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Fingerprints of the Very Early Universe

Tesi di Dottorato

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To my Father

and

Chiara

"... always better then a job !"

" È faticosa la vita dell'attore, sacrifici,
tutto il giorno in teatro, le prove,
ma sempre meglio che lavorare! "

Eduardo de Filippo – Theatre Actor,
1900-1984, Napoli
when requested about what to think
about his activity

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0 Preface

During the work developed in this PhD Thesis, it has been faced a number of problems concerning Theoretical Cosmology especially with respect to the dynamical evolution of the Universe near the cosmological singularity.
Wide space has been devoted to the study of the subtle question concerning the covariance chaoticity of the Bianchi type VIII and IX model, which has led to important issues favourable to the independence of the "chaos" with respect to the choice of the temporal gauge.
Such analysis found its basis either on the standard approach using the Jacobi metric (a scheme allowed by the existence of an energy-like constant of motion), either by a Statistical Mechanics approach in which the Mixmaster evolution is represented as a billiard on a Lobatchevski plane and therefore admitting a Microcanonical ensemble associated to such an energy-like constant.

Furthermore, an important step consisted in searching a physical link between the chaoticity characterizing the system at a classical level and the quantum indeterminism appearing in the Planckian era for such a model.
More precisely, it was constructed the canonical quantization of the model via a Schrödinger approach (equivalent to the Wheeler-DeWitt scheme) and then developed the WKB semiclassical limit to be compared with the classical dynamics.
As an issue, it resulted a correspondence between the continuity equation of the microcanonical distribution function and that one describing the dynamics of the first-order corrections in the wave function for $\hbar \to 0$.

A detailed discussion was pursued in view of clarifying the peculiarity existing to characterize chaos in General Relativity; in particular, it has been provided a critical discussion on the predictability allowed by the fractal basin boundary approach in qualifying the nature of the Mixmaster dynamics; the main issue on this direction relies on the numerical approximations limits when treating iterations of irrational numbers and overall on the potential methods commonly adopted in the dynamical systems approach.

With respect to this, it is emphasized the ambiguity of describing chaos in terms of geodesic deviation when the background metric is a pseudo-Riemannian one; a correct characterization of the Lyapunov exponents required a projection of the connecting vector over a Fermi basis.

The investigation performed about a quasi-isotropic inflationary solution has allowed to confirm how there is no chance for classical inhomogeneous perturbations to survive after the de Sitter phase; such an analysis supports strongly the idea that only quantum fluctuations of the scalar field can provide a satisfactory explanation for the observed spectrum of inhomogeneous perturbations, when requiring the matter to dominate the first order of the solution.

Finally, it has been provided a generic inhomogeneous solution concerning the dynamics of a real self interacting scalar field minimally coupled to gravity in a region of the configuration space where it performs a slow rolling on a plateau of its potential. During the generic inhomogeneous de Sitter phase the scalar field which dominates zero- and first-order of approximation is a function of the spatial coordinates only. This solution specialized nearby the Friedmann-Lemaitre-Robertson-Walker (FLRW) model allows a classical origin for the inhomogeneous perturbations spectrum.

The opposite nature of the results obtained in such a general case with respect to the one proper of the quasi-isotropic solution relies on the negligibility of the matter contribution up to first-two orders of approximation considered in the former case.

The motivations of this work rely on the properties of the Einstein equations general solution near a cosmological singularity which exhibit an oscillatory stochastic behaviour – of which the Bianchi types VIII and IX provide a valuable prototype. This feature of the Very Early Universe is in striking contrast with the Universe as described by the well-tested theory of the Standard Cosmological Model, which is based on the highly symmetric FLRW geometry. The experimental evidence for the homogeneity and isotropy of the present state Universe concerns relatively late stages of evolution. Indeed the good agreement of the light element nucleosynthesis prediction with the observed abundances implies that the Standard Cosmological Model is surely valid since 10^{-3}-10^{-2} seconds after the Big Bang, but yet says nothing about the very early dynamics before this time – indeed recent observations of the cosmic microwave background radiation seem to support the existence of an inflationary scenario.

The FLRW metric solution is unstable when regarded as running backwards in time: from the existence of structures in the Universe, like galaxies and clusters of galaxies, we infer that such a symmetric geometry cannot continue all along up to the initial singularity, even in the presence of an inflationary scenario. The clumpiness of the Universe implies very early perturbations of homogeneity and isotropy, which unavoidably "explode" when approaching the Big Bang. Such an instability, when regarded backwards in time, states the existence of an instant of time t_* before which the evolution of the Universe was described by a "generic" inhomogeneous model – the Belinski, Khalatnikov and Lifshitz (BKL) picture.

This peculiar moment represents a free parameter depending on initial conditions and on specific properties of matter.
In vacuum case such a picture is described by the inequality $L_h \ll L_{in}$, $L_h \sim t$ being the horizon size and L_{in} the characteristic scale of inhomogeneity, respectively.

Thus, in the vacuum case, the moment $t_* = t_{in}$ when the Mixmaster phase (i.e. the oscillatory regime) ends corresponds to $L_h \sim L_{in}$, roughly considerable as a boundary over the BKL approximation.
The reversibility of the Einstein equations implies the BKL picture validity in both directions of time.

The chaotic nature of the evolution – temporally as well as spatially – implies a stationary statistical distribution for the geometry of the Very Early Universe, even if "geometry" has to be intended only in an average sense; the mean values of all the geometrical quantities (lengths, scalar products, etc.) during the oscillatory regime are unstable in the sense that higher moments have the same order of magnitude as the average values and therefore the cosmological background near the singularity is unstable itself.
Exactly the same situation holds for the chaoticity of the Bianchi types VIII and IX models that we will discuss in detail in this work, especially with respect its covariant nature; therefore, many of the results obtained by our analysis have to be yet valid even when they are referred point by point in space for a *generic* inhomogeneous cosmological model.

The same situation holds in the quantum evolution of the inhomogeneous Mixmaster Universe, although in the quantum case the statistical distribution has a different – but somehow related – nature.

The problem of the origin of a stable background and how it could arise out of the chaos and be compatible with the notion of isotropy is solved considering a *bridge solution*, so that the strong correlation between the appearance of a stable background and its isotropic character is displayed as a key feature of the Very Early Cosmology.

The isotropic component of the metric tensor – measuring the volume of the Universe –, is a monotonic function of the time variable and can be chosen as the temporal coordinate while the physical degrees of freedom are entirely contained in the anisotropic components, either on the quantum or on the classical level. A stable background metric can appear only when the anisotropy of the universe is damped enough.

In vacuum inhomogeneous models a classical background can arise from the Planckian epoch of the Universe when the oscillatory regime is over, i.e. in correspondence of the matching between the characteristic scale of inhomogeneity L_{in} with the horizon size L_h.

A classical quasi-isotropic Universe may emerge, up to suitable initial conditions, from general cosmological dynamics, essentially by virtue of an inflationary expansion due to the real scalar field potential term.

It is worth noting how the inflationary scenario provides a mechanism to simultaneously dump out anisotropic (see the bridge solution) and inhomogeneous (see quasi-isotropic model) contributions.

Thesis Outline

In ***Chapter 1*** we discuss the physical and mathematical reasons to consider the framework of the homogeneous cosmological models, with respect to the original work by Luigi Bianchi and the application to Cosmology by Belinski, Khalatnikov and Lifshitz, discussing in the details the dynamical properties of the solutions for the Bianchi types I and IX (which is equivalent to the type VIII) in the natural choice of a synchronous reference

system.
We show how chaotic features emerge even from the original formalism, while appearing an indefinite sequence of Kasner solutions, bringing to light the fundamental problems at the basis of a proper chaoticity characterization.

In ***Chapter 2*** we develop the Hamiltonian formulation of the cosmological problem showing how it can be reduced to the dynamics of a billiard-ball.
In particular, in Section 2.3 is presented an original reformulation of the Bianchi type IX dynamics by using a set of Misner–Chitre-like variables which a generic function of one coordinate.
This permits to overcome the ambiguities of many assessments found in the literature, due to the dependence of the choice of the time parameter and developments are shown in details in the following of the Chapter.
Our reformulation is not affected by such a possibility and permits to discuss the dynamics via a standard Arnowitt-Deser-Misner (ADM) approach in the reduced phase space.
The Jacobi metric obtained induces the derivation of an invariant formulation of the Liouville measure within the *microcanonical ensemble* framework.
This new approach permits to derive, within the potential approximation, an analytic expression for the Lyapunov exponents, independently of the choice of the temporal gauge and a discussion about a correct formulation of the same problem in General Relativity.
We conclude then with some deep criticism about the methods to be used in order to characterize a chaotic system.

In ***Chapter 3***, considering the Universe evolution towards the initial singularity, we deal with the canonical quantization of gravity and apply such a theoretical formalism to the Mixmaster, in order to obtain the corresponding Schrödinger-like equation, equivalent to the Wheeler-DeWitt one.
We then find the wave-function solution, considered in the semiclassical limit, and show how it induces a continuity equation of the same form as

the one discussed within the microcanonical ensemble description of the deterministic approach.

In ***Chapter 4*** we consider the inflationary scenario as the possible way to interpolate the rich and variegate Kasner dynamics of the Very Early Universe discussed so far with an inflationary scenario, in order to reach the present state observable FLRW Universe, via a *bridge solution*. The Einstein-Hamilton-Jacobi equation is solved in presence of a real self-interacting scalar field.

Hence we show how it is possible to have a quasi-isotropic solution of the Einstein equations in presence of the ultrarelativistic matter and a real self-interacting scalar field. In this case, the spatial distributions of both admit an arbitrary form but such a small inhomogeneity is incompatible with structures formation of classical origin.

Finally, we discuss a generic inhomogeneous solution of the Einstein equations in presence of a scalar field minimally coupled to gravity. The maximal degree of generality allowed in this case in terms of free functions allows the leading order of the scalar field to be a spatial function, leaving open the possibility for it to be seed of structure formation.

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1 Anisotropic Homogeneous Cosmological Dynamics

1.1 Theoretical Approach

The general approach by which it is customary to open a description of cosmology is strictly referred to what is the *actually observed* cosmological image of the global universe structure. The observational data of the Cosmic Microwave Background Radiation (CMBR) show how the actual universe agrees in what the Standard Cosmological Model of homogeneity in space and isotropy in evolution provides, as well as also on a large scale of the matter distribution.

The essential feature which we are going to consider is *how long* this can happen and how it can be related with some deeper details of the observations.

First of all, structures exist.

The Standard Cosmological Model (SCM) gives, nevertheless, an adequate description of the present day state of the universe considered on a large scale. Moreover it has to be considered an essential result (Lifshitz and Khalatnikov, 1963) that perturbations which do not affect the uniformity of the distribution of matter are either damped with time or remain constant. The universe evolution is naturally led to increase the degree of homogeneity and isotropy.
In the same way, for the reversibility of the Einstein field equations, the universe evolution *backward* in time reveals the same attitude to increase perturbations. A detailed picture is then needed of the time scales at which various approaches remain valid: as a first step the CMBR observational data offer a picture of how the universe looked like at an age of approximately $3 \cdot 10^5$ years, homogeneous and isotropic, up to a degree of accuracy of the order of one part in 10^4; nucleosynthesis up to 10^{-3} s then the seeds of density perturbations date back to an earlier epoch.

The study of the large scale structure of the universe is most often accomplished by using some cosmological model, i.e. a mathematical idealization with some precise assumptions to tackle the details of problems relying on the description of kinematics and dynamics. For the kinematical approach the focus is on the space-time metric on which some symmetry condition is imposed and on the adopted coordinates; as far as dynamics is concerned one needs to obtain field equations to describe geometry as well as the matter involved, like Einstein's and matter's equations, of the kind

$$\underbrace{G_{\alpha\beta} + \Lambda g_{\alpha\beta}}_{\text{geometry}} = k \underbrace{T_{\alpha\beta}}_{\text{matter}} \tag{1.1.1}$$

where the terms on the left/right-hand side refer to Einstein tensor together with a "cosmological constant" term and to energy-momentum tensor for matter, respectively, to which is coupled a set of constraint and state equations as

$$T^{\alpha\beta}_{\ \ ;\beta} = 0\,, \qquad p = (\gamma - 1)\rho\,, \qquad \text{etc.} \tag{1.1.2}$$

whose details will be described in the following.

1.1.1 Model for Matter

The next clue to the field mathematical structure, lying in the Einstein's equations, is the connection with a model for matter, whose details depend deeply on the astronomical scale distances to which one refers and to the time scale of the universe evolution: if the description of the actual universe is satisfactory as a dust fluid model, with zero pressure at very low temperature, going backward to the Big Bang is necessary to consider a plasma gas before the creation of nuclei, stars, galaxies an other structures; "very" near the Big Bang then quantum vacuum energy dominates and particle physics is the key representation to describe the universe, meanwhile "very very" near the Bang quantum gravitational effects become important requiring quantum cosmology for the geometry (Arnowitt et al., 1959*a*,*b*,*c*), (DeWitt, 1967*a*,*b*,*c*) (Rovelli, 1991*a*,*b*).

1.1.2 Model for Geometry

When equations are too difficult to solve, as it happens in the general form of the Einstein equations, it is customary to impose symmetry to simplify as a *practical approach* in view of the fact that to the highest symmetry there

corresponds a set of equations easier to solve. This way of approach is clearly equivalent to a *philosophical approach* in terms of a "cosmological principle": why should our position and time in the universe be special?
With this in mind, in a symmetry breaking models order, the

(i) *first try* lies in the consideration of a universe which is the same at all places, all times, all directions: homogeneous and isotropic space-time, like Einstein static universe allowed by $\Lambda \neq 0$ with symmetry as $[R \times S^3]$, but not the same at all times, requiring expansion.

(ii) The *second try* takes under analysis a model for the universe which is the same at all places and in all directions at a given time, but evolves with time, breaking homogeneity and isotropy in time direction: spatially homogeneous and isotropic space-time, like Friedmann-Lemaitre-Robertson-Walker (FLRW) models.

(iii) The *curiosity* leads to consider a universe not the same in all spatial directions: spatially homogeneous anisotropic universe in which isotropy is broken in space, like Bianchi models (see below and Bianchi (1897); Belinski et al. (1970)).

(iv) Finally *more curiosity* considers a universe not the same at every place at cosmological distance scales: inhomogeneous universe, breaking homogeneity in space.

What we have here considered as *curiosity* clearly is based on the physical property of the quasi-isotropic solution to Einstein equations (Lifshitz and Khalatnikov, 1963) which describes trajectories close to the isotropic and homogeneous FRLW space-time: perturbations to energy density are damped when considering evolution forward in time but, reversely, this corresponds to a divergence in the evolution towards the singularity, hence supporting the necessity of considering a more general solution then the symmetric (homogeneous and isotropic) one able to describe well only the recent stages of universe evolution.
The mathematical degree of difficulty is the counterpart of the symmetries that can be imposed on the space-time properties (see for example Capozziello et al. (1996)).

As a possible perspective one has to suppose that, perhaps, observable universe is only a small part of a inhomogeneous anisotropic space-time which seems very symmetric at large scales, as indicated by the Cosmic Microwave Background Radiation (CMBR).

The general way to approach a universe model is then based on assuming a highly symmetric universe model for the “largest scale” structure of the universe, whose dynamics can be handled exactly, qualitatively or numerically, by means of ordinary differential equations.

The structure at smaller distance scales which breaks this symmetry evolves by a much more complicated dynamics (partial differential equations). Only by perturbation techniques (coupled with harmonic analysis) can they be handled, but this is good enough to study certain questions like *galaxy formation* and *clustering*, since these are relatively small deviations from the large scale behaviour of the universe.

Classical relativity together with the observed expansion tells us there should be an initial cosmological singularity of some kind (Big Bang?) but the conditions of high energy and density near such an event invalidate the classical theory there. One needs quantum gravity, which still is not definitely formulated, so semiclassical calculations or possibly unrealistic model quantum calculations are performed.

At very early times the theory of matter becomes extremely important, so GUTS (Grand Unified TheorieS) and their consequences, string theories, higher dimensional space-times, etc.... all complicate the picture, offering the possibility to understand why is the universe the way it is.
The main feature that anyway can be accomplished by Friedmann universe solution is essentially non stationarity coupled to the existence of a singular point.

1.2 Lie Algebra

We will not discuss the details of the FLRW models to whom a wide literature is devoted, but will simply recall their main feature as a starting point for our forward analysis: they are Riemannian spaces of constant curvature and maximally symmetric as can be classified considering the isometries of the space:

i) *homogeneous*, say there exists a translation which can move a point to any other point of the space, without changing any relative distances or angles, i.e. so that the metric (and therefore the geometry) of the space is invariant;

ii) they are *isotropic*, say at any given point there exists a rotation about the point which maps any given direction to any other direction; when expressed as a linear transformation (of the directions) in an orthonormal basis, the matrix of such a rotation lies in the special orthogonal group of the same dimension of the space.

The line element can be summarized

$$ds^2 = -dt^2 + R^2(t) \begin{pmatrix} d\Sigma^2 \\ \delta_{\alpha\beta} dx^\alpha dx^\beta \\ d\Gamma_-^2 \end{pmatrix} \qquad \begin{matrix} k = 1 \\ k = 0 \\ k = -1 \end{matrix} \tag{1.2.1}$$

where $R(t)$ is the scale factor and terms in brackets represent the line elements of spherical, flat and hyperbolic spaces corresponding to the curvature sign in the last column.

Homogeneity and isotropy determine completely the space metric, apart from the curvature sign which is the only arbitrary parameter.

1.2.1 Killing Vector Fields

More useful than the group of motions of various spaces are the Lie algebras of Killing vector fields which generate the groups of motions via infinitesimal motions. They yield conserved quantities and make it possible to classify homogeneous spaces.

Suppose a group of transformations

$$x^\mu \to \bar{x}^\mu = f^\mu(x, a) \tag{1.2.2}$$

on a space M (eventually a manifold), where $\{a^a\}_{a=1,..,r}$ are r independent variables which parametrize the group and let a_0 correspond to the identity transformation

$$f^\mu(x, a_0) = x^\mu \,. \tag{1.2.3}$$

Consider an infinitesimal transformation corresponding to $a_0 + \delta a$, i.e. one which is very close to the identity transformation

$$x^\mu \to \bar{x}^\mu = f^\mu(x, a_0 + \delta a) \approx \underbrace{f^\mu(x, a_0)}_{=x^\mu} + \underbrace{\left(\frac{\partial f^\mu}{\partial a^a}\right)(x, a_0)\,\delta a^a}_{\equiv \xi_a^\mu(x)} \tag{1.2.4}$$

i.e.

$$x^\mu \to \bar{x}^\mu \approx x^\mu + \xi_a^\mu(x)\,\delta a^a = (1 + \delta a^a \xi_a)\, x^\mu\,, \tag{1.2.5}$$

where r first-order differential operators $\{\xi_a\}$ are defined by $\xi_a = \xi_a^\mu \frac{\partial}{\partial x^\mu}$ corresponding to the r vector fields with components $\{\xi_a^\mu\}$. These are the "generating vector fields" and when the group is a group of motions, they are called *Killing vector fields*, satisfying also $L_\xi g = 0$ in terms of the Lie derivative.

The generating vector fields have the interpretation that, under the infinitesimal transformation corresponding to $a_0 + \delta a$, all points of the space M are translated by a distance $\delta x^\mu = \delta a^a \xi_a^\mu$ in the coordinates $\{x^\mu\}$ and moreover

$$\bar{x}^\mu \approx (1 + \delta a^a \xi_a)\, x^\mu \approx e^{\delta a^a \xi_a} x^\mu\,. \tag{1.2.6}$$

In fact, the finite transformations of the group may be represented as

$$\bar{x}^\mu \to \bar{x}^\mu = e^{\theta^a \xi_a} x^\mu \tag{1.2.7}$$

where $\{\theta^a\}$ are r new parameters on the group.
Generating vector fields form a Lie algebra, i.e. a real r-dimensional vector space with basis $\{\xi_a\}$, which is closed under commutation, i.e. the commutators of the basis elements can be expressed as constant linear combinations of themselves

$$[\xi_a, \xi_b] \equiv \xi_a \xi_b - \xi_b \xi_a = \pm C^c{}_{ab} \xi_c \tag{1.2.8}$$

where $C^c{}_{ab}$ are the structure constants of the Lie algebra ((+) refers to left-invariant groups, while (−) to right-invariant ones).
Suppose $\{e_a\}$ is a basis of the Lie algebra g of a group G

$$[e_a, e_b] = C^c{}_{ab} e_c \tag{1.2.9}$$

and define

$$\gamma_{ab} = C^c{}_{ad} C^d{}_{bc} = \gamma_{ba} \tag{1.2.10}$$

which is symmetric by definition. This provides a natural inner product on the Lie algebra

$$\gamma_{ab} \equiv e_a \cdot e_b = \gamma\,(e_a, e_b)\;; \tag{1.2.11}$$

When $\det(\gamma_{ab}) \neq 0$ this inner product is non-degenerate and groups for which this is true are called semi-simple.

The r vector fields $\{e_a\}$ may be used instead of the coordinate basis $\left\{\frac{\partial}{\partial a^a}\right\}$ as a basis in which to express an arbitrary vector field on G. This basis is a frame.

Let us consider briefly the action of an abstract group G with a given multiplication law

$$a_1 a_2 = \phi(a_1, a_2) \ , \tag{1.2.12}$$

next suppose this group acts on a manifold M as a group of transformations

$$x^\mu \to f^\mu(x, a) \equiv f^\mu_a(x) \ . \tag{1.2.13}$$

In order for the transformations $\{f_a \mid \ a \in \ G\}$ to form a group, the product of two transformations must correspond to the product of group elements

$$x \to f_{a_1}(f_{a_2}(x)) = (f_{a_1} \circ f_{a_2})(x) = f_{a_1 a_2}(x) \tag{1.2.14}$$

i.e. the group property

$$f_{a_1} \circ f_{a_2} = f_{a_1 a_2} \ , \tag{1.2.15}$$

together with the identity element a_0 and identity transformation f_{a_0}

$$a_0 a = a a_0 = a \ , \tag{1.2.16}$$

$$f_{a_0} \circ f_a = f_{a_0 a} = f_a \ . \tag{1.2.17}$$

1.2.2 Homogeneous Spaces

Let us define the orbit of x

$$f_G(x) = \{f_a(x) \mid a \in G\} \tag{1.2.18}$$

as the set of all points that can be reached from x under the group of transformations.
The isotropy group at x is

$$G_x = \{a \in G \mid f_a(x) = x\} \tag{1.2.19}$$

as the subgroup of G which leaves x fixed.

Suppose $G_x = \{a_0\}$ and $f_G(x) = M$, i.e. every transformation of G moves the point x and every point in M can be reached from x by a unique transformation. Since $G|G_x = \{a a_0 \,| a \in G\} = G$, G is diffeomorphic to M and one may identify the two spaces.
If g is a metric on M invariant under G, it corresponds to a left-invariant one on G, specified entirely by the inner products of the basis left-invariant vectors fields e_a. For three dimensions one obtains the family of spatially homogeneous

space sections of the spatially homogeneous space-times.
Given a basis $\{e_a\}$ of the Lie algebra of a three dimensional Lie group G, with structure constants ${C^c}_{ab}$, the spatial metric at each moment of time is specified by the spatially constant inner products

$$e_a \cdot e_b = g_{ab}(t) \ , \tag{1.2.20}$$

which are six functions of time. The Einstein equations, as we will apply, become ordinary differential equations for these six functions, plus whatever functions of time are necessary to describe the matter of the universe.

In four dimensions one obtains the homogeneous space-times. Einstein's equations then become a set of algebraic equations for $g_{\alpha\beta}$ and ${C^\alpha}_{\beta\gamma}$ which may not have solutions for every group (in general, for non vacuum space-times, other constants describing the matter occur in the equations).

For both homogeneous and spatially homogeneous space-times, one needs only consider a representative group from each equivalence class of isomorphic Lie groups of dimension four and three respectively. In three dimensions the classification of inequivalent three dimensional Lie groups is called the Bianchi classification (Bianchi, 1897) and determines the various symmetry types possible for homogeneous three spaces just as $(k = +1, 0, -1)$ classifies the symmetry types possible for homogeneous and isotropic three spaces (FRW).

1.2.3 Bianchi Classification

Scheme from Geometry

In the work done by Bianchi (1897), in 1897 following the works by Riemann (1868) and Lie and Engel (1888), a Lie group is a continuous and finite group G_r, generated by r infinitesimal transformations. The problem to determine which spaces possess a continuous group of movements is reduced to the classification of all possible forms for ds^2 corresponding to which, under G_r, ds^2 is self-transformed.

The relation between the number of transformations r and the dimension of the space n, with the maximum degree of freedom is given by

$$r = \frac{n(n+1)}{2} \tag{1.2.21}$$

giving constant curvature spaces. For ordinary manifolds $n = 2$ was already a solved problem, giving only the two possibilities

$$r = 1 \, , \qquad r = 3 \tag{1.2.22}$$

i.e. self-similar manifolds and constant curvature ones, respectively ($r = 3 \rightarrow k = \pm 1, 0$).
The work by Bianchi spans the case $n = 3$ for different degrees of freedom.
The main difference between the $n = 2$ and $n = 3$ cases lies essentially in the fact that a manifold admitting a transitive group of movements is necessarily a constant curvature one, that is to say that if a point can be moved to any other, then the manifold can also rotate. In dimension three this situation does not require the constancy of the curvature, admitting 3 or 4 parameters.

For $n > 3$ the classification becomes rapidly more involved.
Let us introduce the Killing (KILLING, 1892) notation for the differential form

$$ds^2 = \sum_{i,k=1}^{n} a_{ik} dx_i dx_k \tag{1.2.23}$$

and

$$Xf = \sum_{r=1}^{n} \xi_r \frac{\partial f}{\partial x_r} \tag{1.2.24}$$

for the infinitesimal transformation over the function f of the n variables x_r, ξ_r being n unknown functions of the coordinates to be determined in terms of

$$\begin{aligned} X(ds^2) = & \sum_{i,k} X(a_{ik}) dx_i dx_k + \\ & + \sum_{r,k} a_{rk} dX(dx_r) dx_k + \sum_{i,r} a_{ir} dX(dx_r) dx_i \end{aligned} \tag{1.2.25}$$

The symmetry conditions to be imposed will distinguish the number of the r independent infinitesimal transformations Xf admitted by the differential form (1.2.23) in a finite number

$$r \leq \frac{n(n+1)}{2} \tag{1.2.26}$$

and these r transformations

$$X_1 f, X_2 f, \ldots, X_r f, \tag{1.2.27}$$

will generate the continuous group G_r of movements over the chosen n-dimensional space S_n. In general, if two infinitesimal movements over the space S_n have trajectories in common, then they must coincide. The different behaviour of the movements under different composite transformations is what will characterize different spaces.

If we set in general

$$X_\alpha f = \sum_{i=1}^{n} \xi_i^{(\alpha)} \frac{\partial f}{\partial x_i}\,, \qquad (\alpha = 1, 2, \ldots, n)\,, \tag{1.2.28}$$

we will have the composition formula

$$(X_\alpha\, X_\beta) = \sum_\gamma c_{\alpha\beta\gamma} X_\gamma f\,, \tag{1.2.29}$$

$c_{\alpha\beta\gamma}$ being the composition constants. The explicit form of a_{ik} is given by a system of total differential, linear and homogeneous; in terms of $\xi_r^{(\alpha)}(x)$ it is determined by the symmetry imposed on the transformation group.
To any group of transformations G_3, transitive over three variables, there always correspond three-dimensional spaces admitting such a G_3 as a group of movements. With this in mind, for *real* groups it is straightforward to consider the classification first given by Lie and Scheffers (1893) for the integrable groups divided in *types* as

$$\text{I} \qquad (X_1\, X_3) = (X_2\, X_3) = 0\,, \tag{1.2.30a}$$

$$\text{II} \qquad (X_1\, X_3) = 0\,,\ (X_2\, X_3) = X_1\, f\,, \tag{1.2.30b}$$

$$\text{III} \qquad (X_1\, X_3) = X_1\, f\,,\ (X_2\, X_3) = 0\,, \tag{1.2.30c}$$

$$\text{IV} \qquad (X_1\, X_3) = X_1\, f\,,\ (X_2\, X_3) = X_1 f + X_2 f\,, \tag{1.2.30d}$$

$$\text{V} \qquad (X_1\, X_3) = X_1\, f\,,\ (X_2\, X_3) = X_2 f\,, \tag{1.2.30e}$$

$$\text{VI} \qquad (X_1\, X_3) = X_1\, f\,,\ (X_2\, X_3) = h\, X_2 f\,, \tag{1.2.30f}$$

with $(h \neq 0, 1)$, and

$$\text{VII} \qquad (X_1\, X_3) = X_2\, f\,,\ (X_2\, X_3) = -X_1\, f + h\, X_2 f\,, \tag{1.2.30g}$$

where the constant h satisfies $0 \leq h < 2$ while for all seven types

$$(X_1\, X_2) = 0\,. \tag{1.2.31}$$

Considering the case in which the group G_3 is non-integrable, Lie adds also the following:

$$\text{Type VIII}\ \ (X_1\, X_2) = X_1 f\,,\ (X_1\, X_3) = 2X_2 f\,,\ (X_2\, X_3) = X_3 f\,, \tag{1.2.32}$$

to which Bianchi added the ninth one

$$\text{Type IX}\ \ (X_1\, X_2) = X_3 f\,,\ (X_2\, X_3) = X_1\, f\,,\ (X_3\, X_1) = X_2 f\,, \tag{1.2.33}$$

which differs from the type VIII for the fact that the last one has no two-parameter real subgroup. In particular, integration of type VIII requires the elliptic Weierstrass function $\mathfrak{p}(x; g_2, g_3)$, with elliptic invariants g_2, g_3 specified for the model (Weierstrass, 1886), while type IX admits also a G_4 group of movements, containing a three-parameter transitive subgroup G_3.

Scheme from Cosmology

In this Section we will specify the calculation outlined in the previous one in a way well suited for cosmology.
The homogeneity means that the metric properties are identical at all points of the space.

Consider the group of transformations of coordinates which transform the space into itself, i.e. leave the metric unchanged: if the line element before the transformation has the form

$$dl^2 = \gamma_{\alpha\beta}\left(x^1, x^2, x^3\right) dx^\alpha dx^\beta \tag{1.2.34}$$

then it is transformed into

$$dl^2 = \gamma_{\alpha\beta}\left(x'^1, x'^2, x'^3\right) dx'^\alpha dx'^\beta \tag{1.2.35}$$

where $\gamma_{\alpha\beta}$ has the same form in the new coordinates.

In the general case of a non Euclidean homogeneous three-dimensional space, there are three independent differential forms which are invariant under the transformations of the group of motions. However they do not represent the total differential of any function of the coordinates. We shall write them as

$$e^a_\alpha dx^\alpha \tag{1.2.36}$$

where the Latin index here enumerates three independent vectors, functions of the coordinates, to be used as basis vectors. Hence the metric (1.2.35) is re-expressed as

$$dl^2 = \eta_{ab}\left(e^a_\alpha dx^\alpha\right)\left(e^b_\beta dx^\beta\right) \tag{1.2.37}$$

so that the metric tensor reads as

$$\gamma_{\alpha\beta} = \eta_{ab} e^a_\alpha e^b_\beta, \tag{1.2.38}$$

where η_{ab} is function of time only, symmetric in ab and in contravariant components we have

$$\gamma^{\alpha\beta} = \eta^{ab} e^{\alpha}_{a} e^{\beta}_{b}, \tag{1.2.39}$$

where η^{ab} should be viewed as the components of the inverse matrix so that $\eta^{ac}\eta_{cb} = \delta^{a}_{b}$ and

$$e^{a}_{\alpha} e^{\alpha}_{b} = \delta^{a}_{b}\,, \qquad e^{a}_{\alpha} e^{\beta}_{a} = \delta^{\alpha}_{\beta}\,. \tag{1.2.40}$$

The tensor η_{ab} will be used to operate over indices in the new basis according to

$$\eta_{ab} e^{a}_{i} = e_{bi}\,, \quad \eta^{ab} e_{ai} = e^{b}_{i} \tag{1.2.41}$$

and for any generic co-vector A_j or tensorial T_{ij} expression when projected on the tetradic basis e^{α}_{a} obtaining the tetradic components

$$A_a = e_{aj} A^j = e^{j}_{a} A_j\,, \quad A^i = e^{i}_{a} A^a = e^{ai} A_a \tag{1.2.42}$$

$$A^a = \eta^{ab} A_b = e^{a}_{j} A^j = e^{aj} A_j \tag{1.2.43}$$

and in general

$$T_{ab} = e^{i}_{a} e^{j}_{b} T_{ij} = e^{i}_{a} T_{ib}\,, \tag{1.2.44}$$

$$T_{ij} = e^{a}_{i} e^{b}_{j} T_{ab} = e^{a}_{i} T_{aj}\,. \tag{1.2.45}$$

Clearly, in these formulae the indices i, j are used to show the tetradic algebra and enumerate the Lorentz ones, labelled otherwise with greek letters.

The relationship between the covariant and contravariant expression for the three basis vectors is

$$\begin{aligned} \mathbf{e}_1 &= \frac{1}{v}\left[\mathbf{e}^2 \wedge \mathbf{e}^3\right] \\ \mathbf{e}_2 &= \frac{1}{v}\left[\mathbf{e}^3 \wedge \mathbf{e}^1\right] \\ \mathbf{e}_3 &= \frac{1}{v}\left[\mathbf{e}^1 \wedge \mathbf{e}^2\right] \end{aligned} \tag{1.2.46}$$

where $\mathbf{e}^a$ and $\mathbf{e}_a$ are to be understood formally as Cartesian vectors with components e^{a}_{α} and e^{α}_{a} while v represents

$$v = \mid e^{a}_{\alpha} \mid = e^1 \cdot \left[e^2 \wedge e^3\right]\,. \tag{1.2.47}$$

The determinant of the metric tensor (1.2.38) is given by

$$\gamma = \eta v^2 \tag{1.2.48}$$

with η the matrix η_{ab} determinant.
The invariance of the differential form (1.2.35) and the expression (1.2.36) means that

$$e^{(a)}_{\alpha}(x)\,dx^{\alpha} = e^{(a)}_{\alpha}(x')\,dx'^{\alpha} \tag{1.2.49}$$

where now we put tetradic indices in brackets to avoid misunderstandings and $e^{(a)}_{\alpha}$ on both sides of (1.2.49) are the same functions expressed in terms of the old and the new coordinates.
The algebra for the differential forms permits to rewrite (1.2.49) as

$$\frac{\partial x'^{\beta}}{\partial x^{\alpha}} = e^{\beta}_{(a)}(x')\,e^{(a)}_{\alpha}(x)\ . \tag{1.2.50}$$

This is a system of differential equations which define the change of coordinates $x'^{\beta}(x)$ in terms of given basis vectors.
Integrability over the system (1.2.50) is rewritten in terms of the Schwartz condition

$$\frac{\partial^2 x'^{\beta}}{\partial x^{\alpha}\partial x^{\gamma}} = \frac{\partial^2 x'^{\beta}}{\partial x^{\gamma}\partial x^{\alpha}} \tag{1.2.51}$$

which, explicitly, leads to

$$\begin{aligned}\left[\frac{\partial e^{\beta}_{(a)}(x')}{\partial x'^{\delta}}e^{\delta}_{(b)}(x') - \frac{\partial e^{\beta}_{(b)}(x')}{\partial x'^{\delta}}e^{\delta}_{(a)}(x')\right] e^{(b)}_{\gamma}(x)\,e^{(a)}_{\alpha}(x) = \\ = e^{\beta}_{(a)}(x')\left[\frac{\partial e^{(a)}_{\gamma}(x)}{\partial x^{\alpha}} - \frac{\partial e^{(a)}_{\alpha}(x)}{\partial x^{\gamma}}\right].\end{aligned} \tag{1.2.52}$$

Multiplying both sides of (1.2.52) by $e^{\alpha}_{(d)}(x)e^{\gamma}_{(c)}(x)e^{(f)}_{\beta}(x')$ and differentiating, the left-hand side becomes

$$\begin{aligned}e^{(f)}_{\beta}(x')\left[\frac{\partial e^{\beta}_{(d)}(x')}{\partial x'^{\delta}}e^{\delta}_{(c)}(x') - \frac{\partial e^{\beta}_{(c)}(x')}{\partial x'^{\delta}}e^{\delta}_{(d)}(x')\right] = \\ = e^{\beta}_{(c)}(x')\,e^{\delta}_{(d)}(x')\left[\frac{\partial e^{(f)}_{\beta}(x')}{\partial x'^{\delta}} - \frac{\partial e^{(f)}_{\delta}(x')}{\partial x'^{\gamma}}\right]\end{aligned} \tag{1.2.53}$$

and the right hand side the same function expressed in terms of x.
Since x and x' are arbitrary, both sides must be constant, then last equation reduces to

$$\left(\frac{\partial e^{(c)}_{\alpha}}{\partial x^{\beta}} - \frac{\partial e^{(c)}_{\beta}}{\partial x^{\alpha}}\right) e^{\alpha}_{(a)}e^{\beta}_{(b)} = C^{c}{}_{ab} \tag{1.2.54}$$

(now brackets around indices are superfluous): this provides the expression in terms of the group structure constants ${C^c}_{ab}$. Multiplying (1.2.54) by $e^{\gamma}_{(c)}$ finally leads to

$$e^{\alpha}_{(a)}\frac{\partial e^{\gamma}_{(b)}}{\partial x^{\alpha}} - e^{\beta}_{(b)}\frac{\partial e^{\gamma}_{(a)}}{\partial x^{\beta}} = {C^c}_{ab}e^{\gamma}_{(c)}\,. \tag{1.2.55}$$

By construction, we have the antisymmetry property from (1.2.53) or (1.2.54)

$${C^c}_{ab} = -{C^c}_{ba}\,. \tag{1.2.56}$$

To use the notation of the previous Section,

$$X_a = e^{\alpha}_{(a)}\frac{\partial}{\partial x^{\alpha}} \tag{1.2.57}$$

and (1.2.55) rewrites

$$[X_a, X_b] \equiv X_a X_b - X_b X_a = {C^c}_{ab}X_c\,. \tag{1.2.58}$$

Homogeneity is expressed as the Jacobi identity

$$[[X_a, X_b]\,, X_c] + [[X_b, X_c]\,, X_a] + [[X_c, X_a]\,, X_b] = 0 \tag{1.2.59}$$

and explicitly

$${C^f}_{ab}{C^d}_{cf} + {C^f}_{bc}{C^d}_{af} + {C^f}_{ca}{C^d}_{bf} = 0\,. \tag{1.2.60}$$

With this formalism, Einstein equations for a homogeneous universe can be written in the form of a system of ordinary differential equations which involve only functions of time, once all three-dimensional vectors and tensors are expanded as stated before. To obtain the equations ruling the dynamics it is not necessary to use the explicit coordinate dependence of the basis vectors. Such choice in fact is not unique as

$$e_{(a)} = A_{(b)(a)}e^{(b)} \tag{1.2.61}$$

yields again a set of basis vectors.

Let us introduce the two-index structure constants which will be used later as

$${C^c}_{ab} = \varepsilon_{abd}C^{dc} \tag{1.2.62}$$

where $\varepsilon_{abc} = \varepsilon^{abc}$ is the Levi-Civita tensor ($\varepsilon_{123} = +1$); the Jacobi identity (1.2.60) becomes

$$\varepsilon_{bcd}C^{cd}C^{ba} = 0. \tag{1.2.63}$$

The problem of classification of all homogeneous spaces reduces to the problem of finding all inequivalent sets of structure constants. Condition (1.2.63) leads to

$$
\begin{aligned}
[X_1, X_2] &= -aX_2 + n_3 X_3 \\
[X_2, X_3] &= n_1 X_1 \\
[X_3, X_1] &= n_2 X_2 + aX_3
\end{aligned}
\tag{1.2.64}
$$

where $a > 0$ and (n_1, n_2, n_3) are constants related to the structure constants and in particular the n_i are reducible to unity. Similarly to what found in the previous Section we finally find the Bianchi classification as in the Table 1.2.3.

Type	**a**	$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$
I	0	0	0	0
II	0	1	0	0
VII	0	1	1	0
VI	0	1	-1	0
IX	**0**	**1**	**1**	**1**
VIII	**0**	**1**	**1**	**-1**
V	1	0	0	0
IV	1	0	0	1
VII	a	0	1	1
III ($a = 1$) / VI ($a \neq 1$)	a	0	1	-1

Table 1.1: Inequivalent structure constants corresponding to the Bianchi classification

Not all anisotropic dynamics are compatible with a satisfactory SCM but, as shown in the early Seventies, some can be represented, under suitable conditions, as a FLRW model plus a gravitational waves packet (Lukash, 1974), (Grischchuk et al., 1975).
The interest in the IX model – the so-called Mixmaster (Misner, 1969)– relies on the property to have invariant geometry under the $SO(3)$ group, shared with the closed FLRW universe. For such a inhomogeneous dynamics, the line element allows a decomposition as

$$
ds^2 = {ds_0}^2 - \delta_{(a)(b)} G_{ik}^{(a)(b)} dx^i dx^k \tag{1.2.65}
$$

where ds_0 denotes the line element of an isotropic universe having positive constant curvature, $G_{ik}^{(a)(b)}$ is a set of spatial tensors and $\delta_{(a)(b)}(t)$ are amplitude functions, resulting small sufficiently far from the singularity. The tensors introduced in (1.2.65) satisfy the equations

$$G_{ik\ ;l}^{(a)(b);l} = -(n^2-3)G_{ik}^{(a)(b)}, \quad G_{\ ;k}^{(a)(b)k} = 0, \quad G_{\ i}^{(a)(b)i} = 0\,, \tag{1.2.66}$$

in which the Laplacian is referred to the geometry of the sphere of unit radius.

1.3 Synchronous Reference and Einstein's Equations

1.3.1 Generality of the Solution

Let us come back to the application of the progressive symmetry reduction scheme to cosmological models in the homogeneous anisotropic case treated independently by what discussed so far by Belinski, Khalatnikov and Lifshitz in 1969 (Belinski et al., 1970).
The discussion will take always into account the question about generality of the solution considered, in the sense that the general solution is the one which allows completely arbitrary conditions (distribution of matter and gravitational field) at any chosen initial moment of time. Then the criterion for the degree of generality of a solution is the number of arbitrary functions of the space coordinates contained in it, bearing in mind also that some functions are arbitrary by virtue of the arbitrariness of the choice of the reference system allowed by the equations, and hence we will focus our attention only over "physically arbitrary" functions, independently of the choice of reference system.

The (1.1.1) equations read a bit more explicitly

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{1.3.1}$$

1.3.2 Synchronous Reference

The most general properties of the cosmological solutions, namely those which involve their singularities, do not depend on the presence or absence of matter. In connection with this it is not necessary to use the comoving reference system, widely used elsewhere, but the natural choice is a system subject to the conditions for the components of the metric tensor g_{ab} $(a, b = 0 \dots 3)$

$$g_{0\alpha} = 0\,, \qquad g_{00} = 1\,, \qquad (\alpha = 1 \dots 3)\,. \tag{1.3.2}$$

In particular, the first in (1.3.2) is the condition which allows the synchronization of clocks at different points of space, while the second relation sets the time coordinate $x^0 = t$ as the proper time at each point of space. The elementary interval in such a system is given by the expression

$$ds^2 = dt^2 - dl^2\,, \tag{1.3.3}$$

where

$$dl^2 = \gamma_{\alpha\beta}\left(t, x^\gamma\right) dx^\alpha dx^\beta\,, \tag{1.3.4}$$

in which the three-dimensional tensor $\gamma_{\alpha\beta}$ defines the space metric.
In the synchronous reference system, lines of equal times are geodesic lines in the four-space, as implied by the splitting definition. Indeed the four-vector

$$u^i = \frac{dx^i}{ds} \tag{1.3.5}$$

which is tangent to the world line ($x^\gamma = const.$), has components $u^0 = 1$, $u^\alpha = 0$ and automatically satisfies the geodesic equations

$$\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = \Gamma^i_{00} = 0\,, \quad (i, k, l = 0, ..3)\ . \tag{1.3.6}$$

The choice of such a reference is always possible and moreover the choice is not unique, since a metric of the form (1.3.4) allows for any transformation of the three space coordinates which does not involve time as

$$\begin{cases} t' &= t \\ x^{\alpha\prime} &= x^{\alpha\prime}\left(x^\beta\right) \end{cases} . \tag{1.3.7}$$

In the reference defined and with a metric as in (1.3.3) the Einstein equations are written in mixed components as

$$R^0_0 \;=\; -\frac{1}{2}\frac{\partial}{\partial t}\kappa^\alpha_\alpha - \frac{1}{4}\kappa^\beta_\alpha \kappa^\alpha_\beta = 8\pi G\left(T^0_0 - \frac{1}{2}T\right) \tag{1.3.8a}$$

$$R^0_\alpha \;=\; \frac{1}{2}\left(\kappa^\beta_{\alpha;\beta} - \kappa^\beta_{\beta;\alpha}\right) = 8\pi G T^0_\alpha \tag{1.3.8b}$$

$$R^\beta_\alpha \;=\; -P^\beta_\alpha - \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\kappa^\beta_\alpha\right) = 8\pi G\left(T^\beta_\alpha - \frac{1}{2}\delta^\alpha_\beta T\right) \tag{1.3.8c}$$

where

$$\kappa_{\alpha\beta} = \frac{\partial \gamma_{\alpha\beta}}{\partial t}, \qquad \gamma \equiv \mid \gamma_{\alpha\beta} \mid , \tag{1.3.9}$$

$T_{\alpha\beta}$ is the energy-momentum tensor describing the system and $P_{\alpha\beta}$ is the three-dimensional Ricci tensor obtained through the metric $\gamma_{\alpha\beta}$ which is used to raise and lower indices within the spatial sections.

The metric $\gamma_{\alpha\beta}$ allows to construct the three-dimensional Ricci tensor $P_\alpha^\beta = \gamma^{\beta\gamma}P_{\alpha\gamma}$ as

$$P_{\alpha\beta} = \partial_\gamma \lambda^\gamma_{\alpha\beta} - \partial_\alpha \lambda^\gamma_{\beta\gamma} + \lambda^\gamma_{\alpha\beta}\lambda^\delta_{\gamma\delta} - \lambda^\gamma_{\alpha\delta}\lambda^\delta_{\beta\gamma} \tag{1.3.10}$$

in which appear the pure spatial Christoffel symbols

$$\lambda^\gamma_{\alpha\beta} \equiv \frac{1}{2}\gamma^{\gamma\delta}(\partial_\alpha\gamma_{\delta\beta} + \partial_\beta\gamma_{\alpha\delta} - \partial_\delta\gamma_{\alpha\beta}) \tag{1.3.11}$$

also used to form the covariant derivative $(\quad)_{;\alpha}$.

From (1.3.8a) it is straightforward to derive, even in the isotropic case, the Landau-Raychaudhuri theorem (Raychaudhuri, 1955), stating that the metric determinant γ has to vanish in a finite instant of time.

The singularity with respect to the time variable is a physical one, characterized by scalar quantities, such as the density of matter and the invariants of the curvature tensor, which are becoming infinite.

1.4 BKL Approach to the Mixmaster Chaos

The Einstein equations (1.3.8a)-(1.3.8c) for a homogeneous universe can be written in the form of a system of ordinary differential equations which involve only functions of time, expanding all three dimensional terms on the tetradic basis built in the previous Section. In empty space such projections take the form

$$R^0_0 = -\frac{1}{2}\dot{\kappa}^a_a - \frac{1}{4}\kappa^b_a\kappa^a_b \tag{1.4.1a}$$

$$R^0_a = -\frac{1}{2}\kappa^c_b\left(C^b{}_{ca} - \delta^b_a C^d{}_{dc}\right) \tag{1.4.1b}$$

$$R^a_b = -\frac{1}{2\sqrt{\eta}}\frac{\partial}{\partial t}\left(\sqrt{\eta}\kappa^b_a\right) - P^a_b \tag{1.4.1c}$$

where we obtain

$$\kappa_{ab} = \dot{\eta}_{ab} \tag{1.4.2a}$$

$$\kappa^b_a = \dot{\eta}_{ac}\eta^{cb}\,. \tag{1.4.2b}$$

On denoting with the dot differentiation with respect to t, the projection

$$P_{ab} = \eta_{bc} P^c_a \tag{1.4.3}$$

of the three-dimensional Ricci tensor becomes

$$P_{ab} = -\frac{1}{2}\Big(C^{cd}{}_{b} C_{cda} + C^{cd}{}_{b} C_{dca} \; - \; \frac{1}{2} C_b{}^{cd} C_{acd} + + \; C^c{}_{cd} C^{\;d}_{ab} \; + \; C^c{}_{cd} C_{ba}{}^{d} \Big) . \tag{1.4.4}$$

At this stage the Einstein equations have reduced to a much simpler differential system, involving only ordinary derivative with respect to the temporal variable t

In this part we will first discuss the Kasner solution which will be generalized in the discussion of the dynamics of Bianchi types VIII and IX.

1.4.1 Kasner Solution

The simplest and paradigmatic solution of the Einstein equations (1.3.8a)-(1.3.8b) in the framework of the Bianchi classification is the type I model, first obtained by Kasner (1921) which is appropriate to the gravitational field when considering (1.3.8a)-(1.3.8b) in empty space.
The simultaneous vanishing of the three structure constants and consequently also of the three-dimensional Ricci tensor provides

$$\left.\begin{array}{l} e^a_\alpha = \delta^a_\alpha \\ C^c_{ab} \equiv 0 \end{array}\right\} \Longrightarrow P_{ab} = 0 \, . \tag{1.4.5}$$

Then (1.4.1a)-(1.4.1c) describe uniform space and reduce to the system

$$\dot{\kappa}^a_a + \frac{1}{2}\kappa^b_a \kappa^a_b \;=\; 0 \tag{1.4.6}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\kappa^b_a\right) \;=\; 0 \, , \tag{1.4.7}$$

whose solution is a Euclidean metric depending on time as

$$dl^2 = t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \, . \tag{1.4.8}$$

Here p_1, p_2, p_3 are three arbitrary numbers, so-called Kasner indices, satisfying the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \, , \tag{1.4.9}$$

therefore only one of these numbers is independent. Except for the cases $(0,0,1)$ and $(-\frac{1}{3},\frac{2}{3},\frac{2}{3})$, such indices are never equal, one of them being negative and two positive; in the peculiar case $p_1 = p_2 = 0, p_3 = 1$ the metric is reducible to a Galilean form by the transformation

$$t\sinh x^3 = \xi\,, \quad t\cosh x^3 = \tau\,, \tag{1.4.10}$$

i.e. with a fictitious singularity in a flat space time. Once that Kasner indices have been ordered according to

$$p_1 < p_2 < p_3\,, \tag{1.4.11}$$

their corresponding variation ranges are

$$-\frac{1}{3} \leq p_1 \leq 0\,, \quad 0 \leq p_2 \leq \frac{2}{3}\,, \quad \frac{2}{3} \leq p_3 \leq 1\,. \tag{1.4.12}$$

In parametric form we have the representation

$$\begin{aligned} p_1(u) &= \frac{-u}{1+u+u^2} \\ p_2(u) &= \frac{1+u}{1+u+u^2} \\ p_3(u) &= \frac{u(1+u)}{1+u+u^2} \end{aligned} \tag{1.4.13}$$

as the parameter u varies in the range

$$1 \leq u < +\infty. \tag{1.4.14}$$

The values $u < 1$ lead to the same range as

$$p_1\left(\frac{1}{u}\right) = p_1(u)\;, \quad p_2\left(\frac{1}{u}\right) = p_3(u)\;, \quad p_3\left(\frac{1}{u}\right) = p_2(u)\;. \tag{1.4.15}$$

This solution clearly corresponds to a completely uniform but anisotropic space, when in the Friedmann solution all distances had a contraction towards the singularity with the same power law in any direction.
In a generalized solution the limiting metric expressed by the principal terms as expansion in powers of t has the form analogous to (1.4.8), hence as (1.3.3) with line element

$$dl^2 = \left(a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta\right) dx^\alpha dx^\beta \tag{1.4.16}$$

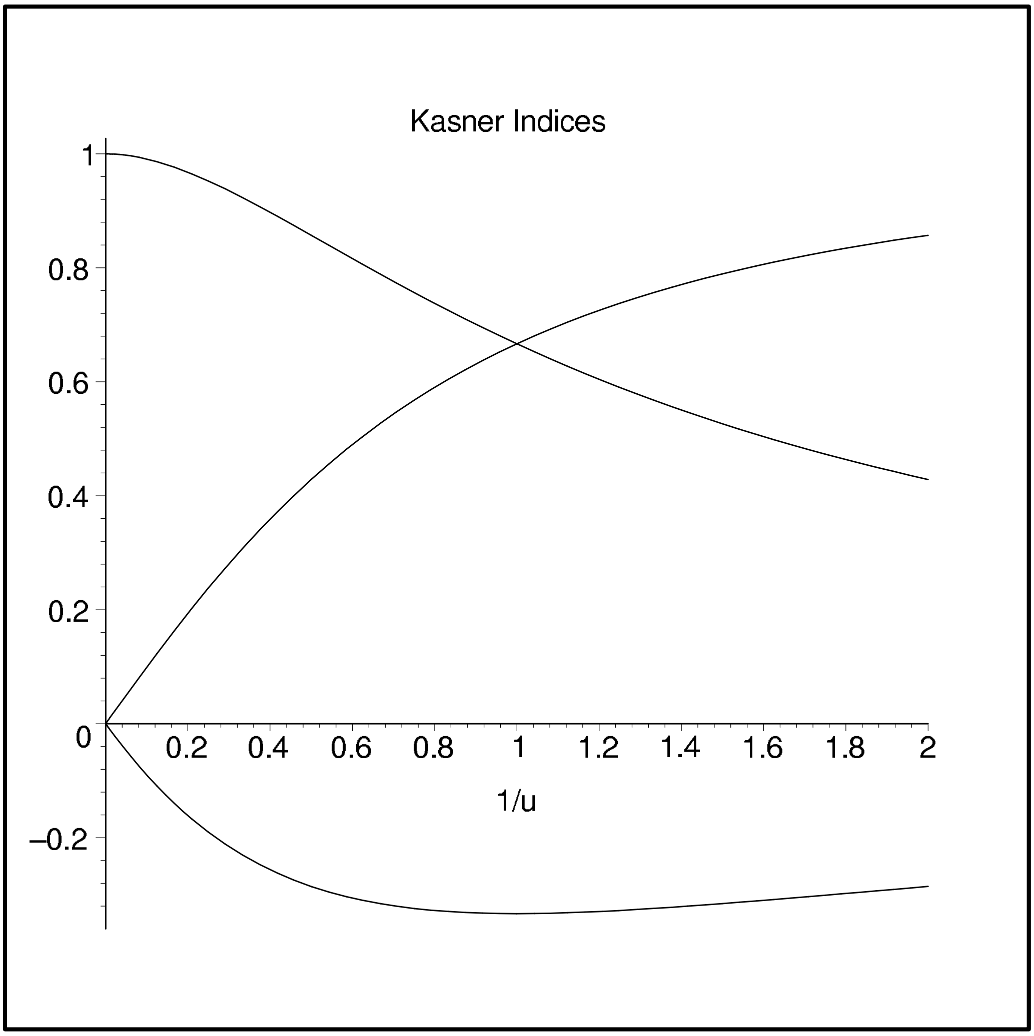

Figure 1.1: Evolution of Kasner indexes in terms of the parameter u.

where

$$a = t^{p_l}\,, \quad b = t^{p_m}\,, \quad c = t^{p_n}\,. \tag{1.4.17}$$

The three-dimensional vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$ define the directions along which the spatial distances vary with time according to the power laws (1.4.17). These vectors and the Kasner indices are functions of the spatial coordinates: their relation with the structure constants defined in (1.2.55) is given in Section 1.2.3. Since the exponents in (1.4.17) are all different, the spatial metric (1.4.16) is always anisotropic.
The degree of generality is not diminished by the presence of matter: this can be brought into the metric (1.4.16)-(1.4.17) with all the four new functions of the coordinates which are necessary to provide the initial distribution of density of matter and the three components of the velocity of its motion. The behaviour of matter in the neighbourhood of a singular point is determined by the equations of motion of matter in a given gravitational field as hydrodynamic equations (see (4.6.15) below).

1.4.2 Piecewise Representation of the Model

The Kasner solution is compatible with the situation in which the Ricci tensor appearing in the Einstein equations $P_{\alpha\beta}$ is of higher order in $1/t$ with respect to all other terms involved. However, since one of the Kasner exponents is negative, terms of order higher than t^{-2} appear in the tensor $P_{\alpha\beta}$. In such a case the discussion of solutions has to be extended to the general anisotropic case, in the search of a general behaviour of the universe towards the initial singularity. In fact, the Kasner regime outlined relies on a restriction over the phase space of the solution (not discussed here in the details, see Belinski et al. (1970), §3) which causes an instability with perturbations which violate this condition. A general solution is, by definition, completely stable, i.e. application of any perturbation is equivalent to a change in the initial conditions at some moment of time and, since the general solution satisfies arbitrary initial conditions, the perturbation cannot change the form of the solution. Nevertheless, the cited restriction over the Kasner solution makes it unstable with respect to perturbations violating it: the transition to a new state cannot be treated as small ans lies outside the region of infinitesimal perturbations.

The Einstein equations in a synchronous reference system and for a generic homogeneous cosmological model in empty space are given by the system

$$-R_l^l = \frac{(\dot{a}bc)^{\cdot}}{abc} + \frac{1}{2a^2b^2c^2}\left[\lambda^2 a^4 - \left(\mu b^2 - \nu c^2\right)^2\right] \qquad = 0 \tag{1.4.18a}$$

$$-R_m^m = \frac{\left(a\dot{b}c\right)^{\cdot}}{abc} + \frac{1}{2a^2b^2c^2}\left[\mu^2 b^4 - \left(\lambda a^2 - \nu c^2\right)^2\right] \qquad = 0 \tag{1.4.18b}$$

$$-R_n^n = \frac{(ab\dot{c})^{\cdot}}{abc} + \frac{1}{2a^2b^2c^2}\left[\nu^2 c^4 - \left(\lambda a^2 - \mu b^2\right)^2\right] \qquad = 0 \tag{1.4.18c}$$

and

$$-R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0 \tag{1.4.19}$$

where l, m, n label the three spatial directions and the other off-diagonal components of the four-dimensional Ricci tensor vanish identically as a consequence of the choice of the diagonal form for the metric tensor according to $\gamma_{11} = a^2,\ \gamma_{22} = b^2,\ \gamma_{33} = c^2$. The constants λ, μ, ν correspond to the structure constants C_{11}, C_{22}, C_{33} respectively, introduced earlier in Section 1.2.3, which for the Bianchi type IX read as $\mu = \nu = \lambda = 1$ while for type VIII $\mu = \nu = -\lambda = 1$. All these equations contain only functions of time, reflecting homogeneity of space, being exact ones, without any restriction for the closeness to the singular point $t = 0$. Through the notation

$$a = e^{\alpha}\,, \quad b = e^{\beta}\,, \quad c = e^{\gamma} \tag{1.4.20}$$

and the new temporal variable τ

$$dt = abc\ d\tau \tag{1.4.21}$$

(1.4.18) and (1.4.19) simplify to

$$2\alpha_{\tau\tau} = \left(\mu b^2 - \nu c^2\right)^2 - \lambda^2 a^4 \tag{1.4.22a}$$

$$2\beta_{\tau\tau} = \left(\lambda a^2 - \nu c^2\right)^2 - \mu^2 b^4 \tag{1.4.22b}$$

$$2\gamma_{\tau\tau} = \left(\lambda a^2 - \mu b^2\right)^2 - \nu^2 c^4 \tag{1.4.22c}$$

$$\frac{1}{2}\left(\alpha + \beta + \gamma\right)_{\tau\tau} = \alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau \tag{1.4.23}$$

where subscript τ denotes derivation with respect to τ. Manipulating system (1.4.22) and using (1.4.23), one obtains the first integral

$$\begin{aligned}\alpha_\tau\beta_\tau + &\alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau = \\ &= \frac{1}{4}\Big(\lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 + \\ &\quad - 2\lambda\mu a^2 b^2 - 2\lambda\nu a^2 c^2 - 2\mu\nu b^2 c^2\Big)\end{aligned} \tag{1.4.24}$$

involving only first derivatives. The Kasner regime (1.4.17) is the solution of equations (1.4.22) corresponding to the case when all terms on the right hand side of (1.4.22) can be neglected. However, such a situation cannot persist indefinitely ($t \to 0$) since there are always such terms on the right-hand side of (1.4.22) which are increasing.
Let's consider for instance the case in which the negative power corresponds to the function $a(t)$ (that is to say $p_l = p_1$): the perturbation of the Kasner regime results from the terms $\lambda^2 a^4$ while the other terms decrease with decreasing t, in fact

$$p_1 < 0 \to p_1 = -|p_1|\,, \quad \begin{cases} \alpha \sim -|p_1| \ln t \\ a \sim \dfrac{1}{t^{|p_1|}} \end{cases} \nearrow \quad \text{for}\, t \to 0 \tag{1.4.25}$$

and along other directions

$$\begin{array}{ll} p_2 > 0 \to p_2 = |p_2|\,, & \beta \sim |p_2| \ln t \searrow \\ p_3 > 0 \to p_3 = |p_3|\,, & \gamma \sim |p_3| \ln t \searrow \end{array} \quad \text{for}\, t \to 0\,. \tag{1.4.26}$$

Preserving on the right-hand side of equations (1.4.22) only the increasing terms we obtain

$$\begin{aligned} \alpha_{\tau\tau} &= -\frac{1}{2} e^{4\alpha} \\ \beta_{\tau\tau} &= \gamma_{\tau\tau} = \frac{1}{2} e^{4\alpha}\,. \end{aligned} \tag{1.4.27a}$$

The solution of these equations describes the evolution of the metric from its initial state (1.4.17). Since in the following we will consider the dynamics of Bianchi types VIII and IX only (in particular the last one) we will set $\lambda = 1$ without loss of generality in the properties discussed. Let

$$p_l = p_1\,, \quad p_m = p_2\,, \quad p_n = p_3\,, \tag{1.4.28}$$

so that

$$a \sim t^{p_1}\,, \quad b \sim t^{p_2}\,, \quad c \sim t^{p_3}\,, \tag{1.4.29}$$

and then

$$\begin{aligned} abc &= \Lambda t \\ \tau &= \frac{1}{\Lambda} \ln t + \text{const.} \end{aligned} \tag{1.4.30}$$

where Λ is a constant, therefore the initial conditions to (1.4.27) can be formulated as

$$\alpha_\tau = \Lambda p_1\,, \quad \beta_\tau = \Lambda p_2\,, \quad \gamma_\tau = \Lambda p_3\,, \qquad \text{for} \quad \tau \to \infty\,. \tag{1.4.31}$$

The system (1.4.27) with (1.4.31) is integrated to

$$a^2 = \frac{2\mid p_1 \mid \Lambda}{\cosh\left(2\mid p_1\mid \Lambda\tau\right)} \tag{1.4.32a}$$

$$b^2 = {b_0}^2 \exp\left[2\Lambda\left(p_2 - \mid p_1 \mid\right)\tau\right]\cosh\left(2\mid p_1\mid \Lambda\tau\right) \tag{1.4.32b}$$

$$c^2 = {c_0}^2 \exp\left[2\Lambda\left(p_3 - \mid p_1 \mid\right)\tau\right]\cosh\left(2\mid p_1\mid \Lambda\tau\right) \tag{1.4.32c}$$

where b_0 and c_0 are integration constants. Let us consider solutions (1.4.32) in the limit $\tau \to \infty$: towards the singularity

$$a \sim \exp\left[-\Lambda p_1 \tau\right] \tag{1.4.33a}$$

$$b \sim \exp\left[\Lambda\left(p_2 + 2p_1\right)\tau\right] \tag{1.4.33b}$$

$$c \sim \exp\left[\Lambda\left(p_3 + 2p_1\right)\tau\right] \tag{1.4.33c}$$

$$t \sim \exp\left[\Lambda\left(1 + 2p_1\right)\tau\right] \tag{1.4.33d}$$

that is to say, in terms of t

$$a \sim t^{p'_l}\,, \quad b \sim t^{p'_m}\,, \quad c \sim t^{p'_n}\,, \quad abc = \Lambda' t \tag{1.4.34}$$

where the primed exponents are related to the un-primed ones by

$$p'_l = \frac{\mid p_1 \mid}{1 - 2\mid p_1 \mid}\,, \qquad p'_m = -\frac{2\mid p_1\mid - p_2}{1 - 2\mid p_1 \mid}\,, \tag{1.4.35a}$$

$$p'_n = \frac{p_3 - 2\mid p_1 \mid}{1 - 2\mid p_1 \mid}\,, \qquad \Lambda' = \left(1 - 2\mid p_1 \mid\right)\Lambda\,. \tag{1.4.35b}$$

Summarizing these results, we see the effect of the perturbation over the Kasner regime: a Kasner epoch is replaced by another one so that the negative power of t is transferred from **l** to **m** direction, i.e. if in the original solution p_l is negative, in the new solution $p'_m < 0$. The previously increasing perturbation $\lambda^2 a^4$ in (1.4.18) is damped and eventually vanishes. The other terms involving μ^2 instead of λ^2 will grow, therefore permitting the replacement of a Kasner epoch by another. Such rules of rotation in the perturbing property can be summarized with the rules

$$\left.\begin{array}{r} p_l = p_1(u) \\ p_m = p_2(u) \\ p_n = p_3(u) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} p'_l = p_2\left(u-1\right) \\ p'_m = p_1\left(u-1\right) \\ p'_n = p_3\left(u-1\right) \end{array}\right. \tag{1.4.36}$$

which constitutes the so-called *BKL map*, the greater of the two positive powers remains positive.
The following changes according to (1.4.36) accompanied by a bouncing of the negative power between the direction **l** and **m** continue as long as the integral part of the initial value of u is not exhausted, i.e. until u becomes less than one. At that point, according to (1.4.15), the value $u < 1$ is turned into $u > 1$; at this moment either the exponent p_l or p_m is negative and p_n becomes the smaller one of the two positive numbers, say $p_n = p_2$. The next sequence of changes will bounce the negative power between the directions **n** and **l** or **n** and **m**.

1.4.3 Dynamics and Gauss Map

For an arbitrary, irrational initial value of u the changes (1.4.36) repeat indefinitely. In the case of an exact solution the exponents p_l, p_m and p_n lose their literary meaning. In general, it has no meaning to consider any well defined, for example rational, value of u. Only the conclusions which refer to the general (irrational) u have real meaning.

The evolution of the model towards the singularity consists of successive periods, so-called *eras*, in which distances along two axes oscillate and along the third axis decrease monotonically while the volume decreases to a law near to $\sim t$. In the transition from one era to another, the axes along which the distances decrease monotonically are interchanged. The order in which the pairs of axes are interchanged and the order in which eras of different lengths (number of Kasner epochs contained in it) follow each other acquire a stochastic character. Successive eras 'condense' towards a singularity. Such general qualitative properties are not changed in the case of space filled in with matter, however the role of the solution would change: the model so far discussed would be considered as the principal terms of the limiting form of the metric as $t \to 0$.

The discussion needs some more comments. To every s-th era there corresponds a decreasing sequence of values of the parameter u. This sequence, from the starting era has the form $u^{(s)}_{max}$, $u^{(s)}_{max} - 1$, $u^{(s)}_{max} - 2, \ldots, u^{(s)}_{min}$. We can introduce the notation

$$u^{(s)} = k^{(s)} + x^{(s)} \tag{1.4.37}$$

then

$$u_{min}^{(s)} = x^{(s)} < 1\,, \qquad u_{max}^{(s)} = k^{(s)} + x^{(s)}\,, \tag{1.4.38}$$

where $u_{max}^{(s)}$ is the greatest value of u for an assigned era and $k^{(s)} = \left[u_{max}^{(s)}\right]$ (square brackets denote the greatest integer less or equal to $u_{max}^{(s)}$). The number $k^{(s)}$denotes the era length, i.e. the number of Kasner epoch contained into it. For the next era we obtain

$$u_{max}^{(s+1)} = \frac{1}{x^{(s)}}\,, \quad k^{(s+1)} = \left[\frac{1}{x^{(s+1)}}\right]\,. \tag{1.4.39}$$

For large u the Kasner exponents approaching the values $(0, 0, 1)$ correspond to

$$p_1 \approx -\frac{1}{u}\,, \quad p_2 \approx \frac{1}{u}\,, \quad p_1 \approx 1 - \frac{1}{u^2}\,, \tag{1.4.40}$$

and the transition to the next era is governed by the fact that not all terms in the Einstein equations are negligible and some terms are comparable: in such a case, the transition is accompanied by a long regime of small oscillations lasting until the next era, whose details are not relevant for the purposes of this work (for details see BELINSKI ET AL. (1970)), after which a new series of Kasner epochs begins. The probability λ of all possible values of $x^{(0)}$ which lead to a dynamical evolution towards this specific case, expressed as a fraction of the unit interval, is strongly converging to a number $\lambda \ll 1$. If the initial value of $x^{(0)}$ is outside this special interval for λ, the special case cannot occur; if $x^{(0)}$ lies in this interval, a peculiar evolution in small oscillations can occur, but after this period the model begins to evolve regularly with a new initial value $x^{(0)}$, which can fall only accidentally in the dangerous interval (with probability λ). The repetition of this situation can lead to dangerous cases only with probabilities $\lambda, \lambda^2, \ldots$ which converge asymptotically to zero.

If the sequence begins with $k^{(0)}+x^{(0)}$, the lengths $k^{(1)}, k^{(2)}, \ldots$ are the numbers appearing in the expansion for $x^{(0)}$ in terms of the continuous fraction

$$x^{(0)} = \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \cfrac{1}{k^{(3)} + \ldots}}}\,, \tag{1.4.41}$$

which is finite if related to a rational number, but in general is an infinite one.

For the infinite sequence of positive numbers u ordered as (1.4.39) admitting the expansion (1.4.41) it is possible to note that

i) a rational number would have a finite expansion;

ii) periodic expansion represents quadratic irrational numbers (i.e. numbers which are roots of quadratic equations with integral coefficients)

iii) irrational numbers have infinite expansion.

All terms $k^{(1)}, k^{(2)}, k^{(3)}, \ldots$ in the first two cases having the exceptional property to be bounded in magnitude are related to a set of numbers $x^{(0)} < 1$ of zero measure in the interval $(0, 1)$.

An alternative to the numerical approach in terms of continuous fractions is the statistic distribution of the eras' sequence for the numbers $x^{(0)}$ over the interval $(0, 1)$, governed by some probability law. For the series $x^{(s)}$ with increasing s these distributions tend to a stationary one $w(x)$, independent of s, in which the initial conditions are completely forgotten

$$w(x) = \frac{1}{(1+x)\mathrm{ln}2}\,. \tag{1.4.42}$$

Consider, instead of a well defined initial value as in (1.4.37) with $s = 0$, a probability distribution for $x^{(0)}$ over the interval $(0, 1)$, $W_0(x)$ for $x^{(0)} = x$ in such an interval. Then also the numbers $x^{(s)}$ are distributed with some probability. Let $w_s(x)dx$ be the probability that the last term in the s-th series $x^{(s)} = x$ lies in the interval dx. Then the last term of the previous series must lie in the interval between $1/(k+1)$ and $1/k$, in order for the length of the s-th series to be k.
Then the probability for the series to have length k is given by

$$W_s(k) = \int_{\frac{1}{1+k}}^{\frac{1}{k}} w_{s-1}(x)dx\,. \tag{1.4.43}$$

The fact that the last term of the $(s+1)$-st series $x^{(s)} = x$ can be generated by the initial term of the same series $u_{max}^{(s+1)} = x + k$ (where $k = 1, 2, \ldots$) and correspond to the numbers $x^{(s)} = 1/(k+x)$ from the preceding series for each pair of subsequent series can be re-expressed in terms of the recurrence formula relating the distribution $w_{s+1}(x)$ to $w_s(x)$

$$w_{s+1}(x)dx = \sum_{k=1}^{\infty} w_s\left(\frac{1}{k+x}\right)\left|d\frac{1}{(k+x)}\right|\,, \tag{1.4.44}$$

or, simplifying the differential interval,

$$w_{s+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w_s\left(\frac{1}{k+x}\right)\,. \tag{1.4.45}$$

If for increasing n the w_{s+n} distribution (1.4.45) tends to a stationary one, independent of s, $w(x)$ has to satisfy

$$w(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w\left(\frac{1}{k+x}\right) . \tag{1.4.46}$$

A normalized solution to (1.4.46) is

$$w\,(x) = \frac{1}{(1+x)\ln 2} . \tag{1.4.47}$$

The corresponding stationary distribution of the lengths of the series k is obtained substituting (1.4.47) in (1.4.43) and evaluating the integral

$$W\,(k) = \int_{\frac{1}{1+k}}^{\frac{1}{k}} w\,(x)\,dx = \frac{1}{\ln 2}\ln\frac{(k+1)^2}{k\,(k+2)} . \tag{1.4.48}$$

Finally, k and x being not independent, they must admit a stationary cumulative probability distribution

$$w\,(k,x) = \frac{1}{(k+x)\,(k+x+1)\ln 2} \tag{1.4.49}$$

which, for $u = k + x$, rewrites as

$$w\,(u) = \frac{1}{u\,(u+1)\ln 2} \tag{1.4.50}$$

say a stationary distribution for the parameter u.

The formalism developed so far and the method used give clear indications on how the singularity in the general solution of the gravitational equations is of the same type as the one discussed here. In fact, the oscillatory approach to the singular point is caused by that type of perturbation which makes the Kasner solution unstable. In the same way, it is shown that there is no direct connection between the singular point and the finiteness or infiniteness of the universe, in view of the existence of open and closed homogeneous models with oscillatory singular points.

The infinite number of oscillations between any finite moment of the world time t and the moment $t = 0$ modify the notion of finiteness of time, as relies in the approach in term of the logarithmic one τ in terms of which the whole evolution is stretched to $-\infty$.

The nature of this singularity and its generality is peculiar: for a singular point in the future, one has to assume arbitrary initial conditions at any preceding moment of time while for a singularity in the past there is no way to connect straightforwardly the initial conditions on the equations as related to the present state of the universe with arbitrary conditions in the past, without the assumption of some mechanism to connect the various eras of the universe evolution.

Let us outline one other important task related to homogeneous dynamics marking a deep difference between isotropic and anisotropic ones (Misner, 1969): the standard model metric for the radiation dominated early phase FRW is

$$ds^2 = \eta^2(d\eta^2 - d\vec{l}^2) \tag{1.4.51}$$

then the coordinate time $\Delta\eta$ required for a light signal with $ds^2 = 0$ to connect two regions of spatial separation Δx is

$$\Delta x = \Delta\eta \,, \tag{1.4.52}$$

so that at a fixed epoch $\eta_0 > 0$ no causal interactions subsequent to the singularity at $\eta = 0$ have occurred between regions of spatial separation $\Delta x > \eta_0$, leaving unexplained, as the so-called horizon paradox, the strong homogeneity of the last scattering background radiation. Quite a different situation occurs, for example, in the closed Bianchi type IX: considering the propagation of a light signal during a long era, i.e. when evolution is in the dynamical regime of small oscillations, in which the distances vary according to the law $\sim t$, the spatial element is given by $dl^2 = t^2 d\vec{l}^2$ and the corresponding four-dimensional elementary interval by

$$ds^2 = \eta^2(d\eta^2 - d\vec{l}^2) \tag{1.4.53}$$

similar to the previous one, but with $\eta = \ln t$. Then, in contrast to the (1.4.51), now η varies from $-\infty$ to $+\infty$: therefore, for every given moment of time, any finite distance can be covered by a signal in the time available since the singularity. Thus, during a long era, the world horizon opens in one spatial direction. The duration of such eras is finite while their number is infinite: in the present model the causal connection between events could be possible in the whole volume of the universe.

The effect of matter on the evolution of the universe will become dominant after some time and this effect will lead gradually to more isotropic models of space, approaching the Friedmann model.

Non Diagonal Case

The metric $\gamma_{\alpha\beta}$ considered so far to describe the dynamics was supposed to be diagonal and the Einstein equations were found always to be compatible, but this was a very peculiar situation: the most general case of homogeneous spaces and inhomogeneous ones allows for new qualitative properties of the oscillatory modes. The universality of the alternation law of Kasner exponents is confirmed but moreover associated to a rotation of the axes themselves, with the asymptotic laws contained as a limit in the general solution (Belinski et al., 1982).
The metric $\gamma_{\alpha\beta}$ related to the spatial line element

$$dl^2 = \gamma_{ab}(t) \left(e^a_\alpha dx^\alpha\right) \left(e^b_\beta dx^\beta\right) \tag{1.4.54}$$

is expected to have *six* unknown functions of time which reduce to *three* (a, b, c) in the diagonal case

$$dl^2 = \left(a^2 e^1_\alpha e^1_\beta + b^2 e^2_\alpha e^2_\beta + c^2 e^3_\alpha e^3_\beta\right) dx^\alpha dx^\beta \,. \tag{1.4.55}$$

In this case the symmetry of the space types did not lead to inconsistencies because the off-diagonal components of the Ricci tensor vanished identically while the rest of the Einstein equations for te field in empty space constituted a consistent system. Non-diagonality of the metric tensor implies some properties peculiar of this more general case. On considering such a non diagonal situation, an exact solution exists only for a matter filled space, otherwise the equations R^0_α would automatically lead to the disappearance of the off-diagonal components of $\gamma_{\alpha\beta}$ (Belinski et al., 1982).
In the non-diagonal metric case, the metric is solved for the spatial line element as (1.4.16)

$$\gamma_{\alpha\beta} = a^2 L_\alpha L_\beta + b^2 M_\alpha M_\beta + c^2 N_\alpha N_\beta \tag{1.4.56}$$

where $L_\alpha, M_\alpha, N_\alpha$ are constant coefficients in terms of which the solution to the Einstein equations is analogous to (1.4.16)

$$dl^2 = \left(a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta\right) dx^\alpha dx^\beta \tag{1.4.57}$$

where

$$\mathbf{l} = L_a \mathbf{e}^a\,, \quad \mathbf{m} = M_a \mathbf{e}^a\,, \quad \mathbf{n} = N_a \mathbf{e}^a \tag{1.4.58}$$

in terms of which is then written the temporal variation for the *Kasner axes* $\mathbf{l}, \mathbf{m}, \mathbf{n}$. Transformation (1.4.58) is not orthogonal, then $\mathbf{l}, \mathbf{m}, \mathbf{n}$ cannot be chosen as frame vectors. Introducing the angles defining the Cartesian vectors $\mathbf{L}, \mathbf{M}, \mathbf{N}$ and their relative orientations as

$$\mathbf{L} = (1, 0, 0)\,, \tag{1.4.59}$$

$$\mathbf{M} = (\cos\theta_m, \sin\theta_m, 0)\,, \tag{1.4.60}$$

$$\mathbf{N} = (\cos\theta_n, \sin\theta_n \cos\phi_n, \sin\theta_n \sin\phi_n) \tag{1.4.61}$$

it is possible to find the dependence of the directions evolution in terms of the parameter u labelling each Kasner epoch as

$$\frac{\tan\theta'_m}{\tan\theta_m} = -\frac{2u-1}{2u+1}\,, \quad \phi'_n = \phi_n\,, \quad \frac{\tan\theta'_n}{\tan\theta_n} = \frac{u-2}{u+2}\,. \tag{1.4.62}$$

In the diagonal case the Kasner axes are rigidly fixed to the frame vectors and *do not change* when the Kasner epochs are replaced. In the non-diagonal case the Kasner axes *are not fixed* and their directions change from one epoch to another according to (1.4.62), the law of replacement of the Kasner exponents (1.4.36) being the same as the diagonal case. By (1.4.62) we see that $|\theta'_m/\theta_m| < 1$ and $|\theta'_n/\theta_n| < 1$, which means that with each replacement of the epochs the Kasner axes approach each other and similarly for the change from one era to the next one. Kasner axes with an irregular evolution approach the common direction $\mathbf{C}$ which is defined as a three-vector whose components are *exact* solutions of the $\alpha - \beta$ Einstein equations summarized as components of the vectorial equation given by

$$p_l\,[\mathbf{L} \times [\mathbf{M} \times \mathbf{N}]] + p_m\,[\mathbf{M} \times [\mathbf{N} \times \mathbf{L}]] + \\ + p_n\,[\mathbf{N} \times [\mathbf{L} \times \mathbf{M}]] = \mathbf{C} \tag{1.4.63}$$

where $\mathbf{C} = (C_1, C_2, C_3)$ and all vector operations are performed in terms of Cartesian vectors. Equations (1.4.62) can be rewritten in the suggestive form

$$\mathbf{l}' = \mathbf{l}\,, \tag{1.4.64a}$$

$$\mathbf{m}' = \mathbf{m} - \mathbf{l}\frac{4p_1 \cot\theta_m}{p_2 + 3p_1}\,, \tag{1.4.64b}$$

$$\mathbf{n}' = \mathbf{n} - \mathbf{l}\frac{4p_1}{p_3 + 3p_1}\frac{\cot\theta_n}{\sin\phi_n}\,. \tag{1.4.64c}$$

Summarizing, the dynamical evolution is characterized by an irregular approach to the direction **C**.
The role of matter, as in the diagonal case, is reduced only to a change in the relations imposed on the spatial functions entering the solution, with no specific new physical implications. Nevertheless, in contrast with the diagonal case, we have sketched how in the general homogeneous model exists the preferred direction given by the vector **C** which is connected to the evolution of matter by the corresponding $0-\alpha$ Einstein equation as

$$C_a = -\frac{4}{3}\epsilon^{(0)} u_a^{(0)} \left(u_b^{(0)} N^b \right) \tag{1.4.65}$$

being C_a three arbitrary constants, $\epsilon^{(0)}, u_a^{(0)}$ constants related to the energy density and four velocity respectively. Homogeneous models with rotating axes require the presence of matter while in empty space exist only homogeneous models with fixed axes.

To conclude the discussion on homogeneous cosmological models we spend a few words on the general solution for non-homogeneous geometries. Similarly to the former case, the line element and the expression for the metric in terms of Kasner indices read as (1.4.16) and (1.4.29), where the frame vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$ are also functions of the coordinates and arbitrary ones, subjected only to the conditions imposed by the $0-\alpha$ components of the Einstein equations, not close or similar to the expressions introduced in the homogeneous case. In the non-homogeneous space there is no reason to introduce a fixed set of frame vectors, also independent of the Kasner axes.
The rules for the process of replacement of two Kasner epochs is similar to the (1.4.18) as

$$-R_l^l = \frac{(\dot{a}bc)^{\cdot}}{abc} + \lambda^2 \frac{a^2}{2b^2c^2} = 0 \tag{1.4.66a}$$

$$-R_m^m = \frac{\left(a\dot{b}c\right)^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0 \tag{1.4.66b}$$

$$-R_n^n = \frac{(ab\dot{c})^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0 \tag{1.4.66c}$$

while equation (1.4.19) remains the same. The difference from the corresponding former set explored for the homogeneous models relies in the expression for λ

$$\lambda = \frac{\mathbf{l} \cdot \nabla \wedge \mathbf{l}}{\mathbf{l} \cdot [\mathbf{m} \times \mathbf{n}]} \tag{1.4.67}$$

which is no longer a constant but a function of the coordinates: since the set (1.4.66a) is still a system of differential equations with respect to time, this difference doesn't affect the solution of the alternation rule of Kasner exponents deriving from it remains the same as (1.4.34) and following (1.4.35a)-(1.4.35b). Despite of this, during the evolution the turning of the Kasner axes is reduced to the appearance in the final epoch of off-diagonal projections of the metric tensor g_{lm}, g_{ln}, g_{mn}.

The general solution of homogeneous models contains the largest possible number of unremovable arbitrary constants. This solution is contained as a particular case in the general inhomogeneous solution. The homogeneous model can also admit some particular solutions of the equations, with eventually a lesser number of arbitrary constants, but such particular homogeneous solutions are not contained in the general inhomogeneous one, belonging to certain inhomogeneous classes of solutions of a lesser degree of generality. The metric involves the spatial displacements only because the spatial derivatives don't influence the character of the solution: metric changes with time in every point of the space with its own Kasner exponents and axes.

The choice adopted of considering all quantities in the synchronous reference system in which the singularity is attained simultaneously at $t = 0$ in the entire space doesn't affect the generality of the solution, which by itself is the property guaranteeing its robustness.

The spontaneous stochastization of the solution, as outlined in Section 1.4.2 and 1.4.3 and more deeply discussed in the following Chapter 2, means that on sufficient recession towards the singularity from the moment $t = t_0 > 0$ at which initial conditions are imposed these conditions are forgotten and the evolution admits a statistical description, as discussed in Section 1.4.3. The stability of the solution with respect of the choice of initial conditions as well as its generality remains as the notion describing the existence of the oscillatory regime and the described asymptotic stochastic properties.

2 Hamiltonian Formulation and Mixmaster Chaos

2.1 Mixmaster Dynamics in Misner Variables

Up to now we have considered as "chaotic" the dynamics of a system if, after some "time" of evolution the initial conditions are "forgotten": we have used so many quotes because we will show how such common terms for the description of dynamical properties have to be accurately specified for a task in General Relativity.
The sense of chaoticity has appeared for the discussion of the parameter u as a multiple fraction expansion; irrationality of the initial value reflects evolution towards a singularity and mixing properties of the general solution. The complexity of the sequence in terms of quasi-similar eras has been described through the choice of a very specific time gauge and we will show, in a different framework, how the chaoticity features are independent of such a choice (Imponente and Montani, 2001; Imponente and Tavakol, 2003; Imponente and Montani, 2004a, 2002b, 2003c).
Apart from the conclusions of the previous Section 1.4.2, the structure of the Einstein field equations prevents from saying anything about relevant classes of solution *not* having asymptotic behaviour corresponding to the BKL scenario, essentially due to the choice of a local system of coordinates. In virtue of this we take the motivations for the next Chapters, checking the covariance or the properties found up to now for the Bianchi models of type VIII and IX close to the singularity and their link far from it.

2.1.1 Towards Continuous Kasner Dynamics

In order to implement the formalism to our purposes we start allowing a more general time dependence for the parameters p_i, preserving the simple conditions (1.4.9) (Arnowitt et al., 1962). For a metric of the kind (1.3.2), (1.3.4), (1.4.8), such exponents satisfy the relations

$$p_1 \equiv \frac{d\ln g_{11}}{d\ln g} \tag{2.1.1a}$$

$$p_2 \equiv \frac{d\ln g_{22}}{d\ln g} \tag{2.1.1b}$$

$$p_3 \equiv \frac{d\ln g_{33}}{d\ln g} \tag{2.1.1c}$$

in order to parametrize the spatial 3×3 part as

$$g_{ij} = e^{2\alpha}\left(e^{2\beta}\right)_{ij} \tag{2.1.2a}$$

or, equivalently,

$$(\ln g)_{ij} = 2\alpha\delta_{ij} + 2\beta_{ij} \tag{2.1.2b}$$

where β_{ij} is a three-dimensional matrix with null trace of the kind $\mathrm{diag}(\beta_{11}, \beta_{22}, \beta_{33})$ and the exponential matrix has to be intended as a power series of matrices, so that

$$\det e^{2\beta} = e^{2\ \mathrm{tr}\beta} = 1 \tag{2.1.3}$$

and

$$g = e^{6\alpha}\,; \tag{2.1.4}$$

from (2.1.4) follow the relations

$$\sqrt{g} = e^{3\alpha} \tag{2.1.5a}$$

and

$$\ln g = 6\alpha\,, \tag{2.1.5b}$$

considering the structure formalism

$$p_{ij} = \frac{d\,(\ln g)_{ij}}{d\ln\det g}\,. \tag{2.1.6}$$

From equations (2.1.2a) and (2.1.5a) follows

$$p_{ij} = \frac{1}{3}\left[\delta_{ij} + \left(\frac{d\beta_{ij}}{d\alpha}\right)\right] \tag{2.1.7}$$

so that the first Kasner condition (1.4.9) rewrites as

$$1 = \sum_i p_i \equiv \text{tr}\ p_{ij} = 1 + \frac{1}{3}\text{tr}\left(\frac{d\beta}{d\alpha}\right) \tag{2.1.8}$$

which is an identity in view of being the trace $\text{tr}\beta_{ij} = 0$.
The second Kasner relation (1.4.9) rewrites

$$\text{tr}\left(p^2\right) = 1 \tag{2.1.9}$$

which, by virtue of (2.1.7) becomes

$$\frac{1}{9}\text{tr}\left(1 + 2\frac{d\beta_{ij}}{d\alpha} + \left(\frac{d\beta_{ij}}{d\alpha}\right)^2\right) = \frac{1}{3} + \frac{1}{9}\left(\frac{d\beta_{ij}}{d\alpha}\right)^2 = 1\,, \tag{2.1.10}$$

and then

$$\left(\frac{d\beta_{ij}}{d\alpha}\right)^2 = 6 \tag{2.1.11}$$

which is no longer an identity but a consequence of the Einstein equations in empty space.

2.1.2 Bianchi I Example

For example, in the simple Bianchi I case, with the choice of a matrix β_{ij} diagonal, metric is explicitly

$$ds^2 = -dt^2 + e^{2\alpha}\left(e^{2\beta}\right)_{ij} dx^i dx^j\,. \tag{2.1.12}$$

Following on our discussion, Einstein equations $0-0$ and ij within this system of coordinates are

$$-3\frac{d^2\alpha}{dt^2} = {}^{(3)}R + 9\left(\frac{d\alpha}{dt}\right)^2 \tag{2.1.13a}$$

$$\frac{{}^{(3)}R}{3}\delta^\tau_\sigma - e^{-3\alpha}\frac{d}{dt}\left[e^{3\alpha}\left(\frac{d\beta^\tau_\sigma}{dt}\right)\right] = 8\pi\left(T^\tau_\sigma - 4\pi T\delta^\tau_\sigma\right)\,, \tag{2.1.13b}$$

respectively, being ${}^{(3)}R$ the three-dimensional curvature scalar.

First and second derivatives of the function α with respect of the synchronous time t re-express as

$$3\ddot{\alpha} = -6\dot{\alpha}^2 - \frac{1}{2}\text{tr}\left(\frac{d\beta_{ij}}{d\alpha}\right)^2 - \frac{R}{2} \tag{2.1.14}$$

$$3\dot{\alpha}^2 - \frac{1}{2}\text{tr}\left(\frac{d\beta_{ij}}{dt}\right)^2 = 8\pi T^0_0\,, \tag{2.1.15}$$

and finally inserted in the Einstein equations lead to

$$\left(\frac{d\alpha}{dt}\right)^2 = \frac{8\pi}{3}\left[T_0^0 + \frac{1}{16\pi}\mathrm{tr}\left(\frac{d\beta_{ij}}{dt}\right)^2\right] \tag{2.1.16}$$

$$e^{3\alpha}\frac{d}{dt}\left(e^{-3\alpha}\frac{d\beta^i_j}{dt}\right) = 8\pi\left(T^i_j - \frac{1}{3}\,{}^{(3)}T\delta^i_j\right) \tag{2.1.17}$$

where ${}^{(3)}T$ is the three-dimensional part of the trace of the tensor T^a_b. The choice of the metric in a diagonal form, as discussed in details in the previous Section 1.3 gives a redundant character to the other non-diagonal terms of the energy-momentum tensor. The second Kasner condition

$$\sum {p_i}^2 = 1 \tag{2.1.18}$$

in terms of the new variables β_{ij} then becomes

$$\left(\frac{d\beta_{ij}}{dt}\right)^2 = 6\,. \tag{2.1.19}$$

2.1.3 Misner Approach to Mixmaster

Once seen how to perform the convenient change of variables for such a diagonal case, we step further to approach with this formalism the Mixmaster case, say the Bianchi types VIII and IX.
The matrix β_{ij} has only two independent components and we adopt the parametrization in terms of the anisotropy parameters

$$\beta_{11} = \beta_+ + \sqrt{3}\beta_- \tag{2.1.20a}$$
$$\beta_{22} = \beta_+ - \sqrt{3}\beta_- \tag{2.1.20b}$$
$$\beta_{33} = -2\beta_+\,. \tag{2.1.20c}$$

Then the Kasner relation (2.1.19) becomes

$$\left(\frac{d\beta_+}{d\alpha}\right)^2 + \left(\frac{d\beta_-}{d\alpha}\right)^2 = 1\,. \tag{2.1.21}$$

The variables $\beta_\pm$ together with α are the *Misner coordinates*. The relation (2.1.21) in terms of the Kasner exponents now is

$$\frac{d\beta_+}{d\alpha} = \frac{1}{2}(1-3p_3) \tag{2.1.22a}$$
$$\frac{d\beta_-}{d\alpha} = \frac{1}{2}\sqrt{3}\,(p_1-p_2) \tag{2.1.22b}$$

and with the u parameter

$$\frac{d\beta_+}{d\alpha} = -1 + \frac{3}{2}\frac{1}{1+u+u^2} \tag{2.1.23a}$$

$$\frac{d\beta_-}{d\alpha} = -\frac{1}{2}\sqrt{3}\frac{1+2u}{1+u+u^2}\,. \tag{2.1.23b}$$

Such quantities represent the *anisotropy velocity* β'

$$\beta' \equiv \left(\frac{d\beta_+}{d\alpha}, \frac{d\beta_-}{d\alpha}\right) \tag{2.1.24}$$

which is a measure for the variation in the anisotropy amount with respect to the expansion parametrized by the α parameter. The volume of the universe behaves as $e^{3\alpha}$ and tends to zero towards the singularity and the temporal parameter itself is directly related.
Eventually the presence of matter as well the effects of the spatial curvature can lead the norm $||\beta'||$ to a deviation from the Kasnerian unity.

In order to develop a general metric for a homogeneous space-time we rewrite the line element in the general form

$$ds^2 = -N(\eta)^2 d\eta^2 + e^{2\alpha}(e^{2\beta})_{ij}\sigma^i\sigma^j \tag{2.1.25}$$

where $N(\eta)$ denotes the *lapse function* which measures the lapse of proper time between hypersurfaces corresponding to different values of the temporal parameter η, σ^i are the 1-forms associated to the angular basis for the rotation group associated to the homogeneity constraint.
In order to discuss homogeneous spaces, the cosmological problem reduces to the equations involving the functions α, N, β_{ij} in terms of the independent temporal parameter η, independently of the spatial coordinates. Explicitly, dual 1-forms associated to the Bianchi types VIII and IX are, respectively,

$$(VIII) \quad \begin{aligned} \sigma^1 &= -\sinh\psi\sinh\theta d\phi + \cosh\psi d\theta \\ \sigma^2 &= -\cosh\psi\sinh\theta d\phi + \sinh\psi d\theta \\ \sigma^3 &= \cosh\theta d\phi + d\psi \end{aligned} \tag{2.1.26a}$$

$$(IX) \quad \begin{aligned} \sigma^1 &= \sin\psi\sin\theta d\phi + \cos\psi d\theta \\ \sigma^2 &= -\cos\psi\sin\theta d\phi + \sin\psi d\theta \\ \sigma^3 &= \cos\theta d\phi + d\psi\,. \end{aligned} \tag{2.1.26b}$$

For example, the Einstein equation involving $T^0_{\ 0}$ with $N = 1$ reads

$$3\dot{\alpha}^2 - \frac{1}{2}\left(\frac{\beta_{ij}}{dt}\right)^2 + \frac{1}{2}P^{\alpha}_{\ \alpha} = 8\pi T^0_{\ 0} \tag{2.1.27}$$

being P^{α}_{α} the three-dimensional curvature tensor and then

$$3\left(\dot{\alpha}^2 - \dot{\beta_+}^2 - \dot{\beta_-}^2\right) + \frac{1}{2}\left({}^3R_B\right) = 8\pi T^0_{\ 0} \tag{2.1.28}$$

where ${}^3R_B$ is the curvature scalar for the three-dimensional spatial surface corresponding to $t =$ const. and index B refers to the symmetry properties for the Bianchi cosmological models. In such a term lies the peculiar difference between the nine types of the Bianchi classification, to be evaluated through expressions (1.4.4) in terms of the structure constants.

For the models referring to types VIII and IX such curvature scalar reads, respectively,

$$ {}^3R_{VIII} = \frac{1}{2}e^{-2\alpha}\left(4e^{-2\beta_3} - 2\text{tr } e^{-2\beta} - \text{tr } e^{4\beta}\right) \tag{2.1.29a}$$

$$ {}^3R_{IX} = \frac{1}{2}e^{-2\alpha}\text{tr}\left(2e^{-2\beta} - e^{4\beta}\right) \tag{2.1.29b}$$

and the trace operation has to be intended over the exponential of diagonal matrices, without ambiguity.
Equation (2.1.28) with (2.1.29b) or (2.1.29a) can be interpreted as a contribution of anisotropy energy, connected to the term $T^0_{\ 0}$, to the volume expansion $\dot{\alpha}^2$, so that it appears as a potential term together with the kinetic ones $\dot{\beta}^2_{\pm}$. Close to the singularity, this term is negligible for small values of the anisotropy parameters $\beta_{\pm}$.
Finally, equation (2.1.28) has to be regarded as a fundamental constraint over the field equations.

2.2 Lagrangian Formulation

The variational principle that we are going to use specified for the Bianchi cosmological models has the general expression

$$I = -\frac{1}{16\pi}\int \mathcal{L}\, d^4x = -\frac{1}{16\pi}\int R\sqrt{-g}\, d\Omega \tag{2.2.1}$$

where $\mathcal{L}$ is the four-dimensional Lagrangian density. Integration with respect to the spatial coordinates leads to the integral form

$$\delta \int_{\eta_1}^{\eta_2} L \, d\eta = 0 \tag{2.2.2}$$

in which η_1 and η_2 ($\eta_2 > \eta_1$) are two values of the temporal coordinate. Integration for the Bianchi type VIII is considered over a spatial volume $(4\pi)^2$, in order to have the same integration constant used for the type IX (and keep a uniform formalism) using

$$\int \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \int \sin\theta d\phi \wedge d\theta \wedge d\psi = (4\pi)^2 \ . \tag{2.2.3}$$

With this choice, Lagrangian L is written

$$L = -\frac{6\pi}{N} e^{3\alpha} \left[{\alpha'}^2 - {\beta_+'}^2 - {\beta_-'}^2 \right] + N \frac{\pi}{2} e^{\alpha} U^{(B)} \left(\beta_+, \beta_- \right) \tag{2.2.4}$$

where $()' = \frac{d}{d\eta}$, $U^{(B)}$ is a function linear in $R^{(B)}$ with a potential role. The variational principle rewrites explicitly

$$\delta I = \delta \int \Big(p_\alpha \alpha' + p_+ {\beta_+}' + p_- {\beta_-}' - N\mathcal{H} \Big) d\eta = 0 \tag{2.2.5}$$

in which $\mathcal{H}$ represents a *super Hamiltonian* given in detail as

$$\mathcal{H} = \frac{e^{-3\alpha}}{24\pi} \Big(-p_\alpha^2 + p_+^2 + p_-^2 + \mathcal{V} \Big) \tag{2.2.6}$$

and the potential term $\mathcal{V}$ as

$$\mathcal{V} = -12\pi^2 e^{4\alpha} U^{(B)} \left(\beta_+, \beta_- \right) \tag{2.2.7}$$

having $U^{(B)}$ specified for the two Bianchi models under study as

$$U^{VIII} = e^{-8\beta_+} + 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} \Big(\cosh(4\sqrt{3}\beta_-) - 1 \Big) \tag{2.2.8a}$$

$$U^{IX} = e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} \Big(\cosh(4\sqrt{3}\beta_-) - 1 \Big) \ . \tag{2.2.8b}$$

From Lagrangian (2.2.4) it is standard to derive the conjugate momenta

$$p_\alpha = \frac{\partial L}{\partial \alpha'} = -\frac{12\pi}{N} e^{3\alpha} \alpha' \tag{2.2.9a}$$

$$p_\pm = \frac{\partial L}{\partial {\beta_\pm}'} = \frac{12\pi}{N} e^{3\alpha} {\beta_\pm}' \ . \tag{2.2.9b}$$

2.2.1 ADM Hamiltonian

In order to obtain Einstein equations, the variational principle requires δI to be null for arbitrary and independent variations of $p_\pm, p_\alpha, \beta_\pm, \alpha, N$. Variation with respect to N leads to the fundamental equation

$$\mathcal{H} = 0 \tag{2.2.10}$$

which behaves as a constraint for the Hamilton equations. Such a standard formulation treated with the method introduced by Arnowitt, Deser and Misner (the so-called ADM) permits to reduce the variational principle to the canonical Hamiltonian form. The procedure prescribes the choice of one of the field variables, or one of the momenta, as a temporal coordinate and subsequently solving the constraint (2.2.10) with respect to the corresponding conjugate quantity.
It is customary, as in this general approach, to set $t = \alpha$ and solve $\mathcal{H} = 0$ as

$$\mathcal{H}_{ADM} = -p_\alpha = \sqrt{p_+^2 + p_-^2 + \mathcal{V}}\,. \tag{2.2.11}$$

Within this equation it is defined a relation between the temporal gauge described by the function N and the dynamical quantity $\mathcal{H}_{ADM}$.

Through (2.2.11) we explicit p_α in the action integral, so that the reduced variational principle in a canonical form reads

$$\delta I_{\text{reduced}} = 0 \tag{2.2.12}$$

being $I_{reduced}$ written as

$$I_{reduced} = \int \left(p_+ d\beta_+ + p_- d\beta_- - \mathcal{H}_{ADM} d\alpha\right) \tag{2.2.13}$$

together with the equation defining the temporal gauge.

2.2.2 Mixmaster Dynamics

In the present Section we will write in general the approach to the Mixmaster dynamics and later in it will be applied to prove specific properties, such as chaoticity in a covariant approach, with respect to the temporal gauge and subsequent statistical effects.

The Hamiltonian introduced so far differs from the typical expression of classical mechanics for the non positive definiteness of the kinetic term, i.e. the sign in front of p_α^2, and for the peculiar form of the potential as a function of α (say *time*) and $\beta_\pm$, reduced to the study of a function of the kind $V(\beta_+, \beta_-)$.

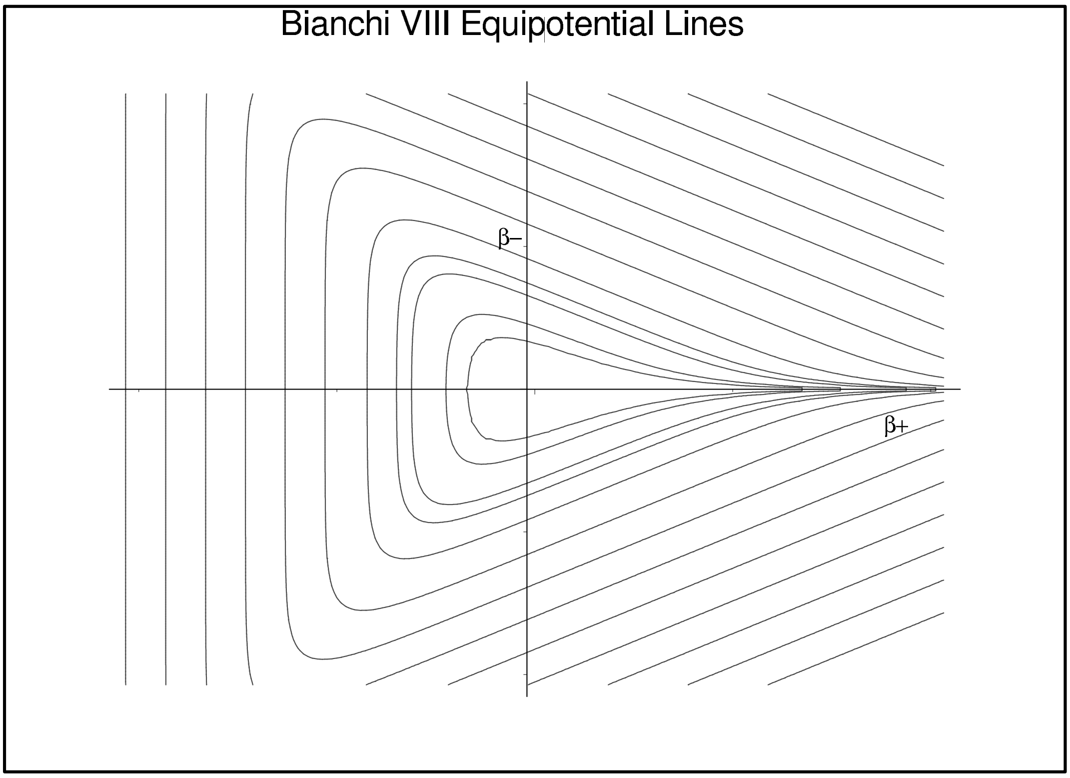

Figure 2.1: Equipotential lines of the Bianchi type VIII model in the β_+, β_- plane.

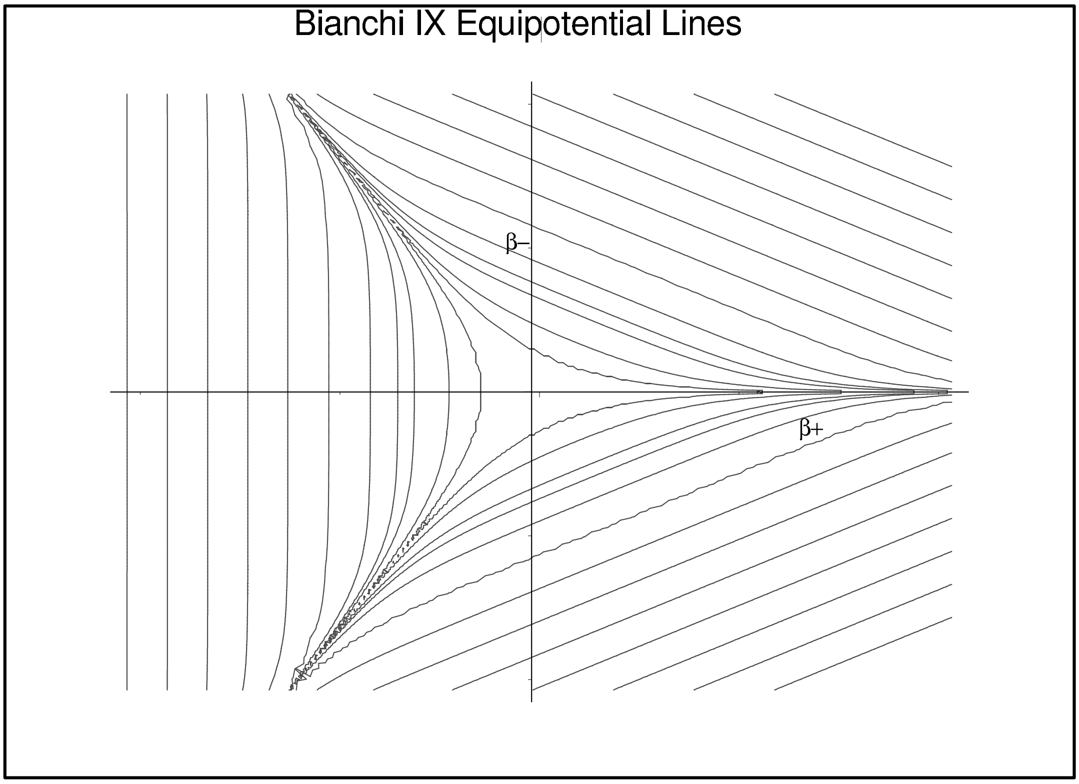

Figure 2.2: Equipotential lines of the Bianchi type IX model in the β_+, β_- plane.

Hamiltonian approach permits to derive the equations of motion as

$$\alpha' = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad p'_\alpha = -\frac{\partial \mathcal{H}}{\partial \alpha}, \tag{2.2.14a}$$

$$\beta'_\pm = \frac{\partial \mathcal{H}}{\partial p_\pm}, \quad p'_\pm = -\frac{\partial \mathcal{H}}{\partial \beta_\pm}. \tag{2.2.14b}$$

This set considered with the explicit form of the potential (see Figure 2.1 and 2.1), can be interpreted as the motion of a "point-particle" in a potential. The term $\mathcal{V}$ is proportional to the curvature scalar and describes the anisotropy of the universe, i.e. in the regions of the configuration space where it can be negligible, the dynamics resembles the pure Kasner behaviour, corresponding to $|\beta'| = 1$. In general, it is necessary a detailed study of the potential form and each Bianchi model has its own potential form, depending on the specific structure constants.

U behaves, for the Bianchi types VIII and IX, as a potential wall with the same symmetries of an equilateral triangle in the plane β_+, β_-.

Asymptotically close to the origin, i.e. $\beta_\pm = 0$, equipotential lines for the

Bianchi type IX are circles

$$U^{IX}(\beta_+,\beta_-) \simeq -3 + \left({\beta_+}^2 + {\beta_-}^2\right) + o\left(\beta^3\right), \qquad (2.2.15)$$

while for Bianchi type VIII are ellipses

$$U^{VIII}(\beta_+,\beta_-) \simeq \left(40{\beta_+}^2 + 24{\beta_-}^2\right) - 8\left(\beta_+ + \beta_-\right) + 5 + o\left(\beta^3\right). \qquad (2.2.16)$$

The expressions for large values of β are common for both types

$$U(\beta) \simeq e^{-8\beta_+}, \qquad \beta_+ \longrightarrow -\infty \qquad (2.2.17)$$

or

$$U(\beta) \simeq 48{\beta_-}^2 e^{4\beta_+}, \qquad \beta_+ \longrightarrow +\infty \qquad (2.2.18)$$

when

$$\mid \beta_- \mid \ll 1\,. \qquad (2.2.19)$$

In the figures 2.1 2.2 are represented some of the equipotential lines $U(\beta) =$ const., for which corresponding to $\Delta\beta \sim 1$ the potential value has an increment of a factor $e^8 \sim 3 \times 10^3$.
Universe evolution is described as the motion of a point-like particle under the influence of such potentials. Such evolution corresponds to bounces on the potential walls evolving towards the singularity. The behaviour of the anisotropy parameters $\beta_\pm$ in this regime consists of a Kasner epoch followed by a bounce and then a new epoch with different Kasner parameters, in correspondence with the description given in Section 1.4.2 according to the BKL approach.

2.3 Covariant Mixmaster Chaos

2.3.1 The Fundamental Questions on the Mixmaster

We have discussed the oscillatory regime whose properties characterize the behaviour of the Bianchi types VIII and IX cosmological models in the BKL formalism (Belinski et al., 1970, 1982; Misner, 1969) near a physical singularity, in which it is outlined the appearance of chaotic properties: firstly, the dynamics evolution for Kasner exponents characterized the sequence of Kasner epochs, each one described by its own line element, with the epochs sequence nested in multiple eras. Secondly, the use of parameter u and its relation to dynamical functions offered the statistical treatment connected to each Kasner

era, finding an appropriate expression for the distribution over the space of variation: the entire evolution has been decomposed in a discrete mapping in terms of the rational/irrational initial value attributed to u.
The limits of this approach reside essentially in the non-continuous evolution of the description towards the initial singularity and the lack of an assessment of chaoticity in accordance with the indicators commonly used in the theory of dynamical systems, say in terms of the criterion based on the estimate of Lyapunov exponents.
A wide literature faced over the years this subject in order to provide the best possible understanding of the resulting chaotic dynamics (Belinski, 2000; Berger, 1990).

The research activity developed overall in two different, but related, directions:

(i) on one hand the dynamical analysis was devoted to remove the limits of the BKL approach due to its discrete nature (by analytical treatments Barrow (1982); Chernoff and Barrow (1983); Burd et al. (1991); Cornish and Levin (1997a,b); Kirillov and Montani (1997a); Montani (2000a) and numerical simulations (Berger, 1991, 1994; Berger et al., 1997),

(ii) on the other one to get a better characterization of the Mixmaster chaos (especially in view of its properties of covariance (Ferraz et al., 1991; Ferraz and Francisco, 1992; Szydlowski and Krawiec, 1993; Hobill et al., 1994).

The first line of investigation provided satisfactory representations of the Mixmaster dynamics in terms of continuous variables (Bogoyavlenskii and Novikov, 1973), mainly studying the properties of the BKL map and its reformulation as a Poincaré one Barrow (1981).

Parallely to these studies has been performed detailed numerical descriptions allowing to make precise validity tests on the obtained analytical results Berger (1994).

The efforts (Biesiada, 1995; Contopoulos et al., 1994; Rugh, 1994) to develop a precise characterization of the chaoticity observed in the Mixmaster dynamics found non-trivial difficulties due to the impossibility, or in the best cases the ambiguity, to apply the standard chaos indicators to relativistic systems. However, chaotic properties summarized so far were questioned when numerical evolution of the Mixmaster equations yielded zero Lyapunov exponents (Burd et al., 1990; Hobill et al., 1989; Francisco and Matsas, 1988). Nevertheless, exponential divergence of initially nearby trajectories was found by other numerical studies yielding positive Lyapunov numbers. This issue was understood when by Berger (1991) and Francisco and Matsas (1991) was shown numerically and analytically how such calculations depend on the choice of the time variable and parallely to the failure in the conservation of the Hamiltonian constraint in the numerical analysis by Zardecki (1983), but was still debated by Hobill et al. (1994).

In particular, the first clear distinction between the direct numerical study of the dynamics and the map approximation, stating the appearance of chaos and its relation with the increase of entropy has been introduced by Burd et al. (1991). The puzzle consisted simulations providing even in the following years zero (Ferraz and Francisco, 1992) Lyapunov numbers, claiming that the Mixmaster universe is non chaotic with respect to the intrinsic time (associated with the function α introduced for the Hamiltonian formalism) but chaotic with respect to the synchronous time (the temporal parameter t). The non-zero claims (Szydlowski and Krawiec, 1993) about Lyapunov exponents, using different time variables, have been obtained reducing the universe dynamics to a geodesic flow on a pseudo-Riemannian manifold: on average, local instability has been discussed for the BKL approximations. Nevertheless, a geometrized model of dynamics defining an average rate of separation of nearby trajectories in terms of a geodesic deviation equation in a Fermi basis has been interpreted for detection of chaotic behaviour as a principal Lyapunov exponent. A non definitive result was given: the principal Lyapunov exponents result always positive in the BKL approximations but, if the period of oscillations in the long phase (the evolution of long oscillations, say when the particle enters the corners of the potential) is infinite, the principal Lyapunov exponent tends to zero.

Such contrasting results have found a clear explanation realizing the non-covariant nature of these indicators and their inapplicability to hyperbolic manifolds (Gurzadyan and Kocharyan, 1986). The existence of such difficulties prevented, up to now, to say a definitive word about the covariance of the Mixmaster chaos, with particular reference to the possibility of removing the observed chaotic features by a suitable choice of the time variable, apart from the indication provided by Cornish and Levin (1997a,b) (a detailed discussion about the method based on a fractality property used in such work will be discussed in details Imponente and Tavakol (2003)).

Interest in these covariance aspects has increased in recent years in view of the contradictory and often dubious results that have emerged on this topic. The confusion which arises regarding the effect of a change of the time variable in this problem depends on some special properties of the Mixmaster model when represented as a dynamical system, in particular the vanishing of the Hamiltonian and its non-positive definite kinetic terms (a typical feature of a gravitational system). These peculiar features prevent the direct application of the most common criteria provided by the theory of dynamical systems for characterizing chaotic behaviour (for a review, see Hobill et al. (1994)).

Although a whole line of research opened up, following this problem of covariant characterization for the chaos in the Mixmaster model (Pullin, 1991; Szydlowski and Szczesny, 1994), the first widely accepted indications in favour of covariance were derived with a fractal formalism by Cornish and Levin (1997*a*,*b*) (see also Motter and Letelier (2001*b*)). Indeed the requirement of a complete covariant description of the Mixmaster chaoticity when viewed in terms of continuous dynamical variables, due to the discrete nature of the fractal approach, leaves this subtle question open and prevents a general consensus in this sense from being reached.

2.3.2 Anisotropy Parameters and Misner-Chitre-like Variables

A valuable framework of analysis of the Mixmaster evolution, able to join together the two points of view of map approach and continuous dynamics evolution, relies on a Hamiltonian treatment of the equations in terms of Misner-Chitre variables (Chitre, 1972). This formulation allows to individualize the existence of an asymptotic (energy-like) constant of motion when performed an ADM reduction. By this result the stochasticity of the Mixmaster can be treated either in terms of the statistical mechanics (by the microcanonical ensemble) (Imponente and Montani, 2002*a*), either by its characterization as isomorphic to a billiard on a two-dimensional Lobachevsky space (Arnold, 1989*b*) and such scheme can be constructed independently of the choice of a time variable, simply providing very general Misner-Chitre-like coordinates (Imponente and Montani, 2001, 2002*b*).

Let us define the anisotropy parameters H_i $(i = 1, 2, 3)$ as in (Kirillov, 1993; Kirillov and Melnikov, 1995), (Montani, 1995; Kirillov and Montani, 1997*a*) as the functions

$$H_1 = \frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3\alpha} \tag{2.3.1a}$$

$$H_2 = \frac{1}{3} + \frac{\beta_+ - \sqrt{3}\beta_-}{3\alpha} \tag{2.3.1b}$$

$$H_3 = \frac{1}{3} - \frac{2\beta_+}{3\alpha}\,, \tag{2.3.1c}$$

excluding the pathological cases when two or three anisotropy parameters H_i coincide.

We then introduce the Misner-Chitre variables $\{\tau, \zeta, \theta\}$ as

$$\alpha = -e^{\tau}\cosh\zeta \tag{2.3.2a}$$

$$\beta_+ = e^{\tau}\sinh\zeta\cos\theta \tag{2.3.2b}$$

$$\beta_- = e^{\tau}\sinh\zeta\sin\theta \tag{2.3.2c}$$

where $0 \leq \zeta < \infty$, $0 \leq \theta < 2\pi$, and τ plays the role of a "radial" coordinate coming out from the origin of the $\beta_\pm$ space (MISNER ET AL., 1973). In terms of these variables it is possible to study the first interesting approximation of the potential (2.2.8) as independent of τ towards the singularity, i.e. for $\alpha \to -\infty$.

To discuss the contrasting results concerning chaoticity and dynamical properties which arose from numerics, it is necessary to introduce a slight modification to the set (2.3.2) via the Misner-Chitre–like coordinates $\{\Gamma(\tau), \xi, \theta\}$ through the transformation

$$\alpha = -e^{\Gamma(\tau)}\xi \tag{2.3.3a}$$

$$\beta_+ = e^{\Gamma(\tau)}\sqrt{\xi^2 - 1}\cos\theta \tag{2.3.3b}$$

$$\beta_- = e^{\Gamma(\tau)}\sqrt{\xi^2 - 1}\sin\theta \tag{2.3.3c}$$

where $1 \leq \xi < \infty$, and $\Gamma(\tau)$ stands for a *generic* function of τ: Chitre took simply $\Gamma(\tau) \equiv \tau$ and set also $\xi = \cosh\zeta$.

This modified set of variables permits to express the anisotropy parameters (2.3.1) H_i $(i = 1, 2, 3)$ as *independent* of the variable Γ

$$H_1 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}\left(\cos\theta + \sqrt{3}\sin\theta\right) \tag{2.3.4a}$$

$$H_2 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}\left(\cos\theta - \sqrt{3}\sin\theta\right) \tag{2.3.4b}$$

$$H_3 = \frac{1}{3} + 2\frac{\sqrt{\xi^2 - 1}}{3\xi}\cos\theta\,. \tag{2.3.4c}$$

All dynamical quantities, if expressed in terms of (2.3.4) will be independent of τ *too.*

2.3.3 The Invariant Measure

The first results after the pioneeristic ones obtained by Lifshitz et al. (1970) over the statistical distribution function describing the system rely in the work of Barrow Barrow (1981); Chernoff and Barrow (1983), stating how it is not necessary that random or chaotic behaviour in dynamical systems depends on the distribution of initial data: moreover, also simple recursive systems, like iterated maps over the unit interval, are very sensitive to the initial data so that the evolution itself has to be considered unpredictable. In particular, the first analysis of the BKL map in terms of u has been reduced to a Poincaré one, for the sequence of Kasner states coded as a one dimensional map with a sensitive dependence on initial conditions: two Mixmaster universes beginning arbitrarily close to each other will diverge exponentially fast as they evolve. For such map is has been derived the invariant normalized measure $\mu_0(\theta)$ simply as

$$\mu_0 = \frac{1}{(\theta+1)\ln 2}, \qquad x_k = (\theta+k)^{-1} \tag{2.3.5}$$

where x_k and k are the parameters introduced in Section 1.4.3. This quantity permits to derive the information loss under the iteration in the u sequence, evaluated for the measure μ_0, represents exactly the metric (or K–) entropy that, for the system under consideration, is evaluated to be

$$h(T,\mu_0) = \frac{\pi^2}{6(\ln 2)^2} \tag{2.3.6}$$

where T specifies the mapping and the positive value of h gives a chaotic indication for the system evolution, still in the map framework (Moser and ans S. Varadhan, 1975*a*,*b*): this is an ergodic characterization for the Mixmaster universe. In Chernoff and Barrow (1983) it has been constructed the invariant measure for the system with a peculiar application to all degrees of freedom involved in the Mixmaster dynamics, regarding the special case of the finite expansion of Kasner parameters with continuous fraction as a non-physically interesting one. In Kirillov and Montani (1997*a*) the use of Misner-Chitre variables (Chitre, 1972) (see above Section 2.3.9) has permitted a reduction of the invariant measure in the *continuous* approach containing explicit informations about durations of Kasner eras, while the measure in the case of BKL map didn't. Complete measure theoretic distributions, describing statistical properties of the BKL map were constructed already in LIFSHITZ ET AL. (1970); KHALATNIKOV ET AL. (1983). In fact, the treatment in KIRILLOV AND MONTANI (1997*a*) poses a direct correspondence between the anisotropy functions Q_1, Q_2, Q_3 (KIRILLOV, 1993; KIRILLOV AND MELNIKOV, 1995; MONTANI, 1995) and the Kasner exponents as

$$\begin{aligned} Q_1(u,v) &= \frac{-u}{u^2+u+1+v^2} \\ Q_2(u,v) &= \frac{1+u}{u^2+u+1+v^2} \\ Q_3(u,v) &= \frac{u(u+1)+v^2}{u^2+u+1+v^2} \end{aligned} \tag{2.3.7}$$

where the v parametrizes the inverse sequence of Kasner eras and the Q_i satisfy

$$\sum_{a=1}^{3} Q_a = \sum_{a=1}^{3} (Q_a)^2 = 1\,, \tag{2.3.8}$$

similarly to (1.4.9).

2.3.4 The Hamiltonian Equations

The main advantage relying in the reformulation of the dynamics as a chaotic scattering process consists of replacing the discrete BKL map by a geodesic flow in a space of continuous variables (CHERNOFF AND BARROW, 1983; KIRILLOV AND MONTANI, 1997*a*)– (BARROW, 1982; MONTANI, 2000*b*), (IMPONENTE AND MONTANI, 2003*c*).

The canonical variational principle (2.2.2) describing the dynamics in the Misner variables (MISNER, 1969; MISNER ET AL., 1973) has explicitly the Lagrangian L written

$$L = \frac{6D}{N}\left[-\alpha'^2 + {\beta_+}'^2 + {\beta_-}'^2\right] - \frac{N}{D} V\left(\alpha, \beta_+, \beta_-\right). \tag{2.3.9}$$

in which the metric determinant is $D \equiv e^{3\alpha}$ and the function representing the potential $V\left(\alpha, \beta_+, \beta_-\right)$ reads as

$$\begin{aligned} V = \frac{1}{2}\Big(D^{4H_1} + D^{4H_2} + D^{4H_3}\Big) + \\ - D^{2H_1+2H_2} \pm D^{2H_2+2H_3} \pm D^{2H_3+2H_1}, \end{aligned} \tag{2.3.10}$$

where $(+)$ and $(-)$ refer respectively to Bianchi type VIII and IX.

In terms of (2.3.3) the Lagrangian (2.3.9) becomes

$$L = \frac{6D}{N}\left[\frac{\left(e^{\Gamma}\xi'\right)^2}{\xi^2-1} + \left(e^{\Gamma}\theta'\right)^2\left(\xi^2-1\right) - \left(e^{\Gamma}\right)'^2\right] + \\ - \frac{N}{D}V\left(\Gamma\left(\tau\right),\xi,\theta\right). \tag{2.3.11}$$

In terms of $\Gamma(\tau)$, ξ and θ we have D as

$$D = \exp\left\{-3\xi e^{\Gamma(\tau)}\right\}\ , \tag{2.3.12}$$

and since $D \to 0$ towards the singularity, independently of its particular form, the only property required for Γ is to approach infinity in this limit.
The Lagrangian (2.3.9) leads to the conjugate momenta

$$p_\tau = -\frac{12D}{N}\left(e^{\Gamma}\frac{d\Gamma}{d\tau}\right)^2\tau' \tag{2.3.13a}$$

$$p_\xi = \frac{12D}{N}\frac{e^{2\Gamma}}{\xi^2-1}\xi' \tag{2.3.13b}$$

$$p_\theta = \frac{12D}{N}e^{2\Gamma}\left(\xi^2-1\right)\theta' \tag{2.3.13c}$$

which by a Legendre transformation make the variational principle assume the form (Imponente and Montani, 2001)

$$\delta\int\left(p_\xi\xi' + p_\theta\theta' + p_\tau\tau' - \frac{Ne^{-2\Gamma}}{24D}\mathcal{H}\right)d\eta = 0, \tag{2.3.14}$$

where

$$\mathcal{H} = -\frac{{p_\tau}^2}{\left(\frac{d\Gamma}{d\tau}\right)^2} + {p_\xi}^2\left(\xi^2-1\right) + \frac{{p_\theta}^2}{\xi^2-1} + 24Ve^{2\Gamma}. \tag{2.3.15}$$

2.3.5 Dynamics in the Reduced Phase Space

By variating (2.3.14) with respect to N we get

$$\frac{\delta I}{\delta N} = 0 \Rightarrow \mathcal{H} = 0\,, \tag{2.3.16}$$

hence such constraint solved provides the expression for $\mathcal{H}_{ADM}$

$$-p_\tau \equiv \frac{d\Gamma}{d\tau}\mathcal{H}_{ADM} = \frac{d\Gamma}{d\tau}\sqrt{\varepsilon^2 + 24Ve^{2\Gamma}}, \tag{2.3.17}$$

where

$$\varepsilon^2 \equiv \left(\xi^2-1\right){p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2-1}\,. \tag{2.3.18}$$

in terms of this constraint, principle (2.3.14) reduces to the simpler form

$$\delta \int \left(p_\xi \xi' + p_\theta \theta' - \Gamma' \mathcal{H}_{ADM}\right) d\eta = 0\,. \tag{2.3.19}$$

This variational principle (2.3.19) provides the Hamiltonian equations for ξ' and θ' (Imponente and Montani, 2001)

$$\xi' = \frac{\Gamma'}{\mathcal{H}_{ADM}} \left(\xi^2 - 1\right) p_\xi \tag{2.3.20a}$$

$$\theta' = \frac{\Gamma'}{\mathcal{H}_{ADM}} \frac{p_\theta}{(\xi^2 - 1)}\,. \tag{2.3.20b}$$

The first of (2.3.13a) and (2.3.17) lead to the time gauge relation

$$N\left(\eta\right) = \frac{12D}{\mathcal{H}_{ADM}} e^{2\Gamma} \frac{d\Gamma}{d\tau} \tau'\,, \tag{2.3.21}$$

our analysis remains fully independent of the choice of the time variable until the form of Γ and τ' is not fixed.
By the choice $\Gamma' = 1$ the principle (2.3.19) reduces to the two-dimensional one (Imponente and Montani, 2001)

$$\delta \int \left(p_\xi \xi' + p_\theta \theta' - \mathcal{H}_{ADM}\right) d\eta = 0\,, \tag{2.3.22}$$

where

$$\mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + U}\,, \qquad U \equiv 24 V e^{2\tau}\,; \tag{2.3.23}$$

moreover, the choice $\tau' = 1$ for the temporal gauge lets the lapse function as

$$N_{ADM}\left(\tau\right) = \frac{12D}{\mathcal{H}_{ADM}} e^{2\tau}\,. \tag{2.3.24}$$

The reduced principle (2.3.22) provides the Hamiltonian equations (Imponente et al., 2002)

$$\xi' = \frac{(\xi^2 - 1)}{\mathcal{H}_{ADM}} p_\xi\,, \tag{2.3.25}$$

$$\theta' = \frac{1}{\mathcal{H}_{ADM}} \frac{p_\theta}{(\xi^2 - 1)}\,, \tag{2.3.26}$$

$$p'_\xi = -\frac{\xi}{\mathcal{H}_{ADM}} \left[{p_\xi}^2 - \frac{{p_\theta}^2}{(\xi^2 - 1)^2}\right] - \frac{1}{2\mathcal{H}_{ADM}} \frac{\partial U}{\partial \xi}\,, \tag{2.3.27}$$

$$p'_\theta = -\frac{1}{2\mathcal{H}_{ADM}} \frac{\partial U}{\partial \theta}\,, \tag{2.3.28}$$

where, because of the choice of the time gauge, $(\ )' = \frac{d}{d\tau}$.

2.3.6 Billiard Induced from the Asymptotic Potential

The Hamiltonian equations are equivalently viewed through the two time variables Γ and τ then, for this Section only, we choose the natural time gauge $\tau' = 1$, so that the variational principle (2.3.19) in terms of the time variable Γ reads

$$\delta \int \left(p_\xi \frac{d\xi}{d\Gamma} + p_\theta \frac{d\theta}{d\Gamma} - \mathcal{H}_{ADM} \right) d\Gamma = 0. \tag{2.3.29}$$

Nevertheless for any choice of time variable τ (i.e. $\tau = \eta$), there exists a corresponding function $\Gamma(\tau)$ (i.e. a set of MCl variables leading to the scheme (2.3.29)) defined by the (invertible) relation

$$\frac{d\Gamma}{d\tau} = \frac{\mathcal{H}_{ADM}}{12D} N(\tau) e^{-2\Gamma}. \tag{2.3.30}$$

The metric determinant D vanishes asymptotically approaching the initial singularity: in fact, by applying the Landau-Raichaudhury theorem[1] near the initial singularity (which occurs by convention at $T = 0$, where T now denotes the synchronous time, i.e. $dT = -N(\tau) d\tau$), for $T \to 0$ we have $d\ln D/dT > 0$; in terms of the adopted variable τ

$$D \to 0 \quad \Rightarrow \quad \Gamma(\tau) \to \infty, \tag{2.3.31}$$

then by (2.3.12) and (2.3.30) we have

$$\frac{d\ln D}{d\tau} = \frac{d\ln D}{dT}\frac{dT}{d\tau} = -\frac{d\ln D}{dT} N(\tau) \tag{2.3.32}$$

and therefore D vanishes monotonically even in the generic time gauge as soon as $d\Gamma/d\tau > 0$ for increasing τ according to (2.3.30).

Thus, approaching the initial singularity, the limit $D \to 0$ for the Mixmaster potential (2.3.10) implies for the second three terms to be negligible with respect to the first ones (excluding the particular cases when two or three anisotropy parameters H_i coincide).

Furthermore, by (2.3.17) holds the important relation

$$\frac{d(\mathcal{H}_{ADM}\Gamma')}{d\eta} = \frac{\partial(\mathcal{H}_{ADM}\Gamma')}{\partial\eta} \Longrightarrow \Longrightarrow \frac{d(\mathcal{H}_{ADM}\Gamma')}{d\Gamma} = \frac{\partial(\mathcal{H}_{ADM}\Gamma')}{\partial\Gamma}, \tag{2.3.33}$$

[1] This theorem, based on the mathematical assumptions underlying the dynamics of the Einstein equations, states that in a synchronous reference frame there always exists a value of the time at which the metric determinant vanishes and that in this time variable the zero is approached monotonically (Landau and Lifshitz, 1975).

i.e. explicitly

$$\frac{\partial \mathcal{H}_{ADM}}{\partial \Gamma} = \frac{e^{2\Gamma}}{2\mathcal{H}_{ADM}} 24 \left(2V + \frac{\partial V}{\partial \Gamma} \right) . \tag{2.3.34}$$

In this reduced Hamiltonian formulation, the functional $\Gamma(\eta)$ plays simply the role of a parametric function of time and we recall how actually the anisotropy parameters H_i $(i = 1, 2, 3)$ are functions of the variables ξ, θ only (see 2.3.4).

By the expressions (2.3.10, 2.3.4, 2.3.12) and (2.3.34), we see that

$$V \sim D^{4H_i} \sim \exp\left(-e^{\Gamma}\right) \Rightarrow \frac{\partial V}{\partial \Gamma} = O\left(e^{\Gamma} V\right) , \tag{2.3.35}$$

denoting by $O()$ terms of the same order of the enclosed ones. With these asymptotic values, right hand side of (2.3.34) is over-exponentially depressed.

Let's define Π_H, as the region in the phase space where all the H_i are simultaneously greater than 0, the potential term $U \equiv e^{2\Gamma} V$ can be modelled by the potential walls

$$U_\infty = \Theta_\infty \left(H_1\left(\xi, \theta\right)\right) + \Theta_\infty \left(H_2\left(\xi, \theta\right)\right) + \Theta_\infty \left(H_3\left(\xi, \theta\right)\right) \tag{2.3.36}$$

$$\Theta_\infty(x) = \begin{cases} +\infty & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x > 0 \end{cases}$$

therefore in Π_H the ADM Hamiltonian becomes (asymptotically) an integral of motion

$$\forall \{\xi, \theta\} \in \Pi_H \quad \begin{cases} \dfrac{\partial \mathcal{H}_{ADM}}{\partial \Gamma} = 0 = \dfrac{\partial E}{\partial \Gamma} \\ \mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + 24\, U} \cong \varepsilon = E = \text{const.} . \end{cases} \tag{2.3.37}$$

(see Fig.2.3).

In the region Π_H where the potential vanishes, equation (2.3.37) permits to conclude that the ADM Hamiltonian asymptotically approaches an integral of the motion.

2.3.7 Equivalence of the ADM Reduction

It is interesting to check if and how the ADM formalism is consistent with respect to the *complete* dynamical scheme, i.e. to obtain a statement about the properties characterizing the Mixmaster when expressed in terms of the four degrees of freedom $\{N, \alpha, \beta_+, \beta_-\}$ (Misner variables) as well as in the *reduced* Hamiltonian formulation relying on $\{\beta_+, \beta_-\}$ only and induced by the ADM prescription: we will compare the dynamic equations obtained without and

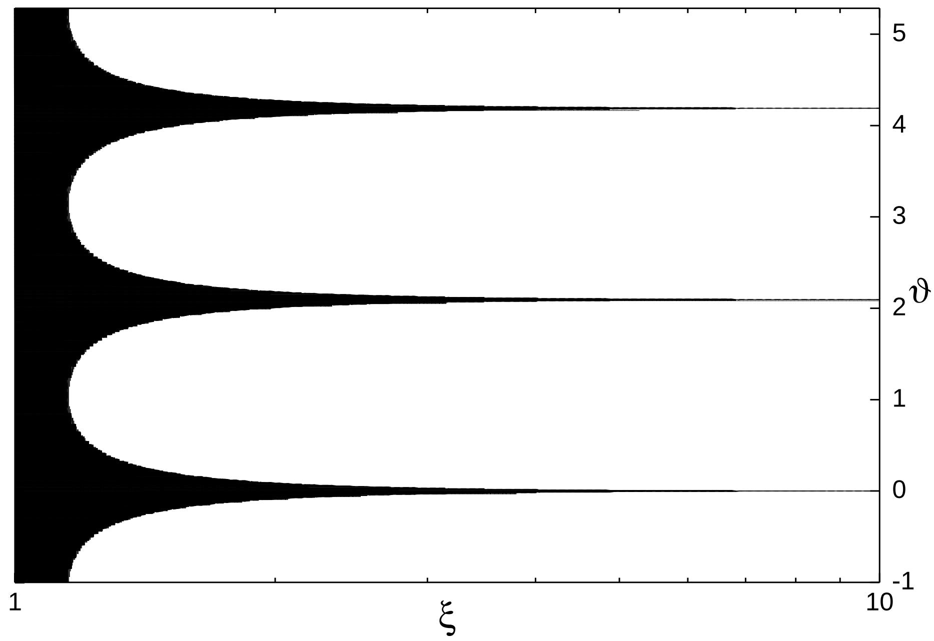

Figure 2.3: Asymptotic potential domain in the plane θ, ξ.

with such framework for the Bianchi I (with zero potential) and Bianchi VIII (or IX, as well for any with non-zero potential) model, respectively.

As a first instance, let us recall the variational principle for the simple case of the Bianchi type I – as in (2.2.5) considered with $\mathcal{V}=0$ – in order to have

$$\delta I = \delta \int \Big(p_\alpha \alpha' + p_+ {\beta_+}' + p_- {\beta_-}' - N\mathcal{H}\Big) d\eta = 0 \tag{2.3.38}$$

where the super Hamiltonian reads

$$\mathcal{H} = \frac{e^{-3\alpha}}{24\pi}\Big(- p_\alpha^2 + p_+^2 + p_-^2 \Big)\,. \tag{2.3.39}$$

For such a dynamical system, the equations of motion for the coordinates expressed in terms of the temporal variable η read as

$$\alpha' = -\frac{N}{24\pi} e^{-3\alpha} p_\alpha \tag{2.3.40a}$$

$${\beta_\pm}' = \frac{N}{24\pi} e^{-3\alpha} p_\pm \tag{2.3.40b}$$

and

$${p_\alpha}' = -\frac{N}{8\pi} e^{-3\alpha}\Big(- p_\alpha^2 + p_+^2 + p_-^2\Big) \tag{2.3.41a}$$

$${p_\pm}' = 0 \tag{2.3.41b}$$

for the conjugate momenta, respectively. In particular, the variation of the action (2.3.38) with respect to N provides the constraint

$$\frac{\delta I}{\delta N} = \frac{e^{-3\alpha}}{24\pi}\Big(- p_\alpha^2 + p_+^2 + p_-^2\Big) = 0\,; \tag{2.3.42}$$

hence, by virtue of the first of (2.3.41) induces

$${p_\alpha}' = 0\,, \tag{2.3.43}$$

and consequently the three conjugate momenta are constant. These equations represent the evolution of the system and are easily integrable starting by the (2.3.40a), which in particular suggests how the variable α can assume the role of time-parameter due to its direct dependence on η. In fact, the ratio of the two equations in (2.3.40) leads to the equations of motion

$$\frac{{\beta_\pm}'}{\alpha'} = \frac{d\beta_\pm}{d\alpha} = -\frac{p_\pm}{p_\alpha} \tag{2.3.44}$$

where, by virtue of (2.3.42), p_α can be expressed in terms of the remaining momenta as

$$-p_\alpha = \sqrt{{p_+}^2 + {p_-}^2} \tag{2.3.45}$$

in which it is customary to adopt the choice of the plus sign in front of the square root.
Let us now apply the ADM prescription: if we put straightforwardly (2.3.45), the momentum conjugate to α can be re-expressed to define $\mathcal{H}_{ADM}$ as

$$\mathcal{H}_{ADM} \equiv \sqrt{{p_+}^2 + {p_-}^2}\,, \tag{2.3.46}$$

hence we are free to choose the gauge in which $\alpha' = 1$, i.e. $\alpha = \eta$ and consequently

$$N_{ADM} = \frac{24\pi e^{3\alpha}}{p_\alpha}\,. \tag{2.3.47}$$

The variational principle (2.3.38) can be reduced to the one

$$\delta I = \delta \int d\alpha \Big(p_+ \frac{d\beta_+}{d\alpha} + p_- \frac{d\beta_-}{d\alpha} - \sqrt{{p_+}^2 + {p_-}^2} \Big) d\eta = 0\,. \tag{2.3.48}$$

The Hamilton equations provided by (2.3.48) read as

$$\dot\beta_\pm = \frac{p_\pm}{\sqrt{{p_+}^2 + {p_-}^2}}\,, \tag{2.3.49}$$

and

$$\frac{dp_\pm}{d\alpha} = 0\,, \tag{2.3.50}$$

where the over-dot here denotes the derivative with respect to the *new* temporal parameter α, i.e. exactly the same as in (2.3.44).

By this calculations, we see that in the dynamical case with potential equal to zero the information is reduced, nevertheless the two systems (before and after ADM reduction) have an equivalent dynamics, provided that one of the variables has assumed the temporal parameter role.

In the discussion performed through the present Chapter concerning the dynamics, its chaoticity and statistical mechanics issues, we always had an infinite potential barrier and therefore we dealt with a free motion interrupted by bounces on the potential walls: in the calculations, the explicit expression for $\mathcal{V}$ never appears.

Let us apply the ADM prescription to the case with potential, in the approximation used in Section 2.3 to Bianchi types VIII or IX and in terms of the Misner-Chitre variables (2.3.2).
Firstly, let us recall the variational principle (2.3.14) which, with the choice $\Gamma(\tau) \equiv \tau$, reads as

$$\delta \int \left(p_\xi \xi' + p_\theta \theta' + p_\tau \tau' - \frac{N e^{-2\tau}}{24D} \mathcal{H} \right) d\eta = 0, \tag{2.3.51}$$

where now $\mathcal{H}$ is given by

$$\mathcal{H} = -{p_\tau}^2 + {p_\xi}^2 \left(\xi^2 - 1\right) + \frac{{p_\theta}^2}{\xi^2 - 1} + 24 V e^{2\tau} . \tag{2.3.52}$$

Hence we derive the equations of motion for τ, ξ, θ in terms of η

$$\tau' = -\frac{N e^{-2\tau}}{12D} p_\tau \tag{2.3.53a}$$
$$\xi' = \frac{N e^{-2\tau}}{12D} \left(\xi^2 - 1\right) p_\xi \tag{2.3.53b}$$
$$\theta' = \frac{N e^{-2\tau}}{12D} \frac{p_\theta}{\xi^2 - 1} \tag{2.3.53c}$$

$${p_\tau}' = \frac{N e^{-2\tau}}{12D} \mathcal{H} \tag{2.3.54a}$$
$${p_\xi}' = \frac{N e^{-2\tau}}{24D} \left[\partial_\xi \left(\frac{1}{D} \right) \mathcal{H} + \left(\frac{1}{D} \right) \partial_\xi \mathcal{H} \right] \tag{2.3.54b}$$
$${p_\theta}' = \frac{N e^{-2\tau}}{24D} \left[\partial_\theta \left(\frac{1}{D} \right) \mathcal{H} + \left(\frac{1}{D} \right) \partial_\theta \mathcal{H} \right] \tag{2.3.54c}$$

together with the constraint $\mathcal{H} = 0$ which permits to simplify the right-hand side of (2.3.54), and straightforwardly to obtain

$$-p_\tau = \sqrt{{p_\xi}^2 \left(\xi^2 - 1\right) + \frac{{p_\theta}^2}{\xi^2 - 1} + 24 V e^{2\tau}} . \tag{2.3.55}$$

The consequence of the cited constraint induces the equation

$${p_\tau}' = 0 \tag{2.3.56}$$

together with the natural choice $\tau' = 1$ and $\tau = \eta$, and moreover

$$N_{ADM} = -12 D e^{2\tau} \frac{1}{p_\alpha} . \tag{2.3.57}$$

We then compute the Hamilton equations after the ADM reduction

$$\xi' = -\left(\xi^2-1\right)\frac{p_\xi}{p_\tau} \tag{2.3.58a}$$

$$\theta' = -\frac{1}{\xi^2-1}\frac{p_\theta}{p_\tau}; \tag{2.3.58b}$$

a similar system is obtained by the ratio between (2.3.53b)-(2.3.53c) and (2.3.53a) through the use of the gauge choice (2.3.57), reading as

$$\frac{\xi'}{\tau'} = \frac{d\xi}{d\tau} = -\left(\xi^2-1\right)\frac{p_\xi}{p_\tau} \tag{2.3.59a}$$

$$\frac{\theta'}{\tau'} = \frac{d\theta}{d\tau} = -\frac{1}{\xi^2-1}\frac{p_\theta}{p_\tau}; \tag{2.3.59b}$$

in this system, the dynamics is naturally re-expressed in terms of the coordinate τ, which assumes the role of the temporal variable η.
The same remains valid for the relations regarding the momenta (2.3.54b) and (2.3.54c).

Hence we have shown for the simple case of the Bianchi type I as well as for the more interesting type VIII and IX that the dynamics, together with the induced constraint, leads to the *same* system of equations in the standard Hamiltonian approach and in the ADM one.

2.3.8 The Jacobi Metric and the Billiard Representation

Since above we have shown that asymptotically to the singularity ($\Gamma \to \infty$, i.e. $\alpha \to -\infty$) $d\mathcal{H}_{ADM}/d\Gamma = 0$, i.e. $\mathcal{H}_{ADM} = \epsilon = E = const.$, the variational principle (2.3.29) reduces to

$$\delta \int \left(p_\xi d\xi + p_\theta d\theta - E d\Gamma\right) = \delta \int \left(p_\xi d\xi + p_\theta d\theta\right) = 0\,, \tag{2.3.60}$$

where we dropped the third term on the left hand side since it behaves as an exact differential.

By following the standard Jacobi procedure (Arnold, 1989*b*) to reduce our variational principle to a geodesic one, we set $x^{a\prime} \equiv g^{ab}p_b$, and by the Hamiltonian equation (2.3.20) we obtain the metric (Imponente and Montani, 2001, 2002*a*)

$$g^{\xi\xi} = \frac{\Gamma'}{E}\left(\xi^2-1\right)$$

$$g^{\theta\theta} = \frac{\Gamma'}{E}\frac{1}{\xi^2-1}\,. \tag{2.3.61}$$

By these and by the fundamental constraint relation obtained rewriting (2.3.18)

$$\left(\xi^2 - 1\right) {p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2 - 1} = E^2 \,, \tag{2.3.62}$$

we get

$$g_{ab} x^{a\prime} x^{b\prime} = \frac{\Gamma'}{E}\left[\left(\xi^2 - 1\right) {p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2 - 1}\right] = \Gamma' E \,. \tag{2.3.63}$$

By the definition

$$x^{a\prime} = \frac{dx^a}{ds}\frac{ds}{d\eta} \equiv u^a \frac{ds}{d\eta} \,, \tag{2.3.64}$$

equation (2.3.63) is rewritten

$$g_{ab} u^a u^b \left(\frac{ds}{d\eta}\right)^2 = \Gamma' E \,, \tag{2.3.65}$$

which leads to the key relation

$$d\eta = \sqrt{\frac{g_{ab} u^a u^b}{\Gamma' E}}\; ds \,. \tag{2.3.66}$$

Indeed the expression (2.3.66) together with $p_\xi \xi' + p_\theta \theta' = E\Gamma'$ allows us to put the variational principle (2.3.60) in the geodesic form

$$\begin{aligned} \delta \int \Gamma' E \; d\eta = \delta \int \sqrt{g_{ab} u^a u^b \Gamma' E}\; ds = \\ = \delta \int \sqrt{G_{ab} u^a u^b}\; ds = 0 \end{aligned} \tag{2.3.67}$$

where the metric $G_{ab} \equiv \Gamma' E g_{ab}$ satisfies the normalization condition $G_{ab} u^a u^b = 1$ and therefore

$$\frac{ds}{d\eta} = E\Gamma' \Rightarrow \frac{ds}{d\Gamma} = E \,, \tag{2.3.68}$$

where we take the positive root since we require that the curvilinear coordinate s increases monotonically with increasing value of Γ, i.e. approaching the initial cosmological singularity.

Summarizing, in the region Π_H the considered dynamical problem reduces to a geodesic flow on a two dimensional Riemannian manifold described by the line element (Imponente and Montani, 2001)

$$ds^2 = E^2 \left[\frac{d\xi^2}{\xi^2 - 1} + \left(\xi^2 - 1\right) d\theta^2\right] \,. \tag{2.3.69}$$

The above metric has negative curvature, since the associated curvature scalar reads $R = -\frac{2}{E^2}$; therefore the point-universe moves over a negatively curved bidimensional space on which the potential wall (2.3.36) cuts the region Π_H. By a way completely independent of the temporal gauge we provided a satisfactory representation of the system as isomorphic to a billiard on a Lobachevsky plane (Arnold, 1989*b*).

From a geometrical point of view, the domain defined by the potential walls is not strictly closed, since there are three directions corresponding to the three corners in the traditional Misner picture from which the point universe could in principle escape (see Fig.2.3).

However, as discussed in Section 1.4.2 for the Bianchi models under consideration, the only case in which an asymptotic solution of the field equations exists with this behaviour corresponds to two scale factors equal to each other (i.e. $\theta = 0$); but, as shown by Belinski and Khalatnikov (1969), these cases are dynamically unstable and therefore correspond to sets of zero measure in the space of the initial conditions. Thus, it has no sense to speak of a probability to reach certain configurations and the domain is *de facto* dynamically closed.

The bounces (billiard configuration) against the potential walls together with the geodesic flow instability, with a formalism true for any Bianchi type model, on a closed domain of the Lobachevsky plane imply the point-universe to have a stochastic feature. Indeed the types VIII and IX are the only Bianchi models having a compact configuration space, hence the claimed compactness of the domain bounded by the potential walls guarantees that the geodesic instability is upgraded to a real stochastic behaviour. On the other hand, the possibility to speak of a stochastic scattering is justified by the constant negative curvature of the Lobachevsky plane and therefore these two notions (compactness and curvature) are both necessary for our considerations.

2.3.9 Invariant Measure

Here we show how the derivation of an invariant measure for the Mixmaster model (performed by Kirillov and Montani (1997*a*); Montani (2000*b*) within the framework of the statistical mechanics) can be extended to a generic time gauge (Imponente and Montani, 2002*a*, 2004*d*) (more directly than in previous approaches relying on fractal methods by Cornish and Levin (1997*a*,*b*)) provided the Misner-Chitre-like variables used so far. We have seen how the (ADM) reduction of the variational problem asymptotically close to the cosmological singularity permits to modelize the Mixmaster dynamics by a two-dimensional point-universe randomizing in a closed domain with fixed “energy” (just the ADM kinetic energy) (2.3.37); the key point addressed here is that we consider an approximation dynamically induced by the asymptotic vanishing of the metric determinant.

From the statistical mechanics point of view, such a stochastic motion within closed domain Π_H of the phase-space, induces a suitable ensemble representation which, in view of the existence of the "energy-like" constant of motion, has to have the natural feature of a *microcanonical* one. Therefore the stochasticity of this system can be described in terms of the Liouville invariant measure

$$d\varrho = \text{const} \times \delta\left(E - \varepsilon\right) d\xi d\theta dp_\xi dp_\theta \tag{2.3.70}$$

characterizing the *microcanonical ensemble*, having denoted by $\delta\left(x\right)$ the Dirac function.
The particular value taken by the constant ε ($\varepsilon = E$) cannot influence the stochastic property of the system and must be fixed by the initial conditions. This useless information from the statistical dynamics is removable integrating over all admissible values of ε. Introducing the natural variables (ε, ϕ) in place of (p_ξ, p_θ) by

$$\begin{aligned} p_\xi &= \frac{\varepsilon}{\sqrt{\xi^2 - 1}} \cos\phi & \text{(2.3.71a)} \\ p_\theta &= \varepsilon\sqrt{\xi^2 - 1} \sin\phi\,, & \text{(2.3.71b)} \\ & 0 \leq \phi < 2\pi \end{aligned}$$

the integration removes the Dirac function, leading to the uniform (normalized) invariant measure (Imponente and Montani, 2002*a*)

$$d\mu = d\xi d\theta d\phi \frac{1}{8\pi^2}\,. \tag{2.3.72}$$

The approximation on which our analysis is based (i.e. the potential wall model) is reliable since it is dynamically induced no matter what time variable τ is adopted.

2.3.10 Stationary Statistical Distribution

In order to outline how the existence of this invariant measure (i.e. a stationary probability distribution for the reduced phase space) is independent of the adopted temporal gauge, we derive it as a solution of the continuity equation relative to the whole phase space $\Pi \equiv \{\xi, \theta, \phi, \varepsilon\}$, i.e. clearly,

$\Pi = \Pi_H \otimes S^1_\phi \otimes \{R^+ \cup \{0\}\}$; by other words, we consider the probability distribution $\Xi = \Xi(\Gamma, \xi, \theta, \phi, \varepsilon)$ and write down the continuity equation (Imponente and Montani, 2002*a*)

$$\begin{aligned}\partial_\Gamma \Xi = &- \partial_\xi \left(\Xi \frac{d\xi}{d\Gamma}\right) - \partial_\theta \left(\Xi \frac{d\theta}{d\Gamma}\right) + \\ &- \partial_\phi \left(\Xi \frac{d\phi}{d\Gamma}\right) - \partial_\varepsilon \left(\Xi \frac{d\varepsilon}{d\Gamma}\right) .\end{aligned} \tag{2.3.73}$$

With a rather simple algebra (see Montani (2000*b*, 2001)), the Hamiltonian equations in terms of $(\xi, \theta, \phi, \varepsilon)$ read

$$\frac{d\xi}{d\Gamma} = \frac{f_\xi}{h} , \tag{2.3.74a}$$

$$\frac{d\theta}{d\Gamma} = \frac{f_\theta}{h} , \tag{2.3.74b}$$

$$\frac{d\phi}{d\Gamma} = \frac{1}{h}\left[f_\phi + \partial_\phi \left(\frac{d\ln\varepsilon}{d\tau}\right)\right] , \tag{2.3.74c}$$

$$\frac{d\ln\varepsilon}{d\Gamma} = -\frac{1}{2\varepsilon^2}\left(\partial_\xi U \sqrt{\xi^2-1}\cos\phi + \partial_\theta U \frac{\sin\phi}{\sqrt{\xi^2-1}}\right) , \tag{2.3.74d}$$

where we introduced the notations

$$h \equiv \frac{\mathcal{H}_{ADM}}{\varepsilon} , \qquad f_\xi \equiv \sqrt{\xi^2-1}\cos\phi , \tag{2.3.75a}$$

$$f_\theta \equiv \frac{\sin\phi}{\sqrt{\xi^2-1}} , \qquad f_\phi \equiv -\frac{\xi\sin\phi}{\sqrt{\xi^2-1}} . \tag{2.3.75b}$$

The probability distribution Ξ has to verify the boundary condition of vanishing at $\partial\Pi$, i.e. $\varepsilon = 0$, $\varepsilon = \infty$ and on the border of $\Pi_H \otimes S^1_\phi$.

2.3.11 Continuity Equation in the Reduced Phase Space

Let us consider the probability distribution $w = w(\Gamma, \xi, \theta, \phi)$, restricted to the reduced phase space $\Pi_H \otimes S^1_\phi$, defined as

$$w(\Gamma, \xi, \theta, \phi) \equiv \int_0^\infty \Xi(\Gamma, \xi, \theta, \phi, \varepsilon) d\varepsilon , \tag{2.3.76}$$

and take the corresponding integral in (2.3.73), to get

$$\partial_\Gamma w = -\int_0^\infty \left[\partial_\xi \left(\Xi \frac{d\xi}{d\Gamma}\right) + \partial_\theta \left(\Xi \frac{d\theta}{d\Gamma}\right) + \partial_\phi \left(\Xi \frac{d\phi}{d\Gamma}\right)\right] d\varepsilon . \tag{2.3.77}$$

In the limit $\Gamma \to \infty$ ε becomes an integral of motion $d\varepsilon/d\Gamma = 0$, assuming the constant value E (fixed by the initial conditions).

As a direct consequence, the probability distribution Ξ approaches the limit $w \times \delta(E - \varepsilon)$ and therefore, using the identities

$$\partial_\xi f_\xi + \partial_\theta f_\theta + \partial_\phi f_\phi = 0\,, \qquad \partial^2{}_\phi \left(\frac{d\ln\varepsilon}{d\tau}\right) = -\frac{d\ln\varepsilon}{d\tau}\,, \tag{2.3.78}$$

we write down the Liouville theorem for the reduced phase space $\Gamma_H \otimes S^1_\phi$

$$\begin{aligned}\partial_\Gamma w + \frac{d\xi}{d\Gamma}\partial_\xi w + \frac{d\theta}{d\Gamma}\partial_\theta w + \frac{d\phi}{d\Gamma}\partial_\phi w =\\ =\partial_\Gamma w + \sqrt{\xi^2 - 1}\cos\phi\partial_\xi w +\\ + \frac{\sin\phi}{\sqrt{\xi^2-1}}\partial_\theta w - \frac{\xi\sin\phi}{\sqrt{\xi^2-1}}\partial_\phi w = 0\,.\end{aligned} \tag{2.3.79}$$

Multiplying (2.3.79) by $d\Gamma/d\tau$ it rewrites in the gauge free form

$$\begin{aligned}\partial_\tau w + \frac{d\xi}{d\tau}\partial_\xi w + \frac{d\theta}{d\tau}\partial_\theta w + \frac{d\phi}{d\tau}\partial_\phi w =\\ =\partial_\tau w + \sqrt{\xi^2 - 1}\cos\phi\partial_\xi w +\\ + \frac{\sin\phi}{\sqrt{\xi^2-1}}\partial_\theta w - \frac{\xi\sin\phi}{\sqrt{\xi^2-1}}\partial_\phi w = 0\,,\end{aligned} \tag{2.3.80}$$

where $w = w(\tau, \xi, \theta, \phi)$.
When using a generic time variable τ, the right-hand side of equation (2.3.80) in general is no longer (asymptotically) vanishing, but yet negligible with respect to the left-hand one. The same result is obtainable by writing the Hamiltonian equations and the continuity one directly in term of τ.
The invariant measure (2.3.72) is a stationary solution of the continuity equation (2.3.80) (Imponente and Montani, 2002*a*).
The key point is that any stationary solution of the Liouville theorem like (2.3.70), remains valid for any choice of the time variable τ. Clearly the knowledge of the invariant measure (2.3.72) provides a satisfactory statistical representation of the system for any choice of time variable, since it allows one calculate the asymptotic average values (as well as higher order moments) of any dynamical variable involved in the problem.

By the above argument, the claim made at the beginning of this Section which joins together, on one hand the existence of a formulation for the system dynamics (based on the use of generic MCl variables) leaving open the choice of the temporal gauge N and, on the other one, the derivation of the Mixmaster invariant measure as gauge independent solution of the Liouville theorem restricted to $\Pi_H \otimes S^1_\phi$.

We remark that when one approaches the singularity $\Gamma(\tau) \to \infty$ (i.e. $\mathcal{H}_{ADM} \to E$), the time gauge relation (2.3.30) simplifies to

$$\frac{d\Gamma}{d\tau} = N(\tau)\,\frac{Ee^{-2\Gamma+3\xi e^{\Gamma}}}{12}e^{-2\Gamma}\,. \tag{2.3.81}$$

According to the analysis presented by Montani (2000*b*, 2001), by virtue of (2.3.20) and (2.3.68) the asymptotic functions $\xi(\Gamma), \theta(\Gamma), \phi(\Gamma)$ during free geodesic motion are governed by the equations

$$\frac{d\xi}{d\Gamma} = \sqrt{\xi^2-1}\cos\phi \tag{2.3.82a}$$

$$\frac{d\theta}{d\Gamma} = \frac{\sin\phi}{\sqrt{\xi^2-1}} \tag{2.3.82b}$$

$$\frac{d\phi}{d\Gamma} = -\frac{\xi\sin\phi}{\sqrt{\xi^2-1}}\,. \tag{2.3.82c}$$

Once the solution $\xi(\Gamma)$ is obtained in the parametric form

$$\xi(\phi) = \frac{\rho}{\sin^2\phi} \tag{2.3.83a}$$

$$\Gamma(\phi) = \frac{1}{2}\text{arctanh}\left(\frac{1}{2}\frac{\rho^2+a^2\cos^2\phi}{a\rho\cos\phi}\right)+b \tag{2.3.83b}$$

$$\rho \equiv \sqrt{a^2+\sin^2\phi} \qquad a,b = \text{const.} \in \Re$$

equation (2.3.81) reduces to a simple differential equation for the function $\Gamma(\tau)$.

However the global behaviour of ξ along the whole geodesic flow is described by the invariant measure (2.3.72) and therefore the relation (2.3.81) acquires a stochastic character:if we assign one of the two functions $\Gamma(\tau)$ or $N(\tau)$ with an *arbitrary* analytic functional form, then the other one will exhibit stochastic behaviour. Finally, by retaining the same dynamical scheme adopted in the construction of the invariant measure, we see how the one-to-one correspondence between any lapse function $N(\tau)$ and the associated set of MCl variables (2.3.3) guarantees covariance with respect to the time-gauge of the Mixmaster universe stochastic behaviour, when viewed in the framework of statistical mechanics.

2.4 Invariant Lyapunov Exponent

In order to characterize the dynamical instability of the billiard in terms of an invariant treatment (with respect to the choice of the coordinates ξ, θ), let us introduce the following (orthonormal) tetradic basis (Imponente and Montani, 2001, 2004*b*)

$$v^i = \left(\frac{1}{E}\sqrt{\xi^2-1}\cos\phi,\ \frac{1}{E}\frac{\sin\phi}{\sqrt{\xi^2-1}}\right) \tag{2.4.1a}$$

$$w^i = \left(-\frac{1}{E}\sqrt{\xi^2-1}\sin\phi,\ \frac{1}{E}\frac{\cos\phi}{\sqrt{\xi^2-1}}\right) \tag{2.4.1b}$$

Indeed the vector v^i is nothing else than the geodesic field, i.e.

$$\frac{Dv^i}{ds} = \frac{dv^i}{ds} + \Gamma^i_{kl} v^k v^l = 0\,, \tag{2.4.2}$$

while the vector w^i is parallely transported along the geodesic, according to the equation

$$\frac{Dw^i}{ds} = \frac{dw^i}{ds} + \Gamma^i_{kl} v^k w^l = 0\,, \tag{2.4.3}$$

where by Γ^i_{kl} we denoted the Christoffel symbols constructed by the metric (2.3.69). Projecting the geodesic deviation equation along the vector w^i (its component along the geodesic field v^i does not provide any physical information about the system instability), the corresponding connecting vector (tetradic) component Z satisfies the following equivalent equation

$$\frac{d^2Z}{ds^2} = \frac{Z}{E^2}\,. \tag{2.4.4}$$

This expression, as a projection on the tetradic basis, is a scalar one and therefore completely independent of the choice of the variables. Its general solution reads

$$Z(s) = c_1 e^{\frac{s}{E}} + c_2 e^{-\frac{s}{E}}\,, \qquad c_{1,2} = \text{const.}\,, \tag{2.4.5}$$

and the invariant Lyapunov exponent (Lyapunov, 1907) defined by Pesin (1977)

$$\lambda_v = \sup \lim_{s\to\infty} \frac{\ln\left(Z^2 + \left(\frac{dZ}{ds}\right)^2\right)}{2s}\,, \tag{2.4.6}$$

in terms of the form (2.4.5) takes the value (Imponente and Montani, 2001, 2004*b*,*a*)

$$\lambda_v = \frac{1}{E} > 0\,. \tag{2.4.7}$$

When the point-universe bounces against the potential walls, it is reflected from a geodesic to another one thus making each of them unstable. Though up to the limit of our potential wall approximation, this result shows without any ambiguity that, independently of the choice of the temporal gauge, the Mixmaster dynamics is isomorphic to a well described chaotic system. Equivalently, in terms of the BKL representation, the free geodesic motion corresponds to the evolution during a Kasner epoch and the bounces against the potential walls to the transition between two of them. By itself, the positive Lyapunov number (2.4.7) is not enough to ensure the system chaoticity, since its derivation remains valid for any Bianchi type model; the crucial point is that for the Mixmaster (type VIII and IX) the potential walls reduce the configuration space to a compact region (Π_H), ensuring that the positivity of λ_v implies a real chaotic behaviour (i.e. the geodesic motion fills the entire configuration space).

Summarizing, our analysis shows that for any choice of the time variable, we are able to give the above stochastic representation of the Mixmaster model, provided the factorized coordinate transformation in the configuration space

$$\alpha = -e^{\Gamma(\tau)} a\left(\theta, \xi\right) \tag{2.4.8a}$$

$$\beta_+ = e^{\Gamma(\tau)} b_+\left(\theta, \xi\right) \tag{2.4.8b}$$

$$\beta_- = e^{\Gamma(\tau)} b_-\left(\theta, \xi\right) , \tag{2.4.8c}$$

where $\Gamma, a, b_\pm$ denote generic functional forms of the variables τ, θ, ξ.

It is worth noting that the success of our analysis, in showing the time gauge independence of the Mixmaster chaos, relies on the use of a standard ADM reduction of the variational principle (which reduces the system by one degree of freedom) and overall because, adopting Misner-Chitre-like variables, the asymptotic potential walls are fixed in time. The difference between our approach and the one presented in Szydlowski and Lapeta (1993); Szydlowski (1993) (see also for a critical analysis Burd and Tavakol (1993)) consists effectively in these features, though in those works is even faced the problem of the Mixmaster chaos covariance with respect to the choice of generic configuration variables.

2.5 Lyapunov Exponents in General Relativity

Chaos in the Einstein equations, as discussed in detail in Imponente and Montani (2001, 2002*a*); Imponente et al. (2002); Imponente and Mon-

TANI (2004b), is expected to be a typical property of multidimensional non-linear dynamical systems, nevertheless such a genuine feature within various approaches lead to an increase in the doubts instead of giving solid corner-stones about it (CHERNOFF AND BARROW, 1983; KIRILLOV AND MONTANI, 1997a; MONTANI, 2000b; CORNISH AND LEVIN, 1997a,b; BERGER, 1994; HOBILL ET AL., 1994; BERGER, 2000) etc.

Such contrasting results were due to the absence of the chaos descriptions for relativistic systems via the standard methods of dynamical systems theory (ARNOLD, 1989b,a): indeed the classical treatment is quite different from General Relativity and cosmological problems because in the last one one deals with pseudo-Riemannian spaces and hence with the impossibility of using the theory of Lyapunov exponents which were developed only for Riemannian spaces (as calculated after the ADM reduction in Section 2.4).
This is a fundamental issue since on Lorentzian spaces nearby geodesics can diverge while the connecting vector (whose norm is involved in the definition of Lyapunov numbers, see Section 2.4 above and Section 2.6.2 below) can remain finite and even vanish. Such risk is overcome applying to the study of the chaotic properties of Mixmaster models the covariant definition of Lyapunov exponents for N-dimensional pseudo-Riemannian manifolds introduced in GURZADYAN AND KOCHARYAN (1986, 1987a,b).

2.5.1 The Hamiltonian Reduction to the Geodesic Flow

The dynamical evolution of the Mixmaster cosmological model has been formulated in Section 2.2 in terms of the variational principle (2.2.1) which has led to the Hamiltonian $\mathcal{H}$ that we rewrite in a form suitable for the present discussion as

$$\mathcal{H} = \frac{e^{-3\alpha}}{24\pi}\big(-p_\alpha^2 + p_+^2 + p_-^2\big) - \frac{\pi}{2}Ne^{\alpha}U^{(B)}(\beta_+, \beta_-) \tag{2.5.1}$$

and the corresponding conjugate momenta are the same as (2.2.9a). The action takes the form

$$I = \int p_a dx^a - N\mathcal{H}d\eta, \tag{2.5.2}$$

where the sum over the index a runs through the variables α, β_+, β_-, and the variation carried out with respect to N, as discussed in Section 2.2, leads to the constraint equation $\mathcal{H} = 0$.
Thus we have a dynamical system with a Hamiltonian

$$\mathcal{H} = \frac{1}{2}g^{ab}p_a p_b + V_N(x) \tag{2.5.3}$$

where

$$
\begin{aligned}
V_N(x) &\equiv N\,V(\alpha,\beta_+,\beta_-) = N\frac{\pi}{2}e^{\alpha}U(\beta_+,\beta_-)\,, && \text{(2.5.4a)}\\
g^{ab} &= N\frac{e^{-3\alpha}}{12\pi}\eta^{ab}\,, \quad \eta^{ab} = \mathrm{d}iag(-1,1,1)\,. && \text{(2.5.4b)}
\end{aligned}
$$

In a generic temporal gauge we define the following quantities

$$
\begin{aligned}
v^a &\equiv x^{a\prime} = p_b g^{ab}\,, \quad u^a = \frac{dx^a}{ds}\,, && \text{(2.5.5a)}\\
p_a &= g_{ab}v^b\,, \quad g_{ab}g^{bc} = \delta_a^c\,, && \text{(2.5.5b)}
\end{aligned}
$$

where s is an arbitrary parameter along the geodesic. Following GURZADYAN AND KOCHARYAN (1987*b*), we denote the two regions of the configuration space

$$
W^+ = \{x \mid V_N(x) > 0\}\,, \quad W^- = \{x \mid V_N(x) < 0\}\,, \tag{2.5.6}
$$

and let $\gamma_{(ext)}$ be a solution of the Hamiltonian equations, extremizing the action $\delta I(\gamma_{(ext)}) = 0$. We have

$$
\begin{aligned}
\mathrm{ext}(I)\mid_{\mathcal{H}=0} &= \mathrm{ext}\left(\int p_a dx^a \mid_{\mathcal{H}=0,\ p_a=g_{ab}v^b}\right) = \\
&= \mathrm{ext}\left(\int g_{ab}v^a v^b d\eta \mid_{\mathcal{H}(g_{ab}v^b,x)=0}\right)\,, && \text{(2.5.7)}
\end{aligned}
$$

where $ext()$ represents the extremizing of the quantity enclosed in brackets. As shown by GURZADYAN AND KOCHARYAN (1987*b*), in both cases when g is Riemannian and pseudo-Riemannian metric one can write in the region W^-

$$
g_{ab}v^a v^b = -2V_N > 0 \tag{2.5.8}
$$

and

$$
d\eta = \left(\frac{g_{ab}u^a u^b}{-2V_N}\right)^{1/2} ds; \tag{2.5.9}
$$

then the last integral in (2.5.7) becomes equal to

$$
\mathrm{ext}\int(-2V_N)d\eta = \mathrm{ext}\int\sqrt{G_{ab}u^a u^b}ds
$$

where $G_{ab} = -V_N g_{ab}$ and s is chosen in a way to satisfy the condition $\parallel u \parallel^2 = G_{ab}u^a u^b = 1$, and finally results

$$
ds = \pm\sqrt{2}V_N d\eta \tag{2.5.10}
$$

where the positive and negative signs correspond to the growth of the curvilinear abscissa when the dynamical system evolves forward or backward in synchronous time $dt = N(\eta)d\eta$, respectively; though for the study of the Mixmaster dynamics we are interested to the backward choice, in this Section only we retain the positive one in agreement with the standard behaviour of chaotic systems.
Thus the Hamiltonian system is reduced to a geodesic flow in the region W^-

$$\mathcal{H} = \frac{1}{2}g^{ab}p_a p_b + V_N \quad \longleftrightarrow \quad \Big\{G_{ab} = -V_N g_{ab}, \quad ds = \sqrt{2}(-V_N)d\eta, \ \parallel u \parallel^2 = 1\Big\} . \tag{2.5.11}$$

Analogously in W^+ one obtains

$$\mathcal{H} \longleftrightarrow \Big\{G_{ab} =\mid V_N \mid g_{ab}, \ ds = \sqrt{2} \mid V_N \mid d\eta, \ \parallel u \parallel^2 = -\mathrm{sign}V_N\Big\} . \tag{2.5.12}$$

Thus the Hamiltonian system is represented by a geodesic flow on a pseudo-Riemannian manifold

$$\mathcal{H} \longleftrightarrow \Big\{G_{ab} = G\eta_{ab}, \ ds = \sqrt{2} \mid V_N \mid d\eta, \ \parallel u \parallel^2 = -\mathrm{sign}V_N\Big\} , \quad G \equiv 12\pi e^{3\alpha} \mid V \mid \tag{2.5.13}$$

where now s denotes a curvilinear coordinate along the geodesic. It is important to observe that the metric $G_{ab} = 12\pi e^{3\alpha}\eta_{ab}$ is independent on the lapse function N, so leading to a time-gauge independent formulation of the dynamical problem.

Summarizing, after the geodesic reduction the dynamical content of the above Hamiltonian formulation is summarized by the line element

$$ds^2 = G\eta_{\mu\nu}dx^\mu dx^\nu = G(\alpha, \beta_+, \beta_-)\left(d\alpha^2 - d\beta_+^2 - d\beta_-^2\right) , \tag{2.5.14}$$

with $\mu, \nu = 0, 1, 2$.

2.6 Geodesic Deviation and Lyapunov Exponents

2.6.1 Geodesic Deviation and the Fermi Basis

In this formalism the Hamiltonian equation corresponds to the geodesic one for the four-velocity $u^\mu \equiv \frac{dx^\mu}{ds}$

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0, \tag{2.6.1}$$

where the Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$ correspond to the above (2.5.14) (conformally flat) metric tensor. The stability of the Hamiltonian equations solutions with respect to different initial conditions is therefore, in this formalism, understandable on the basis of the geodesic deviation equation (each geodesic corresponds to the same Hamiltonian solution having different initial data).
Consider the equation of geodesic deviation (Jacobi) for the four-vector ξ^μ

$$\frac{D^2\xi^\mu}{ds^2} + R^{\mu}{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma = 0. \tag{2.6.2}$$

For dynamical systems defined on Riemannian manifolds this equation is the fundamental one to analyse the statistical properties of the geodesic flows (see e.g. Anosov (1967); Arnold (1989a)). However, in pseudo-Riemannian manifolds as the one (2.5.14) or (2.5.10) the norm of the connecting vector $\parallel \xi^\alpha \parallel$ cannot be a correct characteristic of the geodesic flow instability, since it can eventually vanish (for instance in the case of a null vector) even when the connected geodesics are diverging.
Consider the component of ξ^μ orthogonal to the 4-velocity u^μ (satisfying $u^\mu u_\mu = -1$)

$$Z^\mu = \xi^\mu + (\xi^\nu u_\nu)u^\mu, \tag{2.6.3}$$

and substitute it into the expression

$$\frac{DZ^\mu}{ds} = \frac{D\xi^\mu}{ds} + \frac{D\xi^\nu}{ds}u_\nu u^\mu, \tag{2.6.4}$$

where $\frac{Du^\mu}{ds} = 0$ since u^μ satisfies the geodesic equation. For the second covariant derivative we get

$$\frac{D^2Z^\mu}{ds^2} = \frac{D^2\xi^\mu}{ds^2} + \frac{D^2\xi^\nu}{ds^2}u_\nu u^\mu, \tag{2.6.5}$$

so that

$$\frac{D^2Z^\mu}{ds^2} + R^{\mu}{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma + R^{\tau}{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma u_\tau u^\mu = 0, \tag{2.6.6}$$

but the last term in the left-hand side is zero due to the antisymmetry of $R_{\mu\nu\rho\sigma}$ with respect to $\mu\nu$. From (2.6.3) and (2.6.6) we then obtain

$$\frac{D^2Z^\mu}{ds^2} + R^{\mu}{}_{\nu\rho\sigma}u^\nu u^\rho \Big(Z^\sigma - (\xi^\varepsilon u_\varepsilon)u^\sigma\Big) = 0, \tag{2.6.7}$$

where the last term in the left-hand side is again zero for the same reason. Thus Z^μ also satisfies the geodesic deviation equation

$$\frac{D^2Z^\mu}{ds^2} + R^{\mu}{}_{\nu\rho\sigma}u^\nu u^\rho Z^\sigma = 0. \tag{2.6.8}$$

This result (in which the Riemann tensor is calculated by the given metric (2.5.14)) is coupled to the geodesic one (2.6.1) and the solution to the system (2.6.1, 2.6.8) is simplified by the use of a Fermi basis.

Indeed, let us introduce a basis $\{e^a_\alpha\}$ in which the time-like vector coincides with the geodesic field ($e^0_\mu \equiv u_\mu$), i.e. parallely transported along the geodesic $\gamma(s)$, and require for the other vectors to be Fermi transported, hence the other vectors satisfy (Imponente et al., 2002)

$$\frac{De^a_\alpha}{ds} = u^l \nabla_l e^a_\alpha = 0, \quad a, \alpha, l = 0, 1, 2\,. \tag{2.6.9}$$

When we project equations (2.6.1, 2.6.8) on this basis, the geodesic one disappears while the geodesic deviation equation acquires a simpler form.

Each vector $Z \epsilon T_{\gamma(s)} W$ ($T_{\gamma(s)}$ is the tangent space to W) can be expressed in terms of the Fermi basis

$$Z^m(s) = Z^\mu e^m_\mu \quad m = 0, 1, 2\,, \tag{2.6.10}$$

then by means of (2.6.3) and (2.6.10) we project (2.6.7) onto the Fermi vectors, thus getting a scalar equation for the triadic components

$$\frac{d^2 Z^m}{ds^2} + R^m_{\ nrs} u^n u^r Z^s = 0\,, \tag{2.6.11}$$

since $\frac{D^2 Z^m}{ds^2} = \frac{d^2 Z^m}{ds^2}$ and

$$R^a_{\ bcd} = R^\mu_{\ \nu\rho\sigma} e^a_\mu e^\nu_b e^\rho_c e^\sigma_d\,. \tag{2.6.12}$$

which is invariant with respect to space-time coordinates transformations, while it is covariant with respect to Lorentzian transformations of the triadic index.

Because of the identically vanishing of the 0 component in (2.6.12), the Lorentzian (invariant) norm of the triad vector Z^a, $\sqrt{\eta_{ab} Z^a Z^b}$, coincides with the Euclidean one which is positively defined. This means that we have arrived at our goal: namely the possibility of the introduction of the notion of Lyapunov exponents.

2.6.2 Lyapunov Exponents

For an Hamiltonian system described by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \quad i = 0, 1, 2\ , \tag{2.6.13}$$

the variations of the canonical variables δq , δp satisfy

$$\dot{\delta q_i} = \frac{\partial^2 H}{\partial q_m \partial p_i}\delta q_m + \frac{\partial^2 H}{\partial p_m \partial p_i}\delta p_m \tag{2.6.14a}$$

$$\dot{\delta p_i} = -\frac{\partial^2 H}{\partial q_m \partial q_i}\delta q_m - \frac{\partial^2 H}{\partial p_m \partial q_i}\delta p_m, \tag{2.6.14b}$$

since δq_i δp_i are functions of time.

The Lyapunov exponents are defined (e.g. PESIN (1977); GURZADYAN AND KOCHARYAN (1987c)) as

$$\lambda(q, p, \delta q, \delta p) = \lim_{t\to\infty} \frac{\ln \mid \delta \mid}{t}, \quad \delta = \sqrt{\sum_{i=1}^{3} \left(\delta {q_i}^2 + \delta {p_i}^2\right)}\,, \tag{2.6.15}$$

in terms of which, as proved in the theory of dynamical systems (ARNOLD, 1989a), the existence of non-zero ones on compact manifolds indicates the mixing (chaotic) property of the dynamical system.

If $Z = Z^a e_a$ is the deviation vector of the geodesics the Lyapunov exponents can be rewritten as PESIN (1977); GURZADYAN AND KOCHARYAN (1987b)

$$\lambda(\gamma(0), u) = \lim_{s\to\infty} \frac{\ln\left(Z^2 + \left(\frac{dZ}{ds}\right)^2\right)}{2s}\,. \tag{2.6.16}$$

The notion of Lyapunov exponents involves only the Euclidean (i.e. Lorentzian) norm of the vector Z^a and its derivative, so that such norms can be computed in any suitable reference frame, i.e. independently of the choice of the space-time coordinates as well as the specific triad vectors, which are fixed up to a Lorentzian transformation with respect to the triadic index depending point by point in the configuration space on the curvilinear abscissa s.

2.6.3 The Jacobi Equation for the Mixmaster Model

The Jacobi equation (2.6.11) in the case under study reads

$$\frac{d^2 Z^a}{ds^2} + \omega^a Z^a = 0\,, \tag{2.6.17}$$

where the index $a = 0, 1, 2$ runs for α, β_+, β_- and the non-zero components of the Riemann tensor in triadic form are

$$\omega^1 = {R^1}_{001} = -\frac{1}{G}\left[-\frac{d^2 F}{d\alpha^2} + \frac{d^2 F}{d{\beta_-}^2} + \left(\frac{dF}{d\beta_+}\right)^2\right] \tag{2.6.18a}$$

$$\omega^2 = {R^2}_{002} = -\frac{1}{G}\left[-\frac{d^2 F}{d\alpha^2} + \frac{d^2 F}{d{\beta_+}^2} + \left(\frac{dF}{d\beta_-}\right)^2\right] \tag{2.6.18b}$$

where $G_{ab} = G\eta_{ab}$, $G = \det G_{ab} = 12\pi e^{3\alpha} \mid V \mid$, $F = \frac{1}{2}\ln G$. We can consider the equations for Z^1 and Z^2, ($Z^0 \equiv 0$), distinguishing the two cases: $\vec{Z} = (0, Z^1, 0)$, $\vec{Z} = (0, 0, Z^2)$, i.e. $\vec{Z}$ along the first or the second vectors of the basis.

2.6.4 Equations of Motion and Instability

For Bianchi VIII and IX we have obtained a potential term $V_N(x)$ as in (2.5.13) and (2.5.4a)

$$V_N(x) = -e^{3\alpha}\pi R^{(B)} N \tag{2.6.19}$$

where the three-curvature $R^{(B)}$ of the Bianchi models has the explicit form given by (2.1.29a),(2.1.29b) and the potential is (2.2.8).

The constraint $\mathcal{H} = 0$ (2.3.16) leads to

$$-{p_\alpha'}^2 + {p_+'}^2 + {p_-'}^2 = 12\pi^2 e^{4\alpha} U^{(B)} \,, \tag{2.6.20}$$

leaving the choice of a temporal gauge.
The Hamiltonian equations yield

$$\alpha' = \frac{\partial \mathcal{H}}{\partial p_\alpha} = -N\frac{e^{-3\alpha}}{12\pi} p_\alpha \tag{2.6.21a}$$

$$\beta_\pm = \frac{\partial \mathcal{H}}{\partial p_\pm} = N\frac{e^{-3\alpha}}{12\pi} p_\pm \tag{2.6.21b}$$

$$p_\alpha' = -\frac{\partial \mathcal{H}}{\partial \alpha} = \frac{\pi}{2} N e^{\alpha} U \tag{2.6.21c}$$

$$p_\pm' = -\frac{\partial \mathcal{H}}{\partial \beta_\pm} = \frac{\pi}{2} N e^{\alpha} \frac{\partial U}{\partial \beta_\pm} \,. \tag{2.6.21d}$$

One can introduce the arc-length parameter s along each time-like geodesic related to the generic time variable η by $ds = \pm\sqrt{2} \mid V \mid N d\eta$. Thus we reduce the Mixmaster dynamics to a geodesic flow on the pseudo-Riemannian manifold with metric (2.5.14). This corresponds to time-like geodesics in the type VIII case where V is always positive, and sufficiently far from isotropy in the type IX one (where $V > 0$).
The orthonormal coordinate frame $\{e_{(a)}\}$ has a dual one

$$\{e^{(a)}\}_{a=0,1,2} = (12\pi e^{3\alpha} \mid V \mid)^{1/2} \{d\alpha, d\beta^+, d\beta^-\} \,. \tag{2.6.22}$$

The system of first-order differential equations (2.6.21) in terms of the curvilinear coordinate s can finally be reduced to the following one (Imponente

ET AL., 2002), independent of the temporal gauge, i.e. of the lapse function $N(\eta)$

$$\frac{d\alpha}{ds} = \mp\frac{e^{-3\alpha}}{12\pi}\frac{1}{\sqrt{2}\mid V\mid}p_\alpha \tag{2.6.23a}$$

$$\frac{d\beta_\pm}{ds} = \pm\frac{e^{-3\alpha}}{12\pi}\frac{1}{\sqrt{2}\mid V\mid}p_\pm \tag{2.6.23b}$$

$$\frac{dp_\alpha}{ds} = \pm 2\pi e^{\alpha}\frac{1}{\sqrt{2}\mid V\mid}U \tag{2.6.23c}$$

$$\frac{dp_\pm}{ds} = \pm\frac{\pi}{2}e^{\alpha}\frac{1}{\sqrt{2}\mid V\mid}\frac{\partial U}{\partial\beta_\pm}\,, \tag{2.6.23d}$$

where U is the specific $U^{(B)}$, so that the cosmological singularity at $t \to 0$ lies in the limit $\alpha \to -\infty$, corresponding to the lower signs in these formulas. These Hamiltonian system has to be coupled to the Jacobi equation (2.6.2) in order to get the Lyapunov numbers (2.6.16).

The stability properties of the Hamiltonian dynamics (2.6.23) are described by the geodesic deviation properties of the corresponding metric (2.5.14); however, we consider the projection Z^i of the connecting vector ξ^i to represent the flow (in)stability, onto the space-like platform orthogonal to the time-like geodesics flow. Let us introduce the Fermi frame with its dual one $\{{}^F e^{(a)}\}$ along a time-like geodesic, having the time-like element ${}^F e_{(0)}$ equal to the unit tangent along the geodesic, in order for the projected connecting vector in this frame to have components $Z^{(A)} = \xi^i\ {}^F e_i^{(A)}, (A = 1, 2)$; in this frame the component $Z^{(0)}$ vanishes by construction.

We can finally re-express the stability problem in terms of the system (2.6.23d) coupled to $(a = 0, 1, 2)$

$$\frac{dA^{(a)}{}_{(b)}}{ds} + e^i_{(b)}A^{(a)}{}_{(d)}\frac{De_i^{(d)}}{ds} = 0\,, \tag{2.6.24a}$$

$${}^F e_i^{(a)} = A^{(a)}{}_{(b)}\,(s)\,e_i^{(b)}\,, \tag{2.6.24b}$$

$$\frac{d^2Z^{(A)}}{ds^2} + {}^F R^A{}_{00D}Z^{(D)} = 0\,, \tag{2.6.24c}$$

together with (2.6.16) where the 3×3 matrix $A^{(a)}{}_{(d)}$ defines the Lorentz transformation from the dual frame $e^{(a)}$ to the Fermi dual frame ${}^F e^{(a)}$, ${}^F R^a{}_{bcd}$ are the Fermi frame components of the Riemann tensor and finally λ denotes the Lyapunov exponent calculated in the Euclidean norm $\|\cdot\|$. The characterization via λ of the Mixmaster instability is completely covariant with respect to both the choice of the time variable (the scheme is independent of the lapse function N) as well as of the choice of generic configuration variables.

2.7 On Occurrence of Fractal Boundaries

In order to give an invariant characterization of the dynamics chaoticity have been proposed many methods along the years, but not all approaches have been reached an undoubtable consensus. A very interesting one, relying on techniques considering fractality of the basin of initial conditions evolution has been proposed in 1997 by Cornish and Levin (1997a) which has opened a whole line of debate. The conflict among different approaches has been tackled by using an observer-independent fractal method, nevertheless leaving open some questions about the conjectures lying at the basis of it.
As we have seen in Section 1.4.3, the asymptotic behaviour towards the initial singularity of a Bianchi type IX trajectory depends on whether or not we have a rational or irrational initial condition for the parameter u in the BKL map.

In such a scheme, it has been considered the effect of the Gauss map together with the evolution of the equations of motion in order to "uncover" dynamical properties about the possible outcoming configurations with the varying of the corresponding initial conditions.
Nevertheless, such an approach has led to some doubts regarding the reliability of the method itself.

In fact, let us observe that rationals initial conditions are dense and yet constitutes a set of measure zero and, moreover, they correspond to *fictitious* singularities (Belinski et al., 1970; Misner, 1969). The nature of this initial set needs to be compared with the one regarding the *complete* set of initial conditions, having finite measure over a finite interval: the conclusions obtained after the dynamical evolution are not necessarily complementary between the two initial assumptions (Imponente and Tavakol, 2003).

The approach used in Cornish and Levin (1997a) is based on the method firstly stated in Bleher et al. (1988) where it is shown how fractal boundaries can occur for some solutions involving chaotic systems. The space of initial conditions is spanned giving rise to different exit behaviours whose borders have fractal properties: this constitutes a *conjecture* as a typical property of chaotic Hamiltonian dynamics with multiple exit modes.

For the case of the Bianchi IX model potential (see Figure 2.2) the opening are obtained widening the three corners, on the basis that the point representing the evolution spends much of the time there nearby.
This method has three essential fallacies:

1. the case-points chosen as representatives within this framework are the ones whose dynamics proceeds never reaching the singularity;

2. the "most frequent" dynamical evolution is the one in which the point enters the corner with the velocity *not parallelly* oriented towards the corner's bisecting line and, after some oscillations, it is sent back in the middle of the potential;

3. the artificial opening up of the potential corners adopted in the basin boundary approach could be creating the fractal nature of it.

In particular, the third observation is supported by the existence of strange attractors that *are not* chaotic, as counter-exampled by Grebogi et al. (1984) and discussed by Heagy and Hammel (1994). The choice of the method adopted to characterize the property of chaos or it absence is very relevant, especially when based on the presence of fractal boundaries in the dynamics underlying Bianchi IX models. This is important to be checked, first of all, because the result of Cornish and Levin (1997a) relies on the conjecture as in Bleher et al. (1988) that opening gates in a chaotic Hamiltonian system can result in the presence of fractal basin boundaries (which needs in principle to be checked in the case of Bianchi IX), not satisfying the necessity of a general statement concerning chaos: even the opening of the corners does not solve the question about what happens when taking the limit of closing them and if there is an universal behaviour (for general systems).
Secondly, it is needed to integrate the Bianchi IX flow and this operation is not necessarily commuting with the statement regarding the remaining (and equally relevant) part of the set of initial conditions constituted by the irrational numbers, which needs to be checked (Imponente and Tavakol, 2003).

Motter and Letelier (2001a) in the criticism to the paper of Cornish and Levin (1997a), find some conceptual flaws in their conclusion. They claim the same results with more accurate comprehension of the global chaotic transient and afford calculations involving a more stable constraint check and a higher order integrator.

Again is followed the same criteria used by CORNISH AND LEVIN (1997*a*) to get the same results. The informations obtained following the Farey map approach are not relevant (only rational values of u are led to the three peculiar outcomes) because the corresponding invariant set contains almost every point of phase space. But they claim it is possible to get strict indications of chaos with the Hamiltonian exit method (BLEHER ET AL., 1988; SCHNEIDER AND NEUFELD, 2002; DE MOURA AND GREBOGI, 2002, 2001; BLEHER ET AL., 1989).

Firstly one has to fix the width ($\longleftrightarrow u_{exit}$) of the open corners, then let the system evolve. The future invariant set leads to a box-counting dimension D_0 (estimated from the uncertainty exponent method OTT (1993)) coherent with previous results, which is, by construction, a function of the width itself. The value of D_0 found, equal numerically to 1.87, is dependent on the change done to the original potential, and converges to the value of 2, which is an indicator of *non*-chaoticity DE OLIVEIRA ET AL. (2002). Any of such fundamental property, if outlined in a specific case, must be jointed through a limit procedure to the general case.

PIANIGIANI AND YORKE (1979) study the evolution of a ball on a billiard table with smooth obstacles so that all trajectories are unstable with respect to initial data. This is a system energy conserving and then they open a small hole on such table in order to allow the ball to go through. Such two differences have not been taken account of.

The map counterpart of such a system requires a mapping T twice differentiable, while the Farey map is not smooth for $0 < u < 1$ either for $u \to \infty$ and hence with this procedure is forced a system, *supposing* it to be chaotic, to show peculiar features.

In the work by SCHNEIDER AND NEUFELD (2002) Schneider et al. it is supposed to show the existence of a chaotic saddle, whose signature is the chaotic basin.

The paper by DE OLIVEIRA ET AL. (2002) declares the absence of such points, in a model with $\Lambda = 0$, hence we infer the inapplicability of that method to discover a supposed unknown feature of a dynamical system.
They stress too that the limit (not unnatural) for $\Lambda \to 0$ doesn't matches: it doesn't permit to characterize the chaos in mixmaster vacuum model: a continuous change in a parameter of the theory affects heavily the method's applicability, mainly while the study of Bianchi IX dynamics is of interest towards the

initial singularity, where the BKL approach applies: in such approximation, the domain walls close to a circle.
Hence there are objections which are subject for interesting further investigation

(*i*) has the opening of a polygonal domain the same effect as the opening a circle (which has curvature)?

(*ii*) Is the system truly independent from temporal reparameterization?

(*iii*) Is the opening independent of temporal (either spatial) reparameterization?

(*iv*) Could exist a temporal reparameterization whose effect is to close the artificial openings?

(*v*) How to interpret this eventuality?

Even if this is not relevant for dynamical system in classical mechanics, in General Relativity it is.

Cornish and Levin claim that they open a non compact domain

Misner introduced the variables (β_+, β_-) (Misner, 1969) in the search for new insights on mixmaster dynamics, in view of the solution of the horizon paradox and making a detailed study of the potential corners: billiard balls escaping the potential represent *exceptional* cases of homogeneous cosmologies. Then the Hamiltonian treatment, introduced by Misner in 1969, provides, regardless of the BKL map, the most general evolution.
In such a case remains to deepen the study of the potential domain, near the origin of the (β_+, β_-) plane, where equipotential lines are closed curves, $V < 1$, and between two close equipotential lines, where V increases steeply (i.e. to an increase of $\Delta\beta \approx 2$ it corresponds $\Delta V \approx 10^3$).

The cuncurrent motion of the point particle, as well as of the potential walls, are described by two velocity terms, $d\beta/d\Omega$ and $d\beta_{wall}/d\Omega$ (in Misner formalism $\Omega \equiv \alpha$), whose directions change in an ergodic way in the corresponding phase-space.

The corners of this triangular potential are flared open:

$$V(\beta) \approx 4\beta_-^2 e^{2\beta_+} + 1 \qquad \beta_+ \to +\infty \qquad |\ \beta_-\ | \ll 1 \tag{2.7.1}$$

but equipotentials

$$\beta_- \approx e^{-\beta_+} \tag{2.7.2}$$

narrow down exponentially: is the corner opening an artefact?

Because of the very steep potential rising for large β, little Ω time is spent with the point bouncing against the potential wall and most of the time is spent in free motion, where V can be neglected. If the system point finds itself running towards a corner rather then a wall of the potential, the velocities of the particle and of the walls are in first approximation equal and Kasner behaviour with this parameters will last for a long Ω time.
These directions of β motion are those required to remove horizons in a particular direction.

When the point-universe velocity is closely parallel to one of the three corner axes, horizons along the corresponding one of the three expansion axes of the universe would approach the circumference of the universe. If such a communication phase ($u = -1, 0, \infty$) persisted as $\Omega \to \infty$, then casual influence could circumnavigate the universe only in one direction (Taub metric), but with $\beta_- \equiv 0$ (TAUB, 1951). This is an unstable case and cannot occur: say $\mid \beta_- \mid \ll 1$; the point, after some time, starts oscillating in the corner and is drifted away towards the centre of the potential, to resume bouncing on the flat walls as the small β_- approximation breaks down.
Almost all solutions come arbitrarily close to the values $u = -1, 0, \infty$, as states giving communication along the three corresponding expansion axes of the universe, but such behaviour is unstable.

Summarizing, many questions remain open:

(*i*) Is the generality of the solution found affected by the choices done for the exits?

(*ii*) If one let the system evolve for a very long time, is the fractal boundary picture dependent on the criteria one choose to select the initial data?

(*iii*) (If possible) the choice of initial data from a grid of different *irrational* values for u, would lead to *the same* fractal basin boundaries?

(*iv*) During the course of evolution the solution will come close to Kasner exponents $(0, 0, 1)$, for all the times when $u \approx 0$ (or $u \approx \infty$). Assuming the precise value $u = 0$ is a particular case which leads to the elimination of the physical singularity; the general solution is, by definition, stable.

(v) In the specific Bianchi IX case, whose evolution depends on a numeric-dependent property (rational/irrational initial value), does it make sense to use this method?

(vi) Is there, in numerics, some possibility to distinguish a rational by an irrational number?

The case in which the evolution gets close to the Kasner one (the small oscillation evolution) lasts for a finite amount of time until the dynamics becomes general again: the probability for such cases to occur tends asymptotically to zero and then the the claim of adding exits to a chaotic billiard does not solve, by itself, the problem in how to know it is chaotic – and justifying the method itself.

For all the criticism here outlined we consider an analytical approach crucial to distinguish among chaos indicators relying on numerical properties not well-manageable via numerical simulations.

3 Canonical Quantization of the Mixmaster

3.1 Canonical Quantization of Gravity

The Mixmaster dynamics discussed so far in the details regarding chaoticity and with its properties about statistical approach is of great interest looking for a the connection with the dynamics quite close to the singularity: in the very beginning, close to the Planck epoch ($t \ll 10^{-43}$) the Universe could have performed a still different scenario which needs to be connected with the stage of well formulated theories as the BKL study of the Einstein equations as well as the chaotic behaviour.

The framework for the description of the earlier stages of the universe evolution relies in the quantum cosmology approach whose later stages are connected via a semi-classical approach to the classical–general–relativity dynamics (Lifshitz and Khalatnikov, 1963).

However, before tackling such a semi-classical approximations (Imponente and Montani, 2003*b*, 2002*c*), it is necessary to choose a way by which *quantize* the gravitational degrees of freedom (Pullin, 1991; Zinn-Justin, 1996): a first approach to construct an appropriate theory is based on a canonical quantization procedure through the standard correspondence between canonical variables and operators. The application of such prescription leads to a Schrödinger-like equation for the wave function describing the universe, governing in a unitary expression the geometry of the space-time as well as any other matter or scalar fields involved. This is the Wheeler-DeWitt equation (WDE) (DeWitt, 1967*a*) (Kuchar, 1981) which we later will specify for the Mixmaster model in Section 3.3.1 as in Imponente and Montani (2002*c*, 2003*b*).

The Hamiltonian formulation discussed so far needs to be reformulated in a general way, in order to underline the effects of the $3+1$ split on the slicing of the space-time: the dynamical degrees of freedom are the spatial components of the metric (see also Tomonaga (1946)).

As a different approach to proceed towards quantization, instead of considering spaces with continuously varying curvature, it is possible to deal with spaces where the curvature is restricted to subspaces of codimension two. This approach of discretization was first introduced by Regge (1961), see also (Regge, 1997), while for a more recent proposals via spin-foam methods see (Oriti, 2001).

3.1.1 Hamiltonian Formulation of the Geometrodynamics

The canonical formalism implemented to the gravitational field quantization, leads to the Wheeler-DeWitt equation (WDE) (DeWitt, 1967*a*; Misner et al., 1973), which consists of a functional approach where the states of the theory are represented by wave functionals taken on the three-geometries and, in view of the requirement of general covariance, they do not possess any real time dependence.

Let us introduce a four-dimensional manifold M^4 over which is defined the metric tensor $g_{\mu\nu}(x^\rho)$ and consider the temporal parameter t. Spatial hypersurfaces can be parametrized as $y^\mu(x^i)$ $(\mu = 0, \dots, 3)$ $(i = 1, \dots, 3)$ and in each point it is possible to define a basis via the tangential vectors e^μ_i and the normal vector n^μ, which satisfy

$$g_{\mu\nu} e^\mu_i n^\nu = 0 \, , \quad g_{\mu\nu} n^\mu n^\nu = -1 \, . \tag{3.1.1}$$

Set up a one-parameter family of hypersurfaces depending on the parameter t as $y^\mu = y^\mu(t, x^i)$ in order to reduce the originary four-dimensional manifold to a direct product of a three-dimensional one by the real axis as $\mathcal{M}^4 = \Sigma^3 \otimes \Re$. The deformation vector connecting two points having common spatial coordinate and belonging to adjacent surfaces can be split in the basis (3.1.7) introduced as

$$N^\mu = \partial_t y^\mu(t, x^i) = N n^\mu + N^i e^\mu_i \tag{3.1.2}$$

where the functions N and N^i are the *lapse-function* (identically to the (2.1.25) introduced in Section 2.1.3) and the shift vector, respectively. The geometric interpretation is straightforwardly obtained by the definition: $d\tau = N dt$ is the proper distance between the point P_1 on Σ_t and P_2 on Σ_{t+dt} lying on the intersection of the normal vector coming out from Σ in P_1, while the shift vector N^i represents the distance between the point P_2 and the corresponding P_3, which represents corresponding point with the same spatial coordinates as P_1 belonging to Σ_{t+dt}. The spatial metric induced by the four-dimensional one is the projection of (3.1.1)

$$h_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j = g_{ij} + n_i n_j \,. \tag{3.1.3}$$

From the definition of co-vectors

$$n_\mu = g_{\mu\nu} n^\nu \,, \quad e^i_\mu = h^{ij} g_{\mu\nu} e^\nu_j \,, \quad N_i = h_{ij} N^j \tag{3.1.4}$$

follows the orthonormality condition $e^\mu_i e^j_\mu = \delta^j_i$.
Evaluating

$$dy^\mu = \partial_t y^\mu dt + \partial_i y^\mu dx^i \tag{3.1.5}$$

we can express the distance between any two points from Σ_t to Σ_{t+dt}

$$ds^2 = g_{\mu\nu} dy^\mu dy^\nu \tag{3.1.6}$$

in terms of (3.1.2) as

$$ds^2 = -\left(N dt\right)^2 + h_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right) \,, \tag{3.1.7}$$

where, for this Section only, we adopt the standard signature (-+++) (DeWitt, 1967*a*). The comparison of (3.1.6) with (3.1.7) gives an expression of the metric and its inverse as functions of N, N^i and h_{ij} explicitly

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N_j h^{ij} & N_j \\ N_i & h_{ij} \end{pmatrix} \tag{3.1.8a}$$

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{pmatrix} , \tag{3.1.8b}$$

in consequence of which holds the relation among determinants $\sqrt{-g} = N\sqrt{h}$.
If we introduce the extrinsic curvature

$$K_{ij} \equiv \frac{1}{2N}\left[N_{i|j} + N_{j|i} - \frac{\partial h_{ij}}{\partial t}\right] \tag{3.1.9}$$

where the covariant differentiation $(\ )_|$ is performed with respect to the induced connection on Σ_t, the Ricci scalar in four dimensions multiplied by $\sqrt{-g}$ in terms of the current functions reads

$$\begin{aligned}\sqrt{-g}\mathcal{R}(g) &= \sqrt{h}N\left(K_{ij}K^{ij}-K^2+\ {}^3\mathcal{R}\right)+\\ &- \underbrace{2\left(\sqrt{h}K\right)_{,t}}_{\text{surface term}}+\underbrace{2\left(\sqrt{h}KN^i-\sqrt{h}h^{ij}N_{|j}\right)_{|i}}_{\text{divergence term}}\end{aligned} \tag{3.1.10}$$

where ${}^3\mathcal{R}$ is the Ricci scalar on the spatial surface and the trace $K \equiv K^i{}_i = h^{ij}K_{ij}$.
In suitable units ($\hbar = c = 16\pi G = 1$), we can than express the gravitational Lagrangian density using (3.1.10) as

$$\mathcal{L}^{tot}\left[g_{\mu\nu}\right] = \sqrt{-g}\mathcal{R} \tag{3.1.11a}$$

$$\mathcal{L}\left[N, N^i, h_{ij}\right] = \sqrt{h}N\left[K_{ij}K^{ij}-K^2+\ {}^3\mathcal{R}\right] \tag{3.1.11b}$$

Hence by virtue of (3.1.11) we obtain the global action

$$\mathcal{S}^{tot} = \int_{\mathcal{M}^4} dtd^3x\mathcal{L}^{tot}\left[g_{\mu\nu}\right], \tag{3.1.12}$$

and by virtue of (3.1.12) we have

$$\mathcal{S}^g = \mathcal{S}^{tot} + 2\int_{\Sigma^3} d^3x\sqrt{h}K, \tag{3.1.13}$$

where the second integral is a surface term (York, 1972; Gibbons and Hawking, 1977) which is present if the space-time is compact, we dropped the integral of the divergence term and $\mathcal{S}^g$ is the Einstein–Hilbert action contribution which explicitly reads

$$\begin{aligned}\mathcal{S}^g &= \int_{\Sigma^3\times\Re} dtd^3x\mathcal{L}\left[N, N^i, h_{ij}\right] =\\ &= \int_{\Sigma^3\times\Re} dtd^3xN\sqrt{h}\left(K_{ij}K^{ij}-K^2+\ {}^3\mathcal{R}\right)\end{aligned} \tag{3.1.14}$$

which, as already used in a different form earlier, is well suited to obtain a manageable form for the Hamiltonian terms. In view of (3.1.9) we see that in (3.1.14) there are only spatial derivatives of h_{ij} as well there are no time derivatives of N and N_i, hence the conjugate momenta (standardly defined) read

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = -\sqrt{h}\left(K^{ij}-h^{ij}K\right) \tag{3.1.15a}$$

and

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \tag{3.1.15b}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{N}_i} = 0\,, \tag{3.1.15c}$$

respectively. The primary constraints (3.1.15b) and (3.1.15c) give N and N_i the role of Lagrange multipliers not being dynamical variables.
The prescription to obtain the Hamiltonian formulation require to perform the Legendre transformation (written generically in q^i coordinates and p_i momenta)

$$H \equiv \int d^3x \sum_i \left(p_i \dot{q}^i - \mathcal{L}\left[q^i\right]\right) \tag{3.1.16}$$

hence straightforwardly we have

$$\begin{aligned} H &= \int d^3x \left(\pi^{ij}\dot{h}_{ij} + \pi^i \dot{N}_i + \pi \dot{N} - \mathcal{L}\right) = \\ &= \int d^3x \left(N\mathcal{H}_G + N_i \mathcal{H}^i\right)\,, \end{aligned} \tag{3.1.17}$$

where

$$\begin{aligned} \mathcal{H}_G &\equiv \sqrt{h}\Big(K_{ij}K^{ij} - K^2 - {}^3\mathcal{R}\Big) = \\ &= \frac{1}{2\sqrt{h}}\Big(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\Big)\pi^{ij}\pi^{kl} - \sqrt{h}\ {}^3\mathcal{R} = \\ &\equiv G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\ {}^3\mathcal{R} \end{aligned} \tag{3.1.18}$$

and

$$\mathcal{H}^i \equiv -2\pi^{ij}{}_{|j}\,. \tag{3.1.19}$$

In the expression (3.1.18) the *super-metric* introduced by DeWitt (1967*a*) as

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\Big(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\Big) \tag{3.1.20}$$

satisfies

$$G_{ijkl}G^{klmn} = \delta^{mn}_{ij} = \frac{1}{2}\Big(\delta^m_i\delta^n_j + \delta^m_j\delta^n_i\Big) \tag{3.1.21}$$

and integrating over the three-space permits to define a "distance" between the metric h_{ij} and its variation $h_{ij} + \delta h_{ij}$ as

$$\delta s^2 = \int d^3x G^{ijkl}\delta h_{ij}\delta h_{kl}\,. \tag{3.1.22}$$

In view of the *primary* constraints (3.1.15b) and (3.1.15c), the Poisson brackets giving the Hamilton equation for the evolution in time of π and π^i always vanish

$$\dot{\pi} = -\{H, \pi\} = \frac{\partial H}{\partial N} = 0\,, \tag{3.1.23a}$$

$$\dot{\pi}^i = -\left\{H, \pi^i\right\} = \frac{\partial H}{\partial N_i} = 0\,, \tag{3.1.23b}$$

Since now N and N^i are, in principle, dynamical variables, they have to be varied, so leading to the *secondary* constraints $H^g = 0$ and $H_i^g = 0$ which are equivalent to the $\mu - 0$-components of the Einstein equations and therefore play the role of constraints for the Cauchy data.
Indeed behaving like Lagrange multipliers, the lapse function and the shift vector have not a real dynamics and their specification corresponds to assign a particular slicing of $\mathcal{M}^4$, i.e. a system of reference.

From (3.1.23) follows *(i)* that H is independent of N and N_i and consequently these are not dynamical variables and *(ii)* the secondary constraints over the functional forms

$$\mathcal{H}_G = \mathcal{H}^i = 0\,. \tag{3.1.24}$$

In particular, the first equality in (3.1.24) explicitly reads

$$\mathcal{H}_G = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\ \ {}^3\mathcal{R} = 0 \tag{3.1.25}$$

3.1.2 Wheeler-DeWitt Equation

In order to proceed towards the quantization of the still classical formalism, the constraint equation obtained become equations between operators and moreover commutation relations are taken to correspond to Dirac-bracket relations.
Hence we firstly consider the constraint equation $\mathcal{H}_G = 0$ as a zero-energy Schrödinger equation for the wave function of the Universe described by the state vector $\Psi[h_{ij}]$ in terms of the canonical variables h_{ij}, π_{ij}

$$\mathcal{H}_G\left(\pi_{ij}, h_{ij}\right)\Psi[h_{ij}] = 0\,; \tag{3.1.26}$$

secondly we proceed following the canonical quantization prescription implementing the canonical variables to operators acting on this wave functional

$$h_{ij} \to \hat{h}_{ij}\,, \tag{3.1.27a}$$

and the *super-momentum* with the derivative with respect to the "coordinate" to which is conjugate as

$$\pi^{ij} \to \hat{\pi}^{ij} \equiv -i\frac{\delta}{\delta h_{ij}} \tag{3.1.27b}$$

and finally obtaining the Wheeler-DeWitt (DeWitt, 1967*a*) equation in its general form

$$\left[G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \sqrt{h}\ \ {}^3\mathcal{R}\right]\Psi[h_{ij}] = 0\,. \tag{3.1.28}$$

The effect given by the ordering of the operators when setting up together the terms in (3.1.28) is known as the *normal-ordering* one: when written for an explicit metric, the momenta are multiplied by functions of the coordinates and the effect given by the order of calculation is crucial. We will find in the following Section 3.3 a result relying on a specific choice, nevertheless there is no any *a priori* rule to solve the problem.

Finally, the interpretation to $\Psi[h_{ij}]$ is clearly fastened to the role of time, since we are discussing a general-relativity system and we will dedicate the next Section 3.1.3 to some considerations.

Our application will study the semiclassical limit of the wave function to find a correspondence with some classical quantity.

The primary constraints, in general, are weakly zero (Esposito, 1994): when working out Poisson brackets on phase space involving functions that are compositions of quantum constraints, it is mandatory to follow the Dirac prescription (Dirac, 1964), setting to zero such functional forms only after these brackets have been computed, hence also the appearance of the normal-ordering ambiguity.

In order to build a quantization procedure assume the states be represented by a wave functional $\Psi(\{h_{ij}\}, \phi)$ (the notation $\{h_{ij}\}$ means all the three-geometries connected by a three-diffeomorphism). Let us define the operators $\hat{h}_{ij}$ and $\hat{\pi}^{kl}$ over the state vector represented by the functional $\Psi(\{h_{ab}\})$ as

$$\hat{h}_{ij}(x)\Psi(\{h_{ab}\}) = h_{ij}\Psi(\{h_{ab}\})\,, \tag{3.1.29a}$$

$$\hat{\pi}^{kl}(x)\Psi(\{h_{ab}\}) = -i\frac{\delta\Psi(\{h_{ab}\})}{\delta h_{kl}}\,, \tag{3.1.29b}$$

where $\{h_{ab}\}$ underlines the dependence of the wave functional by the three-geometries and not by the three-metric, a property related to the invariance under the transformations induced by the group of spatial diffeomorphisms.

The equations regarding the two constraints read explicitly

$$\hat{\mathcal{H}}_G(x)\Psi = -:\ G_{ijkl}(x)\frac{\delta^2\Psi}{\delta h_{ij}(x)\delta h_{kl}(x)}\ :\ -\sqrt{h}\ {}^3\mathcal{R}\Psi = 0\,, \tag{3.1.30a}$$

$$\hat{\mathcal{H}}_l(x)\Psi = 2i\ h_{lk}\nabla_j\frac{\delta\Psi}{\delta h_{jk}(x)} = 0\,, \tag{3.1.30b}$$

where in the first row (3.1.30a) we have rewritten the Wheeler-DeWitt equation equivalent to (3.1.28) in which the double dots ":. . . :" underline the necessity of a choice for the operators ordering, eventually requiring for the constraints either to satisfy some algebra (as discussed below) or to be satisfied under coordinate transformations over the Riemannian manifold Σ^3; in the second row, the super-momentum constraint equation (3.1.30b) defines the diffeomorphisms.
In the Wheeler-DeWitt equation there is no explicit time dependence and this feature, known as *frozen formalism*, reflects the absence of a proper parameter measuring time in General Relativity and the consequent need to use variables inner to the system, such as even the matter.
In particular, a wave functional satisfying (3.1.30b) is invariant under the transformations induced by the group $Diff(\Sigma^3)$, justifying the notation $\Psi = \Psi(\{h_{ab}\})$. For this purpose, let us consider a generic spatial diffeomorphism $x'^i = x'^i(x^k)$ which for an infinitesimal transformation reads

$$x'^i = x^i + \xi^i\,, \tag{3.1.31}$$

whose effect is an infinitesimal variation of the metric h_{ij}

$$h'_{ij}(t,x'^i) = h_{ij}(t,x^i)+\delta h_{ij}(t,x^i)\,, \qquad \delta h_{ij} = -\nabla_j\xi_i - \nabla_i\xi_j\,. \tag{3.1.32}$$

The dependence of the functional Ψ on the metric together (3.1.32) induces the variation

$$\Psi(h'_{ij}) = \Psi(h_{ij}) + \int_{\Sigma^3} d^3x\ \delta h_{ij}\frac{\delta\Psi}{\delta h_{ij}}\,, \tag{3.1.33}$$

and the cited invariance expresses as $\Psi(h'_{ij}) = \Psi(h_{ij})$, say

$$\int_{\Sigma^3} d^3x\ \delta h_{ij}\frac{\delta\Psi}{\delta h_{ij}} = 0\,. \tag{3.1.34}$$

The variation δh_{ij} is related to the infinitesimal vector ξ_k by (3.1.32), then (3.1.34) transforms to

$$-2\int_{\Sigma^3} d^3x\nabla_j\xi_i\frac{\delta\Psi}{\delta h_{ij}} = 2\int_{\Sigma^3} d^3x h_{ik}\xi^k\nabla_j\frac{\delta\Psi}{\delta h_{ij}} = 0 \tag{3.1.35}$$

where we have used the property of the product derivation together with the vanishing of the integral of the divergence term over a compact space; the arbitrariness of the transformation induced by ξ^k leads to the vanishing of the integrand

$$h_{ik}\nabla_j \frac{\delta\Psi}{\delta h_{ij}} = 0 \tag{3.1.36}$$

which, apart from the eventual multiplying constant, is exactly the expression for the super-momentum (3.1.30b).

Some limits of the Wheeler-DeWitt equation (Isham and Kuchar, 1985; Isham, 1992) are shared with the equations related to analogous approaches to a canonical quantization and can be briefly summarized as

(*i*) the products of differential operators evaluated in the same points involved in the equations lead to a theory containing divergences when acting over a variety of state functionals, requiring some regularization procedures;

(*ii*) the physical interpretation of the wave function is not as straight as in classical quantum mechanics, essentially with respect to the notion of time and the corresponding evolution, unless considered as an external degree of freedom;

(*iii*) last but not least remark, the *procedure* to find a solution for such a differential equation in not well defined, leaving the possibility to different algorithms of solution; if the eigenvalue equation refers to a null one it should be considered the problem of suitable boundary conditions induced by the theory itself.

The operator algebra relies on the canonical re-writing of the Poisson brackets for the dynamical variables as commutation relations among operators. For such commutator defined over the three-dimensional manifold Σ^3 we have

$$\left[\hat{h}_{ab}(x), \hat{h}_{ab}(x')\right] = 0\,, \tag{3.1.37a}$$

$$\left[\hat{p}^{ab}(x), \hat{p}^{ab}(x')\right] = 0\,, \tag{3.1.37b}$$

$$\left[\hat{h}_{ab}(x), \hat{p}^{cd}(x')\right] = i\hbar\delta^{cd}_{ab}\delta(x-x')\,, \tag{3.1.37c}$$

without considering any specific time dependence in a Schrödinger approach. As in the Dirac quantization prescription, constraints have been imposed in an operatorial form via the substitution of each canonical variable with the corresponding operator as constraints on the allowed physical states as

$$\hat{\mathcal{H}}\left(x;\hat{h},\hat{p}\right)\Psi = 0 \tag{3.1.38a}$$

$$\hat{\mathcal{H}}_i\left(x;\hat{h},\hat{p}\right)\Psi = 0 \tag{3.1.38b}$$

which are the equation discussed above (3.1.30a) and (3.1.30b).
Moreover, it has to be required that no new constraints come out from the commutation relations, i.e. the operatorial constraints generate an algebra formally identical to the classical one, simply following the standard replacing of the Poisson brackets with the commutators multiplied by $(i\hbar)^{-1}$ (or eventually set $\hbar = 1$) obtaining

$$\frac{1}{i}\left[\hat{\mathcal{H}}_i(x),\hat{\mathcal{H}}_k(y)\right] = \hat{\mathcal{H}}_k(x)\delta_{,i}(x-y) - \hat{\mathcal{H}}_i(y)\delta_{,k}(y-x)\,, \tag{3.1.39a}$$

$$\frac{1}{i}\left[\hat{\mathcal{H}}_k(x),\hat{\mathcal{H}}(y)\right] = \hat{\mathcal{H}}(x)\delta_{,k}(x-y)\,, \tag{3.1.39b}$$

$$\frac{1}{i}\left[\hat{\mathcal{H}}(x),\hat{\mathcal{H}}(y)\right] = \hat{\mathcal{H}}^k(x)\delta_{,k}(x-y) - \hat{\mathcal{H}}^i(y)\delta_{,i}(y-x)\,. \tag{3.1.39c}$$

At a classical level, the constraints are equivalent to the dynamical equations if they are satisfied on every spatial hypersurface of a four-dimensional metric $g_{\mu\nu}$ and hence such metric is compatible with the vacuum Einstein equations.

3.1.3 The Problem of Time and the Constraints

The first formulation obtained by DeWitt (DeWitt, 1967*a*,*b*,*c*) for the quantization of gravity, as it stands in the previous Section, has been followed by a huge number of papers (for example Halliwell (1988) has dedicated a whole work to list the bibliography as well as Hartle (1988) to introduce a detailed discussion) and still manifests its open problems (Hartle, 1991): in particular, the meaning of time is peculiar in view of the special role played in any physical theory (Halliwell, 1991), as well as the proper meaning of the wave-function of the Universe (Hartle and Hawking, 1983).
In Newtonian mechanics, for example, the time has the role of a parameter external to the system and this reflects directly in quantum mechanics to build the temporal evolution of an operator as it stands in the Schrödinger-like time dependent equation

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi\,, \tag{3.1.40}$$

which is a functional differential equation: ψ is the wave function and $\hat{H}$ the suitably defined operator corresponding to the Hamiltonian obtained considering a first class constraint.

Let us call with $\hat{C}_l$ the additional constraints: these induce the supplementary conditions which, at the next step, have to be imposed as to the wave function

$$\hat{C}_l \psi = 0\,, \tag{3.1.41}$$

as well as, for $l \neq m$,

$$\hat{C}_m \psi = 0\,. \tag{3.1.42}$$

Multiplying both equation for different operators and subtracting we find

$$\left[\hat{C}_l, \hat{C}_m\right] \psi = 0 \tag{3.1.43}$$

which is not *a priori* obviously satisfied: in the classical theory, by definition, the Poisson bracket of any two primary constraints provides again a linear combination of primary constraints, while in this quantum theory the condition

$$\left[\hat{C}_l, \hat{C}_m\right] = c_{lmn} \hat{C}_n \tag{3.1.44}$$

has to hold as an additional one. In fact, the coefficients c_{lmn} depend on all the field variables and, in general, do not commute with the $\hat{C}_l$ in quantum theory, hence one has to check that they appear on the left of (3.1.44), and no extra terms occur.

The time variable in General Relativity is no longer a parameter but one of the four coordinates, over which are defined transformations, tensors, covariant derivatives and so on: as a consequence, time would be a candidate for the quantization algorithms to obtain operators. The foliation introduced in Section 3.1.1 is the simplest way to give to the temporal coordinate the role of characterizing the various hypersurfaces $\Sigma_t, \Sigma_{t+dt}, \ldots$ and the starting point to tackle quantization.

Conceptually, the non-operatorial role of time is reflected in the notion of measurement for an observable, in the definition of a scalar product, in the constraints on its evolution conservation and finally in the commutation relations in Hilbert space for observables. The first limit posed to the Newtonian time relies in the impossibility of a representation as a physical observable. The question raised by the General Relativity scenario comes out from the property of a covariant formulation with respect to changes of space-time coordinates and such choice freedom has to be compatible with normal quantum theory.

A large number of unsatisfactory features (DeWitt, 1999) flows out from the WDE hyperbolic nature, strongly supporting the idea that is impossible any straightforward extension to the gravitational phenomena of procedures well-tested only in limited ranges of energies; however, in some contexts, like the very early cosmology (Hartle, 1988; Kolb and Turner, 1990) (where a suitable internal time variable is provided by the universe volume) the WDE can give interesting information about the origin of our classical universe, (Kirillov and Montani, 1997a), which may be expected to remain qualitatively valid even for the outcoming of a more consistent approach (Imponente and Montani, 2003f,d).

Over the last ten years the canonical quantum gravity found its best improvement in a reformulation of the constraints problem in terms of the so-called *Ashtekar variables*, leading to the *loop quantum gravity theory* (Rovelli, 1998; QG9, 1992); this more recent approach overcomes some of the WDE limits, like the problem of constructing an appropriate Hilbert space, but under many aspects is yet a theory in progress.

3.2 The Multi-time and Schrödinger Approach

In this section we provide a schematic formulation of the so-called *multi-time approach* and of its smeared Schrödinger version.

The multi-time formalism is based on the idea that many gravitational degrees of freedom appearing in the classical geometrodynamics have to be not quantized because are not real physical ones; indeed we have $10\ \times\infty^3$ variables, i.e. the values of the functions (N, N^i, h_{ij}) at each point of the hypersurface Σ^3, but it is well-known that the gravitational field possesses only $4\times\infty^3$ physical degrees of freedom (in fact the gravitational waves have, at each point of the space, only two independent polarizations and satisfy second order equations).
As a first instance, let us rewrite the canonical action (3.1.17) taking account of (3.1.24) as

$$\mathcal{S}^g = \int_{\mathcal{M}^4}\left(\pi^{ij}\partial_t h_{ij} - N\mathcal{H}^g - N^i\mathcal{H}^g_i\right)d^3x dt\,. \tag{3.2.1}$$

The first step is therefore to extract the real canonical variables by the transformation

$$h_{ij}\,,\pi^{ij} \quad\Rightarrow\quad \xi^\mu\,,\pi_\mu \qquad H_r\,,P^r \tag{3.2.2}$$

$(\mu = 0,1,2,3), (r = 1,2)$ where H_r, P^r are the four real degrees of freedom, while ξ^μ, π_μ are embedding variables.

In terms of this new set of canonical variables, the gravity-"matter" action (3.2.1) rewrites

$$\mathcal{S}^g = \int_{\mathcal{M}^4} \left(\pi_\mu \partial_t \xi^\mu + P^r \partial_t H_r - N H^g - N^i H^g_i \right) d^3x dt \,, \tag{3.2.3}$$

where $H^g = H^g(\xi^\mu, \pi_\mu, H_r, P^r)$ and $H^g_i = H^g_i(\xi^\mu, \pi_\mu, H_r, P^r)$.

An ADM reduction of the dynamical problem is obtained provided that it is possible to solve the Hamiltonian constraint with respect to the momenta π_μ

$$\pi_\mu + h_\mu(\xi^\mu, H_r, P^r) = 0 \,, \tag{3.2.4}$$

reducing the action (3.2.1) to the form

$$\mathcal{S}^g = \int_{\mathcal{M}^4} \left(P^r \partial_t H_r - h_\mu \partial_t \xi^\mu \right) d^3x dt \,. \tag{3.2.5}$$

The Hamiltonian equations lost with the ADM reduction fix the lapse function and the shift vector as soon as the functions $\partial_t \xi^\mu$ are assigned.
A choice of particular relevance is to set $\partial_t \xi^\mu = \delta^\mu_0$ which leads to

$$\mathcal{S}^g = \int_{\mathcal{M}^4} \left\{ P^r \partial_t H_r - h_0 \right\} d^3x dt \,. \tag{3.2.6}$$

The canonical quantization of the model goes through replacing all Poisson brackets with the corresponding commutators; assuming that a wave functional $\Psi = \Psi(\xi^\mu, H_r)$ represents the states of the quantum system, the evolution is described by (let us take $\hbar \neq 1$)

$$i\hbar \frac{\delta \Psi}{\delta \xi^\mu} = \hat{h}_\mu \Psi \,, \tag{3.2.7}$$

where $\hat{h}_\mu$ are the operator version of the classical Hamiltonian densities.
Denoting by $\hat{\mathcal{J}}$ the quantum counterpart to the smeared Hamiltonian

$$\mathcal{J} = \int_{\mathcal{M}^4} \left(h_\mu \partial_t \xi^\mu \right) d^3x dt \,, \tag{3.2.8}$$

the smeared formulation reduces the multi-time approach to the Schrödinger equation

$$i\hbar \partial_t \Psi = \hat{\mathcal{J}} \Psi \,, \qquad \Psi = \Psi(t, H_r) \,. \tag{3.2.9}$$

For example, it is useful to implement such approach in an explicit mini-superspace model like a Bianchi type IX Universe for which the classical action using Misner variables (α, β_+, β_-) as in MISNER ET AL. (1973) (see Section2.1.3) the classical action describing this system reads

$$\mathcal{S} = \int \Big[p_\alpha \dot{\alpha} + p_{\beta_+}\dot{\beta}_+ + p_{\beta_-}\dot{\beta}_- + \\ - cNe^{-3\alpha}\Big(- p_\alpha^2 + p_{\beta_+}^2 + p_{\beta_-}^2 + V(\alpha, \beta_\pm)\Big)\Big] dt\,, \tag{3.2.10}$$

where c is a constant, clearly $\dot{(\,)} \equiv d(\,)/dt$ and the precise form of the potential term V has been shown in Section 2.2 but is not relevant for the present discussion.

In this model the Hamiltonian density is independent of the spatial coordinates, hence the multi-time approach and its smeared Schrödinger version overlap.

The multi-time approach requires preliminarily to perform an ADM reduction of the dynamics (3.2.10). The Hamiltonian constraint obtained by varying N is solved to find

$$-p_\alpha \equiv h_{ADM} = \sqrt{p_{\beta_+}^2 + p_{\beta_-}^2 + V}\,. \tag{3.2.11}$$

therefore the action (3.2.10) rewrites as

$$S = \int \Big(p_{\beta_+}\dot{\beta}_+ + p_{\beta_-}\dot{\beta}_- - \dot{\alpha} h_{ADM}\Big) dt\,. \tag{3.2.12}$$

Here α plays the role of the embedding variable (indeed it is related to the Universe volume), while $\beta_\pm$ are the real gravitational degrees of freedom describing the Universe anisotropy.

By the Hamilton equation lost when varying p_α in (3.2.10) in the ADM reduction we get

$$\dot{\alpha} = -2cNe^{-3\alpha} p_\alpha = 2cNe^{-3\alpha} h_{ADM}\,, \tag{3.2.13}$$

hence, by setting $\dot{\alpha} = 1$, we fix the lapse function as

$$N = \frac{e^{3\alpha}}{2ch_{ADM}} \quad . \tag{3.2.14}$$

Concluding, the quantum dynamics in the multi-time approach is summarized by

$$i\hbar\partial_\alpha \Psi = \sqrt{-\hbar^2\left(\partial_{\beta_+}^2 + \partial_{\beta_-}^2\right) + V}\;\Psi\,, \qquad \Psi = \Psi(\alpha, \beta_\pm)\,. \tag{3.2.15}$$

In this approach the variable α, corresponding to the volume of the Universe, assumed the role of a “time”-coordinate and therefore the quantum dynamics cannot prevent the Universe from reaching the cosmological singularity ($\alpha \to -\infty$).

3.3 Mixmaster Canonical Quantization

The highly symmetric Standard Cosmological Model (SCM) and its agreement with observational data (CMBR and nucleosynthesis of light elements) does not prevent a more general dynamics in the very early stages, followed by isotropization only in a later phase up to a complete agreement with actual experimental data.

The intrinsic nature of the chaotic Mixmaster dynamics and its very early appearance in the Universe evolution lead to believe in the existence of a relation with the quantum behaviour the system performs during the Planckian era.
A precise meaning to this relation relies in the construction of the semiclassical limit of the Schödinger approach to the canonical quantization of the Arnowitt-Deser-Misner (ADM) dynamics whose corresponding probability distribution we will show to coincide with the (deterministic) microcanonical one in Section 2.3.9-2.3.11.

Within the framework developed in Section 2.3 above (Imponente and Montani, 2001), we will show following Imponente and Montani (2003*b*, 2002*c*) the existence of a direct correspondence between the classical and quantum dynamics outlined by the common form of the continuity equation for the statistical distribution and the one for the first order approximation in the semiclassical expansion.

We have seen how the dynamics is described by the canonical variational principle

$$\delta I = \delta \int L \, d\eta = 0 \,, \tag{3.3.1}$$

where details have been already specified (see Section 2.3).

By the use of the MCl variables $\{\xi, \theta, \tau\}$ defined in (2.3.2) and a standard ADM reduction (based on the vanishing nature of the original Hamiltonian) we have seen how the asymptotic properties of the potential term has led in Π_H (see Figure 2.3)) the ADM Hamiltonian to become asymptotically an integral of motion

$$\forall \{\xi, \theta\} \in \Pi_H \qquad \mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + U} \cong \varepsilon = E = \text{const.} \,, \tag{3.3.2}$$

independently from the time parameter.

The billiard representation (Section 2.3.8) has reduced the dynamics to a point-universe moving over a negatively curved two-dimensional space (Lobachevsky plane (Arnold, 1989*b*)). The invariant Lyapunov exponent calculated as (2.4.7) (Kirillov and Montani, 1997*a*; Montani, 2000*b*) and by Imponente and Montani (2002*a*, 2004*d*) in Section 2.4 has given to the bounces against the potential walls and the instability of the geodesic flow a stochastic feature. The "energy-like" constant of motion ($\varepsilon = E$) has been shown to correspond, by a *microcanonical ensemble*, to a uniform invariant measure reading as (2.3.72)

$$d\mu = w_\infty\left(\xi,\theta,\phi\right)d\xi d\theta d\phi \equiv \frac{1}{8\pi^2}d\xi d\theta d\phi \tag{3.3.3}$$

hence over the reduced phase space $\{\xi,\theta\}\otimes S^1_\phi$ the distribution w_∞ behaves like the step-function (Section 2.3.9)

$$w_\infty\left(\xi,\theta,\phi\right) = \begin{cases} \dfrac{1}{8\pi^2} & \forall\ \ \{\xi,\theta,\phi\}\in\Pi_H\otimes S^1_\phi \\ 0 & \forall\ \ \{\xi,\theta,\phi\}\notin\Pi_H\otimes S^1_\phi \end{cases}. \tag{3.3.4}$$

For the sake of clarity, we rewrite here the equations in that Chapter describing the free geodesic motion over Π_H in the asymptotic limit $U\to U_\infty \Rightarrow \varepsilon = E =$ const., (Montani, 2000*b*)

$$\frac{d\xi}{d\tau} = \sqrt{\xi^2-1}\cos\phi\,, \tag{3.3.5a}$$

$$\frac{d\theta}{d\tau} = \frac{\sin\phi}{\sqrt{\xi^2-1}}\,, \tag{3.3.5b}$$

$$\frac{d\phi}{d\tau} = -\frac{\xi\sin\phi}{\sqrt{\xi^2-1}}\,. \tag{3.3.5c}$$

and the *stationary* continuity equation for the distribution function $w_\infty(\xi,\theta,\phi)$ describing the ensemble representation

$$\sqrt{\xi^2-1}\cos\phi\frac{\partial w_\infty}{\partial\xi} + \frac{\sin\phi}{\sqrt{\xi^2-1}}\frac{\partial w_\infty}{\partial\theta} - \frac{\xi\sin\phi}{\sqrt{\xi^2-1}}\frac{\partial w_\infty}{\partial\phi} = 0\,, \tag{3.3.6}$$

reduced on the configuration space Π_H to

$$\varrho\left(\xi,\theta\right) \equiv \int_0^{2\pi} w_\infty\left(\xi,\theta,\phi\right)d\phi\,, \tag{3.3.7}$$

and where the two dimensional continuity equation reads

$$\sqrt{\xi^2-1}\cos\phi\frac{\partial\varrho_\infty}{\partial\xi} + \frac{\sin\phi}{\sqrt{\xi^2-1}}\frac{\partial\varrho_\infty}{\partial\theta} = 0\,. \tag{3.3.8}$$

The *microcanonical* solution on the configuration space $\{\xi,\theta\}$ reads

$$\varrho_\infty(\xi,\theta) = \begin{cases} \frac{1}{4\pi} & \forall \quad \{\xi,\theta\} \in \Pi_H \\ 0 & \forall \quad \{\xi,\theta\} \notin \Pi_H \end{cases} . \tag{3.3.9}$$

3.3.1 Semiclassical Limit of the Quantum Mixmaster Dynamics

The intrinsic chaos resulting from Imponente and Montani (2001) and Imponente and Montani (2002*a*) appears close enough to the Big Bang, with the indeterministic quantum dynamics the model performs in the *Planckian era*, in the sense of a semiclassical limit for a canonical quantization of the model.

The asymptotical principle corresponding to (3.3.2) describes a two dimensional anholonomic Hamiltonian system, which can be quantized by a natural Schrödinger approach

$$i\hbar\frac{\partial\psi}{\partial\tau} = \hat{\mathcal{H}}_{ADM}\psi\,, \tag{3.3.10}$$

$\psi = \psi(\tau,\xi,\theta)$ being the wave function for the point-universe when $\hat{\mathcal{H}}_{ADM}$ is implemented (see (3.3.2)) to an operator, i.e.

$$\xi \to \hat{\xi}\,, \qquad\qquad \theta \to \hat{\theta}\,, \tag{3.3.11a}$$

$$p_\xi \to \hat{p}_\xi \equiv -i\hbar\frac{\partial}{\partial\xi}\,, \qquad\qquad p_\theta \to \hat{p}_\theta \equiv -i\hbar\frac{\partial}{\partial\theta}\,, \tag{3.3.11b}$$

for which the only non vanishing canonical commutation relations are

$$\left[\hat{\xi},\hat{p}_\xi\right] = i\hbar\,, \qquad \left[\hat{\theta},\hat{p}_\theta\right] = i\hbar\,. \tag{3.3.12}$$

The equation (3.3.10) rewrites explicitly, in the asymptotic limit $U \to U_\infty$ (Imponente and Montani, 2003*b*, 2002*c*)

$$\begin{aligned} i\frac{\partial\psi}{\partial\tau} &= \sqrt{\hat{\varepsilon}^2 + \frac{U_\infty}{\hbar^2}}\;\psi = \\ &= \Bigg[-\sqrt{\xi^2-1}\frac{\partial}{\partial\xi}\sqrt{\xi^2-1}\frac{\partial}{\partial\xi} + \\ &\qquad - \frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\theta}\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\theta} + \frac{U_\infty}{\hbar^2}\Bigg]^{1/2}\psi\,, \end{aligned} \tag{3.3.13}$$

where we took an appropriate symmetric normal ordering prescription and we kept U_∞ to stress that the potential cannot be neglected on the entire configuration space $\{\xi,\theta\}$ and, being infinity out of Π_H, it requires as boundary condition for ψ to vanish outside the potential walls

$$\psi\left(\partial\Pi_H\right) = 0\,. \tag{3.3.14}$$

The *quantum* equation (3.3.13) is equivalent to the Wheeler-DeWitt one for the same Bianchi model (3.2.15), once separated the positive- and negative-frequency solutions (KUCHAR, 1981; KUCHAR AND TORRE, 1991), with the advantage that now τ is a real time variable and the equivalence can be trivially checked by taking the square of the operators on both sides of the equation.

Since the potential walls U_∞ are time independent and the domain turns out to be closed, we infer that the energy-spectrum be a discrete one, hence a solution of (3.3.13) has the form

$$\psi\left(\tau,\xi,\theta\right) = \sum_{n=1}^{\infty} c_n e^{-iE_n\tau/\hbar}\varphi_n\left(\xi,\theta\right) \tag{3.3.15}$$

where c_n are constant coefficients; for the quantum point-universe restricted in the finite region Π_H, via the position (3.3.15) in (3.3.13) we find the eigenvalue problem

$$\left[-\sqrt{\xi^2-1}\frac{\partial}{\partial\xi}\sqrt{\xi^2-1}\frac{\partial}{\partial\xi} - \frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\theta}\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\theta}\right]\varphi_n = $$
$$= \left(\frac{{E_n}^2 - U_\infty}{\hbar^2}\right)\varphi_n \equiv \frac{{E_{\infty n}}^2}{\hbar^2}\varphi_n\,. \tag{3.3.16}$$

In what follows we search the semiclassical solution of this equation regarding the eigenvalue $E_{\infty n}$ as a finite constant (i.e. we consider the potential walls as finite) and only at the end of the procedure we will take the limit for U_∞ (2.3.36).

We infer that in the semiclassical limit when $\hbar \to 0$ and the *occupation number* n tends to infinity (but $n\hbar$ approaches a finite value) the wave function φ_n approaches a function φ as

$$\lim_{\substack{n\to\infty\\ \hbar\to 0}} \varphi_n\left(\xi,\theta\right) = \varphi\left(\xi,\theta\right)\,, \qquad \lim_{\substack{n\to\infty\\ \hbar\to 0}} E_{\infty n} = E_\infty\,. \tag{3.3.17}$$

The expression φ is taken as a semiclassical expansion up to the first order, i.e.

$$\varphi\left(\xi,\theta\right) = \sqrt{r\left(\xi,\theta\right)}\exp\left\{i\frac{S\left(\xi,\theta\right)}{\hbar}\right\}\,, \tag{3.3.18}$$

where r and S are functions to be determined.
Substituting (3.3.18) in (3.3.16) and separating the real from the complex part we get two independent equations, i.e.

$$E_\infty{}^2 = \underbrace{\left(\xi^2-1\right)\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{\xi^2-1}\left(\frac{\partial S}{\partial \theta}\right)^2}_{\text{classical term}} + \\ -\frac{\hbar^2}{\sqrt{r}}\left[\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}+\frac{1}{\xi^2-1}\frac{\partial^2}{\partial \theta^2}\right]\sqrt{r} \qquad (3.3.19)$$

where we multiplied both sides by $\hbar^2$ and, respectively,

$$\underbrace{\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}\left(\sqrt{\xi^2-1}\, r\frac{\partial S}{\partial \xi}\right)+\frac{1}{\xi^2-1}\frac{\partial}{\partial \theta}\left(r\frac{\partial S}{\partial \theta}\right)}_{O(1/\hbar)} = 0\,. \qquad (3.3.20)$$

In the limit $\hbar \to 0$ the second term of (3.3.19) is negligible, meanwhile the first one reduces to the Hamilton-Jacobi equation

$$\left(\xi^2-1\right)\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{\xi^2-1}\left(\frac{\partial S}{\partial \theta}\right)^2 = E_\infty{}^2\,. \qquad (3.3.21)$$

The solution of (3.3.21) can be easily checked to be

$$S\left(\xi,\theta\right) = \int\left(\frac{1}{\sqrt{\xi^2-1}}\sqrt{E_\infty^2-\frac{k^2}{\xi^2-1}}\, d\xi + k\, d\theta\right), \qquad (3.3.22)$$

$$k = \text{const.}.$$

We observe that (3.3.21), through the identifications

$$\frac{\partial S}{\partial \xi} = p_\xi\,, \quad \frac{\partial S}{\partial \theta} = p_\theta \quad \Longleftrightarrow \quad S = \int\left(p_\xi d\xi + p_\theta d\theta\right)\,, \qquad (3.3.23)$$

is reduced to the algebraic relation

$$\left(\xi^2-1\right)p_\xi{}^2 + \frac{1}{\xi^2-1}p_\theta{}^2 = E_\infty{}^2\,. \qquad (3.3.24)$$

The constraint (3.3.24) is nothing more than the asymptotic one $\mathcal{H}^2_{ADM} = E^2 =$ const. and can be solved by setting

$$\frac{\partial S}{\partial \xi} = p_\xi \equiv \frac{E_\infty}{\sqrt{\xi^2-1}}\cos\phi\,, \qquad (3.3.25a)$$

$$\frac{\partial S}{\partial \theta} = p_\theta \equiv E_\infty\sqrt{\xi^2-1}\sin\phi\,, \qquad (3.3.25b)$$

where $\phi \in [0, 2\pi[$ is a momentum-function related to ξ and θ by the dynamics, $\varphi(\tau) = \varphi(\xi(\tau), \theta(\tau))$. On the other hand, by (3.3.22) we get

$$p_\xi = \frac{1}{\sqrt{\xi^2 - 1}} \sqrt{E_\infty^2 - \frac{k^2}{\xi^2 - 1}} \tag{3.3.26a}$$

$$p_\theta = k\,; \tag{3.3.26b}$$

apart from the bounces against the potential walls the equations of motion (3.3.5) describe the whole evolution of the system and permit to check the compatibility of these expressions with (3.3.25) providing

$$\frac{d\xi}{d\phi} = -\frac{\xi^2}{\xi^2 - 1}\mathrm{ctg}\varphi \qquad \Rightarrow \qquad \sqrt{\xi^2 - 1}\sin\varphi = c\,, \qquad c = \mathrm{const.}. \tag{3.3.27}$$

The required compatibility comes from the identification $k = E_\infty c$. Since

$$\lim_{U \to U_\infty} E_\infty = \begin{cases} E & \forall\, \{\xi, \theta\} \in \Pi_H \\ i\infty & \forall\, \{\xi, \theta\} \notin \Pi_H \end{cases} \tag{3.3.28}$$

we see by (3.3.22) that the solution $\varphi(\xi, \theta)$ vanishes, as it should occur in presence of infinite potential walls, outside Π_H.
The substitution in (3.3.20) of the positions (3.3.25) leads to the new relation

$$\sqrt{\xi^2 - 1}\cos\phi \frac{\partial r}{\partial \xi} + \frac{\sin\phi}{\sqrt{\xi^2 - 1}} \frac{\partial r}{\partial \theta} = 0\,. \tag{3.3.29}$$

This equation *coincides* with (3.3.8), provided the identification $r \equiv \varrho_\infty$ is made; this is the correspondence between the statistical and the semiclassical quantum analysis, ensuring the quantum chaos of the Bianchi IX model approaches its deterministic one in the considered limit.
Any constant function is a solution of (3.3.29), but the normalization condition requires $r = 1/4\pi$ and therefore we finally get

$$\lim_{\substack{n \to \infty \\ \hbar \to 0}} |\, \varphi_n \,|^2 = |\, \varphi \,|^2 \equiv \varrho_\infty = \begin{cases} \dfrac{1}{4\pi} & \forall \quad \{\xi, \theta\} \in \Pi_H \\ 0 & \forall \quad \{\xi, \theta\} \notin \Pi_H \end{cases}\,, \tag{3.3.30}$$

i.e. the limit for the quantum probability distribution as $n \to \infty$ and $\hbar \to 0$ associated to the wave function

$$\psi(\tau, \theta, \xi) = \varphi(\xi, \theta)\, e^{-i\frac{E}{\hbar}\tau} = \\ = \sqrt{r}\, \exp\left[i \int \Big(p_\xi d\xi + p_\theta d\theta - E_\infty d\tau\Big)\right] \tag{3.3.31}$$

coincides with the classical statistical distribution on the microcanonical ensemble.
Though this formalism of correspondence remains valid for all Bianchi models, only the types VIII and IX admit a normalizable wave function $\varphi(\xi, \theta)$, being confined in Π_H, and a continuity equation (3.3.8) with a real statistical meaning (Imponente and Montani, 2002*a*, 2003*b*).
Since referred to stationary states $\varphi_n(\xi, \theta)$, the considered semiclassical limit has to be intended in view of a "macroscopic" one and is not related to the temporal evolution of the model (Kirillov and Montani, 1997*b*).

4 Inhomogeneous Inflationary Dynamics

4.1 The Standard Cosmological Model

So far we have discussed the Very Early Universe, underlining some peculiar features regarding the evolution towards the initial singularity, on the basis of the general behaviour expected by Einstein equation especially in relation with the instability of density perturbations when evolving backwards in time (LIFSHITZ AND KHALATNIKOV, 1963).

We will discuss how to connect the Mixmaster dynamics and its properties as seen in IMPONENTE AND MONTANI (2003*c*) with an inflationary scenario (IMPONENTE AND MONTANI, 2003*e*,*a*), as well as how such a scheme can be connected with a quasi-isotropic solution of the Einstein equations (IMPONENTE AND MONTANI, 2003*f*,*d*, 2004*c*), in order to recover the homogeneous and isotropic Universe so far observed.

In the present Section, we need to sketch what is expected from such a rich dynamics, say the scenario which is considered to be well representing of actual Universe, i.e. the Standard Cosmological Model, based upon the assumption spatial homogeneity and isotropy. The observed Universe manifests a significant degree of inhomogeneity at the scale of galaxies, clusters, etc., nevertheless observational evidences on sufficiently large scales (of the order of 100 Mpc) show that inhomogeneities are smoothed out and the properties of isotropy and homogeneity are reached as in the FLRW model. Even the early Universe exhibits an isotropy and an homogeneity by far higher than now, as testified by the extreme uniformity (of the order of 10^{-4}) of the Cosmic Microwave Background Radiation (CMBR).

The Robertson-Walker Metric

The most general metric which is spatially homogeneous and isotropic is the FLRW one (1.2.1), which in terms of the *comoving coordinates* (t, r, θ, ϕ) reads

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{4.1.1}$$

where the scale factor $R(t)$ is a generic function of time only and, for an appropriate rescaling of the coordinates, the factor $k = 0, \pm 1$ distinguishes the sign of constant spatial curvature. Such coordinates represent a reference frame participating in the expansion of the Universe: an observer at rest will remain at rest; leaving the effects of the expansion to the cosmic scale factor $R(t)$.

4.1.1 The Friedmann Equation

The FLRW dynamics is reduced to the time dependence of the scale factor $R(t)$ once solved the Einstein equations (1.3.1) with a stress-energy tensor $T_{\mu\nu}$ for all the fields present (matter, radiation, etc.) which then must be diagonal, with the spatial components equal to each other for the homogeneity and isotropy constraints. A simple realization of it is the one of a perfect fluid, characterized by a space-independent energy density $\rho(t)$ and pressure $p(t)$

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)\,, \tag{4.1.2}$$

and in this case the $0-0$ component of the Einstein equations reads

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{4.1.3}$$

which is the *Friedmann equation*, while the $i-i$ components are

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p\,. \tag{4.1.4}$$

The difference between (4.1.3) and (4.1.4) leads to

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)\,, \tag{4.1.5}$$

which is solved for $R(t)$ once provided an equation of state, i.e. a relation between ρ and p.

When the Universe was radiation-dominated, as in the early period, the radiation component provided the greatest contribution to its energy density and for

a photon gas we have $p = \rho/3$.
The present time Universe is on the contrary matter-dominated: the "particles" (i.e. the galaxies) have only long-range (gravitational) interactions and can be treated as a pressureless gas ("dust"): the equation of state is $p = 0$.

In order to find an equation providing the temporal evolution of ρ we are going to we have to use the energy tensor conservation. Nevertheless, such energy tensor appearing in the right-hand side of the Einstein field equation describes the complete local energy due to all non-gravitational fields, while gravitational energy has a non-local contribution. An unambiguous formulation for such a non-local expression is found only in the expressions used at infinity for an asymptotically flat space-time (Penrose, 1982; Arnowitt et al., 1962). This is due to the property of the mass-energy term to be only one component of the energy-momentum tensor which can be reduced only in a peculiar case to a four-vector expression which can not be summed in a natural way.

Bearing in mind such difficulties, such a conservation law ${T^{\mu\nu}}_{;\nu} = 0$ leads to

$$d(\rho R^3) = -p d(R^3) \tag{4.1.6}$$

i.e. the first law of thermodynamics in an expanding Universe which, inserted in the equation of state leads to a differential equation for ρ and hence giving the dependence of the energy density on the scale factor. If we take its solution together with the Friedmann equation (4.1.3) with $k = 0$ is summarized by

$$\text{RADIATION} \qquad \rho \ \propto R^{-4}\,, \quad R \propto t^{1/2} \tag{4.1.7}$$

$$\text{MATTER} \qquad \rho \ \propto R^{-3}\,, \quad R \propto t^{2/3} \tag{4.1.8}$$

where as long as the Universe is not curvature-dominated the choice of $k = 0$ is not relevant. The special equation of state $p = -\rho$ leads to $\rho =$ const. and $R \propto e^{At}$, i.e. a phase of exponential expansion, equivalent to adding a constant term to the right-hand side of the Einstein equation mimicking a cosmological constant: this is exactly what the inflationary paradigm proposes to overcome the paradoxes of the standard model outlined in the next Section 4.1.2.
The Hubble parameter $H \equiv \dot{R}/R$ and the critical density $\rho_c \equiv 3H^2/8\pi G$ make it possible to rewrite (4.1.3) as

$$\frac{k}{H^2 R^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1, \tag{4.1.9}$$

where Ω is the ratio of the density to the critical one $\Omega \equiv \rho/\rho_c$; since $H^2 R^2$ is always positive, the relation between the sign of k and the sign of $(\Omega - 1)$ reads

$$\begin{array}{rclll} k = +1 & \Rightarrow & \Omega > 1 & \text{CLOSED} \\ k = 0 & \Rightarrow & \Omega = 1 & \text{FLAT} \\ k = -1 & \Rightarrow & \Omega < 1 & \text{OPEN} \end{array} \tag{4.1.10}$$

The Friedmann equation describes how the matter distribution influences the curvature of the Universe, coherently with the meaning of the Einstein equations coupling the matter with geometry.

4.1.2 Shortcomings of the Standard Model: Horizon and Flatness Paradoxes

Despite the simplicity of the Friedmann solution (in view also of the thermodynamic property which can be studied in detail) some paradoxes occur when taking into account the problem of *initial conditions*. The observed Universe has to match very specific physical conditions in the very early epoch, but Collins and Hawking (1973)showed that the set of initial data that can evolve to a Universe similar to the present one is of zero measure and the standard model tells *nothing* about initial conditions.

Flatness

Let us assume that all particle species present in the early Universe have the same temperature as the photon bath, i.e. $T_i = T_\gamma$ and are far from any mass threshold. Then the average number of degrees of freedom for the photons and fermions bath g^* is a constant and $T \propto R^{-1}$. The average energy density

$$\rho = \frac{\pi^2}{30} g^* T^4 \tag{4.1.11}$$

substituted in the (4.1.3) reads

$$\left(\frac{\dot{T}}{T}\right)^2 + \epsilon(T) T^2 = \frac{4\pi^3}{45} G g^* T^4 \,, \tag{4.1.12}$$

where

$$\epsilon(T) \equiv \frac{k}{R^2 T^2} = k \left[\frac{2\pi^2}{45} \frac{g^*}{S}\right]^{2/3} \,, \tag{4.1.13}$$

$S = R^3 s$ is total entropy per comoving volume and the entropy density reads

$$s = \frac{2\pi^2}{45} g^* T^3 \,. \tag{4.1.14}$$

Today $\rho \simeq \rho_c$, then by taking $\rho < 10\rho_c$ in (4.1.9) we have

$$\left|\frac{k}{R^2}\right| < 9H^2 \,. \tag{4.1.15}$$

For $k = \pm 1$ (the case $k = 0$ is regained in the limit $R \to \infty$), we have for today $R > \frac{1}{3}H^{-1} \approx 3 \cdot 10^9$ years and $T_\gamma \simeq 2.7$ K; from (4.1.13), the present photon contribution to the entropy has the lower bound

$$S_\gamma > 3 \cdot 10^{85} \tag{4.1.16}$$

expressed in units of the Boltzmann constant $k_B = 1.3806 \cdot 10^{-16}$ erg/K. The relativistic particles present today together with photons are the three neutrino species and their contribution to the total entropy is of the same order of magnitude

$$S > 10^{86} \,, \tag{4.1.17}$$

and finally with

$$|\epsilon| < 10^{-58} g^{*2/3} \tag{4.1.18}$$

we gain

$$\left|\frac{\rho - \rho_c}{\rho}\right| = \frac{45}{4\pi^3}\frac{m_P^2}{g^* T^2}|\epsilon| < 3 \cdot 10^{59} g^{*-1/3}\left(\frac{m_P}{T}\right)^2 \,, \tag{4.1.19}$$

where m_P is the Planck mass $(\hbar c/G)^{1/2} = 2.1768 \cdot 10^{-5}$g $= 1.2211 \cdot 10^{19}$ GeV. When taking $T = 10^{17}$ GeV, all species in the standard model of particle interactions – 8 gluons, $W^\pm$, Z^0, 3 generations of quark and leptons – are relevant and relativistic: then $g^* \approx 100$ and finally

$$\left.\left|\frac{\rho - \rho_c}{\rho}\right|\right|_{T=10^{17}\mathrm{GeV}} < 10^{-55} \,. \tag{4.1.20}$$

A flat Universe today requires a curvature of the original one close to unity up to a part in 10^{55}. A little displacement from flatness at the beginning – for example 10^{-30} – would produce an actual Universe either very open or very closed, so that $\Omega = 1$ is a very unstable condition: this is the *flatness problem.* The natural time scale for cosmology is the Planck time ($\sim 10^{-43}$ sec): in a time of this order a typical closed Universe would reach maximum size while an open one would become curvature dominated. The actual Universe has survived 10^{60} Planck times without neither recollapsing nor becoming curvature dominated.

Horizon

Neglecting the ϵT^2 term, (4.1.12) is solved by

$$T^2 = \left(\frac{4\pi^3}{45} g^*\right)^{-1/2} \frac{m_P}{2\,t}\,. \tag{4.1.21}$$

A light signal emitted at $t = 0$ travelled during a time t the physical distance

$$l(t) = R(t) \int_0^t \frac{dt'}{R(t')} = 2\,t \tag{4.1.22}$$

in a radiation-dominated Universe with $R \propto t^{1/2}$, measuring the physical horizon size, i.e. the linear size of the greatest region causally connected at time t. The distance (4.1.22) has to be compared with the radius $L(t)$ of the region which will evolve in our currently observed region of the Universe. Conservation of entropy for $s \propto T^3$ gives

$$L(t) = \left(\frac{s_0}{s(t)}\right)^{1/3} L_0\,, \tag{4.1.23}$$

where s_0 is the present entropy density and $L_0 \sim H^{-1} \simeq 10^{10}$years is the radius of the currently observed region of the Universe. The ratio of the volumes provides

$$\frac{l^3}{L^3} = 4 \cdot 10^{-89} g *^{-1/2} \left(\frac{m_P}{T}\right)^3 \tag{4.1.24}$$

and, as above, for $g^* \sim 100$ and $T \sim 10^{17}$GeV we obtain

$$\left.\frac{l^3}{L^3}\right|_{T=10^{17}\text{GeV}} \sim 10^{-83}\,. \tag{4.1.25}$$

The currently observable Universe is composed of several regions which have *not* been in causal contact for the most part of their evolution, preventing an explanation about the present days Universe smoothness. In particular, the CMBR is uniform up to 10^{-4}. Moreover, we have at the time of recombination, i.e. when the photons of the CMBR last scattered, the ratio $l^3/L^3 \sim 10^5$: the present Hubble volume consists of about 10^5 causally disconnected regions at recombination and no process could have smoothed out the temperature differences between these regions without violating causality. The particle horizon at recombination subtends an angle of only $0.8°$ in the sky today, while the CMBR is uniform across the sky.

4.2 The Inflationary Paradigm

The basic ideas of the theory of inflation rely firstly on the original work by Guth (1981), i.e. the *old inflation*, which provides a phase in the Universe evolution of inflationary expansion; then formulation of *new inflation* by Linde (1983*a*), introduces the slow-rolling phase in inflationary dynamics; finally, many models have sprung from the original theory.

4.2.1 Old Inflation: the Original Idea

In Guth (1981)describes a scenario capable of avoiding the horizon and flatness problems: both paradoxes would disappear dropping the assumption of adiabaticity and in such a case the entropy per comoving volume S would be related as

$$S_0 = Z^3 S_{\rm early} \tag{4.2.1}$$

where S_0 and $S_{\rm early}$ refer to the values at present and at very early times, for example at $T = T_0 = 10^{17}$ GeV, and Z is some large factor.
With this in mind, the right-hand side of (4.1.19) is multiplied by a factor Z^2 and the value of $|\rho - \rho_c|/\rho$ would be of the order of unity if

$$Z > 3 \cdot 10^{27} \tag{4.2.2}$$

getting rid of the flatness problem.
The right-hand side of (4.1.23) is multiplied by Z^{-1}: for any given temperature the length scale of the early Universe is smaller by a factor Z than previously evaluated, and for Z sufficiently large the initial region which has evolved in our observed one would have been smaller than the horizon size at that time.
Let us evaluate Z considering that the right-hand side of (4.1.23) is multiplied by Z^3: if

$$Z > 5 \cdot 10^{27} \tag{4.2.3}$$

the horizon problem disappears.
Making some *ad hoc* assumptions the model accounts for the horizon and flatness paradoxes while a suitable theory needs a physical process capable of such a large entropy production. A simple solution relies on the assumption that at very early times the energy density of the Universe was dominated by a scalar field $\phi(\vec{x}, t)$, i.e. $\rho = \rho_\phi + \rho_{\rm rad} + \rho_{\rm mat} + \ldots$ with $\rho_\phi \gg \rho_{\rm rad,\,mat,\,etc}$ and hence $\rho \simeq \rho_\phi$.

The quantum field theory Lagrangian density for such a field is

$$\mathcal{L} = \frac{\partial^\mu \phi \, \partial_\mu \phi}{2} - V(\phi) \tag{4.2.4}$$

and the corresponding stress-energy density

$$T^\mu_{\ \nu} = \partial^\mu \phi \, \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu \tag{4.2.5}$$

by which for a spatially homogeneous and isotropic Universe the form of a perfect fluid leads to (LIDDLE, 1989)

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{1}{2R^2} \nabla^2 \phi \, , \tag{4.2.6a}$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{1}{6R^2} \nabla^2 \phi \, . \tag{4.2.6b}$$

Spatial homogeneity would induce a slow variation of ϕ with position, hence the spatial gradients are negligible and the ratio $\omega = p/\rho$ reads

$$\frac{p_\phi}{\rho_\phi} \simeq \frac{\frac{\dot{\phi}^2}{2} + V(\phi)}{\frac{\dot{\phi}^2}{2} - V(\phi)} \, . \tag{4.2.7}$$

If the field is at a minimum of the potential, $\dot{\phi} = 0$, and (4.2.7) becomes an equation of state

$$p_\phi = -\rho_\phi \tag{4.2.8}$$

giving rise to a phase of exponential growth of $R \propto e^{Ht}$, the *inflationary* or *de Sitter phase*.
The field evolution is very different when in vacuum or in a thermal bath and such a coupling can be summarized by adding a term $-(1/2)\lambda T^2 \phi^2$ to the Lagrangian. The potential $V(\phi)$ is replaced by the *finite-temperature* effective potential

$$V_T(\phi) = V(\phi) + \frac{1}{2} \lambda T^2 \phi^2 \, . \tag{4.2.9}$$

In the old inflation, $V(\phi)$ appearing in (4.2.9) has the form of a Georgi-Glashow $SU(5)$ theory

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{1}{3}(A+B)\phi^3 + \frac{1}{2} AB \phi^2 \tag{4.2.10}$$

with $A > 2B > 0$, and possesses a local minimum at $\phi = 0$ and a global minimum at $\phi = A$, separated by a barrier with a maximum at $\phi = B$. The

temperature-dependent term $-(1/2)\lambda T^2\phi^2$ leaves the local minimum unchanged and raises the global minimum as well as the maximum – the former by a larger amount than the latter.
At sufficiently high temperature $V_T(\phi)$ has only one global minimum at $\phi = 0$ and as long as T decreases a second minimum develops at $\phi = \sigma(T)$, with $V(0) < V(\sigma)$, and $\phi = 0$ is still the true minimum of the potential. At a certain critical temperature T_c the two minima are exactly degenerate as $V(0) = V(\sigma)$ and at temperatures below T_c, $V(\sigma) < V(0)$ and $\phi = 0$ is no longer the true minimum of the potential.

Let us consider when at some initial time, corresponding to $T = T_i > T_c$, the field ϕ is trapped in the minimum at $\phi = 0$ (*false vacuum*) with constant energy density, given by (4.2.6a) $V_T(0) \simeq T_C^4$. The temperature lowers with the Universe expansion up to the critical value T_c: the scalar field begins dominating the Universe and a second minimum of the potential develops at $\phi = \sigma$. The inflationary phase is characterized by

$$R(t) \propto e^{Ht} \tag{4.2.11}$$

where the Hubble parameter $H \equiv \dot{R}/R$ is a constant; $\phi = 0$ becomes a metastable state, since there exists a more energetically favourable one at $\phi = \sigma$ (*true vacuum*).
The potential barrier lying within cannot prevent a non-vanishing probability per unit time that the field performs a first order phase transition via quantum tunnelling to the true vacuum state, proceeding along by bubble nucleation: bubbles of the true vacuum phase are created expanding outward at the speed of light in the surrounding "sea" of false vacuum, until all the Universe has undergone the phase transition.
If the rate of bubble nucleation is low, the time to complete the phase transition can be very long if compared to the expansion time scale: when the transition ends, the Universe has cooled to a temperature T_f many orders of magnitude lower than T_c. On the contrary, when approaching the true vacuum, the field ϕ begins to oscillate around this position on a time scale short if compared to the expansion one, releasing all its vacuum energy in the form of ϕ-particles, the quanta of the ϕ field. The oscillations are damped by particles decay and when such products thermalize, the Universe is reheated to a temperature T_r of the order of T_c. This represents the release of the latent heat associated with the phase transition after which the scalar field is no longer the dominant

component of the Universe: inflation comes to an end and the standard FLRW cosmology is recovered.

The non-adiabatic reheating process releases an enormous amount of entropy whose density is increased by a factor of the order of $(T_r/T_f)^3 \simeq (T_c/T_f)^3$, while R remains nearly constant and the entropy is increased by a factor $(T_c/T_f)^3$ too. Flatness and horizon paradoxes are solved if the Universe super-cools of 28 or more orders of magnitude during inflation. Even if this looks very difficult to achieve, it is enough that the transition takes place in about a hundred Hubble times: the inflationary expansion is adiabatic and the entropy density $s \sim R^{-3}$. Since $s \propto T^3$, then $T \propto R^{-1} \propto e^{-Ht}$ and finally

$$\frac{T_c}{T_f} = e^{H\Delta t} \tag{4.2.12}$$

where Δt is the duration of the de Sitter phase. From the requirement $(T_c/T_f) > 10^{28}$ it follows

$$\Delta t > 50 H^{-1}\,. \tag{4.2.13}$$

The critical temperature is estimated to be of order of the energy typically involved for GUT spontaneous symmetry-breaking phase transition, 10^{14} GeV, and

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}V(0) \simeq \frac{T_c^4}{m_P^2} \to H^{-1} \simeq 10^{-34}\text{s}\,. \tag{4.2.14}$$

The inflation removes the discussed paradoxes if the transition takes place in a time $t \approx 10^{-32}$s, nevertheless leaving some open problems regarding its dynamics:

- (i) in the old scenario inflation never ends, due to smallness of the tunnelling transition rate, so that the nucleation of true vacuum bubbles is rare;
- (ii) the energy released during the reheating is stored in the bubbles kinetic energy so that the reheating proceeds via bubble collisions which remain too rare due to low transition rate to produce sufficient reheating: the phase transition is never completed;
- (iii) such a discontinuous process of bubble nucleation via quantum tunnelling should produce a lot of inhomogeneities which aren't actually observed.

4.2.2 New Inflation: the Slow Rolling Model

In 1982, both LINDE (1983*a*) and ALBRECHT AND STEINHARDT (1982) proposed a variant of Guth's model, now referred to as *new inflation* or *slow-rolling inflation*, in order to avoid the shortcomings of the old inflation. Their original idea is to consider a different mechanism of symmetry breaking, the so-called Coleman-Weinberg (CW) mechanism. The potential of the CW model for a gauge boson field with a vanishing mass reads

$$V(\phi) = \frac{B\sigma^4}{2} + B\phi^4 \left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] , \tag{4.2.15}$$

where B is connected to the fundamental constants of the theory and is $\simeq 10^{-3}$, while σ gives the energy associated with the symmetry breaking process and is $\simeq 2 \cdot 10^{15}$ GeV.

The finite-temperature effective potential is obtained as above in (4.2.9) by adding a term of the form $(1/2)\lambda T^2\phi^2$. Expression (4.2.15) can be generalized by adding a mass term of the form $-(1/2)m^2\phi^2$. Defining a temperature-dependent mass

$$m_T \equiv \sqrt{-m^2 + \lambda T^2} , \tag{4.2.16}$$

the temperature-dependent potential becomes

$$V_T(\phi) = \frac{B\sigma^4}{2} + B\phi^4 \left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] + \frac{1}{2}m_T^2\phi^2 . \tag{4.2.17}$$

The quantity m_T^2 can be used to parametrize the potential (4.2.17):

1. when $m_T^2 > 0$, the point $\phi = 0$ is a minimum of the potential, while when $m_T^2 < 0$ it is a maximum;
2. when $m_T^2 < \dfrac{4\sigma^2}{e} \simeq 1.5\sigma^2$, a second minimum develops for some $\bar{\phi} > 0$; initially this minimum is higher than the one at 0, but when m_T becomes lower than a certain value m_T^* $(0 < m_T^* < 1.5\sigma^2)$ it will eventually become the global minimum of the potential.

If at some initial time the ϕ-field is trapped in the minimum at $\phi = 0$, as the temperature lowers the true minimum of the potential can eventually disappear. In this case, as m_T approaches 0, the potential barrier becomes low and can be easily overcome by *thermal* (not quantum) tunnelling, i.e. due to classical (thermal) fluctuations of the ϕ field around its minimum; the barrier can

disappear completely when $m_T = 0$. Independently of what really happens, the phase transition doesn't proceed via quantum tunnelling –a very discontinuous and a strongly first order process– but it evolves either by a *weakly* first order (thermal tunnelling) or second order (barrier disappearing at $m_T = 0$).
Hence the transition occurs rather smoothly, avoiding the formation of undesired inhomogeneities: inflation is not yet started, so the requirement for the field to take a long time to escape the false vacuum is not necessary; the transition rate can be very close to unity, and completed without problem.
When the ϕ-field has passed the barrier (if any), it begins to evolve towards its true minimum. However, the potential (4.2.17) has a very interesting feature: if the coefficient of the logarithmic term is sufficiently high, the potential is very flat around 0, and then the field ϕ "slow rolls" rather than falling abruptly in the true vacuum state: during this slow roll phase the inflation takes place, lasting enough to produce the required supercooling, as seen in the previous Section. When the field reaches the minimum, it begins to oscillate around it thus originating the reheating, exactly as seen in the previous Section.
The problems of Guth's originary model are skipped moving the inflationary phase *after* the field has escaped the false vacuum state, by adding the slow-rolling phase.
Virtually all models of inflation are based upon this principle.

4.2.3 Alternative Models

Many models in the following years have used the idea of an inflationary phase in the early Universe, but none of them is fully satisfactory, nor completely free from any drawback. For instance, in new inflation, we have to keep the potential very flat near the origin in order for sufficient inflation to occur, thus requiring very unnatural fine tuning of the parameters. Moreover, the density perturbations produced are too large and disagree with the observed uniformity of the cosmic microwave background.

Chaotic

Among the many inflationary models sprung from the original theory, one of the first has been suggested by LINDE (1983*b*)the *chaotic inflation*, characterized by the simple potential

$$V(\phi) = \lambda\phi^4 \,. \qquad (4.2.18)$$

The field is initially displaced from its minimum at $\phi = 0$, but its initial value ϕ_{in} is not homogeneous across the Universe, assuming different values in different regions, following a chaotic distribution (hence the name). This model has nothing to do with spontaneous symmetry breaking and GUT, showing that inflation is a concept much more general than previously thought.

Power Law

In *power law inflation* (Liddle, 1989), the potential

$$V(\phi) = V_0 e^{\lambda \phi} \tag{4.2.19}$$

typical of higher dimensional theories gives rise to an expansion phase in which the scale factor evolves as $R(t) \propto t^q$, with $q > 1$. This permits to better control the evolution of density perturbations, avoiding an excessive growth.

Double

In *double inflation* (Silk and Turner, 1987), inflation is obtained not introducing some new scalar field, but modifying Einstein's equations. This is common in modern quantum gravity theories, in which higher order terms such as $\mathcal{R}^2$ ($\mathcal{R}$ is the Ricci scalar) are added to the gravitational Lagrangian and their presence is similar to consider more than one (chaotic) scalar field. This yields two inflationary epochs and two perturbation spectra, yielding firstly very-large scale structures (i.e. clusters of galaxies) and then small large-scale structures (i.e. galaxies), in a *top-down* structure formation scenario.

All these models suffer from a fine tuning problem, being all based on the slow rolling idea, and on the potential flatness close to the origin.
With the spirit of Guth's original work in the late 80's some models have reconsidered a meta-stable vacuum decay to the true vacuum state via a first order phase transition. In such *extended inflation*, the phase transition is implemented in an extended theory of gravitation as in double inflation, seeming to avoid the shortcomings of old inflation but unfortunately, until few years ago there existed very little experimental data to test these models.
The situation is rapidly changing and the new experiments measuring the CMBR could distinguish which model is viable and which is not.

4.2.4 Testing Inflation

A theory as inflation involving energy scales far beyond those testable in particle accelerators leads nevertheless to some “inescapable” predictions, providing some severe tests for the model as, among others

1. $\Omega_0 \cong 1$, i.e. the density has the critical value;
2. homogeneity and isotropy;
3. Harrison-Zeldovich spectrum for density perturbations;
4. Gaussian fluctuations of the CMBR temperature.

$\Omega_0 = 1$: a Flat Universe

Inflation has been developed to explain why the Universe is not yet curvature dominated and having critical density: such a scenario requires very peculiar initial conditions in the early Universe to obtain now a value of Ω differing significantly from 1. Since the smoothness and flatness problems are solved by the same amount of inflation, then $\Omega_0 = 1$ implies that a Universe after the de Sitter phase is very homogeneous and isotropic, even though it began quite differently. The present determinations of Ω_0 are the kinematical ones, relying on the Hubble diagram, the galaxy-number count red-shift test, the rotation curves of galaxies, etc. However, these methods simply constrain Ω_0 in the interval [0.1, 0.2]: this supports the flatness problem, but not yet confirms inflation.
For recent developments on the Wilkinson Microwave Anisotropy Probe (WMAP) temperature correlation interpretation has been recently proposed an approach concerning the questions about curvature and topology (Luminet et al., 2003), accounting for WMAP's observations without fine-tuning parameters using as a geometrical model for finite space the Poincaré dodechaedral one.

A more precise test is given by the analysis of the CMBR angular power spectrum, which reflects the power of the cosmic microwave background temperature fluctuations for a given angular scale, parametrized by the multipole l. The two point angular correlation function is expanded in a series of Legendre polynomials with power spectrum C_l

$$C(\theta) \equiv \left\langle \frac{\Delta T(\hat{n}_1)}{\bar{T}} \frac{\Delta T(\hat{n}_2)}{\bar{T}} \right\rangle = \frac{1}{4\pi} \sum_l (2l+1) C_l P_l(\cos\theta) \qquad (4.2.20)$$

where $\bar{T}$ is the mean temperature of the CMBR ($\bar{T} \cong 2.75$K), the average is taken over all pairs of points separated by an angle θ such that $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$, and P_l is the Legendre polynomial of order l.
In the case of Gaussian temperature fluctuations, all the information is stored in the C_l themselves.
The power spectrum behaviour strongly depends on the model and on the cosmological parameters and a large literature exists on this topic. In particular, in the context of a CDM model, the theory predicts a dominant peak at angular scales of about 1° ($l \approx 200$) for $\Omega_0 = 1$ and the results from the most recent experiments measuring the CMBR spectrum (de Bernardis, 2002; Netterfield et al., 2002; Lee et al., 2001) strongly agree with this prediction; from the peak position, the BOOMERanG collaboration finds $\Omega_0 = 1.03 \pm 0.06$, while DASI finds $\Omega_0 = 1.00 \pm 0.04$ (Halverson, 2001; Pryke et al., 2002).

The temperature fluctuations are approximatively proportional to their seeds density fluctuations and the CMBR can trace the distribution of matter at the epoch of decoupling. Since the CMBR is homogenous and isotropic in about a part in 10^4, apart from extrinsic factors, the Universe should exhibit a similar degree of homogeneity and isotropy.

4.3 The Bridge Solution

An inflationary scenario is of crucial importance to understand how an anisotropic universe like the one described by the Bianchi IX type cosmological solution can approach an isotropic universe when the volume expands enough.
In fact, during the inflation, the dominant term in the Einstein equations corresponds to the effective cosmological constant associated with the false-vacuum energy; such a term is an isotropic one and when dominates it produces an exponential decay of the universe anisotropies.
In this Section, following Imponente and Montani (2004*c*, 2003*e*) we show how it is possible to interpolate a Kasner-like behaviour with an isotropic stage of evolution. We will refer to this scheme as the *bridge solution* by Kirillov and Montani (2002), because it allows to match the chaotic dynamics of the Bianchi IX model together with the later isotropic dynamics of the SCM.

With respect to this, let us observe that in the presence of an effective cosmological constant the action of the Bianchi IX framework takes the form

$$\delta I = \delta \int \Big(p_\alpha \alpha' + p_+ {\beta_+}' + p_- {\beta_-}' + p_\phi \phi' - N\mathcal{H} \Big) d\eta = 0 \tag{4.3.1}$$

where we recall that $\mathcal{H}$ reads as

$$\mathcal{H} = \frac{e^{-3\alpha}}{24\pi} \Big(-p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + \mathcal{V} + e^{6\alpha}\Lambda \Big) \tag{4.3.2}$$

and the Bianchi IX potential is

$$\mathcal{V} = -12\pi^2 e^{4\alpha} U^{(B)} \left(\beta_+, \beta_-\right) . \tag{4.3.3}$$

The variation of the action (4.3.1) with respect to N provides the super-hamiltonian constraint to be $\mathcal{H} = 0$.
Near the Big Bang, $\alpha \to -\infty$, the Bianchi IX potential (4.3.3) turns out to be negligible with respect to the cosmological constant term and then, by replacing the conjugate momenta as

$$p_X \to \frac{\partial I}{\partial X}, \qquad X = \alpha, \beta_\pm, \phi \tag{4.3.4}$$

we get the Hamilton-Jacobi equation

$$-\left(\frac{\partial I}{\partial \alpha}\right)^2 + \left(\frac{\partial I}{\partial \beta_+}\right)^2 + \left(\frac{\partial I}{\partial \beta_-}\right)^2 + \left(\frac{\partial I}{\partial \phi}\right)^2 + e^{3\alpha}\Lambda = 0\,. \tag{4.3.5}$$

The general solution of (4.3.5) takes the form

$$I(\chi^r, \alpha) = \sum_r K_r \chi^r + \frac{2}{3} K_\alpha + \frac{K}{3} \ln \left| \frac{K_\alpha - K}{K_\alpha + K} \right| , \tag{4.3.6}$$

where $\chi_r = \{\beta_+, \beta_-, \phi\}$, K_r are constants of integration and

$$K = \sqrt{\sum_r K_r^2} \tag{4.3.7}$$

in which the index r is the label for $\beta_\pm, \phi$, while

$$K_\alpha(K_r, \alpha) = \pm \sqrt{\sum_r K_r^2 + 6\Lambda \exp\left(3\alpha\right)} , \tag{4.3.8}$$

by which we adopt the negative sign in order to describe universe expansion; in fact, we have

$$\frac{\partial \alpha}{\partial t} = -\frac{N p_\alpha}{3e^{3\alpha/2}} . \tag{4.3.9}$$

According to the Hamilton-Jacobi method, we differentiate with respect to the quantities K^r and then, by putting the resulting expressions equal to arbitrary constant functions as

$$\frac{\delta I}{\delta K^r} = \chi_0^r = \text{const.}\,, \tag{4.3.10}$$

we find the solutions describing the trajectories of the system to be

$$\chi^r(\alpha) = \chi_0^r + \frac{K_r}{3\,|K|}\ln\left|\frac{K_\alpha - K}{K_\alpha + K}\right|\,. \tag{4.3.11}$$

Let us now consider the two opposite limits:

$\alpha \to -\infty$ we find the solution

$$\chi^r(\alpha) = \chi_0^r - \frac{K_r}{K}(\alpha - \alpha_0) \tag{4.3.12}$$

corresponding to a Kasner-like behaviour which can be regarded as the *last epoch* of the oscillatory regime;

$\alpha \to +\infty$ we get the isotropic stage of evolution

$$\chi^r(\alpha) \to \chi_0^r\,. \tag{4.3.13}$$

In fact, when the anisotropies $\beta_\pm$ approach constant values, they are no longer dynamical degrees of freedom and the solution looks homogeneous and isotropic. In the same limit, the scalar field frozens to a constant value too and it disappears from the dynamics as soon as the inflation starts.

Our analysis provides an interpolation between the two relevant stages of the universe evolution and is a convincing feature in favour of the idea that inflation can isotropize the universe. In this sense, the inflationary scenario constitutes the natural mechanism by which the chaotic dynamics of the Bianchi IX model can be smoothed out towards a closed FLRW dynamics.

4.4 Quasi-isotropic Cosmological Solution

The study of the general properties of the Cosmological solution to the Einstein field equations has led to outline many properties of modern Cosmology, in particular with respect to the chaoticity (as discussed in details in Chapter 2), as well as with the questions regarding the existence of a singularity in a general theory framework. The presence of a singularity with respect to time is not a necessary property of cosmological models, however this result does not exclude the possibility that in some restricted classes of cosmological solutions of the Einstein equations. In order to discuss any model is necessary to match the mathematical properties with some deep physical requirements.

In order to consider the eventuality that the present state Universe – as looking homogeneous and isotropic from experimental observations at large scales–, it is interesting to investigate its gravitational stability. The perturbations of the distribution of matter not affecting uniformity are damped with time or remain constant (LIFSHITZ AND KHALATNIKOV, 1963) in the isotropic model. Hence, the evolution backwards in time of small density perturbations is of particular interest when considering cosmological models more general then the homogeneous and isotropic one, being the assumption of uniformity justified only at an approximate level.

The Friedmann solution is a particular case of a class of solutions in which space contracts in a quasi-isotropic way, in the sense that linear distances change with the same time-dependence in all directions, such a solution existing only in a space filled with matter.

When considered the isotropic solution in the synchronous reference frame, isotropy and homogeneity are reflected in the vanishing of the off-diagonal metric components $g_{0\alpha}$. The approach to zero of such functions depends upon the equation of state of matter: for the ultrarelativistic equation $p = \epsilon/3$, the metric is linear in t, hence the metric $g_{\alpha\beta}$ is supposed to be expandable in integral powers of t.
In the case ultra-relativistic matter equation of state $p = \epsilon/3$, the Einstein equations reduce to the partial differential system

$$\frac{1}{2}\partial_t k^{\alpha}_{\alpha} + \frac{1}{4}k^{\beta}_{\alpha}k^{\alpha}_{\beta} = \frac{\epsilon}{3}\left(4u_0u^0 + 1\right) \tag{4.4.1a}$$

$$\frac{1}{2}\left(k^{\beta}_{\alpha;\beta} - k^{\beta}_{\beta;\alpha}\right) = \frac{4}{3}\epsilon u_\alpha u^0 \tag{4.4.1b}$$

$$\frac{1}{2\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}k^{\beta}_{\alpha}\right) + P^{\beta}_{\alpha} = \frac{\epsilon}{3}\left(u_\alpha u^\beta + \delta^{\beta}_{\alpha}\right), \tag{4.4.1c}$$

where, as stated earlier, $P^{\beta}_{\alpha} = \gamma^{\beta\gamma}P_{\alpha\gamma}$ represents the three-dimensional Ricci tensor obtained by the metric $\gamma_{\alpha\beta}$ and u_i $(i = 0, 1, 2, 3)$ denotes the matter four-velocity vector field.

Let us consider a spatial metric of the form

$$g_{\alpha\beta} = ta_{\alpha\beta} + t^2 b_{\alpha\beta} + \dots, \tag{4.4.2}$$

whose inverse reads as

$$g^{\alpha\beta} = t^{-1}a^{\alpha\beta} - b^{\alpha\beta} + \dots\,, \tag{4.4.3}$$

being $a^{\alpha\beta}$ the inverse tensor to $a_{\alpha\beta}$ which is the one used for the operations of rising and lowering indices, as well as for the covariant differentiation. Once (4.4.2) is substituted in the field equations (4.4.1), to leading order we find the energy density

$$\epsilon = \frac{3}{4t^2} - \frac{b}{2t} \tag{4.4.4a}$$

$$u_\alpha = \frac{t^2}{2}\left(b_{;\alpha} - b^{\beta}{}_{\alpha;\beta}\right)\,. \tag{4.4.4b}$$

The three-dimensional Christoffel symbols and the tensor $P_{\alpha\beta}$ are, to first approximation, independent of time and (4.4.1c) gives

$$P_\alpha{}^\beta + \frac{3}{4}b_\alpha{}^\beta + \frac{5}{12}\delta_\alpha{}^\beta b = 0 \tag{4.4.5}$$

and then

$$b_\alpha{}^\beta = -\frac{4}{3}P_\alpha{}^\beta + \frac{5}{18}\delta_\alpha{}^\beta P\,. \tag{4.4.6}$$

The six functions $a_{\alpha\beta}$ are arbitrary and once these are given the coefficients $b_{\alpha\beta}$ are determined by (4.4.6) and hence also the density of matter and its velocity can be derived. When $t \to 0$ the distribution of matter becomes homogeneous and its density approaches a value which is coordinate independent. The expression giving the distribution of velocity follows from (4.4.4b) explicitly as

$$u_\alpha = \frac{t^2}{9}b_{;\alpha}\,. \tag{4.4.7}$$

Such a framework is completed considering that an arbitrary transformation of the spatial coordinates (for example to reduce $a_{\alpha\beta}$ to a diagonal form) leaves to three the number of arbitrary functions allowed in this quasi-isotropic solution, while the fully isotropic model is recovered in the specific choice of $a_{\alpha\beta}$ corresponding to the space of constant curvature $P_{\alpha\beta} = \text{const.} \times \delta_{\alpha\beta}$.

4.5 Quasi-isotropic Solution Towards Singularity

In this Section we show how, following MONTANI (1999, 2000*b*), in the asymptotic limit to the cosmological singularity, the quasi-isotropic universe dynamics in presence of ultrarelativistic matter and a real self-interacting scalar field, while in the next Section, following Imponente and Montani (2003*f*) we will see the opposite limit far from the singularity and its generalization (Imponente and Montani, 2003*d*).

In particular, the presence of the scalar field kinetic term allows the existence of a quasi-isotropic solution characterized by an arbitrary spatial dependence of the energy density associated with the ultrarelativistic matter. To leading order, there is no direct relation between the isotropy of the universe and the homogeneity of the ultrarelativistic matter in it distributed.

However, as discussed deeply in Chapter 1 and 2 the general behaviour of the universe near the initial Big-Bang is characterized by a completely disordered dynamics and an increasing degree of anisotropy, up to develop a fully turbulent regime.

Hence, the contrast between such a general tendency to anisotropy and the evidence that in the forward evolution since a given age the universe should have performed an highly symmetric behaviour, is a problem related to properties of the universe evolution at very different stages of anisotropy.

The quasi-isotropic solution allows, far enough from the initial singularity, the oscillatory regime (Belinski et al., 1970, 1982) to be decomposed in terms of a quasi-isotropic component plus suitable wawe-like small corrections. An analogous decomposition has been obtained in Grischchuk et al. (1975) for the Bianchi type IX model as a homogeneous prototype of the general inhomogeneous case.

Here we summarize the feature acquired by a quasi-isotropic solution (i.e. one in which the three spatial directions are dynamically equivalent) in presence of ultrarelativistic matter and a real self-interacting scalar field. Then a quasi-isotropic model solution exists and is characterized, asymptotically to the Big-Bang, by an arbitrary distribution of the ultrarelativistic matter and in which the spatial curvature component has no dynamical role in the first two orders of approximation. The presence of the scalar field kinetic term, close enough to the singularity, modifies deeply the general cosmological solution, leading to the appearance of a dynamical regime characterized, point by point in space, by the monotonical collapse of the three spatial directions (Belinski and Khalatnikov, 1973; Kirillov and Kochnev, 1987).

Let us consider a synchronous reference frame in which the line element reads as

$$ds^2 = c^2dt^2 - \gamma_{\alpha\beta}(t, x^\gamma)dx^\alpha dx^\beta \,, \tag{4.5.1}$$

the matter is described by a perfect fluid with ultrarelativistic equation of state $p = \frac{\epsilon}{3}$ and the scalar field $\phi(t, x^\gamma)$ admits a potential term $V(\phi)$; the Einstein equations reduce to the partial differential system

$$\frac{1}{2}\partial_t k^\alpha_\alpha + \frac{1}{4}k^\beta_\alpha k^\alpha_\beta = \chi\left[-(4{u_0}^2 - 1)\frac{\epsilon}{3} - \frac{1}{2}(\partial_t\phi)^2 + V(\phi)\right] \tag{4.5.2a}$$

$$\frac{1}{2}\left(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}\right) = \chi\left(\frac{4}{3}\epsilon u_\alpha u_0 + \frac{1}{c}\partial_\alpha\phi\partial_t\phi\right) \tag{4.5.2b}$$

$$\begin{aligned}\frac{1}{2\sqrt{\gamma}}\partial_t(\sqrt{\gamma}k^\beta_\alpha) + P^\beta_\alpha &= \\ &= \chi\left[\gamma^{\beta\sigma}\left(\frac{4}{3}\epsilon u_\alpha u_\sigma + \partial_\alpha\phi\partial_\sigma\phi\right) + \left(\frac{\epsilon}{3} + V(\phi)\right)\delta^\beta_\alpha\right]\end{aligned} \tag{4.5.2c}$$

where, as usual, χ is the Einstein constant $\chi = \dfrac{8\pi G}{c^4}$ (with $c = 1$), obvious notation for derivatives (see also (1.3.9) and following formulas for details).

The partial differential equation describing the scalar field $\phi(t, x^\gamma)$ dynamics, deeply coupled to the Einstein ones reads as

$$\partial_{tt}\phi + \frac{1}{2}k^\alpha_\alpha\partial_t\phi - \gamma^{\alpha\beta}\phi_{;\alpha\beta} + \frac{dV}{d\phi} = 0 \tag{4.5.3}$$

and finally the hydrodynamic equations accounting for the matter evolution are explicitly (Lifshitz and Khalatnikov, 1963)

$$\frac{1}{\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}\epsilon^{3/4}u_0\right) + \frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\epsilon^{3/4}u^\alpha) = 0 \tag{4.5.4a}$$

$$\begin{aligned}4\epsilon\left(\frac{1}{2}\partial_t{u_0}^2 + u^\alpha\partial_\alpha u_0 + \frac{1}{2}k_{\alpha\beta}u^\alpha u^\beta\right) &= \\ &= \left(1 - {u_0}^2\right)\partial_t\epsilon - u_0u^\alpha\partial_\alpha\epsilon\end{aligned} \tag{4.5.4b}$$

$$\begin{aligned}4\epsilon\left(u_0\partial_t u_\alpha + u^\beta\partial_\beta u_\alpha + \frac{1}{2}u^\beta u^\gamma\partial_\alpha\gamma_{\beta\gamma}\right) &= \\ &= -u_\alpha u_0\partial_t\epsilon + \left(\delta^\beta_\alpha - u_\alpha u^\beta\right)\partial_\beta\epsilon\end{aligned} \tag{4.5.4c}$$

Any kind of matter described by a perfect fluid energy-momentum tensor with equation of state $p = c\epsilon$ (here c is a generic constant and not the speed of light), $c \neq 0$, is dynamically equivalent to a scalar field $\psi(t, x^\gamma)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{c}+1\right)} \tag{4.5.5}$$

once identified

$$\epsilon \equiv \frac{1}{2c}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{c}+1\right)}, \tag{4.5.6a}$$

$$p \equiv \frac{1}{2}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{c}+1\right)}, \tag{4.5.6b}$$

$$u_i \equiv \frac{\partial_i\psi}{\sqrt{g^{ik}\partial_i\psi\partial_k\psi}} \tag{4.5.6c}$$

where g_{ik} $(i,k=0,1,2,3)$ is the four-dimensional covariant metric. The considered (Klein-Fock) scalar field ϕ $(c=1)$ corresponds to a perfect fluid with equation of state $p=\epsilon$, as well as the ultrarelativistic matter $(p=\frac{\epsilon}{3})$ is dynamically equivalent to a scalar field ψ, described by the above Lagrangian density in the case $c=\frac{1}{3}$.

The Einstein equations follow by the variational principle

$$\delta S=\delta\int\sqrt{-g}\left\{R-\chi\left[g^{ik}\partial_i\phi\partial_k\phi+\left(g^{ik}\partial_i\psi\partial_k\psi\right)^2\right]\right\}d^4x \tag{4.5.7}$$

where R is the four-dimensional curvature scalar.

The quasi-isotropic solution, as seen in Section 4.4 (and in Lifshitz and Khalatnikov (1963)), near the cosmological singularity refers to a Taylor expansion of the three-dimensional metric time dependence as (see 4.4.2)

$$\gamma_{\alpha\beta}(t,x^\gamma)=\sum_{n=0}^{\infty}a^{(n)}{}_{\alpha\beta}(x^\gamma)\left(\frac{t}{t_0}\right)^n \tag{4.5.8}$$

where

$$a^{(n)}_{\alpha\beta}(x^\gamma)\equiv\left.\frac{\partial^n\gamma_{\alpha\beta}}{\partial t^n}\right|_{t=0}{t_0}^n \tag{4.5.9}$$

in which t_0 is an arbitrarily fixed instant of time $(t\ll t_0)$ and the existence of the singularity implies $a^{(0)}_{\alpha\beta}\equiv 0$. In Lifshitz and Khalatnikov (1963) only the first two terms appear, i.e. $\gamma_{\alpha\beta}=a^{(1)}_{\alpha\beta}\frac{t}{t_0}+a^{(2)}_{\alpha\beta}\left(\frac{t}{t_0}\right)^2$).

The presence of the scalar field permits to relax the assumption of expandability in integer powers.

In order to introduce in a quasi isotropic scenario (eventually inflationary, see below Section 4.7) small inhomogeneous corrections to the leading order, we require a three-dimensional metric tensor having the following structure

$$\begin{aligned}\gamma_{\alpha\beta}\left(t,x^\gamma\right)&=a^2(t)\xi_{\alpha\beta}\left(x^\gamma\right)+b^2(t)\theta_{\alpha\beta}\left(x^\gamma\right)+\mathcal{O}\left(b^2\right)=\\&=a^2(t)\Big[\xi_{\alpha\beta}\left(x^\gamma\right)+\eta(t)\theta_{\alpha\beta}\left(x^\gamma\right)+\mathcal{O}\left(\eta^2\right)\Big]\end{aligned} \tag{4.5.10}$$

where we defined $\eta \equiv \dfrac{b^2}{a^2}$ and suppose that η satisfies the condition

$$\lim_{t\to\infty} \eta(t) = 0\,. \tag{4.5.11}$$

In the limit of the considered approximation, the inverse three-metric reads

$$\gamma^{\alpha\beta}\left(t, x^\gamma\right) = \frac{1}{a^2(t)}\Big(\xi^{\alpha\beta}\left(x^\gamma\right) - \eta(t)\theta^{\alpha\beta}\left(x^\gamma\right) + \mathcal{O}\left(\eta^2\right)\Big)\,, \tag{4.5.12}$$

where $\xi^{\alpha\beta}$ denotes the inverse matrix of $\xi_{\alpha\beta}$ and assumes a metric role, i.e. we have

$$\xi^{\beta\gamma}\xi_{\alpha\gamma} = \delta^\beta_\alpha\,, \qquad \theta^{\alpha\beta} = \xi^{\alpha\gamma}\xi^{\beta\delta}\theta_{\gamma\delta}\,. \tag{4.5.13}$$

The covariant and contravariant three-metric expressions lead to the important explicit relations

$$k^\beta_\alpha = 2\frac{\dot a}{a}\delta^\beta_\alpha + \dot\eta\theta^\beta_\alpha \quad \Rightarrow \quad k^\alpha_\alpha = 6\frac{\dot a}{a} + \dot\eta\theta\,, \qquad \theta \equiv \theta^\alpha_\alpha\,. \tag{4.5.14}$$

Since the fundamental equality $\partial_t(\ln\gamma) = k^\alpha_\alpha$ holds, then we immediately get

$$\gamma = ja^6e^{\eta\theta} \quad \Rightarrow \quad \sqrt{\gamma} = \sqrt{j}a^3e^{\frac{1}{2}\eta\theta} \sim \sqrt{j}a^3\left(1 + \frac{1}{2}\eta\theta + \mathcal{O}(\eta^2)\right)\,, \tag{4.5.15}$$

once defined $j \equiv \det\xi_{\alpha\beta}$.

The Landau-Raychaudhury theorem applied to the present case implies the condition

$$\lim_{t\to 0} a(t) = 0\,. \tag{4.5.16}$$

The set of field equations 4.5.2 is solved retaining only the terms linear in η and its time derivatives and neglecting all terms containing spatial derivatives of the dynamical variables, in order to obtain asymptotic solutions in the limit $t \to 0$ and then checking the self-consistence of the approximation scheme. The possibility to neglect the potential term $V(\phi)$ is not ensured by the field equations but is based on the idea that, in an inflationary scenario, the scalar field potential energy becomes dynamically relevant only during the "slow-rolling phase", far from the singularity, while asymptotically the kinetic term dominates. With this in mind, it is possible to find the solution for $a(t)$

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{3}}\,, \tag{4.5.17}$$

where t_0 is an integration constant, for $\eta(t)$

$$\eta(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} , \tag{4.5.18}$$

and u_α

$$u_\alpha(t, x^\gamma) = v_\alpha(x^\gamma)\left(\frac{t}{t_0}\right)^{\frac{1}{3}} + \mathcal{O}\left(\frac{t}{t_0}\right) , \tag{4.5.19}$$

respectively, in order to obtain for the tensor $\theta_{\alpha\beta}(x^\gamma)$ the expression

$$\theta_{\alpha\beta} = \frac{\rho}{3+4v^2}\left[\left(1-2v^2\right)\xi_{\alpha\beta} + 10 v_\alpha v_\beta\right] \quad \Rightarrow \quad \theta = \rho\,, \tag{4.5.20}$$

where $\rho(x^\gamma)$ denotes an arbitrary function of the spatial coordinates.
The energy density of the ultrarelativistic matter is found, to leading order, to have the expression

$$\epsilon(t, x^\gamma) = \frac{5\rho(x^\gamma)}{3\chi\left[3 + 4v^2(x^\gamma)\right] {t_0}^{2/3} t^{4/3}} + \mathcal{O}\left(\frac{t}{t_0}\right) , \tag{4.5.21}$$

which permits to integrate the scalar field equation (4.5.3) to obtain

$$\phi(t, x^\gamma) = \sqrt{\frac{2}{3\chi}}\left[\ln\left(\frac{t}{t_0}\right) - \frac{3}{4}\left(\frac{t}{t_0}\right)^{\frac{2}{3}}\rho(x^\gamma) + \sigma(x^\gamma)\right] + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{4.5.22}$$

where $\sigma(x^\gamma)$ is an arbitrary function of the spatial coordinates.
Finally, equation (4.5.2c) yields the expression for the functions v_α in terms of ρ and of the spatial gradient $\partial_\alpha\sigma$ as

$$v_\alpha = -\frac{3(3+4v^2)}{10\rho\sqrt{1+v^2}} t_0 \partial_\alpha \sigma\,, \tag{4.5.23a}$$

$$v^2 = \frac{24\tau^2 - 1 + \sqrt{1-12\tau^2}}{2(1-16\tau^2)} \tag{4.5.23b}$$

where τ represents the quantity

$$\tau \equiv \frac{3t_0}{10\rho}\sqrt{\xi^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma}\,. \tag{4.5.23c}$$

The particular and simple case $\sigma \equiv 0$, in correspondence to which $v^2 \equiv 0$

($v_\alpha \equiv 0$) leads to the solutions

$$\theta_{\alpha\beta} = \frac{1}{3}\rho(x^\gamma)\xi_{\alpha\beta} \tag{4.5.24a}$$

$$\epsilon(t, x^\gamma) = \frac{5}{9\chi}\rho(x^\gamma)\frac{1}{{t_0}^{2/3}t^{4/3}} + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{4.5.24b}$$

$$\phi(t, x^\gamma) = \sqrt{\frac{2}{3\chi}}\ln\left(\frac{t}{t_0}\right) - \frac{3}{4}\sqrt{\frac{2}{3\chi}}\left(\frac{t}{t_0}\right)^{2/3}\rho(x^\gamma) + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{4.5.24c}$$

$$u_\alpha(t, x^\gamma) = \frac{3}{8}\partial_\alpha\ln\big(\rho(x^\gamma)\big)t + \mathcal{O}\left(\frac{t}{t_0}\right). \tag{4.5.24d}$$

Finally we obtain the three-dimensional metric tensor as

$$\gamma_{\alpha\beta}(t, x^\gamma) = \left(\frac{t}{t_0}\right)^{2/3}\left[1 + \left(\frac{t}{t_0}\right)^{2/3}\frac{\rho(x^\gamma)}{3}\right]\xi_{\alpha\beta}(x^\gamma) + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{4.5.24e}$$

On the basis of equations (4.5.24) the hydrodynamic ones (4.5.4) reduce to an identity in the considered approximation.

The solution here shown is completely self-consistent to the first-two orders in time and contains five physically arbitrary functions of the spatial coordinates: three out of the six functions $\xi_{\alpha\beta}$ (the remaining three of them can be fixed by pure spatial coordinates transformations), the spatial scalar $\rho(x^\gamma)$ and the function $\sigma(x^\gamma)$.

The independence among the functions $\xi_{\alpha\beta}$, ρ and σ implies the existence of a quasi-isotropic dynamics in correspondence to an arbitrary spatial distribution of ultrarelativistic matter.
The kinetic term of the scalar field behaves, to leading order, as $\sim a^{-6}$ and therefore, in the limit $a \to 0$, dominates over the ultrarelativistic energy density which diverges only as $\sim a^{-4}$ with respect to which the spatial curvature term $\sim a^{-2}$ is negligible.

Concluding, for a generic equation of state $p = c\epsilon$ the corresponding matter energy density behaves asymptotically as $\sim a^{-3(1+c)}$, but the above dynamical scheme is not qualitatively affected when considering values of c in the range $-\frac{1}{3} < c < 1$ instead of the ultrarelativistic case $c = \frac{1}{3}$.

4.6 Quasi-isotropic Inflationary Solution

In this Section we find a solution for a quasi-isotropic inflationary Universe which allows to introduce in the problem a certain degree of inhomogeneity

(Imponente and Montani, 2003f,a). We consider a model which generalizes the (flat) FLRW one by introducing a first order inhomogeneous term, whose dynamics is induced by an effective cosmological constant. The three-metric tensor consists by a dominant term, corresponding to an isotropic-like component, while the amplitude of the first order one is controlled by a "small" function $\eta(t)$.

In a Universe filled with ultra relativistic matter and a real self-interacting scalar field, we discuss the resulting dynamics, up to first order in η, when the scalar field performs a slow roll on a plateau of a symmetry breaking configuration and induces an effective cosmological constant.

We show how the spatial distribution of the ultra relativistic matter and of the scalar field admits an arbitrary form but nevertheless, due to the required inflationary e-folding, it cannot play a serious dynamical role in tracing the process of structures formation (via the Harrison–Zeldovich spectrum). As a consequence, we find reinforced the idea that the inflationary scenario is incompatible with a classical origin of the large scale structures.

4.6.1 Quasi-isotropic Inflation and Density Perturbation

As we have seen, the inflationary model (Guth, 1981; Coleman and Weinberg, 1973) is, up to now, the most natural and complete scenario to make account of the problems outstanding in the Standard Cosmological Model, like the horizons and flatness paradoxes (Kolb and Turner, 1990) (for pioneer works on inflationary scenario and the spectrum of gravitational perturbation, see also Starobinsky (1980, 1979)); indeed such a dynamical scheme, on one hand is able to justify the high isotropy of the cosmic microwaves background radiation (characterized by temperature fluctuations $\mathcal{O}(10^{-4})$ de Bernardis (2002)) and, on the other one, provides a mechanism for generating a (scale invariant) spectrum of inhomogeneous perturbations (via the scalar field quantum fluctuations).
Moreover, as shown in Kirillov and Montani (2002); Starobinsky (1983), a slow-rolling phase of the scalar field allows to connect the generic inhomogeneous Mixmaster dynamics (Belinski et al., 1970; Imponente and Montani, 2002a) with a later quasi-isotropic Universe evolution, in principle compatible with the actual cosmological picture, (van Elst et al., 1995).
With respect to this, we investigate the dynamics performed by small inhomogeneous corrections to a leading order metric, during inflationary expansion.

The model presented in Imponente and Montani (2003*f*) has the relevant feature to contain inhomogeneous corrections to a flat FLRW Universe, which in principle could take a role to understand the process of structure formation, even in presence of an inflationary behaviour; however, a careful analysis of our result prevents this possibility in view of the strong inflationary e-folding, so confirming the expected incompatibility between an inflationary scenario and a classical origin of the Universe clumpyness.

In what follows, we will use the quasi isotropic solution which was introduced in Lifshitz and Khalatnikov (1963) (see Section 4.4) as the simplest, but rather general, extension of the FLRW model; for a discussion of the quasi isotropic solution in the framework of the "long-wavelength" approximation, see Tomita and Deruelle (1994) while for the implementation of such a solution after inflation (Khalatnikov et al., 1983; Belinsky et al., 1985) to generic equation of state and to the case of two ideal hydrodynamic fluid see, respectively, Khalatnikov et al. (2002) and Khalatnikov et al. (2003). In the previous Section 4.5 this solution is discussed in the presence of a real scalar field kinetic energy, which leads to a power-law solution for the three-metric, and predicts interesting features for the ultra relativistic matter dynamics.

We analyse here the opposite dynamical scheme, when the scalar field undergoes a slow-rolling phase since the effective cosmological constant dominates its kinetic energy. We provide, up to the first two orders of approximation and in a synchronous reference, a detailed description of the three-metric, of the scalar field and of the ultra relativistic matter dynamics, showing that the volume of the Universe expands exponentially and induces a corresponding exponential decay (as the inverse fourth power of the cosmic scale factor), either of the three-metric corrections, as well as of the ultra relativistic matter (the same behaviour characterizes roughly even the scalar field inhomogeneities). It is remarkable that the spatial dependence of such component is described by a function which remains an arbitrary degree of freedom; in spite of such freedom in fixing the primordial spectrum of inhomogeneities, due to the inflationary e-folding, we show there is no chance that, after the de Sitter phase, such relic perturbations can survive enough to trace the large scale structures formation by an Harrison–Zeldovich spectrum.
This behaviour suggests that the spectrum of inhomogeneous perturbations (Ma and Bertschinger, 1995) cannot arise directly by the classical field nature,

but by its quantum dynamics.

Finally, we recall that the presence of the kinetic term of a scalar field, here regarded as negligible, induces, near enough to the singularity, a deep modification of the general cosmological solution, leading to the appearance of a dynamical regime, during which, point by point in space, the three spatial directions behave monotonically (Belinski and Khalatnikov, 1973; Berger, 2000).

4.6.2 Inhomogeneous Perturbations from an Inflationary Scenario

The theory of inflation is based on the idea that during the Universe evolution a phase transition takes place (for instance associated with a spontaneous symmetry breaking of a Grand-Unification model of strong and electroweak interactions) which induces an effective cosmological constant dominating the expansion dynamics. As a result, an exponential expansion of the Universe arises and, under a suitable fine-tuning of the parameters, it is able to "stretch" so strongly the geometry that the *horizon* and *flatness paradoxes* of the SCM are naturally solved.
In the *new inflation* theory (see Section 4.2.2), the Universe undergoes a de-Sitter phase when the scalar field performs a "slow-rolling" behaviour over a very flat region of the potential between the false and true vacuum. The exponential expansion ends with the scalar field falling down in the potential well associated to the real vacuum and the scalar field dies via damped (by the expansion of the Universe and particles creation) oscillations which reheat the cold Universe left by the de-Sitter expansion (the relativistic particles temperature is proportional to the inverse scale factor). Indeed, the decay of this super-cooled bosons condensate into relativistic particles – as a typical irreversible process – generates a huge amount of entropy, which allows to account for the present high value ($\sim \mathcal{O}(10^{88})$) of the Universe entropy per comoving volume.

Apart from the transition across the potential barrier between false and true vacuum, which takes place in general via a tunnelling, the whole inflationary dynamics can be satisfactorily described via a classical *uniform* scalar field $\phi = \phi(t)$. The assumption that the field behaves in a classical way is supported by its bosonic and cosmological nature, but the existence of quantum fluctuations of the field within the different inflationary "bubbles" leads to relax the hypothesis of dealing with a perfectly uniform scalar field.

In general, when analysing density perturbations, it is convenient to introduce the dimensionless quantity (Kolb and Turner, 1990; Padmanabham, 1993)

$$\delta\rho(t,x^\gamma) \equiv \frac{\Delta\rho(t,x^\gamma)}{\bar{\rho}} = \frac{\rho-\bar{\rho}}{\bar{\rho}}\,, \tag{4.6.1}$$

where $\bar{\rho}$ denotes the mean density and $\gamma = 1,2,3$. The best formulation of the density perturbations theory is obtained expanding $\delta\rho$ in its Fourier components, or modes,

$$\delta\rho_k = \frac{1}{(2\pi)^3}\int d^3x\, e^{ik_\alpha x^\alpha}\,\delta\rho(t,x^\gamma)\,. \tag{4.6.2}$$

As long as the perturbations are in the linear regime, i.e. $\delta\rho_k \ll 1$, it is possible to follow appropriately the dynamics of each mode with wave number k, which corresponds to a wavelength $\lambda = \frac{2\pi}{k}$; however, in an expanding Universe, the *physical* size of the perturbations evolves via the *cosmic scale factor* which, in order to avoid ambiguities, from now on, we write as $a(t)$.

Since in the Standard Cosmological Model, the "Hubble radius" scales as $H^{-1} \propto t$, while $a(t) \propto t^n$ with $n < 1$, then every perturbation, now inside the Hubble radius, was outside it at some earlier time. We stress how the perturbations with a physical size, respectively smaller or greater than the Hubble radius, have a very different dynamics, the former ones being affected by the action of the microphysics processes.

In the case of an inflationary scenario, the situation is quite different. Since during the de Sitter phase the Hubble radius remains constant, while the cosmic scale factor "blows up" exponentially; hence, all cosmologically interesting scales have crossed the horizon twice, i.e. the perturbations begin sub-horizon sized, cross the Hubble radius during inflation and later cross back again inside the horizon.
This feature has a strong implication on the initial spectrum of density perturbations predicted by inflation. We present a qualitative argument to understand how this spectrum can be generated.

During inflation, the density perturbations are expected to arise from the quantum mechanical fluctuations of the scalar field ϕ; these are, as usual, decomposed in their Fourier components $\delta\phi_k$, i.e.

$$\delta\phi_k = \frac{1}{(2\pi)^3}\int d^3x\, e^{ik_\alpha x^\alpha}\,\delta\phi(t,x^\gamma)\,. \tag{4.6.3}$$

The spectrum of quantum mechanical fluctuations of the scalar field is defined as

$$(\Delta\phi)^2_k \equiv \frac{1}{\mathcal{V}}\frac{k^3}{2\pi^2}\left|\delta\phi_k\right|^2 \,, \tag{4.6.4}$$

where $\mathcal{V}$ denotes the comoving volume. For a massless minimally coupled scalar field in a de Sitter space-time, which approximates very well the real physical situation during the Universe exponential expansion, it is well known that (see Kolb and Turner (1990))

$$(\Delta\phi)^2_k = \left(\frac{H}{2\pi}\right)^2 \,, \tag{4.6.5}$$

then the mean square fluctuation of ϕ, $(\Delta\phi)^2$, takes the following form

$$(\Delta\phi)^2 = \frac{1}{(2\pi)^3 V}\int d^3k\,\left|\delta\phi_k\right|^2 = \int \left(\frac{H}{2\pi}\right)^2 d(\ln k)\,. \tag{4.6.6}$$

Since H is constant during the de Sitter phase of the Universe, each mode k contributes roughly the same amplitude to the mean square fluctuation. Indeed, the only dependence on k takes place in the logarithmic term, but the modes of cosmological interest lay between 1 Mpc and 3000 Mpc (it is commonly adopted the convention to set the actual cosmic scale factor equal to unity), corresponding to a logarithmic interval of less than an order of magnitude.
Thus we can conclude that any mode k crosses the horizon having almost a constant amplitude $\delta\phi_k \simeq H/2\pi$. A delicate question concerns the mechanism by which such quantum fluctuations of the scalar field achieve a classical nature(Polarski and Starobinsky, 1996); here we simply observe how each mode k, once reached a classical stage, is governed by the dynamics

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0\,; \tag{4.6.7}$$

according to this equation, super-horizon modes $k \ll aH$ (i.e. $\lambda_{phys} \gg H^{-1}$) admit the trivial dynamics (4.6.7) with $\delta\phi_k \sim$ const.. This simple analysis implies the important feature that any mode re-enters the horizon with roughly the same amplitude it had at the first horizon crossing. The spectrum of perturbations so generated is then induced into the relativistic energy density coming from the reheating phase, associated with the bosons decay; since that moment the evolution of the perturbation spectrum follows a standard paradigm.

The density perturbations discussed so far are related to the scalar field by

$$\delta\rho = \delta\phi\frac{\partial V}{\partial\phi}\,, \tag{4.6.8}$$

where, because of the potential is very flat during inflation, $\partial V/\partial\phi$ is approximatively constant and then we have

$$\delta\rho \simeq \text{const.} \times \delta\phi \tag{4.6.9}$$

The spectrum of density perturbations has a Harrison-Zeldovich form, characterized by constant amplitude: this is a very generic prediction of inflation, based on the features of the potential flatness common to nearly all inflationary models. On the other hand, the spectrum amplitude is model dependent and accurate measures could discriminate between the various models.

Finally, we discuss another feature of the quantum mechanical fluctuations generated by the inflation, regarding their Gaussian distribution: as long as the field ϕ is minimally coupled, it has a low self interaction and each mode fluctuates independently; hence, since the fluctuations we actually observe are the sum of many of its quantum ones, their distribution can be expected to be Gaussian (as it should be for the sum of many independent variables).

This is reflected on the distribution of temperature fluctuations of the CMBR as a powerful test inflation. In a detailed analysis by WU (2001),82 (even if not independent) hypothesis tests for Gaussianity are implemented, showing how the MAXIMA map is consistent with Gaussianity on angular scales between 10' and 5°, where deviations are most likely to occur.

4.6.3 Geometry, Matter and Scalar Field Equations

We have already discussed in Section 1.3 the line element in a synchronous reference frame of coordinates (t, x^γ) (in units $c = 1$) which we re-write here for simplicity as

$$ds^2 = dt^2 - \gamma_{\alpha\beta}(t, x^\gamma)dx^\alpha dx^\beta \,, \tag{4.6.10}$$

where $\alpha, \beta, \gamma = 1, 2, 3$.

Let us describe the matter by a perfect fluid with ultra relativistic equation of state $p = \dfrac{\epsilon}{3}$ (p and ϵ denote respectively the fluid pressure and energy density) and a scalar field $\phi(t, x^\gamma)$ with a potential term $V(\phi)$.
In what follows, we write the Einstein equations as

$$R_i^k = \chi \sum_{(z)=m,\phi} \left(T_i^{k(z)} - \frac{1}{2}\delta_i^k T_l^{l(z)} \right) \tag{4.6.11}$$

where χ denotes the Einstein constant $\chi = 8\pi G$ (G being the Newton constant) and $T_i^{k(m)}$ and $T_i^{k(\phi)}$ indicate, respectively, the energy-momentum tensor of the matter and the scalar field.

We have to add to the set (1.3.8) used in Chapter 1 those interactions and explicitly reduce such partial differential equations to the system

$$\frac{1}{2}\partial_t k^\alpha_\alpha + \frac{1}{4}k^\beta_\alpha k^\alpha_\beta = \chi\left[-\left(4{u_0}^2 - 1\right)\frac{\epsilon}{3} - (\partial_t\phi)^2 + V(\phi)\right] \tag{4.6.12a}$$

$$\frac{1}{2}\left(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}\right) = \chi\left(\frac{4}{3}\epsilon u_\alpha u_0 + \partial_\alpha\phi\partial_t\phi\right) \tag{4.6.12b}$$

$$\frac{1}{2\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}k^\beta_\alpha\right) + P^\beta_\alpha = \chi\left[\gamma^{\beta\gamma}\left(\frac{4}{3}\epsilon u_\alpha u_\gamma + \partial_\alpha\phi\partial_\gamma\phi\right) + \right.$$
$$\left. + \left(\frac{\epsilon}{3} + V(\phi)\right)\delta^\beta_\alpha\right], \tag{4.6.12c}$$

(for details, see equations (1.3.10) and (1.3.11) in Section 1.3) where the vector field u^i $(i = 0, \ldots, 3)$ represents the matter four-velocity, the three-dimensional Ricci tensor $P^\beta_\alpha = \gamma^{\beta\gamma}P_{\alpha\gamma}$ is constructed via the metric $\gamma_{\alpha\beta}$ which is also used to form the covariant derivative $(\quad)_{;\alpha}$; we recall also the notations

$$\gamma \equiv \det\gamma_{\alpha\beta}\,, \qquad k_{\alpha\beta} \equiv \partial_t\gamma_{\alpha\beta}\,, \qquad k^\beta_\alpha = \gamma^{\beta\sigma}k_{\alpha\sigma}\,. \tag{4.6.13}$$

The dynamics of the scalar field $\phi(t, x^\gamma)$ is described by a partial differential equation, coupled to the above Einsteinian system, which in a synchronous reference reads

$$\partial_{tt}\phi + \frac{1}{2}k^\alpha_\alpha\partial_t\phi - \gamma^{\alpha\beta}\phi_{;\alpha\beta} + \frac{dV}{d\phi} = 0 \tag{4.6.14}$$

where we adopted the obvious notation $\partial_{tt}(\,) \equiv \dfrac{\partial^2(\,)}{\partial t^2}$. The hydrodynamic equations, taking into account for the matter evolution, in a synchronous reference and for the ultra relativistic case, possess the structure

$$\frac{1}{\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}\epsilon^{3/4}u_0\right) + \frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\epsilon^{3/4}u^\alpha) = 0 \tag{4.6.15a}$$

$$4\epsilon\left(\frac{1}{2}\partial_t{u_0}^2 + u^\alpha\partial_\alpha u_0 + \frac{1}{2}k_{\alpha\beta}u^\alpha u^\beta\right) =$$
$$= \left(1 - {u_0}^2\right)\partial_t\epsilon - u_0 u^\alpha\partial_\alpha\epsilon \tag{4.6.15b}$$

$$4\epsilon\left(u_0\partial_t u_\alpha + u^\beta\partial_\beta u_\alpha + \frac{1}{2}u^\beta u^\gamma\partial_\alpha\gamma_{\beta\gamma}\right) =$$
$$= -u_\alpha u_0\partial_t\epsilon + \left(\delta^\beta_\alpha - u_\alpha u^\beta\right)\partial_\beta\epsilon\,. \tag{4.6.15c}$$

In view of the chosen feature for (4.6.11), equation (4.6.15a) doesn't contain spatial gradients of the three-metric tensor and of the scalar field. This scheme is completed by observing how it can be made covariant with respect to coordinate transformations of the form

$$t' = t + f(x^\gamma)\,, \qquad x^{\alpha\prime} = x^{\alpha\prime}(x^\gamma) \tag{4.6.16}$$

f being a generic space dependent function.

4.7 Quasi Isotropic Inflationary Solution

In order to introduce in a quasi isotropic (inflationary) scenario small inhomogeneous corrections to the leading order, we require a three-dimensional metric tensor having the structure as in (4.5.10)-(4.5.13) and (4.5.14)-(4.5.15).

We shall analyse the field equations (4.6.12a)-(4.6.12c) retaining only terms linear in η and its time derivatives. Equations (4.6.12a)-(4.6.12c) are analysed via the standard procedure of constructing asymptotic solutions in the limit $t \to \infty$, by verifying *a posteriori* the self-consistency of the approximation scheme, i.e. that the neglected terms were really of higher order in time (Imponente and Montani, 2003*f*).

In the quasi-isotropic approach, we assume that the scalar field dynamics, in the plateau region, is governed by a potential term as

$$V(\phi) = \Lambda + K(\phi)\,, \qquad \Lambda = \text{const.} \tag{4.7.1}$$

where Λ is the dominant term and $K(\phi)$ is a small correction to it. The role of K, as shown in the following, is to contain inhomogeneous corrections via the ϕ-dependence; the functional form of K can be any one of the most common inflationary potentials, as they appear near the flat region for the evolution of ϕ.

What follows remains valid, for example, for the relevant cases of the quartic and Coleman–Weinberg expressions already introduced in some detail in Section 4.2.2

$$K(\phi) = \begin{cases} -\dfrac{\lambda}{4}\phi^4\,, & \lambda = \text{const.} \\ B\phi^4\left[\ln\left(\dfrac{\phi^2}{\sigma^2}\right) - \dfrac{1}{2}\right]\,, & \sigma = \text{const.}\,, \end{cases} \tag{4.7.2}$$

viewed as corrections to the constant Λ term, although explicit calculations are developed below only for the first case.

Our inflationary solution is obtained under the standard requirements

$$\frac{1}{2}\left(\partial_t\phi\right)^2 \ll V\left(\phi\right) \tag{4.7.3a}$$

$$\mid \partial_{tt}\phi \mid \ll \mid k^\alpha_\alpha \partial_t\phi \mid\,. \tag{4.7.3b}$$

The above approximations and the substitution of (4.5.14) reduce the scalar field equation (4.6.14) to the form

$$\left(3\frac{\dot{a}}{a}+\frac{1}{2}\dot{\eta}\theta\right)\partial_t\phi-\lambda\phi^3=0 \tag{4.7.4}$$

where we assumed that the contribution of the ϕ spatial gradient is negligible.

Similarly, the quasi-isotropic approach (in which the inhomogeneities become relevant only for the next-to-leading order), once neglecting the spatial derivatives, in (4.6.15a), leads to

$$\begin{aligned}&\sqrt{\gamma}\epsilon^{3/4}u_0=l(x^\gamma)\ \Rightarrow\\&\qquad\Rightarrow\epsilon\sim\frac{l^{4/3}}{j^{2/3}a^4{u_0}^{4/3}}\left(1-\frac{2}{3}\eta\theta+\mathcal{O}(\eta^2)\right)\end{aligned} \tag{4.7.5}$$

where $l(x^\gamma)$ denotes an arbitrary function of the spatial coordinates.

Let us now face, in the same approximation scheme, the analysis of the Einstein equations (4.6.12a)-(4.6.12c). Taking into account (4.7.3a), up to the first order in η, equation (4.6.12a) reads

$$3\frac{\ddot{a}}{a}+\left[\frac{1}{2}\ddot{\eta}+\frac{\dot{a}}{a}\dot{\eta}\right]\theta-\chi\Lambda=-\chi\frac{\epsilon}{3}\left(3+4u^2\right) \tag{4.7.6}$$

having set

$$u^2\equiv\frac{1}{a^2}\xi^{\alpha\beta}u_\alpha u_\beta\quad\Rightarrow\quad u_0=\sqrt{1+u^2}\,. \tag{4.7.7}$$

Equation (4.6.12c) reduces to the form

$$\begin{aligned}&\frac{2}{3}\left(a^3\right)_{,tt}\delta^\beta_\alpha+\left(a^3\ \eta_{,t}\right)_{,t}\theta^\beta_\alpha+\frac{1}{3}\left[\left(a^3\right)_{,t}\eta\right]_{,t}\theta\delta^\beta_\alpha+aA^\beta_\alpha=\\&\qquad=\chi\left[\frac{1}{a^2}\left(\xi^{\beta\gamma}-\eta\theta^{\beta\gamma}\right)\frac{4}{3}\epsilon u_\alpha u_\gamma+\right.\\&\qquad\qquad\left.+\left(\frac{\epsilon}{3}+\Lambda\right)\delta^\beta_\alpha\right]2a^3\left(1+\frac{\eta\theta}{2}\right)\,,\end{aligned} \tag{4.7.8}$$

where we adopted the notation $(\ )_{,t}\equiv d/dt$ and $(\ )_{,tt}\equiv d^2/dt^2$ for simplicity of writing. In this expression, the spatial curvature term reads, to leading order, as

$$P^\beta_\alpha(t,x^\gamma)=\frac{1}{a^2(t)}A^\beta_\alpha(x^\gamma)\,, \tag{4.7.9}$$

where $A_{\alpha\beta}(x^\gamma)=\xi_{\beta\gamma}A^\gamma_\alpha$ denotes the Ricci tensor corresponding to $\xi_{\alpha\beta}(x^\gamma)$.

The trace of (4.7.8) provides the additional relation

$$2\left(a^3\right)_{,tt} + \left(a^3\eta\right)_{,tt}\theta + aA^\alpha_\alpha =$$
$$= \chi\left[\frac{\epsilon}{3}\left(3+4u^2\right) + 3\Lambda\right]2a^3\left(1+\frac{\eta\theta}{2}\right). \qquad (4.7.10)$$

Comparing (4.7.6) with the trace (4.7.10), via their common term $(3+4u^2)\epsilon/3$, and estimating the different orders of magnitude, we get the following equations:

$$\left(a^3\right)_{,tt} + 3a^2 a_{,tt} - 4\chi a^3\Lambda = 0 \qquad (4.7.11)$$

$$A_{\alpha\beta} = 0 \qquad (4.7.12)$$

$$3\left(a^3\eta\right)_{,tt} + 3a^3\eta_{,tt} + 2\left(a^3\right)_{,t}\eta_{,t} +$$
$$+9a^2\eta\, a_{,tt} - 12\chi a^3\Lambda\eta = 0\,. \qquad (4.7.13)$$

Since (4.7.12) implies the vanishing of the the three-dimensional Ricci tensor and this condition corresponds to the vanishing of the Riemann tensor too, then we can conclude that the obtained Universe is flat up to the leading order, i.e.

$$\xi_{\alpha\beta} = \delta_{\alpha\beta} \qquad \Rightarrow \qquad j = 1. \qquad (4.7.14)$$

Equation (4.7.11) admits the expanding solution

$$a(t) = a_0 \exp\left(\frac{\sqrt{3\chi\Lambda}}{3}t\right) \qquad (4.7.15)$$

a_0 being the initial value of the scale factor amplitude, taken at the instant $t = 0$ when the de Sitter phase starts.

Expression (4.7.15) for $a(t)$, when substituted in (4.7.13) yields the differential equation for η

$$\ddot{\eta} + \frac{4}{3}\sqrt{3\chi\Lambda}\,\dot{\eta} = 0\,, \qquad (4.7.16)$$

whose only solution, satisfying the limit (4.5.11), reads

$$\eta(t) = \eta_0 \exp\left(-\frac{4}{3}\sqrt{3\chi\Lambda}\,t\right) \qquad \Rightarrow \qquad \eta = \eta_0\left(\frac{a_0}{a}\right)^4, \qquad (4.7.17)$$

and, of course, we require $\eta_0 \ll a_0$.

Equations (4.7.5) and (4.7.6), in view of the solutions (4.7.15) for $a(t)$ and (4.7.17) for $\eta(t)$, are matched with consistency, by posing

$$u_\alpha(t, x^\gamma) = v_\alpha(x^\gamma) + \mathcal{O}\left(\eta^2\right)$$
$$(u_0)^2 = 1 + \mathcal{O}\left(\frac{1}{a^2}\right) \approx 1\,, \qquad (4.7.18)$$

and respectively

$$\epsilon = -\frac{4}{3}\Lambda\eta\theta\,, \tag{4.7.19}$$

which implies $\theta < 0$ for each point of the allowed domain of the spatial coordinates. The comparison between (4.7.5) and (4.7.19) leads to the explicit expression also for $l(x^\gamma)$ in terms of θ

$$l(x^\gamma) = \left(\frac{4}{3}\Lambda\eta_0{a_0}^4\right)^{3/4}(-\theta)^{3/4}\,. \tag{4.7.20}$$

Defining the auxiliary tensor with unit trace $\Theta_{\alpha\beta}(x^\gamma) \equiv \theta_{\alpha\beta}/\theta$, the above analysis permits, from (4.7.8), to obtain for it the expression

$$\Theta_\alpha^\beta = \frac{\delta_\alpha^\beta}{3}\,. \tag{4.7.21}$$

By (4.7.4), the explicit form for a, once expanded in η, yields the first two leading orders of approximation for the scalar field

$$\phi\left(t, x^\gamma\right) = \mathcal{C}\sqrt{\frac{t_r}{t_r - t}}\left(1 - \frac{1}{4\sqrt{3\chi\Lambda}}\frac{\eta}{t_r - t}\theta\right)\,, \tag{4.7.22}$$
$$t_r = \frac{\sqrt{3\chi\Lambda}}{\mathcal{C}^2 2\lambda}\,,$$

where $\mathcal{C}$ is an integration constant; finally, equation (4.6.12c) provides v_α in terms of θ

$$v_\alpha = -\frac{3}{4}\frac{1}{\sqrt{3\chi\Lambda}}\partial_\alpha \ln \mid \theta \mid\,. \tag{4.7.23}$$

On the basis of (4.7.21)-(4.7.23), the hydrodynamic equations (4.6.15a)-(4.6.15c) reduce to an identity, to leading order of approximation; in fact such equations contain the energy density of the ultra relativistic matter, which is known only to first order (the higher one of the Einstein equations). Therefore it makes no sense to take into account higher order contributions, coming from those equations.

As soon as $(t_r - t)$ is sufficiently large, it can be easily checked that the solution here constructed is completely self-consistent to the all calculated orders of approximation in time and contains one physically arbitrary function of the spatial coordinates, $\theta(x^\gamma)$ which, indeed, being a three-scalar, is not affected by spatial coordinate transformations.

In particular, the terms quadratic in the spatial gradients of the scalar field are of order

$$\left(\partial_\alpha\phi\right)^2 \approx \mathcal{O}\left(\frac{\eta^2}{a^2}\frac{1}{\left(t_r - t\right)^3}\right) \tag{4.7.24}$$

and therefore can be neglected with respect to all the inhomogeneous ones. Such a solution fails when t approaches t_r and therefore its validity requires that the de Sitter phase ends (with the fall of the scalar field in the true potential vacuum) when t is yet much smaller than t_r (see below).

4.7.1 Physical considerations

The peculiar feature of the solution constructed above lies in the independence of the function θ which, from a cosmological point of view, implies the existence of a quasi-isotropic inflationary solution in correspondence with an arbitrary spatial distribution of ultra-relativistic matter and of the scalar field.

We get an inflationary picture from which the Universe outcomes with the appropriate standard features, but in presence of a suitable spectrum of *classical* perturbations as due to the small inhomogeneities which can be modelled according to an Harrison–Zeldovich spectrum; in fact, expanding the function θ in Fourier series as

$$\theta(x^\gamma) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \tilde{\theta}\left(\vec{k}\right) e^{i\vec{k}\cdot\vec{x}} d^3k\,, \qquad (4.7.25)$$

we can impose an Harrison–Zeldovich spectrum by requiring

$$\mid \tilde{\theta} \mid^2 = \frac{Z}{\mid k \mid^{3/2}}\,, \qquad Z = \text{const.}\,. \qquad (4.7.26)$$

However, the following three points have to be taken into account to give a complete picture for our analysis:

(i) limiting (as usual) our attention to the leading order, the validity of the slow-rolling regime is ensured by the natural conditions

$$\mathcal{O}\left(\sqrt{\chi\Lambda}(t-t_r)\right) \ll 1\,, \qquad \lambda \gg \mathcal{O}(\chi^2\Lambda)\,, \qquad (4.7.27)$$

which respectively translate (4.7.3b) and (4.7.3a);

(ii) denoting by t_i and t_f respectively the beginning and the end of the de-Sitter phase, we should have $t_r \gg t_f$ and the validity of our solution is guaranteed if

(a) the flatness of the potential is preserved, i.e. $\lambda\phi^4 \ll \Lambda$: such a requirement coincides, as it should, with the second of inequalities (4.7.27);

(b) given Δ as the width of the flat region of the potential, we require that the de-Sitter phase ends before t becomes comparable with t_r, i.e.

$$\phi(t_f) - \phi(t_i) \sim \sqrt{\frac{\sqrt{\chi\Lambda}}{\lambda}\frac{t_f - t_i}{t_r^{3/2}}} \sim \mathcal{O}(\Delta)\,, \tag{4.7.28}$$

where we expanded the solution to first order in $t_{i,f}/t_r$; via the usual position $(t_f - t_i) \sim \mathcal{O}(10^2)/\sqrt{\chi\Lambda}$, the relation (4.7.28) becomes a constraint for the integration constant t_r.

(iii) In order to get an inflationary scenario, able to overcome the shortcomings present in the Standard Cosmological Model, we need an exponential expansion sufficiently strong. For instance we have to require that a region of space, corresponding to a cosmological horizon $\mathcal{O}(10^{-24}cm)$ when the de-Sitter phase starts, now covers all the actual Hubble horizon $\mathcal{O}(10^{26}cm)$; the redshift at the end of the de Sitter phase is $z \sim \mathcal{O}(10^{24})$, then we should require $a_f/a_i \sim e^{60} \sim \mathcal{O}(10^{26})$. Let us estimate the density perturbations (inhomogeneities) at the (matter-radiation) decoupling age ($z \sim \mathcal{O}(10^4)$) as $\delta_{in} \sim \mathcal{O}(10^{-4})$; if we start by this same value at the beginning of inflation (δ^i_{in}), we arrive at the end with $\delta^f_{in} \sim (\eta_f/\eta_i)\delta^i_{in} \sim \mathcal{O}(10^{-100})$. Though these inhomogeneities increase as z^2 once they are at scale greater than the horizon, nevertheless they reach only $\mathcal{O}(10^{-60})$ at the decoupling age. This result provides support to the idea that the spectrum of inhomogeneous perturbations cannot have a classical origin in presence of an inflationary scenario.

In the considerations above developed, we regard the ratio of the inhomogeneous terms ϵ_f and ϵ_i as the quantity $\delta\rho$ and now we show how this assumption is (roughly) correct: after the reheating the Universe is dominated by a homogeneous (apart from the quantum fluctuations) relativistic energy density ρ_r to which is superimposed the relic ϵ_f after inflation; therefore we have

$$\delta\rho = \frac{\epsilon_f}{\rho_r} = \frac{\epsilon_f}{\epsilon_i}\frac{\epsilon_i}{\rho_r} = \left(\frac{a_i}{a_f}\right)^4 \frac{\epsilon_i}{\rho_r}\,. \tag{4.7.29}$$

Hence our statement follows as soon as we observe that the inhomogeneous relativistic energy density before the inflation ϵ_i and the uniform one ρ_r, generated by the reheating process, differ by only some orders of magnitude.

4.8 Generic Inflationary Solution

In this Section, following Imponente and Montani (2003d) we provide a generic inhomogeneous solution concerning the dynamics of a real self interacting scalar field minimally coupled to gravity in a region of the configuration space where it performs a slow rolling on a plateau of its potential. During the generic inhomogeneous de Sitter phase the scalar field dominant term is a function only of the spatial coordinates. This solution specialized nearby the FLRW model allows a classical origin for the inhomogeneous perturbations' spectrum.

4.8.1 Generality Requirements

When referred to a homogeneous and isotropic FLRW model, the de Sitter phase of the inflationary scenario rules out so strongly the small inhomogeneous perturbations, that makes them unable to become seeds for the later structures formation (Tomita and Deruelle, 1994; Imponente and Montani, 2003f). This picture emerges sharply within the inflationary paradigm and it is at the ground level of the statements according to which the cosmological perturbations arise from the scalar field quantum fluctuations (Polarski and Starobinsky, 1996).

Though this argument is well settled down and is very attractive even because the predicted quantum spectrum of inhomogeneities takes the Harrison-Zeldovich form, nevertheless the question remains open whether, in more general contexts, it is possible that classical inhomogeneities can survive up to a level to be relevant for the origin of the actual Universe large scale structures.

Indeed here we investigate the behavior of a generic cosmological model (MacCallum, 1979; Belinski et al., 1982) which undergoes a de Sitter phase (Starobinsky, 1983; Kirillov and Montani, 2002) and show how such general scheme allows the scalar field to retain, at the end of the exponential expansion, a generic inhomogeneous term on its leading order (for connected topic see Barrow (1987)).

Thus our analysis provides relevant information either with respect to the morphology of a generic inflationary model, either stating that within this framework the scalar field is characterized by an arbitrary spatial function which plays the role of its leading order. We consider a generic inhomogeneous model in the same way as Belinski, Khalatnikov and Lifshitz (BKL) (Belinski et al., 1982) (see also Belinski and Khalatnikov (1973); Montani (2000b)) did when they

discussed the chaotic cosmologies, i.e. we refer to a cosmological model which contains a number of physically (gauge independent) arbitrary spatial functions, sufficient to assign a generic Cauchy problem on a non-singular space-like hypersurface.
The model taken into account as in Imponente and Montani (2003*d*, 2004*d*) refers to the coupled dynamics of a cosmological model with a real self-interacting scalar field and the solution we construct concerns the phase of the evolution when the potential associated to the scalar field performs a plateau behaviour and the Universe evolution is dominated by the effective cosmological constant associated with the plateau level over the true vacuum state of the theory. We are in a position to neglect the contribution due to the non-relativistic matter because it would be relevant only for higher order terms and becomes increasingly negligible as the exponential expansion develops.

4.8.2 Generic Inflationary Model

The Einstein equations in the presence of a self interacting scalar field $\{\phi(t,x^\gamma), V(\phi)\}$ without involving ultra-relativistic matter read

$$\frac{1}{2}\partial_t k^\alpha_\alpha + \frac{1}{4}k^\beta_\alpha k^\alpha_\beta = \chi\big[-(\partial_t\phi)^2 + V(\phi)\big] \tag{4.8.1a}$$

$$\frac{1}{2}\big(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}\big) = \chi\big(\partial_\alpha\phi\,\partial_t\phi\big) \tag{4.8.1b}$$

$$\frac{1}{2\sqrt{\gamma}}\partial_t\big(\sqrt{\gamma}\,k^\beta_\alpha\big) + P^\beta_\alpha = \chi\Big[\gamma^{\beta\sigma}\,\partial_\alpha\phi\,\partial_\sigma\phi + V(\phi)\,\delta^\beta_\alpha\Big]\,, \tag{4.8.1c}$$

and is still valid notation (4.6.13). This system is coupled to the dynamics of the scalar field $\phi(t,x^\gamma)$ by (4.6.14).
In what follows we will consider the three fundamental statements:

(*i*) the three metric tensor is taken in the general factorized form

$$\gamma_{\alpha\beta}(t,x^\gamma) = \Gamma^2(t,x^\gamma)\xi_{\alpha\beta}(x^\gamma) \tag{4.8.2}$$

where $\xi_{\alpha\beta}$ is a generic symmetric three-tensor and therefore contains six arbitrary functions of the spatial coordinates, while Γ is to be determined by the dynamics. The inverse metric reads

$$\gamma^{\alpha\beta}(t,x^\gamma) = \frac{1}{\Gamma^2(t,x^\gamma)}\xi^{\alpha\beta}(x^\gamma)\,, \quad \xi^{\alpha\gamma}\xi_{\gamma\beta} = \delta^\alpha_\beta\,; \tag{4.8.3}$$

(*ii*) the self interacting scalar field dynamics is described by a potential term which satisfies all the features of an inflationary one, say a symmetry breaking configuration characterized by a relevant plateau region;

(*iii*) the inflationary solution is constructed under the assumptions (4.7.3a) and (4.7.3b).

Our analysis concerns the evolution of the cosmological model when the scalar field slow rolls on the plateau and the corresponding potential term is described as

$$V(\phi) = \Lambda_0 - \lambda U(\phi)\,, \tag{4.8.4}$$

where Λ_0 behaves as an effective cosmological constant of the order $10^{15} - 10^{16}$ GeV and λ ($\ll 1$) is a coupling constant associated with the perturbation $U(\phi)$.
Since the scalar field moves on an almost flat plateau, we infer that in the lower order of approximation $\phi(t, x^\gamma) \sim \alpha(x^\gamma)$ (see below (4.8.10)) and therefore the potential reduces to a space-dependent effective cosmological constant

$$\Lambda(x^\gamma) \equiv \Lambda_0 - \lambda U(\alpha(x^\gamma))\,. \tag{4.8.5}$$

In this scheme the $0-0$ (4.8.1a) and $\alpha - \beta$ (4.8.1c) components of the Einstein equations reduce respectively, under condition *(iii)* and neglecting all the spatial gradients, to the simple ones

$$3\,\partial_{tt}\ln\Gamma + 3\,(\partial_t \ln\Gamma)^2 = \chi\Lambda(x^\gamma) \tag{4.8.6a}$$
$$(\partial_{tt}\ln\Gamma)\delta^\alpha_\beta + 3\,(\partial_t \ln\Gamma)^2\delta^\alpha_\beta = \chi\Lambda(x^\gamma)\delta^\alpha_\beta\,. \tag{4.8.6b}$$

A simultaneous solution for Γ of both equations (4.8.6a) and (4.8.6b) takes the form

$$\Gamma(x^\gamma) = \Gamma_0(x^\gamma)\exp\left[\sqrt{\frac{\chi\Lambda(x^\gamma)}{3}}\Big(t - t_0(x^\gamma)\Big)\right]\,, \tag{4.8.7}$$

where $\Gamma_0(x^\gamma)$ and $t_0(x^\gamma)$ are integration functions. Under the same assumptions and taking into account (4.8.7) for Γ, the scalar field equation (4.6.14) can be re-expressed now as

$$3H(x^\gamma)\partial_t\phi - \lambda W(\phi) = 0\,, \tag{4.8.8}$$

where we naturally defined

$$H(x^\gamma) = \partial_t \ln\Gamma = \sqrt{\frac{\chi}{3}\Lambda(x^\gamma)}\,, \qquad W(\phi) = \frac{dU}{d\phi}\,. \tag{4.8.9}$$

We search a solution of the dynamical equation (4.8.8) in the form

$$\phi(t,x^\gamma) = \alpha(x^\gamma) + \beta(x^\gamma)\Big(t - t_0(x^\gamma)\Big)\,. \tag{4.8.10}$$

Inserting expression (4.8.10) in (4.8.8) and considering it at the lowest order, we get the relation

$$3H\beta = \lambda W(\alpha)\,, \qquad W(\alpha) = \left.\frac{dU}{d\phi}\right|_{\phi=\alpha}. \tag{4.8.11}$$

This equation allows to express β in terms of α

$$\beta = \frac{\lambda W(\alpha)}{\sqrt{3\chi\Lambda_0 - \lambda U(\alpha)}}\,. \tag{4.8.12}$$

Of course the validity of solution (4.8.12) takes place in the limit

$$t - t_0(x^\gamma) \ll \left|\frac{\alpha}{\beta}\right| = \left|\frac{\alpha}{W(\alpha)}\sqrt{3\chi\frac{\Lambda_0}{\lambda^2} - \frac{U(\alpha)}{\lambda}}\right| \tag{4.8.13}$$

where the ratio Λ_0/λ^2 takes, in general, very large values.
At this point it remains to solve the $0-\alpha$ component (4.8.1b) of the Einstein equations. In view of (4.8.7) and (4.8.10) through (4.8.12) this provides the relation

$$-2\sqrt{\frac{\chi}{3}}\partial_\gamma\left(\sqrt{\Lambda}\right) = \chi(\partial_\gamma\alpha)\,\beta = \sqrt{\frac{\chi}{3\Lambda}}\,\lambda\,\partial_\gamma U \tag{4.8.14}$$

or, simplifying easily,

$$\partial_\gamma\left(\Lambda + \lambda U\right) = 0\,, \tag{4.8.15}$$

which is reduced to an identity by (4.8.5) for $\Lambda(x^\gamma)$.
The validity of the obtained inflationary solution is guaranteed by considering that all spatial gradients, either of the three-metric field or of the scalar one, behave like Γ^{-2} and therefore decay exponentially. If we take into account the coordinate characteristic lengths L and l for the inhomogeneous scales respectively regarding the functions Γ_0 and $\xi_{\alpha\beta}$, i.e.

$$\partial_\gamma\Gamma_0 \sim \frac{\Gamma_0}{L}\,, \qquad \partial_\gamma\,\xi_{\alpha\beta} \sim \frac{\xi_{\alpha\beta}}{l}\,, \tag{4.8.16}$$

then negligibility of the spatial gradients at the initial instant t_0 leads to the inequalities for the physical quantities

$$\Gamma_0 l \;=\; l_{\text{phys}} \gg H^{-1}\,, \tag{4.8.17a}$$

$$\Gamma_0 L \;=\; L_{\text{phys}} \gg H^{-1}\,. \tag{4.8.17b}$$

These conditions state that all inhomogeneities have to be much greater then the physical horizon H^{-1}; such estimates do not involve the $t_0(x^\gamma)$ spatial gradients, since we expect the inflation starts almost simultaneously everywhere and then they are negligible.

The assumption made on the negligibility of the spatial gradients at the beginning of inflation is required (as well known) by the existence of the de Sitter phase itself; however, spatial gradients with a passive dynamical role allow to deal with a fully inhomogeneous solution. This feature simply means that, to leading order, space point dynamically decouple.

The analysis is completed by stressing that the condition (4.7.3a) becomes

$$W^2(\alpha) \ll \chi \left(\frac{\Lambda}{\lambda}\right)^2 , \tag{4.8.18}$$

or equivalently by (4.8.5)

$$\lambda^2 W^2(\alpha) \ll \chi \left(\Lambda_0 - \lambda U(\alpha)\right)^2 \tag{4.8.19}$$

which, neglecting all terms in λ^2, simply states that the dominant contribution in $\Lambda(x^\gamma)$ is provided by Λ_0, i.e.

$$\lambda U(\alpha) \ll \Lambda_0 ; \tag{4.8.20}$$

meanwhile (4.7.3b) is always naturally satisfied. By other words, we get the only important restriction on the spatial function $\alpha(x^\gamma)$ which reads

$$|\alpha| \ll \left|U^{-1}\left(\Lambda_0/\lambda\right)\right| . \tag{4.8.21}$$

As is well known, to get a satisfactory exponential expansion able to overcome the SCM shortcomings, we require that at each point of space the condition

$$H(t_f - t_i) \sim \mathcal{O}(10^2) \tag{4.8.22}$$

holds, where t_i and t_f denote respectively the instants when the de Sitter phase starts and ends. We may take $t_i \equiv t_0$ and t_f must satisfy the inequality

$$t_f \ll t^* \equiv t_0 + \left|\frac{\alpha}{\beta}\right| . \tag{4.8.23}$$

Hence we have

$$H(t_f - t_i) \ll H(t^* - t_0) = H \left|\frac{\alpha}{\beta}\right| \tag{4.8.24a}$$

or equivalently

$$H(t_f - t_i) \ll \frac{\Lambda_0}{\lambda}\left|\frac{\alpha}{W(\alpha)}\right| , \tag{4.8.24b}$$

where we made use of (4.8.12). Being Λ_0/λ a very large quantity, no serious restrictions appear for the e-folding of the model.

A fundamental feature of our analysis relies on the generic nature of the obtained solution; in fact, once satisfied all dynamical equations, there still remain nine arbitrary spatial functions, i.e. six for $\xi_{\alpha\beta}(x^\gamma)$, and then $\Gamma_0(x^\gamma)$, $\alpha(x^\gamma)$ and $t_0(x^\gamma)$.
However, taking into account the possibility to choose an arbitrary gauge via the choice of the spatial coordinates, we have to kill three degrees of freedom; so finally there remain six physically arbitrary functions: four corresponding to gravity degrees of freedom and two related to the scalar field.
This picture corresponds exactly to the possibility of specifying a generic Cauchy problem for the dynamics, on a spatial non singular hypersurface.

4.8.3 Coleman–Weinberg Model

Let us specify our solution in the case of the Coleman–Weinberg zero-temperature potential (Coleman and Weinberg, 1973)

$$V(\phi) = \frac{B\sigma^4}{2} + B\phi^4\left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] \tag{4.8.25}$$

where $B \simeq 10^{-3}$ is connected to the fundamental constants of the theory, while $\sigma \simeq 2{\cdot}10^{15}$GeV gives the energy associated with the symmetry breaking process. In the region $\mid \phi \mid \ll \mid \sigma \mid$ the potential (4.8.25) approaches a plateau behavior profile similar to (4.8.4) and acquires the form

$$V(\phi) \simeq \frac{B\sigma^4}{2} - \frac{\lambda}{4}\phi^4 , \qquad \lambda \simeq 80B \simeq 0.1 . \tag{4.8.26}$$

This is effectively reducible to (4.8.4) when

$$\Lambda_0 = \frac{B\sigma^4}{2} , \quad U(\phi) = \frac{\phi^4}{4} , \qquad W(\phi) = \phi^3 , \tag{4.8.27}$$

and the relations (4.8.12) and (4.8.18) are rewritten as

$$\beta = \frac{\lambda\alpha^3}{3H} \tag{4.8.28a}$$

and

$$\alpha^3\left(\alpha + \sqrt{\frac{8}{3\chi}}\right) \ll \frac{\Lambda_0}{\lambda} \simeq \frac{\sigma^4}{160} , \tag{4.8.28b}$$

respectively. The inequality in (4.8.28b) is equivalent to fulfill the initial assumption

$$\Lambda_0 \gg \lambda U(\alpha) \sim \frac{\lambda}{4}\alpha^4\,, \tag{4.8.29}$$

like in (4.8.20).
The restriction (4.8.21) affects the free function α so that

$$|\alpha| \ll \sqrt[4]{\frac{\Lambda_0}{\lambda}} \sim \sigma\,. \tag{4.8.30}$$

4.8.4 Towards the FLRW Universe

Though, in view of (4.8.7) and (4.8.17) the Universe described by our solution at the end of the de Sitter phase is actually homogeneous and isotropic on the horizon scale, nevertheless we consider the FRW case in order to match the standard literature on the inflation.

Let us now specify as in Imponente and Montani (2003*d*) the solution found nearby the FLRW case by requiring the following conditions over all the quantities involved in the dynamical problem.

(a) The spatial metric tensor $\xi_{\alpha\beta}$ specifies to

$$\xi_{\alpha\beta}(x^\gamma) \quad \to \quad h_{\alpha\beta}(\theta^\gamma)\,, \tag{4.8.31}$$

where $h_{\alpha\beta}$ denotes the FRW spatial part of the three metric and $\{\theta^\gamma\}$ are the usual three angular coordinates;

(b) the leading order of the scalar field ϕ must be independent of the spatial coordinates and therefore we have to require

$$\alpha(x^\gamma) \quad \to \quad \alpha_0 + \delta\alpha(\theta^\gamma)\,, \tag{4.8.32}$$

where $\alpha_0 =$ const. and $\delta\alpha/\alpha_0 \ll 1$;

(c) the quantities t_0 and Γ_0 must be independent of the spatial coordinates, i.e. $\{t_0, \Gamma_0\} =$ const..

Hence, we easily get

$$\Lambda \sim \tilde{\Lambda}_0 - \lambda W(\alpha_0)\,\delta\alpha \tag{4.8.33}$$

where

$$\Lambda \sim \tilde{\Lambda}_0 - \lambda W(\alpha_0)\,\delta\alpha\,, \quad \tilde{\Lambda}_0 \equiv \Lambda_0 - \lambda U(\alpha_0) \sim \Lambda_0\,. \tag{4.8.34}$$

By this expression (4.8.34) and expanding Γ to first order in $\delta\alpha$ we get

$$\Gamma(t, x^\gamma) \simeq \tilde{\Gamma}(t)\left[1 - \frac{\tilde{\lambda}}{\tilde{\Lambda}_0}(t - t_0)\,\delta\alpha(\theta^\gamma)\right] \tag{4.8.35a}$$

$$\tilde{\Gamma}(t) \equiv \Gamma_0 \exp\left[\sqrt{\frac{\chi}{3}\tilde{\Lambda}_0}\,(t - t_0)\right], \tag{4.8.35b}$$

$$\tilde{\lambda} \equiv \lambda\frac{3}{2\chi}W(\alpha_0)\,.$$

The physical implications of this nearly homogeneous model with respect to the density perturbation spectrum rely on the dominant behaviour of the potential term over the energy density ρ_ϕ associated to the scalar field during the de Sitter phase and therefore

$$\Delta \equiv \left|\frac{\delta\rho_\phi}{\rho_\phi}\right| \sim \left|\frac{d\mathrm{ln}V}{d\phi}\delta\phi\right| \simeq \left|\frac{\lambda}{\Lambda_0}W(\alpha_0)\,\delta\alpha(\theta^\gamma)\right|\,. \tag{4.8.36}$$

In particular, for the Coleman-Weinberg case, (4.8.36) reduces to

$$\Delta_{CW} \simeq \frac{160}{\sigma^4}{\alpha_0}^3\,\delta\alpha(\theta^\gamma)\,. \tag{4.8.37}$$

However, to get information about the problem of computing the physically relevant perturbations after the re-entry of scales in the horizon, we have to deal with the gauge invariant quantity ζ (Guth and Pi., 1982; Bardeen et al., 1983) which when the perturbations leave the horizon has the form

$$\zeta = \frac{\delta\rho}{\rho + p} \cong 3\chi\frac{\Lambda_0}{\lambda}\frac{\delta\alpha}{W(\alpha)} \tag{4.8.38}$$

being $\rho + p = (\partial_t\phi)^2$; for the CW case we have

$$\zeta_{CW} = 3\chi\frac{\sigma^4}{160}\,\frac{\delta\alpha}{{\alpha_0}^3}\,. \tag{4.8.39}$$

Since ζ remains constant during the super-horizon evolution of the perturbations, then at re-entry to the causal scale in the matter-dominated era, we get $\zeta_{MD} \sim \delta\rho/\rho \sim \zeta_{CW}$.

By restoring physical units and assuming $\alpha_0 \lesssim 10^{-4}\sigma/\sqrt{\hbar c}$ in agreement with (4.8.30), then in order to have perturbations $\delta\rho/\rho \sim 10^{-4}$ at the horizon re-entry during the matter-dominated age, it is required $\delta\alpha/\alpha_0 \lesssim 10^{-2}$.

Hence the expression (4.8.39) explains how the perturbation spectrum after the de Sitter phase can still arise from classical inhomogeneous terms. Indeed, the function $\delta\alpha(x^\gamma)$ is arbitrary and a Harrison-Zeldovich spectrum can be chosen for it by assigning its Fourier transform as

$$|\delta\alpha(k)|^2 \propto \frac{\text{const.}}{k^3}\,. \tag{4.8.40}$$

Thus, the pre-inflationary inhomogeneities of the scalar field remain almost of the same amplitude during the de Sitter phase as a consequence of the linear form of the scalar field solution (4.8.10). Hence we get that the Harrison-Zeldovich spectrum can be a pre-inflationary picture of the density perturbations and it survives to the de Sitter phase, becoming a classical seed for structure formation. The existence of such a classical spectrum is not related with the quantum fluctuations of the scalar field whose effect is an independent contribution to the classical one.

The merit of the analysis in the present Section relies on having provided a dynamical framework within which classical inhomogeneous perturbations to a real scalar field minimally coupled with gravity can survive even after the de Sitter expansion of the universe stretched the geometry; the key feature underlying this result consists *(i)* of constructing a generic inhomogeneous model for which the leading order of the scalar field is provided by a spatial function and then *(ii)* of showing how the general case contains as a limit a model close to the FRW one.

Conclusions

We have studied in the details the Hamiltonian formulation of the Mixmaster cosmological model and its property of chaoticity, reducing the system, via an ADM prescription, to the dynamics of a "billiard-ball"; in fact, asymptotically, it exists an energy-like constant of the motion which allows to recognize the point-Universe moving over a Lobatchevsky plane on which the potential cuts a closed domain. Lyapunov exponents have been calculated for this system and they result to be positive, independently of the choice of the temporal gauge. Thus, we have shown the covariance of chaos and linked it to a natural treatment in Statistical Mechanics.

The evolution towards the singularity has been analysed from a Quantum point of view in terms of a natural Schrödinger-like formulation; in fact, the ADM Hamiltonian picture provides a valuable scheme to quantize the real physical degrees of freedom of the Bianchi IX Cosmology, i.e. its anisotropies. It has been shown the coincidence between the Liouville theorem, as restricted to the configuration space and the continuity equation coming from the semiclassical limit of the wave function.

Such a rich dynamics of the Universe performed at "the very beginning" has been shown to be compatible with a quasi-isotropic Cosmology through a *bridge solution* based on an inflationary phase of expansion.

In a much more general context a *generic* cosmological solution to the Einstein equations in presence of a scalar field has been investigated, in order to show the possibility of a classical origin of density perturbations in our Universe, compatible with the inhomogeneous seeds for structure formation.

Such result supports the idea that the classical origin of density perturbations requires a dominant role of the scalar field up to first-two orders in the cosmological dynamics; in fact, our analysis of the quasi-isotropic solution, in which the matter dominates the first dynamical order, is a proof that under such hypothesis no classical perturbations survive to the de Sitter phase.

All these issues call attention for further developments, especially in view of the gravitational waves associated with the relic anisotropies. The possibility for a detection of such waves is an intriguing scenario of investigation; in fact, the knowledge of the amplitude and spectrum of a cosmic microwave background would provide important insights on the initial conditions for the dynamical regimes here discussed.

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