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Asymptotic symmetries and modular covariance in General Relativity and gauge theories

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Abstract

The first part of this thesis is devoted to the study of asymptotic symmetries in the theory of general relativity. We investigate properties of the Bondi-Metzner-Sachs (BMS) group in four dimensions, that is the asymptotic symmetry group of a certain class of asymptotically flat spacetimes. Particular emphasis is given on the construction of surface charges associated to BMS symmetries and on their connection with the soft graviton theorems, that are important cornerstones of the recently explored infrared structure of asymptotically flat gravity. Furthermore, in the context of asymptotically locally AdS_3 spacetimes, we define conformally flat boundary conditions and, consequently, analyze the corresponding asymptotic symmetry and surface charge algebras. We construct a new sector of Weyl charges and examine the features of the holographic boundary theory. In the second part of this work, we deepen the notion of modular invariance and temperature dualities in quantum field theory. We show that the partition function of certain theories living on partially compactified manifolds exhibits modular covariance, allowing to derive interesting high-/low-temperature dualities. Moreover, we apply these results to the case of electromagnetism and linearized gravity in the Casimir effect setup. Even if the two subjects studied in this thesis are not directly related, they both share and make use of techniques in conformal field theory and strongly rely on the role of boundary conditions in gauge theories.

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Part I

Asymptotic Symmetries in General Relativity, the BMS group and asymptotically locally AdS_3

1 Introduction

Gauge theories play a pivotal role in the understanding of fundamental laws governing nature. On the one hand, they comprise electromagnetism, weak and strong interactions and, as such, they provide the description of the Standard Model of particle physics, which is one of the most experimentally tested theories. On the other hand, the theory of general relativity (GR) also falls in this class. In particular, the recent direct detection of gravitational waves and the observation of both supermassive and intermediate mass black holes, entirely predicted by GR, is shedding new light on the nature of strongly gravitating systems existing in the universe.

On the theoretical side, gauge theories exhibit a deep mathematical structure that allows for an elegant interplay between physics and geometry. The common feature shared by all gauge theories is that they involve, in their description, a certain number of redundant degrees of freedom. Equivalently, they are characterised by some local symmetries of the dynamics, called gauge symmetries. Thus in most cases a gauge fixing procedure, through which the above mentioned gauge invariance is used to eliminate some of the redundancies, is needed. The remaining allowed gauge transformations that do not spoil the gauge fixing are referred to as residual gauge transformations.

Together with gauge invariance, the other necessary ingredient that defines the physical content of a gauge theory is the set of boundary conditions that must be imposed on the fields and that selects, through the equations of motion, the allowed solutions. Such conditions specify in a unique way the value of the fields and their derivatives, or of a combination of the two, on the boundary of the system. Choosing boundary conditions sharply distinguishes between different physical situations one wants to investigate. Indeed, if two theories are governed by the same dynamics, *i.e.* by the same set of equations of motion, assigning different boundary conditions can lead, in general, to completely different solutions. Thus, they encode the nature of the phenomena one is willing to describe.

From the above considerations it follows that a natural requirement to ask for gauge theories is that gauge transformations preserve the choice of boundary conditions. In general, among these transformations, some are true redundancies of the theory and are thus called trivial. However, there are some that change the physical state of the system, *i.e.* that act non-trivially on the solution space. The latter, named asymptotic symmetries, are of great relevance in physics and form a group, called *asymptotic symmetry group*. More precisely, after having performed a gauge fixing, they can be defined as the set of residual gauge transformations of the theory preserving the given boundary conditions and that carry, through Noether's theorem, associated non-vanishing charges. The latter are objects of interest since they encode information about the physical and observable quantities, such as the energy or the electric charge.

1.1 Asymptotically flat spacetimes and the BMS_4 group

The study of asymptotic symmetries in GR started with the seminal works of Bondi, Metzner, Sachs and van der Burg [1–3]. In these works, it has been shown that, under suitable boundary conditions on the gravitational field, the asymptotic symmetry group of asymptotically flat spacetimes in four dimensions, which comprise a certain class of solutions of Einstein's equations with vanishing cosmological constant, is given by an enhancement of the Poincaré group, called the BMS_4 group.

In particular, the BMS_4 group, in its *global* version, is a semidirect product between the Lorentz group and the infinite-dimensional group of supertranslations, which is a suitable enhancement of the usual translations:

$$\textit{Global}\ \mathrm{BMS}_4 = \mathrm{Lorentz} \ltimes \mathrm{Supertranslations}.$$

More recently, the BMS group has been further enlarged with the inclusion of superrotations on the celestial sphere, which enhance also the Lorentz group [4–6]. These works have led to the definition of *extended* BMS group:

$$\textit{Extended}\ \mathrm{BMS}_4 = \mathrm{Superrotations} \ltimes \mathrm{Supertranslations}.$$

We mention that there exists also a *generalized* BMS_4 group [7–9], in which superrotations are substituted by smooth diffeomorphisms on the 2-sphere ($\mathrm{Diff}(S^2)$)

$$\textit{Generalized}\ \mathrm{BMS}_4 = \mathrm{Smooth\ Diff}(S^2) \ltimes \mathrm{Supertranslations}.$$

1.2 The Infrared triangle and the "soft hair" proposal

Recently, there has been a renewed interest in the infrared structure of asymptotically flat spacetimes. In particular, it has been shown that the quantum Ward identities associated to BMS_4 invariance of the gravitational $\mathcal{S}$-matrix [7–12] are equivalent to the leading orders of the soft graviton theorem [13–17]. Moreover, these aspects were shown to be just two of the three corners of a triangular equivalence relation, the third corner consisting of the gravitational memory effect [18–20]. These connections are sometimes referred to as the *infrared triangle* [21]. One of the strengths of this result relies in its universality, since similar relations are shared by other gauge theories. It is intriguing how the infrared triangle has been used to discuss the black hole information paradox. The existence of an infinite number of conserved charges associated with BMS_4 symmetries could equip the black hole with the soft hair needed to support correlations between the interior of the black hole and the emitted Hawking quanta, during the evaporation process [22].

1.3 Asymptotically AdS spacetimes and holography

The study of asymptotic symmetries has become particularly relevant in holography [23–26]. According to the holographic principle, a theory of quantum gravity living in the bulk of the spacetime is dual to a quantum field theory living on its lower-dimensional boundary. In particular, the AdS/CFT correspondence proposes that a gravitational theory living in an asymptotically AdS spacetime is dual to a Conformal

Field Theory living on its boundary and there exists a well-developed dictionary which translates between observables on the two sides of the holographic correspondence. Then, such a holographic dictionary states that the asymptotic symmetries of the theory living in the bulk should exactly match the global symmetries of the boundary theory.

The first result in this direction has been obtained by Brown and Henneaux [27] and it is considered one of the earliest precursors of holography. They have proven that the asymptotic symmetry algebra of asymptotically AdS_3 spacetimes consists in two commuting copies of the Witt algebra and that the corresponding charge algebra is centrally extended by the so-called *Brown-Henneaux central charge*

$$c = \frac{3\ell}{2G},$$

ℓ being the AdS_3 radius and G the Newton constant. This result is compatible with the AdS_3/CFT_2 correspondence, according to which there exists a two-dimensional CFT theory boundary theory whose global symmetries comprise the infinite-dimensional conformal group in two dimensions.

1.4 Publications and original results

1. F. A. and G. Esposito, "On the structure and applications of the Bondi-Metzner-Sachs group", *Int. J. Geom. Meth. Mod. Phys.* **15** (2018) no. 02, 1830002, arXiv: 1709.05134 [gr-qc],

2. G. Esposito and F. A., "From parabolic to loxodromic BMS transformations", *Gen. Rel. Grav.* **50** (2018) no. 11, 141, arXiv: 1806.06246 [gr-qc],

3. F. A. and M. Arzano, "A fuzzy bipolar celestial sphere", *JHEP* **07** (2019) 028, arXiv: 1901.01167 [gr-qc],

4. F. A. and M. Arzano, "Note on the symplectic structure of asymptotically flat gravity and BMS symmetries", *Phys. Rev. D* **100** (2019) no. 4, 044028, arXiv: 1901.01167 [gr-qc],

5. F. A., G. Barnich, L. Ciambelli, P. Mao and R. Ruzziconi "Weyl Charges in Asymptotically Locally AdS_3 Spacetimes ", *Phys. Rev. D* **103** (2021) 046003, arXiv: 2010.15452 [hep-th]

2 Symmetries and asymptotic symmetries

In this chapter, we start in 2.1 by briefly reviewing the Noether theorems, which relate the notion of symmetry to that of conserved quantities within a certain theory. We distinguish between the case of global symmetries and gauge symmetries, that lead to the first and second Noether theorems, respectively. These theorems are fundamental in physics, for they provide a procedure to construct conserved quantities, related to important observables of the system under consideration. In 2.2 we discuss more in detail to what extent it is possible to draw the same conclusions and what are the difficulties in the case of generally covariant theories. In 2.3 we review the second Noether theorem for such theories with some emphasis on the construction of surface charges and in 2.4 we will give the notion of asymptotic symmetries. In 2.5 we provide some details on the charge algebra, while in 2.6, we explicitly discuss the case of general relativity.

The main literature used in this part is [5, 28–36]. Useful reviews can also be found in [37–39].

2.1 First and second Noether theorems

Consider a theory described by the action

$$S[\phi^i] = \int \mathrm{d}^n x L[\phi^i, \partial_\mu \phi^i, \partial_\mu \partial_\nu \phi^i, ...], \tag{2.1}$$

where L is the Lagrangian. Here latin indices $i, j, ..$ run from 1 to N, where N is the number of fields in the theory. An infinitesimal variation $\delta\phi^i$ yields a variation δL of the Lagrangian,

$$\delta L = \delta\phi^i \frac{\partial L}{\partial \phi^i} + \partial_\mu \delta\phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \partial_\mu \partial_\nu \delta\phi^i \frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi^i)} + ... \equiv \delta\phi^i \frac{\delta L}{\delta \phi^i} + \partial_\mu \theta^\mu(\phi^i, \delta\phi^i), \tag{2.2}$$

where we have defined

$$\frac{\delta L}{\delta \phi^i} \equiv \frac{\partial L}{\partial \phi^i} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi^i)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi^i)} \right) + ... \tag{2.3}$$

to be the *Euler-Lagrange equations of motion* for the field ϕ^i and where θ^μ comprises all the terms that come from using repeatedly the Leibniz rule. Note that in (2.1), since we are directly interested in general relativity, we assume that the Lagrangian may depend on higher derivatives of the fields ϕ^i. In the form notation [1], reviewed in Appendix A, equation (2.3) can be rewritten as

$$\delta \mathbf{L} = \delta\phi^i \frac{\delta \mathbf{L}}{\delta \phi^i} + d\mathbf{\Theta}[\phi^i, \delta\phi^i], \tag{2.4}$$

where $\mathbf{\Theta}$ is the *presymplectic potential* $(n-1)$ form. An infinitesimal transformation of the fields $\delta_\lambda \phi$, for some spacetime function λ that at this stage we deliberately leave arbitrary, is defined to be a symmetry for the theory described by $\mathbf{L}$ if

$$\delta_\lambda \mathbf{L} = d\mathbf{B}[\phi^i, \delta_\lambda \phi^i], \tag{2.5}$$

for some $(n-1)$-form $\mathbf{B}$. We are going to refer to λ as the *generator* of the symmetry.

[1] Throughout the remainder of this work, we will interchangeably use the vectors and the forms notation.

We first consider the case in which λ is a constant. In this case, we talk about *global symmetries*. Comparing equations (2.4) and (2.5), we obtain

$$\delta_\lambda \phi^i \frac{\delta \mathbf{L}}{\delta \phi^i} + d\mathbf{\Theta}[\phi^i, \delta_\lambda \phi^i] = d\mathbf{B}[\phi^i, \delta_\lambda \phi^i]. \tag{2.6}$$

Defining the *Noether current* $(n-1)$-form as

$$\mathbf{J}[\phi^i, \delta_\lambda \phi^i] = \mathbf{B}[\phi^i, \delta_\lambda \phi^i] - \mathbf{\Theta}[\phi^i, \delta_\lambda \phi^i], \tag{2.7}$$

we get that, *on-shell*, *i.e.* provided that the equations of motion are satisfied $\frac{\delta \mathbf{L}}{\delta \phi^i} = 0$, the Noether current is conserved

$$d\mathbf{J}[\phi^i, \delta_\lambda \phi^i] = \delta_\lambda \phi^i \frac{\delta \mathbf{L}}{\delta \phi^i} \approx 0, \tag{2.8}$$

where the symbol $\approx$ stays for equal on-shell. Note that this statement holds regardless of the nature of the symmetry. The variation $\delta_\lambda \phi$ can be any symmetry, spacetime or internal. For instance, the above described procedure for a Lorentz covariant Lagrangian yields the standard definition of stress-energy tensor whereas for a global $U(1)$-invariant Lagrangian gives the electric current. Given a Cauchy $(n-1)$-hypersurface Σ, we define the *charge* associated to the symmetry $\delta_\lambda \phi$ as

$$Q_{\Sigma} = \int_\Sigma \mathbf{J}[\phi^i, \delta_\lambda \phi^i]. \tag{2.9}$$

Provided that the fields fall rapidly enough on the $(n-2)$-boundary $\partial\Sigma$, Q_{Σ} does not depend on the particular choice of Σ. In fact, if $\mathcal{M}$ is a spacetime region bounded by two Cauchy hypersurfaces Σ_1 and Σ_2, *i.e.* $\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$, we have, using Stokes theorem

$$0 \approx \int_\mathcal{M} d\mathbf{J}[\phi^i, \delta_\lambda \phi^i] = \int_{\Sigma_1} \mathbf{J}[\phi^i, \delta_\lambda \phi^i] - \int_{\Sigma_2} \mathbf{J}[\phi^i, \delta_\lambda \phi^i] = Q_{\Sigma_1} - Q_{\Sigma_2}, \tag{2.10}$$

from which the result follows. If Σ is a constant time hypersurface, *i.e.* is described by the equation $x^0 = \text{const}$, (2.10) states that the charge Q is conserved in time. This procedure is sometimes referred to as the *first Noether theorem* and it states that to each global symmetry it corresponds a conserved current and hence a conserved charge. Note however that we can define a new current $\mathbf{J}'$ as

$$\mathbf{J}'[\phi^i, \delta_\lambda \phi^i] = \mathbf{J}[\phi^i, \delta_\lambda \phi^i] + d\mathbf{Q}[\phi^i, \delta_\lambda \phi^i] + \mathbf{E}\left[\frac{\delta \mathbf{L}}{\delta \phi^i}\right], \tag{2.11}$$

for an arbitrary form $(n-2)$-form $\mathbf{Q}$ and a $(n-1)$-form $\mathbf{E}$ proportional to the equations of motion. Then

$$d\mathbf{J}'[\phi^i, \delta_\lambda \phi^i] = d\mathbf{J}[\phi^i, \delta_\lambda \phi^i]. \tag{2.12}$$

These arguments suggest that the first Noether should be more correctly stated as follows: there exists an isomorphism between the equivalence class of conserved currents, where two currents $\mathbf{J}$ and $\mathbf{J}'$ are equivalent if (2.11) holds, and the equivalence class of global symmetries, where two global symmetries are defined to be equivalent if they differ by some, physically irrelevant, gauge transformation, *i.e.* by a symmetry whose generator λ arbitrarily depends on the spacetime coordinates. We now proceed to investigate the latter.

Assume that the generator $\lambda = \lambda(x)$ is a suitably differentiable spacetime function. In this case $\delta_\lambda \phi^i$ is referred to as *gauge symmetry* or as *local symmetry*. For simplicity, we assume that ϕ^i transforms as [28]

$$\delta_\lambda \phi^i = f^i[\phi^j]\lambda + f^{i\nu}[\phi^j]\partial_\nu \lambda. \tag{2.13}$$

and that the Lagrangian depends only on the first derivatives of ϕ^i[2]. Following the procedure outlined in (2.2)-(2.3), we obtain, for θ^μ,

$$\theta^\mu[\phi^i, \delta_\lambda \phi^i] = \frac{\partial L}{\partial(\partial_\mu \phi^i)}\delta_\lambda \phi^i = \frac{\partial L}{\partial(\partial_\mu \phi^i)}\left[f^i(\phi^j)\lambda + f^{i\nu}(\phi^j)\partial_\nu \lambda\right]. \tag{2.14}$$

Similarly, we assume that the $(n-1)$-form $\mathbf{B}$ in (2.5) can be expanded as

$$B^\mu[\phi^i, \delta_\lambda \phi^i] = g^\mu[\phi^i]\lambda + g^{\mu\nu}[\phi^i]\partial_\mu \lambda. \tag{2.15}$$

Substituting in (2.6), we get

$$\partial_\mu[J^\mu[\phi^i]\lambda + J^{\mu\nu}[\phi^i]\partial_\nu \lambda] = \frac{\delta L}{\delta \phi^j}\left[f^j[\phi^i]\lambda + f^{j\nu}[\phi^i]\partial_\nu \lambda\right], \tag{2.16}$$

where we have defined

$$J^\mu[\phi^i] \equiv g^\mu[\phi^i] - \frac{\partial L}{\partial(\partial_\mu \phi^j)} f^j[\phi^i], \qquad J^{\mu\nu}[\phi^i] \equiv g^{\mu\nu}[\phi^i] - \frac{\partial L}{\partial(\partial_\mu \phi^j)} f^{j\nu}[\phi^i]. \tag{2.17}$$

The vector J^μ defined in the first of (2.17) is nothing but the usual Noether current, for note that in the case of constant λ, (2.16) reduces to (2.8). Arbitrariness of λ in equation (2.16) implies the following identities

$$\partial_\mu J^\mu[\phi^j] = \frac{\delta L}{\delta \phi^i} f^i[\phi^j], \qquad J^\nu[\phi^j] + \partial_\mu J^{\mu\nu}[\phi^j] = \frac{\delta L}{\delta \phi^i} f^{i\nu}[\phi^j], \qquad J^{\mu\nu}[\phi^j] = 0. \tag{2.18}$$

Putting these equations together, we arrive at the so-called *Noether identities*:

$$\triangle\left[\frac{\delta L}{\delta \phi^i}\right] \equiv \frac{\delta L}{\delta \phi^i} f^i[\phi^j] - \partial_\mu\left(\frac{\delta L}{\delta \phi^i} f^{i\mu}[\phi^j]\right) = 0. \tag{2.19}$$

These identities hold off-shell and represent a constraint that has to be satisfied by the equations of motion $\frac{\delta L}{\delta \phi^i}$, so that they are not all independent and therefore the Cauchy problem is not well-posed. This is a common feature of gauge theories. Furthermore, taking a variation δ_λ of the action yields

$$\delta_\lambda S[\phi^i] = \int \mathrm{d}^n x \, \frac{\delta \mathbf{L}}{\delta \phi^i}\delta_\lambda \phi^i = \int \mathrm{d}^n x \, \triangle\left[\frac{\delta \mathbf{L}}{\delta \phi^i}\right]\lambda. \tag{2.20}$$

Hence, the two integrands must differ by the exterior derivative of a $(n-1)$-form $\mathbf{S}$, *i.e.*

$$\frac{\delta \mathbf{L}}{\delta \phi^i}\delta_\lambda \phi^i - \triangle\left[\frac{\delta \mathbf{L}}{\delta \phi^i}\right]\lambda = d\mathbf{S}[\phi^j, \delta_\lambda \phi^j]. \tag{2.21}$$

[2]Generalizations to the case of a Lagrangian depending on higher derivatives of the field and to that of $\delta_\lambda \phi^i$ involving higher derivatives of λ are discussed *e.g.* [29].

Taking into account Noether identities in (2.19), equation (2.21) yields

$$\frac{\delta \mathbf{L}}{\delta \phi^i}\delta_\lambda \phi^i = d\mathbf{S}[\phi^j, \delta_\lambda \phi^j], \tag{2.22}$$

The $(n-1)$-form $\mathbf{S}$ is referred to as *weakly vanishing Noether current* and it satisfies

$$\mathbf{S}[\phi^i, \delta_\lambda \phi^i] \approx 0, \qquad d\mathbf{S}[\phi^i, \delta_\lambda \phi^i] \approx 0. \tag{2.23}$$

Comparing the second of (2.23) with (2.8), we see that

$$d(\mathbf{J}[\phi^i, \delta_\lambda \phi^i] - \mathbf{S}[\phi^i, \delta_\lambda \phi^i]) \approx 0, \tag{2.24}$$

and thus, assuming trivial De-Rham cohomology,

$$\mathbf{J}[\phi^i, \delta_\lambda \phi^i] = \mathbf{S}[\phi^i, \delta_\lambda \phi^i] + d\mathbf{Q}[\phi^i, \delta_\lambda \phi^i] \approx d\mathbf{Q}[\phi^i, \delta_\lambda \phi^i], \tag{2.25}$$

for some $(n-2)$-form $\mathbf{Q}$. This shows that in gauge theories the Noether current is always formed by a "bulk" part that vanishes on-shell and the divergence of an *arbitrary* skew-symmetric tensor. A direct consequence of this result is that, when trying to define the charge associated to a certain gauge symmetry using the same procedure used for global symmetries, it is given by

$$Q_{\Sigma} = \int_{\Sigma} \mathbf{J}[\phi^i, \delta_\lambda \phi^i] \approx \int_{\Sigma} d\mathbf{Q}[\phi^i, \delta_\lambda \phi^i] = \int_{\partial\Sigma} \mathbf{Q}[\phi^i, \delta_\lambda \phi^i], \tag{2.26}$$

where $\partial\Sigma$ is the $(n-2)$-dimensional boundary of Σ. The charge Q defined in (2.26) is completely arbitrary, for $\mathbf{Q}$ is any, totally unconstrained $(n-2)$-form.

The solution to this problem [29] consists in focusing on "lower-degree conservation laws", *i.e.* on conservation laws of the form $d\mathbf{k} \approx 0$, for a $(n-2)$-form $\mathbf{k}$. In fact, assuming that there is a procedure to uniquely determine a $(n-2)$-form $\mathbf{k}$, we could define a charge as

$$Q_S = \int_S \mathbf{k}[\phi^i, \delta_\lambda \phi^i] \tag{2.27}$$

for some $(n-2)$-hypersurface S. Then, similarly to what happens in (2.10), we see that if $\partial\Sigma = S_1 \cup S_2$,

$$0 \approx \int_{\Sigma} d\mathbf{k}[\phi^i, \delta_\lambda \phi^i] = \int_{S_1} \mathbf{k}[\phi^i, \delta_\lambda \phi^i] - \int_{S_2} \mathbf{k}[\phi^i, \delta_\lambda \phi^i] = Q_{S_1} - Q_{S_2}, \tag{2.28}$$

i.e. the charges defined in (2.27) are conserved. This is the core of the so-called *second Noether theorem*, which states that there exists an isomorphism between the equivalence class of gauge symmetries that satisfy $\delta_\lambda \phi^i \approx 0$[3], two gauge symmetries being equivalent if they are equal on-shell, and the equivalence class of $(n-2)$-forms $\mathbf{k}$ which are closed on-shell $d\mathbf{k} \approx 0$, where two $(n-2)$-forms $\mathbf{k}$ and $\mathbf{k}'$ are defined to be equivalent if

$$\mathbf{k}'[\phi^i, \delta_\lambda \phi^i] = \mathbf{k}[\phi^i, \delta_\lambda \phi^i] + d\mathbf{I}[\phi^i, \delta_\lambda \phi^i] + \mathbf{E}\left[\frac{\delta \mathbf{L}}{\delta \phi^i}\right], \tag{2.29}$$

for an arbitrary $(n-3)$-form $\mathbf{I}$ and a $(n-2)$-form $\mathbf{E}$ proportional to the equations of motion.

[3]This subset of gauge transformations are called *field symmetries*.

Before proceeding to analyze in more detail the case of generally covariant theories, it is instructive to explicitly carry out the example of electromagnetism. In this case the dynamical fields are $\phi^i \equiv A^\mu$ and gauge transformations of A^μ read

$$\delta_\lambda A^\mu = \partial^\mu \lambda, \tag{2.30}$$

so that $f^\mu[A^\rho] = 0$ and $f^{\mu\nu}[A^\rho] = \eta^{\mu\nu}$. The Lagrangian of the theory and its variation corresponding to a variation δA^μ are

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \delta L = -F^{\mu\nu} \partial_\mu \delta A_\nu = \partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\mu (F^{\mu\nu} \delta A_\nu), \tag{2.31}$$

so that $\theta^\mu[A^\rho, \delta A^\rho] = -F^{\mu\nu}\delta A_\nu$, $B^\mu[A^\rho, \delta A^\rho] = 0$ and the Euler-Lagrange equations of motion are $\frac{\delta L}{\delta A_\nu} = \partial_\mu F^{\mu\nu}$. Following the prescription in (2.7), the Noether current J^μ can be easily constructed as

$$J^\mu[A^\rho, \delta A^\rho] = -\theta^\mu[A^\rho, \delta_\lambda A^\rho] = F^{\mu\nu} \partial_\nu \lambda. \tag{2.32}$$

The Noether identities are

$$\triangle \left[\frac{\delta L}{\delta A_\nu} \right] = -\partial_\mu \partial_\nu F^{\nu\mu} = 0, \tag{2.33}$$

and note how they hold off-shell because of the anti-symmetry of $F^{\mu\nu}$. We can now construct the weakly vanishing Noether current S^ν as

$$\frac{\delta L}{\delta A_\nu} \delta_\lambda A_\nu = \partial_\mu F^{\mu\nu} \partial_\nu \lambda = \partial_\nu \big(\lambda \partial_\mu F^{\mu\nu} \big) - \lambda \partial_\nu \partial_\mu F^{\mu\nu} \overset{(2.33)}{=} \partial_\nu S^\nu[A^\rho, \delta A^\rho], \tag{2.34}$$

where

$$S^\mu[A^\rho, \delta A^\rho] = \lambda \partial_\nu F^{\nu\mu}. \tag{2.35}$$

Note that $S^\mu \approx 0$ and $\partial_\mu S^\mu \approx 0$. Taking now the difference $J^\mu - S^\mu$, we get

$$J^\mu[A^\rho, \delta A^\rho] - S^\mu[A^\rho, \delta A^\rho] = F^{\mu\nu} \partial_\nu \lambda - \lambda \partial_\nu F^{\nu\mu} = \partial_\nu (\lambda F^{\mu\nu}), \tag{2.36}$$

and hence the $(n-2)$-form $\mathbf{Q}$ of equation (2.25) has components $Q^{\mu\nu}$ given by

$$Q^{\mu\nu} = \lambda F^{\mu\nu}. \tag{2.37}$$

Now, in order to apply the second Noether theorem, we need to look at the gauge parameters that are field symmetries, *i.e.* those λ satisfying

$$\delta_\lambda A^\mu = \partial^\mu \lambda \approx 0. \tag{2.38}$$

This equation is trivially solved by $\lambda = c$ with $c \in \mathbb{R}$. Thus, gauge parameters that are field symmetries that do not vanish are just constants. Then, the representative of the equivalence class of conserved $(n-2)$-forms is

$$\mathbf{k}[A^\rho] = c F^{\mu\nu} (\mathrm{d}^{n-2} x)_{\mu\nu}, \tag{2.39}$$

Indeed, it is well-known that the electric charge enclosed in $(n-1)$-hypersurface Σ whose boundary is a codimension 2 hypersurface S can be expressed as

$$Q_S = \int_S \mathbf{k}[A^\rho] = c \int_S \mathbf{F}[A^\rho], \tag{2.40}$$

where $\mathbf{F} = F^{\mu\nu}(\mathrm{d}^{n-2}x)_{\mu\nu}$. The charge in (2.40) is conserved as a consequence of Maxwell's equations.

2.2 Generally covariant theories and asymptotic symmetries

In the case of electromagnetism we have seen that the equation $\delta_\lambda \phi^i = 0$ admits a simple non-trivial solution given by $\lambda = c$ with $c \in \mathbb{R}$. However, when considering more complicated, possibly non-linear theories, gauge parameters that are field symmetries are not as easy to find. In particular, in the case of general relativity gauge transformations are diffeomorphisms and those that are field symmetries are generated by a vector field ξ^μ solving the Killing equation $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \approx 0$. These equations for a general metric $g_{\mu\nu}$ have no solution.

However, also for generally covariant theories there are cases in which the second Noether theorem applies. Let us consider the linearized theory around a certain fixed background solution $\bar{g}_{\mu\nu}$, obtained by expanding the full metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. A crucial observation is that the linearized theory is also a gauge theory, the field $h_{\mu\nu}$ transforming as $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$. A direct consequence is that isometries of the background, *i.e.* those diffeomorphisms generated by a vector field ξ satisfying $\mathcal{L}_\xi \bar{g}_{\mu\nu} = 0$ are field symmetries of the linearized theory. For instance, Pauli-Fierz theory, obtained linearizing around flat spacetime $\eta_{\mu\nu}$, inherits the ten-dimensional Poincaré group of isometries of $\eta_{\mu\nu}$. Even if for the full, non-linear theory involving $g_{\mu\nu}$ it is not possible to find gauge transformations that are field symmetries, for the linearized theory around a background with certain non-trivial isometries, one can still apply Noether second theorem and compute charges. Clearly, in order for this procedure to make sense, one has to work in a regime in which the theory can be reliably described by $h_{\mu\nu}$, *e.g.* in an asymptotic region where gravity is "weak". Notice that defining an asymptotic region where the gravitational field is well described by $h_{\mu\nu}$ implies the introduction of certain *boundary conditions* for the full metric $g_{\mu\nu}$ that in most cases are expressed as fall-offs for the metric components, such as $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} = \mathcal{O}(r^{-1})$, where r is a suitably defined radial coordinate [4]. Under these assumptions, instead of looking for exact field symmetries and thus solve $\mathcal{L}_\xi \bar{g}_{\mu\nu} = 0$, which gives nothing but the isometries of $\bar{g}_{\mu\nu}$, one can look at those diffeomorphisms that preserve only the boundary conditions, *e.g.* $\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(r^{-1})$. These equations are sometimes referred to as *asymptotic Killing equations* and their generators as *asymptotic symmetries* and they are relevant *only* asymptotically. As already mentioned in the introduction 1, also asymptotic symmetries form a group, denoted asymptotic symmetry group, which will be defined later in 2.4. It is worth mentioning that the notion of asymptotic symmetry is the closest analogue to the concept of symmetry for generally covariant theories.

Clearly, exact symmetries of the background $\bar{g}_{\mu\nu}$ always form a subset of the asymptotic symmetries, for if a diffeomorphism is an exact symmetry of the background it automatically preserves boundary conditions. It follows that the asymptotic symmetries corresponding to certain fall-offs can only have *more* generators with respect to the ones of exact symmetries. In many case, perhaps surprisingly, the generators of asymptotic symmetries are infinite in number. The first example of this phenomenon is the global BMS_4 group, found in [3] in the context of asymptotically flat spacetimes.

Developing a procedure that allows to compute charges associated to the generators of asymptotic symmetries as surface integrals (*i.e.* codimension 2 integrals) is of crucial importance, because these definitions would involve only the value of the gravitational field on the asymptotic boundary of the spacetime. In particular, giving a definition of energy of the gravitational field depending only on its boundary value would mean implementing the idea that gravity is holographic (as mentioned in section 1.3) since the spectrum of the bulk theory would be entirely governed by boundary quantities. There exists a huge literature and different approaches to this procedure, the most relevant being [30,40–43], which we will briefly summarize

[4] Notice that, in order to define a radial coordinate, we need to choose a coordinate system, which in general relativity boils down to fix a gauge.

and review in the next sections.

2.3 Second Noether theorem and surface charges for generally covariant theories

Consider a diffeomorphisms-invariant Lagrangian n-form $\mathbf{L}[\phi^i, \partial_\mu \phi^i, \partial_\mu \partial_\nu \phi^i, ...]$ satisfying (2.4) and thus defining a presymplectic potential $(n-1)$-form $\mathbf{\Theta}$. The variation of a field under the action of a diffeomorphism generated by ξ is given by the Lie derivative, *i.e.* $\delta_\xi \phi^i = \mathcal{L}_\xi \phi^i$. Hence, a variation of $\mathbf{L}$ under a diffeomorphism generated by a vector field ξ is given by,

$$\delta_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} \overset{(A.12)}{=} i_\xi (d\mathbf{L}) + d(i_\xi \mathbf{L}) = d(i_\xi \mathbf{L}), \tag{2.41}$$

because $\mathbf{L}$ is a n-form. It is useful to introduce the notation $\mathbf{A}[\phi^i, \mathcal{L}_\xi \phi^i] \equiv \mathbf{A}_\xi[\phi^i]$. Comparing equations (2.41) and (2.5) we see that the $(n-1)$-form $\mathbf{B}_\xi[\phi^i]$ is given by

$$\mathbf{B}_\xi[\phi^i] = i_\xi \mathbf{L}[\phi^i]. \tag{2.42}$$

It implies that the Noether current in (2.7) can be expressed as

$$\mathbf{J}_\xi[\phi^i] = i_\xi \mathbf{L}[\phi^i] - \mathbf{\Theta}_\xi[\phi^i], \tag{2.43}$$

and it satisfies, by construction

$$d\mathbf{J}_\xi[\phi^i] = \frac{\delta \mathbf{L}}{\delta \phi^i} \mathcal{L}_\xi \phi^i \approx 0. \tag{2.44}$$

On the other hand, repeating the procedure of the second Noether theorem of section 2.1, we can define the weakly vanishing Noether current of (2.21) by

$$d\mathbf{S}_\xi[\phi^j] = \frac{\delta \mathbf{L}}{\delta \phi^i} \mathcal{L}_\xi \phi^i \Longrightarrow \mathbf{S}_\xi[\phi^j] \approx 0, \qquad d\mathbf{S}_\xi[\phi^j] \approx 0. \tag{2.45}$$

We have

$$d\mathbf{J}_\xi[\phi^i] = d\mathbf{S}_\xi[\phi^i] \Longrightarrow \mathbf{J}_\xi[\phi^i] = \mathbf{S}_\xi[\phi^i] + d\mathbf{Q}_\xi[\phi^i], \tag{2.46}$$

for a $(n-2)$-form $\mathbf{Q}_\xi$, sometimes referred to as the *Noether-Wald charge*. Furthermore, it can be shown that it is possible to directly obtain the expression of $\mathbf{Q}_\xi$ from the action of an operator I_ξ on $\mathbf{\Theta}_\xi$:

$$\mathbf{Q}_\xi[\phi^i] = -I_\xi \mathbf{\Theta}_\xi[\phi^i], \tag{2.47}$$

where the action of I_ξ on an r-form $\mathbf{A}_\xi$ depending linearly on ξ is given by

$$I_\xi \mathbf{A}_\xi = \frac{1}{n-r} \xi^\nu \frac{\partial}{\partial(\partial_\mu \xi^\nu)} \frac{\partial}{\partial \mathrm{d}x^\mu} \mathbf{A}_\xi. \tag{2.48}$$

Consider now equation (2.46), where we use (2.43) to express $\mathbf{J}$

$$\mathbf{S}_\xi[\phi^i] = i_\xi \mathbf{L}[\phi^i, \delta\phi^i] - \mathbf{\Theta}_\xi[\phi^i] - d\mathbf{Q}_\xi[\phi^i]. \tag{2.49}$$

Taking a variation of both sides of (2.49)[5]

$$\begin{aligned}\delta\mathbf{S}_\xi[\phi^i] &= \delta(i_\xi\mathbf{L}[\phi^i]-\mathbf{\Theta}_\xi[\phi^i])-\delta(d\mathbf{Q}_\xi[\phi^i]) = i_\xi(\delta\mathbf{L}[\phi^i])-\delta\mathbf{\Theta}_\xi[\phi^i]-d\delta\mathbf{Q}_\xi[\phi^i]\\ &= i_\xi\left(\frac{\delta\mathbf{L}}{\delta\phi^i}\delta\phi^i+d\mathbf{\Theta}[\phi^i,\delta\phi^i]\right)-\delta\mathbf{\Theta}_\xi[\phi^i]-d\delta\mathbf{Q}_\xi[\phi^i]\approx i_\xi d\mathbf{\Theta}[\phi^i,\delta\phi^i]-\delta\mathbf{\Theta}_\xi[\phi^i]-d\delta\mathbf{Q}_\xi[\phi^i]\\ &= (\delta_\xi\mathbf{\Theta}[\phi^i,\delta\phi^i]-\delta\mathbf{\Theta}_\xi[\phi^i])-d(i_\xi\mathbf{\Theta}[\phi^i,\delta\phi^i]+\delta\mathbf{Q}_\xi[\phi^i])\equiv-\boldsymbol{\omega}[\phi^i,\delta\phi^i,\mathcal{L}_\xi\phi^i]+d\mathbf{k}_\xi[\phi^i,\delta\phi^i]\end{aligned}\tag{2.50}$$

where we defined the *presymplectic current* $(n-1)$-form $\boldsymbol{\omega}$ as the antisymmetrized variation of $\mathbf{\Theta}[\phi^i,\delta\phi^i]$ with respect to $\delta\phi^i$,

$$\boldsymbol{\omega}[\phi^i,\delta_1\phi^i,\delta_2\phi^i]\equiv\delta_1\mathbf{\Theta}[\phi^i,\delta_2\phi^i]-\delta_2\mathbf{\Theta}[\phi^i,\delta_1\phi^i],\tag{2.51}$$

and where the *Iyer-Wald* $(n-2)$-form $\mathbf{k}_\xi$ [42] is defined up to an exterior derivative as

$$\mathbf{k}_\xi[\phi^i,\delta\phi^i]=-\delta\mathbf{Q}_\xi[\phi^i]-i_\xi\mathbf{\Theta}[\phi^i,\delta\phi^i].\tag{2.52}$$

Equation (2.50) implies that

$$\boldsymbol{\omega}[\phi^i,\delta\phi^i,\mathcal{L}_\xi\phi^i]-d\mathbf{k}_\xi[\phi^i,\delta\phi^i]\approx 0,\tag{2.53}$$

where here by $\approx$ we mean that ϕ^i and $\delta\phi^i$ solve the equations of motion and the linearized equations of motion around ϕ^i, respectively. Before proceeding, let us mention here that (2.4) does not entirely fix $\mathbf{\Theta}$. Indeed (2.4) is left invariant by

$$\mathbf{\Theta}[\phi^i,\delta\phi^i]\longrightarrow\mathbf{\Theta}[\phi^i,\delta\phi^i]+d\mathbf{Y}[\phi^i,\delta\phi^i],\tag{2.54}$$

for some $(n-2)$-form $\mathbf{Y}[\phi^i,\delta\phi^i]$. Under (2.54), $\boldsymbol{\omega}$ transforms as

$$\boldsymbol{\omega}\longrightarrow\boldsymbol{\omega}+d(\delta_1\mathbf{Y}[\phi^i,\delta_2\phi^i]-\delta_2\mathbf{Y}[\phi^i,\delta_1\phi^i])\equiv\boldsymbol{\omega}+d\boldsymbol{\omega}_B,\tag{2.55}$$

and $\mathbf{k}_\xi$ as

$$\mathbf{k}_\xi\longrightarrow\mathbf{k}_\xi+\boldsymbol{\omega}_B.\tag{2.56}$$

We define the *local variation of charge* between the solutions ϕ^i and $\phi^i+\delta\phi^i$ by integrating $\mathbf{k}_\xi$ on a closed codimension 2 surface S,

$$\not\delta Q_\xi[\phi^i,\delta\phi^i]=\int_S\mathbf{k}_\xi[\phi^i,\delta\phi^i],\tag{2.57}$$

where we use $\not\delta$ in order to emphasize that (2.57) may not be an exact differential in the space of fields. In case it is exact, there exists a functional $Q_\xi[\phi^i]$ such that $\not\delta Q_\xi=\delta(Q_\xi[\phi^i])$. A necessary condition for exactness is the *integrability condition*

$$\delta_1\int_S\mathbf{k}_\xi[\phi^i,\delta_2\phi^i]-\delta_2\int_S\mathbf{k}_\xi[\phi^i,\delta_1\phi^i]=0,\qquad\forall\ \delta_1\phi^i,\ \delta_2\phi^i.\tag{2.58}$$

[5]More correctly δ should be taken as the *variational operator* acting on (p,q)-forms on the jet bundle, for which additional structure is required, see *e.g.* [30]

If (2.58) is satisfied, we can choose a path γ in the space of fields connecting $\bar{\phi}^i$ with ϕ^i and define the *surface charge* as

$$Q_\xi[\phi^i, \bar{\phi}^i] = \int_\gamma \int_S \mathbf{k}_\xi[\phi^i, \delta\phi^i] + N_\xi[\bar{\phi}^i], \tag{2.59}$$

where $N_\xi[\bar{\phi}^i]$ is the surface charge associated to the reference $\bar{\phi}^i$. For what concerns (on-shell) conservation of surface charges, we have, using Stokes theorem

$$\begin{aligned} Q_\xi[\phi^i, \bar{\phi}^i]|_{S_1} - Q_\xi[\phi^i, \bar{\phi}^i]|_{S_2} = \int_\gamma \int_{S_1} \mathbf{k}_\xi[\phi^i, \delta\phi^i] - \int_\gamma \int_{S_2} \mathbf{k}_\xi[\phi^i, \delta\phi^i] = \int_\gamma \int_\Sigma d\mathbf{k}_\xi[\phi^i, \delta\phi^i] \\ \approx \int_\gamma \int_\Sigma \boldsymbol{\omega}[\phi^i, \delta\phi^i, \mathcal{L}_\xi \phi^i], \end{aligned} \tag{2.60}$$

where we assumed $\partial\Sigma = S_1 \cup S_2$. Thus, surface charges are conserved if and only if $\boldsymbol{\omega}[\phi^i, \delta\phi^i, \mathcal{L}_\xi\phi^i] \approx 0$. We can further define the *symplectic structure* associated with Σ as

$$\boldsymbol{\Omega}_\Sigma[\phi^i, \delta_1\phi^i, \delta_2\phi^i] = \int_\Sigma \boldsymbol{\omega}[\phi^i, \delta_1\phi^i, \delta_2\phi^i], \tag{2.61}$$

so that equation (2.60) can be equivalently expressed as

$$Q_\xi[\phi^i, \bar{\phi}^i]|_{S_1} - Q_\xi[\phi^i, \bar{\phi}^i]|_{S_2} = \int_\gamma \boldsymbol{\Omega}_\Sigma[\phi^i, \delta\phi^i, \mathcal{L}_\xi\phi^i]. \tag{2.62}$$

As already discussed in section 2.2, in general relativity there are two main cases of interest. The first concerns exact isometries of the background metric, generated by diffeomorphisms ξ satisfying $\mathcal{L}_\xi g = 0$. This condition automatically implies that $\boldsymbol{\omega}[g, \delta g, \mathcal{L}_\xi g] = 0$ and thus the associated surface charges are conserved everywhere in the spacetime, including the bulk. Note that the ambiguity (2.55) does not affect the value of surface charges associated to exact symmetries. The second regards asymptotic symmetries, for which $\mathcal{L}_\xi g \to 0$ as $r \to \infty$. In this case, the presymplectic current vanishes only asymptotically, $\boldsymbol{\omega}[g, \delta g, \mathcal{L}_\xi g] \to 0$ and consequently the conservation laws only hold asymptotically. From now on, we will only be concerned about the latter.

Usually, for asymptotic symmetries, the $(n-2)$-hypersurfaces over which the charges are computed are defined by $r \to \infty$ and constant value of a time coordinate. For asymptotically AdS, they can be constant t sections of the asymptotic cylinder and for asymptotically flat spacetimes they can be celestial spheres, *i.e.* sections of null infinity $\mathscr{I}$ at fixed value of a retarded time coordinate u. In both cases they are spanned by angular coordinates. If the equation $\boldsymbol{\omega}[g, \delta g, \mathcal{L}_\xi g] \to 0$ is not satisfied, we say that there is a *breaking* in the conservation law. Typical examples of breaking appear in the context of radiating spacetimes, *e.g.* spacetimes characterized by a non-vanishing flux of gravitational radiation or energy carried by the matter fields. In this case, the interpretation of (2.60) is that the "time difference" between the charges is taken into account by an outgoing flux of radiation through the asymptotic boundary, as represented in Figure 1.

We note here that finiteness of the above defined Q_ξ is not guaranteed *a priori* and it must be checked for each set of boundary conditions one wants to implement. A general observation is that the set of boundary conditions should be "large" enough to include several physical solutions. However, one must be careful because if they are "too large", they might lead to the above mentioned divergences. In this case, there are two options. One is to check whether the divergences can be understood and eliminated in terms of a renormalization procedure. The other is to go back to the assumptions and try with more stringent boundary conditions. On the other hand, if they are too stringent one might be implicitly excluding the possibility of getting interesting physics.

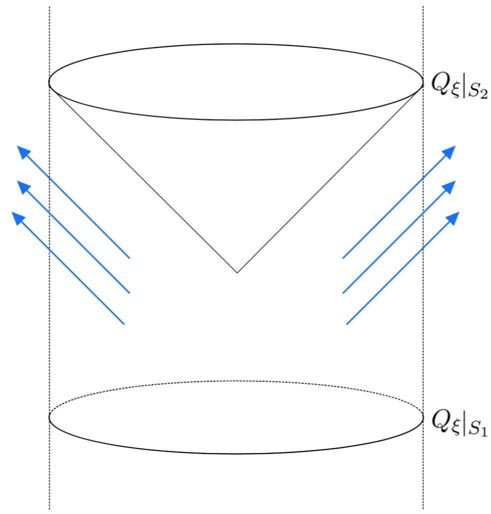

Figure 1: The time difference time between two charges $Q_\xi|_{S_1}$ and $Q_\xi|_{S_2}$ is taken into account by a non-vanishing flux of radiation (blu arrows) through the asymptotic boundary.

2.4 Asymptotic Symmetry Group

Having defined surface charges enables to discuss more concretely what it is meant by asymptotic symmetry group. We consider the set G of field configurations obeying certain boundary conditions. A vector field ξ is an *allowed diffeomorphism* if the variation of a field in G, under the action of ξ, is still in G. In other words, the action of ξ on the fields preserves the choice of boundary conditions. For every allowed diffeomorphism we can compute, using (2.59), its associated surface charge Q_ξ. Assuming it is integrable and finite, there are two possibilities. The first is that Q_ξ vanishes. In this case, we say that ξ is a *trivial diffeomorphism* and it has to be interpreted as a mere change of coordinates that does not carry any physical information or that, equivalently, it just describes a true redundancy of the theory. We want to exclude trivial diffeomorphisms from the definition of asymptotic symmetries. In case the surface charge does not vanish, it means that Q_ξ is physical and it generates non-trivial transformations within G, through the Poisson bracket (see (2.63) below) or that, equivalently, it changes the physical state of the system. These observations lead to define the asymptotic symmetry group as a quotient:

$$\text{Asymptotic Symmetry Group} = \frac{\text{Allowed Diffeomorphisms}}{\text{Trivial Diffeomorphisms}}.$$

Another way to define the asymptotic symmetry group is through a gauge fixing procedure. The idea is to start the analysis by performing a gauge fixing that, depending on the context, can be total or partial. In fact, in GR one can use the freedom in changing the coordinates to reach and fix certain conditions for the metric components. This amounts to eliminate all, or part of the redundant degrees of freedom through which the gauge theory is defined. Such procedure is in most cases equivalent to eliminating the above defined trivial diffeomorphisms. However, having performed a gauge fixing still leaves the freedom to perform residual gauge transformations. Then the asymptotic symmetry group can be defined as the set of those residual transformations that do not spoil the chosen boundary conditions.

2.5 Charge algebra

We assume that the asymptotic symmetry generators ξ_a form an algebra under some Lie bracket, *i.e.* $[\xi_a, \xi_b] = C_{ab}{}^c \xi_c$ with certain structure constants $C_{ab}{}^c = -C_{ba}{}^c$ [6]. The latter is referred to as *asymptotic symmetry algebra*. If the surface charges are integrable and finite, we define their Poisson bracket as

$$\{Q_{\xi_a}[\phi^i, \bar{\phi}^i], Q_{\xi_b}[\phi^i, \bar{\phi}^i]\} \equiv \delta_{\xi_b} Q_{\xi_a}[\phi^i, \bar{\phi}^i] = \int_S \mathbf{k}_{\xi_b}[\phi^i, \delta_{\xi_a}\phi^i]. \tag{2.63}$$

where we used equation (2.59). It can be shown [30, 37, 38] that the charges form, under the Poisson bracket, a projective representation of the asymptotic symmetry algebra,

$$\{Q_{\xi_a}[\phi^i, \bar{\phi}^i], Q_{\xi_b}[\phi^i, \bar{\phi}^i]\} \approx Q_{[\xi_a,\xi_b]}[\phi^i, \bar{\phi}^i] + \mathcal{K}_{\xi_a,\xi_b}[\bar{\phi}^i], \tag{2.64}$$

where $\mathcal{K}_{\xi_a,\xi_b} = -\mathcal{K}_{\xi_b,\xi_a}$ is a background-dependent central charge. $\mathcal{K}_{\xi_1,\xi_2}$ is a 2-cocycle on the asymptotic symmetry algebra

$$\mathcal{K}_{[\xi_a,\xi_b],\xi_c}[\bar{\phi}^i] + \mathcal{K}_{[\xi_c,\xi_a],\xi_b}[\bar{\phi}^i] + \mathcal{K}_{[\xi_b,\xi_c],\xi_a}[\bar{\phi}^i] = 0, \qquad \forall\, \xi_a,\ \xi_b,\ \xi_c. \tag{2.65}$$

The central charge $\mathcal{K}_{\xi_a,\xi_b}$ is non-trivial if it cannot be reabsorbed into a normalization of the charges $N_{[\xi_a,\xi_b]}[\bar{\phi}^i]$.

Equation (2.64) is the *surface charge algebra*. The main assumption behind the proof of (2.64) is that the charges are integrable. If they are not, the theorem (2.64) may not hold. However, in these situations it has been shown that it is still possible to split the local variation of charge into an integrable and a non-integrable part [32, 35] as

$$\not{\delta} Q_\xi[\phi^i, \delta\phi^i] = \delta(Q_\xi^{\text{int}}[\phi^i]) + \Xi_\xi[\phi^i, \delta\phi^i], \tag{2.66}$$

such splitting being non-unique. Modifying the definition (2.63) to

$$\{Q^{\text{int}}_{\xi_a}[\phi^i, \bar{\phi}^i], Q^{\text{int}}_{\xi_b}[\phi^i, \bar{\phi}^i]\} \equiv \delta_{\xi_b} Q^{\text{int}}_{\xi_a}[\phi^i, \bar{\phi}^i] + \Xi_{\xi_b}[\phi^i, \delta_{\xi_a}\phi^i], \tag{2.67}$$

has led to representation theorems similar to (2.64), where the 2-cocycle condition (2.65) has to be suitably modified.

2.6 The case of general relativity

Here we consider explicitly the case of GR. We start from the Einstein-Hilbert Lagrangian with vanishing cosmological constant,

$$\mathbf{L}_{EH}[g] = \frac{\sqrt{-g}}{16\pi G}(g^{\mu\nu} R_{\mu\nu}) \mathrm{d}^n x. \tag{2.68}$$

The variation of $\delta \mathbf{L}_{EH}$ corresponding to $\delta g_{\mu\nu}$ yields

$$\delta \mathbf{L}_{EH} = \frac{\delta \mathbf{L}_{EH}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + d\mathbf{\Theta}[g, h], \tag{2.69}$$

[6] Note that often, in the context of asymptotic symmetries, one uses the so-called "modified Lie bracket" that takes into account a possible field dependendance of the vector fields and of the structure constants, see *e.g.* [5, 34].

where

$$\frac{\delta \mathbf{L}_{EH}}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{16\pi G} G_{\mu\nu} \mathrm{d}^n x, \qquad \mathbf{\Theta}[g,h] = \frac{\sqrt{-g}}{16\pi G}\big(\nabla_\nu h^{\nu\mu} - \nabla^\mu h\big)(\mathrm{d}^{n-1}x)_\mu, \tag{2.70}$$

and where we introduced the notation $\delta g_{\mu\nu} \equiv h_{\mu\nu}$, $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} = -h^{\mu\nu}$, $h = g^{\mu\nu}h_{\mu\nu}$ and ∇ is the covariant derivative with respect to the background metric g. The presymplectic current [40, 43] $(n-1)$-form is

$$\begin{aligned}\boldsymbol{\omega}[g, h_1, h_2] &= \delta_1 \mathbf{\Theta}[g, h_2] - \delta_2 \mathbf{\Theta}[g, h_1] \\ &= \frac{\sqrt{-g}}{16\pi G}\left[\frac{1}{2}h_2 \nabla^\mu h_1 + h_{2\nu\rho}\nabla^\nu h_1^{\mu\rho} - \frac{1}{2}h_2\nabla_\nu h_1^{\nu\mu} - \frac{1}{2}h_2^{\nu\rho}\nabla^\mu h_{1\nu\rho} - \frac{1}{2}h_2^{\mu\rho}\nabla_\rho h_1 - (1\leftrightarrow 2)\right](\mathrm{d}^{n-1}x)_\mu,\end{aligned} \tag{2.71}$$

or, more compactly

$$\boldsymbol{\omega}[g, h_1, h_2] = \frac{\sqrt{-g}}{16\pi G} P^{\mu\nu\rho\sigma\alpha\beta}[h_{2\nu\rho}\nabla_\sigma h_{1\alpha\beta} - (1\leftrightarrow 2)](\mathrm{d}^{n-1}x)_\mu, \tag{2.72}$$

with

$$P^{\mu\nu\rho\sigma\alpha\beta} = g^{\mu\alpha}g^{\beta\nu}g^{\rho\sigma} - \frac{1}{2}g^{\mu\sigma}g^{\nu\alpha}g^{\beta\rho} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta} - \frac{1}{2}g^{\nu\rho}g^{\mu\alpha}g^{\beta\sigma} + \frac{1}{2}g^{\nu\rho}g^{\mu\sigma}g^{\alpha\beta}. \tag{2.73}$$

Under a diffeomorphism generated by ξ the metric transforms as $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ and we thus have for $\mathbf{\Theta}_\xi$,

$$\mathbf{\Theta}_\xi[g] = \frac{\sqrt{-g}}{16\pi G}\left[\nabla_\nu(\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu) + 2R^\mu{}_\nu \xi^\nu\right](\mathrm{d}^{n-1}x)_\mu, \tag{2.74}$$

where we used $\nabla^\mu \nabla_\nu \xi^\mu = \nabla_\nu \nabla^\mu \xi^\nu - R^\mu{}_\nu \xi^\nu$. On the other hand we have

$$i_\xi \mathbf{L}_{EH} = \frac{\sqrt{-g}}{16\pi G}\xi^\mu g^{\nu\rho}R_{\nu\rho}(\mathrm{d}^{n-1}x)_\mu, \tag{2.75}$$

so that the Noether current $(n-1)$-form $\mathbf{J}_\xi$ reads

$$\mathbf{J}_\xi[g] = i_\xi \mathbf{L}_{EH}[g] - \mathbf{\Theta}_\xi[g] = -\frac{\sqrt{-g}}{8\pi G}G^{\mu\nu}\xi_\nu(\mathrm{d}^{n-1}x)_\mu - \frac{1}{16\pi G}\partial_\nu\left[\sqrt{-g}(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu)\right](\mathrm{d}^{n-1}x)_\mu. \tag{2.76}$$

The weakly vanishing Noether current $\mathbf{S}_\xi$ is defined through

$$\frac{\delta \mathbf{L}_{EH}}{\delta g_{\mu\nu}}\mathcal{L}_\xi g_{\mu\nu} = -\frac{\sqrt{-g}}{8\pi G}G^{\mu\nu}\nabla_\nu \xi_\mu \mathrm{d}^n x = \frac{\sqrt{-g}}{8\pi G}(\nabla_\nu G^{\mu\nu})\xi_\mu \mathrm{d}^n x - \frac{1}{8\pi G}\partial_\nu[\sqrt{-g}G^{\mu\nu}\xi_\mu]\mathrm{d}^n x = d\mathbf{S}_\xi[g], \tag{2.77}$$

where we used the Noether identities $\nabla_\nu G^{\mu\nu} = 0$ and

$$\mathbf{S}_\xi[g] \equiv -\frac{\sqrt{-g}}{8\pi G}G^{\mu\nu}\xi_\nu(\mathrm{d}^{n-1}x)_\mu. \tag{2.78}$$

Comparing with (2.76), (2.78) and (2.46) implies that

$$\mathbf{J}_\xi[g] = \mathbf{S}_\xi[g] + d\mathbf{Q}_\xi[g], \qquad \mathbf{Q}_\xi[g] = \frac{\sqrt{-g}}{16\pi G}(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu)(\mathrm{d}^{n-2}x)_{\mu\nu} = \frac{\sqrt{-g}}{8\pi G}\nabla^\mu\xi^\nu(\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{2.79}$$

The $(n-2)$-form $\mathbf{Q}_\xi$ is the *Komar term* and integrated on the $(n-2)$-sphere at infinity gives the mass or angular momentum of stationary or rotationally invariant spacetimes, *i.e.* those admitting $\xi_t = \partial_t$ or $\xi_\phi = \partial_\phi$ as Killing vectors. Indeed, for these spacetimes ξ_t and ξ_ϕ satisfy the Killing equation

$$\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = 0. \tag{2.80}$$

In this case $\mathbf{Q}_\xi$ is conserved on-shell. We have

$$d\mathbf{Q}_\xi[g] = -\frac{1}{16\pi G}\partial_\nu \left[\sqrt{-g}(\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu)\right](\mathrm{d}^{n-1}x)_\mu \overset{(2.80)}{=} -\frac{\sqrt{-g}}{8\pi G}\nabla_\nu \nabla^\nu \xi^\mu (\mathrm{d}^{n-1}x)_\mu, \tag{2.81}$$

where in the last step we used that $T^{\mu\nu} = \nabla^\mu \xi^\nu$ is antisymmetric. We also have, using repeatedly the definition of the Riemann tensor and the Killing equation that $\nabla_\nu \nabla^\nu \xi^\mu = -R^\mu{}_\nu \xi^\nu$ and hence

$$d\mathbf{Q}_\xi[g] = \frac{\sqrt{-g}}{8\pi G} R^\mu{}_\nu \xi^\nu (\mathrm{d}^{n-1}x)_\mu \approx 0. \tag{2.82}$$

The charge $Q_\xi = \int_S \mathbf{Q}_\xi$ is conserved because of (2.82) and proportional to the mass or angular momentum of the spacetime.

We have

$$i_\xi \mathbf{\Theta}[g,h] = \frac{\sqrt{-g}}{8\pi G}\left(\xi^\nu \nabla_\rho h^{\rho\nu} - \xi^\nu \nabla^\mu h\right)(\mathrm{d}^{n-2}x)_{\mu\nu}, \tag{2.83}$$

and

$$\delta \mathbf{Q}_\xi[g] = \frac{\sqrt{-g}}{8\pi G}\left(\frac{1}{2}h\nabla^\mu \xi^\nu - h^{\mu\rho}\nabla_\rho \xi^\nu + \xi_\rho \nabla^\mu h^{\rho\nu}\right)(\mathrm{d}^{n-2}x)_{\mu\nu}, \tag{2.84}$$

so that the Iyer-Wald $(n-2)$-form $\mathbf{k}_\xi[g,h]$ is

$$\mathbf{k}_\xi[g,h] = \frac{\sqrt{-g}}{8\pi G}\left(\xi^\mu \nabla_\rho h^{\nu\rho} - \xi^\mu \nabla^\nu h + \xi_\rho \nabla^\nu h^{\mu\rho} + \frac{1}{2}h\nabla^\nu \xi^\mu - h^{\rho\nu}\nabla_\rho \xi^\mu\right)(\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{2.85}$$

If $g_{\mu\nu}$ and $h_{\mu\nu}$ satisfy Einstein's equations and linearized Einstein's equations around $g_{\mu\nu}$, respectively, it can be shown that $d\mathbf{k}_\xi \approx 0$. As mentioned in (2.56), $\mathbf{k}_\xi$ is still not completely fixed and it is defined up to a $(n-2)$-form $\boldsymbol{\omega}_B$ coming from a boundary contribution. Such term should vanish for exact symmetries because their associated charges are unambiguous. In particular, it should be proportional to the Killing equation, *e.g.*

$$\boldsymbol{\omega}_B[g,h,\delta_\xi g] = \frac{\sqrt{-g}}{16\pi G}h^{\rho\mu}\left(\nabla^\mu \xi_\rho + \nabla_\rho \xi^\mu\right)(\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{2.86}$$

With this choice we get the *Abbot-Deser charge* $(n-2)$-form $\mathbf{k}'_\xi[g,h]$ [44]

$$\begin{aligned}\mathbf{k}'_\xi[g,h] &= \mathbf{k}_\xi[g,h] + \boldsymbol{\omega}_B[g,h,\delta_\xi g]\\ &= \frac{\sqrt{-g}}{8\pi G}\left(\xi^\mu \nabla_\rho h^{\nu\rho} - \xi^\mu \nabla^\nu h + \xi_\rho \nabla^\nu h^{\mu\rho} + \frac{1}{2}h\nabla^\nu \xi^\mu - \frac{1}{2}h^{\rho\nu}\nabla_\rho \xi^\mu + \frac{1}{2}h^{\nu\rho}\nabla^\mu \xi_\rho\right)(\mathrm{d}^{n-2}x)_{\mu\nu}.\end{aligned} \tag{2.87}$$

However, $\mathbf{k}'_\xi$ in (2.87) can be obtained directly from the weakly vanishing Noether $(n-1)$-current $\mathbf{S}_\xi$ by acting on it with the "Anderson's homotopy operator" $I^{n-1}_{\delta\phi^i}$,

$$I^{n-1}_{\delta\phi^i} = \left[\frac{1}{2}\delta\phi^i \frac{\partial}{\partial(\partial_\mu \phi^i)} - \frac{1}{3}\delta\phi^i \partial_\nu \frac{\partial}{\partial(\partial_\mu \partial_\nu \phi^i)} + \frac{2}{3}\partial_\nu \delta\phi^i \frac{\partial}{\partial(\partial_\mu \partial_\nu \phi^i)} + \dots\right]\frac{\partial}{\partial \mathrm{d}x^\mu}, \tag{2.88}$$

where ... comprise terms with higher derivatives of ϕ^i, as[7]

$$\mathbf{k}'_\xi[\phi^i, \delta\phi^i] = I^{n-1}_{\delta\phi^i}\mathbf{S}_\xi[\phi^i]. \tag{2.89}$$

The $(n-2)$-form $\mathbf{k}'_\xi$ in equation (2.89) is the *Barnich-Brandt charge* [30] and, as mentioned, in the case of general relativity matches with the Abbot-Deser charge. The local variation of charge in general relativity is thus

$$\delta\!\!\!/ Q_\xi[g,h] = \int_S \mathbf{k}'_\xi[g,h]. \tag{2.90}$$

In the remainder of the first part of this work, we will make large use of the above considered quantities. In particular, we will explicitly compute the symplectic structure in (2.72) for asymptotically flat spacetimes in four dimensions and, correspondingly, we will construct the supertranslation charges using (2.62) and verify that the breaking in their conservation law is ultimately due to a non-vanishing flux of gravitational radiation through null infinity. This will play a fundamental role in understanding the interplay between supertranslations symmetries and the Weinberg's soft graviton theorem. Moreover, we will use the Barnich-Brandt prescription in (2.90) in the context of asymptotically locally AdS_3 spacetimes to compute the surface charges associated to boundary diffeomorphisms and to diffeomorphisms generating Weyl transformations and we will prove that they satisfy the surface charge algebra in (2.64) and exhibit its associated central extension.

[7] Note that equation (2.89) yields a prescription to compute $(n-2)$-conserved forms for *any* gauge theory and it is not restricted to the case of general relativity.

3 Asymptotically flat spacetimes and the BMS$_4$ group

Flat spacetime has an interesting and useful group of isometries, but, for a generic spacetime, the isometry group is simply the identity and hence provides no significant information. Yet symmetry groups have important role to play in physics; in particular, the Poincaré group, describing isometries of Minkowski spacetime plays a crucial role in the standard definitions of energy-momentum and angular momentum. For this reason alone it would seem important to look for a generalization of the concept of isometry group that can apply in a useful way to suitable curved spacetimes. In this chapter we introduce the notion of asymptotic flatness, which encompasses all those solutions of Einstein's equations that suitably approach, at infinity, Minkowski spacetime and we study their asymptotic symmetries, given by the BMS$_4$ group.

In sections 3.1 and 3.2 we start by reviewing the notion of conformal compactification and conformal infinity and we outline how this is an essential feature to capture the geometrical structure underlying the notion of asymptotic flatness. We proceed in sections 3.3 and 3.4 to introduce the retarded Bondi gauge and consequently the notion of asymptotic flatness in terms of certain fall-offs for the metric components; we solve Einstein's equations corresponding to these boundary conditions. In sections 3.5 and 3.6 we identify the generators of the $\mathfrak{bms}_4$ algebra by solving the asymptotic Killing equations and we comment on the mathematical differences between global, local and generalized $\mathfrak{bms}_4$ algebra, providing explicit realizations of the first two.

3.1 Conformal infinity and the asymptotic structure of flat spacetime

Throughout this chapter, the notion of asymptotic flatness will be of crucial importance. Intuitively, this notion specifies how a certain class of spacetimes, "at infinity", approach Minkowski spacetime. Before introducing the details of asymptotic flatness, it is instructive to investigate the asymptotic structure of Minkowski spacetime itself and we do it by reviewing the notion of conformal compactification and conformal infinity, originally introduced by Penrose [45–47].

Let us consider Minkowski line element in spherical coordinates $x^\mu = (t, r, x^A)$, where the index $A = 1, 2$ labels angular coordinates,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{3.1}$$

with γ_{AB} standard metric on the unit 2-sphere. Notice that, as $r \to \infty$, the line element in (3.1) is characterized by a second order pole, due to the factor r^2 multiplying γ_{AB}. Therefore Minkowski spacetime is singular in this limit. In other words, the value $r = \infty$ has to be excluded from the range of the radial coordinate. We expect similar singularities to occur for all asymptotically flat spacetimes, because their line element, at leading order in the radial coordinate, should approach (3.1). We introduce the *retarded time* and *advanced time* null coordinates (u, v) defined as $u = t - r$ and $v = t + r$, with $v \geq u$ such that curves with $u = \text{const}$ and $v = \text{const}$ represent outgoing and ingoing light rays, respectively. The ranges of these coordinates are $-\infty < u, v < \infty$. We can further define new coordinates (p, q) as $u = \tan p$ and $v = \tan q$ so that $-\pi/2 < p, q < \pi/2$. Notice that the effect of introducing (p, q) is to map "points at infinity" to points at a finite distance. However, $p, q = \pm\pi/2$ are still singular for the metric. Indeed (3.1), in (p, q) coordinates reads

$$\mathrm{d}s^2 = \frac{1}{(\cos q \cos p)^2}\Big(- \mathrm{d}p\mathrm{d}q + \frac{\sin^2(q-p)}{4}\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B\Big). \tag{3.2}$$

In order to eliminate these divergences, we perform a Weyl rescaling with conformal factor $\Omega(q,p) = 2\cos q \cos p$, so that [8]

$$\mathrm{d}\tilde{s}^2 \equiv \Omega^2(p,q)\mathrm{d}s^2 = -4\mathrm{d}p\mathrm{d}q + \sin^2(q-p)\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B. \tag{3.3}$$

The points $p, q = \pm\pi/2$ are not singular for the rescaled line element $\mathrm{d}\tilde{s}^2$ and we can define an *unphysical spacetime* $(\tilde{\mathcal{M}}, \mathrm{d}\tilde{s}^2)$ by adding to the *physical spacetime* $(\mathcal{M}, \mathrm{d}s^2)$ the points at infinity $p, q = \pm\pi/2$. Consequently, for the unphysical spacetime, we can extend the range of the coordinates (p, q) to $-\pi/2 \leq p, q \leq \pi/2$. As mentioned, one of the main reasons for introducing the unphysical spacetime manifold $\tilde{\mathcal{M}}$ is that the infinity of $\mathcal{M}$ gets mapped to a finite hypersurface in $\tilde{\mathcal{M}}$, which we will denote by $\mathscr{I}$, so that asymptotic properties of the fields in $\mathcal{M}$ can be investigated by studying the behavior of the rescaled fields on $\mathscr{I}$. We can thus write $\tilde{\mathcal{M}} = \mathcal{M} \cup \mathscr{I}$. Furthermore, null geodesics in $\mathcal{M}$ correspond to null geodesics in $\tilde{\mathcal{M}}$ because conformal transformations map null vectors to null vectors so that the light-cone structure is preserved under (3.3).

Introducing now time and space coordinates (t', r') as $p = (t'-r')/2$ and $q = (t'+r')/2$ we have

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + \sin^2 r' \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{3.4}$$

We introduce the following points in $\tilde{\mathcal{M}}$:

- *Future timelike infinity* i^+, defined by $(t', r') = (\pi, 0)$. All the images in $\tilde{\mathcal{M}}$ of timelike geodesics terminate at this point;
- *Past timelike infinity* i^- defined by $(t', r') = (-\pi, 0)$. All the images in $\tilde{\mathcal{M}}$ of timelike geodesics originate at this point;
- *Spacelike infinity* i^0 defined by $(t', r') = (0, \pi)$. All the images in $\tilde{\mathcal{M}}$ of spacelike geodesics originate and terminate at this point.

We also introduce the following hypersurfaces in $\tilde{\mathcal{M}}$:

- *Future null infinity* $\mathscr{I}^+$ defined by the equation $t' = \pi - r'$. It is the null hypersurface where all the outgoing null geodesics terminate. In the conformal diagram 2, it connects i^+ to i^0;
- *Past null infinity* $\mathscr{I}^-$ defined by the equation $t' = \pi + r'$. It is the null hypersurface where all the outgoing null geodesics originate. In the conformal diagram, it connects i^- to i^0.

An intuitive picture of the structure of the unphysical spacetime in (t', r') is given by the *Penrose diagram* 2. In the picture, null geodesic are described by $p =$ const and $q =$ const straight lines at 45°, starting from $\mathscr{I}^-$, reflecting at $r' = 0$ and terminating on $\mathscr{I}^+$ while timelike curves start from i^- and terminate on i^+. Spacelike curves start and terminate at i^0. Notice that because $\mathscr{I}^+$ is null, its induced metric is degenerate, $\mathrm{d}s^2_{\mathscr{I}^+} = 0\,\mathrm{d}u^2 + \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$. The topology of $\mathscr{I}^+$ is $\mathbb{R} \times \mathbb{S}^2$, the $\mathbb{R}$ and $\mathbb{S}^2$ components being

[8] Let us point out that there is however an ambiguity in the choice of the conformal factor Ω in (3.3) if the only requirement is to eliminate the second order pole. Indeed, one could have chosen $\Omega' = e^{\omega}\Omega$ with ω smooth function independent of the radial coordinate. This will be relevant in section 5, in the context of asymptotically AdS spacetimes.

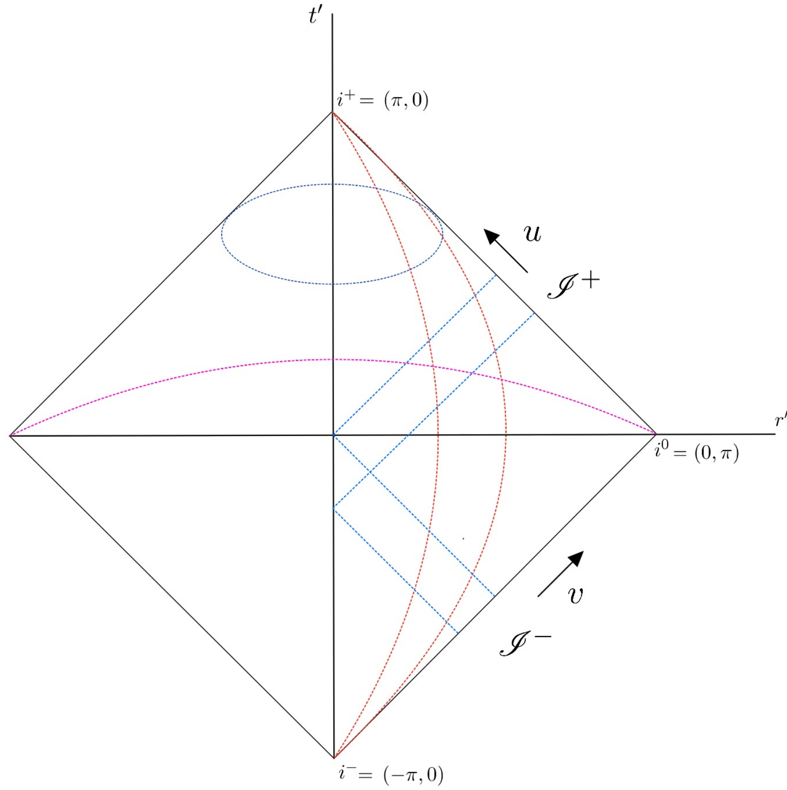

Figure 2: A Penrose diagram for $\mathcal{M}$. Note that the angular coordinates x^A have been suppressed so that each point represents a 2-sphere.

parametrized by u and x^A, respectively. A *celestial sphere* is a section $\mathbb{S}^2$ of $\mathscr{I}^+$ defined by a constant value of the retarded time u. It will be useful to introduce the future and past boundaries of $\mathscr{I}^+$, denoted by $\mathscr{I}^+_+$ and $\mathscr{I}^+_-$, respectively. It is possible to make similar considerations and give analogous definitions for the asymptotic structure on past null infinity $\mathscr{I}^-$.

3.2 Asymptotic flatness in a coordinate-independent way

In the previous section, we analyzed the asymptotic properties of Minkowski spacetime. In particular, we have shown that, starting from flat spacetime $(\mathcal{M}, \mathrm{d}s^2)$, it is possible to construct an unphysical spacetime $(\tilde{\mathcal{M}}, \mathrm{d}\tilde{s}^2)$ with $\mathrm{d}\tilde{s}^2 = \Omega^2 \mathrm{d}s^2$ and $\tilde{\mathcal{M}} = \mathcal{M} \cup \mathscr{I}$. The function Ω vanishes on the null boundary $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^- \cup i^- \cup i^0 \cup i^+$, where $\mathscr{I}^\pm \simeq \mathbb{R} \times \mathbb{S}^2$. It is reasonable to ask that, whatever is the definition of asymptotic flatness, it should encompass some of the above mentioned properties. There are several definitions in the literature of what it is actually meant by asymptotic flatness. Some of them consist in specifying a coordinate system (*i.e.* a gauge) and giving certain fall-offs for the metric, *e.g.* $g - \eta = \mathcal{O}(r^{-1})$. A study of these boundary conditions will be extensively given in the next sections. Here, we briefly mention that there exist an "alternative" definition of asymptotic flatness, which has the advantage of not relying on the choice of a coordinate system, but rather on the definition a global structure. We present it here and for a more detailed analysis we refer the reader to [45–53].

A spacetime $(\mathcal{M}, \mathrm{d}s^2)$ is *asymptotically flat* if there exists a spacetime $(\tilde{\mathcal{M}}, \mathrm{d}\tilde{s}^2)$ with boundary $\mathscr{I} = \partial\tilde{\mathcal{M}}$

and a diffeomorphism of $\mathcal{M}$ onto the interior $\tilde{\mathcal{M}}/\mathscr{I}$ (so that $\mathcal{M}$ can be identified with its image in $\tilde{\mathcal{M}}$ under the diffeomorphism) such that

1. There exists a smooth function Ω on $\tilde{\mathcal{M}}$ with:
   (a) $\mathrm{d}\tilde{s}^2 = \Omega^2 \mathrm{d}s^2$ on $\mathcal{M}$;
   (b) $\Omega = 0$ on $\mathscr{I}^+$;
   (c) $n_\mu \equiv \nabla_\mu \Omega$ is nowhere vanishing on $\mathscr{I}$;
2. $\mathscr{I}$ has two connected components, $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ such that $\mathscr{I}^\pm \simeq \mathbb{R} \times \mathbb{S}^2$ [9][10];
3. The metric $g_{\mu\nu}$ satisfies $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $\Omega^{-2} T_{\mu\nu}$ has a smooth limit on $\mathscr{I}$;
4. The integral curves of n^μ are complete on $\mathscr{I}$ for any choice [11] of the conformal factor for which $\mathscr{I}$ is divergence-free (*i.e.* $\nabla_\mu n^\mu = 0$).

It can be shown that: conditions 1b and 1c imply that Ω falls on $\mathscr{I}$ as $1/r$ as one recedes along null directions; conditions 1 and 3 imply that $\mathscr{I}$ is a null hypersurface with normal vector $n^\mu = \nabla^\mu \Omega$ satisfying $n^\mu n_\mu|_{\mathscr{I}} = 0$ and the *Bondi condition* $\nabla_\mu n_\nu|_{\mathscr{I}} = 0$; condition 4, weaker than the one originally considered in [46] which was excluding black holes solutions, ensures that the action that the full BMS_4 group is well-defined on $\mathscr{I}$ and not only that of its Lie algebra.

We will not further pursue this approach and explore its consequences here. However it worth pointing out that it is possible to derive the asymptotic symmetries of the class of spacetimes satisfying 1-4 as the subgroup of diffeomorphisms preserving a certain geometrical structure.

3.3 Bondi gauge and residual Killing vectors

We now proceed to introduce a suitable coordinate system adapted to study the asymptotic properties of the gravitation field at future null infinity. This set of coordinates will be essential to describe asymptotic flatness.

We define the *retarded Bondi gauge* as the set of coordinates $x^\mu = (u, r, x^A)$ with $A = 1, 2$ such that:

- The hypersurfaces given by the equations $u = \text{const}$ are null. It implies that the orthogonal covector to such hypersurfaces $k_\mu = \partial_\mu u = \delta^u_\mu$ satisfies $g^{\mu\nu} k_\mu k_\mu = 0$. Consequently $g^{uu} = 0$.
- The angular coordinates x^A are constant along the null rays, *i.e.*

$$k^\mu \partial_\mu x^A = g^{\mu\nu} \delta^u_\nu \partial_\mu x^A = g^{u\mu} \delta^A_\mu = g^{uA} = 0, \tag{3.5}$$

  for $A = 1, 2$.

[9] It is not completely clear if 2 is actually a consequence of the other conditions. A first proof of 2 can indeed be found in [54]. However, as remarked in [55], the arguments used in [54] were incorrect and more rigorous proofs can be found in [56, 57].

[10] Note also that the points $i^\pm$ and i^0 are excluded from this definition. In fact, for asymptotically flat spacetimes that are not flat, these points might be singular for the conformal geometry [50].

[11] Bear in mind that Ω is ambiguous, as remarked in 8.

- The radial coordinate r, which varies along null rays is an areal coordinate [3] so that

$$\det[g_{AB}] = r^4 \det[\gamma_{AB}] \equiv r^4 b(x^A), \tag{3.6}$$

e.g. in spherical coordinates (θ, ϕ) and in complex stereographic coordinates $(z, \bar{z})$ where $z = \tan\left(\frac{\theta}{2}\right)e^{i\phi}$ we have $b(\theta,\phi) = \sin^2\theta$ and $b(z,\bar{z}) = 4/(1+z\bar{z})^2$. Sometimes equation (3.6) is substituted by the weaker condition [5]

$$\partial_r\left(\frac{\det[g_{AB}]}{r^4}\right) = 0, \tag{3.7}$$

ensuring that r is a luminosity distance. The solution of (3.7) is $\det[g_{AB}] = r^4\bar{b}(u, x^A)$, allowing for an extra u dependence. We will use the more general (3.7) instead of (3.6).

As mentioned, the above defined set of coordinates is suited to describe the behavior of the metric at future null infinity $\mathscr{I}^+$. Similarly, we could have introduced the *advanced Bondi gauge*, consisting in specifying a set of coordinates (v, r, x^A) describing the behavior of the fields at past null infinity $\mathscr{I}^-$. All the considerations in the remainder can be easily extended to the case of advanced gauge.

The gauge fixing conditions $g^{uA} = 0 = g^{uu}$ are equivalent to $g_{rr} = 0 = g_{rA}$ and hence the most general line element in the retarded Bondi gauge can be written as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = e^{2\beta}\frac{V}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + g_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u), \tag{3.8}$$

where V, U^A, β and g_{AB} satisfying (3.7) are six arbitrary functions to be determined. Note that we have used all the freedom coming from the gauge invariance of the theory. Indeed, a general symmetric metric tensor in four dimensions is defined by ten arbitrary functions. Having fixed the Bondi gauge has reduced this number to six. However, there is still some residual freedom which consists in diffeomorphisms generated by a ξ solving

$$\mathcal{L}_\xi g_{rr} = 0, \qquad \mathcal{L}_\xi g_{rA} = 0, \qquad \mathcal{L}_\xi \partial_r\left(\frac{\det[g_{AB}]}{r^4}\right) = 0. \tag{3.9}$$

The solution to (3.9) for ξ is

$$\xi^u = f, \tag{3.10}$$

$$\xi^A = Y^A - \partial_B f \int_r^\infty \mathrm{d}r' (e^{2\beta} g^{BA}), \tag{3.11}$$

$$\xi^r = -\frac{r}{2}\left[D_A Y^A - D_A\left(\partial_B f \int_r^\infty \mathrm{d}r'(e^{2\beta}g^{BA})\right) - U^A\partial_A f - 2\omega\right]. \tag{3.12}$$

where $f(u, x^A), Y^A(u, x^B)$ and $\omega(u, x^B)$ are independent of the radial coordinate and where D_A is the covariant derivative associated to the unit metric γ_{AB} on the 2-sphere. It follows that the residual diffeomorphisms preserving the Bondi gauge are parametrized by four functions (f, Y^A, ω) of (u, x^A).

3.4 Asymptotic flatness and solution of Einstein's equations

Following the gauge fixing procedure outlined in 2.4, we now proceed to solve (vacuum) Einstein's equations $R_{\mu\nu} = 0$ introducing certain boundary conditions. We require that the angular metric g_{AB} admits an analytical expansion in the radial coordinate r as

$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + \tilde{D}_{AB} + \mathcal{O}(r^{-1}), \tag{3.13}$$

where the first term accounts for the second order pole discussed in 3.1 and the symmetric tensors $C_{AB}(u, x^D)$, referred to as the *asymptotic shear*, and $\tilde{D}_{AB}(u, x^D)$ are independent of the radial coordinate. From now on the angular indices $A, B, ..$ will be lowered and raised with γ_{AB}. Condition (3.7) implies that

$$C^A{}_A = 0, \qquad \tilde{D}_{AB} = \frac{1}{4}\gamma_{AB}C^{DE}C_{DE} + D_{AB}, \qquad D^A{}_A = 0, \tag{3.14}$$

with D_{AB} symmetric and traceless tensor. We are interested in describing asymptotic flatness and hence we require that $\mathrm{d}s^2$ in (3.8) approaches, at large values of r, the Minkowski line element in (3.1). It implies the following fall-offs for β, V and U^A:

$$\beta = \mathcal{O}(r^{-2}), \qquad \frac{V}{r} = -1 + \mathcal{O}(r^{-1}), \qquad U^A = \mathcal{O}(r^{-2}). \tag{3.15}$$

The set of boundary conditions in (3.13) and (3.15) defines *asymptotically flat spacetimes*[12].

We now proceed to present the solution of Einstein's equations corresponding to these boundary conditions [1, 2, 5, 39, 58, 59]. They split as follows:

- *Main equations*:

$$R_{rr} = 0, \qquad R_{rA} = 0, \qquad g^{AB}R_{AB} = 0, \qquad R_{AB} - \frac{1}{2}g_{AB}R = 0. \tag{3.16}$$

  Sometimes the first three of (3.16) are referred to as *hypersurface equations* because they determine the radial evolution of β, V and U^A, while the last of (3.16) is referred to as *standard equation*.

- *Trivial equation*:

$$R_{ur} = 0. \tag{3.17}$$

- *Supplementary equations*:

$$R_{uA} = 0, \qquad R_{uu} = 0. \tag{3.18}$$

The first three main equations in (3.16) yield the following fall-offs for β, U^A and V, respectively

$$\beta = -\frac{1}{32r^2}C^{AB}C_{AB} - \frac{1}{12r^3}C^{AB}D_{AB} + \mathcal{O}(r^{-4}), \tag{3.19}$$

$$U^A = -\frac{1}{2r^2}D_C C^{AC} - \frac{2}{3r^3}\left[N^A + \left(\frac{1}{3} + \log r\right)D_B D^{BA} - \frac{1}{2}C^{AB}D_C C^{CB}\right] + \mathcal{O}(r^{-4}), \tag{3.20}$$

$$V = -1 + \frac{2m}{r} + \mathcal{O}(r^{-2}). \tag{3.21}$$

[12] For more general boundary conditions see *e.g.* [39].

Note the appearance of the integration constants $N^A(u,x^B)$ and $m(u,x^A)$ in (3.20) and (3.21). In principle there could be an integration constant also in (3.19), $\beta_0(u,x^B)$ which is however absent because of the first condition in (3.15). $N^A(u,x^B)$ and $m(u,x^A)$ are referred to as *angular momentum aspect* and *Bondi mass aspect*, respectively, and they give information about the angular density of angular momentum and energy of the spacetime, as we will further comment later on.

The last equation in (3.16) gives the retarded time evolution of all the tensors appearing in the analytic expansion (3.13) of g_{AB} in terms of β, U^A and V. However the time derivative of the shear tensor is not fixed by the last of (3.16). It means that, introducing the *Bondi news tensor* $N_{AB}(u,x^C)$ as

$$N_{AB} \equiv \partial_u C_{AB}, \tag{3.22}$$

it must be given as part of the free data. Taking into account the traceless condition in (3.14), it consists in two arbitrary functions that have to be interpreted as the two degrees of freedom of the gravitational field. Further, it can be shown that the last of (3.16) fixes

$$\partial_u D_{AB} = 0. \tag{3.23}$$

If it is required the absence of logarithmic terms in (3.20) expansion, then one should require

$$D_B D^{BA} = 0, \tag{3.24}$$

which, taking into account (3.23) is solved in stereographic coordinates by

$$D^{zz} = f(\bar{z})(1+z\bar{z})^4, \qquad D^{\bar{z}\bar{z}} = g(z)(1+z\bar{z})^4. \tag{3.25}$$

with f and g arbitrary functions. Notice that for non-vanishing f and g, (3.25) introduce singularities on the 2-sphere spanned by $(z,\bar{z})$. In our analysis we will not be concerned about these issues and from now on we will simply impose $D_{AB} = 0$.

It can be shown that the trivial equation in (3.17), once the main equations are solved, is automatically satisfied and hence it gives no additional information. For what concerns the supplementary equations in (3.18), assuming that all other equations are solved, they have to be solved only at order r^{-2}. So far we were only concerned about vacuum Einstein's equations. However, it can be shown that adding a non-vanishing stress energy tensor $T_{\mu\nu}$ with certain fall-offs motivated by the behavior of radiative scalar-field solutions in Minkowski spacetime (see *e.g.* condition 3 and [60]) will influence the supplementary equations, that constraint the retarded time evolution of N^A and m, respectively, as

$$\begin{aligned}\partial_u N_A = \partial_A m &+ \frac{1}{16}\partial_A\big(C_{BC}N^{BC}\big) - \frac{1}{4}N_{BC}D_A C^{BC} - \frac{1}{4}D_B\big(D^B D^C C_{CA} - D_A D_C C^{BC}\big)\\ &- \frac{1}{4}\big(C^{CB}N_{AC} - C_{AC}N^{CB}\big) - 8\pi G\hat{T}_{uA},\end{aligned} \tag{3.26}$$

$$\partial_u m = \frac{1}{4}D_A D_B N^{AB} - \frac{1}{8}N_{AB}N^{AB} - 4\pi G\hat{T}_{uu}, \tag{3.27}$$

where $T_{uA} = r^{-2}\hat{T}_{uA}(u,x^B) + \mathcal{O}(r^{-3})$ and $T_{uu} = r^{-2}\hat{T}_{uu}(u,x^A) + \mathcal{O}(r^{-3})$. Defining the *Bondi mass* as $M = \int_{S^2} \mathrm{d}^2x\sqrt{\gamma}m$ and assuming the null energy condition $\hat{T}_{uu} \geq 0$, an interesting application of equation (3.27) is the *Bondi mass loss formula*,

$$\partial_u M = -\frac{1}{8}\int_{S^2} \mathrm{d}^2x\sqrt{\gamma}N_{AB}N^{AB} - 4\pi G\int_{S^2}\mathrm{d}^2x\sqrt{\gamma}\hat{T}_{uu} \leq 0, \tag{3.28}$$

according to which if a system emits gravitational radiation, *i.e.* there is a non-vanishing news tensor, the Bondi mass always decreases in time.

The above analysis shows that for asymptotically flat spacetimes, within our setup, a generic set of initial data on $\mathscr{I}^+$ is $\chi_{\mathscr{I}^+} = \{m(u_0, x^A), C_{AB}(u_0, x^C), N_{AB}(u, x^C), N_A(u_0, x^B), ...\}$, for an arbitrary initial point u_0 on $\mathscr{I}^+$ and where ... comprise all the infinite tower of subleading multipoles in the g_{AB} expansion of (3.13) at u_0 [13]. The subset $\chi'_{\mathscr{I}^+} = \{m(u_0, x^A), C_{AB}(u_0, x^C), N_{AB}(u, x^C), N_A(u_0, x^B)\} \subset \chi_{\mathscr{I}^+}$ forms a set of initial data on $\mathscr{I}^+$ at first and second subleading order in the luminosity distance r.

Putting all these results together, the line element (3.8) admits the following asymptotic expansion around $\mathscr{I}^+$,

$$\begin{aligned} \mathrm{d}s^2 = &- \left(1 - \frac{2m}{r} + \mathcal{O}(r^{-2})\right)\mathrm{d}u^2 - 2\left(1 - \frac{C_{AB}C^{AB}}{16r^2} + \mathcal{O}(r^{-3})\right)\mathrm{d}u\mathrm{d}r + r^2\left(\gamma_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(r^{-2})\right)\mathrm{d}x^A\mathrm{d}x^B \\ &+ \left(D^C C_{AC} + \frac{4N_A}{3r} - \frac{C_{AB}D_C C^{CB}}{3r} + \mathcal{O}(r^{-2})\right)\mathrm{d}x^A\mathrm{d}u, \end{aligned} \tag{3.29}$$

where the retarded time evolutions of N_A and m are governed by (3.26) and (3.27), respectively.

3.5 Asymptotic symmetries and asymptotic symmetry algebra

We now focus on the asymptotic symmetries of the above defined class of spacetimes, *i.e.* the subset of the residual Killing vectors in equations (3.10)-(3.12) preserving the fall-off conditions in (3.29). In particular, we ask that

$$\mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \qquad \mathcal{L}_\xi g_{uA} = \mathcal{O}(1), \qquad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r), \qquad \mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1}). \tag{3.30}$$

These equations constrain the form of $f(u, x^A), Y^A(u, x^A)$ and $\omega(u, x^A)$ appearing in (3.10)-(3.12) as

$$\partial_u Y^A = 0, \qquad \partial_u f = \frac{1}{2}D_A Y^A, \qquad D_A Y_B + D_B Y_A = \gamma_{AB} D_C Y^C, \qquad \omega = 0. \tag{3.31}$$

Solving the first two equations in (3.31) yields

$$Y^A = Y^A(x^B), \qquad f = T(x^A) + \frac{u}{2}D_A Y^A, \tag{3.32}$$

with $T(x^A)$ arbitrary function on the 2-sphere, while the third implies that Y^A is a conformal Killing vector of γ_{AB}. The solution, in $(z, \bar{z})$ coordinates, is

$$Y^z = Y^z(z), \qquad Y^{\bar{z}} = Y^{\bar{z}}(\bar{z}). \tag{3.33}$$

The asymptotic symmetries of asymptotically flat spacetimes are thus generated by a vector field ξ parametrized by (T, Y^A), where T and Y^A are an arbitrary function and a conformal Killing vector on the

[13] Equivalently, introducing $E_{AB}(u_0, r, x^C) \equiv g_{AB}(u_0, r, x^C) - r^2\gamma_{AB}(x^C) - rC_{AB}(u_0, x^C)$, we have $\chi_{\mathscr{I}^+} = \{m(u_0, x^A), C_{AB}(u_0, x^B), N_{AB}(u, x^C), N_A(u_0, x^A), E_{AB}(u_0, x^C)\}$.

2-sphere and whose components ξ^μ are given by

$$\xi^u = T + \frac{u}{2} D_A Y^A, \tag{3.34}$$

$$\xi^A = Y^A - \partial_B \left(T + \frac{u}{2} D_C Y^C\right) \int_r^\infty \mathrm{d}r' (e^{2\beta} g^{BA}), \tag{3.35}$$

$$\xi^r = -\frac{r}{2}\left[D_A Y^A - D_A\left[\partial_B \left(T + \frac{u}{2} D_C Y^C\right) \int_r^\infty \mathrm{d}r' (e^{2\beta} g^{BA})\right] - U^A \partial_A \left(T + \frac{u}{2} D_C Y^C\right)\right]. \tag{3.36}$$

Note that ξ is field-dependent because it depends on the metric components.

3.5.1 Asymptotic symmetry algebra on $\mathscr{I}^+$

Let us start by considering the algebra realized by the vector ξ on $\mathscr{I}^+ = \mathbb{R} \times \mathbb{S}^2$, given by $\xi = \xi_T + \xi_Y$, with

$$\xi_T = T \frac{\partial}{\partial u}, \qquad \xi_Y = \frac{u}{2} D_A Y^A \frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial x^A}. \tag{3.37}$$

The standard Lie bracket between vector fields yield

$$[\xi_{T_1}, \xi_{T_2}] = 0, \qquad [\xi_{Y_1}, \xi_{Y_2}] = \xi_{\hat{Y}}, \qquad [\xi_{Y_1}, \xi_{T_2}] = \xi_{\hat{T}_{12}}, \tag{3.38}$$

where

$$\hat{Y}^B = Y_1^A \partial_A Y_2^B - Y_2^A \partial_A Y_1^B, \qquad \hat{T}_{12} = Y_1^A \partial_A T_2 - \frac{1}{2} T_2 D_A Y_1^A. \tag{3.39}$$

The vector fields ξ_T and ξ_Y in (3.37) are the generators of *supertranslations* and of conformal transformations (global or local, as we will discuss in section 3.6), respectively. Note that the supertranslations form an abelian algebra. The full algebra reads

$$[\xi_1, \xi_2] = [\xi_{Y_1} + \xi_{T_1}, \xi_{Y_2} + \xi_{T_2}] = \xi_{\hat{Y}} + \xi_{\hat{T}_{12}} - \xi_{\hat{T}_{21}} \equiv \hat{\xi} = \xi_{\hat{Y}} + \xi_{\hat{T}}, \tag{3.40}$$

where

$$\hat{T} = Y_1^A \partial_A T_2 - Y_2^A \partial_A T_1 + \frac{1}{2}\left(T_1 D_A Y_2^A - T_2 D_A Y_1^A\right), \tag{3.41}$$

and $\hat{Y}$ is again given by the first in (3.39). The Lie algebra satisfied by ξ is denoted $\mathfrak{bms}_4$ (Bondi-Metzner-Sachs) and, as it is clear from the structure of the bracket (3.38), it is a semidirect sum of the algebra of conformal vector fields on the 2-sphere with that of abelian supertranslations. The corresponding Lie group is called the BMS_4 group. Note that there is a non-trivial action of ξ_Y on ξ_T, giving the structure of semidirect sum. We can abstractly define the Lie bracket of $\mathfrak{bms}_4$ as

$$[(Y_1, T_1), (Y_2, T_2)] = (\hat{Y}, \hat{T}), \tag{3.42}$$

where $\hat{Y}$ and $\hat{T}$ are given by the first of (3.39) and (3.41).

3.5.2 Extension of $\mathfrak{bms}_4$ in the bulk

We now turn to consider the vector field in (3.34)-(3.36) defined on the whole spacetime. As mentioned, it depends on the metric through the functions β, g^{AB} and U^A. In other words, the gauge parameters are field-dependent. We thus consider the *modified Lie bracket* [5], defined as

$$[\xi_1(g), \xi_2(g)]_M = [\xi_1(g), \xi_2(g)] + \delta_{\xi_1}\xi_2(g) - \delta_{\xi_2}\xi_1(g), \tag{3.43}$$

where $\delta_{\xi_1}\xi_2(g) = \xi_2(\delta_{\xi_1}g)$, where $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$. The last two terms in (3.43) take into account the above discussed field dependence. We start by splitting again $\xi = \xi_T + \xi_Y$ with

$$\xi_T = T\frac{\partial}{\partial u} - \partial_B T\int_r^\infty \mathrm{d}r'(e^{2\beta}g^{BA})\frac{\partial}{\partial x^A} + \frac{r}{2}\left[D_A\left(\partial_B T\int_r^\infty \mathrm{d}r'(e^{2\beta}g^{BA})\right) + U^A\partial_A T\right]\frac{\partial}{\partial r}, \tag{3.44}$$

$$\begin{aligned}\xi_Y = {}& \frac{u}{2}D_A Y^A\frac{\partial}{\partial u} + \left[Y^A - \frac{u}{2}\partial_B\big(D_C Y^C\big)\int_r^\infty \mathrm{d}r'(e^{2\beta}g^{BA})\right]\frac{\partial}{\partial x^A}\\ & - \frac{r}{2}\left[D_A Y^A - \frac{u}{2}D_A\left[\partial_B\big(D_C Y^C\big)\int_r^\infty \mathrm{d}r'(e^{2\beta}g^{BA})\right] - \frac{u}{2}U^A\partial_A\big(D_C Y^C\big)\right]\frac{\partial}{\partial r}.\end{aligned} \tag{3.45}$$

It is then possible to show that the modified Lie bracket between vector fields $\xi_1 = \xi_{Y_1} + \xi_{T_1}$ and $\xi_2 = \xi_{Y_2} + \xi_{T_2}$ in (3.44) yields again

$$[\xi_1, \xi_2]_M = \hat{\xi} = \xi_{\hat{Y}} + \xi_{\hat{T}}, \tag{3.46}$$

where $\hat{Y}$ and $\hat{T}$ are given by the first of (3.39) and (3.41). In other words the full spacetime vector fields of the form (3.44)-(3.45) provide a representation of $\mathfrak{bms}_4$ when equipped with the modified Lie bracket of (3.43). This shows that $\mathfrak{bms}_4$, though it was originally defined at null infinity, it is actually faithfully represented everywhere in the bulk of the spacetime throught the modified Lie bracket.

3.6 Global, local and generalized $\mathfrak{bms}_4$

So far, we have shown that the generators of $\mathfrak{bms}_4$ depend on an arbitrary function T and a conformal Killing vector Y^A on the 2-sphere. In particular, since an arbitrary function on the 2-sphere is specified by an infinite number of parameters, the supertranslations subalgebra is infinite-dimensional. However, both for T and Y^A we have not yet specified the space of functions under consideration. Here we discuss three choices which lead to different definitions of $\mathfrak{bms}_4$.

The first is to consider only the globally defined, invertible conformal transformations on the 2-sphere into itself. This choice corresponds to the *global* $\mathfrak{bms}_4$ algebra. In this approach the conformal group generated by Y^A is restricted to its six-dimensional globally defined component $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}^2$ isomorphic to the connected component of the Lorentz group and, correspondingly, the smooth function T on the 2-sphere is expanded into spherical harmonics. The global $\mathfrak{bms}_4$ algebra can then be defined as the semidirect sum between the algebra of the Lorentz group $\mathfrak{so}(3,1)$ and that of supertranslations, denoted by $\mathfrak{s}$. This is the choice originally considered in [3] and successively in [45, 61].

Another choice, proposed in [4–6], consists in allowing Y^A to generate not only the globally well-defined Lorentz group, but all possible conformal transformations of the Riemann sphere into itself, including the singular ones. In this case, Laurent series are used to expand the components $(Y^z(z), Y^{\bar{z}}(\bar{z}))$ of the conformal Killing vector Y^A and they form two copies of the infinite-dimensional Witt algebra. The corresponding transformations are referred to as *superrotations*. Consequently, in order for the $\mathfrak{bms}_4$ algebra to be well-defined, one needs to include also singular supertranslations. This choice defines the *local* $\mathfrak{bms}_4$ algebra, which is then defined as the semidirect sum between superrotations and supertranslations. It is worth pointing out that including singular conformal transformations has played a crucial role in the developing of two-dimensional conformal field theories and, in the context of asymptotically flat spacetimes, they have been interpreted in terms of cosmic strings [62].

More recently, a third choice has been proposed in [7,8]. It was shown that, allowing the boundary metric on the asymptotic 2-sphere to fluctuate and thus relaxing the boundary condition outlined in (3.13), it is possible to substitute in a consistent way the above mentioned superrotations by smooth diffeomorphisms on the 2-sphere (Diff(S^2)). Correspondingly, the semidirect sum between smooth Diff(S^2) and supertranslations defines the so-called *generalized* $\mathfrak{bms}_4$ algebra. The relevance of this proposal relies in the fact that, differently from what happens for singular superrotation, the Ward Identities associated to smooth Diff(S^2)-invariance of the gravitational $\mathcal{S}$-matrix, reproduce the subleading soft graviton theorem [15].

Note that in each of the three above spelled cases the Poincaré algebra $\mathfrak{iso}(3,1)$, realizing isometries of flat spacetime, is a subalgebra of $\mathfrak{bms}_4$. We now proceed to show the explicit realizations of the global and local $\mathfrak{bms}_4$.

3.6.1 Realization of the global $\mathfrak{bms}_4$ algebra

We start by considering a realization of the global $\mathfrak{bms}_4$ on $\mathscr{I}^+ \simeq \mathbb{R} \times \mathbb{S}^2$. The generators of the Lorentz algebra are rotations (L_1, L_2, L_3) and boosts (R_1, R_2, R_3) satisfying, through the ordinary Lie bracket between vector fields, the standard commutation relations

$$[L_i, L_j] = \epsilon_{ijk} L_k, \qquad [R_i, R_j] = -\epsilon_{ijk} L_k, \qquad [L_i, R_j] = \epsilon_{ijk} R_k. \tag{3.47}$$

Introducing coordinates (u, θ, ϕ) to parametrize $\mathscr{I}^+$, L_i and R_i admit a representation in terms of differential operators as [51,63]

$$L_1 = i\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \tag{3.48}$$

$$L_2 = i\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right), \tag{3.49}$$

$$L_3 = -i\frac{\partial}{\partial\phi}, \tag{3.50}$$

$$R_1 = -i\left(\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - u\sin\theta\cos\phi\frac{\partial}{\partial u}\right), \tag{3.51}$$

$$R_2 = -i\left(\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} - u\sin\theta\sin\phi\frac{\partial}{\partial u}\right), \tag{3.52}$$

$$R_3 = i\left(\sin\theta\frac{\partial}{\partial\theta} + u\cos\theta\frac{\partial}{\partial u}\right). \tag{3.53}$$

On the other hand, expanding the function $T(\theta,\phi)$ in the first of (3.37) in spherical harmonics $Y_{lm}(\theta,\phi)$, the generators of supertranslations and their Lie bracket read

$$P_{lm} = Y_{lm}(\theta,\phi)\frac{\partial}{\partial u}, \qquad [P_{lm}, P_{l'm'}] = 0. \tag{3.54}$$

The algebra formed by the generators (3.48)-(3.53) and (3.54) is better expressed by introducing the standard ladder operators $L_\pm = L_1 \pm iL_2$ and $R_\pm = R_1 \pm iR_2$ and it reads

$$[L_+, P_{lm}] = \sqrt{l(l+1)-m(m+1)}P_{l,m+1}, \tag{3.55}$$

$$[L_+, P_{lm}] = \sqrt{l(l+1)-m(m-1)}P_{l,m-1}, \tag{3.56}$$

$$[L_3, P_{lm}] = mP_{lm}, \tag{3.57}$$

$$[R_+, P_{lm}] = -i(l-1)\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}P_{l+1,m+1} - i(l+2)\sqrt{\frac{(l-m-1)(l-m)}{4l^2-1}}P_{l-1,m+1}, \tag{3.58}$$

$$[R_-, P_{lm}] = i(l-1)\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}P_{l+1,m-1} + i(l+2)\sqrt{\frac{(l+m-1)(l+m)}{4l^2-1}}P_{l-1,m-1}, \tag{3.59}$$

$$[R_3, P_{lm}] = i(l-1)\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}P_{l+1,m} - i(l+2)\sqrt{\frac{(l+m)(l-m)}{4l^2-1}}P_{l-1,m}. \tag{3.60}$$

The set of commutation relations (3.47),(3.54) and (3.55)-(3.60) define the global $\mathfrak{bms}_4$ algebra. As remarked, it is a semidirect sum $\mathfrak{bms}_4 = \mathfrak{so}(3,1) \oplus_\sigma \mathfrak{s}$, where $\mathfrak{s}$ is the abelian ideal of supertranslations [14] and σ denotes the action of the Lorentz algebra on it [51]. The Poincaré algebra $\mathfrak{iso}(3,1)$ is the subalgebra spanned by the set $\{P_{00}, P_{1m}, L_i, R_i\}$ as clear from the factor $(l-1)$ multiplying $P_{l+1,m}$ in (3.58)-(3.60).

Now we briefly comment on the *finite* global BMS_4 transformations of the coordinates. Ordinary four-translations act on bulk coordinates x^μ as $x'^\mu = x^\mu + \delta x^\mu$, δx^μ being a constant. We now ask what is the corresponding effect on the coordinates (u, x^A) on $\mathscr{I}^+$. For large values of the radial coordinate r, for asymptotically flat spacetimes, we have $u \sim t - r$. An infinitesimal time translation $t' = t + \delta t$ and a spatial displacement $\delta\vec{x}$ produce a shift $\delta u = \delta t - \frac{\vec{x}\cdot\delta\vec{x}}{r}$.Therefore, for an infinitesimal four-translation δx^μ we have, using spherical coordinates $x^A = (\theta,\phi)$,

$$\delta u = \delta t - \delta x^1 \cos\phi\sin\theta - \delta x^2 \sin\phi\sin\theta - \delta x^3\cos\theta \equiv \sum_{l\in\{0,1\}}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\theta,\phi), \tag{3.61}$$

with

$$\alpha_{00} = \sqrt{4\pi}\delta x^0, \qquad \alpha_{10} = -\sqrt{\frac{4\pi}{3}}\delta x^3, \qquad \alpha_{1,\pm1} = \pm\sqrt{\frac{2\pi}{3}}(\delta x^1 \mp i\delta x^2). \tag{3.62}$$

Then, the effect of a supertanslation (3.54) can be easily expressed as the generalization of (3.61) to arbitrary values of l:

$$u' = u + \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\theta,\phi), \tag{3.63}$$

[14] Indeed $[P_{lm}, \mathfrak{g}] \in \mathfrak{s}$, $\forall\ \mathfrak{g} \in \mathfrak{bms}_3$

with α_{lm} satisfying the reality condition $\alpha_{lm} = (-1)^m \alpha^*_{l,-m}$. The retarded time u gets shifted as $u' = u + T(\theta, \phi)$ for an arbitrary smooth function $T(\theta, \phi)$, whose expansion in spherical harmonics is that of (3.63). It means that each point on the celestial sphere gets shifted by a certain amount depending on its angular coordinates, as pictorically represented in Figure 3.

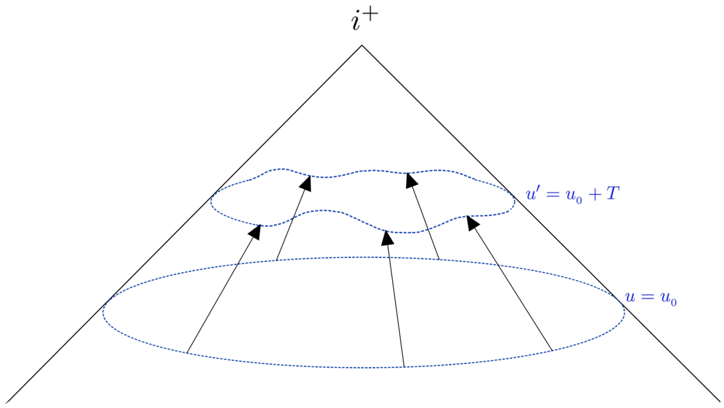

Figure 3: A supertranslation maps the celestial sphere $u = u_0$ to the one $u' = u_0 + T$.

In order to describe the action of Lorentz transformations on $\mathscr{I}^+$, it is now convenient to parametrize the latter with coordinates $(u, z, \bar{z})$ and to use the isomorphism $\mathrm{SO}(3,1) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}^2 = \mathrm{PSL}(2,\mathbb{C})$, according to which to each Lorentz transformation it corresponds a fractional linear transformation of $(z, \bar{z})$ [64],

$$z' = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}^2. \tag{3.64}$$

Rotations of an angle φ and boosts of rapidity χ about an axis $\hat{n}' = (\cos\phi' \sin\theta', \sin\phi' \sin\theta', \cos\theta')$ are described by the following $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ matrices

$$L_{\hat{n}'}(\varphi) = \pm \begin{pmatrix} \cos\frac{\varphi}{2} - i\cos\theta' \sin\frac{\varphi}{2} & -i\sin\theta' \sin\frac{\varphi}{2} e^{-i\phi'} \\ -i\sin\theta' \sin\frac{\varphi}{2} e^{i\phi'} & \cos\frac{\varphi}{2} + i\cos\theta' \sin\frac{\varphi}{2} \end{pmatrix}, \tag{3.65}$$

$$R_{\hat{n}'}(\chi) = \pm \begin{pmatrix} \cosh\frac{\chi}{2} - \cos\theta' \sinh\frac{\chi}{2} & -\sin\theta' \sinh\frac{\chi}{2} e^{-i\phi'} \\ -\sin\theta' \sinh\frac{\chi}{2} e^{i\phi'} & \cosh\frac{\chi}{2} + \cos\theta' \sinh\frac{\chi}{2} \end{pmatrix}. \tag{3.66}$$

Notice that $L_{\hat{n}'}(\varphi)$ is also an SU(2) transformation while $R_{\hat{n}'}(\chi)$ is not. Indeed, for any rotation the conformal factor is $K(z, \bar{z}) = 1$, because rotations are pure isometries of the 2-sphere, while boosts are only conformal symmetries. In the complex plane spanned by $(z, \bar{z})$ the former are rotations while the latter are dilations.

Under (3.64) the induced (degenerate) line element on $\mathscr{I}^+$ transforms as

$$\mathrm{d}s'^2_{\mathscr{I}^+} = \frac{4}{(1+z'\bar{z})^2}\mathrm{d}z'\mathrm{d}\bar{z}' = K^2(z,\bar{z})\mathrm{d}s^2_{\mathscr{I}^+}, \tag{3.67}$$

with

$$K(z,\bar{z}) = \frac{1+z\bar{z}}{|az+b|^2+|cz+d|^2}. \tag{3.68}$$

It can be easily shown [65] that the retarded time u, under (3.64) transforms as

$$u' = K(z,\bar{z})u. \tag{3.69}$$

The finite form of global BMS$_4$ transformations on the coordinates $(u, z, \bar{z})$ is given by (3.63), (3.64) and (3.69).

3.6.2 A fuzzy celestial sphere

As shown in (3.54), the generators of supertranslations are indexed by the angular momentum l of spherical harmonics on the celestial sphere and are infinite in number because one can have infinite angular resolution on such sphere. Here we show how a cut-off in the angular modes of the celestial sphere can be consistently introduced using techniques of non-commutative geometry. In particular, we introduce a non-commutative deformation of the algebra of spherical harmonics analogous to the one used in the literature to describe a non-commutative analogue of the 2-sphere, the so-called *fuzzy sphere* [66].

The celestial sphere, as a smooth manifold, is a 2-sphere and the commutative algebra of smooth functions defined on it, which we denote by $C(S^2)$, is generated by the spherical harmonics $\{Y_{lm}(\theta,\phi)\}$ that provide an orthonormal and complete basis with inner product given by

$$\int \mathrm{d}\Omega\, Y^*_{l_1m_1}(\theta,\varphi)Y_{l_2m_2}(\theta,\varphi) = \delta_{l_1l_2}\delta_{m_1m_2}. \tag{3.70}$$

Thus, any smooth function $f(\theta,\phi) \in C(S^2)$ can be expanded as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}Y_{lm}(\theta,\phi)\,, \tag{3.71}$$

with the coefficients f_{lm} given by

$$f_{lm} = \int \mathrm{d}\Omega\, Y^*_{lm}(\theta,\varphi)f(\theta,\phi). \tag{3.72}$$

The product of two spherical harmonics can expressed in terms of a linear combination of spherical harmonics using the Clebsch-Gordan coefficients:

$$Y_{l_1m_1}Y_{l_2m_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2}\sum_{m=-l}^{l}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}C^{l0}_{l_10l_20}C^{lm}_{l_1m_1l_2m_2}Y_{lm}. \tag{3.73}$$

Note that such product is commutative, since $C^{l0}_{l_10l_20}C^{lm}_{l_1m_1l_2m_2} = C^{l0}_{l_20l_10}C^{lm}_{l_2m_2l_1m_1}$ and that the maximum value of the angular momentum l is given by $l_{\max} = l_1 + l_2$.

The action of the Lorentz algebra generators given in (3.48)-(3.53) on the algebra of spherical harmonics $Y_{lm}(\theta,\phi)$ is given by [67]:

$$L_\pm(Y_{lm}) = \sqrt{l(l+1)-m(m\pm 1)}Y_{l,m+1}, \tag{3.74}$$
$$L_z(Y_{lm}) = mY_{lm}, \tag{3.75}$$
$$R_+(Y_{lm}) = \mp il\sqrt{\frac{(l\pm m+1)(l\pm m+2)}{(2l+1)(2l+3)}}Y_{l+1,m\pm 1} \mp i(l+1)\sqrt{\frac{(l\mp m-1)(l\mp m)}{4l^2-1}}Y_{l-1,m+1}, \tag{3.76}$$
$$R_3(Y_{lm}) = il\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}Y_{l+1,m} - i(l+1)\sqrt{\frac{(l+m)(l-m)}{4l^2-1}}Y_{l-1,m}. \tag{3.77}$$

Note that since the total angular momentum L^2 does not commute with the boosts, the action of a boost on a spherical harmonic changes in general its total angular momentum l and always maps the harmonic with given l to one with $l+1$. This is not an issue because, has already remarked, l can be any arbitrarily large integer number because one can have infinite angular resolution on the ordinary sphere.

The first step in order to obtain a non-commutative deformation of the celestial sphere is to deform the algebra of spherical harmonics (3.73) by introducing the so-called *fuzzy spherical harmonics*, which can be thought of as the algebra of functions on a non-commutative space known as the fuzzy sphere [66,68–78]. This deformation is concretely realized in terms of a "quantization map" between the commutative algebra of functions on the 2-sphere $C(S^2)$ and the algebra of $N\times N$ complex matrices $M_N(\mathbb{C})$,

$$\Omega_N : C(S^2) \to M_N(\mathbb{C})\,; \qquad \Omega_N[Y_{lm}(\theta,\varphi)] = \begin{cases} \hat{Y}^{(N)}_{lm} & l<N \\ 0 & l\geq N \end{cases}\,, \tag{3.78}$$

where the mapping between the spherical harmonics $Y_{lm}(\theta,\phi)$ and the matrices $\hat{Y}^{(N)}_{lm}$ is explicitly realized as:

$$\hat{Y}^{(N)}_{lm} = \frac{2^l}{l!}\left[\frac{N(N-1-l)!}{(N+l)!}\right]^{\frac{1}{2}} (\mathbf{J}^{(N)}\cdot\nabla)^l \left(r^l Y_{lm}(\theta,\varphi)\right), \tag{3.79}$$

with $\mathbf{J}^{(N)} = (J^{(N)}_1, J^{(N)}_2, J^{(N)}_3)$ and $J^{(N)}_i$ are the N-dimensional spin matrices with spin j_N

$$[J^{(N)}_i, J^{(N)}_j] = i\epsilon_{ijk}J^{(N)}_k, \qquad J^{(N)2} = j_N(j_N+1)\mathbb{I}^{(N)}, \qquad 2j_N+1=N\,. \tag{3.80}$$

$\hat{Y}^{(N)}_{lm}$ are referred to as fuzzy spherical harmonics and they are irreducible tensor operators of rank l. Introducing ladder operators $J^{(N)}_\pm = J^{(N)}_1 \pm iJ^{(N)}_2$, their adjoint action $\triangleright$ on the fuzzy spherical harmonics is given by

$$J^{(N)}_\pm \triangleright \hat{Y}^{(N)}_{lm} \equiv [J^{(N)}_\pm, \hat{Y}^{(N)}_{lm}] = \sqrt{(l\mp m)(l\pm m+1)}\hat{Y}^{(N)}_{l,m\pm 1}, \tag{3.81}$$

$$J^{(N)}_3 \triangleright \hat{Y}^{(N)}_{lm} \equiv [J^{(N)}_3, \hat{Y}^{(N)}_{lm}] = m\hat{Y}^{(N)}_{lm}. \tag{3.82}$$

Furthermore the action of the the Casimir $J^{(N)2}$ is,

$$J^{(N)2} \triangleright \hat{Y}^{(N)}_{lm} = \left[J^{(N)}_+, \left[J^{(N)}_-, \hat{Y}^{(N)}_{lm}\right]\right] + \left[J^{(N)}_3, \left[J^{(N)}_3, \hat{Y}^{(N)}_{lm}\right]\right] - \left[J^{(N)}_3, \hat{Y}^{(N)}_{lm}\right]$$

$$= l(l+1)\hat{Y}^{(N)}_{lm} \equiv - \triangle \hat{Y}^{(N)}_{lm}, \tag{3.83}$$

where we introduced the fuzzy Laplacian $\triangle$. This is the non-commutative analogue of the ordinary angular Laplacian and its eigenmatrices are the fuzzy harmonics. Its spectrum is truncated at $l = l_{\max} = 2j_N = N-1$. Note that the operation $\triangleright$ is a derivation, that is the non-commutative analogue of a vector field. The product of $\hat{Y}^{(N)}_{l_1m_1}$ and $\hat{Y}^{(N)}_{l_2m_2}$ can be expanded as a linear combination of $\hat{Y}^{(N)}_{lm}$ using 6j-symbols

$$\hat{Y}^{(N)}_{l_1m_1}\hat{Y}^{(N)}_{l_2m_2} = \sum_{l=0}^{2j_N}(-1)^{2j_N+l}\sqrt{\frac{(2l_1+1)(2l_2+1)(2j_N+1)}{4\pi}} \begin{Bmatrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{Bmatrix} C^{lm}_{l_1m_1l_2m_2}\hat{Y}^{(N)}_{lm}. \tag{3.84}$$

Notice that the 6j-symbols of (3.84) automatically vanish if the triangular conditions $|l_1 - l_2| < l < l_1 + l_2$ and $0 < l < 2j_N + 1$ are not satisfied. It means that l can assume values up to $l_{\max} = 2j_N = N-1$, in contrast to what happens for the product of ordinary spherical harmonics (3.73). From the antisymmetry properties of the 6j-symbols, we derive the commutator

$$\begin{aligned}\left[\hat{Y}^{(N)}_{l_1,m_1}, \hat{Y}^{(N)}_{l_2m_2}\right] &= \sum_{l=0}^{2j_N}(-1)^{2j_N+l}\sqrt{\frac{(2l_1+1)(2l_2+1)(2j_N+1)}{4\pi}} \begin{Bmatrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{Bmatrix} \\ &\times C^{lm}_{l_1m_1l_2m_2}\hat{Y}^{(N)}_{lm}[1-(-1)^{l_1+l_2-l}].\end{aligned} \tag{3.85}$$

Using the asymptotic behavior of the 6j-symbols [79] for large values of N

$$\begin{Bmatrix} l_1 & l_2 & l \\ j_N & j_N & j_N \end{Bmatrix} \approx \frac{(-1)^{2j+l}}{\sqrt{(2l+1)(2j_N+1)}} C^{l0}_{l_10l_20}, \tag{3.86}$$

we have that

$$\lim_{N\to\infty} \Omega_N^{-1}\left(\hat{Y}^{(N)}_{l_1m_1}\hat{Y}^{(N)}_{l_2m_2}\right) = Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi)\,. \tag{3.87}$$

and thus the commutator (3.85) vanishes in the large-N limit, leading to the the usual commutative algebra of spherical harmonics. On $M_N(\mathbb{C})$ we can introduce a scalar product $(,)_{(N)}$ as

$$\left(\hat{Y}^{(N)}_{l_1m_1}, \hat{Y}^{(N)}_{l_2m_2}\right)_{(N)} = \frac{4\pi}{N}\mathrm{Tr}\left(\hat{Y}^{(N)\dagger}_{l_1m_1}\hat{Y}^{(N)}_{l_2m_2}\right) = \delta_{l_1l_2}\delta_{m_1m_2}. \tag{3.88}$$

Since there are $\sum_{l=0}^{2j_N}(2l+1) = N^2$ independent fuzzy spherical harmonics the set $\left\{\hat{Y}^{(N)}_{lm}\right\}$, equipped with (3.88) is an orthonormal basis in $M_N(\mathbb{C})$. Any element $\hat{f}^{(N)} \in M_N(\mathbb{C})$ can thus be expanded as

$$\hat{f}^{(N)} = \sum_{l=0}^{2j_N}\sum_{m=-l}^{l}\left(\hat{Y}^{(N)\dagger}_{lm}, \hat{f}^{(N)}\right)_{(N)}\hat{Y}^{(N)}_{lm}\,. \tag{3.89}$$

Again, note that this expansion is truncated at $l_{\max}$, in contrast to what happens in (3.73). The quantization map (3.78) can be extended by linearity to arbitrary functions of (θ, ϕ)

$$\Omega_N : f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta,\phi) \to \hat{f}^{(N)} = \sum_{l=0}^{2j_N} \sum_{m=-l}^{l} f_{lm} \hat{Y}^{(N)}_{lm}. \tag{3.90}$$

The set $C_N(S^2) \subset C(S^2)$ of truncated functions on the 2-sphere, *i.e.* the set of functions whose expansion in terms of the spherical harmonics includes only terms with $l < N$ as $f^{(N)}(\theta,\phi) = \sum_{l=0}^{2j_N} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta,\phi)$ is a vector space, but not an algebra with the standard definition of pointwise product of two functions, since the product of two spherical harmonics of order say $N-1$ has spherical components of order larger than $N-1$, as remarked before. However, we can equip this vector space with a non-commutative $\star$-product via the Weyl-Wigner map:

$$\left(f^{(N)} \star g^{(N)}\right)(\theta,\phi) = \sum_{l=0}^{2j_N} \sum_{m=-l}^{l} \left(\hat{Y}^{(N)\dagger}_{lm}, \hat{f}^{(N)} \hat{g}^{(N)}\right)_{(N)} Y_{lm}(\theta,\phi), \tag{3.91}$$

turning $C_N(S^2)$ into a non-commutative algebra. This non-commutative algebra of functions can be interpreted as functions on the fuzzy sphere. An important feature introduced by the non-commutativity is that we now have a cut-off on the allowed values of the angular momentum $l_{\max}$ in a way which is compatible with the multiplicative structure on the space of non-commutative spherical harmonics.

In what follows we show how the non-commutative deformation of spherical harmonics just presented can be extended in order to include an action of the Lorentz algebra which is compatible with the presence of a maximal allowed value of the angular momentum. In order to do that, we start by looking at the finite $N = N_1 N_2$-dimensional representations of the Lorentz algebra, that can be constructed in terms of the spin matrices as

$$L_i^{(N)} = J_i^{(N_1)} \otimes \mathbb{1}^{(N_2)} + \mathbb{1}^{(N_1)} \otimes J_i^{(N_2)}, \qquad R_i^{(N)} = i\left(J_i^{(N_1)} \otimes \mathbb{1}^{(N_2)} - \mathbb{1}^{(N_1)} \otimes J_i^{(N_2)}\right). \tag{3.92}$$

It is easy to prove that these matrices satisfy the $\mathfrak{so}(3,1)$ Lie algebra of (3.47). For both sets of spin matrices $J_i^{(N_1)}$ and $J_i^{(N_2)}$ we can construct their associated fuzzy spherical harmonics $\hat{Y}^{(N_1)}_{lm}$ and $\hat{Y}^{(N_2)}_{lm}$ which are $N_1 \times N_1$ and $N_2 \times N_2$ matrices, respectively, and they satisfy all the properties discussed previously. We can therefore construct the matrices

$${}^{l_1 l_2}\hat{Y}^{(N)}_{LM} = \sum_{\substack{m_1 \\ m_2}} C^{LM}_{l_1 m_1 l_2 m_2} \hat{Y}^{(N_1)}_{l_1 m_1} \otimes \hat{Y}^{(N_2)}_{l_2 m_2}. \tag{3.93}$$

The matrices ${}^{l_1 l_2}\hat{Y}^{(N)}_{LM}$ are irreducible tensors of rank L and are eigenmatrices of $J^{(N_1)2} \otimes \mathbb{I}^{(N_2)}$ and $\mathbb{I}^{(N_1)} \otimes J^{(N_2)2}$ with eigenvalues $l_1(l_1+1)$ and $l_2(l_2+1)$, respectively. The allowed values of the total angular momentum are $L = l_{\min}, ..., l_{\max}$, and $M = m_1 + m_2 = -L, ..., L$ with $l_{\min} = |l_1 - l_2|$ and $l_{\max} = l_1 + l_2$ as follows from the rules for the addition of two angular momenta. Note that, since $l_{1\max} = N_1 - 1$ and $l_{2\max} = N_2 - 1$, the value of L is never grater than $L_{\max} = N_1 + N_2 - 2$. The set $\left\{{}^{l_1 l_2}\hat{Y}^{(N)}_{LM}\right\}$ is an orthonormal basis in $M_N(\mathbb{C})$ with a scalar product analogous to the one of the fuzzy spherical harmonics, given by

$$\left({}^{l_1 l_2}\hat{Y}^{(N)}_{L_1 M_1}, {}^{l'_1 l'_2}\hat{Y}^{(N)}_{L_2 M_2}\right)_{(N)} = \frac{(4\pi)^2}{N} \mathrm{Tr}\left({}^{l_1 l_2}\hat{Y}^{(N)\dagger}_{L_1 M_1} {}^{l'_1 l'_2}\hat{Y}^{(N)}_{L_2 M_2}\right) = \delta_{L_1 L_2} \delta_{M_1 M_2} \delta_{l_1 l'_1} \delta_{l_2 l'_2}. \tag{3.94}$$

The algebra realized by the matrices (3.93) is given in terms of 6j- and 9j-symbols $\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$ as

$$
{}^{l_1'l_2'}\hat{Y}^{(N)}_{L'M'}\,{}^{l_1''l_2''}\hat{Y}^{(N)}_{L''M''} = \sum_{\substack{LM\\ l_1l_2}} \frac{\sqrt{N}}{4\pi}\sqrt{(2l_1+1)(2l_1'+1)(2l_1''+1)(2L'+1)(2l_2+1)(2l_2'+1)(2l_2''+1)(2L''+1)}
$$
$$
\times \begin{Bmatrix} l_1' & l_1'' & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{Bmatrix}\begin{Bmatrix} l_2' & l_2'' & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{Bmatrix}(-1)^{2j_{N_1}+2j_{N_2}+l_1+l_2}C^{LM}_{L'M'L''M''}\begin{Bmatrix} l_1' & l_2' & L' \\ l_1'' & l_2'' & L'' \\ l_1 & l_2 & L \end{Bmatrix}{}^{l_1l_2}\hat{Y}^{(N)}_{LM}. \tag{3.95}
$$

For large values of $N = N_1N_2$ we have

$$
\begin{Bmatrix} l_1' & l_1'' & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{Bmatrix}\begin{Bmatrix} l_2' & l_2'' & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{Bmatrix} \approx \frac{(-1)^{j_{N_1}+j_{N_2}+l_1+l_2}}{\sqrt{N(2l_1+1)(2l_2+1)}}C^{l_10}_{l_1'0l_1''0}C^{l_20}_{l_2'0l_2''0}, \tag{3.96}
$$

so that the algebra becomes

$$
{}^{l_1'l_2'}\hat{Y}^{(N)}_{L'M'}\,{}^{l_1''l_2''}\hat{Y}^{(N)}_{L''M''} \approx \sum_{\substack{LM\\ l_1l_2}} \frac{1}{4\pi}\sqrt{(2l_1'+1)(2l_1''+1)(2L'+1)(2l_2'+1)(2l_2''+1)(2L''+1)}
$$
$$
\times C^{l_10}_{l_1'0l_1''0}C^{l_20}_{l_2'0l_2''0}C^{LM}_{L'M'L''M''}\begin{Bmatrix} l_1' & l_2' & L' \\ l_1'' & l_2'' & L'' \\ l_1 & l_2 & L \end{Bmatrix}{}^{l_1l_2}\hat{Y}^{(N)}_{LM}, \tag{3.97}
$$

which is exactly the algebra satisfied by commutative bipolar spherical harmonics (see *e.g.* [67]), as one would expect. For these reason, the matrices of (3.93) can be thought of as *fuzzy bipolar spherical harmonics*. Their commutator is given by

$$
\left[{}^{l_1'l_2'}\hat{Y}^{(N)}_{L'M'}, {}^{l_1''l_2''}\hat{Y}^{(N)}_{L''M''}\right] = \sum_{\substack{LM\\ l_1l_2}} \frac{\sqrt{N}}{4\pi}\sqrt{(2l_1+1)(2l_1'+1)(2l_1''+1)(2L'+1)(2l_2+1)(2l_2'+1)(2l_2''+1)(2L''+1)}
$$
$$
\times \begin{Bmatrix} l_1' & l_1'' & l_1 \\ j_{N_1} & j_{N_1} & j_{N_1} \end{Bmatrix}\begin{Bmatrix} l_2' & l_2'' & l_2 \\ j_{N_2} & j_{N_2} & j_{N_2} \end{Bmatrix}(-1)^{2j_{N_1}+2j_{N_2}+l_1+l_2}C^{LM}_{L'M'L''M''}\begin{Bmatrix} l_1' & l_2' & L' \\ l_1'' & l_2'' & L'' \\ l_1 & l_2 & L \end{Bmatrix}
$$
$$
\times [1-(-1)^{l_1+l_2+l_1'+l_2'+l_1''+l_2''}]{}^{l_1l_2}\hat{Y}^{(N)}_{LM}\,.
$$

These equations define our non-commutative algebra of spherical harmonics on the fuzzy celestial sphere. It can be shown [63] that Lorentz algebra acts as

$$
L^{(N)}_{\pm} \triangleright {}^{l_1l_2}\hat{Y}^{(N)}_{LM} = \sqrt{(L\mp M)(L\pm M+1)}\,{}^{l_1l_2}\hat{Y}^{(N)}_{L,M\pm1}, \tag{3.98}
$$

$$
L^{(N)}_{3} \triangleright {}^{l_1l_2}\hat{Y}^{(N)}_{LM} = M\,{}^{l_1l_2}\hat{Y}^{(N)}_{LM}, \tag{3.99}
$$

$$R_{\pm}^{(N)} \triangleright {}^{l_1 l_2}\hat{Y}_{LM}^{(N)} = \pm \frac{i}{L}\sqrt{\frac{(L \mp M)(L \mp M - 1)[L^2 - (l_{\min})^2][(l_{\max}+1)^2 - L^2)]}{(4L^2-1)}}\,{}^{l_1 l_2}\hat{Y}_{L-1,M\pm 1}^{(N)}$$
$$+ i\frac{l_{\min}(l_{\max}+1)}{L(L+1)}\sqrt{(L \mp M)(L \pm M + 1)}\,{}^{l_1 l_2}\hat{Y}_{L,M\pm 1}^{(N)}$$
$$\mp \frac{i}{(L+1)}\sqrt{\frac{(L \pm M + 1)(L \pm M + 2)[(L+1)^2 - (l_{\min})^2][(l_{\max}+1)^2 - (L+1)^2]}{(2L+1)(2L+3)}}\,{}^{l_1 l_2}\hat{Y}_{L+1,M\pm 1}^{(N)}, \quad (3.100)$$

$$R_{3}^{(N)} \triangleright {}^{l_1 l_2}\hat{Y}_{LM}^{(N)} = \frac{i}{L}\sqrt{\frac{(L + M)(L - M)[L^2 - (l_{\min})^2][(l_{\max}+1)^2 - L^2)]}{(4L^2-1)}}\,{}^{l_1 l_2}\hat{Y}_{L-1,M}^{(N)}$$
$$+ i\frac{M l_{\min}(l_{\max}+1)}{L(L+1)}\,{}^{l_1 l_2}\hat{Y}_{LM}^{(N)}$$
$$+ \frac{i}{(L+1)}\sqrt{\frac{(L + M + 1)(L - M + 1)[(L+1)^2 - (l_{\min})^2][(l_{\max}+1)^2 - (L+1)^2]}{(2L+1)(2L+3)}}\,{}^{l_1 l_2}\hat{Y}_{L+1,M}^{(N)}. \quad (3.101)$$

Note that the coefficients of the ${}^{l_1 l_2}\hat{Y}_{L+1,M+q}^{(N)}$ terms automatically vanish if L equals $l_{\max}$ and thus the action of boosts is compatible with the existence of a cut-off in the value of L. Equivalently, the action of Lorentz boosts on the non-commutative spherical harmonics cannot produce harmonics labelled with an arbitrarily high angular momentum. Therefore the matrix generalization of ordinary spherical harmonics on the celestial sphere introduced in (3.93) carries a non-trivial representation of the Lorentz algebra and equations (3.98)-(3.101) can be thought of as the non-commutative analogue of (3.74)-(3.77). This suggests that, since the generators of global supertranslations are proportional to the spherical harmonics on the celestial sphere, as in (3.54), it could be possible to introduce a "fuzzy version" of the global $\mathfrak{bms}_4$ algebra [15] characterized by a non-abelian sub-algebra of supertranslations having a *finite* number of generators. Such construction might be relevant in the context of the the soft-hair proposal [21, 22, 82–84]. The latter proposes that the charges associated with BMS$_4$ symmetries [85, 86] could equip the black hole with the *soft hair*[16] needed to support correlations between the interior of the black hole and the emitted Hawking quanta. However, one of the obstacles in making such identification concrete is that the actual degrees of freedom which can be associated to BMS$_4$ charges are *too many*, in fact infinite, while the Bekenstein-Hawking entropy, albeit large, is finite and proportional to the black hole area divided by the Planck length squared. Thus non-commutativity, or the fuzziness of the celestial sphere, could be the ingredient needed to provide a consistent cut-off mechanism for supertranslations modes.

3.6.3 Realization of local $\mathfrak{bms}_4$

In this section we consider a local realization of the $\mathfrak{bms}_4$ algebra. As mentioned, the local $\mathfrak{bms}_4$ differs from the global $\mathfrak{bms}_4$ by the choice of the function T and the vector field Y^A on the 2-sphere. Instead of using spherical harmonics, we use Laurent series and therefore we define l_n and $\bar{l}_n$ to be ξ_Y with $Y^z = -z^{n+1}, Y^{\bar{z}} = 0$ and $Y^z = 0, Y^{\bar{z}} = -\bar{z}^{n+1}$, respectively and T_{nm} to be ξ_T with $T = 2z^n\bar{z}^m/(1+z\bar{z})$.

[15] For recent attempts at generalizing $\mathfrak{bms}_4$ using quantum groups techniques see [80, 81].

[16] The actual meaning of soft will be clarified in detail in sections 4.3-4.5.

Explicitly, we have

$$l_n = -z^{n+1}\frac{\partial}{\partial z} + \frac{\bar{z}z^{n+1}(1-n)-(n+1)z^n}{1+z\bar{z}}\frac{u}{2}\frac{\partial}{\partial u}, \tag{3.102}$$

$$\bar{l}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}} + \frac{\bar{z}^{n+1}z(1-n)-(n+1)\bar{z}^n}{1+z\bar{z}}\frac{u}{2}\frac{\partial}{\partial u}, \tag{3.103}$$

$$T_{nm} = \frac{2z^n\bar{z}^m}{1+z\bar{z}}\frac{\partial}{\partial u}. \tag{3.104}$$

Then, using (3.40)-(3.41) we get the commutation relations

$$[l_n, l_m] = (n-m)l_{n+m}, \qquad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \qquad [l_m, \bar{l}_m] = 0, \tag{3.105}$$

$$[l_n, T_{ml}] = \left(\frac{n+1}{2} - m\right)T_{m+n,l}, \qquad [\bar{l}_n, T_{ml}] = \left(\frac{n+1}{2} - l\right)T_{m,l+n}, \tag{3.106}$$

$$[T_{mn}, T_{pq}] = 0. \tag{3.107}$$

Note that (3.105) are two commuting copies of the Witt algebra. The local $\mathfrak{bms}_4$ is defined by (3.105)-(3.107) and it contains $\mathfrak{iso}(3,1)$ as a subalgebra. In particular, the $\mathfrak{so}(3,1)$ component is generated by l_n and by $\bar{l}_n$, for $n = -1, 0, 1$, whereas the four-translations are realized by T_{mn} with $m, n = 0, 1$. There is however another way to define the local $\mathfrak{bms}_4$ algebra, which we briefly point out here. We start by defining a tensor density of weight $(h, \bar{h})$ on the 2-sphere as

$$F = F(z,\bar{z})dz^h d\bar{z}^{\bar{h}}. \tag{3.108}$$

Under an infinitesimal conformal transformation

$$z' = f(z) = z + \epsilon X(z), \qquad \bar{z}' = \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}\bar{X}(\bar{z}), \tag{3.109}$$

we have

$$\delta_X F = X\partial_z F + hF\partial_z X, \qquad \delta_{\bar{X}} F = \bar{X}\partial_{\bar{z}} F + \bar{h}F\partial_{\bar{z}}\bar{X}. \tag{3.110}$$

We denote the vector space of tensor densities of weight $(h, \bar{h})$ on the 2-sphere by $\mathcal{F}_{h,\bar{h}}(S^2)$. We can associate to each supertranslations generator T_{ml} a density of weight $(-\frac{1}{2}, -\frac{1}{2})$ as

$$T_{ml} = z^m\bar{z}^l dz^{-\frac{1}{2}}d\bar{z}^{-\frac{1}{2}}, \tag{3.111}$$

and therefore we have that, under a conformal transformation generated by $X_n = -z^{n+1}$ and $\bar{X}_n = -\bar{z}^{n+1}$,

$$\delta_{X_n} T_{ml} = \left(\frac{n+1}{2} - m\right)T_{m+n,l}, \qquad \delta_{\bar{X}_n} T_{ml} = \left(\frac{n+1}{2} - l\right)T_{m,l+n}. \tag{3.112}$$

Hence, the local $\mathfrak{bms}_4$ algebra is the semidirect sum of the algebra generated by vector fields $l_n = -z^{n+1}\frac{\partial}{\partial z}$ and $\bar{l}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$ with the abelian ideal $\mathcal{F}_{-\frac{1}{2},-\frac{1}{2}}(S^2)$, the bracket being induced by the module action (3.110), *i.e.* $[l_n, T_{ml}] \equiv \delta_{X_n} T_{ml}$ and $[\bar{l}_n, T_{ml}] \equiv \delta_{\bar{X}_n} T_{ml}$, as clear from (3.106).

4 BMS$_4$ charges, gravitational scattering and soft degrees of freedom

In this chapter, we start in section 4.1 by investigating the action of $\mathfrak{bms}_4$ on the set $\chi'_{\mathscr{I}^+}$ with particular emphasis on the action of supertranslations on the asymptotic shear tensor C_{AB}. In fact, the transformation of the latter comprises an interesting homogeneous term that breaks supertranslations invariance of gravitational vacua. We show that, when focusing on the scattering problem in Christodoulou-Klainerman spaces discussed in section 4.2, it turns out that the above considered homogeneous term is encoded in a natural "soft" contribution associated to the supertranslations charges, computed in section 4.3. The final result is that the Weinberg's soft graviton theorem, reviewed in section 4.4, is a consequence of the gravitational scattering invariance under supertranslations and vice-versa. We comment on this last point in 4.5. We conclude by touching upon other related developments, including superrotations and the gravitational memory effect in section 4.6.

4.1 Action of $\mathfrak{bms}_4$ on solution space and spontaneous breaking of gravitational vacua

In this section we compute the action of $\mathfrak{bms}_4$ on the solution space $\chi'_{\mathscr{I}^+}$, defined at the end of section 3.4. As we will see, the coordinates in $\chi'_{\mathscr{I}^+}$ transform non-trivially under $\mathfrak{bms}_4$. It means that data differing by $\mathfrak{bms}_4$ tranformations are physically inequivalent, despite being diffeomorphic. To find the action, we need to compute the Lie derivative along ξ of the on-shell metric (3.29). Denoting $\delta_{(T,Y)} = \delta_\xi$, with ξ decomposed as $\xi = \xi_T + \xi_Y$ according to (3.44) and (3.45), we find [5][17]

$$\delta_{(T,Y)} m = \Big(f\partial_u + \mathcal{L}_Y + \frac{3}{2} D_A Y^A\Big) m + \frac{1}{4} N^{AB} D_A D_B f + \frac{1}{2} D_A N^{AB} D_B f + \frac{1}{8} C^{AB} D_A D_B D_C Y^C, \tag{4.1}$$

$$\delta_{(T,Y)} C_{AB} = \Big(f\partial_u + \mathcal{L}_Y - \frac{1}{2} D_C Y^C\Big) C_{AB} - 2 D_A D_B f + \gamma_{AB} D_C D^C f, \tag{4.2}$$

$$\delta_{(T,Y)} N_{AB} = \Big(f\partial_u + \mathcal{L}_Y\Big) N_{AB} - \Big(D_A D_B D_C Y^C - \frac{1}{2}\gamma_{AB} D_C D^C D_D Y^D\Big), \tag{4.3}$$

$$\begin{aligned}\delta_{(T,Y)} N_A = {} & \Big(f\partial_u + \mathcal{L}_Y + D_B Y^B\Big) N_A + 3m D_A f - \frac{3}{16} N_{BC} C^{BC} D_A f + \frac{1}{2} N^{BC} C_{AC} D_B f \\ & - \frac{1}{32} C_{CD} C^{CD} D_A D_B Y^B + \frac{1}{4} C_{AB} D^B D_C D^C f - \frac{3}{4} D_B f \Big(D^B D^C C_{AC} - D_A D_C C^{BC}\Big) \\ & + \frac{1}{2}\Big(D_A D_B f - \frac{1}{2}\gamma_{AB} D_C D^C f\Big) D_D C^{DB} + \frac{3}{8} D_A\Big(C^{CB} D_C D_B f\Big),\end{aligned} \tag{4.4}$$

where $f = T + u/2 D_A Y^A$ as in (3.32). Note that the action of $\delta_{(T,Y)}$ preserves the tracelessness of C_{AB} and N_{AB}. Under a supertranslation T, equation (4.2) in $(z,\bar{z})$ coordinates reads

$$\delta_T C_{zz} = T\partial_u C_{zz} - 2D_z^2 T, \tag{4.5}$$

[17] Note that, for a non-vanishing D_{AB} one finds $\delta_{(T,Y)} D_{AB} = \mathcal{L}_Y D_{AB}$ so that if one starts with vanishing D_{AB}, it remains zero after the action of asymptotic symmetries.

where $\delta_T \equiv \delta_{(T,0)}$. The homogeneous term in (4.5), for $T_{nm} \equiv 2z^n\bar{z}^m/(1+z\bar{z})$, becomes

$$D_z^2 T_{nm} = (n-1)n\frac{2z^{n-2}\bar{z}^m}{1+z\bar{z}} = (n-1)nT_{n-2,m}, \tag{4.6}$$

so that if $n = 0, 1$, *i.e.* if T is a four-translation, $D_z^2 T$ vanishes and it also does the homogeneous term in (4.5). Equivalently, spherical harmonics with $l = 0, 1$ are zero-modes of the operator $2D_AD_B - \gamma_{AB}D_CD^C$. Under a superrotation

$$\delta_Y C_{zz} = \Big(\mathcal{L}_Y - \frac{1}{2}D_zY^z - \frac{1}{2}D_{\bar{z}}Y^{\bar{z}}\Big)C_{zz} - uD_z^2(D_zY^z + D_{\bar{z}}Y^{\bar{z}}). \tag{4.7}$$

If $Y_n^z(z) \equiv -z^{n+1}$ and $Y_m^{\bar{z}}(\bar{z}) \equiv -\bar{z}^{m+1}$, the homogeneous term in (4.7) is proportional to[18]

$$D_z^3 Y_n^z + D_z^2 D_{\bar{z}} Y_m^{\bar{z}} = -n(n^2-1)z^{n-2} = n(n^2-1)Y_{n-3}^z. \tag{4.8}$$

If $n = \pm 1, 0$, *i.e.* if Y^A is an ordinary Lorentz transformation, the homogeneous term in (4.7) vanishes again. We conclude that when ξ is an element of the Poincaré algebra $\mathfrak{iso}(3,1)$, if we start with vanishing asymptotic shear tensor $C_{AB} = 0$, after an infinitesimal Poincaré transformation it will still be zero. However, when considering supertranslations that are not translations and superrotations that are not Lorentz transformations, if we start with a $C_{AB} = 0$, then the action of these transformations produces a non-zero C_{AB}. Recently, this has been interpreted as the fact that supertranslations and superrotations spontaneously break the vacuum of Minkowski spacetime [10]. To better illustrate this concept, let us focus only on supertranslations and let us assume to start with a vacuum Minkowski configuration, for which $m = 0$, $C_{AB} = 0$, $N_{AB} = 0$ and $N_A = 0$. Equations (4.1)-(4.4) are telling us that a supertranslation cannot create gravitational radiation, inertial mass, or angular momentum. The *only* effect of a supertranslation (which is not a translation) on the vacuum is to shift C_{AB} by a homogeneous term. Being a symmetric traceless tensor on the 2-sphere C_{AB} can be decomposed as

$$C_{AB} = -2D_AD_BC + \gamma_{AB}D_CD^CC + \epsilon_{C(A}D_{B)}D^C\psi, \tag{4.9}$$

$C(u, x^A)$ and $\psi(u, x^A)$ being a scalar and pseudosclar, respectively. The former will play a fundamental role and sometimes it is referred to as *supertranslations memory field* [35,38,87,88]. Comparing with (4.2), under a supertranslation, it simply transforms as

$$\delta_T C = T. \tag{4.10}$$

The proposed interpretation is as follows: there exists an infinite number of gravitational vacua labelled by the field C and fixing it breaks spontaneously their supertranslations invariance. In particular, C is the Goldstone boson associated with this breaking. In sections 4.3 and 4.5 we will deeper investigate these properties and we will argue that the different vacua are related by the insertion of soft gravitons.

[18] One can also show that $D_{\bar{z}}^2(D_zY^z + D_{\bar{z}}Y^{\bar{z}}) = m(m^2-1)Y_{m-3}^{\bar{z}}$.

4.2 Christodoulou-Klainerman spaces and the scattering problem

We have seen that, in order to solve Einstein's equations at first and second subleading order in the luminosity distance r, we have to specify the set of data $\chi'_{\mathscr{I}^+} = \{m(u_0, x^A), C_{AB}(u_0, x^C), N_{AB}(u, x^C), N_A(u_0, x^B)\}$. In this section we focus on a particular set of solutions, namely the ones reverting to the vacuum in the far past and in the far future. In particular, we focus on the class of spacetimes studied by Christodoulou and Klainerman (CK) in [89], where the non-linear stability of Minkowski spacetime was proved. These conditions are also reviewed in [10, 38]. CK spacetimes are characterized by certain conditions on the data in $\chi'_{\mathscr{I}^+}$. The Bondi news falls on the past and future boundaries of $\mathscr{I}^+$ as

$$N_{AB} = \mathcal{O}(|u|^{-(1+\epsilon)}) \quad \text{when } u \to \pm\infty, \quad \epsilon > 0. \tag{4.11}$$

We also require that $m|_{\mathscr{I}^+_+} = 0$ (so that we are excluding black holes formation), $m|_{\mathscr{I}^+_-}$ and $N_A|_{\mathscr{I}^+_\pm}$ to be finite. This last condition implies that $\partial_u N_A|_{\mathscr{I}^+_\pm} = 0$ and, from (3.26), further assuming that $\hat{T}_{uA}|_{\mathscr{I}^+_\pm} = 0$, we obtain

$$D_B\big(D^B D^C C_{CA} - D_A D_C C^{BC}\big)|_{\mathscr{I}^+_\pm} = 0. \tag{4.12}$$

Taking into account the decomposition (4.9) of C_{AB}, (4.12) reduces the number of degrees of freedom of $C_{AB}|_{\mathscr{I}^+_\pm}$ from two to one. Indeed, (4.12) annihilates the part of C_{AB} containing C and it is easy to show that it implies, for ψ

$$\epsilon_{AB} D^B D_C D^C (D_D D^D + 2)\psi|_{\mathscr{I}^+_\pm} = 0. \tag{4.13}$$

Expanding ψ into spherical harmonics as $\psi|_{\mathscr{I}^+_\pm} = \sum_{l,m} \psi_{lm} Y_{lm}$, we see that (4.13) reduces to

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm} l(l+1)[l(l+1) - 2]\epsilon_{AB} D^B Y_{lm} = 0. \tag{4.14}$$

which implies $\psi_{lm} = 0$ for $l > 1$. However, spherical harmonics with $l \leq 1$ are also annihilated by the operator $\epsilon_{C(A} D_{B)} D^C$, as can be easily checked, and moreover they do not break the supertranslations invariance of the gravitational vacua. From these considerations, it follows that the solution of (4.12) is [19]

$$C_{AB}|_{\mathscr{I}^+_\pm} = -2 D_A D_B C|_{\mathscr{I}^+_\pm} + \gamma_{AB} D_C D^C C|_{\mathscr{I}^+_\pm}. \tag{4.15}$$

In other words, the asymptotic shear on the past and future boundaries of future null infinity is completely determined by the value of the supertranslations memory field there. Note that, equations (4.15) is telling us that, interpreting (4.2) as a gauge transformation, the field C_{AB} is pure gauge on $\mathscr{I}^+_\pm$.

Introducing similar conditions for the fields living on $\mathscr{I}^-$ and choosing the initial values u_0 and v_0 to be $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$, the relevant set of data in CK spacetimes are

$$\chi'_{\mathscr{I}^+} = \{m(x^A)|_{\mathscr{I}^+_-}, C(x^A)|_{\mathscr{I}^+_-}, N_{AB}(u, x^C), N_A(x^B)|_{\mathscr{I}^+_-}\}, \tag{4.16}$$

$$\tilde{\chi}'_{\mathscr{I}^-} = \{\tilde{m}(x^A)|_{\mathscr{I}^-_+}, \tilde{C}(x^A)|_{\mathscr{I}^-_+}, \tilde{N}_{AB}(v, x^C), \tilde{N}_A(x^B)|_{\mathscr{I}^-_+}\}. \tag{4.17}$$

In particular, we have reduced the problem of solving Einstein's equations to the following procedure:

[19] In (4.15) and in the following we denote $\Phi(x^A)|_{\mathscr{I}^+_\pm} \equiv \lim_{u\to\pm\infty} \Phi(u, x^A)$ and $\tilde{\Phi}(x^A)|_{\mathscr{I}^-_\pm} \equiv \lim_{v\to\pm\infty} \tilde{\Phi}(v, x^A)$, for any pair of fields $\Phi(u, x^A)$ and $\tilde{\Phi}(v, x^A)$ living on $\mathscr{I}^+$ and $\mathscr{I}^-$.

1. Assign the fields $C|_{\mathscr{I}^+_-}$ and $\tilde{C}|_{\mathscr{I}^-_+}$ and suitable Bondi news tensors N_{AB} on $\mathscr{I}^+$ and $\tilde{N}_{AB}$ on $\mathscr{I}^-$ to get the time evolutions of C_{AB} and $\tilde{C}_{AB}$ on $\mathscr{I}^+$ and $\mathscr{I}^-$ as

$$C_{AB}(u, x^C) = -2D_A D_B C(x^A)|_{\mathscr{I}^+_-} + \int_{-\infty}^{u} \mathrm{d}u N_{AB}(u, x^C), \tag{4.18}$$

$$\tilde{C}_{AB}(v, x^C) = -2D_A D_B \tilde{C}(x^A)|_{\mathscr{I}^-_+} + \int_{\infty}^{v} \mathrm{d}v \tilde{N}_{AB}(v, x^C); \tag{4.19}$$

2. Assign initial values $N_A(x^B)|_{\mathscr{I}^+_-}$, $\tilde{N}_A(x^B)|_{\mathscr{I}^-_+}$, $m(x^A)|_{\mathscr{I}^+_-}$ and $\tilde{m}(x^A)|_{\mathscr{I}^-_+}$ on $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ and integrate (3.26), (3.27) and their $\mathscr{I}^-$ counterparts to get the angular momentum aspect and the Bondi mass aspect everywhere on $\mathscr{I}^+$ and $\mathscr{I}^-$.

In principle, solving Einstein's equations on $\mathscr{I}^+$ and on $\mathscr{I}^-$ are two decoupled, independent problems. A direct consequence of this is that there exist two different asymptotic symmetry groups, preserving fall-offs conditions of the fields on $\mathscr{I}^+$ and $\mathscr{I}^-$ and acting separately on the data (4.16) and (4.17). We denote these groups by BMS_4^+ and BMS_4^-, respectively. Here by different we mean that, *a priori*, the functions (T, Y^A) and $(\tilde{T}, \tilde{Y}^A)$ characterizing their generators are independent. However, when focusing on the so-called *scattering problem*, which consists (in CK spacetimes) in finding a map relating the set in (4.17) to that on (4.16), the former being thought of as initial data of the latter, the fact that they can be acted upon with two different asymptotic symmetry groups is an issue. Indeed, for a scattering problem, data on $\mathscr{I}^-$ and on $\mathscr{I}^+$ can be thought of as in and out states, respectively, and we need a prescription to relate them in order to get a unique solution for the geometry on the entire spacetime (or, at least, in a neighborhood of the whole $\mathscr{I}$). But since the symmetry groups preserving the intrinsic structure at $\mathscr{I}^\pm$ are given by $\mathrm{BMS}_4^\pm$, such prescription would be spoiled by their non-trivial action. Another way to phrase it is that initial data on $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ are defined up to BMS_4^+ and BMS_4^- transformations and thus, the total group $\mathrm{BMS}_4^+ \times \mathrm{BMS}_4^-$ is somehow "too large" and it must be reduced to a *diagonal* BMS_4^0 component, preserving the given prescription. Moreover, only such BMS_4^0 will be the true, physical symmetry of the scattering problem in GR.

The Lorentz and CPT invariant conditions proposed in [10] are the so called *antipodal matching conditions*, which in angular coordinates $x^A = (\theta, \phi)$ read as

$$m(\theta, \phi)|_{\mathscr{I}^+_-} = \tilde{m}(\pi - \theta, \pi + \phi)|_{\mathscr{I}^-_+}, \qquad C(\theta, \phi)|_{\mathscr{I}^+_-} = \tilde{C}(\pi - \theta, \pi + \phi)|_{\mathscr{I}^-_+}, \tag{4.20}$$

$$N_A(\theta, \phi)|_{\mathscr{I}^+_-} = \tilde{N}_A(\pi - \theta, \pi + \phi)|_{\mathscr{I}^-_+}. \tag{4.21}$$

which are preserved under the diagonal BMS_4^0 transformations generated by gauge parameters (T, Y^A) and $(\tilde{T}, \tilde{Y}^A)$ obeying

$$T(\theta, \phi) = \tilde{T}(\pi - \theta, \pi + \phi), \qquad Y^A \theta, \phi) = \tilde{Y}^A(\pi - \theta, \pi + \phi). \tag{4.22}$$

It is worth noting that antipodal matching conditions at spatial infinity are obeyed by a large class of solutions, including Schwarzchild and Kerr, and they also play a crucial role in the derivation of infinite-dimensional asymptotic symmetry groups, including BMS_4, at spatial infinity [90–92] using the Hamiltonian approach.

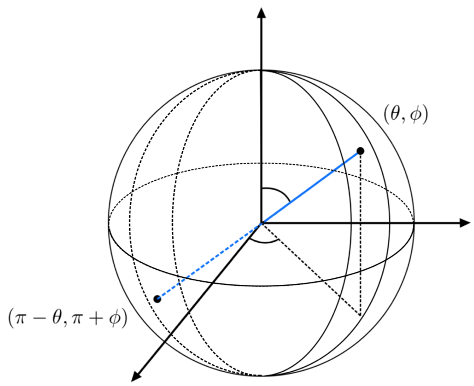

Figure 4: A light ray crossing the past celestial sphere at (θ, ϕ) will come out of the future celestial sphere at $(\pi - \theta, \pi + \phi)$. The map relating the in and out points is the antipodal map.

4.3 Symplectic structure on $\mathscr{I}^+$, Poisson bracket and supertranslations charges

We are now interested in the symplectic geometry of radiative modes of the gravitational field at null infinity. As outlined in section 2.3, computing the symplectic structure is an essential ingredient to study properties of the asymptotic charges, such as their conservation, and to introduce Poisson bracket between canonical variables. The description of the symplectic structure of the gravitational field at future null infinity $\mathscr{I}^+$ for asymptotically flat spacetimes was originally carried out in a series of works by Ashtekar and collaborators [93–96]. Here we define the symplectic structure induced on null infinity $\mathbf{\Omega}_{\mathscr{I}^+}$ as

$$\mathbf{\Omega}_{\mathscr{I}^+} = \lim_{\Sigma \to \mathscr{I}^+} \mathbf{\Omega}_\Sigma, \tag{4.23}$$

where $\mathbf{\Omega}_\Sigma$ is the symplectic structure of (2.61) associated to the presymplectic current of (2.71)-(2.72),

$$\mathbf{\Omega}_\Sigma[g, h_1, h_2] \equiv \frac{1}{16\pi G} \int_\Sigma (\mathrm{d}^3x)_\mu \, \omega^\mu[g, h_1, h_2], \tag{4.24}$$

with ω^μ given by

$$\omega^\mu[g, h_1, h_2] = \left[\frac{1}{2} h_2 \nabla^\mu h_1 + h_{2\nu\rho} \nabla^\nu h_1^{\mu\rho} - \frac{1}{2} h_2 \nabla_\nu h_1^{\nu\mu} - \frac{1}{2} h_2^{\nu\rho} \nabla^\mu h_{1\nu\rho} - \frac{1}{2} h_2^{\mu\rho} \nabla_\rho h_1 - (1 \leftrightarrow 2)\right], \tag{4.25}$$

and where, from the line element (3.29) we find

$$h_{uu} = \frac{2}{r}\delta m + \mathcal{O}(r^{-2}), \quad h_{zz} = r\delta C_{zz} + \mathcal{O}(r^0), \quad h_{uz} = \frac{1}{2} D^z \delta C_{zz} + \mathcal{O}(r^{-1}), \quad h_{ur} = \mathcal{O}(r^{-2}), \tag{4.26}$$

and correspondingly

$$h^{rr} = \frac{2}{r}\delta m + \mathcal{O}(r^{-2}), \quad h^{rz} = -\frac{1}{2r^2} D_z \delta C^{zz} + \mathcal{O}(r^{-3}), \quad h^{zz} = \frac{1}{r^3}\delta C^{zz} + \mathcal{O}(r^{-4}), \quad h^{ur} = \mathcal{O}(r^{-2}), \tag{4.27}$$

For a null hypersurface in Bondi coordinates we have that $\sqrt{-g}(\mathrm{d}^3x)_\mu = \delta^r_\mu[r^2\gamma_{z\bar{z}} + \mathcal{O}(r)]\mathrm{d}u\wedge\mathrm{d}z\wedge\mathrm{d}\bar{z}$ and we find

$$\begin{aligned}\sqrt{-g}\omega^\mu|_{\mathscr{I}^+}(\mathrm{d}^3x)_\mu &= \gamma_{z\bar{z}}\lim_{r\to\infty} r^2\left[\frac{1}{2r^2}\delta_1 C_{zz}\delta_2 N^{zz} + \frac{1}{2r^2}\delta_1 C_{\bar{z}\bar{z}}\delta_2 N^{\bar{z}\bar{z}} + \mathcal{O}(r^{-3}) - (1\leftrightarrow 2)\right]\mathrm{d}u\wedge\mathrm{d}z\wedge\mathrm{d}\bar{z}\\ &= \frac{1}{2}\gamma_{z\bar{z}}\left[\delta_1 C_{zz}\delta_2 N^{zz} + \delta_1 C_{\bar{z}\bar{z}}\delta_2 N^{\bar{z}\bar{z}} - (1\leftrightarrow 2)\right]\mathrm{d}u\wedge\mathrm{d}z\wedge\mathrm{d}\bar{z}.\end{aligned} \tag{4.28}$$

In terms of the $\wedge$-product between field variations, the symplectic structure is

$$\mathbf{\Omega}_{\mathscr{I}^+} = \frac{1}{32\pi G}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\big(\delta C_{zz}\wedge\delta N^{zz} + \delta C_{\bar{z}\bar{z}}\wedge\delta N^{\bar{z}\bar{z}}\big). \tag{4.29}$$

So far, in (4.29), we have not specified the boundary conditions of the asymptotic shear tensor C_{AB} on $\mathscr{I}^+_\pm$. However, here we are interested in explicitly understanding the contributions of the boundary values of C_{AB} to $\mathbf{\Omega}_{\mathscr{I}^+}$ [97]. Hence, we assume

$$\lim_{u\to\pm\infty} C_{zz}(u,z,\bar{z}) = \varphi^\pm_{zz}(z,\bar{z}), \tag{4.30}$$

where $\varphi^\pm_{zz}(z,\bar{z})$ are smooth, non vanishing functions on the 2-sphere. Integrating (3.24) yields

$$\int_{-\infty}^{\infty}\mathrm{d}u\,N_{zz} = \varphi^+_{zz} - \varphi^-_{zz} \equiv \Delta\varphi_{zz}. \tag{4.31}$$

This last equation can be seen as the $\lim_{\omega\to 0} N^\omega_{zz}$, where N^ω_{zz} is the Fourier transform of N_{zz}. A non-vanishing $\Delta\varphi_{zz}$, measuring the difference between the asymptotic shear C_{zz} at $\mathscr{I}^+_+$ and $\mathscr{I}^+_-$, can thus be associated to the existence of soft (*i.e.* zero energy) gravitons (see *e.g.* [10, 11, 52]).

It will be fundamental to introduce the new field $\hat{C}_{zz}$, defined as

$$\hat{C}_{zz}(u,z,\bar{z}) \equiv \frac{1}{2}\left[\int_{-\infty}^{u}\mathrm{d}u' N_{zz}(u',z,\bar{z}) - \int_{u}^{\infty}\mathrm{d}u' N_{zz}(u',z,\bar{z})\right]. \tag{4.32}$$

We have

$$C_{zz}(u,z,\bar{z}) = \frac{1}{2}\Delta\varphi_{zz}(z,\bar{z}) + \varphi^-_{zz}(z,\bar{z}) + \hat{C}_{zz}(u,z,\bar{z}). \tag{4.33}$$

In (4.33) we are choosing $\Delta\varphi_{zz}$ and φ^-_{zz} as independent variables, but we could have equally chosen φ^+_{zz} and φ^-_{zz}. Note also that

$$\lim_{u\to\pm\infty}\hat{C}_{zz} = \pm\frac{1}{2}\Delta\varphi_{zz}. \tag{4.34}$$

In equation (4.33), we have divided the asymptotic shear tensor C_{zz} into a "bulk" contribution $\hat{C}_{zz}$ and a pure boundary part, comprising $\Delta\varphi_{zz}$ and φ^-_{zz}. Working with $\hat{C}_{zz}$ rather than with C_{zz} has several advantages. First, it is the fastest way to highlight the role of boundary degrees of freedom. Further, it simplifies the calculation of the symplectic form, for it has the property

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}u\,\delta\hat{C}_{\bar{z}\bar{z}}\wedge\delta N^{\bar{z}\bar{z}} &= -\int_{-\infty}^{\infty}\mathrm{d}u\,\delta N_{\bar{z}\bar{z}}\wedge\delta\hat{C}^{\bar{z}\bar{z}} + \delta\hat{C}_{\bar{z}\bar{z}}\wedge\delta\hat{C}^{\bar{z}\bar{z}}\Big|_{-\infty}^{\infty} = -\int_{-\infty}^{\infty}\mathrm{d}u\,\delta N_{\bar{z}\bar{z}}\wedge\delta\hat{C}^{\bar{z}\bar{z}}\\ &= \int_{-\infty}^{\infty}\mathrm{d}u\,\delta\hat{C}_{zz}\wedge\delta N^{zz},\end{aligned} \tag{4.35}$$

where we used (4.34) and the antisymmetry of the wedge product in the last step. Substituting (4.32) in (4.29), we obtain, after some algebra

$$\boldsymbol{\Omega}_{\mathscr{I}^+} = \frac{1}{16\pi G}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\,\delta\hat{C}_{zz}\wedge\delta N^{zz} + \frac{1}{32\pi G}\int\gamma_{z\bar{z}}\mathrm{d}^2z\left(\delta\varphi^-_{zz}\wedge\delta\Delta\varphi^{zz} + \delta\varphi^-_{\bar{z}\bar{z}}\wedge\delta\Delta\varphi^{\bar{z}\bar{z}}\right). \tag{4.36}$$

We thus see that the symplectic structure splits into a bulk and a boundary part. We now focus on CK spacetimes considered in the previous section. First notice that, because of the fall-off (4.11) of the Bondi news, the integral over u in (4.36) converges. Further, the boundary values of the asymptotic shear tensor are given by

$$\varphi^-_{zz} = -2D_z^2C|_{\mathscr{I}^+_-} \equiv D_z^2C, \qquad \varphi^+_{zz} = -2D_z^2C|_{\mathscr{I}^+_+} \equiv D_z^2C^+, \qquad \Delta\varphi_{zz} = D_z^2(C^+ - C) \equiv D_z^2N. \tag{4.37}$$

with C and C^+ real functions on the 2-sphere. Furthermore we also assume the variations δN_{zz} and δC_{zz} of N_{zz} and C_{zz} to be CK so that

$$\delta N_{zz} = \mathcal{O}(|u|^{-(1+\epsilon)}) \quad \text{when } u\to\pm\infty, \quad \epsilon>0. \tag{4.38}$$

$$\delta C_{zz}|_{\mathscr{I}^+_+} = D_z^2\delta C^+, \qquad \delta C_{zz}|_{\mathscr{I}^+_-} = D_z^2\delta C, \tag{4.39}$$

Thus, the symplectic structure in (4.36) reads

$$\boldsymbol{\Omega}_{\mathscr{I}^+} = \frac{1}{16\pi G}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\,\delta\hat{C}_{zz}\wedge\delta N^{zz} + \frac{1}{16\pi G}\int\gamma_{z\bar{z}}\mathrm{d}^2z\,D_z^2\delta C\wedge D^{2z}\delta N. \tag{4.40}$$

From (4.40) we can easily read off the non-vanishing Poisson bracket:

$$\{N_{\bar{z}\bar{z}}(u,z,\bar{z}),\hat{C}_{ww}(u',w,\bar{w})\} = 16\pi G\delta^{(2)}(z-w)\delta(u-u')\gamma_{z\bar{z}}, \tag{4.41}$$

$$\{D_{\bar{z}}^2N(z,\bar{z}),D_w^2C(w,\bar{w})\} = 16\pi G\delta^{(2)}(z-w)\gamma_{z\bar{z}}. \tag{4.42}$$

These bracket match those derived in earlier works, see *e.g.* [11], however in our approach it is not necessary to add *ad hoc* boundary terms in the symplectic form to obtain the desired result, as suggested in [7, 98]. In fact the bulk-bulk and boundary-boundary Poisson bracket are obtained directly from the definition of the symplectic form and from the splitting (4.32) we introduced.

We are now interested in computing the supertranslations charges. From (2.62), the key quantity to compute is $\boldsymbol{\Omega}_{\mathscr{I}^+}[g,h,\delta_T h]$ and therefore we need explicit expressions for $\delta_T\hat{C}_{zz}$, $\delta_T C$ and $\delta_T N$. Evaluating (4.2) on $\mathscr{I}^+_\pm$ yields, due to the fall-off of the Bondi news

$$\delta_T C_{zz}|_{\mathscr{I}^+_+} = -2D_z^2T = D_z^2\delta_T C^+, \qquad \delta_T C_{zz}|_{\mathscr{I}^+_-} = -2D_z^2T = D_z^2\delta_T C, \tag{4.43}$$

so that, up to irrelevant $l=0,1$ spherical harmonics $\delta_T C^+ = \delta_T C = -2T$ and thus $\delta_T N = 0$. Consequently, for the bulk part we have

$$\delta_T\hat{C}_{zz} = TN_{zz}. \tag{4.44}$$

Substituting equations (4.43) and (4.44) into $\boldsymbol{\Omega}_{\mathscr{I}^+}[g,h,\delta_T h]$ we find, integrating by parts and using the vanishing of N_{zz} on the boundaries of $\mathscr{I}^+$

$$\boldsymbol{\Omega}_{\mathscr{I}^+}[g,h,\delta_T h] = \delta\left\{-\frac{1}{16\pi G}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\,\mathrm{d}u\,TN_{zz}N^{zz} + \frac{1}{8\pi G}\int\gamma_{z\bar{z}}\mathrm{d}^2z\,D_z^2TD^{2z}N\right\}. \tag{4.45}$$

From equation (2.62), integrating on a path γ in the space of fields we get the the difference between supertranslations charges at $\mathscr{I}^+_\pm$ as

$$Q_T|_{\mathscr{I}^+_+} - Q_T|_{\mathscr{I}^+_-} = -\frac{1}{16\pi G}\int_{\mathscr{I}^+} \gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\, T N_{zz}N^{zz} + \frac{1}{8\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z\, D_z^2 T D^{2z} N, \tag{4.46}$$

where we denoted $Q_{\xi_T} \equiv Q_T$. The interpretation of the first term in equation (4.46) is that there is a breaking in the conservation law of supertranslations charges whenever there is a non-vanishing flux of gravitational radiation through null infinity. Again, note that the second term depends on the difference between the boundary values of C_{zz} and it is thus related to the presence of soft gravitons. Further, this term depends on $D_z^2 T$ and hence, from the discussion after (4.6), it vanishes for ordinary translations. If we define now the supertranslations charges on a 2-sphere S at a fixed value of the retarded time u as

$$Q_T|_S = \frac{1}{4\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z\, T m|_S, \tag{4.47}$$

we get, using the evolution equation (3.27) for the Bondi mass aspect[20]

$$Q_T|_{\mathscr{I}^+_+} - Q_T|_{\mathscr{I}^+_-} = \int_{-\infty}^{\infty}\mathrm{d}u\frac{\mathrm{d}}{\mathrm{d}u}Q_T|_S = \frac{1}{4\pi G}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\, T\partial_u m \tag{4.48}$$

$$= -\frac{1}{16\pi G}\int_{\mathscr{I}^+} \gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\, T N_{zz}N^{zz} + \frac{1}{8\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z\, D_z^2 T D^{2z} N, \tag{4.49}$$

which is compatible with (4.46). Hence, the supertranslations charges are given by the codimension 2 integrals in (4.47) and, remarkably, they match exactly the ones firtsly found in [32] using (2.90). Taking into account that for CK spacetimes $m|_{\mathscr{I}^+_+} = 0$, we get $Q_T|_{\mathscr{I}^+_+} = 0$ and thus

$$Q_T|_{\mathscr{I}^+_-} = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u\, T N_{zz}N^{zz} - \frac{1}{8\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z D_z^2 T D^{2z} N \equiv Q_T^{\mathcal{H}}|_{\mathscr{I}^+_-} + Q_T^{\mathcal{S}}|_{\mathscr{I}^+_-}. \tag{4.50}$$

The supertranslations charges $Q_T|_{\mathscr{I}^+_-}$ thus split into a *hard* and a *soft* part, $Q_T^{\mathcal{H}}$ and $Q_T^{\mathcal{S}}$, quadratic and linear in the fields, respectively. Note that in the case of an ordinary four-translation the soft term vanishes whereas in the case of a pure supertranslation it does not and its contribution is proportional to the soft mode in (4.31). Using the Poisson bracket derived in (4.41) and (4.42), it is straightforward to show that Q_T canonically generates supertranslations, *i.e.*

$$\{Q_T|_{\mathscr{I}^+_-}, \hat{C}_{zz}\} = \{Q_T^{\mathcal{H}}|_{\mathscr{I}^+_-}, \hat{C}_{zz}\} = T N_{zz}, \qquad \{Q_T|_{\mathscr{I}^+_-}, D_z^2 C\} = \{Q_T^{\mathcal{S}}|_{\mathscr{I}^+_-}, D_z^2 C\} = -2D_z^2 T, \tag{4.51}$$

and hence

$$\{Q_T|_{\mathscr{I}^+_-}, C_{zz}\} = \delta_T C_{zz} = T N_{zz} - 2D_z^2 T. \tag{4.52}$$

These relations and in particular the soft charge $Q_T^{\mathcal{S}}$ play a central role in the connection between the Ward identities associated to supertranslations invariance of the gravitational $\mathcal{S}$-matrix and the Weinberg's soft graviton theorem. Before further exploring this connection, we make a brief digression on the latter.

[20]For simplicity, we are considering vacuum Einstein's equations, so that $\hat{T}_{uu} = 0$.

4.4 An aside: soft theorems

Consider a quantum mechanical scattering process characterized by m incoming particles with four-momenta $(p_k^{\text{in}})^\mu$ with $k = 1, ..., m$ and n outgoing particles with four-momenta $(p_j^{\text{out}})^\mu$ with $j = 1, ..., n$ and denote its scattering amplitude by $\mathcal{M}_N(p)$, where $N = m + n$ and $p = (p_1^{\text{in}}, ..., p_m^{\text{in}}, p_1^{\text{out}}, ..., p_n^{\text{out}})$. Now consider the same process with an additional emission of a soft photon with momentum q^μ. By soft here we mean that we are in the limit $|\vec{q}| = q^0 \to 0$ and therefore we are allowed to Taylor expand quantities depending on q^0 around $q^0 = 0$ and to keep only the leading terms. We denote this new amplitude by $\mathcal{M}^\mu_{N+1}(p, q)$. The setup is represented in Figure 5.

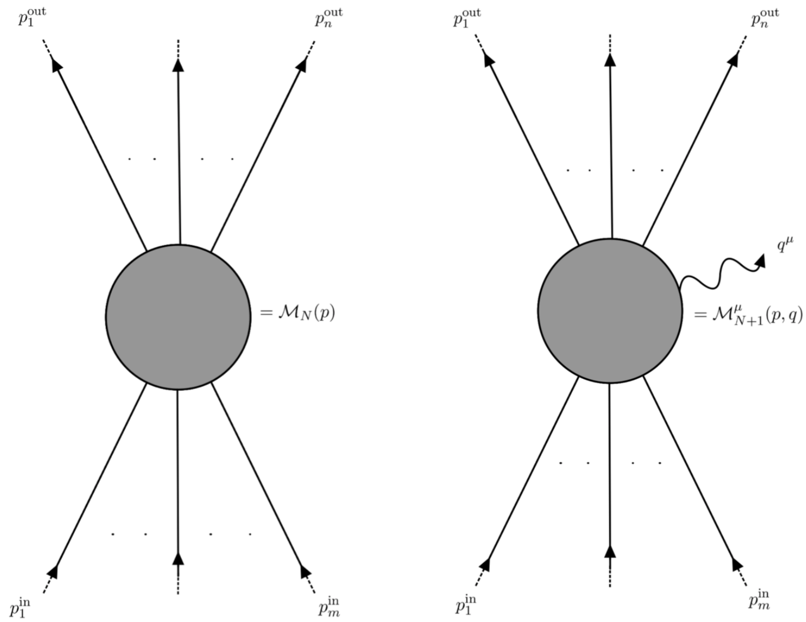

Figure 5: The first diagram represents a scattering process whose amplitude is $\mathcal{M}_N(p)$. The second is the same process with an additional emission of a soft photon, whose amplitude is $\mathcal{M}^\mu_{N+1}(p, q)$.

In principle, in order to compute $\mathcal{M}^\mu_{N+1}(p, q)$, we should consider all diagrams in which the photon line is attached to a "hard" particle line, external or internal. However, in the soft limit, the Feynman diagrams that will contribute to $\mathcal{M}^\mu_{N+1}(p, q)$ are only those where the photon line is attached to an external leg, as in 6, whereas internal lines will not contribute. Indeed, the only divergences come from on-shell propagators associated to physical particles. We thus consider the contributions to $\mathcal{M}^\mu_{N+1}(p, q)$ coming from each term of the sums in Figure 6. So far we made no assumptions on the nature of the incoming and outgoing particles involved in the process and on their interaction with the electromagnetic field. Here, for simplicity, we assume that the interaction of every particle with the electromagnetic field is governed by a scalar QED Lagrangian

$$\mathcal{L}_i = |D_\mu \phi_i|^2 - m_i^2 |\phi_i|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad D_\mu = \partial_\mu - i e_i A_\mu, \qquad i = 1, ..., N. \tag{4.53}$$

where we denoted by e_i the electric charge of the scalars. The interaction term is thus $J_i^\mu A_\mu$ where

$$J_i^\mu = i e_i (\phi_i^\dagger \partial^\mu \phi_i - (\partial^\mu \phi_i^\dagger) \phi_i). \tag{4.54}$$

The corresponding Feynman rules are in Figure 7. In the soft limit, the j-th contribution in the first

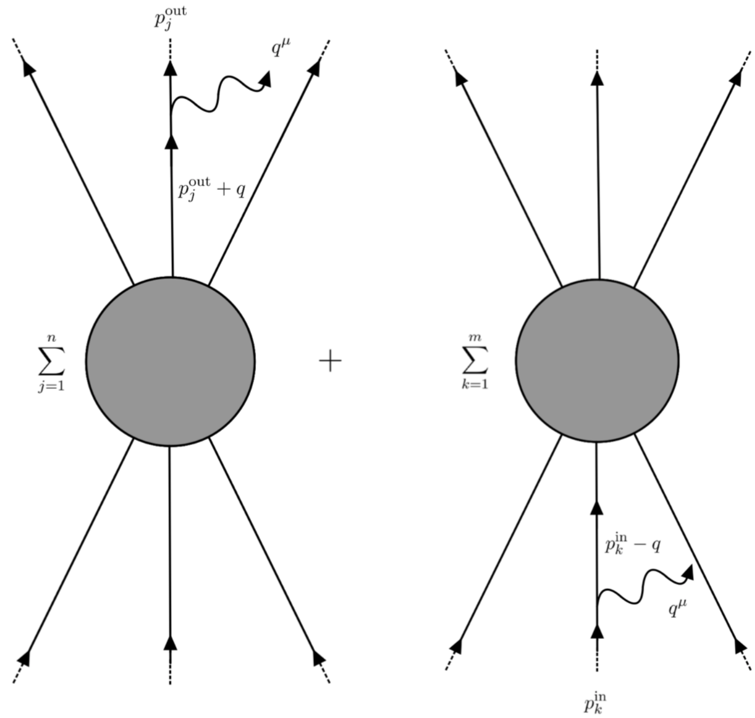

Figure 6: The second diagram in Figure 5, in the soft limit, is equal to the above sum of diagrams.

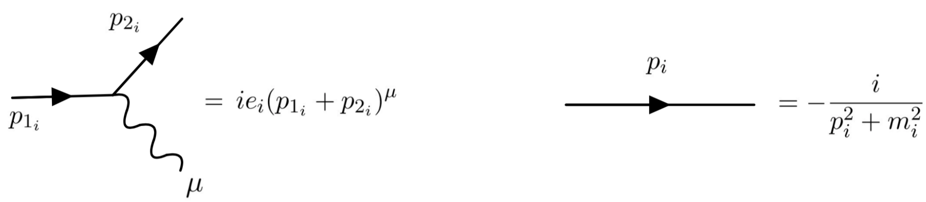

Figure 7: Feynman rules for the scalar QED Lagrangian of (4.68).

diagram of Figure 6 is thus given by

$$M_j^\mu(p,q) \equiv \frac{e_j(2p_j^{\rm out}+q)^\mu}{(p_j^{\rm out}+q)^2+m_j^2} M_N(p^{\rm in}, p_1^{\rm out}, ..., p_j^{\rm out}+q, ..., p_n^{\rm out}) = \left[\frac{e_j(p_j^{\rm out})^\mu}{p_j^{\rm out}\cdot q} + \mathcal{O}(|\vec{q}|^0)\right] M_N(p), \tag{4.55}$$

and, similarly, the k-th contribution in the second diagram of Figure 6 by

$$M_k^\mu(p,q) \equiv \frac{e_k(2p_k^{\rm in}-q)^\mu}{(p_k^{\rm in}-q)^2+m_k^2} M_N(p_1^{\rm in}, ..., p_k^{\rm in}-q, ..., p_m^{\rm in}, p^{\rm out}) = -\left[\frac{e_k(p_k^{\rm in})^\mu}{p_k^{\rm in}\cdot q} + \mathcal{O}(|\vec{q}|^0)\right] M_N(p). \tag{4.56}$$

It follows immediately that the total amplitude $M_{N+1}^\mu(q,p)$, sum of (4.55) and (4.56), in the soft limit factorizes as

$$M_{N+1}^\mu(p,q) = \left[\sum_{i=1}^{N} \frac{\eta_i e_i p_i^\mu}{p_i\cdot q} + \mathcal{O}(|\vec{q}|^0)\right] M_N(p), \quad \eta_i = \begin{cases} 1 & \text{outgoing} \\ -1 & \text{incoming} \end{cases} \tag{4.57}$$

This is the *soft photon theorem*, originally found by Low [99], which relates the scattering amplitude of an arbitrary quantum process involving soft photons to the same amplitude without the soft photons insertions. The proportionality factor in (4.57), at leading order in the soft expansion, is the so-called *soft* *factor* and it has a simple pole at $|\vec{q}| = 0$.

We derived (4.57) in the particularly simple case of scalar QED. However, it holds regardless of the spin of the hard particles. For instance, we could have considered fermions instead of scalar fields and the factorization of (4.57) would still be true. This property is sometimes referred to as *universality*. In other words, we just need to specify that there is a soft photon: the details of the matter coupled to the electromagnetic field are not important.

The soft photon theorem connects gauge invariance with the conservation of the total electric charge of the system [13]. Indeed, if the emitted photon has polarization vector $\epsilon^\mu(q)$ the amplitude is obtained contracting $M^\mu_{N+1}(p,q)$ with $\epsilon^*_\mu(q)$.

$$M_{N+1}(p,q) = M^\mu_{N+1}(p,q)\epsilon^*_\mu(q) = \left[\sum_{i=1}^N \frac{\eta_i e_i p_i^\mu \epsilon^*_\mu(q)}{p_i \cdot q} + \mathcal{O}(|\vec{q}|^0)\right] M_N(p). \tag{4.58}$$

Gauge transformations act on $\epsilon^\mu(q)$ as $\delta_\Lambda \epsilon^\mu(q) = \Lambda q^\mu$, for any complex Λ. Therefore, asking $M_{N+1}(p,q)$ to be a physical, gauge invariant, amplitude, *i.e.* $\delta_\Lambda M_{N+1}(p,q) = 0$ is equivalent to

$$M^\mu_{N+1}(p,q) q_\mu = 0 \Longrightarrow \sum_{\text{incoming}} e_i = \sum_{\text{outgoing}} e_i, \tag{4.59}$$

that is the total electric charge conservation. In other words, low energy photons can only couple to a charge that is conserved by all scattering processes that have non-vanishing probability of happening.

It is possible to show that there exists a gravitational analogue of the soft photon theorem, for which the scattering amplitude $M_{N+1}(q,p)$ of a process involving N hard particles and the emission of a soft graviton with polarization tensor $\epsilon_{\mu\nu}(q)$ is

$$M_{N+1}(p,q) = M^{\mu\nu}_{N+1}(p,q)\epsilon^*_{\mu\nu}(q) = \left[\sum_{i=1}^N \frac{\eta_i f_i p_i^\mu p_i^\nu \epsilon^*_{\mu\nu}(q)}{p_i \cdot q} + \mathcal{O}(|\vec{q}|^0)\right] M_N(p). \tag{4.60}$$

where $M_N(p)$ is again the same amplitude without the soft graviton insertion. This equation is referred to as the *Weinberg's soft graviton theorem* [13, 14]. In (4.60), we denoted by f_i the individual gravitational coupling constants, which may in principle depend on the particle species. We can think of f_i as being defined by the gravitational analogue of the electromagnetic interaction in Figure 7. In fact, such diagram should yield a Lorentz tensor $N^{\mu\nu}(p_{1_i}, p_{2_i})$ that in the soft limit becomes $N^{\mu\nu}(p_i)$, with $p_{1_i} = p_{2_i} \equiv p_i$. Having the structure of a Lorentz tensor, the only possibility is to have (in the simple case of a spinless hard particle) $N^{\mu\nu}(p_i) = (\alpha p_i^\mu p_i^\nu + \beta \eta^{\mu\nu} p_i^2) f_i(p_i^2)$ for some constants α and β. If the particle is on-shell, $p_i^2 = -m_i^2$ and thus $f_i(p_i^2 = -m_i^2)$ is a constant, which we choose to be proportional to $i f_i$.

Under a gauge transformation $\delta_\Lambda \epsilon^{\mu\nu}(q) = \Lambda^\mu q^\nu + \Lambda^\nu q^\mu$ and hence $\delta_\Lambda M^{N+1}(p,q) = 0$ implies that

$$M^{\mu\nu}_{N+1}(p,q) q_\nu = 0 \Longrightarrow \sum_{\text{incoming}} f_i p_i^\mu = \sum_{\text{outgoing}} f_i p_i^\mu. \tag{4.61}$$

This equation, together with the standard energy-momentum conservation, is telling us that $f_i = f \equiv 1$, $\forall i = 1, ..., N$, *i.e.* all particles must have the same gravitational coupling constant. Therefore the *equivalence principle*, stating that at low energies gravity couples to all forms of energy-momentum with the same strength, regardless of "chemical composition" of the matter, is a consequence of energy-momentum conservation, which is ultimately a consequence of the gauge invariance of the theory. It is also worth remarking that the factorization (4.60) is again universal and thus independent of the nature of the hard particles. For instance, they could have been also gravitons themselves. This last observation implies that gravity not only does not distinguish between "ordinary" matter, but it couples with the same strength also to itself. This feature is related to the *strong equivalence principle* for which, whenever gravitational self-interactions are concerned, the energy and momentum stored in gravitational fields are indistinguishable from those of ordinary matter.

In [13], a generalization of the soft photon and graviton theorem to the case of higher spins has been obtained,

$$M_{N+1}^{(J)}(q,p) = \left[\sum_{i=1}^{N} \frac{\eta_i g_i^{(J)} [p_i^\mu \epsilon_\mu^*(q)]^J}{p_i \cdot q} + \mathcal{O}(|\vec{q}|^0) \right] M_N(p), \tag{4.62}$$

where J is the spin of the soft particle, $\epsilon_{\mu_1 \ldots \mu_J}(q)$ its polarization tensor and where the product in the numerator $p_i^{\mu_1} \ldots p_i^{\mu_J} \epsilon^*_{\mu_1 \ldots \mu_J}(q)$ has been decomposed using $\epsilon_{(\mu_1 \ldots \mu_J)}(q) = \epsilon_{\mu_1}(q) \ldots \epsilon_{\mu_J}(q)$. In this case, gauge invariance requires that when one of the $\epsilon_\mu^*(q)$ in the numerator is replaced by q_μ, the corresponding amplitude $M_{N+1}^{(J)}(q,p)$ vanishes and thus

$$\sum_{\text{incoming}} g_i^{(J)} [p_i \cdot \epsilon^*(q)]^{J-1} = \sum_{\text{outgoing}} g_i^{(J)} [p_i \cdot \epsilon^*(q)]^{J-1}. \tag{4.63}$$

If the energy-momentum conservation holds, the only way for the above equation to be satisfied is that $g_i^{(J)} = 0$, $\forall\, i = 1, \ldots, N$. The conclusion is that fields with spin greater than two cannot interact at zero frequency and hence they cannot generate macroscopic fields.

Before proceeding to explore the connection of these theorems with supertranslations symmetry, we briefly mention that, both for electromagnetism and gravity, the subleading $\mathcal{O}(|\vec{q}|^0)$ terms have also been computed in [100] and [15], respectively. In the case of gravity and of spinless hard particles, the subleading term reads [21]

$$M_{N+1}(p,q) = \left[\sum_{i=1}^{N} \frac{\eta_i p_i^\mu}{p_i \cdot q} \left(p_i^\nu - i q_\rho J_i^{\nu\rho} \right) \epsilon^*_{\mu\nu}(q) + \mathcal{O}(|\vec{q}|) \right] M_N(p), \tag{4.64}$$

where

$$J_i^{\nu\rho} = i \left(p_i^\nu \frac{\partial}{\partial p_{i\rho}} - p_i^\rho \frac{\partial}{\partial p_{i\nu}} \right), \tag{4.65}$$

is the orbital angular momentum of the i-th particle. Gauge invariance of (4.65) requires, beside conservation of energy-momentum, that

$$\sum_{\text{incoming}} J_i^{\nu\rho} = \sum_{\text{outgoing}} J_i^{\nu\rho}, \tag{4.66}$$

which is the conservation of the total orbital angular momentum of the system.

[21] See also *e.g.* [16, 17] for a derivation of the subleading and subsubleading terms from gauge invariance and for a generalization to gluons and hard particles with spin.

4.5 Conservation laws and Ward identities from antipodal matching conditions

Now we return to the role played by supertranslations charges in the scattering problem outlined in section 4.2, using the antipodal matching conditions. We follow closely the discussion in [10, 11, 21].

Generalizing the results obtained above to the case of past null infinity $\mathscr{I}^-$, the $\mathrm{BMS}_4^\pm$ supertranslations charges at $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ read

$$Q_T|_{\mathscr{I}^+_-} = \frac{1}{4\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z T m|_{\mathscr{I}^+_-}, \qquad Q_{\tilde{T}}|_{\mathscr{I}^-_+} = \frac{1}{4\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2z \tilde{T}\tilde{m}|_{\mathscr{I}^-_+}. \tag{4.67}$$

Suppose now that the CK space under consideration is a solution of the scattering problem and that the antipodal matching conditions in (4.20) and (4.22) hold, allowing to single out BMS_4^0. These conditions imply that in going from $\mathscr{I}^-$ to $\mathscr{I}^+$ the supertranslations charges are conserved [22]

$$Q_T^+ = Q_T^- \tag{4.68}$$

Note that (4.68) is a quite different conservation law with respect to the one discussed in (2.60) and (2.62) of section 2.3. Indeed, the former is an equality between quantities defined on $\mathscr{I}^+$ and $\mathscr{I}^-$, relating charges of BMS_4^+ to those of BMS_4^- through the diagonal BMS_4^0 identification as represented in Figure 8, while the latter states the independence of the charges of $\mathrm{BMS}_4^\pm$ on the particular section of $\mathscr{I}^\pm$.

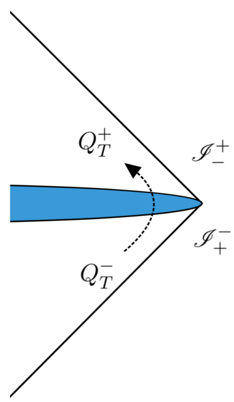

Figure 8: Conserved supertranslations charges in going from $\mathscr{I}^-_+$ to $\mathscr{I}^+_-$ in a scattering event.

Returning to (4.68), it expresses an infinite number of conservation laws, because the function T is arbitrary. We now comment on the physical meaning of these laws.

First suppose that T is a spherical harmonic Y_{lm} with $l \leq 1$ and hence it represents an ordinary four-translation. In this case the soft term $Q_T^{\mathcal{S}}$ in (4.50) vanishes. In particular, for $l = 0$, the hard term is proportional to the standard ADM mass, which is the Bondi mass measured at $u = -\infty$, plus a possible matter contribution in the case of a non-vanishing $\hat{T}_{uu}$. Hence, equation (4.68) expresses the total energy conservation law in a scattering process. Similarly, when T comprises spherical harmonics with $l = 1$, (4.68) is the ADM momentum conservation. When T is a pure supertranslation with harmonics with $l > 1$ the soft term does not vanish and the hard term has an unfamiliar form because the charges piercing null infinity are weighted by an arbitrary function that depends on the angle at which they "exit" the

[22] Here and in the following we introduce the notation $Q_T^+ \equiv Q_T|_{\mathscr{I}^+_-}$ and $Q_T^- \equiv Q_T|_{\mathscr{I}^-_+}$ for BMS_4^0 supertranslations charges.

spacetime. Before continuing, it is useful to introduce the splitting into hard and soft term also for BMS_4^- supertranslations charges. Using the evolution equation for m on $\mathscr{I}^-$,

$$\partial_v \tilde{m} = \frac{1}{4}\tilde{N}_{zz}\tilde{N}^{zz} + \frac{1}{4}\big(D_z^2\tilde{N}^{zz} + D_{\bar{z}}^2\tilde{N}^{\bar{z}\bar{z}}\big), \tag{4.69}$$

and that $m|_{\mathscr{I}^-_-} = 0$ in CK spacetimes, we have

$$Q_{\tilde{T}}|_{\mathscr{I}^-_+} = \frac{1}{16\pi G}\int_{\mathscr{I}^-} \gamma_{z\bar{z}} \mathrm{d}^2 z \mathrm{d}v \tilde{T}\tilde{N}_{zz}\tilde{N}^{zz} + \frac{1}{8\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2 z D_z^2 \tilde{T} D^{2z}\tilde{N} \equiv Q_{\tilde{T}}^{\mathcal{H}}|_{\mathscr{I}^-_+} + Q_{\tilde{T}}^{\mathcal{S}}|_{\mathscr{I}^-_+}, \tag{4.70}$$

similarly to (4.50). Choosing $T(z,\bar{z}) = \delta^2(z-w)$ the conservation law (4.68) becomes

$$\tag{4.71}$$

$$\int_{-\infty}^{\infty} \mathrm{d}u \big[N_{ww}N^{ww} - 2\big(D_w^2 N^{ww} + D_{\bar{w}}^2 N^{\bar{w}\bar{w}}\big)\big] = \int_{-\infty}^{\infty} \mathrm{d}v \big[\tilde{N}_{ww}\tilde{N}^{ww} + 2\big(D_w^2 \tilde{N}^{ww} + D_{\bar{w}}^2 \tilde{N}^{\bar{w}\bar{w}}\big)\big], \tag{4.72}$$

which means that the total energy is conserved at any angle. Notice that the soft terms provide a non-vanishing contribution to the expression of local energy on the asymptotic 2-sphere.

For later convenience we define the *total soft charge operator* for the diagonal BMS_4^0 group as

$$Q_T^{\mathcal{S}} \equiv Q_T^{\mathcal{S}+} - Q_T^{\mathcal{S}-} = -\frac{1}{8\pi G}\int \gamma_{z\bar{z}}\mathrm{d}^2 z D_z^2 T\left[\int_{-\infty}^{\infty}\mathrm{d}u N^{zz} + \int_{-\infty}^{\infty}\mathrm{d}v \tilde{N}^{zz}\right], \tag{4.73}$$

where we used the explicit definition of the soft mode N in terms of the Bondi news, *i.e.* $\int_{-\infty}^{\infty}\mathrm{d}u N_{zz} = D_z^2 N$ on $\mathscr{I}^+$. In the following we will show how the above construction is crucial in proving the equivalence of the Ward identities associated to supertranslations invariance of the gravitational $\mathcal{S}$-matrix with the Weinberg's soft graviton theorem presented in the previous section. To this aim, consider the expansion of the line element around $\mathscr{I}^+$ obtained in (3.29). The full metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ with $\kappa^2 = 32\pi G$ and we are interested in the scattering processes driven by the quantized asymptotic fluctuations $h_{\mu\nu}$. At late times, on $\mathscr{I}^+$, the field $h_{\mu\nu}^{\text{out}}$ can be regarded as free and can thus be approximated by a standard mode expansion

$$h_{\mu\nu}^{\text{out}}(x) = \sum_{\alpha=\pm}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{2\omega_q}[\epsilon_{\mu\nu}^{\alpha *}(\vec{q})a_\alpha^{\text{out}}(\vec{q})e^{iq\cdot x} + \epsilon_{\mu\nu}^{\alpha}(\vec{q})a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq\cdot x}], \tag{4.74}$$

where $q^0 = \omega_q = |\vec{q}|$, $\alpha = \pm$ are the two helicites of the graviton and

$$[a_\alpha^{\text{out}}(\vec{q}), a_\beta^{\text{out}}(\vec{q'})^\dagger] = \delta_{\alpha\beta}(2\omega_q)(2\pi)^3\delta^{(3)}(\vec{q} - \vec{q'}). \tag{4.75}$$

Outgoing gravitons with momentum q satisfying $q^\mu q_\mu = 0$ and helicity α arrive, at late times, at a point $(z,\bar{z})$ on the celestial sphere at $\mathscr{I}^+$. Indeed, there is a natural map from null vectors q^μ to points $(z,\bar{z})$ on the sphere towards which the null vector is directed. This map is given by [23]

$$q^\mu = \frac{\omega_q}{1+z\bar{z}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) = (\omega_q, \vec{q}). \tag{4.76}$$

[23]Note that $\vec{q} = \omega_q \hat{x}$, where Cartesian coordinates (t, x^1, x^2, x^3) are related to $(u, r, z, \bar{z})$ by

$$t = u + r, \qquad x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \qquad x^3 = \frac{r(1-z\bar{z})}{1+z\bar{z}}.$$

The above equation shows that the three independent components of the four-momentum q^μ can be equivalently parametrized by $(\omega_q, z, \bar{z})$. The polarization tensors may be written as $\epsilon^{\pm\mu\nu} = \epsilon^{\pm\mu}\epsilon^{\pm\nu}$, with

$$\epsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \qquad \epsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(z, 1, i, -z), \tag{4.77}$$

and they obey the orthogonality and the tracelessness conditions $\epsilon^{\pm\mu\nu}q_\nu = 0 = \epsilon^{\pm\mu}{}_\mu$. From (3.29), the asymptotic shear tensor can be defined as

$$C_{zz} = \kappa \lim_{r\to\infty} \frac{1}{r} h^{\text{out}}_{zz}, \tag{4.78}$$

and, after using (4.74) and the stationary phase approximation, we get

$$C_{zz} = -\frac{i\kappa}{4\pi^2(1+z\bar{z})^2}\int_0^\infty \mathrm{d}\omega_q [a^{\text{out}}_+(\omega_q\hat{x})e^{-i\omega_q u} - a^{\text{out}}_-(\omega_q\hat{x})^\dagger e^{i\omega_q u}]. \tag{4.79}$$

The "out" creation and annihilation operators in (4.79) involve the three momentum $\omega_q\hat{x}$ pointing towards the point $(z, \bar{z})$ on the celestial sphere and therefore they annihilate positive helicity gravitons and create negative helicity gravitons headed to the point $(z, \bar{z})$. The Fourier transform $N^\omega_{zz} = \int_{-\infty}^{\infty} \mathrm{d}u\, \partial_u C_{zz} e^{i\omega u}$ of N_{zz} yields, for $\omega > 0$,

$$N^\omega_{zz} = -\frac{\kappa\omega a^{\text{out}}_+(\omega\hat{x})}{2\pi(1+z\bar{z})^2}, \qquad N^{-\omega}_{zz} = -\frac{\kappa\omega a^{\text{out}}_-(\omega\hat{x})^\dagger}{2\pi(1+z\bar{z})^2}. \tag{4.80}$$

The zero mode of N^ω_{zz} can be defined in a Hermitian way as

$$N^0_{zz} \equiv \lim_{\omega\to 0^+} \frac{1}{2}(N^\omega_{zz} + N^{-\omega}_{zz}) = -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0^+} [\omega a^{\text{out}}_+(\omega\hat{x}) + \omega a^{\text{out}}_-(\omega\hat{x})^\dagger]. \tag{4.81}$$

A similar construction at early times on $\mathscr{I}^-$ leads to

$$\tilde{N}^0_{zz} = -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0^+} [\omega a^{\text{in}}_+(\omega\hat{x}) + \omega a^{\text{in}}_-(\omega\hat{x})^\dagger]. \tag{4.82}$$

We can construct the total soft graviton operator as

$$\mathcal{O}_{zz} \equiv N^0_{zz} + \tilde{N}^0_{zz}, \tag{4.83}$$

so that the quantized total soft charge of (4.73) can be expressed in terms of $\mathcal{O}_{zz}$ as

$$Q^{\mathcal{S}}_T = -\frac{1}{8\pi G}\int \gamma_{z\bar{z}} \mathrm{d}^2 z D^2_z T \mathcal{O}^{zz}, \tag{4.84}$$

and it is therefore related to an operator that creates and annihilates incoming and outgoing soft gravitons.

So far, we were only concerned about a free gravitational field. However, in order to make a connection with the Weinberg's soft graviton theorem, we shall include also interactions with matter. For simplicity, we consider the case of n scalar massless particles with late times momenta $(p^{\text{out}}_j)^\mu$ and energies E^{out}_j, with $j = 1, .., n$ and m scalar massless particles with early times momenta $(p^{\text{in}}_k)^\mu$ and energies E^{in}_j, with $k = 1, .., m$ and we assume that they are the out and in states of a scattering process governed by an $\mathcal{S}$-matrix, as represented in Figure 9 or, equivalently, in the first diagram of Figure 5. Note that this is precisely the setup in which the soft graviton theorem applies.

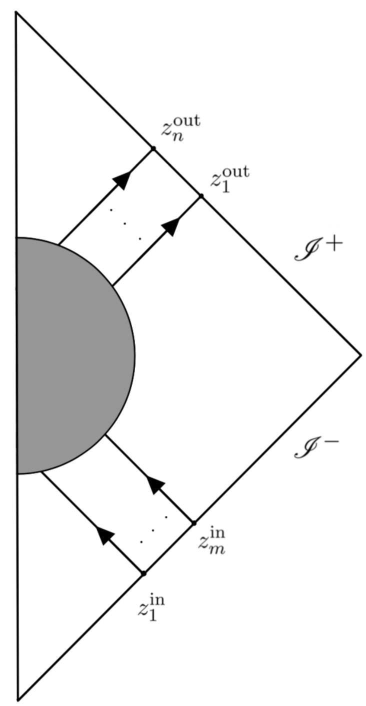

Figure 9: A scattering between m incoming and n outgoing massless particles in an asymptotically flat spacetime.

Similarly to (4.76), four-momenta of the outgoing particles that will cross the point $(z_j^{\rm out}, \bar{z}_j^{\rm out})$ admit the parametrization

$$(p_j^{\rm out})^\mu = \frac{E_j^{\rm out}}{1+z_j^{\rm out}\bar{z}_j^{\rm out}}\left(1, z_j^{\rm out}+\bar{z}_j^{\rm out}, -i(z_j^{\rm out}-\bar{z}_j^{\rm out}), 1-z_j^{\rm out}\bar{z}_j^{\rm out}\right). \tag{4.85}$$

The asymptotic out state of these n massless scalar particles can be represented as $|{\rm out}\rangle \equiv \left|z_1^{\rm out}, ..., z_n^{\rm out}\right\rangle$ and the $T=1$ supertranslation charge, that is the Hamiltonian, acts on it as

$$Q_1^+\,|{\rm out}\rangle = \sum_{j=1}^{n} E_j^{\rm out}\,|{\rm out}\rangle\,. \tag{4.86}$$

For the early times asymptotic states of m incoming particles we have, with $k=1,...,m$,

$$(p_k^{\rm in})^\mu = \frac{E_k^{\rm in}}{1+z_k^{\rm in}\bar{z}_k^{\rm in}}\left(1, z_k^{\rm in}+\bar{z}_k^{\rm in}, -i(z_k^{\rm in}-\bar{z}_k^{\rm in}), 1-z_k^{\rm in}\bar{z}_k^{\rm in}\right), \tag{4.87}$$

and

$$|{\rm in}\rangle \equiv \left|z_1^{\rm in}, ..., z_m^{\rm in}\right\rangle, \qquad Q_1^-\,|{\rm in}\rangle = \sum_{k=1}^{m} E_k^{\rm in}\,|{\rm in}\rangle\,. \tag{4.88}$$

As remarked, the supertranslation charge obtained by taking $T=1$ is the Hamiltonian H of the system and the $\mathcal{S}$-matrix governing the scattering process, for $t\to\pm\infty$, is proportional to $\exp(iHt)$. Since supetranslations commute with themselves, they commute with the $\mathcal{S}$-matrix and therefore they must be symmetries of the gravitational scattering. Thus

$$\langle \text{out}| Q_T^+ \mathcal{S} - \mathcal{S} Q_T^- |\text{in}\rangle = 0. \tag{4.89}$$

When acting on in and out states with the hard part of the supertranslations charges we have, from (4.86) and (4.88)

$$Q_T^{\mathcal{H}-} \left| z_1^{\text{in}}, ..., z_m^{\text{in}} \right\rangle = \sum_{k=1}^{m} T(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) E_k^{\text{in}} \left| z_1^{\text{in}}, ..., z_m^{\text{in}} \right\rangle, \tag{4.90}$$

$$\left\langle z_1^{\text{out}}, ..., z_n^{\text{out}} \right| Q_T^{\mathcal{H}+} = \sum_{j=1}^{n} T(z_j^{\text{out}}, \bar{z}_j^{\text{out}}) E_j^{\text{out}} \left\langle z_1^{\text{out}}, ..., z_n^{\text{out}} \right|, \tag{4.91}$$

and hence, when acting with the full supertranslations charges, including the soft part, we have

$$Q_T^- |\text{in}\rangle = Q_T^{\mathcal{S}-} |\text{in}\rangle + \sum_{k=1}^{m} E_k^{\text{in}} T(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) |\text{in}\rangle, \tag{4.92}$$

$$\langle \text{out}| Q_T^+ = \langle \text{out}| Q_T^{\mathcal{S}+} + \sum_{j=1}^{n} E_j^{\text{out}} T(z_j^{\text{out}}, \bar{z}_j^{\text{out}}) \langle \text{out}|. \tag{4.93}$$

Equation (4.89) for the BMS_4^0 invariance of the $\mathcal{S}$-matrix now reads as

$$\langle \text{out}| : Q_T^{\mathcal{S}} \mathcal{S} : |\text{in}\rangle = \left(\sum_{k=1}^{m} E_k^{\text{in}} T(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) - \sum_{j=1}^{n} E_j^{\text{out}} T(z_j^{\text{out}}, \bar{z}_j^{\text{out}}) \right) \langle \text{out}| \mathcal{S} |\text{in}\rangle, \tag{4.94}$$

where

$$\langle \text{out}| : Q_T^{\mathcal{S}} \mathcal{S} : |\text{in}\rangle \equiv \langle \text{out}| Q_T^{\mathcal{S}+} \mathcal{S} - \mathcal{S} Q_T^{\mathcal{S}-} |\text{in}\rangle, \tag{4.95}$$

denotes the time ordered product and $Q_T^{\mathcal{S}}$ is the total soft charge defined in (4.73). This identity relates the $\mathcal{S}$-matrix elements between two asymptotic states, one of which has an additional insertion of a total soft charge operator. Choosing $T = \frac{1}{z-w}$ and integrating by parts we have, for the total soft charge in (4.84)

$$Q^{\mathcal{S}}_{\frac{1}{z-w}} \equiv P_w = \frac{1}{8\pi G} \int \gamma_{z\bar{z}} \mathrm{d}^2 z \partial_{\bar{z}} \frac{1}{z-w} \partial_{\bar{z}} O^{\bar{z}\bar{z}} = \frac{1}{4G} \gamma^{w\bar{w}} \partial_{\bar{w}} O_{ww}, \tag{4.96}$$

where we used

$$\partial_{\bar{z}} \frac{1}{z-w} = 2\pi \delta^{(2)}(z-w). \tag{4.97}$$

The operator P_z in (4.96) is denoted *soft graviton current*, it involves zero-frequency integrals over $\mathscr{I}^{\pm}$ and thus it creates and annihilates soft gravitons. The identity (4.94) reads, in this case

$$\langle \text{out}| : P_w \mathcal{S} : |\text{in}\rangle = \sum_{i=1}^{N} \left(\frac{E_i^{\text{out}}}{w - z_i^{\text{out}}} - \frac{E_i^{\text{in}}}{w - z_i^{\text{in}}} \right) \langle \text{out}| \mathcal{S} |\text{in}\rangle. \tag{4.98}$$

This equation relates the $\mathcal{S}$-matrix element with a soft graviton current insertion to the $\mathcal{S}$-matrix element without such insertion. It can be shown [11] that, rewriting the the soft graviton theorem of (4.62) in $(z, \bar{z})$ coordinates implies (4.98) and vice-versa.

4.6 Further developments and outlooks

The above analysis shows that the Ward identities associated to supertranslations invariance of the $\mathcal{S}$-matrix associated to a scattering process are equivalent to the Weinberg's soft graviton theorem. At this stage, a natural question to ask concerns the role of the full BMS_4 group in the gravitational scattering and whether similar arguments can be applied to supperrotations or smooth Diff(S^2). It turns out [12] that the subleading soft graviton theorem of (4.64) implies local $\mathfrak{bms}_4$ superrotations invariance of the gravitational $\mathcal{S}$-matrix but the vice-versa is not true. A complete equivalence has been obtained when considering instead the generalized $\mathfrak{bms}_4$ smooth Diff(S^2) [7, 8]. However, it must be pointed out that there are some difficulties to construct Diff(S^2) charges and, in particular, the method outlined in 4.3 for supertranslations cannot be directly applied in this framework due to a divergence in the symplectic structure caused by a linear u term in $\delta_{(0,Y)}C_{AB}$ of (4.2) as also remarked in [53,97]. A possible resolution of this issue through a renormalization procedure, based on the ambiguity (2.55) in defining $\boldsymbol{\omega}$, has been proposed in [35]; a drawback of this approach is that the symplectic current turn out to be a not local and not covariant functional of the fields.

As remarked in section 1.3, the discussed equivalence between soft theorems and asymptotic symmetries is part of a larger triangular equivalence relation that includes also the gravitational memory effect [18–20, 101–104]. In particular, the so-called *displacement memory effect* predicts that a couple of inertial observers moving on timelike curves in a neighbourhood of future null infinity $\mathscr{I}^+$, after a burst of gravitational waves localized in a finite retarded time interval $\Delta u = u_f - u_i$, will experience a permanent shift in their separation. Such separation crucially depends on the retarded time difference of the asymptotic shear tensor $\Delta C_{AB} = C_{AB}(u_f) - C_{AB}(u_i)$. Assuming that first and after the transit of the waves the spacetime is in a vacuum configuration it implies that ΔC_{AB} is proportional to ΔC, where C is the supertranslations memory field of (4.9) and it depends on the moments of the radiation energy flux. However, as argued in section 4.1, different vacua are related by supertranslations and their action shifts the value of C, as in (4.10). Therefore, performing a supertranslation on a vacuum configuration has the same effect of a burst of gravitational waves. Further, it has been shown that the Weinberg's soft graviton theorem and the displacement memory effect are related by a Fourier transform. It is worth mentioning that is has also been conjectured a subleading *spin memory effect* that can be connected to the action of superrotations on the vacua and to the subleading soft graviton theorem.

The interconnection discovered between these infrared properties of gravity is stimulating growing interest in the context of flat space holography and the so-called *celestial CFT* [105–119]. It would be in fact appealing to describe scattering processes in four-dimensional asymptotically flat spacetimes in terms of correlators on the celestial sphere. The idea, in the case of massless particles, is to represent in- and out- states by the insertion of local operators $O_k(z, \bar{z})$ on the point $(z, \bar{z})$ of the celestial sphere, where k is a set of quantum numbers describing the properties of the particles under consideration, *e.g.* their energy. The set $(z, \bar{z}, k)$ replaces the components p^μ of the momenta of the particles under consideration as explicitly seen in (4.85) and (4.87). Futher, the operators $O_k(z, \bar{z})$ must transform covariantly under $\text{SL}(2,\mathbb{C})/\mathbb{Z}^2$, which is the Lorentz group acting on the celestial sphere as in (3.64) or, as discussed in the case of local $\mathfrak{bms}_4$, under the full, infinite-dimensional, local conformal group. More in general, n-point scattering amplitudes of gravity could be described by n-point correlation functions on the celestial sphere

$$\langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \longrightarrow \langle O_1(z_1, \bar{z}_2), ..., O_n(z_n, \bar{z}_n) \rangle \, . \tag{4.99}$$

This is another way of interpreting the scattering problem, equivalent to the usual one, which is familiar with the language of CFT on the 2-sphere. Understanding in detail these features also in the quantum regime, could provide a microscopic realization of the holographic principle in four-dimensional asymptotically flat quantum gravity.

5 Asymptotically locally AdS_3 spacetimes

In this chapter, we focus on three-dimensional GR and, in particular, on solutions with negative cosmological constant (AdS_3) whose holographic nature is well-established [26, 120, 121]. The absence of bulk propagating degrees of freedom makes this theory a privileged playground to understand the role of boundary conditions in gravity. Indeed, the dynamics can be described by a pure boundary theory, as shown by the Chern-Simons formulation [122–130], reviewed in Appendix C.

In the seminal work by Brown and Henneaux (BH) [27] it was shown that the asymptotic symmetry algebra of AdS_3, under certain boundary conditions encompassing BTZ black holes [131–133], consists in two commuting copies of the Virasoro algebra with central charges $c^{\pm} = c = \frac{3\ell}{2G}$, ℓ being the AdS_3 radius and G the Newton constant. This result is considered as a precursor of the AdS/CFT correspondence [23–25], which, applied to three-dimensional general relativity, conjectures the existence of a dual Confomal Field Theory (CFT) living on the two-dimensional boundary. Remarkably, the value of c has been used to microscopically derive the Bekenstein-Hawking entropy of the BTZ black hole, using the Cardy formula [134,135].

Here, along these ideas, we relax BH boundary conditions allowing the boundary metric to be conformally flat, the conformal factor being dynamical and explore the consequences on the asymptotic symmetry algebra and, ultimately, on the dual theory [136]. Thus, the set of boundary conditions we implement in our analysis encompasses all those spacetimes approaching a general boundary conformal structure, known in the literature as *asymptotically locally* AdS_3 spacetimes.

We start in sections 5.1 and 5.2 by fixing the Fefferman-Graham (FG) gauge, introducing conformally flat boundary conditions and, correspondingly, we compute the asymptotic Killing vectors preserving these choices, *i.e.* the asymptotic symmetry group. We show that the latter comprises, besides the usual left and right Witt sectors, a new sector corresponding to Weyl rescalings of the boundary metric. In section 5.3 we solve Einstein's equations in the conformally flat parametrization and extract the action of the asymptotic symmetries on the solution space whereas in section 5.4 we explicitly analyze the properties of the asymptotic symmetry algebra. In section 5.5 we compute the surface charges associated to the solution space and to the asymptotic symmetries generators and we show that the charge algebra is centrally extended in both the Witt and the Weyl sector. In section 5.6, we touch upon some features of the boundary holographic theory. In particular, we show that, under our choice of boundary conditions, the variational problem is not well-defined due to the presence of the Weyl anomaly. Further, we construct the boundary Weyl currents and we show that their non-conservation can be interpreted in terms of an anomalous Ward-Takahashi identity for the boundary Weyl transformations. section 5.7 contains a short summary and some perspectives.

5.1 Fefferman–Graham gauge, residual diffeomorphisms and their algebra

The FG gauge [137,138] in three spacetime dimensions consists in choosing coordinates $x^\mu = (\rho, x^a)$, where $\rho \geq 0$ is a radial coordinate and $x^a = (t, \phi)$. The three gauge-fixing conditions for the metric are

$$g_{\rho\rho} = \frac{\ell^2}{\rho^2}, \qquad g_{\rho a} = 0, \tag{5.1}$$

where $\ell^2 = -1/\Lambda$ is the AdS$_3$ radius. The boundary is located at $\rho = 0$ and the bulk at $\rho > 0$. The line element takes the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\ell^2}{\rho^2}\mathrm{d}\rho^2 + \gamma_{ab}(\rho, x)\mathrm{d}x^a \mathrm{d}x^b. \tag{5.2}$$

Solving Einstein's equations for (5.2) with boundary condition $\gamma_{ab}(\rho, x) = \mathcal{O}(\rho^{-2})$ yields

$$\gamma_{ab}(\rho, x) = \rho^{-2} g^{(0)}_{ab}(x) + g^{(2)}_{ab}(x) + \rho^2 g^{(4)}_{ab}(x), \tag{5.3}$$

with

$$g^{ab}_{(0)} g^{(2)}_{ab} = -\frac{\ell^2}{2} R^{(0)}, \qquad D^a_{(0)} g^{(2)}_{ab} = -\frac{\ell^2}{2} D^{(0)}_b R^{(0)}, \qquad g^{(4)}_{ab} = \frac{1}{4} g^{(2)}_{ac} g^{cd}_{(0)} g^{(2)}_{db}. \tag{5.4}$$

We denote by $R^{(0)}$ and $D^{(0)}_a$ the Ricci scalar and the covariant derivative associated to $g^{(0)}_{ab}$, respectively. The leading term $g^{(0)}_{ab}$ of the expansion (5.3) as $\rho \to 0$ is usually referred to as the *boundary metric*. From now on the indices will be raised and lowered using this metric.

Defining the *holographic stress-energy tensor* as [139, 140]

$$T_{ab} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{ab} + \frac{\ell^2}{2} g^{(0)}_{ab} R^{(0)}\right), \tag{5.5}$$

the first two equations of (5.4) imply

$$T_a{}^a = \frac{c}{24\pi} R^{(0)}, \qquad D^{(0)}_a T^{ab} = 0, \tag{5.6}$$

where $c = \frac{3\ell}{2G}$ is the BH central charge [27]. The first equation in (5.6) states that, for a general $g^{(0)}_{ab}$, the trace of the tensor T_{ab} defined in (5.5) is non-vanishing and proportional to the scalar curvature $R^{(0)}$ of $g^{(0)}_{ab}$, with a proportionality constant determined by the BH central charge. This is a signal that the dual CFT living on the boundary is Weyl anomalous, as we will further comment in section 5.6. The full solution space χ is therefore characterized by five functions, three contained in $g^{(0)}_{ab}$ and two in $g^{(2)}_{ab}$ or, equivalently, in T_{ab}. Furthermore, these last two functions satisfy the dynamical constraints (5.4) or, equivalently, the second equation in (5.6). In the following, we will write $\chi = \{g^{(0)}_{ab}, g^{(2)}_{ab}\}$.

The residual gauge diffeomorphisms are those preserving the FG gauge conditions in (5.1). They are thus generated by the vector field ξ satisfying

$$\mathcal{L}_\xi g_{\rho\rho} = 0, \qquad \mathcal{L}_\xi g_{\rho a} = 0, \qquad \mathcal{L}_\xi \gamma_{ab} = \mathcal{O}(\rho^{-2}). \tag{5.7}$$

The solution of these equations is

$$\xi^\rho = \rho\sigma(x), \qquad \xi^a = Y^a(x) - \ell^2 \partial_b \sigma(x) \int_0^\rho \frac{\mathrm{d}\rho'}{\rho'} \gamma^{ab}(\rho', x). \tag{5.8}$$

In this expression, $\sigma(x)$ and $Y^a(x)$ are arbitrary integration constants. Note that ξ^a depends on the metric field γ^{ab}. Therefore, as in (3.43), we use the modified Lie bracket to study their algebra. On defining

$$\hat{\xi}^\rho = \rho\hat{\sigma}(x), \qquad \hat{\sigma}(x) = Y^a_1(x)\partial_a \sigma_2(x) - Y^a_2(x)\partial_a \sigma_1(x), \tag{5.9}$$

and

$$\hat{\xi}^a = \hat{Y}^a(x) - \ell^2 \partial_b \hat{\sigma}(x) \int_0^\rho \frac{\mathrm{d}\rho'}{\rho'} \gamma^{ab}(\rho', x), \qquad \hat{Y}^a(x) = Y_1^b(x) \partial_b Y_2^a(x) - Y_2^b(x) \partial_b Y_1^a(x), \tag{5.10}$$

it is possible to show that the algebra is closed off-shell:

$$\big[\xi_1, \xi_2\big]_M = \hat{\xi}. \tag{5.11}$$

To prove this we used that $\big[\xi_1, \xi_2\big]_M^a$, *i.e.* the a component of $\big[\xi_1, \xi_2\big]_M = \big[\xi_1, \xi_2\big]_M^\rho \partial_\rho + \big[\xi_1, \xi_2\big]_M^a \partial_a$, satisfies the differential equation $\partial_\rho \big[\xi_1, \xi_2\big]_M^a = -\frac{\ell^2}{\rho^2} \gamma^{ab} \partial_b \big[\xi_1, \xi_2\big]_M^\rho$ with boundary condition $\lim_{\rho\to 0} \big[\xi_1, \xi_2\big]_M^a = \hat{Y}^a$. On-shell, the residual diffeomorphism generator (5.8) admits the following expansion in powers of ρ,

$$\xi^a = Y^a - \frac{\rho^2}{2} \ell^2 g_{(0)}^{ab} \partial_b \sigma + \frac{\rho^4}{4} \ell^2 g_{(0)}^{ac} g_{cd}^{(2)} g_{(0)}^{db} \partial_b \sigma + \mathcal{O}(\rho^6). \tag{5.12}$$

Acting with the Lie derivative along ξ on the on-shell line element (5.2) we find the general variation of solution space

$$(\mathcal{L}_\xi g_{\mu\nu}) \, \mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\ell^2}{\rho^2} \mathrm{d}\rho^2 + \left(\rho^{-2} \delta_\xi g_{ab}^{(0)} + \delta_\xi g_{ab}^{(2)} + \rho^2 \delta_\xi g_{ab}^{(4)} \right) \mathrm{d}x^a \mathrm{d}x^b, \tag{5.13}$$

with

$$\delta_\xi g_{ab}^{(0)} = \mathcal{L}_Y g_{ab}^{(0)} - 2\sigma g_{ab}^{(0)}, \qquad \delta_\xi g_{ab}^{(2)} = \mathcal{L}_Y g_{ab}^{(2)} - \ell^2 D_a^{(0)} D_b^{(0)} \sigma. \tag{5.14}$$

The first equation in (5.14) is telling us that a general variation of the boundary metric under the action of residual gauge diffeomorphisms has two independent contributions, one coming from σ and the other from Y^a.

5.2 Boundary conditions and boundary gauge fixing

As stressed above, once the boundary data $g_{ab}^{(0)}$ is assigned, the full solution space, comprising also the two functions in $g_{ab}^{(2)}$, is completely determined. That is, for every arbitrary choice of the boundary metric, solving (5.4) yields a complete solution of Einstein's equations. Before solving them explicitly, we now proceed to impose boundary conditions, motivated by the Penrose conformal compactification described in section 3.1.

Notice from (5.3) that the boundary data for the full metric g are located at infinite distance, due to the second order pole. Multiplying g by Ω^2, with Ω a positive function with a simple zero on the boundary, such a pole is eliminated and an induced metric on the boundary may be defined. There is however an ambiguity in the choice of Ω, as remarked in 8. The replacement $\Omega \to \Omega' = e^\omega \Omega$, with ω a smooth function independent of the radial coordinate, induces a conformal transformation $g^{(0)} \to e^{2\omega} g^{(0)}$ of the boundary metric. This freedom allows one to define only an equivalence class of conformally related boundary metrics, $\big[g^{(0)}\big]$, rather than a metric [137–139, 141–143]. In BH, a particular representative of the equivalence class is picked up, namely the flat Minkowski metric η, and kept fixed under the action of the asymptotic symmetry algebra. This set of Dirichlet boundary conditions defines the so-called asymptotically (globally) AdS_3 spacetimes ($AAdS_3$). Here we focus on asymptotically locally AdS_3 ($AlAdS_3$) spacetimes [144–147], with no restriction on their boundary conformal structure and thus we assume the boundary metric to be conformally flat, the conformal factor being an arbitrary smooth function independent of the radial coordinate.[24] Hence, we impose the more general condition

$$g_{ab}^{(0)}(x) = e^{2\varphi(x)}\eta_{ab}. \tag{5.15}$$

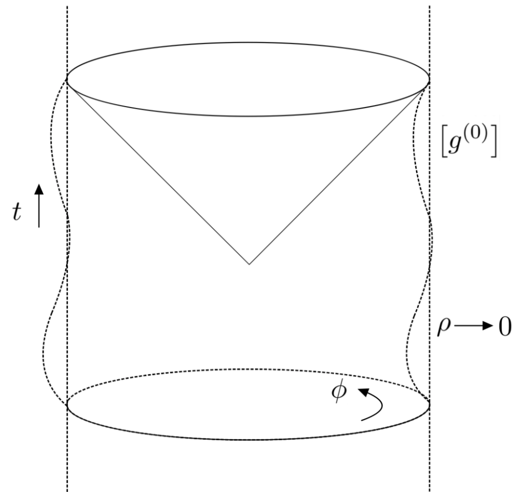

Figure 10: The boundary metric $g^{(0)}$ of asymptotically locally AdS_3 is not rigid, but left free to fluctuate through its conformal factor. Here (ρ, t, ϕ) are FG coordinates.

Notice that every two-dimensional metric is conformally flat. That is, we can always use boundary diffeomorphisms to fix two components of the boundary metric in order to reach (5.15). This will constrain the form of the vector fields Y^a appearing in (5.8). Therefore (5.15) is a natural case to investigate. Note that an arbitrary variation of the boundary metric now reduces to an arbitrary variation of its conformal factor, *i.e.* $\delta g_{ab}^{(0)} = 2(\delta\varphi)g_{ab}^{(0)}$.

Equation (5.14) becomes then

$$\delta_\xi g_{ab}^{(0)} = \mathcal{L}_Y g_{ab}^{(0)} - 2\sigma g_{ab}^{(0)} = 2(\delta_\xi\varphi)g_{ab}^{(0)}. \tag{5.16}$$

This implies that Y^a is a conformal Killing vector of $g_{ab}^{(0)}$

$$\mathcal{L}_Y g_{ab}^{(0)} = D_a^{(0)}Y_b + D_b^{(0)}Y_a = 2\Omega_Y g_{ab}^{(0)}, \qquad \Omega_Y = \frac{1}{2}D_a^{(0)}Y^a. \tag{5.17}$$

where $\Omega_Y = \delta_\xi\varphi - \sigma$. Thence

$$\delta_\xi g_{ab}^{(0)} = 2(\Omega_Y - \sigma)g_{ab}^{(0)}. \tag{5.18}$$

Introducing light-cone coordinates $x^\pm = t/\ell \pm \phi$ we have $g_{ab}^{(0)}\mathrm{d}x^a\mathrm{d}x^b = -e^{2\varphi(x)}\mathrm{d}x^+\mathrm{d}x^-$ and (5.17) is solved by the usual chiral vectors

$$Y^+ = Y^+(x^+), \qquad Y^- = Y^-(x^-), \qquad \Omega_Y = \frac{1}{2}\left(\partial_- Y^- + \partial_+ Y^+\right) + Y^+\partial_+\varphi + Y^-\partial_-\varphi. \tag{5.19}$$

[24]The case in which the conformal factor admits a chiral splitting has been extensively analyzed in previous works [148,149].

Consistently, the effect of residual gauge symmetries on the boundary metric is to induce a shift in its conformal factor, given in (5.18) by $\delta_\xi\varphi = \Omega_Y - \sigma$. Often, in the literature, the full set of transformations induced by ξ is referred to as Penrose-Brown-Henneaux (PBH) [150] diffeomorphisms.

The standard BH boundary conditions [27] $\delta_\xi\varphi = 0$ are a subclass of our boundary conditions obtained by fixing $\sigma = \Omega_Y$. With this choice the effect of the conformal isometries generated by Y^a exactly compensates the effect of the Weyl rescalings generated by σ, as clear from the first of (5.14). Furthermore, also the boundary conditions studied in [148] are encompassed in our analysis, as we show in Appendix B.

5.3 Solution space

In the conformally flat parametrization it is possible to explicitly solve Einstein's equations for $g^{(2)}_{ab}$, given by the first two equations in (5.4), see *e.g.* [5]. The first is an algebraic equation for $g^{(2)}_{+-}$ and it yields

$$g^{(2)}_{+-} = \ell^2 \partial_+\partial_-\varphi, \tag{5.20}$$

where we used that, for $g^{(0)}$ in (5.15), $R^{(0)} = 8e^{-2\varphi}\partial_+\partial_-\varphi$. The second implies

$$\partial_\mp g^{(2)}_{\pm\pm} = -\ell^2\left(2\partial_\pm\varphi\partial_\pm\partial_\mp\varphi - \partial^2_\pm\partial_\mp\varphi\right), \tag{5.21}$$

whose solutions are

$$g^{(2)}_{\pm\pm} = \ell^2\left[\Xi_{\pm\pm}(x^\pm) + \partial^2_\pm\varphi - (\partial_\pm\varphi)^2\right], \tag{5.22}$$

where $\Xi_{\pm\pm}(x^\pm)$ are two arbitrary functions of $x^\pm$. The holographic stress-energy tensor (5.5) reads

$$T_{+-} = -\frac{\ell}{8\pi G}\partial_+\partial_-\varphi, \qquad T_{\pm\pm} = \frac{\ell}{8\pi G}\left[\Xi_{\pm\pm}(x^\pm) + \partial^2_\pm\varphi - (\partial_\pm\varphi)^2\right]. \tag{5.23}$$

While the most general solution space described in 5.1 is characterized by five independent functions of x^+ and x^-, the solution space in the conformally flat gauge of (5.33) is given by $\varphi(x^+, x^-)$ and the two chiral functions $\Xi_{\pm\pm}(x^\pm)$. Thus, the solution space we are interested in is $\chi = \{\Xi_{++}(x^+), \Xi_{--}(x^-), \varphi(x^+, x^-)\}$. Note that the presence of an arbitrary φ prevents a complete chiral splitting of the solution space and that, equivalently, the holographic stress-energy tensor components $T_{\pm\pm}$ in (5.23) are not chiral nor anti-chiral. This is one of the main differences with respect to [148].

A generic variation of the solution space is generated by σ and $Y^\pm$, so we symbolically write $\delta_\xi\chi = \delta_{(\sigma,Y^\pm)}\chi$. Using (5.14) and (5.17) we compute

$$\delta_{(\sigma,0)}\varphi = -\sigma, \qquad \delta_{(\sigma,0)}\Xi_{\pm\pm} = 0, \tag{5.24}$$

and

$$\delta_{(0,Y^\pm)}\varphi = \partial_- Y^- + \partial_+ Y^+ + 2(Y^+\partial_+\varphi + Y^-\partial_-\varphi), \tag{5.25}$$

$$\delta_{(0,Y^\pm)}\Xi_{\pm\pm} = Y^\pm\partial_\pm\Xi_{\pm\pm} + 2\Xi_{\pm\pm}\partial_\pm Y^\pm - \frac{1}{2}\partial^3_\pm Y^\pm. \tag{5.26}$$

Before proceeding to compute the asymptotic symmetry algebra, it is convenient to trade σ for a parameter ω defined as

$$\omega = \Omega_Y - \sigma. \tag{5.27}$$

Note that ω depends on the derivatives of φ and therefore it is field-dependent, contrarily to σ. Using ω, (5.24)-(5.25) can be more compactly written as

$$\delta_{(\omega,Y^{\pm})}\varphi = \omega, \qquad \delta_{(\omega,Y^{\pm})}\Xi_{\pm\pm} = Y^{\pm}\partial_{\pm}\Xi_{\pm\pm} + 2\Xi_{\pm\pm}\partial_{\pm}Y^{\pm} - \frac{1}{2}\partial_{\pm}^{3}Y^{\pm}. \tag{5.28}$$

The conformal factor φ transforms only under the action ω while $\Xi_{\pm\pm}$ transform as the components of an anomalous two-dimensional CFT stress-energy tensor under the action of Y^a [151]. Thanks to the redefinition of the residual diffeomorphisms generators (5.27) we have isolated the total part of the asymptotic symmetries that induces a Weyl rescaling of the boundary metric. Hence, From now on we will refer to the sector of the asymptotic symmetries generated by ω as the *Weyl sector*. Note that another more straightforward way to introduce ω is to require that the residual vector fields of (5.8) asymptotically induce a Weyl rescaling of the boundary metric, *i.e.*

$$\mathcal{L}_{\xi}\gamma_{ab} = 2\omega\rho^{-2}g^{(0)}_{ab} + O(\rho^0). \tag{5.29}$$

This equation leads to

$$D^{(0)}_a Y_b + D^{(0)}_b Y_b = 2(\omega+\sigma)g^{(0)}_{ab}, \tag{5.30}$$

which implies (5.27).

Note that from the definition (5.5) of T_{ab} and from (5.14) it follows that, under a residual Weyl transformation, T_{ab} transforms as

$$\delta_{(\omega,0)}T_{ab} = \frac{c}{12\pi}(D^{(0)}_a D^{(0)}_b\omega - g^{(0)}_{ab}\square^{(0)}\omega). \tag{5.31}$$

Hence, if we required that the vector field generating Weyl transformations satisfied

$$\delta_{(\omega,0)}T_a{}^a = -2\omega T_a{}^a - \frac{c}{12\pi}\square^{(0)}\omega \equiv -2\omega T_a{}^a, \tag{5.32}$$

then the trace of T_{ab}, or equivalently $R^{(0)}$, would transform as a Weyl scalar of weight -2. This condition automatically implies that ω is an harmonic function [25]

$$\square\omega = 0, \tag{5.33}$$

whose general solution is

$$\omega = \omega^+(x^+) + \omega^-(x^-). \tag{5.34}$$

In the following, we will refer to this peculiar situation as the ω-*chiral case*. Note that requiring the gauge parameter ω to satisfy (5.34) implies that φ can vary under the action of the asymptotic symmetry group only as

$$\delta_{(\omega,Y^{\pm})}\varphi = \omega^+(x^+) + \omega^-(x^-). \tag{5.35}$$

Therefore, under (5.33), even if the solution space does not admit a chiral splitting, its variation $\delta_\xi\chi$ can be decomposed into two sectors with definite chiralities, $\delta_{\xi^{\pm}}\chi = \{\delta_{(\omega^{\pm},Y^{\pm})}\Xi_{\pm\pm}, \delta_{(\omega^{\pm},Y^{\pm})}\varphi\}$.

[25] Note that $\square^{(0)} = e^{-2\varphi}\square$, where $\square = \partial_a\partial^a$.

5.4 Asymptotic symmetry algebra

The on-shell residual diffeomorphisms generator in light-cone coordinates is

$$\xi^\rho = \rho\sigma(x), \qquad \xi^\pm = Y^\pm(x^\pm) + \rho^2\ell^2 e^{-2\varphi}\partial_\mp\sigma + \rho^4\ell^2 e^{-4\varphi}\left[g^{(2)}_{+-}\partial_\mp\sigma + g^{(2)}_{\pm\pm}\partial_\pm\sigma\right] + \mathcal{O}(\rho^6), \tag{5.36}$$

whereas the algebra is

$$\left[\xi_1,\xi_2\right]^\rho_M = \hat{\xi}^\rho = \rho\hat{\sigma}, \qquad \hat{\sigma} = Y_1^+\partial_+\sigma_2 + Y_1^-\partial_-\sigma_2 - (2\leftrightarrow 1), \tag{5.37}$$

$$\left[\xi_1,\xi_2\right]^\pm_M = \hat{\xi}^\pm = \hat{Y}^\pm + \rho^2\ell^2 e^{-2\varphi}\partial_\mp\hat{\sigma} + \mathcal{O}(\rho^4), \qquad \hat{Y}^\pm = Y_1^\pm\partial_\pm Y_2^\pm - (2\leftrightarrow 1). \tag{5.38}$$

This algebra is a semidirect sum: by denoting an element of the algebra as the pair $(\sigma, Y^\pm)$, the modified Lie bracket between two elements is $\left[(\sigma_1, Y_1^\pm),(\sigma_2, Y_2^\pm)\right]_M = (\hat{\sigma},\hat{Y}^\pm)$, where $\hat{\sigma}$ and $\hat{Y}^\pm$ are given in (5.37) and (5.38).

We now reformulate the algebra in terms of the parameter ω introduced in (5.27). The on-shell generator is

$$\xi^\rho = \rho\left(\Omega_Y - \omega\right), \tag{5.39}$$

$$\xi^\pm = Y^\pm + \rho^2\ell^2 e^{-2\varphi}\partial_\mp\left(\Omega_Y - \omega\right) + \rho^4\ell^2 e^{-4\varphi}\left[g^{(2)}_{+-}\partial_\mp\left(\Omega_Y - \omega\right) + g^{(2)}_{\pm\pm}\partial_\pm\left(\Omega_Y - \omega\right)\right] \tag{5.40}$$

$$+ \mathcal{O}(\rho^6). \tag{5.41}$$

Notice that, since ω is field-dependent, this reformulation introduces a field dependence in ξ^ρ, which was previously absent. This implies that we need to use the modified Lie bracket also for this component. We now obtain

$$\left[\xi_1,\xi_2\right]^\rho_M = \hat{\xi}^\rho = \rho\left(\Omega_{\hat{Y}} - \hat{\omega}\right), \qquad \hat{Y}^\pm = Y_1^\pm\partial_\pm Y_2^\pm - Y_2^\pm\partial_\pm Y_1^\pm, \qquad \hat{\omega} = 0, \tag{5.42}$$

and, as before,

$$\partial_\rho\left(\left[\xi_1,\xi_2\right]^\pm_M\right) = -\frac{\ell^2}{\rho^2}g^{ab}\partial_b\left(\left[\xi_1,\xi_2\right]^\rho_M\right), \qquad \lim_{\rho\to 0}\left(\left[\xi_1,\xi_2\right]^\pm_M\right) = \hat{Y}^\pm. \tag{5.43}$$

Integrating these equations leads to

$$\left[\xi_1,\xi_2\right]^\pm_M = \hat{\xi}^\pm = \hat{Y}^\pm + \rho^2\ell^2 e^{-2\varphi}\partial_\mp\left(\Omega_{\hat{Y}} - \hat{\omega}\right) + \rho^4\ell^2 e^{-4\varphi}\left[g^{(2)}_{+-}\partial_\mp\left(\Omega_{\hat{Y}} - \hat{\omega}\right) + g^{(2)}_{\pm\pm}\partial_\pm\left(\Omega_{\hat{Y}} - \hat{\omega}\right)\right] + \mathcal{O}(\rho^6), \tag{5.44}$$

where $\hat{Y}^\pm$ and $\hat{\omega}$ are defined in (5.42). With this set of independent generators, the asymptotic symmetry algebra is thus a direct sum of two copies of the Witt algebra with the abelian ideal of Weyl rescalings. Denoting an element of the algebra as the pair $(\omega, Y^\pm)$, the modified Lie bracket between two elements is $\left[(\omega_1, Y_1^\pm),(\omega_2, Y_2^\pm)\right]_M = (0,\hat{Y}^\pm)$. From now on we will work in the ω-parametrization, for it allows to disentangle the asymptotic symmetry algebra.

In order to describe the asymptotic symmetry algebra in a basis, we first introduce the decomposition $\xi = \zeta_\omega + \xi_{Y^+} + \xi_{Y^-}$ where $\zeta_\omega = \xi|_{Y^\pm=0}$, $\xi_{Y^+} = \xi|_{\omega=0=Y^-}$ and $\xi_{Y^-} = \xi|_{\omega=0=Y^+}$. From the mode expansions $Y^\pm_n = e^{inx^\pm}$ we have, for $\hat{Y}^\pm$ in (5.42)

$$\hat{Y}^\pm = Y^\pm_n \partial_\pm Y^\pm_m - (n \leftrightarrow m) = (n-m) e^{i(n+m)x^\pm}, \tag{5.45}$$

and thus we gather

$$\left[\xi^\pm_n, \xi^\pm_m\right]_M = i(n-m)\xi^\pm_{n+m}, \qquad \left[\xi^\pm_n, \xi^\mp_m\right]_M = 0, \tag{5.46}$$

where we replaced the $Y^\pm$ subscript by the mode number $\xi_{Y^\pm_n} \equiv \xi^\pm_n$. We thus have two copies of the Witt algebra, which is expected since for $\omega = 0$ we reach BH boundary conditions, where this algebra has already been derived [27].

Expanding $\omega_{pq} = e^{ipx^+} e^{iqx^-}$ [26] the sectors of the algebra involving Weyl rescalings read

$$\left[\zeta_{pq}, \zeta_{rs}\right]_M = 0, \qquad \left[\xi^\pm_n, \zeta_{rs}\right]_M = 0. \tag{5.47}$$

where we defined $\zeta_{\omega_{pq}} \equiv \zeta_{pq}$.

In the particular subclass of ω satisfying (5.34), *i.e.* the ω-chiral case, we can consider the algebra of left and right Weyl sectors separately. Expanding $\omega^\pm_p = e^{ipx^\pm}$ we denote $\zeta_{\omega^\pm_p} \equiv \zeta^\pm_p$. The sectors of the algebra involving Weyl rescalings now read

$$\left[\zeta^\pm_p, \zeta^\pm_q\right]_M = 0, \qquad \left[\zeta^\pm_p, \zeta^\mp_q\right]_M = 0, \qquad \left[\xi^\pm_n, \zeta^\pm_p\right]_M = 0, \qquad \left[\xi^\pm_n, \zeta^\mp_p\right]_M = 0. \tag{5.48}$$

5.5 Surface charges and their algebra

We now proceed to study asymptotic surface charges under the boundary conditions spelled above, using the Barnich-Brandt prescription in (2.90),

$$\delta\!\!\!/ Q_\xi[h,g] = \int_S \mathbf{k}'_\xi[g,h] = \frac{1}{16\pi G}\int_0^{2\pi} \mathrm{d}\phi\, k^{\rho t}_\xi[g,h], \tag{5.49}$$

where

$$k^{\mu\nu}_\xi[g,h] = \sqrt{-g}\Big[\xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\nu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu + \frac{1}{2} h^{\nu\sigma}(\nabla^\mu \xi_\sigma - \nabla_\sigma \xi^\mu) - (\mu \leftrightarrow \nu)\Big]. \tag{5.50}$$

Here $h_{\mu\nu} = \delta g_{\mu\nu}$ are the on-shell variations of the metric, S in (5.49) is on the circle at infinity spanned by ϕ at a fixed time t and hence $k^{\rho t}_\xi[g,h]$ is evaluated on the boundary, *i.e.* at $\rho = 0$. The charges associated to $\xi_Y = \xi_{Y^+} + \xi_{Y^-}$ are integrable and found to be

$$Q_{\xi_Y}[g] = \frac{\ell}{8\pi G}\int_0^{2\pi} \mathrm{d}\phi \left(Y^- \Xi_{--} + Y^+ \Xi_{++}\right). \tag{5.51}$$

[26]Note that if ω is arbitrary we cannot decompose it into chiral and anti-chiral parts and therefore we need to use the most general decomposition.

These are the usual conserved charges associated to conformal transformations, found also with BH boundary conditions. The $Q_{\xi_Y}[g]$ in (5.51) are computed with respect to the background metric $\bar{g}$ defined by $\Xi_{\pm\pm} = 0$, which is the BTZ black hole with vanishing mass and angular momentum. The ones computed with respect to the global AdS_3 background can be obtained by shifting $\Xi_{\pm\pm} \to \Xi_{\pm\pm} + \frac{1}{4}$ in (5.51) [152].

For the Weyl sector we find integrable charges given by

$$Q_{\zeta_\omega}[g] = \frac{\ell^2}{8\pi G}\int_0^{2\pi} \mathrm{d}\phi(\varphi\partial_t\omega - \omega\partial_t\varphi). \tag{5.52}$$

These additional interesting charges are finite, integrable but not conserved. The non-conservation is accounted for by the presence of a non-vanishing symplectic flux through the boundary, as we will emphasize in section 5.6.

It can be shown that the same set of charges in (5.51) and (5.52) can be obtained using the Iyer-Wald prescription in (2.85). While the charges in (5.52) are the most general Weyl charges in our setup, we now restrict attention to the ω-chiral case (5.34), *i.e.* $\omega = \omega^+ + \omega^-$. Correspondingly, the Weyl charges decompose as

$$Q_{\zeta_\omega}[g] = -\frac{\ell}{4\pi G}\int_0^{2\pi} \mathrm{d}\phi\left(\omega^+\partial_+\varphi + \omega^-\partial_-\varphi\right) \equiv Q_{\zeta_{\omega^+}}[\varphi] + Q_{\zeta_{\omega^-}}[\varphi], \tag{5.53}$$

where we have integrated by parts. Hence, they split into two pieces, generating the chiral and anti-chiral transformations of φ. We now proceed to compute the charge algebra.

Consistently with (2.64), it can be shown that the surface charges, under the Poisson bracket, form a projective representation of the asymptotic symmetry algebra with modified Lie bracket, *i.e.*

$$\{Q_{\xi_1}[g], Q_{\xi_2}[g]\} = \delta_{\xi_2}Q_{\xi_1}[g] \approx Q_{[\xi_1,\xi_2]_M}[g] + \mathcal{K}_{\xi_1,\xi_2}[g], \tag{5.54}$$

where $\mathcal{K}_{\xi_1,\xi_2}$ is the central charge. We find, for $\xi_i = \zeta_{\omega_i} + \xi_{Y_i}$,

$$\mathcal{K}_{\xi_1,\xi_2} = \mathcal{K}_{\xi_{Y_1},\xi_{Y_2}} + \mathcal{K}_{\zeta_{\omega_1},\zeta_{\omega_2}}, \tag{5.55}$$

with

$$\mathcal{K}_{\xi_{Y_1},\xi_{Y_2}} = \frac{1}{8\pi G}\int_0^{2\pi} \mathrm{d}\phi\big(\partial_\phi Y_1^t\partial_\phi^2 Y_2^\phi - \partial_\phi Y_2^t\partial_\phi^2 Y_1^\phi\big), \tag{5.56}$$

and

$$\mathcal{K}_{\zeta_{\omega_1},\zeta_{\omega_2}} = \frac{\ell^2}{8\pi G}\int_0^{2\pi} \mathrm{d}\phi\big(\omega_2\partial_t\omega_1 - \omega_1\partial_t\omega_2\big). \tag{5.57}$$

It can be easily shown that $\mathcal{K}_{\xi_1,\xi_2}$ satisfies the cocycle condition (2.65). Indeed, it is automatically satisfied for the Weyl sector and the mixed sector, while in the Witt sectors it is proved as usual. Furthermore, since the Virasoro central charge is non-trivial and any 2-cocycles of an Abelian algebra cannot be a coboundary, (5.57) is fully non-trivial. The central charge (5.56) in the Y sector is the well-known BH result [27], while (5.57) is a new result. The total central charge in (5.55) has therefore two independent contributions, one coming from the ordinary BH central charge and an additional one coming from the sector of Weyl rescalings of the boundary metric. Evaluating (5.56) on the basis $Y^\pm_n = e^{inx^\pm}$ we get

$$\mathcal{K}_{Y^\pm_n, Y^\pm_m} = -im^3 \frac{c^\pm}{12}\delta_{n+m,0}, \qquad \mathcal{K}_{Y^\pm_n, Y^\mp_m} = 0, \qquad c^\pm = c = \frac{3\ell}{2G}. \tag{5.58}$$

On the other hand, evaluating (5.57) on the basis $\omega_{pq} = e^{ipx^+}e^{iqx^-}$ for the modes decomposition of the Weyl sector yields

$$\mathcal{K}_{\zeta_{pq},\zeta_{rs}} = -i(r-q)c_W \omega_{q+s,q+s}\delta_{p+r,q+s}, \qquad c_W = \frac{\ell}{2G}. \tag{5.59}$$

The total charge algebra then reads

$$\big\{Q_{\xi^\pm_n}[g], Q_{\xi^\pm_m}[g]\big\} = i(n-m)Q_{\xi^\pm_{n+m}}[g] - im^3\frac{c}{12}\delta_{n+m,0}, \tag{5.60}$$
$$\big\{Q_{\xi^\pm_n}[g], Q_{\xi^\mp_m}[g]\big\} = 0, \tag{5.61}$$
$$\big\{Q_{\zeta_{pq}}[g], Q_{\zeta_{rs}}[g]\big\} = -i(r-q)c_W e^{2i(q+s)\frac{t}{\ell}}\delta_{p+r,q+s}, \tag{5.62}$$
$$\big\{Q_{\xi^\pm_n}[g], Q_{\zeta_{pq}}[g]\big\} = 0. \tag{5.63}$$

This algebra is the direct sum of two Virasoro sectors and the centrally extended Weyl sector. We note that the Weyl central charge is explicitly time dependent. As such, we are dealing here with a one-parameter family of algebras, labelled by the time slice t at which the charges are computed.

In the ω-chiral case, the central charge for the Weyl left- and right-movers simplifies to

$$\mathcal{K}_{\zeta^\pm_p,\zeta^\pm_q} = ipc^\pm_W\delta_{p+q,0}, \qquad \mathcal{K}_{\zeta^\pm_p,\zeta^\mp_q} = 0, \qquad c^\pm_W = c_W = \frac{\ell}{2G}. \tag{5.64}$$

The charge algebra then reads

$$\big\{Q_{\xi^\pm_n}[g], Q_{\xi^\pm_m}[g]\big\} = i(n-m)Q_{\xi^\pm_{n+m}}[g] - im^3\frac{c}{12}\delta_{n+m,0}, \tag{5.65}$$
$$\big\{Q_{\xi^\pm_n}[g], Q_{\xi^\mp_m}[g]\big\} = 0, \tag{5.66}$$
$$\big\{Q_{\zeta^\pm_p}[g], Q_{\zeta^\pm_q}[g]\big\} = ipc_W\delta_{p+q,0}, \tag{5.67}$$
$$\big\{Q_{\zeta^\pm_p}[g], Q_{\zeta^\mp_q}[g]\big\} = 0, \tag{5.68}$$
$$\big\{Q_{\xi^\pm_n}[g], Q_{\zeta^\pm_p}[g]\big\} = 0, \tag{5.69}$$
$$\big\{Q_{\xi^\pm_n}[g], Q_{\zeta^\mp_p}[g]\big\} = 0. \tag{5.70}$$

In this subcase the Weyl central charge does not depend on time and therefore the one-parameter family of algebras reduces to a Kac-Moody current algebra. The algebra (5.65)-(5.70), up to redefinition of generators, is the same as the one found in [148], as reviewed in Appendix B.

5.6 Holographic aspects

Thanks to the AdS/CFT dictionary, we know that the bulk gravity theory is dual to a boundary conformal field theory. As long as the former is in the classical limit, the latter is strongly coupled. Therefore, little is known about it: we cannot construct its perturbative action but we still have access to non-perturbative features such as quantum symmetries expressed in terms of Ward-Takahashi identities of the path integral [153, 154]. The goal of this section is to show that there is a breaking (see *e.g.* the discussion in section 2.3) in the conservation law of the Weyl current, which has a holographic dual counterpart as a boundary anomalous Ward-Takahashi identity [155]. Before proceeding let us briefly review the emergence of the Weyl anomaly in the context of holographic renormalization, pioneered by Skenderis and collaborators [139, 146, 156–164].

The renormalized action for GR in asymptotically locally AdS_3 spacetimes is defined as $S[g] = \lim_{\epsilon\to 0} S_\epsilon[g]$ where $S_\epsilon[g]$ is the regularized action, given by

$$S_\epsilon[g] = \frac{1}{16\pi G}\int_{\mathcal{M}_\epsilon} \mathrm{d}^3x\sqrt{-g}\left(R - \frac{2}{\ell^2}\right) + \frac{1}{16\pi G}\int_{\partial\mathcal{M}_\epsilon} \mathrm{d}^2x\sqrt{-\gamma}\left(2K - \frac{2}{\ell} + \frac{\ell}{4}R^{(0)}\log\epsilon\right), \tag{5.71}$$

where K is the trace of the extrinsic curvature of the constant ρ hypersurface and the last two terms are the standards counterterms. The renormalized action $S[g]$ is therefore defined by first introducing a cut-off at $\rho = \epsilon$ that allows the divergences to cancel and then by taking the limit $\epsilon \to 0$. An on-shell variation of $S[g]$ yields[27]

$$\delta S[g] = \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{-g^{(0)}}T^{ab}\delta g^{(0)}_{ab} = \frac{c}{24\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{-g^{(0)}}R^{(0)}\delta\varphi, \tag{5.72}$$

where in the last step we have explicitly used the conformally flat parametrization. Hence, with our choice of boundary conditions, the variational problem is not well-defined [146, 163]. Specifying δ to be the variation (5.28) induced by a Weyl diffeomorphism so that $\delta_\omega g^{(0)}_{ab} = 2\omega g^{(0)}_{ab}$, we get

$$\delta_\omega S[g] = \frac{c}{24\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{-g^{(0)}}R^{(0)}\omega \equiv \int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{-g^{(0)}}\mathcal{A}\,\omega, \qquad \mathcal{A} = \frac{c}{24\pi}R^{(0)}, \tag{5.73}$$

which is the standard expression for the Weyl anomaly in AlAdS_3 spacetimes.[28]. Note that we define $\mathcal{A}$ to be the integrand coefficient of $\omega(x)$ in $\delta_\omega S[g]$ [139]. In other words, regularizing the theory implies a specific choice of radial foliation and therefore it explicitly breaks Weyl invariance, causing the emergence of a Weyl anomaly. The latter can be seen in the on-shell variational principle of the renormalized bulk action. When specified to a variation of the conformal factor of the boundary metric, the corresponding variation of the on-shell renormalized action gives the Weyl anomaly, which is then interpreted as the trace anomaly of the boundary stress tensor [140, 173, 174]. Typically, in order to achieve a well-defined variational problem, Dirichlet boundary conditions are imposed on the metric [146, 163, 175]. However, such a condition is too restrictive when working with a conformal class of boundary metrics [146]. Therefore we cannot insist that the variational problem be well defined.

From (5.72) we get, in terms of the wedge product $\wedge$ between field variations, the symplectic structure induced on the boundary $\partial\mathcal{M}$ [38, 97, 176]

$$\boldsymbol{\Omega}_{\partial\mathcal{M}}[g, h_1, h_2] = \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\delta\left(\sqrt{-g^{(0)}}T^{ab}\right)\wedge\delta g^{(0)}_{ab} = -\frac{1}{8\pi G}\int_0^{2\pi}\mathrm{d}\phi\int_{t_1}^{t_2}\mathrm{d}t\,(\Box\delta\varphi\wedge\delta\varphi)\,, \tag{5.74}$$

[27] For the Chern-Simons formulation of the variational problem, see Appendix C.

[28] For intrinsic field-theoretical studies of Weyl anomalies see [165–172].

where we have used the conformally flat parametrization. Integrating by parts in t we have, discarding total ϕ derivatives, that

$$\mathbf{\Omega}_{\partial\mathcal{M}}[\varphi,\delta_1\varphi,\delta_2\varphi] = -\frac{\ell^2}{8\pi G}\int_0^{2\pi} \mathrm{d}\phi\,[\delta\varphi\wedge\partial_t\delta\varphi]_{t_1}^{t_2}\,. \tag{5.75}$$

This shows that the time difference of Weyl charges is equal to the symplectic flux contracted with a Weyl generating vector field,

$$\mathbf{\Omega}_{\partial\mathcal{M}}[\varphi,\delta\varphi,\delta_\omega\varphi] = \delta\left\{\frac{\ell^2}{8\pi G}\int_0^{2\pi}\mathrm{d}\phi\,[\omega\partial_t\varphi-\varphi\partial_t\omega]_{t_1}^{t_2}\right\} = \delta\big\{Q_{\zeta_\omega}[g]|_{t_1} - Q_{\zeta_\omega}[g]|_{t_2}\big\}, \tag{5.76}$$

so that, integrating on a path γ in the space of fields, we get

$$\Delta_t Q_{\zeta_\omega}[g] = \int_\gamma \mathbf{\Omega}_{\partial\mathcal{M}}[\varphi,\delta\varphi,\delta_\omega\varphi]. \tag{5.77}$$

Therefore the Weyl charges are not conserved but integrable, as already mentioned and their non conservation is compensated by a symplectic flux through the boundary [41–43, 177–179].

We proceed now to reduce the theory to the ω-chiral case and comment on some holographic aspects in this framework. The presence of an anomaly indicates that, in the dual theory, a current is not conserved at the quantum level [29]. Therefore, the first step is to construct a Weyl current [182]. This procedure is well-known for the Virasoro sector, where the currents combine in the stress tensor of the boundary dual theory, and its conservation is interpreted in the bulk as Einstein's equations, while in the boundary as the Ward-Takahashi identity for the transformations generated by $\xi^a_{(0,Y^\pm)}$. In a similar fashion, given that the condition $\Box\omega = 0$ ensures we deal with two generators (5.53) of chiral and anti-chiral Weyl transformations, we can define two Weyl currents. Starting from (5.50), since the charges are integrable we can define a functional $K_{\zeta_\omega}[g]$ such that $k_{\zeta_\omega}[g,h] = \delta(K_{\zeta_\omega}[g])$. For $K_{\zeta_\omega}[g]$ we obtain the splitting

$$K^{\rho a}_{\zeta_\omega}[g] = K^{\rho a}_{\zeta_{\omega^+}}[g] + K^{\rho a}_{\zeta_{\omega^-}}[g]. \tag{5.78}$$

We now use the ambiguity (2.29) in defining $K^{\mu\nu}_\xi[g]$,

$$K'^{\rho a}_{\zeta_{\omega^+}}[g] = K^{\rho a}_{\zeta_{\omega^+}}[g] + \partial_b I^{[ba]}_{\zeta_{\omega^+}}[g], \qquad K'^{\rho a}_{\zeta_{\omega^-}}[g] = K^{\rho a}_{\zeta_{\omega^-}}[g] + \partial_b I^{[ba]}_{\zeta_{\omega^-}}[g]. \tag{5.79}$$

Choosing $I^\pm_{\zeta_{\omega^\pm}}$ as

$$I^{+-}_{\zeta_{\omega^+}}[\varphi] = -\frac{\ell}{8\pi G}\varphi\omega^+, \qquad I^{+-}_{\zeta_{\omega^-}}[\varphi] = \frac{\ell}{8\pi G}\varphi\omega^-, \tag{5.80}$$

we obtain for $K'^{\rho a}_\xi[\varphi]$

$$K'^{\rho+}_{\zeta_{\omega^+}}[\varphi] = 0, \qquad K'^{\rho-}_{\zeta_{\omega^+}}[\varphi] = -\frac{\ell}{4\pi G}\omega^+\partial_+\varphi, \qquad K'^{\rho+}_{\zeta_{\omega^-}}[\varphi] = -\frac{\ell}{4\pi G}\omega^-\partial_-\varphi, \qquad K'^{\rho-}_{\zeta_{\omega^-}}[\varphi] = 0. \tag{5.81}$$

[29] see *e.g.* [155, 180, 181] for reviews

These are precisely the integrands of the Weyl charges in (5.53). Introducing two Weyl currents $J^a_{\omega^+}[\varphi]$ and $J_{\omega^-}[\varphi]$ for the two chirality sectors as

$$K'^{\rho a}_{\zeta_{\omega^+}}[\varphi] = \sqrt{-g^{(0)}}\omega^+ J^a_{\omega^+}[\varphi], \qquad K'^{\rho a}_{\zeta_{\omega^-}}[\varphi] = \sqrt{-g^{(0)}}\omega^- J^a_{\omega^-}[\varphi], \tag{5.82}$$

such that the currents are tensors ($K'^{\mu\nu}_{\xi}[g]$ is a tensor density) and they do not depend on the gauge parameters ω^+ and ω^-. Their explicit expressions, using $g^{(0)}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -e^{2\varphi(x)}\mathrm{d}x^+\mathrm{d}x^-$, are

$$J^+_{\omega^+}[\varphi] = 0, \qquad J^-_{\omega^+}[\varphi] = -\frac{\ell e^{-2\varphi}}{2\pi G}\partial_+\varphi, \qquad J^+_{\omega^-}[\varphi] = -\frac{\ell e^{-2\varphi}}{2\pi G}\partial_-\varphi, \qquad J^-_{\omega^-}[\varphi] = 0. \tag{5.83}$$

We eventually compute the boundary covariant divergence of these two currents and we find:

$$D^{(0)}_a J^a_{\omega^+} = -\mathcal{A}, \qquad D^{(0)}_a J^a_{\omega^-} = -\mathcal{A}, \tag{5.84}$$

where $\mathcal{A}$ is the anomaly integrand coefficient defined in (5.73). We have thus shown that the Weyl currents are not conserved due to the presence of the anomaly [155]. The boundary Weyl symmetry is broken, for the bulk counterpart Weyl charges are not conserved and this process is driven by the anomaly coefficient: for flat boundary metrics the current is conserved [148], as we thoroughly review in Appendix B.

5.7 Summary and future directions

The above analysis shows that the asymptotic symmetries corresponding to conformally flat boundary conditions include a new abelian sector of Weyl rescalings of the boundary metric whose associated surface charges are integrable but not conserved and that their charge algebra is characterised by a new, non-trivial, central charge. As already mentioned, the holographic AdS/CFT dictionary predicts that bulk asymptotic symmetries are dual to boundary global symmetries of a putative field theory. In our setup, after constructing new suitable Weyl boundary currents compatible with the surface charges, we proved that the holographic counterpart of the bulk analysis is consistently described by a anomalous Ward-Takahashi identity of the dual holographic theory. In other words, the effect of having a non-flat boundary metric is found to be equivalent to coupling the two-dimensional boundary CFT with a fixed background field.

We have not addressed the holographic interpretation corresponding to the most general variation of the boundary metric (*i.e.* ω not satisfying (5.33)), which is certainly worth exploring. In this regard, a different choice of gauge in the bulk may be more suited, *e.g.* [183]. In particular, this raises the question on how Weyl charges explicitly depend on the gauge condition [178, 179, 184, 185]. Another possible development is the extension of these results to higher dimensions. Specifically, it is tempting to speculate that similar patterns can be unravelled in even-boundary dimensions. On the other hand, it would also be of related interest to investigate Weyl charges in odd-boundary dimensions. Furthermore, a suitable flat limit [152, 186] of these results might be relevant for the flat holography program [187–189] and the recent developments in celestial CFT. Eventually, on the macroscopic side of holography, *i.e.*, in the fluid/gravity correspondence, it would be interesting to study the role of these boundary conditions from the fluid perspective [190].

Part II

Partition functions and modular covariance

6 Introduction

Conformal field theory (CFT) has been an extraordinarily powerful tool in the development of theoretical physics over the last decades. It has allowed for interplays between a wide spectrum of areas in physics, leading to rich results. Among others, it has found applications that range from string theory and holography to statistical and condensed matter physics. In particular, two-dimensional CFT, whose study started with the seminal paper by Belavin, Polyakov and Zamolodchikov [191], appears as a natural and fundamental feature of string theory and has been extensively used to describe two-dimensional critical phenomena, as the classification of critical behaviors of the two-dimensional Ising model.

One of the central ideas of two-dimensional CFT is modular invariance on a Euclidean torus background $\mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_L$ [192, 193], defined as the product between the compact euclidean time and a compact spatial variable of periods β and L, respectively. Indeed, a torus can be defined by specifying two linearly independent vectors on the plane, or equivalently two complex numbers (ω_1, ω_2), called periods of the lattice, and then identifying points that differ by an integer combination of these. Therefore, a torus is equivalent to a plane with periodic boundary conditions. The requirement of a torus background imposes powerful constraints on the theory. In the first place, by conformal invariance, the properties of a CFT living on such background cannot depend separately on the two periods but only on the ratio between them, called modular parameter $\tau = \omega_2/\omega_1 = \tau_1 + i\tau_2$. Secondly, a given torus can be specified by any other pair of lattice vectors which are integer combinations of (ω_1, ω_2) and this implies, in turn, that the partition function of a CFT on the torus must be modular invariant, *i.e.* it must be invariant under $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2$ transformations of the modular parameter τ. As a direct consequence, two-dimensional CFTs living on the torus exhibit a peculiar duality, mapping its partition function at a given $\tau_2 = L/\beta$ to itself at the inversely related temperature $1/\tau_2 = \beta/L$, *i.e.* there exists a symmetry under the swapping of the above defined circles. It has been shown that the partition function of a two-dimensional massless scalar field theory on the torus can be written in terms of the so-called Dedekind's η function. The latter is an important example of quasi-modular form. In general, modular forms are an important chapter of mathematics and they play a crucial role in physics.

The above described dualities have been widely used and have led to strong analytic results. In this regard it is worth to mention the Kramers-Wanniers duality [194, 195], relating the high- and low-temperature behaviors of the free energy of the two-dimensional Ising model, that can be used to exactly determine its critical point. The Cardy formula [135], which gives a prescription to compute the degeneracy of high energy states of the CFT, also heavily relies on this duality. In a slightly different context, the Cardy formula has been used to microscopically reproduce the Bekenstein entropy of certain classes of black holes [134, 196]. Remarkably, modular transformations have been used in the framework of Seiberg-Witten solution [197] of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, where they act on a modular parameter that depends both on the vacuum theta angle and the gauge coupling constant, giving access to the non-perturbative regime of the theory.

The question of whether modular transformations and thus temperature dualities can be realized in different contexts is naturally an intriguing one and definitely worth exploring.

6.1 Modular covariance and Eisenstein series

Investigating the fate of modular invariance for CFTs living in spacetime dimensions greater than two and on partially compactified manifolds is certainly an interesting question, see *e.g.* [198–203] for several successful attempts in this direction.

It turns out [204] that, in the simple case of a $(d+1)$-dimensional massless scalar field living on $\mathbb{T}^2 \times \mathbb{R}^{d-1}$, the above described modular invariance of the partition function gets naturally replaced by modular covariance. In particular, the Dedekind's η function appearing in the case of the two-dimensional CFT on $\mathbb{T}^2$ is substituted by real analytic Eisenstein series [205–208]. They are important objects to consider in the context of modular forms and feature prominently in theoretical physics, for instance in the context of string theory amplitudes, gravitational instantons and Feynman integrals.

Similarly to modular invariance, modular covariance allows to derive certain generalized temperature dualities, that smoothly extend those of two-dimensional CFTs, implying an higher-dimensional analogue of the Cardy formula. In fact, in these models the high-temperature limit of the thermal entropy and of the density of states can be expressed uniquely in terms of the central charge of the theory. Interestingly, it has been also shown that in the low-temperature regime the entropy of the system does not scale as the entire volume of the manifold, as one might expect, but only as the volume of its non-compact $\mathbb{R}^{d-1}$ component. The Fourier analysis of the real analytic Eisenstein series allows to trace back the microscopic origin of this behavior and to show that is due the the zero mode of the scalar field in the compact spatial direction.

6.2 Gauge theories in the Casimir setup

Modular transformations already appear naturally in the context of the quantized electromagnetic field between two perfectly conducting plates, *i.e.* the well-studied setup of the Casimir effect [209] at finite temperature [210–216] While a temperature inversion symmetry for the partition function had already been derived originally in [217] (see also [218–222]), this result can be enhanced to transformations of a suitably defined modular parameter under the full modular group [223]. In this case, it turns out that the partition function of the theory transforms covariantly under modular transformations with weight 2. A crucial role in this derivation is played by the Casimir boundary conditions, which ultimately make the electromagnetic field between two perfectly conducting plates equivalent to a massless scalar field living on $\mathbb{T}^2 \times \mathbb{R}^2$, where $\mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_{2L}$, L being the separation between the two plates. The temperature dualities corresponding to this modular symmetry allow to easily relate the high- and low-temperature limits of thermodynamic quantities and also in this case, at low temperatures, the entropy scales as the area of the metallic plates [224, 225].

A similar analysis [226] has shown that also in the case of linearized gravity around flat space, *i.e.* Pauli-Fierz theory, the same considerations apply. Even if the gauge structure of the spin 2 theory is significantly more complicated than that of spin 1, imposing the analogue of perfectly conducting boundary conditions on the linearized gravitational field leads, after defining a suitable modular parameter, to a modular covariant partition function. In particular, it turns out that it is equal to that of the spin 1 case, showing that the true physical propagating degrees of freedom of free Pauli-Fierz theory in the Casimir setup, after having eliminated all the gauge redundancies, have the same nature as those of electromagnetism.

6.3 Publications and original results

1. F. A. and G. Barnich, "Modular invariance in finite temperature Casimir effect", *JHEP* **10** (2020) 134, arXiv: 2007.1133 [hep-th],

2. F. A., G. Barnich and M. Bonte "Gravitons in a Casimir box", *JHEP* (2021) 216, arXiv: 2007.1133 [hep-th],

3. F. A., G. Barnich, M. Bonte and A. Kleinschmidt "Generalized modular invariance and temperature dualities on $\mathbb{R}^{d-1} \times \mathbb{T}^2$". To appear soon.

7 Path integrals and partition functions: an introduction

In this chapter we review some techniques to compute the partition function $Z(\beta)$ of a quantum field theory. Having access to the latter is essential to capture statistic and thermodynamic properties of a theory at finite temperature. As we will see in the next chapters, understanding the symmetries of $Z(\beta)$ leads to very interesting physical consequences. In particular, as remarked in the Introduction, modular symmetries of the partition function imply the so-called temperature dualities, for which the low- and high-temperature regime of the theory are related. In this chapter, we will only be concerned about the very simple model of a massless scalar field in the large volume limit and we will outline how different techniques to compute $Z(\beta)$ illustrate different but complementary aspects of the problem. The study of this toy model has a twofold purpose. On the one hand, it will help establishing a useful notation that will be used throughout all this part. On the other hand, the results contained in this chapter will be a useful benchmark for the case of more realistic models, *e.g.* electromagnetism and linearized gravity which will be dealt with in chapter 9.

We start in section 7.1 by recalling the path-integral representation of the partition function and in 7.1.1, 7.1.2 and 7.1.3 we describe the zeta function, heath kernel and canonical techniques, respectively, to compute $Z(\beta)$. In section 7.2 we explicitly make use of these methods to obtain thermodynamic properties of a real massless scalar field in $(d+1)$-dimensions. We conclude, in section 7.3, with some comments about the independence of the partition function in the large volume limit of the choice of boundary conditions and about the electromagnetism partition function.

A non-exhaustive list of original literature for this chapter is [227–231]. Useful books and reviews are [232–244] and references therein contained.

7.1 Path integral representation of the partition function

Let us start by considering a non-relativistic particle of unit mass, described by the classical Lagrangian action

$$S[q] = \int_{t_i}^{t_f} \mathrm{d}t \left[\frac{\dot{q}^2}{2} - V(q)\right]. \tag{7.1}$$

Here by q we denoted the coordinate and by $V(q)$ the potential acting on the particle. The classical trajectory $q(t)$ of the particle, according to Hamilton's principle, is the one extremizing $S[q]$. This procedure gives the standard Lagrange's equations of motion $\ddot{q} = V'(q)$. Equivalently, one can introduce the momentum p associated to the coordinate q as $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}$ and compute the first order Hamiltonian action as

$$S_H[q,p] = \int_{t_i}^{t_f} \mathrm{d}t \big[p\dot{q} - H(p,q)\big], \qquad H(p,q) = \frac{p^2}{2} + V(q). \tag{7.2}$$

Extremizing $S_H[q,p]$ yields the standard Hamilton's equations of motion $\dot{q} = \partial_p H$ and $\dot{p} = -\partial_q H$.

In quantum mechanics, we introduce coordinate and momentum operators $\hat{q}$ and $\hat{p}$ satisfying $[\hat{q},\hat{p}] = i\hbar$ [30]

[30] In this chapter we do *not* adopt the standard convention $\hbar = 1$.

with associated eigenstates $|q\rangle$ and $|p\rangle$ such that $\hat{q}|q\rangle = q|q\rangle$ and $\hat{p}|p\rangle = p|p\rangle$. They form an orthonormal and complete set of eigenstates:

$$\langle q|q'\rangle = \delta(q-q'), \qquad \langle p|p'\rangle = \delta(p-p'), \qquad \hat{\mathbb{1}} = \int \mathrm{d}q|q\rangle\langle q|, \qquad \hat{\mathbb{1}} = \int \mathrm{d}p|p\rangle\langle p|. \tag{7.3}$$

In the Heisenberg picture, $\hat{q}$ and of $\hat{p}$ are time dependent $\hat{q}(t) = \hat{U}^\dagger(t)\hat{q}\hat{U}(t)$ and $\hat{p}(t) = \hat{U}^\dagger(t)\hat{p}\hat{U}(t)$, with $\hat{U}(t) = \exp\{-i\hat{H}t/\hbar\}$ time evolution operator, and they have eigenstates $|q;t\rangle = \hat{U}^\dagger(t)|q\rangle$ and $|p,t\rangle = \hat{U}^\dagger(t)|p\rangle$, so that $\hat{q}(t)|q;t\rangle = q|q;t\rangle$ and $\hat{p}(t)|q;t\rangle = p|p;t\rangle$. An object of interest is the quantum amplitude for the particle at q_f at time t_f, starting from q_i at t_i,

$$\langle q_f;t_f|q_i;t_i\rangle = \langle q_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|q_i\rangle, \tag{7.4}$$

where in the last step we have explicitly used the time evolution opertor. We now briefly show how (7.4) can be defined in terms of a path integral [234, 241, 245]. The first step is to split the interval (t_i, t_f) into $N+1$ small intervals of length $\delta t = (t_f - t_i)/(N+1)$ with N very large, so that, using Trotter's formula

$$e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)} = \underbrace{e^{-\frac{i}{\hbar}\hat{H}\delta t}...e^{-\frac{i}{\hbar}\hat{H}\delta t}}_{N+1 \text{ times}}, \qquad N\to\infty. \tag{7.5}$$

Then we insert N times the identity in coordinate space $\hat{\mathbb{1}} = \int_{-\infty}^{\infty}\mathrm{d}q_k|q_k\rangle\langle q_k|$ and $N+1$ times the identity in momentum space $\hat{\mathbb{1}} = \int_{-\infty}^{\infty}\mathrm{d}p_k|p_k\rangle\langle p_k|$, so that the amplitude admits the representation

$$\langle q_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|q_i\rangle = \int_{-\infty}^{\infty}\left(\prod_{k=1}^{N+1}\mathrm{d}p_k\right)\left(\prod_{k=1}^{N}\mathrm{d}q_k\right)\langle q_f|e^{-\frac{i}{\hbar}\hat{H}\delta t}|p_{N+1}\rangle\langle p_{N+1}|q_N\rangle...\langle q_k|e^{-\frac{i}{\hbar}\hat{H}\delta t}|p_k\rangle\langle p_k|q_{k-1}\rangle...$$

$$...\langle q_1|e^{-\frac{i}{\hbar}\hat{H}\delta t}|p_1\rangle\langle p_1|q_i\rangle. \tag{7.6}$$

Using $\langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}pq}$ and $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}pq}$ each factor in (7.6) can be computed as

$$\langle q_k|e^{-\frac{i}{\hbar}\hat{H}\delta t}|p_k\rangle\langle p_k|q_{k-1}\rangle = \frac{1}{2\pi\hbar}e^{\frac{i}{\hbar}\left[p_k(q_k-q_{k-1})-H(p_k,q_k)\delta t\right]}. \tag{7.7}$$

Note that, in order to derive this equation it is essential to assume that the classical Hamiltonian is of the form $H(p,q) = f(p) + V(q)$ so that there is no ambiguity in defining the quantum operator $\hat{H}(\hat{p},\hat{q})$. More generally, the classical Hamiltonian may contain cross-terms as $p^n q^m$. In this case, once we specify an ordering for $\hat{H}(\hat{p},\hat{q})$, we move, using the commutation relations between $\hat{p}$ and $\hat{q}$, all the powers of $\hat{q}$ to the left of those of $\hat{p}$. The function $H(p,q)$ we get in (7.7) using this procedure is called the *p*-*q* symbol of the operator $\hat{H}(\hat{p},\hat{q})$. We get, defining $q_0 \equiv q_i$ and $q_{N+1} \equiv q_f$,

$$\langle q_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|q_i\rangle = \int\prod_{k=1}^{N+1}\frac{\mathrm{d}p_k}{2\pi\hbar}\prod_{k=1}^{N}\mathrm{d}q_k e^{\frac{i}{\hbar}\sum_{k=1}^{N+1}\left[p_k\left(\frac{q_k-q_{k-1}}{\delta t}\right)-H(p_k,q_k)\right]\delta t}. \tag{7.8}$$

It is convenient to regard the q_k and p_k appearing in (7.8) as defining a skeletonized path in phase space and indeed they can be thought of as specific values of some sufficiently smooth interpolating functions $q(t)$ and $p(t)$ such that $q(k\delta t) = q_k$ and $p(k\delta t) = p_k$, as shown in Figure 11.

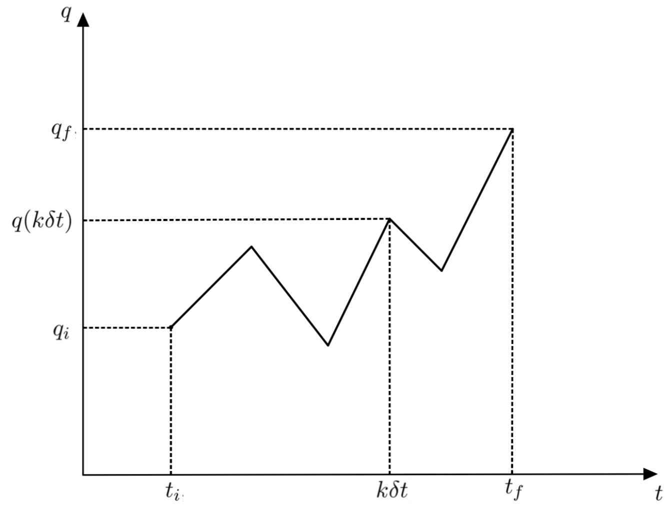

Figure 11: The skeletonized path connecting q_i with q_f, through the interpolating function $q(t)$.

In the $N \to \infty$ limit we get, for the exponent in (7.8)

$$\sum_{k=1}^{N+1} \left[p_k \left(\frac{q_k - q_{k-1}}{\delta t} \right) - H(p_k, q_k) \right] \delta t \xrightarrow{N \to \infty} \int_{t_i}^{t_f} \mathrm{d}t \left[p\dot{q} - H(p,q) \right] = S_H[q,p], \tag{7.9}$$

and for the measure we define, as usual [31]

$$\int \prod_{k=1}^{N+1} \frac{\mathrm{d}p_k}{2\pi\hbar} \prod_{k=1}^{N} \mathrm{d}q_k \xrightarrow{N \to \infty} \int_{q(t_i)=q_i}^{q(t_f)=q_f} \frac{\mathcal{D}p(t)\mathcal{D}q(t)}{2\pi\hbar}, \tag{7.10}$$

so that, in the large-N limit the amplitude can be expressed as

$$\langle q_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | q_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \frac{\mathcal{D}p(t)\mathcal{D}q(t)}{2\pi\hbar} e^{\frac{i}{\hbar} S_H[q,p]}. \tag{7.11}$$

Equation (7.11) is a *Hamiltonian path integral* representation for the amplitude and it is an integral over all possible paths $q(t)$ and $p(t)$ in phase space starting at $q(t_i) = q_i$ and terminating at $q(t_f) = q_f$. Note that there is no restriction at all on the integration over the momenta $p(t)$, contrarily to that over $q(t)$. The Hamiltonian is quadratic in p and hence one can explicitly perform each Gaussian integration over

[31] The mathematically rigorous definition of the path integral measure is a well-known problem. Note however that, as remarked in [234], the measure $\mathcal{D}p(t)\mathcal{D}q(t)$ should be, formally, a product over time of the phase space Liouville measures at each time.

the momenta p_k,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p_k}{2\pi\hbar} e^{\frac{i}{\hbar}\left[p_k(q_k-q_{k+1})-\frac{p_k^2}{2}\delta t\right]} = \frac{1}{\sqrt{2\pi i\hbar\delta t}} e^{\frac{i}{\hbar}\left[\frac{(q_k-q_{k-1})^2}{2\delta t}\right]}, \tag{7.12}$$

so that the amplitude in (7.8) is

$$\langle q_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|q_i\rangle = \frac{1}{(2\pi i\hbar\delta t)^{\frac{N+1}{2}}} \int \prod_{k=1}^{N} \mathrm{d}q_k e^{\frac{i}{\hbar}\sum_{k=1}^{N+1}\left[\frac{(q_k-q_{k-1})^2}{2\delta t^2}-V(q_k)\right]\delta t}, \tag{7.13}$$

which, in the large-N limit, up to an overall (divergent) factor that does not contain dynamical information, can be more conveniently rewritten as

$$\langle q_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|q_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) e^{\frac{i}{\hbar}S[q]}, \tag{7.14}$$

which is the standard *Feynman path integral* [246] representation of a quantum mechanical amplitude. Note that, since the integral over momenta is Gaussian, performing the integration (7.13) is equivalent to find the value $p_* = p_*(q, \dot{q})$ extremizing $S_H[p, q]$ and evaluating $S_H[q, p]$ at p_*. With this procedure, one finds $p_* = \dot{q} = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ and hence $S_H[q, p_*] = S[q]$, with $S[q]$ given in (7.1), in agreement with (7.14). The above arguments can be easily generalized to matrix elements between states $\langle q_f; t_f|$ and $|q_i; t_i\rangle$ of time-ordered products of operators $\hat{\mathcal{O}}[\hat{p}(t), \hat{q}(t)]$, *i.e.*

$$\langle q_f; t_f|T\{\hat{\mathcal{O}}_A[\hat{p}(t_A), \hat{q}(t_A)]\hat{\mathcal{O}}_B[\hat{p}(t_B), \hat{q}(t_B)]...\}|q_i; t_i\rangle$$

$$= \int_{q(t_i)=q_i}^{q(t_f)=q_f} \frac{\mathcal{D}p(t)\mathcal{D}q(t)}{2\pi\hbar} \mathcal{O}_A[p(t_A), q(t_A)]\mathcal{O}_B[p(t_B), q(t_B)]...e^{\frac{i}{\hbar}S_H[p,q]}. \tag{7.15}$$

The expressions we obtained for the amplitude admit an insightful interpretation. While in classical mechanics there is only one path associated to the motion of the particle, in quantum mechanics all possible paths satisfying the right boundary conditions play a role. Indeed, as $\hbar \to 0$, the dominant paths contributions in (7.11) and (7.14) are given by the stationary points of $S_H[p, q]$, solutions to Hamilton's equations or, equivalently, by the the stationary points of $S[q]$, solutions to Lagrange's equations. Expressions (7.11) and (7.14) for the amplitude are therefore equivalent. The advantage of one or the other depends on whether one chooses to work in the Hamiltonian or Lagrangian formalism, respectively.

In general, it is difficult to use the path integral (7.11)-(7.14) for numerical computations and amplitudes are better computed in Euclidean space where one introduces Euclidean time $\tau = it$. The analytic continuation of (7.4) to imaginary times reads

$$\langle q_f|e^{-\frac{1}{\hbar}\hat{H}(\tau_f-\tau_i)}|q_i\rangle, \tag{7.16}$$

so that the rapid oscillations of the factor $e^{\frac{i}{\hbar}S[q]}$ are replaced by a an exponential suppression. To get a formula for (7.16), we can just substitute $t = -i\tau$ in the previous expressions and we obtain

$$\langle q_f|e^{-\frac{1}{\hbar}\hat{H}(\tau_f-\tau_i)}|q_i\rangle = \int_{q(\tau_f)=q_f}^{q(\tau_i)=q_i} \frac{\mathcal{D}p(\tau)\mathcal{D}q(\tau)}{2\pi\hbar} e^{-\frac{1}{\hbar}S_H^E[p,q]} = \int_{q(\tau_f)=q_f}^{q(\tau_i)=q_i} \mathcal{D}q(\tau) e^{-\frac{1}{\hbar}S^E[q]}, \tag{7.17}$$

where we defined the first order Euclidean action and the Euclidean action as

$$S_H^E[p,q] = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[-ip\frac{\mathrm{d}q}{\mathrm{d}\tau} + H(p,q) \right], \qquad S^E[q] = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q) \right]. \tag{7.18}$$

In quantum statistical mechanics the canonical partition function is defined as

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int \mathrm{d}q \langle q|e^{-\beta\hat{H}}|q\rangle, \qquad \beta = \frac{1}{k_B T}, \tag{7.19}$$

where the trace is taken over a basis in the Hilbert space. The matrix element appearing in (7.19) is the same as in (7.16) with the substitution $\tau_f - \tau_i = \hbar\beta$ and where the initial and last point are equal $q_i = q_f = q$, *i.e.* the path $q(\tau)$ has to be periodic. Therefore, also the canonical partition function admits a path integral representation:

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int_{\substack{q(0)=q(\hbar\beta)\\ p(0)=p(\hbar\beta)}} \frac{\mathcal{D}p(\tau)\mathcal{D}q(\tau)}{2\pi\hbar} e^{-\frac{1}{\hbar}S_H^E[p,q]} = \int_{q(0)=q(\hbar\beta)} \mathcal{D}q(\tau) e^{-\frac{1}{\hbar}S^E[q]}, \tag{7.20}$$

with $S_H^E[q,p]$ and $S^E[q]$ given in (7.18) with the replacements $\tau_i = 0$ and $\tau_f = \hbar\beta$. Remarkably, the imaginary time path integral in (7.20) can be thought of as a partition function in classical statistical mechanics if one interprets the Euclidean time coordinate τ as a spatial coordinate. Therefore, one can think of quantum mechanics in Euclidean time as classical mechanics in one higher spatial dimension. In particular, denoting the size of this additional dimension as $L = \hbar\beta$, the zero temperature limit of the quantum theory corresponds to the usual infinite volume classical partition function in the Euclidean time/spatial direction.

So far, we were only concerned about quantum mechanics. However, when one turns to quantum field theory similar considerations apply. The conjugate variables q and p are replaced by the field $\phi(t,x)$ and its canonical momentum $\Pi(t,x)$ satisfying canonical commutation relations $[\hat{\phi}(t,x'), \hat{\Pi}(t,x)] = i\hbar\delta(x-x')$. The partition function of a quantum field theory can be represented by a path integral over the fields $\phi(\tau,x)$ and $\Pi(\tau,x)$ as

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int_{\substack{\phi(0,x)=\phi(\hbar\beta,x)\\ \Pi(0,x)=\Pi(\hbar\beta,x)}} \frac{\mathcal{D}\Pi(\tau,x)\mathcal{D}\phi(\tau,x)}{2\pi\hbar} e^{-\frac{1}{\hbar}S_H^E[\phi,\Pi]} = \int_{\phi(0,x)=\phi(\hbar\beta,x)} \mathcal{D}\phi(\tau,x) e^{-\frac{1}{\hbar}S^E[\phi]}. \tag{7.21}$$

It is important to bear in mind that, while the partition function of quantum mechanics is a finite function of the parameters upon which the Lagrangian depends, in quantum field theory this is not true. The major issues come from UV divergences, that have to be taken care of by a renormalization procedure.

The partition function in (7.21) contains all the information about the statistic properties of finite temperature quantum field theory [232, 233, 238, 242–244]. From this quantity other important observables such as the the free energy $F(\beta)$, the entropy $S(\beta)$ or the internal energy $E(\beta)$ may be derived as

$$F(\beta) = -\frac{1}{\beta}\log Z(\beta), \qquad S(\beta) = k_B(1-\beta\partial_\beta)\log Z(\beta), \qquad E(\beta) = -\partial_\beta \log Z(\beta). \tag{7.22}$$

Since the partition function will be the main object of interest throughout next sections and chapters, we now briefly outline few methods to explicitly compute it.

7.1.1 Zeta function technique

Consider a real, elliptic and self-adjoint operator $\hat{A}$ with a discrete set $\{\lambda_k\}_{k\in I}$ of real eigenvalues $\lambda_k \geq 0$ $\forall\, k \in I$ associated to a complete set of orthonormal eigenfunctions $\{\phi_k\}_{k\in I}$, satisfying

$$\hat{A}\phi_k(x) = \lambda_k\phi_k(x), \qquad \int \mathrm{d}^{d+1}x\, \phi_k(x)\phi_{k'}(x) = \delta_{kk'}, \qquad \sum_{k\in I}\phi_k(x)\phi_k(x') = \delta^{d+1}(x-x'). \tag{7.23}$$

Here and from now on we denote the Euclidean time by $\tau \equiv x^{d+1}$. It is important to define a new index set $I' \subseteq I$, such that $k \in I'$ if an only if $\lambda_k \neq 0$. In other words, the set $\{\lambda_k\}_{k\in I'}$ is equal to the set $\{\lambda_k\}_{k\in I}$, up to the zero modes of $\hat{A}$. This is crucial to define the *generalized zeta function* [230, 235, 237, 240] associated to the operator $\hat{A}$ as

$$\zeta_{\hat{A}}(s) \equiv \mathrm{Tr}\,\hat{A}^{-s} = \sum_{k\in I'}\lambda_k^{-s}. \tag{7.24}$$

Here and in the following the superscript $'$ means that we are excluding the zero modes from the sums/products involving the eigenvalues of $\hat{A}$. The determinant of the operator $\hat{A}$ can be written in terms of the generalized zeta function associated to $\hat{A}$ as

$$\det{}'\,\hat{A} = \prod_{k\in I'}\lambda_k = e^{-\zeta'_{\hat{A}}(s)\big|_{s=0}}. \tag{7.25}$$

We are interested in having a formula for $Z(\beta)$ in terms of $\zeta_{\hat{A}}(s)$. To this aim, we assume that the Euclidean action appearing in (7.21) can be rewritten as

$$S^E[\phi] = \frac{1}{2}\int \mathrm{d}^{d+1}x\, \phi(x)\hat{A}\phi(x), \qquad \hat{A} = \frac{\delta^2 S^E[\phi]}{\delta\phi(x)\delta\phi(x')}. \tag{7.26}$$

The field ϕ can be expanded in the basis $\{\phi_k\}_{k\in I}$ as

$$\phi(x) = \sum_{k\in I} a_k\phi_k(x), \qquad a_k = \int \mathrm{d}^{d+1}x\, \phi(x)\phi_k(x). \tag{7.27}$$

Therefore, the Euclidean action in (7.26) becomes

$$S^E[\phi] = \frac{1}{2}\sum_{k\in I} a_k^2 \lambda_k, \tag{7.28}$$

and the measure transforms as $\mathcal{D}\phi = \mu \prod_{k\in I} \mathrm{d}a_k$, where μ is a normalization constant with the dimension of a mass, so that the partition function can be written as

$$Z(\beta) = \prod_{k\in I} \mu \int \mathrm{d}a_k e^{-\frac{1}{2\hbar}\lambda_k a_k^2} = \prod_{k\in I} \mu \sqrt{\frac{2\pi\hbar}{\lambda_k}} = \left(\det \frac{\hat{A}}{2\pi\hbar\mu^2}\right)^{-\frac{1}{2}}. \tag{7.29}$$

Clearly, the potential presence of vanishing eigenvalues of $\hat{A}$ creates issues in the above expression. However, in most of the cases that will be considered throughout this work, we will argue that these zero modes do not actually contribute to the partition function. This is due to the fact that the we will ultimately consider operators on partially compactified manifold, such as $\mathbb{R}^p \times \mathbb{T}^q$, whose spectrum is therefore not discrete. An exception is represented by the case of the Bose field on the Euclidean torus, described in detail in section 8.1, where we will explicitly comment on how to take care of the zero mode.

Assuming that $I' = I$ in (7.29) and using (7.25) we eventually we arrive at [32]

$$\log Z(\beta) = \frac{1}{2}\zeta'_{\hat{A}}(s)\big|_{s=0} + \frac{1}{2}\log\!\left(2\pi\hbar\mu^2\right)\zeta_{\hat{A}}(0). \tag{7.30}$$

In the above hypothesis, this equation reduces the problem of computing a path integral to that of computing the zeta function associated to a certain operator $\hat{A}$ appearing in the Euclidean action. Note that, in general, to use the zeta function technique we need to know, even numerically, the eigenvalues of the operator $\hat{A}$ defining the Euclidean action and the action must be quadratic in the dynamical fields. There are however situations where this last condition is not fulfilled, *e.g.* when one deals with path integrals on curved spacetimes. In these cases, if we have access to a background solution ϕ_0 of the classical equations of motion, it can be argued that the major contributions to the partition function come from field configurations near the background, *i.e.* we can expand ϕ as

$$\phi(x) = \phi_0(x) + \tilde{\phi}(x), \tag{7.31}$$

and, correspondingly the Euclidean action as

$$S^E[\phi] = S^E[\phi_0] + S^E_{(2)}[\tilde{\phi}] + \dots\,, \tag{7.32}$$

where $\tilde{\phi}$ is a fluctuation around the background solution ϕ_0, $S^E_{(2)}[\tilde{\phi}]$ is the term quadratic in $\tilde{\phi}$, given by

$$S^E_{(2)}[\tilde{\phi}] = \frac{1}{2}\int \mathrm{d}^{d+1}x' \int \mathrm{d}^{d+1}x\, \tilde{\phi}(x) \frac{\delta^2 S^E[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_0(x)} \tilde{\phi}(x'), \tag{7.33}$$

[32]In order to get (7.30) we used $\zeta_{\hat{A}/\alpha}(s) = \alpha^s \zeta_{\hat{A}}(s)$ and thus $\zeta'_{\hat{A}/\alpha}(s)|_{s=0} = \zeta'_{\hat{A}}(s)|_{s=0} + \log\alpha\, \zeta_{\hat{A}}(0)$.

and ... denote higher order corrections. Note that in (7.32) the term linear in $\tilde{\phi}$ is absent because we assumed that ϕ_0 solves the classical equations of motion and thus it is a stationary point for the action. It follows that

$$\log Z(\beta) = -\frac{1}{\hbar} S^E[\phi_0] + \log \int \mathcal{D}\tilde{\phi}\, e^{-\frac{1}{\hbar} S^E_{(2)}[\tilde{\phi}]} + \text{higher order contributions.} \tag{7.34}$$

Now, since $S^E_{(2)}[\tilde{\phi}]$ is quadratic in the fields, we can apply the zeta function technique to the second term in (7.34) and therefore the relevant zeta function to compute is the one associated with the operator appearing in (7.33).

7.1.2 Heat kernel technique and Schwinger proper time

Consider again the operator $\hat{A}$ satisfying the same properties listed previously. The set of equations (7.23) can be more conveniently rewritten introducing using the bra and ket abstract notation as

$$\hat{A}|k\rangle = \lambda_k |k\rangle, \qquad \langle k|k'\rangle = \delta_{kk'}, \qquad \sum_{k\in I} |k\rangle\langle k| = \hat{\mathbb{1}}. \tag{7.35}$$

We define the *heat kernel operator* [236, 239] associated to $\hat{A}$ as [33]

$$\hat{K}_{\hat{A}}(\tau) \equiv e^{-\hat{A}\tau}. \tag{7.36}$$

The matrix elements of $\hat{K}_{\hat{A}}(\tau)$ between coordinates states $\langle x|$ and $|x'\rangle$ define the *heat kernel* associated to $\hat{A}$:

$$K_{\hat{A}}(x, x'; \tau) \equiv \langle x|\hat{K}_{\hat{A}}(\tau)|x'\rangle = \langle x|e^{-\hat{A}\tau}|x'\rangle. \tag{7.37}$$

Differentiating this equation with respect to τ yields

$$-\frac{\partial}{\partial\tau} K_{\hat{A}}(x, x'; \tau) = \hat{A} K_{\hat{A}}(x, x'; \tau), \tag{7.38}$$

where $\hat{A}$ is taken to act on the first argument of $K_{\hat{A}}(x, x'; \tau)$. When $\hat{A}$ is the spatial Laplacian $\partial_i\partial^i$, equation (7.38) is just the standard heat equation, which is the Euclidean time counterpart of the Schrödinger equation with zero potential energy and $\hbar = 1$. Furthermore, inserting a resolution of the identity in (7.36) we get

$$K_{\hat{A}}(x, x'; \tau) = \sum_{k\in I} \phi_k(x)\phi_k(x') e^{-\lambda_k \tau}, \tag{7.39}$$

[33] Note that in (7.36) τ is *not* the Euclidean time, to which we will refer to as x^{d+1}.

so that, in the limit $\tau \to 0$, using the last of (7.23) $K_{\hat{A}}(x, x'; \tau)$ satisfies

$$K_{\hat{A}}(x, x'; 0) = \delta^{d+1}(x - x'). \tag{7.40}$$

Equations (7.38) and (7.40) ensure that any solution of the equation $-\partial_\tau \psi(x, \tau) = \hat{A}\psi(x, \tau)$ with initial condition $\psi(x, 0) = \psi_0(x)$ can be expressed using the heat kernel associated to $\hat{A}$ as

$$\psi(\tau, x) = \int \mathrm{d}^{d+1}x'\, K_{\hat{A}}(x, x'; \tau)\psi_0(x'). \tag{7.41}$$

From (7.29) it follows that [34]

$$\log Z(\beta) = -\frac{1}{2}\mathrm{Tr}\,\log\frac{\hat{A}}{2\pi\mu^2}, \qquad \hat{A} = \frac{\delta^2 S^E[\phi]}{\delta\phi(x)\delta\phi(x')}. \tag{7.42}$$

Now consider the following general identity [35]

$$\log z = -\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\left(e^{-\tau z} - e^{-\tau}\right). \tag{7.43}$$

Hence, from (7.42) and generalizing (7.43) to the case of operators, we get, up to a constant independent of the dynamics

$$\log Z(\beta) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{Tr}\, e^{-\hat{A}\tau} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{Tr}\,\hat{K}_{\hat{A}}(\tau). \tag{7.44}$$

The integration variable τ in the above formula is called *Schwinger proper time* or the parameter of the *world-line* [227, 247, 248]. Note that in this approach the operator whose trace is being evaluated is treated as the Hamiltonian for evolution in the Schwinger proper time direction and therefore the result in (7.44) is reminiscent of a one-dimensional field theory living in the one-dimensional space of proper time.

We now briefly show that it is indeed possible to relate the zeta function introduced previously to the trace of the heat kernel appearing in (7.44) by a Mellin transform. We start from the standard representation of the gamma function

$$\Gamma(s) = \lambda_k^s \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} e^{-\lambda_k \tau}, \qquad \mathfrak{Re}(s) > 0. \tag{7.45}$$

---

[34] Note that the partition function can also be expressed as $Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \mathrm{Tr}\,\hat{K}_{\hat{H}}(\beta)$, where $\hat{K}_{\hat{H}}(\tau)$ is the heat kernel associated to the standard Hamiltonian $\hat{H}$ of the system. This approach will be followed in the context of the canonical approach in 7.1.3.

[35] Indeed, the Taylor series around $z = 1$ of the integrand is

$$-\frac{1}{\tau}\left(e^{-\tau z} - e^{-\tau}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\tau^{n-1}(z-1)^n}{n!} e^{-t},$$

and the integral over t yields $\Gamma(n) = (n-1)!$ so that the right hand side of (7.43) is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}$ which is exactly the Taylor series of $\log z$ around $z = 1$.

Hence

$$\zeta_{\hat{A}}(s) = \sum_{k\in I'} \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} \sum_{k\in I'} e^{-\lambda_k \tau} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1}\, \mathrm{Tr}\,\big(\hat{K}_{\hat{A}}(\tau) - \hat{P}\big), \tag{7.46}$$

where $\hat{P}$ denotes the orthogonal projector operator into the kernel of $\hat{A}$, satisfying $\mathrm{Tr}\,\hat{P} = \dim\{\mathrm{Ker}\hat{A}\}$.

7.1.3 Canonical approach

The canonical approach consists in computing the partition function by directly evaluating the trace $\mathrm{Tr}\, e^{-\beta\hat{H}}$ in the Fock space. Clearly, in order to do so, one must have access to the full spectrum of the Hamiltonian of the theory,

$$\hat{H}|n\rangle = E_n|n\rangle, \qquad \langle n|n'\rangle = \delta_{nn'}, \qquad \sum_{n\in I} |n\rangle\langle n| = \hat{\mathbb{1}}. \tag{7.47}$$

Therefore, one can choose to evaluate the trace on the above set of eigenstates

$$Z(\beta) = \sum_{n\in I} \langle n|e^{-\beta\hat{H}}|n\rangle = \sum_{n\in I} e^{-\beta E_n}. \tag{7.48}$$

Note that the operator $e^{-\beta\hat{H}}$ provides a tool to determine the structure of the vacuum quantum state. Indeed, if $\hat{H}$ is bounded from below, the ground state energy E_0 is given by

$$E_0 = \lim_{\beta\to\infty} \left[-\frac{1}{\beta} \log Z(\beta) \right]. \tag{7.49}$$

Usually, in free quantum field theories, the Fock space is the direct sum of infinitely many Hilbert spaces associated to harmonic oscillators and thus the Hamiltonian is the sum of the Hamiltonians of each mode,

$$\hat{H} = \sum_i \hat{H}_i, \tag{7.50}$$

and thus the partition function is a product of the partition functions associated to each oscillator,

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \prod_i \mathrm{Tr}\, e^{-\beta\hat{H}_i} = \prod_i Z_i(\beta). \tag{7.51}$$

We will see that this method to compute $Z(\beta)$ is often related to the zeta function and heat kernel techniques by a standard Fourier series.

7.2 Thermodynamics of a free massless scalar in the large volume limit and the black body result

Before concluding this chapter, it is instructive to explicitly use the three methods described above to compute the partition function of a free massless scalar field ϕ in $(d+1)$ spacetime dimensions described by the action

$$S[\phi] = -\frac{1}{2}\int_V \mathrm{d}^{d+1}x\, \partial_\mu\phi\partial^\mu\phi, \tag{7.52}$$

in the limit of large volume. The indices μ and i label spacetime and spatial indices, respectively so that $\mu = 0, ..., d$, and $i = 1, ..., d$. In (7.52), we take the spatial volume V of the system to be $V = \prod_i L_i$. We can choose without loss of generality periodic boundary conditions in all directions,

$$\phi(t, x^1, ..., x^i, ..., x^d) = \phi(t, x^1, ..., x^i + L_i, ..., x^d), \qquad i = 1, ...d, \tag{7.53}$$

bearing in mind that in the large volume limit the result will not depend at all on this choice, see *e.g.* the discussion in section 7.4. Therefore, the field ϕ and its conjugate momentum $\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$ can be decomposed in the orthonormal basis $\{e_{k_i}\}$ of eigenfunctions of the Laplacian on the d-dimensional torus $\mathbb{T}^d = \mathbb{S}^1_{L_1} \times .. \times \mathbb{S}^1_{L_d}$ associated with the choice of periodic boundary conditions (7.53), given by

$$e_{k_i}(x) = \frac{1}{\sqrt{V}}e^{ik_ix^i}, \qquad k_i = \frac{2\pi}{L_i}n_i, \qquad n_i \in \mathbb{Z}, \quad \forall\, i = 1, ..., d, \tag{7.54}$$

satisfying the orthonormality and completeness conditions

$$\big(e_{k_i}, e_{k'_i}\big) = \int_V \mathrm{d}^d x\, e^*_{k_i}(x)e_{k'_i}(x) = \prod_i \delta_{n_in'_i}, \qquad \sum_{n_i\in\mathbb{Z}^d} e^*_{k_i}(x)e_{k_i}(x') = \delta^d(x - x'). \tag{7.55}$$

The fields are decomposed in the above basis as

$$\phi(t, x) = \frac{1}{\sqrt{V}}\sum_{n_i\in\mathbb{Z}^d}\phi_{k_i}(t)e^{ik_ix^i}, \qquad \Pi(t, x) = \frac{1}{\sqrt{V}}\sum_{n_i\in\mathbb{Z}^d}\Pi_{k_i}(t)e^{ik_ix^i}, \tag{7.56}$$

with $\phi_{k_i} = (e_{k_i}, \phi)$ and $\Pi_{k_i} = (e_{k_i}, \Pi)$. The reality of ϕ and Π implies that $\phi_{k_i} = \phi^*_{-k_i}$ and $\Pi_{k_i} = \Pi^*_{-k_i}$ and the equal-time Poisson bracket are

$$\{\phi(t, x), \Pi(t, x')\} = \delta^d(x - x'), \qquad \{\phi_{k_i}(t), \Pi^*_{k'_i}(t)\} = \prod_i \delta_{n_in'_i}. \tag{7.57}$$

The first order action is

$$S_H[\phi, \Pi] = \int \mathrm{d}t\Big[\int_V \mathrm{d}^d x\, \Pi\dot{\phi} - H[\phi, \Pi]\Big], \tag{7.58}$$

where the Hamiltonian is given by

$$H[\phi, \Pi] = \frac{1}{2}\int_V \mathrm{d}^d x\big(\Pi^2 + \partial_i\phi\partial^i\phi\big) = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}(\Pi_{k_i}\Pi^*_{k_i} + \omega^2_{k_i}\phi_{k_i}\phi^*_{k_i}), \tag{7.59}$$

with $\omega_{k_i} = \sqrt{k_i k^i}$. The oscillators variables are defined for $k_i \neq 0$ as usual

$$a_{k_i}(t) = \sqrt{\frac{\omega_{k_i}}{2}}\Big(\phi_{k_i}(t) + \frac{i}{\omega_{k_i}}\Pi_{k_i}(t)\Big), \qquad \{a_{k_i}(t), a^*_{k'_i}(t)\} = -i\prod_i \delta_{n_i n'_i}, \tag{7.60}$$

so that the Hamiltonian in terms of a_{k_i} and $a^*_{k_i}$ is, on denoting $p \equiv \Pi_0$ [36]

$$H = \frac{p^2}{2} + \frac{1}{2}{\sum_{n_i\in\mathbb{Z}^d}}' \omega_{k_i}(a^*_{k_i}a_{k_i} + a_{k_i}a^*_{k_i}), \tag{7.61}$$

whereas the conjugate pair (ϕ,Π) is given by

$$\phi(t,x) = \frac{q}{\sqrt{V}} + \frac{1}{\sqrt{V}}{\sum_{n_i\in\mathbb{Z}^d}}' \frac{1}{\sqrt{2\omega_{k_i}}}(a_{k_i}(t)e^{ik_ix^i} + \text{c.c}), \tag{7.62}$$

$$\Pi(t,x) = \frac{p}{\sqrt{V}} - \frac{i}{\sqrt{V}}{\sum_{n_i\in\mathbb{Z}^d}}' \sqrt{\frac{\omega_{k_i}}{2}}(a_{k_i}(t)e^{ik_ix^i} - \text{c.c}), \tag{7.63}$$

where we denoted $q \equiv \phi_0$. In the next chapters, another observable will be of relevance, namely the momentum of the field along the x^i direction. It is given by

$$P_i[\phi,\Pi] = -\int_V \mathrm{d}^dx\,\Pi\,\partial_i\phi = -i{\sum_{n_i\in\mathbb{Z}^d}}' k_i\phi_{k_i}\Pi^*_{k_i}, \qquad \{P_i,\phi(t,x)\} \overset{(7.57)}{=} \partial_i\phi(t,x). \tag{7.64}$$

In terms of a_{k_i} and $a^*_{k_i}$,

$$P_i = \frac{1}{2}{\sum_{n_i\in\mathbb{Z}^d}}' k_i(a^*_{k_i}a_{k_i} + a_{k_i}a^*_{k_i}). \tag{7.65}$$

7.2.1 Functional approach and zeta function

Performing a Wick rotation $t = -i\tau$ and denoting $\tau \equiv x^{d+1}$, the first order Euclidean action reads

$$S^E_H[\phi,\Pi] = \int_0^{\hbar\beta}\mathrm{d}x^{d+1}\int_V \mathrm{d}^dx\Big[-i\Pi\partial_{d+1}\phi + \frac{1}{2}\Pi^2 + \frac{1}{2}\partial_i\phi\partial^i\phi\Big], \tag{7.66}$$

[36] Note that the classical Hamiltonian in (7.61) could be equally written as $H = \frac{p^2}{2} + {\sum}'_{n_i\in\mathbb{Z}^d}\omega_{k_i}a^*_{k_i}a_{k_i}$ since $a^*_{k_i}$ are just complex numbers at this stage. However, the form (7.61) will be useful when we will discuss ordering issues and the Casimir energy.

where the fields satisfy, besides (7.53), periodic boundary conditions in the Euclidean time variable as

$$\phi(x, x^{d+1}) = \phi(x, x^{d+1} + \hbar\beta), \qquad \Pi(x, x^{d+1}) = \Pi(x, x^{d+1} + \hbar\beta). \tag{7.67}$$

Therefore, we can mode expand ϕ and Π also in Euclidean time as

$$\phi(x^A) = \frac{1}{\sqrt{\hbar\beta}} \sum_{n_{d+1}\in\mathbb{Z}} e^{ik_{d+1}x^{d+1}} \phi_{k_{d+1}}(x), \qquad \Pi(x^A) = \frac{1}{\sqrt{\hbar\beta}} \sum_{n_{d+1}\in\mathbb{Z}} e^{ik_{d+1}x^{d+1}} \Pi_{k_{d+1}}(x), \tag{7.68}$$

where we introduced the index $A = 1, ..., d+1$, $k_{d+1} = (2\pi/\hbar\beta)n_{d+1}$ are the *Matsubara frequencies* and

$$\phi_{k_{d+1}}(x) = \frac{1}{\sqrt{\hbar\beta}} \int_0^{\hbar\beta} \mathrm{d}x^{d+1}\, \phi(x^A) e^{-ik_{d+1}x^{d+1}}, \qquad \Pi_{k_{d+1}}(x) = \frac{1}{\sqrt{\hbar\beta}} \int_0^{\hbar\beta} \mathrm{d}x^{d+1}\, \Pi(x^A) e^{-ik_{d+1}x^{d+1}} \tag{7.69}$$

satisfying $\phi^*_{k_{d+1}}(x) = \phi_{-k_{d+1}}(x)$ and $\Pi^*_{k_{d+1}}(x) = \Pi_{-k_{d+1}}(x)$.

Following the procedure outlined after (7.14) we find the stationary points of S^E_H as

$$0 = \left.\frac{\delta S^E_H}{\delta \Pi}\right|_{\Pi_*} = -i\partial_{d+1}\phi + \Pi_* \Longrightarrow \Pi_* = i\partial_{d+1}\phi, \tag{7.70}$$

and therefore we need to evaluate the path integral

$$Z(\beta) = \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S^E[\phi]}, \qquad S^E[\phi] = -\frac{1}{2}\int_0^{\hbar\beta} \mathrm{d}x^{d+1} \int_V \mathrm{d}^d x\, \phi\Delta\phi, \tag{7.71}$$

where $\Delta = \partial_A\partial^A$ is the Laplacian on the manifold $\mathcal{M} = \mathbb{S}^1_{\hbar\beta} \times \mathbb{T}^d$ and hence, from (7.26) we have $\hat{A} = -\Delta$. Here, the $\mathbb{S}^1_{\hbar\beta}$ component is the thermal cycle with period $\hbar\beta$. In the large volume limit the spatial torus decompactifies to $\mathbb{R}^d$. A set of normalized eigenfunctions of $-\Delta$ is given by $\{e_{k_A}\}$,

$$e_{k_A}(x) = \frac{1}{\sqrt{\hbar\beta V}} e^{ik_A x^A} \tag{7.72}$$

satisfying

$$-\Delta e_{k_A}(x) = \lambda_{n_A} e_{k_A}(x), \qquad \lambda_{n_A} = \left(\frac{2\pi}{\hbar\beta}n_{d+1}\right)^2 + \sum_{i=1}^{d}\left(\frac{2\pi}{L_i}n_i\right)^2. \tag{7.73}$$

Therefore, for the zeta function we have

$$\zeta_{-\Delta}(s) = {\sum_{n_A\in\mathbb{Z}^{d+1}}}' \lambda_{n_A}^{-s}. \tag{7.74}$$

In the large volume limit we can replace the sums over n_i into integrals as $\sum_{n_i\in\mathbb{Z}} \to \frac{L_i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k_i$ and we get

$$\zeta_{-\Delta}(s) = \frac{V}{(2\pi)^d}\int_{-\infty}^{\infty} \mathrm{d}^d k_i \sum_{n_{d+1}\in\mathbb{Z}} \left[\left(\frac{2\pi}{\hbar\beta}n_{d+1}\right)^2 + k_i k^i\right]^{-s}. \tag{7.75}$$

Note that in going from (7.74) to (7.75) we lost information about the zero mode λ_0. Indeed, when taking some directions to be large the spectrum of the Laplacian becomes continuous and the zero mode becomes a zero measure set which will not contribute to $Z(\beta)$. In order to perform the integral in (7.75) we go the hyperspherical coordinates and get, after changing the integration variable

$$\zeta_{-\Delta}(s) = \frac{V}{(2\pi)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\left(\frac{2\pi}{\hbar\beta}\right)^{d-2s}\int_0^{\infty} \mathrm{d}k\, k^{d-1} \sum_{n_{d+1}\in\mathbb{Z}} \left[n_{d+1}^2 + k^2\right]^{-s}, \tag{7.76}$$

where the factor $\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the area of the $(d-1)$-dimensional unit ball. The $n_{d+1} = 0$ term in the above integral, when regulated with an IR cut-off ϵ, yields

$$\int_{\epsilon}^{\infty} \mathrm{d}k\, k^{d-1-2s} = -\frac{\epsilon^{d-2s}}{d-2s}, \qquad \mathfrak{Re}(s) > \frac{d}{2}. \tag{7.77}$$

Since the derivative with respect to s of this expression at $s = 0$ vanishes in the limit $\epsilon \to 0$, we can discard this contribution. The remaining integral gives

$$\int_0^{\infty} \mathrm{d}k\, k^{d-1} {\sum_{n_{d+1}\in\mathbb{Z}}}' \left[n_{d+1}^2 + k^2\right]^{-s} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(s-\frac{d}{2}\right)}{\Gamma(s)}\zeta(2s-d), \qquad \mathfrak{Re}(s) > \frac{d}{2}, \tag{7.78}$$

where $\zeta(s) = \sum_{n>0} n^{-s}$ is a representation of the Riemann zeta function for $\mathfrak{Re}(s) > 1$. Using the reflection formula,

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{\frac{2z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z), \tag{7.79}$$

at $z = 2s - d$ we have, for $\zeta_{-\Delta}(s)$,

$$\zeta_{-\Delta}(s) = \frac{V}{(\hbar\beta)^{d-2s}}\frac{2^{1-2s}}{\pi^{\frac{d+1}{2}}}\frac{\Gamma\left(\frac{d+1}{2}-s\right)\zeta(d+1-2s)}{\Gamma(s)}. \tag{7.80}$$

Using $\Gamma(s)^{-1} = s + \mathcal{O}(s^2)$, it follows that $\zeta_{-\Delta}(0) = 0$ and hence the full partition function can be written as

$$\log Z(\beta) = \frac{1}{2}\zeta'_{-\Delta}(s)\bigg|_{s=0} = \frac{V}{(\hbar\beta)^d}\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}. \tag{7.81}$$

This result is known as the (scalar) *black body result*. Using (7.22) we obtain

$$F(\beta) = -\frac{V}{\hbar^d \beta^{d+1}} \frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \qquad S(\beta) = \frac{k_B V}{(\hbar\beta)^d} \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{7.82}$$

that are the free energy and the entropy one would obtain from an infinite set of non-interacting harmonic oscillators at thermal equilibrium in a bath at fixed temperature $T = 1/k_B\beta$. Note that the entropy scales as the volume of the system, as it should being an extensive quantity. As we will see later on in chapter 8, when we allow for one spatial dimension to be "small", there will be a certain regime where the entropy does not scale as the entire volume, but as the volume of the large spatial directions.

7.2.2 Heat kernel approach

We are interested in computing the (Euclidean) transition amplitude from x' to x, *i.e.*

$$K_{-\Delta}(x, x'; \tau) = \langle x| e^{-\tau \hat{A}} |x'\rangle, \qquad \Delta = \partial_A \partial^A \equiv \sum_A \hat{H}_A. \tag{7.83}$$

We have for $K_{-\Delta}(x, x'; \tau)$

$$K_{-\Delta}(x, x'; \tau) = \prod_A \langle x| e^{-\tau \hat{H}_A} |x'\rangle \equiv \prod_A K_{\hat{H}_A}(x, x'; \tau). \tag{7.84}$$

Therefore, we can focus on a single factor $K_{\hat{H}_A}(x, x'; \tau)$ of the product in (7.84). Each term is a Hamiltonian of a free particle of mass $m = 1/2$ given by

$$\hat{H}_A = -\frac{\partial^2}{\partial x^{A2}} = \hat{p}_A^2, \qquad \left[\hat{x}^A, \hat{p}_A\right] = i. \tag{7.85}$$

In the following we omit the index A for notational simplicity. Repeating the steps (7.5)-(7.6) and substituting $\delta t = -i\delta\tau$, we arrive at

$$\langle x| e^{-\tau \hat{H}} |x'\rangle = \int_{-\infty}^{\infty} \left(\prod_{k=1}^{N+1} \mathrm{d}p_k\right) \left(\prod_{k=1}^{N} \mathrm{d}x_k\right) \langle x| e^{-\frac{1}{\hbar}\delta\tau \hat{H}} |p_{N+1}\rangle \langle p_{N+1}|x_N\rangle ... \langle x_k| e^{-\frac{1}{\hbar}\delta\tau \hat{H}} |p_k\rangle \langle p_k|x_{k-1}\rangle ...$$

$$... \langle x_1| e^{-\frac{1}{\hbar}\delta\tau \hat{H}} |p_1\rangle \langle p_1|x'\rangle. \tag{7.86}$$

We find, using $\langle p|x\rangle = \frac{1}{\sqrt{2\pi}} e^{-ipx}$ and $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$ that each term appearing in the product in (7.86) is

$$\langle x_k| e^{-\frac{1}{\hbar}\delta\tau \hat{H}} |p_k\rangle \langle p_k|x_{k-1}\rangle = \frac{1}{2\pi\hbar} e^{-\frac{1}{\hbar} p_k^2 \delta\tau} e^{ip_k(x_k - x_{k-1})}, \tag{7.87}$$

and thus

$$\langle x|e^{-\tau\hat{H}}|x'\rangle = \int_{-\infty}^{\infty}\left(\prod_{k=1}^{N+1}\frac{\mathrm{d}p_k}{2\pi\hbar}\right)\left(\prod_{k=1}^{N}\mathrm{d}x_k\right)e^{-\frac{1}{\hbar}p_{N+1}^2\delta\tau}e^{ip_{N+1}(x-x_N)}...e^{-\frac{1}{\hbar}p_k^2\delta\tau}e^{ip_k(x_k-x_{k-1})}$$

$$...e^{-\frac{1}{\hbar}p_1^2\delta\tau}e^{ip_1(x_1-x')}. \tag{7.88}$$

We can now perform all the $N+1$ integrals over p_k. We get

$$\langle x|e^{-\tau\hat{H}}|x'\rangle = \left(\frac{\hbar}{4\pi\delta\tau}\right)^{\frac{N+1}{2}}\int_{-\infty}^{\infty}\left(\prod_{k=1}^{N}\mathrm{d}x_k\right)e^{-\frac{\hbar}{4\delta\tau}(x-x_N)^2}...e^{-\frac{\hbar}{4\delta\tau}(x_k-x_{k-1})^2}...e^{-\frac{\hbar}{4\delta\tau}(x_1-x')^2}. \tag{7.89}$$

We start by doing the integral over x_1:

$$\int_{-\infty}^{\infty}\mathrm{d}x_1\, e^{-\frac{\hbar}{4\delta\tau}(x_2-x_1)^2}e^{-\frac{1}{4\delta\tau}(x_1-x')^2} = \sqrt{\frac{2\pi\delta\tau}{\hbar}}e^{-\frac{\hbar}{8\delta\tau}(x_2-x')^2}. \tag{7.90}$$

Then we perform it over x_2

$$\int_{-\infty}^{\infty}\mathrm{d}x_2\, e^{-\frac{\hbar}{4\delta\tau}(x_3-x_2)^2}e^{-\frac{\hbar}{8\delta\tau}(x_2-x')^2} = \sqrt{\frac{8\pi\delta\tau}{3\hbar}}e^{-\frac{\hbar}{12\delta\tau}(x_3-x)^2}. \tag{7.91}$$

The k-th integral to perform is

$$\int_{-\infty}^{\infty}\mathrm{d}x_k\, e^{-\frac{\hbar}{4\delta\tau}(x_{k+1}-x_k)^2}e^{-\frac{\hbar}{4k\delta\tau}(x_k-x')^2} = \sqrt{\frac{4\pi k\delta\tau}{(1+k)\hbar}}e^{-\frac{\hbar}{4(k+1)\delta\tau}(x_{k+1}-x')^2}. \tag{7.92}$$

The final result is

$$\langle x_|e^{-\tau\hat{H}}|x'\rangle = \sqrt{\frac{\hbar}{4\pi\delta\tau}}e^{-\frac{\hbar}{4(N+1)\delta\tau}(x-x')^2}\prod_{k=1}^{N}\sqrt{\frac{k}{k+1}} = \sqrt{\frac{\hbar}{4\pi\delta\tau(N+1)}}e^{-\frac{\hbar}{4(N+1)\delta\tau}(x-x')^2}. \tag{7.93}$$

Since $\tau = (N+1)\delta\tau$, we find, for $A = i$

$$\langle x_i|e^{-\tau\hat{H}_i}|x_i'\rangle = \sqrt{\frac{\hbar}{4\pi\tau}}e^{-\frac{\hbar}{4\tau}(x_i-x_i')^2}. \tag{7.94}$$

When $A = d+1$ we have to take into account that the particle starting from x'_{d+1} can arrive at x_{d+1} wrapping around the thermal cycle of length $\hbar\beta$ an integer number n of times. Since in the path integral we sum over all possible paths we get

$$\langle x_{d+1}|e^{-\tau\hat{H}_{d+1}}|x'_{d+1}\rangle_{\mathbb{S}^1_{\hbar\beta}} = \sum_{n\in\mathbb{Z}}\langle x_{d+1}+n\hbar\beta|e^{-\tau\hat{H}_{d+1}}|x'_{d+1}\rangle$$

$$= \sqrt{\frac{\hbar}{4\pi\tau}}\sum_{n\in\mathbb{Z}} e^{-\frac{\hbar}{4\tau}(x_{d+1}-x'_{d+1}+n\hbar\beta)^2}. \tag{7.95}$$

The full heat kernel can be written as

$$K_{-\Delta}(x,x';\tau) = \prod_A\langle x|e^{-\tau\hat{H}_A}|x'\rangle = \left(\frac{\hbar}{4\pi\tau}\right)^{\frac{d+1}{2}} e^{-\frac{\hbar}{4\tau}(x_i-x'_i)(x^i-x'^i)}\sum_{n\in\mathbb{Z}} e^{-\frac{\hbar}{4\tau}(x_{d+1}-x'_{d+1}+n\hbar\beta)^2}, \tag{7.96}$$

whose trace is

$$\mathrm{Tr}\,\hat{K}_{-\Delta}(\tau) = \int_0^{\hbar\beta}\mathrm{d}x^{d+1}\int_V \mathrm{d}^d x\, K_{-\Delta}(x,x;\tau) = \hbar\beta V\left(\frac{\hbar}{4\pi\tau}\right)^{\frac{d+1}{2}}\sum_{n\in\mathbb{Z}} e^{-\frac{\hbar}{4\tau}n^2\hbar^2\beta^2}. \tag{7.97}$$

Using (7.44), for the partition function we obtain, dropping the $n=0$ mode contribution,

$$\log Z(\beta) = \frac{\hbar^{\frac{d+3}{2}}\beta V}{2^{d+2}\pi^{\frac{d+1}{2}}}\sum_{n\in\mathbb{Z}}{}' \int_0^\infty \mathrm{d}\tau\,\tau^{-\frac{d+3}{2}} e^{-\frac{\hbar}{4\tau}n^2\hbar^2\beta^2}. \tag{7.98}$$

The integral yields

$$\int_0^\infty \mathrm{d}\tau\,\tau^{-\frac{d+3}{2}} e^{-\frac{\hbar}{4\tau}n^2\hbar^2\beta^2} = \left(\frac{4}{\hbar^3\beta^2 n^2}\right)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right), \qquad d>1. \tag{7.99}$$

Hence, the result for the partition function is

$$\log Z(\beta) = \frac{V}{(\hbar\beta)^d}\frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{7.100}$$

which is in perfect agreement with the result (7.81) found using the zeta function technique.

Before computing the partition function using the canonical approach, it is also instructive to apply to equation (7.95) the Poisson summation formula. The latter states that, given the standard formulae for the Fourier transform of a suitable function $f(x)$,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{d}k\,\tilde{f}(k)e^{ikx}, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{d}x\,f(x)e^{-ikx}, \tag{7.101}$$

then

$$\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \tilde{f}(2\pi m). \tag{7.102}$$

Using

$$f(x) = \sqrt{\frac{\hbar}{4\pi\tau}} e^{-\frac{\hbar}{4\tau}(x_{d+1}-x'_{d+1}+x\hbar\beta)^2}, \qquad \tilde{f}(k) = \frac{1}{\hbar\beta\sqrt{2\pi}} e^{i(x_{d+1}-x'_{d+1})\frac{k}{\hbar\beta}-\frac{\tau}{\hbar}\left(\frac{k}{\hbar\beta}\right)^2}, \tag{7.103}$$

we have, from (7.102)

$$\sqrt{\frac{\hbar}{4\pi\tau}} \sum_{n\in\mathbb{Z}} e^{-\frac{\hbar}{4\tau}(x_{d+1}-x'_{d+1}+n\hbar\beta)^2} = \frac{1}{\hbar\beta}\sum_{m\in\mathbb{Z}} e^{i(x_{d+1}-x'_{d+1})\frac{2\pi m}{\hbar\beta}-\frac{\tau}{\hbar}\left(\frac{2\pi m}{\hbar\beta}\right)^2}. \tag{7.104}$$

In this parametrization, the heat kernel can be written as

$$K_{-\Delta}(x,x';\tau) = \frac{1}{\hbar\beta}\left(\frac{\hbar}{4\pi\tau}\right)^{\frac{d}{2}} e^{-\frac{\hbar}{4\tau}(x_i-x'_i)(x^i-x'^i)} \sum_{m\in\mathbb{Z}} e^{i(x_{d+1}-x'_{d+1})\frac{2\pi m}{\hbar\beta}-\frac{\tau}{\hbar}\left(\frac{2\pi m}{\hbar\beta}\right)^2}, \tag{7.105}$$

and the partition function reads

$$\log Z(\beta) = \frac{V\hbar^{\frac{d}{2}}}{2^{d+1}\pi^{\frac{d}{2}}} {\sum_{m\in\mathbb{Z}}}' \int_0^\infty \mathrm{d}\tau\, \tau^{-\frac{d+2}{2}} e^{-\frac{\tau}{\hbar}\left(\frac{2\pi m}{\hbar\beta}\right)^2}, \tag{7.106}$$

where we have remove the $m = 0$ term. The integral yields

$$\int_0^\infty \mathrm{d}\tau\, \tau^{-\frac{d+2}{2}} e^{-\frac{\tau}{\hbar}\left(\frac{2\pi m}{\hbar\beta}\right)^2} = \left(\frac{2\pi m}{\hbar\beta}\right)^d \frac{1}{\hbar^{\frac{d}{2}}}\Gamma\left(-\frac{d}{2}\right), \qquad d<0, \tag{7.107}$$

and therefore

$$\log Z(\beta) = \frac{V\pi^{\frac{d}{2}}}{(\hbar\beta)^d}\Gamma\left(-\frac{d}{2}\right)\zeta(-d) \overset{(7.79)}{=} \frac{V}{(\hbar\beta)^d}\frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{7.108}$$

which is again the desired result.

7.2.3 Canonical approach

When promoting the variables a_{k_i} and $a^*_{k_i}$ introduced in section 7.1 to operators [37] we have, for the Hamiltonian (7.61),

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hbar}{2}{\sum_{n_i\in\mathbb{Z}^d}}' \omega_{k_i}(\hat{a}^\dagger_{k_i}\hat{a}_{k_i} + \hat{a}_{k_i}\hat{a}^\dagger_{k_i}), \tag{7.109}$$

Here we choose standard normal ordering : : for the Hamiltonian. One could have also chosen symmetric ordering, but this would produce an infinite zero point energy that needs to be renormalized. This is the

[37]The operator $\hat{a}_{k_i}$ is defined as $\hat{a}_{k_i} = \sqrt{\omega_{k_i}/2\hbar}(\hat{\phi}_{k_i} + i/\omega_{k_i}\hat{\Pi}_{k_i})$ so that $[\hat{a}_{k_i}, \hat{a}^\dagger_{k_i}] = \prod_i \delta_{n_i,n'_i}$

phenomenon of the Casimir energy that will be extensively discussed in the next chapters. Here, we just mention that the Casimir energy depends on the inverse of the distance between the sides of the volume and therefore in the large volume limit its contribution can be neglected. We also drop the contribution of the zero mode $\hat{p}$ in (7.109) on which we will return in the next section.

We consider

$$: \hat{H} := \hbar \sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i} \hat{a}^\dagger_{k_i} \hat{a}_{k_i}, \tag{7.110}$$

and a set of eigenstates of $: \hat{H} :$ is given by $|k_i\rangle$ such that $\hat{a}^\dagger_{k_i}|0\rangle = |k_i\rangle$, where $|0\rangle$ is the vacuum state. The partition function is

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta :\hat{H}:} = \prod_{n_i \in \mathbb{Z}^d}{}' \sum_{N_{k_i}=0}^{\infty} e^{-\hbar\beta\omega_{k_i} n_{k_i}} = \prod_{n_i \in \mathbb{Z}^d}{}' \frac{1}{1 - e^{-\hbar\beta\omega_{k_i}}}. \tag{7.111}$$

In (7.111), N_{k_i} is the eigenvalue of the number operator $\hat{N}_{k_i} = \hat{a}^\dagger_{k_i} \hat{a}_{k_i}$. As usual we consider $\log Z(\beta)$ instead of $Z(\beta)$ so that the product in (7.111) over n_i becomes a sum that, in the large volume limit, can be replaced by an integral

$$\begin{aligned} \log Z(\beta) &= -\sum_{n_i}{}' \log\left(1 - e^{-\hbar\beta\omega_{k_i}}\right) = -\frac{V}{(2\pi)^d} \int_{-\infty}^{\infty} \mathrm{d}^d k_i \, \log\left(1 - e^{-\hbar\beta\omega_{k_i}}\right) \\ &= -\frac{V}{(2\pi)^d} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^{\infty} \mathrm{d}k k^{d-1} \, \log\left(1 - e^{-\hbar\beta k}\right) \overset{x=\hbar\beta k}{=} \frac{V}{(\hbar\beta)^d} \frac{1}{d 2^{d-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^{\infty} \mathrm{d}x \frac{x^d}{e^x - 1}, \end{aligned} \tag{7.112}$$

where in the last step we performed an integration by parts. The integral gives

$$\int_0^{\infty} \mathrm{d}x \frac{x^d}{e^x - 1} = \Gamma(d+1)\zeta(d+1), \qquad d > 0, \tag{7.113}$$

so that

$$\log Z(\beta) = \frac{V}{(\hbar\beta)^d} \frac{\Gamma(d+1)\zeta(d+1)}{d 2^{d-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})}. \tag{7.114}$$

We can now use the reduplication formula,

$$\Gamma(z)\sqrt{\pi} = 2^{z-1} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right), \tag{7.115}$$

at $z = d$ and $\Gamma(d+1) = d\Gamma(d)$ so that (7.114) becomes

$$\log Z(\beta) = \frac{V}{(\hbar\beta)^d} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{7.116}$$

as it should. Note that in this approach and in particular from (7.111) it is clear that the partition function is an infinite product of partition functions of single harmonic oscillators with energies ω_{k_i}.

7.3 Contribution of the zero modes

We have seen that, in order to have an agreement between (7.116) and the result obtained with different techniques, we have to drop the $\hat{p}$ contribution in the Hamiltonian (7.109). This can be justified as follows. Note that the $\hat{p}$ term in the Hamiltonian is that of a free particle and, since $\hat{p}$ commutes with $\hat{H}$, the true Fock space of the theory is build upon a family of vacua $|p\rangle$, where p is the real continuous eigenvalue of $\hat{p}$. Therefore, the vacua satisfy

$$\hat{a}_{k_i}|p\rangle = 0, \qquad \hat{p}|p\rangle = p|p\rangle, \qquad p \in \mathbb{R}. \tag{7.117}$$

It means that, *a priori*, when we evaluate the trace (7.111) in the Fock space we should also include a term $\mathrm{Tr}\exp\{-\beta\hat{p}^2/2\}$. In order to compute it, we compactify the field ϕ and we assume that it can take values on a circle of circumference $\mathcal{A}$,

$$\phi \sim \phi + \mathcal{A}. \tag{7.118}$$

Note that, in $\hbar = 1$ units, the canonical dimension of ϕ is $[\phi] = [L]^{\frac{d-1}{2}}$, so that also $[\mathcal{A}] = [L]^{\frac{d-1}{2}}$. Assuming (7.118) implies that the "coordinate" q in (7.62) must also be periodic with period $\sqrt{V}\mathcal{A}$

$$q \sim q + \sqrt{V}\mathcal{A}. \tag{7.119}$$

Hence, when considering the quantized free particle with generalized coordinate $\hat{q}$ and momentum $\hat{p}$, such that $[\hat{q}, \hat{p}] = i$, the wave function of the system $\psi(q)$ must satisfy $\psi(q) = \psi(q + \sqrt{V}\mathcal{A})$. Therefore, $\psi(q)$ admits the expansion

$$\psi(q) = \sum_{n\in\mathbb{Z}} \psi_n \varphi_n(q), \tag{7.120}$$

where the orthonormal basis $\{\varphi_n(q)\}$ is given by

$$\varphi_n(q) = \frac{1}{\sqrt{\mathcal{A}\sqrt{V}}} e^{\frac{2\pi i}{\mathcal{A}\sqrt{V}}nq}, \qquad \int_0^{\mathcal{A}\sqrt{V}} \mathrm{d}q\, \varphi_n^*(q)\varphi_m(q) = \delta_{nm}, \qquad \sum_{n\in\mathbb{Z}} \varphi_n^*(q)\varphi_n(q') = \delta(q - q'). \tag{7.121}$$

Indeed, $\varphi_n(q)$ are eigenfunctions of $\hat{p}$ and denoting $\varphi_n(q) = \langle q|n\rangle$, we have

$$\hat{p}|n\rangle = p_n|n\rangle, \qquad p_n = \frac{2\pi n}{\mathcal{A}\sqrt{V}}, \qquad n \in \mathbb{Z}. \tag{7.122}$$

Compactifying the field ϕ had the effect of discretizying the spectrum (7.117) of the zero mode operator

$\hat{p}$. When we evaluate its contribution to the partition function, we have

$$Z_0(\beta) = \operatorname{Tr} e^{-\beta \hat{H}_0} = \sum_{n\in\mathbb{Z}} \langle n| e^{-\beta \hat{H}_0} |n\rangle = \sum_{n\in\mathbb{Z}} e^{-2\beta\left(\frac{\pi n}{\mathcal{A}\sqrt{V}}\right)^2}, \tag{7.123}$$

where we used $\hat{H}_0 = \frac{\hat{p}^2}{2}$. Taking the large $\mathcal{A}\sqrt{V}$ limit, the previous expression becomes

$$Z_0(\beta) = \frac{\mathcal{A}\sqrt{V}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \, e^{-\frac{\beta p^2}{2}} = \mathcal{A}\sqrt{\frac{V}{2\pi\beta}}. \tag{7.124}$$

Hence $\log Z_0(\beta) \sim \log V$ and it can be neglected as $V \to \infty$ because we have proved that $\log Z(\beta) \sim V$. In the case of a Bose field on the torus, analyzed in section 8.1, we will see that the contribution (7.124) not only cannot be discarded, but it will actually play a crucial role in establishing modular invariance of the result.

7.4 Some comments

So far, we used periodic boundary conditions (7.53) in all spatial directions and claimed that in the large volume limit the result for the partition function is independent of this choice. If one had chosen Dirichlet conditions [38] $\phi(t, x^1, ..., 0, ..., x^d) = 0 = \phi(t, x^1, ..., L_i, ..., x^d)$ the appropriate orthonormal and complete basis in which the fields can be expanded would be

$$e_{k_i}(x) = \sqrt{\frac{2^d}{V}} \prod_i \sin k_i x^i = e^*_{k_i}(x), \qquad k_i = \frac{\pi}{L_i} n_i, \qquad n_i \in \mathbb{N}, \qquad \forall\, i = 1, ..., d. \tag{7.125}$$

All mode expansions are then the same except that sums and products are restricted to $n_i > 0$. In particular there is no $n_i = (0, ..., 0)$ mode that has to be dealt with. The fact that the sums start from $n_i > 0$ is compensated by the large volume limit that now reads $\sum_{n_i > 0} \to \frac{L_i}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k_i$. Equivalently, the sums can be extended from $\mathbb{N}^d$ to $\mathbb{Z}^d/(0, .., 0) = Z^d_*$ because the modes ϕ_{k_i} and Π_{k_i} are even functions of k_i, *i.e.* $\phi_{k_i} = -\phi_{-k_i}$ and $\Pi_{k_i} = -\Pi_{-k_i}$ and one has

$$\sum_{n_i \in \mathbb{N}^d} f_{k_i} f^*_{k_i} = \frac{1}{2^d} \sum_{n_i \in \mathbb{Z}^d_*} f_{k_i} f^*_{k_i}, \tag{7.126}$$

where f_{k_i} is either ϕ_{k_i} or Π_{k_i}.

The standard argument to get the partition function for the electromagnetic field is to multiply the result for $\log Z(\beta)$ in (7.81) by 2 in order to take into account the 2 independent polarizations of the photon. We already point out here that this argument is certainly true in the infinite volume limit. However, when the manifold is partially compactified, *e.g.* in the standard setup of the electromagnetic Casimir effect, one should be careful. Indeed, as we will discuss in section 9.1, the correct electromagnetic result (in $d = 3$)

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[38] Similar consideration apply to Neumann boundary conditions $\partial_i \phi(t, x)|_{x^i=0} = 0 = \partial_i \phi(t, x)|_{x^i=L_i}$ where, up to zero modes, the sines in (7.125) have to be replaced by cosines. See *e.g.* Appendix D.

with Casimir boundary conditions along the x^3 direction in an interval of length L_3 is more correctly obtained from that of a single scalar field with periodic boundary conditions on the double interval $2L_3$. This shows that, since in the large volume limit $\log Z(\beta)$ depends linearly on L_3, the correct electromagnetic result is obtained by multiplying the scalar field result by 2. However, when some spatial direction is taken to be "small" with respect to the other this will be no longer true.

8 Modular covariance, temperature dualities and Eisenstein series

In this chapter, we analyze the consequences of keeping one spatial direction "small " with respect to the others and, in particular, we assume that such direction is compactified on a circle $\mathbb{S}^1_L$ of circumference L. As we show, this leads to the notion of temperature dualities, according to which the high- and low-temperature and the small- and large- volume limits are related and thus not independent. The mathematical tool to deal with these interesting dualities is the modular group $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2$. Modular transformations naturally appear when working with theories living on partially compactified manifolds having a two-dimensional torus component.

We start in section 8.1 with an introduction where we show the standard computation of the Casimir energy of a massless scalar field in $(d+1)$-dimensions and we comment on how the expression for such energy is connected, through a temperature duality, to the black body result of (7.81). In section 8.2 we review the massless two-dimensional scalar field living on $\mathbb{T}^2$ and we show in 8.2.1, following the canonical approach, that the partition function of this theory is modular invariant. Despite the apparent simplicity of this model, the computation of the partition function in two-dimensions is quite subtle because of the presence of zero modes of the Laplacian on $\mathbb{T}^2 = \mathbb{S}^1_{\hbar\beta} \times \mathbb{S}^1_L$ that cannot be neglected. In section 8.3 we deal with the $(d+1)$-dimensional case, where the manifold is $\mathbb{T}^2 \times \mathbb{R}^{d-1}$ and we show that the partition function is modular covariant and that it can be expressed in terms of the real analytic Eisenstein series. In particular, in 8.3.1, 8.3.2 and 8.3.3 we show how to compute such partition function using the zeta function, heat kernel and canonical techniques, respectively, and what are the advantages and drawbacks of each of these methods. In 8.4 we make use of modular transformations and the Fourier series of the real analytic Eisenstein series to derive the high- and low-temperature expansions of the partition function and of the entropy, showing the power of temperature dualities. Remarkably, we prove that in the low-temperature regime the entropy does not scale as the entire volume of the system, but as the volume of the large directions only and we trace back the microscopic origin of this behavior. We conclude in 8.5 with the most general case of a manifold $\mathbb{T}^{q+1} \times \mathbb{R}^p$, showing the transformation properties of the partition function under the $\mathrm{SL}(q+1,\mathbb{Z})$ group, which naturally replaces $\mathrm{SL}(2,\mathbb{Z})$ in the case of higher-dimensional tori.

8.1 Casimir energy

So far, we only considered the partition function of a massless scalar in the large volume limit. We started in (7.53) by imposing periodic boundary conditions in all spatial directions of lengths L_i and then we let $L_i \to \infty$. In other words, we started with a compact manifold $\mathcal{M} = \mathbb{S}^1_{\hbar\beta} \times \mathbb{T}^d$ but then, after taking the large volume limit, we decompactified $\mathcal{M}$ to $\mathbb{S}^1_{\hbar\beta} \times \mathbb{R}^d$. This crucial step allows to pass from sums over the integers n_i labelling the eigenvalues of the discrete spectrum of the Laplacian on $\mathbb{T}^d$ to integrals over the momenta k_i and correspondingly to factorize the volume V of the system in $\log Z(\beta)$, leading to the black body result (7.81).

Here we keep the length of one spatial direction, say $L_d \equiv L$, small with respect to the others, *i.e.* $L \ll L_i$ for $i = 1, ..., d-1$. Hence, we want to investigate the case of a partially compactified manifold $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^{d-1}$, where $\mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_L$. One of the main consequences of this is the emergence of the Casimir energy [209]. Historically, the latter was derived for the electromagnetic field between two perfectly conducting plates separated by a small distance. Here, we derive an analogue expression for the scalar Casimir energy, leaving a full treatment of the electromagnetic case to chapter 9.

We start from the Hamiltonian in (7.109) and we choose symmetric ordering so that, neglecting again the contribution of $\hat{\Pi}_0$, we get

$$\hat{H} = \hbar \sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i} \left(\hat{a}^\dagger_{k_i} \hat{a}_{k_i} + \frac{1}{2} \right). \tag{8.1}$$

The expectation value of the Hamiltonian on the vacuum, *i.e.* the zero point energy is therefore divergent and given by

$$E_0 = \langle 0 | \hat{H} | 0 \rangle = \frac{\hbar}{2} \sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i}. \tag{8.2}$$

As mentioned, we are taking $d-1$ dimensions to be large and hence it is convenient to introduce a new index $a = 1, ..., d-1$ labelling them. By taking the limit of large L_a, we can turn the sums over n_a of (8.2) into integrals,

$$E_0 = \frac{\hbar}{2} \sum_{n_d \in \mathbb{Z}} \frac{\prod_a L_a}{(2\pi)^{d-1}} \int_{-\infty}^{\infty} \mathrm{d}^{d-1} k_a \left[k_a k^a + \left(\frac{2\pi}{L} n_d \right)^2 \right]^{\frac{1}{2}}. \tag{8.3}$$

In order to renormalize this UV divergent expression, we use ζ function regularization. We start by defining $E_0(s)$ as

$$\begin{aligned} E_0(s) &\equiv \mu^{2s} \frac{\hbar}{2} \sum_{n_d \in \mathbb{Z}} \frac{\prod_a L_a}{(2\pi)^{d-1}} \int_{-\infty}^{\infty} \mathrm{d}^{d-1} k_a \left[k_a k^a + \left(\frac{2\pi}{L} n_d \right)^2 \right]^{\frac{1-2s}{2}} \\ &= \mu^{2s} \frac{\hbar}{2} \sum_{n_d \in \mathbb{Z}} \frac{\prod_a L_a}{(2\pi)^{d-1}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^{\infty} \mathrm{d}k k^{d-2} \left[k^2 + \left(\frac{2\pi}{L} n_d \right)^2 \right]^{\frac{1-2s}{2}}, \end{aligned} \tag{8.4}$$

where $2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$ is the area of the $d-2$ dimensional sphere and μ has the dimension of a mass. After the regularization, we have to take the limit $s \to 0$. The $n_d = 0$ term can be dropped by the same arguments of (7.77) and we get, introducing a new integration variable y as $k = \frac{2\pi n_d}{L} y$,

$$E_0(s) = \mu^{2s} \hbar \frac{\prod_a L_a}{L^{d-2s}} \frac{\pi^{\frac{d+1}{2} - 2s} \zeta(2s-d)}{2^{2s-2} \Gamma(\frac{d-1}{2})} \int_0^{\infty} \mathrm{d}y \, y^{d-2} \left(y^2 + 1 \right)^{\frac{1-2s}{2}}. \tag{8.5}$$

Using the reflection formula (7.79) and the integral

$$\int_0^{\infty} \mathrm{d}y \, y^{d-2} (y^2+1)^{\frac{1-2s}{2}} = \frac{1}{2} \frac{\Gamma\left(\frac{d-1}{2}\right) \Gamma\left(s - \frac{d}{2}\right)}{\Gamma\left(s - \frac{1}{2}\right)}, \qquad \mathfrak{Re}(s) > \frac{d}{2}, \tag{8.6}$$

we get,

$$E_0(s) = \mu^{2s}\hbar\frac{\prod_a L_a}{L^{d-2s}}\frac{\Gamma\left(\frac{d+1-2s}{2}\right)}{2^{2s-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{2s-1}{2}\right)}\zeta(d+1-2s). \tag{8.7}$$

Taking the limit $s \to 0$ yields

$$E_0 = -\hbar\frac{\prod_a L_a}{L^d}\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{8.8}$$

which is the Casimir energy of a $(d+1)$-dimensional massless scalar field [235, 249]. Note that E_0 is proportional to $\hbar$ so that it is a pure quantum effect and that, as claimed in section 7.2.3, it vanishes for $L \to \infty$, so that in the large volume limit we can neglect it. Note also that the "core" of the renormalization procedure was the use of (7.79) to do an analytic continuation of the Riemann zeta function.

The correct electromagnetic result for the Casimir setup, in $d = 3$, as will be shown in chapter 9.1 is given by

$$E_0^{\text{em}} = -\hbar\frac{\pi^2 L_1 L_2}{720 L^3}, \tag{8.9}$$

which is in agreement with (8.8) for $d = 3$ and with the replacement $L \to 2L$, as discussed in the last part of section 7.4.

Here, the zeta function technique has been used to renormalize the divergent vacuum energy. We started with a divergent series and, after the regularization procedure, the result (8.8) is finite. The same result can be obtained by means of other different, more "physical", regularization schemes. For instance, it is possible to derive the same expression (8.8) by defining the vacuum energy as

$$E_0 = \lim_{\delta\to 0}\Delta E_0(\delta), \tag{8.10}$$

where $\Delta E_0(\delta)$ [39] is the regularized difference between the vacuum energy obtained by sending $L \to \infty$ and that obtained by keeping L finite, *i.e.*

$$\Delta E_0(\delta) = E_0(\delta) - E_0^{\text{vac}}(\delta), \tag{8.11}$$

where

$$E_0(\delta) = \frac{\hbar}{2}\frac{\prod_a L_a}{(2\pi)^{d-1}}\int_{-\infty}^{\infty} \mathrm{d}^{d-1}k_a \sum_{n_d\in\mathbb{Z}}\left[k_a k^a + \left(\frac{2\pi}{L}n_d\right)^2\right]^{\frac{1}{2}} e^{-\delta\sqrt{k_a k^a + \left(\frac{2\pi}{L}n_d\right)^2}}, \tag{8.12}$$

[39] Sometimes ΔE_0 in (8.11) is referred to as the *Casimir subtraction*.

and

$$E_0^{\mathrm{vac}}(\delta) = \frac{\hbar}{2}\frac{L\prod_a L_a}{(2\pi)^d}\int_{-\infty}^{\infty}\mathrm{d}^d k_i \Big[k_a k^a + k_d^2\Big]^{\frac{1}{2}} e^{-\delta\sqrt{k_i k^i}}. \tag{8.13}$$

Here we have introduced in the expressions for the vacuum energies a UV cut-off function $e^{-\delta k}$ with $\delta > 0$ in order to neglect the contributions coming from arbitrarily high frequency modes. In other words, the presence of finite boundaries forces the field modes to be quantized. However, high energy modes are not affected by the boundaries and they should be neglected in the computation of the free energy. A way to get ride of them is to subtract the empty space result as in (8.11).

Notice that, for $d = 1$, we get $E_0 = -\hbar\pi/6L$ which yields exactly the value of the central charge of a free Bose field on the torus [193]. Indeed, the central charge is related to E_0 as $E_0 = -\frac{\pi c}{6L}$, yielding $c = \hbar$. Generalizing this argument to d spatial dimensions yields the central charge

$$c = \hbar\frac{\prod_a L_a}{L^{d-1}}\frac{6\,\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+3}{2}}}. \tag{8.14}$$

Besides being interesting in itself, the computation just shown of the Casimir energy allows to introduce the notion of temperature dualities. Indeed, we expect that computing the partition function $Z(\beta)$ on $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^{d-1}$ would give a free energy $F(\beta) = -\beta^{-1}\log Z(\beta)$ such that, in the $\beta \to \infty$ limit, it reproduces exactly E_0 (see *e.g.* (7.49)). Therefore, on denoting $Z_{\mathrm{low}}(\beta)$ the partition function in the low-temperature limit, we expect that

$$\log Z_{\mathrm{low}}(\beta) = -\beta E_0 = \frac{\hbar\beta}{L}\frac{\prod_a L_a}{L^{d-1}}\frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{\frac{d+1}{2}}} \equiv \tau_2\frac{\prod_a L_a}{L^{d-1}}\frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{8.15}$$

where we defined a dimensionless quantity $\tau_2 \equiv \hbar\beta/L$. At this stage the introduction of this parameter might seem artificial but, as we will show soon, τ_2 is the (imaginary part of the) so called *modular parameter* $\tau = \tau_1 + i\tau_2$ (or *Teichmüller parameter*) and it is a fundamental geometrical quantity to consider when dealing with partition functions on manifolds having a two-dimensional torus component. Note that sending $\beta \to \infty$ at fixed L, which is a standard low-temperature limit, is equivalent to take $\tau_2 \gg 1$ which, in turn, is equivalent to take $L \to 0$ at fixed β, which is the small-separation in the x^d direction limit. Conversely, the large separation limit $L \to \infty$ at fixed β is equivalent to take $\tau_2 \ll 1$ and hence to send $\beta \to 0$ at fixed L. This naive argument suggests that somehow, taking the high-/low- temperature limits is "equivalent" of taking the large-small L limits. Hence, on denoting $Z_{\mathrm{high}}(\beta)$ the partition function at large volumes and therefore at high temperatures, from the analysis in the previous chapter, we expect it to be governed by the black body result (7.81),

$$\log Z_{\mathrm{high}}(\beta) = \left(\frac{L}{\hbar\beta}\right)^d\frac{\prod_a L_a}{L^{d-1}}\frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{\frac{d+1}{2}}} = \frac{1}{\tau_2^d}\frac{\prod_a L_a}{L^{d-1}}\frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}. \tag{8.16}$$

Comparing (8.15) and (8.16), we can more conveniently write that

$$\log Z_{\text{high}}(\tau_2) = \frac{1}{\tau_2^{d-1}} \log Z_{\text{low}}\left(\frac{1}{\tau_2}\right). \tag{8.17}$$

This equation relates the high- and low-temperature and large- and small L limits of the partition function. Knowing the theory in a certain regime is enough to know the behavior of the theory in the opposite regime. These powerful techniques are referred to as *temperature dualities* and they have led to strong analytic results, as mentioned in the introduction 6. The "temperature inversion" symmetry of (8.17) is just a particular case of a richer symmetry, namely

$$\log Z(\tau', \bar{\tau}') = |c\tau + d|^{d-1} \log Z(\tau, \bar{\tau}), \qquad \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}^2, \tag{8.18}$$

in the particular case $\tau_1 = 0$, $a = 0 = d$ and $b = -1 = -c$. In the following we will rigorously prove that the partition function of a massless scalar on $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^{d-1}$ satisfies (8.18) by a variety of techniques and we will explore the physical consequences of such symmetry.

8.2 The Bose field on the Euclidean torus

Here we consider the thermodynamics of a scalar field ϕ theory in two spacetime dimensions and we assume that the underlying manifold over which ϕ lives is a Euclidean torus, $\mathcal{M} = \mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_L$. Thus, we consider the model described by the Euclidean action [40]

$$S^E[\phi] = \frac{1}{2} \int_0^\beta \mathrm{d}x^2 \int_0^L \mathrm{d}x^1\, \partial_A \phi \partial^A \phi, \qquad A = 1, 2, \tag{8.19}$$

where x^2 is the Euclidean time direction and ϕ satisfies periodic boundary conditions

$$\phi(x^1, x^2) = \phi(x^1 + L, x^2), \qquad \phi(x^1, x^2) = \phi(x^1, x^2 + \beta). \tag{8.20}$$

The properties of this model have been widely investigated, see *e.g.* [173, 192, 193, 250].

We consider the partition function of the theory

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \mathrm{Tr}\, e^{-\frac{2\pi\beta}{L}\left(\hat{L}_0 + \hat{\bar{L}}_0 - \frac{c}{12}\right)}, \tag{8.21}$$

where the operators $\hat{L}_0$ and $\hat{\bar{L}}_0$ are the $n = 0$ Virasoro modes defined in Appendix E and $c = 1$. Equation

[40] From now on we set $\hbar = 1$.

(8.21) can be written as

$$Z(\tau) = \mathrm{Tr}\, e^{2\pi i\tau\left(\hat{L}_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(\hat{\bar{L}}_0 - \frac{c}{24}\right)}, \qquad \tau = i\frac{\beta}{L} = i\tau_2. \tag{8.22}$$

Note that if we considered instead of (8.21) the more general partition function

$$Z(\beta,\mu) = \mathrm{Tr}\, e^{-\beta(\hat{H} - i\mu\hat{P}_1)} = \mathrm{Tr}\, e^{-\frac{2\pi\beta}{L}\left[\hat{L}_0 + \hat{\bar{L}}_0 - \frac{c}{12} - i\mu(\hat{L}_0 - \hat{\bar{L}}_0)\right]}, \tag{8.23}$$

where $\hat{P}_1 = \frac{2\pi}{L}(\hat{L} - \hat{\bar{L}}_0)$ is the standard generator of translations along x^1 defined in (7.64), we would have

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\, e^{2\pi i\tau\left(\hat{L}_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(\hat{\bar{L}}_0 - \frac{c}{24}\right)}, \qquad \tau = \frac{\beta}{L}(\mu + i) = \tau_1 + i\tau_2. \tag{8.24}$$

When comparing equations (8.22) and (8.24), we notice that they are formally given the same expression. However, while in the former τ is a purely imaginary number, in the latter τ has also a non-vanishing real part, whose magnitude is controlled by μ. The geometrical interpretation of adding a real part to the modular parameter is as follows.

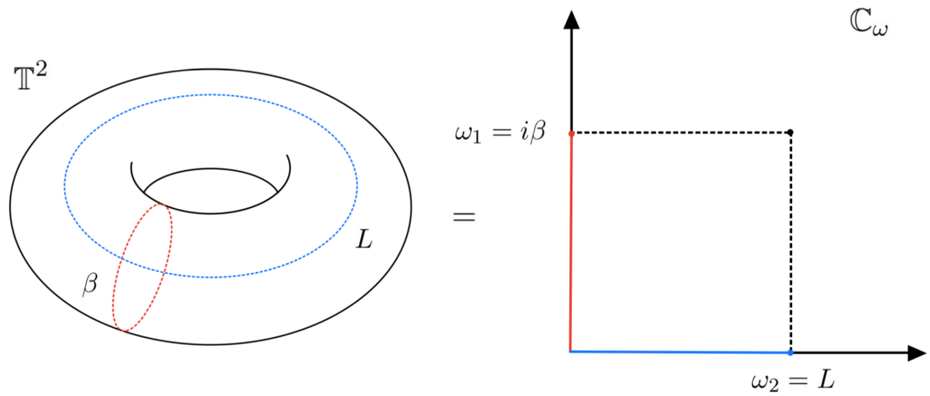

Figure 12: A torus $\mathbb{T}^2$ generated by $\omega_1 = i\beta$ and $\omega_2 = L$ represented in the complex plane $\mathbb{C}_\omega$.

As remarked, the manifold over which ϕ lives is a torus $\mathbb{T}^2$, which is mathematically described by the quotient of the flat space $\mathbb{R}^2$ by the lattice Λ^2 generated by a pair of linearly independent vectors $(\vec{\omega}_1, \vec{\omega}_2)$, denoted *periods* of the lattice, where we identify all the points that differ by an integer combination of $\vec{\omega}_1$ and $\vec{\omega}_2$:

$$\mathbb{T}^2 = \mathbb{R}^2/\Lambda^2, \qquad \Lambda^2 = \{m\vec{\omega}_1 + n\vec{\omega}_2 |\, (n,m) \in \mathbb{Z}^2\}. \tag{8.25}$$

Equivalently, one can choose two complex numbers (ω_1, ω_2) in the complex plane $\mathbb{C}_\omega$, whose imaginary and real axes are chosen to be the Euclidean time and spatial directions, respectively. We define the modular parameter of the torus to be $\tau = \omega_1/\omega_2$. In the case we are considering we take

$$\omega_1 = i\beta, \qquad \omega_2 = L, \qquad \tau = i\frac{\beta}{L}, \tag{8.26}$$

as represented in Figure 12. Such rectangular torus can be obtained by gluing the two parallel sides of the rectangle in Figure 12. Adding a real part to the modular parameter means that, before gluing the two sides parallel to the imaginary axis in $\mathbb{C}_\omega$, we tilt them by an amount depending on μ, as represented in Figure 13. Hence, taking

$$\omega_1 = \beta\mu + i\beta, \qquad \omega_2 = L, \qquad \tau = \frac{\beta}{L}(\mu + i), \tag{8.27}$$

we obtain a skewed torus, denoted by $\mathbb{T}^2_\mu$.

Therefore, we will refer to the quantity in (8.23)-(8.24) as the partition function on the skewed torus.

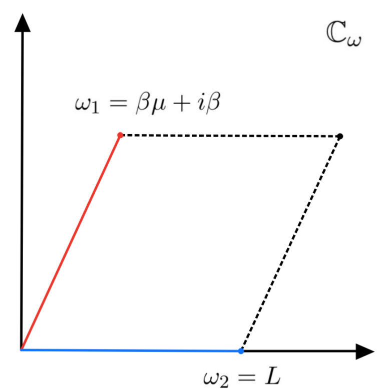

Figure 13: A skewed torus $\mathbb{T}^2_\mu$ generated by $\omega_1 = \beta(\mu + i)$ and $\omega_2 = L$.

Sometimes, we will denote the latter $\mathbb{T}^2_\mu$. Referring to the complex plane $\mathbb{C}_\tau$, we see that a Euclidean time translation of length τ_2 does not end up at the starting point, but is displaced in space by an amount equal to τ_1.

According to the definition (8.25) of $\mathbb{T}^2$, we could have defined the same torus by any other integer combination of ω_1 and ω_2. In other words, if we took new independent lattice vectors (ω_1', ω_2') as

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2, \tag{8.28}$$

they would define the same torus. The matrix in (8.28) has to be invertible because we must be able to express also (ω_1, ω_2) in terms of (ω_1', ω_2') and its determinant has to be one because the area of the unit cell must be the same whatever periods we use. Furthermore, we can reverse the sign of all its entries because the lattice spanned by $(-\omega_1, -\omega_2)$ is equal to the one spanned by (ω_1, ω_2). Under (8.28), the modular parameter transforms as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2. \tag{8.29}$$

Returning to the partition function $Z(\tau,\bar\tau)$ we have

$$Z(\tau,\bar\tau) = \mathrm{Tr}\, q^{\hat L_0 - \frac{c}{24}}\, \bar q^{\hat{\bar L}_0 - \frac{c}{24}}, \qquad q = e^{2\pi i \tau}, \qquad \bar q = e^{-2\pi i \bar\tau}. \tag{8.30}$$

Note that $Z(\tau,\bar\tau)$, because of conformal invariance, does not depend separately on the two periods, but only on their ratio τ and further, from (8.30), we expect an holomorphic/anti-holomorphic factorization for $Z(\tau,\bar\tau)$. Because of the above arguments, we also expect $Z(\tau,\bar\tau)$ to be invariant under modular transformations of the modular parameter τ.

8.2.1 Partition function on $\mathbb{T}^2$ and modular invariance

We now proceed to explicitly compute the partition function $Z(\tau,\bar\tau)$. We use the canonical approach and hence we are interested in the quantized Hamiltonian and momentum, given by [41]

$$\hat H' = {\sum_{n\in\mathbb{Z}}}' \frac{2\pi|n|}{L} \hat a_n^\dagger \hat a_n - \frac{\pi}{6L}, \qquad \hat P = {\sum_{n\in\mathbb{Z}}}' \frac{2\pi n}{L} \hat a_n^\dagger \hat a_n, \tag{8.31}$$

where $\hat H'$ is the Hamiltonian without the zero mode, that we will consider separately. We have

$$\begin{aligned} Z'(\beta,\mu) &= \mathrm{Tr}\, e^{-\beta(\hat H' - i\mu \hat P)} = e^{\frac{\pi\beta}{6L}}\, \mathrm{Tr}\, e^{-\beta \sum'_{n\in\mathbb{Z}} \frac{2\pi}{L}(|n| - i\mu n)\hat a_n^\dagger \hat a_n)} = e^{\frac{\pi\beta}{6L}} {\prod_n}' \sum_{N_n\geq 0} e^{-\frac{2\pi\beta}{L}(|n|-i\mu n)N_n} \\ &= {\prod_n}' \frac{e^{\frac{\pi\beta}{6L}}}{1 - e^{-\frac{2\pi\beta}{L}(|n|-i\mu n)}}. \end{aligned} \tag{8.32}$$

where N_n is the eigenvalue of the number operator $\hat N_n = \hat a_n^\dagger \hat a_n$. As usual, we are interested in $\log Z'(\beta,\mu)$,

$$\log Z'(\beta,\mu) = \frac{\pi\beta}{6L} - {\sum_{n\in\mathbb{Z}}}' \log\left(1 - e^{-\frac{2\pi\beta}{L}(|n|-i\mu n)}\right) \tag{8.33}$$

$$= \frac{\pi\beta}{6L} - \sum_{n\in\mathbb{N}} \log\big(1 - e^{2\pi i \tau n}\big) - \sum_{n\in\mathbb{N}} \log\big(1 - e^{-2\pi i \bar\tau n}\big) \tag{8.34}$$

where we used the Taylor expansion of $\log(1-x)$ around $x=0$ and we introduced $\tau = \frac{\beta}{L}(\mu + i)$. Note that the modular parameter appears naturally during the computation of $Z(\beta,\mu)$. Taking into account the definition of q in (8.30),

$$\log Z(\tau,\bar\tau) = -\log\left(|q|^{\frac{1}{12}}\right) - \sum_{n\in\mathbb{N}} \log(1 - q^n) - \sum_{n\in\mathbb{N}} \log(1 - \bar q^n) \tag{8.35}$$

[41] Here, since $k = \frac{2\pi n}{L}$, we use the convention $a_k \equiv a_n$.

.
We define the *Dedekind's η function* as

$$\eta(q) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1 - q^n). \tag{8.36}$$

Clearly,

$$Z'(\tau, \bar{\tau}) = \frac{1}{|\eta(q)|^2}. \tag{8.37}$$

As mentioned, the partition function must be invariant under the modular group. In order to check this we just need to prove that it is invariant under $\mathcal{T}$ and $\mathcal{S}$ transformations generating the whole $\mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}^2$, defined as

$$\mathcal{T} : \tau' = \tau + 1, \qquad \mathcal{S} : \tau' = -\frac{1}{\tau}, \tag{8.38}$$

satisfying $(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \mathbb{1}$. It can be shown[42] that the Dedekind's eta function transforms as

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau), \tag{8.39}$$

and therefore the $Z'(\tau, \bar{\tau})$ is not modular invariant. This happens because we have excluded the zero mode. Indeed, by taking into account the contribution coming from the continuous eigenvalue of $\hat{\Pi}_0$ we have, from (7.124),

$$Z_0(\beta) = \mathcal{A}\sqrt{\frac{L}{2\pi\beta}} = \mathcal{A}\frac{1}{\sqrt{2\pi\tau_2}}. \tag{8.40}$$

Hence, up to irrelevant factors, the full partition function is

$$Z(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}} \frac{1}{|\eta(q)|^2}, \tag{8.41}$$

consistently with what was originally found in [193] using the zeta function technique. This expression, using (8.38) and that under a modular transformation τ_2 transforms as

$$\tau_2' = \frac{\tau_2}{|c\tau + d|^2}, \tag{8.42}$$

turns out to be exactly modular invariant

$$Z(\tau', \bar{\tau}') = Z(\tau, \bar{\tau}), \qquad \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}^2. \tag{8.43}$$

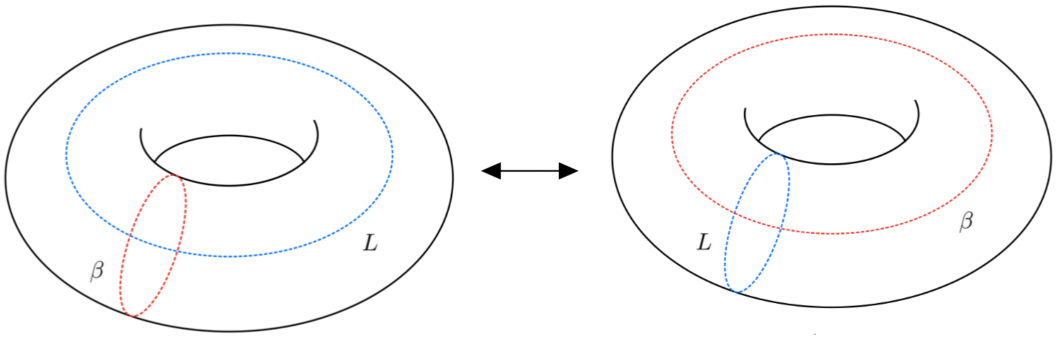

Figure 14: Swapping of thermal and spatial cycles of $\mathbb{T}^2$.

Setting $\mu = 0$ and considering $\mathcal{S}$ transformations, equation (8.43) reduces to

$$Z(\tau_2) = Z\left(\frac{1}{\tau_2}\right). \tag{8.44}$$

It is worth remarking that here the contribution to the partition function of the zero mode of the field is fundamental to achieve modular invariance. Contrarily to the arguments of 7.3, here it cannot be discarded and this is due to the fact that, in this model, there are no large directions. The manifold is just $\mathbb{T}^2$, so that the spectrum of the Laplacian is purely discrete and there is no large volume limit that could allow us to neglect the zero mode.

Swapping the thermal and spatial cycles of the torus, as schematically shown in 14, yields the same partition function, in agreement with what conjectured in (8.17), for $d = 1$. If we consider the high-temperature limit of $\log Z(\tau_2)$, defined by $\tau_2 \ll 1$, we see from (8.44) that it is the same of taking the limit of $\log Z(\tau_2)$ for $\tau_2 \gg 1$, which is the low-temperature limit.

8.3 Partition functions on $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^{d-1}$ and modular covariance

In the previous section we have seen that, provided that the underlying manifold is $\mathbb{T}^2$, the partition function of the theory is exactly modular invariant. We now investigate in detail the case of a partially compactified manifold having a $\mathbb{T}^2$ component, *i.e.* $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^{d-1}$ where we choose $\mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_L$ [204]. In particular, we are interested in the effect of mixing extended and compact dimensions on the partition function

$$Z(\beta, \mu) = \mathrm{Tr}\, e^{-\beta(\hat{H} - i\mu\hat{P}_d)}. \tag{8.45}$$

where $\hat{H}$ and $\hat{P}_d$ are the quantized operators corresponding to the classical observables in (7.61) and (7.65),

[42] See *e.g.* [173] and Appendix F.1, (F.21).

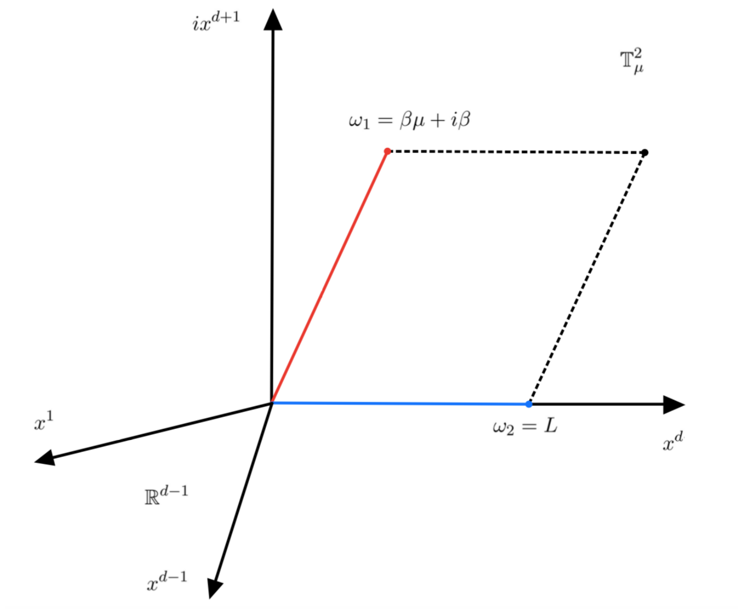

Figure 15: The manifold $\mathbb{T}^2_\mu \times \mathbb{R}^{d-1}$ obtained by tilting the period ω_1.

respectively. Further, we want to investigate what is its modular behavior and to understand to what extent the arguments of the previous section can be applied to this case. Again, by including the observable $\hat{P}_d$ in (8.45), we are tilting the period ω_1 of the torus adding to it a real part of magnitude $\beta\mu$, as represented in Figure 16.

In the following we consider again the case of a massless scalar field and we compute $Z(\beta,\mu)$ using the zeta function, heat kernel and canonical approaches.

8.3.1 Functional approach and zeta function

From the discussion in section 7.2.1 we can infer that $Z(\beta,\mu)$ admits the Hamiltonian path integral representation,

$$Z(\beta,\mu) = \mathrm{Tr}\, e^{-\beta(\hat{H}-i\mu\hat{P}_d)} = \int \frac{\mathcal{D}\Pi\,\mathcal{D}\phi}{2\pi} e^{-\tilde{S}^E_H[\phi,\Pi]}, \tag{8.46}$$

where we integrate over momenta and fields periodic in the Euclidean time direction and where, using (7.64),

$$\begin{aligned}\tilde{S}^E_H[\phi,\Pi] &= \int_0^\beta \mathrm{d}x^{d+1} \int_V \mathrm{d}^d x \Big[-i\Pi\partial_{d+1}\phi + \frac{1}{2}\Pi^2 + \frac{1}{2}\partial_i\phi\partial^i\phi + i\mu\Pi\partial_d\phi \Big] \\ &= S^E_H[\phi,\Pi] - i\mu \int_0^\beta \mathrm{d}x^{d+1} P_d[\phi,\Pi],\end{aligned} \tag{8.47}$$

The stationary points of $\tilde{S}^E_H$ are solutions of

$$0 = \frac{\delta \tilde{S}^E_H}{\delta \Pi}\bigg|_{\Pi_*} = -i\partial_{d+1}\phi + \Pi_* + i\mu\partial_d\phi \Longrightarrow \Pi_* = i\partial_{d+1}\phi - i\mu\partial_d\phi. \tag{8.48}$$

The partition function is then given by the path integral

$$Z(\beta,\mu) = \int \mathcal{D}\phi\, e^{-\tilde{S}^E[\phi]}, \tag{8.49}$$

where $\tilde{S}^E[\phi] = \tilde{S}^E_H[\phi,\Pi_*]$ and hence,

$$\tilde{S}^E[\phi] = -\int_0^\beta \mathrm{d}x^{d+1} \int_V \mathrm{d}^d x \left[\frac{1}{2}(\partial_{d+1}\phi - \mu\partial_d\phi)^2 + \frac{1}{2}\partial_i\phi\partial^i\phi\right]. \tag{8.50}$$

The operator $\hat{A}$ appearing in the above Euclidean action is the ordinary Laplacian with the substitution $\partial_{d+1} \to \partial_{d+1} - \mu\partial_d$, which we denote by Δ_μ. It implies that the eigenvalues of the Euclidean time part of the Laplacian are shifted by $-\mu\frac{2\pi}{L_d}n_d$ with respect to the ones appearing in (7.73) and they are explicitly given by

$$\begin{aligned}\lambda_{n_A} &= \left(\frac{2\pi}{\beta}n_{d+1} - \mu\frac{2\pi}{L}n_d\right)^2 + \left(\frac{2\pi}{L}n_d\right)^2 + \sum_{a=1}^{d-1}\left(\frac{2\pi}{L_a}n_a\right)^2 \\ &= \left(\frac{2\pi}{\beta}\right)^2 |n_{d+1} - \tau n_d|^2 + \sum_{a=1}^{d-1}\left(\frac{2\pi}{L_a}n_a\right)^2, \qquad \tau = \frac{\beta}{L}(\mu + i).\end{aligned} \tag{8.51}$$

Note again that the modular parameter on $\mathbb{T}^2_\mu$ appears naturally in the expression of the eigenvalues of the Laplacian.

The zeta function is $\zeta_{-\Delta_\mu}(s) = \sum'_{n_A\in\mathbb{Z}^{d+1}} \lambda_{n_A}^{-s}$. Taking the x^a dimensions to be large, we can turn the sums over n_a into integrals and we get, for $\zeta_{-\Delta_\mu}(s)$,

$$\zeta_{-\Delta_\mu}(s) = \frac{\prod_a L_a}{(2\pi)^{d-1}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \left(\frac{2\pi}{\beta}\right)^{-2s+d-1} \int_0^\infty \mathrm{d}y\, y^{d-2} \sum_{(n_d,n_{d+1})\in\mathbb{Z}^2} \left[|n_{d+1} + \tau n_d|^2 + y^2\right]^{-s}, \tag{8.52}$$

where we have changed the integration variable with $y = \frac{\beta}{2\pi}k$. The $(n_d, n_{d+1}) = (0,0)$ mode is not relevant because its contribution, using an IR regulator ϵ, is proportional to

$$\int_\epsilon^\infty \mathrm{d}y\, y^{d-2-2s} = -\frac{\epsilon^{d-2s-1}}{d-2s-1}, \qquad \mathfrak{Re}(s) > \frac{d-1}{2}, \tag{8.53}$$

to which the same arguments of (7.77) apply. Performing the integral in (8.52) yields

$$\sum_{(n_d,n_{d+1})\in\mathbb{Z}^2}' \int_0^\infty \mathrm{d}y\, y^{d-2}[|n_{d+1}+\tau n_d|^2+y^2]^{-s} = \frac{1}{2}\frac{\Gamma(\frac{d-1}{2})\Gamma(s-\frac{d-1}{2})}{\Gamma(s)} \sum_{(n_d,n_{d+1})\in\mathbb{Z}^2}' \frac{1}{|n_{d+1}+\tau n_d|^{2s+1-d}}. \tag{8.54}$$

We define the *real analytic Eisenstein series* $f_s(\tau)$ to be

$$f_s(\tau) = \sum_{(n,m)\in\mathbb{Z}^2}' \frac{\tau_2^s}{|n+\tau m|^{2s}}, \qquad \mathfrak{Re}(s) > 1. \tag{8.55}$$

where $\tau = \tau_1 + i\tau_2$, see *e.g.* [205, 206, 208, 235] for recent reviews. Some properties of interest of $f_s(\tau)$ are also reviewed in Appendix F.2. We have

$$\zeta_{-\Delta_\mu}(s) = \frac{\prod_a L_a}{L^{d-1-2s}\tau_2^{\frac{d-1}{2}-s}} \frac{\pi^{\frac{d-1}{2}-2s}}{2^{2s}} \frac{\Gamma(s-\frac{d-1}{2})}{\Gamma(s)} f_{s-\frac{d-1}{2}}(\tau). \tag{8.56}$$

Using the functional equation (F.30)

$$\Gamma(z) f_z(\tau) = \pi^{2z-1}\Gamma(1-z) f_{1-z}(\tau), \tag{8.57}$$

at $z = s - \frac{d-1}{2}$, we get

$$\zeta_{-\Delta_\mu}(s) = \frac{\prod_a L_a}{L^{d-1-2s}\tau_2^{\frac{d-1}{2}-s}} \frac{1}{2^{2s}\pi^{\frac{d+1}{2}}} \frac{\Gamma(\frac{d+1}{2}-s)}{\Gamma(s)} f_{\frac{d+1}{2}-s}(\tau). \tag{8.58}$$

Expanding $\Gamma(s)^{-1} = s + \mathcal{O}(s^2)$ we get for the partition function

$$\log Z(\tau,\bar\tau) = \frac{1}{2}\zeta'_{-\Delta_\mu}(s)\Big|_{s=0} = \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}} \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} f_{\frac{d+1}{2}}(\tau). \tag{8.59}$$

This is the final result for the partition function on $\mathbb{T}^2_\mu \times \mathbb{R}^{d-1}$. It is worth pointing out that here the renormalization procedure is achieved using the functional relation (8.57) that allows to do an analytic continuation of the real analytic Eisenstein series, whereas in section 8.1, in order to renormalize the vacuum energy, it was just necessary to use the Riemann zeta function reflection formula. Note also that, contrarily to what happens for the result (7.81) on $\mathbb{S}^1_\beta \times \mathbb{R}^d$, the total volume is replaced by the volume associated to the large directions and the Riemann zeta function is replaced by the real analytic Eisenstein series. The advantage of expressing $\log Z(\tau,\bar\tau)$ in terms of the latter is that it makes its modular properties very transparent. Indeed, using (8.42) and that

$$f_s(\tau') = f_s(\tau), \qquad \tau' = \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2, \tag{8.60}$$

i.e. the modular invariance of $f_s(\tau)$, it follows that under a modular transformation the partition function transforms as

$$\log Z(\tau', \bar{\tau}') = |c\tau + d|^{d-1} \log Z(\tau, \bar{\tau}). \tag{8.61}$$

Hence, $\log Z(\tau, \bar{\tau})$ is modular covariant and transforms with a weight $d - 1$, in agreement with (8.18). In the $\mu = 0$ case and when considering $\mathcal{S}$ transformations (8.61) implies (8.17). However, here we can drop the labels "high" and "low" and extend (8.17) to all orders in τ_2,

$$\log Z(\tau_2) = \frac{1}{\tau_2^{d-1}} \log Z\left(\frac{1}{\tau_2}\right). \tag{8.62}$$

Again, (8.62) is telling us that taking the high-temperature limit of $\log Z(\tau_2)$ is the same, up to an overall weight factor, of taking its low-temperature limit. Note that, if we are interested in the low- and high-

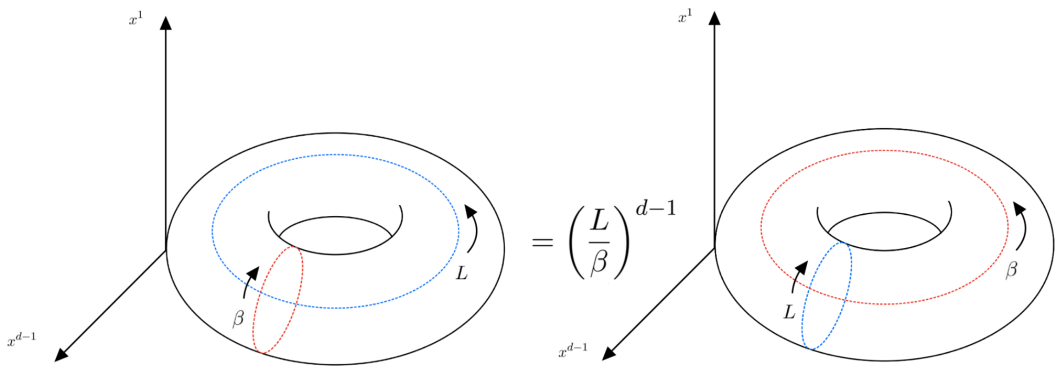

Figure 16: The effect on the partition function of swapping the thermal and spatial cycles in $(d + 1)$-dimensions is to get a weight factor as in (8.62).

temperature limits, $\tau_2 \gg 1$ and $\tau_2 \ll 1$, the leading contributions are due to $n_d = 0$ and $n_{d+1} = 0$ terms in the double sum appearing in the Eisenstein series and they are given by

$$\log Z(\tau, \bar{\tau}) = \frac{1}{2}\tau_2 \frac{\prod_a L_a}{L^{d-1}} \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} {\sum_{n_{d+1}\in\mathbb{Z}}}' \frac{1}{|n_{d+1}|^{d+1}} + \mathcal{O}(\tau_2^0) = -\beta E_0 + \mathcal{O}(\tau_2^0), \tag{8.63}$$

$$\log Z(\tau, \bar{\tau}) = \frac{1}{2}\frac{1}{\tau_2^d} \frac{\prod_a L_a}{L^{d-1}} \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} {\sum_{n_d\in\mathbb{Z}}}' \frac{1}{|n_d|^{d+1}} + \mathcal{O}(\tau_2^{1-d}) = \frac{V}{\beta^d} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}} + \mathcal{O}(\tau_2^{1-d}), \tag{8.64}$$

respectively, where E_0 is the Casimir energy in (8.8). Hence the above computation of the partition function "naturally" contains the renormalized Casimir energy and the standard black body result. Applying an $\mathcal{S}$ transformation to $\log Z(\tau, \bar{\tau})$ implies a swapping of the indices $n_{d+1} \leftrightarrow n_d$ in the Eisenstein series:

$$f_s\left(-\frac{1}{\tau}\right) = {\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}}' \frac{1}{|\tau|^{d+1}} \frac{\tau_2^{d+1}}{|n_{d+1}-\tau^{-1}n_d|^{d+1}} = {\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}}' \frac{1}{|n_d+\tau n_{d+1}|^{d+1}} = f_s(\tau). \tag{8.65}$$

Therefore we see that the swapping of the thermal and spatial cycles is mathematically implemented through an $\mathcal{S}$ transformation and this is explains why the the Casimir and black body results are "dual" and mapped one into the other through an inversion of the modular parameter. We will return more in the detail on the low- and high-temperature limits and their connection with $\mathcal{S}$ transformations in section 8.4, using the Fourier analysis.

Note that for $d = 1$ equations (8.61) and (8.62) imply (8.43) and (8.44) proved in the previous section. However, one must be careful because, in $d = 2$ the partition function cannot be expressed in terms of the real analytic Eisenstein series at $s = 1$. In other words, the expression

$$\log Z(\tau,\bar{\tau}) = \frac{1}{2\pi} f_1(\tau), \tag{8.66}$$

is meaningless because $f_s(\tau)$ is not absolutely convergent for $s = 1$, and hence not convergent since for real s the Eisenstein series coincides with the series of its absolute values. This is reviewed in Appendix F.3. Two spacetime dimensions is a peculiar case that deserves a special treatment, for it is the only case where the zero mode plays a role and where we cannot express the partition function in terms of the real analytic Eisenstein series, but we have to use the Dedekind's η function instead.

8.3.2 Heat kernel approach

In (7.96), we derived an expression for the heat kernel $K_{\hat{A}}(x,x';t)$ [43]on $\mathbb{S}^1_\beta \times \mathbb{R}^d$ as the product of single heat kernels $K_{\hat{H}_A}$ associated to the Hamiltonians $\hat{H}_A$ of free particles of mass $m = 1/2$. In that expression, the sum over $n \in \mathbb{Z}$ was due to the fact that a particle starting from x'_{d+1} could arrive at x_{d+1} by wrapping an integer number n of times around the thermal cycle of periodicity β, as clear from (7.95). If we now momentarily set $\mu = 0$, generalizing this argument to two compact dimensions is quite easy. In this case, in addition to the sum over n we would have an additional sum over an integer m labelling the number of times that the particle, starting from x'_d, wraps around the spatial cycle of length L in order to arrive to x_d. Concretely, for a Hamiltonian

$$\hat{H} = \hat{p}_{d+1}^2 + \hat{p}_d^2, \tag{8.67}$$

the heat kernel would be

[43] In this section, we denote by t the Schwinger proper time in order to avoid confusion with the modular parameter τ.

$$\begin{aligned}\langle x_{d+1}, x_d | e^{-t\hat{H}} | x'_{d+1}, x'_d \rangle_{\mathbb{T}^2} &= \langle x_{d+1} | e^{-t\hat{p}^2_{d+1}} | x'_{d+1} \rangle_{\mathbb{S}^1_\beta} \langle x_d | e^{-t\hat{p}^2_d} | x'_d \rangle_{\mathbb{S}^1_L} \\ &= \sum_{(n,m)\in\mathbb{Z}^2} \langle x_{d+1} + n\beta | e^{-t\hat{p}^2_{d+1}} | x'_{d+1} \rangle \langle x_d + mL | e^{-t\hat{p}^2_d} | x'_d \rangle \\ &\overset{(7.95)}{=} \frac{1}{4\pi t} \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}(x_{d+1} - x'_{d+1} + n\beta)^2} e^{-\frac{1}{4t}(x_d - x'_d + mL)^2}, \end{aligned} \tag{8.68}$$

i.e. it admits a complete factorization. However, we have seen in the previous section that when turning on μ, the operator appearing in the Euclidean action is $-\Delta_\mu$,

$$-\Delta_\mu = -\partial_a \partial^a - (\partial_{d+1} - \mu \partial_d)^2 - \partial_d^2. \tag{8.69}$$

Correspondingly, in the heat kernel formalism, the presence of a non-vanishing μ creates a cross-term between the Hamiltonian $\hat{H}_d$ and $\hat{H}_{d+1}$ as

$$\hat{H} = \hat{p}^2_{d+1} + \hat{p}^2_d \xrightarrow{\mu \neq 0} \hat{H}_\mu = (\hat{p}_{d+1} - \mu\, \hat{p}_d)^2 + \hat{p}^2_d. \tag{8.70}$$

which is the Hamilitonian of a particle moving on the skewed torus $\mathbb{T}^2_\mu$. The relevant heat kernel that we have to compute is now, instead of (8.68)

$$\langle x_{d+1}, x_d | e^{-t\hat{H}_\mu} | x'_{d+1}, x_d \rangle_{\mathbb{T}^2_\mu}, \tag{8.71}$$

that does not directly admit a factorization. We remark here that the periodicity of the coordinates is

$$x_{d+1} \sim x_{d+1} + n\beta, \qquad x_d \sim x_d + mL, \qquad (n,m) \in \mathbb{Z}^2. \tag{8.72}$$

We define new coordinates (X_{d+1}, X_d) as

$$X_{d+1}(x_{d+1}, x_d) = x_{d+1} \qquad X_d(x_{d+1}, x_d) = x_d + \mu\, x_{d+1}. \tag{8.73}$$

Their periodicity is inherited from that of (x_{d+1}, x_d)

$$X_{d+1} \sim X_{d+1} + n\beta, \qquad X_d \sim X_d + mL + n\mu\beta. \tag{8.74}$$

These coordinates have the effect of rectangularize the skewed torus. Indeed

$$\hat{p}_{d+1} - \mu\hat{p}_d = \frac{\partial}{\partial x_{d+1}} - \mu\frac{\partial}{\partial x_d} = \frac{\partial X_{d+1}}{\partial x_{d+1}}\frac{\partial}{\partial X_{d+1}} + \frac{\partial X_d}{\partial x_{d+1}}\frac{\partial}{\partial X_d} - \mu\frac{\partial X_{d+1}}{\partial x_d}\frac{\partial}{\partial X_{d+1}} - \mu\frac{\partial X_d}{\partial x_d}\frac{\partial}{\partial X_d}$$
$$= \frac{\partial}{\partial X_{d+1}} = \hat{P}_{d+1}, \tag{8.75}$$

$$\hat{p}_d = \frac{\partial}{\partial x_d} = \frac{\partial X_{d+1}}{\partial x_d}\frac{\partial}{\partial X_{d+1}} + \frac{\partial X_d}{\partial x_d}\frac{\partial}{\partial X_d} = \frac{\partial}{\partial X_d} = \hat{P}_d, \tag{8.76}$$

so that the Hamiltonian, in the new coordinates, is

$$\hat{H}_\mu = \hat{P}_{d+1}^2 + \hat{P}_d^2, \tag{8.77}$$

Thus, we have

$$\begin{aligned}
\langle x_{d+1}, x_d | e^{-t\hat{H}_\mu} | x'_{d+1}, x'_d \rangle_{\mathbb{T}^2_\mu} &= \sum_{(n,m)\in\mathbb{Z}^2} \langle X_{d+1} + n\beta | e^{-t\hat{P}^2_{d+1}} | X'_{d+1} \rangle \langle X_d + mL + n\mu\beta | e^{-t\hat{P}^2_d} | X'_d \rangle \\
&= \frac{1}{4\pi t} \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}[(X_{d+1} - X_{d+1} + n\beta)^2 + (X'_d - X_d + mL + n\mu\beta)^2]} \\
&= \frac{1}{4\pi t} \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}\{(x'_{d+1} - x_{d+1} + n\beta)^2 + [x'_d - x_d + mL + \mu(x'_{d+1} - x_{d+1} + n\beta)]^2\}},
\end{aligned} \tag{8.78}$$

Taking also into account the contribution of the large dimensions x^a, the full heat kernel is

$$\begin{aligned}
K_{-\Delta_\mu}(x, x'; t) &= \left(\frac{1}{4\pi t}\right)^{\frac{d+1}{2}} e^{-\frac{1}{4t}(x_a - x'_a)(x^a - x'^a)} \\
&\times \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}\{(x'_{d+1} - x_{d+1} + n\beta)^2 + [x'_d - x_d + mL + \mu(x'_{d+1} - x_{d+1} + n\beta)]^2\}},
\end{aligned} \tag{8.79}$$

The trace of the heat kernel is

$$\begin{aligned}
\operatorname{Tr} \hat{K}_{-\Delta_\mu}(t) &= \int_0^\beta \mathrm{d}x^{d+1} \int_V \mathrm{d}^d x \, K_{-\Delta_\mu}(x, x; t) \\
&= \beta L \prod_a L_a \left(\frac{1}{4\pi t}\right)^{\frac{d+1}{2}} \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}[n^2\beta^2 + (mL + n\mu\beta)^2]} \\
&= \beta L \prod_a L_a \left(\frac{1}{4\pi t}\right)^{\frac{d+1}{2}} \sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{L^2}{4t}|n\tau + m|^2}, \qquad \tau = \frac{\beta}{L}(\mu + i).
\end{aligned} \tag{8.80}$$

The partition function is therefore given by

$$\log Z(\tau,\bar\tau) = \frac{\beta L \prod_a L_a}{2^{d+2}\pi^{\frac{d+1}{2}}} \sideset{}{'}\sum_{(n,m)\in\mathbb{Z}^2} \int_0^\infty \mathrm{d}t\, t^{-\frac{d+3}{2}} e^{-\frac{L^2}{4t}|n\tau+m|^2}, \tag{8.81}$$

where we dropped the contribution of the zero mode $n = 0 = m$. The integral gives, analogously to (7.99),

$$\int_0^\infty \mathrm{d}t\, t^{-\frac{d+3}{2}} e^{-\frac{L^2}{4t}|n\tau+m|^2} = \left(\frac{4}{L^2}\right)^{\frac{d+1}{2}} \left(\frac{1}{|n\tau+m|^2}\right)^{\frac{d+1}{2}} \Gamma\Big(\frac{d+1}{2}\Big), \qquad d>1. \tag{8.82}$$

The full partition function reads

$$\log Z(\tau,\bar\tau) = \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}} \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} f_{\frac{d+1}{2}}(\tau), \tag{8.83}$$

This expression agrees completely with (8.59) found with the zeta function technique.

Again, it is possible to use the Poisson summation formula (7.102) to the double sum appearing in (8.78). The result can be shown to be

$$\begin{aligned} &\frac{1}{4\pi t}\sum_{(n,m)\in\mathbb{Z}^2} e^{-\frac{1}{4t}\{(x'_{d+1}-x_{d+1}+n\beta)^2+[x'_d-x_d+mL+\mu(x'_{d+1}-x_{d+1}+n\beta)]^2\}} \\ &= \frac{1}{\beta L}\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2} e^{i(x_{d+1}-x'_{d+1})\frac{2\pi n_{d+1}}{\beta}+i(x_d-x'_d)\frac{2\pi n_d}{L}-t\left(\frac{2\pi}{\beta}\right)^2|n_{d+1}+n_d\tau|^2}. \end{aligned} \tag{8.84}$$

Having applied the Poisson summation formula, the full heat kernel can be written as

$$\begin{aligned} K_{-\Delta_\mu}(x,x';t) &= \frac{1}{\beta L}\left(\frac{1}{4\pi t}\right)^{\frac{d-1}{2}} e^{-\frac{1}{4t}(x_a-x'_a)(x^a-x'^a)} \\ &\quad\times \sum_{(n_{d+1},n_d)\in\mathbb{Z}^2} e^{i(x_{d+1}-x'_{d+1})\frac{2\pi n_{d+1}}{\beta}+i(x_d-x'_d)\frac{2\pi n_d}{L}-t\left(\frac{2\pi}{\beta}\right)^2|n_{d+1}+n_d\tau|^2}, \end{aligned} \tag{8.85}$$

and hence the partition function is

$$\log Z(\tau,\bar\tau) = \frac{\prod_a L_a}{2^d\pi^{\frac{d-1}{2}}} \sideset{}{'}\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2} \int_0^\infty \mathrm{d}t\, t^{-\frac{d+1}{2}} e^{-t\left(\frac{2\pi}{\beta}\right)^2|n_{d+1}+n_d\tau|^2}. \tag{8.86}$$

The integral gives

$$\int_0^\infty \mathrm{d}t\, t^{-\frac{d+1}{2}} e^{-t\left(\frac{2\pi}{\beta}\right)^2|m+l\tau|^2} = \left(\frac{2\pi}{\beta}\right)^{d-1} \Gamma\Big(-\frac{d-1}{2}\Big)|n_{d+1}+\tau n_d|^{d-1}, \qquad d<1 \tag{8.87}$$

and thus

$$\log Z(\tau,\bar{\tau}) = \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}}\frac{\Gamma(\frac{1-d}{2})}{\pi^{\frac{1-d}{2}}} f_{\frac{1-d}{2}}(\tau) \overset{(8.57)}{=} \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}}\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} f_{\frac{d+1}{2}}(\tau), \tag{8.88}$$

giving again the correct result.

8.3.3 Canonical approach

Here we repeat the same computation of $Z(\beta,\mu)$ following the canonical approach. Remarkably, the result obtained with this method will be expressed directly in terms of the Fourier transform of the real analytic Eisenstein series displayed in (F.25). We start from the operators $\hat{H}'$ and $\hat{P}_d$, given by

$$\hat{H}' = {\sum_{n_i\in\mathbb{Z}^d}}' \omega_{k_i}\hat{a}^\dagger_{k_i}\hat{a}_{k_i} + E_0, \qquad \hat{P}_d = {\sum_{n_i\in\mathbb{Z}^d}}' k_d\,\hat{a}^\dagger_{k_i}\hat{a}_{k_i}, \tag{8.89}$$

where E_0 is the Casimir energy given in (8.8). We have

$$Z(\beta,\mu) = e^{-\beta E_0}{\prod_{n_i}}'\sum_{N_{k_i}\geq 0} e^{-\beta(\omega_{k_i}-i\mu k_d)N_{k_i}} = e^{-\beta E_0}{\prod_{n_i}}'\frac{1}{1-e^{-\beta\left(\omega_{k_i}-i\mu k_d\right)}}. \tag{8.90}$$

Hence, taking $\log Z(\beta,\mu)$ and turning the sums over n_a into integrals we get

$$\log Z(\beta,\mu) = -\beta E_0 - \frac{\prod_a L_a}{(2\pi)^{d-1}}\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\,k^{d-2}\log\left[1-e^{-\beta(\sqrt{k^2+k_d^2}-i\mu k_d)}\right]. \tag{8.91}$$

Introducing a new integration variable as $z=\beta\sqrt{k^2+k_d^2}$ we have

$$\log Z(\beta,\mu) = -\beta E_0 - \frac{\prod_a L_a}{\beta^{d-1}}\frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}}\int_{\beta|k_d|}^\infty \mathrm{d}z z(z^2-\beta^2k_d^2)^{\frac{d-3}{2}}\log\left[1-e^{-z+i\mu\beta k_d}\right]. \tag{8.92}$$

Now we expand the $\log(1-x)$ around $x=0$, with $x=e^{-z+i\mu\beta k_d}$,

$$\log\left[1-e^{-z+i\mu\beta k_d}\right] = -\sum_{l\in\mathbb{N}}\frac{e^{-lz+il\mu\beta k_d}}{l}, \tag{8.93}$$

and thus, using $y=lz$

$$\log Z(\beta,\mu) = -\beta E_0 + \frac{\prod_a L_a}{\beta^{d-1}}\frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\frac{e^{il\mu\beta k_d}}{l^d}\int_{l\beta|k_d|}^\infty \mathrm{d}y\,y(y^2-l^2\beta^2k_d^2)^{\frac{d-3}{2}}e^{-y}. \tag{8.94}$$

The integral in the above expression yields

$$\int_{l\beta|k_d|}^{\infty} \mathrm{d}y\, y(y^2 - l^2\beta^2k_d^2)^{\frac{d-3}{2}} e^{-y} = 2^{\frac{d-2}{2}} \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}} (l\beta|k_d|)^{\frac{d}{2}} K_{\frac{d}{2}}(l\beta|K_d|), \tag{8.95}$$

where $K_n(x)$ is the modified Bessel function of the second kind [251]. We have, using $k_d = (2\pi/L)n_d$,

$$\log Z(\beta,\mu) = -\beta E_0 + \frac{\prod_a L_a}{L^{\frac{d}{2}}\beta^{\frac{d-2}{2}}} \sum_{n_d\in\mathbb{Z}} \sum_{l\in\mathbb{Z}}{}' \left|\frac{n_d}{l}\right|^{\frac{d}{2}} K_{\frac{d}{2}}\left(\frac{2\pi\beta|ln_d|}{L}\right) e^{2\pi i n_d l \frac{\beta\mu}{L}}, \tag{8.96}$$

where we have extended the sum to $l \in \mathbb{Z}_*$ using (F.27). For later convenience, we isolate the term $n_d = 0$ in the above sum. Using that, for small value of x and for $n > 0$,

$$K_n(x) = \Gamma(n)2^{n-1}x^{-n} + \mathcal{O}(x^{-n+1}), \qquad x > 0, \tag{8.97}$$

the $n_d = 0$ term in equation (8.96) gives a contribution

$$\frac{\prod_a L_a}{L^{\frac{d}{2}}\beta^{\frac{d-2}{2}}} 2^{\frac{d-2}{2}}\Gamma\left(\frac{d}{2}\right)\left(\frac{2\pi\beta}{L}\right)^{-\frac{d}{2}} \sum_{l\in\mathbb{Z}}{}' |l|^{-d} = \frac{\prod_a L_a}{\beta^{d-1}\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\zeta(d). \tag{8.98}$$

Taking into account the explicit value (8.8) of the Casimir energy, the previous equation can be factorized as

$$\begin{aligned}\log Z(\tau,\bar\tau) = \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}}\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\Bigg[&2\zeta(d+1)\tau_2^{\frac{d+1}{2}} + 2\sqrt{\pi}\frac{\Gamma(\frac{d}{2})\zeta(d)}{\Gamma(\frac{d+1}{2})}\tau_2^{\frac{1-d}{2}} \\ &+ 2\tau_2^{\frac{1}{2}}\frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}\sum_{n_d\in\mathbb{Z}}{}'\sum_{l\in\mathbb{Z}}{}'\left|\frac{n_d}{l}\right|^{\frac{d}{2}} K_{\frac{d}{2}}\left(2\pi|ln|\tau_2\right)e^{2\pi i n_d l\tau_1}\Bigg],\end{aligned} \tag{8.99}$$

where $\tau = \frac{\beta}{L}(\mu + i)$. Comparing the expression between square bracket with (F.25), we see that it is exactly the Fourier series of the real analytic Eisenstein series $f_{\frac{d+1}{2}}(\tau)$, as claimed in section 7.2.3. We can conclude again that

$$\log Z(\tau,\bar\tau) = \frac{1}{2}\frac{\prod_a L_a}{L^{d-1}\tau_2^{\frac{d-1}{2}}}\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} f_{\frac{d+1}{2}}(\tau). \tag{8.100}$$

In Appendix F.5 we consider the $d = 3$ case and we explicitly show the equivalence between the series (8.99) and the Eisenstein series $f_2(\tau)$.

We point out here that the main advantage of working with the Fourier series (8.99) instead of the real analyitic Eisenstein series is that it allows to trace back the contribution of each field mode to the partition function and, as explained in the next section, it will be extremely useful to take the high- and low-temperature limits. For the moment let us just mention that the $n_d = 0$ contribution in (8.98) is exactly a black body contribution for a scalar field living in one dimension less. In other words, it can be obtained from the black body result computed previously in (7.81) by substituting $d \to d-1$. Indeed, from the mode expansion of ϕ in (7.56), isolating the *zero mode of the field in the compact direction*, we get

$$\phi(t,x) = \frac{1}{\sqrt{V}} \sum_{n_a \in \mathbb{Z}^{d-1}} \phi_{k_a}(t) e^{ik_a x^a} + {\sum_{n_d \in \mathbb{Z}}}' \sum_{n_a \in \mathbb{Z}^{d-1}} \ldots \tag{8.101}$$

The first term in the above expression is nothing but the mode expansion of a scalar field living in $(d-1)$ spatial large dimensions x^a and therefore its thermodynamics is described by the black body result.

8.4 Generalized high-/low- temperature dualities

Let us start by considering the low-temperature limit of the partition function, *i.e.* the $\tau_2 \gg 1$ limit of $Z(\tau, \bar{\tau})$. In this limit we expect, from the discussion in section 8.1 and from equation (8.15), $\log Z(\tau, \bar{\tau})$ to be proportional, through β, to the scalar Casimir energy E_0 in (8.8). In order to show that, it is convenient to use the Fourier series of the partition function in (8.99) and the asymptotic expansion for large positive values of x of the modified Bessel $K_n(x)$,

$$K_n(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \mathcal{O}(x^{-1})\right]. \tag{8.102}$$

Therefore, we see that all the terms appearing in the double sum in (8.99) are exponentially suppressed for large values of τ_2. There are only two terms left,

$$\begin{aligned} \log Z(\tau, \bar{\tau}) &= \frac{\prod_a L_a}{L^{d-1}} \left[\tau_2 \frac{\Gamma(\frac{d+1}{2}) \zeta(d+1)}{\pi^{\frac{d+1}{2}}} + \frac{1}{\tau_2^{d-1}} \frac{\Gamma(\frac{d}{2}) \zeta(d)}{\pi^{\frac{d}{2}}} \right] + \mathcal{O}(e^{-\tau_2}) \\ &\equiv \log Z_{\text{low}}(\tau, \bar{\tau}) + \mathcal{O}(e^{-\tau_2}). \end{aligned} \tag{8.103}$$

The leading term is indeed the Casimir contribution, as claimed, and the first subleading term is the one coming from the $n_d = 0$ term of the field mode expansion which is the lower-dimensional scalar field introduced previously.

In order to have access to the high-temperature limit, defined by $\tau_2 \ll 1$, we make explicit use of the modular properties of the partition function. We start by considering an $\mathcal{S}$ transformation obtained by taking $a = 0 = d$ and $b = -1 = -c$ in (8.61),

$$\tau' = -\frac{1}{\tau}, \qquad \tau_1' = -\frac{\tau_1}{|\tau|^2}, \qquad \tau_2' = \frac{\tau_2}{|\tau|^2}. \tag{8.104}$$

under which the partition function transforms as

$$\log Z(\tau', \bar{\tau}') = \log Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = |\tau|^{d-1} \log Z(\tau, \bar{\tau}). \tag{8.105}$$

Inverting this relation and using the Fourier series representation for $\log Z(\tau, \bar{\tau})$, we get

$$\begin{aligned}\log Z(\tau, \bar{\tau}) = \frac{\prod_a L_a}{L^{d-1}} \Bigg[& \frac{\tau_2}{|\tau|^{d+1}} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}} + \frac{|\tau|^{d-1}}{\tau_2^{d-1}} \frac{\Gamma(\frac{d}{2})\zeta(d)}{\pi^{\frac{d}{2}}} \\ & + \frac{1}{|\tau|\tau_2^{\frac{d-2}{2}}} \sum_{n_d \in \mathbb{Z}}{}' \sum_{l \in \mathbb{Z}}{}' \left|\frac{n_d}{l}\right|^{\frac{d}{2}} K_{\frac{d}{2}}\left(\frac{2\pi |ln| \tau_2}{|\tau|^2}\right) e^{-\frac{2\pi i n_d l \tau_1}{|\tau|^2}} \Bigg]. \end{aligned} \tag{8.106}$$

In the high-temperature limit we take $\tau_2 \ll 1$ and hence $|\tau| = \tau_2 |\mu + i| \ll 1$. It implies that the terms in the double sum in (8.106) are again exponentially suppressed because of the asymptotic behavior of the modified Bessel function in (8.102). Therefore, we get, in the high-temperature limit

$$\begin{aligned}\log Z(\tau, \bar{\tau}) &= \frac{\prod_a L_a}{L^{d-1}} \left[\frac{\tau_2}{|\tau|^{d+1}} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}} + \frac{|\tau|^{d-1}}{\tau_2^{d-1}} \frac{\Gamma(\frac{d}{2})\zeta(d)}{\pi^{\frac{d}{2}}} \right] + \mathcal{O}(e^{-\frac{1}{\tau_2}}) \\ &\equiv \log Z_{\text{high}}(\tau, \bar{\tau}) + \mathcal{O}(e^{-\frac{1}{\tau_2}}). \end{aligned} \tag{8.107}$$

The leading term of this expression is

$$\frac{V}{\beta^d (1+\mu^2)^{\frac{d+1}{2}}} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}} \xrightarrow{\mu \to 0} \frac{V}{\beta^d} \frac{\Gamma(\frac{d+1}{2})\zeta(d+1)}{\pi^{\frac{d+1}{2}}}, \tag{8.108}$$

which gives, as expected, the scalar black body result for vanishing μ. The subleading term is independent of the temperature and depends only on the parameter μ. Clearly we have

$$\log Z_{\text{high}}(\tau, \bar{\tau}) = \frac{1}{|\tau|^{d-1}} \log Z_{\text{low}}\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) \xrightarrow{\mu \to 0} \log Z_{\text{high}}(\tau_2) = \frac{1}{\tau_2^{d-1}} \log Z_{\text{low}}\left(\frac{1}{\tau_2}\right). \tag{8.109}$$

Notice that taking the high-temperature limit has become possible *after* having used the modular properties of the partition function. In particular, we have used first its behavior under $\mathcal{S}$ and then the Fourier series. The analysis shows that once we have access to the low-temperature limit, we can directly infer, using (8.109) the high-temperature regime and vice-versa. These are the so-called generalized temperature dualities. As already pointed out, the black body result, under the duality, maps to its opposite regime counterpart, which is the Casimir term. Analogously, the subleading term in the low-temperature expansion, which is due to the $n_d = 0$ term of the field mode expansion, is mapped through the duality to the subleading, temperature-independent term in the high-temperature expansion (8.108). Even if the subleading contribution in the high-temperature expansion can be simply understood by applying the inversion symmetry argument to the lower-dimensional scalar, it would however be wrong to think that it also comes from the $n_d = 0$ mode in the field expansion. In fact, even if we are able to trace back this term in the low-temperature limit, we cannot do the same in the high-temperature one. Inverting the temperature implies a re-summation and a re-shuffling of all the modes, through the Poisson resummation formula. In fact, the full partition function is covariant under the symmetry, but its Fourier coefficients are not separately covariant.

Before proceeding to analyze the low- and high- temperature behavior of the entropy of the system, it is useful to introduce the *completed Riemann zeta function* as

$$\xi(d) \equiv \frac{\Gamma\left(\frac{d}{2}\right)\zeta(d)}{\pi^{\frac{d}{2}}}. \tag{8.110}$$

Note that most of the formulae shown in this chapter can be more conveniently rewritten in terms of $\xi(d)$. This happens because it is a dimensionless quantity characterizing the geometry of the system. In order to make contact with [201], we introduce $\epsilon_{\text{vac}}(d) \equiv \xi(d+1)$. The Casimir, black body and their subleading terms in the expansions of $\log Z(\tau, \bar{\tau})$ can be expressed in terms of $\epsilon_{\text{vac}}(d)$.

The low-temperature expansion of the entropy is given by

$$S_{\text{low}}(\tau_2) = \left(1 - \tau_2 \frac{\partial}{\partial \tau_2}\right) \log Z_{\text{low}}(\tau_2) = \frac{\prod_a L_a}{L^{d-1}\tau_2^{d-1}} d\, \epsilon_{\text{vac}}(d-1) = \frac{\prod_a L_a}{\beta^{d-1}} d\, \epsilon_{\text{vac}}(d-1). \tag{8.111}$$

Hence, in the low-temperature regime the entropy is governed, up to exponentially suppressed terms, by the lower-dimensional scalar contribution, *i.e.* by the zero mode of the field in the compact direction. In fact, in (8.111) the Casimir term drops being linear in τ_2. Equation (8.111) shows that the entropy at low-temperatures does not scale as the entire volume of the system, but only as the volume of the large directions. Indeed, the length L of the compact direction does not appear in (8.111).

In the high temperature regime, up to exponentially suppressed terms we have for $\mu = 0$,

$$S_{\text{high}}(\tau_2) = \left(1 - \tau_2 \frac{\partial}{\partial \tau_2}\right) \log Z_{\text{high}}(\tau_2) = \frac{\prod_a L_a}{L^{d-1}\tau_2^{d}} (d+1)\, \epsilon_{\text{vac}}(d) = \frac{V}{\beta^d}(d+1)\epsilon_{\text{vac}}(d), \tag{8.112}$$

in complete agreement with [201]. Note that, differently from the low-temperature case, here the entropy scales as the entire volume V of the system, as one would expect in the large L limit.

Using $Z_{\text{high}}(\beta)$ we obtain the asymptotic microcanonical density of states as [44]

$$\rho(E) = \int_0^\infty \mathrm{d}\beta\, Z_{\text{high}}(\beta) e^{\beta E} = \int_0^\infty \mathrm{d}\beta\, e^{\frac{V \epsilon_{\text{vac}}(d)}{\beta^d}} e^{\beta E}. \tag{8.113}$$

[44] In equation (8.113) we neglect in $Z_{\text{high}}(\beta)$ the subleading term.

We can use the saddle point approximation to evaluate this integral. The stationary point of the exponent in (8.113) is given by β^* such that $f'(\beta^*) = 0$, where

$$f(\beta) = \frac{V}{\beta^d}\epsilon_{\text{vac}}(d) + \beta E \Longrightarrow \beta^* = \left(\frac{dV\epsilon_{\text{vac}}(d)}{E}\right)^{\frac{1}{d+1}}. \tag{8.114}$$

Thus

$$\rho(E) \approx e^{\frac{V\epsilon_{\text{vac}}(d)}{\beta^{*d}}} e^{\beta^* E} \Longrightarrow \log\rho(E) \approx \frac{d+1}{d^{\frac{d}{d+1}}}[\epsilon_{\text{vac}}(d)V]^{\frac{1}{d+1}} E^{\frac{d}{d+1}}. \tag{8.115}$$

Since $\epsilon_{\text{vac}}(d+1)$ is related to the central charge c in (8.14) by

$$\epsilon_{\text{vac}}(d) = \frac{\pi L^d}{6V}c, \tag{8.116}$$

both the high-temperature entropy and the asymptotic density of states in (8.112) and (8.116) can be expressed in terms of c as

$$S_{\text{high}}(\beta) = \left(\frac{L}{\beta}\right)^d (d+1)\frac{\pi}{6}c, \qquad \log\rho(E) \approx \left(\frac{EL}{d}\right)^{\frac{d}{d+1}} (d+1)\left(\frac{\pi c}{6}\right)^{\frac{1}{d+1}}. \tag{8.117}$$

Equations (8.112),(8.115) and (8.117) can be though of as higher-dimensional generalizations of the Cardy formula to the case of partially compactified manifolds [135, 199], expressing the high-temperature entropy and the asymptotic density of states in terms of a dimensionless number $\epsilon_{\text{vac}}(d)$ characterizing the Casimir energy and in terms of the central charge c of the theory [252].

8.5 Higher-dimensional tori and SL$(n, \mathbb{Z})$ Eisenstein series

In this section, we consider the most general case where there are p large and q small spatial directions, such that $p + q = d$. Here it is convenient to introduce indices $a = 1, ..., p$ and $\alpha = (i, d+1)$ with $i = p+1, ...d$ labelling the large spatial directions and an the small spatial and Euclidean time directions, respectively. We assume $L_a \gg L_i$, $\forall\, a$ and $\forall\, i$ and consider the partition function

$$Z(\beta, \mu_1, ..., \mu_q) = \text{Tr}\,(e^{-\beta(\hat{H} - i\sum_i \mu_i \hat{P}_i)}), \tag{8.118}$$

on a partially compactified manifold $\mathcal{M} = \mathbb{T}^{q+1} \times \mathbb{R}^p$, where $\mathbb{T}^{q+1} = \mathbb{S}^1_\beta \times \mathbb{S}^1_{L_{p+1}} \times ... \times \mathbb{S}^1_{L_d}$ [198, 201]. Mathematically, the torus $\mathbb{T}^{q+1}$ is defined to be the quotient of $\mathbb{R}^{q+1}$ by the lattice Λ^{q+1} generated by a set of $q+1$ linearly independent vectors $\vec{\omega}_\alpha \in \mathbb{R}^{q+1}$,

$$\mathbb{T}^{q+1} = \mathbb{R}^{q+1}/\Lambda^{q+1}, \qquad \Lambda^{q+1} = \{ \sum_\alpha m_\alpha \vec{\omega}_\alpha | \, m_\alpha \in \mathbb{Z}^{q+1} \}. \tag{8.119}$$

We choose $\vec{\omega}_\alpha$ to be

$$\vec{\omega}_i = (0, .., L_i, ..., 0), \qquad \vec{\omega}_{d+1} = \beta(\mu_1, \mu_2, ..., \mu_d, 1). \tag{8.120}$$

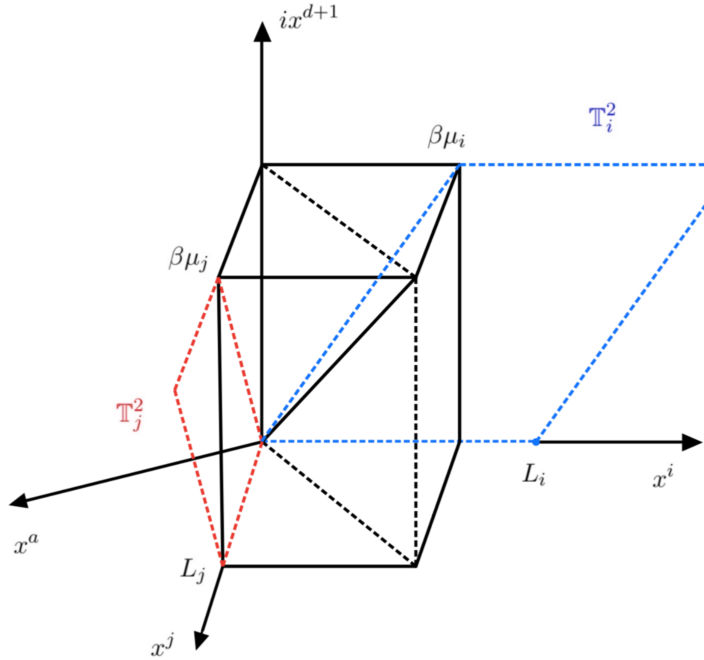

Figure 17: Geometry of the manifold associated with the choice of $\vec{\omega}_\alpha$ in (8.120).

Using the lattice vectors, we can define a metric on $\mathbb{T}^{q+1}$ as $\tilde{g}_{\alpha\beta} = \vec{\omega}_\alpha \cdot \vec{\omega}_\beta$. We have

$$\tilde{g}_{ij} = 0, \qquad \tilde{g}_{ii} = L_i^2, \qquad \tilde{g}_{i,d+1} = \beta\mu_i L_i, \qquad \tilde{g}_{d+1,d+1} = \beta^2, \tag{8.121}$$

and $\sqrt{\det[\tilde{g}_{\alpha\beta}]} = V_{q+1}$, where $V_{q+1} = \beta L_{p+1} ... L_d$ is the volume of the $(q+1)$-dimensional torus. The torus we are considering is not rectangular, but the Euclidean time lattice vector $\vec{\omega}_{d+1}$ has non vanishing components on the spatial directions x^α as represented in Figure 17. When turning off the parameters μ_α, the torus becomes rectangular. It is also important to introduce the inverse lattice vectors $\vec{K}^\alpha$ are defined by

$$\vec{K}^\alpha \cdot \vec{\omega}_\beta = \delta^\alpha_\beta. \tag{8.122}$$

It is easy to show that the $\hat{K}^\alpha$ corresponding to the choice of $\vec{\omega}_\alpha$ in (8.120) are given by

$$\vec{K}^i = \frac{1}{L_i}(0, .., 1, ..., 0, -\mu_i), \qquad \vec{K}^{d+1} = \frac{1}{\beta}(0, ..., 0, 1). \tag{8.123}$$

The matrix $\tilde{g}^{\alpha\beta} = \vec{K}^\alpha \cdot \vec{K}^\beta$, by construction, is the inverse of $\tilde{g}_{\alpha\beta}$ and its components are explicitly given by

$$\tilde{g}^{ij} = \frac{\mu_i \mu_j}{L_i L_j}, \qquad \tilde{g}^{ii} = \frac{1}{L_i^2}(1 + \mu_i^2), \qquad \tilde{g}^{i,d+1} = -\frac{\mu_i}{L_i \beta}, \qquad \tilde{g}^{d+1,d+1} = \frac{1}{\beta^2}, \tag{8.124}$$

whereas its determinant is $\sqrt{\det[\tilde{g}^{\alpha\beta}]} = V_{q+1}^{-1}$. It is convenient to introduce a normalized metric $g^{\alpha\beta} = (V_{q+1})^{\frac{2}{q+1}}\, \tilde{g}^{\alpha\beta}$ so that $\det[g_{\alpha\beta}] = 1 = \det[g^{\alpha\beta}]$.

Repeating similar arguments to those of section 8.3.1 the partition function in (8.118) admits the path integral representation

$$Z(\beta, \mu_1, ..., \mu_q) = \int \mathcal{D}\phi \, e^{-\tilde{S}^E[\phi]}, \tag{8.125}$$

where

$$\tilde{S}^E[\phi] = -\int_0^\beta \mathrm{d}x^{d+1} \int_V \mathrm{d}^d x \left[\frac{1}{2}\big(\partial_{d+1}\phi - \sum_i \mu_i \partial_i \phi\big)^2 + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{2}\partial_a\phi\partial^a\phi\right], \tag{8.126}$$

and hence the relevant zeta function is, taking p dimensions to be large and thus turning the sum over the integers n_a into integrals,

$$\zeta(s) = \frac{\prod_a L_a}{(2\pi)^p} \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \int_0^\infty \mathrm{d}k\, k^{p-1} \sideset{}{'}\sum_{n_\alpha \in \mathbb{Z}^{q+1}} \left[\left(\frac{2\pi n_{d+1}}{\beta} - \sum_i \mu_i \frac{2\pi n_i}{L_i}\right)^2 + \sum_i \left(\frac{2\pi n_i}{L_i}\right)^2 + k^2\right]^{-s}, \tag{8.127}$$

where we have already excluded the $n_\alpha = 0$ modes. Integrating over k we get

$$\zeta(s) = \frac{\prod_a L_a \pi^{\frac{p}{2}-2s}\Gamma(s - \frac{p}{2})}{2^{2s}\Gamma(s)} \sideset{}{'}\sum_{n_\alpha \in \mathbb{Z}^{q+1}} \left[\left(\frac{n_{d+1}}{\beta} - \sum_i \mu_i \frac{n_i}{L_i}\right)^2 + \sum_i \left(\frac{n_i}{L_i}\right)^2\right]^{\frac{p}{2}-s} \tag{8.128}$$

It is immediate to verify that the above equation can be more compactly rewritten in terms of $g^{\alpha\beta}$ as

$$\zeta(s) = \frac{\prod_a L_a \pi^{\frac{p}{2}-2s}\Gamma(s - \frac{p}{2})}{2^{2s}\Gamma(s)(V_{q+1})^{\frac{p-2s}{q+1}}} \sideset{}{'}\sum_{n_\alpha \in \mathbb{Z}^{q+1}} \left[n_\alpha\, g^{\alpha\beta} n_\beta\right]^{\frac{p}{2}-s}. \tag{8.129}$$

We define the SL$(n,\mathbb{Z})$ *Eisenstein series* $f_s(n;g)$ [205] to be [45]

$$f_s(n;g) = \sum_{m_\alpha \in \mathbb{Z}^n}{}' [m_\alpha\, g^{\alpha\beta} m_\beta]^{-s}, \qquad \mathfrak{Re}(s) > \frac{n}{2}, \tag{8.130}$$

where $g^{\alpha\beta}$ is the metric on a n-dimensional torus normalized to have unit determinant. For $n = 2$, using (8.124), the metric $g^{\alpha\beta}$ on $\mathbb{T}^2$ can be written as

$$g^{\alpha\beta} = \begin{pmatrix} \frac{\beta}{L}(\mu^2+1) & \mu \\ \mu & \frac{L}{\beta} \end{pmatrix} = \begin{pmatrix} \frac{|\tau|^2}{\tau_2} & \frac{\tau_1}{\tau_2} \\ \frac{\tau_1}{\tau_2} & \frac{1}{\tau_2} \end{pmatrix}, \tag{8.131}$$

once we take into account $\tau = \frac{\beta}{L}(\mu + i)$ so that

$$f_s(2;\tau) = \sum_{(n,m)\in\mathbb{Z}^2}{}' \frac{\tau_2^s}{|n+m\tau|^{2s}}, \tag{8.132}$$

is the real analytic Eisenstein series of (8.55), that therefore can be interpreted as the SL$(2,\mathbb{Z})$ Eisenstein series, according to the definition (8.130). The SL$(n,\mathbb{Z})$ Eisenstein series satisfies the functional equation

$$\Gamma(s) f_s(n;g) = \pi^{2s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right) f_{\frac{n}{2}-s}(n;g). \tag{8.133}$$

Note how this equation reduces to the one satisfied by the real analytic Eisenstein series $f_s(\tau)$ in (8.57) for $n = 2$ and to the reflection formula in (7.79) satisfied by Riemann zeta function $\zeta(2s)$ for $n = 1$. The zeta function in (8.129) can be written in terms of the SL$(q+1,\mathbb{Z})$ Eisenstein series $f_{s-\frac{p}{2}}(q+1;g)$:

$$\zeta(s) = \frac{\prod_a L_a \pi^{\frac{p}{2}-2s}\Gamma(s-\frac{p}{2})}{2^{2s}\Gamma(s)(V_{q+1})^{\frac{p-2s}{q+1}}} f_{s-\frac{p}{2}}(q+1;g). \tag{8.134}$$

so that, applying as usual (8.133),

$$\zeta(s) = \frac{\prod_a L_a \Gamma(\frac{p+q+1}{2}-s)}{2^{2s}\Gamma(s)\pi^{\frac{p+q+1}{2}}(V_{q+1})^{\frac{p-2s}{q+1}}} f_{\frac{p+q+1}{2}-s}(q+1;g). \tag{8.135}$$

Taking into account that $p+q = d$, we get for the partition function

$$\log Z(g) = \frac{1}{2}\zeta'(s)|_{s=0} = \frac{1}{2}\frac{\prod_a L_a \Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}(V_{q+1})^{\frac{p}{q+1}}} f_{\frac{d+1}{2}}(q+1;g). \tag{8.136}$$

[45] It is possible to consider also a GL$(n,\mathbb{Z})$ Eisenstein series $f_s(n;\tilde{g})$ where $\tilde{g}_{\alpha\beta}$ has not a unit determinant, but $\sqrt{\det[\tilde{g}_{\alpha\beta}]} = V_n$. Here we choose to work with a metric with normalized determinant.

This is the final result for the partition function on $\mathbb{T}^{q+1} \times \mathbb{R}^p$. Note that, even if it was derived it in the particular case of a metric $\tilde{g}^{\alpha\beta}$ in (8.124), equation (8.136) holds for an arbitrary manifold whose $\mathbb{T}^{q+1}$ component has a volume V_{q+1}, regardless of the choice of the lattice vectors. The particular torus geometry we are considering is motivated by the specific form of the partition function in (8.118). Equation (8.136) includes all the main results for the partition function shown so far as particular cases. For $p = d-1$ and $q = 1$ it correctly reduces to the partition function on $\mathbb{T}^2 \times \mathbb{R}^{d-1}$ in (8.59) covariant under $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2$ transformations of the modular parameter. For $p = d$ and $q = 0$ it reproduces the black body result in (7.81).

The $\mathrm{SL}(n,\mathbb{Z})$ Eisenstein series is a natural object to consider on $\mathbb{T}^n$. Indeed, it is invariant under $\mathrm{SL}(n,\mathbb{Z})$ transformations of the lattice vectors defining $\mathbb{T}^n$, *i.e.* under the diffeomorphisms of the torus into itself. Indeed, consider an $\mathrm{SL}(n,\mathbb{Z})$ transformation of the lattice vectors $\vec{\omega}_\alpha$,

$$\vec{\omega}'_\alpha = S_\alpha{}^\beta\,\vec{\omega}_\beta, \qquad g'^{\alpha\beta} = (S^{-1})^\alpha{}_\gamma g^{\gamma\eta}(S^{-1})^\beta{}_\eta, \qquad S_\alpha{}^\beta \in \mathrm{SL}(n,\mathbb{Z}). \tag{8.137}$$

Correspondingly the $\mathrm{SL}(n,\mathbb{Z})$ Eisenstein series transforms as

$$\begin{aligned} f_s(n;g') &= \sum_{m_\alpha\in\mathbb{Z}^n}{}' [m_\alpha\, g'^{\alpha\beta} m_\beta]^{-s} = \sum_{m_\alpha\in\mathbb{Z}^n}{}' [m_\alpha (S^{-1})^\alpha{}_\gamma g^{\gamma\eta}(S^{-1})^\beta{}_\eta m_\beta]^{-s} = \sum_{m'_\alpha\in\mathbb{Z}^n}{}' [m'_\gamma g^{\gamma\eta}\, m'_\eta]^{-s} \\ &= f_s(n;g). \end{aligned} \tag{8.138}$$

We are now interested in introducing a set of q modular parameters τ_i for each torus $\mathbb{T}^2_i = \mathbb{S}^1_\beta \times \mathbb{T}^2_{L_i}$ and to understand how $\mathrm{SL}(q+1,\mathbb{Z})$ transformations act on them and, correspondingly, on the partition function in (8.136). We define τ_i as

$$\tau_i = \tau_{i_1} + i\tau_{i_2} = \frac{\beta}{L_i}(\mu_i + i). \tag{8.139}$$

The components of the rescaled inverse metric $g^{\alpha\beta}$ can be expressed in terms of τ_i as follows:

$$g^{ij} = \frac{\mu_i\mu_j(\beta L_{p+1}...L_d)^{\frac{2}{q+1}}}{L_iL_j} = \frac{\tau_{i_1}\tau_{j_1}}{(\tau_{1_2}...\tau_{q_2})^{\frac{2}{q+1}}}, \tag{8.140}$$

$$g^{ii} = \frac{(1+\mu_i^2)(\beta L_{p+1}...L_d)^{\frac{2}{q+1}}}{L_i^2} = \frac{|\tau_i|^2}{(\tau_{1_2}...\tau_{q_2})^{\frac{2}{q+1}}}, \tag{8.141}$$

$$g^{i,d+1} = \frac{\mu_i(\beta L_{p+1}...L_d)^{\frac{2}{q+1}}}{L_i\beta} = \frac{\tau_{i_1}}{(\tau_{1_2}...\tau_{q_2})^{\frac{2}{q+1}}}, \tag{8.142}$$

$$g^{d+1,d+1} = \frac{(\beta L_{p+1}...L_d)^{\frac{2}{q+1}}}{\beta^2} = \frac{1}{(\tau_{1_2}...\tau_{q_2})^{\frac{2}{q+1}}}. \tag{8.143}$$

Note that in the case $q = 1$ the above equations imply that

$$g^{11} = \frac{|\tau|^2}{\tau_2}, \qquad g^{12} = \frac{\tau_1}{\tau_2}, \qquad g^{22} = \frac{1}{\tau_2}, \tag{8.144}$$

in agreement with (8.131). The volume V_{q+1} can be expressed in terms of τ_i as

$$V_{q+1} = \beta L_{p+1}...L_d = \left(\tau_{1_2}...\tau_{q_2}\right)^{\frac{1}{q}} (L_{p+1}....L_d)^{\frac{q+1}{q}}. \tag{8.145}$$

Therefore,

$$\log Z(\tau_i, \bar{\tau}_i) = \frac{1}{2} \frac{\prod_a L_a \Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}} (L_{p+1}...L_d)^{\frac{p}{q}} (\tau_{1_2}...\tau_{q_2})^{\frac{p}{q(q+1)}}} f_{\frac{d+1}{2}}(q+1; g). \tag{8.146}$$

The transformation laws of τ_i under $\mathrm{SL}(q+1, \mathbb{Z})$ is derived in detail in Appendix F.6. The final result in (F.76) for an $\mathrm{SL}(q+1, \mathbb{Z})$ transformation $\vec{\omega}'_\alpha = S_\alpha{}^\beta \vec{\omega}_\beta$ is

$$\log Z(\tau'_i, \bar{\tau}'_i) = \left[\frac{(S_{d+1}{}^{d+1} S_1{}^1 - S_{d+1}{}^1 S_1{}^{d+1})...(S_{d+1}{}^{d+1} S_q{}^q - S_{d+1}{}^q S_q{}^{d+1})}{\left|S_1{}^{d+1}\tau_1 + S_1^1\right|^2 ... |S_q{}^{d+1}\tau_q + S_q{}^q|^2} \right]^{-\frac{p}{q(q+1)}} \log Z(\tau_i, \bar{\tau}_i), \tag{8.147}$$

that correctly reduces to (8.61) in the $q = 1$ case.

8.6 Conclusions

In this chapter, we have shown how real analytic Eisenstein series naturally appear in the computation of partition functions of massless scalar fields living on partially compactified manifolds having some torus components. However, it has to be pointed out that the literature on these topics and, in general, on the dependence of the partion function on the boundary conditions, is substantial. In particular, Casimir energies and partition functions for the configurations considered in this chapter were originally computed in [249] in terms of the Epstein zeta functions (see *e.g.* Appendix F.4) and, moreover, they had also been derived in the context of conformal field theories in higher dimensions in [198], including a discussion on modular invariance and an extension to higher-dimensiional tori. More recently, general considerations on modular invariance in higher dimensions and its relation to Casimir energies have appeared in [202]. Therefore, the results derived here necessarily have large overlaps with existing literature.

In the context of finite temperature Casimir effect, our discussion and techniques follow closely the approach developed in [219–222] in order to derive and understand temperature inversion symmetry, originally discovered in [217] through the method of images, in terms of functional methods and Epstein zeta functions. Our addition here consists in including a chemical potential for the linear momentum in the compact direction, which brings one from Epstein zeta functions with temperature inversion symmetry to the real analytic Eisenstein series with full modular invariance.

From the viewpoint of modular invariance in higher dimensions, as compared to the analysis in [198], our derivation provides the full analytic expression for the partition function in the case of the simplest model of a free massless scalar on a spatial section of the form $\mathbb{T}^q \times \mathbb{R}^p$ for $p+q=d$, in terms of $\mathrm{SL}(q+1,\mathbb{Z})$ Eisenstein series, with an explicit proof on modular transformations built in. As compared to [202], for the model under consideration, there is full control on finite-size corrections, and also a new formula, valid now at low rather than at high temperature, that relates the leading contribution to the entropy to the Casimir energy density of a massless scalar field in one dimension lower.

For what concerns the relevance of Eisenstein series in physical applications, we have shown that, in the context of quantum statistical physics, partitions functions of massless scalars are among the simplest physical observables that are directly expressed through such series. Various approaches to computing these partition functions, such as functional, heat kernel/worldline and canonical quantization methods illustrate complementary aspects of Eisenstein series.

9 Gauge theories in the Casimir setup

In this chapter we consider an application of the techniques shown in the previous chapters to the case of gauge theories in the Casimir setup. In particular, we will focus on spin 1 and spin 2 gauge fields. We start in section 9.1, where we consider the electromagnetic field placed in a slab geometry with two infinite perfectly conducting parallel planes separated by a distance L and we outline what are the correct boundary conditions that have to be imposed on the fields on such planes. In sections 9.2 and 9.3 we analyze in two different but equivalent ways the reduced phase space of the theory in the above described setup. In particular, we show that the full dynamics of the electromagnetic field can be, and has to be, described in terms of that of two single scalar fields, one satisfying Dirichlet conditions and the other Neumann conditions on the metallic plates. In section 9.4 we present an additional, more convenient reformulation of the theory in terms of a "fictitious" single scalar field living on the double volume satisfying periodic boundary conditions on an interval of length $2L$. Correspondingly, in section 9.5, we use this result to exactly compute the partition function of the theory, showing its modular properties and the associated high-/low-temperature dualities. Surprisingly, the momentum of the scalar field in the compact direction, that according to the discussion in section 8.2 has to be included in the partition function in order to consistently add a real part to the modular parameter, does not coincide with the standard momentum of the original electromagnetic field and we show that instead it is related to the spin angular momentum of the photon. We also emphasize that the thermodynamic entropy in the low-temperature regime scales as the area of the conducting plates and we argue that this behavior can ultimately be traced back to the non-triviality of the boundary conditions. In sections 9.6 and 9.7 we prove in detail that, imposing the analogue of perfectly conducting boundary conditions on the linearized gravitational field, the same conclusions can be drawn also in the case of spin 2 fields. We conclude in section 9.8 with some outlooks and future directions.

9.1 Casimir geometry and boundary conditions

The Casimir effect setup consists in considering the electromagnetic field between two perfectly conducting plates separated by a distance L in the x^3 direction, as represented in Figure 18. It is useful to introduce the following index notation: spacetime coordinates are denoted by $x^\mu = (t, x^a, x^3)$, where $\mu = 0, ..., 3$ and $a = 1, 2$. The area of the plates is $A = L_1 L_2$, where L_a are the lengths of the sides of the plates and we assume that $L_a \gg L$. This implies that we can take without loss of generality periodic boundary conditions for the electromagnetic field components in the x^a directions, for in the large L_a limit the result for the thermodynamic quantities we are interested in will not depend at all from this choice.

In order to make connection with the results of the previous chapters, we are interested in a Hamiltonian analysis of the system, so we consider the first order action

$$
\begin{aligned}
S_H[A, \Pi] &= \int \mathrm{d}t \int_V \mathrm{d}^3x \left(\Pi^i \dot{A}_i + A_t \partial_i \Pi^i - \frac{1}{2}\Pi_i \Pi^i - \frac{1}{2} B_i B^i \right) \\
&= \int \mathrm{d}t \int_V \mathrm{d}^3x \big(\Pi^i \dot{A}_i + A_t \partial_i \Pi^i - H[A, \Pi] \big), \qquad H[A, \Pi] = \frac{1}{2} \int_V \mathrm{d}^3x \big(\Pi_i \Pi^i + B_i B^i \big), \qquad (9.1)
\end{aligned}
$$

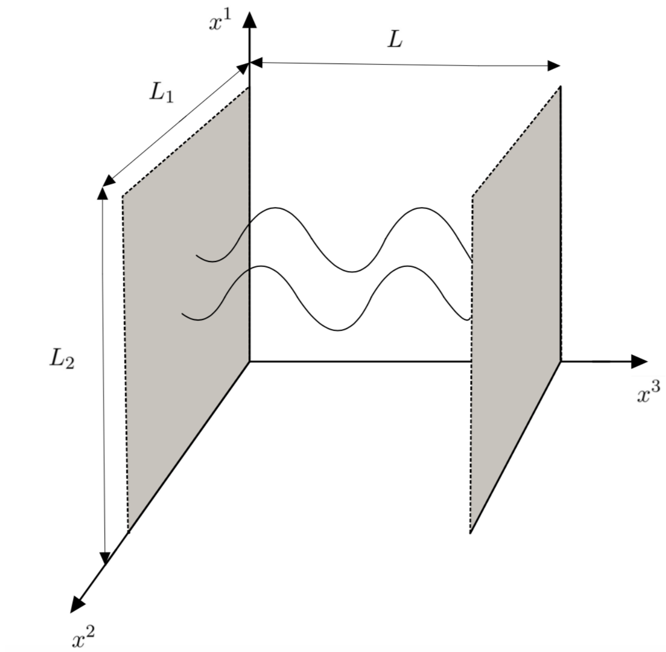

Figure 18: Electromagnetic field between two metallic plates separated by a distance L along the x^3 direction.

where $V = AL_3$, $\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}^i - \partial^i A_t = -E^i$, we denoted by $H[A, \Pi]$ the Hamiltonian and $B^i = \epsilon^{ijk}\partial_j A_k$ is the magnetic field. The Gauss law constraint $\partial_i \Pi^i = 0$ follows from setting to zero the derivative of the Lagrangian with respect to the Lagrangian multiplier A_t. The requirement that the plates are perfectly conducting imposes that the tangential components of the electric field and the normal component of the magnetic field must vanish on the plates surfaces,

$$\Pi^a(t, x^a, 0) = 0 = \Pi^a(t, x^a, L), \qquad B^3(t, x^a, 0) = 0 = B^3(t, x^a, L). \tag{9.2}$$

The standard gauge invariance $\delta_\epsilon A_i = \partial_i \epsilon$ can be used to reach the standard radiation gauge

$$A_t = 0, \qquad \partial_i A^i = 0, \tag{9.3}$$

that leaves the freedom of performing residual gauge transformations parametrized by a time-independent ϵ satisfying $\partial_i \partial^i \epsilon = 0$. Since $B^3 = \epsilon^{ab}\partial_a A_b$, imposing the second in (9.2) implies that A_b is pure gauge on the boundary, *i.e.* there exists a field $\phi(x^a)$ such that $A_a|_{x^3=0,L} = \partial_a \phi$. Using residual gauge freedom it is possible to set $\phi = 0$, so that the set of boundary conditions on the canonical variables (A_a, Π^a) reads

$$\Pi^a(t, x^a, 0) = 0 = \Pi^a(t, x^a, L), \qquad A_a(t, x^a, 0) = 0 = A_a(t, x^a, L). \tag{9.4}$$

These boundary conditions are *Dirichlet* conditions for the canonical fields components parallel to the plates. We need also boundary conditions on the components (A_3, Π^3) perpendicular to the plates.

Asking the Coulomb gauge condition $\partial_i A^i = 0$ and the Gauss law $\partial_i \Pi^i = 0$ to hold on the boundary [223–225, 253, 254] implies that, taking into account (9.4),

$$\partial_3 \Pi^3(t,x)|_{x^3=0} = 0 = \partial_3 \Pi^3(t,x)|_{x^3=L}, \qquad \partial_3 A^3(t,x)|_{x^3=0} = 0 = \partial_3 A^3(t,x)|_{x^3=L}, \tag{9.5}$$

that are *Neumann conditions*. As already mentioned, we also impose periodic boundary conditions in the x^a directions,

$$\Pi^i(t,0,x^3) = \Pi^i(t,L_a,x^3), \qquad A^i(t,0,x^3) = A^i(t,L_a,x^3). \tag{9.6}$$

Having defined the set of boundary conditions, we can now mode expand the field components into the appropriate orthonormal bases. The orthonormal bases for functions satisfying Dirichlet and Neumann conditions in $x^3 = 0$ and in $x^3 = L$ and periodic conditions at $x^a = 0$ and $x^a = L_a$ are $\{e^D_{k_i}\}$ and $\{e^N_{k_i}\}$,

$$\{e^D_{k_i}(x)\} = \left\{\sqrt{\frac{2}{V}}\sin\big(k_3 x^3\big)e^{ik_a x^a}\right\}, \qquad \{e^N_{k_i}(x)\} = \left\{\frac{1}{\sqrt{V}}e^{ik_a x^a}, \sqrt{\frac{2}{V}}\cos\big(k_3 x^3\big)e^{ik_a x^a}\right\}, \tag{9.7}$$

respectively, where $k_a = \frac{2\pi n_a}{L_a}$ and $k_3 = \frac{\pi n_3}{L}$, with $n_a \in \mathbb{Z}$ and $n_3 \in \mathbb{N}$. The orthonormality is

$$(e^D_{k_i}, e^D_{k'_i}) = \int_V \mathrm{d}^3x\, e^{D*}_{k_i}(x) e^D_{k'_i}(x) = \prod_i \delta_{n_i n'_i}, \qquad (e^N_{k_i}, e^N_{k_i}) = \int_V \mathrm{d}^3x\, e^{N*}_{k_i}(x) e^N_{k'_i}(x) = \prod_i \delta_{n_i n'_i}, \tag{9.8}$$

and the completeness

$$\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} e^{D*}_{k_i}(x) e^D_{k_i}(x') = \delta^3(x-x'), \qquad \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}_0} e^{N*}_{k_i}(x) e^N_{k_i}(x') = \delta^3(x-x'). \tag{9.9}$$

Because of the perfectly conducting boundary conditions in (9.4) and (9.5), the fields admit the mode expansions [46],

$$\phi^a(t,x) = i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \phi^a_{k_i}(t)\sin\big(k_3 x^3\big)e^{ik_a x^a}, \tag{9.10}$$

$$\phi^3(t,x) = \sum_{n_a\in\mathbb{Z}^2}\left[\frac{1}{\sqrt{V}}\phi^3_{k_a,0}(t) + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}\phi^3_{k_i}(t)\cos\big(k_3 x^3\big)\right]e^{ik_a x^a}, \tag{9.11}$$

with

$$\phi^a_{k_i}(t) = (e^D_{k_i}, \phi^a), \qquad \phi^3_{k_i}(t) = (e^N_{k_i}, \phi^3), \tag{9.12}$$

[46] see, *e.g.* Appendix D.

where by ϕ^i we denote either A^i or Π^i. The unusual i factor in (9.10) is chosen for later convenience. From now on, since we will not be concerned about the time evolution of the fields, for notational purposes we denote $\phi^i_{k_i}(t)$ simply by $\phi^i_{k_i}$. The reality conditions and parity in the x^3 direction are

$$\phi^a_{k_a,k_3} = -\phi^{*a}_{-k_a,k_3}, \qquad \phi^3_{k_a,k_3} = \phi^{*3}_{-k_a,k_3} \qquad \phi^a_{k_a,k_3} = -\phi^a_{k_a,-k_3}, \qquad \phi^3_{k_a,k_3} = \phi^3_{k_a,-k_3}. \tag{9.13}$$

So far, we only imposed boundary conditions on the fields components. However, since we are dealing with a gauge theory, not all the modes appearing in (9.10)-(9.11) are physical and there is therefore a redundancy. Indeed, these modes do not automatically satisfy the Coulomb gauge and the Gauss law. In the next section we proceed to identify the *reduced phase space* of the theory comprising only the physical degrees of freedom, where we have eliminated all the redundancies.

9.2 Reduced phase space I: algebraic approach

We start by inserting the mode decompositions (9.10)-(9.11) in the Hamiltonian $H[A,\Pi]$ of (9.3). We find $H = H_{n_3\neq 0} + H_{n_3=0}$, where

$$\begin{aligned} H_{n_3\neq 0} = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \Big[& \Pi^a_{k_i}\Pi^{*a}_{k_i} + \Pi^3_{k_i}\Pi^{*3}_{k_i} + k_1^2(A^2_{k_i}A^{*2}_{k_i} + A^3_{k_i}A^{*3}_{k_i}) + k_2^2(A^1_{k_i}A^{*1}_{k_i} + A^3_{k_i}A^{*3}_{k_i}) \\ & + k_3^2(A^1_{k_i}A^{*1}_{k_i} + A^2_{k_i}A^{*2}_{k_i}) - k_2k_3(A^2_{k_i}A^{*3}_{k_i} + A^3_{k_i}A^{*2}_{k_i}) \\ & - k_1k_3(A^1_{k_i}A^{*3}_{k_i} + A^3_{k_i}A^{*1}_{k_i}) - k_1k_2(A^1_{k_i}A^{*2}_{k_i} + A^2_{k_i}A^{*1}_{k_i})\Big], \end{aligned} \tag{9.14}$$

and

$$H_{n_3=0} = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2} \Pi^3_{k_a,0}\Pi^{*3}_{k_a,0} + k_\perp^2 A^3_{k_a,0}A^{*3}_{k_a,0}, \tag{9.15}$$

where $k_\perp = \sqrt{k_a k^a}$. Note that, because of Neumann conditions, there is also a $n_i = 0$ mode, contributing to the Hamiltonian as a free particle. Indeed, on denoting $p \equiv \Pi^3_0$,

$$H_{n_3=0} = H_{n_i=0} + \frac{1}{2}{\sum_{n_a\in\mathbb{Z}^2}}' \Pi^3_{k_a,0}\Pi^{*3}_{k_a,0} + k_\perp^2 A^3_{k_a,0}A^{*3}_{k_a,0}, \qquad H_{n_i=0} = \frac{p^2}{2}. \tag{9.16}$$

As remarked in section 7.3, since we will ultimately be concerned about the partition function in the large area of the plates limit, the contribution of $H_{n_i=0}$ will be negligible. However, if one is interested in considering metallic plates equipped with an electric charge distribution the zero mode p will also give a non-vanishing contribution to the partition function in the large area limit [224, 225].

Applying the Gauss law constraint and the Coulomb gauge condition $\partial_i\phi^i = 0$ to the mode expansions

(9.10)-(9.11) we get,

$$\phi^3_{k_i} = -\frac{k_a}{k_3}\phi^a_{k_i}, \qquad \phi^{*3}_{k_i} = -\frac{k_a}{k_3}\phi^{*a}_{k_i}, \qquad k_3 \neq 0. \tag{9.17}$$

Note that the $k_3 = 0$ modes automatically satisfy the constraint and gauge conditions. Inserting the relations of (9.17) into $H_{n_3\neq 0}$ we get

$$\begin{aligned} H_{n_3\neq 0} = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} & \Pi^1_{k_i}\Pi^{*1}_{k_i}\left(1+\frac{k_1^2}{k_3^3}\right) + \Pi^2_{k_i}\Pi^{*2}_{k_i}\left(1+\frac{k_2^2}{k_3^3}\right) + \Pi^1_{k_i}\Pi^{*2}_{k_i}\frac{k_1k_2}{k_3^2} + \Pi^{*1}_{k_i}\Pi^2_{k_i}\frac{k_1k_2}{k_3^2} \\ & + k^2\left[A^1_{k_i}A^{*1}_{k_i}\left(1+\frac{k_1^2}{k_3^3}\right) + A^2_{k_i}A^{*2}_{k_i}\left(1+\frac{k_2^2}{k_3^3}\right) + A^1_{k_i}A^{*2}_{k_i}\frac{k_1k_2}{k_3^2} + A^{*1}_{k_i}A^2_{k_i}\frac{k_1k_2}{k_3^2}\right], \end{aligned} \tag{9.18}$$

where $k = \sqrt{k_ik^i}$. Contrarily to (9.14), $H_{n_3\neq 0}$ in (9.18) is a quadratic form expressed only in terms of physical modes, satisfying the constraint and the gauge condition. In fact, in this expression appear only the two physical components of the electromagnetic field. It can be more conveniently rewritten as

$$H_{n_3\neq 0} = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \Pi^{*a}_{k_i} M_{aa'}(k_i)\,\Pi^{a'}_{k_i} + k^2 A^{*a}_{k_i} M_{aa'}(k_i) A^{a'}_{k_i}, \qquad M_{aa'}(k_i) = \begin{pmatrix} 1+\dfrac{k_1^2}{k_3^2} & \dfrac{k_1k_2}{k_3^2} \\ \dfrac{k_1k_2}{k_3^2} & 1+\dfrac{k_2^2}{k_3^2} \end{pmatrix}. \tag{9.19}$$

We now proceed to diagonalize the quadratic form in (9.19) and hence the matrix $M_{aa'}(k_i)$. Note that $\det[M_{aa'}(k_i)] = k^2/k_3^2$ and hence it is always non-vanishing as long as $k_i \neq 0$. The eigenvalues of $M_{aa'}(k_i)$ are $\lambda^H = 1$ and $\lambda^E = k^2/k_3^2$. The corresponding properly normalized eigenvectors are

$$\phi^H_{k_i} = \frac{1}{k_\perp}\epsilon_{ab}k^b\phi^a_{k_i}, \qquad \phi^E_{k_i} = \frac{1}{k_3}\frac{k}{k_\perp}k_a\phi^a_{k_i}, \tag{9.20}$$

satisfying the reality and parity conditions

$$\phi^H_{k_a,k_3} = \phi^{*H}_{-k_a,k_3}, \qquad \phi^E_{k_a,k_3} = \phi^{*E}_{-k_a,k_3}, \qquad \phi^H_{k_a,-k_3} = -\phi^H_{k_a,k_3}, \qquad \phi^E_{k_a,-k_3} = \phi^H_{k_a,k_3}, \tag{9.21}$$

inherited from those of $\phi^i_{k_i}$ in (9.13). Further, on denoting

$$\phi^E_{k_a,0} \equiv -\phi^3_{k_a,0}, \tag{9.22}$$

the full Hamiltonian, including the $n_3 = 0$ sector, becomes $H = H^H + H^E$ with

$$H^H = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \Pi^H_{k_i}\Pi^{*H}_{k_i} + k^2A^H_{k_i}A^{*H}_{k_i}, \qquad H^E = \frac{p^2}{2} + \frac{1}{2}{\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}_0}}' \Pi^E_{k_i}\Pi^{*E}_{k_i} + k^2A^E_{k_i}A^{*E}_{k_i}, \tag{9.23}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and where we included the $n_i = 0$ mode in H^E. Hence, the total Hamiltonian is the sum of two Hamiltonians of a scalar field. We denote ϕ^H and ϕ^E the H and E fields and, comparing with (D.5) and (D.12), they satisfy Dirichlet and Neumann conditions in $x^3 = 0, L$, respectively. This comes from the fact that the in H^E there is an additional $n_3 = 0$ mode, coming from the Neumann basis element $e^N_{k_a,0}(x)$, which is absent in H^H. Therefore the physical degrees of freedom of electromagnetism in the Casimir setup are encoded into two scalar fields, one satisfying Dirichlet and the other Neumann conditions. Introducing an index $\alpha = H, E$ the Poisson bracket on the reduced phase space are

$$\{A^\alpha_{k_i}, \Pi^{*\beta}_{k'_i}\} = \delta^{\alpha\beta} \prod_i \delta_{n_i n'_i}, \qquad \{A^E_{k_a,0}, \Pi^{*E}_{k'_a,0}\} = \prod_a \delta_{n_a n'_a}, \qquad \{p, q\} = 1, \tag{9.24}$$

where we introduced $q \equiv A^E_0$. These brackets are automatically satisfied if the modes $\phi^a_{k_i}$ in the full phase space satisfy the Dirac bracket

$$\{A^a_{k_i}, \Pi^{*b}_{k'_i}\}^* = \prod_i \delta_{n_i n'_i} \left(\delta^{ab} - \frac{k^a k^b}{k^2} \right), \tag{9.25}$$

Furthermore, from (9.24) and (9.25) it is possible to derive the other bracket as

$$\{A^3_{k_i}, \Pi^{*3}_{k'_i}\}^* = \prod_i \delta_{n_i n'_i} \frac{k^2_\perp}{k^2}, \qquad \{A^a_{k_i}, \Pi^{*3}_{k'_i}\}^* = \{A^3_{k_i}, \Pi^{*a}_{k'_i}\}^* = -\prod_i \delta_{n_i n'_i} \frac{k^a k^3}{k^2}, \tag{9.26}$$

so that, putting together (9.25) and (9.26) we have

$$\{A^l_{k_i}, \Pi^{*m}_{k'_i}\}^* = \prod_i \delta_{n_i n'_i} \left(\delta^{lm} - \frac{k^l k^m}{k^2} \right). \tag{9.27}$$

9.3 Reduced phase space II: polarization vectors and Bromwich-Borgnis fields

We consider the basis of vectors in momentum space $(\vec{e}_H, \vec{e}_E, \vec{e}_\parallel) = (\vec{e}_\alpha, \vec{e}_\parallel) = (\vec{e}_A)$, where $\alpha = H, E$ as in the previous section and $A = \alpha, \parallel$, satisfying

$$\vec{e}_\parallel = \frac{\vec{k}}{k}, \qquad \vec{e}_H \times \vec{e}_E = \vec{e}_\parallel, \qquad \vec{k} \times \vec{e}_\alpha = k \epsilon_\alpha{}^\beta \vec{e}_\beta, \qquad \vec{k} \cdot \vec{e}_\alpha = 0, \tag{9.28}$$

together with the completeness and orthonormality relations

$$e_A{}^i e^A{}_j = \delta^i_j, \qquad e_A{}^i e^B{}_i = \delta^A_B. \tag{9.29}$$

The basis is defined as long as $k_i \neq 0$. The vectors $\vec{e}_A$ are referred to as *polarization vectors.*

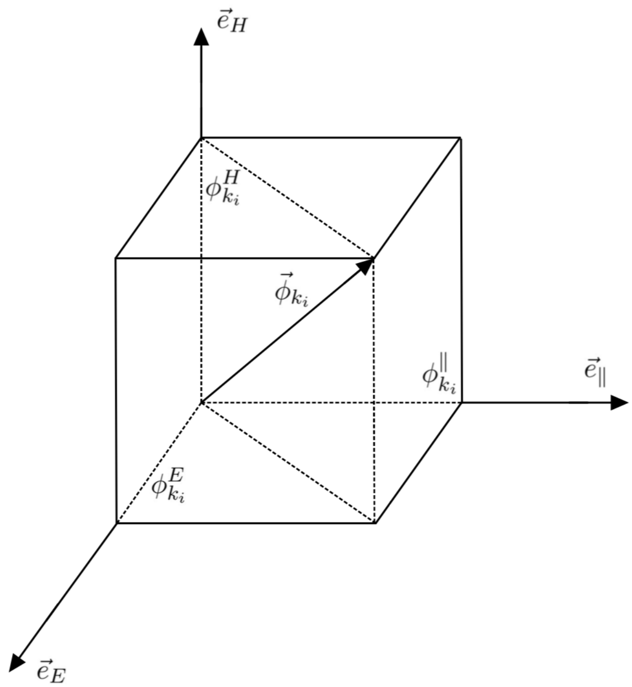

Figure 19: The decomposition of a vector $\vec{\phi}_{k_i}$ in the polarization vector basis ($\vec{e}_A$).

Any vector field $\vec{\phi}_{k_i} = (\phi^a_{k_i}, \phi^3_{k_i})$ in momentum space can be expanded as (see Figure 19)

$$\phi^a_{k_i} = e_\alpha{}^a \phi^\alpha_{k_i} + e_\parallel{}^a \phi^\parallel_{k_i}, \qquad \phi^3_{k_i} = e_\alpha{}^3 \phi^\alpha_{k_i} + e_\parallel{}^3 \phi^\parallel_{k_i}. \tag{9.30}$$

The inverse of these relations is

$$\phi^\alpha_{k_i} = e^\alpha{}_a \phi^a_{k_i} + e^\alpha{}_3 \phi^3_{k_i}, \qquad \phi^\parallel_{k_i} = e^\parallel{}_a \phi^a_{k_i} + e^\parallel{}_3 \phi^3_{k_i}. \tag{9.31}$$

It is easy to show that the Hamiltonian $H_{n_3\neq 0}$ in (9.14) can be more compactly written as

$$H_{n_3\neq 0} = \frac{1}{2} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \Pi_{k_i\, j} \Pi^{*j}_{k_i} + (\vec{k} \times \vec{A}_{k_i})_j (\vec{k} \times \vec{A}^*_{k_i})^j, \tag{9.32}$$

Using the third property in (9.28) and since $\Pi_{k_i\, j}\Pi^{*j}_{k_i} = \Pi_{k_i\, A}\Pi^{*A}_{k_i}$, we have

$$H_{n_3\neq 0} = \frac{1}{2} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \Pi_{k_i\, A} \Pi^{*A}_{k_i} + k^2 A_{k_i\, \alpha} A^{*\alpha}_{k_i}, \tag{9.33}$$

Note that the Gauss Law constraint and the Coulomb gauge condition can be written as

$$\partial_i \phi^i = -\sqrt{\frac{2}{V}} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} k\, \phi^\parallel_{k_i} e^{i k_a x^a} \sin\big(k_3 x^3\big). \tag{9.34}$$

Therefore the reduced phase space in the polarization vectors formalism is defined by $\phi^{\parallel}_{k_i} = 0$. This is compatible with the usual statement according to which the components of the electromagnetic field parallel to the momentum are pure gauge degrees of freedom and hence not physical. The Hamiltonian $H_{n_3 \neq 0}$ in the reduced phase space is

$$H_{n_3 \neq 0} = \frac{1}{2} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} (\Pi_{k_i\,\alpha} \Pi^{*\alpha}_{k_i} + k^2 A_{k_i\,\alpha} A^{*\alpha}_{k_i}). \tag{9.35}$$

The $n_3 = 0$ mode contributes to the magnetic and electric fields as

$$B^{a(0)} = \frac{i}{\sqrt{V}} \sum_{n_a \in \mathbb{Z}^2} \epsilon^{ab} k_b A^3_{k_a,0} e^{i k_a x^a}, \qquad \Pi^{3(0)} = \frac{1}{\sqrt{V}} \sum_{n_a \in \mathbb{Z}} \Pi^3_{k_a,0} e^{i k_a x^a}. \tag{9.36}$$

Thus, on defining analogously to (9.22) the $k_3 = 0$ modes as

$$\phi^3_{k_a,0} \equiv -\phi^E_{k_a,0}, \qquad k_i \neq 0 \tag{9.37}$$

we have

$$H_{n_3=0} = \frac{p^2}{2} + \frac{1}{2} {\sum_{n_a \in \mathbb{Z}^2}}' (\Pi^E_{k_a,0} \Pi^{*E}_{k_a,0} + k^2 A^E_{k_a,0} A^{*E}_{k_a,0}), \tag{9.38}$$

and hence, the full Hamiltonian in the reduced phase space is again

$$H = H^H + H^E, \tag{9.39}$$

where H^α were defined in (9.23). The Poisson bracket on the reduced phase space are thus the same as those in (9.24).

So far, we have not yet specified the form of the $\vec{e}_\alpha$ vectors. Consider the following choice of $\vec{e}_\alpha$ (see *e.g.* [255]),

$$\vec{e}_H = \frac{1}{k_\perp}(k_2, -k_1, 0), \qquad \vec{e}_E = \frac{1}{k k_\perp}(k_1 k_3, k_2 k_3, -k_a k^a). \tag{9.40}$$

It is easy to verify that they satisfy all the properties in (9.28). Using (9.30) and (9.31), we have

$$\phi^H_{k_i} = \frac{\epsilon_{ab} k^b}{k_\perp} \phi^a_{k_i}, \qquad \phi^E_{k_i} = \frac{k_a k_3}{k_\perp k} \phi^a_{k_i} - \frac{k_\perp}{k} \phi^3_{k_i}, \qquad \phi^{\parallel} = \frac{k_i}{k} \phi^i_{k_i}, \tag{9.41}$$

$$\phi^a_{k_i} = \frac{\epsilon^{ab} k_b}{k_\perp} \phi^H_{k_i} + \frac{k^a k_3}{k_\perp k} \phi^E_{k_i} + \frac{k^a}{k} \phi^{\parallel}_{k_i}, \qquad \phi^3_{k_i} = -\frac{k_\perp}{k} \phi^E_{k_i} + \frac{k_3}{k} \phi^{\parallel}_{k_i}, \tag{9.42}$$

The reality and parity conditions are, using (9.13)

$$\phi^H_{k_a,k_3} = \phi^{*H}_{-k_a,k_3}, \qquad \phi^E_{k_a,k_3} = \phi^{*E}_{-k_a,k_3}, \qquad \phi^H_{k_a,-k_3} = -\phi^H_{k_a,k_3}, \qquad \phi^E_{k_a,k_3} = \phi^E_{k_a,-k_3}, \tag{9.43}$$

in agreement with what was found using the algebraic approach in (9.21). Note also that the $k_3 = 0$ modes, using (9.41) are automatically given by $\phi^3_{k_a,0} = -\phi^E_{k_a,0}$, consistently with the definitions in (9.22) and (9.37). Using (9.41) and (9.42) we obtain, in the reduced phase space $\phi^{\parallel}_{k_i} = 0$,

$$\phi^H_{k_i} = \frac{1}{k_\perp}\epsilon_{ab}k^b\phi^a_{k_i}, \qquad \phi^E_{k_i} = \frac{1}{k_3}\frac{k}{k_\perp}k_a\phi^a_{k_i}, \tag{9.44}$$

again in agreement with (9.20). Therefore, using the polarization vectors technique, we recover all the results obtained in section 9.2 with the algebraic method. However, because of the geometric properties of the polarization vectors, it is easier to attach the problem in this approach. This will be fundamental in section 9.1, when we will deal with the linearized gravitational field and we will exclusively use the polarization tensors formalism. To conclude, once again, we see that the physical degrees of freedom of electromagnetism with Casimir boundary conditions can be organized in a way so that the theory is described by a sum of two scalar fields, the H and E fields, satisfying respectively Dirichlet and Neumann conditions in $x^3 = 0, L$.

For $k_i \neq 0$ we define the oscillator variables $a^\alpha_{k_i}$ as

$$a^\alpha_{k_i} = \sqrt{\frac{k}{2}}\left(A^\alpha_{k_i} + \frac{i}{k}\Pi^\alpha_{k_i}\right), \qquad \{a^\alpha_{k_i}, a^{*\beta}_{k'_i}\} = -i\delta^{\alpha\beta}\prod_i \delta_{n_i n'_i}, \tag{9.45}$$

$$a^E_{k_a} = \sqrt{\frac{k_\perp}{2}}\left(A^E_{k_a,0} + \frac{i}{k_\perp}\Pi^E_{k_a,0}\right), \qquad \{a^E_{k_a}, a^{*E}_{k'_a}\} = -i\prod_a \delta_{n_a n'_a}. \tag{9.46}$$

The Hamiltonian in (9.39) reads, using the just defined oscillator variables

$$H = \frac{p^2}{2} + \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k(a^{*H}_{k_i}a^H_{k_i} + a^H_{k_i}a^{*H}_{k_i}) + \frac{1}{2}\sideset{}{'}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}_0} k(a^{*E}_{k_i}a^E_{k_i} + a^E_{k_i}a^{*E}_{k_i}). \tag{9.47}$$

Using (9.42), we have

$$\phi^a = i\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}}\left(\frac{\epsilon^{ab}k_b}{k_\perp}\phi^H_{k_i} + \frac{k^a k_3}{k_\perp k}\phi^E_{k_i} + \frac{k^a}{k}\phi^{\parallel}_{k_i}\right)\sin(k_3x^3)e^{ik_ax^a}, \tag{9.48}$$

$$\phi^3 = \sum_{n_a\in\mathbb{Z}^2}\left[-\frac{1}{\sqrt{V}}\phi^E_{k_a,0} + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}\left(-\frac{k_\perp}{k}\phi^E_{k_i} + \frac{k_3}{k}\phi^{\parallel}_{k_i}\right)\cos(k_3x^3)\right]e^{ik_ax^a}. \tag{9.49}$$

Notice that the map between field modes in (9.42) and consequently that in (9.48) is algebraic in momentum space but non-local in coordinate space, since it involves inverse derivatives. In terms of the oscillators in (9.45) and (9.46) we have

$$A^a = i\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}\left[\left(\frac{\epsilon^{ab}k_b}{\sqrt{2k}k_\perp}a^H_{k_i}e^{ik_ax^a}+\frac{k^ak_3}{k_\perp k\sqrt{2k}}a^E_{k_i}e^{ik_ax^a}-\text{c.c.}\right)+\frac{k^a}{k}A^{\parallel}_{k_i}e^{ik_ax^a}\right]\sin\big(k_3x^3\big), \quad (9.50)$$

$$\Pi^a = \sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}\left[\left(\frac{\sqrt{k}\epsilon^{ab}k_b}{\sqrt{2}k_\perp}a^H_{k_i}e^{ik_ax^a}+\frac{k^ak_3}{\sqrt{2k}k_\perp}a^E_{k_i}e^{ik_ax^a}+\text{c.c.}\right)+i\frac{k^a}{k}\Pi^{\parallel}_{k_i}e^{ik_ax^a}\right]\sin\big(k_3x^3\big), \quad (9.51)$$

and

$$\begin{aligned} A^3 =&\frac{q}{\sqrt{V}}-\frac{1}{\sqrt{V}}\sum_{n_a\in\mathbb{Z}^2}{}' \frac{1}{\sqrt{2k_\perp}}(a^E_{k_a}e^{ik_ax^a}+\text{c.c.}) && (9.52)\\ &+\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2 n_3\in\mathbb{N}}\left[-\frac{k_\perp}{\sqrt{2k}k}(a^E_{k_i}e^{ik_ax^a}+\text{c.c.})+\frac{k_3}{k}A^{\parallel}_{k_i}e^{ik_ax^a}\right]\cos\big(k_3x^3\big), && (9.53)\end{aligned}$$

$$\begin{aligned} \Pi^3 =&\frac{p}{\sqrt{V}}+\frac{i}{\sqrt{V}}\sum_{n_a\in\mathbb{Z}^2}{}' \sqrt{\frac{k_\perp}{2}}(a^E_{k_a}e^{ik_ax^a}-\text{c.c.})\\ &+\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2 n_3\in\mathbb{N}}\left[\frac{ik_\perp}{\sqrt{2k}}(a^E_{k_i}e^{ik_ax^a}-\text{c.c.})+\frac{k_3}{k}\Pi^{\parallel}_{k_i}e^{ik_ax^a}\right]\cos\big(k_3x^3\big). && (9.54)\end{aligned}$$

9.3.1 Bromwich-Brognis fields

We define the real fields

$$\varphi^H = \sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}\frac{1}{\sqrt{2k}k_\perp}(a^H_{k_i}e^{ik_ax^a}+\text{c.c.})\sin\big(k_3x^3\big), \quad (9.55)$$

$$\varphi^E = -\frac{1}{\sqrt{V}}\sum_{n_a\in\mathbb{Z}^2}{}' \frac{1}{\sqrt{2k_\perp}k_\perp^2}(a^E_{k_a}e^{ik_ax^a}+\text{c.c.})-\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}\frac{1}{\sqrt{2k}kk_\perp}(a^E_{k_i}e^{ik_ax^a}+\text{c.c.})\cos\big(k_3x^3\big), \quad (9.56)$$

and the momenta

$$\pi^H = -i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{\sqrt{k}}{\sqrt{2}k_\perp}(a^H_{k_i} e^{ik_a x^a} - \text{c.c.}) \sin\big(k_3 x^3\big), \tag{9.57}$$

$$\pi^E = \frac{i}{\sqrt{V}} {\sum_{n_a\in\mathbb{Z}^2}}' \frac{\sqrt{k_\perp}}{\sqrt{2}k_\perp^2}(a^E_{k_a} e^{ik_a x^a} - \text{c.c.}) + i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{1}{\sqrt{2k}k_\perp}(a^E_{k_i} e^{ik_a x^a} - \text{c.c.}) \cos\big(k_3 x^3\big). \tag{9.58}$$

We denote by ψ^α either φ^α or π^α. ψ^H and ψ^E satisfy Dirichlet and Neumann conditions in $x^3 = 0, L$, respectively. We have

$$A^a = \epsilon^{ab}\partial_b\varphi^H + \partial^a\partial_3\varphi^E + i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k^a A^{\|}_{k_a,k_3} e^{ik_a x^a} \sin\big(k_3 x^3\big), \tag{9.59}$$

$$\Pi^a = \epsilon^{ab}\partial_b\pi^H + \partial^a\partial_3\pi^E + i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k^a \Pi^{\|}_{k_a,k_3} e^{ik_a x^a} \sin\big(k_3 x^3\big), \tag{9.60}$$

$$A^3 = \frac{q}{\sqrt{V}} - (\Delta - \partial_3^2)\varphi^E + \sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2 n_3\in\mathbb{N}} \frac{k_3}{k} A^{\|}_{k_i} e^{ik_a x^a} \cos\big(k_3 x^3\big), \tag{9.61}$$

$$\Pi^3 = \frac{p}{\sqrt{V}} - (\Delta - \partial_3^2)\pi^E + \sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2 n_3\in\mathbb{N}} \frac{k_3}{k} \Pi^{\|}_{k_i} e^{ik_a x^a} \cos\big(k_3 x^3\big), \tag{9.62}$$

$$B^a = -\epsilon^{ab}\partial_b\Delta\varphi^E + \partial^a\partial_3\varphi^H, \tag{9.63}$$

$$B^3 = -(\Delta + \partial_3^2)\varphi^H. \tag{9.64}$$

In the reduced phase space $\phi^{\|}_{k_i} = 0$, one obtains

$$A^a = \epsilon^{ab}\partial_b\varphi^H + \partial^a\partial_3\varphi^E, \qquad \Pi^a = \epsilon^{ab}\partial_b\pi^H + \partial^a\partial_3\pi^E, \tag{9.65}$$

$$A^3 = \frac{q}{\sqrt{V}} - (\Delta - \partial_3^2)\varphi^E, \qquad \Pi^3 = \frac{p}{\sqrt{V}} - (\Delta - \partial_3^2)\pi^E, \tag{9.66}$$

$$B^a = -\epsilon^{ab}\partial_b\Delta\varphi^E + \partial^a\partial_3\varphi^H, \qquad B^3 = -(\Delta + \partial_3^2)\varphi^H, \tag{9.67}$$

which is the original construction in [256,257] (see also [213,229,258] for related modern discussions). Note that from (9.65)-(9.67) $\partial_i\phi^i = 0$ *i.e.* the Coulomb gauge and the Gauss law constraint are automatically satisfied. Also, if ϕ^a and ϕ^3 satisfy Dirichlet and Neumann conditions in $x^3 = 0, L$, it follows immediately that φ^H and φ^E satisfy Dirichlet and Neumann conditions as well. Equations (9.65)-(9.67) emphasize that the true physical degrees of freedom of electromagnetism in the Casimir setup can be encoded in the pair of scalar fields (φ^H, φ^E) whose boundary conditions automatically follow from those of A^i.

9.4 Single scalar field formulation

In sections 9.2 and 9.3 we have seen that the dynamics of electromagnetism in the Casimir setup is equivalently described by a pair of scalar fields, one satisfying Dirichlet and the other Neumann conditions on the metallic plates. In practice, the standard electromagnetic action

$$S[A] = -\frac{1}{4}\int_V \mathrm{d}^4\, x F_{\mu\nu}F^{\mu\nu}, \tag{9.68}$$

after having taken into account the right boundary conditions, fixed the radiation gauge and solved the Gauss law constraint becomes equivalent to

$$S[\phi^H] + S[\phi^E] = -\frac{1}{2}\int_V \mathrm{d}^4 x\, \partial_\mu\phi^H\partial^\mu\phi^H - \frac{1}{2}\int_V \mathrm{d}^4 x\, \partial_\mu\phi^E\partial^\mu\phi^E, \tag{9.69}$$

with boundary conditions

$$\phi^H(t, x^a, 0) = 0 = \phi^H(t, x^a, L), \qquad \partial_3\phi^E(t,x)|_{x^3=0} = 0 = \partial_3\phi^E(t,x)|_{x^3=L}, \tag{9.70}$$

and periodic boundary conditions in the x^a directions. This shows that, when dealing with finite boundaries, the usual statement that polarization effects just double the number of modes can turn out to be inaccurate. Now we proceed to present an alternative reformulation of electromagnetism in the Casimir setup in terms of a *single* scalar field which will be fundamental for the computation of the partition function.

We start by defining new field and momentum modes Φ_{k_i} and Π_{k_i} as

$$\Psi_{k_i} = \frac{\phi^E_{k_i} - i\phi^H_{k_i}}{\sqrt{2}}, \qquad k_3 \neq 0 \qquad\qquad \Psi_{k_a,0} = \phi^E_{k_a,0}, \tag{9.71}$$

where by Ψ_{k_i} we denote either Φ_{k_i} or Π_{k_i}, satisfying the reality condition

$$\Psi^*_{-k_i} = \frac{\phi^{*E}_{-k_a,-k_3} + i\phi^{*H}_{-k_a,-k_3}}{\sqrt{2}} \overset{(9.43)}{=} \frac{\phi^E_{k_i} - i\phi^H_{k_i}}{\sqrt{2}} = \Psi_{k_i}, \qquad \Psi^*_{-k_a,0} = \Psi_{k_a,0}. \tag{9.72}$$

It is immediate to verify that the Hamiltonian in (9.39), in terms of Φ_{k_i} and Π_{k_i} is

$$H = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^3} \Pi_{k_i}\Pi^*_{k_i} + k^2\Phi_{k_i}\Phi^*_{k_i}, \tag{9.73}$$

or,

$$H = \frac{p^2}{2} + {\sum_{n_i\in\mathbb{Z}^3}}' k(a^*_{k_i}a_{k_i} + a_{k_i}a^*_{k_i}), \tag{9.74}$$

in terms of the oscillator variables

$$a_{k_i} = \sqrt{\frac{k}{2}}\Big(\Phi_{k_i} + \frac{i}{k}\Pi_{k_i}\Big), \qquad k_i \neq 0. \tag{9.75}$$

Note that now $n_3 \in \mathbb{Z}$ as well. Comparing H in (9.74) and (9.75) with that in (7.59) and (7.61), we see that it is the Hamiltonian of a single real scalar field $\Phi(t,x)$ satisfying periodic boundary conditions in all directions and, in particular along x^3 with periodicity $2L$. This comes from the simple observation that $k_3 = \frac{\pi}{L}n_3 = \frac{2\pi}{2L}n_3$. Therefore, the field Φ and its conjugate momentum can be decomposed in the basis adapted to the volume $V' = 2V$ as

$$\Phi = \frac{1}{\sqrt{V'}}\sum_{n_i\in\mathbb{Z}^3}\Phi_{k_i}e^{ik_ix^i} = \frac{q}{\sqrt{V'}} + \frac{1}{\sqrt{V'}}\sum_{n_i\in\mathbb{Z}^3}\frac{1}{\sqrt{2k}}(a_{k_i}e^{ik_ix^i} + a^*_{k_i}e^{-ik_ix^i}), \tag{9.76}$$

$$\Pi = \frac{1}{\sqrt{V'}}\sum_{n_i\in\mathbb{Z}^3}\Pi_{k_i}e^{ik_ix^i} = \frac{p}{\sqrt{V'}} - \frac{i}{\sqrt{V'}}\sum_{n_i\in\mathbb{Z}^3}\sqrt{\frac{k}{2}}(a_{k_i}e^{ik_ix^i} - a^*_{k_i}e^{-ik_ix^i}). \tag{9.77}$$

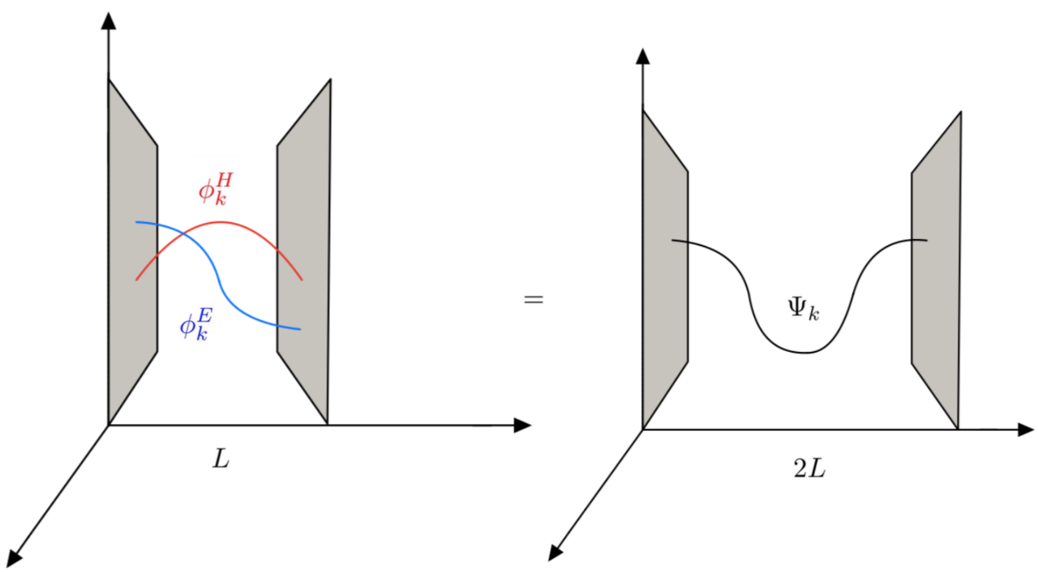

Figure 20: The H (in red) and E (in blue) modes ϕ_k^H and ϕ_k^E satisfying Dirichlet and Neumann conditions in $x^3 = 0, L$ are equivalent to a single Ψ_k satisfying periodic conditions on the double interval of length $2L$.

By "gluing" together the H and E fields, we have further mapped the dynamics of the full electromagnetic field to that of a single scalar field governed by the standard action

$$S[\Phi] = -\frac{1}{2}\int_{V'} \mathrm{d}^4x\, \partial_\mu\Phi\partial^\mu\Phi, \tag{9.78}$$

with

$$\Phi(t, x^a, 0) = \Phi(t, x^a, 2L). \tag{9.79}$$

and periodic boundary conditions along the x^a directions, as represented in Figure 20. Equations (9.78) and (9.79) replace (9.69) and (9.70), respectively.

Before concluding this section it is worth pointing out that, as remarked in the end of section 7.4 and in section 8.1, the correct black-body and Casimir energies for the electromagnetic field can be obtained by those of a scalar field in (7.82) and (8.8) just by setting $d = 3$ and replacing L by $2L$. This is compatible with the single scalar field formulation we have just presented. Further, the black body and the Casimir energies are mapped one into the other by the temperature inversion symmetry considered in chapter 8, as we discuss in the next section.

9.5 Modular covariance in finite temperature Casimir effect

Here we turn to investigate the properties of the partition function of electromagnetism in the above described setup. Having completely reformulated the theory in terms of a single scalar field living on the Euclidean manifold $\mathcal{M} = \mathbb{T}^2 \times \mathbb{R}^2$, where $\mathbb{T}^2 = \mathbb{S}^1_\beta \times \mathbb{S}^1_{2L}$, the result for the partition function follows directly from that in section 8 in (8.59), for $d = 3$ with the substitution $L \to 2L$. Consequently, the relevant modular parameter to consider here is

$$\tau = \frac{\beta}{2L}(\mu + i), \tag{9.80}$$

in terms of which the partition function $Z(\tau, \bar{\tau}) = \mathrm{Tr}\, e^{-\beta(\hat{H} - i\mu\hat{P}_3)}$ reads [223]

$$\log Z(\tau, \bar{\tau}) = \frac{A}{8L^2\tau_2\pi^2} f_2(\tau), \tag{9.81}$$

and it transforms covariantly with weight 2 under modular transformations,

$$\log Z(\tau', \bar{\tau}') = |c\tau + d|^2 \log Z(\tau, \bar{\tau}), \qquad \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}^2. \tag{9.82}$$

This result consistently enhances the temperature inversion symmetry discovered in [217] to the full set of $\mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}^2$ transformations. Indeed, as shown in Appendix F.4, for $\mu = 0$, the modular parameter becomes purely complex, $\tau = i\tau_2$, and the partition function in (9.82) can be expressed in terms of the Epstein zeta function as

$$\log Z(\tau_2) = \frac{A\tau_2}{2\pi^2 L^2}\zeta(2, \tau_2^2, 1), \qquad \log Z(\tau_2) = \frac{1}{\tau_2^2} \log Z\left(\frac{1}{\tau_2}\right). \tag{9.83}$$

Here, the observables used for the computation of $Z(\tau, \bar{\tau})$ are the Hamiltonian $\hat{H}$, that is the quantum operator corresponding to H in (9.74) including the electromagnetic Casimir energy E_0^{em} of (8.9), *i.e.*[47]

$$H[\Phi, \Pi] = \frac{1}{2}\int_{V'} \mathrm{d}^3x (\Pi^2 + \partial_i \Phi \partial^i \Phi), \qquad \hat{H} = \frac{\hat{p}^2}{2} + \sum_{n_i \in \mathbb{Z}^3}{}' \, k \, \hat{a}^\dagger_{k_i} \hat{a}_{k_i} - \frac{\pi^2 A}{720 L^3}, \tag{9.84}$$

and $\hat{P}_3$ is the quantized linear momentum of the field Φ in the compact x^3 direction,

$$P_3[\Phi, \Pi] = -\int_{V'} \mathrm{d}^3x \, \Pi \partial_3 \, \Phi, \qquad \hat{P}_3 = \sum_{n_i \in \mathbb{Z}^3}{}' \, k_3 \hat{a}^\dagger_{k_i} \hat{a}_{k_i}. \tag{9.85}$$

Note however that there is subtlety regarding the interpretation of P_3. While H can be equivalently interpreted as the Hamiltonian of the electromagnetic field in (9.3) or as that of the single scalar field Φ in (9.84), it would be wrong to draw the same conclusions for P_3. Indeed, it is certainly true that it is the linear momentum of the single scalar field along x^3, as emphasised in (9.85) but it does not coincide with the standard momentum of the electromagnetic field in the x^3 direction, which is the observable discussed in almost all other investigation of the Casimir effect. In terms of the electric and magnetic fields, it is given by

$$P_3^{\mathrm{em}} = \int_V \mathrm{d}^3x \, (\vec{E} \times \vec{B})_3, \tag{9.86}$$

and it easy to show that it does not match with P_3. Therefore, a natural question to ask at this stage regards the electromagnetic interpretation of P_3, which is the only observable that allows to consistently add a real part to the modular parameter, enhancing the inversion symmetry of the partition function to full modular covariance. The key is to use the map we have constructed in the previous sections between the original electromagnetic field modes $\phi^i_{k_i}$ and the scalar field modes Ψ_{k_i} in momentum space. As shown in Appendix G.1, the observable P_3 is, in terms of the original electromagnetic variables $\vec{A}$ and $\vec{\Pi}$ is given by the gauge-invariant expression,

$$\begin{aligned} P_3 = \int_V \mathrm{d}^3x \, \partial_3 \vec{B} \cdot \frac{1}{\sqrt{-\Delta}} \vec{\Pi} = \int_V \mathrm{d}^3x (\vec{\nabla} \times \partial_3 \vec{A}) \cdot \frac{1}{\sqrt{-\Delta}} \vec{\Pi} &= \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} k_3 (\Pi^{*H}_{k_i} A^E_{k_i} - \Pi^{*E}_{k_i} A^H_{k_i}) \\ &= i \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} k_3 (a^{*H}_{k_i} a^E_{k_i} - a^{*E}_{k_i} a^H_{k_i}). \end{aligned} \tag{9.87}$$

Note that this observable is usually not considered in the context of the Casimir effect and this is explains why full modular covariance, that can be interpreted as a suitable enhancement of the standard temperature inversion symmetry discovered in [217], has never been previously discussed in the literature. The non-locality of the expression in (9.87) is a direct consequence of the non-locality in coordinate space of the maps (9.41) and (9.42). Comparing the expression of P_3 with that of the spin angular momentum of the electromagnetic field along the x^3 direction

[47]Note that the Casimir energy in (9.84) reproduces exactly the Casimir force [209] $F^{\mathrm{em}} = -\frac{\partial E_0^{\mathrm{em}}}{\partial L} = -\frac{\pi^2 A}{240 L^4}$.

$$J_3 = \int_V \mathrm{d}^3x\, \epsilon_{ab} A^a_\perp \Pi^b_\perp = \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \frac{k_3}{k} (\Pi^{*E}_{k_i} A^H_{k_i} - \Pi^{*H}_{k_i} A^E_{k_i}) = -i \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \frac{k_3}{k} (a^{*H}_{k_i} a^E_{k_i} - a^{*E}_{k_i} a^H_{k_i}), \quad (9.88)$$

where $\phi^i_\perp$ are given in (9.50)-(9.54) having set $\phi^{\|}_{k_i} = 0$, we see that P_3 has the same mode expansion of J_3 up to an overall sign and a multiplication of each term in the sum by k. For what concerns the action of P_3 on the reduced phase space variables, we have

$$\{P_3, \Phi\} = \partial_3 \Phi, \quad (9.89)$$

on the single scalar field Φ and

$$\{P_3, A^H\} = \partial_3 A^E, \qquad \{P_3, A^E\} = \partial_3 A^H, \quad (9.90)$$

on the E and H fields. Note how the action of P_3 on them, besides generating shifts in the x^3 direction, flips their "polarization".

Having the complete result for the partition function in (9.81) in terms of the real Eisenstein series satisfying (9.82), it is possible to extend the considerations done in 8.4 also to the case of electromagnetism in the Casimir setup. In particular, for $\mu = 0$ we get in the low-temperature regime $\tau_2 \gg 1$,

$$\log Z(\tau, \bar{\tau}) = \frac{A}{L^2} \left[\tau_2 \frac{\pi^2}{360} + \frac{1}{\tau_2^2} \frac{\zeta(3)}{8\pi} \right] + \mathcal{O}(e^{-\tau_2}), \quad (9.91)$$

$$S(\beta) = \frac{A}{\beta^2} \frac{\zeta(3)}{2\pi} + \mathcal{O}(e^{-\tau_2}), \quad (9.92)$$

and in the high-temperature regime $\tau_2 \ll 1$,

$$\log Z(\tau, \bar{\tau}) = \frac{A}{L^2} \left[\frac{1}{\tau_2^3} \frac{\pi^2}{360} + \frac{\zeta(3)}{8\pi} \right] + \mathcal{O}(e^{-\frac{1}{\tau_2}}), \quad (9.93)$$

$$S(\beta) = \frac{V}{\beta^3} \frac{4\pi^2}{45} + \mathcal{O}(e^{-\frac{1}{\tau_2}}). \quad (9.94)$$

Note that in the low-temperature regime the entropy scales as the area of the metallic plates and not as the volume of the system. It has already been shown that the $n_3 = 0$ mode of the mode expansion of the scalar field Φ is the one that is responsible for this contribution. Tracing back the origin of this mode from the electromagnetic analysis, it is immediate to check that it comes from the $n_3 = 0$ mode of the expansion of the electromagnetic conjugate pair ϕ^3 in the direction orthogonal to the plates. Ultimately, it comes from the Neumann boundary conditions in (9.5). Hence, the electromagnetic Casimir effect just presented provides, in the low-temperature regime, an example of a microscopic model whose thermodynamic entropy scales as the area and not as the volume of the system and it is due to a specific sector of the theory that comes about because of non-trivial boundary conditions on the metallic plates.

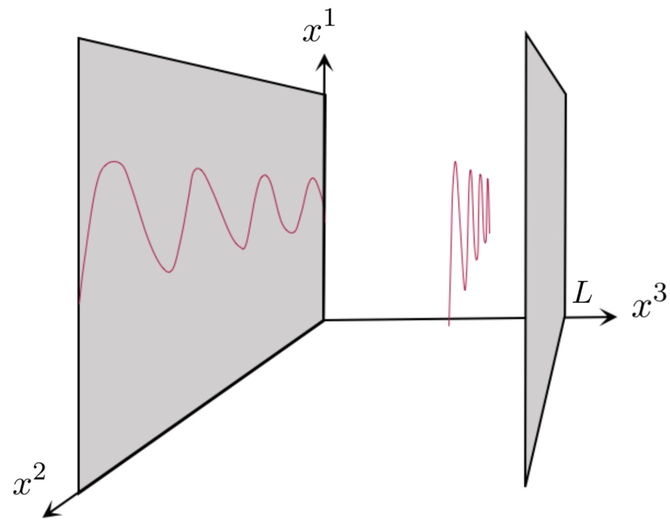

Figure 21: The leading low-entropy contribution in (9.91) is due to the $n_3 = 0$ photons that oscillate on planes parallel to the plates.

9.6 Gravitons in a Casimir box

Now we turn to describe what are the features of the linearized gravitational field around Minkowski spacetime with Casimir-type boundary conditions [226]. Again, we consider the same setup as the one just discussed in the electromagnetic case. We do not address here the problem of whether one may in principle confine gravitons to a box in order to achieve thermal equilibrium [259–262]. Neither we speculate on the physical meaning of perfect conductors for gravitons nor comment on the relation to a recent study of the gravitational Casimir effect at zero temperature with non-idealized boundary conditions [263]. We will limit ourselves to deriving the exact analytic and modular covariant result that allows one to access the qualitatively very different high and low temperature expansions, in line with what has been discussed for electromagnetism.

The Hamiltonian formulation of massless spin 2 fields may be obtained by linearizing the ADM formulation of full general relativity around flat space. The first order action is

$$S_H[h, \Pi] = \int \mathrm{d}t \left[\int_V \mathrm{d}^3x \big(\Pi^{ij} \dot{h}_{ij} - n^i \mathcal{H}_i - n \mathcal{H}_\perp \big) - H[h, \Pi] \right], \tag{9.95}$$

where (h_{ij}, Π^{ij}) is the conjugate pair and where $n_i = h_{0i}$ and $n = -\frac{1}{2} h_{00}$ are the linearized shift and lapse and appear in S_H as Lagrange multipliers implementing the constraints

$$\mathcal{H}_i = -2\partial^j \Pi_{ij} = 0, \qquad \mathcal{H}_\perp = \Delta h - \partial^i \partial^j h_{ij} = 0, \tag{9.96}$$

and where H is the standard Pauli-Fierz Hamiltonian

$$H[h, \Pi] = \int_V \mathrm{d}^3x \left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2 + \frac{1}{4}\partial^l h^{ij} \partial_l h_{ij} - \frac{1}{2}\partial_i h^{ij} \partial^l h_{lj} + \frac{1}{2}\partial^i h \partial^j h_{ij} - \frac{1}{4}\partial^i h \partial_i h \right). \tag{9.97}$$

Indices are lowered and raised with the flat space metric δ_{ij} and its inverse, $h = h^i{}_i$, $\Pi = \Pi^i{}_i$. We choose to work in the gauge[48]

$$h_{00} = 0, \qquad h_{0i} = 0, \qquad \mathcal{G}_i = -2\partial^j h_{ij} = 0, \qquad \mathcal{G}_\perp = \Delta\Pi - \partial^i\partial^j\Pi_{ij} = 0. \tag{9.98}$$

We remark that the above choice does not eliminate all the gauge redundancy and we still have the freedom of performing residual gauge transformations preserving the conditions in (9.98). Note also that the choices $\mathcal{G}_i = 0$ and $\mathcal{G}_\perp = 0$ can be obtained from $\mathcal{H}_i = 0$ and $\mathcal{H}_\perp$ by exchanging the roles of Π^{ij} and h_{ij}. In analogy with the electromagnetic case, we define Casimir-type boundary conditions by requiring that (h_{ab}, Π^{ab}) and (h_{33}, Π^{33}) satisfy Dirichlet conditions, while (h_{a3}, Π^{a3}) satisfy Neumann conditions on the plates. Explicitly,

$$\Pi^{ab}(t, x^a, 0) = 0 = \Pi^{ab}(t, x^a, L), \qquad h_{ab}(t, x^a, 0) = 0 = h_{ab}(t, x^a, L), \tag{9.99}$$

$$\Pi^{33}(t, x^a, 0) = 0 = \Pi^{33}(t, x^a, L), \qquad h_{33}(t, x^a, 0) = 0 = h_{33}(t, x^a, L), \tag{9.100}$$

$$\partial_3\Pi^{a3}(t, x)|_{x^3=0} = 0 = \partial_3\Pi^{a3}(t, x)|_{x^3=L}, \qquad \partial_3 h_{a3}(t, x)|_{x^3=0} = 0 = \partial_3 h_{a3}(t, x)|_{x^3=L}. \tag{9.101}$$

Note that, *a priori*, imposing boundary conditions on the gauge-dependent quantities Π^{ij} and h_{ij} might be dangerous. Indeed, one could spoil (9.99)-(9.101) by acting on them with the residual gauge transformations preserving (9.98). More correctly, boundary conditions should be imposed on gauge-invariant quantities. However, it can be shown that it is always possible to further restrict the form of the parameters ξ^i generating residual gauge transformations in order to reach (9.99)-(9.101). This will be clearer *a posteriori*, once we have identified the reduced phase space of the theory in section 9.2. Therefore, the conjugate variables admit the mode expansions

$$\phi^{ab}(t, x) = \sqrt{\frac{2}{V}} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \phi^{ab}_{k_i} \sin\!\left(k_3 x^3\right) e^{i k_a x^a}, \tag{9.102}$$

$$\phi^{33}(t, x) = \sqrt{\frac{2}{V}} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \phi^{33}_{k_i} \sin\!\left(k_3 x^3\right) e^{i k_a x^a}, \tag{9.103}$$

$$\phi^{a3}(t, x) = -i \sum_{n_a \in \mathbb{Z}^2} \left[\frac{1}{\sqrt{V}} \phi^{a3}_{k_a, 0} + \sqrt{\frac{2}{V}} \sum_{n_3 \in \mathbb{N}} \phi^{a3}_{k_i} \cos\!\left(k_3 x^3\right) \right] e^{i k_a x^a}. \tag{9.104}$$

where we have used the Dirichlet and Neumann bases in (9.7) and where $\phi^{ab}_{k_i} = (e^D_{k_i}, \phi^{ab})$, $\phi^{33}_{k_i} = (e^D_{k_i}, \phi^{33})$ and $\phi^{a3}_{k_i} = (e^N_{k_i}, \phi^{a3})$. The modes satisfy the reality and parity in the x^3 direction conditions

[48] Bear in mind that the gauge parameters ξ^μ act on the conjugate pair (h_{ij}, Π^{ij}) as $\delta_\xi h_{ij} = \partial_i\xi_j + \partial_j\xi_i$ and as $\delta_\xi \Pi^{ij} = (\partial^i\partial^j - \delta^{ij}\Delta)\xi^0$.

$$\phi^{ab}_{k_a,k_3} = \phi^{*ab}_{-k_a,k_3}, \qquad \phi^{33}_{k_a,k_3} = \phi^{*33}_{-k_a,k_3}, \qquad \phi^{a3}_{k_a,k_3} = -\phi^{*a3}_{-k_a,k_3}, \tag{9.105}$$

$$\phi^{ab}_{k_a,k_3} = -\phi^{ab}_{k_a,-k_3}, \qquad \phi^{33}_{k_a,k_3} = -\phi^{33}_{k_a,-k_3}, \qquad \phi^{a3}_{k_a,k_3} = \phi^{a3}_{k_a,-k_3}. \tag{9.106}$$

In terms of the modes $\phi^{ij}_{k_i}$, the Pauli-Fierz Hamiltonian in (9.97) reads

$$H_{\mathrm{PF}} = H_{\mathrm{PF}\,n_3\neq 0} + H_{\mathrm{PF}\,n_3=0}, \tag{9.107}$$

where

$$H_{n_3\neq 0} = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \Big[\Pi^{ij}\Pi^*_{ij} - \frac{1}{2}\Pi\Pi^* + \frac{1}{4}k^2 h_{ij}h^{*ij} - \frac{1}{2}k_i k^l h^{ij}h^*_{lj} + \frac{1}{2}k^i k^j h h^*_{ij} - \frac{1}{4}k^2 h h^*\Big], \tag{9.108}$$

and

$$H_{n_3=0} = \sum_{n_a\in\mathbb{Z}^2} \Big[2\Pi^{a3}\Pi^*_{a3} + \frac{1}{2}k^2_\perp h_{a3}h^{*a3} - \frac{1}{2}k_a k^b h^{a3}h^*_{b3}\Big]. \tag{9.109}$$

In the above expressions, for notational simplicity we omitted the k_i dependence of the Fourier coefficients. Once again, because of the Neumann conditions in (9.101) there is a contribution to $H_{n_3=0}$ coming from the $n_i = 0$ modes, which we denote by $H_{n_i=0}$. Explicitly, introducing $p^a \equiv 2\Pi^{a3}_0$, we have

$$H_{n_3=0} = H_{n_i=0} + {\sum_{n_a\in\mathbb{Z}^2}}' \Big[2\Pi^{a3}\Pi^*_{a3} + \frac{1}{2}k^2_\perp h_{a3}h^{*a3} - \frac{1}{2}k_a k^b h^{a3}h^*_{b3}\Big], \qquad H_{n_i=0} = \frac{p_a p^a}{2}. \tag{9.110}$$

Contrarily to what happens in the electromagnetic case, here the $n_i = 0$ sector is characterized by two free particles.

Now, in order to identify the reduced phase space of the theory we should solve the constraints and gauge conditions. Similarly to the electromagnetic case, this would boil down to find certain relations between the modes $\phi^{ij}_{k_i}$ specifying the constraint hypersurface. Injecting these relations in the Hamiltonian in (9.107) and (9.108), we would find a non-diagonal quadratic form, whose eigenvectors would give the physical degrees of freedom of the theory. Here, we do not pursue this approach and instead we choose to work directly with the polarization tensors.

Consider the basis $(e_A{}^i)$ of polarization vectors defined in (9.28) and (9.40). Let $\Xi = (TTs, T, LT\alpha, LL)$, with $s = (+, \times)$. When $k_i \neq 0$, an orthonormal basis for symmetric tensors $(e_\Xi{}^{ij})$ can be constructed as follows (see, *e.g.* [264, 265]).

1. The *transverse-traceless* (TT) tensors ${e_{TTs}}^{ij}$ are defined as

$$ {e_{TT+}}^{ij} = \frac{1}{\sqrt{2}}({e_H}^i {e_H}^j - {e_E}^i {e_E}^j), \qquad {e_{TT\times}}^{ij} = \frac{1}{\sqrt{2}}({e_H}^i {e_E}^j + {e_E}^i {e_H}^j), \tag{9.111} $$

   and they satisfy

$$ k_i \; {e_{TTs}}^{ij} = 0, \qquad \delta_{ij} \; {e_{TTs}}^{ij} = 0; \tag{9.112} $$

2. the *transverse* (T) tensor ${e_T}^{ij}$ is defined as

$$ {e_T}^{ij} = \frac{1}{\sqrt{2}}\left(\delta^{ij} - {e_\parallel}^i {e_\parallel}^j\right), \tag{9.113} $$

   satisfying

$$ k_i \; {e_T}^{ij} = 0, \qquad \delta_{ij} \; {e_T}^{ij} = \sqrt{2}; \tag{9.114} $$

3. the *longitudinal-traceless* (LT) tensors are

$$ {e_{LT\alpha}}^{ij} = \frac{1}{\sqrt{2}}({e_\parallel}^i {e_\alpha}^j + {e_\alpha}^i {e_\parallel}^j), \tag{9.115} $$

   satisfying

$$ k_i {e_{LT\alpha}}^{ij} = \frac{k}{\sqrt{2}} {e_\alpha}^j, \qquad \delta_{ij} {e_{LT\alpha}}^{ij} = 0; \tag{9.116} $$

4. the *longitudinal* (LL) tensor is defined as

$$ {e_{LL}}^{ij} = {e_\parallel}^i {e_\parallel}^j, \tag{9.117} $$

   and it satisfies

$$ k_i {e_{LL}}^{ij} = k^j, \qquad \delta_{ij} {e_{LL}}^{ij} = 1. \tag{9.118} $$

The above basis basis is orthonormal and complete,

$$ {e_\Xi}^{ij} {e^\Gamma}_{ij} = \delta^\Gamma_\Xi, \qquad {e_\Xi}^{ij} {e^\Xi}_{mn} = \frac{1}{2}(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m). \tag{9.119} $$

The explicit expressions for the basis elements $({e_\Xi}^{ij})$ are given in Appendix H. The decompositions of the Fourier components ϕ^{ij} of the conjugate pair in (e^{ij}_Ξ) and their inverse are

$$ \phi^{ij}_{k_i} = {e_\Xi}^{ij} \phi^\Xi_{k_i}, \qquad \phi^\Xi_{k_i} = {e^\Xi}_{ij} \phi^{ij}_{k_i}. \tag{9.120} $$

From (H.1)-(H.7), it is immediate to verify that the reality conditions on the components $\phi^\Xi_{k_i}$ are, using (9.105)

$$ \phi^{TTs}_{k_a,k_3} = \phi^{*TTs}_{-k_a,k_3}, \qquad \phi^{T}_{k_a,k_3} = \phi^{*T}_{-k_a,k_3}, \qquad \phi^{LT\alpha}_{k_a,k_3} = \phi^{*LT\alpha}_{-k_a,k_3}, \qquad \phi^{LL}_{k_a,k_3} = \phi^{*LL}_{-k_a,k_3}, \tag{9.121} $$

whereas the parity conditions are, using (9.106),

$$\phi^{TT+}_{k_a,k_3} = -\phi^{TT+}_{k_a,-k_3}, \qquad \phi^{TT\times}_{k_a,k_3} = -\phi^{TT\times}_{k_a,-k_3}, \qquad \phi^{T}_{k_a,k_3} = -\phi^{T}_{k_a,-k_3} \tag{9.122}$$

$$\phi^{LTH}_{k_a,k_3} = -\phi^{LTH}_{k_a,-k_3}, \qquad \phi^{LTE}_{k_a,k_3} = \phi^{LTE}_{k_a,-k_3}, \qquad \phi^{LL}_{k_a,k_3} = -\phi^{LL}_{k_a,-k_3}. \tag{9.123}$$

In terms of the fields $\phi^{\Xi}_{k_i}$, the Hamiltonian $H_{n_3\neq 0}$ is

$$\begin{aligned} H_{n_3\neq 0} = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \big[&\pi^{TTs}\pi^*_{TTs} + \pi^{LT\lambda}\pi^*_{LT\lambda} + \frac{1}{2}\pi^{LL}\pi^*_{LL} - \frac{1}{\sqrt{2}}(\pi^T\pi^*_{LL} + \pi^*_T\pi^{LL}) \\ &+ \frac{1}{4}k^2(h_{TTs}h^{*TTs} - h_T h^{*T})\big]. \end{aligned} \tag{9.124}$$

9.7 Reduced phase space and single scalar field formulation

We are now interested in studying the reduced phase space of the theory and therefore in solving the constraints in (9.97) and the gauge conditions in (9.98).

We start by considering the $n_3 \neq 0$ sector. In Fourier space, equations (9.97) and (9.98) are equivalent to

$$\mathcal{F}_a = -2i\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k\left(e^{\alpha}{}_a \frac{\phi^{LT\alpha}_{k_i}}{\sqrt{2}} + e^{\parallel}{}_a \phi^{LL}_{k_i}\right) e^{ik_a x^a} \sin\big(k_3 x^3\big) = 0, \tag{9.125}$$

$$\mathcal{F}_3 = -2\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k\left(e^{\alpha}{}_3 \frac{\phi^{LT\alpha}_{k_i}}{\sqrt{2}} + e^{\parallel}{}_3 \phi^{LL}_{k_i}\right) e^{ik_a x^a} \cos\big(k_3 x^3\big) = 0, \tag{9.126}$$

$$\mathcal{F}_\perp = -\sqrt{\frac{2}{V}} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \sqrt{2}k^2 \phi^{T}_{k_i} e^{ik_a x^a} \sin\big(k_3 x^3\big) = 0, \tag{9.127}$$

where by $\mathcal{F}_i$ and $\mathcal{F}_\perp$ we denoted either $\mathcal{H}_i$ or $\mathcal{G}_i$ and either $\mathcal{H}_\perp$ or $\mathcal{G}_\perp$. Contracting the first two equations with $e_\beta{}^i$ we see immediately that the above set of equations are equivalent to

$$\phi^{LT\alpha}_{k_i} = 0, \qquad \phi^{LL}_{k_i} = 0, \qquad \phi^{T}_{k_i} = 0, \qquad k_i \neq 0. \tag{9.128}$$

These relations are somehow expected because it is well-known that the longitudinal and trace components of gravitons are pure gauge degrees of freedom and hence not physical.

In the $n_3 = 0$ sector we have

$$\mathcal{F}^{(0)\,3} = \frac{1}{\sqrt{V}} \sum_{n_a \in \mathbb{Z}^2}{}' k_a {\phi^{a3}}_{k_a,0} e^{i k_a x^a} = 0. \tag{9.129}$$

The solution is

$$ {\Pi^{a3}}_{k_a,0} = \epsilon^{ab} \frac{k_b}{\sqrt{2} k_\perp} f_{k_a}, \qquad {h^{a3}}_{k_a,0} = \epsilon^{ab} \frac{k_b}{\sqrt{2} k_\perp} g_{k_a}, \qquad k_i \neq 0, \tag{9.130}$$

with f_{k_a} and g_{k_a} complex numbers depending on k_a. The normalization in (9.130) has been choosen for later convenience. Note that the constraint $\mathcal{F}^{(0)\,3}$ reduces the number of degrees of freedom in the $n_3 = 0$ sector from two to one, as clear from equation (9.130). From the reality condition in (9.105), the we have

$$f_{k_a} = f^*_{-k_a}, \qquad g_{k_a} = g^*_{-k_a}. \tag{9.131}$$

Taking into account (9.128) and (9.130), the Hamiltonian in the reduced phase space is given by $H = H_{n_3 \neq 0} + H_{n_3 = 0}$ where, from (9.124) we read

$$H_{n_3 \neq 0} = \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \Pi^{TTs} \Pi^*_{TTs} + \frac{1}{4} k^2 h^{TTs} h^*_{TTs}, \tag{9.132}$$

together with

$$H_{n_3 = 0} = \frac{p_a p^a}{2} + \sum_{n_a \in \mathbb{Z}^2}{}' f f^* + \frac{1}{4} k_\perp^2 g g^*. \tag{9.133}$$

We introduce the following notation for $k_i \neq 0$:

$$\Pi^s_{k_i} \equiv \sqrt{2} \Pi^{TTs}_{k_i}, \qquad h^s_{k_i} \equiv \frac{1}{\sqrt{2}} h^{TTs}_{k_i}, \qquad \Pi^\times_{k_a,0} \equiv \sqrt{2} f_{k_a}, \qquad h^\times_{k_a,0} \equiv \frac{1}{\sqrt{2}} g_{k_a}. \tag{9.134}$$

The full Hamiltonian can be written as $H = H^+ + H^\times$, with

$$H^+ = \frac{1}{2} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \Pi^+_{k_i} \Pi^{*+}_{k_i} + k^2 h^+_{k_i} h^{*+}_{k_i}, \qquad H^\times = \frac{p_a p^a}{2} + \frac{1}{2} \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}_0}{}' \Pi^\times_{k_i} \Pi^{*\times}_{k_i} + k^2 h^\times_{k_i} h^{*\times}_{k_i}. \tag{9.135}$$

The Hamiltonian is again the sum of two Hamiltonians of a single scalar field. The only difference with respect to the Hamiltonian of the electromagnetic field in (9.23) is the presence of an additional zero mode that, in any case, will not contribute to partition function. Hence it would be slightly incorrect to say that the $+$ and $\times$ fields satisfy Dirichlet and Neumann conditions, respectively. The degrees of freedom in the reduced phase space of the theory satisfy the Poisson bracket

$$\{h^s_{k_i},\Pi^{*s'}_{k'_i}\}=\delta^{ss'}\prod_i\delta_{n_in'_i},\qquad \{h^{\times}_{k_a,0},\Pi^{*\times}_{k'_a,0}\}=\prod_a\delta_{n_an'_a},\qquad \{p^a,q^b\}=\delta^{ab}. \tag{9.136}$$

Repeating the same steps in (9.71)-(9.75) replacing E with $\times$ and H with $+$, it is possible to further reformulate the theory in terms of a single scalar field Φ satisfying periodic boundary conditions on the double volume $V'=2V$, its Hamiltonian being

$$H=\frac{p_ap^a}{2}+{\sum_{n_i\in\mathbb{Z}^3}}' k(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i}). \tag{9.137}$$

This shows that, even if Pauli-Fierz theory is described by the gauge invariant action,

$$S[h]=\frac{1}{2}\int_V \mathrm{d}^4x\Big(-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}+\partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\rho}-\partial_\mu h^{\mu\nu}\partial_\nu h+\frac{1}{2}\partial_\rho h\partial^\rho h\Big), \tag{9.138}$$

the dynamics of the linearized gravitational field with Casimir-type boundary conditions, in the reduced phase space and up to zero modes, can be rearranged in terms of a single scalar field Φ governed by the action (9.78) with periodic boundary conditions on the double interval, as in (9.79), equivalently to what happens for electromagnetism. The result for the partition function is again given by

$$\log Z(\tau,\bar\tau)=\frac{A}{8L^2\tau_2\pi^2}f_2(\tau),\qquad \tau=\frac{\beta}{2L}(\mu+i), \tag{9.139}$$

and the same considerations on high-/low- temperature dualities of section 9.5 apply here.

The main difference regards the expression of P_3 that is needed to consistently add a real part to the modular parameter in terms of the original variables (h_{ij},Π^{ij}). Its expression is most transparent in terms of generalized vector calculus operations that feature prominently in the context of the Hamiltonian approach to duality invariance [266–270]. Therefore, we consider the action of the generalized curl $\vec\nabla\times$ and of the operator $\mathcal{P}$ on symmetric spacetime tensor ϕ^{ij}, defined as

$$(\vec\nabla\times\phi)^{ij}\equiv\frac{1}{2}(\epsilon^i{}_{lm}\partial^l\phi^{mj}+\epsilon^j{}_{lm}\partial^l\phi^{mi}), \tag{9.140}$$

$$(\mathcal{P}\,\phi)^{ij}\equiv-\Delta\phi^{ij}+\partial^i\partial_l\phi^{jl}+\partial^j\partial_l\phi^{il}. \tag{9.141}$$

Some properties and explicit expressions for the various components of the action of $\vec\nabla\times$ and of $\mathcal{P}$ on symmetric tensors in Fourier space are given in Appendix G.2. From (G.32) we see that the gauge invariant expression P_3 in terms of the conjugate variables (h_{ij},Π^{ij}) is

$$\begin{aligned}P_3=\int_V\mathrm{d}^3x\,\partial_3(\vec\nabla\times h)_{ij}\frac{1}{\sqrt{-\Delta}^3}(\mathcal{P}\,\Pi)^{ij}&=\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}k_3(\Pi^{*+}_{k_i}h^{\times}_{k_i}-\Pi^{*\times}_{k_i}h^{+}_{k_i})\\&=i\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}k_3(a^{*+}_{k_i}a^{\times}_{k_i}-a^{*\times}_{k_i}a^{+}_{k_i}).\end{aligned} \tag{9.142}$$

Also in the case of linearized gravity the non-locality in coordinate space of the maps in (9.120) reflects into a non-locality of the gauge invariant expression of the observable P_3 in (9.142) and, comparing its expression with that of the spin angular momentum of the graviton in (G.35)

$$J_3 = \int_V \mathrm{d}^3x\, \Pi^{ij}_{TT}(\vec{e}_3 \times h^{TT})_{ij} = \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \frac{k_3}{k}(\Pi^{*\times}_{k_i} h^{+}_{k_i} - \Pi^{*+}_{k_i} h^{\times}_{k_i})$$
$$= -i \sum_{n_a \in \mathbb{Z}^2, n_3 \in \mathbb{N}} \frac{k_3}{k}(a^{*+}_{k_i} a^{\times}_{k_i} - a^{*\times}_{k_i} a^{+}_{k_i}), \tag{9.143}$$

we see that they have the same mode expansion up to a multiplication by k in momentum space of each term in the sum and an overall sign.

9.8 Conclusions and open questions

As shown in detail by the Hamiltonian analysis, the perfectly conducting boundary conditions implemented in this chapter for both spin 1 and spin 2 fields are fully consistent with gauge invariance and are non-trivial, as mentioned for instance in [249]. Further, as highlighted by the polarization vectors and tensors technique, they allow to reshuffle and reorganize the physical degrees of freedom in terms of those of a pair of scalar fields satisfying Dirichlet and Neumann boundary conditions or, equivalently, in terms of a single scalar field with periodic boundary conditions on the double interval $2L$. In turn, after having identified the correct modular parameter and the correct observable to include in the partition function besides the Hamiltonian, this gives access to the exact expression of the partition function in terms of the real analytic Eisenstein series and therefore to its $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}^2$ symmetries. It is worth emphasizing that, evaluating the central charge in equation (8.14) in chapter 8 at $d = 3$ and replacing in that formula $L \to 2L$, we obtain for c

$$c = \frac{A}{L^2}\frac{\pi}{60}, \tag{9.144}$$

that can be used to determine the asymptotic values of both the entropy and the microcanonical density of states in the high-temperature regime as

$$S_{\text{high}}(\beta) = \frac{L^3 c}{\beta}\frac{16\pi}{3}, \qquad \log\rho(E) \approx \left(ELc^{\frac{1}{3}}\right)^{\frac{3}{4}} \frac{\pi^{\frac{1}{4}} 4\sqrt{2}}{3}, \tag{9.145}$$

where we used (8.117). This shows that techniques originally developed in the context of two-dimensional CFTs combined with modular invariance (see *e.g.* [252]) can be consistently applied in the framework of the Casmir effect in order to produce exact results.

The fact that the final result for the partition function is the same for spin 1 and spin 2 is suggesting that, at the level of free field theories, they are both characterized by the same number and nature of propagating degrees of freedom, as also underlined by the reduced phase space analysis. Indeed, a consequence of the boundary conditions we have chosen is that the gravitational Casimir force on the "walls" is the same as the electromagnetic one. Trying to give a meaning to this brings one back to the discussion in the beginning of section 9.6 of what should be the physical nature of the walls implementing the boundary conditions on the gravitational field in (9.99)-(9.101). More generally, in order to distinguish the result for linearized gravity from that for electromagnetism and to begin to discuss potential implications, one needs to consider interactions, and thus bring in Newtons constant, in one way or another. Indeed, none of the results contained in this chapter depend at all on the electromagnetic or gravitational coupling constant. Therefore, it is legitimate to expect that also in the case of more complicated gauge structures, such as those of free higher spin fields, similar patterns can be unravelled.

An open question concerns ADM surface charges for the setup we are considering. This might be relevant in relation to black holes because in the Gibbons-Hawking treatment [271] they play a crucial role. In particular, it would be interesting to see whether they can be understood in terms of the free particles (*i.e.* zero modes) in the spectrum. Recent investigations in this directions have been done *e.g.* in [224, 225]. Another question of interest would be to understand whether traces of modular covariance can emerge by exploring the case of more general, non necessarily flat, boundary geometries and that of curved backgrounds.

A The forms notation

In this appendix we briefly review the form notation. Greek indices $\mu, \nu, ...$ are spacetime indices $\mu = 0, ..., d$ so that $n = d+1$ is the full dimension of spacetime. The index $\mu = 0$ labels a time coordinate. Consider a r-form $A \in \Omega^r(M)$ on a $n = (d+1)$-dimensional Lorentzian manifold (M, g).

$$A = \frac{1}{r!} A_{\mu_1 \ldots \mu_r} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}. \tag{A.1}$$

Then the *Hodge dual* $*$ is a map

$$* : \Omega^r(M) \to \Omega^{n-r}(M), \tag{A.2}$$

such that

$$*A \equiv \mathbf{A} = \frac{\sqrt{-g}}{r!(n-r)!} A^{\mu_1 \ldots \mu_r} \varepsilon_{\mu_1 \ldots \mu_r \mu_{r+1} \ldots \mu_n} \mathrm{d}x^{\mu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_n} \equiv \sqrt{-g} A^{\mu_1 \ldots \mu_r} (\mathrm{d}^{n-r}x)_{\mu_1 \ldots \mu_r}, \tag{A.3}$$

where $\varepsilon_{\mu_1 \mu_2 \ldots \mu_n}$ is the skew-symmetric Levi-Civita pseudo-tensor with the convention $\varepsilon_{01 \ldots d} = 1$ and where we have introduced

$$(\mathrm{d}^{n-r}x)_{\mu_1 \ldots \mu_r} \equiv \frac{1}{r!(n-r)!} \epsilon_{\mu_1 \ldots \mu_r \mu_{r+1} \ldots \mu_n} \mathrm{d}x^{\mu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}. \tag{A.4}$$

Using Hodge duality, we can associate to a Lagrangian density $L = \sqrt{-g}\mathcal{L}$ a n-form $\mathbf{L} = L \mathrm{d}^n x$. Similarly, it is possible to associate to a vector J with components J^μ a $(n-1)$-form $\mathbf{J} = \sqrt{-g} J^\mu (\mathrm{d}^{n-1}x)_\mu$ and to a skew-symmetric tensor with components $K^{[\mu\nu]}$, a $(n-2)$-form $\mathbf{K} = \sqrt{-g} K^{[\mu\nu]} (\mathrm{d}^{n-2}x)_{\mu\nu}$.

The *exterior derivative* is a map d

$$d : \Omega^r(M) \to \Omega^{r+1}(M), \tag{A.5}$$

such that

$$dA = \frac{1}{r!} \partial_\mu A_{\mu_1 \ldots \mu_r} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \ldots \wedge \mathrm{d}x^{\mu_r}. \tag{A.6}$$

Simbolically, d can be written as

$$d = \mathrm{d}x^\mu \partial_\mu. \tag{A.7}$$

The exterior derivatives of a $(n-1)$-form $\mathbf{J}$ and of a $(n-2)$-form $\mathbf{K}$ are given by

$$d\mathbf{J} = \partial_\mu J^\mu \mathrm{d}^n x, \qquad d\mathbf{K} = \partial_\nu K^{\mu\nu} (\mathrm{d}^{n-1}x)_\mu, \tag{A.8}$$

so that, if a vector J^μ and skew-symmetric tensor $K^{[\mu\nu]}$ are conserved, $\partial_\mu J^\mu = 0 = \partial_\nu K^{\mu\nu}$, the corresponding Hodge dual $(n-1)$ and $(n-2)$-forms satisfy $d\mathbf{J} = 0$ and $d\mathbf{K} = 0$. Note that the exterior derivative of a Lagrangian n-form vanishes, *i.e.* $d\mathbf{L} = 0$.

The *interior product* i_ξ with respect to a vector ξ is a map

$$i_\xi : \Omega^r(M) \to \Omega^{r-1}(M), \tag{A.9}$$

such that

$$i_\xi A = \frac{1}{(r-1)!}\xi^\mu A_{\mu\mu_2...\mu_r} dx^{\mu_2} \wedge ... \wedge dx^{\mu_r}. \tag{A.10}$$

Note that i_ξ can also be written as

$$i_\xi = \xi^\mu \frac{\partial}{\partial \mathrm{d}x^\mu}. \tag{A.11}$$

Furthermore, the exterior derivative and the interior product satisfy the *Cartan's magic formula*:

$$\mathcal{L}_\xi = d\, i_\xi + i_\xi\, d, \tag{A.12}$$

where $\mathcal{L}_\xi$ is the Lie derivative along a vector ξ. Applying this formula to a Lagrangian n-form yields

$$\mathcal{L}_\xi \mathbf{L} = d\, i_\xi \mathbf{L}. \tag{A.13}$$

Stokes theorem is

$$\int_{\Sigma_{n-r+1}} d\mathbf{A} = \int_{\partial\Sigma_{n-r}} \mathbf{A}, \tag{A.14}$$

where Σ_{n-r+1} is an $(n-r+1)$-dimensional hypersurface and $\partial\Sigma_r$ its $(n-r)$-dimensional boundary.

B Chiral splitting of the conformal factor

This Appendix is devoted to the comparison of chapter 5 with [148]. There, one requires an additional boundary condition, namely

$$\Box^{(0)}\varphi = 0. \tag{B.1}$$

This implies that the variational principle is well defined. Indeed, the solution of (B.1) is, in light-cone coordinates,

$$\varphi = \varphi^{+}(x^{+}) + \varphi^{-}(x^{-}). \tag{B.2}$$

The boundary line element is thus

$$g^{(0)}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -e^{2\varphi^{+}(x^{+})}\mathrm{d}x^{+}e^{2\varphi^{-}(x^{-})}\mathrm{d}x^{-}. \tag{B.3}$$

Notice in particular that, with these boundary conditions, the boundary metric is flat

$$R^{(0)} = 8e^{-2\varphi}\partial_{+}\partial_{-}\varphi = 0. \tag{B.4}$$

Clearly, in order to preserve (B.1), the parameter ω generating Weyl transformations must be of the form

$$\omega = \omega^{+}(x^{+}) + \omega^{-}(x^{-}), \tag{B.5}$$

i.e. it admits a splitting into a chiral and an anti-chiral part. Note that (B.5) is exactly the ω-chiral condition of (5.34). However, in this framework, it comes automatically from preserving the additional boundary condition in (B.1). Thus, we can repeat the same arguments of section 5.4 and the asymptotic symmetry algebra sector involving Weyl generators is again given by (5.48),

$$\left[\zeta^{\pm}_p, \zeta^{\pm}_q\right]_M = 0, \qquad \left[\zeta^{\pm}_p, \zeta^{\mp}_q\right]_M = 0, \qquad \left[\xi^{\pm}_n, \zeta^{\pm}_p\right]_M = 0, \qquad \left[\xi^{\pm}_n, \zeta^{\mp}_p\right]_M = 0. \tag{B.6}$$

In this setup the Weyl charges become explictly

$$Q_{\zeta_\omega}[g] = -\frac{\ell}{4\pi G}\int_0^{2\pi}\mathrm{d}\phi(\omega^{+}\partial_{+}\varphi^{+} + \omega^{-}\partial_{-}\varphi^{-}) \equiv Q_{\zeta_{\omega^+}}[\varphi^{+}] + Q_{\zeta_{\omega^-}}[\varphi^{-}], \tag{B.7}$$

and therefore they admit a complete chiral/anti-chiral splitting, differently from what happens in (5.53). Furthermore, since the Weyl central charge (5.57) is independent of φ, it is given again by

$$\mathcal{K}_{\zeta^{\pm}_p,\zeta^{\pm}_q} = ipc_W\delta_{q+p,0}, \qquad \mathcal{K}_{\zeta^{\pm}_p,\zeta^{\mp}_q} = 0, \qquad c^{\pm}_W = c_W = \frac{\ell}{2G} \tag{B.8}$$

just as in (5.64). Therefore the centrally extended charge algebra with these boundary conditions is the same as (5.65)-(5.70),

$$\left\{Q_{\xi^{\pm}_n}[g], Q_{\xi^{\pm}_m}[g]\right\} = i(n-m)Q_{\xi^{\pm}_{n+m}}[g] - im^3\frac{c}{12}\delta_{n+m,0}, \tag{B.9}$$

$$\left\{Q_{\xi^{\pm}_n}[g], Q_{\xi^{\mp}_m}[g]\right\} = 0, \tag{B.10}$$

$$\left\{Q_{\zeta^{\pm}_p}[g], Q_{\zeta^{\pm}_q}[g]\right\} = ipc_W\delta_{p+q,0}, \tag{B.11}$$

$$\left\{Q_{\zeta^{\pm}_p}[g], Q_{\zeta^{\mp}_q}[g]\right\} = 0, \tag{B.12}$$

$$\left\{Q_{\xi^{\pm}_n}[g], Q_{\zeta^{\pm}_p}[g]\right\} = 0, \tag{B.13}$$

$$\left\{Q_{\xi^{\pm}_n}[g], Q_{\zeta^{\mp}_p}[g]\right\} = 0. \tag{B.14}$$

Some comments are in order here. First, we remark that the charges obtained here are conserved, integrable and finite. This is expected: the non-conservation of our charges was due to the non-flatness of the boundary metric. Secondly, just as in the ω-chiral case, also here the charge algebra is not explicitly time-dependent. Lastly, note that in our basis the algebra is a direct sum of the Virasoro and the Weyl piece, while in [148] the algebra was represented as a semidirect sum. This is ultimately a consequence of our field-dependent redefinition (5.27).

C Chern-Simons formulation

We reformulate here our results in the Chern-Simons formulation. This has a twofold purpose: it allows on the one hand to compare our results with [149] while on the other hand to perform the Gauss decomposition which outlines the role played by the Weyl anomaly and the absence of propagating bulk degrees of freedom. In particular, we will show that the conformal factor decouples from the other dynamical fields of the theory. This Appendix extends to our boundary conditions results obtained originally in [125, 126] and further discussed in [130, 149].

C.1 Conventions and solution space

Three-dimensional GR with a negative cosmological constant can be described by a Chern-Simons theory for an $\mathfrak{so}(2,2)$ valued connection [123, 272]. In particular, since $\mathfrak{so}(2,2)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$,[49] the Einstein-Hilbert action can be written, up to boundary terms, as the sum of two Chern-Simons actions

$$S_{EH}[A,\bar{A}] = S_{CS}[A] - S_{CS}[\bar{A}], \tag{C.1}$$

where we have denoted by A and $\bar{A}$ the chiral and anti-chiral connections, respectively, and where

$$S_{CS}[A] = -\kappa \int_{\mathcal{M}} \mathrm{d}^3x \mathrm{Tr}\big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\big), \qquad \kappa = \frac{\ell}{16\pi G}. \tag{C.2}$$

Following the conventions used in [149], we choose the generators of $\mathfrak{sl}(2,\mathbb{R})$ as

$$j_+ = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}, \qquad j_- = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \qquad j_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \tag{C.3}$$

so that the Killing form is

$$\mathrm{Tr}\big(j_a\, j_b\big) = \frac{1}{2}\eta_{ab}, \qquad \eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}, \tag{C.4}$$

where the latin indices a and b take the values $+,-,z$. The dreibein $e^a{}_\mu$ satisfy

$$g_{\mu\nu}(x) = e^a{}_\mu(x) e^b{}_\nu(x)\eta_{ab}, \tag{C.5}$$

or, defining the one-forms $e^a = e^a{}_\mu \mathrm{d}x^\mu$,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}e^a e^b = (e^z)^2 + 2e^+e^-. \tag{C.6}$$

The Hodge dual of the spin connection $\omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu$ is defined as

$$\omega^a = -\frac{1}{2}\epsilon^{abc}\omega_{bc}, \qquad \epsilon_{z+-} = -\epsilon^{z+-} = 1, \tag{C.7}$$

[49] We are going to refer to the two copies of $\mathfrak{sl}(2,\mathbb{R})$ as the left or chiral $\mathfrak{sl}(2,\mathbb{R})_L$ and right or anti-chiral $\mathfrak{sl}(2,\mathbb{R})_R$.

whereas the chiral and anti-chiral connections as

$$A=\left(\omega^a+\frac{e^a}{\ell}\right)j_a,\qquad \bar{A}=\left(\omega^a-\frac{e^a}{\ell}\right)j_a. \tag{C.8}$$

The one-forms e^a are chosen to be

$$e^{\pm}=-\frac{1}{\sqrt{2}}\left[\frac{e^{\varphi}}{\rho}\mathrm{d}x^{\pm}-\rho e^{-\varphi}\big(g^{(2)}_{\mp\mp}\mathrm{d}x^{\mp}+g^{(2)}_{+-}\mathrm{d}x^{\pm}\big)\right],\qquad e^z=-\frac{\ell}{\rho}\mathrm{d}\rho, \tag{C.9}$$

and the dual of the spin connection

$$\omega^{\pm}=-\frac{1}{\sqrt{2}\ell}\left[\frac{e^{\varphi}}{\rho}\mathrm{d}x^{\pm}+\rho e^{-\varphi}\big(g^{(2)}_{\mp\mp}\mathrm{d}x^{\mp}+g^{(2)}_{+-}\mathrm{d}x^{\pm}\big)\right],\qquad \omega^z=\partial_-\varphi\mathrm{d}x^- - \partial_+\varphi\mathrm{d}x^+. \tag{C.10}$$

It follows that the left and right connections are given by $A=A_\mu\mathrm{d}x^\mu$ and $\bar{A}=\bar{A}_\mu\mathrm{d}x^\mu$ with

$$A_+=\begin{pmatrix}-\frac{1}{2}\partial_+\varphi & \frac{e^{-\varphi}\rho}{\ell}g^{(2)}_{++}\\ \frac{e^{\varphi}}{\ell\rho} & \frac{1}{2}\partial_+\varphi\end{pmatrix},\quad A_-=\begin{pmatrix}\frac{1}{2}\partial_-\varphi & \frac{e^{-\varphi}\rho}{\ell}g^{(2)}_{+-}\\ 0 & -\frac{1}{2}\partial_-\varphi\end{pmatrix},\quad A_\rho=\begin{pmatrix}-\frac{1}{2\rho} & 0\\ 0 & \frac{1}{2\rho}\end{pmatrix}, \tag{C.11}$$

$$\bar{A}_+=\begin{pmatrix}-\frac{1}{2}\partial_+\varphi & 0\\ \frac{e^{-\varphi}\rho}{\ell}g^{(2)}_{+-} & \frac{1}{2}\partial_+\varphi\end{pmatrix},\quad \bar{A}_-=\begin{pmatrix}\frac{1}{2}\partial_-\varphi & \frac{e^{\varphi}}{\ell\rho}\\ \frac{e^{-\varphi}\rho}{\ell}g^{(2)}_{--} & -\frac{1}{2}\partial_-\varphi\end{pmatrix},\quad \bar{A}_\rho=\begin{pmatrix}\frac{1}{2\rho} & 0\\ 0 & -\frac{1}{2\rho}\end{pmatrix}. \tag{C.12}$$

Note that with BH boundary conditions defined by $\varphi=0$, A_+ is chiral, $A_-=0$ and $\bar{A}_-$ is anti-chiral, $\bar{A}_+=0$.

C.2 Variational problem, Weyl anomaly and WZW reduction

Let us now discuss the action principle and the variatonal problem associated with (C.1). We find it convenient to discuss it in terms of FG coordinates (ρ,t,ϕ). The action contains a pure boundary term that does not change the dynamics and that we ignore. Indeed, we define our starting action as

$$\tilde{S}_{CS}[A]=-\kappa\int_{\mathcal{M}}\mathrm{d}^3x\,\mathrm{Tr}\big(A_\rho\dot{A}_\phi-A_\phi\dot{A}_\rho+2A_tF_{\phi\rho}\big). \tag{C.13}$$

Taking a variation of (C.13) yields

$$\begin{aligned}\delta\tilde{S}_{CS}[A]&=-\kappa\int_{\mathcal{M}}\mathrm{d}^3x\,\mathrm{Tr}\big(2\delta A_rF_{t\phi}-2\delta A_\phi F_{tr}+2\delta A_tF_{\phi r}\big)+2\kappa\int_{\partial\mathcal{M}}\mathrm{d}^2x\,\mathrm{Tr}\big(A_t\delta A_\phi\big)\\ &\approx 2\kappa\int_{\partial\mathcal{M}}\mathrm{d}^2x\,\mathrm{Tr}\big(A_t\delta A_\phi\big),\end{aligned} \tag{C.14}$$

where in the last step we have imposed the equations of motion, $F=dA=0$. In total, considering also the contribution of the anti-chiral sector, we have

$$\delta\tilde{S}_{CS}[A]-\delta\tilde{S}_{CS}[\bar{A}]=2\kappa\int_{\partial\mathcal{M}}\mathrm{d}^2x\,\mathrm{Tr}\big(A_t\delta A_\phi-\bar{A}_t\delta\bar{A}_\phi\big). \tag{C.15}$$

With BH boundary conditions, in order to have a well-defined variational problem, it is sufficient to add to the action the so-called Coussaert-Henneaux-Van Driel boundary term [125],

$$\tilde{S}[A,\bar{A}] = \tilde{S}_{CS}[A] - \tilde{S}_{CS}[\bar{A}] - \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x \mathrm{Tr}\big(A_\phi^2 + \bar{A}_\phi^2\big), \tag{C.16}$$

whose variation cancels exactly the right-hand side of (C.15), since on-shell $A_t = \frac{1}{\ell}A_\phi$ and $\bar{A}_t = -\frac{1}{\ell}\bar{A}_\phi$. However, with our choice of boundary conditions the variation of the action is

$$\delta\tilde{S}[A,\bar{A}] = -\delta\left\{\kappa\ell\int_{\partial\mathcal{M}} \mathrm{d}^2x\,(\partial_t\varphi)^2\right\} + \frac{2\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x(\ell^2\partial_t^2 - \partial_\phi^2)\varphi\delta\varphi. \tag{C.17}$$

The first term is exact and can be hence compensated by adding an additional boundary term to the action. The last term is not exact, due to the Weyl anomaly. With the decomposition

$$A_\mu = A_\mu^a j_a \Longrightarrow A_\phi^z = \bar{A}_\phi^z = -\ell\partial_t\varphi, \tag{C.18}$$

we can eventually write the action as

$$S[A,\bar{A}] = \tilde{S}_{CS}[A] - \tilde{S}_{CS}[\bar{A}] - \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x \mathrm{Tr}\big(A_\phi^2 + \bar{A}_\phi^2\big) + \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x A_\phi^z\bar{A}_\phi^z. \tag{C.19}$$

The variational problem for this action is ill-defined, for the theory is Weyl anomalous. In other words, it is not possible to add more boundary terms to the action in order to achieve $\delta S = 0$.

Let us now perform the reduction to a WZW model [273–276]. Solving the constraints, the spatial components of the connection are given by

$$A_i = G^{-1}\partial_i G, \qquad \bar{A}_i = \bar{G}^{-1}\partial_i\bar{G}, \tag{C.20}$$

for some group elements $G \in \mathrm{SL}(2,\mathbb{R})_L$ and $\bar{G} \in \mathrm{SL}(2,\mathbb{R})_R$. The constraints $F_{\rho\phi} = 0$ and $\bar{F}_{\rho\phi} = 0$ imply that G and $\bar{G}$ have the form

$$G = g(t,\phi)h(\rho,t), \qquad \bar{G} = \bar{g}(t,\phi)\bar{h}(\rho,t), \tag{C.21}$$

as can be easily verified. Furthermore we assume that $\partial_t h(\rho,t)|_{\partial M} = \partial_t\bar{h}(\rho,t)|_{\partial M} = 0$. Plugging this into $S[A,\bar{A}]$, we have, after some algebra,

$$\begin{aligned}\tilde{S}[A,\bar{A}] =& \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x\mathrm{Tr}\big[g^{-1}\partial_\phi g(\ell^{-1}g^{-1}\partial_t g - g^{-1}\partial_\phi g)\big] + \frac{\kappa}{3}\int_{\mathcal{M}} \mathrm{Tr}\big(G^{-1}dG\wedge G^{-1}dG\wedge G^{-1}dG\big)\\ &-\frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x\mathrm{Tr}\big[\bar{g}^{-1}\partial_\phi\bar{g}(\ell^{-1}\bar{g}^{-1}\partial_t\bar{g} + \bar{g}^{-1}\partial_\phi\bar{g})\big] - \frac{\kappa}{3}\int_{\mathcal{M}} \mathrm{Tr}\big(\bar{G}^{-1}d\bar{G}\wedge\bar{G}^{-1}d\bar{G}\wedge\bar{G}^{-1}d\bar{G}\big)\\ &+\frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x A_\phi^z\bar{A}_\phi^z.\end{aligned} \tag{C.22}$$

This is the WZW reduced action.

C.3 Gauss decomposition

Let us focus on the chiral part of the action in (C.22) and consider the following decomposition of g

$$g = \begin{pmatrix}1 & 0\\ \sigma & 1\end{pmatrix}\begin{pmatrix}e^{-\chi/2} & 0\\ 0 & e^{\chi/2}\end{pmatrix}\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{-\chi/2} & \tau e^{-\chi/2}\\ \sigma e^{-\chi/2} & \sigma\tau e^{-\chi/2} + e^{\chi/2}\end{pmatrix}, \tag{C.23}$$

from which it follows that

$$g^{-1}\partial_\mu g = \begin{pmatrix} -e^{-\chi}\tau\partial_\mu\sigma - \frac{1}{2}\partial_\mu\chi & -e^{-\chi}\tau^2\partial_\mu\sigma + \partial_\mu\tau - \tau\partial_\mu\chi \\ e^{-\chi}\partial_\mu\sigma & e^{-\chi}\tau\partial_\mu\sigma + \frac{1}{2}\partial_\mu\chi \end{pmatrix}. \tag{C.24}$$

In terms of the *Gauss fields* (σ, χ, τ), the boundary term is

$$\begin{aligned} &\frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x \mathrm{Tr}\big[g^{-1}\partial_\phi g(\ell^{-1}g^{-1}\partial_t g - g^{-1}\partial_\phi g)\big] \\ &\qquad = \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x \bigg[\frac{1}{2}\chi'(\ell\dot{\chi} - \chi') + \ell e^{-\chi}(\tau'\dot{\sigma} + \dot{\tau}\sigma') - 2e^{-\chi}\tau'\sigma'\bigg]. \end{aligned} \tag{C.25}$$

For the bulk term first note that, decomposing G as

$$G = \begin{pmatrix} 1 & 0 \\ \Sigma & 1 \end{pmatrix}\begin{pmatrix} e^{-X/2} & 0 \\ 0 & e^{X/2} \end{pmatrix}\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \tag{C.26}$$

we gather

$$\mathrm{Tr}\big(G^{-1}dG \wedge G^{-1}dG \wedge G^{-1}dG\big) = -3\epsilon^{\mu\nu\lambda}\partial_\mu\big(e^{-X}\partial_\nu\Sigma\partial_\lambda T\big)\mathrm{d}\rho\wedge\mathrm{d}t\wedge\mathrm{d}\phi. \tag{C.27}$$

Hence, applying Stokes theorem,

$$\frac{\kappa}{3}\int_{\mathcal{M}} \mathrm{Tr}\big(G^{-1}dG \wedge G^{-1}dG \wedge G^{-1}dG\big) = -\kappa\int_{\partial\mathcal{M}} \mathrm{d}^2x e^{-X}(\dot{\Sigma}T' - \Sigma'\dot{T})\big|_{\partial\mathcal{M}}. \tag{C.28}$$

Furthermore, for the matrix $h(\rho, t)$ we have

$$A_\rho = h^{-1}\partial_\rho h = \begin{pmatrix} -\frac{1}{2\rho} & 0 \\ 0 & \frac{1}{2\rho} \end{pmatrix} \Longrightarrow h = \begin{pmatrix} \sqrt{\frac{\ell}{\rho}} & 0 \\ 0 & \sqrt{\frac{\rho}{\ell}} \end{pmatrix}. \tag{C.29}$$

Since $G = g(t,\phi)h(\rho,t)$ we have the equality

$$\begin{pmatrix} e^{-X/2} & Te^{-X/2} \\ \Sigma e^{-X/2} & \Sigma T e^{-X/2} + e^{X/2} \end{pmatrix} = \begin{pmatrix} e^{-\chi/2}\sqrt{\frac{\ell}{\rho}}, & e^{-\chi/2}\sqrt{\frac{\rho}{\ell}}\tau \\ e^{-\chi/2}\sqrt{\frac{\ell}{\rho}}\sigma & e^{-\chi/2}\sqrt{\frac{\rho}{\ell}}(e^{\chi} + \sigma\tau) \end{pmatrix}, \tag{C.30}$$

which gives

$$e^{-X} = \frac{\ell}{\rho}e^{-\chi}, \qquad T = \frac{\rho}{\ell}\tau, \qquad \Sigma = \sigma. \tag{C.31}$$

Hence the second term in (C.28) in terms of Gauss fields is $-\kappa\int_{\partial M}\mathrm{d}^2x e^{-\chi}(\dot{\sigma}\tau' - \sigma'\dot{\tau})$. The full chiral part of the action thus reads

$$\frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x\bigg[\frac{1}{2}\chi'(\ell\dot{\chi} - \chi') + 2e^{-\chi}\sigma'(\ell\dot{\tau} - \tau')\bigg]. \tag{C.32}$$

Assuming also for $\bar{g}$ and $\bar{G}$ the Gaussdecompositions

$$\bar{g} = \begin{pmatrix} 1 & \bar{\sigma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\bar{\chi}/2} & 0 \\ 0 & e^{-\bar{\chi}/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\tau} & 1 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}\bar{\tau}e^{-\bar{\chi}/2} + e^{\bar{\chi}/2} & \bar{\sigma}e^{-\bar{\chi}/2} \\ \bar{\tau}e^{-\bar{\chi}/2} & e^{-\bar{\chi}/2} \end{pmatrix}, \tag{C.33}$$

$$\bar{G} = \begin{pmatrix} 1 & \bar{\Sigma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\bar{X}/2} & 0 \\ 0 & e^{-\bar{X}/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{T} & 1 \end{pmatrix} = \begin{pmatrix} \bar{\Sigma}\bar{T}e^{-\bar{X}/2} + e^{\bar{X}/2} & \bar{\Sigma}e^{-\bar{X}/2} \\ \bar{T}e^{-\bar{X}/2} & e^{-\bar{X}/2} \end{pmatrix}, \tag{C.34}$$

and noting that $\bar{h}(\rho, t)$ is given by

$$\bar{h} = \begin{pmatrix} \sqrt{\frac{\rho}{\ell}} & 0 \\ 0 & \sqrt{\frac{\ell}{\rho}} \end{pmatrix}. \tag{C.35}$$

The procedure used for the chiral part can be repeated for the anti-chiral part of the action that can be written as

$$\frac{\kappa}{\ell} \int_{\partial\mathcal{M}} \mathrm{d}^2x \Big[-\frac{1}{2}\bar{\chi}'(\ell\dot{\bar{\chi}} + \bar{\chi}') - 2e^{-\bar{\chi}}\bar{\sigma}'(\ell\dot{\bar{\tau}} + \bar{\tau}') \Big]. \tag{C.36}$$

Summing (C.32) and (C.36), the total action in terms of the Gauss fields is then

$$S[A, \bar{A}] = \frac{\kappa}{\ell} \int_{\partial\mathcal{M}} \mathrm{d}^2x \Big[\frac{1}{2}\chi'(\ell\dot{\chi} - \chi') + 2e^{-\chi}\sigma'(\ell\dot{\tau} - \tau') - \frac{1}{2}\bar{\chi}'(\ell\dot{\bar{\chi}} + \bar{\chi}') - 2e^{-\bar{\chi}}\bar{\sigma}'(\ell\dot{\bar{\tau}} + \bar{\tau}') + A^z_\phi \bar{A}^z_\phi \Big]. \tag{C.37}$$

Note that it is possible to express A^z_ϕ and $\bar{A}^z_\phi$ in terms of the Gauss fields as

$$A^z_\phi = -2e^{-\chi}\tau\sigma' - \chi' = \bar{A}^z_\phi = 2e^{-\bar{\chi}}\bar{\tau}\bar{\sigma}' + \bar{\chi}'. \tag{C.38}$$

On defining

$$C = \frac{A^z_\phi}{2} = \frac{\bar{A}^z_\phi}{2} = -\frac{\ell}{2}\partial_t\varphi. \tag{C.39}$$

we can rewrite the last term in (C.37) as $4C^2$. We perform now the Hamiltonian analysis of the action (C.37). The canonical momenta $\pi_{\phi_i} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^i}$ are

$$\pi_\chi = \frac{\kappa}{2}\chi', \qquad \pi_\tau = 2\kappa e^{-\chi}\sigma', \qquad \pi_\sigma = 0, \tag{C.40}$$

$$\pi_{\bar{\chi}} = -\frac{\kappa}{2}\bar{\chi}', \qquad \pi_{\bar{\tau}} = -2\kappa e^{-\bar{\chi}}\bar{\sigma}', \qquad \pi_{\bar{\sigma}} = 0, \tag{C.41}$$

together with $\pi_C = 0$. The Hamiltonian density is

$$\mathcal{H} = \dot{\phi}^i\pi_{\phi_i} - \mathcal{L} = \frac{\kappa}{\ell}\Big(\frac{1}{2}\chi'^2 + \frac{1}{2}\bar{\chi}'^2 + 2e^{-\chi}\sigma'\tau' + 2e^{-\bar{\chi}}\bar{\sigma}'\bar{\tau}' - 4C^2 \Big). \tag{C.42}$$

Now we implement our boundary conditions, using the equalities $g^{-1}\partial_\phi g = hA_\phi h^{-1}\big|_{\partial M}$ and $\bar{g}^{-1}\partial_\phi \bar{g} = \bar{h}\bar{A}_\phi \bar{h}^{-1}\big|_{\partial M}$. We obtain the following set of relations:

$$C = -e^{-\chi}\tau\sigma' - \frac{1}{2}\chi' = e^{-\bar{\chi}}\bar{\tau}\bar{\sigma}' + \frac{1}{2}\bar{\chi}', \tag{C.43}$$

$$e^{-\chi}\sigma' = \frac{e^{\varphi}}{\ell^2} = -e^{-\bar{\chi}}\bar{\sigma}', \tag{C.44}$$

$$-e^{-\chi}\tau^2\sigma' + \tau' - \tau\chi' = e^{-\varphi}(g^{(2)}_{++} - g^{(2)}_{+-}) \tag{C.45}$$

$$-e^{-\bar{\chi}}\bar{\tau}^2\bar{\sigma}' - \bar{\tau}' + \bar{\tau}\bar{\chi}' = -e^{-\varphi}(g^{(2)}_{--} - g^{(2)}_{+-}). \tag{C.46}$$

Plugging these equations in the Hamiltonian of (C.42) we have

$$\mathcal{H} = \frac{\kappa}{\ell}\Big(\frac{1}{2}\chi'^2 + \frac{1}{2}\bar{\chi}'^2 - \chi'' - \bar{\chi}'' + \varphi'(\chi' + \bar{\chi}') - 4C^2\Big), \tag{C.47}$$

xx where we note that C cannot be further expressed in terms of other independent fields. Note also that the Hamiltonian can be simply expressed as

$$\mathcal{H} = \frac{2\kappa}{\ell^3}\big(g^{(2)}_{++} + g^{(2)}_{--} - 2g^{(2)}_{+-}\big) = \ell T_{tt}, \tag{C.48}$$

as it is reasonable, using (C.43)-(C.46). Let us consider the equations of motion. The Hamiltonian action is

$$S_H = \int_{\partial\mathcal{M}} \mathrm{d}^2x\,(\pi_{\phi_i}\dot{\phi}_i - \mathcal{H}) = \int_{\partial\mathcal{M}} \mathrm{d}^2x\big(\pi_\chi\dot{\chi} + \pi_{\bar{\chi}}\dot{\bar{\chi}} + \pi_\tau\dot{\tau} + \pi_{\bar{\tau}}\dot{\bar{\tau}} - \mathcal{H}\big), \tag{C.49}$$

and, using equations (C.40) and (C.41), together with the relations (C.43)-(C.46), we get

$$S_H = \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x\Big[\Big(\frac{1}{2}\chi' + \varphi'\Big)\Big(\ell\dot{\chi} - \chi'\Big) - \Big(\frac{1}{2}\bar{\chi}' + \varphi'\Big)\Big(\ell\dot{\bar{\chi}} + \bar{\chi}'\Big) + 4\ell C\dot{\varphi} + 4C^2\Big]. \tag{C.50}$$

It follows from (C.50) that C is proportional to the canonical momentum conjugate to φ. The Poisson bracket

$$\{\varphi(t,\phi), C(t,\phi')\} = \frac{1}{4\kappa}\delta(\phi - \phi') \Longrightarrow \{\varphi(t,\phi)\partial_t\varphi(t,\phi')\} = -\frac{8\pi G}{\ell^2}\delta(\phi - \phi'), \tag{C.51}$$

where we have used (C.39) to express C in terms of $\partial_t\varphi$. Using again (C.39) in (C.51), we get

$$S_H = \frac{\kappa}{\ell}\int_{\partial\mathcal{M}} \mathrm{d}^2x\Big[\Big(\frac{1}{2}\chi' + \varphi'\Big)\Big(\ell\dot{\chi} - \chi'\Big) - \Big(\frac{1}{2}\bar{\chi}' + \varphi'\Big)\Big(\ell\dot{\bar{\chi}} + \bar{\chi}'\Big) - \ell^2\dot{\varphi}^2\Big]. \tag{C.52}$$

The action (C.52) mixes χ and $\bar{\chi}$ with φ, but it is straightforward to show that, introducing new fields

$$\psi = \chi + \varphi, \qquad \bar{\psi} = \bar{\chi} + \varphi, \tag{C.53}$$

it admits a simple rewriting

$$S_H = \frac{\kappa}{\ell} \int_{\partial \mathcal{M}} \mathrm{d}^2 x \left[\frac{1}{2} \psi' (\ell \partial_t - \partial_\phi) \psi - \frac{1}{2} \bar{\psi}' (\ell \partial_t + \partial_\phi) \bar{\psi} - \ell^2 (\partial_t \varphi)^2 + (\partial_\phi \varphi)^2 \right]. \tag{C.54}$$

From (C.54) it is clear that the dynamics of ψ and $\bar{\psi}$ is independent of φ, which is the desired result. The action (C.54) can be shown to be equivalent to a Liouville theory [130,277–279] coupled to an external two-dimensional metric in conformal gauge. Eventually, we stress again that the Chern-Simons construction carried out so far shows that the φ reduced boundary action (*i.e.* the last terms in (C.54)) is completely disentangled from the rest.

D Scalar field with Dirichlet or Neumann boundary conditions

Dirichlet boundary conditions

Consider a scalar field $\phi(t,x)$ in 1+1 spacetime dimensions satisfying Dirichlet conditions in $x=0$ and $x=L$,

$$\phi(t,0) = 0 = \phi(t,L). \tag{D.1}$$

The appropriate basis to expand ϕ is given by the set of eigenfunctions $\{e^D_{k_i}\}$ of the Laplacian ∂_x^2, defined as

$$e^D_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \qquad k = \frac{\pi}{L}n, \qquad n \in \mathbb{N}, \tag{D.2}$$

satisfying the orthonormality and completeness relations

$$(e^D_k, e^D_{k'}) = \int_0^L \mathrm{d}x\, e^D_k(x) e^D_{k'}(x) = \delta_{nn'}, \qquad \sum_{n\in\mathbb{N}} e^D_k(x) e^D_k(x') = \delta(x-x'). \tag{D.3}$$

The field ϕ and its conjugate momentum Π can be expanded in $\{e^D_k\}$ as

$$\phi(t,x) = \sqrt{\frac{2}{L}}\sum_{n\in\mathbb{N}} \phi_k(t)\sin(kx), \qquad \Pi(t,x) = \sqrt{\frac{2}{L}}\sum_{n\in\mathbb{N}} \Pi_k(t)\sin(kx), \tag{D.4}$$

with reals $\phi_k(t) = (e^D_k, \phi) = \phi^*_k(t)$ and $\Pi_k(t) = (e^D_k, \Pi) = \Pi^*_k(t)$. Therefore, the modes are odd functions of k, *i.e.* $\phi_k(t) = -\phi_{-k}(t)$ and $\Pi_k(t) = -\Pi_{-k}(t)$. The Hamiltonian reads

$$H[\phi,\Pi] = \frac{1}{2}\int_0^L \mathrm{d}x(\Pi^2 + \partial_x\phi^2) = \frac{1}{2}\sum_{n\in\mathbb{N}}(\Pi_k^2 + \omega_k^2\phi_k^2), \qquad \omega_k = \frac{\pi}{L}|n|, \tag{D.5}$$

or

$$H = \frac{1}{2}\sum_{n\in\mathbb{N}} \omega_k (a^*_k a_k + a_k a^*_k), \tag{D.6}$$

in terms of the oscillator variables

$$a_k = \sqrt{\frac{k}{2}}\left(\phi_k + \frac{i}{k}\Pi_k\right). \tag{D.7}$$

Neumann boundary conditions

Assume now that $\phi(t,x)$ satisfies Neumann conditions,

$$\partial_x \phi(t,x)|_{x=0} = 0 = \partial_x \phi(t,x)|_{x=L}. \tag{D.8}$$

The basis of eigenfunctions $\{e^N_{k_i}\}$ of the Laplacian is now

$$e^N_0(x) = \frac{1}{\sqrt{L}}, \qquad e^N_k(x) = \sqrt{\frac{2}{L}}\cos(kx) \qquad k = \frac{\pi}{L}n, \qquad n \in \mathbb{N}, \tag{D.9}$$

satisfying

$$(e^N_k, e^D_{k'}) = \int_0^L \mathrm{d}x\, e^N_k(x) e^N_{k'}(x) = \delta_{nn'}, \qquad \sum_{n\in\mathbb{N}_0} e^N_k(x) e^N_k(x') = \delta(x-x'). \tag{D.10}$$

Note that there is an additional zero mode in the basis $\{e^N_k\}$ with respect to the $\{e^D_k\}$. The field ϕ and its conjugate momentum Π can be expanded in $\{e^N_k\}$ as

$$\phi(t,x) = \frac{\phi_0(t)}{\sqrt{L}} + \sqrt{\frac{2}{L}}\sum_{n\in\mathbb{N}} \phi_k(t)\cos(kx), \qquad \Pi(t,x) = \frac{\Pi_0(t)}{\sqrt{L}} + \sqrt{\frac{2}{L}}\sum_{n\in\mathbb{N}} \Pi_k(t)\cos(kx), \tag{D.11}$$

with reals $\phi_k(t) = (e^N_k, \phi) = \phi^*_k(t)$ and $\Pi_k(t) = (e^N_k, \Pi) = \Pi^*_k(t)$. In this case the modes are even functions of k, *i.e.* $\phi_k(t) = \phi_k(t)$ and $\Pi_k(t) = \Pi_{-k}(t)$. The Hamiltonian reads

$$H[\phi,\Pi] = \frac{1}{2}\int_0^L \mathrm{d}x(\Pi^2 + \partial_x\phi^2) = \frac{p^2}{2} + \frac{1}{2}\sum_{n\in\mathbb{N}_0}(\Pi_k^2 + \omega_k^2\phi_k^2), \qquad \omega_k = \frac{\pi}{L}|n|, \tag{D.12}$$

where we denoted $p \equiv \Pi_0$, or

$$H = \frac{p^2}{2} + \frac{1}{2}\sum_{n\in\mathbb{N}} \omega_k(a^*_k a_k + a_k a^*_k), \tag{D.13}$$

in terms of the oscillator variables

$$a_k = \sqrt{\frac{k}{2}}\left(\phi_k + \frac{i}{k}\Pi_k\right), \qquad k \neq 0. \tag{D.14}$$

E Stress energy tensor of a scalar field

Here we consider the stress-energy tensor of a massless scalar field living in $d+1$ spacetime dimensions. The standard definition of the stress-energy tensor yields,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S[\phi]}{\delta g^{\mu\nu}}\bigg|_{g=\eta} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho \phi \partial^\rho \phi, \tag{E.1}$$

where $S[\phi]$ is that in (7.52). Note that $T_{\mu\nu}$ is traceless in two dimensions. Introducing light-cone coordinates (x^+, x^-) as

$$x^\pm = t \pm x^d, \tag{E.2}$$

the Minkowski line element reads

$$ds^2 = -dx^+ dx^- + \delta_{ab} dx^a dx^b, \tag{E.3}$$

so that

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1/2 & 0 & \dots & 0 \\ -1/2 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & \dots & 0 \\ -2 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{E.4}$$

For the energy density we have

$$T_{00} = (T_{++} + T_{--} + 2T_{+-}), \tag{E.5}$$

where

$$T_{++} = \partial_+ \phi \partial_+ \phi, \qquad T_{--} = \partial_- \phi \partial_- \phi, \qquad T_{+-} = \frac{1}{4}\partial_a \phi \partial^a \phi. \tag{E.6}$$

The time evolution of ϕ can be obtained from that of a_{k_i} and $a^*_{k_i}$. We have, from (7.61)

$$\dot{a}_{k_i} = \{a_{k_i}, H\} = -i\omega_{k_i} a_{k_i}, \tag{E.7}$$

so that the time evolution of ϕ is

$$\phi(t,x) = \frac{q}{\sqrt{V}} + \frac{1}{\sqrt{V}} \sum_{n_i \in \mathbb{Z}^d}{}' \frac{1}{\sqrt{2\omega_{k_i}}} (a_{k_i}(0) e^{-i\omega_{k_i} t + i k_d x^d} e^{i k_a x^a} + \text{c.c.}), \tag{E.8}$$

and hence, in terms of $x^\pm$ coordinates

$$\phi(t,x) = \frac{q}{\sqrt{V}} + \frac{1}{\sqrt{V}} \sum_{n_i \in \mathbb{Z}^d}{}' \frac{1}{\sqrt{2\omega_{k_i}}} (a_{k_i} e^{-i\frac{\omega_{k_i}}{2}(x^+ + x^-) + i\frac{k_d}{2}(x^+ - x^-) + i k_a x^a} + \text{c.c.}). \tag{E.9}$$

It is straightforward to calculate the following integrals

$$\int_V \mathrm{d}^d x\, \partial_+\phi\partial_+\phi = \frac{1}{8}\sum_{n_i\in\mathbb{Z}^d}{}' \frac{1}{\omega_{k_i}}\left[(k_d^2-\omega_{k_i}^2)(a_{k_i}a_{-k_i}e^{-2i\omega_{k_i}t}+a^*_{k_i}a^*_{-k_i}e^{2i\omega_{k_i}t})+(k_d-\omega_{k_i})^2(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i})\right],$$

$$\int_V \mathrm{d}^d x\, \partial_-\phi\partial_-\phi = \frac{1}{8}\sum_{n_i\in\mathbb{Z}^d}{}' \frac{1}{\omega_{k_i}}\left[(k_d^2-\omega_{k_i}^2)(a_{k_i}a_{-k_i}e^{-2i\omega_{k_i}t}+a^*_{k_i}a^*_{-k_i}e^{2i\omega_{k_i}t})+(k_d+\omega_{k_i})^2(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i})\right],$$

$$\int_V \mathrm{d}^d x\, \partial_a\phi\partial^a\phi = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}{}' \frac{k_a k^a}{\omega_{k_i}}\left[a_{k_i}a_{-k_i}e^{-2i\omega_{k_i}t}+a^*_{k_i}a^*_{-k_i}e^{2i\omega_{k_i}t}+a_{k_i}a^*_{k_i}+a^*_{k_i}a_{k_i}\right].$$

Using these expressions one obtains the Hamiltonian as

$$H' = \int_V \mathrm{d}^d x\, T_{00} = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}{}' \omega_{k_i}(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i}). \tag{E.10}$$

Further, the other observable of interest is the spatial integral of

$$T_{0d} = (T_{++}-T_{--}), \tag{E.11}$$

which gives again, as it should

$$P_d \equiv -\int_V \mathrm{d}^d x T_{0d} = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}{}' k_d(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i}), \tag{E.12}$$

where P_d is the spatial momentum in the compact x^d direction, defined in (7.64).

On defining

$$L_0 = \frac{L}{2\pi}\int_{\mathrm{V}} \mathrm{d}^d x(T_{--}+T_{+-}), \qquad \bar{L}_0 = \frac{L}{2\pi}\int_{\mathrm{V}} \mathrm{d}^d x(T_{++}+T_{+-}), \tag{E.13}$$

we have

$$L_0 = \frac{L}{8\pi}\sum_{n_i\in\mathbb{Z}^d}{}'(\omega_{k_i}+k_d)(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i}), \qquad \bar{L}_0 = \frac{L}{8\pi}\sum_{n_i\in\mathbb{Z}^d}{}'(\omega_{k_i}-k_d)(a^*_{k_i}a_{k_i}+a_{k_i}a^*_{k_i}), \tag{E.14}$$

and therefore the Hamiltonian in (E.10) and the momentum in (E.13) can be written as

$$H = \frac{2\pi}{L}(L_0+\bar{L}_0), \qquad P_d = \frac{2\pi}{L}(L_0-\bar{L}_0). \tag{E.15}$$

Using symmetric ordering, the operators corresponding to L_0 and $\bar{L}_0$ are

$$\hat{L}_0 = \frac{L}{4\pi}\sum_{n_i\in\mathbb{Z}}{}'(\omega_{k_i}+k_d)\hat{a}^\dagger_{k_i}\hat{a}_{k_i} - \frac{c}{24}, \qquad \hat{\bar{L}}_0 = \frac{L}{4\pi}\sum_{n_i\in\mathbb{Z}}{}'(\omega_{k_i}-k_d)\hat{a}^\dagger_{k_i}\hat{a}_{k_i} - \frac{c}{24}, \tag{E.16}$$

where we used (8.14) for the value of the central charge c,

$$c = \frac{\prod_a L_a}{L^{d-1}} \frac{6\,\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+3}{2}}}. \tag{E.17}$$

With a slightly incorrect notation we will denote by $\hat{L}_0$ and $\hat{\bar{L}}_0$ the terms of (E.16) that do not contain c. Therefore, the operators corresponding to the Hamiltonian and momentum along x^d are written as

$$\hat{H}' = \frac{2\pi}{L}\left(\hat{L}_0 + \hat{\bar{L}}_0 - \frac{c}{12}\right), \qquad \hat{P}_d = \frac{2\pi}{L}\left(\hat{L}_0 - \hat{\bar{L}}_0\right). \tag{E.18}$$

Note that Hamiltonian receives a contribution from the central charge, while the momentum does not.

The $d = 1$ case is peculiar. Indeed, from the equations of motion

$$\partial_+\partial_-\phi = 0 \Longrightarrow \phi = \phi^+(x^+) + \phi^-(x^-), \tag{E.19}$$

we get

$$T_{++} = T_{++}(x^+), \qquad T_{--} = T_{--}(x^-), \qquad T_{+-}(x) = 0. \tag{E.20}$$

Furthermore, the theory is conformally invariant and the conformal group in two dimensions is infinite dimensional and generated by chiral and anti-chiral transformations of coordinates

$$x^{'+} = f(x^+), \qquad x^{'-} = g(x^-), \tag{E.21}$$

and the infinitely many associated conserved charges are given by

$$Q_g^- = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, g(x^-)T_{--}(x^-) \qquad Q_f^+ = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, f(x^+)T_{++}(x^+). \tag{E.22}$$

In the case $g(x^-) = 1 = f(x^+)$ we retrieve (E.13),

$$Q_1^- \equiv \bar{L}_0 = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, T_{--}(x^-) \qquad Q_1^+ \equiv L_0 = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, T_{++}(x^+). \tag{E.23}$$

In general, $f(x^+)$ and $g(x^-)$ can be any pair functions of x^+ and x^- and we can expand them in the basis $f_n(x^+) = \exp\{2\pi i n x^+/L\}$ and $g_n(x^+) = \exp\{2\pi i n x^-/L\}$. We denote the corresponding charges $\bar{L}_n$ and

L_n, respectively:

$$L_n = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, T_{--}(x^-)e^{\frac{2\pi i n}{L}x^-}, \qquad \bar{L}_n = \frac{L}{2\pi}\int_0^L \mathrm{d}x^1\, T_{++}(x^+)e^{\frac{2\pi i n}{L}x^+}. \tag{E.24}$$

satisfying two commuting copies of the Witt algebra under the standard Poisson bracket

$$\{L_n, L_m\} = i(n-m)L_{n+m}, \qquad \{\bar{L}_n, \bar{L}_m\} = i(n-m)\bar{L}_{n+m}, \qquad \{L_n, \bar{L}_m\} = 0. \tag{E.25}$$

Introducing Euclidean time $x^2 = ix^0$, we have $x^- = -(x^1 + ix^2) \equiv -z$ and $x^+ = x^1 + ix^2 \equiv \bar{z}$, and therefore $(L_n)^* = \bar{L}_n$.

F Eisenstein series

In this Appendix we review some properties of the various Eisenstein series used in chapter 8. The main literature for this Appendix is [205, 206, 208, 235, 280, 281], to which we refer for rigorous proofs.

F.1 Holomorphic Eisenstein series

Consider the upper-half complex plane $\mathbb{H}$ defined as $\mathbb{H} = \{z = x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$.

Definition F.1.

A *holomorphic modular form* of weight $w \geq 0$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ transforming as

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^w f(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{F.1}$$

Equation (F.1) implies that $f(z)$ is periodic of period 1, *i.e.* $f(z+1) = f(z)$ and therefore it admits a Fourier expansion of the form

$$f(z) = \sum_{n\in\mathbb{Z}} a(n) q^n, \qquad q \equiv e^{2\pi i z}, \tag{F.2}$$

where

$$a(n) e^{-2\pi n y} = \int_0^1 \mathrm{d}x\, e^{-2\pi i n x} f(x+iy). \tag{F.3}$$

Proposition F.1.

The holomorphic Eisenstein series $E_{2w}(z)$, defined as

$$E_w(z) = \frac{1}{2} \sideset{}{'}\sum_{(n,m)\in\mathbb{Z}^2} \frac{1}{(nz+m)^{2w}}, \tag{F.4}$$

is, for integer $w > 1$, a holomorphic modular form of weight $2w$ satisfying

$$E_w(\infty) = \zeta(2w), \tag{F.5}$$

where $\zeta(n)$ is the Riemann zeta function.

Remark 1.

Notice that, in order to prove the previous proposition, one just need to prove that $E_w(z)$ is invariant under $\mathcal{T}$ and $\mathcal{S}$ transformations defined in (8.38). The invariance under $\mathcal{T}$ is trivial to verify, whereas, for $\mathcal{S}$, one has

$$E_w\left(-\frac{1}{z}\right) = \frac{1}{2}\sum_{(n,m)\in\mathbb{Z}^2}{}' \frac{z^{2w}}{(-n+mz)^{2w}} = z^{2w}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(nz+m)^{2w}} = z^{2w}E_w(z). \tag{F.6}$$

In the last step we used that, for $w > 1$, the double sum in (F.4) is absolutely convergent so the we could exchange the order of the sums. This is no longer true for $w = 1$ as we will discuss below.

Consider the well-known formula

$$\pi\cot\pi z = \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z+m} + \frac{1}{z-m}\right). \tag{F.7}$$

On the other hand we have

$$\pi\cot\pi z = i\pi\frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i\sum_{m=1}^{\infty} q^m, \qquad q = e^{2\pi i z}, \tag{F.8}$$

and hence, comparing (F.7) and (F.8), we get

$$\sum_{m\in\mathbb{Z}}\frac{1}{z+m} = \pi i - 2\pi i\sum_{m=1}^{\infty} q^m. \tag{F.9}$$

Taking $(k-1)$-derivatives with respect to z of both members the above equality yields

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+m)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} n^{k-1}q^m. \tag{F.10}$$

Consider now $E_w(z)$ for integer $w > 1$,

$$\begin{aligned} E_w(z) &= \frac{1}{2}\sum_{(n,m)\in\mathbb{Z}^2}{}' \frac{1}{(nz+m)^{2w}} = \zeta(2w) + \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}}\frac{1}{(nz+m)^{2w}} \\ &\overset{(F.10)}{=} \zeta(2w) + \frac{(-2\pi i)^{2w}}{(2w-1)!}\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} m^{2w-1}q^{m\,n} \\ &\equiv \zeta(2w) + \frac{(2\pi i)^{2w}}{(2w-1)!}\sum_{n\in\mathbb{N}}^{\infty}\sigma_{2w-1}(n)q^n, \end{aligned} \tag{F.11}$$

where we defined

$$\sigma_{2w-1}(n) \equiv \sum_{d|n} d^{2w-1}, \tag{F.12}$$

the sum of the $(2w-1)$-th powers of the positive divisors of n. Equation (F.11) is the Fourier expansion (F.2) of the holomorphic Eisenstein series. Starting with $E_w(z)$, we can define, for $w > 1$ a modular invariant object as

$$F_w(z) = y^w E_w(z), \qquad F_w\left(\frac{az+b}{cz+d}\right) = F_w(z), \tag{F.13}$$

where we used (8.42) for the transformation of y.

Because of the relevance for the case of a scalar field living on $\mathbb{T}^2$ discussed in section 8.2.1 we are interested in the properties of $E_w(z)$ in the case of $w = 1$. The crucial point is that, for $w = 1$, the double sum appearing in (F.4) is not absolutely convergent and therefore we must specify the order of summation. As discussed in Remark 1, this issue prevents $E_1(z)$ from being a modular form. We define

$$E_1(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}{}' \frac{1}{(nz+m)^2} = \zeta(2) + \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} \frac{1}{(nz+m)^2}, \tag{F.14}$$

where $'$ here means the $m \neq 0$ if $n = 0$. It can be shown, similarly to (F.11), that the above defined $E_1(z)$ is a holomorphic function on $\mathbb{H}$ whose Fourier expansion is given by

$$E_1(z) = \zeta(2) - 4\pi^2 \sum_{n\in\mathbb{N}} \sigma_1(n) q^n. \tag{F.15}$$

Let us now investigate the behavior of $E_1(z)$ under $\mathcal{S}$ transformations. We have

$$z^{-2}E_1\left(-\frac{1}{z}\right) = \zeta(2) + \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} \frac{1}{(-n+mz)^2} = \zeta(2) + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \frac{1}{(nz+m)^2} \neq E_1(z). \tag{F.16}$$

Hence, extent to which $E_1(z)$ fails to satisfy the "right" transformation rule is a reflection of the alteration produced by reversing the order of summation. The error is explicitly given by the following Proposition.

Proposition F.2.

$$z^{-2}E_1\left(-\frac{1}{z}\right) = E_1(z) + \frac{\pi}{iz}. \tag{F.17}$$

Consider now the Dedekind's η function defined in (8.36),

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n\in\mathbb{Z}} (1-q^n). \tag{F.18}$$

We have

$$\frac{\eta'(z)}{\eta(z)} = \frac{\mathrm{d}}{\mathrm{d}z}\log(\eta(z)) = \frac{\pi i}{12}\left[1 - 24\sum_{n\in\mathbb{Z}} n\frac{e^{2\pi i n z}}{1-e^{2\pi i n z}}\right] = \frac{\pi i}{12}\left[1 - 24\sum_{n\in\mathbb{N}}\sum_{m\geq 0} n e^{2\pi i n z(m+1)}\right]$$
$$= \frac{\pi i}{12}\left[1 - 24\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} n q^{nm}\right] \overset{(F.15)}{=} \frac{i\pi}{12\zeta(2)}E_1(z). \tag{F.19}$$

Hence, using (F.17)

$$z^{-2}\frac{\eta'(-1/z)}{\eta(-1/z)} = \frac{i\pi}{12\zeta(2)}z^{-2}E_1(-1/z) = \frac{i\pi}{12\zeta(2)}\left[E_1(z) + \frac{\pi}{iz}\right] = \frac{\eta'(z)}{\eta(z)} + \frac{1}{2z}. \tag{F.20}$$

Integrating this equation yields

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\,\eta(z), \tag{F.21}$$

which is the transformation rule of the Dedekind's η function under $\mathcal{S}$ in (8.39).

F.2 Real analytic Eisenstein series

Definition F.2.

We define the *real analytic Eisenstein series* the non-holomorphic function on $\mathbb{H}$

$$f_s(z) = \sum_{(n,m)\in\mathbb{Z}^2}{}' \frac{y^s}{|nz+m|^{2s}}, \tag{F.22}$$

which converges absolutely for $\mathfrak{Re}(s) > 1$.

Under a modular transformation, $f_s(z)$ transforms as

$$f_s\left(\frac{az+b}{cz+d}\right) = f_s(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{F.23}$$

Proposition F.3.

Consider the operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$. *Then*

$$\Delta f_s(z) = s(s-1) f_s(z), \tag{F.24}$$

i.e. $f_s(z)$ *is an eigenvector of of* Δ *with eigenvalue* $s(s-1)$.

The function $f_s(z)$ clearly satisfies $f_s(z) = f_z(z+1)$. Therefore, it admits a Fourier series.

Proposition F.4.

The Fourier series of $f_s(z)$ *is given by*

$$f_s(z) = 2\zeta(2s)y^s + 2\sqrt{\pi}\frac{\Gamma(\frac{2s-1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + 2y^{\frac{1}{2}}\frac{\pi^s}{\Gamma(s)}\sum_{n\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' \left|\frac{n}{m}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|nm|y)e^{2\pi i n m x}, \tag{F.25}$$

or, equivalently, by

$$f_s(z) = 2\zeta(2s)y^s + 2\sqrt{\pi}\frac{\Gamma(\frac{2s-1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + 4y^{\frac{1}{2}}\frac{\pi^s}{\Gamma(s)}\sum_{m\in\mathbb{Z}}{}' |m|^{\frac{1}{2}-s}\sigma_{2s-1}(m)K_{s-\frac{1}{2}}(2\pi|m|y)e^{2\pi i m x}, \tag{F.26}$$

where $K_n(x)$ *is the modified Bessel function of the second kind [251] and* $\sigma_s(n)$ *is defined in* (F.12), *satisfying respectively*

$$K_n(-x) = (-)^{\frac{d}{2}} K_n(x) \tag{F.27}$$

$$K_n(x) = K_{-n}(x), \qquad x > 0, \tag{F.28}$$

$$\sigma_s(n) = |n|^s \sigma_{-s}(n). \tag{F.29}$$

Further, the real analytic Eisenstein series satisfies the functional relation

$$\Gamma(s) f_s(z) = \pi^{2s-1}\Gamma(1-s) f_{1-s}(z). \tag{F.30}$$

F.3 Eisenstein series at $s = 1$ and the two-dimensional scalar field on $\mathbb{T}^2$

We have seen that, in equation (8.33) of section 8.2.1, the partition function on $\mathbb{T}^2_\mu$ without the zero mode is given by

$$\begin{aligned}\log Z'(\tau,\bar\tau) &= \frac{\pi\beta}{6L} + \sum_{l\in\mathbb{N}}\left[\sum_{n\in\mathbb{N}}\frac{e^{2\pi i\tau ln}}{l} + \sum_{n\in\mathbb{N}}\frac{e^{-2\pi i\bar\tau ln}}{l}\right] = \frac{\pi\beta}{6L} + \sum_{l\in\mathbb{N}}\frac{i}{2l}\left[\frac{e^{-\pi il\tau}}{\sin(\pi l\tau)} - \frac{e^{\pi il\bar\tau}}{\sin(\pi l\bar\tau)}\right] - \frac{2}{l} \\ &= \frac{\beta\pi}{6L} + \sum_{l\in\mathbb{N}}\frac{i}{2l}[\cot(\pi l\tau) - \cot(\pi l\bar\tau)] - \frac{1}{l}. \end{aligned} \tag{F.31}$$

Note that the series is convergent. The general term of the series, for large values of n, behaves as $|a_n| \sim e^{-\frac{2\pi\beta}{L}n}$ which is exponentially suppressed. The fact that $n \neq 0$ is essential to achieve the convergence. In the last step in (F.31), we split a convergent series into two divergent series, whose divergent contributions cancel. Using now (F.7) we have

$$\begin{aligned}\log Z'(\tau,\bar\tau) &= \frac{\beta\pi}{6L} - \sum_{l\in\mathbb{N}}\sum_{m\in\mathbb{Z}}\frac{1}{2l\pi i}\left[\frac{1}{(m+l\tau)} - \frac{1}{(m+l\bar\tau)}\right] - \frac{1}{l} = \frac{\beta\pi}{6L} + \sum_{l\in\mathbb{N}}\sum_{m\in\mathbb{Z}}\frac{1}{2\pi i}\frac{(\tau-\bar\tau)}{|m+l\tau|^2} - \frac{1}{l} \\ &= \frac{\beta\pi}{6L} + \frac{1}{2\pi}{\sum_{l\in\mathbb{Z}}}'\sum_{m\in\mathbb{Z}}\frac{\tau_2}{|m+l\tau|^2} - \frac{1}{2|l|} \end{aligned} \tag{F.32}$$

The full series appearing in (F.32) is convergent. However the two series having general terms the ones appearing in (F.32) are clearly divergent. Equivalently, if we defined

$$f_1(\tau) = {\sum_{l\in\mathbb{Z}}}'\sum_{m\in\mathbb{Z}}\frac{\tau_2}{|m+l\tau|^2}, \tag{F.33}$$

it would diverge and the degree of divergence would be such that

$$\sum_{l\in\mathbb{Z}}{\sum_{m\in\mathbb{Z}}}'\frac{\tau_2}{|m+l\tau|^2} - \frac{1}{2|l|} < \infty. \tag{F.34}$$

F.4 From the real analytic Eisenstein series to the Epstein zeta function

Now we show how in the case of $z = iy$, the above listed properties of the real analytic Eisenstein series can be rephrased in terms of those of the Epstein zeta function, defined as

$$\zeta(s; a_1, ..., a_p) = {\sum_{n_1,...,m_p\in\mathbb{Z}^p}}'\frac{1}{(a_1n_1^2 + ... + a_pn_p^2)^s}, \tag{F.35}$$

satisfying the functional relation

$$\Gamma(s)\zeta(s;a_1,...,a_p) = \frac{\pi^{2s-\frac{p}{2}}}{\sqrt{a_1...a_p}}\Gamma\left(\frac{p}{2}-s\right)\zeta\left(\frac{p}{2}-s;\frac{1}{a_1},...,\frac{1}{a_p}\right). \tag{F.36}$$

From equation (F.22), we have, using $z = iy$,

$$f_s(iy) = \sum_{(n,m)\in\mathbb{Z}^2}{}' \frac{y^s}{(ny^2+m^2)^s} = y^s\zeta(s;y^2,1). \tag{F.37}$$

Now we show that the Fourier transform of the real analytic Eisenstein series in (F.25)-(F.26), in the case of real z reduces to a standard application of the well-known Sommerfeld-Watson transform, which we briefly review here. Consider the series

$$S = \sum_{m\in\mathbb{Z}} f(m), \tag{F.38}$$

and the complex function

$$F(z) = \pi f(z)\cot(\pi z), \tag{F.39}$$

having simple poles in $z = m$, with $m \in \mathbb{Z}$, with residues

$$\text{Res}\big[F(z)\big]|_{z=m} = f(m). \tag{F.40}$$

Assuming that

$$f(z) \xrightarrow{|z|\to\infty} \frac{1}{|z|^{1+\epsilon}}, \qquad \epsilon > 0, \tag{F.41}$$

then the integral along a circle C_R of radius R centred at the origin vanishes as $R \to \infty$

$$0 = \oint_{C_\infty} F(z)dz = 2\pi i\left[\sum_{m\in\mathbb{Z}}\text{Res}\big[F(z)\big]|_{z=m} + \sum_i \text{Res}\big[F(z)\big]|_{z=z_i}\right], \tag{F.42}$$

where we used the residue theorem and where z_i are other poles of $F(z)$, at points different from $z = m$. It follows that

$$\sum_{m\in\mathbb{Z}} f(m) = -\sum_i \text{Res}\big[F(z)\big]|_{z=z_i}. \tag{F.43}$$

Consider now explicitly the series

$$S(a)=\sum_{m\in\mathbb{Z}}\frac{1}{a^2+m^2}. \tag{F.44}$$

The function $F(z)=\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}$ has two simple poles in $z=\pm ia$, with residues

$$\mathrm{Res}\left[\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}\right]\Bigg|_{z=\pm ia}=-\frac{\pi}{2a}\coth(\pi a). \tag{F.45}$$

Therefore, using (F.43), we have

$$\sum_{m\in\mathbb{Z}}\frac{1}{a^2+m^2}=\frac{\pi}{a}\coth(\pi a) \tag{F.46}$$

It can be easily shown by induction that

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\frac{1}{(a^2+m^2)^s}&=\frac{(-)^{s-1}}{2^{s-1}(s-1)!}\left(\frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}a}\right)^{s-1}\sum_{m\in\mathbb{Z}}\frac{1}{a^2+m^2}\\&\overset{(\text{F.46})}{=}\frac{(-)^{s-1}}{2^{s-1}(s-1)!}\left(\frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}a}\right)^{s-1}\left(\frac{\pi}{a}\coth(\pi a)\right).\end{aligned} \tag{F.47}$$

The Epstein zeta function $\zeta(s;y^2,1)$ is

$$\zeta(s;y^2,1)=\sum_{(n,m)\in\mathbb{Z}^2}{}'\frac{1}{(n^2y^2+m^2)^s}=\sum_{m\in\mathbb{Z}}{}'\frac{1}{m^{2s}}+\sum_{n\in\mathbb{Z}}{}'\sum_{m\in\mathbb{Z}}\frac{1}{(n^2y^2+m^2)^s}. \tag{F.48}$$

Applying (F.47) to the last term in (F.48), setting $a=ny$,

$$\zeta(s;y^2,1)=2\zeta(2s)+\frac{(-)^{s-1}}{2^{s-1}(s-1)!}\sum_{n\in\mathbb{Z}}{}'\frac{1}{m^{2s-1}}\left[\left(\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)^{s-1}\frac{\pi}{y}\coth(\pi y)\right]. \tag{F.49}$$

This is a general expression of the Epstein zeta function in terms of derivatives of trigonometric functions. It would be tempting to say that (F.49) represents the Fourier series of the Epstein zeta function, by analogy with the Fourier series of the real analytic Eisenstein series in (F.25). However, the Epstein zeta function, unlike the real analytic Eisenstein series, is not periodic and therefore it does not admit a Fourier series. It is worth pointing out that there exists an analogous expansion for the real analytic Eisenstein series, which we give here without proof:

$$\sum_{(n,m)\in\mathbb{Z}^2}{}'\frac{y^s}{|nz+m|^{2s}}=y^s\left\{2\zeta(2s)+\left[\frac{i(-1)^{s-1}}{2^s(s-1)!}\sum_{n\in\mathbb{Z}}{}'\left(\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)^{s-1}\frac{\pi}{y}\cot(\pi mz)+\text{c.c.}\right]\right\}. \tag{F.50}$$

The expression between curly bracket in (F.50) correctly reduces to (F.49) when $z=iy$.

F.5 The explicit resummation of the Fourier series in $d = 3$

In the case $d = 3$ the integral in (8.95) can be explicitly performed and splitting the sum over $n_3 \in \mathbb{Z}$ into $n_3 > 0$, $n_3 = 0$ and $n_3 < 0$, it yields for the partition function

$$\begin{aligned}\log Z(\beta,\mu) &= \frac{L_1L_2\beta\pi^2}{90L_3^3} + \frac{L_1L_2}{2\pi\beta^2}\sum_{l\in\mathbb{N}}\Bigg[\sum_{n_3\in\mathbb{N}}\frac{e^{2\pi l n_3\left(i\frac{\beta\mu}{L_3}-\frac{\beta}{L_3}\right)}}{l^2}\frac{2\pi\beta n_3}{L_3} + \frac{e^{2\pi l n_3\left(i\frac{\beta\mu}{L_3}-\frac{\beta}{L_3}\right)}}{l^3} + \frac{1}{2l^3}\\ &\quad + \sum_{n_3\in\mathbb{N}}\frac{e^{-2\pi l n_3\left(i\frac{\beta\mu}{L_3}+\frac{\beta}{L_3}\right)}}{l^2}\frac{2\pi\beta n_3}{L_3} + \frac{e^{-2\pi l n_3\left(i\frac{\beta\mu}{L_3}+\frac{\beta}{L_3}\right)}}{l^3} + \frac{1}{2l^3}\Bigg]\\ &= \frac{L_1L_2\tau_2\pi^2}{90L_3^2} + \frac{L_1L_2}{2\pi L_3^2\tau_2^2}\sum_{l\in\mathbb{N}}\Bigg[\sum_{n_3\in\mathbb{N}}\frac{e^{2\pi i l n_3\tau}}{l^2}2\pi n_3\tau_2 + \frac{e^{2\pi i l n_3\tau}}{l^3} + \frac{1}{2l^3} + \text{c.c.}\Bigg].\end{aligned} \tag{F.51}$$

We now use the following identities

$$\frac{1}{2} + \sum_{n\in\mathbb{N}} e^{2\alpha n} = \frac{1}{2}\coth\alpha \qquad \sum_{n\in\mathbb{N}} n\, e^{2\alpha n} = \frac{1}{4\sinh^2(\alpha)}, \qquad \mathfrak{Re}(\alpha) < 0, \tag{F.52}$$

so that (F.51) becomes

$$\log Z(\tau,\bar\tau) = \frac{L_1L_2\tau_2}{2\pi^2L_3^2}\left\{\frac{\pi^4}{45} + \frac{\pi}{2}\sum_{l\in\mathbb{N}}\left[-\frac{\pi}{\tau_2^2l^2}\frac{1}{\sin(\pi l\tau)^2} + \frac{i}{\tau_2^3l^3}\cot(\pi l\tau) + \text{c.c.}\right]\right\}. \tag{F.53}$$

Consider now the following chain of identities

$$\sum_{(l,m)\in\mathbb{Z}^2}{}' \frac{1}{|l+m\tau|^4} = \sum_{l\in\mathbb{Z}}{}' l^{-4} + \sum_{m\in\mathbb{Z}}{}'\sum_{l\in\mathbb{Z}}|l+\tau m|^{-4}. \tag{F.54}$$

We have

$$\begin{aligned}&\frac{2}{(l+m\tau)(m\tau-m\bar\tau)^3} + \frac{1}{(l+m\tau)^2(m\tau-m\bar\tau)^2} + \text{c.c.}\\ &= \frac{2(l+m\tau)+m\tau-m\bar\tau}{(l+m\tau)^2(m\tau-m\bar\tau)^3} + \frac{2(l+m\bar\tau)+m\bar\tau-m\tau}{(l+m\bar\tau)^2(m\bar\tau-m\tau)^3} = \frac{2l+3m\tau-m\bar\tau}{(l+m\tau)^2(m\tau-m\bar\tau)^3} + \frac{2l+3m\bar\tau-m\tau}{(l+m\bar\tau)^2(m\bar\tau-m\tau)^3}\\ &= \frac{(2l+3m\tau-m\bar\tau)(l+m\bar\tau)^2-(2l+3m\bar\tau-m\tau)(l+m\tau)^2}{|l+m\tau|^4m^3(\tau-\bar\tau)^3} = \frac{2i\mathfrak{Im}[(2l+3m\tau-m\bar\tau)(l+m\bar\tau)^2]}{|l+m\tau|^4m^3(\tau-\bar\tau)^3}.\end{aligned} \tag{F.55}$$

The expression inside $\mathfrak{Im}$ in the numerator of (F.55) is

$$(2l+3m\tau-m\bar\tau)(l+m\bar\tau)^2 = 2l^3 + 3l^2m(\bar\tau+\tau) + 3m^3|\tau|^3\bar\tau + 6lm^2|\tau|^2 - m^3\bar\tau^3, \tag{F.56}$$

and hence

$$\mathfrak{Im}[2l^3 + 3l^2 m(\bar{\tau} + \tau) + 3m^3|\tau|^3\bar{\tau} + 6lm^2|\tau|^2 - m^3\bar{\tau}^3] = \mathfrak{Im}[m^3\bar{\tau}(3|\tau|^2 - \bar{\tau}^2)] = -4m^3\tau_2^3. \tag{F.57}$$

We get

$$\frac{2}{(l+m\tau)(m\tau - m\bar{\tau})^3} + \frac{1}{(l+m\tau)^2(m\tau - m\bar{\tau})^2} + \text{c.c.} = \frac{-8im^3\tau_2^3}{|l+m\tau|^4 m^3(\tau - \bar{\tau})^3} = \frac{1}{|l+m\tau|^4}. \tag{F.58}$$

Thus we have

$$\sum_{l\in\mathbb{Z}} |l+m\tau|^{-4} = \sum_{l\in\mathbb{Z}} \left[\frac{2}{(l+m\tau)(m\tau - m\bar{\tau})^3} + \frac{1}{(l+m\tau)^2(m\tau - m\bar{\tau})^2} + \text{c.c.}\right]. \tag{F.59}$$

Using

$$\sum_{l\in\mathbb{Z}} \frac{1}{(l+z)} = \pi\cot(\pi z), \qquad \sum_{l\in\mathbb{Z}} \frac{1}{(l+z)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \tag{F.60}$$

we have that (F.59) reads

$$\sum_{l\in\mathbb{Z}} |l+m\tau|^{-4} = \frac{\pi}{4}\left[-\frac{\pi}{m^2\tau_2^2}\frac{1}{\sin^2(\pi m\tau)} + \frac{i}{\tau_2^3 m^3}\coth(\pi m\tau) + \text{c.c.}\right]. \tag{F.61}$$

Eventually, we get, from (F.54)

$$\begin{aligned}\sum_{(l,m)\in\mathbb{Z}^2}{}' \frac{1}{|l+m\tau|^4} &= \frac{\pi^4}{45} + \frac{\pi}{4}\sum_{m\in\mathbb{Z}}{}' \left[-\frac{\pi}{m^2\tau_2^2}\frac{1}{\sin(\pi m\tau)^2} + \frac{i}{m^3\tau_2^3}\cot(\pi m\tau) + \text{c.c.}\right] \\ &= \frac{\pi^4}{45} + \frac{\pi}{2}\sum_{m\in\mathbb{N}} \left[-\frac{\pi}{m^2\tau_2^2}\frac{1}{\sin(\pi m\tau)^2} + \frac{i}{m^3\tau_2^3}\cot(\pi m\tau) + \text{c.c.}\right].\end{aligned} \tag{F.62}$$

from which follows, using (F.53)

$$\log Z(\tau,\bar{\tau}) = \frac{L_1 L_2}{2\pi^2 L_3^2 \tau_2}\sum_{(l,m)\in\mathbb{Z}^2}{}' \frac{\tau_2^2}{|l+\tau m|^4} = \frac{L_1 L_2}{2\pi^2 L_3^2\tau_2} f_2(\tau). \tag{F.63}$$

Therefore in $d=3$ we have explicitly shown the Fourier series in (F.51) to be equivalent to the real analytic Eisenstein series. Consider now the case of a rectangular torus, *i.e.* $\mu = 0$ and hence $\tau_1 = 0$. We have,

$|l+\tau m|^2 = l^2+\tau_2^2 m^2$ and thus

$$\log Z(\tau_2) = \frac{L_1 L_2 \tau_2}{2\pi^2 L_3^3} \sum_{(l,m)\in\mathbb{Z}^2}{}' \frac{1}{(l^2+\tau_2^2 m^2)^2} = \frac{L_1 L_2 \tau_2}{2\pi^2 L_3^3} \zeta(2,\tau_2^2,1), \tag{F.64}$$

where ζ is the Epstein zeta function of (F.35). On the other hand, from (F.53), the partition function for $\tau_1 = 0$ becomes

$$\begin{aligned} \log Z(\tau_2) &= \frac{L_1 L_2 \tau_2}{2\pi^2 L_3^2} \left\{ \frac{\pi^4}{45} + \frac{\pi}{2} \sum_{l\in\mathbb{N}} \left[\frac{\pi}{\tau_2^2 l^2} \frac{1}{\sinh(\pi l \tau_2)^2} + \frac{1}{\tau_2^3 l^3} \coth(\pi l \tau_2) + \text{c.c.} \right] \right\} \\ &= \frac{L_1 L_2 \tau_2}{2\pi^2 L_3^2} \left\{ \frac{\pi^4}{45} + \frac{\pi}{2} \sum_{l\in\mathbb{Z}}{}' \left[\frac{\pi}{\tau_2^2 l^2} \frac{1}{\sinh^2(\pi l \tau_2)} + \frac{1}{\tau_2^3 l^3} \coth(\pi l \tau_2) \right] \right\}. \end{aligned} \tag{F.65}$$

Using the general expansion of the Epstein zeta function in equation (F.49), for $s=2$ yields immediately the connection between (F.64) and (F.65).

F.6 SL$(q+1,\mathbb{Z})$ transformation of the partition function on $\mathbb{T}^{q+1}\times\mathbb{R}^{d-1}$.

The modular parameters τ_i in (8.139), in terms of the lattice vectors $\vec{\omega}_\alpha$ in (8.120) are given by [50]

$$\tau_i = \frac{\left(\vec{\omega}_{d+1}\right)_i + i\left(\vec{\omega}_{d+1}\right)_{d+1}}{\left(\vec{\omega}_i\right)_i + i\left(\vec{\omega}_i\right)_{d+1}}. \tag{F.66}$$

Under SL$(q+1,\mathbb{Z})$ transformations of the lattice vectors

$$\vec{\omega}_i' = S_i{}^j \vec{\omega}_j + S_i{}^{d+1}\vec{\omega}_{d+1}, \qquad \vec{\omega}_{d+1}' = S_{d+1}{}^i \vec{\omega}_i + S_{d+1}{}^{d+1}\vec{\omega}_{d+1}, \tag{F.67}$$

and thus

$$\left(\vec{\omega}_{d+1}'\right)_{d+1} = S_{d+1}{}^i \left(\vec{\omega}_i\right)_{d+1} + S_{d+1}{}^{d+1}\left(\vec{\omega}_{d+1}\right)_{d+1} = S_{d+1}{}^{d+1}\beta, \tag{F.68}$$

$$\left(\vec{\omega}_{d+1}'\right)_i = S_{d+1}{}^j \left(\vec{\omega}_j\right)_i + S_{d+1}{}^{d+1}\left(\vec{\omega}_{d+1}\right)_i = S_{d+1}{}^i L_i + S_{d+1}{}^{d+1}\beta\mu_i, \qquad \text{(no sum on } i) \tag{F.69}$$

$$(\vec{\omega}_i')_{d+1} = S_i{}^j \left(\vec{\omega}_j\right)_{d+1} + S_i{}^{d+1}\left(\vec{\omega}_{d+1}\right)_{d+1} = S_i{}^{d+1}\beta, \tag{F.70}$$

$$(\vec{\omega}_i')_i = S_i{}^j \left(\vec{\omega}_j\right)_i + S_i{}^{d+1}\left(\vec{\omega}_{d+1}\right)_i = S_i{}^i L_i + S_i{}^{d+1}\beta\mu_i. \qquad \text{(no sum on } i). \tag{F.71}$$

[50] Note that in (F.66) the component $\left(\vec{\omega}_i\right)_{d+1}$ is 0, but in general after a modular transformation it will be non-zero.

The modular parameters τ_i transform as

$$\tau_i' = \frac{(\vec{\omega}'_{d+1})_i + i(\vec{\omega}'_{d+1})_{d+1}}{(\vec{\omega}'_i)_i + i(\vec{\omega}'_i)_{d+1}} = \frac{S_{d+1}{}^i L_i + S_{d+1}{}^{d+1}\beta\mu_i + iS_{d+1}{}^{d+1}\beta}{S_i{}^i L_i + S_i{}^{d+1}\beta\mu_i + iS_i{}^{d+1}\beta} = \frac{S_{d+1}{}^{d+1}\tau_i + S_{d+1}{}^i}{S_i{}^{d+1}\tau_i + S_i{}^i}. \tag{F.72}$$

In the case $q = 1$, we recover usual modular transformations $\tau' = \frac{a\tau+b}{c\tau+d}$, with $ad - bc = 1$. However, the transformation in (F.72) for τ_i is not modular since in general $S_{d+1}{}^{d+1}S_i{}^i - S_{d+1}{}^i S_i{}^{d+1} \neq 1$. The imaginary part of τ_i transforms as

$$\tau_{i_2}' = \frac{S_{d+1}{}^{d+1}S_i{}^i - S_{d+1}{}^i S_i{}^{d+1}}{|S_i{}^{d+1}\tau_i + S_i{}^i|^2}\tau_{i_2}, \tag{F.73}$$

and thus

$$\tau_{1_2}'...\tau_{q_2}' = \frac{(S_{d+1}{}^{d+1}S_1{}^1 - S_{d+1}{}^1 S_1{}^{d+1})...(S_{d+1}{}^{d+1}S_q{}^q - S_{d+1}{}^q S_q{}^{d+1})}{\left|S_1{}^{d+1}\tau_1 + S_1^1\right|^2...|S_q{}^{d+1}\tau_q + S_q{}^q|^2}\tau_{1_2}...\tau_{q_2}. \tag{F.74}$$

The volume transforms as

$$\begin{aligned} V_{q+1}' &= \left(\tau_{1_2}'...\tau_{q_2}'\right)^{\frac{1}{q}}(L_1....L_q)^{\frac{q+1}{q}} \\ &= \left[\frac{(S_{d+1}{}^{d+1}S_1{}^1 - S_{d+1}{}^1 S_1{}^{d+1})...(S_{d+1}{}^{d+1}S_q{}^q - S_{d+1}{}^q S_q{}^{d+1})}{\left|S_1{}^{d+1}\tau_1 + S_1^1\right|^2...|S_q{}^{d+1}\tau_q + S_q{}^q|^2}\right]^{\frac{1}{q}} V_{q+1} \end{aligned} \tag{F.75}$$

The full partition function on $\mathbb{T}^{q+1} \times \mathbb{R}^p$, under (F.72), transforms as

$$\log Z(\tau_i', \bar{\tau}_i) = \left[\frac{(S_{d+1}{}^{d+1}S_1{}^1 - S_{d+1}{}^1 S_1{}^{d+1})...(S_{d+1}{}^{d+1}S_q{}^q - S_{d+1}{}^q S_q{}^{d+1})}{\left|S_1{}^{d+1}\tau_1 + S_1^1\right|^2...|S_q{}^{d+1}\tau_q + S_q{}^q|^2}\right]^{-\frac{p}{q(q+1)}} \log Z(\tau_i, \bar{\tau}_i). \tag{F.76}$$

Again, for $p = d - 1$ and $q = 1$ we have

$$\log Z(\tau', \bar{\tau}') = \left[\frac{(ad - bc)}{|c\tau + d|^2}\right]^{-\frac{d-1}{2}} \log Z(\tau, \bar{\tau}) = |c\tau + d|^{d-1} \log Z(\tau, \bar{\tau}), \tag{F.77}$$

as we should.

G The linear momentum observable in terms of gauge fields

G.1 Electromagnetic case

Consider the linear momentum along x^3 of the scalar field Φ defined in (9.85),

$$\begin{aligned}P_3 &= -\int_{V'} \mathrm{d}^3x\, \Pi \partial_3 \Phi = -i \sum_{n_i\in\mathbb{Z}^3} k_3 \Pi^*_{k_i} \Phi_{k_i} \overset{(9.71)}{=} -\frac{i}{2} \sum_{n_i\in\mathbb{Z}^3} k_3 (\Pi^{*E}_{k_i} + i\Pi^{*H}_{k_i})(A^E_{k_i} - iA^H_{k_i}) \\ &\overset{(9.43)}{=} -\frac{1}{2} \sum_{n_i\in\mathbb{Z}^3} k_3 (\Pi^{*E}_{k_i} A^H_{k_i} - \Pi^{*H}_{k_i} A^E_{k_i}) \overset{(9.43)}{=} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\Pi^{*H}_{k_i} A^E_{k_i} - \Pi^{*E}_{k_i} A^H_{k_i}) \\ &\overset{(9.31)}{=} \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 \Pi^{*i}_{k_i} A^j_{k_i} ({e^H}_i {e^E}_j - {e^H}_j {e^E}_i).\end{aligned} \tag{G.1}$$

From the second equation in (9.28) we have

$$\epsilon_{ijk} {e_H}^j {e_E}^k = \frac{k_i}{k} \Longrightarrow {e_H}^i {e_E}^j - {e_H}^j {e_E}^i = \epsilon^{ijk} \frac{k_k}{k}, \tag{G.2}$$

and hence

$$P_3 = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 \Pi^{*i}_{k_i} A^j_{k_i} \epsilon_{ijk} \frac{k^k}{k} = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\vec{\Pi}^*_{k_i} \times \vec{A}_{k_i}) \cdot \hat{k}. \tag{G.3}$$

Now consider the gauge invariant expression

$$\begin{aligned}\int_V \mathrm{d}^3x (\vec{\nabla} \times \partial_3 \vec{A}) \cdot \vec{\Pi} &= -\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\vec{k} \times \vec{A}_{k_i}) \cdot \vec{\Pi}^*_{k_i} = -\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\vec{A}_{k_i} \times \vec{\Pi}^*_{k_i}) \cdot \vec{k} \\ &= \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\vec{\Pi}^*_{k_i} \times \vec{A}_{k_i}) \cdot \vec{k}.\end{aligned} \tag{G.4}$$

Therefore, on comparing (G.4) and (G.3) we get

$$P_3 = \int_V \mathrm{d}^3x (\vec{\nabla} \times \partial_3 \vec{A}) \cdot \frac{1}{\sqrt{-\Delta}} \vec{\Pi} = \int_V \mathrm{d}^3x\, \partial_3 \vec{B} \cdot \frac{1}{\sqrt{-\Delta}} \vec{\Pi}. \tag{G.5}$$

Another equivalent expression for P_3 is, in terms of the oscillators $a^\alpha_{k_i}$ defined in (9.45),

$$P_3 = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (\Pi^{*H}_{k_i} A^E_{k_i} - \Pi^{*E}_{k_i} A^H_{k_i}) = i \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3 (a^{*H}_{k_i} a^E_{k_i} - a^{*E}_{k_i} a^H_{k_i}). \tag{G.6}$$

Now consider the spin angular momentum of light, defined on the reduced phase space as

$$J_i = \int_V \mathrm{d}^3x\, \epsilon_{ijk} A_\perp^j \Pi_\perp^k, \tag{G.7}$$

where $\phi_\perp^i$ are given in (9.50)-(9.54) having set $\phi_{k_i}^{\parallel} = 0$. It is easy to show that

$$J_3 = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k}(\Pi_{k_i}^{*E} A_{k_i}^H - \Pi_{k_i}^{*H} A_{k_i}^E) = -i \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k}(a_{k_i}^{*H} a_{k_i}^E - a_{k_i}^{*E} a_{k_i}^H). \tag{G.8}$$

Hence, we find that and P_3 has the same mode expansion of J_3, up to an overall sign and to multiplication of each term in momentum space by k.

G.2 Pauli-Fierz case

In order to obtain a gauge invariant expression for P_3 in the Pauli-Fierz case, one starts by considering the generalized curl for a symmetric spacetime tensor defined in (9.140),

$$(\vec{\nabla}\times\phi)^{ij} \equiv \frac{1}{2}(\epsilon^i{}_{lm}\partial^l\phi^{mj} + \epsilon^j{}_{lm}\partial^l\phi^{mi}). \tag{G.9}$$

We have, using explicitly the Fourier expansions in (9.102)-(9.104)

$$\begin{aligned}(\vec{\nabla}\times\phi)^{ab} &= \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2}\Bigg[\frac{1}{\sqrt{V}}(\epsilon^a{}_c k^c \phi^{3b}_{k_a,0} + \epsilon^b{}_c k^c \phi^{3a}_{k_a,0}) \\ &\qquad + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}[\epsilon^a{}_c(k^c\phi^{3b}_{k_i} - k^3\phi^{bc}_{k_i}) + \epsilon^b{}_c(k^c\phi^{3a}_{k_i} - k^3\phi^{ac}_{k_i})]\cos(k_3x^3)\Bigg]e^{ik_ax^a} \\ &= \sum_{n_a\in\mathbb{Z}^2}\Bigg[\frac{1}{\sqrt{V}}(\hat{\mathcal{O}}\phi_{k_a,0})^{ab} + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}(\hat{\mathcal{O}}\phi_{k_i})^{ab}\cos(k_3x^3)\Bigg]e^{ik_ax^a},\end{aligned} \tag{G.10}$$

$$\begin{aligned}(\vec{\nabla}\times\phi)^{a3} &= \frac{i}{2}\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}}[\epsilon^a{}_b(k^b\phi^{33}_{k_i} - k^3\phi^{b3}_{k_i}) + \epsilon_{bc}k^b\phi^{ca}_{k_i}]\sin(k_3x^3)e^{ik_ax^a} \\ &= i\sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}}(\hat{\mathcal{O}}\phi_{k_i})^{a3}\sin(k_3x^3)e^{ik_ax^a},\end{aligned} \tag{G.11}$$

$$(\vec{\nabla}\times\phi)^{33} = \frac{1}{2}\sum_{n_a\in\mathbb{Z}^2}\left[\frac{1}{\sqrt{V}}\epsilon_{ab}k^a\phi^{b3}_{k_a,0} + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}\epsilon_{ab}k^a\phi^{b3}_{k_i}\cos\big(k_3x^3\big)\right]e^{ik_ax^a}$$
$$= \sum_{n_a\in\mathbb{Z}^2}\left[\frac{1}{\sqrt{V}}(\hat{\mathcal{O}}\phi_{k_a,0})^{33} + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}(\hat{\mathcal{O}}\phi_{k_i})^{33}\cos\big(k_3x^3\big)\right]e^{ik_ax^a}, \tag{G.12}$$

where the operator $\hat{\mathcal{O}}$ acts on momentum space symmetric tensors $\phi^{ij}_{k_i}$ as

$$(\hat{\mathcal{O}}\,\phi_{k_i})^{ij} \equiv \frac{k}{2}(\epsilon^i{}_{lm}e_\parallel{}^l\phi^{mj}_{k_i} + \epsilon^j{}_{lm}e_\parallel{}^l\phi^{mi}_{k_i}). \tag{G.13}$$

Acting with $\hat{\mathcal{O}}$ on a symmetric tensor rotates its electric and magnetic components and projects out both its trace part (T) and its longitudinal part (LL) :

$$(\hat{\mathcal{O}}\,\phi_{k_i})^{ij} = k\bigg(e_{TT\times}{}^{ij}\phi^{TT+}_{k_i} - e_{TT+}{}^{ij}\phi^{TT\times}_{k_i} + \frac{1}{2}e_{LTE}{}^{ij}\phi^{LTH}_{k_i} - \frac{1}{2}e_{LTH}{}^{ij}\phi^{LTE}_{k_i}\bigg). \tag{G.14}$$

Another operator relevant for our analysis is $\mathcal{P}$, whose action on a symmetric tensor ϕ is defined as

$$(\mathcal{P}\,\phi)^{ij} \equiv -\Delta\phi^{ij} + \partial^i\partial_l\phi^{jl} + \partial^j\partial_l\phi^{il}. \tag{G.15}$$

Similarly to (G.10)-(G.11), it is possible to show that

$$(\mathcal{P}\,\phi)^{ab} = \sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}(\hat{\mathcal{P}}\,\phi_{k_i})^{ab}\sin\big(k_3x^3\big)e^{ik_ax^a}, \tag{G.16}$$

$$(\mathcal{P}\,\phi)^{a3} = -i\sum_{n_a\in\mathbb{Z}^2}\left[\frac{1}{\sqrt{V}}(\hat{\mathcal{P}}\,\phi_{k_i})^{a3} + \sqrt{\frac{2}{V}}\sum_{n_3\in\mathbb{N}}(\hat{\mathcal{P}}\,\phi_{k_i})^{a3}\cos\big(k_3x^3\big)\right]e^{ik_ax^a}, \tag{G.17}$$

$$(\mathcal{P}\,\phi)^{33} = \sqrt{\frac{2}{V}}\sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}}(\hat{\mathcal{P}}\,\phi_{k_i})^{33}\sin\big(k_3x^3\big)e^{ik_ax^a}, \tag{G.18}$$

where the operator $\hat{\mathcal{P}}$ acts on momentum space symmetric tensors $\phi^{ij}_{k_i}$ as

$$(\hat{\mathcal{P}}\,\phi_k)^{ij} \equiv k^2(\phi^{ij}_{k_i} - e_\parallel{}^ie^\parallel{}_l\phi^{lj}_{k_i} - e_\parallel{}^je^\parallel{}_l\phi^{li}_{k_i}). \tag{G.19}$$

Acting with $\hat{\mathcal{P}}$ on a symmetric tensor projects out the transverse components of the longitudinal part:

$$(\hat{\mathcal{P}}\,\phi_k)^{ij} = k^2(e_{TT+}{}^{ij}\phi^{TT+}_{k_i} + e_{TT\times}{}^{ij}\phi^{TT\times}_{k_i} + e_T{}^{ij}\phi^T_{k_i} - e_{LL}{}^{ij}\phi^{LL}_{k_i}). \tag{G.20}$$

It can be shown that the projector onto the transverse-traceless part of the symmetric tensor ϕ^{ij} can be obtained by combining the actions of $\vec{\nabla}\times$ and $\mathcal{P}$:

$$(\mathcal{P}^{TT}\,\phi)^{ij} = -(\mathcal{P}\,\vec{\nabla}\times\phi)^{ij} = -(\vec{\nabla}\times\mathcal{P}\,\phi)^{ij}. \tag{G.21}$$

The observable P_3 is given by

$$\begin{aligned} P_3 &= -\int_{V'} \mathrm{d}^3x\,\Pi\partial_3\Phi = -i\sum_{n_i\in\mathbb{Z}^3} k_3\Pi^*_{k_i}\Phi_{k_i} = \sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}} k_3(\Pi^{*+}_{k_i}h^{\times}_{k_i} - \Pi^{*\times}_{k_i}h^{+}_{k_i}) \\ &= \sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}} k_3\Pi^{*ij}_{k_i}h^{lm}_{k_i}({e^{TT+}}_{ij}{e^{TT\times}}_{lm} - {e^{TT\times}}_{ij}{e^{TT+}}_{lm}). \end{aligned} \tag{G.22}$$

After some algebra, we get

$${e^{TT+}}_{ij}{e^{TT\times}}_{lm} - {e^{TT\times}}_{ij}{e^{TT+}}_{lm} = \frac{1}{2k}\left[\epsilon_{imk}\left(\delta_{jl} - \frac{k_jk_l}{k^2}\right) + \epsilon_{jlk}\left(\delta_{im} - \frac{k_ik_m}{k^2}\right)\right]k^k, \tag{G.23}$$

and hence, substituting in (G.22),

$$P_3 = \sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}} \frac{k_3}{k}\epsilon_{imk}h^{lm}_{k_i}\left(\delta_{jl} - \frac{k_jk_l}{k^2}\right)\Pi^{*ij}_{k_i}k^k. \tag{G.24}$$

Consider

$$(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_l = (k^2\delta_{jl} - k_jk_l)\Pi^{*ij}_{k_i} - k^ik_n\Pi^{*n}_{l_{k_i}}. \tag{G.25}$$

Contracting both sides of this equation with $\epsilon_{imk}k^k$, we obtain

$$\epsilon_{imk}(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_lk^k = \epsilon_{imk}(k^2\delta_{jl} - k_jk_l)\Pi^{*ij}_{k_i}k^k, \tag{G.26}$$

and P_3 can be rewritten as

$$P_3 = \sum_{n_a\in\mathbb{Z}^2,n_3\in\mathbb{N}} \frac{k_3}{k^3}\epsilon_{imk}h^{lm}_{k_i}k^k(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_l. \tag{G.27}$$

Consider

$$(\hat{\mathcal{O}}\,h_{k_i})^l_i = \frac{1}{2}({\epsilon^l}_{km}k^kh^m_{i_{k_i}} + \epsilon_{ikm}k^kh^{ml}_{k_i}). \tag{G.28}$$

Contracting both sides of this equation with $(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_l$ yields

$$(\hat{\mathcal{O}}\,h_{k_i})^l_i(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_l = \epsilon_{ikm}k^k h^{ml}_{k_i}(\hat{\mathcal{P}}\,\Pi^*_{k_i})^i_l, \tag{G.29}$$

and therefore P_3 reads

$$P_3 = -\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k^3}(\hat{\mathcal{O}}\,h_{k_i})_{ij}(\hat{\mathcal{P}}\,\Pi^*_{k_i})^{ij}. \tag{G.30}$$

Now consider the gauge invariant expression

$$\int_V \mathrm{d}^3x\,\partial_3(\vec{\nabla}\times h)_{ij}\frac{1}{\sqrt{-\Delta}^3}(\mathcal{P}\,\Pi)^{ij} = -\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k^3}(\hat{\mathcal{O}}h_{k_i})_{ij}(\hat{\mathcal{P}}\Pi^*_{k_i})^{ij}. \tag{G.31}$$

Comparing with (G.30), we get

$$P_3 = \int_V \mathrm{d}^3x\,\partial_3(\vec{\nabla}\times h)_{ij}\frac{1}{\sqrt{-\Delta}^3}(\mathcal{P}\,\Pi)^{ij}. \tag{G.32}$$

Another equivalent expression for P_3 is, in terms of the $+$ and $\times$ oscillators is

$$P_3 = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3(\Pi^{*+}_{k_i}h^{\times}_{k_i} - \Pi^{*\times}_{k_i}h^{+}_{k_i}) = i\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} k_3(a^{*+}_{k_i}a^{\times}_{k_i} - a^{*\times}_{k_i}a^{+}_{k_i}). \tag{G.33}$$

Now consider the spin angular momentum in linearized gravity, defined on the reduced phase space by

$$J_i = \int_V \mathrm{d}^3x\,\Pi^{ij}_{TT}(\vec{e}_i\times h^{TT})_{ij}. \tag{G.34}$$

It is easy to show that

$$J_3 = \sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k}(\Pi^{*\times}_{k_i}h^{+}_{k_i} - \Pi^{*+}_{k_i}h^{\times}_{k_i}) = -i\sum_{n_a\in\mathbb{Z}^2, n_3\in\mathbb{N}} \frac{k_3}{k}(a^{*+}_{k_i}a^{\times}_{k_i} - a^{*\times}_{k_i}a^{+}_{k_i}). \tag{G.35}$$

Hence, we find again that and P_3 has the same mode expansion of J_3, up to an overall sign and to multiplication of each term in momentum space by k.

H Polarization tensors

Using the expressions of $(e_A{}^i)$, the explicit expressions for the basis elements $(e_\Xi{}^{ij})$ are

$$e_{TT+}{}^{ab} = \frac{1}{\sqrt{2}}\frac{(k^2\epsilon^{ac}k_c\epsilon^{bd}k_d - k_3^2k^ak^b)}{k_\perp^2k^2}, \qquad e_{TT\times}{}^{ab} = \frac{k_3}{\sqrt{2}}\frac{(\epsilon^{ac}k_ck^b + k^a\epsilon^{bc}k_c)}{k_\perp^2k}, \tag{H.1}$$

$$e_{TT+}{}^{a3} = \frac{1}{\sqrt{2}}\frac{k^ak^3}{k^2}, \qquad e_{TT\times}{}^{a3} = -\frac{1}{\sqrt{2}}\frac{\epsilon^{ac}k_c}{k}, \tag{H.2}$$

$$e_{TT+}{}^{33} = -\frac{1}{\sqrt{2}}\frac{k_\perp^2}{k^2}, \qquad e_{TT\times}{}^{33} = 0, \tag{H.3}$$

$$e_T{}^{ij} = \frac{1}{\sqrt{2}}\frac{(k^2\delta^{ij} - k^ik^j)}{k^2}, \tag{H.4}$$

$$e_{LTH}{}^{ab} = \frac{1}{\sqrt{2}}\frac{(\epsilon^{bc}k^ak_c + \epsilon^{ac}k^bk_c)}{kk_\perp}, \qquad e_{LTE}{}^{ab} = \sqrt{2}\frac{k^ak^bk_3}{k^2k_\perp}, \tag{H.5}$$

$$e_{LTH}{}^{a3} = \frac{1}{\sqrt{2}}\frac{\epsilon^{ab}k_bk_3}{kk_\perp}, \qquad e_{LTE}{}^{a3} = \sqrt{2}\frac{k^a(k_3^2 - k_\perp^2)}{k^2k_\perp}, \tag{H.6}$$

$$e_{LTH}{}^{33} = 0, \qquad e_{LTE}{}^{33} = -\sqrt{2}\frac{k_3k_\perp}{k^2}, \tag{H.7}$$

$$e_{LL}{}^{ij} = \frac{k^ik^j}{k^2}. \tag{H.8}$$

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