie algebra $\mathfrak{g}$ is semi-simple, then the Lie bialgebra is always a coboundary [47, 48] and the $r-$matrix satisfy the so called *modified classical Yang Baxter equation*(mCYBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]] = 0, \quad \forall X \in \mathfrak{g}. \tag{1.133}$$

Any $r-$matrix which satisfies (1.132) also satisfies (1.133), while the inverse is generally not true.

We also have the following generalization of the Drinfel'd theorem

Theorem 1.4.2. ***(Drinfel'd theorem for Lie bialgebras)*** *Consider a Lie bialgebra* $(\mathfrak{g}, \delta)$ *and* $\chi \in \mathfrak{g} \otimes \mathfrak{g}$. *If* χ *is such that*

$$ad_X((id \otimes \delta)\chi + cyclic + [[\chi, \chi]]) = 0, \qquad \forall X \in \mathfrak{g} \tag{1.134}$$
$$ad_X(\chi + \chi_{21}) = 0, \tag{1.135}$$

then the map

$$\delta_X = \delta + \partial\chi \quad i.e \quad \delta_\chi X = \delta X + ad_X(\chi), \tag{1.136}$$

defines the co-bracket of a new Lie bialgebra $(\mathfrak{g}, \delta_\chi)$.

Furthermore, if $(\mathfrak{g}, [,], r)$ is a quasi-triangular Lie bialgebra and

$$[[r, \chi]] + [[\chi, r]] + [[\chi, \chi]] = 0, \tag{1.137}$$
$$\mathrm{ad}_X\chi + \chi_{21} = 0, \tag{1.138}$$

then also $(\mathfrak{g}, [,], r+\chi)$ is quasi-triangular. In particular, this means that any quasi-triangular Lie bialgebra $(\mathfrak{g}, [,], r)$ is the result of a twist by $\chi = r$ of the Lie bialgebra $(\mathfrak{g}, \delta = 0)$ with 0 co-bracket. Moreover, if χ is antisymmetric then we get the triangular case.

At the beginning of this section we said that the twist of a Lie bialgebra has to do with quantization, now is time to better explain this. Consider an Hopf algebra $(\mathfrak{g}, 1, ., \varepsilon, \Delta, \mathbb{R}[[t]])$ over the *ring of formal power series* $\mathbb{R}[t]$ endowed with two maps

$$[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \qquad \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}, \tag{1.139}$$

which are antisymmetric in input and output respectively. We say that $(\mathfrak{g}, 1, ., \varepsilon\Delta, \mathbb{C}[[t]])$ is a *quantised enveloping algebra* with formal deformation parameter t if the following relations hold

$$XY - YX = [X, Y] + \mathcal{O}(t), \quad \forall X, Y \in \mathfrak{g}, \tag{1.140}$$

$$\Delta X = X \otimes 1 + 1 \otimes X + \frac{t}{2}\delta X + \mathcal{O}(t^2). \tag{1.141}$$

Also suppose $(\mathfrak{g}, R)$ to be a quasitriangular Hopf algebra whose $R-$matrix satisfy

$$R = 1 + r\ t + \mathcal{O}(t) \tag{1.142}$$

with $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then one easily check that $(\mathfrak{g}, [,], r)$ is a quasitriangular Lie bialgebra with co-bracket $\delta = \partial r$. In other words, to any twist deformation of the universal enveloping algebra $U(\mathfrak{g}$ (as as a quasi-triangular Hopf algebra) is associated a twist deformation of the corresponding Lie algebra $\mathfrak{g}$ (seen as a quasi-triangular Lie algebra).

1.4.4 Bicrossproduct Quantum Group

This section is dedicated to a different method of deformation originally introduced by S. Majid [49] as an attempt to unify quantum physics and gravity at the Planck scale. The main difference with the deformation quantization method is that instead of having some classical pre-existing structure of the space, one considers the algebra to be non-commutative from the start. As a consequence, one obtains theories which already know about the intrinsic non-commutativity of the geometry.

We start with a definition based on the bicross product $\bowtie$ of two Hopf algebras introduced in Sec. 1.4.4. Consider a Lie group $\mathcal{L}$ which admits a factorization $\mathfrak{L} = GM$ int terms of two Lie subgroups G and M. We define $C[M]$ as the commutative algebra of coordinates and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ (the lie algebra of G). Then, we define the

Bicrossproduct Quantum Group: $\quad U(\mathfrak{g}) \blacktriangleright\!\!\lhd C(M)$

as the bicross product of the Hopf algebras $U(\mathfrak{g})$ and $C(M)$. Then, the results of Sec. 1.2.2 and Sec. 1.4.4 give a complete description of Quantum Group of this kind. What we want to focus on is a possible Physical implementation of these tools. Basically, sector $C(M)$ can be interpreted as the set of coordinate functions over a non-commutative space-time (*Quantum Space-time*), on the other hand $U(\mathfrak{g})$ may be interpreted as the deformed commutative transformation (*Quantum Symmetry*) which let the non-commutative space-time invariant. The crucial point here is that the so build quantum group automatically preserve all the dual structure of the Hopf algebra which is made of. As a consequence, one is able to formulate a theory in which quantum mechanics and gravity are mutually dual.

Example: Planck-scale Quantum group $C(q) \bowtie_{h,G_N} C(p)$

Consider the bicrossproduct quantum group $C(q) \bowtie_{h,G_N} C(p)$ generated from the two Hopf algebras $C(\mathbf{q})$ and $C(\mathbf{p})$ with the following relations:

$$[\mathbf{q}, \mathbf{p}] = i\hbar(1 - e^{-\mathbf{q}/G_N}), \tag{1.143}$$

$$\Delta\mathbf{q} = \mathbf{q} \otimes 1 + 1 \otimes \mathbf{q}, \tag{1.144}$$

$$\Delta\mathbf{p} = p \otimes 1 + e^{-\mathbf{q}/G_N} \otimes \mathbf{p}, \tag{1.145}$$

where G_N and $\hbar$ are usually taken to be the Newton and the Planck constant respectively. Thus, (1.143) is a deformation of the usual non-commutative relation btween position $\mathbf{q}$ and conjugate momenta $\mathbf{p}$ in quantum mechanics. However, while (1.144) states that $\mathbf{q}$ a primitive element, (1.145) reveals $\mathbf{p}$ to be not. However, in the limit of "vanishing gravity" $G_N \to 0$ *i.e.* when usual quantum mechanics holds) both p and q become primitive. Moreover, if one restrict to the quantum states that are confined in the region of positive $\mathbf{q}$, then one obtains in the $G_N \to 0$ limit a quantum flat space with the usual Heisenberg algebra. On the other hand, the non-commutativity in the momentum space (due to $e^{-\mathbf{q}/G_N}$) can be interpreted as curvature, this has been originally named *cogravity* by Majid [6]. Interestingly, the curvature of the momenta seems to be related with the non commutativity in position sector. This is a consequence of the duality between $\mathbf{q}$ and $\mathbf{p}$ which is present also in other theories [50, 51] then the $C(q) \bowtie_{h,G_N} C(p)$ models.

1.5 From Hopf algebras to the structure of space-time

In the previous section we showed how to deform the group structure into that of a Quantum Group, which can be seen as a way to generalize the usual concept of symmetry. The whole construction is based on the dual structure of Hopf algebras and their representations.

However, Quantum Group are not only a way to describe the invariance of a system with more generality, but also a way to get information about the structure they act on. For instance, suppose to know the group of symmetries of some unknown space. In particular, consider the translation sector of the Poincaré algebra in four dimensions $\mathcal{T}$ with generators P_μ. Since $\mathcal{T}$ is a Lie algebra with $[P_\mu, P_\nu] = 0, \quad \forall \mu, \nu \in \{0, ..., 3\}$ it is easy to make it into an Hopf algebra $(\mathcal{T}, +, 1, \Delta, \varepsilon, S)$ as

$$\Delta P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu, \;\; \varepsilon(P_\mu) = 0, \quad S(P_\mu) = -P_\mu, \quad \forall \mu \in \{0, ..., 3\} \tag{1.146}$$

Suppose the P_μ's to be dual to x_μ according to

$$\langle P_\mu, x_\nu \rangle := -i \; \eta_{\mu\nu}, \tag{1.147}$$

then the structure of $\mathcal{M}$ can be deduced from the one of $\mathcal{T}$ from the dual compatibility relations. In particular, the maps

$$[x_\mu, x_\nu] = 0, \quad \Delta(x_\mu) = 1 \otimes x_\mu + x_\mu \otimes 1, \quad \varepsilon(x_\mu) = 0, \quad S(x_\mu) = -x_\mu \tag{1.148}$$

are compatible with (1.47)-(1.51) and make $\mathcal{M}$ into an Hopf algebra. Then, (1.147) is just the canonical action $P_\mu \triangleright x_\mu$ of translations $\mathcal{T}$ on the coordinates of a commutative (undeformed) Minkowski space-time.

More in general, a dual pair of two Hopf algebras $\mathcal{H}$ and $\mathcal{H}^*$ can be seen as a generalized phase space in which the duality between the generalized momenta and the generalized coordinates is expressed by (1.47)-(1.51). This last statement also means that given any Hopf algebra of position we are able to reconstruct the corresponding Hopf algebra of momenta and vice versa. Perhaps, this is the main physical motivation for introducing the concept of Hopf algebras at all. In particular, (1.47) and (1.48) tell us how the co-product in the momentum sector shapes the commutation relation in the position sector, thus any deformation on one element of dual pair will also change the Hopf algebra of the other one.

Perhaps, all the above construction may seems to be foremost of mathematical interest, with just some taste of physics here and there; but in fact it is no. Indeed, there are many physical models which can be recast in terms of Hopf algebra, deformation and dual pair. The fact that to a non-commutative coordinate space corresponds a curved momentum space and vice versa is not surprising. As an example, consider to have a physical system whose position space is a $3-$sphere S^3 with the curved coordinate functions $\{s_1, s_2, s_3\}$. In this framework the generators $\{J_1, J_2, J_3\}$ of conjugate momenta close a Lie algebra with $[J_i, J_j] = \frac{i}{R}\epsilon_{ijk}J_j$, where R is the radius of curvature of S^3. In other words, the "role of translation" on the sphere is played by rotations, thus on S^3 the Lie algebra of momenta is just $\mathfrak{su}(2)$. This is exactly what happens on the Bloch sphere [52].

However, this is not just a way to describe old models in a new fashion, but also a tool to work out genuinely new results. In particular, when the deformation of a group has to do with the invariance of a physical theory then the above construction gives information about the space on which the theory is defined on. As an example, if the deformation involves translations, then the above procedure suggests the usual structure of space-time to be replaced with a non-commutative one. In such a space-time the notion of point is lost since it is not possible to fully localize an event, which is very far from what one is used to both in general relativity and quantum field theory [53]. Nevertheless, various models of Quantum Gravity propose the space-time at a very small scale to have some sort of event non-localizability. As we will discuss later, this can be recast in terms of a Quantum Group theory whose deformation parameter κ is related to the Planck scale λ_P.

Chapter 2

A closer look at κ-Minkowski Space-time

2.1 A deformation of causality

It is a fact that, despite the idea of formulating a "theory of everything" is alluring, any physical theory has its own range of attainability, *i.e.* it produces trustful prediction only for systems whose physical quantities do not exceed a certain typical scale. The Standard model of particles interactions is no exception and it is expected to lose its predictivity in presence of new physics, or for processes with energy of the order of the Planck energy E_P. When such a high energy is considered, the gravitational effects are not negligible compared to quantum ones, as they are assumed to be in Standard Model. This suggests any quantum correction to the space-time structure to become relevant for lengths of the order of the Planck length λ_P and Planck time $t_p = \lambda_P/c$. Notice that E_p, λ_P and t_P are defined using only *fundamental physical constants* (namely the reduced Planck constant$\hbar$, the speed of light c, and the gravitational constant G_N), thus they are universal constant too. This means that λ_P must be an observer independent minimum length, which cannot transform as a distance under Lorentz transformations. The theory of *double special relativity* proposed in [12] suggests that λ_P must be regarded as a *fundamental length* just as c is a *fundamental velocity*. However, this is not the only possibility. As it has been observed in [19–21], the Poincaré covariance is still compatible with Planck length beacuse this last is not an observable quantity. In particular, the so called *Lorentz covariant $\kappa-$Minkowski space-time*, is a model which admits a fully Lorentz covariant representation [19].

As a consequence, one is left with a choice: or one assumes λ_P to be a non-observable Lorentz scalar which has the dimension of a length or one modifies the Lorentz transformations is such a way that both λ_p and c are left invariant. The second possibility (which we will discuss) can be implemented as a deformation of the usual Poincaré group $\mathcal{P}$ into a quantum group [54–56]. This is the $\kappa-$Poincaré $\mathcal{P}_\kappa$ originally introduced by J. Lukierski and collaborators in 1991 [16,17]. Notice that to deform Lorentz transformations means also to deform the concept of causality between events. Indeed, the usual Minkowski space-time $\mathcal{M}$ is not the homogeneous space of $\mathcal{P}_\kappa$. Furthermore, if one performs the analysis described in previous section, it turns out that $\mathcal{P}_\kappa$ leaves invariant a deformed (non-commutative) space-time called $\kappa-$Minkowski $\mathcal{M}_\kappa$ [17].

2.1.1 $\kappa-$Poincaré Quantum Group

In this section we describe the $\kappa-$Poincaré $\mathcal{P}_\kappa$ quantum group. We do not carry out the whole deformation procedure but we just report the final results. The Poincaré group $\mathcal{P}$ coincides with the group of isometry of the Minkowski space-time $ISO(3,1)$, whose Lie algebra consist of translation generators P_μ and Lorentz transformation generators $M_{\mu\nu}$ with the following commutation relation

$$[P_\mu, P_\nu] = 0, \tag{2.1}$$
$$[P_\rho, M_{\mu\nu}] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right), \tag{2.2}$$
$$[M_{\rho\sigma}, M_{\mu,\nu}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right). \tag{2.3}$$

As expressed by the last commutator, the Lorentz algebra is a subalgebra of $\mathfrak{iso}(3,1)$. We denote by $J_i = \epsilon_{ijk}M^{jk}/2$ and $K_i = M_{i,0}$ with $ijk \in 1,2,3$ the rotation and boosts generator respectively. It is sufficient to endow $\mathfrak{iso}(3,1)$ with a primitive co-product to make it into a bialgebra. In the so called *standard basis*, the deformed commutation relations of $\mathcal{P}_\kappa$ read

$$\begin{aligned}
&[J_i, P_0] = 0, && [J_i, P_j] = i\epsilon_{ijk}P_k,\\
&[K_i, P_0] = iP_i, && [K_i, P_j] = i\delta_{ij}\sinh(\frac{P_0}{\kappa}),\\
&[J_i, J_j] = i\epsilon_{ijk}J_k, && [J_i, K_j] = i\epsilon_{ijk}K_k,\\
&[P_0, P_i] = 0, && [P_i, P_j] = 0.\\
&[K_i, K_j] = -i\epsilon_{ijk}\left(J_k\cosh\left(\frac{P_0}{\kappa}\right) - \frac{P_k}{4\kappa^2}\mathbf{P}\cdot\mathbf{J}\right), && .
\end{aligned} \tag{2.4}$$

Furthermore, also P_κ is a bialgebra with a more complicated co-product [1]

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \tag{2.5}$$

$$\Delta P_i = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i, \tag{2.6}$$

$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i, \tag{2.7}$$

$$\Delta K_i = K_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes K_i + \frac{1}{2\kappa}\epsilon_{ijk}\left(P_j \otimes J_k e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} J_j \otimes P_k\right). \tag{2.8}$$

Not surprisingly, the co-products ΔP_0 and ΔJ_i's are still primitive since the only undeformed commutators in (2.4) are those featuring time translation P_0 and rotations J_i. As a consequence, despite the fact that three dimensional rotation sector (generators of $SO(3)$) is still a subalgebra of $\mathcal{P}_\kappa$, the whole Lorentz sector is not, due to the presence of P_μ's in the $[K_i, K_j]$ commutator. In the previous chapter we learned that duality between Hopf algebras can be used to introduce a generalized phase space. In particular, from a quantum group of transformation one is able to reconstruct the corresponding deformed space. It follows that given the translation sector $\mathcal{T}_\kappa \subset \mathcal{P}_\kappa$ it make sense to consider its dual $\mathcal{T}^*_\kappa$ as the non-commutative counterpart of the Minkowski space-time. Just as in the example of Sec. 1.5, we have the duality relation

$$\langle x_\mu, P_\nu\rangle = -i\eta_{\mu\nu}, \qquad \forall x_\mu \in \mathcal{T}^*_\kappa,\ \forall P_\nu \in \mathcal{T}_\kappa. \tag{2.9}$$

Using the duality pair axioms (1.47)-(1.51) we are able to reconstruct the Hopf algebra structure of $\mathcal{T}^*_\kappa$ from that of $\mathcal{T}_\kappa$, which is given by commutators (2.5) and (2.6). In particular, from the dual pair relation

$$\begin{aligned}\langle [x_0, x_i], P_j\rangle &= \langle x_0 \otimes x_i - x_i \otimes x_j,\ \Delta P_j\rangle \\ &= \langle x_0 \otimes x_i - x_i \otimes x_j,\ P_j \otimes e^{\frac{P_0}{2k}} + e^{-\frac{P_0}{2\kappa}} \otimes P_j\rangle \\ &= \langle x_0, e^{-\frac{P_0}{2\kappa}}\rangle\langle x_i, P_j\rangle - \langle x_i, e^{-\frac{P_0}{2\kappa}}\rangle\langle x_0, P_j\rangle = \\ &= 2\langle x_0, e^{-\frac{P_0}{2\kappa}}\rangle\langle x_i, P_j\rangle = -\frac{i}{\kappa}\langle x_0, P_0\rangle\langle x_i, P_j\rangle = \langle -\frac{i}{\kappa}x_i, P_j\rangle,\end{aligned} \tag{2.10}$$

where we used $\langle x_0, e^{\pm\frac{P_0}{2k}}\rangle = \pm\frac{i}{2\kappa}$one gets the commutation relation on $\mathcal{T}^*_\kappa$:

$$[x_0, x_i] = -i\lambda x_i, \quad [x_i, x_j] = 0, \qquad \forall x_0, x_i \in \mathcal{T}^*_\kappa, \tag{2.11}$$

with $\lambda = 1/\kappa$. Such a non-commutative space is known as $\kappa-$Minkowski space-time $\mathcal{M}_\kappa$. In other words, in $\mathcal{M}_\kappa$ the space x_i and time x_0 "coordinates" have a non vanishing commutator.

[1] The exponentials appearing in this section are intended as formal power series.

2.1.2 Covariance in $\kappa-$Minkowski

In last section the M_κ space has been constructed by dual pairing $\mathcal{T}^*_\kappa \subset \mathcal{P}_\kappa$. Nevertheless, this is not sufficient to guarantee its physical attainability. Even if we are talking about a "fuzzy" space-time, we still want some notion of covariance to hold. In other words, one supposes that $\mathcal{P}_\kappa$ is the quantum group of symmetries over $\mathcal{M}_\kappa$; just as $\mathcal{P}$ is the symmetry group of the usual Minkowxki space-time $\mathcal{M}^4$. Nevertheless, this last feature does not emerge straight-forewordly from previous sections. Remember that what we are truly interested in is the $\kappa-$deformed enveloping algebra $U(\mathcal{P}_\kappa)$, thus we have a wide freedom in the choice of generators which involves non-linear combinations of them. As a consequence, to any choice of the generators of $U(\mathcal{P}_\kappa)$ corresponds a different form of the commutators (2.4) and co-product (2.5)-(2.8) in the Hopf algebra of $\mathcal{P}_\kappa$. It turns out that the covariance of (2.11) under $\mathcal{P}_\kappa$ is more easily understood in a specific base. In fact, it has been proved in [18] that there exists a basis such that $\mathcal{P}_\kappa$ manifestly shows a *bicross product* structure (see Sec. 1.4.4) of the form $U(so(3,1)) \bowtie \mathcal{T}$. In other words, $\kappa-$Poincaré is just the semi-direct product of the usual Lorentz and translation group endowed with a **deformed** action of $\mathfrak{so}(3,1)$ over $\mathcal{T}$. In order to cast $\mathcal{P}_\kappa$ in the *Majid-Ruegg bicrossproduct basis* one has to perform the following non linear transformation on the generators:

$$P_0 \to -P_0, \qquad P_i \to -P_i e^{-\frac{P_0}{2\kappa}}, \qquad K_i \to K_i\, e^{-\frac{P_0}{2\kappa}} - \tfrac{1}{2\kappa}\epsilon_{ijk}J_jP_k\, e^{\frac{P_0}{2\kappa}} \tag{2.12}$$

In this way one gets an algebra in which the deformation affects the commutators featuring a Lorentz generator and a translation only

$$\begin{array}{ll}
[P_\mu, P_\nu] = 0, & [J_i, P_0] = 0 \\
[J_i, J_j] = i\epsilon_{ijk}J_k, & [K_i, P_0] = iP_i, \\
[J_i, K_j] = i\epsilon_{ijk}K_k, & [J_i, P_j] = i\epsilon_{ijk}P_k, \\
[K_i, K_j] = -i\epsilon_{ijk}J_k, & [K_i, P_j] = i\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) - i\frac{P_iP_j}{\kappa}\right),
\end{array} \tag{2.13}$$

From the first column we see that both the Lorentz sector and the translation sector close a subalgebra with the usual commutation relations. On the other hands, In the second column there are the deformed cross commutator between the two sectors. Also the co-product appears different in bicorssproduct base:

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \tag{2.14}$$

$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i \tag{2.15}$$

$$\Delta P_i = P_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_i \tag{2.16}$$

$$\Delta K_i = K_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes K_i - \frac{1}{\kappa}\epsilon_{ijk}P_j \otimes J_k \tag{2.17}$$

Notice how P_0 and the J_i's are still primitive element. Indeed, this is not a different deformation of the Poincaré group but it is just the $\kappa-$deformation cast with a different choice of the generators instead. Indeed, if one replaces ΔP_i in (2.10) with (2.16) he obtains exactly the same result as in (2.11) *i.e.* the $\mathcal{T}$ sector of $U(so(3,1)) \bowtie \mathcal{T}$ is still dual paired to $\kappa-$Minkowski $\mathcal{M}_\kappa$; just as it was the dual paired to $\mathcal{T}_\kappa$. Now, we rewrite the $\kappa-$Poincaré generators P_i, J_i, K_i in terms of the usual generators $\{p_\mu\}$ of $\mathcal{T}$ and m_i, n_i of $SO(3,1)$ as follows:

$$P_\mu = 1 \otimes p_\mu, \qquad J_i = m_i \otimes 1 \qquad K_i = n_i \otimes 1 \tag{2.18}$$

In other words, we represent the P, J, K as elements of $\mathfrak{so}(3,1) \otimes \mathcal{T}$. Using the notions of Sec. 1.3 it is possible to rewrite the cross commutators in terms of the action $\triangleleft$ of $\mathfrak{so}(3,1)$ on $\mathcal{T}$ as follows [18]:

$$\begin{aligned} [J_i, P_0] &= -1 \otimes (p_0 \triangleleft m_i), \qquad & [K_i, P_0] &= -1 \otimes (p_0 \triangleleft n_i), \\ [J_i, P_j] &= -1 \otimes (p_j \triangleleft m_i), \qquad & [K_i, P_j] &= -1 \otimes (p_j \triangleleft n_i), \end{aligned} \tag{2.19}$$

which by comparison with (2.13) give the $\mathcal{T} \triangleleft so(3,1)$ left action

$$\begin{aligned} p_0 \triangleleft m_i &= 0, \qquad & p_0 \triangleleft n_i &= -ip_i, \\ p_i \triangleleft n_j &= -i\epsilon_{ijk} p_k, \qquad & p_i \triangleleft n_j &= -i\delta_{ij} \left(\tfrac{\kappa}{2}(1 - e^{-\frac{2p_0}{\kappa}}) + \tfrac{p^2}{2\kappa} \right) + i\tfrac{p_i p_j}{\kappa}. \end{aligned} \tag{2.20}$$

Furthermore, we have also the coaction $\beta : so(3,1) \to \mathcal{T} \otimes so(3,1)$ defined as

$$\beta(m_i) = 1 \otimes m_i, \qquad \beta(n_i) = e^{-\frac{p_0}{\kappa}} \otimes n_i + \tfrac{1}{\kappa}\epsilon_{ijk}\, p_j \otimes m_k. \tag{2.21}$$

Then, it can be checked [18] that $(U(so(3,1)) \bowtie \mathcal{T}, \triangleleft, \beta)$ is a bicrossproduct Hopf algebra (in the sense of Sec. 1.3) with antipode

$$\begin{aligned} S(P_0) &= -P_0, \qquad & S(P_i) &= -e^{\frac{P_0}{\kappa}} P_i, \\ S(J_i) &= -J_i \qquad & S(K_i) &= -e^{\frac{P_0}{\kappa}} + \tfrac{1}{\kappa}\epsilon_{ijk} P_j \otimes J_k \end{aligned} \tag{2.22}$$

However, (2.20) does not give any explicit informations about the action of the Lorentz subalgebra on the elements $P_\mu \in U(so(3,1) \bowtie \mathcal{T}$ but only on the p_μ's. We recall that for undeformed Poincaré algebra the adjoint action defined in (1.78) gives just the commutators $[p_i, m_i] = p_i \overset{ad}{\triangleleft} m_i$. Then, since the left action in (2.20) has been deduced from commutator (2.13), the most natural generalization of (2.20) to the whole $U(so(3,1) \bowtie \mathcal{T}$ is given by the adjoint action of the Lorentz sector on translations. In formulae we have $P_\mu \overset{ad}{\triangleleft} J_i = S(J_i) P_\mu J_i = [J_i, P_\mu]$, the same holds for the K_i's. It follows that

$$P_\mu \overset{ad}{\triangleleft} J_i = -p_\mu \triangleleft m_i, \qquad p_\mu \overset{ad}{\triangleleft} K_i = -p_\mu \triangleleft n_i. \tag{2.23}$$

Now that we have defined the action on the translation sector for the whole deformed algebra,we are also able to properly work out the action on its dual paired space, *i.e.* $\kappa-$Minkowski. We have all what we need to show that $\kappa-$Mincowski is covariant under $\kappa-$Poincaré. We start from the translation sector. In this case, the duality relation between $\mathcal{T}$ and $\mathcal{M}_\kappa$ can be used to define a (canonical) action $\mathcal{T} \triangleright \mathcal{M}_K$

$$P_\mu \triangleright x_\nu = \langle (x_\mu)_{(1)}, P_\nu \rangle (x_\mu)_{(2)} = -i\eta_{\mu\nu} \tag{2.24}$$

where we used the Swindler notation $\Delta x_\mu = (x_\mu)_{(1)} \otimes (x_\mu)_{(2)}$. Moreover, the product of two coordinates is $P_\mu \triangleright (x_\alpha x_\beta) = (P_\mu \triangleright x_\alpha)(P_\mu \triangleright x_\beta)$, thus

$$\begin{aligned} P_0 \triangleright (x_0 x_i) &= -ix_i, \qquad P_i \triangleright (x_j x_0) = i\delta_{ij}, \\ P_0 \triangleright (x_i x_0) &= -ix_i, \qquad P_i \triangleright (x_0 x_j) = i(x_0 + i\lambda)\delta_{ij}. \end{aligned} \tag{2.25}$$

From the first line of (2.25) P_0 we have

$$0 = P_0 \triangleright x_0, x_i - P_0 \triangleright x_i x_0 = P_0 \triangleright [x_0, x_i] = P_0 \triangleright (-i\lambda x_i) = 0. \tag{2.26}$$

While using the second line of (2.25) we have

$$\begin{aligned} -i\lambda\delta_{ij} &= P_i \triangleright (-i\lambda x_j) = P_i \triangleright [x_0, x_j] \\ &= P_i \triangleright x_0 x_j - P_i \triangleright x_j x_0 = i\delta_{ij} - i(x_0 + i\lambda)\delta_{ij} = -i\lambda\delta_{ij}, \end{aligned} \tag{2.27}$$

thus we conclude that (2.11) is invariant under the action of the translation sector. At this point we need to know the action of the Lorentz sector on $\mathcal{M}_\kappa$. This can be easily deduced using (1.80) together with the fact that $\mathcal{T}$ and $\mathcal{M}_\kappa$ are dual paired

$$\langle P_\mu \overset{adj}{\triangleleft} L, x_\nu \rangle = \langle P_\mu, L \overset{adj}{\triangleright} x_\nu \rangle, \quad L \in so(3,1), \tag{2.28}$$

which is to say

$$\begin{aligned} J_i \triangleright x_0 &= x_0, \qquad J_j \triangleright x_i = i\epsilon_{ijk} x_k, \\ K_i \triangleright x_0 &= -ix_i, \qquad K_i \triangleright x_j = -\delta_{ij} x_0. \end{aligned} \tag{2.29}$$

We proceed as in (2.25) and apply J, K to $[x_0, x_i]$. Thus from

$$\begin{aligned} J_i \triangleright (x_0 x_i) &= i\epsilon_{ijk} x_0 x_k, \qquad K_i \triangleright (x_0 x_i) = -\delta_{ij} x_0^2 - ix_0 x_i + i\lambda\delta_{ij} x_0 \\ J_i \triangleright (x_i x_0) &= i\epsilon_{ijk} x_k x_0, \qquad K_i \triangleright (x_i x_0) = -\delta_{ij} x_0^2 - ix_0 x_i \end{aligned} \tag{2.30}$$

we have

$$\begin{aligned} \lambda\epsilon_{ijk} x_k &= J_i \triangleright (-i\lambda x_j) = J_i \triangleright [x_0, x_j] \\ &= J_i \triangleright (x_0 x_j) - J_i \triangleright (x_j x_0) = i\epsilon_{ijk}(x_0 x_k - x_k x_0 l) = \lambda\epsilon_{ijk} x_k, \end{aligned} \tag{2.31}$$

$$\begin{aligned} i\lambda\delta_{ij} x_0 &= K_i \triangleright (-i\lambda x_j) \\ &= K_i \triangleright [x_0, x_k] = K_i \triangleright x_0 x_j - K_i \triangleright x_j x_0 = i\lambda\delta_{ij} x_0, \end{aligned} \tag{2.32}$$

which proves that (2.11) is invariant under rotations and boosts.

2.2 Between time and space

In the previous section we showed that for coordinates on the homogeneous space of $\kappa-$Poincaré (2.11) holds covariantly *i.e.* for any "$\kappa-$inertial" observer. Notice that, since both the geometry of space-time $\mathcal{M}_\kappa$, as well as its symmetries $\mathcal{P}_\kappa$ has been deformed, also the concept of observer itself deserves some discussion, which we will later give. For the moment, we assume that x_0 and x_i represent respectively time and position in a given reference frame. Now, we want to discuss what the non-commutativity

$$[x^0, x^i] = i\lambda x^i, \qquad [x^i, x^j] = 0, \tag{2.33}$$

in coordinates means from a physical point of view. In other words, x^0,x^i together with (2.33) close a Lie algebra which is isomorphic to $\mathfrak{an}_3$. This last generates the group AN_3, which is the upper triangular section of the Iwasawa decomposition of $SL_3(\mathbb{C})$ [22,23,57,58]. If one follows the intuitive Einsteinian notion of coordinates x^0, x^i as measurement of time and distances of a given event, then it is quite natural to deduce from (2.33) the impossibility to simultaneously localize an event in both time and space outside of spatial origin (where we have $x^i = 0$). Then, it is quite tempting to derive from (2.33), (in analogy with Heisenberg principle in quantum mechanics) a sort of **non-localizability** principle

$$\Delta x^0 \Delta x^i \geq \frac{\lambda}{2} \mid \langle x^i \rangle \mid, \tag{2.34}$$

where Δx^0, Δx^i are the variance and $\langle x^i \rangle$ is the mean value of some *localizability density* distribution describing the fuzziness of a blurred event (de)localized around $\langle x^0 \rangle$ and $\langle x^i \rangle$). This is better understood once a representation of x^0 and x^i in terms of self adjoint operator has been given together with a more precise definition of what the above introduced *localizability density* is.

2.2.1 States of an algebra

In this section we want will give a precise definition of states. We start with some definitions

$*$-Algebra

Definition 2.2.1. *Given an algebra $\mathcal{A}$ and a field $\mathcal{K}$, we say that $\mathcal{A}$ is a $*-$algebra over $\mathcal{K}$ if there exist a $\mathcal{K}$-bilinear multiplication map and an anti-linear map $* : \mathcal{A} \ni a \to a^* \in \mathcal{A}$ such that*

$$(a^*)^* = a, \qquad \forall a \in \mathcal{A},$$
$$(ab)^* = b^* a^*, \qquad \forall a, b \in \mathcal{A}.$$

Furthermore, if $(\mathcal{A}, \|.\|)$ is Banach algebra over a topological filed $\mathcal{K}$ *i.e.* it is complete under the norm $||.||$, then we say it to be a Banach $*$- algebra. Usually in physics we are interested in Banach $*$-algebras over the complex field ($\mathcal{K} = \mathbb{C}$). In particular, the Connes formulation of non-commutative geometry [59, 60] is based on the notion of C^*-algebra.

Definition 2.2.2. *Given an Banach $*$-algebra $(\mathcal{A}, \|.\|)$ and a field $\mathcal{K}$, we say that it is a C^*-**algebra** if it satisfy the C^*-identity:*

$$\|a^*a\| = \|a\|^2, \qquad \forall a \in \mathcal{A} \tag{2.35}$$

Note that, being a Banach space, a C^*-Algebra is also *complete* under the norm $\| \cdot \|$. In other words for every *Cauchy sequence* $\{a_n\} \in \mathcal{A}$ there exist an element $a \in \mathcal{A}$ such that

$$\lim_{n\to\infty} ||a_n - a|| = 0. \tag{2.36}$$

There exist a theorem due Gelfand and Naimark [61] which states that there is a complete equivalence between compact Hausdorff spaces and commutative unital C^*-algebras. This result allows one to reconstruct a topological space from the space of continuous functions over it. Connes approach [60] is to provide a notion of geometry for non-commutative C^*-algebras by following the previous paradigm, *i.e.* by studying the space of continuous functions defined over it, which (unlike the conepts points) is well defined.

Given a C^*-algebra $\mathcal{A}$, we say that any linear functional $\rho : \mathcal{A} \to \mathbb{C}$ which satisfy

$$\rho(A) \geq 0, \quad \forall a \in \mathcal{A}, \tag{2.37}$$

$$\rho(1) = 1, \tag{2.38}$$

is a **state** of $\mathcal{A}$. The set of all states of a C^*-algebra is called **state space** and its extremal points are called **pure state** [62].

2.2.2 Toward the notions of localization

Classical (localized) Points and Probability densities

In classical mechanics the motion of a point-like particle is completely described in terms of it position $\mathbf{q}$ and momentum $\mathbf{p}$ at any given time. As it is well the $\mathbf{p}$'s and the $\mathbf{q}$'s close the commutative algebra describing coordinate on phase space. Although any point in classical phase space is perfectly localized, it is useful for many physical application to introduce the notion of *probability density* $\rho(\mathbf{p}, \mathbf{q})$ over classical phase space *i.e.* the ρ's have to belong to the space of positive normalized integrable functions $\rho(\mathbf{p}, \mathbf{q}) \in L^1(\mathbb{R}^{2d})$ where d is the space dimension. In this way p and q can be represented as multiplication operators over

$\overline{L^1(\mathbb{R}^{2d})}$, then any normalized $\rho(\mathbf{p}, \mathbf{q})$ is a state of the algebra. It is clear that any function $f \in L^1(\mathbb{R}^{2d})$ cannot be a **pure state** since it will always be possible to decompose it as the sum of two other functions. It follows that pure states are given by Dirac's deltas of the form $\delta(p_0, q_0)$, which can be obtained as a limit of normalized vectors and belongs to the $\partial L^1(\mathbb{R}^2 d)$.

Quantum Phase Space

We now discuss the quantum mechanical case. We have non-commutative phase space whose Lie algebra satisfies $[p, q] = i\hbar$. As it is well known, the Hilbert space (states) H of this algebra is $L^2(\mathbb{R}^d)$. *wave functions*. A pure state then is a *wavefunction i.e.* a normalized vector in $L^2(\mathbb{R}^d)$, while mixed density matrices represent a generic (non pure) state. In this case we represent $\mathbf{p}$ and $\mathbf{q}$ as self-adjoint operators on the Hilbert space $\hat{\mathbf{q}}, \hat{\mathbf{p}} \in \mathrm{OP}\left[L^2(\mathbb{R}^d)\right]$. Notice that since the phase space is a non-commutative geometry there are no state which correspond to perfectly localized points: we only have wave function whose square modulus is postulated to give the probability density to find the particle with a certain position or momentum. In other words the phase space itself has been fuzzyfied and the wave functions describe the blurred (once point-like) particles. Physical observables are represented by hermitian operators and their eigenvalues represent the possible outcome of a measurement. Furthermore, from the above physical interpretation, it follows that for any given couple of hermitian operators A,B one has $\mathbf{\Delta} A \mathbf{\Delta} B \geq\mid [A, B] \mid /2$, which for $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ gives the Hiesenberg indetermination principle

$$\mathbf{\Delta}\hat{\mathbf{q}}\ \mathbf{\Delta}\hat{\mathbf{p}} \geq \frac{\hbar}{2}, \tag{2.39}$$

where $\mathbf{\Delta} x$ is the standard deviation of x and has not to be confused with a co-product. In this case any vector in the Hilbert space is a pure state while a generic mixed state is given by mixed density matrices. The inequality (2.39) states that it is impossible to know both position and momentum of a point-like particle at the same time. From a geometrical point of view this means that it is impossible to fully localize a state in a phase space region which is smaller than $\hbar/2$ as consequence of the "fuzzyfication" of points. Furthermore, the modulus square of wave functions can be regarded has the "localizabilty density" of a state. In this picture the objects which resemble classical point the most are the coherent states, *i.e.* states which saturate (2.39) with the equality which are Gaussian state. In other words, in quantum mechanics the mostly localized states are obtained by Gaussian blurring the points of classical phase space. As it is well known, both $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are represented by unbounded hermitian operators with the following action

$$\forall i, \quad \hat{q}^i\psi(q) = q^i\psi(q), \quad \hat{p}^i\psi(q) = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q), \quad \psi(q) \in L^2(\mathbb{R}^3) \tag{2.40}$$

whose spectrum is the real line. The *eigenvalue problem*

$$(\hat{q}^i - \alpha 1)\psi(q) = 0 \tag{2.41}$$

does not admit any proper eigenvector. Nevertheless, it admits distributions as improper solutions. The $\hat{q}^i$'s commute among themselves thus it is possible to have a simultaneous improper eigenvector of them all, which is given by *Dirac* deltas $\delta(\mathbf{q} - \mathbf{q}')$ picked at a generic vector $\mathbf{q} \in \mathbb{R}^3$. Also the $\hat{p}^i$ commutes and their improper eigenfunctions are given by the plane waves $e^{i(p')^i q_i}$ with $\mathbf{p}' \in \mathbb{R}^3$. However, (2.40) is just one representation of the non-commutative phase space. In particular, since we choose the $\hat{q}^i$ as a complete set of observables, in representation (2.40) the quantum states are the elements of $L^2(\mathbb{R}^3)$ intended as function of the position $\mathbf{q} \in \mathbb{R}^3$. In fact, a different choice of the complete set of commuting observable would produce a different representation. As an examples if one starts from the $\hat{p}$'s he get the following representation

$$\forall i, \quad \hat{q}^i\phi(p) = \mathrm{i}\hbar\frac{\partial}{\partial p^i}\phi(p), \quad \hat{p}^i\phi(p) = p^i\varphi(p), \quad \phi(p) \in L^2(\mathbb{R}^3). \tag{2.42}$$

Notice how the elements of $\mathbb{R}^3$ are regarded as positions $\mathbf{q}$ in (2.40) and as momenta $\mathbf{p}$ in (2.42). Nevertheless, whether one consider the elements of $L^2(\mathbb{R}^3)$ as $\psi(q)$'s or $\phi(p)$'s they must carry the same information. Indeed, the functions of position are connected to those of momenta under an isomorphism $\mathcal{F}$ (Fourier transform)

$$\psi(q) = \mathcal{F}[\phi(p)] = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p\ \phi(p)e^{\frac{i}{\hbar}p\cdot q} \tag{2.43}$$

All the contents of this section are of course well known to any physics. In what follows, we want to study the non-commutative $\kappa-$Minkowski space-time with the same spirit.

2.2.3 Repesentation as Operator on Hilbert Space

In previous section we gave a brief review of a well known example of non-commutative geometry: the quantum phase space. We pose our attention on how the lack of localizability in the non-commutative phase space plays a fundamental role in the probabilistic interpretation of quantum mechanics. Somehow the quantum interpretation of $|\psi|^2$ as probability density is equivalent to the geometrical interpretation as localization density, which is what we are currently interested in.

In the spirit of previous section we want to give a representation to the $\mathcal{M}_\kappa$ Lie algebra (2.33) as operator on a Hilbert space. The representations of the algebra generated by (2.33) are discussed in detail in [63, 64]. In [65–67] Meljanac and Stojic have written (in the Euclidean context) the most general class of operator with the correct characteristic, and shown that they depend on two functions with some constraints. We consider the $\hat{x}^i$ as a maximal set of commuting operators on $L^2(\mathbb{R}^3_x)$, the x subscript keeps track of our interpretation of the $\mathbf{x} \in \mathbb{R}^3_x$ as spatial coordinates. In particular, we focus on the following realization

$$\hat{x}^i \psi(x) = x^i \psi(x), \tag{2.44}$$

$$\hat{x}^0 \psi(x) = \mathrm{i}\lambda \left(\sum_i x^i \partial_{x^i} + \frac{3}{2} \right) \psi(x). \tag{2.45}$$

The $\frac{3}{2}$ factor is necessary to have symmetric operators in 3 dimensions, in d dimensions $\frac{1}{2}(r\partial_r + \partial_r r) = r\partial_r + \frac{d}{2}$. Here, $\hat{x}^0$ plays the role that $\hat{p}$ played in quantum phase space. From (2.45) one recognize $\hat{x}^0$ to be (up to constants) a dilation operator while the space coordinate acts as a multiplication. That fact that we are treating $\hat{x}^0$ and $\hat{x}^i$ so differently may misleadingly suggest that we are renouncing the relativistic equivalence between space and time. However, it has already been broken in algebra (2.33) which is of course not invariant under *ordinary* Lorentz transformations. We remind the readers that we are dealing with a *quantum homogenous space* which is covariance under the $\kappa-$Poincaré whose Lorentz sector (2.4) is far from being the ordinary one. As we will discuss in the next section, the fact that covariance has been deformed implies that the way transformations between different observers are concerned need to be revised.

Since $\hat{x}^0$ acts as a dilation it is quite alluring to swap in $\mathbb{R}^3$ from euclidean (x^1, x^2, x^3) to polar basis (r, θ, ϕ). The polar coordinates $\hat{\theta}, \hat{\varphi}$ do not correspond to well defined self-adjoint operators. Nevertheless, if one defines

$$\hat{r}\cos\hat{\theta} = \hat{x}^3, \qquad \hat{r}\,\mathrm{e}^{\mathrm{i}\hat{\varphi}} = (\hat{x}^1 + \mathrm{i}\hat{x}^2), \tag{2.46}$$

a simple calculation shows that

$$[\hat{x}^0, \cos\hat{\theta}] = [\hat{x}^0, \mathrm{e}^{\mathrm{i}\hat{\varphi}}] = 0\,, \qquad [\hat{x}^0, \hat{r}] = \mathrm{i}\lambda\hat{r}\,. \tag{2.47}$$

In fact, $\hat{x}^0$ commutes with any functions of $\hat{\theta}$ and $\hat{\varphi}$ independent of r, like spherical armonics. Hence in the following we will consider the vectors of $L^2(\mathbb{R}^3_x)$ to be functions of the form $\psi = \sum_{lm} \psi_{lm}(r) Y_{lm}(\theta, \varphi)$. Moreover, since the angular variables commute with everything, we will often focus on the radial parts, and consider functions of r variable alone. In particular, instead of (2.44) and (2.45) we have for the radial parts

$$\hat{r}\ \psi(r) = r\psi(r), \tag{2.48}$$

$$\hat{x}^0\psi(r) = \mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right)\psi(r). \tag{2.49}$$

Although we constructed a symmetric $\hat{x}^0$ operator, we also have to find its self-adjointness domain. Notice that while the angular degrees of freedom can be integrated out, the integration over r may be troublesome. Integrating by parts, one finds:

$$\int \mathrm{d}r r^2\, \psi_1^* \mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right)\psi_2 = \mathrm{i}\lambda\int \mathrm{d}r r^2\, \psi_1^*\frac{3}{2}\,\psi_2 - \int \mathrm{d}r\, \mathrm{i}\lambda\partial_r\left(r^3\psi_1^*\right)\psi_2 + \psi_1^* r^3\psi_2\Big|_0^\infty . \tag{2.50}$$

The boundary term disappears if both ψ_1 and ψ_2 vanish at infinity faster than $r^{-\frac{3}{2}}$, which is true for all square-integrable functions with the measure $\int \mathrm{d}r r^2$. In the origin the condition imposed is weaker than the one imposed by square-integrability. We are interested in the spectrum and the (improper) eigenvectors of $\hat{x}^0$. Monomial in r are formal solutions of the eigenvalue problem for time

$$\mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right)r^\alpha = \mathrm{i}\lambda(\alpha + \frac{3}{2})r^\alpha = \lambda_\alpha r^\alpha, \tag{2.51}$$

with eigenvalues

$$\lambda_\alpha = \mathrm{i}\lambda(\alpha + \frac{3}{2}). \tag{2.52}$$

which belong to real line if and only if

$$\alpha = -\frac{3}{2} + \mathrm{i}\tau, \tag{2.53}$$

with $-\infty < \tau < \infty$ a real number. In complete analogy with the momentum case previously discussed, unless the real part of α is -3/2, the improper eigenfunctions would not be acceptable distributions. The spectrum of the time operator is real and goes from minus infinity to plus infinity. The formal solution to (2.51) are

$$T_\tau = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}, \tag{2.54}$$

These distributions are for time in κ-Minkowski space what plane waves are for momentum in quantum phase space. They are not physical states (vector of $L^2(\mathbb{R}^3_x)$) because their behaviour at the origin and at infinity is bad, but "just about", an epsilon slower at the origin and faster at infinity would do, but then they would not be eigenfunction of $\hat{x}^0$. Nevertheless, their inner product with every vector in the domain of $\hat{x}^0$ is well defined. The (2.54) distribution has the correct dimension of a length to the 3/2. Since λ is a natural scale for the model, it seems natural to introduce a factor $1/\lambda$ in the logarithm to make the argument dimensionless, however the choice is not unique The self adjoint operators $\hat{x}^0$,$\hat{x}^i$ satisfy a non-localizability principle (2.34) as well as their polar counterparts $\hat{r}, \hat{x}^0$ satisfy

$$\Delta\hat{x}^0\Delta\hat{r} \geq \frac{\lambda}{2}|\langle\hat{r}\rangle|, \tag{2.55}$$

where $\langle x^0\rangle$ and Δx^0 are defined as

$$\langle\hat{x}^0\rangle = \mathrm{i}4\pi\lambda\int r^2\mathrm{d}r\ \psi(r)^*\left(r\partial_r + \frac{3}{2}\right)\psi(r), \tag{2.56}$$

$$\Delta\hat{x}^0 = \langle\left(\hat{x}^0 - \langle\hat{x}^0\rangle\right)^2\rangle\,. \tag{2.57}$$

Notice that at this level the (2.56) have just geometrical quantity describing how accurately we are able to localize a state over the non-commutative space-time. If one assumes (in analogy with the quantum phase space) a probabilistic interpretation of the $|\psi(x)|^2$ probability density and of eigenvalues of an operator as possible outcome of a measure of the corresponding observable, then (2.56) become the mean value and variance of x^0 for the a given state ψ. In order to adopt a more familiar language we will assume that this quantum-inspired probabilistic interpretation still holds also for $\kappa-$Minkowski space-time. In this sense we say that a $\psi(x) \in L^2(\mathbb{R}^3)$ is the "state of an event". As an example, the possible outcome of measurement of time over a *eigenstate of time* (2.54) will always give $t = \tau\frac{c}{\lambda}$. Note that if $t = 1s$, then $\tau = 2\cdot 10^{43}$ while for t of the order of Planck time τ is of the order of unity.

2.2.4 Time domain, radial domain and Mellin transformation

Notice that operators $\hat{x}^0$ and $\hat{r}$ are both self adjoint. Hence, both their sets of eigenfunctions is a complete basis for the non angular sector of the space of states. Hence, both r and τ give a complete set of observables. We have two different interpretation of the element in $L^2(\mathbb{R})^3$: the *radial domain* whose states are the $\psi(r,\theta,\phi)$'s, and the *time domain* whose states are $\phi(\tau,\theta,\phi)$. Just as in quantum phase space for the q and p, the time and radial domain must carry exactly the same information. The completeness of the $\{T_\tau\}$ basis allows one to *isometrically* expand a function of the radial domain as

$$\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}\tau r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}\widetilde{\psi}(\tau,\theta,\varphi), \tag{2.58}$$

The integral above defines an isometry from functions $\widetilde{\psi}(\tau,\theta,\varphi)$ expressed in the time domain into $\psi(r,\theta,\varphi)$ in the radial domain. It can be seen as the $\kappa-$Minkowski counterpart of what the Fourier transform is in quantum phase space. The (2.58) is in fact a *Mellin* transform [68].

The Mellin transform

Definition 2.2.3. *Given a locally integrable function $f(x)$ with $x \in (0, \infty)$ defined on the half line $x \in (0, \infty)$ and a complex number $s \in \mathbb{C}$. If the following integral*

$$\mathcal{M}[f, s] = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}x \, x^{s-1} f(x) = \mathcal{F}(s) \tag{2.59}$$

*converges, then we say that $\mathcal{M}[f, s]$ is the **Mellin transform** [68] of $f(x)$.*

In particular, the integral in (2.59) converges for $Re(s) \in (A, B)$ with A and B real numbers such that

$$f(x) = \begin{cases} O\left(x^{-A-\epsilon}\right) & \text{as} \quad \chi \to 0_+ \\ O\left(e^{-B+\epsilon}\right) & \text{as} \quad \chi \to +\infty \end{cases}, \ \forall \epsilon > 0 \ , \ A < B. \tag{2.60}$$

The interval (A, B) is often called the "strip of analyticity" of $\mathcal{M}[f, s]$. There is also an inverse Mellin transform [69] defined as[2]:

$$\mathcal{M}^{-1}[\mathcal{F}(s), x] = \frac{1}{\mathrm{i}\sqrt{2\pi}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \mathrm{d}s \, x^{-s} \mathcal{F}(s), \ A < C < B \tag{2.61}$$

In our case we need a Mellin transform which is also an isometry between square integrable functions of r with measure $r^2\mathrm{d}r$ and functions of τ with measure $\mathrm{d}\tau$. The following transformation does the trick [10] :

$$\psi(r, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\tau \, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau, \theta, \varphi) = \mathcal{M}^{-1}\left[\widetilde{\psi}(\tau, \theta, \varphi), \ r\right], \tag{2.62}$$

$$\widetilde{\psi}(\tau, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}r \, r^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \psi(r, \theta, \varphi) = \mathcal{M}\left[\psi(r, \theta, \varphi), \frac{3}{2} + \mathrm{i}\tau\right]. \tag{2.63}$$

In other words, $\widetilde{\psi}$ is the Mellin transform of ψ with $s = 3/2 + \mathrm{i}\tau$. Hereafter we will often omit the explicit dependence on θ and φ when there is no confusion. The above-defined transformations preserve the norm

$$\int_0^\infty \mathrm{d}r r^2 |\psi(r)|^2 = \int_{-\infty}^{\infty} \mathrm{d}\tau |\widetilde{\psi}(\tau)|^2 \tag{2.64}$$

as we required. Likewise there is also a Parseval identity:

$$\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \mathrm{d}r \, r^2 \overline{\psi}_1(r) \psi_2(r) = \int_{-\infty}^{\infty} \mathrm{d}\tau \overline{\widetilde{\psi}_2}(\tau) \widetilde{\psi}_1(\tau) = \langle \widetilde{\psi}_1 | \widetilde{\psi}_2 \rangle \tag{2.65}$$

As stated at the end of previous section, we assume the usual measurement theory to still hold. Then we have that the average time measured by a particle in the state described by ψ with spherical symmetry is given by:

$$\langle \hat{x}^0 \rangle_\psi = 4\pi \int r^2 \mathrm{d}r \overline{\psi}(r) \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right) \psi(r) \tag{2.66}$$

[2] A more detailed discussion can be found in [70].

and the probability of measuring a given value of τ is given by $|\widetilde{\psi}(\tau)|^2$ for normalised functions. Notice that if ψ is real we have $\langle \hat{x}^0 \rangle_\psi = 0$. In fact

$$\begin{gathered}\int r^3 \mathrm{d}r \overline{\psi}(r) \partial_r \psi(r) = r^3 |\psi|^2 \Big|_0^\infty - \int r^3 \mathrm{d}r \psi(r) \partial_r \overline{\psi}(r) - 3 \int r^2 \mathrm{d}r |\psi(r)|^2 \\ \Downarrow \\ \psi = \overline{\psi} \quad \Rightarrow \quad \int r^3 \mathrm{d}r \overline{\psi}(r) \partial_r \psi(r) = -\frac{3}{2} \int r^2 \mathrm{d}r |\psi(r)|^2 \ ,\end{gathered} \tag{2.67}$$

and the two terms in (2.66) cancel each other. Hence only complex valued functions will have a non vanishing mean value for a measurement of time. One may note an analogy with quantum phase space, where real functions have a vanishing mean value of the momentum. In order to get familiar with this representation later we will give a few examples.

Example: State Localized on a Spherical Shell

Consider the following state, localized on a shell of radius r_0

$$\psi(r) = \delta(r - r_0)/r_0^2. \tag{2.68}$$

Once we transform it via (2.63) into the time domain

$$\widetilde{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} r_0^{-\frac{3}{2}} \left(\frac{r_0}{\lambda}\right)^{\mathrm{i}\tau} = \frac{1}{\sqrt{2\pi}} r_0^{-\frac{3}{2}} \mathrm{e}^{\mathrm{i}\tau \log\left(\frac{r_0}{\lambda}\right)} \tag{2.69}$$

the probability $|\psi(\tau)|^2$ does not depend on τ: this means that all values of time are equally probable, just like in quantum mechanics, where a localised particle has all values of momentum equally probable. Not surprisingly the function $\widetilde{\psi}(\tau)$ in (2.69) is not normalizable. We regularize the delta function by approximating it with a constant function with support on a “thick spherical shell”:

$$\psi(r) = \begin{cases} 0 & r < R_1 \\ \sqrt{\frac{3}{4\pi(R_2^3 - R_1^3)}} & R_1 \leq r \leq R_2 \\ 0 & R_2 < r \end{cases} \tag{2.70}$$

Then, the Mellin transform gives [10] :

$$\widetilde{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{4\pi(R_2^3 - R_1^3)}} \left(\frac{R_2^{\frac{3}{2}+\mathrm{i}\tau} - R_1^{\frac{3}{2}+\mathrm{i}\tau}}{\lambda^{\mathrm{i}\tau}} \right) \frac{2}{3 + 2\mathrm{i}\tau} \ , \tag{2.71}$$

with probability density:

$$|\widetilde{\psi}(\tau)|^2 = \frac{3}{8\pi^2(R_2^3 - R_1^3)} \left[R_2^3 + R_1^3 - 2R_1^{\frac{3}{2}} R_2^{\frac{3}{2}} \cos\left(\tau \log \frac{R_2}{R_1}\right) \right] \frac{4}{9 + 4\tau^2} \ . \tag{2.72}$$

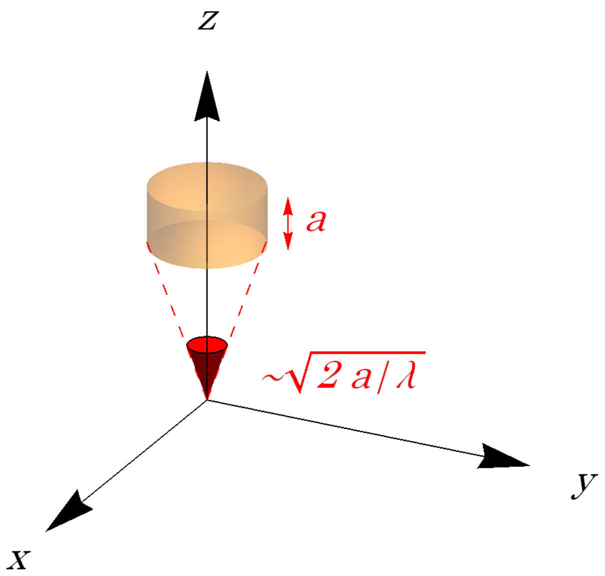


Figure 2.1: *The support of the wavefunction (2.73). [10]*

Since (2.72) is an even function, it is clear that the average value of $\hat{x}^0$ vanishes. The probability density (2.72) now is not constant any more. Instead, it is peaked around $\tau = 0$ and it decreases as τ^{-2} away from the origin. In the limit $R_1 \to R_2$ the Mellin transform (2.71) tends to (be proportional to) the Mellin transform of the delta function (2.69).

Example: Point localised at finite distance from the origin

Consider a state (de)localised in space in a small region of size a around a point at distance z_0 along the z axis. Let such a wave-function to take inside that region a constant value fixes by normalization. In spherical coordinate we have:

$$\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\lambda}{2a\pi\left((a+z_0)^3 - z_0^3\right)}}\,, & z_0 \leq r \leq (z_0+a) \ \vee \ \cos\theta > 1 - \frac{a}{\lambda} \\ 0, & \text{otherwise} \end{cases} \tag{2.73}$$

The shape of the region we are considering is shown in Fig. 2.1. For any nonzero (positive) a the wavefunction is normalized and it is a well defined state of the Hilbert space $L^2(\mathbb{R}^3_x)$. In the limit $a \to 0$ $\psi_{z_0,a}$ it goes to a δ function localised at a distance z_0 from the origin along the positive z axis. We compute the Mellin transform and get:

$$\widetilde{\psi}_{z_0}(\tau,\theta,\varphi) = \frac{\sqrt{3\lambda}}{\pi}\,\frac{(z_0+a)^{\frac{3}{2}+\mathrm{i}\tau} - z_0^{\frac{3}{2}+\mathrm{i}\tau}}{\lambda^{\mathrm{i}\tau}(3+\mathrm{i}2\tau)\sqrt{a\left((a+z_0)^3 - z_0^3\right)}}\,\Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right) \tag{2.74}$$

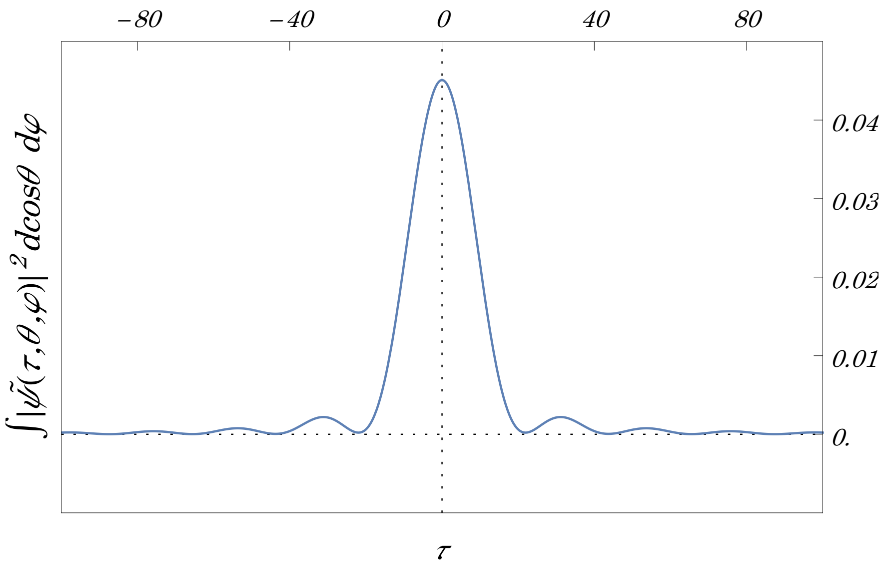


Figure 2.2: *The τ-dependence of the Mellin transform of the wavefunction (2.73). [10]*

whose probability density is:

$$\begin{aligned} |\widetilde{\psi}_{z_0,a}|^2 &= \frac{3\lambda}{\pi^2}\frac{z_0^3 + (z_0+a_0)^3 - 2\,(z_0\,(a+z_0))^{3/2}\cos\left(\tau\log\left(\frac{z_0}{z_0+a}\right)\right)}{(4\tau^2+9)\,a\,((a+z_0)^3 - z_0^3)}\Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right) \\ &= \left[\frac{\lambda}{4\pi^2 z_0} - \frac{\lambda a}{8(\pi^2 z_0^2)} + \mathcal{O}(a^2)\right]\Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right) \end{aligned} \tag{2.75}$$

We can easily carry out the integration in θ, which gives a factor a/λ:

$$\int |\widetilde{\psi}_{z_0,a}|^2 \sin\theta\, \mathrm{d}\theta = \frac{a}{4\pi^2 z_0} - \frac{a^2}{8\lambda(\pi^2 z_0^2)} + \mathcal{O}(a^3) \tag{2.76}$$

By taking the limit $a \to 0$, the Mellin-transformed wavefunction tends to a constant $\frac{\lambda}{4\pi^2 z_0}$ localised at θ in a cone of angle $\arccos(1-\frac{a}{\lambda}) - \pi/2 \sim \sqrt{\frac{2a}{\lambda}}$. Moreover, the angular average tends to a constant which vanishes as $a \to 0$ (because of the normalization). It follows that in the limit the state is not an L^2 function anymore: it becomes a function whose scalar product with any element $L^2(\mathbb{R}^3)$ is zero instead. Not surprisingly, the series expansion for a around 0, and z_0 around ∞ are the same:

$$\begin{aligned} |\widetilde{\psi}_{z_0}|^2 &= \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda\,(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(a^3\right) \\ &= \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda\,(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(z_0^{-4}\right) \end{aligned}$$

This means that a sharp localisation of a particle far away from the origin implies that the particle cannot be localised in time, which is in agreement with the generalised uncertainty principle (2.55).

2.2.5 localising the Origin of Space

We have shown that localising (in space and time) a particle at the origin is different then in regions away from it. Although the non localisability principle (2.55) limit the simultaneous localisabilty of any particle in space and time, it is compatible with a one-parameter family of L^2 functions which tends to a state completely localised at the spatial origin (while in time it might be either completely localised around any value of τ, or it may be nonlocal). Indeed, the presence in relations (2.55) of $\langle \hat{r} \rangle$ on the right hand side suggests that, although there are no general localised states with $\langle r \rangle \neq 0$, it is still possible to have states in the origin $\langle r \rangle = 0$ with perfect localisation $\Delta r = 0$. In analogy with delta functions and plane waves in ordinary quantum mechanics, these states are as the limits of normalized vectors of our Hilbert space. The key is to find functions which saturate the uncertainty bounds *i.e.* which solve (2.55) with the $=$ sign. As discussed in Sec. 2.2.2 for the quantum phase space algebra, these *coherent states* are Gaussians functions. The κ-Minkowski algebra however is not a canonical one and Gaussians are not the state of minimal uncertainty. This role is played by *log-Gaussians* [10] normalized wave-function instead, which we plotted in Fig. 2.3.

$$L(r, r_0) = N\mathrm{e}^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = \frac{\mathrm{e}^{-\left(\frac{\log\left(\frac{r}{r_0}\right)}{\sigma}\right)^2} \mathrm{e}^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}} \,. \tag{2.77}$$

Any of the $L(r, r_0)$ has its minimum value in $r = r_0$. Moreover, they localise at $r = r_0$ as $\sigma \to 0$, and at $r = 0$ as $r_0 \to 0$, for any value of $\sigma \geq 0$.

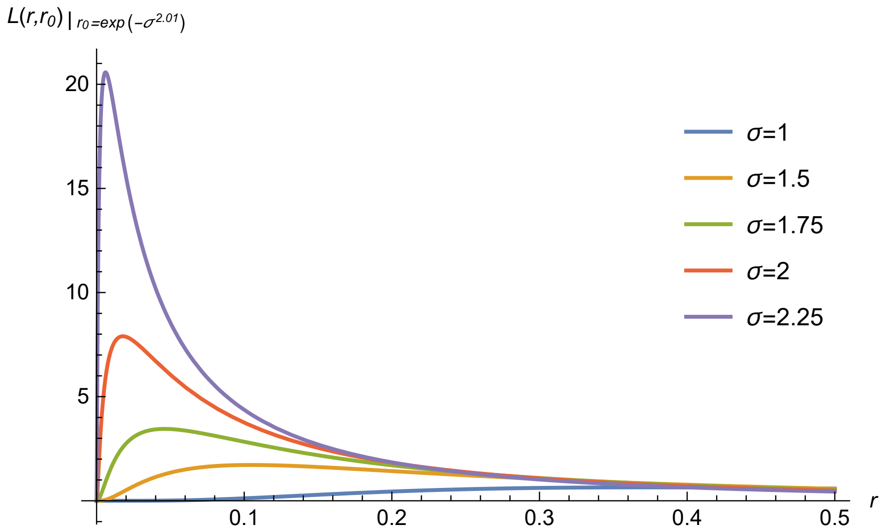


Figure 2.3: *The $\sigma \to \infty$ limit of $L(r, r_0)$ when $\xi = e^{-\sigma^{(2+\epsilon)}}$, for $\epsilon = 0.01$.* [10]

A straightforward calculation shows that the average values of any powers $\hat{r}^n$ is :

$$\langle \hat{r}^n \rangle_L = \mathrm{e}^{\frac{\sigma^2}{8} n(n+6)} r_0^n \,, \tag{2.78}$$

and that they all vanish as $r_0 \to 0$. In order to calculate the quantity $\langle r^n \rangle_L$ it is best to Mellin transform, since the function in τ space is remarkably simpler:

$$\widetilde{L}(\tau, r_0) = \frac{\sigma^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{4}\sigma^2 \tau(\tau - 3\mathrm{i})}}{2\sqrt[4]{2}\pi^{3/4}} \left(\frac{r_0}{\lambda}\right)^{\mathrm{i}\tau}, \tag{2.79}$$

Interestingly, we obtain:

$$\left|\widetilde{L}(\tau, r_0)\right|^2 = \frac{\sigma \mathrm{e}^{-\frac{\sigma^2\tau^2}{2}}}{4\sqrt{2}\pi^{3/2}}, \tag{2.80}$$

hence in τ space the probability density is a Gaussian independent on r_0. It is now easy to

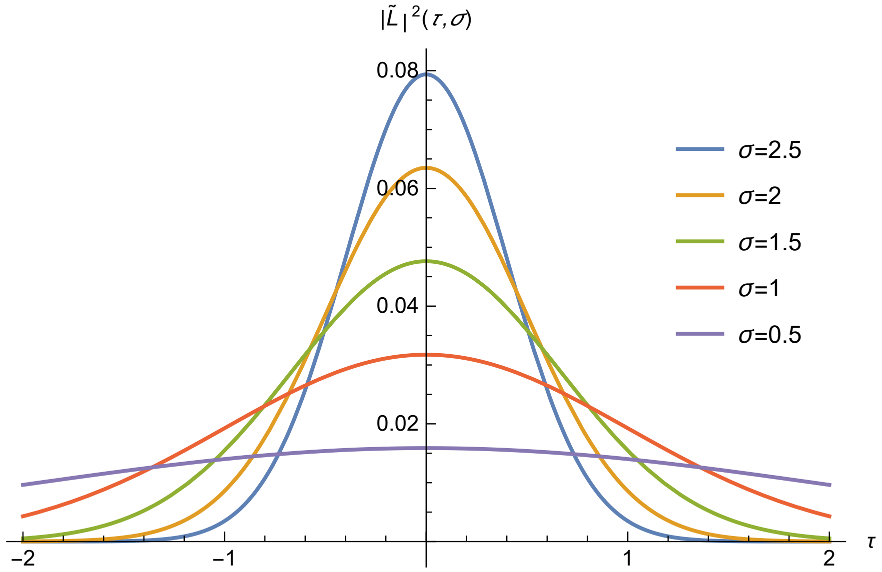


Figure 2.4: *The Gaussians obtained by Mellin transform the log Gaussians $L(r, r_0)$ (2.77) for different values of parameter σ. [71]*

see that

$$\langle (\hat{x}^0)^n \rangle_L = \frac{1}{4\pi} \left(\frac{\lambda}{\sigma}\right)^n \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases} \tag{2.81}$$

Note that there is a double limit $r_0 \to 0$ and $\sigma \to \infty$[3] which gives a state localised both in space (at $r = 0$) and in time. In this example we considered a state time-localised at $\tau = 0$, nevertheless it is easy to time shift the state by multiplying the function by $r^{\mathrm{i}\tau_0}$. Furthermore, one may consider any wavefunction for the temporal part while still keeping the spatial coordinates localised at the origin, just by convoluting this with a function of τ. We call the above introduced state the "eigenstate of the origin" $|o\rangle$ since it is localised at the origin of space-time and it can be obtained as a limit of normalized elements of $L^2(\mathbb{R}^3_x)$.

[3]For example, it is sufficient to take $r_0 = e^{-\sigma^{2+\epsilon}}$ for any $\epsilon > 0$, that all $\langle \hat{r}^n \rangle_L$ in (2.78) and all $\langle (\hat{x}^0)^n \rangle_L$ in (2.81) go to zero as $\sigma \to \infty$.

While we have seen that there is a state corresponding to $|o\rangle$, there is not a normalized vector corresponding to it. Here (and in the following) we are performing the usual abuse of notation made when one uses the ket notation $|x\rangle$ in ordinary quantum mechanics. Moreover, we have a 1-parameter family of states, denoted with $|o_\tau\rangle$, which are localised at the origin of space, but at a non-zero time. Also these states can be obtained as limits of normalized elements of $L^2(\mathbb{R}^3_x)$.

Summarizing, the origin of space has a quite special localisation behaviour which allows to define a family of perfectly localised states all in the same position ($r = 0$) but different time τ. This inhomogeneity between the origin and all the rest of the space seems to suggest that the model intrinsicly admits a somehow privileged point to exist at different time but in precise position in space. Of course, this would be a very unpleasant property from a physical point of view. Nevertheless, we will show the point described by $|o\rangle$ is as unique as it is the origin of coordinates associated to different inertial observers in Special Relativity. As it will be clarified in the next section, each observer will be able to define $|o\rangle$ as the state representing his own origin of coordinates, and he will describe the origin of other observers with a different delocalised state.

2.2.6 Left Coaction of $\mathcal{P}_\kappa$ and Convariance

In Sec. 2.1.1 and Sec. 2.1.2 we showed how the algebra (2.11) emerges as the quantum homogeneous space of the $\kappa-$Poincaré quantum group group [17,72–74]. This object has historical precedence over κ-Minkowski, which was introduced by Majid and Ruegg after recognizing the *bicrossproduct* structure of the κ-Poincaré group [18]. The κ-Poincaré group belongs to a small family of Hopf-algebras obtained from the Poincaré group with deformation parameter has the dimensions of one over energy [75,76]. Furthermore, If one requires undeformed spatial isotropy, there is a unique [76] κ-Poincaré compatible with (2.11).

The κ-Poincaré group can be regarded as the non-commutative algebra of functions $\mathcal{P}_\kappa$, generated by $\Lambda^\mu{}_\nu$ and a^μ such that the *left-coaction*

$$\beta_{(\Lambda,a)}: \quad \mathcal{M}_\kappa \ni \quad x^\mu \longrightarrow \beta_{(\Lambda,a)}(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1 =: x'^\mu \quad \in \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa. \tag{2.82}$$

leaves the commutation relations (2.11) unchanged. Indeed, (2.82) being a left coaction[4], it also is an homomorphism with respect to the non-commutative product of $\mathcal{M}_\kappa$, hence the covariance of the commutation relations (2.11) is preserved

$$[x'^\mu, x'^\nu] = \mathrm{i}\lambda\left(\delta^\mu{}_0\, x'^\nu - \delta^\nu{}_0\, x'^\mu\right) . \tag{2.83}$$

[4]Note that the (2.82) is the co action of the elements of the group over $\mathcal{M}_\kappa$ and it has not to be confused with the action we gave in Sec. 2.1.1 for the group algebra.

This also fixes some of the commutation relations between the κ-Poincaré group coordinates[5]:

$$[a^\mu, a^\nu] = \mathrm{i}\lambda \left(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu\right) , \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0 , \\ [\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda \left[\left(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right] . \tag{2.84}$$

Furthermore, the *co-product* $\Delta : \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa$:

$$\Delta(a^\mu) = a^\nu \otimes \Lambda^\mu{}_\nu + 1 \otimes a^\mu , \qquad \Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu , \tag{2.85}$$

an *antipode* $S : \mathcal{P}_\kappa \to \mathcal{P}_\kappa$

$$S(a^\mu) = -a^\nu(\Lambda^{-1})^\mu{}_\nu , \qquad S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu , \tag{2.86}$$

and a *co-unit* $\varepsilon : \mathcal{P}_\kappa \to \mathbb{C}$,

$$\varepsilon(a^\mu) = 0 , \qquad \varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu , \tag{2.87}$$

of the Hopf algebra $(P_\kappa, 1, \cdot, \Delta, \epsilon, S)$ have to be homorphisms with respect to the commutation relations (2.4). In this way we make sure that our non-commutative algebra of functions on the Poincaré group is compatible with the group structure.

We want to give a representation of the Λ^μ_ν and a^μ's as operators on some Hilbert space. The $\Lambda^\mu{}_\nu$ in (2.84) should not be understood as 16 independent components, but rather as 16 redundant functions satisfying the relations

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{2.88}$$

which reduce the independent components to just 6. Since the $\Lambda^\mu{}_\nu$ commute with each other, they are compatible with the usual representation of the Lorentz group

$$\Lambda^\mu{}_\nu = (exp\omega)^\mu{}_\nu , \qquad \omega^\mu{}_\rho\eta^{\rho\nu} = -\omega^\nu{}_\rho\eta^{\rho\mu} . \tag{2.89}$$

Once again the relation (2.88) reduces the $\omega^\mu{}_\nu$ to 6 independent components which commute with each other:

$$[\omega^\mu{}_\nu, \omega^\rho{}_\sigma] = 0 , \tag{2.90}$$

but do *not* commute with the a^μ's. The structure of the commutation relations (2.84) suggests to represent the a^μ's as vector fields:

$$a^\rho = -\mathrm{i}\lambda \left[\left(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\frac{\partial}{\partial\Lambda^\mu{}_\nu}. \tag{2.91}$$

The exponential relation between $\omega^\mu{}_\nu$ and $\Lambda^\mu{}_\nu$ implies $\frac{\partial}{\partial\Lambda^\mu{}_\nu} = \Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha}$, which suggests the a^μ to be represented as

$$a^\rho = -i\,\lambda \left[\left(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha} . \tag{2.92}$$

[5]The metric used here is $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

on coordinates ω^μ_ν. Interestingly, the above vector fields already 'know' about the commutation relations between the translation operators. In fact, the commutator of two of these vector fields acts on wavefunctions of $\omega^\mu{}_\nu$ as the Lie bracket between the vector fields, and computing this Lie bracket yields $[a^\mu, a^\nu] = \mathrm{i}\lambda\,(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu)$. We found a representation of the κ-Poincaré quantum group as operators on the states $\phi(\omega) \in L^2(so(3,1)$

$$\Lambda^\mu{}_\nu\ \phi(\omega) = (\exp\omega)^\mu{}_\nu \phi(\omega), \tag{2.93}$$

$$a^\rho\ \phi(\omega) = -i\,\lambda\left[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\,\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha}, \tag{2.94}$$

in which the $\Lambda^\mu{}_\nu$'s act as multiplicative operators while the translations operators a^μ act as vector fields. Here, integrability in $L^2(so(3,1)$ is defined by the Haar measure on the Lorentz group.

Unfortunately, the above introduced representation is not a *faithful* one. Indeed, we can write combinations of the $\Lambda^\mu{}_\nu$ and a^ρ operators that are represented into the null operator; as an example

$$\begin{aligned}
&\eta_{\rho\mu}\,(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\ a^\rho \triangleright \phi(\omega) = \\
&\left[\eta_{\rho\beta}\Lambda^\rho{}_\nu\left(\delta^\kappa{}_0 \Lambda^\beta{}_\kappa - \delta^\beta{}_0\right)(\delta^\sigma{}_0 \Lambda^\mu{}_\sigma - \delta^\mu{}_0) + (\delta^\sigma{}_0 \Lambda^\mu{}_\sigma - \delta^\mu{}_0)\left(\delta^0{}_\sigma \Lambda^\sigma{}_\nu - \delta^0{}_\nu\right)\right]\Lambda^\nu{}_\alpha \frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha} = \\
&(\delta^\sigma{}_0 \Lambda^\mu{}_\sigma - \delta^\mu{}_0)\left[\eta_{\rho\beta}\Lambda^\rho{}_\nu\left(\delta^\kappa{}_0 \Lambda^\beta{}_\kappa - \delta^\beta{}_0\right) + \left(\delta^0{}_\sigma \Lambda^\sigma{}_\nu - \delta^0{}_\nu\right)\right]\Lambda^\nu{}_\alpha \frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha} = \\
&(\delta^\sigma{}_0 \Lambda^\mu{}_\sigma - \delta^\mu{}_0)\left[(\eta_{00} - 1)\,\delta^0{}_\nu + (1 - \eta_{00})\,\Lambda^0{}_\nu\right]\Lambda^\nu{}_\alpha \frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha} = 0
\end{aligned} \tag{2.95}$$

where the last line is zero because $\eta_{00} = +1$ in our convention. Since the operator

$$\eta_{\rho\mu}\,(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\ a^\rho\,, \tag{2.96}$$

is non-trivial, (at least in order to admit a good classical limit) some of its expectation values should not be vanishing which is not possible with (2.94) and (2.93). As a consequence the representation (2.94) is not faithful. The simplest way to fix it is to enlarge the representation by writing a direct sum of two representations: the above one and the (at this point familiar) representation (2.44) of κ-Minkowski coordinates, which reproduces the commutation rules between translation operators, but commutes with Lorentz transformations. The so obtained representation acts on a larger Hilbert than $L^3(so(3,1)$ which has three additional coordinates $q^i \in \mathbb{R}$ with $i \in \{1,2,3\}$, and which as a whole is $L^2(SO(3,1) \times \mathbb{R}^3)$. The Lorentz matrices still acts as multiplicative operators (2.93) while the translation operators are represented as follows:

$$
\begin{aligned}
a^\rho \;\; = \;\; & -\mathrm{i}\,\frac{\lambda}{2}\left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha} \\
& +\mathrm{i}\frac{\lambda}{2}\left(\delta^\rho{}_0\, q^i \frac{\partial}{\partial q^i} + \delta^\rho{}_i\, q^i\right) + \frac{1}{2}\mathrm{h.c.}\,,
\end{aligned} \tag{2.97}
$$

where by "h.c." we mean the hermitean conjugate of the previous expression. This ensures that the operator is self-adjoint on some domain. The final form of our representation is

$$
\begin{aligned}
a^\rho \phi(q,\omega) =& \mathrm{i}\lambda\delta^\rho{}_0 \left(\frac{3}{2}\,\phi(q,\omega) + q^i \frac{\partial \phi(q,\omega)}{\partial q^i}\right) + \delta^\rho{}_i\, q^i\, \phi(q,\omega) \\
& - \mathrm{i}\lambda : \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha} \;:\; \phi(q,\omega)\,, \\
\Lambda^\mu{}_\nu \phi(q,\omega) =& \Lambda^\mu{}_\nu(\omega)\phi(\omega) = (exp\omega)^\mu{}_\nu \phi(q,\omega)\,,
\end{aligned} \tag{2.98}
$$

that is,

$$
\begin{aligned}
a^\rho \phi(q,\omega) =& \mathrm{i}\lambda\delta^\rho{}_0 \left(\frac{3}{2}\,\phi(q,\omega) + q^i \frac{\partial \phi(q,\omega)}{\partial q^i}\right) + \delta^\mu{}_i\, q^i\, \phi(q,\omega) \\
& - \frac{\mathrm{i}\lambda}{2}\left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial \phi(q,\omega)}{\partial \omega^\mu{}_\alpha} \\
& - \frac{\mathrm{i}\lambda}{2}\phi(q,\omega)\frac{\partial}{\partial \Lambda^\mu{}_\nu}\left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\,, \\
\Lambda^\mu{}_\nu \phi(q,\omega) =& \Lambda^\mu{}_\nu(\omega)\phi(\omega) = (exp\omega)^\mu{}_\nu \phi(q,\omega)\,.
\end{aligned} \tag{2.99}
$$

It is trivial to check that, since the derivatives with respect to $\omega^\mu{}_\nu$ commute with the functions of q^i, and the derivatives with respect to q^i commute with the functions of $\omega^\mu{}_\nu$, the representation splits into a direct sum of representations, and the commutation relations between a^μ's are satisfied. The representation (2.99) is quite complicated, and its explicit functional form depends on the coordinate system on the Lorentz group we choose. Nevertheless, a concrete realization can be more easily obtained if one restrict to a lower dimensional case, as shown in the example below.

Example: The representation of κ-Poincaré in 1+1 dimensions

We want to obtain an explicit realization of (2.99) for a $(1+1)-$dimensional Lorentian spacetime. This useful exercise both for pedagogical reasons, and in order to have an example that can be worked out explicitly. Since in this case we have a $1-$dimensional Lorentz group, every calculation will be strongly simplified .

The great advantage of working in 1+1 dimensions is that we have an explicit (and simple) coordinatization of the Lorentz group:

$$
\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi\,, \qquad \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi\,, \tag{2.100}
$$

The commutation relations of κ-Poincaré (2.84) become

$$[a^0, a^1] = \mathrm{i}\lambda\, a^1\,, \qquad [\cosh\xi, a^0] = -\mathrm{i}\lambda \sinh^2\xi\,, \qquad [\cosh\xi, a^1] = -\mathrm{i}\lambda\,(\cosh\xi - 1)\sinh\xi\,,$$
$$[\sinh\xi, a^0] = -\mathrm{i}\lambda \sinh\xi\cosh\xi\,, \qquad [\sinh\xi, a^1] = -\mathrm{i}\lambda\,(\cosh\xi - 1)\cosh\xi\,, \tag{2.101}$$

which can be further simplified as

$$[a^0, a^1] = \mathrm{i}\lambda\, a^1\,, \qquad [\xi, a^0] = -\mathrm{i}\lambda\sinh\xi\,, \qquad [\xi, a^1] = \mathrm{i}\lambda\,(1 - \cosh\xi)\;. \tag{2.102}$$

It is evident that a^0 and a^1 act on ξ like vector fields:

$$a^0 = \mathrm{i}\lambda\sinh\xi\,\frac{\partial}{\partial\xi}\,, \qquad a^1 = \mathrm{i}\lambda\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi}\,. \tag{2.103}$$

Then a simple calculation

$$\begin{aligned}
\left[a^0, a^1\right] &= -\lambda^2\left[\sinh\xi\,\frac{\partial}{\partial\xi}\,(\cosh\xi - 1) - (\cosh\xi - 1)\,\frac{\partial}{\partial\xi}\sinh\xi\right]\frac{\partial}{\partial\xi} \\
&= -\lambda^2\left[\sinh^2\xi - (\cosh\xi - 1)\cosh\xi\right]\frac{\partial}{\partial\xi} \\
&= -\lambda^2\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi} = \mathrm{i}\lambda a^1\,.
\end{aligned} \tag{2.104}$$

shows that the above representation of the a^μ's is compatible with $[a^0, a^1]$ commutation relations. However, just as in $(1+3)$ dimensions, this representation is not faithful, because the $(1+1)-$ dimensional counterpart of (2.96)

$$(\cosh\xi - 1)\,a^0 - \sinh\xi\,a^1 = -\mathrm{i}\lambda\,(\cosh\xi - 1)\sinh\xi\,\frac{\partial}{\partial\xi} + \mathrm{i}\lambda\sinh\xi\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi} = 0\,, \tag{2.105}$$

is represented into the null operator. As we learned in the higher dimensional representation, it is sufficient to take the sum of the above representation plus the familiar representation of the κ-Minkowski algebra in $(1+1)$ dimensions:

$$a^0 = \mathrm{i}\lambda q\frac{\partial}{\partial q} + \mathrm{i}\lambda\sinh\xi\,\frac{\partial}{\partial\xi}\,, \qquad a^1 = q + \mathrm{i}\lambda\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi}\,. \tag{2.106}$$

The two commute with each other, and separately satisfy the commutation relations and the Jacobi identity. Hence, they provide a faithful representation of our algebra as operators on the Hilbert space $L^2(SO(1,1)\times\mathbb{R}) \sim L^2(\mathbb{R}^2)$ of square-integrable functions of ξ and q. Note that such a representation is not self-adjoint, but it can be made so by Weyl-ordering it:

$$\begin{aligned}
a^0 &= \frac{\mathrm{i}\lambda}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right) + \frac{\mathrm{i}\lambda}{2}\left(\sinh\xi\,\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi}\sinh\xi\right) \\
a^1 &= q + \frac{\mathrm{i}\lambda}{2}\left((\cosh\xi - 1)\,\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi}\,(\cosh\xi - 1)\right)\,,
\end{aligned} \tag{2.107}$$

which can be written

$$\begin{aligned} a^0 &= \mathrm{i}\lambda\left(\frac{1}{2}+q\frac{\partial}{\partial q}\right)+\mathrm{i}\lambda\left(\frac{1}{2}\cosh\xi+\sinh\xi\,\frac{\partial}{\partial\xi}\right) \\ a^1 &= q+\mathrm{i}\lambda\left(\frac{1}{2}\sinh\xi+(\cosh\xi-1)\,\frac{\partial}{\partial\xi}\right). \end{aligned} \tag{2.108}$$

It is easy to check that the above reproduces the commutation relations (2.84). In the following, we will often refer to the representation given in present section whenever we will give some explicit realization of the upcoming abstract results.

2.3 Observers and Reference Frames

In Sec. 2.2.3 we represented the algebra (2.11) of coordinates in $\kappa-$Minkowski as operators over the Hilbert space $L^2(\mathbb{R}^3)$. We also assumed that the so obtained Lie algebra together with its states describe the *fuzzy* events in κ-Minkowski space-time as well as their delocalisation. Interestingly, in Sec. 2.2.5 we found that the origin of spatial coordinates $|o\rangle$ defines a 1-parameter family perfectly localised state (*i.e.* points) $|o,\tau\rangle$, all at the same position but at different times time τ. At the end of Sec. 2.2.5 we have already noticed how this result seems to admit the existence of an "special" point in space-time. Nevertheless, this is a misleading interpretation as it will be soon clarified by introducing in our model transformations between different reference frames.

In both Sec. 2.2.3 Sec. 2.2.5 we have never explicitly specified the *observer* measuring $\hat{x}^0$ and $\hat{x}^i$. Indeed, since the origin is a perfectly localised point at any time, we have been implicitly assuming that the observer was located there. In ordinary special relativity one changes the observer via a Poincaré transformation. In our case the symmetries are given by the κ-Poincaré quantum group instead. Accordingly, it will be impossible to locate the position of the transformed observer, due the non-commutativity of translations. In the spirit of previous sections, we will consider the algebra generated by the a's and Λ's, and associate to a translated and Lorentz transformed observers a state of this algebra. In other words, also transformation between different observer will be fuzzifyed.

Notice that all the Λ's commute among themselves, therefore they must have common eigenvectors, as a consequence all the uncertainties in localisability come from the translation sector not from the Lorentz one. As a first step, we consider the observer located at the origin, *i.e* the one which corresponds to the identity transformation.

2.3.1 The identity transformation state

The commutation relations (2.84) allows one to define a state $|o\rangle_{\mathcal{P}}$ of $\mathcal{P}_\kappa$ with the property:

$$ {}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f)\,, \tag{2.109}$$

where $f(a,\Lambda)$ is a generic element of the κ-Poincaré algebra (*i.e.* a generic non-commutative function of translations and Lorentz transformation matrices), and ε is the co-unit of the κ-Poincaré algebra defined in (2.87). In other words, the state returns the value of the function on the identity transformation.

The $|o\rangle_{\mathcal{P}}$ state is obtained as limit of vectors in the Hilbert space. It suffices to take a succession of functions which converge to a δ as far as a^μ and the diagonal elements of $\Lambda^\mu{}_\nu$ are concerned, and to zero for the off-diagonal elements of the Λ's.

We interpret this state in the enlarged algebra as describing the Poincaré transformation between two coincident observers, *i.e.* between an observer and a second one located at the origin of the coordinate system of the first observer. It is not difficult to see, looking at (2.84), that $|o\rangle_{\mathcal{P}}$ is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in κ-Minkowski space-time. Nevertheless, $|o\rangle_{\mathcal{P}}$ has not to be confused with $|o\rangle$ introduced in Sec. 2.2.5 : they are states of different algebras.

2.3.2 Physical interpretation

We propose an interpretation for the operators x^μ we have been using all along, and the operators x'^μ that appear in (2.82): they are the coordinate systems associated to two inertial observers, say, Alice and Bob, which are translated and in relative motion with respect to each other. A space-time event (*i.e.* the clicking of a particle detector) seen by Alice will be described by the expectation value of its coordinates $\langle x^\mu\rangle$, their variance $\langle (x^\mu - \langle x^\mu\rangle)^2\rangle$, which measures how localised it is, the skewness $\langle (x^\mu - \langle x^\mu\rangle)^3\rangle$ measuring how asymmetric it is around the expectation value, and all higher moments $\langle (x^\mu - \langle x^\mu\rangle)^n\rangle$ which describe in increasingly finer details the distribution of probability of where the event can be localised in the spirit of Sec. 2.2.2 and Sec. 2.2.3. On the other hand, the same event ,seen by Bob, will be described by the moments of the transformed coordinate operators: $\langle (x'^\mu - \langle x'^\mu\rangle)^n\rangle$, which are in general different from Alice's, unless the transformation that connects Alice and Bob is the identity described in Sect. 2.3.1.

What does it mean to take expectation values of the operators x'^μ and their powers? The x'^μ is obtained by the left coaction (2.82), hence it belongs to the tensor-product algebra $\mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$. A representation of this algebra is given by the direct sum of the representation (2.99) of $\mathcal{P}_\kappa$ and the representation (2.44) of $\mathcal{M}_\kappa$. Clearly the x^μ algebra (Alice's coordinates) is lifted to elements of the kind $\mathbb{1} \otimes \mathcal{M}_\kappa$, where the identity of $\mathcal{P}_\kappa$ is given by $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, $a^\mu = 0$. In this way, we have a representation of $\mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$ as operators on the Hilbert space $\mathcal{H}_\mathcal{P} \times L^2(\mathbb{R}^3_x) \sim L^2(SO(3,1) \times \mathbb{R}^3_q \times \mathbb{R}^3_x)$ defined by the following the action

$$
\begin{aligned}
x'^\mu f(\omega,q,x) =& \mathrm{i}\lambda\, \Lambda^\mu{}_\nu(\omega) \left[\delta^\nu{}_0 \left(\frac{3}{2} f(\omega,q,x) + x^i \frac{\partial f(\omega,q,x)}{\partial x^i} \right) + \delta^\nu{}_i\, x^i\, f(\omega,q,x) \right] \\
&+ \mathrm{i}\lambda \delta^\mu{}_0 \left(\frac{3}{2}\, f(\omega,q,x) + q^i \frac{\partial f(\omega,q,x)}{\partial q^i} \right) + \delta^\mu{}_i\, q^i\, f(\omega,q,x) \\
&- \frac{\mathrm{i}\lambda}{2} \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right) \Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right) \eta^{\mu\rho} \right] \Lambda^\nu{}_\alpha \frac{\partial f(\omega,q,x)}{\partial \omega^\mu{}_\alpha} \\
&- \frac{\mathrm{i}\lambda}{2} f(\omega,q,x) \frac{\partial}{\partial \Lambda^\mu{}_\nu} \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right) \Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right) \eta^{\mu\rho} \right] .
\end{aligned}
$$

In the $(1+1)$ dimensional case (2.102) we have a more intelligible expression for our representation:

$$
\begin{aligned}
x'^0 f(\xi,q^1,x^1) &= \mathrm{i}\lambda\, \cosh\xi \left(\frac{1}{2} f + x^1 \frac{\partial f}{\partial x^1} \right) + \sinh\xi\, x^1\, f + \mathrm{i}\lambda \left(\frac{1}{2} f + q^1 \frac{\partial f}{\partial q^1} \right) \\
&\quad + \mathrm{i}\lambda \left(\frac{1}{2} \cosh\xi\, f + \sinh\xi\, \frac{\partial f}{\partial \xi} \right) , \\
x'^1 f(\xi,q^1,x^1) &= \mathrm{i}\lambda\, \sinh\xi \left(\frac{1}{2} f + x^1 \frac{\partial f}{\partial x^1} \right) + \cosh\xi\, x^1\, f + q^1\, f \\
&\quad + \mathrm{i}\lambda \left(\frac{1}{2} \sinh\xi\, f + (\cosh\xi - 1)\, \frac{\partial f}{\partial \xi} \right) .
\end{aligned}
\tag{2.110}
$$

Our Hilbert space will admit non-entangled states, *i.e.* objects of the kind:

$$
|g,\psi\rangle = |g\rangle \otimes |\psi\rangle \tag{2.111}
$$

with $|g\rangle \in \mathcal{H}_\mathcal{P} = L^2[SO(3,1)] \times \mathbb{R}^3_q$ and $|\psi\rangle \in L^2(\mathbb{R}^3)$. It represents the state of the coordinates x'^μ of a $\kappa-$Poincaré transformed observer. Then, the expectation values of the coordinates of the transformed observer are given by:

$$
\langle x'^\mu \rangle = \langle g| \otimes \langle \psi| \left(\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1 \right) |g\rangle \otimes |\psi\rangle = \langle g|\Lambda^\mu{}_\nu|g\rangle \langle \psi|x^\nu|\psi\rangle + \langle g|a^\mu|g\rangle , \tag{2.112}
$$

(we used the normalization condition $\langle\psi|\psi\rangle = 1$). Similarly, one can calculate all the higher momenta of the coordinates as

$$
\langle x'^{\mu_1} \dots x'^{\mu_n} \rangle = \langle g| \otimes \langle \psi| \left(x'^{\mu_1} \dots x'^{\mu_n} \right) |g\rangle \otimes |\psi\rangle . \tag{2.113}
$$

We invite the reader to take a second look at the relations (2.110), and notice how the coordinates x'^μ of a Poincaré-transformed observer (*e.g.* Bob) act on states describing an event in this observer's reference frame with *two copies* of the now-familiar representation (2.44).

The first acts on the state of the original observer ('Alice'), which, if the state is a product state as in (2.111), is written as a function of $x^i \in \mathbb{R}^3$. The other acts on the state of the Poincaré group coordinates, which, in the product state case, is written as a function of $q^i \in \mathbb{R}^3$ and $\Lambda^\mu{}_\nu \in SO(3,1)$.

2.3.3 Properties of the Transformed States

In this section we provide some general results for the states representing transformed coordinates and observers. The properties described below will not depend on the choice of the representation except for the assumption that the there exist an identity state.

Coordinate origin state $|o\rangle$ Under Transformations

Consider the following state

$$|g,0\rangle = |g\rangle \otimes |o\rangle\,, \tag{2.114}$$

in which the origin $|o\rangle$ undergoes a $\kappa-$Poincaré transformation encoded by the state $|g\rangle$. If we want to know what the *Poincaré-transformed observer* measures with the coordinates centred on his reference frame, we have to apply the operators $x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$ which acts on $L^2(\mathbb{R}^3_x) \times \mathcal{H}_\mathcal{P}$. Hence, we compute the expectation values associated the coordinates x'^μ as follows:

$$\langle x'^\mu \rangle = \langle g| \otimes \langle o|x'^\mu|g\rangle \otimes |o\rangle = \langle g|\Lambda^\mu{}_\nu|g\rangle\langle o|x^\nu|o\rangle + \langle g|a^\mu|g\rangle\langle o|o\rangle\,. \tag{2.115}$$

Since the state $|o\rangle$ is normalized $\langle o|o\rangle = 1$ and the expectation value of x^μ on $|o\rangle$ vanish (see Sec. 2.2.5), the above equation give just

$$\langle x'^\mu \rangle = \langle g|a^\mu|g\rangle\,. \tag{2.116}$$

In other words, the expectation value of the transformed coordinates is completely determined by the expectation value of the translation operators on the chosen κ-Poincaré state $|g\rangle$. This is quite natural: indeed the different observers are comparing just positions and not directions. Now we consider a more general situation. Given an arbitrary monomial in the transformed coordinates $x'^{\mu_1}x'^{\mu_2}\dots x'^{\mu_n}$, its expectation value on the $|g\rangle \otimes |o\rangle$ is:

$$\begin{aligned}\langle x'^{\mu_1}\dots x'^{\mu_n}\rangle =& \langle g|\otimes\langle o|(a^{\mu_1}\otimes 1 + \Lambda^{\mu_1}{}_{\nu_1}\otimes x^{\nu_1})\dots(a^{\mu_n}\otimes 1 + \Lambda^{\mu_n}{}_{\nu_n}\otimes x^{\mu_n})|g\rangle\otimes|o\rangle\\ =&\langle g|a^{\mu_1}\dots a^{\mu_n}|g\rangle\langle o|o\rangle + \langle g|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)|g\rangle\langle o|x^\nu|o\rangle + \dots\\ &+ \langle g|\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\nu_2}(a,\Lambda)|g\rangle\langle o|x^{\nu_1}x^{\nu_2}|o\rangle + \langle g|\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}(a,\Lambda)|g\rangle\langle o|x^{\nu_1}\dots x^{\nu_n}|o\rangle\,.\end{aligned} \tag{2.117}$$

According to Sec. 2.2.5, we have $\langle o|x^{\nu_1}\dots x^{\nu_n}|o\rangle = 0$ for arbitrary n. Hence we have

$$\langle x'^{\mu_1}\dots x'^{\mu_n}\rangle = \langle g|a^{\mu_1}\dots a^{\mu_n}|g\rangle\langle o|o\rangle = \langle g|a^{\mu_1}\dots a^{\mu_n}|g\rangle\,. \tag{2.118}$$

Therefore, we conclude that

The state obtained by transforming the origin state $|o\rangle$ via the $\mathcal{P}_\kappa$ state $|g\rangle$ in the representation of the κ-Poincaré algebra a^μ, $\Lambda^\mu{}_\nu$, is the state which will assign, to any polynomials in the transformed coordinates $x'^\mu = a^\mu\otimes 1 + \Lambda^\mu{}_\nu\otimes x^\nu$, the same expectation value that $|g\rangle$ alone would produce on the corresponding polynomial in a^μ.

In other words, the state of x'^μ is identical to the state of a^μ. For instance, the uncertainty $\Delta x'^\mu$ of the transformed coordinate is a consequence of the uncertainty of the translation operator Δa^μ on the state $|g\rangle$. We stress that, although the new observer is assumed to measure those expectations values, we cannot determine with absolute precision is time and direction because the a^μ close a non-commutative algebra. In other words, we do not know *where* the new observer is, unless he has just time translated the origin, i.e. $|g\rangle = |o_{a^0}\rangle_{\mathcal{P}}$.

Generic State under the Identity Transformation

Consider a generic κ-Minkowski coordinates state $|\psi\rangle\,| \in L^2(\mathbb{R}^3_x)$ which undergoes the identity transformation $|o\rangle_{\mathcal{P}}$. The transformed state is obtained by replacing in (2.111) the state $|g\rangle$ with $|o\rangle_{\mathcal{P}}$. Then the expectation values of a polynomial in the transformed coordinates x'^μ on the transformed state $|o\rangle_{\mathcal{P}}\otimes|\psi\rangle$, gives:

$$\begin{aligned}\langle x'^{\mu_1}\dots x'^{\mu_n}\rangle =& {}_{\mathcal{P}}\langle o|\otimes\langle\psi|(a^{\mu_1}\otimes 1+\Lambda^{\mu_1}{}_{\nu_1}\otimes x^{\nu_1})\dots(a^{\mu_n}\otimes 1+\Lambda^{\mu_n}{}_{\nu_n}\otimes x^{\mu_n})\,|o\rangle_{\mathcal{P}}\otimes|\psi\rangle\\ =& {}_{\mathcal{P}}\langle o|\,a^{\mu_1}\dots a^{\mu_n}\,|o\rangle_{\mathcal{P}}\,\langle\psi|\psi\rangle + {}_{\mathcal{P}}\langle o|\,\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)\,|o\rangle_{\mathcal{P}}\,\langle\psi|x^\nu|\psi\rangle\\ &+{}_{\mathcal{P}}\langle o|\,\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\nu_2}(a,\Lambda)\,|o\rangle_{\mathcal{P}}\,\langle\psi|x^{\nu_1}x^{\nu_2}|\psi\rangle\\ &+\dots+{}_{\mathcal{P}}\langle o|\,\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}(a,\Lambda)\,|o\rangle_{\mathcal{P}}\,\langle\psi|x^{\nu_1}\dots x^{\nu_n}|\psi\rangle\\ =&\epsilon(a^{\mu_1}\dots a^{\mu_n})\langle\psi|\psi\rangle + \epsilon[O^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)]\langle\psi|x^\nu|\psi\rangle+\\ &+\epsilon[\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\nu_2}]\langle\psi\rangle x^{\nu_1}x^{\nu_2}|\psi\rangle+\dots+\epsilon[\mathcal{O}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}(a,\Lambda)]\langle\psi|x^{\nu_1}\dots x^{\nu_n}|\psi\rangle\,,\end{aligned} \tag{2.119}$$

for arbitrary n [10]. The algebra elements $O^{\mu_1\cdots\mu_n}_{\nu_1\dots\nu_m}(a,\Lambda)$ are monomials in a^μ, $\Lambda^\mu{}_\nu$, without a particular ordering. Furthermore, the m-th element contains m Lorentz matrix generators and $n-m$ translation generators. Since the co-unit map ϵ is an homomorphism such that $\epsilon(a^\mu) = 0$, $\epsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu$, we have that $\epsilon[O^{\mu_1\cdots\mu_n}_{\nu_1\dots\nu_m}(a,\Lambda)] = 0$ for any $m\neq n$. On the other hand, for $m=n$ we have

$$\epsilon[O^{\mu_1\cdots\mu_n}_{\nu_1\dots\nu_n}(a,\Lambda)] = \delta^{\mu_1}{}_{\nu_1}\dots\delta^{\mu_n}_{\nu_n}\,, \tag{2.120}$$

hence

$${}_{\mathcal{P}}\langle o|\otimes\langle\psi|x'^{\mu_1}\dots x'^{\mu_n}\,|o\rangle_{\mathcal{P}}\otimes|\psi\rangle = \langle\psi|x^{\mu_1}\dots x^{\mu_n}|\psi\rangle\,, \tag{2.121}$$

This result shows that the identity transformation does not change any expectation value. We conclude that:

Whenever the state of the transformation is the identity $|o\rangle_{\mathcal{P}}$, the original observer and the transformed one, will agree on all measurements of time and position.

For seek of clarity, we specify that the first observer uses the coordinate operators x^μ and the Hilbert space $L^2(\mathbb{R}^3_x)$. The second one uses the coordinate operators x'^μ and the Hilbert space $\mathbb{H}_{\mathcal{P}} \otimes L^2(\mathbb{R}^3_x)$.

κ-Poincaré transformation and uncertainty of coordinates

We want to investigate how the uncertainty in the transformed coordinates $\Delta x'^\mu$ are related to those of the original ones Δx^μ when a state undergoes a generic transformation $|\psi\rangle \to |g\rangle \otimes |\psi\rangle$. The simplest example is given by a pure translation $x'^\mu = 1 \otimes x^\mu + a^\mu \otimes 1$. In this case the variance of x^μ is

$$\begin{aligned}\Delta(x'^\mu)^2 =& \langle (x'^\mu)^2\rangle - \langle x'^\mu\rangle^2 = \langle (x^\mu)^2 + (a^\mu)^2 + x^\mu a^\mu + a^\mu x^\mu\rangle - \langle x^\mu\rangle^2 - \langle a^\mu\rangle^2 - 2\langle x^\mu\rangle\langle a^\mu\rangle \\ =& \Delta(x^\mu)^2 + \Delta(a^\mu)^2 + 2\,\mathrm{cov}(x^\mu, a^\mu)\,.\end{aligned} \tag{2.122}$$

Since a^μ and x^μ belongs to different sides in the tensor product, their covariance

$$\begin{aligned} 2\,\mathrm{cov}(x^\mu, a^\mu) &= \langle g| \otimes \langle\psi|(x^\mu a^\mu + a^\mu x^\mu)|g\rangle \otimes |\psi\rangle - 2\langle\psi|x^\mu|\psi\rangle\langle g|a^\mu|g\rangle \\ &= \langle\psi|x^\mu|\psi\rangle\langle g|a^\mu|g\rangle + \langle g|a^\mu|g\rangle\langle\psi|x^\mu|\psi\rangle - 2\langle\psi|x^\mu|\psi\rangle\langle g|a^\mu|g\rangle = 0\,,\end{aligned} \tag{2.123}$$

just vanishes. As a consequence we have:

$$\Delta(x'^\mu)^2 = \Delta(x^\mu)^2 + \Delta(a^\mu)^2 \geq \Delta(x^\mu)^2\,. \tag{2.124}$$

One is simply adding uncorrelated variables, and their uncertainties get square-summed. Notice that this conclusion is a consequence of the fact that we assumed that transformed states are product states $|g\rangle\otimes|\psi\rangle$. If we allowed for entanglement between the transformation part $|g\rangle$ and the state $|\psi\rangle$ describing the event in the initial reference frame, we would have opened the possibility of reducing the uncertainty of x^μ with a translation. This, however, conflicts with the basic physical intuition that the relationship between inertial observers should be independent of the state of the system that the observers are studying.

Furthermore, if the translation parameter has zero uncertainty then the uncertainty in the coordinates is left unchanged. Of course, this happens only for the identity transformation and for *purely-temporal translation*, which can have zero uncertainty in all of the a^μ's. Nicely, the uncertainty do not depend on time translations. Hence, we conclude that

By translating a state the uncertainty of the coordinates may only increase or remain unchanged; the latter case occurs for identity or pure temporal translations only.

We want give a concrete example to show how the result above influences translation between observers and their perception of localisation. Consider a state which looks uncertain to the observer Alice located at her origin. One may wonder if there would be another observer, Bob, translated with respect to Alice, such that this same state looks perfectly localised for him.

For instance, consider the state $\psi(x^1)$ for x^1 in the $(1+1)-$dimensional model of (2.102). We then perform a translation with wavefunction $\psi(-q^1)$ where ψ has the same functional form as $\psi(x^1)$. One would naively think that the translated state is localised at the origin. Nevertheless, relation (2.124) states that this is not the case. Indeed, calculating the expectation value of $(x'^1)^n = (x^1+a^1)^n$ one gets Newton binomial sum

$$\begin{aligned}\langle (x^1+a^1)^n\rangle &= \sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix}\langle\psi(x^1)|(x^1)^{n-m}|\psi(x^1)\rangle\langle\psi(-q)|(a^1)^m|\psi(-q)\rangle = \\ &= \sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix}\langle\psi|(x^1)^{n-m}|\psi\rangle\langle\psi|(-x^1)^m|\psi\rangle \end{aligned} \tag{2.125}$$

which is different from zero. For example, for $n=2$

$$\langle (x^1+a^1)^2\rangle = \langle (x^1)^2\rangle + 2\langle x^1a^1\rangle + \langle (a^1)^2\rangle = 2\langle (x^1)^2\rangle - 2\langle x^1\rangle^2 = 2\Delta(x^1)^2, \tag{2.126}$$

which shows that the variance is doubled. Thus, the process of translating a state and then "undo" it with a change of observer does not lead to an identification of states. Nevertheless, the symmetry between Alice and Bob is preserved: each has a set of states which is isomorphic, but the *quantum* nature of the transformation in $\mathcal{P}_\kappa$ implies that those states are not transformed into each other by a translation.

Now, consider a general κ-Poincaré transformations, for example the transformation of the spatial coordinate in $(1+1)-$dimensions

$$x'^1 = \cosh\xi\otimes x^1 + \sinh\xi\otimes x^0 + a^1\otimes 1\,. \tag{2.127}$$

The difference between variance of x'^1 and of x^1 gives

$$\begin{aligned}&\Delta(x'^1)^2 = \Delta(x^1)^2 + \Delta(a^1)^2 + \langle x^1\rangle^2\Delta(\cosh\xi)^2 + \langle x^0\rangle^2\Delta(\sinh\xi)^2 \\ &+ \langle\sinh\xi\rangle^2\Delta(x^0)^2 + \Delta(\sinh\xi)^2\Delta(x^0)^2 + \langle\cosh\xi\rangle^2\Delta(x^1)^2 + \Delta(\cosh\xi)^2\Delta(x^1)^2 \\ &+ 2\,\mathrm{cov}(x^1,x^0)\langle\cosh\xi\rangle\langle\sinh\xi\rangle + 2\,\mathrm{cov}(a^1,\sinh\xi)\langle x^0\rangle + 2\,\mathrm{cov}(a^1,\cosh\xi)\langle x^1\rangle \\ &+ 2\,\mathrm{cov}(\cosh\xi,\sinh\xi)(\mathrm{cov}(x^0,x^1) + \langle x^0\rangle\langle x^1\rangle) - \Delta(x^1)^2\,,\end{aligned} \tag{2.128}$$

which can be rewritten as

$$\begin{aligned}\Delta(x'^1)^2 =&\Delta(x^1)^2 + \langle\sinh^2\xi\rangle\left(\Delta(x^0)^2+\Delta(x^1)^2\right)\\&+\Delta[\cosh\xi]^2\langle x^1\rangle^2+\Delta[\sinh\xi]^2\langle x^0\rangle^2+2\mathrm{cov}(\cosh\xi,\sinh\xi)\langle x^0\rangle\langle x^1\rangle\\&+\Delta[a^1]^2+2\mathrm{cov}(\cosh\xi,a^1)\langle x^1\rangle+2\mathrm{cov}(\sinh\xi,a^1)\langle x^0\rangle\\&+2\langle\cosh\xi\sinh\xi\rangle\mathrm{cov}(x^0,x^1)\,.\end{aligned}\tag{2.129}$$

The second and third lines above give the squared uncertainty of the operator $a^1+\sinh\xi\,\langle x^0\rangle+\cosh\xi\,\langle x^1\rangle$, which is positive. Hence, we obtain

$$\begin{aligned}\Delta(x'^1)^2-\Delta(x^1)^2 =&\Delta[a^1+\sinh\xi\,\langle x^0\rangle+\cosh\xi\,\langle x^1\rangle]^2\\&+\langle\sinh^2\xi\rangle\left(\Delta(x^0)^2+\Delta(x^1)^2\right)+2\langle\cosh\xi\sinh\xi\rangle\mathrm{cov}(x^0,x^1)\,.\end{aligned}\tag{2.130}$$

Suppose that $\langle x^0\rangle = \langle x^1\rangle$ so that the first term reduces to the uncertainty of a^1. The covariance of x^0 and x^1 can be rewritten as $2\mathrm{cov}(x^0,x^1)=\Delta(x^0+x^1)^2-\Delta(x^0)^2-\Delta(x^1)^2$, thus

$$\begin{aligned}\Delta(x'^1)^2-\Delta(x^1)^2 =&\Delta(a^1)^2+\left(\langle\sinh^2\xi\rangle-\langle\cosh\xi\sinh\rangle\right)\left(\Delta(x^0)^2+\Delta(x^1)^2\right)\\&+\langle\cosh\xi\sinh\xi\rangle\Delta(x^0+x^1)^2\,.\end{aligned}\tag{2.131}$$

A simple calculation shows that

$$\langle\sinh^2\xi\rangle+\langle\cosh\xi\sinh\xi\rangle=\tfrac{1}{2}\left(\langle e^{2\xi}\rangle-1\right)\,,\tag{2.132}$$

and (2.131) reduces to

$$\begin{aligned}\Delta(x'^1)^2-\Delta(x^1)^2 =&\Delta(a^1)^2+\tfrac{1}{2}\left(\langle e^{2\xi}\rangle-1\right)\left(\Delta(x^0)^2+\Delta(x^1)^2\right)\\&+\langle\cosh\xi\sinh\xi\rangle\Delta(x^0+x^1)^2\,.\end{aligned}\tag{2.133}$$

Notice that a linear combination of x^0 and x^1 can always be made arbitrarily localised, so we can make $\Delta(x^0+x^1)^2$ arbitrarily small. Of course, the same holds for $\Delta(a^1)^2$, without introducing any constraints on the other quantities except the uncertainty of ξ. However, this does not limit our ability to manipulate the state in order to adjust the values of $\langle e^{2\xi}\rangle$ and $\langle\cosh\xi\sinh\xi\rangle$ very much. Indeed, it is possible to have a state such that $\langle e^{2\xi}\rangle<1$ (*e.g.* take the wavefunction over ξ to be supported on the $\xi<0$ region), and $\langle\cosh\xi\sinh\xi\rangle$ is $\mathcal{O}(1)$. In this case, (2.133) will be dominated by $\frac{1}{2}\left(\langle e^{2\xi}\rangle-1\right)(\Delta(x^0)^2+\Delta(x^1)^2)$ which is negative.

We conclude that a states with zero expectation value of x^μ such that the uncertainty of (x^0+x^1) is sufficiently small, can reduce their uncertainty if we perform a κ-Poincaré transformation with sufficiently localised translation and a Lorentz transformation such that $\langle e^{2\xi}\rangle<1$ and $\langle\cosh\xi\sinh\xi\rangle=\mathcal{O}(1)$. This proves that: *Despite a pure translation can only increase the variances of x^μ, under particular circumstances, It is still possible for the uncertainties on coordinates to decrease for some $\kappa-$Poincaré transformation.*

Chapter 3

Deformed Momentum Space

3.1 The Momentum Space of κ-Minkowski

In the previous chapter we proposed a model to describe how different observers perceive localisation of events in a non-commutative space-time ($\kappa-$Minkowski) whose symmetries are given by a quantum group ($\kappa-$Poincaré). We also stressed that the model is strictly kinematic. While the speed of light and the Planck length have been considered in the model, the quantum of action $\hbar$ did not has we do not implemented any quantum mechanical features yet. One expects that the dynamical aspects of the theory would be better understood once a notion of quantized phase space in $\mathcal{M}_\kappa$ will be introduced.

3.1.1 Plane Waves in $\kappa-$Minkowski

As discussed in Sec. 1.5 the momenta conjugate to coordinates over a non-commutative geometry form in general a curved space (the curvature of such a space has been originally named "co-gravity" by Majid [6]). Hence, the $\kappa-$Minkowski space-time is associated with a pseudo Riemannian generalization of the usual vector momentum space of Special and General Relativity [51]. The momenta dwelling in a pseudo-Riemannian geometry can be visualized as follows. Consider the *ordered plane waves*

$$e^{\mathrm{i}k_\mu x^\mu}\,, \qquad k_\mu \in \mathbb{R}^4\,. \tag{3.1}$$

associated to the x^0, x^i non-commutative coordinates. Since the (3.1) provide a basis of functions, we are able to expand functions over it; in this way one is able to discuss field theories over κ-Minkowski [77–81]. Due to the non-commutativity in coordinates, the plane waves (3.1) do not combine in a linear way

$$e^{\mathrm{i}k_\mu x^\mu} e^{\mathrm{i}k_\mu x^\mu} = e^{\mathrm{i}\;\frac{(k_0+q_0)/\kappa}{e^{(k_0+q_0)/\kappa}-1}\left[\left(\frac{e^{k_0/\kappa}-1}{k_0/\kappa}\right)k_i + e^{-k_0/\kappa}\left(\frac{e^{q_0/\kappa}-1}{q_0/\kappa}\right)q_i\right]x^i + \mathrm{i}(k_0+q_0)x^0}\,, \tag{3.2}$$

this has been proven explicitly [63, 82] using only the commutation relations (2.11). A perturbative calculation, using the Baker–Campbell–Haussdorff formula, confirms the above expression order-by-order. Furthermore, we are working with a Lie algebra, thus the exponentials form a subalgebra of the universal enveloping algebra of $\mathfrak{an}_3$ and are closed under product. The usual composition law of plane waves $(k, q) \to k_\mu + q_\mu$ generalizes to

$$(k,q) \longrightarrow p_\mu \ := \begin{cases} p_0 = k_0 + q_0, \\ p_i = \frac{(k_0+q_0)/\kappa}{e^{(k_0+q_0)/\kappa}-1}\left(\left(\frac{e^{k_0/\kappa}-1}{k_0/\kappa}\right)k_i + e^{-k_0/\kappa}\left(\frac{e^{q_0/\kappa}-1}{q_0/\kappa}\right)q_i\right). \end{cases} \tag{3.3}$$

Notice that (3.3) reduces to the previous one in the limit $\kappa \to \infty$; in fact it can be seen as a small deformation for wave vectors much smaller than κ. The non linearity in (3.3) is a consequence of the fact that the Fourier parameters are coordinates on a nonlinear manifold. In particular, since we are working with the $\mathfrak{an}_3$ Lie algebra, the Lie group obtained by exponentiating the Lie algebra of the x^μ is *group*. In fact, it is well known that, exponentializing the generators of a Lie algebra like x^μ, one obtains elements of the associated Lie group, which in our case is the group AN_3 [83–85]. It follows that the composition law between the parameters we used in the exponentials is not linear (because our algebra is not Abelian), and they just codify the group product. As the theory of Lie groups prescribes, these parameters can be considered coordinate systems on the group manifold, and, in general, the group manifold associated to a non-Abelian Lie group is curved. We used *Weyl ordering* in defining (3.1) *i.e.* each monomial in x^0 and x^i have been symmetrized, *e.g.*

$$\begin{aligned} &: (x^0)^2(x^1)^2 := \\ &= \frac{1}{6}[(x^0)^2(x^1)^2 + (x^0)^2(x^1)^2 + x^1x^0x^1x^0 + x^1(x^1)^2x^0 + x^0x^1x^0x^1 + x^0(x^1)^2x^0] \end{aligned} \tag{3.4}$$

and thereby each group element is represented as the exponential of a linear combination of generator. Of course one may chose a different ordering prescription and obtain different factorizations of the group elements. For example, the "time to the right" ordering gives $\exp(\mathrm{i}q_ix^i)\ \exp(\mathrm{i}q_0x^0)$, which is related to the Weyl ordering through a nonlinear relation between the real parameters appearing in the exponentials

$$e^{\mathrm{i}k_\mu x^\mu} = e^{\mathrm{i}\left(\frac{e^{k_0/\kappa}-1}{k_0/\kappa}\right)k_ix^i}e^{\mathrm{i}k_0x^0}. \tag{3.5}$$

This transformation, $(k_0, k_i) \to \left(k_0, \left(\frac{e^{k_0/\kappa}-1}{k_0/\kappa}\right)k_i\right)$ is a general coordinate change, *i.e.*, a diffeomorphism on the group manifold. It is then legitimate to interpret the group manifold associated to the Lie group AN_3 as the momentum space of theories on κ-Minkowski that make use of non-commutative plane waves. This is the case of (quantum) field theories in which ordered plane waves are a basis for scalar fields and solutions of the equations of motion).

3.1.2 Group Orbits as a Tool to Probe Geometry

We know from Lie group theory that if there is a non degenerate Killing form, then there is a natural way to define a bi-invariant metric over the group manifold. Unfortunatly, in our case the group AN_3 is not semi-simple and the Killing form is degenerate, thus the there is no bi-invariant metric. Nevertheless, there is a basis of left-invariant forms and another of right-invariant forms. According to [86], any quadratic form built from symmetrized right-invariant forms will give a right-invariant metric; the same holds for a left-invariant metric. All the right(left)-invariant metrics with the same signature (and same rank) are equivalent modulo diffeomorphisms. Only if one assumes the signature to be Lorentzian and the rank to be maximal (no zero eigenvalues), then there is a unique right-invariant metric and a unique left-invariant one. This is not our case, thus no right- and left-invariant metric is equivalent to each other. Since we have some freedom in choosing right- or left-invariant metrics, we need some more restrictive criterion.

In [50, 51, 83, 84, 87–89] it has been shown that phase space associated with $\mathcal{M}_\kappa$ is compatible with a maximally symmetric geometry of positive curvature, *i.e.* the de Sitter space [90, 91]. Nevertheless, we will later show that this geometry is not the only compatible one.

The curved momentum space of κ-Minkowski has been studied for the first time in [50], using a matrix representation of $\mathfrak{an}_3$. Note that the five-dimensional Lorentz algebra $\mathfrak{so}(4,1)$ has the a subalgebra which is isomorphic to the algebra (2.11). In particular, the isomorphism is realized by

$$x^\mu \sim M_{0\mu} + M_{4\mu}\,, \tag{3.6}$$

where the M_{AB}'s are the standard representation of Lorentz generators as 5×5antisymmetric matrices multiplied by the Minkowski metric. This isomorphism induces the following five-dimensional representation [48] of the non-commutative coordinates in (2.11)

$$\rho(x^0) = -\frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \hat{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}\,, \qquad \rho(x^i) = -\frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_i & 0 \\ \mathbf{e}_i^T & \hat{0} & \mathbf{e}_i^T \\ 0 & -\mathbf{e}_i & 0 \end{pmatrix}\,, \tag{3.7}$$

where we $\mathbf{e}_i^a = \delta_i^a$, $\mathbf{0}$ and $\hat{0}$ are the null vector and matrix in three dimensions respectively. This is a $*$-representation under the involution compatible with the Lorentz group

$$(\rho^\alpha{}_\beta)^* = \eta^{\alpha\lambda}\eta_{\gamma\beta}\overline{\rho^\gamma{}_\lambda}, \tag{3.8}$$

which is to say that rising an index, flipping indices, complex conjugating and lowering back the index), leaves all generators $\rho(x^\mu)$ invariant. This representation also induces a representation

$$G^*(p_\mu) = e^{\mathrm{i}p_i\rho(x^i)}e^{\mathrm{i}p_0\rho(x^0)} = \begin{pmatrix} \cosh\frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} & \frac{\mathbf{p}}{\kappa} & \sinh\frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} \\ e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} & \mathbb{1} & e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} \\ \sinh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} & -\frac{\mathbf{p}}{\kappa} & \cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} \end{pmatrix}, \tag{3.9}$$

of the group elements (plane waves) as 5×5 matrices; we use the "time-to-the-right" ordering in order to get simpler formulas. Since the above representation is transitive and larger than the dimension of the group, we have that all the non-degenerate orbits of the group are diffeomorphic to the group manifold. For example, consider the group manifold obtained by exponentiating the the standard representation of $\mathfrak{su}(2)$ as 2×2 complex matrices acting on the vector space of 2D spinors $\mathbb{C}^2$. One can prove that the non-degenerate orbits of the group are all 3-spheres immersed in $\mathbb{R}^4$ (under the canonical identification $\mathbb{R}^4 \sim \mathbb{C}^2$). Indeed the group manifold of $\mathfrak{su}(2)$ is, topologically, a 3-sphere.

In our case, the 3.9 acts as a matrix on a five dimensional vector space V and the orbit of vector $v \in V$ are defined as

$$(AN_3)\ \ v := \{g \triangleright v \mid g \in AN_3\}\,. \tag{3.10}$$

In our case we consider a $5D$ Minkoski space $\mathcal{M}^5$ as ambient space. Given a fiducial vector $u = (u^0, \mathbf{u}, u^4)$ the group orbits coincide with the locus of points obtained by acting with $G^*(p_\mu)$ upon u^A for all choices of p^μ

$$X^A(p_\mu) = G^*(p_\mu)^A{}_B u^B = \begin{pmatrix} \frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} + (u^0+u^4)e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} + u^0\cosh\frac{p_0}{\kappa} + u^4\sinh\frac{p_0}{\kappa} \\ \mathbf{u} + (u^0+u^4)e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} \\ -\frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} - (u^0+u^4)e^{\frac{p_0}{\kappa}}\frac{\|\mathbf{p}\|^2}{2\kappa^2} + u^0\sinh\frac{p_0}{\kappa} + u^4\cosh\frac{p_0}{\kappa} \end{pmatrix}. \tag{3.11}$$

Furthermore, the $X^A(p_\mu)$ are the parametric representation of a four-dimensional submanifold embedded in $\mathcal{M}^5$ which is diffeomorphic to the group manifold of AN_3 (and to our desired momentum space). Notice that, since the orbits of the Lorentz group are disconnected, the choice of the fiducial vector u^A is not inconsequential. For example, suppose to apply the above construction to the Euclidean rotation group $SO(n)$. Then, any non zero fiducial vector could have been transformed up to a rescaling into any other by a rotation, and there would have been only one kind of orbit *i.e.* only one geometry for momentum space. Nevertheless, we are dealing with $SO(4,1)$, so we cannot transform a space-like fiducial vector into a time-like or light-like one with a Lorentz transformation. Since all $G^*(p_\mu) \in SO(4,1)$, we have

$$X^A(p)X^B(p)\eta_{AB} = u^A u^B \eta_{AB}, \qquad \eta_{AB} = \mathrm{diag}(1,-1,-1,-1,-1), \tag{3.12}$$

for all $p^\mu \in \mathbb{R}^4$. Hence, different values for the Casimir of the group $X^A(p)X^B(p)\eta_{AB}$ correspond to different orbits. It follows that the Lorentz group has three families of non-degenerate orbits depending whether $X^A(p)X^B(p)\eta_{AB}$ is positive, negative, or null. With each one of these three possibilities we associate a different (inequivalent) geometry of the momentum space. In particular, the X^A's coordinates induce the following metric

$$\mathrm{d}s^2 = -\frac{\partial X^A}{\partial p_\mu}\frac{\partial X^B}{\partial p_\nu}\ \eta_{AB}\ \mathrm{d}p_\mu \mathrm{d}p_\nu. \tag{3.13}$$

on the group manifold. Notice that given any fiducial vector, it is always possible to realign the axes of the embedding space via a Lorentz transformation $X'^A = \lambda^A{}_B X^B$ in such a way that the vector $\lambda^A{}_B u^B$ is aligned along one (or two, in the light-like case) of the X'^A axes. In this way we identify three equivalence classes for the choices of fiducial vectors, whose element all give rise to the same geometry: the space-like, light-like and time-like class. Our convention is the following: we align the space-like choice along the 4 axis, the light-like choice along the $1-4$ plane and the time-like choice along the 0 axis. Now, we have to discuss in details what happens for any choice of the equivalent classes of fiducial vectors.

Degenerate Cases

Looking at (3.11) it easy to see that any fiducial vector $u \in \mathcal{M}^5$ such that $u^0 = -u^4$ one has

$$X^A(p_\mu) = \begin{pmatrix} \frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} + u^0 e^{-\frac{p_0}{\kappa}} \\ \\ \mathbf{u} \\ \\ -\frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} - u^0 e^{-\frac{p_0}{\kappa}} \end{pmatrix}. \tag{3.14}$$

where $X^0(p) + X^4(p) = 0$ for all p. In this degenerate case the group orbit reduces to just a straight line in $\mathcal{M}^5$ parametrized by u^o and the components of $\mathbf{u}$. Thus we have a three parameter family of these straight lines depending on the choice of $\mathbf{u}$ and all of them lay in the $3D$ hyperplane $X^0 + X^4 = 0$ immersed in $\mathcal{M}^5$. Moreover, all these lines the induced metric $\mathrm{d}s^2 = 0$ is light-like.

Notice that using (3.12) we get $X^A X^B \eta_{AB} = -||\mathbf{u}||^2$. Since $||\mathbf{u}||^2$ is a non negative quantity, only non time-like fiducial vectors may have a degenerate orbit. Moreover, for a light-like fiducial vector the degenerate orbit occurs if only if $\mathbf{u} = \mathbf{0}$.

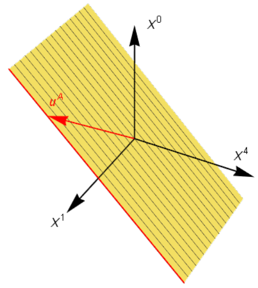


Figure 3.1: *The hyperplane $X^0 + X^4 = 0$ formed by the degenreate orbits obtained from fiducial vectors of the form $u^0 = -u^4$. For all these fiducial vectors the orbit is just a straight line with a degenerated (light-like) induced metric. In particular, we have a three parameter family of orbits depending on components of* $\mathbf{u}$.

Space-like fiducial vector

We consider a space-like fiducial vectors $u^A u^B \eta_{AB} < 0$, this is also the case usually studied in literature. If one considers $u^A = \delta_4^A$ then, using (3.11), one gets the following parametrization of the orbit

$$X(p) = \begin{pmatrix} \sinh \frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}} \frac{\|\mathbf{p}\|^2}{2\kappa^2} \\ \\ e^{\frac{p_0}{\kappa}} \frac{\mathbf{p}}{\kappa} \\ \\ \cosh \frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}} \frac{\|\mathbf{p}\|^2}{2\kappa^2} \end{pmatrix}, \qquad p \in \mathbb{R}^3. \tag{3.15}$$

It is easy to check that (3.15) satisfy $X^A X^B \eta_{AB} = 1$ which is the implicit equation of a one-sheeted $4D$dimensional de Sitter hyperboloid embedded in a $5D$ ambient Minkowski space; this is consistent with the conclusion in [48, 50]. Furthermore, using (3.16) we have

$$\mathrm{d}s^2 = -\frac{1}{\kappa^2}\mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\mathrm{d}\mathbf{p}^2, \tag{3.16}$$

which is the same right-invariant metric as in [86]. One can verify that $X^0 + X^4 > 0$ is verified for all choices of p_μ, and therefore we are actually dealing with half of de Sitter space-time: the half one covered by the flat slicing (the coordinates p_μ corresponding to time-to-the-right ordering of plane waves are what cosmologists call comoving coordinates for de Sitter space-time).

This constraint makes the portion of momentum space covered by the p_μ coordinates non-Lorentz-invariant [84], and one has to choose a slightly different topology for the ambient space in order to restore Lorentz invariance[1]. After one makes the correct topology identification, these two half de Sitter hyperboloids are topologically equivalent since them

[1] For a more detailed discussion see [75, 79]

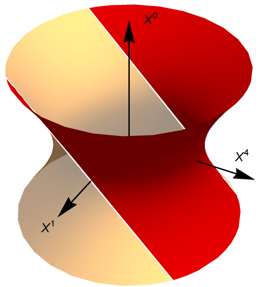


Figure 3.2: *We have coloured in the half one-sheeted deSitter hyperboloid covered by the coordinates X^A over the orbit of a space-like vector u^a. The white straight lines represent degenrated orbits and are the intersection of the embedded de Sitter hyperboloid with the hyperplane $X^0 + X^4$=0.*

both have the topology of a plane. One easily convinces himself that if $u^a = -\delta^A_4$, then the obtained coordinates will satisfy $X^0 + X^4 < 0$, and hence they will cover the other half of the de Sitter manifold. Furthermore, these two half de Sitter hyperboloids are topologically equivalent. Notice that these region are obtained by slicing the de Sitter hyperboloid with the $X^0 + X^4 = 0$ plane.

Light-like fiducial vector

This time we study the class of orbits obtained with a lightlike fiducial vector $u^A u^B \eta_{AB} = 0$. We stress that all light-like vector with $\mathbf{u} \neq 0$ and $u^0 \neq u^4$ will have non degenerate orbits leading to equivalent geometry for the momentum space. As an example, consider the fiducial vector to be $u^a = (\frac{1}{\sqrt{2}}, \mathbf{0}, \frac{1}{\sqrt{2}})$. Using (3.11) adn (3.12) we obtain

$$X^A(p) = \begin{pmatrix} \frac{e^{\frac{p_0}{\kappa}}}{\sqrt{2}} \left(1 + \frac{\|\mathbf{p}\|^2}{\kappa^2}\right) \\ \\ \sqrt{2} e^{\frac{p_0}{\kappa}} \frac{\mathbf{p}}{\kappa} \\ \\ \frac{e^{\frac{p_0}{\kappa}}}{\sqrt{2}} \left(1 - \frac{\|\mathbf{p}\|^2}{\kappa^2}\right) \end{pmatrix}, \qquad p \in \mathbb{R}^3, \tag{3.17}$$

and $X^A X^B \eta_{AB} = 0$. The induced metric

$$\mathrm{d}s^2 = \frac{4}{\kappa^2} e^{\frac{2p_0}{\kappa}} \mathrm{d}\mathbf{p}^2, \tag{3.18}$$

has been computed using (3.16). More in general, for any light-like fiducial vector of the form $\mathbf{u} = 0$, $u^4 = u^0$ with $u^0 > 0$ the momentum manifold is just the limit of the half-one sheeted de Sitter hyperboloid for vanishing cosmological constant; *i.e* the future-oriented light-cone in $\mathcal{M}^5$. The past-oriented light cone is obtained by taking u^0 negative.

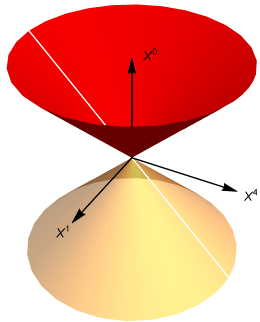


Figure 3.3: *We coloured in red the future-oriented light cone covered by coordinates (3.17). The removed half lines corresponds to degenerate orbits.*

Notice that in Fig. 3.3 we removed a half straight line both from the future oriented and past-oriented light cones. This is because they correspond to a degenerate case. Indeed, if one considers a fiducial vector of the form $(u^0, \mathbf{0}, u^4)$ then one gets a degenerated orbit. In particular, if $u^0 > 0 \;\; (u^0 < 0)$ the parametrization (3.14) satisfies $X^0 + X^4 = 0$ and $X^0 > 0$ $(X^0 < 0)$, which corresponds to the removed half lines. Moreover, each of the two folds of the light-cone is topologically equivalent to a plane.

Time-like fiducial vector

The only remaining class is that of the orbits generated by time-like fiducial vectors. We remind the reader that the time-like condition $u^a u^b \eta_{AB} > 0$ prevents degenerate orbits to occur. The most elementary choice for a time-like fiducial vector is $u^a = \delta_0^A$ and, using (3.11), we obtain

$$X(p) = \begin{pmatrix} \cosh \frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}} \frac{\|\mathbf{p}\|^2}{2\kappa^2} \\ e^{\frac{p_0}{\kappa}} \frac{\mathbf{p}}{\kappa} \\ \sinh \frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}} \frac{\|\mathbf{p}\|^2}{2\kappa^2} \end{pmatrix} \tag{3.19}$$

as a parametrization with $X^A X^B \eta_{AB} = -1$. Then, following the now familiar scheme we work out the induced metric

$$\mathrm{d}s^2 = \frac{1}{\kappa^2} \mathrm{d}p_0^2 + \tfrac{1}{\kappa^2} e^{\frac{2p_0}{\kappa}} \mathrm{d}\mathbf{p}^2 \tag{3.20}$$

using (3.16) over the group manifold; which is given by a two sheeted Riemann hyperboloid this time.

Not surprisingly, coordinates (3.19) maps only one of the two disconnected region; the one with $X^0 > 0$. If one considers a time-like vector with negative u^0, then the coordinate system mapping the other half of the hyperboloid is obtained. There is non need to remove anything this time since there are no degenerate orbits. Moreover, each of the two regions is obviously topologically a plane.

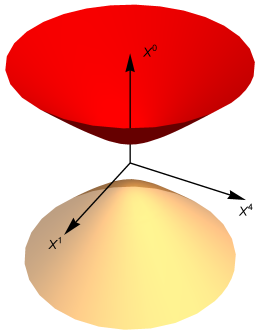


Figure 3.4: *The red coloured region corresponds to the region of the Riemannian hyperboloid covered by coordinates*(3.19).

Summarizing, we showed that there are three classes of possible embedded submanifolds in $\mathcal{M}^5$ which are all diffeomorphic to group manifold of AN_3. In addition to these three families of equivalent four-dimensional momentum spaces, there is also a family of degenerate cases. This corresponds to the choice of fiducial vector with $u^0 = -u^4$, in which case the orbit reduces to a straight line in the $X^0 + X^4 = 0$ hyperplane; each one of those is light-like so that the induced metric vanishes.

For the space-like choice , we reproduce the known result of an embedding in Minkowski space of the patch of de Sitter space that is covered by comoving coordinates/flat slicing. For a light-like fiducial vector, we simply obtain the limit of vanishing cosmological constant of the above case (future-oriented light cone of the ambient Minkowski space). Finally, for a time-like fiducial vector, we get one of the two sheets of a Riemannian hyperbolic space, *i.e.* the positive-frequency mass-shell of a massive particle. The coordinates p_μ in this case cover en entire sheet of the hyperboloid, because the whole sheet lies above the plane $X^0 = -X^4$. The three manifolds we found are diffeomorphic to each other, as they have the same topology as that of a plane. This is to be expected, because they are all diffemorphic to the group manifold of of AN_3. In Table 3.1 we illustrate these results.

3.1.3 Embedding of AN_3 into $SO(3,2)$

Let us discuss a different 5×5 matrix representation then the one given in (3.7). In particular, we introduce the following representation of $\mathfrak{an}_3$

$$\rho'(x^0) = -\frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \hat{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \qquad \rho'(x^1) = \frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & -\mathbf{e}_1 & 0 \\ \mathbf{e}_1^T & \hat{0} & \mathbf{e}_1^T \\ 0 & \mathbf{e}_1 & 0 \end{pmatrix},$$
$$\rho'(x^2) = \frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_2 & 0 \\ \mathbf{e}_2^T & \hat{0} & \mathbf{e}_2^T \\ 0 & -\mathbf{e}_2 & 0 \end{pmatrix}, \qquad \rho'(x^3) = \frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_3 & 0 \\ \mathbf{e}_3^T & \hat{0} & \mathbf{e}_3^T \\ 0 & -\mathbf{e}_3 & 0 \end{pmatrix}, \tag{3.21}$$

as matrices of $\mathfrak{so}(3,2)$. This induces the following isomorphism between coordinates x^μ and the generators $J_{\mu\nu}$ of $\mathfrak{so}(3,2)$ as in:

$$x^0 \sim J_{0,4}\,, \quad x^1 \sim J_{0,1} + J_{4,1}\,, \quad x^2 \sim J_{0,2} + J_{4,2}\,, \quad x^3 \sim J_{0,3} + J_{4,3}\,, \tag{3.22}$$

where the coordinates 0 and 1 have the same signature, opposite to that of coordinates 2, 3 and 4. The difference between (3.7) and (3.21) is merely the form of $\rho(x^1)$. Note that in (3.7) the antisymmetric components are the 4-1 and the 0-1 are symmetric, hence the coordinate 1 has the same nature of coordinate 4 and opposite signature with respect to coordinate 0. This picture is inverted in (3.21): the axis 1 has the same signature as 0. There are only two possible choices for the signature of the 1, 2 and 3: either they have all the same signature, which will be the same of either axis 0 or 4 (which have opposite signatures because $\rho(x^0)$ is symmetric); in this case we have a (3.7). Otherwise, one of the three coordinates has a different signature from the others, and then it is always possible to recast our matrices in the form 3.21 by reshuffling the axes.

It follows that, we can embed $\mathfrak{an}_3$ either into $\mathfrak{so}(4,1)$ or $\mathfrak{so}(3,2)$. However, this choice has some consequences on the corresponding momentum spaces. Consider in fact the exponentiation $G'^*(p_\mu) = e^{\mathrm{i}p_i\rho'(x^i)}e^{\mathrm{i}p_0\rho'(x^0)}$ which in matrix form reads

$$G'^*(p_\mu) = \begin{pmatrix} \mathrm{ch}\left(\frac{p_0}{\kappa}\right) + \frac{p_2^2+p_3^2-p_1^2}{2\kappa^2}\, e^{p_0/\kappa} & \frac{p_1}{\kappa} & -\frac{p_2}{\kappa} & -\frac{p_3}{\kappa} & sh\left(\frac{p_0}{\kappa}\right) + \frac{p_2^2+p_3^2-p_1^2}{2\kappa^2}\, e^{p_0/\kappa} \\ -\frac{e^{p_0/\kappa}p_1}{\kappa} & 1 & 0 & 0 & -\frac{e^{p_0/\kappa}p_1}{\kappa} \\ -\frac{e^{p_0/\kappa}p_2}{\kappa} & 0 & 1 & 0 & -\frac{e^{p_0/\kappa}p_2}{\kappa} \\ -\frac{e^{p_0/\kappa}p_3}{\kappa} & 0 & 0 & 1 & -\frac{e^{p_0/\kappa}p_3}{\kappa} \\ \mathrm{sh}\left(\frac{p_0}{\kappa}\right) - \frac{p_2^2+p_3^2-p_1^2}{2\kappa^2}\, e^{p_0/\kappa} & -\frac{p_1}{\kappa} & \frac{p_2}{\kappa} & \frac{p_3}{\kappa} & \mathrm{ch}\left(\frac{p_0}{\kappa}\right) - \frac{p_2^2+p_3^2-p_1^2}{2\kappa^2}\, e^{p_0/\kappa} \end{pmatrix} \tag{3.23}$$

where $\mathrm{sh}(\cdot)$ and $\mathrm{ch}(\cdot)$ denote $\sinh(\cdot)$ and $\cosh(\cdot)$ respectively. We proceed just as we did in the $SO(4,1)$ case and compute the action of the group on $X'^A(p_\mu) = G^*(p_\mu)^A{}_B u^B$ a generic fiducial vector u and get

$$X'^A(p_\mu) = \begin{pmatrix} u^0 \mathrm{ch}\left(\frac{p_0}{\kappa}\right) + \frac{\left(-p_1^2+p_2^2+p_3^2\right)\left(u^0+u^4\right)e^{\frac{p_0}{\kappa}}}{2\kappa^2} + u^4 \mathrm{sh}\left(\frac{p_0}{\kappa}\right) + \frac{-p_3u^3-p_2u^2+p_1u^1}{\kappa} \\ u^1 - \frac{p_1\left(u^0+u^4\right)e^{\frac{p_0}{\kappa}}}{\kappa} \\ u^2 - \frac{p_2\left(u^0+u^4\right)e^{\frac{p_0}{\kappa}}}{\kappa} \\ u^3 - \frac{p_3\left(u^0+u^4\right)e^{\frac{p_0}{\kappa}}}{\kappa} \\ u^0 \mathrm{sh}\left(\frac{p_0}{\kappa}\right) + \frac{\left(p_1^2-p_2^2-p_3^2\right)\left(u^0+u^4\right)e^{\frac{p_0}{\kappa}}}{2\kappa^2} + u^4 \mathrm{ch}\left(\frac{p_0}{\kappa}\right) + \frac{p_3u^3+p_2u^2-p_1^1u}{\kappa} \end{pmatrix}, \tag{3.24}$$

which is the analogue of (3.11). Hence, the X'^A are the embedding coordinates of a $4D$ submanfiold which is diffeomorphic to the momentum space and the induced metric is

$$\mathrm{d}s^2 = -\eta'_{AB}\frac{\partial X'^A}{\partial p_\mu}\frac{\partial X'^B}{\partial p_\nu}\mathrm{d}p_\mu \mathrm{d}p_\nu \tag{3.25}$$

Note that

$$X'^A X'_A = X'^A(p)X'^B(p)\eta'_{AB}(p) = u^A u^B \eta'_{AB}\,, \quad \eta'_{AB} = \mathrm{diag}(-1,-1,+1,+1,+1). \tag{3.26}$$

allows one to distinguish three inequivalent case depending on the sign of $u^A u^B \eta'_{AB}$. If $u^A u^B \eta'_{AB} < 0$ we get an anti-de Sitter space, which is a one-sheeted hyperboloid whose axis lay along the space-like coordinates. If if $u^A u^B \eta'_{AB} = 0$ we have the light cone the AdS hyperboloid tends to in the limit of vanishing cosmological constant (unless $u^0 = -u^4$, in which case we have a degenerate geometry). Finally, if $u^A u^B \eta'_{AB} > 0$ we have a two-sheeted hyperboloid with signature $(+,+,-,-)$.

Furthermore, the sign of $X'^0 + X'^4$ is fixed and equal to the sign of $u^0 + u^4$ because $X'^0 + X'^4 = e^{\frac{p_0}{\kappa}}(u^0+u^4)$. We assume without loss of generality that $u^0 + u^4 > 0$; the other case mirrors this one. Thus, in the anti-de Sitter case $u^A u^B \eta'_{AB} < 0$ we have that coordinates p_μ cover the *half-space* coordinatization of anti-de Sitter. For example, if $u^A = (1,0,0,0,0)$ the induced metric on the orbit (3.25) is, :

$$\mathrm{d}s^2 = \frac{1}{\kappa^2}\mathrm{d}p_0^2 - \frac{e^{\frac{2p_0}{\kappa}}}{\kappa^2}\left(\mathrm{d}p_1^2 - \mathrm{d}p_2^2 - \mathrm{d}p_3^2\right)\,, \tag{3.27}$$

and by transforming $p_0 = -\kappa \log(y/\kappa)$ we have

$$\mathrm{d}s^2 = \frac{1}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2 - \mathrm{d}p_1^2\right)\,, \tag{3.28}$$

which is the coordinate patch covering half of AdS space-time [92] .

The $u^A u^B \eta'_{AB} = 0$ case will be again a cone, but its intersection with the half-space $X'^0 + X'^4 > 0$ this time will not leave out simply a line, unless we are in the 1+1-dimensional case. In fact, the embedding of $\mathfrak{an}_3$ in $\mathfrak{so}(4,1)$ implied that

$$(X^0)^2 = (X^4)^2 + \sum_{i=1}^{3}(X^i)^2 \geq (X^4)^2\,, \tag{3.29}$$

so that $X^0 = -X^4$ only on the line $X^i = 0$. On the other hand, the embedding of $\mathfrak{an}_3$ into $\mathfrak{so}(3,2)$ gives

$$(X'^0)^2 = (X'^4)^2 - (X'^1)^2 + (X'^2)^2 + (X'^3)^2\,, \tag{3.30}$$

so that the intersection of this submanifold with $X'^0 + X'^4 > 0$ gives a non-zero measure portion of the cone. In the 1+1-dimensional case, however, this difference disappears, because $(X'^0)^2 + (X'^1)^2 = (X'^4)^2$ implies that $(X'^4)^2 \geq (X'^0)^2$, and only the line $X'^1 = 0$ is left out (see Table 3.2).

In the $u^A u^B \eta'_{AB} > 0$ case one has

$$(X'^4)^2 + (X'^2)^2 + (X'^3)^2 > (X'^0)^2 + (X'^1)^2\,, \tag{3.31}$$

which has a quite complicated intersection with $X'^0 + X'^4 > 0$. This case too is greatly simplified by going to 1+1 dimensions. Indeed if we suppress X'^2 and X'^3 we are left with

$$(X'^4)^2 > (X'^0)^2 + (X'^1)^2\,, \tag{3.32}$$

which never intersects the plane $X'^0 = -X'^4$.

Just like in the $SO(4,1)$ case, the isotropy subgrops have to be identified with the subgroups of $SO(3,2)$ that stabilize u^A. In the $u^A u^B \eta'_{AB} < 0$ case, this is the subgroup that stabilizes a time-like vector, and so it is the Lorentz group $SO(3,1)$, this is also compatible with the result in [24]. For u^A light-like, the subgroup is $ISO(2,1)$, *i.e.* the Poincaré group in 2+1 dimensions. Finally, in the $u^A u^B \eta'_{AB} > 0$ case, the group is $SO(2,2)$. The action of these groups on the corresponding momentum spaces are such that a finite transformation can bring a point outside of the coordinate patch covered by the p_μ coordinates, just like in the previous Section for space- and light-like fiducial vectors. This time, however, this phenomenon happens for all choices of fiducial vector.

3.1.4 Isometries of the three new momentum spaces

From the above sections, we obtain the following result. Excluding the degenerate cases, one has four inequivalent choices for the metric:

$$\mathrm{d}s^2 = \mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\left(\mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2\right)\,, \tag{3.33}$$

$$\mathrm{d}s^2 = \mathrm{d}p_0^2 - \frac{e^{2p_0/\kappa}}{\kappa^2}\left(\mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2\right) , \tag{3.34}$$

$$\mathrm{d}s^2 = \mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\left(-\mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2\right) , \tag{3.35}$$

$$\mathrm{d}s^2 = -\mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\left(-\mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2\right) . \tag{3.36}$$

All the above metrics have been encountered with our embeddings of AN_3 into $SO(4,1)$ and $SO(3,2)$. The first is the Riemannian metric of the two-sheeted hyperboloid of $SO(4,1)$, the second is the dS metric of $SO(4,1)$, the third is the AdS metric of $SO(3,2)$ and the last is the signature $(+,+,-,-)$ hyperboloid of $SO(3,2)$.

3.2 Symmetries of the momentum spaces

In the last section we showed that there are three families of inequivalent momentum spaces plus a class of degenerated lower dimensional cases. Now, we want to understand which the possible symmetries of those spaces are. We remind the reader that to each of these classes is associated a families of orbits $(AN_3)\,u$ depending on whether u is a space-like, light-like or time-like vector of the $5D$ Minkowski space. Thus, the symmetries of a momentum space coincide with the symmetries of the corresponding orbit. This can be constructed as the Inönü Wigner group contraction of the global symmetry group of the embedding space with respect to the subgroup which stabilizes the fiducial vector u (little group).

3.2.1 Group Contraction

In 1953 E. Inönü and E. Wigner proposed [93] a method to obtain a new Lie group form another one non-isomorphic to the first. This method consists of a group contraction of the second with respect to one of its continuous subgroup [93–95]. Roughly speaking, the contraction mechanism is cast by introducing a parameter in the structure constant of a Lie algebra, in order to change them in a non trivial singular way, and then taking the limiting procedure (usually as the parameter blows up or vanishes). As an example, the Poincaré group in four dimension can be contracted to the Galilei group by sending the speed of light to infinity [96]. In particular, the d-dimensional Euclidean group $ISO(d)$ can be constructed via Inönü–Wigner contraction of the $d+1$-dimensional rotation group $SO(d+1)$.

The $ISO(d)$ as a Contraction of $SO(d+1)$

It is easy to convince that any $d-$dimensional sphere S^d is indistinguishable from the $d-$dimensional Euclidean plane in a neighbourhood of a point; the same happens if one send the sphere radius to infinite. Since the isometry group of S^d is the Lie group $SO(d+1)$, we are interested in the Lie algebra $\mathfrak{so}(d+1)$

$$\begin{aligned}[M_{AB}, M_{CD}] =& \mathrm{i}\,(\delta_{AC} M_{BD} - \delta_{AD} M_{BC} \\ & + \delta_{BD} M_{AC} - \delta_{BC} M_{AD})\ ,\end{aligned} \tag{3.37}$$

where upper-case latin indices goes from 1 to $d+1$. The Lie algebra $\mathfrak{so}(d)$ can be represented in an embedding $(d+1)$-dimensional Euclidean space as follows:

$$M_{AB} = \mathrm{i}\,(x_A \partial_B - x_B \partial_A)\ . \tag{3.38}$$

Consider a pure translated coordinate system centred at some point, say $t_A = (0, 0, \ldots, 0, r)$ so that $x_A = t_A + y_A$ and $\frac{\partial}{\partial x_A} = \frac{\partial}{\partial y_A}$. The generators which leave the point invariant

$$M_{ab} = \mathrm{i}\,(y_a \partial_b - y_b \partial_a)\ , \tag{3.39}$$

where $a, b \in \{1, \ldots, d\}$, are those generating the little group[2] $SO(d)$. The remaining generators

$$M_{a(d+1)} = \mathrm{i}\left(y_a \partial_{(d+1)} - (\lambda + y_{(d+1)})\partial_a\right)\ . \tag{3.40}$$

are those which move the point.

At this point we introduce the generators $P_a = \frac{1}{r} M_{a(d+1)}$, where r as the dimension of a length, which represent as

$$P_a = M_{a(d+1)} = \mathrm{i}\partial_a + \frac{\mathrm{i}}{r}\left(y_a \partial_{(d+1)} - y_{(d+1)}\partial_a\right)\ . \tag{3.41}$$

it follows that the structure constant are changed and the algebra becomes

$$\begin{aligned}[M_{ab}, M_{cd}] &= \mathrm{i}\,(\delta_{ac} M_{bd} - \delta_{ad} M_{bc} + \delta_{bd} M_{ac} - \delta_{bc} M_{ad})\ , \\ [M_{ab}, P_c] &= \mathrm{i}\,(\delta_{ac} P_b - \delta_{bc} P_a)\ , \qquad [P_a, P_b] = \frac{\mathrm{i}}{r^2} M_{ab}\,.\end{aligned} \tag{3.42}$$

Here, r can be regarded as the radius of the $d+1$-dimensional sphere whose isometries are given by the dimensionful algebra (3.42) in a neighbourhood of point $t_a = (0, 0, ..., r)$. We carry out the contraction procedure by taking the limit of (3.42) as r goes to infinity and obtain

$$\begin{aligned}[M_{ab}, M_{cd}] &= \mathrm{i}\,(\delta_{ac} M_{bd} - \delta_{ad} M_{bc} + \delta_{bd} M_{ac} - \delta_{bc} M_{ad})\ , \\ [M_{ab}, P_c] &= \mathrm{i}\,(\delta_{ac} P_b - \delta_{bc} P_a)\ , \qquad [P_a, P_b] = 0\,.\end{aligned} \tag{3.43}$$

In other words, the $\mathfrak{iso}(d)$ algebra is understood as limit of the isometries of a sphere. Notice that the translation generators represent as derivatives: $P_a = \mathrm{i}\partial_a$.

More in general, the Inönü–Wigner group contraction on a Lie group $\mathcal{G}$ is carried out in terms of one of its little groups (*i.e.* the stabilizer subgroup of some given fiducial vector in the embedding linear representation of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$).

[2]One can write the little group elements as $(t_A t^C - \delta_A^C)(t_B t^D - \delta_B^D) M_{BD}$ and the translation generators as $t^B M_{AB}$.

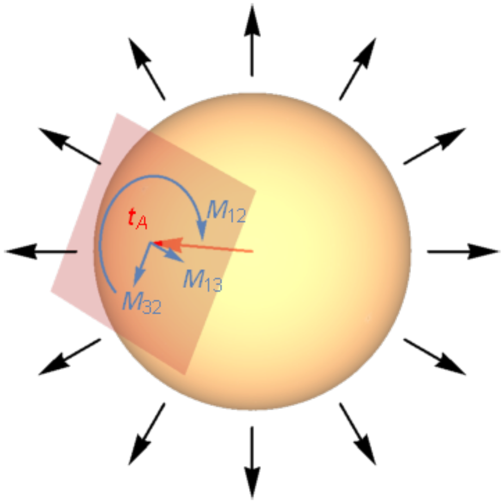


Figure 3.5: *A 2-sphere immersed in* $\mathbb{R}^3$, *with a point of coordinates* t_A *singled out, and the tangent space to the sphere at that point. The generators of the isometry group of the sphere,* $SO(3)$, *split into an isotropy subgroup leaving* t_A *unchanged* (M_{12}) *and a pair of generators that move* t_A (M_{13} *and* M_{32})*. If the sphere is blown out to infinity, these two generators tend to the translation generators of* $ISO(3)$, *while* M_{12} *tends to the rotation generator.*

3.2.2 Contractions of the (A)dS Lie algebra

In the spirit of the above example, we want to obtain the symmetries of the momentum spaces obtained in Sec. 3.1.2 and Sec. 3.1.3. We are now familiar with the fact that each one of those momentum spaces corresponds to the orbit of a certain fiducial vector v under the action of (3.9) or (3.23). Hence, the symmetry of such an orbit can be obtained via group contraction of the symmetry group of the embedding $5D$ space with respect to the little group that stabilizes the fiducial vector v .

In other words, we have to consider the Lie algebra of generators L_{AB} with the following commutation relation

$$[L_{AB}, L_{CD}] = g_{AD}L_{BC} - g_{AC}L_{BD} + g_{BC}L_{AD} - g_{BD}L_{AC}\,, \tag{3.44}$$

where now $A, B, \cdots = 0, \ldots, 4$, and g_{AB} is the metric of the flat embedding space of the $5D$ linear representation of our algebra. In particular, the signature of the metric

$$g_{AB} = \mathrm{diag}(1, -1, -1, -1, -\lambda)\,, \quad \lambda = \pm 1 \tag{3.45}$$

distinguishes between the de Sitter $\mathfrak{so}(4,1)$ the anti-de Sitter $\mathfrak{so}(3,2)$ Lie algebras, these are indicated as dS and AdS respectively [97]. Usually the generators L_{AB} are split as follows:

$$L_{ij} = \epsilon_{ijk}J_k\,, \quad L_{0j} = K_j\,, \quad L_{4j} = M_j\,, \quad L_{04} = B\,, \tag{3.46}$$

hence, the algebra (3.44) reads

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, M_j] = \epsilon_{ijk} M_k, && [J_i, K_j] = \epsilon_{ijk} K_k, \\ &[K_i, K_j] = -\epsilon_{ijk} J_k, && [M_i, M_j] = \lambda \epsilon_{ijk} J_k, && [K_i, M_j] = \delta_{ij} B, \\ &[K_i, B] = M_i, && [M_i, B] = \lambda K_i, && [J_i, B] = 0. \end{aligned} \tag{3.47}$$

We distinguish three classes of inequivalent fiducial vectors: time-like, light-like and space-like. Within these classes, any vector can be transformed into any other with a group transformation and a rescaling (and possibly a reflection).

Contracting $SO(4,1)$ with the little group of a Space-like fiducial vector

We consider the space-like fiducial vector $v_1^A = (0,0,0,0,\alpha)$. It is left invariant under the action of the $L_{\mu\nu}$ generators with $\mu, \nu \in \{0, \dots, 3\}$. In other words, the little group that stabilizes v is generated by the J_i's and the K_i's. On the other hand v is changed by the M_i's and B. Then, following the contraction mechanism, we rescale these last two generators as

$$P_0 = \frac{B}{\alpha}, \qquad P_i = \frac{M_i}{\alpha}, \tag{3.48}$$

where α is the only non-vanishing component of the fiducial vector v. Notice that α plays the same role played by r in Sec. 3.2.1 because (3.48) modify the algebra (3.47)

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k, \\ &[K_i, K_j] = -\epsilon_{ijk} J_k, && [P_i, P_j] = \frac{1}{\alpha^2}\, \epsilon_{ijk} J_k, && [K_i, P_j] = \delta_{ij} P_0, \\ &[K_i, P_0] = P_i, && [M_i, P_0] = \frac{1}{\alpha} K_i, && [J_i, P_0] = 0. \end{aligned} \tag{3.49}$$

We realize the group contraction by sending $\alpha \to \infty$, so that the (3.49) become

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k, \\ &[K_i, K_j] = -\epsilon_{ijk} J_k, && [P_i, P_j] = 0, && [K_i, P_j] = \delta_{ij} P_0, \\ &[K_i, P_0] = P_i, && [P_i, P_0] = 0, && [J_i, P_0] = 0. \end{aligned} \tag{3.50}$$

Not surprisingly we obtain the Poincaré algebra $\mathfrak{iso}(3,1)$. Indeed, since the orbit of the dS group acting on the fiducial vector $(0,0,0,0,\alpha)$ is a de Sitter hyperboloid oriented along the temporal axis, it looks like Minkowski space-time, whose isometry group is $ISO(3,1)$, in a neighbourhood of the fiducial vector.

Contraction of $SO(4,1)$ with the little group of a Light-like fiducial vector

This time we consider the light-like fiducial vector $v_2^A = (\beta, 0, 0, 0, \beta)$ is lightlike whose stabilizing subgroup is generated by the L_{ij} (*i.e.* J_i) and the $N_i^+ := K_i + M_i$. It is changed

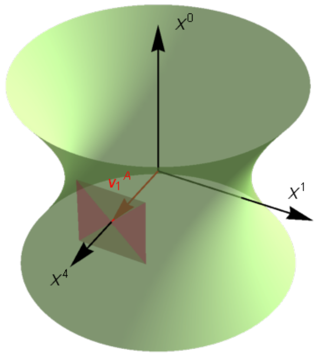


Figure 3.6: *Orbit of the dS group generated by a spacelike fiducial vector v_1^A, with a representation of the tangent space to the orbit at v_1^A.*

by the action of B and $N_i^- := K_i - M_i$, thus we rescale those elements of $\mathfrak{so}(4,1)$ as follows:

$$Q_0 = \frac{B}{\beta}, \qquad Q_i = \frac{N_i^-}{\beta} = \frac{K_i - M_i}{\beta}, \tag{3.51}$$

so that the $\mathfrak{so}(4,1)$ algebra reads

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, N_j^+] = \epsilon_{ijk} N_k^+, && [J_i, Q_j] = \epsilon_{ijk} Q_k, \\ &[N_i^+, N_j^+] = 0, && [N_i^+, Q_j] = -\frac{1}{\beta} 2\epsilon_{ijk} J_k - 2\delta_{ij} Q_0, && [Q_i, Q_j] = 0, \\ &[Q_i, Q_0] = \frac{1}{\beta} Q_i, && [N_i^+, Q_0] = -\frac{1}{\beta} N_i^+, && [J_i, Q_0] = 0. \end{aligned} \tag{3.52}$$

and sending $\beta \to \infty$, we get the algebra:

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, N_j^+] = \epsilon_{ijk} N_k^+, && [J_i, Q_j] = \epsilon_{ijk} Q_k, \\ &[N_i^+, N_j^+] = 0, && [N_i^+, Q_j] = -2\delta_{ij} Q_0, && [Q_i, Q_j] = 0, \\ &[Q_i, Q_0] = 0, && [N_i^+, Q_0] = 0, && [J_i, Q_0] = 0. \end{aligned} \tag{3.53}$$

The brackets in (3.53) define the Lie algebra $\mathfrak{carr}(3,1)$ of the Carroll group [98–102], in which J_i and Q_i are interpreted as spatial rotation and translation generators respectively, N_i^+ plays the role of Carrollian boost and Q_0 is the time translation generator. The Carroll group $\mathrm{Carr}(3,1)$ encodes the symmetries of a manifold with a degenerate metric. Indeed, as we obtained in Sec. 3.1.2, the orbit of a light-like fiducial vector is just the future-oriented fold of the light cone, and the induced metric (3.18) has one zero eigenvalue (and the other eigenvalues have all the same sign).

The Carroll group $\mathrm{Carr}(3,1)$ can be defined as the inhomogeneous group associated to those boost which independently preserve the two metrics $\eta_{\mu\nu} = \mathrm{diag}(0,1,1,1)$ and $\eta^{\mu\nu} = \mathrm{diag}(1,0,0,0)$. In some sense, this is dual to the Galilei boosts [103], which preserve the complementary metrics $\eta_{\mu\nu} = \mathrm{diag}(1,0,0,0)$ and $\eta^{\mu\nu} = \mathrm{diag}(0,1,1,1)$.

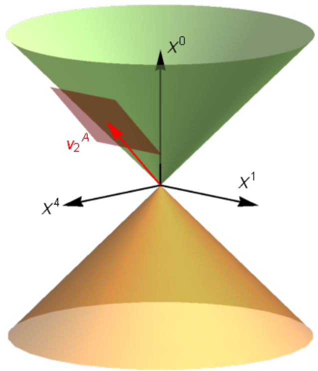


Figure 3.7: *Orbit of the dS group generated by a spacelike fiducial vector v_2^A, with the tangent space to the orbit at v_2^A.*

The name Carroll is a reference to the author of the famous novel *Trough the Looking-glass* [100] because the Carolliann time somehow fits the description of time given to Alice by the Red Queen:

> *"Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else if you run very fast for a long time, as we've been doing."*
>
> *"A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"*

Contracting $SO(4,1)$ with the Little Group of a Time-like fiducial vector

The only case left is that of a time-like vector, say $v_3^A = (\gamma, 0, 0, 0, 0)$. Such a vector is stabilized by the little group of generators L_{ij} (the spatial rotations J_i) and $L_{4i} = M_i$ while it is changed by the action of $L_{0i} = K_i$ and $L_{04} = B$. Repeating the now familiar procedure, we introduce

$$T_i = K_i/\gamma \qquad T_0 = B/\gamma, \tag{3.54}$$

and the algebra $\mathfrak{so}(4,1)$ become

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, M_j] = \epsilon_{ijk} M_k, && [J_i, T_j] = \epsilon_{ijk} T_k, \\
&[T_i, T_j] = -\frac{1}{\gamma^2}\epsilon_{ijk} J_k, && [M_i, M_j] = \epsilon_{ijk} J_k, && [T_i, M_j] = \delta_{ij} T_0, \\
&[T_i, T_0] = M_i, && [M_i, T_0] = T_i, && [J_i, T_0] = 0.
\end{aligned} \tag{3.55}$$

and sending $\gamma \to \infty$ we get

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, M_j] = \epsilon_{ijk} M_k, && [J_i, T_j] = \epsilon_{ijk} T_k, \\ &[T_i, T_j] = 0, && [M_i, M_j] = \epsilon_{ijk} J_k, && [T_i, M_j] = \delta_{ij} T_0, \\ &[T_i, T_0] = 0, && [M_i, T_0] = T_i, && [J_i, T_0] = 0. \end{aligned} \tag{3.56}$$

The Lie algebra above algebra generates the Euclidean group in four dimensions $\mathfrak{iso}(4)$ [88]. with T_μ as translation generators and J_i, M_j as $SO(4)$ generators. Indeed, in Sec. 2.2.5 the orbit of the dS group generated by v_3^A is one of the sheets of the two-sheeted hyperboloid aligned along the X^0 axis. In fact the hyperboloid looks like the Euclidean plane $\mathbb{R}^4$ near it axis.

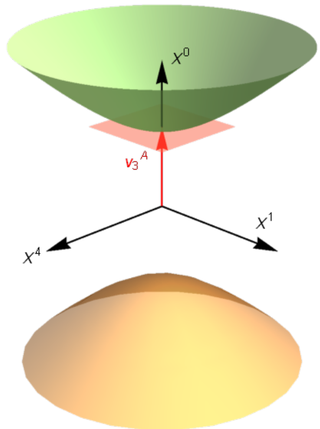


Figure 3.8: *Orbit of the dS group generated by a spacelike fiducial vector v_3^A, with the tangent space to the orbit at v_3^A.*

Contraction of $SO(3,2)$ with a Space-like fiducial vector

In the what follows we will perform on the AdS group $SO(3,1)$ (*i.e.* the group generated by commutators (A)dS algebra (3.47) with $\lambda = -1$ the same kind of contractions we performed on $SO(4,1)$ in the last section. We start with the little group of the space-like fiducial vector laying along the 3 axis $w_1^A = (0,0,0,\alpha,0)$. Its stabilizer is generated by $L_{12} = J_3$, $L_{04} = B$, $L_{41} = M_1$, $L_{42} = M_2$, $L_{01} = K_1$ and $L_{02} = K_2$. When discussing the contraction of AdS we will adopt the following nomenclature for the generators in (3.47):

$$\begin{aligned} J_3 = I_{12}\,, \quad B = I_{34}\,, \quad M_1 = I_{41}\,, \\ M_2 = I_{42}\,, \quad K_1 = I_{31}\,, \quad K_2 = I_{32}\,. \end{aligned} \tag{3.57}$$

The generators that transform w_1^A are L_{3A}, so J_1, J_2, K_3 and M_3. We introduce the following rescaled generators

$$U_a = J_a/\alpha, \qquad U_3 = K_3/\alpha, \qquad U_4 = M_3/\alpha, \tag{3.58}$$

and taking the limit $\alpha \to \infty$ in (3.47) we get:

$$\begin{aligned}
&[I_{\alpha,\beta}, I_{\gamma\delta}] = \gamma_{\alpha\delta} I_{\beta\gamma} - \gamma_{\alpha\gamma} I_{\beta\delta} + \gamma_{\beta\gamma} I_{\alpha\delta} - \gamma_{\beta\delta} I_{\alpha\gamma}\,, \\
&[I_{\alpha,\beta}, U_\gamma] = \gamma_{\alpha\gamma} U_\beta - \gamma_{\beta\gamma} U_\alpha\,, \\
&[U_\alpha, U_\beta] = 0\,,
\end{aligned} \tag{3.59}$$

where $\gamma_{\alpha\beta} = \mathrm{diag}(-1,-1,1,1)$, and the greek indices range from 1 to 44. The contracted algebra (3.59) is $\mathfrak{iso}(2,2)$, describing the isometries of a flat space of signature $(2,2)$, which of course is what a hyperplane parallel to the 0-4 plane is - and that is the tangent space at the fiducial vector to the orbit of w_1^A, a two-sheeted hyperboloid around the axis 3.

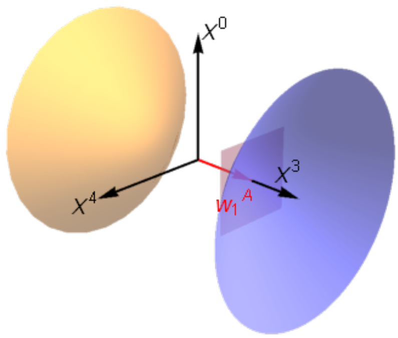


Figure 3.9: *Orbit of the AdS group generated by a spacelike fiducial vector w_1^A, with the tangent space to the orbit at w_1^A.*

Contraction of $SO(3,2)$ with a Light-like fiducial vector

We choose $w_2^A = (0,0,0,\beta,\beta)$ as a fiducial light-like vector this time . The isotropy subgroup is generated by $L_{01} =$, L_{02}, L_{12}, (which close a $\mathfrak{so}(2,1)$ subalgebra), and $L_{03}+L_{04} = K_3+B = N_0$, $L_{13}+L_{14} = -J_2-M_1 = N_1$ and $L_{23}+L_{24} = J_1-M_2 = N_2$. The generators that change w_2^A are $L_{03}-L_{04} = K_3-B$, $L_{13}-L_{14} = -J_2+M_1$, $L_{23}-L_{24} = J_1+M_2$ and $L_{34} = B$. Hence, we rescale

$$V_0 = (K_3-B)/\beta, \quad V_1 = (M_1-J_2)/\beta, \quad V_2 = (J_1+M_2)/\beta, \quad V_3 = B/\beta, \tag{3.60}$$

the commutation relations become

$$\begin{aligned}
&[V_\rho, V_\sigma] = 0, \quad [V_3, V_\rho] = 0, \quad [L_{\rho\sigma}, V_\tau] = h_{\rho\tau} V_\sigma - h_{\sigma\tau} V_\rho, \\
&[L_{\rho\sigma}, V_3] = 0, \quad [N_\rho, V_3] = 0, \quad [N_\rho, V_\sigma] = 0, \\
&[N_\rho, N_\sigma] = 0, \quad [L_{\rho\sigma}, N_\tau] = h_{\rho\tau} N_\sigma - h_{\sigma\tau} N_\rho. \\
&[L_{\rho\sigma}, L_{\tau\lambda}] = h_{\rho\lambda} L_{\sigma\tau} - h_{\rho\tau} L_{\sigma\lambda} + h_{\sigma\tau} L_{\rho\lambda} - h_{\sigma\lambda} L_{\rho\tau},
\end{aligned} \tag{3.61}$$

where $\rho, \sigma, \tau, \lambda, ... = 0, 1, 2$ and $h_{\rho\sigma} = \text{diag}(-1, 1, 1)$. We call the above algebra $\mathfrak{carr}(2,2)$ since it generates a version of the Carroll group in which one of the space-like axes has changed signature. Such a group represent the isometries of a light-like hyperplane in a flat space-time of signature $(2,2)$, which is the description of the tangent space at w_2^A of the orbit of the AdS group generated by w_2^A.

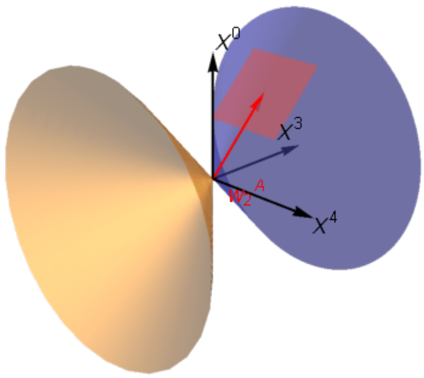


Figure 3.10: *Orbit of the AdS group generated by a spacelike fiducial vector w_2^A, with the tangent space to the orbit at w_2^A.*

Contraction of $SO(3,2)$ with a Time-like fiducial vector

We consider $w_3^A = (\gamma, 0, 0, 0, 0)$ which is a time-like fiducial vector for the $\lambda = -1$ metric. Its stabilizer is generated by L_{ij} (the spatial rotations J_i) and $L_{4i} = M_i$. The generators that change this fiducial vector are $L_{0i} = K_i$ and $L_{04} = B$. Defining

$$S_i = K_i/\gamma \qquad S_0 = B/\gamma, \tag{3.62}$$

and sending $\gamma \to \infty$ we get:

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, M_j] = \epsilon_{ijk} M_k, && [J_i, S_j] = \epsilon_{ijk} S_k, \\
&[S_i, S_j] = 0, && [M_i, M_j] = -\epsilon_{ijk} J_k, && [M_j, S_i] = -\delta_{ij} S_0, \\
&[S_i, S_0] = 0, && [M_i, S_0] = -S_i, && [J_i, S_0] = 0.
\end{aligned} \tag{3.63}$$

This is the Poincaré algebra $\mathfrak{iso}(3,1)$. The orbit of the AdS group acting on the fiducial vector w_3^A is another one-sheeted hyperboloid, oriented so that there is rotational symmetry in the $0-4$ plane. Near the 0 axis, this looks like Minkowski space-time.

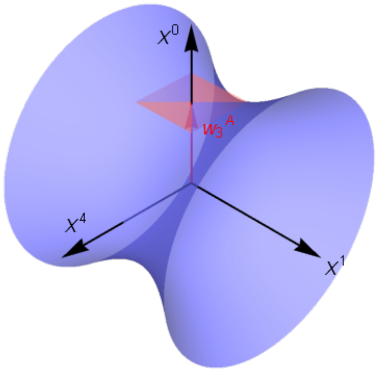


Figure 3.11: *Orbit of the AdS group generated by a spacelike fiducial vector w_3^A, with the tangent space to the orbit at w_3^A.*

Group	Fiducial Vector	Stabilizers	Rescaling	Contrated Algera
$SO(4,1)$	$\begin{pmatrix} 0 \\ \mathbf{0} \\ \alpha \end{pmatrix}$	$J_i,$ K_i	$P_0 = B/\alpha,$ $P_i = M_i/\alpha$	$\mathfrak{iso}(3,1)$
$SO(4,1)$	$\begin{pmatrix} \beta \\ \mathbf{0} \\ \beta \end{pmatrix}$	$J_i,$ $K_i + M_i$	$Q_0 = B/\beta,$ $Q_i = K_i - M_i/\beta$	$\mathfrak{carr}(3,1)$
$SO(4,1)$	$\begin{pmatrix} \gamma \\ \mathbf{0} \\ 0 \end{pmatrix}$	$J_i,$ M_i	$T_0 = B/\gamma,$ $T_i = K_i/\gamma$	$\mathfrak{iso}(4)$
$SO(3,2)$	$\begin{pmatrix} \mathbf{0} \\ \alpha \\ 0 \end{pmatrix}$	$B,\ K_1,\ K_2,$ $J_3,\ M_1,\ M_2$	$U_\alpha = J_\alpha/\alpha,$ $U_3 = K_3/\alpha$ $U_4 = U_4/\alpha$	$\mathfrak{iso}(2,2)$
$SO(3,2)$	$\begin{pmatrix} \mathbf{0} \\ \beta \\ \beta \end{pmatrix}$	$K_3 + B$ $-J_2 - M_1$ $J_1 - M_2$	$V_0 = (K_3 - B)/\beta,$ $V_1 = (M_1 - J_2)/\alpha$ $V_2 = (J_1 + M_2)/\beta$ $U_3 = B/\beta$	$\mathfrak{carr}(2,2)$
$SO(3,2)$	$\begin{pmatrix} \gamma \\ \mathbf{0} \\ 0 \end{pmatrix}$	$J_i,$ M_i	$S_0 = B/\gamma,$ $S_i = K_i/\gamma$	$\mathfrak{iso}(3,1)$

3.3 Lie Bialgebra from $\mathfrak{iso}(p, 4-p)$ and $\kappa-$ Minkowski

In the preceding sections we obtained a 6 classes of inequivalent momentum spaces along with their symmetries. In this section we want to show that any of these symmetries can be made into a Lie bialgebra in such a way that the cocommutator of translation is dual to the the algebra of $\kappa-$Minkowski (see Sec. 1.4.3).

Consider the group of isometries $ISO(p, 4-p)$ of flat space with a metric tensor $g_{\mu\nu}$ of arbitrary signature

$$g_{\mu\nu} = \mathrm{diag}(s_0, s_1, s_2, s_3), \quad s_\mu \in \{1, 0, -1\}. \tag{3.64}$$

The Lie algebra $\mathfrak{iso}(p, 4-p)$ is given by

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\sigma} M_{\nu\rho} - g_{\mu\rho} M_{\nu\sigma} + g_{\nu\rho} M_{\mu\sigma} - g_{\nu\sigma} M_{\mu\rho} \,, \\ [P_\mu, P_\nu] = 0 \,, &\quad [M_{\mu\nu}, P_\rho] = g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu \,, \end{aligned} \tag{3.65}$$

note that we admit also degenerate metrics. We rename those generators as follows:

$$K_i = M_{0i}, \qquad M_{12} = R_3, \qquad M_{23} = R_1, \qquad M_{31} = R_2. \tag{3.66}$$

In [48, 104], it has been shown that any Lie bialgebras built obtained as a deformation of $\mathfrak{iso}(p, q)$ is coboundary. The most generic r-matrix is then of the form

$$r = a^{\mu\nu} P_\mu \wedge P_\nu + b^{\mu\nu\rho} M_{\mu\nu} \wedge P_\rho + c^{\mu\nu\rho\sigma} M_{\mu\nu} \wedge M_{\rho\sigma} \,, \tag{3.67}$$

which in general satisfy the mCYBE (1.133). Moreover, we want the translation sector to be compatible with the algebra of $\kappa-$Minkowski coordinates. Thus we impose the corresponding cocommutators (1.128)

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r] \tag{3.68}$$

to be of the form

$$\delta(P_0) = 0 \,, \quad \delta(P_i) \propto P_i \wedge P_0 \,, \quad \delta(M) = M \wedge P + M \wedge M \,, \tag{3.69}$$

where the last equation is a formal expression indicating that terms of the type $P_\mu \wedge P_\nu$ cannot appear in the cocommutator of $M_{\mu\nu}$. Then the r-matrix reduces to

$$r = s_1 K_1 \wedge P_1 + s_2 K_2 \wedge P_2 + s_3 K_2 \wedge P_3 + + c^{\mu\nu\rho\sigma} M_{\mu\nu} \wedge M_{\rho\sigma} \,. \tag{3.70}$$

. Imposing the co-Jacobi equations, we get:

$$r = s_1 K_1 \wedge P_1 + s_2 K_2 \wedge P_2 + s_3 K_2 \wedge P_3 + \alpha R_1 \wedge R_2 + \beta R_1 \wedge R_3 + \gamma R_2 \wedge R_3 \,, \tag{3.71}$$

where $s_1\alpha^2 + s_2\beta^2 + s_3\gamma^2 = 0$. The terms with α, β, γ are a generalization to arbitrary signature of the twist described in [48, 105], while the other terms generates the κ-Minkowski cocommutators (3.69) for any choice of signature. It follows that for any choice of quadruplet s_0, s_1, s_2, s_4, where $s_\mu \in (-1, 0, +1)$ and $g_{\mu\nu} = \mathrm{diag}(s_0, s_1, s_2, s_3)$, the algebra of isometries $\mathfrak{iso}(p, 4-p)$ of $g_{\mu\nu}$ admits a quantum deformation which is dual to the κ-Minkowski commutation relation; *i.e.* the cocommutator δ of the obtained Lie bialgebra $(\mathfrak{iso}(p, 4-p), [,], \delta)$ satisfy (3.69). We conclude that there are momentum spaces associated to the $\kappa-$Minkowski non commutative space-time with all possible (degenerate or not) signatures. This is compatible with the results of Sec. 3.2.2.

Our physical interpretation of these results is the following. Consider the Quantum Group(s) generated by the above described Lie bialgebras

$$
\begin{array}{ll}
\Delta_R[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, & [\Lambda^\mu{}_\nu, \Lambda^\alpha{}_\beta] = 0, \\
\Delta[a^\mu] = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes \mathbb{1}, & [\Lambda^\mu{}_\nu, a^\gamma] = \frac{\mathrm{i}}{\kappa}\left[\left(\Lambda^\mu{}_\alpha \delta^\alpha_0 - \delta^\mu_0\right)\lambda^\gamma{}_\nu + \left(\Lambda^\alpha{}_\nu \delta^0_\alpha - \delta^0_\nu\right) g^{\mu\gamma}\right], \\
\Delta[\Lambda] = \Lambda^{-1}, & [a^0, a^i] = \frac{\mathrm{i}}{\kappa} a^i \\
\varepsilon = \left\{ \begin{array}{c} \varepsilon[\Lambda^\mu{}_\nu) = \delta^\mu_\nu] \\ \varepsilon[a^\mu] = 0 \end{array} \right. , & [a^i, a^j] = 0
\end{array}
\tag{3.72}
$$

where $\Lambda^\mu{}_\nu$ satisfy the following algebraic rules

$$
\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}, \qquad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} = g_{\mu\nu}, \tag{3.73}
$$

for any choice of matrices $g^{\mu\nu}$ and $g_{\rho\sigma}$. Then the left co-action $\Phi_{\Lambda,a}$

$$
\Phi_{\Lambda,a}[x^\mu] = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes \mathbb{1}, \tag{3.74}
$$

is a homomorphism for (2.11) *i.e.* it leaves $\kappa-$Minkowski space unchanged. In this sense, equations (3.74) and (3.72) are in fact a generalization of equations (2.82) and (2.84) introduced in Sec. 2.2.6.

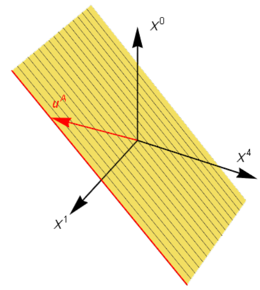
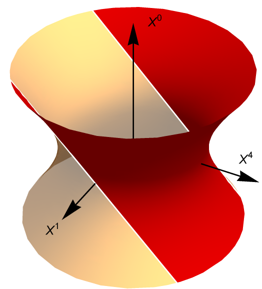
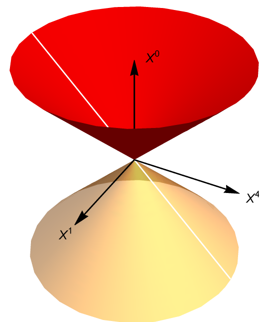
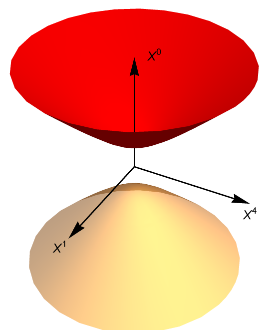
u^A	X^A	$X^A X^B \eta_{AB}$	$\mathrm{d}s^2$	plot
$\begin{pmatrix} u^0 \\ \mathbf{u} \\ -u^0 \end{pmatrix}$	$\begin{pmatrix} \frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} + u^0 e^{-\frac{p_0}{\kappa}}\kappa \\ \mathbf{u} \\ -\frac{\mathbf{p}\cdot\mathbf{u}}{\kappa} - u^0 e^{-\frac{p_0}{\kappa}} \end{pmatrix}$	$\lVert\mathbf{u}\rVert^2$	0	
$\begin{pmatrix} 0 \\ \mathbf{0} \\ 1 \end{pmatrix}$	$\begin{pmatrix} \sinh\frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}}\frac{\lVert\mathbf{p}\rVert^2}{2\kappa^2} \\ e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} \\ \cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{\lVert\mathbf{p}\rVert^2}{2\kappa^2} \end{pmatrix}$	1	$-\frac{1}{\kappa^2}\mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\mathrm{d}\mathbf{p}^2$	
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mathbf{0} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{e^{\frac{p_0}{\kappa}}}{\sqrt{2}}\left(1 + \frac{\lVert\mathbf{p}\rVert^2}{\kappa^2}\right) \\ \sqrt{2}e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} \\ \frac{e^{\frac{p_0}{\kappa}}}{\sqrt{2}}\left(1 - \frac{\lVert\mathbf{p}\rVert^2}{\kappa^2}\right) \end{pmatrix}$	0	$\frac{4}{\kappa^2}e^{\frac{2p_0}{\kappa}}\mathrm{d}\mathbf{p}^2$	
$\begin{pmatrix} 1 \\ \mathbf{0} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \cosh\frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa}}\frac{\lVert\mathbf{p}\rVert^2}{2\kappa^2} \\ e^{\frac{p_0}{\kappa}}\frac{\mathbf{p}}{\kappa} \\ \sinh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{\lVert\mathbf{p}\rVert^2}{2\kappa^2} \end{pmatrix}$	-1	$\frac{1}{\kappa^2}\mathrm{d}p_0^2 + \frac{1}{\kappa^2}e^{\frac{2p_0}{\kappa}}\mathrm{d}\mathbf{p}^2$	

Table 3.1: *First column: norm of the fiducial vector. Second column: components of fiducial vector of choice. Third column: embedding coordinates for the corresponding momentum space. Fourth column: induced metric on momentum space. Last column: plot of the momentum space manifold immersed in the ambient Minkowski space (with coordinates X^2 and X^3 suppressed, one should imagine that each point on the manifold really represents a sphere of radius $|X^1|$.*

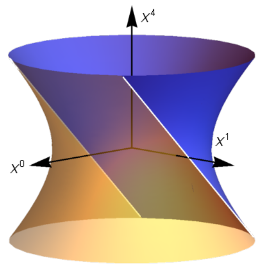
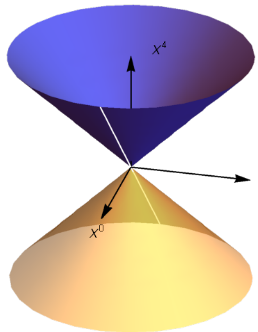
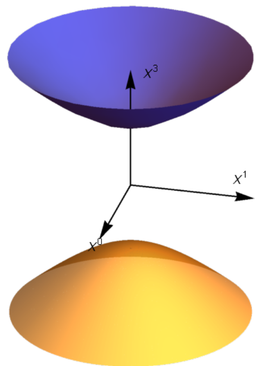
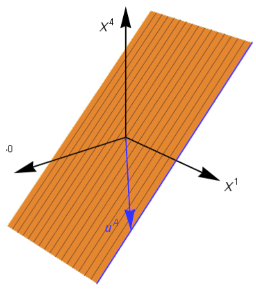
u^A	X'^A	$X'^A X'_A$	ds^2	plot
$\begin{pmatrix} 1 \\ \mathbf{0} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \cosh\left(\frac{p_0}{\kappa}\right) + \frac{(p\cdot p)e^{\frac{p_0}{\kappa}}}{2\kappa^2} \\ -\frac{p_1 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{p_2 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{p_3 e^{\frac{p_0}{\kappa}}}{\kappa} \\ \sinh\left(\frac{p_0}{\kappa}\right) - \frac{(p\cdot p)e^{\frac{p_0}{\kappa}}}{2\kappa^2} \end{pmatrix}$	-1	$\frac{1}{\kappa^2}\mathrm{d}p_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2}\mathrm{d}\mathbf{p}\cdot\mathrm{d}\mathbf{p}$	
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mathbf{0} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{\left(\kappa^2+p\cdot p\right)e^{\frac{p_0}{\kappa}}}{\sqrt{2}\kappa^2} \\ -\frac{\sqrt{2}p_1 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{\sqrt{2}p_2 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{\sqrt{2}p_3 e^{\frac{p_0}{\kappa}}}{\kappa} \\ \frac{\left(\kappa^2-p\cdot p\right)e^{\frac{p_0}{\kappa}}}{\sqrt{2}\kappa^2} \end{pmatrix}$	0	$\frac{2}{\kappa^2}e^{\frac{2p_0}{\kappa}}\mathrm{d}\mathbf{p}\cdot\mathrm{d}\mathbf{p}$	
$\begin{pmatrix} 0 \\ \mathbf{0} \\ 1 \end{pmatrix}$	$\begin{pmatrix} \sinh\left(\frac{p_0}{\kappa}\right) + \frac{(p\cdot p)e^{\frac{p_0}{\kappa}}}{2\kappa^2} \\ -\frac{p_1 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{p_2 e^{\frac{p_0}{\kappa}}}{\kappa} \\ -\frac{p_3 e^{\frac{p_0}{\kappa}}}{\kappa} \\ \cosh\left(\frac{p_0}{\kappa}\right) - \frac{(p\cdot p)e^{\frac{p_0}{\kappa}}}{2\kappa^2} \end{pmatrix}$	1	$\frac{1}{\kappa^2}e^{\frac{2p_0}{\kappa}}\mathrm{d}\mathbf{p}\cdot\mathrm{d}\mathbf{p} - \frac{1}{\kappa^2}\mathrm{d}p_0^2$	
$\begin{pmatrix} u^0 \\ \mathbf{u} \\ -u^0 \end{pmatrix}$	$\begin{pmatrix} u^0 e^{-\frac{p_0}{\kappa}} - \frac{p\cdot u}{\kappa} \\ u^1 \\ u^2 \\ u^3 \\ \frac{p\cdot u}{\kappa} - u^0 e^{-\frac{p_0}{\kappa}} \end{pmatrix}$	$u\cdot u$	0	

Table 3.2: *First column: norm of the fiducial vector. Second column: components of fiducial vector of choice, where* $p\cdot p = -p_1^2 + p_2^2 + p_3^2$*,* $p\cdot u = -p_1 u^1 + p_2 u^2 + p_3 u^3$ *and* $\mathrm{d}p\cdot\mathrm{d}p = -\mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2$*. Third column: embedding coordinates for the corresponding momentum space. Fourth column: induced metric on momentum space. Last column: plot of the momentum space manifold immersed in the ambient Minkowski space in the 1+1-dimensional case. The higher-dimensional cases are impossible to represent on paper, and are significantly more complicated, in that the regions of the submanifolds that are excluded from the coordinate patch, in the second and third lines, are not measure-zero and are rather complicated.*

Chapter 4

Conclusions

In this thesis we studied the non-commutative space-time known as $\kappa-$Minkowski, *i.e.* the homogeneous space associated to the $\kappa-$Poincaré quantum group. Our aim was to improve our understanding of the role of observers in a non-commutative space and their relation with the outcomes of measurements . Thus, inspired by how non-commutative phase-space has been implemented in Quantum Mechanincs, we developed our interpretations of the non-commutativity in coordinates and of the fact that transformation between observers are deformed [10,71]. We were also interested in the geometry dual space paired to $\kappa-$Minkowski. It is a well known result that the space of momenta conjugate to non-commutative coordinate is curve and we are well aware that this topic as already been discussed in a variety of works [11, 50, 51, 88]. Nevertheless, we obtained some new unexpected results from the analysis of momentum space in $\kappa-$Minkowski.

Our physical interpretation of non-commutativity is strictly related to the representation we gave of $\mathcal{M}_\kappa$ and $\mathcal{P}_\kappa$ as operators on the Hilbert space $L^2(\mathfrak{so}(3,1) \times \mathbb{R}^3 \times \mathbb{R}^3)$. In particular, states $|g,\psi\rangle = |g\rangle \otimes |\psi\rangle$ introduced in (2.111) both carry information about the fuzziness of an even and about the observer who his looking at it. The "state of the observer" carries information about the fuzziness of a transformation between different observers in a non-commutative space. In this context, while each observer is able to precisely localise himself, he is not able to localise the other ones. Thus, in order to precisely confront how two distinct observers describe a *fuzzy* event in the non-commutative space-time, one also has to specify how delocalised the transformation between the two observer is. This also give rise to unexpected combinations between the uncertainty due to non commutativity and the one due the deformed transformations. Sometimes the resulting situation just coincide with our intuition. As an examples, given an event localised in the origin, the uncertainties on coordinates increases as the observer is translated further and further away. Nevertheless, there exist tricky configurations where such a combination of uncertainties may produce unexpected results. Indeed, despite pure translations only increase uncertainty of coordinates, this last may decrease under a generic $\kappa-$Poincaré transformation. This shows how the usual probabilistic intuition may lead to misleading predictions.

A future perspective would be to further understand what happens to coordinates uncertainty under a generic transformation. Indeed, although we discussed this topic for a generic $|g\rangle \otimes |\psi\rangle$, we gave just a few out of many possible physical situation. This is far from being merely an exercise in style: only by training our intuition on this unusual framework we are expected to improve our physical interpretation of the model. Furthermore, it would be interesting to provide higher dimensional generalization for the examples given in $(1+1)$ dimensions.

The main result of our work is to propose a new way to think of $\kappa-$Minkowski non commutative geometry and its space of conjugate momenta. The next big challenge would then be to provide a dynamics for the model introduced in Chapter 2. For this purpose, the results obtained in Chapter 3 are crucial. In particular, the physical interpretation of the new momentum spaces we discovered may enlarge our compression of the nature of $\kappa-$Minkowski non-commutative geometry.

The fact that we have so many different possible momentum spaces all dual to $\kappa-$Minkowski introduce us to a world of possibility. In particular, the freedom we have in choosing both the shape and the metric over the momentum manifold suggest $\kappa-$Minkowski to be suitable for a much wider class of physical models then those usually considered in literature. For instance, the fact that there are even momentum spaces with Carollian symmetry group trills the authors ingenuity. Indeed, the duality between the Caroll group and the Galilei group has both physical and mathematical implication [101, 106].

At the very end of Chapter 3 we also showed that $\kappa-$Minkowski Lie algebra is invariant under the left co action of a pletora of fresh new Quantum Groups. It follows that $\mathcal{M}_\kappa$ can be regarded as the homogeneous space of a larger family of quantum symmetries than just the now familiar $\kappa-$Poincarè one. A great challenge will be to discuss the possible physical interpretation for all these configurations whose consequences we can only imagine.

In conclusion, we are left with lots of new (momentum) spaces to be explored, whose physical interpretation may give new and unexpected application of our now familiar $\kappa-$Minkowski space-time $\mathcal{M}_\kappa$.

Acknowledgements

First and foremost, let me thank my advisors Fedele Lizzi and Flavio Mercati, for having motivated and guided me along my Ph.D. studies, as well as for all the contribution to this thesis. I sincerely wish to thank also Patrizia Vitale, for all the advices she gave to me and for her unlimited patience. I must also thank Andrej Borowiec for having hosted my visit at Wroclaw university and for everything I learned there.

I thank the *Univeristy of Naples Federico II* and *INFN* section of Naples for giving me the opportunity to attend the various schools and conferences along the last three years. I also thanks the Ph.D. coordinator Salvatore Capozziello. A special thanks goes to Guido Celentano whose experience and efforts helped me to survive the jungle of bureaucracy.

A very special thank goes to Chiara who has always encouraged and sustained me. Last but not the least I thank my family for all the support they gave me during this unpredictable journey we call Ph.D.

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