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Probabilistic approach to non-local equations

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Contents

Introduction

Diffusion processes, in particular Brownian motion, have been the main modelling tool to describe some form of irregular dynamics of particles in media (see, for instance, [**54**])[1]. However, not all physical phenomena linked with irregular (or chaotic) motion of particles can be described in terms of standard diffusions. This happens in particular when the mean square displacement of the particles follows a different time scale. It is the case of anomalous diffusions (see [**107**]). Among anomalous diffusions, we can recognize subdiffusions (with a mean square displacement whose time scale is given by t^γ with $\gamma < 1$) and in particular fractional diffusions (see [**77**]). With the introduction of fractional diffusion, we have to abandon the Markov property to describe subdiffusive phenomena. Indeed, such kind of phenomena are shown to exhibit a behaviour described by a *time-fractional* Fokker-Planck equation. To model phenomena that happens in a different time scale, we have to rely on fractional calculus, which is a branch of mathematics that, despite the not-so-recent origin (the actual problem of finding an operator D such that D^2 is the classical derivative was posed by de l'Hopital to Leibniz[2] in 1695), has become now a *trendy* argument, since the *useful consequences* Leibniz wished for are now reality (for some history of fractional calculus, we refer to [**52**]).

As the link between Brownian motion and the heat equation is well-known (see [**112**]), the search for processes whose probability laws solved in some sense time-fractional heat equations has been carried on to describe subdiffusions. From this, a new *branch* of stochastic calculus was born: stochastic models of fractional calculus (see for instance [**104**]). As a main tool to describe such kind of processes, Lévy processes, in particular stable subordinators, became the main characters. Indeed, the very good path properties of subordinators (see [**35, 36**]) made them a good candidate to substitute the time scale of the Brownian motion. However, composition of Lévy processes is still a Lévy process (and then a Markov process), thus we cannot expect these new processes to solve a time-fractional heat equation. Just think of the subordinated Brownian motion, that is the stochastic representation of the space-fractional heat equation (i.e. the heat equation in which we consider the fractional Laplacian in place of the standard one). However, the first passage time process of a stable subordinator still provides a good candidate to substitute such time scale. Moreover, the composition between a Lévy process and the generalized

[1]Despite for a formal theory of Brownian motion we have to wait for the works of Einstein [**54**], Smoluchovski [**140**] and Perrin (see, for instance, [**119**]), the idea that Brownian motion is strictly linked with diffusion processes precedes such works. For instance, such idea was already present in [**32**]. For the history of Brownian motion up to Perrin we refer to [**41**].

[2]*It will lead to a paradox*[...]. *From this apparent paradox, one day useful consequences will be drawn.*

left-continuous inverse of a subordinator is not a Markov process. As it can be seen, the composition of the Brownian motion with the inverse of a stable subordinator provides a stochastic representation of the time-fractional heat equation. Thus, the idea is the following: time-changing with the inverse of a stable subordinator *could* transform diffusive processes in subdiffusive processes.

In this context, this kind of time-changed processes became the first tool of investigation of time-fractional differential equations. Starting from the introduction of the fractional Poisson process (see [**93**]) and then to the study of some more complicated time-changed birth-death processes (see for instance [**116, 117**]), to arrive also to fractional Poisson fields ([**8**]), to spectral decomposition for fractional diffusions ([**94**]) and fractional mean-field games ([**45**]).

However, using only stable subordinators was restrictive. Indeed, new models relying on new fractional derivatives were considered. To cite, for instance, a recent result, in [**64**] a fractional derivative with respect to another function (see [**9**]) was used to obtain some new Gompertz models that better fit some particular population growth phenomena.

Working with the limitation of the standard fractional calculus was not enough. Thus, the link between Bernstein functions (see [**132**]) and subordinators has been exploited to define some new *general fractional* derivative operators. Such kind of operators were introduced in [**88**] and then in [**143**], with different approaches. In particular, the latter exploits the link between the density of inverse subordinators and such non-local derivatives induced by general Bernstein functions. From this introduction, different probabilistic models have been considered to discuss the equations of what is usually called *generalized fractional calculus* (see, for instance, [**47, 118, 106**] and many others).

This thesis focuses on some results linked to such generalized fractional calculus. Here, after some preliminaries on Bernstein functions and non-local derivatives, we will consider different topics of generalized fractional calculus. In particular we first explore some existence and uniqueness results for non-linear non-local differential equations (via a fixed point argument) and then we exploit a generalized version of the Grönwall inequality for such kind of operators. Then, we move to some problems related to probabilistic representation of solutions of non-local equations, focusing in particular on some birth-death process with known spectral decomposition and on time-changed non-Markov process, such as the fractional Ornstein-Uhlenbeck process introduced in [**48**]. Moreover, we also explore some properties of the first exit times of time-changed Markov process from open sets and then we show the link between the first passage time of a time-changed drifted Brownian motion and a particular non-local parabolic problem. Such techniques are then used in some applications (in particular to queueing theory). Finally, we explore also some properties linked to non-local operators in space, focusing in particular on isotropic Lévy processes obtained by subordination of the Brownian motion and on fractional operators on a sphere. In particular the work is structured as follows:

- The first chapter presents some preliminaries on Bernstein functions, subordinators and non-local derivatives induced by Bernstein functions. In this chapter we introduce the main tools we will work with.

- The second chapter relies on theoretical results on generalized fractional calculus and stochastic representation of solutions of non-local equations. In particular one can recognize four main topics in such a chapter:
  - Existence, uniqueness and Grönwall inequality for non-local non-linear differential equations.
  - Spectral decomposition of strong solutions of some non-local differential equations in Banach sequence spaces.
  - Theoretical results related to the time-changed fractional Ornstein-Uhlenbeck process and its generalized Fokker-Planck equation.
  - Asymptotic behaviour of first exit times of time-changed processes from open sets and link with some non-local partial differential equations.

  All these arguments are linked together by one main scope: exploiting stochastic representation of non-local differential equations, eventually non-local partial differential equations.
- The third chapter presents two applications of the aforementioned theoretical results. The first one is the introduction of fractional queues, together with some results on some performance parameters. The second one is an application to computational neurophysiology, where time-changed process are shown to exhibit some qualitative properties that better explain the behaviour of spike trains of some particular neurons (see [**124**]). Finally, in such chapter, we provide some simulations procedures to reproduce time-changed processes.
- Finally, in the fourth chapter, we move toward some problems related to non-locality in space. First, we give some asymptotics of the jump functions of subordinated Brownian motions (which are useful to study the properties of potentials for non-local Schrödinger equations that exhibit zero-energy eigenvalues, as done in [**24**]). In the second part of the fourth chapter, we study the eigenvalue problem for the fractional integral on the sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ and for a Marchaud-type integral on $\mathbb{S}^{d-1}$. In particular we show the link between the first non-trivial eigenvalue of the Marchaud-type integral on the sphere and the moments of the length of random segments in the unit ball (which is a first step towards a quantitative version of Groemer-Pfiefer inequality, as in [**17**]).

Common Symbols and Definitions

- We denote $\mathbb{N} = \{1, 2, \dots\}$ while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
- We denote $\mathbb{R}^+ = (0, +\infty)$ while $\mathbb{R}_0^+ = [0, +\infty)$;
- We denote $\mathbb{H} = \{z \in \mathbb{C} : \ \Re(z) > 0\}$ and $\mathbb{H}_0 = \{z \in \mathbb{C} : \ \Re(z) \ge 0\}$;
- We denote $\mathbb{D}_1 = \{z \in \mathbb{C} : \ |z| \le 1\}$;
- We denote $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$;
- We denote $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \ |x| = 1\}$, where we use $|\cdot|$ to denote both the Euclidean norm[3] and the Lebesgue measure of a subset of $\mathbb{R}^d$;
- We denote $B_r(x) = \{x \in \mathbb{R}^d : \ |x| < r\}$ and $\omega_d = |B_1(0)|$;
- In the whole text, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- Given two functions f, g, we denote $f \asymp g$ if and only if there exists a constant $c > 1$ such that $\frac{g}{c} \le f \le cg$;
- Given two functions f, g defined in a neighbourhood of $x_0 \in \mathbb{R}^d$, we denote $f \sim g$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f}{g} = 1$.
- Given a suitable measure μ on $[0, +\infty)$ the Laplace-Stieltjes transform of μ is defined as
$$\mathcal{L}^S[\mu](\lambda) = \int_0^{+\infty} e^{-\lambda t} \mu(dt);$$
- Given a function f of bounded variation defined on $[0, +\infty)$, we denote by $\mathcal{L}^S[f]$ the Laplace-Stieltjes transform of its distributional derivative df (that is a Radon measure);
- Given a suitable function f defined on $[0, +\infty)$ the Laplace transform of f is given by
$$\mathcal{L}[f](\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt;$$
- For any measure space (X, Σ, μ), we denote by $L^p(X)$ for $p \ge 1$ the Banach space of measurable functions $f : X \to \mathbb{R}$ such that
$$\|f\|_{L^p(X)}^p = \int_X |f|^p d\mu < +\infty;$$
- For any measure space (X, Σ, μ), we denote by $L^\infty(X)$ the Banach space of measurable functions $f : X \to \mathbb{R}$ such that
$$\|f\|_{L^\infty(X)} = \inf\{M > 0 : \ \mu(\{x \in X : \ |f(x)| > M\}) = 0\} < +\infty,$$
where we set $\inf \emptyset = +\infty$;

[3] Here we use $|\cdot|$ in place of $\|\cdot\|$ to avoid confusion whenever both the Euclidean norm of a point and any other functional norm is used.

- For any topological (Borel-)measure space (X, Σ, μ) where Σ is the completion of the Borel σ-algebra of X with respect of μ, we denote by $L^p_{\mathrm{loc}}(X)$ for $p \geq 1$ (eventually $p = \infty$) the Banach space of measurable functions $f : X \to \mathbb{R}$ such that for any compact set $K \subseteq X$ the function $f\mathbf{1}_K$ belogs to $L^p(X)$, where $\mathbf{1}_K$ is the indicator function of the compact K;
- For any numerable measure space $(E, \mathcal{P}(E), \mu)$, where $\mathcal{P}(E)$ it the set of all subsets of E, we denote $\ell^p(\mu)$ with $p \geq 1$ the Banach space of all sequences $(f(x))_{x\in E}$ such that
$$\|f\|^p_{\ell^p(\mu)} = \sum_{x\in E} |f(x)|^p \mu(\{x\}) < +\infty;$$
- For any numerable measure space $(E, \mathcal{P}(E), \mu)$, where $\mathcal{P}(E)$ it the set of all subsets of E, we denote $\ell^\infty(\mu)$ with the Banach space of all sequences $(f(x))_{x\in E}$ such that
$$\|f\|_{\ell^\infty(\mu)} = \sup_{x\in E} |f(x)| < +\infty;$$
- For any measurable space (X, Σ, μ) and any Banach space $(Y, |\cdot|)$ we denote by $L^p(X; Y)$ for $p \geq 1$ the Banach space of Bochner-measurable functions $f : X \to Y$ (i.e. $f(x) = \lim_{n\to+\infty} f_n(x)$ almost everywhere in X, with $f_n(X)$ a countable set and $f_n^{-1}(\{y\})$ measurable for any $y \in Y$) such that
$$\|f\|^p_{L^p(X;Y)} = \int_X |f|^p d\mu < +\infty;$$
- For any measurable space (X, Σ, μ) and any Banach space $(Y, |\cdot|)$ we denote by $L^\infty(X; Y)$ the Banach space of Bochner-measurable functions $f : X \to Y$ such that
$$\|f\|_{L^\infty(X;Y)} = \inf\{M > 0 : \ \mu(\{x \in X : \ |f(x)| > M\}) = 0\} < +\infty,$$
where we set $\inf \emptyset = +\infty$;
- With $\mathcal{H}^n$ we denote the Hausdorff measure of dimension n.

CHAPTER 1

Preliminaries

1.1. Bernstein functions

In this section we will introduce the main class of functions we will work with. We mainly refer to [**132**].

Definition 1.1.1. Let $\Phi : (0, +\infty) \to \mathbb{R}$ be a C^∞ function. We say that Φ is a **Bernstein function** if $\Phi(\lambda) \ge 0$ and for any $n \in \mathbb{N}$ and $\lambda > 0$ it holds

$$(-1)^{n-1}\Phi^{(n)}(\lambda) \ge 0.$$

The set of Bernstein functions will be denoted by $\mathcal{BF}$.

Let us also recall the definition of Lévy measure, as given in [**35**].

Definition 1.1.2. For any $d \ge 1$, a measure μ on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ (where with $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ we denote the Borel σ-algebra of $\mathbb{R}^d \setminus \{0\}$) is said to be a **Lévy measure** if and only if

$$\int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1)\mu(dx) < +\infty.$$

The set of Lévy measures will be denoted by $\mathcal{LM}_d$.
Moreover, let us denote by $\mathcal{BLM}$ the subset of $\mathcal{LM}_1$ such that $\mu \in \mathcal{BLM}$ if and only if for any $(a, b) \subseteq (-\infty, 0)$ it holds $\mu(a, b) = 0$ and

$$\int_0^{+\infty} (t \wedge 1)\mu(dt) < +\infty.$$

The link between Bernstein functions and Lévy measures is established by the following representation theorem (see [**132**, Theorem 3.2]).

Theorem 1.1.1 (**Lévy-Khintchine representation theorem**). *A function $\Phi : (0, +\infty) \to \mathbb{R}^+$ belongs to $\mathcal{BF}$ if and only if there exists a triplet $(a_\Phi, b_\Phi, \nu_\Phi) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathcal{BLM}$ such that*

$$\Phi(\lambda) = a_\Phi + b_\Phi\lambda + \int_0^{+\infty} (1 - e^{-\lambda t})\nu_\Phi(dt).$$

Moreover, any $\Phi \in \mathcal{BF}$ determines a unique triplet $(a_\Phi, b_\Phi, \nu_\Phi)$ and vice versa.

Given a Bernstein function $\Phi \in \mathcal{BF}$, the triplet $(a_\Phi, b_\Phi, \nu_\Phi)$ identified by the Lévy-Khintchine representation theorem is called the **characteristic triplet** of Φ. In particular a_Φ is said to be the **killing coefficient**, b_Φ the **drift coefficient** and ν_Φ the Lévy measure of Φ.
From this representation theorem, one achieves, by direct calculations, the following limits for $\Phi \in \mathcal{BF}$

$$\Phi(0+) = a_\Phi, \qquad \lim_{\lambda \to +\infty} \frac{\Phi(\lambda)}{\lambda} = b_\Phi.$$

Concerning the structure of the set $\mathcal{BF}$, let us recall [**132**, Corollary 3.7].

COROLLARY 1.1.2. *$\mathcal{BF}$ is a convex cone closed under pointwise limits and compositions.*

Actually, we will identify Bersntein functions with Laplace exponents of a particular class of Lévy processes, hence we need to understand the behaviour of these functions for complex variables. To do this, we recall [**132**, Proposition 3.5].

PROPOSITION 1.1.3. *Every Bernstein function $\Phi \in \mathcal{BF}$ admits a continuous extension $\Phi : \mathbb{H}_0 \to \mathbb{H}_0$ that is holomorphic on $\mathbb{H}$.*

Now we want to introduce some more regular classes of Bernstein functions. To do this, we first need to introduce the following class of functions (see [**132**, Chapter 1])

DEFINITION 1.1.3. We say a function $f : (0, +\infty) \to \mathbb{R}$ is **completely monotone** if $f \in C^\infty$ and for any $n \in \mathbb{N}_0$ and $\lambda > 0$ it holds $(-1)^n f^{(n)}(\lambda) \ge 0$. We denote by $\mathcal{CM}$ the set of completely monotone functions.

In particular, denoting with $\mathcal{L}^S$ the Laplace-Stieltjes transform operator, the following Theorem (see [**132**, Theorem 1.4]) holds true.

THEOREM 1.1.4 (**Bernstein Theorem**). *If $f \in \mathcal{CM}$ then there exists a unique measure μ on $[0, +\infty)$ such that for any $\lambda > 0$ it holds $f(\lambda) = \mathcal{L}^S[\mu](\lambda)$. Vice versa, if μ is a measure on $[0, +\infty)$ such that for any $\lambda > 0$ it holds $\mathcal{L}^S[\mu](\lambda) < +\infty$, then $\lambda \mapsto \mathcal{L}^S[\mu](\lambda)$ is a completely monotone function.*

Concerning the structure of the set $\mathcal{CM}$, we have the following Proposition (see [**132**, Corollary 1.6])

PROPOSITION 1.1.5. *$\mathcal{CM}$ is a convex cone closed under multiplication and pointwise convergence.*

We can now use completely monotone functions to define a *more regular* class of Bernstein functions (see [**132**, Chapters 6 and 7]).

DEFINITION 1.1.4. We say $\Phi \in \mathcal{BF}$ is a **complete Bernstein function** if its Lévy measure $\nu_\Phi(dt)$ is absolutely continuous with respect to Lebesgue measure and admits a density $\nu_\Phi(t)$ that belongs to $\mathcal{CM}$. The set of all complete Bernstein functions will be denoted as $\mathcal{CBF}$.

Again, we can investigate the structure of the set $\mathcal{CBF}$ (see [**132**, Corollary 7.6])

PROPOSITION 1.1.6. *$\mathcal{CBF}$ is a convex cone closed under pointwise limits and compositions.*

To investigate some other representation theorems for complete Bernstein functions, we need the following definition (see [**132**, Chapter 2]).

DEFINITION 1.1.5. We say a measure $\mathfrak{s}$ on $(0, +\infty)$ is a **Stieltjes measure** if $\int_0^{+\infty} (1+t)^{-1}\, \mathfrak{s}(dt) < +\infty$. We denote the set of Stieltjes measures as $\mathcal{SM}$.
We say a function $f : \mathbb{R}^+ \to \mathbb{R}_0^+$ is a **Stieltjes function** if there exists a triplet $(a_f, b_f, \mathfrak{s}_f) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathcal{SM}$ such that

$$f(\lambda) = \frac{a_f}{\lambda} + b_f + \int_0^{+\infty} \frac{1}{\lambda + t}\, \mathfrak{s}_f(dt).$$

We denote the set of Stieltjes functions as $\mathcal{S}$.

Concerning the structure of the set $\mathcal{S}$, we have the following Proposition (see [**132**, Theorem 2.2]).

PROPOSITION 1.1.7. *$\mathcal{S} \subset \mathcal{CM}$ is a convex cone closed under pointwise limit.*

Let us now stress out the link between complete Bernstein functions and Stieltjes functions (see [**132**, Theorems 6.2 and 7.3])

PROPOSITION 1.1.8. *$\Phi \in \mathcal{CBF}$ if and only if $\lambda \mapsto \frac{\Phi(\lambda)}{\lambda}$ belongs to $\mathcal{S}$. Moreover, if $\Phi \not\equiv 0$, $\Phi \in \mathcal{CBF}$ if and only if $1/\Phi \in \mathcal{S}$.*

From last proposition, we obtain another representation result for complete Bernstein functions. Indeed, for any $\Phi \in \mathcal{CBF}$ there exists a triplet $(a_\Phi, b_\Phi, \mathfrak{s}_\Phi) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathcal{SM}$ such that

$$\Phi(\lambda) = a_\Phi + b_\Phi \lambda + \int_0^{+\infty} \frac{\lambda}{\lambda + t}\, \mathfrak{s}_\Phi(dt).$$

Last class of Bernstein functions we will work with is the following (see [**132**, Chapter 10]).

DEFINITION 1.1.6. A function $\Phi \in \mathcal{BF}$ is said to be a **special Bernstein function** if $\Phi^\star(\lambda) = \frac{\lambda}{\Phi(\lambda)} \in \mathcal{BF}$.
The function $\Phi^\star$ is called the **conjugate Bernstein function** of Φ and the set of special Bernstein function will be denoted by $\mathcal{SBF}$.

Unlike all the other family of functions we introduced up to now, let us stress out that the structure of $\mathcal{SBF}$ is still unknown (see [**132**, Remark 10.20]). However, let us recall [**132**, Proposition 7.1]:

PROPOSITION 1.1.9. *$\Phi \in \mathcal{CBF}$ if and only if $\Phi^\star \in \mathcal{CBF}$.*

Last proposition gives us the following inclusion order for $\mathcal{CBF}$, $\mathcal{SBF}$ and $\mathcal{BF}$:

$$\mathcal{CBF} \subset \mathcal{SBF} \subset \mathcal{BF}\,.$$

In what follows, we will say that $\Phi \in \mathcal{BF}$ is a driftless Bernstein function if its characteristic triple is given by $(0, 0, \nu_\Phi)$ and $\nu_\Phi(0, +\infty) = +\infty$.[1]

1.2. Subordinators and inverse subordinators

Let us recall here the definition of subordinator and of killed subordinator as given in [**35**, Chapter 3].

DEFINITION 1.2.1. A **subordinator** $\sigma(t)$ is an almost surely increasing Lévy process on $\mathbb{R}$.

Let us observe that, being it increasing, it must be also non-negative. Indeed, for any $t \ge 0$, we have $\sigma(t) - \sigma(0) \ge 0$. But, since σ is a Lévy process, we are asking for $\sigma(0) = 0$ almost surely and then $\sigma(t) \ge 0$. The converse also holds true, i.e. if $\sigma(t)$ is a non-negative Lévy process, then it is a subordinator.
Concerning killed subordinators, we have the following definition.

[1] In literature, $\Phi \in \mathcal{BF}$ is a **driftless** Bernstein function if its drift coefficient $b_\Phi = 0$. However, together with the assumption $b_\Phi = 0$ we will always need also $a_\Phi = 0$ and $\nu_\Phi(0, +\infty) = +\infty$, thus we add these requirements in the definition.

DEFINITION 1.2.2. Let $\sigma(t)$ be a subordinator and τ_a be an exponential random variable of parameter $a > 0$ and independent of $\sigma(t)$. Then the **killed subordinator** associated to σ with rate $a > 0$ is defined as

$$\widehat{\sigma}^{(a)}(t) = \begin{cases} \sigma(t) & t \in [0, \tau_a) \\ +\infty & t \in [\tau_a, +\infty) \end{cases}$$

Let us stress out the link between subordinators and Bernstein functions, as a consequence of [**132**, Theorem 5.2 and Proposition 5.5]:

THEOREM 1.2.1. *For any Bernstein function $\Phi \in \mathcal{BF}$ with characteristic triplet $(a_\Phi, b_\Phi, \nu_\Phi)$ there exists a unique subordinator if $a_\Phi = 0$ or killed subordinator if $a_\Phi > 0$ $\sigma_\Phi(t)$ such that*

$$\mathbb{E}[e^{-\lambda \sigma_\Phi(t)}] = e^{-t\Phi(\lambda)}.$$

Viceversa, if σ is a (killed) subordinator, then

$$\Phi(\lambda) = -\log(\mathbb{E}[e^{-\lambda\sigma(1)}]) \in \mathcal{BF}.$$

For any subordinator we can define the inverse subordinator as done in [**36**] as a first passage time process.

DEFINITION 1.2.3. Let $\sigma(t)$ be a subordinator. Then its **inverse subordinator** $L(t)$ is defined as

$$L(t) = \inf\{y > 0 : \ \sigma(y) > t\}.$$

Let us first investigate some regularity property of the random variables $\sigma(t)$ and $L(t)$. Concerning the subordinator $\sigma(t)$, the existence of a density is linked to some regularity property of the Lévy measure. Indeed, by a direct application of [**84**, Theorem 27.10] one can show the following result.

PROPOSITION 1.2.2. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ that is absolutely continuous with respect to the Lebesgue measure. Then, for any $t > 0$, $\sigma_\Phi(t)$ is an absolutely continuous random variable.*

On the other hand, inverse subordinators are generally more regular even if the subordinator is not, as stated in [**103**, Theorem 3.1].

PROPOSITION 1.2.3. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ and let $\sigma_\Phi(t)$ be the associated subordinator with probability law $g_\Phi(ds; t)$. Let $L_\Phi(t)$ be the inverse subordinator. Then, for any $t > 0$, the random variable $L_\Phi(t)$ is absolutely continuous with density $f_\Phi(s; t)$ given by*

$$f_\Phi(s; t) = \int_0^t \nu_\Phi(t - \tau, +\infty) g_\Phi(d\tau; s).$$

From now on we will denote by $g_\Phi(dt; s)$ the law of the random variable $\sigma_\Phi(s)$ (eventually $g_\Phi(dt; s) = g_\Phi(t, s)dt$) and with $f_\Phi(s; t)$ the density of the random variable $L_\Phi(t)$.
Actually, the last proposition implies something more. Indeed, by definition, we have

$$\mathbb{P}(\sigma_\Phi(s) \geq t) = \mathbb{P}(L_\Phi(t) \leq s),$$

thus, since for any $t > 0$ we know that $L_\Phi(t)$ is absolutely continuous whenever $\Phi \in \mathcal{BF}$ is driftless, it holds that the function $s \mapsto \mathbb{P}(\sigma_\Phi(s) \geq t)$ is differentiable in

s and so it is $s \mapsto \mathbb{P}(\sigma_\Phi(s) \le t)$, leading to

$$f_\Phi(s;t) = -\frac{\partial}{\partial s}\mathbb{P}(\sigma_\Phi(s) \le t). \tag{1.2.1}$$

Finally, let us observe that from the last proposition, since $g_\Phi(d\tau, 0) = \delta_0(d\tau)$, it holds $f(0+;t) = \nu_\Phi(t,+\infty)$ for any $t > 0$.
Concerning some pathwise properties of subordinators and inverse subordinators, as first result, let us recall that, from [**84**, Theorem 21.3] it directly follows:

PROPOSITION 1.2.4. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. Then σ_Φ is almost surely strictly increasing.*

Moreover, let us recall [**35**, Chapter 3, Theorems 16 and 17] concerning the Hausdorff dimension of the range of a subordinator.

THEOREM 1.2.5. *Let $\Phi \in \mathcal{BF}$ and σ_Φ be the associated subordinator. Let us define the **upper index** and the **lower index** respectively as*

$$\iota_u := \sup\{\alpha > 0 : \lim_{\lambda \to +\infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\}$$
$$\iota_l := \inf\{\alpha \ge 0 : \lim_{\lambda \to +\infty} \lambda^{-\alpha}\Phi(\lambda) = 0\}$$

and let us denote by $\mathcal{H}\mathrm{dim}$ the Hausdorff dimension of a set in $\mathbb{R}$. Then:

- *For any $s > 0$ it holds $\mathcal{H}\mathrm{dim}(\{\sigma_\Phi(t) :\ t \in [0,s]\}) = \iota_u$ almost surely;*
- *For any $s > 0$ and any measurable set $E \subseteq [0,s]$ it holds*

$$\iota_u\mathcal{H}\mathrm{dim}(E) \le \mathcal{H}\mathrm{dim}(\{\sigma_\Phi(t) :\ t \in E\}) \le \iota_l\mathcal{H}\mathrm{dim}(E).$$

On the other hand, the sample paths of $L(t)$ are generally more regular, as stated in [**35**, Chapter 3, Lemma 17].

THEOREM 1.2.6. *Let $\Phi \in \mathcal{BF}$ with upper index $\iota_u > 0$. Then L_Φ is almost surely locally Hölder continuous of exponent $\iota_u - \varepsilon$ for any $\varepsilon \in (0, \iota_u)$.*

To obtain some other properties, we need to introduce a particular measure associated to σ_Φ, following the definitions given in [**35**].

DEFINITION 1.2.4. Let σ_Φ be a subordinator. Then we call **potential measure** of σ_Φ the Borel measure

$$U_\Phi(A) = \mathbb{E}\left[\int_0^{+\infty} \chi_{\{\sigma_\Phi(t) \in A\}}dt\right], \quad A \in \mathcal{B}(\mathbb{R})$$

where, for any set E, χ_E is the indicator function of E. In particular we define the **renewal function**

$$U_\Phi(x) = U_\Phi([0,x]).$$

Let us stress out that the function $U_\Phi(x)$ is subadditive.

It is easy to check, by definition, that $U_\Phi(t) = \mathbb{E}[L_\Phi(t)]$ and then, by Hölder-continuity property of $L_\Phi(t)$, we know that $t \mapsto U_\Phi(t)$ is a continuous increasing function. In particular this means that the potential measure is diffusive (i.e. does not admit any atomic part). This, together with the fact that any Lévy measure $\nu \in \mathcal{BLM}$ can admit at most a countable number of atoms, leads to the proof of [**35**, Chapter 3, Proposition 2, point *ii*] that we recall here.

PROPOSITION 1.2.7. *For any $t > 0$ it holds*

$$\mathbb{P}(\sigma_\Phi(L_\Phi(t)-) < t = \sigma_\Phi(L_\Phi(t))) = 0.$$

As a direct consequence we get that σ_Φ cannot admit any fixed discontinuity. On the other hand, driftless subordinators are pure jump processes, as stated in [**35**, Chapter 3, Theorem 4]:

PROPOSITION 1.2.8. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. Then*

$$\mathbb{P}(\sigma_\Phi(L_\Phi(t)) > t) = 1.$$

Concerning the asymptotic behaviour of the renewal function, let us introduce the **tail** of the Lévy measure ν_Φ of a Bernstein function $\Phi \in \mathcal{BF}$ as $\bar{\nu}_\Phi(t) = \nu_\Phi(t, +\infty)$ and the **integrated tail** of the Lévy measure ν_Φ

$$I_\Phi(t) = \int_0^t \bar{\nu}_\Phi(\tau)d\tau.$$

Then we can state (see [**35**, Chapter 3, Proposition 1]):

PROPOSITION 1.2.9. *Let $\Phi \in \mathcal{BF}$ with drift coefficient b and Lévy measure ν_Φ. Then*

$$U_\Phi(x) \asymp \frac{1}{\Phi\left(\frac{1}{x}\right)}, \qquad\qquad \frac{\Phi(x)}{x} \asymp I_\Phi\left(\frac{1}{x}\right) + b_\Phi.$$

Now let us investigate some properties concerning the Laplace transform of the density of the inverse subordinators and their moments of any order. First of all, let us observe that, as a consequence of Proposition 1.2.3 we have (as also stated in [**103**, Equation 3.13]) for $\Phi \in \mathcal{BF}$ a driftless Bernstein function, denoting by $\mathcal{L}$ the Laplace transform operator, for any $s, \lambda > 0$,

$$\mathcal{L}_{t \to \lambda}[f_\Phi(s;t)](\lambda) = \frac{\Phi(\lambda)}{\lambda} e^{-s\Phi(\lambda)}.$$

From this formula, we can easily obtain the Laplace-Stieltjes transform of the renewal function $U_\Phi(t)$ (or, directly, of the potential measure) as stated in [**35**]

$$\mathcal{L}^S[U_\Phi](\lambda) = \frac{1}{\Phi(\lambda)}.$$

Actually, we obtain the Laplace-Stieltjes transform of the moment of any order of $L_\Phi(t)$. Indeed, in [**145**, Equation 12], the authors prove the following statement

$$\mathcal{L}^S_{t \to \lambda}[\mathbb{E}[L^\gamma_\Phi(t)]](\lambda) = \frac{\Gamma(1+\gamma)}{\Phi^\gamma(\lambda)}. \tag{1.2.2}$$

Let us give some names to particular class of subordinators (as given in [**132**])

DEFINITION 1.2.5. If $\Phi \in \mathcal{SBF}$, then the associated subordinator σ_Φ is said to be a **special subordinator**. If $\Phi \in \mathcal{CBF}$, then the associated subordinator σ_Φ is said to be a **complete subordinator**.

Concerning special subordinators, one can characterize them by means of their potential measure. Indeed we have the following Theorem (see [**132**, Theorem 10.3 and Equation 10.9])

THEOREM 1.2.10. *Let σ_Φ be a subordinator with potential measure U_Φ and $\Phi \in \mathcal{BF}$ with characteristic triplet $(a_\Phi, b_\Phi, \nu_\Phi)$. Then σ_Φ is a special subordinator if and only if there exists a constant $c_\Phi \ge 0$ and some non-increasing function $u_\Phi : (0, +\infty) \to (0, +\infty)$ with $\int_0^1 u_\Phi(t)dt < +\infty$ such that*

$$U_\Phi(dt) = c_\Phi \delta_0(dt) + u_\Phi(t)dt$$

where δ_0 is a Dirac delta. In such case it holds:

$$c_\Phi = \begin{cases} 0 & b_\Phi > 0 \\ \frac{1}{a_\Phi + \nu_\Phi(0,+\infty)} & b_\Phi = 0. \end{cases}$$

Thus, according to our definition of driftless Bernstein function, we have that if σ_Φ is a driftless special subordinator, the potential measure U_Φ is absolutely continuous with respect to the Lebesgue measure with density u_Φ called **potential density**.

Finally, let us observe that if $\Phi \in \mathcal{CBF}$ is a driftless Bernstein function, then the one-dimensional law of the complete subordinator σ_Φ is absolutely continuous with respect to Lebesgue measure.

1.2.1. α-stable subordinators. A particularly regular case is given by the α-stable subordinator for $\alpha \in (0,1)$. It is the subordinator associated to the complete Bernstein function $\Phi(\lambda) = \lambda^\alpha$. In particular let us recall the following definition (see [**12**, Section 1.2.5]).

DEFINITION 1.2.6. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables, $(b_n)_{n\in\mathbb{N}}$ a sequence of real numbers and $(\sigma_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers. Let us define

$$S_n = \frac{1}{\sigma_n}\left(\sum_{k=1}^{n} Y_k - b_n\right)$$

and suppose that there exists a random variable X such that, for any $x \in \mathbb{R}$,

$$\lim_{n\to+\infty} \mathbb{P}(S_n \le x) = \mathbb{P}(X \le x).$$

In such case X is said to be a **stable random variable**.

From this definition we can see that normal random variables are stable (by Lindeberg–Lévy central limit theorem). However, these are not the only stable random variable. One can achieve a characterization of stable random variables via the following result.

PROPOSITION 1.2.11. *A random variable X is stable if and only if for any $n \in \mathbb{N}$ there exist two constants c_n and d_n such that, denoting by $X_1, \dots, X_n$ n independent copies of X,*

$$\sum_{k=1}^{n} X_k \overset{d}{=} c_n X + d_n.$$

In particular $c_n = \sigma n^{\frac{1}{\alpha}}$ for some constant $\sigma > 0$ and $\alpha \in (0,2]$.

A stable r.v. is said to be **strictly stable** if $d_n = 0$ for any $n \in \mathbb{N}$. Moreover, the exponent $\alpha \in (0,2]$ is said to be the **index of stability**. If $\alpha = 2$, one can show that X must be a normal random variable.

If we want to work with subordinators, we need to exclude the case of stable random variables whose support is not contained in $(0,+\infty)$. Actually, one can show, by a characterization of the characteristic function, that if X is a non-negative stable random variable, then its index of stability is $\alpha \in (0,1)$.

Now we can define the α-stable subordinator.

DEFINITION 1.2.7. A **stable Lévy process** is a Lévy process $X(t)$ such that for any $t > 0$ $X(t)$ is a stable random variable. An **α-stable subordinator** is a stable non-decreasing Lévy process with index of stability $\alpha \in (0,1)$.

Let us denote by $\sigma_\alpha(t)$ an α-stable subordinator. Then, since $\Phi(\lambda)=\lambda^\alpha$, one can easily show that σ_α is self-similar of exponent $\frac{1}{\alpha}$. The Lévy measure of Φ is given by

$$\nu_\alpha(dt)=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}dt.$$

Being in particular $\Phi(\lambda)$ a complete driftless Bernstein function, then $\sigma_\alpha(t)$ is absolutely continuous for any $t>0$. Let us denote by $g_\alpha(s)$ the probability density function of $\sigma_\alpha(1)$. Let us also denote by $L_\alpha(t)$ the inverse α-stable subordinator and $f_\alpha(s;t)$ its one-dimensional probability density function. Then, by using the self similarity property of σ_α, together with Equation (1.2.1), one obtains (as stated in [**105**]),

$$f_\alpha(s;t)=\frac{t}{\alpha}s^{-1-\frac{1}{\alpha}}g_\alpha(ts^{-\frac{1}{\alpha}}). \tag{1.2.3}$$

Finally let us recall that as $s\to+\infty$ we have $g_\alpha(s)\sim\frac{\alpha}{\Gamma(1-\alpha)}s^{-\alpha-1}$.

1.2.2. Tempered α-stable subordinators. It is not difficult to check that $\mathbb{E}[\sigma_\alpha(t)]=+\infty$ for any $t>0$. For this reason, we could look for a *more regular subordinator*, in the sense that we could ask for some subordinator that preserves some properties of the α-stable distribution but at the same time admits finite moments. To do this, we use a tempering procedure.
Indeed, to obtain a *more integrable* subordinator, we could add an exponential term to its Lévy measure. Let us consider a constant $\theta>0$ and let us define the **tempered stable Lévy measure** (see [**104**, Section 7.3])

$$\nu_{\theta,\alpha}(dt)=\frac{\alpha e^{-\theta t}t^{-\alpha-1}}{\Gamma(1-\alpha)}dt.$$

The Bernstein function $\Phi\in\mathcal{CBF}$ with characteristic triplet $(0,0,\nu_{\theta,\alpha})$ is given by

$$\Phi(\lambda)=(\lambda+\theta)^\alpha-\theta^\alpha$$

and the subordinator $\sigma_{\theta,\alpha}(t)$ associated to it is called the **tempered stable subordinator** with stability index α and **tempering parameter** θ. Let us recall that such subordinator is also called **massive relativistic stable subordinator** for reasons that will be explained later.
Concerning the density of the tempered stable subordinator $\sigma_{\theta,\alpha}(t)$, let us stress out the following relation between its density and the density of the stable subordinator, as shown in [**92**, Equation 2.2]:

$$g_{\theta,\alpha}(s;t)=e^{-\lambda s+\lambda^\alpha t}g_\alpha(s;t).$$

Concerning the density of the inverse tempered stable subordinator, [**92**, Theorem 2.1] provides an integral representation from which, setting $\theta=0$, one obtains also an integral representation of the inverse stable density.
The last property we want to stress out here is that, since $g_\alpha(s;t)$ behaves like a power function as $s\to+\infty$, now $g_{\theta,\alpha}(s;t)$ admits an exponential tempering and then we have $\mathbb{E}[\sigma^n_{\theta,\alpha}(t)]<+\infty$ for any $n\in\mathbb{N}$.

1.3. Regular variation

Together with Bernstein functions, another class of functions we will often consider to is the class of regularly varying functions. For the definition we refer to [**79**].

DEFINITION 1.3.1. Let $f : (x_0, +\infty) \to \mathbb{R}^+$ be a measurable function. It is said to be **regularly varying at** ∞ with index $\alpha \in \mathbb{R}$ if for any $t > 0$ it holds

$$\lim_{x \to +\infty} \frac{f(tx)}{f(x)} = t^\alpha.$$

If $\alpha = 0$ then f is said to be **slowly varying** at ∞.
Moreover, f is **regularly varying at** x_0 with index $\alpha \in \mathbb{R}$ if the function $x \mapsto f(x_0 + x^{-1})$ is regularly varying at ∞.

Let us state the following results for functions that are regularly varying at infinity, recalling that such results can be easily extended to the case in which the functions are regularly varying at a point x_0.
From the definition it is obvious that if f is regularly varying of order $\alpha \in \mathbb{R}$ then $f(x)/x^\alpha$ is slowly varying. In particular we have:

PROPOSITION 1.3.1. *Let f be a regularly varying function. Then there exists a slowly varying function ℓ such that $f(x) = x^\alpha \ell(x)$.*

One can show the following representation formula for regularly varying functions f:

$$f(x) = x^\alpha c(x) \exp\left(\int_{x_0}^{x} \frac{\epsilon(y)}{y} dy\right)$$

where c and ϵ are measurable functions such that $\lim_{x \to +\infty} c(x) = c_0 \in (0, +\infty)$ and $\lim_{x \to +\infty} \epsilon(x) = 0$. For a more precise statement of the representation theorem, we refer to [**108**], while for proofs of such formula we refer to the encyclopedic work [**39**].
From the representation formula for $\alpha = 0$, one can show that, for any $p > 0$ and any slowly varying function ℓ, it holds

$$\lim_{x \to +\infty} x^{-p} \ell(x) = 0 \qquad \lim_{x \to +\infty} x^p \ell(x) = +\infty.$$

Moreover, it can be shown that the convergence in the definition of regularly varying function is uniform (with respect to t) in any compact set (see [**39**]).
Let us now state some theorems we will use in the following. First of all, let us recall **Karamata's Theorem**: we will use the *compact* statement of [**108**] referring directly to regularly varying functions, while a complete extensive proof is given in [**39**, Chapter 1, Sections 1.5 − 1.6].

THEOREM 1.3.2 (**Karamata's Theorem**). *Let $f : [x_0, +\infty) \to \mathbb{R}$ be a locally bounded regularly varying function with index $\alpha \in \mathbb{R}$. Then*

- *For any $\sigma \geq -(\alpha + 1)$*

$$\lim_{x \to +\infty} \frac{x^{\sigma+1} f(x)}{\int_{x_0}^{x} t^\sigma f(t) dt} = \sigma + \alpha + 1. \tag{1.3.1}$$

- *For any* $\sigma < -(\alpha+1)$

$$\lim_{x\to+\infty} \frac{x^{\sigma+1} f(x)}{\int_x^{+\infty} t^{\sigma} f(t)dt} = -(\sigma+\alpha+1). \tag{1.3.2}$$

Last property holds also for $\sigma = -(\alpha+1)$ *if* $\int_x^{+\infty} t^{-(\alpha+1)} f(t)dt < +\infty$. *Vice-versa, let* $f : [x_0, +\infty) \to \mathbb{R}$ *be a positive, measurable and locally integrable function. If, for some* $\sigma > -(\alpha+1)$*, property* (1.3.1) *holds, then* f *varies regularly with index* α*. If, for some* $\sigma < -(\alpha+1)$*, property* (1.3.2) *holds, then* f *varies regularly with index* α*.*

Let us also recall the following Tauberian theorem that links the asymptotic behaviour of a monotone function with the one of its Laplace-Stieltjes transform. As for the previous theorem, we refer to the compact statement given in [**108**], while a full proof of this theorem is given in [**39**].

Theorem 1.3.3 (**Karamata's Tauberian theorem**). *Let* U *be a non-decreasing right-continuous function defined on* $[0, +\infty)$*,* ℓ *a slowly varying function,* $c > 0$ *and* $\alpha \geq 0$*. Let us denote by* $\widetilde{u}(\lambda)$ *the Laplace-Stieltjes transform of* U*. Then the following statements are equivalent*

- $U(x) \sim cx^{\alpha} \frac{\ell(x)}{\Gamma(1+\alpha)}$ *as* $x \to +\infty$*;*
- $\widetilde{u}(\lambda) \sim c\lambda^{-\alpha}\ell(1/\lambda)$ *as* $\lambda \to 0^+$*.*

The same holds if we consider $x \to 0^+$ *and* $\lambda \to +\infty$*.*

Moreover, in the case of regularly varying asymptotic behaviour, one can link the asymptotic behaviour of an absolutely continuous function with the behaviour of its density.

Theorem 1.3.4 (**Monotone density theorem**). *Let* $U(x) = \int_0^x u(y)dy$ *or* $U(x) = \int_x^{+\infty} u(y)dy$ *where* u *is ultimately monotone, i.e. there exists* $z > 0$ *such that* u *is monotone in* $(z, +\infty)$*. Let* $c > 0$*,* $\alpha \in \mathbb{R}$ *and* ℓ *a slowly varying function. If* $U(x) \sim cx^{\alpha}\ell(x)$ *for* $x \to +\infty$*, then* $u(x) \sim c\alpha x^{\alpha-1}\ell(x)$*.*

Let us observe that Karamata's Theorem provide a sort of *converse statement* of the latter theorem.
Finally, let us recall the following global bounds, for which we refer to [**39**, Theorem 1.5.6].

Theorem 1.3.5 (**Potter's Theorem**). *Let* $\ell : [0, +\infty) \to \mathbb{R}^+$ *be a slowly varying function,* $A > 1$ *and* $\delta > 0$*. Then there exists* $x_0(A, \delta) > 0$ *such that for any* $x, y \geq x_0(A, \delta)$ *it holds*

$$\frac{\ell(y)}{\ell(x)} \leq A \max\left\{ \left(\frac{x}{y}\right)^{\delta}, \left(\frac{x}{y}\right)^{-\delta} \right\}. \tag{1.3.3}$$

Moreover, if ℓ *is bounded away from* 0 *and* ∞ *on any compact subset of* $[0, +\infty)$*, then for any* $\delta > 0$ *there exists* $A(\delta) > 1$ *such that relation* (1.3.3) *holds for any* $x, y > 0$*.*

Let us finally recall that a definition of *regular variation of infinite index* is still available. These kind of functions are called **rapidly varying functions** (see [**39**, Section 2.4]) and they can be characterized in some sense as the left-continuous inverse of slowly varying functions (see [**39**, Theorem 2.4.7]). As for this class of functions Tauberian theorems are not known, we need to introduce a different class of functions to study some rapid behaviour at 0^+.

1.3.1. Rapidly decreasing functions at 0^+. In order to work with some rapid behaviour, in [**28**] we had to introduce a class of functions that satisfied in some sense our requirement and for which a Tauberian theorem could be shown.

Definition 1.3.2. A function $f : [0,+\infty) \to [0,+\infty)$ is said to be **rapidly decreasing at 0^+** if for any $\alpha > 0$ it holds

$$\lim_{t\to 0^+} \frac{f(t)}{t^\alpha} = 0.$$

One can easily prove the following characterization of smooth rapidly decreasing functions (see [**28**, Lemma 2.4.1])

Proposition 1.3.6. *Let $f : [0,+\infty) \to [0,+\infty)$ be a function such that there exists $\delta > 0$ for which $f \in C^\infty(0,\delta)$. Then the following assertions are equivalent:*

- *f is rapidly decreasing at 0^+;*
- *$f \in C^\infty([0,\delta))$ and $f^{(n)}(0) = 0$ for any $n \geq 0$;*
- *For any $n \geq 0$, $f^{(n)}$ is rapidly decreasing at 0^+.*

In order to show a Tauberian theorem for rapidly decreasing functions, we need to recall the Initial-Value Theorem for the Laplace transform (see [**46**, Section 17.8]).

Theorem 1.3.7 (**Initial Value Theorem**). *Let $f : [0,+\infty) \to [0,+\infty)$ be Laplace transformable and let $\widetilde{f}$ be its Laplace transform. Then it holds*

$$\lim_{t\to 0^+} f(t) = \lim_{\lambda\to+\infty} \lambda \widetilde{f}(\lambda)$$

supposed that at least one of the two involved limits exists.

This Theorem, together with the formula for the Laplace transform of the derivative of a function, are the main tool to show the following result (which is [**28**, Lemma 2.4.2])

Theorem 1.3.8 (**Tauberian Theorem for rapidly decreasing functions**). *Let $f \in C^\infty(0,\delta)$ be Laplace transformable together with all its derivatives. Denote with $\widetilde{f}$ the Laplace transform of f. Then f is rapidly decreasing at 0^+ if and only if for any $\alpha > 0$ it holds $\lim_{\lambda\to\infty} \lambda^\alpha \widetilde{f}(\lambda) = 0$.*

Proof. Let us first show that if f is rapidly decreasing then for any $\alpha > 0$ it holds $\lim_{\lambda\to\infty} \lambda^\alpha \widetilde{f}(\lambda) = 0$. It is easy to observe that one has only to prove such property for $\alpha \in \mathbb{N}$. First of all, let us consider $\alpha = 1$. Then we have, by the Initial-Value Theorem and the characterization given in Proposition 1.3.6

$$\lim_{\lambda\to+\infty} \lambda \widetilde{f}(\lambda) = f(0+) = 0.$$

Now let us consider $\alpha = n > 1$. Then we can write

$$\lim_{\lambda\to+\infty} \lambda^n \widetilde{f}(\lambda) = \lim_{\lambda\to+\infty} \lambda\lambda^{n-1} \widetilde{f}(\lambda) = \lim_{t\to 0^+} f^{(n-1)}(t) = 0$$

where we used again the characterization of smooth rapidly decreasing functions and the Initial-Value Theorem.

To show the converse let us work by induction. First of all, let us observe that by hypothesis

$$\lim_{t\to 0^+} f(t) = \lim_{\lambda\to+\infty} \lambda\widetilde{f}(\lambda) = 0.$$

Now let us suppose we have shown that $f^{(n)}(0+)$ exists and it is equal to 0. Then we have that the Laplace transform of $f^{(n+1)}$ is given by $\lambda^{n+1}\widetilde{f}$. Thus, by using the Initial-Value Theorem, we have

$$\lim_{t\to 0^+} f^{(n+1)}(t) = \lim_{\lambda\to+\infty} \lambda^{n+2}\widetilde{f}(\lambda) = 0$$

concluding the proof by the characterization given in Proposition 1.3.6. □

1.3.2. Regular variation and Lévy processes. Now let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non-negative random variable X on it. We denote by $F_X(x) = \mathbb{P}(X \le x)$ its distribution function and $\overline{F}_X(x) = 1 - F(x)$ its tail or survival function. For the following definition we refer to [**108**].

DEFINITION 1.3.3. The random variable X is said to be **regularly varying of index** $\alpha \ge 0$ if and only if $\overline{F}_X$ is a regularly varying of index $-\alpha$.

Concerning regularly varying random variables, all the Theorems we have stated before apply on their tails. All these restatement are given in [**108**, Proposition 1.3.2]. Here, let us only recall one of the properties we are going to use.

PROPOSITION 1.3.9. *If X is a regularly varying non-negative random variable with index $\alpha > 0$ then:*

- *For any $\beta < \alpha$ it holds $\mathbb{E}[X^\beta] < +\infty$;*
- *For any $\beta > \alpha$ it holds $\mathbb{E}[X^\beta] = +\infty$.*

Now let us consider a subordinator σ_Φ and let $X = \sigma_\Phi(1)$. Actually, the asymptotic behaviour of the tail of any variable $\sigma_\Phi(t)$ for $t \ge 0$ can be reconstructed starting from $X = \sigma_\Phi(1)$ by means of Lévy-Khintchine representation.
Concerning the regular variation of X, let us recall the following Theorem, that is [**39**, Theorem 8.2.1].

THEOREM 1.3.10. *Let $X = \sigma_\Phi(1)$ and $\overline{F}_X(x)$ be its tail. Let ν_Φ be the Lévy measure of $\sigma_\Phi(t)$ and $\overline{\nu}_\Phi(x) = \nu_\Phi(x, +\infty)$ be the tail of the Lévy measure. Then X is regularly varying of index $\alpha \ge 0$ if and only if $\overline{\nu}_\Phi$ is regularly varying of index $-\alpha$. In particular, in such case we have $\overline{F}_X(x) \sim \overline{\nu}_\Phi(x)$.*

Let us recall the case of the α-stable subordinator $\sigma_\alpha(t)$. For such subordinator we have $\overline{\nu}_\alpha(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$ that is actually regularly varying of index $-\alpha$. Thus we have that $\sigma_\alpha(1)$ is regularly varying and then, by Proposition 1.3.9 we know that $\sigma_\alpha(t)$ admits moments up to order α.
Concerning the renewal function U_Φ of a subordinator σ_Φ we have the following direct consequence of both the Tauberian theorem and the monotone density theorem (see [**35**])

PROPOSITION 1.3.11. *Let Φ be regularly varying at ∞ (resp. at 0^+) with index $\alpha \in [0,1]$ and characteristic triplet $(a_\Phi, b_\Phi, \nu_\Phi)$. Then the renewal function U_Φ of the subordinator σ_Φ satisfies the following asymptotic relation as $x \to 0^+$ (resp. at ∞):*

$$U_\Phi(x) \sim \frac{1}{\Gamma(1+\alpha)\Phi\left(\frac{1}{x}\right)}.$$

Moreover, if $\alpha \in [0,1)$, then, as $t \to 0^+$ (resp. as $t \to +\infty$) it holds

$$\bar{\nu}_\Phi(t) \sim \frac{\Phi\left(\frac{1}{t}\right)}{\Gamma(1-\alpha)}.$$

Together with the asymptotic behaviour, one can also get Hölder regularity of the renewal function.

PROPOSITION 1.3.12. *Let $\Phi \in \mathcal{BF}$ be regularly varying at infinity of index $\gamma \in (0,1)$ and let U_Φ be the renewal function of the associated subordinator σ_Φ. Then, for any $\varepsilon \in (0,\gamma)$ it holds $U_\Phi \in C^{\gamma-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^+_0)$. Moreover, if the killing coefficient $a > 0$, then $U_\Phi \in C^{\gamma-\varepsilon}(\mathbb{R}^+_0)$.*

PROOF. Let us consider, without loss of generality, $t > s > 0$. Since U_Φ is increasing and subadditive, we have

$$\frac{|U_\Phi(t) - U_\Phi(s)|}{|t-s|^{\gamma-\varepsilon}} \le \frac{U_\Phi(t-s)}{|t-s|^{\gamma-\varepsilon}}.$$

By using Proposition 1.2.9 we know that there exists a constant C such that

$$\frac{U_\Phi(t-s)}{|t-s|^{\gamma-\varepsilon}} \le \frac{C}{\Phi\left(\frac{1}{t-s}\right)|t-s|^{\gamma-\varepsilon}}.$$

Now let us define

$$\ell(t) = \Phi(t)t^{-\gamma}$$

and observe that $\ell(t)$ is slowly varying at infinity. In particular, this implies

$$\lim_{t \to 0} t^{-\varepsilon}\ell(t^{-1}) = +\infty$$

and then we know there exists $\delta > 0$ such that if $t \in (0,\delta)$ then $t^{-\varepsilon}\ell(t^{-1}) > 1$.
Now fix an interval $[a,b] \subset \mathbb{R}^+_0$ and consider $a \le s < t \le b$. Moreover, define $K = b - a$. Without loss of generality, we can suppose $K \ge \delta$.
Now, let us observe that $t - s \in (0,K]$. If $t - s \in (0,\delta)$, by definition of δ we have

$$\frac{U_\Phi(t-s)}{|t-s|^{\gamma-\varepsilon}} \le \frac{C}{\Phi\left(\frac{1}{t-s}\right)|t-s|^{\gamma-\varepsilon}} < C.$$

Now let us suppose $t - s \in [\delta, K]$ and set $m = \min_{\lambda \in \left[\frac{1}{K},\frac{1}{\delta}\right]} \Phi(\lambda)$. Then we have

$$\frac{U_\Phi(t-s)}{|t-s|^{\gamma-\varepsilon}} \le \frac{C}{\Phi\left(\frac{1}{t-s}\right)|t-s|^{\gamma-\varepsilon}} \le \frac{C}{m\delta^{\gamma-\varepsilon}}.$$

Thus we have that U_Φ is Hölder continuous of exponent $\gamma - \varepsilon$ in $[a,b]$, with Hölder constant given by $C\max\left\{1, \frac{1}{m\delta^{\gamma-\varepsilon}}\right\}$. Let us stress out that the Hölder constant does not depend on the compact set itself, but on its diameter K, since, being $\Phi(\lambda)$ increasing, we have $m = \Phi\left(\frac{1}{K}\right)$.
On the other hand, if the killing coefficient $a > 0$, we know there exists a constant $K > 0$ such that for any $\lambda \in \left(0, \frac{1}{K}\right)$ it holds $\Phi(\lambda) > \frac{a}{2} > 0$. Let us consider $t > s > 0$. If $t - s \le K$, then t, s are contained in the same compact set of diameter K, thus there exists a constant $\widetilde{C} > 0$ such that $U_\Phi(t-s) \le \widetilde{C}|t-s|^{\gamma-\varepsilon}$. If $t-s > K$, then $\Phi(1/(t-s)) > a/2$ and we have

$$U_\Phi(t-s) \le \frac{2C}{aK^{\gamma-\varepsilon}}|t-s|^{\gamma-\varepsilon}$$

concluding the proof. $\square$

REMARK 1.3.13. Let us observe that the Hölder continuity property of U_Φ in this case is coherent with the Hölder continuity property of L_Φ. Indeed, if Φ is regularly varying at ∞ with index $\gamma \in (0,1)$, then the upper index $\iota_u = \gamma$ and also L_Φ is almost surely Hölder continuous of exponent $\gamma - \varepsilon$.

Hölder continuity property implies also a power control on the growth of U_Φ.

COROLLARY 1.3.14. *Let $\Phi \in \mathcal{BF}$ be regularly varying at infinity of index $\gamma \in (0,1)$ and let U_Φ be the renewal function of the associated subordinator σ_Φ. Then, for any $\varepsilon \in (0,\gamma)$ and any $T > 0$ there exists a constant $C(\varepsilon, T)$ such that for any $t \in [0,T]$ it holds*

$$U_\Phi(t) \le C(\varepsilon, T) t^{\gamma-\varepsilon}.$$

PROOF. Let us consider $t \in [0,T]$. Then, by local Hölder continuity and the fact that $U(0) = 0$, we obtain

$$U_\Phi(t) = U_\Phi(t) - U_\Phi(0) \le C(\varepsilon, T) t^{\gamma-\varepsilon}.$$

$\square$

Finally, let us consider the special case. Indeed, if $\Phi \in \mathcal{SBF}$ is a driftless Bernstein function, we achieve also the asymptotic behaviour of the potential density $u_\Phi(t)$.

PROPOSITION 1.3.15. *Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at ∞ with index $\alpha \in (0,1)$. Let σ_Φ be the associated subordinator and U_Φ the renewal function, with potential density u_Φ. Then, as $t \to 0^+$ it holds*

$$u_\Phi(t) \sim \frac{1}{t\Gamma(\alpha)\Phi(1/t)}.$$

PROOF. By Proposition 1.3.11 we have

$$U_\Phi(t) \sim \frac{1}{\alpha\Gamma(\alpha)\Phi(1/t)}.$$

Now let us define $\ell(\lambda) = \frac{\Phi(\lambda)}{\lambda^\alpha}$ and $\widetilde{\ell}(t) = \frac{1}{\ell(1/t)}$. Then $\widetilde{\ell}(t)$ is slowly varying at 0^+ and

$$U_\Phi(t) \sim \frac{t^\alpha}{\alpha\Gamma(\alpha)}\widetilde{\ell}(t).$$

Hence, since $u_\Phi(t)$ is non-increasing, by the monotone density theorem, we have

$$u_\Phi(t) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}\widetilde{\ell}(t),$$

concluding the proof. $\square$

1.4. Convolutionary derivatives

In this section we will introduce the main non-local operators we will work with. These operators generalize the so-called *fractional derivatives* of Caputo-Dzhrbashyan type. They were first introduced in [**88**] for complete Bernstein functions, but we will refer to the weaker but more general approach given in [**143**]. In particular we first refer to [**143**, Definition 2.1], but asking for $b = 0$ for simplicity.

DEFINITION 1.4.1. Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ. Let us denote by $\bar{\nu}_\Phi(t) = \nu_\Phi(t, +\infty)$ its tail. Then the **Riemann-Liouville type convolutionary derivative induced by** Φ on an absolutely continuous function $u : (0, +\infty) \to \mathbb{R}$ is defined as

$$\mathcal{D}^\Phi\, u(t) = \frac{d}{dt}\int_0^t u(\tau)\bar{\nu}_\Phi(t-\tau)d\tau.$$

The **Caputo-Dzhrbashyan type convolutionary derivative induced by** Φ on an absolutely continuous function $u : (0, +\infty) \to \mathbb{R}$ is instead defined as

$$\partial^\Phi u(t) = \int_0^t u'(\tau)\bar{\nu}_\Phi(t-\tau)d\tau.$$

We will refer to them directly as non-local derivatives.

Let us give two important examples:

- If $\Phi(\lambda) = \lambda^\alpha$, then we have that $\bar{\nu}_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$. Thus the Riemann-Liouville type convolutionary derivative induced by Φ is given by

$$\mathcal{D}^\alpha\, u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t u(\tau)(t-\tau)^{-\alpha}d\tau,$$

  while the Caputo-Dzhrbashyan one is given by

$$\partial^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t u'(\tau)(t-\tau)^{-\alpha}d\tau.$$

  In this case we obtain the classical Riemann-Liouville and Caputo-Dzhrbashyan derivatives (see, for instance, [**104**, Chapter 2]).
- If we consider $\Phi(\lambda) = (\lambda+\theta)^\alpha - \theta^\alpha$, then we have $\bar{\nu}_{\theta,\alpha} = \frac{\alpha}{\Gamma(1-\alpha)}\Gamma(-\alpha; t)$, where $\Gamma(\alpha; t)$ is the upper incomplete Gamma function defined as

$$\Gamma(\alpha; t) = \int_t^{+\infty} s^{\alpha-1}e^{-s}ds.$$

  Hence we obtain

$$\mathcal{D}^{\theta,\alpha}\, u(t) = \frac{\alpha\theta^\alpha}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t u(\tau)\Gamma(-\alpha; t-\tau)d\tau$$

  and

$$\partial^{\theta,\alpha} u(t) = \frac{\alpha\theta^\alpha}{\Gamma(1-\alpha)}\int_0^t u'(\tau)\Gamma(-\alpha; t-\tau)d\tau,$$

  that are the tempered Riemann-Liouville and Caputo-Dzhrbashyan derivatives. The Marchaud-type tempered fractional derivative is, for instance, discussed in [**104**, Section 7.4].

These two examples are classical in fractional calculus and then help us understand why we refer to the calculus with respect to these operators as generalized fractional calculus (as done in [**88**]).
First of all, one could ask when such operators can be inverted. As shown in [**106**], this can be done in the special case.

DEFINITION 1.4.2. Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function with associated subordinator σ_Φ and potential density u_Φ. Then we define the **Riemann-Lioville type integral induced by** Φ on any sufficiently regular measurable function $u : \mathbb{R}^+ \to \mathbb{R}$ as

$$\mathcal{I}^\Phi\, u(t) = \int_0^t u(\tau) u_\Phi(t-\tau) d\tau.$$

In particular it holds, for absolutely continuous functions,

$$\mathcal{I}^\Phi\, \mathcal{D}^\Phi\, u(t) = u(t), \qquad \partial^\Phi\, \mathcal{I}^\Phi\, u(t) = u(t).$$

Let us also stress out what are the Laplace transform of these convolutionary derivatives. Indeed, it is easy to see that if u is an absolutely continuous function, it holds (whenever the involved Laplace transform exist)

$$\begin{aligned}\mathcal{L}[\mathcal{D}^\Phi\, u](\lambda) &= \Phi(\lambda)\, \mathcal{L}[u](\lambda),\\ \mathcal{L}[\partial^\Phi u](\lambda) &= \Phi(\lambda)\, \mathcal{L}[u](\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0+),\\ \mathcal{L}[\mathcal{I}^\Phi\, u](\lambda) &= \frac{1}{\Phi(\lambda)}\, \mathcal{L}[u](\lambda).\end{aligned}$$

Finally, let us stress out that the domain of $\mathcal{D}^\Phi$ actually contains absolutely continuous functions (as it can be applied to less regular functions). Moreover, on absolutely continuous functions, the following relation holds

$$\partial^\Phi u = \mathcal{D}^\Phi (u - u(0+)). \tag{1.4.1}$$

Thus one can define the **regularized Caputo-Dzhrbashyan type convolutionary derivative induced by** Φ on a function u belonging to the domain of $\mathcal{D}^\Phi$ such that $u(0+) < +\infty$ by means of Equation (1.4.1). Now let us focus on two main problems on such non-local derivatives:

- The link between non-local derivatives and inverse subordinators;
- The eigenvalue problem for non-local derivatives.

1.4.1. Cauchy problems for the density of inverse subordinators. Let us consider $\Phi \in \mathcal{BF}$ a driftless Bernstein function, σ_Φ the associated subordinator and L_Φ its inverse.
Let us first focus on the case in which σ_Φ is absolutely continuous. It has been shown in [**143**, Theorem 4.1]:

THEOREM 1.4.1. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ absolutely continuous with respect to Lebesgue measure. Then the one-dimensional probability density function $g_\Phi(s;t)$ of $\sigma_\Phi(t)$ satisfies the following Cauchy problem*

$$\begin{cases} \partial_t g_\Phi(s;t) = -\partial_s^\Phi g_\Phi(s;t) & s>0,\, t>0 \\ g_\Phi(0;t) = 0 & t>0 \\ g_\Phi(s;0) = \delta_0(s) & s \ge 0 \end{cases} \tag{1.4.2}$$

On the other hand, less hypotheses are needed to ensure that $L_\Phi(t)$ is absolutely continuous. Thus, let us recall [**143**, Theorem 4.1] for L_Φ, but let us also remark that, despite the hypotheses on ν_Φ, on can achieve the same result in a *mild sense* (i.e. in terms of Laplace transforms) without asking for such hypotheses.

THEOREM 1.4.2. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ absolutely continuous with respect to Lebesgue measure. Then the one-dimensional probability density function $f_\Phi(s;t)$ of $L_\Phi(t)$ satisfies the following Cauchy problem*

$$\begin{cases} \partial_s f_\Phi(s;t) = -\partial_t^\Phi f_\Phi(s;t) & s>0,\, t>0 \\ f_\Phi(0;t) = \overline{\nu}_\Phi(t) & t>0 \\ f_\Phi(s;0) = \delta_0(s) & s \ge 0. \end{cases} \tag{1.4.3}$$

Let us stress out that last result leads to an extension of heat-like equations. This is doable by means of the theory of semigroups. Indeed, it is already well known that the theory of semigroups can be used to describe the solution of linear evolution equations (see, for instance, [**58**]). On the other hand, semigroups are important objects in the study of Markov processes (see, for instance, [**83**]). Combining the both of these approaches to semigroup theory, one can provide, on one hand, stochastic representation of solutions of some partial differential equations while, on the other hand, characterize Markov processes via integro-differential equations, called backward and forward Kolmogorov equations. To give a simple example, if we consider the Cauchy problem for the classical heat equation

$$\begin{cases} \partial_t g(x,t) = \frac{1}{2}\Delta g(x,t) & x \in \mathbb{R}^d,\ t>0 \\ g(x,0) = f(x) & x \in \mathbb{R}^d \end{cases} \tag{1.4.4}$$

where we suppose, for ease of the example, $f \in C^2(\mathbb{R}^d)$. In such case we can define the heat semigroup as the family of operators $(T_t)_{t>0}$ acting on $L^2(\mathbb{R}^d)$ such that

$$T_t f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} f(y)dy.$$

The name semigroup follows from the fact that $T_tT_s = T_{t+s}$, that is called semigroup property. The generator of T_t is given by the operator $\frac{1}{2}\Delta$ with operator core $C^2(\mathbb{R}^d)$, that means, in short terms, that the function $g(x,t) = T_tf(x)$ is solution of the Cauchy problem (1.4.4) for any $f \in C^2(\mathbb{R}^d)$. On the other hand, if we denote

$$p(t,y;x) = \frac{1}{(t2\pi)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}},$$

that is the transition density of a d-dimensional Brownian motion $B(t)$, we have

$$T_t f(x) = \int_{\mathbb{R}^d} p(t,y;x) f(y)dy = \mathbb{E}[f(B(t))|B(0)=x],$$

which gives a stochastic representation of the solution of (1.4.4) in terms of the Brownian motion. For some more precise arguments we refer to [**78**][2] With this idea in mind, one could try to extend the theory of semigroups to the case of non-local differential equations. This is the main objective of the following Theorem (see [**33**] in the case $\Phi(\lambda) = \lambda^\alpha$, [**143**, Theorem 5.1] and [**47**] in the general case).

[2]Some easy arguments to derive backward and forward Kolmogorov equations for general diffusion processes are given in [**83**], however the theory covers more general processes. Stochastic representation results are also given, for instance, in the context of Schrödinger equations, with the name of Feynman-Kac formulae, see [**97**].

THEOREM 1.4.3. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ absolutely continuous with respect to the Lebesgue measure. Let also T_t be a strongly continuous C_0 semigroup on a Banach space $(X, \|\cdot\|_X)$. For any $u \in X$ define the Bochner integral*

$$\mathcal{T}_t u = \mathbb{E}[T_{L_\Phi(t)}u] = \int_0^{+\infty} T_s u f_\Phi(s;t).$$

Then

- *$(\mathcal{T}_t)_{t>0}$ is a uniformly bounded family of linear operators on X;*
- *$(\mathcal{T}_t)_{t>0}$ is strongly continuous on X.*

Moreover, let A be the generator of T_t and $D(A)$ its domain. If $u \in D(A)$, then $q(t) = \mathcal{T}_t u$ solves the following Cauchy problem

$$\begin{cases} \partial_t^\Phi q(t) = Aq(t) & t > 0 \\ q(0) = u \end{cases} \tag{1.4.5}$$

where the integrals involved in ∂_t^Φ are to be interpreted in Bochner sense.

As a direct consequence of such Theorem, we get the following Corollary.

COROLLARY 1.4.4. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function with Lévy measure ν_Φ. Let $M(t)$ be a Feller process with (topological) state space $(X, \mathfrak{X})$ and generator A and $L_\Phi(t)$ be independent from $M(t)$. Let $f \in D(A)$, where $D(A)$ is the domain of A on $C_0(X)$ and $C_0(X)$ is the space of continuous functions on X vanishing at infinity. Let $M_\Phi(t) = M(L_\Phi(t))$ and $u(t,x) = \mathbb{E}[f(M_\Phi(t))|M_\Phi(0) = x]$. Then u solves the following Cauchy problem*

$$\begin{cases} \partial_t^\Phi u(t,x) = Au(t,x) & t > 0 \\ u(0,x) = f(x) \end{cases} \tag{1.4.6}$$

PROOF. Just apply the previous theorem to the strongly continuous C_0 semigroup $T_t f = \mathbb{E}[f(M(t))|M(0) = x]$, defined on the Banach space $(C_0(X), \|\cdot\|_{C_0(X)})$, where the norm is the usual supremum norm. □

REMARK 1.4.5. Let us remark that better regularity of solutions can be obtained in the case $\Phi \in \mathcal{CBF}$, as shown in [**33**] for $\Phi(\lambda) = \lambda^\alpha$ and in [**21**] in the general case, and $\Phi \in \mathcal{BF}$ but $b_\Phi > 0$, as shown in [**47**].

The previous Corollary establishes a link between time-changed Feller processes and linear Cauchy problems that are non-local in time (in particular heat-like equations). This link will be better investigated in Chapter 2.

1.4.2. Eigenfunctions of ∂^Φ and the relaxation equation. Now we want to investigate the solutions of the Cauchy problem

$$\begin{cases} \partial^\Phi u(t) = \lambda u(t) & t > 0 \\ u(0) = 1 \end{cases} \tag{1.4.7}$$

for $\lambda \in \mathbb{R}$. In the case $\lambda = 0$, an obvious solution is given by the constant function $u(t) \equiv 1$. Indeed, despite $\mathcal{D}^\Phi 1(t) = \bar{\nu}_\Phi(t)$, it still holds $\partial^\Phi 1 = 0$.
A first particular case of such equation for $\lambda < 0$ has been studied for $\Phi \in \mathcal{CBF}$. In particular let us recall [**88**, Theorem 2].

THEOREM 1.4.6. *Let $\Phi \in \mathcal{CBF}$ be a driftless unbounded Bernstein function such that* $\lim_{\lambda \to 0^+} \frac{\Phi(\lambda)}{\lambda} = +\infty$. *Then, for any $\lambda < 0$, the Cauchy problem* (1.4.7) *admits a unique solution $\mathfrak{e}_\Phi(t;\lambda)$ that is completely monotone with respect to $-\lambda > 0$ and $t > 0$ and continuous for $t \in [0, +\infty)$.*

Actually, under suitable hypotheses on the Lévy measure ν_Φ, one can show that the previous theorem is actually a characterization. To do this, let us first introduce the following notation.

DEFINITION 1.4.3. We say ν_Φ satisfies **Orey's condition** (see [**115**]) if there exist $\gamma \in (0,2)$, $C > 0$ and $r_0 > 0$ such that for any $r \in (0, r_0)$ it holds

$$\int_0^r s^2 \nu_\Phi(ds) < C r^\gamma$$

This condition, together with the absolute continuity of ν_Φ, implies the infinite differentiability of the density of the subordinator $\sigma_\Phi(t)$ (see [**115**]). With this in mind, let us recall a part of [**106**, Theorem 2.1]

THEOREM 1.4.7. *Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function with Lévy measure ν_Φ absolutely continuous with respect to Lebesgue measure and satisfying Orey's condition. Then, for any $\lambda < 0$, the Cauchy problem* (1.4.7) *admits a unique exponentially bounded solution $\mathfrak{e}_\Phi(t;\lambda)$ that is completely monotone with respect to $-\lambda > 0$ and continuous for $t \in [0, +\infty)$. Moreover, $\mathfrak{e}_\Phi(t;\lambda)$ is completely monotone for $t > 0$ if and only if $\Phi \in \mathcal{CBF}$.*

Concerning the case $\lambda > 0$, a first result for $\Phi \in \mathcal{CBF}$ has been shown in [**89**].

THEOREM 1.4.8. *Let us consider the function $p_0(z)$ such that $\Phi(p_0(z)) = z$ for any $z > 0$. Then for any $\lambda > 0$ the solution $\mathfrak{e}_\Phi(t;\lambda)$ of* (1.4.7) *admits a holomorphic continuation in the complex sector $\Sigma_v = \{re^{i\theta} : \ r > 0, \ \theta \in (-v, v)\}$ for some $v \in \left(0, \frac{\pi}{2}\right)$.*

In any case, it is not difficult to show that

$$\mathfrak{e}_\Phi(t;\lambda) = \mathbb{E}[e^{\lambda L_\Phi(t)}] \tag{1.4.8}$$

that is to say the moment generating function of $L_\Phi(t)$ if $\lambda > 0$ or the Laplace transform of the density of $L_\Phi(t)$ if $\lambda < 0$. Let us stress out that such function is well-defined for any $\lambda \in \mathbb{R}$ and $t \geq 0$ (see [**15**, Lemma 4.1]). Moreover, for $\Phi(\lambda) = \lambda^\alpha$, the eigenfunctions are known. Indeed we have $\mathfrak{e}_\alpha(t;\lambda) = E_\alpha(\lambda t^\alpha)$ (see [**38**]) where E_α are the **Mittag-Leffler function** defined as

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ z \in \mathbb{C}\,.$$

Let us now show with a different approach that $\mathfrak{e}_\Phi(t;\lambda)$ is well-defined. If we use (1.4.8) as definition of $\mathfrak{e}_\Phi(t;\lambda)$ in place of the fact that it should be solution of (1.4.7), we can observe that for $\lambda \leq 0$ $\mathfrak{e}(t;\lambda)$ is always well-defined for any $t > 0$ and any $\Phi \in \mathcal{BF}$ (without any other hypotheses), while this is true for a t belonging to a compact set $[0, t_0(\lambda))$ for $\lambda > 0$ and any unbounded Bernstein function Φ. Last statement follows from the bound (see [**35**])

$$\mathbb{E}[L(t)^n] \leq \frac{n!}{\Phi^n\left(\frac{1}{t}\right)}$$

and $t_0(\lambda)$ is defined in such a way that $\lambda < \Phi\left(\frac{1}{t_0(\lambda)}\right)$. However, we can show the following technical Lemma.

LEMMA 1.4.9. *Fix $\lambda > 0$ and let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. Then $\mathfrak{e}_\Phi(t;\lambda)$ is well defined for any $t > 0$, Laplace transformable with abscissa of convergence given by $p_0(\lambda)$ and in $L^\infty_{\rm loc}(\mathbb{R}^+_0)$.*

PROOF. First of all, let us observe that, since we are working with non-negative functions, we have, by Fubini's theorem and considering $\eta > p_0(\lambda)$ (and then $\Phi(\eta) - \lambda > 0$)

$$\begin{aligned}\mathcal{L}[\mathfrak{e}_\Phi(t;\lambda)](\eta) &= \int_0^{+\infty} e^{-\eta t}\int_0^{+\infty} e^{\lambda s} f_\Phi(s;t)dsdt\\ &= \int_0^{+\infty} e^{\lambda s}\int_0^{+\infty} e^{-\eta t} f_\Phi(s;t)dtds\\ &= \frac{\Phi(\eta)}{\eta}\int_0^{+\infty} e^{-(\Phi(\eta)-\lambda)s}ds = \frac{\Phi(\eta)}{\eta(\Phi(\eta)-\lambda)}.\end{aligned}$$

Moreover, it is easy to see that if $\eta \le p_0(\lambda)$, then $\mathcal{L}[\mathfrak{e}_\Phi(t;\lambda)](\eta) = +\infty$. Thus, we obtain that $\mathrm{abs}(\mathfrak{e}_\Phi(\cdot;\lambda)) = p_0(\lambda)$.
Now let us show that $\mathfrak{e}_\Phi(\cdot;\lambda) \in L^1_{\rm loc}(\mathbb{R}^+_0)$. Consider any compact set $[a,b] \subseteq \mathbb{R}^+_0$ and fix any $\eta > p_0(\lambda)$. Then we have

$$\int_a^b \mathfrak{e}_\Phi(t;\lambda)dt \le e^{-\eta b}\int_0^{+\infty} e^{-\eta t}\,\mathfrak{e}_\Phi(t;\lambda)dt = \frac{\Phi(\eta)e^{-\eta b}}{\eta(\Phi(\eta)-\lambda)} < +\infty.$$

Now, let us observe that this implies that $\mathfrak{e}_\Phi(t;\lambda) < +\infty$ for any $t > 0$; indeed, since $L_\Phi(t)$ is almost surely increasing, $\mathfrak{e}_\Phi(t;\lambda)$ is increasing and if $\mathfrak{e}_\Phi(t_0;\lambda) = +\infty$ so it is for any $t > t_0$ and then $\mathfrak{e}_\Phi(t_0;\lambda) \notin L^1_{\rm loc}(\mathbb{R}^+_0)$, which is a contradiction.
Finally, let us consider any compact set $[a,b] \subseteq \mathbb{R}^+_0$. Then, since $\mathfrak{e}_\Phi(t;\lambda)$ is increasing we have, for any $t \in [a,b]$;

$$\mathfrak{e}_\Phi(t;\lambda) \le \mathfrak{e}_\Phi(b;\lambda) < +\infty,$$

concluding the proof. □

REMARK 1.4.10. Let us observe that if $\lambda \le 0$, then $\mathfrak{e}_\Phi(t;\lambda) \le 1$ and then it is obviously $L^\infty(\mathbb{R}^+_0)$, with $\mathrm{abs}(\mathfrak{e}(\cdot;\lambda)) \le 0$.

Thus, we can argue in a sort of *converse way*: instead of showing that (1.4.7) admits a unique solution, we show that $\mathfrak{e}_\Phi(t;\lambda)$ (whenever it exists for any $t > 0$ and is locally bounded) is the unique solution of (1.4.7). This is the spirit of [**15**, Proposition 4.3]:

PROPOSITION 1.4.11. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. Then $\mathfrak{e}_\Phi(t;\lambda)$ is the unique Laplace transformable solution of* (1.4.7).

PROOF. Since $\mathfrak{e}_\Phi(t;\lambda)$ belongs to $L^1_{\rm loc}(\mathbb{R}^+)$, then we can define the function $F(t) = \int_0^t \mathfrak{e}_\Phi(s;\lambda)ds$ that is an absolutely continuous function. By a simple application of [**13**, Theorem 1.4.3], we know that $F(t)$ is Laplace transformable and, if z_0 is the abscissa of convergence of $\mathfrak{e}_\Phi(t;\lambda)$, then $\mathrm{abs}(F) \le z_0$.
Taking the Laplace transform $\bar{f}(z)$ of $f(t) = \mathfrak{e}_\Phi(t;\lambda)$ as $z \ge z_0$, we obtain (as shown before),

$$\bar{f}(z) = \frac{\Phi(z)}{z(\Phi(z)-\lambda)},$$

that can be rewritten as

$$\frac{\Phi(z)}{z}\left(\bar{f}(z)-\frac{1}{z}\right)=\frac{\lambda}{z}\bar{f}(z). \tag{1.4.9}$$

Let us also observe that

$$\int_0^t \bar{\nu}_\Phi(t-s)|f(s)-1|ds=\int_0^t \bar{\nu}_\Phi(s)|f(t-s)-1|ds \le |f(t)-1|I_\Phi(t)$$

where we recall that $I_\Phi(t)$ is the integrated tail of the Lévy measure. In particular this means that the function $F_\Phi(t)=(\nu_\Phi*(f(\cdot)-1))(t)$ is well defined and belongs to L^∞_{loc}. Moreover, being the convolution product of two Laplace transformable functions, it is Laplace transformable (with abscissa of convergence $\text{abs}(F_\Phi)\le z_0$). Now we can consider the inverse Laplace transform of equation (1.4.9) to obtain

$$F_\Phi(t)=\lambda F(t).$$

Since $F(t)$ is the integral of a $L^1_{\text{loc}}(\mathbb{R}^+)$ function, it is absolutely continuous and then also F_Φ is absolutely continuous. Taking the derivative (almost everywhere) on both sides, we obtain

$$\partial^\Phi f(t)=\lambda f(t),$$

thus $f(t)$ is a solution of (1.4.7).
Next step is to show the uniqueness. However, considering any other Laplace transformable solution, arguing as before, we have that $G(t)=\int_0^t g(s)ds$ is also Laplace transformable. Moreover, taking the Laplace transform on both sides of the relation

$$\int_0^t \bar{\nu}_\Phi(t-s)(g(s)-1)ds=\lambda\int_0^t g(s)ds \tag{1.4.10}$$

we obtain, after some algebraic manipulation,

$$\mathcal{L}[g](z)=\frac{\Phi(z)}{z(\Phi(z)-\lambda)}=\bar{f}(z).$$

The injectivity of the Laplace transform concludes the proof. □

For $\lambda<0$ we can actually obtain a bound on $\lambda\,\mathfrak{e}_\Phi(t;-\lambda)$ for fixed $t>0$, as done in [**22**, Proposition 3.2].

Proposition 1.4.12. *Fix $t>0$. Then there exists a constant $K(t)$ such that for any $\lambda\in[0,+\infty)$ it holds*

$$\lambda\,\mathfrak{e}_\Phi(t;-\lambda)\le K(t)$$

Proof. Let us observe that $\mathfrak{e}_\Phi(t;-\lambda)$ is completely monotone in λ for fixed $t>0$ and $\mathfrak{e}_\Phi(t;0)=1$, thus one only has to check that $\lim_{\lambda\to+\infty}\lambda\,\mathfrak{e}_\Phi(t;-\lambda)<+\infty$. This can be done by means of the initial value theorem. Indeed $\mathfrak{e}_\Phi(t;-\lambda)$ is the Laplace transform of $f_\Phi(s;t)$ in s, thus it holds

$$\lim_{\lambda\to+\infty}\lambda\,\mathfrak{e}_\Phi(t;-\lambda)=f(0+;t)=\bar{\nu}_\Phi(t)<+\infty,$$

concluding the proof. □

In the case $\Phi(\lambda)=\lambda^\alpha$ the constant $K(t)$ can be explicitly computed. Indeed we have, as in [**20**, Lemma 4.2]:

PROPOSITION 1.4.13. *Fix $t>0$. Then for $\lambda \ge 0$ it holds*

$$\lambda E_\alpha(-\lambda t^\alpha) \le \frac{\Gamma(1+\alpha)}{t^\alpha}. \tag{1.4.11}$$

PROOF. Let us recall, as shown in [**138**], that

$$E_\alpha(-\lambda t^\alpha) \le \frac{1}{1+\frac{t^\alpha}{\Gamma(1+\alpha)}\lambda}.$$

Consider the function $f(\lambda)=\frac{\lambda}{1+\frac{t^\alpha}{\Gamma(1+\alpha)}\lambda}$. We have $f(0)=0$ and $\lim_{\lambda\to+\infty} f(\lambda) = \frac{\Gamma(1+\alpha)}{t^\alpha}$. Finally, let us observe that

$$f'(\lambda) = \frac{1}{\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}\lambda\right)^2} > 0,$$

concluding the proof since $f(\lambda) \le f(+\infty)$ for any $\lambda \ge 0$. □

REMARK 1.4.14. Let us consider a fractional Poisson process $N_\alpha(t)$ of rate $\lambda > 0$ as introduced in [**93**], i.e. a counting process with i.i.d. inter-jump times $(T_i)_{i\in\mathbb{N}}$ distributed as a random variable T whose cumulative distribution function is given by

$$\mathbb{P}(T\le t) = \begin{cases} 1-E_\alpha(-\lambda t^\alpha) & t\ge 0 \\ 0 & t<0. \end{cases}$$

It has been shown in [**93**] that $\mathbb{E}[N_\alpha(t)] = \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}$. Hence, we can rewrite inequality (1.4.11) as

$$\mathbb{P}(T>t) \le \frac{1}{\mathbb{E}[N_\alpha(t)]},\ t>0.$$

Concerning the asymptotic behvaiour of $\mathfrak{e}(t;-\lambda)$, for $\lambda>0$, with respect to $t\to+\infty$, let us observe that we can link it to the behaviour at 0^+ of Φ (see [**22**, Proposition 6.4]).

PROPOSITION 1.4.15. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function.*

(1) *Suppose Φ is regularly varying at 0^+ with order $\alpha\in(0,1)$. Then, for fixed $\lambda>0$, $\mathfrak{e}_\Phi(t;-\lambda)$ is regularly varying at $+\infty$ with order $-\alpha$;*
(2) *Suppose $\lim_{\lambda\to 0^+}\frac{\Phi(\lambda)}{\lambda} = l \in (0,+\infty)$. Then, for fixed $\lambda>0$, $\mathfrak{e}_\Phi(t;-\lambda)$ is integrable in $(0,+\infty)$.*

PROOF. Let us define the function $J(t)=\int_0^t \mathfrak{e}_\Phi(s;-\lambda)ds$ and take the Laplace-Stieltjes transform of J. We have

$$\overline{J}(z) = \mathcal{L}^S[J](z) = \frac{\Phi(z)}{z(\Phi(z)+\lambda)}.$$

Now let us suppose that Φ is regularly varying at 0^+ with order α. Then $\overline{J}(z)$ is regularly varying at 0^+ of order $\alpha-1$. By Karamata's Tauberian theorem, we have that $J(t)$ is regularly varying at infinity with order $1-\alpha$ and finally, by monotone density theorem, we have property (1).
Concerning property (2), if $\lim_{z\to 0}\frac{\Phi(z)}{z}=l$, then, being $\Phi(z)$ driftless, we have

$$\lim_{z\to 0^+}\overline{J}(z) = \frac{l}{\lambda}.$$

Still by Karamata's Tauberian Theorem, we get

$$\lim_{t\to+\infty} J(t) = \frac{l}{\lambda},$$

concluding the proof. □

If we focus on the special Bernstein functions case, we can also obtain a series decomposition of $\mathfrak{e}_\Phi(t;\lambda)$ in terms of a sequence of functions constructed starting from the renewal function. Indeed, let us define, in what follows, $U_{\Phi,k}(t) = \mathbb{E}[(L_\Phi(t))^k]$. Moreover, for $\Phi \in \mathcal{SBF}$ a driftless Bernstein function, let us define the following sequence of functions:

$$\begin{cases} u_0^*(t) \equiv 1; \\ u_1^*(t) = U_\Phi(t); \\ u_{k+1}^*(t) = \int_0^t u_\Phi(t-s)u_k^*(s)ds \quad k \ge 1, \end{cases} \tag{1.4.12}$$

where u_Φ is the potential density.
Then, before obtaining the series representation, let us show the following technical Lemma (i.e. [**15**, Lemma 4.4]).

LEMMA 1.4.16. *Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at ∞ with index $\gamma \in (0,1)$. Then, for any $\lambda > 0$ the function series*

$$\sum_{k=1}^{+\infty} \lambda^k u_k^*(t)$$

is normally convergent in any set of the form $[0,T]$ for any $T > 0$.

PROOF. First of all, let us fix $T > 0$, $\varepsilon \in (0,\gamma)$ and $\beta = \gamma - \varepsilon$. By Corollary (1.3.14) and Proposition 1.3.15 there exist two constants $C_1, C_2 > 0$ such that, for any $t \in [0,T]$,

$$U_\Phi(t) \le C_1 t^\beta, \qquad\qquad u_\Phi(t) \le C_2 t^{\beta-1}.$$

Let us suppose the following claim holds true:

- For any $k \ge 1$ it holds

$$u_k^*(t) \le C_1 C_2^{k-1} \frac{\beta}{\Gamma(k\beta+1)}(\Gamma(\beta)t^\beta)^k. \tag{1.4.13}$$

Then we have

$$\sum_{k=1}^{+\infty} \lambda^k u_k^*(t) \le \frac{C_1\beta}{C_2} \sum_{k=1}^{+\infty} \frac{(\lambda C_2 \Gamma(\beta) T^\beta)^k}{\Gamma(k\beta+1)}$$

where the series on the right-hand side converges since the power series $\sum_{k=1}^{+\infty} \frac{y^k}{\Gamma(k\beta+1)}$ admits $+\infty$ as radius of convergence.
Now we only need to show the claim. Equation (1.4.13) obviously holds true for $k = 1$. Suppose it holds true for some $k \ge 1$. Then we have, by definition

$$u_{k+1}^*(t) \le C_1 C_2^k \frac{\beta\Gamma(\beta)^k}{\Gamma(k\beta+1)} \int_0^t (t-s)^{\beta-1} s^{k\beta} ds.$$

By using the change of variables $w = \frac{s}{t}$ and the definition of Euler's Beta function as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 (1-w)^{a-} w^{b-1} dw,$$

we get

$$u_{k+1}^*(t) \le C_1 C_2^k \frac{\beta\Gamma(\beta)^k}{\Gamma(k\beta+1)} t^{(k+1)\beta} B(\beta, k\beta+1) = C_1 C_2^k \frac{\beta\Gamma(\beta)^{k+1}}{\Gamma((k+1)\beta+1)} t^{(k+1)\beta},$$

concluding the proof. □

Now that we have shown this technical lemma, we have the following series expansion of $\mathfrak{e}_\Phi(t;\lambda)$ (see [**15**, Theorem 4.5])

THEOREM 1.4.17. *Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at ∞ with index $\gamma \in (0,1)$. Then, for any $\lambda \in \mathbb{R}$, it holds*

$$\mathfrak{e}_\Phi(t;\lambda) = \sum_{k=0}^{+\infty} \lambda^k u_k^*(t), \quad t > 0. \tag{1.4.14}$$

PROOF. Since this is trivial for $\lambda = 0$, let us consider $\lambda \neq 0$. Let us first work with $\lambda > 0$. By monotone convergence theorem, it is easy to show that

$$\mathfrak{e}_\Phi(t;\lambda) = \sum_{k=0}^{+\infty} \frac{\lambda^k U_{\Phi,k}(t)}{k!}.$$

Now, recalling Equation (1.2.2) and using Laplace transform in place of the Laplace-Stieltjes one, we have, again by monotone convergence theorem,

$$\mathcal{L}[\mathfrak{e}_\Phi(t;\lambda)](z) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{z\Phi^k(z)}.$$

Let us suppose the following claim holds true:

- For any $k \ge 1$ it holds

$$\mathcal{L}[u_k^*(t)](z) = \frac{1}{z\Phi^k(z)}. \tag{1.4.15}$$

Then we have, again by monotone convergence theorem,

$$\mathcal{L}\left[\sum_{k=0}^{+\infty} \lambda^k u_k^*(t)\right] = \sum_{k=0}^{+\infty} \frac{\lambda^k}{z\Phi^k(z)} = \mathcal{L}[\mathfrak{e}_\Phi(t;\lambda)](z),$$

concluding the proof by injectivity of the Laplace transform. For $\lambda < 0$, one has only to observe that

$$\sum_{k=0}^{+\infty} \frac{|\lambda|^k U_{\Phi,k}(t)}{k!} = \sum_{k=1}^{+\infty} |\lambda|^k u_k^*(t) < +\infty$$

and then use dominated convergence theorem in place of the monotone convergence one.

Now we only need to show the claim. For $k = 1$ this follows from the fact that $\mathcal{L}^S[U(t)](z) = \frac{1}{\Phi(z)}$ and then $\mathcal{L}[U_\Phi](z) = \frac{1}{z\Phi(z)}$. Let us suppose Equation (1.4.15) holds true for some $k \ge 1$. Then

$$\mathcal{L}[u_{k+1}^*](z) = \mathcal{L}[u_\Phi](z)\,\mathcal{L}[u_k^*](z) = \frac{1}{z\Phi^{k+1}(z)},$$

concluding the proof. □

We will use this series representation in Chapter 2 to show a generalization of Grönwall's inequality.

1.5. Bochner subordination and Phillips formula

In the previous Section we considered some one-dimensional non local derivatives that arise after time-changing via inverse subordinator. However, since also the subordinators are themselves non-negative and increasing, one could use a time-change directly via a subordinator. Let us consider the approach described in [**40**].

DEFINITION 1.5.1. Let $M(t)$ be a Feller process on $\mathbb{R}^d$ and $\Phi \in \mathcal{BF}$ a Bernstein function. Let σ_Φ be the associated subordinator and let us suppose it is independent from M. Then we define the **subordinated process** $M^\Phi(t) = M(\sigma_\Phi(t))$.
Let $(T_t)_{t\ge 0}$ be a one-parameter strongly continuous C_0 semigroup on $C^\infty(\mathbb{R}^d)$ with the supremum norm. Denote by $g_\Phi(ds;t)$ the probability law of $\sigma_\Phi(t)$. Then we define the **subordinated semigroup** as

$$T_t^\Phi u = \int_0^{+\infty} T_s u g_\Phi(ds;t).$$

Since we are integrating over a Banach space, this whole procedure takes the name of **Bochner subordination**. Let us stress out the link between subordinated Feller processes and subordinated semigroups (see [**40**, Lemma 4.5]).

PROPOSITION 1.5.1. *Let $M(t)$ be a Feller process. The subordinated process $M^\Phi(t)$ is still a Feller process. Moreover, if T_t is the semigroup induced by the Feller process $M(t)$, T_t^Φ is the semigroup induced by $M^\Phi(t)$.*

Now we need to investigate what happens to the generators of such semigroups. In this direction, let us refer to [**40**, Theorem 4.6].

THEOREM 1.5.2 (**Phillips' formula**). *Let $M(t)$ a Feller process with generator A whose domain is $D(A)$ and consider a Bernstein function $\Phi \in \mathcal{BF}$ with characteristic triple (a, b, ν_Φ). Then the generator A^Φ of $M^\Phi(t)$ is given by*

$$A^\Phi u = -au + bAu + \int_0^{+\infty} (T_t u - u)\nu_\Phi(dt).$$

Moreover, $D(A)$ is an operator core for A^Φ.

First of all, let us observe that another notation for A^Φ is $\Phi(A)$. We shall use this second notation.
Let us focus on a standard case, i.e. choosing as Feller process the Brownian motion $B(t)$ on $\mathbb{R}^d$. Then it is well known that $A = -\Delta$ (up to a constant, hence we are considering a Brownian motion with variance $2t$ in place of t). If we consider a subordinated Brownian motion $B^\Phi(t)$, it is easy to see that the Levy measure of such process is given by $\mu(dx) = j_\Phi(|x|)dx$ where

$$j_\Phi(r) = \int_0^{+\infty} \widetilde{p}(r;t)\nu_\Phi(dt) \tag{1.5.1}$$

where $\widetilde{p}(r;t) = p(x;t)$ for any $x \in \mathbb{R}^d$ such that $|x| = r$ and $p(x;t)$ is the heat-kernel in $\mathbb{R}^d$, i.e.

$$p(x;t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}.$$

The function $j_\Phi(r)$ is called the **jump function** of $B^\Phi(t)$. Now, let us write the semigroup in terms of resolvent operators, i.e. $T_t u - u = (e^{t\Delta} - 1)u$ and consider a driftless Bernstein function $\Phi \in \mathcal{BF}$. We have that

$$-\Phi(-\Delta)u = \int_0^{+\infty}(e^{t\Delta}-1)u\nu_\Phi(dt).$$

However, we know that $e^{t\Delta}$ is a convolution semigroup, hence we obtain the following representation of $\Phi(-\Delta)$ in terms of the jumping function (see [**24**, Proposition 2.6]).

Proposition 1.5.3. *Let $f:\mathbb{R}^d \to \mathbb{R}$ with $f \in L^\infty(\mathbb{R}^d)$ and let us denote $\mathrm{D}_h^2 f(x) = f(x+h)-2f(x)+f(x-h)$ and $\mathrm{D}_h^1 f(x) = f(x+h)-f(x)$. Suppose $x \in \mathbb{R}^d$ such that $|\mathrm{D}_h^2 f(x)| \le C|h|^2$ for $|h| \le R_1$. Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. Then*

$$-\Phi(-\Delta)f = \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}_h^2 f(x)j(|h|)dh = \lim_{\varepsilon\to 0^+}\int_{B_\varepsilon^c}\mathrm{D}_h^1 f(x)j(|h|)dh$$

where B_ε is a ball centered in 0 with radius $\varepsilon > 0$ and $B_\varepsilon^c = \mathbb{R}^d \setminus B_\varepsilon$.

Proof. First of all, let us rewrite $-\Phi(-\Delta)$ as

$$-\Phi(-\Delta)f(x) = \int_0^{+\infty}\int_{\mathbb{R}^d}(f(x)-f(y))p(x-y;t)dy\nu_\Phi(dt)$$

where we also used the fact that $\int_{\mathbb{R}^d}p(x-y;t)dx = 1$. Now let us consider the inner integral. We have in particular

$$\begin{aligned}\int_{\mathbb{R}^d}(f(y)-f(x))p(x-y;t)dy &= \frac{1}{2}\int_{\mathbb{R}^d}(f(x+h)-f(x))p(h;t)dh \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^d}(f(x-h)-f(x))p(-h;t)dh \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}_h^2 f(y)\widetilde{p}(|h|;t)dh.\end{aligned}$$

Now we want to show that we can use Fubini's theorem. To do this, let us observe that

$$\int_{\mathbb{R}^d}|\mathrm{D}_h^2 f(y)|\int_0^{+\infty}\widetilde{p}(|h|;t)\nu_\Phi(dt)dh = \int_{\mathbb{R}^d}|\mathrm{D}_h^2 f(y)|j(|h|)dh$$

Let us split the integral in two parts. We have

$$\int_{B_{R_1}}|\mathrm{D}_h^2 f(y)|j(|h|)dh \le C\int_{B_{R_1}}|h|^2 j(|h|)dh < +\infty$$

since $j(|h|)dh$ is a Lévy measure. On the other hand we have

$$\int_{B_{R_1}^c}|\mathrm{D}_h^2 f(y)|j(|h|)dh \le 4\,\|f\|_{L^\infty(\mathbb{R}^d)}\int_{B_{R_1}^c}j(|h|)dh < +\infty$$

as before, since $j(|h|)dh$ is a Lévy measure.
Thus we can use Fubini's theorem to achieve

$$-\Phi(-\Delta)f = \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}_h^2 f(x)j(|h|)dh.$$

Concerning the second relation, let us observe that, by dominated convergence theorem, we have

$$-\Phi(-\Delta)f = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{B_\varepsilon^c}\mathrm{D}_h^2 f(x)j(|h|)dh.$$

However, we have

$$\begin{aligned}\int_{B_\varepsilon^c} \mathrm{D}_h^2 f(x) j(|h|) dh &= \int_{B_\varepsilon^c} \mathrm{D}_h^1 f(x) j(|h|) dh + \int_{B_\varepsilon^c} \mathrm{D}_{-h}^1 f(x) j(|h|) dh \\ &= 2\int_{B_\varepsilon^c} \mathrm{D}_h^1 f(x) j(|h|) dh,\end{aligned}$$

concluding the proof. □

REMARK 1.5.4. Last Proposition still holds if we substitute in place of $|h|^2$ a modulus of continuity $\beta(|h|)$ such that there exists $R_2 > 0$ for which $\int_0^{R_2} r^{d-1}\beta(r) j(r) dr < +\infty$.

Let us also observe that for some particular choices of Φ one achieves some quite known operators:

- For $\Phi(\lambda) = \lambda^\alpha$, denoting by $\beta = 2\alpha$, one obtains the fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$. The representation obtained in the previous theorem coincides with the one given in [**1**, Equation 2.2];
- For $\Phi(\lambda) = (\lambda + (c^2 m)^{\frac{1}{\alpha}})^\alpha - c^2 m$ we obtain the relativistic Laplacian, which is a commonly used operator for the study of non-local Schrödinger equations in relativistic context.

We will go into details in the properties of the jumping functions in Chapter 4.

CHAPTER 2

Non-local operators in time and time-changes

2.1. Existence and uniqueness of solutions for some time non-local Cauchy problems

In Subsection 1.4.2 we introduced the relaxation equation for the non-local derivative ∂^Φ induced by a driftless Bernstein function $\Phi \in \mathcal{BF}$. However we are interested in more general Cauchy problems of the form

$$\begin{cases} \partial^\Phi f(t) = F(t, f(t)) & t \in [0,T] \\ f(0) = f_0. \end{cases} \tag{2.1.1}$$

In particular, we want to prove an existence and uniqueness theorem for such problem. First of all, by interpreting all the integrals as Bochner integrals (see [**13**]), we can consider $F : [0,T] \times X \to X$ and $f : [0,T] \to X$ with $(X, \|\cdot\|_X)$ a Banach space. Moreover, we want to transform the Cauchy problem (2.1.1) in an integral equation. To do this, we show the following Lemma (see [**15**, Lemma 3.1]).

LEMMA 2.1.1. *Let $\Phi \in \mathcal{SBF}$ be a special driftless Bernstein function and $(X, \|\cdot\|_X)$ a Banach space. Consider $F : [0,T] \times X \to X$. Then $f : [0,T] \to X$ is a solution of the abstract Cauchy problem* (2.1.1) *if and only if*

$$f(t) = f_0 + \mathcal{I}^\Phi F(\cdot, f(\cdot))(t), \qquad t \in [0,T]. \tag{2.1.2}$$

PROOF. Let us consider $f : [0,T] \to X$ a solution of (2.1.1). Then, by Equation (1.4.1) and the fact that $\mathcal{I}^\Phi \mathcal{D}^\Phi$ is the identity operator, we obtain (2.1.2) by applying $\mathcal{I}^\Phi$ on both sides of the first equation of (2.1.1) and then using $f(0) = f_0$. Vice versa, if we suppose f satisfies (2.1.2), then, since $\mathcal{I}^\Phi F(\cdot, f(\cdot))(0) = 0$, we have $f(0) = f_0$. Moreover, applying ∂^Φ to both sides of (2.1.2) we obtain the first equation of (2.1.1). □

The fact we are able to transform the non-local Cauchy problem in an integral equation permits us to recognize solutions of (2.1.1) as fixed point of some Picard operator. This has been done for instance for $\Phi(\lambda) = \lambda^\alpha$ in [**150**, Section 3.4]. In general, under suitable hypotheses on Φ and F, one can show the following local existence and uniqueness theorem (see [**15**, Theorem 3.2]).

THEOREM 2.1.2. *Let $\Phi \in \mathcal{SBF}$ and $F : [0,T] \times X \to X$ where $(X, \|\cdot\|_X)$ is a Banach space. Let us suppose the following conditions hold:*

- *Φ is regularly varying at ∞ with index $\gamma \in (0,1)$;*
- *For any ball B_R in X centered in 0 with radius R there exists a constant $C_R > 0$ such that $\|F(t,x)\|_X \le C_R$ for almost any $t \in [0,T]$ and for any $x \in B_R$;*

- *For any ball B_R in X centered in 0 with radius R there exists a constant $L_R > 0$ such that $\|F(t,x) - F(t,z)\|_X \le L_R \|x - z\|_X$ for almost any $t \in [0,T]$ and for any $x, z \in B_R$.*

Fix $R > 0$. Then, for any $f_0 \in B_R$ there exists a constant $T_1 > 0$ such that the Cauchy problem (2.1.1) *admits a unique solution $f \in C^{\gamma-\varepsilon}([0,T_1]; B_R(f_0))$ for any $\varepsilon \in (0,\gamma)$.*

To show this theorem, we first need some preliminary definitions and Lemmas. First of all, let us observe that, if we set $J = [0,T]$, then the space $C(J;X)$ is a Banach space when equipped with the supremum norm, i.e.

$$f \in C(J;X) \mapsto \|f\|_{C(J;X)} = \max_{t \in J} \|f(t)\|_X .$$

Now let us consider $\eta > 0$ and let us observe that

$$e^{-\eta T} \max_{t \in J} \|f(t)\|_X \le \max_{t \in J} e^{-\eta t} \|f(t)\|_X \le \max_{t \in J} \|f(t)\|_X .$$

Thus, let us define the Bielecki-type norm

$$\|f\|_\eta = \max_{t \in J} e^{-\eta t} \|f(t)\|_X$$

and observe that $\|\cdot\|_{C(J;X)}$ and $\|\cdot\|_\eta$ are equivalent. Hence $(C(J;X), \|\cdot\|_\eta)$ is still a Banach space equivalent to $(C(J;X), \|\cdot\|_{C(J;X)})$. Now let us introduce the following operator on $C(J;X)$:

$$A_\Phi f(t) = f_0 + \mathcal{I}^\Phi F(\cdot, f(\cdot))(t).$$

By Lemma 2.1.1, we know that any solution of (2.1.1) is a fixed point of A_Φ.
Now let us show the following Lemma (that is [**15**, Lemma 3.3 and 3.4]).

Lemma 2.1.3. *Under the hypotheses of Theorem 2.1.2 for fixed $\varepsilon \in (0,\gamma)$, $f_0 \in X$ and $R > 0$, there exists T_1 such that, setting $J_1 = [0,T_1]$, $A_\Phi : C(J_1; B_R(f_0)) \to C^{\gamma-\varepsilon}(J_1; B_R(f_0))$ is well-defined.*

Proof. Let us first show that if $f \in C(J; B_R(f_0))$, then $A_\Phi f \in C^{\gamma-\varepsilon}(J;X)$. To do this, define $\widetilde{R} = R + |f_0|$, fix $\delta > 0$ and $\varepsilon \in (0,\gamma)$. Then we have

$$\begin{aligned}\|A_\Phi f(t+\delta) - A_\Phi f(t)\|_X \le & \int_0^t |u_\Phi(t-s) - u_\Phi(t+\delta-s)| \, \|F(s,f(s))\|_X \, ds \\ & + \int_t^{t+\delta} u_\Phi(t+\delta-s) \, \|F(s,f(s))\|_X \, ds \\ & := I_1(t) + I_2(t).\end{aligned}$$

Let us first consider $I_2(t)$. Since $f : J \to B_R(f_0) \subseteq B_{\widetilde{R}}$ and $\|F(s,x)\|_X \le C_{\widetilde{R}}$ for any $s \in J$ and $x \in B_{\widetilde{R}}$, it holds

$$I_2(t) \le C_{\widetilde{R}} U_\Phi(\delta),$$

where U_Φ is the renewal function of Φ. Now, by Corollary 1.3.14 and since $\delta < T$, we have that there exists a constant $C > 0$ such that

$$I_2(t) \le C\delta^{\gamma-\varepsilon}.$$

Now let us consider I_1. Since u is non-increasing, we have

$$I_1(t) \le C_{\widetilde{R}}(U(t+\delta) - U(t) + U(\delta)).$$

As before, by Corollary 1.3.14 and Proposition 1.3.12, there exists a constant $C > 0$ such that
$$I_1(t) \le C\delta^{\gamma-\varepsilon},$$
concluding that $A_\Phi f \in C^{\gamma-\varepsilon}(J, X)$.
Now we want to show that there exists T_1 such that $\max_{t\in[0,T_1]} \|A_\Phi f(t) - f_0\|_X \le R$. To do this, let us observe that, being $U_\Phi(t)$ Hölder-continuous with $U_\Phi(0) = 0$, there exists $T_1 > 0$ depending on $\widetilde{R}$ such that $C_{\widetilde{R}} U_\Phi(T_1) < R$. Thus, since U_Φ is increasing, we have, for any $t \in [0, T_1]$,
$$\|A_\Phi f(t) - f_0\|_X \le C_{\widetilde{R}} U_\Phi(t) \le C_{\widetilde{R}} U_\Phi(T_1) < R.$$
Taking the maximum on $[0, T_1]$ we conclude the proof. □

Now we are ready to prove the main Theorem of this section (we incorporate in the proof of the Theorem also the proof of [**15**, Proposition 3.6]).

PROOF OF THEOREM 2.1.2. We need to show that $A_\Phi : (C(J_1; B_R(f_0)), \|\cdot\|_\eta) \to (C(J_1; B_R(f_0)), \|\cdot\|_\eta)$ is a contraction for some $\eta > 0$. To do this, let us consider $f, g \in C(J_1; B_R(f_0))$ and observe that, by uniform local Lipschitz-continuity of F,
$$\begin{aligned}\|A_\Phi f(t) - A_\Phi g(t)\|_X &\le \int_0^t u(t-s)\,\|F(s, f(s)) - F(s, g(s))\|_X\, ds \\ &\le L_{\widetilde{R}} \int_0^t u_\Phi(t-s)\,\|f(s) - g(s)\|_X\, ds \\ &\le L_{\widetilde{R}} \|f - g\|_\eta \int_0^t u_\Phi(t-s) e^{\eta s} ds.\end{aligned}$$
Now, by Proposition 1.3.15, we know that u_Φ is regularly varying at 0^+ of index $\gamma - 1$, thus, if we consider $\varepsilon_1 \in (0, \gamma)$, there exists a constant $C > 0$ such that for any $t \in [0, T_1]$ it holds $u(t) \le Ct^{\gamma-1-\varepsilon_1}$. Now consider $p \in \left(1, \frac{1}{1+\varepsilon_1-\gamma}\right)$, denote by p' the conjugate exponent of p and use Hölder inequality:
$$\begin{aligned}\|A_\Phi f(t) - A_\Phi g(t)\|_X &\le CL_{\widetilde{R}} \|f - g\|_\eta \int_0^t (t-s)^{\gamma-1-\varepsilon_1} e^{\eta s} ds \\ &\le CL_{\widetilde{R}} \|f - g\|_\eta \left(\int_0^t (t-s)^{p(\gamma-1-\varepsilon_1)} ds\right)^{\frac{1}{p}} \left(\int_0^t e^{p'\eta s} ds\right)^{\frac{1}{p'}} \\ &\le CL_{\widetilde{R}} \|f - g\|_\eta \frac{T_1^{\frac{p(\gamma-1-\varepsilon_1)+1}{p}}}{(p(\gamma-1-\varepsilon_1)+1)^{\frac{1}{p}}} \left(\frac{1}{p'\eta}\right)^{\frac{1}{p'}} e^{\eta t}.\end{aligned}$$
Multiplying both sides of last inequality by $e^{-\eta t}$ and taking the supremum on J_1 we get
$$\|A_\Phi f - A_\Phi g\|_\eta \le CL_R \|f - g\|_\eta \frac{T_1^{\frac{p(\gamma-1-\varepsilon_1)+1}{p}}}{(p(\gamma-1-\varepsilon_1)+1)^{\frac{1}{p}}} \left(\frac{1}{p'\eta}\right)^{\frac{1}{p'}}.$$
Now, let us observe that
$$\lim_{\eta\to+\infty} CL_R \frac{T_1^{\frac{p(\gamma-1-\varepsilon_1)+1}{p}}}{(p(\gamma-1-\varepsilon_1)+1)^{\frac{1}{p}}} \left(\frac{1}{p'\eta}\right)^{\frac{1}{p'}} = 0,$$

hence there exists η_* such that

$$L^* := CL_R \frac{T_1^{\frac{p(\gamma-1-\varepsilon_1)+1}{p}}}{(p(\gamma-1-\varepsilon_1)+1)^{\frac{1}{p}}}\left(\frac{1}{p'\eta_*}\right)^{\frac{1}{p'}} < 1.$$

From last relation we get

$$\|A_\Phi f - A_\Phi g\|_{\eta_*} \le L^* \|f-g\|_{\eta_*}$$

and then A_Φ is a contraction. Thus, by contraction theorem (see [**85**]) it admits a unique fixed point in $C(J_1; B_R(f_0))$. Let f be such fixed point. Then, since $f = Af$, it holds $f \in C^{\gamma-\varepsilon}(J_1; B_R(f_0))$ for any $\varepsilon \in (0,\gamma)$, concluding the proof. $\square$

Now we have a *more or less* general result of local existence and uniqueness. We want to understand if under some cases one can show global existence and uniqueness of the solution. Let us restrict this study to the affine autonomous case.

2.1.1. The affine autonomous case: global uniqueness. Before going into details, let us introduce some other notation. Let us define by $L(X,X)$ the space of bounded linear operators $F: X \to X$ equipped with the norm $\|F\|_{L(X,X)} = \sup_{\|x\|_X=1}\|Fx\|_X$. Moreover, let us consider the renewal function U_Φ and let us define its left-continuous inverse on $u>0$:

$$U_\Phi^{\leftarrow}(u) = \min\{x>0 : U_\Phi(x) \ge u\}.$$

It is not difficult to show that, being U_Φ continuous, $U_\Phi(U_\Phi^{\leftarrow}(u)) = u$.
Now we want to focus on abstract Cauchy problems of the form

$$\begin{cases} \partial^\Phi f(t) = \xi + Ff(t) & t \in [0,T] \\ f(0) = f_0, \end{cases} \tag{2.1.3}$$

where $F \in L(X,X)$ and $\xi, f_0 \in X$. For such kind of problems, we can show the following Proposition (see [**15**, Corllary 3.7]).

Proposition 2.1.4. *Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at infinity with index $\gamma \in (0,1)$, $F \in L(X,X)$ and $\xi, f_0 \in X$. Then there exists a time horizon $T>0$ depending only on $\|\xi\|_X, \|f_0\|_X$ and $\|F\|_{L(X,X)}$ such that the problem* (2.1.3) *admits a unique continuous solution $f \in C(J,X)$ where $J=[0,T]$.*

Proof. Let us consider the Picard operator A_Φ and fix $R = \|f_0\|_X + 1$. Arguing as in Lemma 2.1.3 we have, for any $T>0$, $t \in [0,T)$ and $\delta>0$ such that $t+\delta \in (0,T]$,

$$\|A_\Phi f(t+\delta) - A_\Phi f(t)\|_X \le (\|\xi\|_X + \|F\|_{L(X,X)} R)(U_\Phi(t+\delta) - U_\Phi(t) + 2U_\Phi(\delta))$$

and then, sending $\delta \to 0^+$, we have that $A_\Phi f \in C([0,T];X)$.
Concerning boundedness, we have

$$\|A_\Phi f(t)\|_X \le (\|\xi\|_X + \|F\|_{L(X,X)} R) U_\Phi(T);$$

thus, if we consider

$$T = U_\Phi^{\leftarrow}\left(\frac{R}{2(\|\xi\|_X + \|F\|_{L(X,X)} R)}\right),$$

we have

$$\|A_\Phi f(t)\|_X < R.$$

Now let us show that all the other quantities involved in the proof of Theorem 2.1.2 can be chosen only depending on T. To do this, let us first observe that setting $\varepsilon_1 = \frac{\gamma}{2}$, there exists a constant $C(T)$ such that $u(t) \le C(T)t^{\frac{\gamma-2}{2}}$ for any $t \in [0,T]$. Then, to conclude the proof, one can fix

$$p = \frac{4-\gamma}{2(2-\gamma)},$$

$$\eta_* = \left(2C(T)\left\|F\right\|_{L(X,X)}\left(\frac{4}{8-\gamma}\right)^{\frac{2(2-\gamma)}{4-\gamma}} T^{\frac{(2-\gamma)(8-\gamma)}{2(4-\gamma)}}\left(\frac{\gamma}{4-\gamma}\right)^{\frac{4-\gamma}{\gamma}}\right)^{\frac{\gamma}{4-\gamma}}$$

to achieve, for any $f, g \in C(J; B_R)$,

$$\left\|A_\Phi f - A_\Phi g\right\|_{\eta_*} \le \frac{1}{2}\left\|f-g\right\|_{\eta_*}$$

and conclude the proof. □

REMARK 2.1.5. After we fixed $T>0$, Theorem 2.1.2 guarantees $(\gamma-\varepsilon)$-Hölder regularity of the solution for any $\varepsilon \in (0,\gamma)$.

Thus, if the linear operator is continuous, any affine autonomous Cauchy problem admits a local solution. However, we want to show something more: indeed, we did not need to chose T_1 after we already had $T>0$ and then we can ask whenever the existence interval can be extended. For the classical fractional differential equations, such problem is considered in [**18**, Corollary 2] and then reconsidered in [**15**, Corollary 3.9].
Before giving the proof, we need to introduce some other operators. Let us denote:

$${}_{t_0}\mathcal{D}^\Phi f(t) = \frac{d}{dt}\int_{t_0}^t \bar{\nu}_\Phi(t-\tau)f(\tau)d\tau$$

and ${}_{t_0}\partial^\Phi f(t) = {}_{t_0}\mathcal{D}^\Phi(f(t)-f(t_0))$. For this kind of non-local derivative, we can show again Theorem 2.1.2 using t_0 in place of 0 for the initial datum.
Concerning the link between ∂^Φ and ${}_{t_0}\partial^\Phi$, let us observe that, for $t>t_0$,

$$\begin{aligned}
\partial^\Phi f(t) &= \frac{d}{dt}\left(\int_0^{t_0}\bar{\nu}_\Phi(t-\tau)(f(\tau)-f(0))d\tau + \int_{t_0}^t \bar{\nu}_\Phi(t-\tau)(f(\tau)-f(0))d\tau\right)\\
&= \frac{d}{dt}\left(\int_0^{t_0}\bar{\nu}_\Phi(t-\tau)(f(\tau)-f(0))d\tau + \int_{t_0}^t \bar{\nu}_\Phi(t-\tau)(f(\tau)-f(t_0))d\tau\right)\\
&\quad + (f(t_0)-f(0))\bar{\nu}_\Phi(t-t_0)\\
&= \frac{d}{dt}\int_0^{t_0}\nu_\Phi(t-\tau)(f(\tau)-f(0))d\tau + {}_{t_0}\partial^\Phi f(t) + (f(t_0)-f(0))\bar{\nu}_\Phi(t-t_0).
\end{aligned}$$

Suppose $f \in C([0,+\infty); B_R)$ is a solution of (2.1.3). Let us set

$$g(t) := \frac{d}{dt}\int_0^{t_0}\bar{\nu}_\Phi(t-\tau)(f(\tau)-f(0))d\tau + (f(t_0)-f(0))\bar{\nu}_\Phi(t-t_0) = \partial^\Phi f(t) - {}_{t_0}\partial^\Phi f(t).$$

Then we have that f solves also

$${}_{t_0}\partial^\Phi f(t) = \xi + Ff(t) - g(t) \qquad t \in [t_0, t_0+\Delta T]$$

which, for $f(t_0)$ assigned, admits a unique solution for a certain ΔT. This is actually the spirit of the proof of [**15**, Corollary 3.9] which ensures uniqueness of global solutions:

COROLLARY 2.1.6. *Under the hypotheses of Proposition 2.1.4, if $\nu_\Phi(dt)$ is absolutely continuous with respect to the Lebesgue measure and $f \in C([0,+\infty), X)$ is a solution of* (2.1.3)*, then it is the unique solution.*

REMARK 2.1.7. Observe that $g(t)$ is uniquely determined by the values of $f(t)$ for $t \in [0, t_0]$.

The same result in the non-linear case is given in [**15**, Proposition 6.7]

2.2. Generalized Grönwall inequality for special non-local time derivatives

Now let us consider $X = \mathbb{R}$. Let us recall that, for classical ODEs, an indispensable tool to show continuity with respect to initial datum and other parameters is what is usually called Grönwall's Inequality (see [**11**]). In the easiest case, such inequality compares the behaviour of a solution of an integral inequality with the behaviour of the exponential functions (which are eigenfunctions of the classical derivative). In [**149**] a similar inequality has been shown in the context of fractional differential equations, where the *comparing function* is the Mittag-Leffler function (which is the eigenfunction of the Caputo-Dzhrbashyan derivative). Thus we expect, if we want to generalize Grönwall's Inequality to the non-local case, that the *comparing functions* will be given by $\mathfrak{e}_\Phi(t;\lambda)$, which are the eigenfunctions of the generalized Caputo-Dzhrbashyan type derivatives.
To do this, we first need to introduce the following special functions (see [**86**]).

DEFINITION 2.2.1. Let $\alpha \in \mathbb{H}$ and $\beta \in \mathbb{C}$. Then the **two-parameters Mittag-Leffler function** is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, \qquad t \in \mathbb{C}\,.$$

Let us observe that $E_{\alpha,1} = E_\alpha$.

The main theorem of this section is the following (see [**15**, Theorem 5.1]).

THEOREM 2.2.1. *Fix $T > 0$ and let $a, g \in L^1([0,T])$ be non-negative functions with g non-decreasing and consider $f \in L^1([0,T])$. Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at ∞ with index $\gamma \in (0,1)$. Suppose the following integral inequality holds:*

$$f(t) \le a(t) + g(t)\,\mathcal{I}^\Phi f(t) \qquad t \in [0,T]. \tag{2.2.1}$$

Let us denote by B^0 the identity operator and define, for any function $x \in L^1([0,T])$,

$$Bx(t) = g(t)\int_0^t u_\Phi(t-s)x(s)ds,$$

where u_Φ is the potential density. Then:

- *For any $t \in [0,T]$, it holds*

$$f(t) \le \sum_{k=0}^{+\infty} B^k a(t);$$

- *For any $\varepsilon \in (0,\gamma)$, setting $\beta = \gamma - \varepsilon$ there exists a constant C_2 non depending on f such that, for any $t \in [0,T]$,*

$$f(t) \le a(t) + C_2\Gamma(\beta)g(t)\int_0^t E_{\beta,\beta}(C_2\Gamma(\beta)g(T)(t-s))(t-s)^{\beta-1}a(s)ds;$$

- *If a is non-decreasing we have, for any $t \in [0,T]$,*

$$f(t) \le a(t)\,\mathfrak{e}_\Phi(t, g(T)).$$

To show this theorem, we first need to stress out some properties of the operator B (see [**15**, Lemmas 5.2 to 5.6]).

LEMMA 2.2.2. *Fix $T > 0$ and let $g \in L^1([0,T])$ be non-negative and non-decreasing. Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at ∞ with index $\gamma \in (0,1)$. Then:*

(1) *If $f_1, f_2 \in L^1([0,T])$ are such that $f_1(t) \le f_2(t)$ almost everywhere in $[0,T]$, then $Bf_1(t) \le Bf_2(t)$ for any $t \in [0,T]$;*
(2) *For any $\varepsilon \in (0,\gamma)$, setting $\beta = \gamma - \varepsilon$, there exists a constant $C_2 > 0$ such that for any $k \ge 1$ and any non-negative function $f \in L^1([0,T])$ it holds, for any $t \in [0,T]$,*

$$B^k f(t) \le \frac{(C_2\Gamma(\beta)g(t))^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}f(s)ds; \tag{2.2.2}$$

(3) *For any $k \ge 1$ and $t \in [0,T]$, it holds*

$$B^k 1(t) \le (g(t))^k u_k^*(t) \tag{2.2.3}$$

where $u_k^*(t)$ are defined in (1.4.12);
(4) *For any function $f \in L^1([0,T])$ the series $\sum_{k=1}^{+\infty} B^k f(t)$ normally converges for any $t \in [0,T]$;*
(5) *For any function $f \in L^1([0,T])$ it holds $\lim_{k\to+\infty} B^k f(t) = 0$ uniformly in $[0,T]$;*
(6) *For any non-negative functions $f_i \in L^1([0,T])$ with $i = 1,2$ such that f_1 is non-decreasing, it holds $B^k(f_1f_2)(t) \le f_1(t)B^k(f_2)(t)$ for any $t \in [0,T]$.*

PROOF. Concerning property (1), it is obvious since g and u_Φ are non-negative. Let us show property (2) by induction. First of all, let us observe that, by Proposition 1.3.15, we know that, for fixed $\varepsilon \in (0,\gamma)$, there exists a constant $C_2 > 0$ such that $u(t) \le C_2 t^{\gamma-1-\varepsilon} = C_2 t^{\beta-1}$ for any $t \in [0,T]$. Then we have

$$Bf(t) = g(t)\int_0^t u_\Phi(t-s)f(s)ds \le C_2 g(t)\int_0^t (t-s)^{\beta-1}f(s)ds.$$

Now let us suppose that (2.2.2) holds true for some $k \ge 1$. Then, by property (1) and Fubini's theorem, we have

$$\begin{aligned}
B^{k+1}f(t) = B(B^k f(t)) &\le B\left(\frac{(C_2 g(\cdot)\Gamma(\beta))^k}{\Gamma(k\beta)}\int_0^{\cdot}(\cdot - s)^{\beta-1}f(s)ds\right)(t) \\
&= g(t)\frac{(C_2\Gamma(\beta))^k}{\Gamma(k\beta)}\int_0^t u_\Phi(t-s)g(s)\int_0^s (s-\tau)^{k\beta-1}f(\tau)d\tau ds \\
&\le \frac{(C_2 g(t)\Gamma(\beta))^{k+1}}{\Gamma(\beta)\Gamma(k\beta)}\int_0^t (t-s)^{\beta-1}\int_0^s (s-\tau)^{k\beta-1}f(\tau)d\tau ds \\
&= \frac{(C_2 g(t)\Gamma(\beta))^{k+1}}{\Gamma(\beta)\Gamma(k\beta)}\int_0^t f(\tau)\int_\tau^s (t-s)^{\beta-1}(s-\tau)^{k\beta-1}ds d\tau \\
&= \frac{(C_2 g(t)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)}\int_0^t f(\tau)(t-\tau)^{(k+1)\beta-1}d\tau.
\end{aligned}$$

Let us also show property (3) by induction. Indeed we have

$$B1(t) = g(t)\int_0^t u_\Phi(t-s)ds = g(t)U_\Phi(t).$$

Now let us suppose (2.2.3) holds true for some $k \ge 1$. Then we have, by property (1)

$$\begin{aligned}
B^{k+1}1(t) = B(B^k 1(\cdot))(t) &\le B((g(\cdot))^k u_k^*(\cdot))(t) \\
&= g(t)\int_0^t u_\Phi(t-s)(g(s))^k u_k^*(s)ds \le (g(t))^{k+1}u_{k+1}^*(t).
\end{aligned}$$

Now let us show property (4). First of all, observe that $|Bf(t)| \le B|f|(t)$. Thus we can consider $f \ge 0$ without loss of generality. Then we have, by property (2),

$$\begin{aligned}
\sum_{k=1}^{+\infty} B^k f(t) &\le \sum_{k=1}^{+\infty}\frac{(C_2\Gamma(\beta)g(t))^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}f(s)ds \\
&\le \frac{\|f\|_{L^1([0,T])}}{T}\sum_{k=1}^{+\infty}\frac{(C_2\Gamma(\beta)g(T)T^\beta)^k}{\Gamma(k\beta)} < +\infty.
\end{aligned}$$

Thus we have normal convergence of the series $\sum_{k=0}^{+\infty} B^k f(t)$ for any $t \in [0,T]$. Moreover, this implies $B^k f \to 0$ as $k \to +\infty$ uniformly in $[0,T]$, thus showing also property (5).
Finally, let us show property (6) by induction. We have

$$B(f_1 f_2)(t) \le g(t)f_1(t)\int_0^t u_\Phi(t-s)f_2(s)ds = f_1(t)B(f_2)(t).$$

Now suppose that for some $k \ge 1$ it holds $B^k(f_1 f_2) \le f_1 B^k(f_2)$. Then we have, by property (1),

$$B^{k+1}(f_1 f_2) = B(B^k(f_1 f_2)) \le B(f_1 B^k(f_2)) \le f_1 B^{k+1}(f_2).$$

□

Now we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Let us first rewrite (2.2.1) as

$$f(t) \le a(t) + Bf(t). \tag{2.2.4}$$

We want to show that, for any $n \in \mathbb{N}$ and $t \in [0,T]$,

$$f(t) \le a(t) + \sum_{k=1}^{n-1} B^k a(t) + B^n f(t). \tag{2.2.5}$$

Let us show it by induction. For $n = 1$ we obtain again equation (2.2.1). Suppose inequality (2.2.5) holds for some $n \in \mathbb{N}$. Then applying B^n on both sides of (2.2.4) and using property (1) of the previous Lemma, we obtain

$$B^n f(t) \le B^n a(t) + B^{n+1} f(t). \tag{2.2.6}$$

Using this inequality in (2.2.5) we obtain the proof.
Now let us observe that, being $a, f \in L^1([0,T])$, by properties (4) and (5) of the previous Lemma we obtain $\sum_{k=1}^{+\infty} B^k a(t) < +\infty$ and $\lim_{n\to+\infty} B^n f(t) = 0$, thus, taking the limit as $n \to +\infty$ in inequality (2.2.5), we obtain

$$f(t) \le a(t) + \sum_{k=1}^{+\infty} B^k a(t), \tag{2.2.7}$$

that is the first assertion. By properties (3) and (4) of the previous Lemma, we also have

$$\begin{aligned}\sum_{k=1}^{+\infty} B^k a(t) &\le \int_0^t \sum_{k=0}^{+\infty} \frac{(C_2\Gamma(\beta)g(t)(t-s)^\beta)^{k+1}}{\Gamma(k\beta+\beta)}(t-s)^{\beta-1}a(s)ds \\ &\le C_2\Gamma(\beta)g(t)\int_0^t E_{\beta,\beta}(C_2\Gamma(\beta)g(T)(t-s))(t-s)^{\beta-1}a(s)ds,\end{aligned}$$

obtaining the second assertion.
Concerning the third one, let us reconsider (2.2.7) and, by using properties (3) and (6) of the previous Lemma, we obtain

$$B^k a(t) \le a(t)B^k 1(t) \le a(t)(g(T))^k u_k^*(t).$$

Thus, by Equation (1.4.14), we have

$$f(t) \le a(t)\left(1 + \sum_{k=1}^{+\infty} g(T)^k u_k^*(t)\right) = a(t)\,\mathfrak{e}_\Phi(t,(g(T))),$$

concluding the proof. □

2.2.1. Consequences of the Generalized Grönwall Inequality. Now let us take into account some consequences of the generalized Grönwall Inequality. First of all, let us show the following bound on the distance between the solutions.

Proposition 2.2.3. *Let $F_i : [0,T_i] \times \mathbb{R} \to \mathbb{R}$ with $i = 1,2$ and $\Phi \in \mathcal{SBF}$ a driftless Bernstein function such that hypotheses of Theorem 2.1.2 are verified. Let $f_i : [0,T_i] \to \mathbb{R}$ ($i = 1,2$) be solutions of*

$$\begin{cases}\partial^\Phi f_i(t) = F_i(t, f_i(t)) & t \in (0,T_i] \\ f_i(0) = f_0^i\end{cases} \tag{2.2.8}$$

with $\|f_i\|_{C([0,T_i])} \le R$ *for some* $R > \max\{|f_0^1|, |f_0^2|\}$.
Set $T = \min\{T_1, T_2\}$ *and suppose that for any* $R > 0$ *there exists a constant* $M_R > 0$ *such that*

$$|F_1(t,x) - F_2(t,x)| \le M_R \ \forall t \in [0,T], \ \forall x \in [-R,R]$$

Then it holds

$$\|f_1 - f_2\|_{C([0,T])} \le (|f_0^1 - f_0^2| + M_R U_\Phi(T))\, \mathfrak{e}_\Phi(T, L_R).$$

Proof. Define $h : t \in [0,T] \mapsto |f_1(t) - f_2(t)| \in \mathbb{R}$. As f_i is solution of (2.2.8) and $f_i(t) \le R$ for $t \in [0,T]$, we have

$$h(t) \le |f_0^1 - f_0^2| + \int_0^t u_\Phi(t-s)|F_1(s, f_1(s)) - F_2(s, f_2(s))| ds.$$

Now, since $f_i(s) \in [-R, R]$, we have

$$|F_1(s, f_1(s)) - F_2(s, f_2(s))| \le L_R h(s) + M_R$$

and then

$$h(t) \le |f_0^1 - f_0^2| + M_R U_\Phi(t) + L_R \mathcal{I}^\Phi h(t).$$

Finally, by Theorem 2.2.1, being U_Φ non-decreasing, we conclude the proof. □

With the same spirit, we can now investigate continuous dependence with respect to the initial datum and some parameters in a certain *parameter space*. To do this, let us first stress out that if F and Φ satisfy the hypotheses of Theorem 2.1.2, then the *guaranteed* existence and uniqueness interval is given by $[0,T]$ where $T = U_\Phi^{\leftarrow}\left(\frac{R}{C_{\widetilde{R}}}\right)$ where $R > |f_0|$ and $\widetilde{R} = R + |f_0| + 1$.

2.2.1.1. *Continuous dependence on the initial datum.* Let us first investigate the continuous dependence on the initial datum. To do this, let us first observe that if we fix $f_0 \in \mathbb{R}$ and $\delta \in (0,1)$, then there exists a common interval of *guaranteed* existence as the initial datum varies in $(f_0 - \delta, f_0 + \delta)$ (see [**15**, Proposition 6.1]).

Lemma 2.2.4. *Let* $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfy the hypotheses of Theorem 2.1.2. Fix* $R = |f_0| + 1$ *and* $\delta \in (0,1)$. *Then there exists* $T_1 > 0$ *such that for any* $\widetilde{f}_0 \in (f_0 - \delta, f_0 + \delta)$ *the problem*

$$\begin{cases} \partial^\Phi f(t) = F(t, f(t)) & t \in (0, T_1], \\ f(t) = \widetilde{f}_0 \end{cases} \tag{2.2.9}$$

admits a unique solution in $\bigcap_{\varepsilon \in (0,\gamma)} C^{\gamma - \varepsilon}([0, T_1])$.

Proof. We only have to observe that $|\widetilde{f}_0| \le |f_0| + \delta < |f_0| + 1 = R$, thus we can choose $T_1 = U_\Phi^{\leftarrow}\left(\frac{R}{C_R}\right)$. □

Remark 2.2.5. Here we considered only $\delta \in (0,1)$, but, obviously, we can *revert* in some sense the argument. Indeed, consider a compact set $K \subseteq \mathbb{R}$. Without loss of generality we can consider $K = [a,b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Then define $f_0 = \frac{a+b}{2}$, $\delta = \frac{b-a}{2}$ and $R > |f_0| + \delta = b$. Then we can choose $T_1 = U_\Phi^{\leftarrow}\left(\frac{R}{C_R}\right)$ for such choice of R to obtain that the Cauchy problem (2.2.9) admits solution in $[0, T_1]$ for any $\widetilde{f}_0 \in [a,b]$.

Now we can show the continuity of the solutions with respect to initial datum, as done in [**15**, Proposition 6.2].

PROPOSITION 2.2.6. *Fix $f_0 \in \mathbb{R}$ and $\delta \in (0,1)$ and set $R = |f_0|+1$ and $T > 0$ as in Lemma 2.2.4. Suppose $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $\Phi \in \mathcal{SBF}$ satisfy the hypotheses of Theorem 2.1.2. Define the function $\Psi : \widetilde{f}_0 \in (f_0-\delta, f_0+\delta) \to \Psi(\cdot;\widetilde{f}_0) \in C^0([0,T])$ where*

$$\begin{cases} \partial^\Phi \Psi(t;\widetilde{f}_0) = F(t,\Psi(t;\widetilde{f}_0)) & t \in [0,T] \\ \Psi(0;\widetilde{f}_0) = \widetilde{f}_0. \end{cases}$$

Then Ψ is Lipschitz with constant $L_\Psi \le \mathfrak{e}_\Phi(T;L_R)$.

PROOF. Fix $f_0^1, f_0^2 \in (f_0-\delta, f_0+\delta)$ and define $h(t) = |\Psi(t;f_0^1) - \Psi(t;f_0^2)|$. Then we have

$$h(t) \le |f_0^1 - f_0^2| + \int_0^t u_\Phi(t-s)|F(s,\Psi(s;f_0^1)) - F(s,\Psi(s;f_0^2))|ds.$$

Now, by our choice of T, we have that $|\Psi(s;f_0^i)| \le R$, thus we obtain

$$h(t) \le |f_0^1 - f_0^2| + L_R\,\mathcal{I}^\Phi\, h(t).$$

By using the third part of Theorem 2.2.1 and the fact that $\mathfrak{e}_\Phi(t,L_R)$ is increasing in t we conclude the proof. $\square$

2.2.1.2. *Continuous dependence on a parameter.* Now let us suppose the non-local Cauchy problem we are considering depends on some additional parameters, varying in some metric space. As before, we first want to show that we can choose a constant $T > 0$ such that the solutions of the parametric Cauchy problem exists in $[0,T]$ as the parameters belong to some set (see [**15**, Proposition 6.4]).

LEMMA 2.2.7. *Let (V,d) be a metric space, $F : [0,T]\times\mathbb{R}\times V \to \mathbb{R}$ and $\Phi \in \mathcal{SBF}$. Suppose, for any fixed $v \in V$, $F(\cdot,\cdot;v)$ and Φ satisfy the hypotheses of Theorem 2.1.2. Moreover, suppose that for any fixed $R, r > 0$ and $v_0 \in V$, there exists a constant $L(r,R,v_0)$ such that $\forall v \in B_r(v_0)$ it holds*

$$|F(t,x;v) - F(t,x;v_0)| \le L(r,R,v_0)d(v,v_0) \qquad \forall x \in [-R,R],\ \forall t \in [0,T].$$

Fix $v_0 \in V$. Then there exists a constant $T_1(r,f_0,v_0) > 0$ such that the Cauchy problem

$$\begin{cases} \partial^\Phi f(t) = F(t,f(t);v) & t \in (0,T_1], \\ f(t) = f_0 \end{cases} \tag{2.2.10}$$

admits a unique solution for any $v \in B_R(v_0)$.

PROOF. Fix $R = |f_0| + 1$ and consider $x \in [-R,R]$. Then we have, by the hypotheses of Theorem 2.1.2,

$$|F(t,x;v_0)| \le C_R(v_0).$$

Now consider any $v \in B_r(v_0)$. Then we have

$$|F(t,x;v)| \le |F(t,x;v_0)| + |F(t,x;v) - F(t,x;v_0)| \le C_R(v_0) + rL(r,R,v_0).$$

Thus, the constant T_1 we are searching for is given by

$$T_1 = U_\Phi^{\leftarrow}\left(\frac{R}{C_R(v_0) + rL(r,R,v_0)}\right).$$

$\square$

Now we are ready to show continuity with respect to the parameters in (V,d) (see [**15**, Proposition 6.5]).

PROPOSITION 2.2.8. *Let (V,d) be a metric space, $F:[0,T]\times\mathbb{R}\times V\to\mathbb{R}$ and $\Phi\in\mathcal{SBF}$. Suppose, for fixed $v\in V$, $F(\cdot,\cdot;v)$ and Φ satisfy the hypotheses of Theorem 2.1.2. Moreover, suppose that for any fixed $R,r>0$ and $v_0\in V$, there exists a constant $L(r,R,v_0)$ such that $\forall v\in B_r(v_0)$ it holds*

$$|F(t,x;v)-F(t,x;v_0)|\le L(r,R,v_0)d(v,v_0) \qquad \forall x\in[-R,R],\ \forall t\in[0,T].$$

Fix $v_0\in V$, $r>0$ and $f_0\in\mathbb{R}$. Consider $R=|f_0|+1$ and let $T_1>0$ as in Lemma 2.2.7. Define the function $\Psi: v\in B_r(v_0)\mapsto\Psi(\cdot;v)\in C^0([0,T_1])$ where

$$\begin{cases}\partial^\Phi\Psi(t;v)=F(t,\Psi(t;v);v) & t\in(0,T_1]\\ \Psi(0;v)=f_0.\end{cases} \tag{2.2.11}$$

Then Ψ is continuous in v_0. In particular, it holds

$$\|\Psi(\cdot;v)-\Psi(\cdot;v_0)\|_{C([0,T_1])}\le L(r,R,v_0)U_\Phi(T_1)\,\mathfrak{e}_\Phi(T_1;L_R)d(v,v_0), \qquad \forall v\in B_r(v_0).$$

PROOF. For any $v\in B_r(v_0)$, define the function

$$h(t)=|\Psi(t;v)-\Psi(t;v_0)|$$

and observe that

$$h(t)\le\int_0^t u_\Phi(t-s)|F(s,\Psi(s;v);v)-F(s,\Psi(s;v_0);v_0)|ds.$$

By definition of $T_1>0$ we have that $\Psi(s;v)\in[-R,R]$ for any $s\in[0,T_1]$ and $v\in B_r(v_0)$, thus we obtain

$$|F(s,\Psi(s;v);v)-F(s,\Psi(s;v_0);v_0)|\le L(r,R,v_0)d(v,v_0)+L_Rh(s).$$

Hence we achieve

$$h(t)\le L(r,R,v_0)d(v,v_0)U_\Phi(T_1)+L_R\,\mathcal{I}^\Phi\,h(t).$$

The third part of Theorem 2.2.1 and the fact that $\mathfrak{e}_\Phi(t;L_R)$ is increasing conclude the proof. $\square$

2.3. Non-local Cauchy problems in ℓ^2 and birth-death polynomials

In Section 2.1 we gave some conditions to obtain local existence and uniqueness of solutions of non-local abstract Cauchy problems in the form (2.1.1) (hence also in the non-linear case) for some Bernstein function $\Phi\in\mathcal{SBF}$ that is regularly varying at infinity. However, in Subsection 1.4.2 we studied the eigenvalue problem for the non-local convolutionary derivative ∂^Φ for any $\Phi\in\mathcal{BF}$. Thus, one could ask if such results on the eigenfunctions of ∂^Φ can be used to obtain solutions of problems of the form (2.1.1) where the right-hand side of the equation is a linear function. This is done, for instance, in the local case by means of Fourier series for the heat equation, which, after giving an L^2 initial datum, can be seen as an abstract Cauchy problem in a L^2 space.
In this section we will consider some non-local difference-differential equations that can be seen as abstract Cauchy problems in some ℓ^2 spaces. In particular such Cauchy problem arise from a specific class of birth-death processes that we will call *solvable*, and are strictly linked to classical orthogonal polynomials of discrete variable. In particular, we will show existence and uniqueness of the solutions

of these particular equations by means of a spectral decomposition in terms of suitable orthogonal polynomials and then, in the next section, we will provide some stochastic representation of the solutions.

2.3.1. Solvable birth-death processes. The theory of birth-death polynomials and spectral decomposition of birth-death processes is presented in the seminal papers [**80, 82**]. Here we focus on a particular class of birth-death processes whose spectral measure is actually the invariant measure and the spectrum of the generator is purely discrete, real and non-positive.
Let us first introduce some notation. Let $E \subseteq \mathbb{N}_0$ be a finite or at most countable set and $N(t)$ a time-homogeneous continuous-time Markov chain with state space E. Let us denote by

$$p(t,x;y) = \mathbb{P}(N(t+s) = x | N(s) = y), \qquad x, y \in E, \ t \geq 0$$

the transition probabilities and $P(t) = (p(t,x;y))_{x,y \in E}$ the transition matrix. Then $P(t)$ can be seen as a semigroup acting on a suitable Banach sequence space $\mathfrak{b}$ and we can consider its generator $\mathcal{G}$. $N(t)$ is a birth-death process if and only if there exist two non-negative functions $b, d : E \to \mathbb{R}_0^+$ such that for any $f \in D(\mathcal{G})$

$$\mathcal{G} f(x) = (b(x) - d(x))\delta^+ f(x) + d(x)\delta^2 f(x), \qquad x \in E,$$

where

$$\delta^+ f(x) = f(x+1) - f(x) \qquad\qquad \delta^- f(x) = f(x) - f(x-1)$$

and

$$\delta^2 f(x) = \delta^- \delta^+ f(x) = \delta^+ \delta^- f(x) = f(x+1) - 2f(x) + f(x-1).$$

Let us observe that $\delta^+ = \mathrm{D}^1_1$, $\delta^- = \mathrm{D}^1_{-1}$ and $\delta^2 = \mathrm{D}^2_1$, according to the notation introduced in Section 1.5. The functions b and d are called respectively **birth** and **death rates**. Moreover, if we suppose $N(t)$ is irreducible, then E has to be a segment in $\mathbb{N}$ (i.e. if $n_1, n_2 \in E$ and $n_1 \leq n \leq n_2$, then $n \in E$). We can always suppose that $\min E = 0$ and $N(t)$ does not admit a cemetery (i.e. $d(0) = 0$). In particular, we define the following class of birth-death processes (as done in [**22**])

Definition 2.3.1. We say that a birth-death process $N(t)$ is **solvable** if

- $N(t)$ is irreducible and recurrent;
- $N(t)$ admits an invariant and stationary measure $\mathbf{m}$ on E;
- The function $m(x) = \mathbf{m}(\{x\})$ solves the discrete Pearson equation

$$\delta^+(d(\cdot)m(\cdot))(x) = (b(x) - d(x))m(x) \qquad x \in E; \tag{2.3.1}$$

- d is a polynomial of degree at most 2 and $b - d$ is a polynomial of degree at most 1;
- $\mathcal{G}$ is a diagonalizable operator with non-positive eigenvalues $(\lambda_n)_{n \in E}$, such that $\lambda_0 = 0$ and $\lambda_n < 0$ for any $n \geq 1$, and its eigenfunctions $(P_n)_{n \in E}$ are classical orthogonal polynomials of discrete variable whose orthogonality measure is actually $\mathbf{m}$.

Observe that any invariant measure m satisfies Equation (2.3.1). We refer to it as discrete Pearson equation since d and b are polynomials (in analogy to the Pearson equation, see [**62**].)

First of all, let us remark that if $N(t)$ is a solvable birth-death process, then the eigenvalues of $\mathcal{G}$ can be obtained by using the formula

$$\lambda_n = n\delta^+(b(\cdot)-d(\cdot))(x) + \frac{1}{2}n(n-1)\delta^2 d(x), \tag{2.3.2}$$

where, since d is a polynomial with degree at most 2 and $b-d$ is a polynomial with degree at most 1, $\delta^2 d(x)$ and $\delta^+(b-d)(x)$ do not depend on x. Concerning the semigroup $P(t)$, it can be seen as defined on $\ell^2(\mathbf{m})$ and then $D(\mathcal{G}) = \ell^2(\mathbf{m})$. Moreover, if $E = \mathbb{N}_0$, let us observe that equation (2.3.1) can be rewritten as

$$d(x+1)m(x+1) = b(x)m(x), \tag{2.3.3}$$

hence it is easy to see that a solution m exists if and only if

$$\sum_{x=0}^{+\infty}\prod_{k=0}^{x}\frac{b(k)}{d(k+1)} < +\infty. \tag{2.3.4}$$

In particular, since both b and d must be polynomials, then $\lim_{x\to+\infty}\frac{b(x)}{d(x+1)}$ exists and condition (2.3.4) implies that

$$\lim_{x\to+\infty}\frac{b(x)}{d(x+1)} \le 1.$$

However, we can easily exclude the cases in which $\lim_{x\to+\infty}\frac{b(x)}{d(x+1)} = 1$. Indeed this could happen if and only if $b(x)$ and $d(x)$ are polynomials of the same degree and with the same director coefficient. However, if b and d are polynomials of degree 0 or 1, then we should have $\lambda_n = 0$ for any $n \in \mathbb{N}_0$, which is absurd. If b and d are polynomials of degree 2, then, since we need $\lambda_n < 0$ for any $n \ge 1$, the director coefficient of d must be negative, which is absurd since in such case there exists $x_0 \in E$ for which, for any $x \ge x_0$, it holds $d(x) < 0$. Thus we conclude that

$$\lim_{x\to+\infty}\frac{b(x)}{d(x+1)} < 1. \tag{2.3.5}$$

Concerning the orthogonal polynomials P_n, let us observe that the orthogonality relation can be written as

$$\sum_{x\in E} P_n(x)P_m(x)m(x) = \mathfrak{d}_n^2\,\delta_{n,m}$$

where $\mathfrak{d}_n = \|P_n\|_{\ell^2(\mathbf{m})}$ and $\delta_{n,m}$ is the Kronecker delta symbol. The orthonormal polynomials will be denoted as $Q_n = \frac{P_n}{\mathfrak{d}_n}$. Moreover, the function $\widetilde{m}(x) = \frac{1}{\mathfrak{d}_x^2}$ defined on E induces a measure on E. Then, by using Gram-Schmidt orthogonalization procedure on the monomials $(x^n)_{n\in E}$, one obtains the family of polynomials $\widetilde{P}_n(x)$ that are orthogonal with respect to the measure $\widetilde{\mathbf{m}}$ on E defined by $\widetilde{\mathbf{m}}(\{x\}) = \widetilde{m}(x)$. In particular it holds, for any $n, m \in E$,

$$\sum_{x\in E}\widetilde{P}_n(x)\widetilde{P}_m(x)\widetilde{m}(x) = \frac{1}{m(n)}\delta_{n,m}.$$

For this reason, the family of polynomials $(\widetilde{P}_n)_{n\in E}$ is called the **dual family** of $(P_n)_{n\in E}$ (see [**113**]). We say that a family of classical orthogonal polynomials of discrete variable $(P_n)_{n\in E}$ is self-dual if the dual family coincides with $(P_n)_{n\in E}$ and $P_n(x) = P_x(n)$ for any $n, x \in E$.

We can also introduce the **forward operator**, that for a birth-death process $N(t)$ is given by

$$\mathcal{F} f(x) = -\delta^-[(b(\cdot) - d(\cdot))f(\cdot)](x) + \delta^2[d(\cdot)f(\cdot)](x). \tag{2.3.6}$$

An interesting property concerning solvable birth-death processes is that the forward operator admits the same eigenvalues of the generator and the eigenfunctions are given by mQ_n (as shown in [**22**, Lemma 2.1]):

LEMMA 2.3.1. *Let $N(t)$ be a solvable birth-death process, $\mathcal{F}$ be its forward operator and $\mathbf{m}$ be its stationary measure. Let $(Q_n)_{n\in E}$ be the associated family of orthonormal polynomials. Then, for any $n, x \in E$ it holds*

$$\mathcal{F}(m(\cdot)Q_n(\cdot))(x) = \lambda_n m(x)Q_n(x)$$

where $(\lambda_n)_{n\in E}$ are the eigenvalues of the generator $\mathcal{G}$.

PROOF. Before proceeding with the proof, let us recall the discrete Leibniz rule for the difference operators $\delta^{\pm}$:

$$\begin{aligned}
\delta^+(fg)(x) &= f(x+1)\delta^+g(x) + g(x)\delta^+f(x)\\
\delta^-(fg)(x) &= f(x)\delta^-g(x) + g(x-1)\delta^-f(x).
\end{aligned}$$

Moreover, let us use the notation $\delta_z^{\pm}$ and δ_z^2 to refer to the variable we are working on.

We have, by definition of $\mathcal{F}$

$$\begin{aligned}
\mathcal{F}(m(\cdot)Q_n(\cdot))(x) &= -\delta_z^-((b(z) - d(z))m(z)Q_n(z))(x) + \delta_z^2(d(z)m(z)Q_n(z))(x)\\
&= \delta_z^-[-(b(z) - d(z))m(z)Q_n(z) + \delta_y^+(d(y)m(y)Q_n(y))(z)](x).
\end{aligned}$$

Now we can use the discrete Leibniz rule for δ^+ to achieve

$$\begin{aligned}
\mathcal{F}(m(\cdot)Q_n(\cdot))(x) &= \delta_z^-[-(b(z) - d(z))m(z)Q_n(z) + Q_n(z)\delta_y^+(d(y)m(y))(z)\\
&\qquad + \delta^+Q_n(z)(d(z+1)m(z+1))](x)\\
&= \delta_z^-[\delta^+Q_n(z)(d(z+1)m(z+1))](x),
\end{aligned}$$

where the first inner summand is actually 0 by the discrete Pearson equation (2.3.1). Now let us use the discrete Leibniz rule for δ^- to achieve

$$\begin{aligned}
\mathcal{F}(m(\cdot)Q_n(\cdot))(x) &= \delta^2Q_n(x)(d(x)m(x))(x) + \delta^+Q_n(x)\delta_z^-(d(z+1)m(z+1))(x)\\
&= \delta^2Q_n(x)(d(x)m(x))(x) + \delta^+Q_n(x)\delta^+(d(\cdot)m(\cdot))(x)\\
&= m(x)\,\mathcal{G}\,Q_n(x) = \lambda_n m(x)Q_n(x),
\end{aligned}$$

where we also used the relation $\delta^+f(x) = \delta_z^-(f(z+1))(x)$ and, again, the discrete Pearson equation (2.3.1). $\square$

Concerning the existence of the moments of $N(t)$, in the stationary case we have that $N(t)$ admits moments of any order. This is obvious when E is finite. However, to show this property for $E = \mathbb{N}_0$, we need the following proposition.

PROPOSITION 2.3.2. *Let $N(t)$ be a solvable birth-death process with stationary measure $\mathbf{m}$ and state space $E = \mathbb{N}_0$. Then there exists a constant $\rho < 1$ and a state $x_0 \in E$ such that for any $x \geq x_0$ it holds*

$$m(x) \leq \rho^{x-x_0}m(x_0). \tag{2.3.7}$$

PROOF. Let us reconsider the discrete Pearson equation written as in (2.3.3):

$$m(x+1) = \frac{b(x)}{d(x+1)} m(x).$$

Let us recall that, by (2.3.5), it holds $\lim_{x\to+\infty} \frac{b(x)}{d(x+1)} = l < 1$. Fix $\rho \in (l,1)$ and observe that there exists a state $x_0 \in \mathbb{N}_0$ such that $\frac{b(x)}{d(x+1)} < \rho$ for any $x \geq x_0$. Then we obtain

$$m(x+1) < \rho m(x).$$

Thus, inequality (2.3.7) follows inductively from the previous one. □

From this Proposition we have directly the following Corollary.

COROLLARY 2.3.3. *Let $N(t)$ be a solvable birth-death process with stationary measure $\mathbf{m}$ such that $N(0)$ admits distribution $\mathbf{m}$. Then $N(t)$ admits moments of any order for any $t \geq 0$.*

The most important result concerning solvable birth-death processes is strictly linked with the backward and forward Kolmogorov equations. Let us first consider the backward one

$$\begin{cases} \frac{du}{dt}(t,y) = \mathcal{G}\, u(t,y) & t \geq 0,\ y \in E \\ u(0,y) = g(y) & y \in E \end{cases} \tag{2.3.8}$$

and let us denote $\mathfrak{u} : t \in [0,+\infty) \mapsto u(t,\cdot) \in \ell^2(\mathbf{m})$ the solution map. Let us recall the definition of strong solution of (2.3.8).

DEFINITION 2.3.2. We say that u is a **strong solution** of (2.3.8) if:

- $u(t,\cdot)$ belongs to $\ell^2(\mathbf{m})$ for any $t \geq 0$ (and then $\mathfrak{u}$ is well defined);
- $\mathfrak{u} \in C([0,+\infty); \ell^2(\mathbf{m})) \cap C^1((0,+\infty); \ell^2(\mathbf{m}))$;
- the equations in (2.3.8) hold pointwise.

The same definition holds for the forward problem

$$\begin{cases} \frac{dv}{dt}(t,x) = \mathcal{F}\, v(t,x) & t \geq 0,\ x \in E \\ v(0,x) = f(x) & x \in E. \end{cases} \tag{2.3.9}$$

For solvable birth-death processes the following spectral decomposition theorem holds.

THEOREM 2.3.4. *Let $N(t)$ be a solvable birth-death process with state space E, invariant measure $\mathbf{m}$, generator $\mathcal{G}$, forward operator $\mathcal{F}$ (with eigenvalues $(\lambda_n)_{n\in E}$) and associated family of orthonormal polynomials $(Q_n)_{n\in E}$. Then the following assertions hold true:*

- *The transition probability function $p(t,x;y)$ admits the following spectral representation:*

$$p(t,x;y) = m(x) \sum_{n\in E} e^{\lambda_n t} Q_n(x) Q_n(y) \qquad x,y \in E,\ t \geq 0;$$

- *If $g \in \ell^2(\mathbf{m})$ with decomposition $g(y) = \sum_{n\in E} g_n Q_n(y)$ for any $y \in E$ where $(g_n)_{n\in E}$ belongs to $\ell^2(E)$, then the Cauchy problem (2.3.8) admits a unique strong solution given by*

$$u(t,y) = \sum_{n\in E} e^{\lambda_n t} Q_n(y) g_n = \sum_{x\in E} p(t,x;y) g(x) \qquad y \in E,\ t \geq 0.$$

In particular, $p(t,x;y)$ is the fundamental solution of $\frac{\partial u}{\partial t}(t,y) = \mathcal{G}\, u(t,y)$. Moreover, denoting by $\mathbb{E}_y[\cdot] = \mathbb{E}[\cdot|N(0)=y]$, then one obtains

$$u(t,y) = \mathbb{E}_y[g(N(t))];$$

- *If $f/m \in \ell^2(\mathbf{m})$ with decomposition $f(x) = m(x)\sum_{n\in E} f_n Q_n(x)$ for any $x \in E$ where $(f_n)_{n\in E}$ belongs to $\ell^2(E)$, then the Cauchy problem (2.3.9) admits a unique strong solution given by*

$$v(t,x) = m(x)\sum_{n\in E} e^{\lambda_n t} Q_n(x) f_n = \sum_{y\in E} p(t,x;y) f(y) \qquad x \in E,\ t \geq 0.$$

In particular, $p(t,x;y)$ is the fundamental solution of $\frac{\partial v}{\partial t}(t,x) = \mathcal{F}\, u(t,x)$. Moreover, if $f \geq 0$ and $\|f\|_{\ell^1(E)} = 1$, denoting by $\mathbb{P}_f(\cdot)$ the probability measure obtained by $\mathbb{P}$ conditioning with the fact that $N(0)$ admits distribution f, then one obtains

$$v(t,x) = \mathbb{P}_f(N(t) = x).$$

The previous Theorem gives us spectral decompositions and stochastic representations of the solutions of the backward and the forward Kolmogorov equations. We can use such decomposition to obtain informations, for instance, on the moments of the process $N(t)$. In particular, we can obtain informations on its covariance.

Proposition 2.3.5. *Let $N(t)$ be a solvable birth-death process with invariant measure $\mathbf{m}$. Let us denote by $\mathrm{Cov}_m(\cdot,\cdot)$ the covariance operator conditioned under the fact that $N(0)$ admits distribution $\mathbf{m}$. Then there exists a constant $a_1 \in \mathbb{R}$ such that for any $t, s \geq 0$ it holds*

$$\mathrm{Cov}_m(N(t), N(s)) = a_1^2 e^{\lambda_1 |t-s|}.$$

In particular, being $N(t)$ a second-order stationary process if $N(0)$ admits distribution $\mathbf{m}$, it is short-range dependent.

Proof. First of all, let us observe that since $\mathbf{m}$ admits moments of any order, $\mathrm{Cov}_m(N(t), N(s))$ is well-defined. Since $N(t)$ is stationary it holds for any $t \geq s$

$$\mathrm{Cov}_m(N(t), N(s)) = \mathrm{Cov}_m(N(t-s), N(0)) = \mathbb{E}_m[N(t-s)N(0)] - \mathbb{E}_m[N(0)]^2. \tag{2.3.10}$$

Let us denote by $\iota(x) = x$. Since ι is a polynomial of degree 1, it can be written as a linear combination of $Q_0 = 1$ and $Q_1(x)$, obtaining

$$\iota(x) = a_0 + a_1 Q_1(x)$$

for some constants a_0 and a_1. Now let us evaluate $\mathbb{E}_m[N(0)]$. We have, by definition,

$$\mathbb{E}_m[N(0)] = \sum_{x\in E} x m(x) = a_0 + a_1 \sum_{x\in E} Q_1(x) m(x) = a_0 \tag{2.3.11}$$

since $\sum_{x\in E} Q_1(x)m(x) = \sum_{x\in E} Q_0 Q_1(x) m(x) = 0$ by orthogonality relation. On the other hand, we have

$$\mathbb{E}_m[N(t)N(0)] = \sum_{x\in E} \mathbb{E}_x[N(t)N(0)]\, \mathbb{P}_m(N(0) = x) = \sum_{x\in E} x\, \mathbb{E}_x[N(t)] m(x).$$

By Theorem 2.3.4 we have that

$$\mathbb{E}_x[N(t)] = \mathbb{E}_x[\iota(N(t))] = a_0 + a_1 Q_1(x) e^{\lambda_1 t}.$$

Thus, we get

$$\mathbb{E}_m[N(t)N(0)] = \sum_{x\in E}(a_0+a_1Q_1(x))(a_0+a_1Q_1(x)e^{\lambda_1 t})m(x) = a_0^2+a_1^2e^{\lambda_1 t}. \tag{2.3.12}$$

Substituting equalities (2.3.11) and (2.3.12) in (2.3.10) we conclude the proof. □

Let us now make a classification of such solvable birth-death processes. Let us focus on the case $E=\mathbb{N}_0$. First of all, let us observe that since $\lim_{x\to+\infty}\frac{b(x)}{d(x+1)}<1$, we have $\deg b(x)\le \deg d(x)$. If $\deg d(x)=0$, then also $\deg b(x)=0$, which is absurd since in such case $\lambda_n=0$ for any $n>0$. Thus we have that $\deg d(x)\ge 1$. Let us consider then $\deg d(x)=1$. Thus we have $\deg b(x)\le 1$.

- If $\deg b(x)=0$ and the director coefficient of d is positive, then we are considering an immigration-death process, as described in [**7**]. In such case the orthogonal polynomials are Charlier polynomials (see [**113, 133**]). Indeed, since $d(0)=0$, it must be $d(x)=d_0x$ and $b(x)=b_0$ for some constants $d_0>0$ and $b_0>0$. Setting $\rho=\frac{b_0}{d_0}$, the invariant measure $m(x)=e^{-\rho}\frac{\rho^x}{x!}$ is a Poisson distribution and the orthogonal polynomials are actually Charlier polynomials of parameter ρ.
- If $\deg b(x)=1$, to obtain a state space $E=\mathbb{N}_0$, one has $b(x)=b_0(x+\beta)$ and $d(x)=d_0x$ with $b_0,d_0>0$. The invariant measure is given by
$$m(x)=\frac{(\beta)_x\rho^x}{x!(1-\rho)^\beta}$$
where $(\beta)_x=\frac{\Gamma(\beta+x)}{\Gamma(\beta)}$. In such case we are considering a Meixner process as discussed in [**81**] and the orthogonal polynomials are Meixner polynomials of parameters $\rho=\frac{b_0}{d_0}$ and β.

Concerning the case $\deg d(x)=2$, one has to observe that, to achieve $\lambda_n<0$, the director coefficient of d must be negative. Then we should have $d(x)<0$ for x sufficiently big, which is absurd. We can conclude the following proposition.

PROPOSITION 2.3.6. *Let $N(t)$ be a solvable birth-death process with state space E. Then, one of the following properties holds true:*
- *E is finite;*
- *$N(t)$ is an immigration-death process;*
- *$N(t)$ is a Meixner process.*

In particular E is either finite or the orthogonal polynomials P_n are self-dual.

2.3.2. Strong solutions of the non-local Kolmogorov equations. Now we can focus on the following non-local Cauchy problem

$$\begin{cases}\partial_t^\Phi u(t,y)=\mathcal{G}\,u(t,y) & t>0,\ y\in E\\ u(0,y)=g(y) & y\in E,\end{cases} \tag{2.3.13}$$

for some suitable initial datum g, where $\mathcal{G}$ is the generator of some solvable birth-death process $N(t)$ with state space E. Let us define the function $\mathfrak{u}:t\in[0,+\infty)\mapsto u(t,\cdot)\in\ell^2(\mathbf{m})$ as before. Now we can give the definition of strong solution for the Cauchy problem (2.3.13).

DEFINITION 2.3.3. We say u is a **strong solution** of (2.3.13) if:
- $u(t,\cdot)$ belongs to $\ell^2(\mathbf{m})$ for any $t\ge 0$ (and then $\mathfrak{u}$ is well defined);

- $\mathfrak{u} \in C([0,+\infty);\ell^2(\mathbf{m}))$;
- $\partial_t^\Phi u(t,y)$ exists for any $t>0$ and $y \in E$;
- $\partial_t^\Phi u(t,\cdot)$ belongs to $\ell^2(\mathbf{m})$ (and then $\partial_t^\Phi \mathfrak{u}$ is well defined);
- $\partial_t^\Phi \mathfrak{u} \in C((0,+\infty);\ell^2(\mathbf{m}))$;
- the equations in (2.3.13) hold pointwise.

The idea is to apply some spectral decomposition technique to obtain strong solutions of the non-local Cauchy problem (2.3.13). Before doing this, let us give some heuristics to show what is the expected *form* of the solutions. Suppose we want to find a solution $u(t,y)=T(t)\varphi(y)$ by separation of variables. Then the first equation of (2.3.13) can be *decoupled*, leading to two eigenvalue problems:

$$\begin{cases} \mathcal{G}\,\varphi(y)=\lambda\varphi(y) & y\in E \\ \partial^\Phi T(t)=\lambda T(t) & t>0. \end{cases}$$

Now, since we know that $\mathcal{G}$ is the generator of some solvable birth-death process, if we consider the family of orthonormal polynomials $(Q_n)_{n\in E}$ associated to it, we have that $\varphi = Q_n$ up to a multiplicative constant and $\lambda=\lambda_n$ for some $n\in N_0$. Then, concerning the second equation, it is a relaxation equation for the non-local derivative ∂^Φ with $\lambda_n \le 0$, thus we have, up to a multiplicative constant, $T(t)=\mathfrak{e}_\Phi(t;\lambda_n)$. In general we expect our solution to be a linear combination of these *simple ones*, i.e.

$$u(t,y)=\sum_{n\in E} u_n Q_n(y)\,\mathfrak{e}_\Phi(t;\lambda_n)$$

for some coefficients u_n. Now let us suppose the initial datum $g\in\ell^2(\mathbf{m})$. Then we have $g=\sum_{n\in E} g_n Q_n(y)$ for some coefficients g_n. However, since $\mathfrak{e}_\Phi(0;\lambda_n)=1$ and $(Q_n)_{n\in E}$ constitute an orthonormal system in $\ell^2(\mathbf{m})$, we obtain, from $u(0,y)=g(y)$, $u_n=g_n$ for any $n\in E$. We conclude that the expected solution is of the form

$$u(t,y)=\sum_{n\in E} g_n Q_n(y)\,\mathfrak{e}_\Phi(t;\lambda_n),$$

where g_n are the coordinates of $g\in\ell^2(\mathbf{m})$ with respect to the orthonormal basis $(Q_n)_{n\in E}$.

As we will see in the following, if E is finite this heuristic argument is formal. The real problem arises when E is infinite and then the sums are actually series. In this case we need to show the convergence of the involved series and the fact that we can exchange the operators ∂^Φ and $\mathcal{G}$ with the summation operator.

To do this, we first need to identify in some sense the fundamental solution. Thus, let us prove the following Lemma (see [**22**, Lemma 4.1]).

LEMMA 2.3.7. *Let $N(t)$ be a solvable birth-death process with state space E, generator $\mathcal{G}$, invariant measure $\mathbf{m}$ and family of associated orthogonal polynomials $(P_n)_{n\in E}$. Then the summation*

$$p_\Phi(t,x;y):=m(x)\sum_{n\in E}\mathfrak{e}_\Phi(t;\lambda_n)Q_n(x)Q_n(y) \tag{2.3.14}$$

absolutely converges for any fixed $t\ge 0$ and $x,y\in E$.

PROOF. Let us first observe that if E is finite, then the summation is finite. Thus, let us work in the case $E=\mathbb{N}_0$ (and $N(t)$ is either an immigration-death process or a Meixner process). By using the definition $\widetilde{m}(n) = \frac{1}{\mathfrak{d}_n^2}$ together with the self-duality of $(P_n)_{n\geq 0}$ we get

$$\begin{aligned} p_\Phi(t,x;y) &= m(x)\sum_{n=0}^{+\infty} \widetilde{m}(n)\, \mathfrak{e}_\Phi(t;\lambda_n) P_n(x) P_n(y) \\ &= m(x)\sum_{n=0}^{+\infty} \widetilde{m}(n)\, \mathfrak{e}_\Phi(t;\lambda_n) P_x(n) P_y(n). \end{aligned}$$

Let us denote by $\mathrm{root}(x)$ the set of the roots of the polynomial $P_x(n)$. Then, by fundamental theorem of algebra, the cardinality of $\mathrm{root}(x)$ is at most x. In particular we can define

$$n_0 = \lceil \max(\mathrm{root}(x) \cup \mathrm{root}(y)) \rceil + 1$$

and observe that the series (2.3.14) absolutely converges if and only if the series

$$\sum_{n=n_0}^{+\infty} \widetilde{m}(n)\, \mathfrak{e}_\Phi(t;\lambda_n) P_x(n) P_y(n)$$

absolutely converges. Now let us observe (see, for instance, [**113**, Table 2.3]) that the director coefficient of $P_x(n)$ is positive if x is even and negative if x is odd. Thus we have, by also using $\mathfrak{e}_\Phi(t;\lambda_n) \leq 1$ since $\lambda_n \leq 0$,

$$\sum_{n=n_0}^{+\infty} |\widetilde{m}(n)\, \mathfrak{e}_\Phi(t;\lambda_n) P_x(n) P_y(n)| \leq (-1)^{x+y} \sum_{n=n_0}^{+\infty} \widetilde{m}(n) P_x(n) P_y(n).$$

Now let us observe that the series $\sum_{n=n_0}^{+\infty} \widetilde{m}(n) P_x(n) P_y(n)$ converges if and only if the series $\sum_{n=0}^{+\infty} \widetilde{m}(n) P_x(n) P_y(n)$ converges. In particular, we have

$$\sum_{n=0}^{+\infty} \widetilde{m}(n) P_x(n) P_y(n) = \frac{1}{m(x)} \delta_{x,y} < +\infty,$$

concluding the proof. □

Now that we have shown the convergence of the series that will be our fundamental solution, let us show a technical Lemma concerning the convergence of some useful series (see [**22**, Lemma 4.2]).

LEMMA 2.3.8. *Let $N(t)$ be a solvable birth-death process with state space $E = \mathbb{N}_0$, generator $\mathcal{G}$, invariant measure $\mathbf{m}$ and family of associated classical orthogonal polynomials $(P_n)_{n\in E}$. Let $g \in \ell^2(\mathbf{m})$ such that $g(x) = \sum_{n\in E} g_n Q_n(x)$ for $x \in E$ with $(g_n)_{n\geq 0} \in \ell^2$. Then*

(1) *For any $x \in E$ it holds*
$$\sum_{n=0}^{+\infty} |g_n Q_n(x)| \leq \frac{\|g\|_{\ell^2(\mathbf{m})}}{\sqrt{m(x)}};$$
(2) *For any fixed $x \in E$ the sum $\sum_{n=0}^{+\infty} \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(x)$ normally converges for $t \in [0,+\infty)$;*
(3) *For any fixed $x \in E$ and $T_1 > 0$ the sum $\sum_{n=0}^{+\infty} \lambda_n\, \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(x)$ normally converges for $t \in [T_1,+\infty)$.*

PROOF. Let us first show property (1). Let us observe that, since $(Q_n)_{n\in E}$ is an orthonormal basis of $\ell^2(\mathbf{m})$, we get $\sum_{n\ge 0} g_n^2 = \|g\|^2_{\ell^2(\mathbf{m})}$. By using Cauchy-Schwartz inequality and the self-duality relation (let us recall that if $E=\mathbb{N}_0$, then $N(t)$ is either an immigration-death process or a Meixner process) we get

$$\left(\sum_{n\ge 0}|g_nQ_n(x)|\right)^2 \le \|g\|^2_{\ell^2(\mathbf{m})}\sum_{n\ge 0}Q_n^2(x) = \|g\|^2_{\ell^2(\mathbf{m})}\sum_{n\ge 0}\widetilde{m}(n)P_x^2(n) = \frac{\|g\|^2_{\ell^2(\mathbf{m})}}{m(x)}.$$

To show property (2), let us just recall that since $\lambda_n \le 0$ then $\mathfrak{e}_\Phi(t,\lambda_n)\le 1$ and

$$\sum_{n\ge 0}|g_n\,\mathfrak{e}_\Phi(t,\lambda_n)Q_n(x)| \le \sum_{n\ge 0}|g_nQ_n(x)|.$$

Finally, concerning property (3), let us observe that for $t\ge T_1$ it holds $\mathfrak{e}_\Phi(t,\lambda_n)\le \mathfrak{e}_\Phi(T_1,\lambda_n)$. Thus, by using Proposition 1.4.12, we get

$$\sum_{n\ge 0}|\lambda_n g_n\,\mathfrak{e}_\Phi(t,\lambda_n)Q_n(x)| \le \sum_{n\ge 0}|\lambda_n g_n\,\mathfrak{e}_\Phi(T_1,\lambda_n)Q_n(x)| \le K(T_1)\sum_{n\ge 0}|g_nQ_n(x)|,$$

concluding the proof. □

Now we are ready to show the two main results of this section. The first one concerns strong solution of (2.3.13) (see [**22**, Theorem 4.3]).

THEOREM 2.3.9. *Let $N(t)$ be a solvable birth-death process with state space E, generator $\mathcal{G}$, invariant measure $\mathbf{m}$ and family of associated classical orthogonal polynomials $(P_n)_{n\in E}$. Let $g\in\ell^2(\mathbf{m})$ with $g=\sum_{n\in E}g_nQ_n$ in $\ell^2(\mathbf{m})$. Then the Cauchy problem* (2.3.13) *admits a unique strong solution*

$$u(t,y)=\sum_{n\in E}\mathfrak{e}_\Phi(t,\lambda_n)g_nQ_n(y) \qquad t\ge 0,\ y\in E \tag{2.3.15}$$

with $\|\mathfrak{u}\|_{C([0,+\infty);\ell^2(\mathbf{m}))} = \sup_{t\ge 0}\|u(t,\cdot)\|_{\ell^2(\mathbf{m})} \le \|g\|_{\ell^2(\mathbf{m})}$. Moreover, $p_\Phi(t,x;y)$ is the fundamental solution of (2.3.13)*, in the sense that it is the unique strong solution of* (2.3.13) *for $g(y)=\delta_{x,y}$ as $x,y\in E$ and, for any $g\in\ell^2(\mathbf{m})$, it holds*

$$u(t,y)=\sum_{x\in E}p_\Phi(t,x;y)g(x).$$

PROOF. By Lemma 2.3.8 we know that the summation in (2.3.15) is well defined. Moreover, we have, for the single summand,

$$\begin{aligned}\mathcal{G}[\mathfrak{e}_\Phi(t,\lambda_n)g_nQ_n](y) &= \mathfrak{e}_\Phi(t,\lambda_n)g_n\,\mathcal{G}\,Q_n(y) = \lambda_n\,\mathfrak{e}_\Phi(t,\lambda_n)g_nQ_n(y)\\ &= \partial_t^\Phi\,\mathfrak{e}_\Phi(t,\lambda_n)g_nQ_n(y) = \partial_t^\Phi[\mathfrak{e}_\Phi(\cdot,\lambda_n)g_nQ_n(y)].\end{aligned}$$

In particular this ensures that u is a strong solution of (2.3.13) whenever E is finite.

Let us consider the case in which $E=\mathbb{N}_0$. First of all, let us show that $\mathfrak{u}$ is well-defined. To do this, define the partial sum

$$u_N(t,y)=\sum_{n=0}^{N}\mathfrak{e}_\Phi(t,\lambda_n)g_nQ_n(y) \qquad t\ge 0,\ y\in E$$

for some $N \in \mathbb{N}$. Consider $N < M$ in $\mathbb{N}$ and observe that, since $\mathfrak{e}_\Phi(t,\lambda_n) \le 1$ being $\lambda_n \le 0$, it holds

$$\|u_N(t,\cdot) - u_M(t,\cdot)\|^2_{\ell^2(\mathbf{m})} \le \sum_{n=N+1}^{M} g_n^2,$$

that, since $(g_n)_{n\ge 0} \in \ell^2$, ensures that the series in (2.3.15) converges in ℓ^2.
Let us consider the integrated tail I_Φ of the Lévy measure ν_Φ that is an increasing and non-negative function. In particular we have

$$\int_0^t (u(\tau,y) - u(0+,y))\overline{\nu}_\Phi(t-\tau)d\tau = \int_0^t (u(\tau,y) - u(0+,y))dI_\Phi(t-\tau).$$

In Lemma 2.3.8 we have shown that the series defining $u(t,y)$ normally converges for fixed y, thus we can exchange summation and integral sign by [**126**, Theorem 7.16], obtaining

$$\int_0^t (u(\tau,y) - u(0+,y))\overline{\nu}_\Phi(t-\tau)d\tau = \sum_{n=0}^{+\infty}\left(\int_0^t (\mathfrak{e}_\Phi(\tau,y)-1)\overline{\nu}_\Phi(t-\tau)d\tau\right) g_n Q_n(y).$$

Now we want to show that we can take the derivative in t term by term. To do this, let us observe that

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{\partial}{\partial t}\left(\int_0^t (\mathfrak{e}_\Phi(\tau,y)-1)\overline{\nu}_\Phi(t-\tau)d\tau\right) g_n Q_n(y) &= \sum_{n=0}^{+\infty}\partial_t^\Phi\, \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(y)\\ &= \sum_{n=0}^{+\infty}\lambda_n\, \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(y),\end{aligned}$$

where the series on the right-hand side normally converges in any compact interval $[T_1,T_2]$ with $T_1 > 0$. Thus we can exchange the derivative operator with respect to the summation one, obtaining

$$\partial_t^\Phi u(t,y) = \sum_{n=0}^{+\infty}\lambda_n\, \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(y) = \sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, Q_n(y).$$

Now we have to show that we can exchange the operator $\mathcal{G}$ with the summation operator. To do this, we just have to observe that, since the involved series normally converge for $t \in [0,+\infty)$, we have

$$\begin{aligned}\delta^+\sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, Q_n(y) &= \sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, \delta^+ Q_n(y),\\ \delta^2\sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, Q_n(y) &= \sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, \delta^2 Q_n(y).\end{aligned}$$

Thus, we finally obtain

$$\partial_t^\Phi u(t,y) = \sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n\, \mathcal{G}\, Q_n(y) = \mathcal{G}\sum_{n=0}^{+\infty}\mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(y) = \mathcal{G}\, u(t,y).$$

Now that we have shown that the first equation of (2.3.13) holds pointwise, we have to check for the second one. To do this, just observe that $\mathfrak{e}_\Phi(0,\lambda_n)=1$ and then

$$u(0,y)=\sum_{n=0}^{+\infty} g_n Q_n(y)=g(y).$$

Let us also observe that, arguing as we did for u, we have $\partial_t^\Phi u(t,\cdot)\in\ell^2(\mathbf{m})$ and $\partial_t^\Phi \mathfrak{u}$ is well-defined.
Now we have to show that $\mathfrak{u}$ is continuous in $[0,+\infty)$. Let us show continuity in 0^+, since for any other $t>0$ the proof is analogous. Let us consider $n(\varepsilon)\ge 0$ such that $\sum_{n=n(\varepsilon)}^{+\infty} g_n^2\le\varepsilon$. Then we have

$$\|\mathfrak{u}(t)-g\|^2_{\ell^2(\mathbf{m})}\le\sum_{n=1}^{n(\varepsilon)}(1-\mathfrak{e}_\Phi(t,\lambda_n))^2 g_n^2+\varepsilon.$$

Sending $t\to 0^+$ and then $\varepsilon\to 0^+$ we obtain the assertion.
Concerning the continuity of $\partial_t^\Phi \mathfrak{u}$ in $(0,+\infty)$, let us fix $t_0>0$ and $t_1\in(0,t_0)$. Consider $t\ge t_1$ and observe that

$$\left\|\partial_t^\Phi \mathfrak{u}(t)-\partial_t^\Phi \mathfrak{u}(t_0)\right\|^2_{\ell^2(\mathbf{m})}\le\sum_{n=1}^{n(\varepsilon)}(\mathfrak{e}_\Phi(t,\lambda_n)-\mathfrak{e}_\Phi(t_0,\lambda_n))^2 g_n^2+K(t_1)\varepsilon,$$

obtaining the claim by sending $t\to t_0$ and $\varepsilon\to 0^+$.
Uniqueness follows from the fact that $(Q_n)_{n\in E}$ is an orthonormal basis of $\ell^2(\mathbf{m})$ (both in the finite and countably infinite case). Moreover, since $(Q_n)_{n\in E}$ is an orthonormal basis of $\ell^2(\mathbf{m})$, we have

$$\|\mathfrak{u}(t)\|^2_{\ell^2(\mathbf{m})}=\sum_{n\in E}\mathfrak{e}_\Phi^2(t,\lambda_n)g_n^2\le\|g\|_{\ell^2(\mathbf{m})}$$

and then, taking the supremum, we obtain the desired bound on the norm of $\mathfrak{u}$.
Now let us show that $p_\Phi(t,x;y)$ is the fundamental solution. To do this, let us observe that, since all the involved sums are normally convergent in compact sets containing t, we can use Fubini's theorem to obtain

$$\begin{aligned}\sum_{x\in E}p_\Phi(t,x;y)g(x)&=\sum_{x\in E}m(x)\left(\sum_{n\in E}\mathfrak{e}_\Phi(t,\lambda_n)Q_n(x)Q_n(y)\right)g(x)\\&=\sum_{n\in E}Q_n(y)\,\mathfrak{e}(t,\lambda_n)\left(\sum_{x\in E}m(x)Q_n(x)g(x)\right)\\&=\sum_{n\in E}Q_n(y)\,\mathfrak{e}(t,\lambda_n)g_n=u(t,y).\end{aligned}$$

Finally, fix $z\in E$ and consider $g(x)=\delta_{z,x}$. Then we have

$$u(t,y)=\sum_{x\in E}p_\Phi(t,x;y)g(x)=p_\Phi(t,z;y),$$

concluding the proof of the Theorem. $\square$

Now we want to do the same thing with the non-local forward equation

$$\begin{cases}\partial_t^\Phi v(t,x)=\mathcal{F}\,v(t,x) & t>0,\ x\in E\\ v(0,x)=f(x) & x\in E,\end{cases}\tag{2.3.16}$$

where f is some suitable initial datum and $\mathcal{F}$ is the forward operator of some solvable birth-death process. We define the function $\mathfrak{v}$ as done for u and we refer to the definition of strong solution as given in 2.3.3. Now we are ready to show the second main result of this section, concerning strong solutions of (2.3.16) (see [**22**, Theorem 4.4]).

THEOREM 2.3.10. *Let $N(t)$ be a solvable birth-death process with state space E, forward operator $\mathcal{F}$, invariant measure $\mathbf{m}$ and family of associated classical orthogonal polynomials $(P_n)_{n\in E}$. Let $f/m \in \ell^2(\mathbf{m})$ such that $f = m\sum_{n\in E} f_n Q_n$ in $\ell^2(\mathbf{m})$. Then the non-local Cauchy problem* (2.3.16) *admits a unique strong solution*

$$v(t,x) = m(x)\sum_{n\in E} \mathfrak{e}_\Phi(t,\lambda_n) g_n Q_n(x), \qquad t \geq 0,\ x \in E \tag{2.3.17}$$

satisfying the following norm estimates:

- $\|\mathfrak{v}\|_{C([0,+\infty);\ell^2(\mathbf{m}))} = \sup_{t\geq 0} \|v(t,\cdot)\|_{\ell^2(\mathbf{m})} \leq \|f/m\|_{\ell^2(\mathbf{m})}$;
- $\sup_{t\geq 0} \|v(t,\cdot)/m(\cdot)\|_{\ell^2(\mathbf{m})} \leq \|f/m\|_{\ell^2(\mathbf{m})}$.

Moreover, $p_\Phi(t,x;y)$ is the fundamental solution of (2.3.16), *in the sense that it is the strong solution of* (2.3.16) *as $f(x) = \delta_{y,x}$ for fixed $y \in E$ and for any $f/m \in \ell^2(\mathbf{m})$ it holds*

$$v(t,x) = \sum_{y\in E} p_\Phi(t,x;y) f(y).$$

PROOF. Once we have checked that a single summand of (2.3.17) is a solution of the first equation of (2.3.16), the proof is analogous to the one of Theorem 2.3.9, except for the first norm bound. Let us then observe that

$$\mathcal{F}_z[m(z)\,\mathfrak{e}_\Phi(t,\lambda_n)Q_n(z)] = \lambda_n\,\mathfrak{e}_\Phi(t,\lambda_n)m(z)Q_n(z) = \partial_t^\Phi[m(z)\,\mathfrak{e}_\Phi(t,\lambda_n)Q_n(z)].$$

Concerning the first norm estimate, let us recall that $\mathbf{m}$ is a probability measure on E, thus $0 \leq m(x) \leq 1$ for any $x \in E$. Then it holds

$$\begin{aligned}
\|v(t,\cdot)\|^2_{\ell^2(\mathbf{m})} &= \sum_{x\in E} m^3(x)\left(\sum_{n\in E}\mathfrak{e}_\Phi(t,\lambda_n) f_n Q_n(x)\right)^2 \\
&\leq \sum_{x\in E} m(x)\left(\sum_{n\in E}\mathfrak{e}_\Phi(t,\lambda_n) f_n Q_n(x)\right)^2 \\
&= \sum_{n\in E}\mathfrak{e}^2_\Phi(t,\lambda_n) f_n^2 \leq \|f/m\|^2_{\ell^2(\mathbf{m})},
\end{aligned}$$

completing the proof. □

In the next section we will focus on stochastic representation of the strong solutions we obtained here.

2.4. Non-local solvable birth-death processes

To give a stochastic representation of the strong solutions of the non-local Cauchy problems (2.3.13) and (2.3.16), we need to introduce a particular class of time-changed processes.

DEFINITION 2.4.1. Let $N(t)$ be a solvable birth-death process and $\Phi \in \mathcal{BF}$ a driftless Bernstein function. Then the **non-local solvable birth-death process** induced by $N(t)$ and Φ is defined as

$$N_\Phi(t) = N(L_\Phi(t)), \ t \ge 0$$

where $L_\Phi(t)$ is an inverse subordinator associated to Φ independent of $N(t)$.

Let us observe that in the local case, the transition probability function $p(t, x; y) = \mathbb{P}(N(t) = x | N(0) = y)$ is the fundamental solution of the Cauchy problems related to the backward and forward Kolmogorov equations. Concerning the non-local, case, the process $N_\Phi(t)$ is not a Markov process, but only a semi-Markov one. However, we can still define the **transition probability function** as:

$$p_\Phi(t, x; y) = \mathbb{P}(N_\Phi(t) = x | N_\Phi(0) = y), \qquad t \ge 0, \ x, y \in E.$$

Let us observe that we have used the same notation as the fundamental solution of the non-local problems (2.3.13) and (2.3.16). Indeed we can show the following Theorem (see [**22**, Theorem 5.1]).

THEOREM 2.4.1. *Let $N_\Phi(t)$ be a non-local solvable birth-death process with state space E. Then the transition probability function $p_\Phi(t, x; y)$ coincides with the summation in Equation* (2.3.14).

PROOF. Let us first observe that $N_\Phi(0) = N(0)$ by the fact that $L_\Phi(0) = 0$ almost surely. Thus, by conditioning, we easily get

$$p_\Phi(t, x; y) = \int_0^{+\infty} p(s, x; y) f_\Phi(s; t) ds, \ t \ge 0, \ x, y \in E$$

where $f_\Phi(\cdot; t)$ is the density of $L_\Phi(t)$.
Now, by Theorem 2.3.4, we know that

$$p_\Phi(t, x; y) = \int_0^{+\infty} m(x) \sum_{n \in E} e^{\lambda_n s} Q_n(x) Q_n(y) f_\Phi(s; t) ds,$$

where $(Q_n)_{n \in E}$ is the family of orthonormal polynomials associated to the solvable birth-death process $N(t)$, of which $N_\Phi(t)$ is the time-changed process, and $\mathbf{m}$ is its invariant measure. Thus, if E is finite, recalling that by definition $\mathfrak{e}_\Phi(t, \lambda_n) = \mathbb{E}[e^{\lambda L_\Phi(t)}]$, we conclude the proof.
Let us consider the case in which $E = \mathbb{N}_0$. We need to change the order of integral and series. To do this, let us consider again $n_0 = \lceil \max(\mathrm{root}(x) \cup \mathrm{root}(y)) \rceil + 1$ as done in the proof of Lemma 2.3.7. Then we have

$$\begin{aligned} p_\Phi(t, x; y) = \int_0^{+\infty} m(x) \sum_{n=0}^{n_0} e^{\lambda_n s} Q_n(x) Q_n(y) f_\Phi(s; t) ds \\ + \int_0^{+\infty} m(x) \sum_{n=n_0}^{+\infty} e^{\lambda_n s} Q_n(x) Q_n(y) f_\Phi(s; t) ds. \end{aligned}$$

Since the first summation is finite, we can exchange the summation sign with the integral one. Concerning the second summation, let us observe that

$$Q_n(x) Q_n(y) = \widetilde{m}(n) P_x(n) P_y(n)$$

that are of fixed sign for $n > n_0$ and we can use Fubini's theorem to exchange the integral sign with the summation one. Thus, we finally get

$$p_\Phi(t,x;y) = m(x)\sum_{n=0}^{+\infty} Q_n(x)Q_n(y)\int_0^{+\infty} e^{\lambda_n s} f_\Phi(s;t)ds,$$

concluding the proof. □

The previous Theorem gives us a stochastic representation of the fundamental solution of both problems (2.3.13) and (2.3.16). By using such result, we can easily exploit stochastic representations for strong solutions of (2.3.13) and (2.3.16) (see [**22**, Proposition 5.2]).

Corollary 2.4.2. *Let $N_\Phi(t)$ be a non-local solvable birth-death process with state space E. Then*

(1) *For any $g \in \ell^2(\mathbf{m})$ the function $u(t,y) = \mathbb{E}_y[g(N_\Phi(t))]$ is the unique strong solution of* (2.3.13)*;*
(2) *For any $f \in \ell^1$ such that $f/m \in \ell^2(\mathbf{m})$, $f \ge 0$ and $\|f\|_{\ell^1} = 1$, the function $v(t,x) = \mathbb{P}_f(N_\Phi(t) = x)$ is the unique strong solution of* (2.3.16)*.*

Proof. Concerning assertion (1), it follows from the fact that

$$u(t,y) = \sum_{x \in E} g(x)p_\Phi(t,x;y)$$

and Theorem 2.3.9. Instead, concerning assertion (2), we have

$$v(t,x) = \sum_{y \in E} f(y)p_\Phi(t,x;y),$$

thus Theorem 2.3.10 concludes the proof. □

Now that we have such representation, we can use the spectral decomposition provided in Theorem 2.3.10 to show that $\mathbf{m}$ is also the invariant measure and the limit distribution for $N_\Phi(t)$ (see [**22**, Corollary 5.3]).

Proposition 2.4.3. *Let $N_\Phi(t)$ be a non-local solvable birth-death process with state space E. Then*

(1) *If $N_\Phi(0)$ admits distribution $\mathbf{m}$, then $N_\Phi(t)$ is first-order stationary with distribution $\mathbf{m}$ for any $t \ge 0$;*
(2) *If $N_\Phi(0)$ admits distribution f such that $f/m \in \ell^2(\mathbf{m})$, then $\lim_{t\to+\infty} \mathbb{P}_f(N_\Phi(t) = x) = m(x)$.*

Proof. Let us first show assertion (1). To do this, let us observe that $1 \in \ell^2(\mathbf{m})$ and then $v(t,x) = \mathbb{P}_m(N_\Phi(t) = x)$ is the unique strong solution of (2.3.16) with initial datum $f \equiv m$. Now we need to determine m_n such that $\sum_{n\in E} m_n Q_n(x) \equiv 1$. However, $\deg(Q_n) = n$ for any $n \in E$, thus the unique possibility is that $m_0 = 1$ while $m_n = 0$ for any $n \in E \setminus \{0\}$. Finally, by Equation (2.3.17) and the fact that $\lambda_0 = 0$, we get

$$v(t,x) = m(x)\sum_{n\in E} m_n\, \mathfrak{e}_\Phi(t,\lambda_n)Q_n(x) = m(x).$$

Concerning property (2), let us consider $f \in \ell^1$ with $f \geq 0$ and $\sum_{x\in E} f(x) = 1$, such that $f/m \in \ell^2(\mathbf{m})$ and suppose $N_\Phi(0)$ admits distribution f. Consider the decomposition $f/m = \sum_{n\in E} f_n Q_n(x)$ and let us determine f_0. We have

$$f_0 = \sum_{x\in E} m(x)\frac{f(x)}{m(x)}Q_0(x) = \sum_{x\in E} f(x) = 1.$$

Thus, by Equation (2.3.17), we get

$$v(t,x) = \mathbb{P}_f(N_\Phi(t) = x) = m(x) + \sum_{\substack{n\in E\\ n\geq 1}} f_n\, \mathfrak{e}_\Phi(t,\lambda_n)Q_n(x)$$

Now let us also observe that $f \in \ell^2(\mathbf{m})$ since $f/m \in \ell^2(\mathbf{m})$ and $0 \leq f(x) \leq f(x)/m(x)$ for any $x \in E$. Thus in particular the series $\sum_{\substack{n\in E\\ n\geq 1}} f_n\, \mathfrak{e}_\Phi(t,\lambda_n)Q_n(x)$ normally converges for $t \in [0,+\infty)$. Thus we can exchange the limit sign with the summation one. Let us determine $\lim_{t\to+\infty} \mathfrak{e}_\Phi(t,\lambda_n)$. We know that $\lim_{t\to+\infty} L_\Phi(t) = +\infty$ almost surely (driftless subordinators are unbounded) and $\mathfrak{e}_\Phi(t,\lambda_n)$ is decreasing (since $\lambda_n < 0$ for any $n \geq 1$) with $\mathfrak{e}_\Phi(t,\lambda_n) \leq 1$. Thus, by monotone convergence theorem, we have

$$\lim_{t\to+\infty} \mathfrak{e}_\Phi(t,\lambda_n) = \mathbb{E}[\lim_{t\to+\infty} e^{\lambda_n L_\Phi(t)}] = 0.$$

Hence we finally get

$$\lim_{t\to+\infty} v(t,x) = m(x) + \sum_{\substack{n\in E\\ n\geq 1}} f_n \lim_{t\to+\infty} \mathfrak{e}_\Phi(t,\lambda_n)Q_n(x) = m(x).$$

$\square$

In particular we have shown that the process $N_\Phi(t)$ is first-order stationary if $N_\Phi(0)$ admits $\mathbf{m}$ as distribution. However, we can show that it is not even second-order stationary in such case, since it is not stationary in wide sense. To do this, we need a preliminary technical Lemma (see [**29**, Theorem 2]).

LEMMA 2.4.4. *Consider $\Phi \in \mathcal{BF}$ a driftless Bernstein function. For any $t, s \geq 0$ define the measure*

$$F^{(2)}_\Phi(t,s,A) = \mathbb{P}((L_\Phi(t), L_\Phi(s)) \in A),\ \forall A \in \mathcal{B}(\mathbb{R}^2).$$

Then for any $\lambda > 0$ and $t \geq s > 0$ it holds

$$\begin{aligned}(2.4.1)\qquad \int_0^{+\infty}\int_0^{+\infty} & e^{-\lambda|u-v|}F^{(2)}_\Phi(t,s,dudv) \\ &= \lambda\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)dU_\Phi(y) - 2 + 2\,\mathfrak{e}_\Phi(s;-\lambda) + \mathfrak{e}_\Phi(t;-\lambda).\end{aligned}$$

PROOF. Denote $G(u,v) = e^{-\lambda|u-v|}$. Let us first observe that, by monotone convergence theorem,

(2.4.2)

$$\int_0^{+\infty}\int_0^{+\infty} G(u,v)F^{(2)}_\Phi(t,s,dudv) = \lim_{a\to+\infty}\lim_{b\to+\infty}\int_0^a\int_0^b G(u,v)F^{(2)}_\Phi(t,s,dudv).$$

To work with the integral in the right-hand side, we want to use the bivariate integration by parts formula (as given in [**70**, Lemma 2.2]), obtaining
(2.4.3)

$$\int_0^a \int_0^b G(u,v)F_\Phi^{(2)}(t,s,dudv) = I_1(a,b)+I_2(a,b)+I_3(a,b)+F_\Phi^{(2)}(t,s,[0,a]\times[0,b]),$$

where

$$\begin{aligned} I_1(a,b) &= \int_0^a F_\Phi^{(2)}(t,s,[u,a]\times[0,b])G(du,0),\\ I_2(a,b) &= \int_0^b F_\Phi^{(2)}(t,s,[0,a]\times[v,b])G(0,dv),\\ I_3(a,b) &= \int_0^a \int_0^b F_\Phi^{(2)}(t,s,[u,a]\times[v,b])G(du,dv). \end{aligned}$$

Observe that

$$\begin{aligned} G(du,v) &= (-\lambda e^{-\lambda(u-v)}\chi_{u\ge v}(u,v) + \lambda e^{-\lambda(v-u)}\chi_{u<v}(u,v))du\\ G(u,dv) &= (\lambda e^{-\lambda(u-v)}\chi_{u>v}(u,v) - \lambda e^{-\lambda(v-u)}\chi_{u\le v}(u,v))dv \end{aligned}$$

thus, for $u \in [0,a]$, it holds $G(du,0) = -\lambda e^{-\lambda u}du$. Hence we can use monotone convergence theorem to obtain

$$\begin{aligned} \lim_{a,b\to+\infty} I_1(a,b) &= -\int_0^{+\infty} F_\Phi^{(2)}(t,s,[u,+\infty)\times[0,+\infty))\lambda e^{-\lambda u}du\\ &= \int_0^{+\infty} \mathbb{P}(L_\Phi(t)\ge u)d(e^{-\lambda u}) \qquad (2.4.4)\\ &= -1 + \int_0^{+\infty} e^{-\lambda u} f_\Phi(u;t)du = \mathfrak{e}_\Phi(t;-\lambda) - 1. \end{aligned}$$

Arguing in the same way we get

$$\lim_{a,b\to+\infty} I_2(a,b) = \mathfrak{e}_\Phi(s;-\lambda) - 1. \qquad (2.4.5)$$

Moreover, we have

$$\lim_{a,b\to+\infty} F_\Phi^{(2)}(t,s,[0,a]\times[0,b]) = 1. \qquad (2.4.6)$$

Finally, let us observe that $F_\Phi^{(2)}(t,s,\cdot)$ is a probability measure on $\mathbb{R}^+\times\mathbb{R}^+$ and $G(u,v)$ is bounded by 1, thus the integral on the left-hand side of (2.4.2) is finite. Combining inequalities (2.4.4), (2.4.5) and (2.4.6), we know that $\lim_{a,b\to+\infty} I_3(a,b) = I_4$ is finite. Now let us consider

$$I_4 = \iint_{\mathbb{R}^+\times\mathbb{R}^+} \mathbb{P}(L_\Phi(t)\ge u, L_\Phi(s)\ge v)G(du,dv) = I_5+I_6+I_7 \qquad (2.4.7)$$

where

$$\begin{aligned} I_5 &= \iint_{u<v} \mathbb{P}(L_\Phi(t)\ge u, L_\Phi(s)\ge v)G(du,dv)\\ I_6 &= \iint_{u=v} \mathbb{P}(L_\Phi(t)\ge u, L_\Phi(s)\ge v)G(du,dv)\\ I_7 &= \iint_{u>v} \mathbb{P}(L_\Phi(t)\ge u, L_\Phi(s)\ge v)G(du,dv). \end{aligned}$$

Let us first work with the integral in I_6. Let us observe that $G(du,dv)$ admits a jump part on $u=v$. In particular we have, on $u=v$, $G(du,dv)=2\lambda du$. Moreover, since L_Φ is increasing, we have that $L_\Phi(s)\ge u$ implies $L_\Phi(t)\ge u$. Thus, we obtain

$$I_6 = 2\lambda \int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge u)du = 2\lambda U_\Phi(s). \tag{2.4.8}$$

Now let us consider I_5. As before, we have $L_\Phi(s)\ge v$ implies $L_\Phi(t)\ge u$, thus we get, since $G(u,v)$ is C^2 in the region of $\mathbb{R}^+\times\mathbb{R}^+$ such that $u<v$,

$$\begin{aligned}
I_5 &= -\int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)\lambda e^{-\lambda v}\left(\int_0^v \lambda e^{\lambda u}du\right)dv \\
&= -\int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)\lambda(1-e^{-\lambda v})dv \\
&= -\lambda\int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)dv + \int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)d(e^{-\lambda v}) \\
&= -\lambda U_\Phi(s) + \mathfrak{e}_\Phi(s;-\lambda) - 1.
\end{aligned} \tag{2.4.9}$$

Concerning I_7, things are more complicated. First of all, let us define the set

$$A(t,s) = \{(x,y)\in\mathbb{R}^2:\ y\in[0,s],\ x\in[0,t-y]\}.$$

Let us denote by $g_\Phi(du;t)$ the law of $\sigma_\Phi(t)$. Then we have, for $u>v$, since σ_Φ is a Lévy process,

$$\begin{aligned}
\mathbb{P}(L_\Phi(t)\ge u, L_\Phi(s)\ge v) &= \mathbb{P}(\sigma_\Phi(u)\le t, \sigma_\Phi(v)\le s) \\
&= \mathbb{P}(\sigma_\Phi(u)-\sigma_\Phi(v)+\sigma_\Phi(v)\le t, \sigma_\Phi(v)\le s) \\
&= \int_0^s\left(\int_0^{t-y} g_\Phi(dx;u-v)\right)g_\Phi(dy;v) \\
&= \iint_{A(t,s)} g_\Phi(dx;u-v)g_\Phi(dy;v).
\end{aligned} \tag{2.4.10}$$

Now we can substitute such formula in the definition of I_7 to achieve

$$\begin{aligned}
I_7 &= -\iint_{A(t,s)}\iint_{u>v} \lambda^2 e^{-\lambda(u-v)}dudv g_\Phi(dx;u-v)g_\Phi(dy;u) \\
&= \lambda\iint_{A(t,s)}\int_0^{+\infty} g(dy;v)\left(\int_v^{+\infty}(-\lambda e^{-\lambda(u-v)})g_\Phi(dx;u-v)du\right)dv,
\end{aligned}$$

where the order of the integrals has been exchanged by using the properties of mixture measures.
Now let us use the change of variables $u-v=w$ in the inner integral to obtain

$$I_7 = \lambda\iint_{A(t,s)}\left(\int_0^{+\infty} g(dy;v)dv\right)\left(\int_0^{+\infty}(-\lambda e^{-\lambda w})g_\Phi(dx;w)dw\right).$$

Now we have decoupled the two inner integrals. Let us define the mixture measures

$$I_8(dy) = \int_0^{+\infty} g_\Phi(dy;v)dv \qquad I_9(dx) = \int_0^{+\infty}(-\lambda e^{-\lambda w})g_\Phi(dx;w)dw.$$

To better understand what are these measures, let us consider their Laplace-Stieltjes transforms. Concerning $I_8(dy)$, we have

$$\mathcal{L}^S[I_8](z) = \int_0^{+\infty} e^{-zy} \int_0^{+\infty} g(dy;v)dv = \int_0^{+\infty} e^{-v\Phi(z)}dv = \frac{1}{\Phi(z)} = \mathcal{L}^S[dU_\Phi](z),$$

thus we have $I_8(dy) = dU_\Phi(y)$, that is well-defined since U_Φ is locally of bounded variation. Rewriting I_7, we have

$$I_7 = \lambda \int_0^s \left(\int_0^{t-y} I_9(dx) \right) dU_\Phi(y).$$

Now let us determine $\int_0^{t-y} I_9(dx)$. We have

$$\begin{aligned} \int_0^{t-y} I_9(dx) &= \int_0^{t-y} \int_0^{+\infty} (-\lambda e^{-\lambda w}) g_\Phi(dx;w)dw \\ &= \int_0^{+\infty} (-\lambda e^{-\lambda w}) \left(\int_0^{t-y} g_\Phi(dx;w) \right) dw \\ &= \int_0^{+\infty} \mathbb{P}(\sigma_\Phi(w) \le t-y) d(e^{-\lambda w}) - 1 \\ &= \int_0^{+\infty} e^{-\lambda w} f_\Phi(t-y;w)dw - 1 = \mathfrak{e}_\Phi(t-y;-\lambda) - 1. \end{aligned}$$

Hence we get

$$I_7 = \lambda \int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)dU_\Phi(y) - \lambda U_\Phi(s). \tag{2.4.11}$$

Substituting Equation (2.4.8), (2.4.9) and (2.4.11) in (2.4.7) we obtain

$$I_4 = \mathfrak{e}_\Phi(s;-\lambda) - 1 + \lambda \int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)dU_\Phi(y). \tag{2.4.12}$$

Finally, substituting (2.4.4), (2.4.5), (2.4.6) and (2.4.12) in (2.4.3) we conclude the proof. □

Concerning the autocovariance function we get then

PROPOSITION 2.4.5. *Let $N_\Phi(t)$ be a non-local solvable birth-death process with state space E, invariant measure $\mathbf{m}$ and family of associated classical orthogonal polynomials $(P_n)_{n\in E}$. Suppose that $\iota = a_0 + a_1 Q_1$ (where $\iota(x) = x$). Then it holds, for any $t \ge s$,*

$$\mathrm{Cov}_m(N_\Phi(t), N_\Phi(s)) = a_1^2 \left(-\lambda_1 \int_0^s \mathfrak{e}_\Phi(t-y;\lambda_1)dU_\Phi(y) - 2 + 2\,\mathfrak{e}_\Phi(s;\lambda_1) + \mathfrak{e}_\Phi(t;\lambda_1) \right).$$

In particular

$$\mathrm{Cov}_m(N_\Phi(t), N_\Phi(0)) = a_1^2\, \mathfrak{e}_\Phi(t;\lambda_1). \tag{2.4.13}$$

PROOF. It directly follows from $\mathrm{Cov}_m(N(t), N(s)) = a_1^2 e^{\lambda_1|t-s|}$ and Lemma 2.4.4. □

We have that $N_\Phi(t)$ is not second-order stationary. However, if we want to study how memory affects the process, we need to adapt some definitions to our non-stationary case. In particular, referring to [**34**, Lemmas 2.1 and 2.2], we give the following definitions:

DEFINITION 2.4.2. Set $\gamma(n) = \mathrm{Cov}_m(N_\Phi(n), N_\Phi(0))$ for $n \in \mathbb{N}$.

- $N_\Phi(t)$ is said to be **long-range dependent with respect to the initial datum** if $\gamma(n) \sim \ell(n)n^{-\alpha}$ where ℓ is a slowly varying function and $\alpha \in (0,1)$;
- $N_\Phi(t)$ is said to be **short-range dependent with respect to the initial datum** if $\sum_{n=1}^{+\infty} |\gamma(n)| < +\infty$.

In particular we get the following result (see [**22**, Corllary 6.5])

COROLLARY 2.4.6. *Let $N_\Phi(t)$ be a non-local solvable birth-death process. Then:*

- *If Φ is regularly varying at 0^+ with order $\alpha \in (0,1)$, then $N_\Phi(t)$ is long-range dependent with respect to the initial datum;*
- *If $\lim_{z\to 0^+} \frac{\Phi(z)}{z} = l \in (0,+\infty)$, then $N_\Phi(t)$ is short-range dependent with respect to the initial datum.*

PROOF. The first statement is a direct consequence of Propositions 2.4.5 and 1.4.15. The second statement also follows from the same propositions with the application of the integral criterion for the convergence of the series. □

We are currently working on a non-local extension of the spectral decomposition of Pearson diffusions [**62**] by using the techniques used in [**94**] and [**95**]. The extension of such techniques to Student diffusions is quite complicated, as the eigenfunctions of the absolutely continuous part of the spectrum of the generator are expressed by means of technically difficult formulas (see [**96**]) or, as in the case of skew Student distributions, such eigenfunctions are not explicitly known (see [**31**]). See [**21**] for technical details.

2.5. The Time-Changed fractional Ornstein-Uhlenbeck (TCfOU) process

As we saw in the previous sections, time-changing a time-homogeneous Feller process generally leads to a substitution of a non-local derivative in place of the classical one in the Kolmogorov equations of the process. Even if the process is not uniquely determined by these equations (since we lose Markov property), we can provide a stochastic representation of the solutions of some non-local difference-differential Cauchy problems or some non-local (in time) parabolic equations (in general, for non-local heat-like equations whose operator in the right-hand side is the generator of a Feller process, as stated in Corollary 1.4.4). However, there are different processes that, despite not being Feller process (even non Markov), are still associated to some partial differential equations.
This is the case of Gaussian processes. Indeed, if we consider a centred one-dimensional Gaussian process $G(t)$ with differentiable variance function $V(t) = \mathbb{E}[G^2(t)]$ and such that $G(0) = 0$ almost surely, then its probability density function $p(x;t)$ is solution of the non-autonomous heat-like equation:

$$\frac{\partial p}{\partial t}(x;t) = \frac{1}{2}V'(t)\frac{\partial^2 p}{\partial x^2}(x;t) \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ .$$

This leads to a *natural* question: what happens if we apply a time-change to $G(t)$? The answer for this question is given for instance in [**75**], in the case of the inverse stable subordinators and the inverse of a mixture of stable subordinators (leading to a distributed order fractional derivative). However, the proofs of [**75**, Theorems

3 and 4] both rely on a property that can be resumed as: *The Laplace transform of the product of two function is the complex convolution of the Laplace transform of the function.* Actually, this is true if both the Laplace transform can be inverted by using Paley-Wiener inversion for holomorphic Fourier transforms (thus if these Laplace transforms are L^2 on vertical lines in the region of convergence). Hence, we are relying on the regularity of both the variance of the Gaussian process and the its probability density function. Here we want to consider the case in which one of the two involved functions cannot be inverted by means of Paley-Wiener inversion (but we can still use complex inversion formula), thus the operator on the right-hand side is generally much more complicated with respect to the one presented in [**75**].
To consider an interesting example, let us work with a modified version of a quite regular process: it has been shown in [**67**] that, for the general time-changed Ornstein-Uhlenbeck process, the spectral decomposition theorems given in [**94**] for fractional Pearson diffusions still hold, using a different function in place of the Mittag-Leffler one. Here, let us consider the fractional Ornstein Uhlenbeck process introduced in [**48**]. In this section we will focus on the main properties of the time-changed fractional Ornstein Uhlenbeck (TCfOU) process and then, in the next section, we will follow the path to the construction of the generalized Fokker-Planck equation for such process.

2.5.1. Properties of the fractional Ornstein-Uhlenbeck process. Let us first give the definition of fractional Brownian motion (fBm, see [**109**]) and fractional Ornstein-Uhlenbeck (fOU) process (see [**48**]).

Definition 2.5.1. The **fractional Brownian motion** $B_H(t)$ with Hurst index $H \in (0, 1)$ is a centred Gaussian process such that $B_H(0) = 0$ almost surely and the auto-covariance function is given by

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The **fractional Ornstein-Uhlenbeck process** $U_H(t)$ with initial point 0, Hurst index $H \in (0, 1)$ and relaxation parameter $\theta > 0$ is a centered Gaussian process that is solution of the following stochastic differential equation driven by a fractional Brownian noise:

$$dU_H(t) = -\frac{U_H(t)}{\theta}dt + dB_H(t), \qquad U_H(0) = 0$$

and can be expressed as

$$U_H(t) = e^{-\frac{t}{\theta}} \int_0^t e^{\frac{s}{\theta}} dB_H(s)$$

for $t \geq 0$, where the integral is a path-wise Riemann-Stieltjes integral (since the integrand is a C^1 function and then one can integrate by parts, see [**147**]).

From now on, let us consider $H \in \left(\frac{1}{2}, 1\right)$. By a direct application of the fractional Itô isometry (see [**37**]), we have the following variance function:

$$V_{2,H}(t) := \mathbb{E}[U_H^2(t)] = H(2H-1)\theta^{2H} \int_0^{\frac{t}{\theta}} \int_0^{\frac{t}{\theta}} e^{-(s+u)}|u-s|^{2H-2} duds. \tag{2.5.1}$$

Moreover, let us denote by $V_{n,H}(t) := \mathbb{E}[|U_H(t)|^n]$. Since U_H is a Gaussian process we have

$$V_{2n,H}(t) = \frac{(2H\theta^{2H}(2H-1))^n\Gamma\left(\frac{2n+1}{2}\right)}{\sqrt{\pi}}\left(\int_0^{\frac{t}{\theta}}\int_0^{\frac{t}{\theta}} e^{-(s+u)}|u-s|^{2H-2}duds\right)^n.$$

Concerning the behaviour of $V_{2,H}$ at infinity, the following limit has been shown in [**91**] and in [**25**] with different strategies:

$$V_{2,H}(\infty) := \lim_{t\to+\infty} V_{2,H}(t) = \theta^{2H} H\Gamma(2H).$$

It will be extremely useful to recall the following representation formula for $V_{2,H}$ given in [**91**]:

$$V_{2,H}(t) = H\left(\int_0^t e^{-\frac{z}{\theta}} z^{2H-1}dz + e^{-\frac{2t}{\theta}}\int_0^t e^{\frac{z}{\theta}} z^{2H-1}dz\right). \tag{2.5.2}$$

Now, let us observe that, by Equation (2.5.1), it is easy to observe that $V_{2,H}(t)$ is increasing, hence $V_{2,H}(t) \le V_{2,H}(\infty)$ for any $t \ge 0$. Being $V_{2,H}(t)$ bounded, the Laplace transform is well-defined for any $\lambda \in \mathbb{H}$. Now let us determine this Laplace transform, as done in [**27**, Lemma 5.1].

LEMMA 2.5.1. *The abscissa of convergence of $V_{2,H}$ is 0 and, for any $\lambda \in \mathbb{H}$, the Laplace transform of $V_{2,H}$ is given by*

$$\mathcal{L}[V_{2,H}](\lambda) = \frac{2H\theta^{2H}\Gamma(2H)}{\lambda(\theta\lambda+2)(\theta\lambda+1)^{2H-1}},$$

where, for $\beta > 0$, $z^\beta = e^{\beta \mathrm{Log}(z)}$ and Log *is the principal value of the complex logarithm.*

PROOF. Without loss of generality, we can consider $\lambda \in \mathbb{R}^+$ and then define $\mathcal{L}[V_{2,H}]$ on the whole semi-plane $\mathbb{H}$ by holomorphic extension. Since all the involved quantities are non-negative, we can use Fubini's theorem, together with Equation (2.5.2), and the change of variables $y = \left(\lambda + \frac{1}{\theta}\right) z$ to get

$$\begin{aligned}\mathcal{L}[V_{2,H}](\lambda) &= H\int_0^{+\infty} z^{2H-1}\left(e^{-\frac{z}{\theta}}\int_z^{+\infty} e^{-\lambda t}dt + e^{\frac{z}{\theta}}\int_z^{+\infty} e^{-\left(\lambda+\frac{2}{\theta}\right)t}dt\right)dz \\ &= \frac{2H\theta^{2H}}{\lambda(\theta\lambda+2)(\theta\lambda+1)^{2H-1}}\int_0^{+\infty} e^{-y}y^{2H-1}dy = \frac{2H\theta^{2H}\Gamma(2H)}{\lambda(\theta\lambda+2)(\theta\lambda+1)^{2H-1}}.\end{aligned}$$

□

Concerning the asymptotics of $V_{2,H}$ at 0^+, it is not difficult to check that $V_{2,H} \sim t^{2H}$ as $t \to 0^+$. Indeed we have

$$\lim_{t\to0^+}\frac{\int_0^t e^{-\frac{z}{\theta}} z^{2H-1}dz}{t^{2H}} = \lim_{t\to0^+}\frac{e^{-\frac{t}{\theta}}}{2H} = \frac{1}{2H}$$

while

$$\lim_{t\to0^+}\frac{\int_0^t e^{\frac{z}{\theta}} z^{2H-1}dz}{e^{\frac{2t}{\theta}}t^{2H}} = \lim_{t\to0^+}\frac{e^{\frac{t}{\theta}}}{\left(2H+\frac{2}{\theta}t\right)e^{\frac{2t}{\theta}}} = \frac{1}{2H}$$

thus

$$\lim_{t\to0^+}\frac{V_{2,H}(t)}{t^{2H}} = 1.$$

Now, since we want to work with the Fokker-Planck equation of $U_H(t)$, we need to show that $V_{2,H}$ is a C^1 function. This is easy, by working directly with Equation (2.5.2). In particular, we can also obtain some asymptotic results for $V'_{2,H}$ (see [**27**, Lemma 5.2 and Corollary 5.1]).

LEMMA 2.5.2. *The function $V_{2,H}$ belongs to $C^1([0,+\infty))$. Moreover it holds $V'_{2,H} \sim 2H(2H-1)\theta e^{-\frac{t}{\theta}}t^{2H-2}$ as $t \to +\infty$, $V'_{2,H} \sim 2Ht^{2H-1}$ as $t \to 0^+$. Finally, $V'_{2,H} \in L^2(0,+\infty)$.*

PROOF. Let us first observe that, differentiating Equation (2.5.2) and then integrating by parts, we get

$$V'_{2,H}(t) = 2H(2H-1)e^{-\frac{2}{\theta}t}\int_0^t e^{\frac{z}{\theta}}z^{2H-2}dz$$

thus we already have $V_{2,H} \in C^1(0,+\infty)$. Concerning the (right) differentiability in 0, we have

$$\lim_{t\to 0^+}\frac{V_{2,H}(t)}{t} = 0$$

since $V_{2,H}(t) \sim t^{2H}$ and $2H > 1$, being $H > \frac{1}{2}$. Hence we have $V'_{2,H}(0) = 0$ and $V_{2,H} \in C^1([0,+\infty))$.
Now, concerning the asymptotics of $V'_{2,H}$, we have

$$\lim_{t\to 0^+}\frac{V'_{2,H}(t)}{t^{2H-1}} = \lim_{t\to 0^+}\frac{2H(2H-1)\int_0^t e^{\frac{z}{\theta}}z^{2H-2}dz}{t^{2H-1}e^{\frac{2}{\theta}t}} = \lim_{t\to 0^+}\frac{2H(2H-1)e^{-\frac{t}{\theta}}}{2H-1+\frac{2}{\theta}t} = 2H,$$

and

$$\begin{aligned}\lim_{t\to 0^+}\frac{V'_{2,H}(t)}{e^{-\frac{t}{\theta}}t^{2H-2}} &= \lim_{t\to+\infty}\frac{2H(2H-1)\int_0^t e^{\frac{z}{\theta}}z^{2H-2}dz}{e^{\frac{t}{\theta}}t^{2H-2}} \\ &= \lim_{t\to+\infty}\frac{2H(2H-1)\theta}{1+\theta(2H-2)t^{-1}} = 2H(2H-1)\theta.\end{aligned}$$

Finally, the fact that $V'_{2,H} \in L^2(0,+\infty)$ follows from the fact that $V'_{2,H} \in C([0,+\infty))$ and $V'_{2,H}(t) \sim 2H(2H-1)\theta t^{2H-2}e^{-\frac{t}{\theta}}$ as $t \to +\infty$. □

Now, let us focus on the Laplace transform of $V'_{2,H}$. We have the following result (see [**27**, Lemma 5.3]).

LEMMA 2.5.3. *The abscissa of convergence of $V'_{2,H}$ is $-1/\theta$ and*

$$\mathcal{L}[V'_{2,H}](\lambda) = \frac{2H\theta^{2H}\Gamma(2H)}{(\theta\lambda+2)(\theta\lambda+1)^{2H-1}}.$$

Moreover, for any $c > -\frac{1}{\theta}$ the function $\omega \in \mathbb{R} \mapsto \mathcal{L}[V'_{2,H}](c+i\omega)$ belongs to $L^p(\mathbb{R})$ for any $p \geq 1$.

PROOF. From the asymptotics we obtained in the previous Lemma, it is easy to see that $V'_{2,H}$ is of exponential order and the abscissa of convergence $\operatorname{abs}(V'_{2,H}) \leq -\frac{1}{\theta}$. Moreover, for any $\lambda \in \mathbb{H}$ we have

$$\mathcal{L}[V'_{2,H}](\lambda) = \lambda\,\mathcal{L}[V_{2,H}](\lambda) = \frac{2H\theta^{2H}\Gamma(2H)}{(\theta\lambda+2)(\theta\lambda+1)^{2H-1}}$$

that admits a unique holomorphic extension up to $\Re(\lambda) > -\frac{1}{\theta}$. Thus we have $\mathrm{abs}(V_{2,H}) = -\frac{1}{\theta}$. Now let us fix $c > -\frac{1}{\theta}$ and observe that

$$|\mathcal{L}[V'_{2,H}](c+i\omega)| = \frac{2H\theta^{2H}\Gamma(2H)}{|c\theta+2+\omega\theta i||c\theta+1+\omega\theta i|^{2H-1}}.$$

For $\omega \in [-1,1]$, being the function $\omega \mapsto |\mathcal{L}[V'_{2,H}](c+i\omega)|$ continuous, it is also bounded. For $|\omega| > 1$ we have

$$|\mathcal{L}[V'_{2,H}](c+i\omega)| = \frac{2H\theta^{2H}\Gamma(2H)}{|\omega|^2|\frac{c\theta+2}{\omega}+\theta i||\frac{c\theta+1}{\omega}+\theta i|^{2H-1}} \leq \frac{2H\Gamma(2H)}{|\omega|^2},$$

that is in $L^p(\mathbb{R}\setminus[-1,1])$ for any $p \geq 1$, concluding the proof. □

Since we have some informations on $V_{2,H}$, we can also deduce some properties of the one-dimensional probability density function $p_H(x;t)$ of $U_H(t)$. Indeed, being $U_H(t)$ a Gaussian process with variance $V_{2,H}(t)$ and zero mean, it holds

$$p_H(x;t) = \frac{1}{\sqrt{2\pi V_{2,H}(t)}}e^{-\frac{x}{2V_{2,H}(t)}}, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}^+ \,. \tag{2.5.3}$$

Starting from this equation, one obtains Laplace transformability of p_H (with respect to t) and some properties of the Laplace transform of p_H on vertical lines (see [**27**, Lemma 6.1 and Corollary 6.1]).

Lemma 2.5.4. *For any $x \in \mathbb{R}^*$ there exists a constant $C_H(x)$ such that $\sup_{t\in(0,+\infty)} p_H(t,x) \leq C_H(x)$. Moreover, $p_H(x;t)$ is Laplace transformable with respect to t for fixed $x \in \mathbb{R}^*$ with abscissa of convergence $\mathrm{abs}(p_H(x;\cdot)) \leq 0$. Moreover, for any $c > 0$ the function $\omega \in \mathbb{R} \mapsto \mathcal{L}[p_H(x;\cdot)](c+i\omega)$ is bounded.*

Proof. Let us first show that for $x \in \mathbb{R}^*$ the function $p_H(x;\cdot)$ is bounded. Since $V_{2,H}(t) \sim t^{2H}$ as $t \to 0^+$, there exist two positive constants $C_1(H)$ and $C_2(H)$ such that, for any $t \in [0,1]$,

$$\sqrt{2\pi V_{2,H}(t)} \geq C_1(H)t^H \qquad\qquad 2V_{2,H}(t) \leq C_2(H)t^{2H}.$$

Hence in particular we have, for any $t \in (0,1]$ and $x \in \mathbb{R}^*$,

$$0 \leq p_H(x;t) \leq \frac{1}{C_1(H)t^H}e^{-\frac{x^2}{C_2(H)t^{2H}}}.$$

Taking the limit as $t \to 0^+$ we obtain that for any $x \in \mathbb{R}^*$ it holds $p_H(x;0+) = 0$. Moreover

$$\lim_{t\to+\infty} p_H(x;t) = \frac{1}{\sqrt{2\pi V_{2,H}(\infty)}}e^{-\frac{x}{2V_{2,H}(\infty)}} < +\infty.$$

Thus we can extend by continuity the function $p_H(x;\cdot)$ to $[0,+\infty]$ and Weierstrass' theorem completes the proof.

Concerning the Laplace transform, since $p_H(x;\cdot)$ is bounded, it holds $\mathrm{abs}(p_H(x;\cdot)) \leq 0$. Moreover, if we fix $c > 0$, we have

$$|\mathcal{L}[p_H(x;\cdot)](c+i\omega)| \leq \int_0^{+\infty} e^{-ct}p_H(x;t)dt \leq \frac{C_H(x)}{c}$$

concluding the proof. □

Remark 2.5.5. Let us observe that, since $V_{2,H}(t) \sim t^{2H}$ as $t \to 0^+$, we have that $p_H(0;t)$ is also Laplace transformable with $\mathrm{abs}(p_H(0;\cdot)) = 0$.

Finally, let us work with the time-derivative of $p_H(x;t)$. Indeed we can show that $p_H(x;\cdot) \in C^1(0,+\infty)$ and it is also Lipschitz. Thus, as a consequence, we will obtain Laplace transformability of $\frac{\partial p_H}{\partial t}(x;t)$ (see [**27**, Lemma 6.3]).

LEMMA 2.5.6. *For any $x \in \mathbb{R}^*$ the function $p_H(x;\cdot) \in C^1(0,+\infty) \cap W^{1,\infty}(0,+\infty)$. Moreover, $\frac{\partial p_H}{\partial t}(x;\cdot)$ is Laplace transformable with* $\operatorname{abs}\left(\frac{\partial p_H}{\partial t}(x;\cdot)\right) \le 0$ *and then, for any $\lambda \in \mathbb{H}$, it holds*

$$\mathcal{L}\left[\frac{\partial p_H}{\partial t}(x;\cdot)\right](\lambda) = \lambda\, \mathcal{L}[p_H(x;\cdot)](\lambda).$$

PROOF. Let us first of all observe that, from Equation (2.5.3), we easily get

$$\frac{\partial p_H}{\partial t}(x;t) = \frac{V'_{2,H}(t)}{2}\left(\frac{-V_{2,H}(t)+x^2}{V^2_{2,H}(t)\sqrt{2\pi V_{2,H}(t)}}\right)e^{-\frac{x^2}{2V_{2,H}(t)}}.$$

Moreover, it is easy to see that $\lim_{t\to+\infty} \frac{\partial p_H}{\partial t}(x;t) = 0$. Now let us consider $t \in [0,1]$. Then, by Lemma 2.5.2, we know there exists a positive constant $C_1(H)$ such that $V'_{2,H}(t) \le 2C_1(H)t^{2H-1}$ for any $t \in [0,1]$. Moreover, there exist two positive constants $C_2(H)$ and $C_3(H)$ such that

$$\left|\frac{-V_{2,H}(t)+x^2}{V^2_{2,H}(t)\sqrt{2\pi V_{2,H}(t)}}\right| e^{-\frac{x^2}{2V_{2,H}(t)}} \le C_2(H)\frac{t^{2H}+x^2}{t^{5H}}e^{-\frac{x^2}{C_3(H)t^{2H}}},$$

thus, we have

$$\left|\frac{\partial p_H}{\partial t}(x;t)\right| \le C_1(H)C_2(H)\frac{t^{2H}+x^2}{t^{3H+1}}e^{-\frac{x^2}{C_3(H)t^{2H}}}.$$

Taking the limit as $t \to 0^+$ we achieve, for any $x \in \mathbb{R}^*$, $\lim_{t\to 0^+} \frac{\partial p_H}{\partial t}(x;t) = 0$. Thus we can extend $\frac{\partial p_H}{\partial t}(x;\cdot)$ on $[0,+\infty]$ and Weierstrass' theorem completes the proof. $\square$

Now that we have these properties on the fOU process before we apply the time-change, we can define the TCfOU process and then consider some basic properties.

2.5.2. Definition and first properties of the Time-Changed fractional Ornstein-Uhlenbeck process. Now let us define the time-changed fractional Ornstein-Uhlenbeck process, as done in [**27**].

DEFINITION 2.5.2. Let us consider $\Phi \in \mathcal{BF}$ a driftless Bernstein function, $U_H(t)$ a fOU process and $L_\Phi(t)$ the inverse of the subordinator associated to Φ. Supposing that L_Φ and U_H, are independent we define the **time-changed fractional Ornstein-Uhlenbeck** (TCfOU) process as $U_{H,\Phi}(t) := U_H(L_\Phi(t))$.

Before going into details of the main characteristics of the process $U_{H,\Phi}(t)$, let us state the following easy technical lemma.

LEMMA 2.5.7. *Let $f : \mathbb{R}^+_0 \to \mathbb{R}$ be a measurable function and $\Phi \in \mathcal{BF}$ a driftless Bernstein function. Define $f_\Phi(t) = \mathbb{E}[f(L_\Phi(t))]$.*

- *If there exists a constant M such that for any $t \ge 0$, $f(t) \le M$, then also $f_\Phi(t) \le M$;*

- *If f is non-negative and Laplace transformable with* $\operatorname{abs}(f)\le 0$*, then also* $\operatorname{abs}(f_\Phi)\le 0$ *and it holds*

$$\mathcal{L}[f_\Phi](\lambda)=\frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[f](\Phi(\lambda));$$

- *If f is increasing (resp. decreasing) also f_Φ is increasing (resp. decreasing).*

Proof. The first property is obvious by monotonicity of the expectation operator. Concerning the the second property, we have, by Fubini's theorem,

$$\begin{aligned}\mathcal{L}[f_\Phi](\lambda)&=\int_0^{+\infty}f(y)\int_0^{+\infty}e^{-\lambda t}f_\Phi(y;t)dtdy\\&=\frac{\Phi(\lambda)}{\lambda}\int_0^{+\infty}e^{-y\Phi(\lambda)}f(y)dy=\frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[f](\Phi(\lambda)).\end{aligned}$$

Finally, concerning last property, we have, since f is increasing and L_Φ is almost surely increasing, for $t\ge s$,

$$f_\Phi(t)-f_\Phi(s)=\mathbb{E}[f(L(t))-f(L(s))]\ge 0.$$

□

As a first step, let us prove that the process $U_{H,\Phi}$ admits all non-negative finite order moments (see [**27**, Lemma 3.1])

Proposition 2.5.8. *Let* $V_{n,H,\Phi}(t)=\mathbb{E}[|U_{H,\Phi}(t)|^n]$*. Then*

$$V_{n,H,\Phi}(t)=\int_0^{+\infty}V_{n,H}(s)f_\Phi(s;t)ds.$$

Moreover, setting, for $n\ge 1$, $V_{n,H}(\infty)=\sqrt{\frac{2^n}{\pi}}\Gamma\left(\frac{n+1}{2}\right)\sqrt{V_{2,H}^n(\infty)}$*, we have that, for any $n\ge1$, $V_{n,H,\Phi}(t)$ is increasing and* $\lim_{t\to+\infty}V_{n,H,\Phi}(t)=V_{n,H}(\infty)$.

Proof. Let us first observe that, by conditioning and using the fact that U_H and L_Φ are independent:

$$V_{n,H,\Phi}(t)=\int_0^{+\infty}\mathbb{E}[U_H(s)|L_\Phi(t)=s]f_\Phi(s,t)ds=\int_0^{+\infty}V_{n,H}(s)f_\Phi(s;t)ds.$$

Thus, in particular, we can write $V_{n,H,\Phi}(t)=\mathbb{E}[V_{n,H}(L_\Phi(t))]$. First of all, since $V_{n,H}(t)$ is increasing, this implies that also $V_{n,H,\Phi}(t)$ is increasing. Now, let us observe, by final value theorem (see [**46**]), it holds

$$\lim_{\lambda\to0^+}\lambda\,\mathcal{L}[V_{n,H}](\lambda)=V_{n,H}(\infty).$$

Hence, we have, by Lemma 2.5.7, since $\Phi(0)=0$,

$$\lim_{\lambda\to0^+}\lambda\,\mathcal{L}[V_{n,H,\Phi}](\lambda)=\lim_{\lambda\to0^+}\Phi(\lambda)\,\mathcal{L}[V_{n,H}](\Phi(\lambda))=V_{n,H}(\infty).$$

By final value theorem again, we have $\lim_{t\to+\infty}V_{n,H,\Phi}(t)=V_{n,H}(\infty)$. □

Now we want to obtain some information on the one-dimensional law of $U_{H,\Phi}$. First of all, let us show that $U_{H,\Phi}$ admits a one-dimensional probability density function, via the characteristic function (see [**27**, Proposition 4.1 and Corollary 4.1]).

PROPOSITION 2.5.9. *If for any $t>0$ it holds $\mathbb{E}[L_\Phi^{-H}(t)]<+\infty$, then $U_{H,\Phi}(t)$ is absolutely continuous and its probability density function $p_{H,\Phi}(x;t)$ satisfies:*

$$p_{H,\Phi}(x;t)=\int_0^{+\infty}p_{H,\Phi}(x;s)f_\Phi(s;t)ds. \tag{2.5.4}$$

Moreover, if for some $n\in\mathbb{N}$, it holds $\mathbb{E}[L_\Phi^{-(n+1)H}(t)]<+\infty$, then $p_{H,\Phi}(x;t)$ is differentiable in x n times.

PROOF. Let us consider the characteristic functions $\varphi_H(z;t)=\mathbb{E}[e^{izU_H(t)}]$ and $\varphi_{H,\Phi}(z;t)=\mathbb{E}[e^{izU_{H,\Phi}(t)}]$. Arguing as in Proposition 2.5.8 we have

$$\varphi_{H,\Phi}(z;t)=\int_0^{+\infty}\varphi_H(z;s)f_\Phi(s;t)ds.$$

Since $U_H(t)$ is Gaussian with zero mean and variance $V_{2,H}(t)$, it holds

$$\varphi_H(z;t)=e^{-\frac{z^2}{2}V_{2,H}(t)}$$

thus we have

$$\varphi_{H,\Phi}(z;t)=\int_0^{+\infty}e^{-\frac{z^2}{2}V_{2,H}(s)}f_\Phi(s;t)ds.$$

Now we want to show that $z\mapsto\varphi_{H,\Phi}(z;t)$ belongs to $L^1(\mathbb{R})$, since, in such case, a simple application of Lévy's inversion theorem implies the existence of $p_{H,\Phi}$. To do this, let us use Fubini's theorem, since all the integrands involved are non-negative, to obtain

$$\int_{\mathbb{R}}\varphi_{H,\Phi}(z;t)dz=(2\pi)^{\frac{1}{2}}\int_0^{+\infty}\frac{1}{\sqrt{V_{2,H}(s)}}f_\Phi(s;t)ds.$$

Now, since $V_{2,H}$ is increasing, we have

$$\int_1^{+\infty}\frac{1}{\sqrt{V_{2,H}(s)}}f_\Phi(s;t)ds\le\frac{1}{\sqrt{V_{2,H}(1)}}.$$

On the other hand, since as $s\to 0$ it holds $V_{2,H}\sim s^{2H}$, there exists a constant $C_1(H)>0$ such that for any $s\in[0,1]$ it holds $V_{2,H}(s)\ge C_1(H)s^{2H}$ and then

$$\int_1^{+\infty}\frac{1}{\sqrt{V_{2,H}(s)}}f_\Phi(s;t)ds\le\frac{1}{\sqrt{C_1(H)}}\mathbb{E}[L_\Phi^{-H}(t)]<+\infty.$$

Thus $\varphi_{H,\Phi}(\cdot;t)\in L^1(\mathbb{R})$ and then there exists $p_{H,\Phi}(x;t)$. Moreover, Lévy's inversion theorem gives us a formula to obtain $p_{H,\Phi}(x;t)$:

$$p_{H,\Phi}(x;t)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-izx}\varphi_{H,\Phi}(z;t)dz=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-izx}\int_0^{+\infty}\varphi_H(z;s)f_\Phi(s;t)dsdz.$$

Applying again Fubini's theorem and Lévy's inversion theorem we get

$$p_{H,\Phi}(x;t)=\int_0^{+\infty}\frac{1}{2\pi}\int_{\mathbb{R}}e^{-izx}\varphi_H(z;s)dzf_\Phi(s;t)ds=\int_0^{+\infty}p_H(x;s)f_\Phi(s;t)ds.$$

Finally, suppose that $\mathbb{E}[L_\Phi^{-(n+1)H}(t)]<+\infty$. Setting $C_n=\int_0^{+\infty}z^ne^{-\frac{z^2}{2}}dz$, we have

$$\int_0^{+\infty}z^n\varphi_{H,\Phi}(z;t)dz=C_n\int_0^{+\infty}(V_{2,H}(s))^{-\frac{n+1}{2}}f_\Phi(s;t)ds.$$

Arguing as before and using the fact that the function $|z|^n\varphi_{H,\Phi}(z;t)$ is even, we have that $z \mapsto z^n\varphi_{H,\Phi}(z;t)$ belongs to $L^1(\mathbb{R})$ and then, by [**127**, Theorem 9.2], it is enough to obtain n-times differentiability of $p_{H,\Phi}(x;t)$ in x. $\square$

REMARK 2.5.10. Let us observe that the hypotheses $\mathbb{E}[L_\Phi^{-2H}(t)] < +\infty$ gives a sufficient condition for differentiability on the whole real line. However, this is actually a sufficient condition to achieve differentiability in 0, since for any $x \neq 0$ it holds

$$\frac{\partial p_H}{\partial x}(x;t) = -\frac{x}{\sqrt{2\pi V_{2,H}^3(t)}}e^{-\frac{x^2}{2V_{2,H}(t)}}.$$

Thus it is easy to check that if $x \in (a,b)$ such that $0 \notin (a,b)$, then $\frac{\partial p_H}{\partial x}(x;t)$ is dominated by a $L^1(0,+\infty)$ function depending on t and then we can differentiate under the integral sign in (2.5.4).
Moreover, we have

$$\frac{\partial p_{H,\Phi}}{\partial x}(x;t) = -\int_0^{+\infty}\frac{x}{\sqrt{2\pi V_{2,H}^3(s)}}e^{-\frac{x^2}{2V_{2,H}(s)}}f_\Phi(s;t)ds.$$

The same holds for the second derivative.
Let us observe that if $H \ge 1/2$, then $\mathbb{E}[L_\Phi^{-(n+1)H}(t)] = +\infty$ for any $t > 0$. However, despite we fixed the case $H > 1/2$, the proof of the previous Proposition easily adapts to the case $H < 1/2$, in which we can gain some regularity. Indeed, if we consider a tempered α-stable inverse subordinator, then $p_{H,\Phi}$ admits any derivative in x up to $n = \lfloor \frac{1}{H} - 1\rfloor - 1$, since we will not have integrability problems near 0 (where $f_\Phi(0+;t) = \bar{\nu}_\Phi(t) > 0$) while the asymptotic exponential bound given in [**10**, Lemma 4.6] gives us integrability at infinity.

Let us give the main example.

EXAMPLE 2.5.1. If we consider $\Phi(\lambda) = \lambda^\alpha$ and then $L_\alpha(t)$ an inverse α-stable subordinator, then we have for any $n \in \mathbb{N}_0$, by using Equation (1.2.3) and the change of variables $z = ts^{-1-\frac{1}{\alpha}}$,

$$\mathbb{E}[L_\alpha^{-(n+1)H}(t)] = \frac{t}{\alpha}\int_0^{+\infty} s^{-(n+1)H-1-\frac{1}{\alpha}}g_\alpha(ts^{-\frac{1}{\alpha}})ds = t^{-\alpha(n+1)H}\,\mathbb{E}[\sigma_\alpha^{\alpha(n+1)H}(1)].$$

However, $\alpha(n+1)H < \alpha$ if and only if $H < \frac{1}{n+1}$. If we are considering $H \ge 1/2$, then the only case is $n = 0$. Thus we obtain the existence of $p_{H,\alpha}(x;t)$ but not its differentiability in $x = 0$. In particular, as we stated before, $p_{H,\alpha}(x;t)$ is differentiable for any $x \neq 0$ and its derivative is given by

$$\frac{\partial p_{H,\alpha}}{\partial x}(x;t) = -\int_0^{+\infty}\frac{x}{\sqrt{2\pi V_{2,H}^3(s)}}e^{-\frac{x^2}{2V_{2,H}(s)}}f_\alpha(s;t)ds.$$

Now let us make some observation concerning Bernstein functions before exploiting an important property of the Laplace transform of $p_{H,\Phi}$. We know by Proposition 1.1.3 that any Bernstein function Φ admits an homolorphic extension on $\mathbb{H}$. In particular this means that Φ is locally invertible in $\mathbb{H}\backslash Z$ where Z is at most countable (since the zeros of the derivative Φ' are at most countable, being Φ analytic). If, by contradiction, for any $c > 0$ there exists $z \in \mathbb{R}$ such that $\Phi'(c+iz) = 0$, then the zeros of Φ' are not countable. Thus there exists a whole

vertical line $r_c := \{c+iz,\ z \in \mathbb{R}\}$ such that $\Phi'(c+iz) \neq 0$. In particular we have that Φ is invertible on such vertical line. Moreover, this argument still holds if we ask for $c \in (a,b)$ whenever $a < b$.
Thus, we have that Φ is invertible on any vertical line r_c for $c \in \mathbb{R}^+ \setminus Z_{\mathbb{R}}$ where $Z_{\mathbb{R}}$ is the projection of Z on the real axis.
On the other hand, we have, by Lemma 2.5.7, that $p_{H,\Phi}$ is bounded and Laplace transformable for $\lambda \in \mathbb{H}$ and

$$\mathcal{L}[p_{H,\Phi}(x;\cdot)](\lambda) = \frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[p_H(x;\cdot)](\Phi(\lambda)).$$

Now let $U \subset \mathbb{H} \setminus Z$ be an open set in which Φ is invertible. Then we have, for any $\lambda \in \Phi(U)$:

$$\frac{\Phi^{-1}(\lambda)}{\lambda}\,\mathcal{L}[p_{H,\Phi}(x;\cdot)](\Phi^{-1}(\lambda)) = \mathcal{L}[p_H(x;\cdot)](\lambda).$$

Now let us consider another local invertibility open set $V \subseteq \mathbb{H} \setminus Z$ and $\lambda \in \Phi(U) \cap \Phi(V)$ if $\Phi(U) \cap \Phi(V) \neq \emptyset$. Let Φ_*^{-1} be the local inverse on V. Since the right-hand side of the previous relation does not depend on Φ, we have

$$\frac{\Phi^{-1}(\lambda)}{\lambda}\,\mathcal{L}[p_{H,\Phi}(x;\cdot)](\Phi^{-1}(\lambda)) = \frac{\Phi_*^{-1}(\lambda)}{\lambda}\,\mathcal{L}[p_{H,\Phi}(x;\cdot)](\Phi_*^{-1}(\lambda)).$$

In particular we have show the following result.

Lemma 2.5.11. *The quantity* $\frac{\Phi^{-1}(\lambda)}{\lambda}\,\mathcal{L}[p_{H,\Phi}(x;\cdot)](\Phi^{-1}(\lambda))$ *is well defined for any* $\lambda \in \Phi(\mathbb{H} \setminus Z)$. *In particular it is independent of the choice of the local inverse map.*

Concerning the time-derivative of $p_{H,\Phi}(x;t)$, as exploited in Equation (2.5.4), it strictly depends on the regularity of the density of the inverse subordinator.
In case $\Phi(\lambda) = \lambda^\alpha$, for instance, we are able to show the following result (see [**26**, Proposition 3.5]).

Proposition 2.5.12. *Let* $\alpha \in (0,1)$ *and* $x \neq 0$. *Then* $p_{H,\alpha}(x;\cdot) \in C^1(0,+\infty)$.

Proof. By using the change of variables $w = ts^{-\frac{1}{\alpha}}$ we have

$$p_{H,\alpha}(x;t) = \int_0^{+\infty} p_H\left(x;\left(\frac{t}{w}\right)^\alpha\right) g_\alpha(w)dw. \tag{2.5.5}$$

Let us fix $t > 0$ and $0 < t_1 < t_2$ such that $t \in (t_1, t_2)$. Fix $x \neq 0$. First of all, let us observe that

$$\frac{d}{dt}p_H\left(x;\left(\frac{t}{w}\right)^\alpha\right) = \alpha t^{\alpha-1}w^{-\alpha}\frac{\partial p_H}{\partial y}(x;y)_{|y=\left(\frac{t}{w}\right)^\alpha}$$

and

$$\frac{\partial p_H}{\partial y}(x;y) = \frac{V_{2,H}'(y)(x^2 - V_{2,H}(y))}{\sqrt{8\pi V_{2,H}^3(y)}}e^{-\frac{x^2}{2V_{2,H}(y)}}.$$

By using Lemma 2.5.2 and the fact that $V_{2,H} \sim t^{2H}$ as $t \to 0^+$, we have

$$\left|\frac{\partial p_H}{\partial y}(x;y)\right| \le C_1 y^{-3H-1}e^{-\frac{x^2}{C_2 y^{2H}}}$$

and then

$$\left|\frac{d}{dt}p_H\left(x;\left(\frac{t}{w}\right)^\alpha\right)\right| \le \alpha C_1 t_1^{-3\alpha H-1}w^{3H\alpha}e^{-\frac{w^{3H\alpha}x^2}{C_2 t_2^{2\alpha H}}}$$

as $w > t$. Defining

$$C_3 := \sup_{w>t_1} w^{3H\alpha} e^{-\frac{w^{3H\alpha} x^2}{C_2(H) t_2^{2\alpha H}}} \qquad \text{and} \qquad C_4 := \alpha C_1(x,H) t_1^{-3\alpha H - 1} C_3,$$

we have

$$\left| \frac{d}{dt} p_H \left(x; \left(\frac{t}{w} \right)^\alpha \right) \right| \le C_4$$

for $w \ge t$.
On the other hand, for $y \ge 1$, we have, by using the asymptotics as $y \to +\infty$ given in 2.5.2,

$$\left| \frac{\partial p_H}{\partial y}(x;y) \right| \le C_5 y^{2-2H} e^{-\frac{y}{\theta}}$$

and then, as $w < t$,

$$\left| \frac{d}{dt} p_H \left(x; \left(\frac{t}{w} \right)^\alpha \right) \right| \le \alpha C_5 t_*^{(3-2H)\alpha H - 1} w^{(2H-3)\alpha} e^{-\frac{t_1^\alpha}{\theta w^\alpha}},$$

where

$$t_* = \begin{cases} t_1 & (3-2H)\alpha - 1 < 0 \\ t_2 & (3-2H)\alpha - 1 \ge 0. \end{cases}$$

As before, setting

$$C_6 := \sup_{w \in (0,t_2)} w^{\alpha(2H-3)} e^{-\frac{t_1^\alpha}{\theta w^\alpha}} \qquad \text{and} \qquad C_7 := \alpha C_5 t_*^{(3-2H)\alpha H - 1} C_6,$$

we have, for any $w < t$,

$$\left| \frac{d}{dt} p_H \left(x; \left(\frac{t}{w} \right)^\alpha \right) \right| \le C_7.$$

Finally, taking $C_8 = \max\{C_6, C_7\}$, it holds, for any $w \in (0, +\infty)$,

$$\left| \frac{d}{dt} p_H \left(x; \left(\frac{t}{w} \right)^\alpha \right) \right| \le C_8.$$

Being $g_\alpha(w)dw$ a probability measure, this is enough to guarantee that $p_{H,\alpha}(x;t)$ is differentiable in t as $x \neq 0$ by differentiating under the integral sign. □

Now let give some final considerations concerning the existence of a limit distribution. As $H = 1/2$, it has been shown in [**67**] that despite the time-changed Ornstein-Uhlenbeck process is not Gaussian anymore, its limit distribution is still Gaussian. The adopted technique relies on the spectral decomposition, that here is missing. However, in some cases, we are still able to prove that the process admits a Gaussian limit distribution. For instance, let us refer to [**26**, Propositiomn 3.4] for the α-stable case.

Proposition 2.5.13. *It holds*

$$\lim_{t \to +\infty} p_{H,\alpha}(x;t) = \frac{1}{\sqrt{2\pi V_{2,H}(\infty)}} e^{-\frac{x^2}{2V_{2,H}(\infty)}}.$$

Proof. Starting from equation (2.5.5), let us consider the function $h_1(t) = (2\pi t)^{-1/2} e^{-x^2/2t}$ and let us observe that it is bounded for $x \neq 0$. Hence we easily conclude that we can take the limit under the integral sign by dominated convergence theorem.

Let us now consider $x=0$ and suppose $t>t_0>0$. Let us observe that the function $h_2(t)=\frac{V_{2,H}(t)}{\sqrt{t^{2H}\wedge 1}}$ is positive on $(0,+\infty)$. Moreover $\lim_{t\to 0} h_2(t)=1>0$ and $\lim_{t\to+\infty} h_2(t)=V_{2,H}(\infty)>0$, thus there exists $C_1=\inf_{t>0} h_2(t)>0$. Setting $C_2=(2\pi C_1)^{-1/2}$ and defining $h_3(w)=C_2 h_2((t_0/w)^{\alpha})$, we have

$$p_H\left(0,\left(\frac{t}{w}\right)^{\alpha}\right)\le h_3(w).$$

Finally, observe that $h_3(\cdot)g_\alpha(\cdot)\in L^1(0,+\infty)$, since g_α is a probability density function and h_3 is bounded near 0, while $\mathbb{E}[\sigma_\alpha^{H\alpha}(1)]<+\infty$ being $H\alpha<\alpha$. Thus, even in this case, we can take the limit inside the integral sign, concluding the proof. □

2.6. The Generalized Fokker-Planck equation of the TCfOU process

In this section we will introduce the Generalized Fokker-Planck equation for the TCfOU process and will discuss some issues concerning uniqueness and regularity of its solutions. To do this, we first need to study a particular class of functions that will be introduced in the following.

2.6.1. Inverse-subordinated functions and weighted inverse-subordinated functions. We have already seen that, generally, functions of the form $f_\Phi(t)=\mathbb{E}[f(L_\Phi(t))]$ inherit different properties from both the Bernstein function Φ and the transformed function f. Thus, let us consider this kind of transformation from an operator point of view. Before doing this, let us stress out that, despite in [**26**, Section 4] the operators are called *subordination operators*, here the name could create confusion with Bochner subordination, hence we will opt for a name related to the fact we are time-changing with an inverse-subordinator.

DEFINITION 2.6.1. Let $\Phi\in\mathcal{BF}$ be a driftless Bernstein function and $f_\Phi(s;t)$ the density of the associated inverse subordinator. Then we denote by $S_\Phi: L^\infty(0,+\infty)\to L^\infty(0,+\infty)$ the **inverse-subordination operator** as

$$S_\Phi v(t)=\mathbb{E}[v(L_\Phi(t))]=\int_0^{+\infty} v(s)f_\Phi(s;t)ds.$$

Moreover, we define the **weighted inverse-subordination operator** as an operator $S_{\Phi,H}: L^\infty(0,+\infty)\to L^\infty(0,+\infty)$ such that

$$S_{\Phi,H}v(t)=\mathbb{E}[V_{2,H}'(L_\Phi)v(L_\Phi(t))]=\int_0^{+\infty} V_{2,H}'(s)v(s)f_\Phi(s;t)ds.$$

In particular we denote by $\mathcal{S}_\Phi$ the range of S_Φ and a function $v_\Phi\in\mathcal{S}_\Phi$ will be called an **inverse-subordinated function**.

Let us state some basic properties of the operators S_Φ and $S_{\Phi,H}$ ([**26**, Lemma 4.1]).

PROPOSITION 2.6.1. *Both the operators S_Φ and $S_{\Phi,H}$ are continuous and bounded. Moreover*

$$\|S_\Phi\|_{L(L^\infty(0,\infty),L^\infty(0,\infty))}\le 1$$

and

$$\|S_{\Phi,H}\|_{L(L^\infty(0,\infty),L^\infty(0,\infty))}\le \left\|V_{2,H}'\right\|_{L^\infty(0,+\infty)}.$$

PROOF. Concerning S_Φ, it is easy to see that $\|S_\Phi v\|_{L^\infty(0,+\infty)} \le \|v\|_{L^\infty(0,+\infty)}$ by Lemma 2.5.7. Concerning $S_{\Phi,H}$, one has only to check that

$$S_{\Phi,H}v(t) = S_\Phi(V'_{2,H}v)(t)$$

and then Lemma 2.5.7 concludes the proof. □

Concerning the Laplace transform, let us work first only on S_Φ (see [**26**, Proposition 4.2]).

PROPOSITION 2.6.2. *Let $v \in L^\infty(0,+\infty)$ and $\Phi \in \mathcal{BF}$ be a driftless unbounded Bernstein function. Then* $\mathrm{abs}(S_\Phi v) \le 0$ *and, for any $\lambda \in \mathbb{H}$,*

$$\mathcal{L}[S_\Phi v](\lambda) = \frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[v](\Phi(\lambda)).$$

In particular, S_Φ and $S_{\Phi,H}$ are injective.

PROOF. First of all, let us observe that the Laplace transform relation follows from Lemma 2.5.7 and the fact that $v \in L^\infty(0,+\infty)$. Concerning injectivity, let us consider only real $\lambda > 0$. Φ is invertible in $(0,+\infty)$ and, being Φ driftless and unbounded, $\lim_{\lambda\to 0}\Phi(\lambda) = 0$ and $\lim_{\lambda\to+\infty}\Phi(\lambda) = +\infty$. Thus $\Phi^{-1} : (0,+\infty) \to (0,+\infty)$. Since S_Φ is linear, we only have to show that $\mathrm{Ker}(S_\Phi) = \{0\}$. Thus, suppose $S_\Phi v = 0$. Taking the Laplace transform on both sides we have $\frac{\Phi(\lambda)}{\lambda}\mathcal{L}[v](\Phi(\lambda)) = 0$. Since $\frac{\Phi(\lambda)}{\lambda} \neq 0$ for any $\lambda > 0$ (if Φ is not constant, then it must be strictly increasing), we have $\mathcal{L}[v](\Phi(\lambda)) = 0$. Now, considering $\lambda = \Phi^{-1}(\eta)$ for any $\eta > 0$, we obtain $\mathcal{L}[v](\eta) = 0$ for any $\eta > 0$. Thus, being the Laplace transform injective, it holds $v \equiv 0$.
With a similar argument, for $S_{\Phi,H}v = 0$ we have $\mathcal{L}[V'_{2,H}v](\eta) = 0$ for any $\eta > 0$. Thus, by injectivity of the Laplace transform we have $V'_{2,H}v \equiv 0$. Since we have shown that $V'_{2,H} > 0$ for any $t \in (0,+\infty)$, it holds $v \equiv 0$, concluding the proof. □

Now let us consider the action of the operators S_Φ and $S_{\Phi,H}$ on functions of more variables. In particular, let $I \subseteq \mathbb{R}$ be a closed bounded interval. Then we say that a function $v \in L^\infty(\mathbb{R}^+; C^k(I))$ if $v : I \times \mathbb{R}^+ \to \mathbb{R}$, $\frac{\partial^k v}{\partial x^k}(x;t)$ is well defined for any $x \in I$ and continuous for fixed $t \in \mathbb{R}^+$ and

$$\|v\|_{L^\infty(\mathbb{R}^+;C^k(I))} = \sup_{t\in\mathbb{R}^+}\sum_{j=0}^{k}\left\|\frac{\partial^k v}{\partial j^k}(\cdot;t)\right\|_{L^\infty(I)} < +\infty.$$

With this norm, the space $L^\infty(\mathbb{R}^+; C^k(I))$ is a Banach space. Moreover, as a direct consequence of dominated convergence theorem, we obtain the following result.

PROPOSITION 2.6.3. *Fix $I \subseteq \mathbb{R}$ a closed bounded interval. The operators $S_\Phi : L^\infty(\mathbb{R}^+; C^k(I)) \to L^\infty(\mathbb{R}^+; C^k(I))$ and $S_{\Phi,H} : L^\infty(\mathbb{R}^+; C^k(I)) \to L^\infty(\mathbb{R}^+; C^k(I))$ are well-defined, linear and bounded.*

Let us stress out that such proposition easily follows from the actual statement of [**26**, Lemma 4.1], in which the inverse subordination operators are defined via Bochner integrals.
In the special case of the inverse α-stable subordinator, let us denote the operators as S_α and $S_{\alpha,H}$. As another direct consequence of dominated convergence theorem, together with the expression of $f_\alpha(s;t)$, we get the following easy result (see [**26**, Proposition 4.6]).

PROPOSITION 2.6.4. *Suppose $v \in C^1(\mathbb{R}^+)$. If there exist two constants $C > 0$ and $\beta \in \left(\frac{\alpha-1}{\alpha}, 2\right)$ such that $|v'(t)| \le Ct^{-\beta}$, then $S_\alpha v \in C^1(\mathbb{R}^+)$ and*

$$\frac{d}{dt}S_\alpha v(t) = \alpha t^{-1} S_\alpha(zv'(z))(t).$$

PROOF. We can rewrite

$$S_\alpha v(t) = \int_0^{+\infty} v\left(\left(\frac{t}{w}\right)^\alpha\right) g_\alpha(w)dw.$$

If we derive under the integral sign we have

$$\frac{d}{dt}v\left(\left(\frac{t}{w}\right)^\alpha\right) = \alpha t^{\alpha-1}w^{-\alpha}v'\left(\left(\frac{t}{w}\right)^\alpha\right).$$

By hypothesis, considering $t \in [t_1, t_2]$ for some $0 < t_1 < t_2$, we have

$$\left|\frac{d}{dt}v\left(\left(\frac{t}{w}\right)^\alpha\right)\right| \le \alpha t_1^{\alpha-1}w^{\beta\alpha-\alpha}t_*^{-\beta\alpha},$$

where $t_* = \begin{cases} t_1 & \beta > 0 \\ t_2 & \beta \le 0. \end{cases}$ Thus, since $w^{\beta\alpha-\alpha}g_\alpha(w)$ belongs to $L^1(0, +\infty)$, we can differentiate under integral sign to obtain

$$\begin{aligned}
\frac{d}{dt}S_\alpha v(t) &= \int_0^{+\infty} \alpha t^{\alpha-1}w^{-\alpha}v'\left(\left(\frac{t}{w}\right)^\alpha\right) g_\alpha(w)dw \\
&= \int_0^{+\infty} \alpha t^{\alpha-1}t^{-\alpha}zv'(z)\frac{1}{\alpha}tz^{-\frac{1}{\alpha}-1}g_\alpha(tz^{-\frac{1}{\alpha}})dz \\
&= \alpha t^{-1}\int_0^{+\infty} zv'(z)f(z;t)dz,
\end{aligned}$$

concluding the proof. □

2.6.2. The weighted Laplace transform. Now we need to introduce another operator. For any function $v \in L^\infty(\mathbb{R}^+)$ we consider

$$L_H v(\lambda) = \mathcal{L}[V'_{2,H}v](\lambda) \qquad \lambda \in \mathbb{H},$$

that we call a **weighted Laplace transform**. As one can easily deduce from Proposition 2.6.2, there is a link between the weighted Laplace transform and the weighted inverse subordination operator (see [**26**, Corollary 4.3]).

PROPOSITION 2.6.5. *Let $v \in L^\infty(\mathbb{R}^+)$. Then we have, for any $\lambda \in \mathbb{H}$,*

$$\mathcal{L}[S_{\Phi,H}v](\lambda) = \frac{\Phi(\lambda)}{\lambda}L_H v(\Phi(\lambda)).$$

PROOF. It easily follows from

$$\mathcal{L}[S_{\Phi,H}v](\lambda) = \mathcal{L}[S_\Phi(V'_{2,H}v)](\lambda) = \frac{\Phi(\lambda)}{\lambda}\mathcal{L}[V'_{2,H}v](\Phi(\lambda)) = \frac{\Phi(\lambda)}{\lambda}L_H(\Phi(\lambda)).$$

□

Moreover, if we consider a function $v \in L^\infty(\mathbb{R}^+; C^k(I))$ for some compact interval $I \subset \mathbb{R}$, being $\frac{\partial^k}{\partial x^k}$ a closed operator, it holds $\frac{\partial^k}{\partial x^k}L_H v(\cdot; x) = L_H\left(\frac{\partial^k}{\partial x^k}v(\cdot; x)\right)$, by means of [**13**, Proposition 1.7.6].

Now let us observe that under suitable hypotheses on v we can write a different representation of the operator L_H (see [**26**, Proposition 4.4]).

PROPOSITION 2.6.6. *Fix $c_1 < 0 < c_2$ with $c_1 - c_2 > -1/\theta$ and let $v \in L^\infty(\mathbb{R}^+)$. Suppose one of the following properties hold:*

a) v is Lipschitz and $x \in \mathbb{R} \mapsto \mathcal{L}[v](c_2 + ix)$ is in $L^1(\mathbb{R})$;

b) v belongs to $L^2(\mathbb{R}^+)$ and $x \in \mathbb{R} \mapsto \mathcal{L}[v](c_2 + ix)$ is in $L^2(\mathbb{R})$.

Then it holds

$$\begin{aligned} L_H v(\lambda) = \frac{1}{4\pi^2} \int_0^{+\infty} e^{-\lambda t} \lim_{R \to +\infty} \int_{-\infty}^{+\infty} e^{(c_1 + iw)t} \\ \times \int_{-R}^{R} \mathcal{L}[V'_{2,H}](c_1 - c_2 + i(w-u))\, \mathcal{L}[v](c_2 + iu) du dw dt. \end{aligned}$$

PROOF. Let us define the function

$$I(r,R) = \int_{-r}^{r} e^{(c_1+iw)t} \int_{-R}^{R} \mathcal{L}[V'_{2,H}](c_1 - c_2 + i(w-u))\, \mathcal{L}[v](c_2+iu) du dw.$$

Since, for fixed c_1, c_2, all the involved functions are bounded, we can use Fubini's theorem to achieve

$$\begin{aligned} I(r,R) &= \int_{-R}^{R} e^{(c_2+iu)t}\, \mathcal{L}[v](c_2+iu) \int_{-r}^{r} e^{(c_1-c_2+i(w-u))t}\, \mathcal{L}[V'_{2,H}](c_1-c_2+i(w-u)) dw du \\ &= \int_{-R}^{R} e^{(c_2+iu)t}\, \mathcal{L}[v](c_2+iu) \int_{-r-u}^{r-u} e^{(c_1-c_2+iy)t}\, \mathcal{L}[V'_{2,H}](c_1-c_2+iy) dy du. \end{aligned}$$

Now, let us observe that for $u \in [-R, R]$ we have $|\,\mathcal{L}[v](c_2+iu)| < C$ for some constant C, thus it holds

$$\begin{aligned} &\left| e^{(c_2+iu)t}\, \mathcal{L}[v](c_2+iu) \int_{-r-u}^{r-u} e^{(c_1-c_2+iy)t}\, \mathcal{L}[V'_{2,H}](c_1-c_2+iy) dy \right| \\ &\qquad \le C e^{c_1 t} \int_{-\infty}^{+\infty} |\,\mathcal{L}[V'_{2,H}](c_1-c_2+iy)| dy < +\infty, \end{aligned}$$

since $\mathcal{L}[V'_{2,H}]$ belongs to L^1 with respect to any vertical line. Hence, by dominated convergence theorem, we can take the limit inside the integral sign. Moreover, since $\mathcal{L}[V'_{2,H}]$ belongs to L^2 with respect to any vertical line and $V'_{2,H}$ belongs to L^2, we can apply Paley-Wiener theorem (see [**127**, Theorem 19.2]) to obtain

$$\lim_{r \to +\infty} I(r,R) = 2\pi V'_{2,H}(t) \int_{-R}^{R} e^{(c_2+iu)t}\, \mathcal{L}[v](c_2+iu) du.$$

Moreover, we have

$$\frac{1}{4\pi^2} \lim_{R\to+\infty} \lim_{r \to +\infty} I(r,R) = V'_{2,H}(t) v(t),$$

where we used the complex inversion theorem (see [**13**, Theorem 2.3.4]) under hypothesis *a*) or Paley-Wiener theorem under hypothesis *b*).
Taking the Laplace transform on both sides of last identity we complete the proof. □

REMARK 2.6.7. Let us observe that if we can take the limit as $R \to +\infty$ inside the integral, we actually obtain that the inner integrals represent a convolution product between the Laplace transform of $V'_{2,H}$ and v on a vertical line, as obtained in [**75**].

Now that we have this new particular form for L_H, let us consider another operator, which transforms the action of L_H be means on Φ. To do this, let us fix c_1, c_2 as in the previous proposition, but such that Φ is invertible on the vertical line r_{c_2}. Now, for any function $\bar{v} : \mathbb{H} \to \mathbb{C}$, we can define the operator

(2.6.1)

$$\widehat{L}_{\Phi,H}\bar{v}(\lambda) = \frac{1}{4\pi^2}\int_0^{+\infty} e^{-\Phi(\lambda)t} \lim_{R\to+\infty}\int_{-\infty}^{+\infty} e^{(c_1+iw)t} \times \int_{-R}^{R} \mathcal{L}[V_{2,H}'](c_1-c_2+i(w-u))\frac{\Phi^{-1}(c_2+iu)}{c_2+iu}\bar{v}(\Phi^{-1}(c_2+iu))dudwdt$$

provided that the involved integrals exist and are finite.
Despite the complicated form of the operator, let us remark that for inverse subordinated functions it is much easier to evaluate (see [**26**, Proposition 4.5]).

PROPOSITION 2.6.8. *Let $v_\Phi = S_\Phi v$ and let $\bar{v}_\Phi = \mathcal{L}[v_\Phi]$. Suppose the hypotheses of Proposition 2.6.6 hold. Then, for any $\lambda \in \mathbb{H}$,*

$$\widehat{L}_{\Phi,H}\bar{v}_\Phi(\lambda) = L_H v(\Phi(\lambda)).$$

PROOF. By Proposition 2.6.2 we know that for any $\lambda \in \mathbb{H}$ it holds

$$\bar{v}_\Phi(\lambda) = \frac{\Phi(\lambda)}{\lambda}\mathcal{L}[v](\Phi(\lambda)).$$

Let us consider $\lambda = c_2 + iu$ with c_2 defined as before. Then Φ is invertible on r_{c_2} and we have

$$\frac{\Phi^{-1}(c_2+iu)}{c_2+iu}\bar{v}_\Phi(\Phi^{-1}(c_2+iu)) = \mathcal{L}[v](c_2+iu).$$

By using last identity in definition (2.6.1) we get

$$\widehat{L}_{\Phi,H}\bar{v}_\Phi(\lambda) = \frac{1}{4\pi^2}\int_0^{+\infty} e^{-\Phi(\lambda)t} \lim_{R\to+\infty}\int_{-\infty}^{+\infty} e^{(c_1+iw)t} \times \int_{-R}^{R} \mathcal{L}[V_{2,H}'](c_1-c_2+i(w-u))\,\mathcal{L}[v](c_2+iu)dudwdt = L_H v(\Phi(\lambda)).$$

$\square$

2.6.3. The generalized Fokker-Planck equation. Now let us consider a function $v : I \times \mathbb{R}^+ \to \mathbb{R}$, where I is some compact interval of $\mathbb{R}$, belonging to $L^\infty(\mathbb{R}^+; C^2(I))$ and let us define, whenever it exists, the operator

$$\mathcal{F}_{\Phi,H}\,v(x;t) = \mathcal{L}^{-1}_{\lambda\to t}\left[\frac{\Phi(\lambda)}{\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_H\,\mathcal{L}[v(x;\cdot)](\lambda)\right](t),$$

where the operator $\widehat{L}_H$ acts on the variable t. Let us denote by $D(\mathcal{F}_{\Phi,H}, I)$ the domain of the operator $\mathcal{F}_{\Phi,H}$. Now we have all the ingredients we need to introduce the generalized Fokker-Planck equation of the TCfOU process.

DEFINITION 2.6.2. We define the **generalized Fokker-Planck equation of the TCfOU** (in $I \times (0,T]$ for $I \subseteq \mathbb{R}$ any interval and $T > 0$, eventually $T = +\infty$) as:

$$\partial_t^\Phi v(x;t) = \frac{1}{2}\mathcal{F}_{\Phi,H}\,v(x;t) \qquad (x;t) \in I \times (0,T]. \tag{2.6.2}$$

Given the equation, now we have to establish what is a solution for such equation. Thus let us first give the following definition of *classical solution* (which is actually the analogous of a Caratheodory solution of a Banach-space valued Ordinary differential equation, see for instance [**49**]).

DEFINITION 2.6.3. We say that a function $v \in L^\infty([0,T];C^2(I))$ is a **classical solution** of Equation (2.6.2) if:

- $v \in D(\mathcal{F}_{\Phi,H}, I)$;
- $v(x;\cdot)$ belongs to the domain of ∂_t^Φ for any $x \in I$;
- Identity (2.6.2) holds for any $x \in I$ and almost any $t \in [0,T]$.

A classical solution is said to be a **strong solution** if $v(x;\cdot) \in C^1((0,T]) \cup W^{1,1}(0,T)$ for any $x \in I$.

On the other hand, due to the definition of $\mathcal{F}_{\Phi,H}$ in terms of composition of inverse Laplace transform, multiplication operator and Laplace transform, classical solutions could be difficult to work with[1], hence we need a weaker form of solution.

DEFINITION 2.6.4. We say that a function $v \in L^\infty(\mathbb{R}^+;C^0(I))$ is a **mild solution** of Equation (2.6.2) (with $T = +\infty$) if, denoting $\bar{v}(x;\lambda) = \mathcal{L}[v(x;\cdot)](\lambda)$ for any $\lambda \in \mathbb{H}$:

- $\bar{v}(x;\cdot)$ belongs to the domain of $\widehat{L}_{\Phi,H}$ for any $x \in I$;
- It holds, for any $\lambda \in \mathbb{H}$ and $x \in I$

$$\Phi(\lambda)\bar{v}(x;\lambda) - \frac{\Phi(\lambda)}{\lambda}v(x;0) = \frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_{\Phi,H}\bar{v}(x;\lambda). \tag{2.6.3}$$

Finding mild solutions is easier than finding classical solutions, thus we want to exploit some criterion to determine whenever a mild solution is more regular than we expect it to be. To do this, we need to introduce other two notions of solutions for the Fokker-Planck equation of the fOU:

$$\partial_t v(x;t) = \frac{1}{2}V'_{2,H}(t)\frac{\partial^2}{\partial x^2}v(x;t). \tag{2.6.4}$$

Being this a non-autonomous heat-like equation, the notions of strong and weak solutions are already known. However, we need a notion of solution that considers the equation as a Banach-space valued ordinary differential equation, and another notion that works on Laplace transform of solutions.

DEFINITION 2.6.5. We say that a function $v \in L^\infty(\mathbb{R}^+;C^2(I))$ is a **classical solution** of Equation (2.6.4) (with $T = +\infty$) if:

- $\partial_t v(x;t)$ exists for almost any $t > 0$ and for any $x \in I$;
- $\partial_t v(x;\cdot)$ belongs to $L^1_{\mathrm{loc}}(0,+\infty)$;
- Identity (2.6.4) holds for any $x \in I$ and almost any $t > 0$.

We say that a function $v \in L^\infty(\mathbb{R}^+;C^0(I))$ is a **mild solution** of Equation (2.6.4) (with $T = +\infty$) if, denoting by $\bar{v}(x;\lambda) = \mathcal{L}[v(x;\cdot)](\lambda)$ for any $\lambda \in \mathbb{H}$:

- It holds, for any $\lambda \in \mathbb{H}$ and $x \in I$

$$\lambda\bar{v}(x;\lambda) - v(x;0) = \frac{1}{2}\frac{\partial^2}{\partial x^2}L_H v(x;\lambda). \tag{2.6.5}$$

[1] For instance, it could be difficult to check if a function is a classical solution, even if it solves the equation in terms of Lapalce transforms

In particular we will consider inverse-subordinated solutions. Now we want to study some features of the inverse-subordinated solutions of the generalized Fokker-Planck equation. In particular we want to address the following issues:

- **Gain of regularity**: we want to investigate what are the hypotheses under which an inverse-subordinated mild solution is also a classical solution;
- **Isolation of mild solutions**: we want to show that, under a certain partial order on inverse-subordinated functions, inverse-subordinated mild solutions are non-comparable;
- **Uniqueness of strong solutions**: we want to show uniqueness of strong solutions via a weak maximum principle.

In particular in each step we will take in consideration as main example of inverse-subordinated solution of the generalized Fokker-Planck equation the function $p_{\Phi,H}(x;t)$. Indeed we will show the following properties:

- $p_{\Phi,H}(x;t)$ is a mild solution of (2.6.2) for $I=\mathbb{R}^*$;
- $p_{\Phi,H}(x;t)$ is a classical solution of (2.6.2) for $I=\mathbb{R}^*$;
- $p_{\alpha,H}(x;t)$ is the unique strong solution of (2.6.2) for $I=\mathbb{R}^*$ with boundary-initial values $p_{\alpha,H}(x;0)=0$, $p_{\alpha,H}(\pm\infty;t)=0$ and

$$p_{\alpha,H}(0;t)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}(V_{2,H}(s))^{-1/2}f_\alpha(s;t)ds.$$

2.6.4. Gain of regularity. We now address the issue of the gain of regularity of inverse-subordinated mild solutions. First of all, we need to link mild solutions of (2.6.4) with the ones of (2.6.2). We have the following result (see [**26**, Proposition 5.2]).

PROPOSITION 2.6.9. *Let $v\in L^\infty(\mathbb{R}^+;C^2(I))$ and $v_\Phi=S_\Phi v$. Then the following properties are equivalent:*

- *v is a mild solution of* (2.6.4)*;*
- *v_Φ is a mild solution of* (2.6.2)*.*

PROOF. Let us first observe that by definition of v and v_Φ and by Proposition 2.6.8, we have that v_Φ belongs to the domain of $\hat{L}_{\Phi,H}$. Let us show that if v is a mild solution of (2.6.4), then v_Φ is a mild solution of (2.6.2), since the converse is analogous.

Let us consider $\lambda>0$ in $\mathbb{R}$ (since, one we have shown the identity (2.6.3) for $\lambda>0$, it holds for any $\lambda\in\mathbb{H}$ since all the involved functions are holomorphic on $\mathbb{H}$). Recalling that equation (2.6.5) holds for any $\lambda>0$, then it also holds if we substitute $\Phi(\lambda)$ in place of λ. Multiplying everything by $\frac{\Phi(\lambda)}{\lambda}$ we achieve

$$\Phi(\lambda)\left(\frac{\Phi(\lambda)}{\lambda}\mathcal{L}[v(x;\cdot)](\Phi(\lambda))\right)-\frac{\Phi(\lambda)}{\lambda}v(x;0)=\frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}L_Hv(x;\Phi(\lambda)).$$

We conclude the proof by applying Propositions 2.6.2 and 2.6.8 and recalling that $v_\Phi(x;0)=v(x;0)$. □

From this proposition we already obtain the following Corollary concerning $p_{\Phi,H}(x;t)$.

COROLLARY 2.6.10. *$p_{\Phi,H}$ is a mild solution of* (2.6.2) *for $I=\mathbb{R}^*$.*

PROOF. We have $p_{\Phi,H} = S_\Phi p_H$, where $p_H \in L^\infty(\mathbb{R}^+; C^2(I))$ as $I = \mathbb{R}^*$. In particular, p_H is a strong solution of (2.6.4), thus it is also a mild solution. $\square$

Another link we can exploit concerns the weighted inverse-subordination operator $S_{\Phi,H}$ and the Fokker-Planck operator $\mathcal{F}_{\Phi,H}$ under some regularity of the inverse-subordinated function (see [**26**, Lemma 5.3]).

LEMMA 2.6.11. *Let $v \in L^\infty(\mathbb{R}^+; C^0(I))$ and $v_\Phi = S_\Phi v$ be a mild solution of* (2.6.2)*. Moreover, suppose that $v_\Phi \in D(\mathcal{F}_{\Phi,H}, I)$ and $\mathcal{F}_{\Phi,H}\, v_\Phi(\cdot, t) \in C^0(I)$ for any fixed $t > 0$. Then it holds $S_{\Phi,H} v(\cdot, t) \in C^2(I)$ and, for any $x \in I$ and almost any $t \in I$,*

$$\mathcal{F}_{\Phi,H}\, v_\Phi(x;t) = \frac{\partial^2}{\partial x^2} S_{\Phi,H} v(x;t).$$

PROOF. Let us consider $\lambda > 0$ without loss of generality (in place of $\lambda \in \mathbb{H}$). Since v_Φ is a mild solution of (2.6.2), from (2.6.3) and Proposition 2.6.8 we obtain that $\frac{\Phi(\lambda)}{\lambda} L_H v(\cdot; \Phi(\lambda)) \in C^2(I)$. Now, since v_Φ belongs to the domain of $\mathcal{F}_{\Phi,H}$, we have that $\frac{\partial^2}{\partial x^2} \frac{\Phi(\lambda)}{\lambda} L_H v(x; \Phi(\lambda))$ is the Laplace transform of some function.
However, let us observe that $S_{\Phi,H} v \in L^\infty(\mathbb{R}^+; C^0(I))$ and

$$\mathcal{L}[S_{\Phi,H} v(x; \cdot)] = \frac{\Phi(\lambda)}{\lambda} L_H v(x; \Phi(\lambda)),$$

thus, being $\frac{\partial^2}{\partial x^2} : C^2(I) \to C^0(I)$ a closed operator, we have, by [**13**, Proposition 1.7.6], that $S_{\Phi,H} v(\cdot, t) \in C^2(I)$ for any $t > 0$ and

$$\frac{\partial^2}{\partial x^2} S_{\Phi,H} v(x,t) = \mathcal{L}^{-1}_{\lambda \to t}\left[\frac{\partial^2}{\partial x^2} \frac{\Phi(\lambda)}{\lambda} L_H v(x; \Phi(\lambda))\right](t) = \mathcal{F}_{\Phi,H}\, v_\Phi(x,t),$$

concluding the proof. $\square$

REMARK 2.6.12. Let us observe that the same holds for $v \in L^\infty(\mathbb{R}^+; C^0(I))$ mild solution of (2.6.4) if we consider $\mathcal{F}_H\, v(x;t) := \mathcal{L}^{-1}\left[\frac{\partial^2}{\partial x^2} L_H v(x; \cdot)\right](t)$ in place of $\mathcal{F}_{\Phi,H}$ and $S_H v := V'_{2,H} v$ in place of $S_{\Phi,H}$.

The previous Lemma and Remark are the main tools to obtain the gain of regularity result. Indeed we can show the following Theorem (see [**26**, Theorem 5.5]).

THEOREM 2.6.13. *Let $v \in L^\infty(\mathbb{R}^+; C^0(I))$ and $v_\Phi = S_\Phi v$ be a mild solution of* (2.6.2)*. Suppose for fixed $t > 0$ it holds $v \in D(\mathcal{F}_H, I)$ and $\mathcal{F}_H\, v(\cdot; t) \in C^0(I)$. Moreover, suppose that $\mathcal{F}_H\, v(x; \cdot) \in L^\infty(0, +\infty)$ for any fixed $x \in I$. Then v_Φ is a classical solution of* (2.6.2)*.*

PROOF. By Lemma 2.6.11 and Remark 2.6.12 we know that $S_H v(\cdot; t) \in C^2(I)$ and then $v(\cdot; t) \in C^2(I)$ since S_H is a multiplication operator.
Now let us observe that, by Proposition 2.6.9, we know that v is mild solution of (2.6.4), thus it holds, after some algebraic manipulations of (2.6.5),

$$\mathcal{L}[v(x; \cdot)](\lambda) = \frac{1}{\lambda} v(x; 0) + \frac{1}{2\lambda} \frac{\partial^2}{\partial x^2} L_H v(x; \lambda).$$

Since $v(\cdot;t) \in C^2(I)$ and $\frac{\partial^2}{\partial x^2} : C^2(I) \to C^0(I)$ is a closed operator, we have, by [**13**, Proposition 1.7.6],

$$\frac{\partial^2}{\partial x^2} L_H v(x;\lambda) = L_H \left(\frac{\partial^2}{\partial x^2} v(x;\cdot) \right)(\lambda).$$

Now, by hypotheses, we know that $\mathcal{F}_H\, v(x;\cdot) \in L^\infty(0,+\infty)$ thus we can define

$$F(x;t) = \frac{1}{2}\int_0^t \mathcal{F}_H\, v(x;s)ds$$

and take the Laplace transform to obtain, by definition of $\mathcal{F}_H$,

$$\mathcal{L}[F(x;\cdot)](\lambda) = \frac{1}{2\lambda} L_H \left(\frac{\partial^2}{\partial x^2} v(x;\cdot) \right)(\lambda).$$

Hence we get

$$\mathcal{L}[v(x;\cdot)](\lambda) = \mathcal{L}[v(x;0) + F(x;\cdot)](\lambda)$$

and, by injectivity of the Laplace transform,

$$v(x;t) = v(x;0) + \frac{1}{2}\int_0^t V_{2,H}'(s) \frac{\partial^2}{\partial x^2} v(x;s)ds.$$

In particular, $v(x;\cdot)$ is absolutely continuous and, taking the derivative in t, v is classical solution of (2.6.4).
Now let us consider the function $v_\Phi(x;t) - v(x;0)$ and, observing that $\bar{\nu}_\Phi$ is Laplace transformable with Laplace transform $\frac{\Phi(\lambda)}{\lambda}$, we have that

$$\mathcal{L}[\bar{\nu}_\Phi * (v_\Phi(x;\cdot) - v(x;0))] = \frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[v_\Phi(x;\cdot)](\lambda) - \frac{\Phi(\lambda)}{\lambda^2} v(x;0).$$

Now, since $\partial_t v(x;t) = \frac{1}{2}\mathcal{F}_H\, v(x;t)$ and $\mathcal{F}_H\, v(x;\cdot) \in L^\infty(0,+\infty)$, also $\partial_t v(x;\cdot) \in L^\infty(0,+\infty)$ and we can apply S_Φ to it. By Proposition 2.6.2 we have

$$\begin{aligned}
\mathcal{L}_{t\to\lambda}\left[\int_0^t S_\Phi \partial_t v(x;s)ds\right] &= \frac{\Phi^2(\lambda)}{\lambda^2}\,\mathcal{L}[v(x;\cdot)](\Phi(\lambda)) - \frac{\Phi(\lambda)}{\lambda^2} v(x;0) \\
&= \frac{\Phi(\lambda)}{\lambda}\,\mathcal{L}[v_\Phi(x;\cdot)](\lambda) - \frac{\Phi(\lambda)}{\lambda^2} v(x;0) \\
&= \mathcal{L}[\bar{\nu}_\Phi * (v_\Phi(x;\cdot) - v(x;0))].
\end{aligned}$$

By injectivity of the Laplace transform we obtain

$$\bar{\nu}_\Phi * (v_\Phi(x;\cdot) - v(x;0))(t) = \int_0^t S_\Phi \partial_t v(x;s)ds$$

and then we can differentiate on both sides to achieve, for almost any $t \in \mathbb{R}^+$,

$$\partial_t^\Phi v_\Phi(x;t) = S_\Phi \partial_t v(x;t).$$

However, we also have, being v_Φ a mild solution of (2.6.2),

$$\begin{aligned}
\mathcal{L}\left[S_\Phi \partial_t v(x;\cdot)\right](\lambda) &= \Phi(\lambda)\,\mathcal{L}[v_\Phi(x;\cdot)](\lambda) - \frac{\Phi(\lambda)}{\lambda} v(x;0) \\
&= \frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_H\,\mathcal{L}[v_\Phi](x;\lambda).
\end{aligned}$$

Thus we have that $\frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_H\,\mathcal{L}[v_\Phi](x;\lambda)$ is the Laplace transform of something and then we can take the inverse Laplace transform to obtain

$$S_\Phi \partial_t v_\Phi(x;t) = \frac{1}{2}\,\mathcal{F}_{\Phi,H}\,v_\Phi(x;t).$$

Finally we get

$$\partial_t^\Phi v_\Phi(x;t) = \frac{1}{2}\,\mathcal{F}_{\Phi,H}\,v_\Phi(x;t),$$

concluding the proof. $\square$

As a direct consequence, we have the following Corollary.

COROLLARY 2.6.14. *$p_{\Phi,H}$ is a classical solution of* (2.6.2) *for $I = \mathbb{R}^*$.*

PROOF. It easily follows from the fact that p_H is a strong solution of (2.6.4) and $V'_{2,H}(\cdot)\frac{\partial^2}{\partial x^2}p_H(x;\cdot) \in L^\infty(0,+\infty)$ for any $x \in \mathbb{R}^+$. $\square$

2.6.5. Isolation of mild solutions. Now let us focus on some uniqueness issues. Concerning mild solutions, we are not able to show uniqueness, but we can prove a form of *isolation* of the solutions, in terms of a partial order.

DEFINITION 2.6.6. Let $v_\Phi, w_\Phi \in L^\infty(\mathbb{R}^+; C^0(I))$ with $v_\Phi = S_\Phi v$ and $w_\Phi = S_\Phi w$. We say that $v_\Phi \preceq w_\Phi$ if and only if:

- $v \le w$ in $I \times \mathbb{R}^+$;
- There exist two constants $\varepsilon, M > 0$ such that for any $x \in I$ the function $(w - v)(x;\cdot)$ is increasing in $[0,\varepsilon]$ and decreasing in $[M,+\infty)$.

In particular $\preceq$ is a partial order on the set of inverse-subordinated functions, that is well defined by injectivity of the operator S_Φ.

Now let us consider the Cauchy problem

$$\begin{cases} \Phi(\lambda)\bar{v}_\Phi(x;\lambda) - \frac{\Phi(\lambda)}{\lambda}v_\Phi(x;0) = \frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_{\Phi,H}\bar{v}_\Phi(x;\lambda) & (x;\lambda) \in I \times \mathbb{R}^+ \\ v_\Phi(x;0) = f(x) & x \in I \\ \widehat{L}_{\Phi,H}\bar{v}_\Phi(a;\lambda) = g_1(\lambda) & \lambda > 0 \\ \frac{\partial}{\partial x}\widehat{L}_{\Phi,H}\bar{v}_\Phi(a;\lambda) = g_2(\lambda) & \lambda > 0, \end{cases} \tag{2.6.6}$$

where $I = [a,b]$, which is the *natural* Cauchy problem associated to mild solutions of equation (2.6.2). Now we can prove the following isolation result (see [**26**, Theorem 6.1]).

THEOREM 2.6.15. *Let $I = [a,b]$, $v, w \in L^\infty(\mathbb{R}^+; C^0(I))$ and consider $v_\Phi = S_\Phi v$ and $w_\Phi = S_\Phi w$ such that, denoting $\bar{v}_\Phi = \mathcal{L}[v_\Phi]$ and $\bar{w}_\Phi = \mathcal{L}[w_\Phi]$, these are solutions of the Cauchy problem* (2.6.6)*. If $w_\Phi \preceq v_\Phi$, then $w_\Phi = v_\Phi$.*

PROOF. First of all, let us observe that, since all the operators involved are linear, $(v_\Phi - w_\Phi)$ is still a mild solution of (2.6.2). Let us set $h_\Phi(x;t) = (v_\Phi(x;t) - w_\Phi(x;t))$. It holds $h_\Phi(x;0) = 0$ and then

$$\Phi(\lambda)\bar{h}_\Phi(x;\lambda) = \frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_{\Phi,H}\bar{h}_\Phi(x;\lambda),$$

where $\bar{h}_\Phi(x;\lambda) := \mathcal{L}[h_\Phi(x;\cdot)](\lambda)$. Since S_Φ is linear, we can define $h(x;t) = v(x;t) - w(x;t)$ to obtain $h_\Phi = S_\Phi h$. Setting $\bar{h}(x;\lambda) := \mathcal{L}[h(x;\cdot)](\lambda)$, we have

$$2\Phi(\lambda)\bar{h}(x;\Phi(\lambda)) = \frac{\partial^2}{\partial x^2}L_H h(x;\Phi(\lambda)). \tag{2.6.7}$$

Now we want to transform the previous second order differential equation in a system of first order ones and then write it in vector form. Let us define

$$f(x;\lambda) = \frac{\partial}{\partial x} L_H h(x;\Phi(\lambda)) \qquad \text{and} \qquad g(x;\lambda) = (L_H h(x;\Phi(\lambda)), f(x;\lambda))$$

to rewrite (2.6.7) in the equivalent form

$$\frac{\partial}{\partial x} g(x;\lambda) = (f(x;\lambda), 2\Phi(\lambda)\bar{h}(x;\Phi(\lambda))).$$

Now let us observe that $f(a;\lambda) = 0$ and $L_H h(a;\Phi(\lambda)) = 0$, thus we have

$$g(x;\lambda) = \int_a^x \frac{\partial}{\partial x} g(y;\lambda) dy$$

and then

$$|g(x;\lambda)| \le \int_a^x \left| \frac{\partial}{\partial x} g(y;\lambda) \right| dy. \tag{2.6.8}$$

Considering $L_H h(x;\Phi(\lambda))$, we have

$$\begin{aligned} L_H h(x;\Phi(\lambda)) = & \int_0^\varepsilon e^{-\Phi(\lambda)t} h(x;t) V'_{2,H}(t) dt \\ & + \int_\varepsilon^M e^{-\Phi(\lambda)t} h(x;t) V'_{2,H}(t) dt \\ & + \int_M^{+\infty} e^{-\Phi(\lambda)t} h(x;t) V'_{2,H}(t) dt \\ & := I_1 + I_2 + I_3. \end{aligned}$$

Now let us observe that $\min_{t\in[\varepsilon,M]} V'_{2,H}(t) = m > 0$, thus there exists a constant $C_1 > 0$ such that

$$I_2 \ge C_1 \int_\varepsilon^M e^{-\Phi(\lambda)t} h(x;t) dt.$$

Concerning I_1, we get

$$I_1 = \frac{1 - e^{-\Phi(\lambda)\varepsilon}}{\Phi(\lambda)} \int_0^\varepsilon V'_{2,H}(t) h(x;t) d\left(\frac{1 - e^{-\Phi(\lambda)t}}{1 - e^{-\Phi(\lambda)\varepsilon}} \right)$$

where $d\left(\frac{1-e^{-\Phi(\lambda)t}}{1-e^{-\Phi(\lambda)\varepsilon}}\right)$ is a probability measure on $[0,\varepsilon]$. Thus we can use Chebyshev's integral inequality (see [**110**]), since we can suppose $V'_{2,H}$ and $h(x;\cdot)$ to be comonotone in $[0,\varepsilon]$. Setting $C_2 = \frac{\Phi(\lambda)}{1-e^{-\Phi(\lambda)\varepsilon}} \int_0^\varepsilon e^{-\Phi(\lambda)t} V'_{2,H}(t) dt > 0$, we obtain

$$I_1 \ge C_2 \int_0^\varepsilon e^{-\Phi(\lambda)t} h(x;t) dt.$$

Arguing in the same way for I_3, we have that there exists a constant $C_3 > 0$ such that

$$I_3 \ge C_3 \int_M^{+\infty} e^{-\Phi(\lambda)t} h(x;t) dt.$$

Thus, taking $C_4 = \min_{i=1,2,3} C_i > 0$, we obtain

$$L_H h(x;\Phi(\lambda)) \ge C_4 \bar{h}(x;\Phi(\lambda)).$$

Now let us define $k(x;\lambda) = \left|\frac{\partial}{\partial x} g(x;\lambda)\right|$ and observe that

$$k(x;\lambda) = \sqrt{4\Phi^2(\lambda)\bar{h}^2(x;\Phi(\lambda)) + f^2(x;\lambda)}.$$

On the other hand we have, setting $C_5 = \min\left\{\frac{C_4^2}{4\Phi^2(\lambda)}, 1\right\} > 0$,

$$|g(x;\lambda)| = \sqrt{(L_H h(x;\Phi(\lambda)))^2 + f^2(x;\lambda)} \geq C_5 k(x;\lambda).$$

Plugging this inequality in Equation (2.6.8) and setting $C_6 = C_5^{-1}$, we have

$$k(x;\lambda) \leq C_6 \int_a^x k(y;\lambda)dy.$$

By Grönwall's Inequality (see [**11**]) we conclude that $k(x;\lambda) = 0$. This implies that $\bar{h}(x;\Phi(\lambda)) = 0$. Now, considering $\lambda > 0$, we have that Φ is invertible on the real line, thus we conclude that $\bar{h}(x;\lambda) = 0$ for any $\lambda > 0$. Finally, by injectivity of the Laplace transform, we obtain $h(x;t) = 0$ for any $t > 0$ and $x \in I$, that is what we wanted to prove. □

2.6.6. Uniqueness of strong solutions and the weak maximum principle. Here we want to prove uniqueness of strong solutions of the generalized Fokker-Planck equation. To do this, we need to study the value of non-local derivatives at extremal points of a function. Let us recall that this has been done for instance in [**100**] and [**5**] for the Caputo derivative and in [**6**] for the Riemann-Liouville derivative. In our case, we have an adaptation of the Caputo derivative case (see [**26**, Proposition 2.2]).

PROPOSITION 2.6.16. *Let $\Phi \in \mathcal{BF}$ be regularly varying at infinity of index $\alpha \in (0,1)$. Supose $f : [0,T] \to \mathbb{R}$ and t_0 is a maximum point for f. If $f \in W^{1,1}(0,t_0) \cap C^1((0,t_0])$, then $\partial^\Phi f(t_0) \geq 0$.*

PROOF. Let us consider the function $g(\tau) = f(t_0) - f(\tau)$ for $\tau \in [0,T]$ and observe that $\partial^\Phi g(t_0) = -\partial^\Phi f(t_0)$. Let us consider $\varepsilon > 0$ and write

$$\partial^\Phi f(t_0) = \int_0^\varepsilon \bar{\nu}_\Phi(t_0-\tau)g'(\tau)d\tau + \int_\varepsilon^{t_0} \bar{\nu}_\Phi(t_0-\tau)g'(\tau)d\tau := I_1 + I_2.$$

Concerning I_2, by a change of variable we have

$$I_2 = \int_0^{t_0-\varepsilon} \bar{\nu}_\Phi(z)g'(t_0-z)dz.$$

Now, since we know that $g \in C^1([\varepsilon,t_0])$ for any $\varepsilon > 0$, we can use dominated convergence theorem to write

$$I_2 = -\lim_{a\to 0}\int_a^{t_0-\varepsilon} \bar{\nu}_\Phi(z)dg(t_0-z).$$

Let us observe that $\bar{\nu}_\Phi$ is monotone and finite in $[a, t_0-\varepsilon]$, hence it is of bounded variation and we can use integration by parts (see [**147**]) to obtain

$$\int_a^{t_0-\varepsilon} \bar{\nu}_\Phi(z)dg(t_0-z) = \bar{\nu}_\Phi(t_0-\varepsilon)g(\varepsilon) - \bar{\nu}_\Phi(a)g(t_0-a) - \int_a^{t_0} g(t_0-\tau)d\nu_\Phi(\tau).$$

However, since $g(t_0) = 0$ and $g \in C^1([\varepsilon,t_0])$, we know that $g(t_0-a) \leq C|a|$. On the other hand, by Karamata's Tauberian theorem, we know that $\bar{\nu}_\Phi(t) \sim \frac{\Phi(1/t)}{\Gamma(1-\alpha)}$ as $t \to 0^+$. Hence we have that $\lim_{a\to 0}\bar{\nu}_\Phi(a)g(t_0-a) = 0$. Moreover, g is non negative, hence, by monotone convergence theorem, we have

$$I_2 = -\bar{\nu}_\Phi(t_0-\varepsilon)g(\varepsilon) - \int_0^{t_0-\varepsilon} g(t_0-\tau)d\nu_\Phi(\tau).$$

Now let us suppose $\varepsilon < \varepsilon_0$ for some fixed $\varepsilon_0 > 0$. Then, setting $C = \int_0^{t_0-\varepsilon_0} g(t_0-\tau)d\nu_\Phi(\tau)$, we get $I_2 \le -C$.
Concerning I_1, we have

$$I_1 \le \bar{\nu}_\Phi(t_0-\varepsilon_0)\int_0^\varepsilon |g'(\tau)|d\tau.$$

Since $g' \in L^1(0,\varepsilon)$, we know there exists $\varepsilon < \varepsilon_0$ such that $I_1 \le \frac{C}{2}$. Thus, choosing $\varepsilon > 0$ as mentioned, we have

$$\partial^\Phi g(t_0) \le -\frac{C}{2} \le 0,$$

concluding the proof. □

REMARK 2.6.17. With a regularization procedure it can be shown that such property holds even if f does not belong to C^1.

Now let us observe that for strong inverse-subordinated solutions $v_\Phi = S_\Phi v$ of (2.6.2), it holds

$$\mathcal{F}_{\Phi,H}\, v_\Phi = \frac{\partial^2}{\partial x^2} S_{\Phi,H} v.$$

This gives us a hint on what is missing to achieve a weak maximum principle for such equation: we need to show that maximum points of v_Φ are also maximum points of $S_{\Phi,H}v$, as done in [**26**, Lemma 6.2].

LEMMA 2.6.18. *Let $v \in L^\infty(\mathbb{R}^+; C^0(I))$, $v_\Phi = S_\Phi v$ and $v_{\Phi,H} = S_{\Phi,H}v$. Then the following assertions are equivalent:*

i $(x_0,t_0) \in I \times \mathbb{R}^+$ *is a maximum point of* v_Φ*;*
ii $(x_0,t_0) \in I \times \mathbb{R}^+$ *is a maximum point of* $v_{\Phi,H}$*.*

PROOF. The implication $ii \Rightarrow i$ is obvious since $\sup_{t>0} V'_{2,H}(t) > 0$. Thus let us show $i \Rightarrow ii$.
To do this, let us fix $(x,t) \in I \times \mathbb{R}^+$. First of all let us suppose that there exists an increasing sequence $R_n \to +\infty$ and a decreasing sequence $\delta_n \to 0$ such that

$$\int_{\delta_n}^{R_n} (v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds \ge 0.$$

Then, since $\min_{t\in[\delta_n,R_n]} V'_{2,H}(t) > 0$, we have

$$\begin{aligned}
v_{\Phi,H}(x_0;t_0) &- v_{\Phi,H}(x;t)\\
&= \int_0^{\delta_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&+ \int_{\delta_n}^{R_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&+ \int_{R_n}^{+\infty} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\ge \int_0^{\delta_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&+ \int_{R_n}^{+\infty} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds.
\end{aligned}$$

Taking the limit as $n \to +\infty$ we obtain $v_{\Phi,H}(x_0;t_0) - v_{\Phi,H}(x;t) \ge 0$.
Now let us suppose such sequences do not exist. Then there exists δ_0, R_0 such that for any $\delta < \delta_0$ and $R > R_0$ it holds

$$\int_\delta^R (v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds < 0.$$

However, since (x_0,t_0) is a maximum point for v_Φ, taking the limit as $\delta \to 0$ and $R \to +\infty$ we have

$$\int_0^{+\infty} (v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds = 0.$$

Since $\inf_{t\in(0,+\infty)} V'_{2,H}(t) = 0$, we can consider δ_0 so small and R_0 so big to obtain $\inf_{t\in(\delta_0,R_0)} V'_{2,H}(t) < 1$. Now consider any decreasing sequence $\delta_n \to 0$ such that $\delta_n < \delta_0$ and any increasing sequence $R_n \to +\infty$ such that $R_n > R_0$. We have

$$\begin{aligned}
&v_{\Phi,H}(x_0;t_0) - v_{\Phi,H}(x;t)\\
&\qquad= \int_0^{\delta_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\qquad+ \int_{\delta_n}^{R_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\qquad+ \int_{R_n}^{+\infty} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\qquad\ge \int_0^{\delta_n} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\qquad+ \int_{\delta_n}^{R_n} (v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds\\
&\qquad+ \int_{R_n}^{+\infty} V'_{2,H}(s)(v(x_0;s)f_\Phi(s;t_0) - v(x;s)f_\Phi(s;t))ds.
\end{aligned}$$

Taking the limit as $n \to \infty$ we conclude the proof. $\square$

Now we are ready to show the actual weak maximum principle (see [**26**]).

THEOREM 2.6.19 (**Weak maximum principle**). *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function that is regularly varying at ∞ of index $\alpha \in (0,1)$ and consider $v_\Phi = S_\Phi v$ a strong solution of (2.6.2) in $[a,b]\times\mathbb{R}^+$. Fix $T > 0$ and define $\mathcal{O} = [a,b]\times[0,T]$. Suppose that $S_\Phi\left(\frac{T-t}{T}\chi_{[0,T]}(t)\right)$ belongs to $C^1((0,T])\cap W^{1,1}(0,T)$. Let $\partial_p\mathcal{O}$ be the parabolic boundary of $\mathcal{O}$, i.e.*

$$\partial_p \mathcal{O} = ([a,b]\times\{0\}) \cup (\{a,b\}\times[0,T]).$$

Then it holds

$$\max_{(x,t)\in\mathcal{O}} v_\Phi(x;t) = \max_{(x,t)\in\partial_p\mathcal{O}} v_\Phi(x;t)$$

PROOF. First of all, let us observe that for any constant $C \in \mathbb{R}$ it holds $v_\Phi + C = S_\Phi(v+C)$, $\mathcal{F}_{\Phi,H}(v_\Phi + C) = \mathcal{F}_{\Phi,H}\, v_\Phi$ and $\partial^\Phi(v_\Phi + C) = \partial^\Phi v_\Phi$. Thus if v_Φ is an inverse-subordinated strong solution of (2.6.2), so it is also $v_\Phi + C$. In conclusion,

we can suppose, without loss of generality, that $v_\Phi \ge 0$.
Let us also recall that

$$\mathcal{F}_{\Phi,H}\, v_\Phi = \frac{\partial^2}{\partial x^2} S_{\Phi,H} v.$$

Now let us suppose v_Φ admits a maximum point (x_0, t_0) belonging to $\mathring{\mathcal{O}} \cup ((a,b) \times \{T\})$ and that $M = \max_{(x,t)\in\partial_p \mathcal{O}} v_\Phi(x;t) < v_\Phi(x_0;t_0)$. Fix $\delta = v_\Phi(x_0;t_0) - M > 0$ and define for any $(x,t) \in \mathcal{O}$ the function

$$w_\Phi(x;t) = v_\Phi(x;t) + \frac{\delta}{2} S_\Phi \left(\frac{T-\tau}{T} \chi_{[0,T]}(\tau) \right)(t)$$

where $\chi_{[0,T]}(\tau)$ is the indicator function of the interval $[0,T]$. Since $\frac{T-t}{T} \in [0,1]$ as $t \in [0,T]$, it holds

$$v_\Phi(x;t) \le w_\Phi(x;t) \le v_\Phi(x;t) + \frac{\delta}{2}$$

for any $(x,t) \in \mathcal{O}$.
For any $(x,t) \in \partial_p \mathcal{O}$ it holds

$$w_\Phi(x_0;t_0) \ge v_\Phi(x_0;t_0) = \delta + M \ge \delta + v_\Phi(x;t) \ge \frac{\delta}{2} + w_\Phi(x;t)$$

hence, since $(x_0;t_0) \notin \partial_p \mathcal{O}$, w_Φ admits a maximum point $(x_1;t_1) \in \mathring{\mathcal{O}} \cup ((a,b) \times \{T\})$.
Now let us also recall that

$$v_\Phi(x;t) = w_\Phi(x;t) - \frac{\delta}{2} S_\Phi \left(\frac{T-\tau}{T} \chi_{[0,T]}(\tau) \right)(t). \tag{2.6.9}$$

Set $g(t) = \frac{T-t}{T}\chi_{[0,T]}(t)$ and $g_\Phi(t) = S_\Phi g(t)$. We want to determine $\partial^\Phi g_\Phi(t)$. To do this, let us suppose a priori that $\partial^\Phi g_\Phi(t)$ admits a Laplace transform. Then we have, since $g_\Phi(0) = 1$,

$$\mathcal{L}[\partial^\Phi g_\Phi] = \Phi(\lambda)\,\mathcal{L}[g_\Phi] - \frac{\Phi(\lambda)}{\lambda} = -\frac{\Phi(\lambda)(1-e^{-\Phi(\lambda)T})}{T\lambda\Phi(\lambda)}.$$

On the other hand, it holds

$$\mathcal{L}[S_\Phi \chi_{[0,T]}] = \frac{\Phi(\lambda)(1-e^{-\Phi(\lambda)T})}{\lambda\Phi(\lambda)},$$

thus we have

$$\partial^\Phi g_\Phi(t) = -\frac{1}{T}\int_0^T f_\Phi(s;t)ds.$$

Using this equality, together with identity (2.6.9), we get

$$\partial^\Phi v_\Phi(x;t) = \partial^\Phi w_\Phi(x;t) + \frac{\delta}{2T}\int_0^T f_\Phi(s;t)ds. \tag{2.6.10}$$

If we define $w(x;t) = v(x;t) + g(t)$, we obtain that $w_\Phi = S_\Phi w$. Moreover, since g does not depend on x, we have that $\frac{\partial^2}{\partial x^2} S_{\Phi,H} w(x;t) = \frac{\partial^2}{\partial x^2} S_{\Phi,H} v(x;t)$. Thus we can rewrite equation (2.6.2) as

$$\partial^\Phi w_\Phi(x;t) + \frac{\delta}{2T}\int_0^T f_\Phi(s;t)ds - \frac{1}{2}\frac{\partial^2}{\partial x^2} S_{\Phi,H} w(x;t) = 0.$$

Now let us observe that (x_1, t_1) is a maximum point for w_Φ belonging to $\mathring{\mathcal{O}} \cup ((a,b) \times \{T\})$, hence, by Proposition 2.6.16, we know that $\partial^\Phi w_\Phi(x_1;t_1) \ge 0$. Moreover,

(x_1,t_1) is also a maximum point for $S_{\Phi,H}w_(x;t)$, hence $\frac{\partial^2}{\partial x^2}S_{\Phi,H}w(x_1;t_1) \le 0$. Thus we have

$$\partial^\Phi w_\Phi(x_1;t_1) + \frac{\delta}{2T}\int_0^T f_\Phi(s;t_1)ds - \frac{1}{2}\frac{\partial^2}{\partial x^2}S_{\Phi,H}w(x_1;t_1) \ge \frac{\delta}{2T}\int_0^T f_\Phi(s;t_1)ds > 0,$$

which is a contradiction. □

As for regular parabolic equations, weak maximum principle directly implies both uniqueness of the solution of a boundary-initial value problem and continuous dependence from boundary-initial values. Moreover, it is easy to see that all we did can be extended to the case in which $a = -\infty$ or $b = +\infty$ as soon as we introduce some limit condition.

However, we have shown that $p_\Phi(x;t)$ is a classical solution, but we do not know if in general $p_\Phi(x;t)$ is C^1 in t. A case in which this is known is given by $\Phi(\lambda) = \lambda^\alpha$, thus we can conclude $p_\alpha(x;t)$ is a strong solution of (2.6.2).

COROLLARY 2.6.20. *$p_{\alpha,H}(x;t)$ is the unique strong solution of* (2.6.2) *for* $I = \mathbb{R}^*$ *with boundary-initial values* $p_{\alpha,H}(x;0) = 0$, $p_{\alpha,H}(\pm\infty;t) = 0$ *and*

$$p_{\alpha,H}(0;t) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}(V_{2,H}(s))^{-1/2}f_\alpha(s;t)ds.$$

2.7. First exit times of time-changed Markov processes

Now that we have introduced some concepts related to time-changed processes, let us focus on some features of these processes. In particular, here we will discuss the asymptotic behaviour (respectively at infinity and at 0^+) of the survival function and the distribution function of the exit time $\mathfrak{T}$ of a time-changed Markov process from an open set. Let us state that the majority of the results we are considering here are actually valid for any almost surely non-negative random variable T independent of the subordinator σ_Φ.

2.7.1. Some first properties of exit times. Let us consider a Markov process $M(t)$ with (topological) state space $(\Sigma, \mathfrak{G})$ and $\Phi \in \mathcal{BF}$ a driftless Bernstein function. For the whole section we will fix $S \in \mathfrak{G}$ and define

$$T = \inf\{y \ge 0 :\ M(y) \notin S\}.$$

In the following $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot|M(0) = x)$. Moreover, to avoid trivialities, we will always consider points $x \in S$ such that $\mathbb{P}_x(T > 0) > 0$ (i.e. T is not degenerate at 0) and $\mathbb{P}_x(T < +\infty) = 1$ (i.e. T is almost surely finite). Consider $\sigma_\Phi(t)$ a subordinator associated to Φ and independent of M and let $L_\Phi(t)$ be its inverse. The time-changed process will be denoted by $M_\Phi(t) := M(L_\Phi(t))$. Moreover, let us denote

$$\mathfrak{T} = \inf\{y \ge 0 :\ M_\Phi(y) \notin S\}.$$

As preliminary step, let us give some alternative representation of $\mathbb{P}_x(\mathfrak{T} > y)$.

LEMMA 2.7.1. *It holds*

$$\mathbb{P}_x(\mathfrak{T} > y) = \mathbb{P}_x(\sigma_\Phi(T) > y) = \mathbb{P}_x(T > L_\Phi(y)) \tag{2.7.1}$$

PROOF. Let us first observe that M_Φ admits the following alternative representation

$$M_\Phi(t) = M(y), \qquad \sigma_\Phi(y-) \le t < \sigma_\Phi(y).$$

From this representation we obtain that $\mathfrak{T} = \sigma_\Phi(T-)$ on any path. Now let us also recall that σ_Φ does not admit any fixed discontinuity and T and σ_Φ are independent. Thus we have

$$\begin{aligned}\mathbb{P}_x(\mathfrak{T} > y) = \mathbb{P}_x(\sigma_\Phi(T-) > y) &= \mathbb{E}_x[\mathbb{P}_x(\sigma_\Phi(T-) > y|T)] \\ &= \mathbb{E}_x[\mathbb{P}_x(\sigma_\Phi(T) > y|T)] = \mathbb{P}_x(\sigma_\Phi(T) > y),\end{aligned}$$

where the central equality holds since

$$\begin{aligned}\mathbb{P}_x(\sigma_\Phi(T-) > y|T = t) &= \mathbb{P}_x(\sigma_\Phi(t-) > y|T = t) = \mathbb{P}_x(\sigma_\Phi(t-) > y) \\ &= \mathbb{P}_x(\sigma_\Phi(t) > y) = \mathbb{P}_x(\sigma_\Phi(t) > y|T = t) = \mathbb{P}_x(\sigma_\Phi(T) > y|T = t).\end{aligned}$$

Furthermore, since $L_\Phi(t)$ is almost surely continuous and increasing, we get $L_\Phi(\sigma_\Phi(t)) = t$ and then

$$\mathbb{P}_x(\mathfrak{T} > y) = \mathbb{P}_x(\sigma_\Phi(T) > y) = \mathbb{P}_x(T > L_\Phi(y)),$$

concluding the proof. $\square$

This representation will be the main tool in our considerations.
Before going into the details of the asymptotic behaviour, let us show some properties concerning the regularity of the random variable $\mathfrak{T}$.

2.7.2. Smoothness of $\mathfrak{T}$. First of all, we are interested in the absolute continuity of $\mathfrak{T}$. An interesting thing we should underline is the fact that such absolute continuity could not depend at all on the absolute continuity of T. Indeed, the following Theorem (see [**28**, Theorem 2.8, Propositions 2.9 and 2.10]) gives a sufficient condition on absolute continuity and differentiability of the density of $\mathfrak{T}$.

THEOREM 2.7.2. *Suppose $\overline{\nu}_\Phi$ is absolutely continuous and ν_Φ satisfies Orey's condition (defnition 1.4.3). Moreover, suppose there exists $\varepsilon > 0$ such that $\mathbb{E}[T^{-1}\chi_{[0,\varepsilon]}(T)] < +\infty$. Then $\mathfrak{T}$ is an absolutely continuous random variable with respect to $\mathbb{P}_x(\cdot)$. Moreover, if there exists $n \in \mathbb{N}$ such that $\mathbb{E}[T^{-n-1}\chi_{[0,\varepsilon]}(T)] < +\infty$, the probability density function $p_{\mathfrak{T}}(t)$ is differentiable up to n-th derivative with bounded derivatives.*

PROOF. By Proposition 1.2.2 we know that the subordinator $\sigma_\Phi(t)$ is absolutely continuous for any $t > 0$. Let us denote by $g_\Phi(s; t)$ its density. Then, by (2.7.1) and the independence of σ_Φ and T, we have

$$\mathbb{P}_x(\mathfrak{T} \le y) = \mathbb{P}_x(\sigma_\Phi(T) \le y) = \int_0^{+\infty} \mathbb{P}_x(\sigma_\Phi(s) \le y)\mu_T(ds) \tag{2.7.2}$$

where μ_T is the law of T. Now, let us observe that $\mathbb{P}_x(\sigma_\Phi(s) \le y)$ is differentiable and its derivative is given by $g_\Phi(y; s)$. Let us consider the Lévy symbol of σ_Φ, given by $\Psi(\lambda) := \Phi(-i\lambda) = \int_0^{+\infty}(1 - e^{i\lambda\tau})\nu_\Phi(d\tau)$ and the characteristic function $\varphi_\Phi(\lambda; t) := \mathbb{E}[e^{i\lambda\sigma_\Phi(t)}] = e^{-t\Psi(\lambda)}$. This function is in $L^1(\mathbb{R})$ and we can express $g_\Phi(y; s)$, by Lévy inversion theorem, as

$$g_\Phi(y; s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\varphi_\Phi(\lambda; s)d\lambda.$$

Now let us recall that the complex extension of Φ still maps complex numbers with non-negative real part in complex numbers with non-negative real part. Thus $\Re(\Psi(\lambda)) \ge 0$. Now let us use Orey's condition. Indeed, as shown in [**115**], such condition implies that there exists a constant $c>0$ such that for $|\lambda|$ sufficiently big it holds

$$|\varphi_\Phi(\lambda;1)|\le e^{-\frac{c}{4}|\lambda|^{2-\gamma}}$$

where $\gamma\in(0,2)$ is the exponent of Orey's condition. Thus we have that $\Re(\Psi(\lambda))\ge \frac{c}{4}|\lambda|^{2-\gamma}$ and

$$|\varphi_\Phi(\lambda;t)|\le e^{-t\frac{c}{4}|\lambda|^{2-\gamma}}$$

for large values of $|\lambda|$. Suppose in particular the bound holds for $|\lambda|>M$. First of all, we obtain that

$$\begin{aligned}|g_\Phi(y;s)|&\le \frac{M}{\pi}+\frac{1}{\pi}\int_M^{+\infty}e^{-s\frac{c}{4}\lambda^{2-\gamma}}d\lambda\\ &=\frac{M}{\pi}+\frac{4}{\pi cs}\int_{\frac{sc}{4}M^{2-\gamma}}^{+\infty}w^{\frac{1}{2-\gamma}-1}e^{-w}d\lambda\\ &=\frac{M}{\pi}+\frac{4}{\pi cs}\Gamma\left(\frac{1}{2-\gamma};\frac{sc}{4}M^{2-\gamma}\right),\end{aligned}$$

that is integrable by hypothesis. Thus we can differentiate under integral sign in (2.7.2) to obtain

$$p_{\mathfrak{T}}(y)=\int_0^{+\infty}g_\Phi(s;y)\mu_T(ds).$$

Concerning differentiability, let us observe that we can re-write

$$p_{\mathfrak{T}}(y)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\xi s}\int_0^{+\infty}\varphi_\Phi(\xi;s)\mu_T(ds)d\xi$$

and then consider the incremental ratio. To show that we can differentiate under integral sign, one has to show that the incremental ratio is dominated by an integrable function. In particular, we have

$$\left|\frac{e^{-i\xi s}-e^{-i\xi s'}}{s-s'}\int_0^{+\infty}\varphi_\Phi(\xi;s)\mu_T(ds)\right|\le\int_0^{+\infty}|\xi|e^{-s\Re\Psi(\xi)}\mu_T(ds)$$

and then, using the same estimates as before (together with Fubini's theorem), we prove that the right-hand side is in $L^1(\mathbb{R})$. The same holds for successive derivatives. □

As a direct consequence we obtain the following Corollary.

COROLLARY 2.7.3. *Suppose $\overline{\nu}_\Phi$ is absolutely continuous and ν_Φ satisfies Orey's condition. If $t\mapsto\mathbb{P}_x(T\le t)$ is rapidly decreasing at 0^+, then $\mathfrak{T}$ is an absolutely continuous random variable with respect to $\mathbb{P}_x(\cdot)$ and the probability density function $p_{\mathfrak{T}}$ is infinitely differentiable with bounded derivatives.*

2.7.3. The asymptotic behaviour of the survival function. Now let us introduce some notation. Set $F(t)=\mathbb{P}_x(T\le t)$, the cumulative distribution function of T, and $\bar F(t)=1-F(t)$, the survival function. Moreover, set $\mathfrak{F}(t)=\mathbb{P}_x(\mathfrak{T}\le t)$, the cumulative distribution function of $\mathfrak{T}$, and $\bar{\mathfrak{F}}(t)=1-\mathfrak{F}(t)$, the survival function. First of all, we need the following technical Lemma (see [**28**, Lemma 2.1]).

LEMMA 2.7.4. *Let X be a non-negative random variable and Y be an exponential random variable of parameter $\lambda > 0$ independent of X. Then*

$$\mathbb{P}_x(X > Y) = \mathbb{E}_x[1 - e^{-\lambda X}]. \tag{2.7.3}$$

PROOF. By using a conditioning argument we get

$$\mathbb{P}_x(X > Y) = \mathbb{E}_x[\mathbb{P}_x(X > Y|X)] = \mathbb{E}_x[1 - e^{-\lambda X}].$$

□

Now we are ready to show the main result concerning the asymptotic behaviour of the survival function $\bar{\mathfrak{F}}$ (see [**28**, Theorem 2.2]).

THEOREM 2.7.5. *Fix $x \in S$ such that the function $g(s) := \mathbb{E}_x[1 - e^{-sT}]$ is regularly varying at zero with index $\beta \in [0, 1]$. Moreover, suppose Φ is regularly varying at zero with index $\alpha \in [0, 1)$. Then*

$$\bar{\mathfrak{F}}(t) \sim \frac{1}{\Gamma(1-\alpha\beta)} g\left(\Phi\left(\frac{1}{t}\right)\right) \qquad \text{as } t \to +\infty. \tag{2.7.4}$$

In particular $\mathfrak{F}$ is regularly varying at infinity with index $\alpha\beta$.

PROOF. Let us define the function

$$J(t) = \int_0^t \bar{\mathfrak{F}}(s) ds$$

and set $\bar{J}(\lambda) = \mathcal{L}^S[J](\lambda)$. By using Equation (2.7.1) we obtain

$$\begin{aligned}
\bar{J}(\lambda) &= \int_0^{+\infty} e^{-\lambda s}\, \mathbb{P}_x(T > L(s)) ds \\
&= \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} \bar{F}(w) f_\Phi(w; s) dw ds \\
&= \int_0^{+\infty} \bar{F}(w) \int_0^{+\infty} e^{-\lambda s} f_\Phi(w; s) ds dw \\
&= \frac{1}{\lambda} \int_0^{+\infty} \Phi(\lambda) e^{-s\Phi(\lambda)} \bar{F}(s) ds.
\end{aligned}$$

Now let us define Y_Φ an exponential random variable independent of T and with rate $\Phi(\lambda)$. Then we have

$$\bar{J}(\lambda) = \frac{1}{\lambda}\, \mathbb{P}_x(T > Y_\Phi).$$

Finally, by Equation (2.7.3), we obtain

$$\bar{J}(\lambda) = \frac{1}{\lambda}\, \mathbb{E}_x[1 - e^{-\Phi(\lambda)T}] = \frac{1}{\lambda} g(\Phi(\lambda)).$$

In particular we have that $g \circ \Phi$ is regularly varying at zero with index $\alpha\beta$, thus there exists a slowly varying function ℓ such that

$$g(\Phi(\lambda)) = \lambda^{\alpha\beta} \ell(\lambda),$$

and then

$$\bar{J}(\lambda) = \lambda^{\alpha\beta - 1} \ell(\lambda).$$

Now, by Karamata's Tauberian theorem 1.3.3, we have

$$J(t) \sim \frac{t^{1-\alpha\beta}}{\Gamma(2-\alpha\beta)} \ell(1/t) \qquad \text{as } t \to +\infty.$$

Moreover, $\bar{\mathfrak{F}}$ is monotone, thus we can use Monotone density theorem to get

$$\bar{\mathfrak{F}}(t) \sim \frac{t^{-\alpha\beta}}{\Gamma(1-\alpha\beta)}\ell(1/t) \qquad \text{as } t \to +\infty,$$

concluding the proof. □

Let us observe that the previous theorem does not rely directly on the distribution function of T (but it actually does via the function g). Let us give a sufficient condition for g to be regularly varying at zero (see [**28**, Corollary 2.3]).

COROLLARY 2.7.6. *If for $x \in S$ it holds $\mathbb{E}^x[T] = C < +\infty$ and Φ is regularly varying at zero with index $\alpha \in [0,1)$, then it holds*

$$\bar{\mathfrak{F}}(t) \sim \frac{C}{\Gamma(1-\alpha)}\Phi(1/t) \qquad \text{as } t \to +\infty. \tag{2.7.5}$$

PROOF. Let us observe that $\frac{1-e^{-sT}}{s} \le T$ for any $s > 0$. Thus, being T of finite mean, we can use Dominated Convergence Theorem to obtain

$$\lim_{s \to 0} \frac{g(s)}{s} = \mathbb{E}_x[T] = C,$$

concluding the proof, by using Theorem 2.7.5. □

If we consider $M(t)$ to be a 1-dimensional Markov process and $S = (-\infty, c)$ for some $c > 0$, there are several examples of processes such that $\mathbb{E}_0[T] < +\infty$. In particular, in the class of the Gauss-Markov processes, we can consider the drifted Brownian motion or the Ornstein-Uhlenbeck process. In [**28**], finite-mean conditions for Gauss-Markov processes are discussed by means of some comparison principles.

A different case is the one of the driftless Brownian motion $M(t) = B(t)$. Indeed it is well known that, for $S = (-\infty, c)$, $\mathbb{E}_0[T] = +\infty$. However, if we consider the subordinator $\sigma_{\frac{1}{2}}(t)$, it is also known that

$$T = \sigma_{\frac{1}{2}}(\sqrt{2}c)$$

and then we have $g(s) = \mathbb{E}_0[1 - e^{-sT}] = 1 - e^{-c\sqrt{2s}}$ which is regularly varying at 0 with index 1/2. Thus, in this case, we get, if Φ is regularly varying at zero with index $\alpha \in [0,1)$,

$$\bar{\mathfrak{F}}(t) \sim \frac{1 - e^{-c\sqrt{2\Phi(1/t)}}}{\Gamma(1-\alpha/2)} \qquad \text{as } t \to +\infty.$$

2.7.4. The asymptotic behaviour of the distribution function. Concerning the behaviour of the distribution function $\mathfrak{F}$ at zero, we need some properties on the behaviour of F at 0. Indeed, let us show the following result (see [**28**, Theorem 2.13]).

THEOREM 2.7.7. *Fix $x \in S$ such that the function F is regularly varying at zero with index $\rho > 0$. Moreover, suppose that Φ is regularly varying at zero with index $\alpha > 0$. Then*

$$\mathfrak{F}(t) \sim \frac{\Gamma(1+\rho)}{\Gamma(1+\alpha\rho)} F\left(\frac{1}{\Phi\left(\frac{1}{t}\right)}\right) \qquad \text{as } t \to 0^+.$$

In particular, $\mathfrak{F}$ is regularly varying at 0 with index $\alpha\rho$.

PROOF. Let us denote by $\widetilde{F}$ and $\widetilde{\mathfrak{F}}$ the Laplace-Stieltjes transforms of F and $\mathfrak{F}$. Since F is regularly varying at 0, by Karamata's Tauberian theorem we have

$$\widetilde{F}(\lambda) \sim F(1/\lambda)\Gamma(1+\rho) \qquad \text{as } \lambda \to +\infty.$$

Moreover, by using Equation (2.7.1) and denoting by $\mu_T(ds)$ the law of T, we have

$$\begin{aligned}\widetilde{\mathfrak{F}}(\lambda) &= \int_0^{+\infty} e^{-\lambda t} d\,\mathbb{P}_x(\sigma(T) \le t) \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda t} g_\Phi(s;dt)\mu_T(ds) \\ &= \int_0^{+\infty} e^{-s\Phi(\lambda)}\mu_T(ds) = \widetilde{F}(\Phi(\lambda)).\end{aligned}$$

In particular we get

$$\widetilde{\mathfrak{F}}(\lambda) \sim \Gamma(1+\rho)F(1/\Phi(\lambda)) \qquad \text{as } \lambda \to +\infty.$$

Now let us observe that $F(1/\Phi(\lambda))$ is regularly varying at infinity with index $\alpha\rho$. Thus, we can use again Karamata's Tauberian theorem to conclude the proof. $\square$

However, there are different cases in which $F(t)$ is shown to be not only rapidly varying at 0, but in particular rapidly decreasing at 0. For instance, as shown in [**28**], this is the case of 1-dimensional Gauss-Markov processes $M(t)$ and open sets $S = (-\infty, c)$ (choosing $x = 0$). Thus, it could be useful to determine some asymptotic result even in this case. This is done by using the following result (see [**28**, Theorem 2.16]).

THEOREM 2.7.8. *Fix $x \in S$ such that F is rapidly decreasing at 0^+ and C^∞. Suppose $\bar{\nu}_\Phi$ is absolutely continuous and ν_Φ satisfies Orey's condition. Moreover, let Φ vary regularly at infinity with index $\alpha > 0$. Then $\mathfrak{T}$ is absolutely continuous and its probability density function $p_{\mathfrak{T}}(t)$ is rapidly decreasing at 0^+.*

PROOF. First of all, let us observe that T is absolutely continuous with C^∞ density $p_T(t)$ that is rapidly decreasing at 0^+. Moreover, by Corollary 2.7.3 we know that $\mathfrak{T}$ is absolutely continuous with C^∞ density $p_{\mathfrak{T}}(t)$. As before, we have $\widetilde{\mathfrak{F}}(\lambda) = \widetilde{F}(\Phi(\lambda))$. In particular, $\widetilde{\mathfrak{F}}$ and $\widetilde{F}$ are the Laplace transform respectively of $p_{\mathfrak{T}}$ and p_T.
Now let us observe that there exists a slowly varying function (at infinity) ℓ such that

$$\Phi(\lambda) = \lambda^\alpha \ell(\lambda).$$

Fix $k > 0$ and $\beta > 0$ such that $\alpha\beta > k$. We have

$$\lambda^k \widetilde{\mathfrak{F}}(\lambda) = \frac{1}{\lambda^{\beta\alpha-k}\ell^\beta(\lambda)}\Phi^\beta(\lambda)\widetilde{F}(\Phi(\lambda)).$$

The Tauberian Theorem for rapidly decreasing functions ensure that $\lim_{\lambda\to+\infty}\Phi^\beta(\lambda)\widetilde{F}(\Phi(\lambda)) = 0$. On the other hand, since $\alpha\beta - k > 0$, we have $\lim_{\lambda\to+\infty}\lambda^{\beta\alpha-k}\ell^\beta(\lambda) = +\infty$. Thus we get $\lim_{\lambda\to+\infty}\lambda^k\widetilde{\mathfrak{F}}(\lambda) = 0$. Since $k > 0$ was arbitrary and $\mathfrak{F}$ belongs to C^∞, we conclude the proof by the Tauberian Theorem for rapidly decreasing functions. $\square$

It can be checked by hand that this is the case of the Brownian motion with non-negative drift when we set $S = (-\infty, c)$.

2.8. The first passage time of the time-changed Brownian motion with non-negative drift through a constant threshold

In the previous section we obtained some asymptotics for the survival functions and the distribution functions of first exit time from open sets of time-changed Markov processes. In particular, we have seen that the first passage time $\mathfrak{T}_0^c$ of the time-changed Brownian motion with drift $\delta \ge 0$ starting from 0 through the constant threshold $c > 0$ and $\Phi \in \mathcal{BF}$ driftless and regularly varying at 0 with order $\alpha \in [0,1)$ satisfies the following asymptotic relations as $t \to +\infty$:

$$\bar{\mathfrak{F}}(t) \sim \begin{cases} \frac{1-e^{-c\sqrt{2\Phi(1/t)}}}{\Gamma(1-\alpha/2)} & \delta = 0 \\ \frac{c}{\delta\Gamma(1-\alpha)}\Phi(1/t) & \delta > 0, \end{cases}$$

where, to obtain the second asymptotic relation, we just use Corollary 2.7.6 with the fact that $\mathbb{E}_0[T] = \frac{c}{\delta}$. Moreover, if Φ is regularly varying at infinity, then we also have that $\mathfrak{F}(t)$ is rapidly decreasing at 0^+.

Now we need to consider also the dependence of the distribution function $\mathfrak{F}$ with respect to the starting point x. Here, we want to show that, after some transformations, the distribution function is the unique solution of a non-local Partial Differential Equation.

Thus, let us define by T^c the first passage time of the Brownian motion with non-negative drift $B^\delta(t)$ (with $\delta \ge 0$) through the threshold $c > 0$. In particular, we have

$$T^c(\omega) = \begin{cases} \inf\{t > 0 : \ B^\delta(t,\omega) \ge c\} & B^\delta(0,\omega) < c \\ \inf\{t > 0 : \ B^\delta(t,\omega) \le c\} & B^\delta(0,\omega) > c \\ 0 & B^\delta(0,\omega) = c. \end{cases}$$

Here let us focus on the case $B^\delta(0,\omega) < c$. We have, conditioning to $B^\delta(0) = x < c$,

$$p_{T^c}(t;x) = \frac{c-x}{\sqrt{2\pi t^3}} e^{-\frac{(c-x-\delta t)^2}{2t}}, \qquad t \ge 0,$$

and then the distribution function is given by

(2.8.1)
$$\mathbb{P}_x(T^c \le t) = \int_0^t \frac{c-x}{\sqrt{2\pi s^3}} e^{-\frac{(c-x-\delta s)^2}{2s}} ds = \frac{e^{2\delta(c-x)}}{\sqrt{\pi}} \int_0^t \frac{(c-x)}{s\sqrt{2s}} e^{-\frac{(c-x)^2}{2s} - \frac{\delta^2 s}{2}} ds.$$

Now let us set $\frac{c-x}{\sqrt{2s}} = z$ to obtain

$$\frac{e^{2\delta(c-x)}}{\sqrt{\pi}} \int_0^t \frac{(c-x)}{s\sqrt{2s}} e^{-\frac{(c-x)^2}{2s} - \frac{\delta^2 s}{2}} ds = \frac{2e^{2\delta(c-x)}}{\sqrt{\pi}} \int_{\frac{c-x}{\sqrt{2t}}}^{+\infty} e^{-z^2 - \frac{\delta^2(c-x)^2}{4z^2}} dz.$$

From this relation, we easily get that

$$\lim_{x \to c^-} \mathbb{P}_x(T^c \le t) = 1$$

as it was expected. Moreover, it is easy to check that the function $w(z;t) = \frac{z}{\sqrt{2\pi t^3}} e^{-\frac{(z-\delta t)^2}{2t}}$ is decreasing for $z > z_1(t)$ where

$$z_1(t) = \frac{\delta t + \sqrt{\delta^2 t^2 + 4t}}{2}.$$

Since z_1 is a continuous function, for any fixed $t > 0$ there exists $s_* \in [0,t]$ such that $z_1(s_*) = \max_{s \in [0,t]} z_1(s)$. Now, if we send $x \to -\infty$, we have that $c - x \to +\infty$

and then we can suppose, for fixed $t > 0$, that $c - x > z_1(s_*)$. Thus, in particular, we get for fixed $t > 0$, by monotone convergence theorem,

$$\lim_{x \to -\infty} \mathbb{P}_x(T^c \leq t) = 0.$$

We can actually say much more. Indeed, specifying [**101**, Theorem 2.1] to our case, we have the following result.

THEOREM 2.8.1. *Fix $c > 0$ and consider $x = c - \frac{1}{y}$. Define $v(t; y) = \mathbb{P}_x(T^c \leq t)$ and consider the operator*

$$\mathcal{A} = \frac{y^4}{2}\frac{\partial^2}{\partial y^2} + [\delta^2 y^2 + y^3]\frac{\partial}{\partial y}. \tag{2.8.2}$$

Then $v(t; y)$ is the unique strong solution of

$$\begin{cases} \frac{\partial v}{\partial t}(t; y) = \mathcal{A}\, v(t; y) & t > 0,\ y > 0 \\ v(0, y) = 0 & y > 0 \\ \lim_{y \to +\infty} v(t, y) = 1 & t > 0 \\ v(t, 0) = 0 & t > 0. \end{cases} \tag{2.8.3}$$

Here we do not intend *strong solution* in the sense of abstract Cauchy problem, but just that v is continuous in $[0, +\infty) \times [0, +\infty)$, differentiable once with respect to the variable t, twice with respect to y with continuous derivatives and the Equation (2.8.3) hold pointwise. Moreover, uniqueness of the solution follows from the classical weak maximum principle for parabolic problems. It is also interesting to observe that the previous Theorem do not only provide a result of existence of the strong solution to the parabolic problem (2.8.3), but also a stochastic representation of it by actually exhibiting the unique strong solution. In this section we will prove an analogous result for $\mathfrak{T}^c$, i.e. the first passage time of the time-changed process $B^\delta_\Phi(t)$ through the fixed threshold $c \in R$.

Now we want to focus on the following non-local parabolic problem

$$\begin{cases} \partial^\Phi_t u(t; y) = \mathcal{A}\, u(t; y) & t > 0,\ y > 0 \\ u(0, y) = 0 & y > 0 \\ \lim_{y \to +\infty} u(t, y) = 1 & t > 0 \\ u(t, 0) = 0 & t > 0. \end{cases} \tag{2.8.4}$$

where $\Phi \in \mathcal{BF}$ is a driftless Bernstein function and $\mathcal{A}$ is defined in (2.8.2). Before working on the actual equation, let us give the definition of strong solution.

DEFINITION 2.8.1. A function $u(t; y)$ defined for $t \geq 0$ and $y \geq 0$ is a classical solution of the non-local parabolic problem (2.8.4) if and only if:

- $u \in C([0, +\infty) \times [0, +\infty))$;
- u is twice differentiable with respect to y and $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2} \in C((0, +\infty) \times (0, +\infty))$;
- $\partial^\Phi_t u(t; y)$ is well defined for any $t > 0$ and $y > 0$;
- $\partial^\Phi_t u(t; y) \in C((0, +\infty) \times (0, +\infty))$;
- The Equations in (2.8.4) hold pointwise.

Moreover we say that u is a strong solution if it is a classical solution and u is differentiable in t with continuous derivative.

In the following we will need some upper bounds on the solution v of (2.8.3).

LEMMA 2.8.2. *Let $v(t;y)$ be defined as in Theorem 2.8.1. Consider $0 < y_1 < y_2$ and $\mathcal{I} = [y_1, y_2]$. Then there exists a constant $C(\mathcal{I})$ such that*

$$\sup_{(t,y)\in(0,+\infty)\times[y_1,y_2]} \left(|v(t;y)| + \left|\frac{\partial v}{\partial y}(t;y)\right| + \left|\frac{\partial^2 v}{\partial y^2}(t;y)\right| \right) \le C(\mathcal{I}).$$

PROOF. By using the first equality of Equation (2.8.1) and the stochastic representation of v as a distribution function for a first passage time, we obtain that

$$v(t;y) = \int_0^t w(s;y)ds$$

where

$$w(s;y) = \frac{1}{y\sqrt{2\pi s^3}} e^{-\frac{(1-\delta s y)^2}{2sy^2}}.$$

Now let us define the function

$$m_{\mathcal{I}}(s) = \min\left\{ \frac{(1-\delta s y_1)^2}{y_1^2}, \frac{(1-\delta s y_2)^2}{y_2^2} \right\}.$$

It is easy to see that as $s \to +\infty$ it holds

$$m_{\mathcal{I}}(s) \sim \delta^2 s^2. \tag{2.8.5}$$

Now let us observe that

$$w(s;y) \le \frac{1}{y_1\sqrt{2\pi s^3}} e^{-\frac{m_{\mathcal{I}}(s)}{2s}} =: h_0^{\mathcal{I}}(s),$$

where $h_0^{\mathcal{I}} \in L^1(0,+\infty)$, by using Equation (2.8.5) and the fact that $m_{\mathcal{I}}(0) = \frac{1}{y_2^2}$. If we differentiate $w(s;y)$ with respect to y, we easily get

$$\left|\frac{\partial w}{\partial y}(s;y)\right| \le \frac{1}{y_1^2\sqrt{2\pi s^3}} e^{-\frac{m_{\mathcal{I}}(s)}{2s}} \left(1 + \frac{1}{sy_1^2} + \frac{\delta}{y_1}\right) =: h_1^{\mathcal{I}}(s).$$

Now, since $h_1^{\mathcal{I}} \in L^1(0,+\infty)$, still by Equation (2.8.5), we have

$$\frac{\partial v}{\partial y}(t;y) = \int_0^t \frac{\partial w}{\partial y}(s;y)ds, \ y \in \mathcal{I}.$$

Now we can differentiate again with respect to y to get

$$\left|\frac{\partial^2 w}{\partial y^2}(s;y)\right| \le \frac{1}{y_1^3\sqrt{2\pi s^3}} e^{-\frac{m_{\mathcal{I}}(s)}{2s}} \left(\frac{1}{2} + \frac{4}{3sy_1} + \frac{2\delta}{sy_1^5} + \frac{1}{sy_1^4} + \left|\delta^2 - \frac{5}{4s}\right| \frac{1}{y_1^2}\right) =: h_2^{\mathcal{I}}(s).$$

As before, $h_2^{\mathcal{I}} \in L^1(0,+\infty)$ and then we obtain

$$\frac{\partial^2 v}{\partial y^2}(t;y) = \int_0^t \frac{\partial^2 w}{\partial y^2}(s;y)ds, \ y \in \mathcal{I}.$$

Finally, let us define

$$C(\mathcal{I}) = \int_0^{+\infty} (h_0^{\mathcal{I}}(s) + h_1^{\mathcal{I}}(s) + h_2^{\mathcal{I}}(s))ds$$

to conclude the proof. □

Moreover, as we stated before, to prove uniqueness we need a weak maximum principle. Thus, now we want to prove a generalized version of [**100**]. To do this, let us first introduce some notation.

DEFINITION 2.8.2. Let us define $\mathbb{R}^n_\infty = \mathbb{R}^+_0 \times \mathbb{R}^n$ and $\mathbb{R}^n_T = [0,T] \times \mathbb{R}^n$ for any $T > 0$. Given an open set $\mathcal{O} \subset \mathbb{R}^n_T$, the **parabolic interior** of $\mathcal{O}$ is given by the following property:

$$(t_0, x_0) \in \mathcal{O}^* \Leftrightarrow \exists r > 0 : \ B_r(t_0, x_0) \cap \{(t,x) \in \mathbb{R}^N_T : \ t \le t_0\} \subset \mathcal{O}$$

where $B_r(t_0,x_0)$ is the ball in $\mathbb{R}^{n+1}$ centered in (t_0,x_0) with radius $r > 0$.
We define the **parabolic boundary** of $\mathcal{O}$ as $\partial_p \mathcal{O} = \bar{\mathcal{O}} \setminus \mathcal{O}^*$ where $\bar{\mathcal{O}}$ is the closure of $\mathcal{O}$. Given an opens set $\mathcal{O} \subset \mathbb{R}^n_T$, we define the **space projection** as $\mathcal{O}_{\mathbb{R}^n} = \{x \in \mathbb{R}^n : \exists t \in [0,T] : \ (t,x) \in \mathcal{O}\}$ and the **time projection** as $\mathcal{O}_{[0,T]} = \{t \in [0,T] : \ \exists x \in \mathbb{R}^n : \ (t,x) \in \mathcal{O}\}$. Moreover, let us define the **space section** $\mathcal{O}_t = \{x \in \mathbb{R}^n : \ (t,x) \in \mathcal{O}\}$ for any $t \in \mathcal{O}_{[0,T]}$ and the **time section** $\mathcal{O}_x = \{t \in [0,T] : \ (t,x) \in \mathcal{O}\}$. Given a function $u : \bar{\mathcal{O}} \to \mathbb{R}$, we say that $u \in C^{2,1}(\bar{\mathcal{O}})$ if and only if

- $u \in C(\bar{\mathcal{O}})$;
- $\forall t \in \mathcal{O}_{[0,T]}$ it holds $u(t,\cdot) \in C^2((\mathcal{O}^*)_t)$;
- $\forall x \in \mathcal{O}_{\mathbb{R}^n}$ it holds $u(\cdot,x) \in C^1((\mathcal{O}^*)_x) \cap W^{1,1}(\mathcal{O}_x)$;
- $\forall x \in \mathcal{O}_{\mathbb{R}^n}$ such that $0 \notin \bar{\mathcal{O}}_x$ it holds $u(\cdot,x) \in C^1((\bar{\mathcal{O}})_x)$.

Now we can state the following weak maximum principle.

THEOREM 2.8.3 (**Weak maximum principle**). *Let $\mathcal{O} \subseteq \mathbb{R}^n_T$ be a connected bounded open set of $\mathbb{R}^n_T$ of the form $\mathcal{O} = \widetilde{\mathcal{O}} \times [0,T]$ where $\widetilde{\mathcal{O}}$ is a bounded connected open set of $\mathbb{R}^n$. Let $u \in C^{2,1}(\bar{\mathcal{O}})$. Let u be a solution of*

$$\partial_t^\Phi u(t,x) = \mathcal{A}\, u(t,x), \ (t,x) \in \mathcal{O}$$

where

$$\mathcal{A} = p_2(x)\Delta + \langle p_1(x), \nabla \rangle$$

with $p_2 : \widetilde{\mathcal{O}} \to \mathbb{R}^+$ and $p_1 : \widetilde{\mathcal{O}} \to \mathbb{R}^n$. Finally, suppose that $\Phi \in \mathcal{BF}$ is a driftless Bernstein function regularly varying at infinity with order $\alpha \in (0,1)$. Then it holds

$$u(t,x) \le \max_{(s,y) \in \partial_p \mathcal{O}} u(s,y), \quad \forall (t,x) \in \bar{\mathcal{O}}.$$

PROOF. First of all, without loss of generality, we can suppose that $u(t,x) \ge 0$ for any $(t,x) \in \mathcal{O}$.
Let $M_1 = \max_{(s,y)\in\partial_p \mathcal{O}} u(s,y)$ and suppose by contradiction there exists a point $(t_0,x_0) \in \mathcal{O}^*$ such that $u(t_0,x_0) = M_2 > M_1$. Set $\varepsilon = M_2 - M_1$ and consider for $(t,x) \in \bar{\mathcal{O}}$ the function

$$w(t,x) = u(t,x) + \frac{\varepsilon}{2}\frac{T-t}{T}.$$

Now, since $T - t \le T$, we have

$$u(t,x) \le w(t,x) \le u(t,x) + \frac{\varepsilon}{2}$$

for any $(t,x) \in \mathcal{O}$. In particular in (t_0,x_0) we have

$$w(t_0,x_0) \ge u(t_0,x_0) = \varepsilon + M_1.$$

Now, by definition of M_1, considering $(t,x) \in \partial_p \mathcal{O}$, we get

$$w(t_0,x_0) \ge \varepsilon + u(x,t) \ge \frac{\varepsilon}{2} + w(x,t).$$

However, by definition, $w \in C^0(\bar{\mathcal{O}})$, thus it must admit a maximum point (t_1,x_1). The previous inequality implies that $(t_1,x_1) \notin \partial_p \mathcal{O}$. w is derivable in t_1, thus

$\partial_t^\Phi w(t_1,x_1) \ge 0$. Moreover, x_1 is an inner point of Ω_{t_1} and then $w(t_1,\cdot)$ is twice differentiable in x_1. In particular this leads to $\nabla w(t_1,x_1)=0$ and $\Delta w(t_1,x_1) \le 0$. Not let us observe that

$$\begin{aligned}\partial_t^\Phi u(x,t) &= \partial_t^\Phi w(x,t) + \frac{\varepsilon}{2T} I_\Phi(t),\\ \nabla w(x,t) &= \nabla u(x,t),\\ \Delta w(x,t) &= \Delta u(x,t),\end{aligned}$$

where $I_\Phi(t)$ is the integrated tail of the Lévy measure ν_Φ. Thus, in (t_1,x_1), we get

$$0 = \partial_t^\Phi u(t_1,x_1) - \mathcal{A}\, u(t_1,x_1) = \partial_t^\Phi w(t_1,x_1) + \frac{\varepsilon}{2T} I_\Phi(t_1) - p_2(x_1)\Delta w(t_1,x_1) > 0$$

which is a contradiction. □

Now we are ready to exploit a classical solution of Equations (2.8.4) by directly showing the stochastic representation of it. The next Theorem is a refined version of [**4**, Theorem 2].

THEOREM 2.8.4. *Fix $c>0$ and consider $x = c - \frac{1}{y}$. Define $u(t;y) = \mathbb{P}_x(\mathfrak{T}^c \le t)$ where $\mathfrak{T}^c = \inf\{t>0:\ B^\delta_\Phi(t) \ge c\}$ and $B^\delta_\Phi(t) = B^\delta(L_\Phi(t))$ with L_Φ independent of B^δ. Let $\mathcal{A}$ be the operator defined in* (2.8.2)*. Moreover, let $\Phi \in \mathcal{BF}$ be driftless and regularly varying at infinity with order $\alpha \in (0,1)$ with $\bar{\nu}_\Phi$ absolutely continuous and ν_Φ satisfying Orey's condition. Then $u(t;y)$ is the unique strong solution of Equation* (2.8.4)*.*

PROOF. Let us first observe that, by Equation (2.7.1) and Theorem 2.8.1, we have

$$u(t;y) = \int_0^{+\infty} v(s;y) f_\Phi(s;t)ds,$$

where $v(s;y)$ is the unique strong solution of (2.8.3). By definition of $u(t;y)$, we immediately get $u(0;y)=0$ for any $y>0$.
Now let us observe that $v(s;y) \le 1$, thus we can use dominated convergence theorem to obtain

$$\lim_{y\to+\infty} u(t;y) = \int_0^{+\infty} \lim_{y\to+\infty} v(s;y) f_\Phi(s;t)ds = 1.$$

The same holds as $y \to 0^+$. Moreover, we easily have that $u(t;y)$ is continuous in y by dominated convergence theorem. Smoothness of u with respect to t is guaranteed by Corollary 2.7.3. On the other hand, u is twice differentiable with respect to y with continuous derivatives by differentiation under the integral sign, that can be done thanks to the estimates in Lemma 2.8.2. Now we only have to show that the equation actually holds. To do this, let us observe that since u is smooth, we can take the Laplace transform of $\partial_t^\Phi u(t;y)$. Thus we have, denoting $\bar{u}$ the Laplace transform of u with respect to t

$$\mathcal{L}_t[\partial_t^\Phi u(t;y)](\lambda) = \Phi(\lambda)\bar{u}(\lambda;y).$$

On the other hand, we have, by integrating by parts,

$$\Phi(\lambda)\bar{u}(\lambda;y) = \frac{\Phi^2(\lambda)}{\lambda}\int_0^{+\infty} v(s;y)e^{-s\Phi(\lambda)}ds = \frac{\Phi(\lambda)}{\lambda}\int_0^{+\infty} \frac{\partial v}{\partial s}(s;y)e^{-s\Phi(\lambda)}ds.$$

Taking the inverse Laplace transform, it holds

$$\partial_t^{\Phi} u(t;y) = \int_0^{+\infty} \frac{\partial v}{\partial s}(s;y) f_{\Phi}(s;t).$$

Now let us recall that v is strong solution of (2.8.3) and that we can exchange $\mathcal{A}$ with the integral sign thanks to the estimates given in Lemma 2.8.2. We finally have

$$\partial_t^{\Phi} u(t;y) = \int_0^{+\infty} \frac{\partial v}{\partial s}(s;y) f_{\Phi}(s;t) = \int_0^{+\infty} \mathcal{A}\, v(s;y) f_{\Phi}(s;t) = \mathcal{A}\, u(t;y).$$

Uniqueness easily follows from the weak maximum principle. $\square$

CHAPTER 3

Some applications of time-changed processes

In this chapter we will focus on applications of the previously presented results. The main ones are given in the context of queueing theory. Indeed, the latter provides a fruitful soil to study and improve techniques related to methods based on probability generating function to solve difference-differential Cauchy problems and, at the same time, its peculiar models provide interesting examples in which the semi-Markov character of time-changed processes arises.
Discrete models are not the only models we will consider. As application of the time-changed Ornstein-Uhlenbeck process (as presented in [**67**]) with an additional deterministic drift term, we will construct a non-local Leaky Integrate-and-Fire (LIF) model. LIF models are one of the simplest models of neuronal activity. However, as we will see, their simplicity leads also to a *predictable* behaviour of the firing times of the modelled neuron, which is not always the most realistic result. Indeed, we will give an explicit example in which our model seems to fit better than the classical Markov one. Let us state that the considerations are only qualitative due to a lack of data.
Finally, in the context of applications, some simulation procedures are presented, with a particular attention to the generalization of Gillespie's algorithm to time-changed continuous time Markov chains.

3.1. Basics on queueing theory

As we stated before, we will mainly focus on queueing theory. Let us introduce the basic ideas of queueing theory, following the lines of [**134**]. First of all, let us consider a **queueing system**, that is to say a service facility linked to a **source** (from which users arrive) and to a **destination** (to which users are transferred). A queueing system can eventually handle more than one user per time unit. However, when there are more users than the ones the system can handle, a **waiting line** (or **queue**) is formed. Users can arrive at random times and services are not necessarily completed in a fixed time. Queueing systems model a lot of different things, from the more concrete ones (as a supermarket or a post office) to the more complex ones (for instance the scheduler of an operating system [**137**] or telephone traffic, which was the actual first usage queueing models [**56**]).
A queueing system is characterized by the following *components*:

- **The input process**: i.e. the way users arrive. In particular we will consider single arrivals (and not group arrivals, that are still possible), infinite source and infinite capacity of the system, thus the input process is completely defined by the distribution of the **inter-arrival times** T_n, which are the times between the arrival of the $(n-1)$-th user and the n-th user. In particular, denoting by A_n the **arrival times**, i.e. the time instants in which the n-th user enters the system, one has $T_n = A_n - A_{n-1}$ for any $n \in \mathbb{N}$, where $A_0 = 0$. In particular one asks that the inter-arrival times T_n are independent from one another.

- **The queue discipline**: i.e. the rule applied to decide which user in the waiting line will be served. In case of the CPU scheduler, for instance, this is the main feature that establishes the differences between schedulers (e.g. the usage of preemptive algorithms, or also the introduction of a priority value, see [**137**]). We will always consider **First-In First-Out** (FIFO) queues, i.e. queues in which the service is executed in arrival order. These are also called **First-Come First-Served** (FCFS) queues. Let us stress out that one can also combine this discipline with others. For instance, still talking about schedulers, the FCFS scheduler can be modelled as a classical FIFO queue, while the *Round Robin* (RR) scheduler still works with a FIFO discipline, but interrupts the current service if it takes too long and put the user again in the back of the queue (this procedure is called *preemption*).
- **Service mechanism**: i.e. the arrangement and characteristic of the servers. In the model we will work with, we will always consider one server, thus the service mechanism will be described in terms of the **service times** S_n, that is to say the time the n-th user takes to complete the service. As for inter-arrival times, also service times are asked to be independent.

A schematization of a queueing system is given in Figure 3.1.
The description of the queueing system is made by using the **system state process** $N(t)$, which is a stochastic process that counts the number of users in the system at time $t \geq 0$. It is sometimes called the **queue length process**, but, as we will see, in some cases it is useful to make a distinction between the two terminologies. Usually, one can describe much more easily the **performance parameters** of the queueing in terms of the steady state of $N(t)$ (if it exists). However, in our case, we will focus on the **transient behaviour** of the queues. The main motivation of this choice will be clear in the following section.
For the transient behaviour, the main performance parameters are given by

- **The busy period**: i.e. the duration B of a time interval from the moment in which a user comes in the empty service to the moment in which the system is empty again.
- **The virtual waiting time**: i.e. the duration $W(t)$ of time a user has to spend in the system if it enters it at time $t \geq 0$.

Finally, let us recall that in general there is a convenient notation to abbreviate the description of a queue, called **Kendall notation**. We will actually focus on the notation of the form $A/B/C$ where A describes the inter-arrival distribution, B the service time distribution and C the number of servers. Concerning A and B, we will use the following symbols:

- M stands for an exponential distribution;
- E_k stands for an Erlang distribution of shape parameter $k \in \mathbb{N}$.

When $A = B = M$, then the queue is called **Markovian** (since the process $N(t)$ is a continuous time Markov chain).

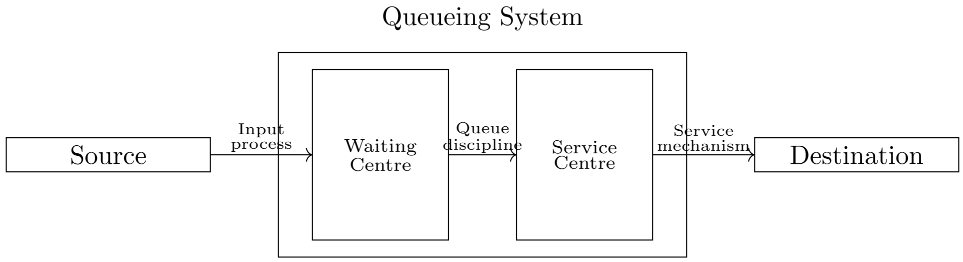

FIGURE 1. The schematization of a queueing system

3.1.1. An example: the transient behaviour of the $M/M/1$ queue. Let us consider a standard $M/M/1$ queue, schematized as in Figure 2. Then its inter-arrival times are given by $T_n \sim \operatorname{Exp}(\lambda)$ and the service times are given by $S_n \sim \operatorname{Exp}(\mu)$. It is not difficult to see that the process $N(t)$ is a birth-death process with transition matrix

$$Q = \begin{pmatrix} -\lambda & \mu & 0 & 0 & \cdots & 0 \\ \lambda & -(\lambda+\mu) & \mu & 0 & \cdots & 0 \\ 0 & \lambda & -(\lambda+\mu) & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

where, for a continuous-time Markov chain $X(t)$ with state space $E \subseteq \mathbb{Z}$, the **transition matrix** is an operator Q defined on some sequence space such that, if we define $P(t) = (p(t,n))_{n \in E}$ the sequence of the state probabilities $p(t,n) = \mathbb{P}_i(X(t) = n)$ for some $i \in E$, it is solution of

$$P'(t) = QP(t)$$

(where $P(t)$ is interpreted as a column vector). As we can see, Q is a matrix representation of the forward operator $\mathcal{F}$.
Thus, in the case of the $M/M/1$ queue model, the state probabilities $(p(t,n))_{n\in\mathbb{N}_0}$ (with $i = 0$) satisfy the following difference-differential problem

$$\begin{cases} \frac{dp}{dt}(t,0) = -\lambda p(t,0) + \mu p(t,1) & t > 0 \\ \frac{dp}{dt}(t,n) = -(\lambda+\mu)p(t,n) + \mu p(t,n+1) + \lambda p(t,n-1) & t > 0,\ n \geq 1 \\ p(0,0) = 1 \\ p(0,n) = 0 & n \geq 1. \end{cases} \tag{3.1.1}$$

We will not actually solve this system here and now, but let us show the method of solution of such system (as in [**129**]). The idea is the following: let us consider the **probability generating function**

$$G(z,t) = \sum_{n=0}^{+\infty} p(t,n) z^n$$

that is well defined for $t \geq 0$ and $z \in \mathbb{D}_1$. For fixed t this is an analytic function on $\mathbb{D}_1$ (since it is actually a power series), thus its coefficient are uniquely determined. This means that if we are able to determine $G(z,t)$, then we automatically have

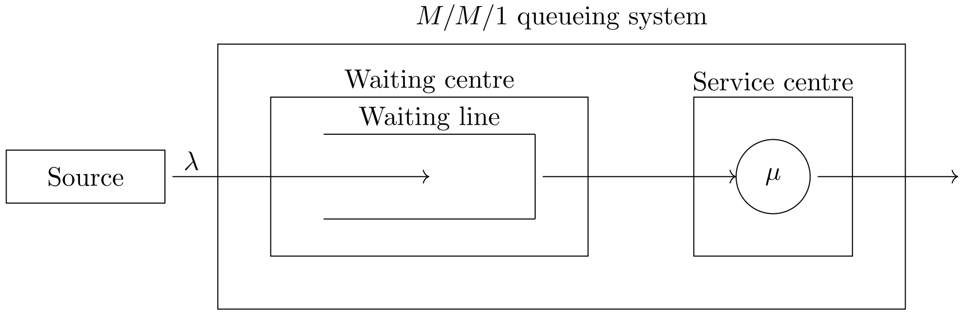

FIGURE 2. A Schematization of an $M/M/1$ queue

$p(t,n) = \frac{1}{n!}\frac{\partial^n G}{\partial z^n}(0,t)$. Now, multiplying the second equation of (3.1.1) by z^n and then summing everything, we get the following partial differential equation:

$$z\frac{\partial G}{\partial t}(z,t) = (1-z)[(\mu - \lambda z)G(z,t) - \mu p(t,0)] \qquad z \in \mathbb{D}_1,\ t \geq 0. \tag{3.1.2}$$

Consider now $z > 0$ and observe that, by monotone convergence theorem, one can take the Laplace transform (in t) inside the summation operator in the definition of $G(z,t)$. Denoting by $\bar{G}(z,\lambda)$ the Laplace transform of $G(z,t)$, one obtains an algebraic expression of $\bar{G}(z,\lambda)$ in terms of z, λ and $\bar{p}(\lambda,0) = \mathcal{L}[p(\cdot,0)](\lambda)$ by taking the transform on the whole equation (3.1.2).
As next step, one should notice that since such transform must converge at least for $z \in (0,1)$, then the zeros of the numerator and the ones of the denominator of the aforementioned algebraic expression must coincide: this gives an explicit form of $\bar{p}(\lambda,0)$ and then, taking the inverse transform, one obtains $p(t,0)$. On the other hand, after we have explicitly $\bar{p}(\lambda,0)$, we also have expressed explicitly $\bar{G}(z,\lambda)$ and then we can take the inverse transform and use the formula

$$p(t,n) = \frac{1}{n!}\frac{\partial^n G}{\partial z^n}(0,t)$$

to obtain all the state probabilities. However, this method can be actually difficult to apply, since the inverse transform are not trivial even in this easy case. To give an idea, let us observe that the solution of such difference-differential system is given by (setting $\rho = \lambda/\mu$)

$$p(t,n) = e^{-(\lambda+\mu)t}\left[\rho^{\frac{n}{2}} I_n(2\sqrt{\lambda\mu}t) + \rho^{\frac{n-1}{2}} I_{n+1}(2\sqrt{\lambda\mu}t) + (1-\rho)\rho^n \sum_{k=n+2}^{+\infty} \rho^{-\frac{k}{2}} I_k(2\sqrt{\lambda\mu}t)\right],$$

where the functions I_n are the modified Bessel functions of the first kind (see [**72**])

$$I_n(z) = \sum_{k=0}^{+\infty} \frac{1}{k!\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k}.$$

3.2. The fractional $M/M/1$ queue

In this section we will introduce the fractional $M/M/1$ queue as described in [**44**]. However, we will do it *reverting the approach*, i.e. first introducing the process and then discussing the equation. This is due to the fact that for non-Markov processes (and this process is a semi-Markov one), knowing the fractional version of the forward Kolmogorov equation and its solution, does not characterize the process. Moreover, to adapt the treatment to Kendall notation, let us fix $\alpha \in (0,1)$ and denote this queue model as $M_\alpha/M_\alpha/1$. So we know that there is 1 server, but we still have to define the service mechanism and the input process. To do this, we will make use of the classical $M/M/1$ queue. Consider $N(t)$ the state process of an $M/M/1$ queue with parameters λ and μ and $\sigma_\alpha(t)$ an α-stable subordinator independent of it, with inverse $L_\alpha(t)$. Then the $M_\alpha/M_\alpha/1$ state process is defined as $N_\alpha(t) := N(L_\alpha(t))$. Let us denote

$$p_\alpha(t,n) = \mathbb{P}_0(N_\alpha(t) = n)$$

the state probabilities and

$$G_\alpha(z,t) = \sum_{n=0}^{+\infty} p_\alpha(t,n) z^n$$

the probability generating function (for $t \geq 0$, $n \in \mathbb{N}_0$ and $z \in \mathbb{D}_1$). Let us first give a system of fractional difference-differential equations whose solution is $P_\alpha(t) = (p_\alpha(t,n))_{n \in \mathbb{N}_0}$ (see [**44**, Theorems 2.1 and 2.2]).

Theorem 3.2.1. *Let $N_\alpha(t)$ be the state process of a $M_\alpha/M_\alpha/1$ queue. Then the state probabilities $p_\alpha(t,n)$ are the unique solution of*

(3.2.1)

$$\begin{cases} \partial^\alpha p_\alpha(t,0) = -\lambda p_\alpha(t,0) + \mu p_\alpha(t,1) & t > 0 \\ \partial^\alpha p_\alpha(t,n) = -(\lambda+\mu) p_\alpha(t,n) + \mu p_\alpha(t,n+1) + \lambda p_\alpha(t,n-1) & t > 0, \ n \geq 1 \\ p_\alpha(0,0) = 1 \\ p_\alpha(0,n) = 0 & n \geq 1. \end{cases}$$

Moreover, the Laplace transform $\bar{G}_\alpha(z,\lambda)$ of the probability generating function $G_\alpha(z,t)$ is given by

$$\bar{G}_\alpha(z,\lambda) = \lambda^{\alpha-1} \frac{z - (1-z)[z_2(\lambda)][1-z_2(\lambda)]^{-1}}{-\lambda[z - z_1(\lambda)][z - z_2(\lambda)]}$$

where $z_i(\lambda)$ for $i = 1,2$ are the roots of the polynomial

$$w_\lambda(z) = \lambda^\alpha z - (1-z)(\mu - \lambda z),$$

with $z_2(\lambda) \in \mathbb{D}_1$.

It is also not difficult to check that

$$p_\alpha(t,n) = \int_0^{+\infty} p(s,n) f_\alpha(s;t) ds$$

and

$$G_\alpha(z,t) = \int_0^{+\infty} G(z,s) f_\alpha(s;t) ds$$

where $p(t,n)$ and $G(z,t)$ are respectively the state probabilities and the probability generating function of a $M/M/1$ queue and $f_\alpha(s;t)$ is the density of the inverse α-stable subordinator $L_\alpha(t)$. Using such relations it is possible to achieve an explicit formula for the state probabilities $p_\alpha(t,n)$. However, we first need to introduce a special function as defined in [**122**].

DEFINITION 3.2.1. Let $\alpha \in \mathbb{H}$, $\beta, \rho \in \mathbb{C}$. Then the **Prabhakar function** of parameters α, β, ρ is defined as

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{(\rho)_k z^k}{\Gamma(\alpha k + \beta) k!}, \ z \in \mathbb{C}$$

where $(\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}$ if ρ is not a non-positive integer (if ρ is a positive integer, then one could directly write $(\rho)_k = \rho(\rho+1)\cdots(\rho+k)$, obtaining 0 for $k \ge \rho$).

Concerning Prahbakar functions, the following Laplace transform formula is known (see [**76**, Formula 11.8]):

$$\mathcal{L}[t^{\beta-1}E^{\rho}_{\alpha,\beta}(wt^{\alpha})](\lambda) = \frac{\lambda^{\alpha\rho-\beta}}{(\lambda^{\alpha}-w)^{\rho}}, \quad \lambda \in \mathbb{H}\,. \tag{3.2.2}$$

Let us also remark that $E^{1}_{\alpha,\beta} = E_{\alpha,\beta}$. Now let us express the state probabilities $p_\alpha(t,n)$ in terms of Prahbakar functions, recalling [**44**, Theorem 2.4].

THEOREM 3.2.2. *For any $n \in \mathbb{N}_0$, $t \ge 0$ and $\alpha \in (0,1)$ it holds*

$$\begin{aligned} p_\alpha(t,n) = {}& \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n + \left(\frac{\lambda}{\mu}\right)^n \sum_{r=0}^{+\infty}\sum_{m=0}^{k+r} \frac{r-m}{r+m}\binom{r+m}{r} \\ & \times \lambda^r \mu^{m-1} t^{\alpha(r+m)-\alpha} E^{r+m}_{\alpha,\alpha(r+m)-\alpha+1}(-(\lambda+\mu)t^{\alpha}). \end{aligned} \tag{3.2.3}$$

Now that we have obtained explicitly the state probabilities, let us focus on some features of this model. First of all, let us give some actual interpretation to the symbol M_α. To do this, we first need to introduce a family of probability distribution functions (that we will do in full generality, while here we will use just one of them).

DEFINITION 3.2.2. Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function. We say that a random variable X is distributed as a **Φ-exponential random variable** with rate $\lambda > 0$ if $X \ge 0$ almost surely and, for $t \ge 0$, the probability distribution function $F_X(t)$ of X is given by

$$F_X(t) = 1 - \mathfrak{e}_{\Phi}(t; -\lambda).$$

Let us denote such distribution as $\mathrm{Exp}_{\Phi}(\lambda)$. Moreover, if $\Phi(z) = z^{\alpha}$, X is said to be **Mittag-Leffler distributed** (see [**121**]). We denote the Mittag-Leffler distribution with fractional parameter $\alpha \in (0,1)$ and rate λ as $ML_\alpha(\lambda)$.

Before proceeding with the next result, let us give some remarks. First of all, since semi-Markov processes still admits Markov properties on jump times, given a time-changed continuous-time Markov chain $X_\Phi(t)$, we can still define its **jump chain**, i.e. the Markov chain $X_n = X_\Phi(J_n)$ where J_n are the jump times of X. In particular, after a time-change, the jump chain remains unchanged: this is due to the fact that we are only changing the temporal scale in which the events occur. Moreover, let us observe that the new inter-arrival times are still independent each other and so do the service times.
Moreover, the fact that the jump chain remains unchanged and that the process still preserves Markov property at the jump times, we can consider the possibility of *modifying the process* to obtain some information. For instance, suppose we want to obtain the distribution of the inter-arrival times. Thus, we can suppose there are already n users in the queue (and the n-th is just arrived) and no one

is served ($\mu = 0$). Moreover, we can fix an absorbing state in $n+1$. Thus we have a new process $N^b_\alpha(t)$ with state space $E = \{n, n+1\}$ and state probabilities $b_\alpha(t,m) = \mathbb{P}(N^b_\alpha(t) = m)$ for $m = n, n+1$. In particular, let us observe that such process describes the arrive of a single user without taking in consideration the service mechanism. It is not difficult to see that $b_\alpha(t,n)$ and $b_\alpha(t,n+1)$ are solutions of the Cauchy problem

$$\begin{cases} \partial^\alpha b_\alpha(t,n) = -\lambda b_\alpha(t,n), & t>0 \\ \partial^\alpha b_\alpha(t,n+1) = \lambda b_\alpha(t,n), & t>0 \\ b_\alpha(0,n) = 1, \\ b_\alpha(0,n+1) = 0. \end{cases}$$

The first equation can be easily solved obtaining $b_\alpha(t,n) = E_\alpha(-\lambda t^\alpha)$ and then

$$b_\alpha(t,n+1) = 1 - b_\alpha(t,n) = 1 - E_\alpha(-\lambda t^\alpha).$$

We have shown part of the following result (as done in [**18**]).

Proposition 3.2.3. *Let T_n and S_n be the inter-arrival and the service times of a $M_\alpha/M_\alpha/1$ queue. Then*

$$T_n \sim ML_\alpha(\lambda) \qquad\qquad S_n \sim ML_\alpha(\mu).$$

Moreover, let $\mathcal{S}_n$ be the soujourn times of $N_\alpha(t)$ in non-zero states. Then

$$\mathcal{S}_n \sim ML_\alpha(\lambda+\mu)$$

hence T_n and S_n are not independent.

Let us observe that the last statement follows from the fact that Mittag-Leffler functions do not exhibit semigroup properties (see [**55**]) and then, if T_n and S_n were independent, the distribution of $\mathcal{S}_n = \min\{T_n, S_n\}$ should not be a $ML_\alpha(\lambda+\mu)$. Finally, let us give the distribution of a performance parameter: the busy period. In particular, let us show the following Theorem (see [**18**, Theorem 2]).

Theorem 3.2.4. *Let B_α the busy period random variable of a $M_\alpha/M_\alpha/1$ queue and let $F_{B_\alpha}(t) = \mathbb{P}(B_\alpha \le t)$. Then it holds*

$$F_{B_\alpha}(t) = 1 - \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m} t^{\alpha(n+2m-1)} E^{n+2m}_{\alpha,\alpha(n+2m-1)+1}(-(\lambda+\mu)t^\alpha), \ \ t \ge 0$$

where

$$C_{n,m} = \binom{n+2m}{m}\frac{n}{n+2m}\lambda^{n+m-1}\mu^m. \tag{3.2.4}$$

Proof. Let us consider a $M/M/1$ queue $N(t)$ with parameters λ, μ and the induced $M_\alpha/M_\alpha/1$ queue $N_\alpha(t)$. Now let us modify both processes. Let us suppose that both processes start from $N(0) = N_\alpha(0) = 1$. Moreover let us suppose 0 is an absorbing state (thus the process stops as it reaches 0). Let us denote such new processes as $\bar{N}(t)$ and $\bar{N}_\alpha(t)$. Observe that $\bar{N}_\alpha$ and N_α behave in the same way up to reaching 0. After the first time N_α reaches 0, it holds $\bar{N}_\alpha(t) = 0$. Thus we know that $\bar{N}_\alpha(t) = 0$ if and only if the first busy period of N_α lasts less than t, i.e.

$$\bar{p}_\alpha(t,0) = \mathbb{P}_1(\bar{N}_\alpha(t) = 0) = \mathbb{P}(B_\alpha \le t).$$

The same holds for N and $\bar{N}$. Let us denote with B the busy period of a $M/M/1$ queue with parameters λ, μ. However, we know that (see [**50**])

$$\bar{p}(t,n) = \mathbb{P}_1(\bar{N}(t) = n) = nt^{-1}\lambda^{\frac{n}{2}-1}\beta^{-\frac{n}{2}}e^{-(\lambda+\mu)t}I_n(2\sqrt{\lambda\mu}t), \qquad n \in \mathbb{N}.$$

Setting $C_{n,m}$ as in Equation (3.2.4) we have, by exploiting the series representation of I_n,

$$\bar{p}(t,n) = \sum_{m=0}^{+\infty} \frac{C_{n,m}}{(n+2m-1)!} t^{n+2m-1} e^{-(\lambda+\mu)t}, \qquad n \in \mathbb{N}.$$

Finally, we get

$$\bar{p}(t,0) = 1 - \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{n,m}}{(n+2m-1)!} t^{n+2m-1} e^{-(\lambda+\mu)t}.$$

Let us in particular recall that all the summands in the series are positive.
Moreover, by definition, we have that $\bar{N}_\alpha(t) = \bar{N}(L_\alpha(t))$, thus, it holds (by a simple conditioning argument)

$$\bar{p}_\alpha(t,n) = \int_0^{+\infty} \bar{p}(s,n) f_\alpha(s;t)ds, \qquad n \in \mathbb{N}_0,$$

and then,

$$\bar{p}_\alpha(t,0) = 1 - \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{n,m}}{(n+2m-1)!} \int_0^{+\infty} s^{n+2m-1} e^{-(\lambda+\mu)s} f_\alpha(s;t)ds.$$

Now let us denote $\pi_\alpha(z,n) = \mathcal{L}[\bar{p}_\alpha(\cdot,n)]$ and observe that, since $\mathcal{L}[f_\alpha(s;\cdot)](z) = z^{\alpha-1}e^{-sz^\alpha}$, we get

$$\begin{aligned}\pi_\alpha(z,0) &= \frac{1}{z} - \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{n,m}}{(n+2m-1)!} z^{\alpha-1} \int_0^{+\infty} s^{n+2m-1} e^{-(\lambda+\mu+z^\alpha)s}ds \\ &= \frac{1}{z} - \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m} \frac{z^{\alpha-1}}{(\lambda+\mu+z^\alpha)^{n+2m}}.\end{aligned}$$

Finally, taking the inverse Laplace transform and using equation (3.2.2) we conclude the proof. □

We could also argue on the virtual waiting time of the $M_\alpha/M_\alpha/1$ queue. However, we will discuss this case directly in Section 3.5

3.3. The fractional $M/M/1$ queue with acceleration of service

Now let us complicate the model a bit. In particular let us assume that the server accelerates linearly with respect to the number of customers in the service. In particular what happens is that the arrival rate $\lambda > 0$ remains constant, while the service rate $\mu(n) = n\mu$ depends on the number of customers. Thus in the classical case we get a process $N(t)$ that is an immigration-death process (see [**133**]), while, in the time-changed case, the $M_\alpha/M_\alpha/1$ queue with acceleration of service $N_\alpha(t) = N(L_\alpha(t))$ is a fractional immigration-death process, as better described in [**20, 22**]. By using the results of Section 2.4 we directly obtain the following result.

THEOREM 3.3.1. *The state probabilities $p_\alpha(t,n) = \mathbb{P}_0(N_\alpha(t) = n)$ are the unique strong solutions (in $\ell^2(\mathbf{m})$ where $\mathbf{m}$ is a Poisson measure with parameter λ/μ) of the following Cauchy problem:*

$$\begin{cases} \partial^\alpha p_\alpha(t,0) = -\lambda p_\alpha(t,0) + \mu p_\alpha(t,1) & t>0 \\ \partial^\alpha p_\alpha(t,n) = \lambda p_\alpha(t,n-1) - (\lambda + n\mu) p_\alpha(t,n) & \\ \qquad\qquad + (n+1)\mu p_\alpha(t,n+1) & n \in \mathbb{N},\ t>0 \\ p_\alpha(0,0) = 1 & \\ p_\alpha(0,n) = 0 & n \in \mathbb{N}. \end{cases}$$

Here we have such theorem as a direct consequence of the more general theory illustrated in Section 2.4. On the other hand, in [**23**], we have proved the same result by using the probability generating function.
It is easy to check that the inter-arrival times are still $ML_\alpha(\lambda)$-distributed random variables. However, the fact that the service rate is state-dependent leads to some difficulty in determining the service times. Indeed, we can only obtain the distribution if we suppose two services happen directly one after another. To obtain such result, we need some further notation. In particular we denote by E_k the time instant in which the k-th exit occurs and with U_k the time instant in which the k-th event occurs (setting $U_0 = 0$). Moreover let us introduce the stochastic process

$$U(t) = \max\{U_k :\ U_k \le t\}$$

that is to say that $U(t)$ is the last time an event occurs before time $t>0$. Hence we have the following result

PROPOSITION 3.3.2. *Let S_{k+1} be the $(k+1)$-th service time. Then*

$$\mathbb{P}(S_{k+1} \le t | U(E_k + t) = E_k,\ N((E_k+t)-) = n) = 1 - E_\alpha(-n\mu t^\alpha).$$

We omit the proof since it is identical to the one we gave for the inter-arrival times of the $M_\alpha/M_\alpha/1$ queue (by using also the conditioning to ensure all the modifications we apply to the process do not alter the nature of the process itself). In the same way, we can use a conditioned distribution for the sojourn times.

PROPOSITION 3.3.3. *Let $\mathcal{S}_{k+1}$ be the $(k+1)$-th sojourn time. Then*

$$\mathbb{P}(\mathcal{S}_{k+1} \le t | N((U_k+1)-) = n) = 1 - E_\alpha(-(\lambda + n\mu)t^\alpha).$$

Now let us consider the virtual waiting times, that are actually the main quantities affected by the semi-Markov property of the queue. To study them, we need to introduce some other quantities. First of all, denoting by A_n the n-th arrival time, let us define the following stochastic processes:

$$A(t) = \max\{A_n :\ A_n \le t\} \qquad\qquad E(t) = \max\{E_n :\ E_n \le t\}$$

for $t>0$, that are respectively the last arrival time and the last exit time before t (where we set $A_0 = 0$ and $E_0 = 0$).
Moreover, we need to show the following Lemma.

LEMMA 3.3.4. *It holds*

$$\mathcal{L}_{s\to z}[E_\alpha(-n\mu(s+\Delta t))] = e^{z\Delta t} \sum_{k=0}^{+\infty} \frac{(-n\mu)^k}{\Gamma(k\alpha+1)} z^{-k\alpha-1} \Gamma(k\alpha+1, z\Delta t).$$

PROOF. Let us recall that the series representation of E_α absolutely converges, thus we can take the Laplace transform under the summation sign. In particular we have

$$\begin{aligned}\mathcal{L}_{s\to z}[E_\alpha(-n\mu(s+\Delta t)^\alpha)] &= e^{z\Delta t}\int_{\Delta t}^{+\infty} E_\alpha(-n\mu w^\alpha)e^{-wz}dw \\ &= z^{-1}e^{\Delta tz}\int_{\Delta t}^{+\infty} E_\alpha(-n\mu z^{-\alpha}u^\alpha)e^{-u}du \\ &= z^{-1}e^{\Delta tz}\sum_{k=0}^{+\infty}\frac{(-n\mu z^{-\alpha})^k}{\Gamma(k\alpha+1)}\int_{z\Delta t}^{+\infty} u^{k\alpha}e^{-u}du \\ &= e^{\Delta tz}\sum_{k=0}^{+\infty}\frac{(-n\mu)^k}{\Gamma(k\alpha+1)}z^{-k\alpha-1}\Gamma(k\alpha+1,z\Delta t),\end{aligned}$$

concluding the proof. □

We also need to introduce another probability distribution function, as done in [**19**].

DEFINITION 3.3.1. Let X be a non-negative random variable with distribution

$$F_X(t) = 1 - \frac{E_\alpha(-\lambda(t_0+t)^\alpha)}{E_\alpha(-\lambda t_0^\alpha)}.$$

Then we say that X is a **residual Mittag-Leffler** random variable with fractional parameter $\alpha \in (0,1)$, rate $\lambda > 0$ and starting time t_0 and we denote it by $RML_\alpha(\lambda, t_0)$.

The name is due to the following easy lemma.

LEMMA 3.3.5. *Let X be a $ML_\alpha(\lambda)$-distributed random variable with respect to the probability measure $\mathbb{P}$. Let $\mathbb{Q} = \mathbb{P}(\cdot|X \ge t_0)$. Then $X - t_0$ is $RML_\alpha(\lambda, t_0)$-distributed with respect to $\mathbb{Q}$.*

PROOF. By definition of conditional probability we obtain

$$\mathbb{Q}(X - t_0 \le t) = \frac{\mathbb{P}(t_0 \le X \le t + t_0)}{\mathbb{P}(X \ge t_0)} = \frac{E_\alpha(-\lambda t_0^\alpha) - E_\alpha(-\lambda(t_0+t)^\alpha)}{E_\alpha(-\lambda t_0^\alpha)},$$

concluding the proof. □

Now we can express a formula concerning a particular conditional virtual waiting time (see [**23**]).

PROPOSITION 3.3.6. *Define the function*

$$F_W(s;t,t_0,n) = \mathbb{P}(W(t) \le s | A(t+s) = t,\ E(t) = t_0,\ N(t-) = n+1),$$

for $s, t, t_0 \ge 0$ with $t_0 \le t$ and $n \in \mathbb{N}_0$, where $W(t)$ is the virtual waiting time for a $M_\alpha/M_\alpha/1$ queue with acceleration of the service. Then

- *If $n = 0$ we have $F_W(s;t,t_0,0) = 1 - E_\alpha(-\mu s^\alpha)$;*
- *If $n \neq 0$, then the Laplace-Stieltjes transform of F_W with respect to s is given by*

$$\mathcal{L}^S[F_W(\cdot;t,t_0,n)](z) = \left(1 - \frac{e^{\Delta tz}\sum_{k=0}^{+\infty}\frac{(-(n+1)\mu)^k}{\Gamma(k\alpha+1)}z^{-k\alpha}\Gamma(k\alpha+1,z\Delta t)}{E_\alpha(-(n+1)\mu\Delta t^\alpha)}\right)\prod_{j=1}^{n}\frac{j\mu}{z^\alpha + j\mu},$$

where $\Delta t = t - t_0$.

Proof. For $n = 0$, the virtual waiting time corresponds to the service time of a single user when there are no other users in the queue, thus the first property follows from Proposition 3.3.2. Concerning the second one, since $N(t-) = n+1$ and no other user enters the queue up to $t+s$, we know that

$$W(t) = \sum_{j=1}^{n} S_j + (S_{n+1} - \Delta t)$$

where $\Delta t = t - t_0$. Now, under the conditioning we imposed, we know that $S_j \sim ML_\alpha(j\mu)$. On the other hand, we know that the user that is in service at time $t > 0$ started its service at time $t_0 < t$. Thus we are conditioning to the fact that $S_{n+1} \geq t - t_0$ and then we have, by Lemma 3.3.5, that $S_{n+1} \sim RML_\alpha((n+1)\mu, t_0)$. Finally, taking in consideration the fact that the variables S_j are independent and using Lemma 3.3.4 together with the definition of $RLM_\alpha(\lambda, t_0)$, we conclude the proof. □

3.4. The fractional $M/M/1$ queue with catastrophes

Let us consider again a $M/M/1$ system state process $N(t)$ with parameters $\lambda, \mu > 0$. Let us suppose that such queue system is subject to catastrophes, whose effect is to instantaneously empty the queue, and that, under the conditioning that the queue is not empty, the time of inter-occurrence of catastrophes is exponentially distributed with rate $\xi > 0$. Let us denote by $N^\xi(t)$ the new system state process. This process has been widely studied in [**53**], where the following Theorem is proved.

Theorem 3.4.1. *Let $N^\xi(t)$ be a $M/M/1$ queue with catastrophes and denote $p^\xi(t,n) = \mathbb{P}_0(N^\xi(t) = n)$ the state probabilities. Then the state probabilities are the unique solution of the difference-differential Cauchy problem*

$$\begin{cases} \frac{dp^\xi}{dt}(t,0) = -(\lambda+\xi)p^\xi(t,0) + \mu p^\xi(t,1) + \xi & t > 0 \\ \frac{dp^\xi}{dt}(t,n) = -(\lambda+\xi+\mu)p^\xi(t,n) + \mu p^\xi(t,n+1) + \lambda p^\xi(t,n-1) & n \in \mathbb{N},\ t > 0 \\ p^\xi(0,0) = 1 \\ p^\xi(0,n) = 0,\ n \in \mathbb{N}\,. \end{cases}$$

Moreover, $N^\xi(t)$ admits stationary state with distribution given by

$$q_n = \left(1 - \frac{1}{z_1}\right)\left(\frac{1}{z_1}\right)^n,\ n \in \mathbb{N}_0 \tag{3.4.1}$$

where $z_1 > 1$ is a solution of the equation

$$\lambda z^2 - (\lambda+\mu+\xi)z + \mu = 0.$$

Finally, let $\widetilde{N}(t)$ be a $M/M/1$ system state process with arrival rate λz_1 and service rate $\frac{\mu}{z_1}$ and N^ξ be a random variable independent of $\widetilde{N}(t)$ with distribution $(q_n)_{n\in\mathbb{N}_0}$. Then $N^\xi(t) \overset{d}{=} \min\{\widetilde{N}(t), N^\xi\}$ for any $t \geq 0$.

Now we can define the $M_\alpha/M_\alpha/1$ queue with catastrophes. Indeed, let us consider a system state process $N^\xi(t)$ of a $M/M/1$ queue with catastrophes and let $L_\alpha(t)$ be the inverse of an α-stable subordinator independent of $N^\xi(t)$. Then the $M_\alpha/M_\alpha/1$ system state process with catastrophes is defined as $N^\xi_\alpha(t) := N^\xi(L_\alpha(t))$.

First of all, let us give an alternative representation of the considered system state process.

PROPOSITION 3.4.2. *Let $\widetilde{N}_\alpha(t)$ be a $M_\alpha/M_\alpha/1$ system state process with arrival rate λz_1 and service rate μ/z_1, defined as $\widetilde{N}_\alpha(t) := \widetilde{N}(L_\alpha(t))$ for $\widetilde{N}(t)$ a $M/M/1$ queue with the same parameters. Moreover, let N^ξ be a random variable independent of $N(t)$ and $L_\alpha(t)$ with distribution $(q_n)_{n\in\mathbb{N}_0}$ where q_n is given by Equation* (3.4.1). *Then it holds*

$$N^\xi_\alpha(t) \overset{d}{=} \min\{\widetilde{N}_\alpha(t), N^\xi\}.$$

PROOF. Let us recall that $N^\xi(t) \overset{d}{=} \min\{\widetilde{N}(t), N^\xi\}$. Thus we get

$$\begin{aligned}\mathbb{P}_0(N^\xi_\alpha(t)=n) &= \int_0^{+\infty} \mathbb{P}_0(N^\xi(s)=n) f_\alpha(s;t)ds \\ &= \int_0^{+\infty} \mathbb{P}_0(\min\{\widetilde{N}(t), N^\xi\}=n) f_\alpha(s;t)ds \\ &= \mathbb{P}_0(\min\{\widetilde{N}_\alpha(t), N^\xi\}=n),\end{aligned}$$

concluding the proof. □

Now that we have this alternative representation, we can use it to determine a system of fractional difference-differential equations whose unique solutions are the state probabilities $p^\xi_\alpha(t,n) = \mathbb{P}(N^\xi_\alpha(t)=n)$, as done in [**18**, Theorem 3].

THEOREM 3.4.3. *Let N^ξ_α be a $M_\alpha/M_\alpha/1$ system state process with catastrophes with rates $\lambda, \mu, \xi > 0$. Then the state probabilities $p^\xi_\alpha(t,n)$ are the unique solutions of*

(3.4.2)

$$\begin{cases} \partial^\alpha p^\xi_\alpha(t,0) = -(\lambda+\xi)p^\xi_\alpha(t,0) + \mu p^\xi_\alpha(t,1) + \xi & t>0 \\ \partial^\alpha p^\xi_\alpha(t,n) = -(\lambda+\xi+\mu)p^\xi_\alpha(t,n) + \mu p^\xi_\alpha(t,n+1) + \lambda p^\xi_\alpha(t,n-1) & n \in \mathbb{N},\ t>0 \\ p^\xi_\alpha(0,0) = 1 \\ p^\xi_\alpha(0,n) = 0,\ n \in \mathbb{N}\,. \end{cases}$$

PROOF. Let $\widetilde{N}^\xi_\alpha(t)$ and N^ξ be the $M/M/1$ process and the random variable defined in Proposition 3.4.2. Then we have

$$\begin{aligned} p^\xi_\alpha(t,n) &= \mathbb{P}_0(N^\xi=n)\,\mathbb{P}_0(\widetilde{N}_\alpha(t)\ge n) + \mathbb{P}_0(N^\xi>n)\,\mathbb{P}_0(\widetilde{N}_\alpha(t)=n) \\ &= q_n \sum_{k=n}^{+\infty} \widetilde{p}_\alpha(t,k) + \left(\sum_{k=n+1}^{+\infty} q_n\right)\widetilde{p}_\alpha(t,n) \\ &= q_n \sum_{k=n}^{+\infty} \widetilde{p}_\alpha(t,k) + \left(\frac{1}{z_1}\right)^{n+1}\widetilde{p}_\alpha(t,n), \end{aligned} \tag{3.4.3}$$

where we used the definition of q_n given in Equation (3.4.1). Now let us recall that $\widetilde{p}_\alpha(t,n)$ are the unique solution of (3.2.1). Working with $p^\xi_\alpha(t,0)$, we have

$$p^\xi_\alpha(t,0) = q_0 + \frac{\widetilde{p}_\alpha(t,0)}{z_1}.$$

Hence, $p^\xi_\alpha(t,0)$ admits a fractional derivative and

$$\partial^\alpha p^\xi_\alpha(t,0) = \frac{\partial^\alpha \widetilde{p}_\alpha(t,0)}{z_1} = -\lambda\widetilde{p}_\alpha(t,0) + \frac{\mu}{z_1^2}\widetilde{p}_\alpha(t,1).$$

On the other hand, we have

$$p^{\xi}_{\alpha}(t,1) = q_1(1-\widetilde{p}_{\alpha}(t,0)) + \frac{\widetilde{p}_{\alpha}(t,0)}{z_1^2}.$$

Thus, it holds, by recalling that $q_0 = 1-\frac{1}{z_1}$, $q_1 = \frac{q_0}{z_1}$ and $\lambda z_1^2-(\lambda+\mu+\xi)z_1+\mu = 0$,

$$\begin{aligned}
-(\lambda+\xi)p^{\xi}_{\alpha}(t,0) + \mu p^{\xi}_{\alpha}(t,1) + \xi &= \xi - (\lambda+\xi)\left(q_0 + \frac{\widetilde{p}_{\alpha}(t,0)}{z_1}\right) \\
&\quad + \mu\left(q_1(1-\widetilde{p}_{\alpha}(t,0)) + \frac{\widetilde{p}_{\alpha}(t,0)}{z_1^2}\right) \\
&= -\frac{\lambda z_1^2-(\lambda+\mu+\xi)z_1+\mu}{z_1^2}(1-\widetilde{p}_{\alpha}(t,0)) - \lambda\widetilde{p}_{\alpha}(t,0) + \frac{\mu}{z_1}\widetilde{p}_{\alpha}(t,1) \\
&= -\lambda\widetilde{p}_{\alpha}(t,0) + \frac{\mu}{z_1}\widetilde{p}_{\alpha}(t,1) = \partial^{\alpha}p^{\xi}_{\alpha}(t,0).
\end{aligned}$$

Now let us consider $n \in \mathbb{N}$ and let us re-write Equation (3.4.3) as

$$p^{\xi}_{\alpha}(t,n) = q_n\left(1-\sum_{k=0}^{n-1}\widetilde{p}_{\alpha}(t,k)\right) + \left(\frac{1}{z_1}\right)^{n+1}\widetilde{p}_{\alpha}(t,n).$$

We have that $p^{\xi}_{\alpha}(t,n)$ admits a fractional derivative and

$$\begin{aligned}
\partial^{\alpha}p^{\xi}_{\alpha}(t,n) &= -q_n\sum_{k=0}^{n-1}\partial^{\alpha}\widetilde{p}_{\alpha}(t,k) + \left(\frac{1}{z_1}\right)^{n+1}\partial^{\alpha}\widetilde{p}_{\alpha}(t,n) \\
&= q_n\left(\lambda z_1 + \frac{\mu}{z_1}\right)\sum_{k=1}^{n-1}\widetilde{p}_{\alpha}(t,k) + q_n\lambda z_1\widetilde{p}_{\alpha}(t,0) - q_n\lambda z_1\sum_{k=0}^{n-2}\widetilde{p}_{\alpha}(t,k) \\
&\quad - q_n\frac{\mu}{z_1}\sum_{k=1}^{n}\widetilde{p}_{\alpha}(t,k) - \left(\frac{1}{z_1}\right)^{n+1}\left(\lambda z_1 + \frac{\beta}{z_1}\right)\widetilde{p}_{\alpha}(t,n) \\
&\quad + \lambda\left(\frac{1}{z_1}\right)^{n}\widetilde{p}_{\alpha}(t,n-1) + \mu\left(\frac{1}{z_1}\right)^{n+2}\widetilde{p}_{\alpha}(t,n+1).
\end{aligned}$$

On the other hand, recalling that $q_{n-1} = z_1q_n$,

$$\begin{aligned}
&-(\lambda+\mu+\xi)p^{\xi}_{\alpha}(t,n) + \lambda p^{\xi}_{\alpha}(t,n-1) + \xi p_{\alpha}(t,n+1) \\
&\quad = -(\lambda+\mu+\xi)\left(q_n\left(1-\sum_{k=0}^{n-1}\widetilde{p}_{\alpha}(t,k)\right) + \left(\frac{1}{z_1}\right)^{n+1}\widetilde{p}_{\alpha}(t,n)\right) \\
&\qquad + \lambda\left(q_{n-1}\left(1-\sum_{k=0}^{n-2}\widetilde{p}_{\alpha}(t,k)\right) + \left(\frac{1}{z_1}\right)^{n}\widetilde{p}_{\alpha}(t,n-1)\right) \\
&\qquad + \mu\left(q_{n+1}\left(1-\sum_{k=0}^{n}\widetilde{p}_{\alpha}(t,k)\right) + \left(\frac{1}{z_1}\right)^{n+2}\widetilde{p}_{\alpha}(t,n+1)\right) \\
&\quad = \frac{\lambda z_1^2-(\lambda+\mu+\xi)z_1+\mu}{z_1}q_n\left(1-\sum_{k=0}^{n-1}\widetilde{p}_{\alpha}(t,k)\right) + \partial^{\alpha}p^{\xi}_{\alpha}(t,n) = \partial^{\alpha}p^{\xi}_{\alpha}(t,n),
\end{aligned}$$

concluding the proof, since the initial conditions are verified by definition of p^{ξ}_{α}. □

Now that we have the Equations (3.4.2) it is not difficult to check the following statement.

PROPOSITION 3.4.4. *Let T_n be an inter-arrival time, S_n be a service time, Θ_α be the first time of occurrence of a catastrophe and $\mathcal{S}_n$ be a soujourn time in a non-zero state. Then we have*

$$\mathbb{P}(T_n \le t) = 1 - E_\alpha(-\lambda t^\alpha), \qquad \mathbb{P}(\Theta_\alpha \le t| \min_{s\in[0,t]} N(s) > 0) = 1 - E_\alpha(-\xi t^\alpha),$$
$$\mathbb{P}(S_n \le t) = 1 - E_\alpha(-\mu t^\alpha), \qquad \mathbb{P}(\mathcal{S}_n \le t) = 1 - E_\alpha(-(\lambda+\mu+\xi) t^\alpha),$$

where the relations hold for any $n \in \mathbb{N}$.

Let us observe that $\Theta \sim ML_\alpha(\xi)$ only if we suppose that the queue has not been empty (for instance, setting $\mu = 0$ and starting from $N_\alpha^\xi(0) = 1$). However, this is enough to give some information on the busy period. Indeed, let us denote by B_α^ξ the busy period of a $M_\alpha/M_\alpha/1$ queue with catastrophes with parameters $\lambda, \mu, \xi > 0$ and with B_α the busy period of a $M_\alpha/M_\alpha/1$ queue without catastrophes and with the same parameters. Then we can recognize $\mathbb{P}(B^\xi \le t) = \mathbb{P}_1(\min_{s\in[0,t]} N_\alpha^\xi(s) = 0)$, i.e. the system state process starting from 1 reached 0 before t. This could happen in two different situations:

- The first catastrophes occurred before $t \ge 0$;
- The first catastrophes did not occur before $t \ge 0$, but the process reaches zero without the help of any catastrophe.

This means in particular that

$$\begin{aligned}\mathbb{P}(B^\xi \le t) &= \mathbb{P}_1(\Theta_\alpha \le t) + \mathbb{P}_1(B_\alpha^\xi \le t|\Theta_\alpha > t)\,\mathbb{P}_1(\Theta_\alpha > t)\\ &= \mathbb{P}_1(\Theta_\alpha \le t) + \mathbb{P}_1(B_\alpha \le t)\,\mathbb{P}_1(\Theta_\alpha > t).\end{aligned}$$

A direct application of Theorem 3.2.4 leads to the following result.

THEOREM 3.4.5. *Let $F_{B_\alpha^\xi}(t) := \mathbb{P}(B_\alpha^\xi \le t)$. Then*

$$F_{B_\alpha^\xi}(t) = 1 - E_\alpha(-\xi t^\alpha) \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m} t^{\alpha(n+2m-1)} E^{n+2m}_{\alpha,\alpha(n+2m-1)+1}(-(\lambda+\mu)t^\alpha), \ t \ge 0$$

where $C_{n,m}$ are defined in Equation (3.2.4).

However, we still have to determine the distribution of the first occurrence of a catastrophe Θ when the queue starts from $N_\alpha^\xi(0) = 0$. This can be done by means of the following Theorem (see [**18**, Theorem 6]).

THEOREM 3.4.6. *Let Θ_α be the first occurrence time of a catastrophe in a $M_\alpha/M_\alpha/1$ queue with catastrophes with parameters $\lambda, \mu, \xi > 0$. Then it holds*

$$\mathbb{P}_0(\Theta_\alpha \le t) = 1 - \sum_{j=1}^{+\infty}\sum_{m=0}^{+\infty} C_{m,j} t^{\alpha(2m+j-1)} E^{2m+j}_{\alpha,\alpha(2m+j-1)+1}[-(\lambda+\mu+\xi)t^\alpha],$$

where

$$C_{m,j} = \frac{j}{2m+j}\,\frac{(\mu+\xi)^j - \lambda^j}{\mu+\xi-\lambda}\binom{2m+j}{m}(\lambda\mu)^m. \tag{3.4.4}$$

PROOF. Let us consider the system state process $N^\xi(t)$ fo a $M/M/1$ queue with catastrophes with parameters $\lambda, \alpha, \xi > 0$ and the corresponding time-changed process $N_\alpha^\xi(t)$. Let us modify both processes in the following way:

- We add the state -1 to $\mathbb{N}_0$;
- Now a catastrophes does not empty the queue, but sends the system state process directly to -1;
- -1 is an absorbing state.

We obtain in this way two new processes $\bar{N}^{\xi}(t)$ and $\bar{N}^{\xi}_{\alpha}(t)$ such that $\bar{N}^{\xi}_{\alpha}(t) = \bar{N}^{\xi}(L_{\alpha}(t))$. Now let us denote $r^{\xi}_{\alpha}(t,n) = \mathbb{P}_0(\bar{N}^{\xi}_{\alpha}(t) = n)$ and $r^{\xi}(t,n) = \mathbb{P}_0(\bar{N}_{\alpha}(t) = n)$ for $n \in \mathbb{N}_0 \cup \{-1\}$. Let us now give an interpretation of $r^{\xi}(t,-1)$. We have that if $\bar{N}^{\xi}(t) = -1$, then a catastrophes must have occurred before t and vice-versa, if a catastrophes occurs before t, then $\bar{N}^{\xi}(t) = -1$. In particular we get that, denoting by Θ the first occurrence of a catastrophe for a $M/M/1$ queue with catastrophes with parameters $\lambda, \mu, \xi > 0$,

$$\mathbb{P}_0(\Theta \le t) = r^{\xi}(t,-1).$$

Arguing in the same way for $\bar{N}^{\xi}_{\alpha}(t)$, we get

$$\mathbb{P}_0(\Theta_{\alpha} \le t) = r^{\xi}_{\alpha}(t,-1) = \int_0^{+\infty} r^{\xi}(s,-1) f_{\alpha}(s;t)ds = \int_0^{+\infty} \mathbb{P}_0(\Theta \le t) f_{\alpha}(s;t)ds.$$

Now, from [**53**, Theorem 3.1], we have

$$\mathbb{P}_0(\Theta \le t) = 1 - \sum_{j=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{m,j}}{(2n+j-1)!} t^{2m+j-1} e^{-(\lambda+\mu+\xi)t},$$

where $C_{m,j}$ are defined in Equation (3.4.4). By monotone convergence theorem we get

$$\mathbb{P}_0(\Theta_{\alpha} \le t) = 1 - \sum_{j=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{m,j}}{(2n+j-1)!} \int_0^{+\infty} s^{2m+j-1} e^{-(\lambda+\mu+\xi)s} f_{\alpha}(s;t)ds.$$

Then, let us take the Laplace transform on both sides to get

$$\begin{aligned}
\mathcal{L}_{t\to z}[\mathbb{P}_0(\Theta_{\alpha} \le t)](z) &= \frac{1}{z} - \sum_{j=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{C_{m,j}}{(2n+j-1)!} z^{\alpha-1} \int_0^{+\infty} s^{2m+j-1} e^{-(\lambda+\mu+\xi+z^{\alpha})s} ds \\
&= \frac{1}{z} - \sum_{j=1}^{+\infty}\sum_{m=0}^{+\infty} C_{m,j} \frac{z^{\alpha-1}}{(\lambda+\mu+\xi+z^{\alpha})^{2m+j}}.
\end{aligned}$$

Taking the inverse Laplace transform and using Equation (3.2.2) we conclude the proof. □

Now let us consider again the state probabilities p^{ξ}_{α}. We have shown that they are solutions of some fractional difference-differential Cauchy problem, but we did not write them in an explicit way. Let us indeed state the following Theorem (see [**18**, Theorem 4]).

THEOREM 3.4.7. *Let z_2 be the other solution of $\lambda z^2 - (\lambda+\mu+\xi)z + \mu = 0$. Then we have*

$$p_\alpha^\xi(t,n) = q_n + \sum_{m=0}^{+\infty}\sum_{r=0}^{m+n} C^1_{n,m,r} t^{\alpha(m+r-1)} E^{m+r}_{\alpha,\alpha(m+r-1)}(-(\lambda+\mu+\xi)t^\alpha) \tag{3.4.5}$$

$$+ \sum_{m=0}^{+\infty}\sum_{r=m+n+1}^{+\infty} C^2_{n,m,r} t^{\alpha(m+r-1)} E^{m+r}_{\alpha,\alpha(m+r-1)}(-(\lambda+\mu+\xi)t^\alpha) \tag{3.4.6}$$

where

$$\begin{aligned} C^1_{n,m,r} &= \frac{z_1-1}{(z_1-z_2)z_1^{n+m+1-r}}\binom{m+r}{r}\frac{m-r}{m+r}\mu^m\lambda^{r-1} \\ C^2_{n,m,r} &= \frac{1-z_2}{(z_1-z_2)z_2^{n+m+1-r}}\binom{m+r}{r}\frac{r-m}{m+r}\mu^m\lambda^{r-1}. \end{aligned} \tag{3.4.7}$$

PROOF. Let us recall that in [**53**] it has been shown that

$$\begin{aligned} p^\xi(t,n) = q_n &+ \sum_{m=0}^{+\infty}\sum_{r=0}^{m+n}\frac{C^1_{n,m,r}}{(m+r-1)!}t^{m+r-1}e^{-(\lambda+\mu+\xi)t} \\ &+ \sum_{m=0}^{+\infty}\sum_{r=m+n+1}^{+\infty}\frac{C^2_{n,m,r}}{(m+r-1)!}t^{m+r-1}e^{-(\lambda+\mu+\xi)t}, \end{aligned}$$

where $C^i_{n,m,r}$ for $i=1,2$ are defined in Equation (3.4.7). Thus we have, by monotone convergence theorem

$$\begin{aligned} p_\alpha^\xi(t,n) &= \int_0^{+\infty} p^\xi(s,n)f_\alpha(s;t)ds \\ &= q_n + \sum_{m=0}^{+\infty}\sum_{r=0}^{m+n}\frac{C^1_{n,m,r}}{(m+r-1)!}\int_0^{+\infty} s^{m+r-1}e^{-(\lambda+\mu+\xi)s}f_\alpha(s;t)ds \\ &\quad + \sum_{m=0}^{+\infty}\sum_{r=m+n+1}^{+\infty}\frac{C^2_{n,m,r}}{(m+r-1)!}\int_0^{+\infty} s^{m+r-1}e^{-(\lambda+\mu+\xi)s}f_\alpha(s;t)ds. \end{aligned}$$

Now let us denote by $\pi_\alpha^\xi(z,n)$ the Laplace transforms of $p_\alpha^\xi(t,n)$ to achieve

$$\begin{aligned} \pi_\alpha^\xi(t,n) &= \frac{q_n}{z} + \sum_{m=0}^{+\infty}\sum_{r=0}^{m+n}\frac{C^1_{n,m,r}}{(m+r-1)!}z^{\alpha-1}\int_0^{+\infty} s^{m+r-1}e^{-(\lambda+\mu+\xi+z^\alpha)s}ds \\ &\quad + \sum_{m=0}^{+\infty}\sum_{r=m+n+1}^{+\infty}\frac{C^2_{n,m,r}}{(m+r-1)!}z^{\alpha-1}\int_0^{+\infty} s^{m+r-1}e^{-(\lambda+\mu+\xi+z^\alpha)s}ds \\ &= \frac{q_n}{z} + \sum_{m=0}^{+\infty}\sum_{r=0}^{m+n} C^1_{n,m,r}\frac{z^{\alpha-1}}{(\lambda+\mu+\xi+z^\alpha)^{m+r}} \\ &\quad + \sum_{m=0}^{+\infty}\sum_{r=m+n+1}^{+\infty} C^2_{n,m,r}\frac{z^{\alpha-1}}{(\lambda+\mu+\xi+z^\alpha)^{m+r}}. \end{aligned}$$

Finally, taking the inverse Laplace transform and using Equation (3.2.2) we conclude the proof. $\square$

Let us now consider some simple one-dimensional limit theorems concerning the process $N^\xi_\alpha(t)$. Indeed, let us recall that as $\xi \to 0^+$, then $N^\xi(t)$ converges in one-dimensional distributions to $N(t)$ (where $N(t)$ is the system state process of a $M/M/1$ queue without catastrophes). The same happens for $N^\xi_\alpha(t)$.

Proposition 3.4.8. *Let $N^\xi_\alpha(t)$ be the system state process of a $M_\alpha/M_\alpha/1$ queue with catastrophes with parameters $\lambda, \mu, \xi > 0$ and let $N_\alpha(t)$ be the system state process of a $M_\alpha/M_\alpha/1$ queue with parameters $\lambda, \mu > 0$. Then we have $N^\xi_\alpha(t) \overset{d}{\to} N_\alpha(t)$ as $\xi \to 0^+$.*

Proof. Let us consider $N^\xi(t)$ a system state process of a $M/M/1$ queue with catastrophes with parameters $\lambda, \mu, \xi > 0$ and $N(t)$ a system state process of a $M/M/1$ queue without catastrophes. Moreover, let $L_\alpha(t)$ be the inverse of an α-stable subordinator independent of both $N(t)$and $N^\xi(t)$. Let us recall that $\lim_{\xi\to 0} p^\xi(t,n) = p(t,n)$ where $p^\xi(t,n)$ and $p(t,n)$ are the state probabilities respectively of $N^\xi(t)$ and $N(t)$. Now let $N^\xi_\alpha(t) = N^\xi(L_\alpha(t))$ and $N_\alpha(t) = N(L_\alpha(t))$ with state probabilities respectively $p^\xi_\alpha(t,n)$ and $p_\alpha(t,n)$. We have

$$p^\xi_\alpha(t,n) = \int_0^{+\infty} p^\xi(s,n) f_\alpha(s;t)ds.$$

In particular, since $0 \le p^\xi(s,n) \le 1$ for any fixed n and $\xi > 0$, we can use dominated convergence theorem to obtain

$$\begin{aligned}\lim_{\xi\to 0^+} p^\xi_\alpha(t,n) = \int_0^{+\infty} \lim_{\xi\to 0^+} p^\xi(s,n) f_\alpha(s;t)ds \\ = \int_0^{+\infty} p(s,n) f_\alpha(s;t)ds = p_\alpha(t,n),\end{aligned}$$

concluding the proof. ☐

Now let us use this limit theorem to obtain a new representation of $p_\alpha(t,n)$. First of all, let us recall that both z_1 and z_2 depend on ξ. However, being the roots of a second degree polynomial whose coefficient depend with continuity on ξ, they are both continuous functions of ξ. As $\mu \le \lambda$ we have

$$\lim_{\xi\to 0^+} z_1(\xi) = 1 \qquad\qquad \lim_{\xi\to 0^+} z_2(\xi) = \frac{\beta}{\alpha}.$$

For this reason, also q_n and $C^i_{n,m,r}$ are continuous functions of ξ. In particular it holds

$$\lim_{\xi\to 0^+} q_n(\xi) = 0 \qquad \lim_{\xi\to 0^+} C^1_{n,m,r}(\xi) = 0 \qquad \lim_{\xi\to 0^+} C^2_{n,m,r}(\xi) = C^3_{n,m,r},$$

where

$$C^3_{n,m,r} = \lambda^{n+m}\mu^{r-n-1}\binom{m+r}{m}\frac{r-m}{m+r}. \tag{3.4.8}$$

Moreover, it is not difficult to see that for fixed t and n, the series in (3.4.5) normally converges and then we can take the limit inside the series. Thus we have proved the following proposition.

PROPOSITION 3.4.9. *Let $N_\alpha(t)$ be the system state process of a $M_\alpha/M_\alpha/1$ queue with parameters $\lambda \ge \mu > 0$. Then it holds*

$$p_\alpha(t,n) = \sum_{m=0}^{+\infty} \sum_{r=m+n+1}^{+\infty} C^3_{n,m,r} t^{\alpha(m+r-1)} E^{m+1}_{\alpha,\alpha(m+r-1)+1}(-(\lambda+\mu)t^\alpha), \tag{3.4.9}$$

where $C^3_{n,m,r}$ is defined in Equation (3.4.8).

Let us observe that, if we do the same with $\lambda < \mu$, we obtain again (3.2.3). Finally let us observe that in this case the virtual waiting time does not provide any useful information from the user's point of view, since if a catastrophe occurs then the user is not served.

3.5. The fractional $M/E_k/1$ queue

Now let us move to a different model. Let us suppose that the inter-arrival times are still exponentially distributed of parameter λ, but that the service is divided in k subsequent phases, each one with exponential service time with rate $k\mu$. What we obtain is that the complete service time is not exponentially distributed, but its distribution is an Erlang one with shape parameter $k \in \mathbb{N}$ and rate μ, that we will denote as $Erl_k(\mu)$. As one can see, an $M/E_k/1$ queue is not a Markovian queue. In particular the system state process $N^s(t)$ (let us recall that this process counts the number of users in the system at time $t > 0$) is not Markov, being the service times not exponential. However, we can still obtain a Markov representation of such queueing system. This can be done in two equivalent ways:

- We can introduce a **phase state process** $N^p(t)$ that counts the remaining number of phases the user in service has still to execute and is set to 0 when the system is empty. In such a way, the coupled **state-phase process** $N^c(t) = (N^s(t), N^p(t))$ is a actually a bivariate continuous time Markov chain (the sojourn times are now exponentials of parameter $\lambda + k\mu$) with state space $E^c = \{(n,s) : \ n \in \mathbb{N}, \ 1 \le s \le k\} \cup \{(0,0)\}$;
- We can directly count the number of phases the system has to execute in place of the number of users. In this way, each user will contribute as k phases (like a group entrance). In this case, the phase counter process is called **queue length process** $N^l(t)$ and is actually a Markov process. However, it jumps backward of just one unit, but forward of k units, thus it is not really a birth-death process, despite the similarity.

As we already stated, the approaches are equivalent. Actually, one can pass from the coupled process $N^c(t)$ to the queue length one $N^l(t)$ by using the following transformation

$$N^l(t) = \begin{cases} k(N^s(t)-1) + N^p(t) & (N^s(t), N^p(t)) \neq (0,0) \\ 0 & (N^s(t), N^p(t)) = (0,0). \end{cases}$$

It is not difficult to see that such mapping is actually a bijection from E to $\mathbb{N}$ and then the procedure can be inverted. Here we will use $N^l(t)$ to describe our system. Now let us consider $p(t,l) = \mathbb{P}_0(N^l(t) = l)$. Let us denote by $l : E \to \mathbb{N}$ the map

$$l(s,p) = \begin{cases} k(s-1) + p & (s,p) \neq (0,0) \\ 0 & (s,p) = (0,0). \end{cases}$$

In [**98, 99**] this system has been considered as particular case of group arrival systems, leading to the following forward equations:

(3.5.1)
$$\begin{cases} \frac{dp}{dt}(t,0) = -\lambda p(t,0) + k\mu p(t,1), & t > 0 \\ \frac{dp}{dt}(t,l) = -(\lambda + k\mu)p(t,l) + k\mu p(t,l+1), & 0 \le l \le k-1, t > 0 \\ \frac{dp}{dt}(t,l) = -(\lambda + k\mu)p(t,l) + k\mu p(t,l+1) + \lambda p(t,l-k), & l \ge k, t > 0 \\ p(0,0) = 1 \\ p(0,l) = 0 & l \in \mathbb{N}. \end{cases}$$

Concerning the solution of such system, we need to introduce the following special functions.

Definition 3.5.1. The **generalized modified Bessel functions of two parameters** (see [**99**]) is defined as

$$I_n^k(z) = \sum_{r=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{n+r(k+1)}}{r!\Gamma(n+rk+1)}, \qquad z \in \mathbb{C}, n > 0, \; k \in \mathbb{N}.$$

The **generalized modified Bessel functions of three parameters** (see [**73**]) is defined as

$$I_{(n,s)}^k(z) = \sum_{r=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{n+k-s+r(k+1)}}{(k(r+1)-s)!\Gamma(n+r+1)}, \qquad z \in \mathbb{C}, n > 0, \; k \in \mathbb{N}, \; s = 1, \dots, k.$$

In [**74**] the solutions of (3.5.1) have been expressed in terms of the generalized modified Bessel functions. In particular we have an explicit formulation only for $p(t,0)$, given by

$$p(t,0) = \sum_{m=1}^{+\infty} m \left(\frac{\lambda}{k\mu}\right)^{-\frac{m}{k+1}} \frac{I_m^k\left(2\left(\frac{\lambda}{k\mu}\right)^{\frac{1}{k+1}} k\mu t\right)}{k\mu t} e^{-(\lambda+k\mu)t}. \tag{3.5.2}$$

Concerning $p(t,l)$ for $l \ge 1$ we have integral representation in terms of $p(t,0)$. To give it, let us denote $q(l) = (q_1(l), q_2(l))$ where $q_2(l)$ is the remainder of the Euclidean division of l with respect to k if it is not 0 and k if it is 0, while $q_1(l) = \frac{l-q_2(l)}{k} + 1$. Let us observe that in such way $q(l)$ belongs to E^c. On E^c we can define the strict lexicographic order, i.e.

$$(n_1, s_1) < (n_2, s_2) \Leftrightarrow n_1 < n_2 \text{ or } (n_1 = n_2 \text{ and } s_1 < s_2)$$

and then we can define the order $\le$ from this strict order. With such definition, $(E, \le)$ is well ordered and then we can define a **successor**. In particular we define the successor of (n,s) a

$$(n,s) + 1 = \begin{cases} (n, s+1) & s \ne k, 0 \\ (n+1, 1) & s = k \\ (1,1) & (n,s) = (0,0). \end{cases}$$

By using these definitions, we have

$$\begin{aligned} p(t,l) &= \left(\frac{\lambda}{k\mu}\right)^{\frac{l}{k+1}} I^k_{q(l)}\left(2(\lambda(k\mu)^k)^{\frac{1}{k+1}}t\right)e^{-(\lambda+k\mu)t} \\ &+ k\mu\left(\frac{\lambda}{k\mu}\right)^{\frac{l}{k+1}}\int_0^t p(\tau,0)I^k_{q(l)}(2(\lambda(k\mu)^k)^{\frac{1}{k+1}}(t-\tau))e^{-(\lambda+k\mu)(t-\tau)}d\tau \\ &- k\mu\left(\frac{\lambda}{k\mu}\right)^{\frac{l+1}{k+1}}\int_0^t p(\tau,0)I^k_{q(l)+1}(2(\lambda(k\mu)^k)^{\frac{1}{k+1}}(t-\tau))e^{-(\lambda+k\mu)(t-\tau)}d\tau. \end{aligned} \tag{3.5.3}$$

Now we want to introduce the fractional version of this process. To do this, we first need to introduce another probability distribution function.

Definition 3.5.2. Let X be a non-negative random variable with distribution function

$$F_X(t) = 1 - \sum_{n=0}^{k-1}\frac{(\lambda t^\alpha)^n}{n!}E^{(n)}_\alpha(-\lambda t^\alpha)$$

where $E^{(n)}_\alpha$ is the n-th derivative of the Mittag-Leffler function. Then we say that X is a **generalized Erlang** random variable **with fractional index α, shape parameter k and rate λ** (see [**102**]) and we denote it by $Erl_{k,\alpha}(\lambda)$.

It particular it can be easily shown that any $Erl_{k,\alpha}(\lambda)$ random variable is sum of k independent $ML_\alpha(k\lambda)$ random variables. Thus in particular if $X \sim Erl_{k,\alpha}(\lambda)$, its Laplace transform is given by

$$\mathbb{E}[e^{-zX}] = \frac{(k\lambda)^k}{(k\lambda+z^\alpha)^k},\ z\in C.$$

Now let us consider the queue length process $N^l(t)$ of a $M/E_k/1$ queue. Then we define the queue length process of a $M_\alpha/E_{k,\alpha}/1$ queue as $N^l_\alpha(t) = N^l(L_\alpha(t))$ where $L_\alpha(t)$ is the inverse of an α-stable subordinator independent of $N^l(t)$.
As a first step, let us find the forward equations for the state probabilities $p_\alpha(t,l) = \mathbb{P}_0(N^l_\alpha(t) = l)$ for $l \in \mathbb{N}_0$, as shown in [**19**, Theorem 3.1].

Theorem 3.5.1. *The state probabilities $p_\alpha(t,l)$ are solution of the following fractional difference-differential Cauchy problem*

$$\begin{cases} \partial^\alpha p_\alpha(t,0) = -\lambda p_\alpha(t,0) + k\mu p_\alpha(t,1), & t>0 \\ \partial^\alpha p_\alpha(t,l) = -(\lambda+k\mu)p_\alpha(t,l) + k\mu p_\alpha(t,l+1), & 0\le l\le k-1, t>0 \\ \partial^\alpha p_\alpha(t,l) = -(\lambda+k\mu)p_\alpha(t,l) + k\mu p_\alpha(t,l+1) + \lambda p_\alpha(t,l-k), & l\ge k, t>0 \\ p_\alpha(0,0) = 1 \\ p_\alpha(0,l) = 0 & l\in\mathbb{N}. \end{cases} \tag{3.5.4}$$

Proof. Let us consider the probability generating function

$$G_\alpha(z,t) = \sum_{n=0}^{+\infty} p_\alpha(t,n)z^n,\ z\in\mathbb{D}_1\,.$$

First we want to show that the system (3.5.4) is equivalent to

$$\begin{cases} z\partial^\alpha_t G_\alpha(z,t) = (1-z)[G_\alpha(z,t)(k\mu - \lambda(z+\cdot+z^k)) - k\mu p_\alpha(t,0)] & t>0,\ z\in\mathbb{D}_1 \\ G_\alpha(z,0) = 1 & z\in\mathbb{D}_1\,. \end{cases} \tag{3.5.5}$$

Let us suppose that $p_\alpha(t,n)$ are solutions of (3.5.4). Then multiplying both the second and the third equations by z^l and summing everything we obtain the first equation of (3.5.5), while the initial datum follows by definition.
Vice versa, if $G(z,t)$ is solution of (3.5.5), then we have

$$\begin{aligned}\sum_{n=0}^{+\infty}\partial^\alpha p_\alpha(t,n)z^{n+1} = k\mu\sum_{n=0}^{+\infty}p_\alpha(t,n)z^n - \lambda\sum_{r=1}^{k}\sum_{n=0}^{+\infty}p_\alpha(t,n)z^{n+r} - k\mu p_\alpha(t,0)\\ - k\mu\sum_{n=0}^{+\infty}p_\alpha(t,n)z^{n+1} + \lambda\sum_{r=1}^{k}\sum_{n=0}^{+\infty}p_\alpha(t,n)z^{n+r+1} + k\mu p_\alpha(t,0)z\end{aligned}$$

Now let us define $p_\alpha(t,n)=0$ for any $n<0$. Thus we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\partial^\alpha p_\alpha(t,n-1)z^{n} = k\mu\sum_{n=0}^{+\infty}p_\alpha(t,n)z^n - \lambda\sum_{r=1}^{k}\sum_{n=r}^{+\infty}p_\alpha(t,n-r)z^{n} - k\mu p_\alpha(t,0)\\ - k\mu\sum_{n=1}^{+\infty}p_\alpha(t,n-1)z^{n} + \lambda\sum_{r=1}^{k}\sum_{n=r+1}^{+\infty}p_\alpha(t,n-r-1)z^{n} + k\mu p_\alpha(t,0)z\\ = (k\mu p_\alpha(t,1)-\lambda p_\alpha(t,0))z + \sum_{n=2}^{k}(k\mu p_\alpha(t,n)-(\lambda+k\mu)p_\alpha(t,n-1))z^n\\ + \sum_{n=k+1}^{+\infty}(k\mu p_\alpha(t,n)-(\lambda+k\mu)p_\alpha(t,n-1)+\lambda p_\alpha(t,n-k-1))z^n,\end{aligned}$$

and then we obtain the system (3.5.4) by identity of power series.
Hence, we only need to show that the probability generating function $G_\alpha(z,t)$ solves (3.5.5). Taking the Laplace transform and denoting by $\bar{G}_\alpha(z,\xi)$ and $\pi_\alpha(\xi,0)$ the Laplace transforms respectively of $G_\alpha(z,t)$ and $p_\alpha(t,0)$, we have that (3.5.5) is equivalent to

$$z\xi^\alpha\bar{G}_\alpha(z,\xi)-z\xi^{\alpha-1} = (1-z)[\bar{G}_\alpha(z,\xi)(k\mu-\lambda(z+\cdots+z^k))-k\mu\pi_\alpha(\xi,0)]. \tag{3.5.6}$$

Now let us consider $p(t,n)$ the state probabilities of the queue length process $N^l(t)$ of the $M/E_k/1$ queue such that $N^l_\alpha(t)=N^l(L_\alpha(t))$ and let us define the probability generating function

$$G(z,t)=\sum_{n=0}^{+\infty}p(t,n)z^n. \tag{3.5.7}$$

Then it is easy to check that

$$G_\alpha(z,t)=\int_0^{+\infty}G(z,s)f_\alpha(s;t)ds, \qquad p_\alpha(t,0)=\int_0^{+\infty}p(s,0)f_\alpha(s;t)ds,$$

and the Laplace transforms are given by

$$\bar{G}_\alpha(z,\xi)=\xi^{\alpha-1}\int_0^{+\infty}G(z,s)e^{-s\xi^\alpha}ds, \quad \pi_\alpha(t,0)=\xi^{\alpha-1}\int_0^{+\infty}p(s,0)e^{-s\xi^\alpha}ds.$$

Substituting these relations in (3.5.6) and dividing by $\xi^{\alpha-1}$ we get the equivalent equation

$$z\xi^\alpha \int_0^{+\infty} G(z,s)e^{-s\xi^\alpha}ds - z = (1-z)\left[(k\mu - \lambda(z+\cdots+z^k))\int_0^{+\infty} G(z,s)e^{-s\xi^\alpha}ds \right.$$
$$\left. -k\mu\int_0^{+\infty} p(s,0)e^{-s\xi^\alpha}ds\right].$$

Finally, observing that

$$\xi^\alpha \int_0^{+\infty} G(z,s)e^{-s\xi^\alpha}ds = \int_0^{+\infty} \frac{\partial G}{\partial s}(z,s)e^{-s\xi^\alpha}ds + 1,$$

we get the equivalent equation

$$\int_0^{+\infty}\left[z\frac{\partial G}{\partial s}(z,s) - (1-z)[(k\mu - \lambda(z+\cdots+z^k))G(z,s) - k\mu p(s,0)]\right]e^{-s\xi^\alpha}ds = 0,$$

that is verified since $G(z,t)$ is solution of

$$\begin{cases} z\frac{\partial G}{\partial t}(z,t) = (1-z)[G(z,t)(k\mu - \lambda(z+\cdot+z^k)) - k\mu p(t,0)] & t>0,\ z\in\mathbb{D}_1 \\ G(z,0)=1 & z\in\mathbb{D}_1\,. \end{cases}$$

□

Now that we have the forward equations, we can argue as we did before to show the following features:

- The inter-arrival times T_n are $ML_\alpha(\lambda)$-distributed;
- The phase-service times P_n are $ML_\alpha(k\mu)$-distributed;
- The total service times S_n are $Erl_{k,\alpha}(\mu)$ distributed;
- The soujourn times $\mathcal{S}^l_n$ of the queue-length process $N^l(t)$ are $ML_\alpha(k\mu+\lambda)$ distributed if the queue is not empty, $ML_\alpha(\lambda)$-distributed otherwise.
- Inter-arrival and phase-service times are not independent.

We want to determine an explicit form of the state probabilities $p_\alpha(t,n)$. To do this, let us first consider the case of $p_\alpha(t,0)$ (see [**19**, Theorem 5.1]).

Theorem 3.5.2. *It holds*

$$p_\alpha(t,0) = \sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} C^0_{m,r} t^{\alpha(\delta^0_{m,r}-1)} E^{\delta^0_{m,r}}_{\alpha,\alpha(\delta^0_{m,r}-1)+1}(-(\lambda+k\mu)t^\alpha) \tag{3.5.8}$$

where

(3.5.9)
$$C^0_{m,r} = \frac{m}{m+r(k+1)}\binom{m+r(k+1)}{m+rk}\lambda^r(k\mu)^{m+rk-1} \qquad \delta^0_{m,r} = m+r(k+1).$$

Proof. Let us write $p(t,0)$ in terms of power series. We have, by using the definition of modified Bessel function of two parameters and equation (3.5.2),

$$p(t,0) = \sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} \frac{C^0_{m,r}}{(m+r(k+1)-1)!} t^{m+r(k+1)-1} e^{-(\lambda+k\mu)t},$$

where $C^0_{m,r}$ is defined in (3.5.9). In particular, by monotone convergence theorem, it holds

$$p_\alpha(t,0)=\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty}\frac{C^0_{m,r}}{(m+r(k+1)-1)!}\int_0^{+\infty}s^{m+r(k+1)-1}e^{-(\lambda+k\mu)s}f_\alpha(s;t)ds,$$

and then, taking the Laplace transform, we get

(3.5.10)

$$\begin{aligned}\pi_\alpha(z,0)&=\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty}\frac{C^0_{m,r}}{(m+r(k+1)-1)!}z^{\alpha-1}\int_0^{+\infty}s^{m+r(k+1)-1}e^{-(\lambda+k\mu+z^\alpha)s}ds\\&=\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty}C^0_{m,r}\frac{z^{\alpha-1}}{(\lambda+k\mu+z^\alpha)^{m+r(k+1)}}.\end{aligned}$$

We conclude the proof by taking the inverse Laplace transform and using Equation (3.2.2). ☐

To obtain $p_\alpha(t,l)$ for $l\geq 1$, we first need the following technical Lemma (see [**19**, Lemma 5.4]).

LEMMA 3.5.3. *For any $l\in\mathbb{N}$ it holds*

$$\int_0^{+\infty}\int_0^y p(s,0)(y-s)^n e^{-(\lambda+k\mu)(y-s)-yz^\alpha}dsdy=\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty}C^0_{m,r}\frac{n!}{(\lambda+k\mu+z^\alpha)^{m+r(k+1)+n+1}}\tag{3.5.11}$$

where $C^0_{m,r}$ is defined in (3.5.9).

PROOF. By a direct application of Fubini's theorem and the change of variables $w=y-s$, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^y p(s,0)(y-s)^n e^{-(\lambda+k\mu)(y-s)-yz^\alpha}dsdy\\&\qquad=\int_0^{+\infty}\int_s^{+\infty}p(s,0)(y-s)^n e^{-(\lambda+k\mu)(y-s)-yz^\alpha}dyds\\&\qquad=\int_0^{+\infty}\int_0^{+\infty}p(s,0)w^n e^{-(\lambda+k\mu+z^\alpha)w-sz^\alpha}dwds\\&\qquad=\left(\int_0^{+\infty}p(s,0)e^{-sz^\alpha}ds\right)\left(\int_0^{+\infty}w^n e^{-(\lambda+k\mu+z^\alpha)w}dw\right)\\&\qquad=\frac{\pi_\alpha(z,0)n!}{z^{\alpha-1}(\lambda+k\mu+z^\alpha)^{n+1}}.\end{aligned}$$

We conclude the proof by substituting $\pi_\alpha(z,0)$ with the right-hand side of Equation (3.5.10). ☐

Now we are ready to give a (*more or less*) explicit formula for $p_\alpha(t,l)$ (see [**19**, Theorem 5.5]).

THEOREM 3.5.4. *For any $l \geq 1$ it holds*

$$\begin{aligned} p_\alpha(t,l) &= \sum_{j=0}^{+\infty} A_j^l t^{\alpha(a_j^l-1)} E_{\alpha,\alpha(a_j^l-1)+1}^{a_j^l}(-(\lambda+k\mu)t^\alpha) \\ &+ \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} B_{j,m,r}^l t^{\alpha(b_{j,m,r}^l-1)} E_{\alpha,\alpha(b_{j,m,r}^l-1)+1}^{b_{j,m,r}^l}(-(\lambda+k\mu)t^\alpha) \\ &- \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} C_{j,m,r}^l t^{\alpha(c_{j,m,r}^l-1)} E_{\alpha,\alpha(c_{j,m,r}^l-1)+1}^{b_{j,m,r}^l}(-(\lambda+k\mu)t^\alpha) \end{aligned}$$

where, setting $q_3(l) = q_1(l) - q_2(l)$,

$$\begin{aligned} A_j^l &= \binom{q_3(l)+k+kj+j}{q_1(l)+j}\lambda^{q_1(l)+j}(k\mu)^{k(j+1)-q_2(l)}, & a_j^l &= q_3(l)+(j+1)(k+1), \\ B_{j,m,r}^l &= k\mu C_{m,r}^0 A_j^l, & b_{j,m,r}^l &= \delta_{m,r}^0 + a_j^l, \\ C_{j,m,r}^l &= k\mu C_{m,r}^0 A_j^{l+1}, & c_{j,m,r}^l &= \delta_{m,r}^0 + a_j^{l+1}, \end{aligned}$$

and $C_{m,r}^0$ and $\delta_{m,r}^0$ are defined in (3.5.9).

PROOF. Let us recall that

$$p_\alpha(t,l) = \int_0^{+\infty} p(s,l) f_\alpha(s;t)ds,$$

thus, by monotone convergence theorem, writing the generalized Bessel functions in terms of power series and taking the Laplace transform, we get

$$\begin{aligned} \pi_\alpha(z,l) &= \sum_{j=0}^{+\infty} \frac{A_j^l}{(a_j^l-1)!} z^{\alpha-1} \int_0^{+\infty} y^{a_j^l-1} e^{-(\lambda+k\mu+z^\alpha)y} dy \\ &+ \sum_{j=0}^{+\infty} \frac{k\mu A_j^l}{(a_j^l-1)!} \int_0^{+\infty}\int_0^y p(s,0)(y-s)^{a_j^l-1} e^{-(\lambda+k\mu)(y-s)-yz^\alpha} dsdy \\ &- \sum_{j=0}^{+\infty} k\mu \frac{A_j^{l+1}}{(a_j^{l+1}-1)!} \int_0^{+\infty}\int_0^y p(s,0)(y-s)^{a_j^{l+1}-1} e^{-(\lambda+k\mu)(y-s)-yz^\alpha} dsdy. \end{aligned}$$

By using Equation (3.5.11), we obtain

$$\begin{aligned} \pi_\alpha(z,l) &= \sum_{j=0}^{+\infty} A_j^l \frac{z^{\alpha-1}}{(\lambda+k\mu+z^\alpha)^{a_j^l}} \\ &+ \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} B_{j,m,r}^l \frac{z^{\alpha-1}}{(\lambda+k\mu+z^\alpha)^{b_{j,m,r}^l}} \\ &- \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty}\sum_{r=0}^{+\infty} C_{j,m,r}^l \frac{z^{\alpha-1}}{(\lambda+k\mu+z^\alpha)^{c_{j,m,r}^l}}. \end{aligned}$$

Finally, taking the inverse Laplace transform and using formula (3.2.2) we conclude the proof. □

Let us observe that for $k = 1$ we obtain another representation of the state probabilities of a $M_\alpha/M_\alpha/1$ queue.

Now let us work with some performance parameters. First of all, let us determine the distribution of busy period (see [**19**, Theorem 6.3])

THEOREM 3.5.5. *Let B_α be the duration of a busy period of a $M_\alpha/E_{k,\alpha}/1$ queueing system. Then it holds*

$$\mathbb{P}(B_\alpha \le t) = \sum_{r=0}^{+\infty} k\mu C^0_{k,r} t^{\alpha\delta^0_{k,r}} E^{\delta^0_{k,r}}_{\alpha,\alpha\delta^0_{k,r}+1}(-(\lambda+k\mu)t^\alpha),$$

where $C^0_{k,r}$ and $\delta^0_{k,r}$ are defined in (3.5.9).

PROOF. Let $N^l(t)$ be the queue length process of a $M/E_k/1$ queueing system and $N^l_\alpha(t) = N^l(L_\alpha(t))$ the respective queue length process of a $M_\alpha/E_{k,\alpha}/1$ queueing system. Let us modify both processes by setting 0 as an absorbing state and let us denote the modified processes as $\bar{N}^l(t)$ and $\bar{N}^l_\alpha(t)$. First of all, it still holds $\bar{N}^l(L_\alpha(t)) = \bar{N}^l_\alpha(t)$. Moreover, by definition, denoting by B the busy period of a $M/E_k/1$ queuing system, we have

$$\mathbb{P}(B \le t) = \mathbb{P}_1(\bar{N}^l(t) = 0) \qquad\qquad \mathbb{P}(B_\alpha \le t) = \mathbb{P}_1(\bar{N}^l_\alpha(t) = 0)$$

thus it holds

$$\begin{aligned}\mathbb{P}(B_\alpha \le t) &= \mathbb{P}_1(\bar{N}^l_\alpha(t) = 0)\\ &= \int_0^{+\infty} \mathbb{P}_1(\bar{N}^l(s) = 0) f_\alpha(s;t)ds = \int_0^{+\infty} \mathbb{P}(B \le s) f_\alpha(s;t)ds.\end{aligned}$$

Let us recall that in [**99**] it has been shown that

$$\begin{aligned}\mathbb{P}(B \le t) &= \sum_{r=0}^{+\infty} \frac{k\lambda^r(k\mu)^{k(r+1)}}{r!(rk+k)!} \int_0^t s^{k+r(k+1)-1} e^{-(\lambda+k\mu)s} ds\\ &= \sum_{r=0}^{+\infty} \frac{k\mu C^0_{k,r}}{(k+r(k+1)-1)!} \int_0^t s^{k+r(k+1)-1} e^{-(\lambda+k\mu)s} ds\end{aligned}$$

thus, by monotone convergence theorem, we have

$$\mathbb{P}(B_\alpha \le t) = \sum_{r=0}^{+\infty} \frac{k\mu C^0_{k,r}}{(k+r(k+1)-1)!} \int_0^{+\infty} \int_0^y s^{k+r(k+1)-1} e^{-(\lambda+k\mu)s} f_\alpha(y;t)ds.$$

Let us take the Laplace transform to obtain

$$\begin{aligned}&\mathcal{L}_{t\to z}[\mathbb{P}(B_\alpha \le t)](z)\\ &\quad= \sum_{r=0}^{+\infty} \frac{k\mu C^0_{k,r}}{(k+r(k+1)-1)!} z^{\alpha-1} \int_0^{+\infty} \int_0^y s^{k+r(k+1)-1} e^{-(\lambda+k\mu)s} e^{-yz^\alpha} dsdy\\ &\quad= \sum_{r=0}^{+\infty} \frac{k\mu C^0_{k,r}}{(k+r(k+1)-1)!} z^{\alpha-1} \int_0^{+\infty} s^{k+r(k+1)-1} e^{-(\lambda+k\mu)s} \int_s^{+\infty} e^{-yz^\alpha} dyds\\ &\quad= \sum_{r=0}^{+\infty} \frac{k\mu C^0_{k,r}}{(k+r(k+1)-1)!} z^{-1} \int_0^{+\infty} s^{k+r(k+1)-1} e^{-(\lambda+k\mu+z^\alpha)s} ds\\ &\quad= \sum_{r=0}^{+\infty} k\mu C^0_{k,r} \frac{z^{-1}}{(\lambda+k\mu+z^\alpha)^{k+r(k+1)}}.\end{aligned}$$

Taking the inverse Laplace transform and using Equation (3.2.2) we conclude the proof. □

Finally, we can argue concerning the virtual waiting times. As for the $M_\alpha/M_\alpha/1$ queue with acceleration of service, here we have to handle in some way the semi-Markov property of the process. To do this, let us set E^p_n as the time instants in which a phase is completed (i.e. the time instants in which the process $N^l(t)$ jumps backward). Let $E^p(t) = \max\{E^p_n : \ E^p_n \le t\}$ be the last phase-time before t setting $E^p_0 = 0$. Thus we have now the following result (see [**19**, Section 6.3]).

Proposition 3.5.6. *Define the function*

$$F_W(s;t,t_0,n) = \mathbb{P}(W(t) \le s | E^p(t) = t_0, \ N^l(t-) = n+1)$$

for $s, t, t_0 \ge 0$ with $t_0 \le t$ and $n \in \mathbb{N}_0$, where $W(t)$ is the virtual waiting time for a $M_\alpha/E_{k,\alpha}/1$ queue. Then

- *If $n = 0$ we have*

$$F_W(s;t,t_0,0) = 1 - \sum_{n=0}^{k-1} \frac{(k\mu t^\alpha)^n}{n!} E^{(n)}_\alpha(-k\mu t^\alpha);$$

- *If $n \ge 1$, then the Laplace-Stieltjes transform of F_W with respect to s is given by*

$$\mathcal{L}^S[F_W(\cdot;t,t_0,n)](z) = \frac{(k\mu)^n}{(k\mu + z^\alpha)^n} \left[1 - \frac{e^{z\Delta t} \sum_{j=0}^{+\infty} \frac{(-k\mu)^j}{\Gamma(j\alpha+1)} z^{-\alpha j} \Gamma(\alpha j + 1, z\Delta t)}{E_\alpha(-k\mu(\Delta t)^\alpha)}\right],$$

where $\Delta t = t - t_0$.

Proof. Let us first consider the case in which $n = 0$. Then the virtual waiting time $W(t)$ (conditioned with $N(t-) = 1$, which is the only necessary conditioning in this case) coincides with the service time, that is a $Erl_{k,\alpha}(\mu)$-distributed random variable.
Let us now work with $n \ge 1$. Under our conditioning we can split $W(t)$ as a sum of $n+1$ random variables:

$$W(t) = \sum_{j=1}^{n} W_j + W_{n+1}(t) \tag{3.5.12}$$

where each W_j represents the time our user has to wait due to the completion of one of the $n+1$ remaining phases. In particular, let us count such phases backward, in the sense that $W_{n+1}(t)$ is the first phase to be completed, W_n the second and so on. Thus we have that W_j for $1 \le j \le n$ are all $ML_\alpha(k\mu)$-distributed independent random variables. Let us split again $W(t)$ as

$$W(t) = \widetilde{W} + W_{n+1}(t)$$

then $\widetilde{W} = \sum_{j=1}^{n} W_j$ is a $Erl_{n,\alpha}\left(\frac{k}{n}\mu\right)$-distributed random variable. Now let us consider W_{n+1}. Then we can write $W_{n+1}(t) = S_{n+1} - \Delta t$, where S_{n+1} is a phase-service time. In particular we know that in general $S_{n+1} \sim ML_\alpha(k\mu)$. However, our conditioning implies $S_{n+1} \ge \Delta t$, thus Lemma 3.3.5 implies that $S_{n+1} - \Delta t$ is a $RLM_\alpha(k\mu, t_0)$-distributed random variable. Thus we conclude that $W_{n+1}(t)$ is a $RLM_\alpha(k\mu, t_0)$-distributed random variable independent of $\widetilde{W}$ and then the Laplace transform of $W(t)$ (i.e. the Laplace-Stieltjes transform of F_W with respect to s)

is the product of the Laplace transforms of $\widetilde{W}$ and $W_{n+1}(t)$. This concludes the proof. □

If, in the previous result, we consider the case $k = 1$, we obtain the same result on the virtual waiting time for the $M_\alpha/M_\alpha/1$ queue.
Let us finally stress out that such techniques can be applied also to other more complicated models, as for instance epidemics models, as done in [**14**].

3.6. A Semi-Markov Leaky Integrate-and-Fire model

Let us now consider an example of model with continuous state space that relies on the time-change of some Markov process. To do this we first need to introduce some classical models and then we will discuss our semi-Markov model.

3.6.1. The Leaky Integrate-and-Fire models. The model we are going to describe concerns the behaviour of a single neuronal cell. Obviously, different models could be coupled to obtain the behaviour of a *local circuit* of neurons (see [**135**]), but we will focus on a single one. To understand such models we need to introduce a bit of terminology. First of all, one can imagine the membrane of a neuron as a little circuit, subject to the presence of ions in the ambient. In particular, we will describe the membrane in terms of its **potential difference** (or just **potential**) $V(t)$. If the neuron is not stimulated, it reaches a *fixed* potential value called **resting potential** V_L. If it is stimulated by a current $I(t)$ (that can be a net current due to other neurons or an *in vitro* stimulus), then $V(t)$ varies. If $V(t)$ exceeds a certain value V_{th}, it **depolarizes** and sends a signal, called **action potential**. The act of sending a signal is generally called a **spike**. Let us just stress out that, since this is a text on mathematics and not on neurophysiology, this is just a *long story short*: this is just a(n) (over-)simplified description of the (actually more complex) mechanism of synaptic transmission. For more details, check [**135**].
One of the main single-neuron models has been introduced by Lapicque in 1907 (see [**3**]). Despite the fact that the synaptic behaviour is *evidently non-linear* (see [**135**]), Lapicque's model, called the Leaky Integrate-and-Fire model, is a quite useful linear approximation of such behaviour. Following the lines of [**51**] for the passive membrane model, let us denote by R_m the membrane resistance and with C_m the membrane capacitance. Let us consider a *small section* of the membrane of area A. Each section can be modelled as an RC-circuit as in Figure 3. Let us observe that we are not distinguishing between the action of any particular pump of the membrane[1]. Suppose the neuron is subject to an external current $I_e(t)$. If $I_e(t) \equiv 0$, then we want the membrane potential $V(t)$ to decay to a *steady state* V_L. Such effect is produced by considering a node of constant potential V_L connected to the membrane. If $I_e(t) \neq 0$, then each section of area A is subject to a current $\frac{I_e(t)}{A}$. The whole membrane can be seen as composed of a family of circuits given as in Figure 3, thus in particular by a family of parallel capacitors and resistors. Thus we can define the specific membrane capacitance as $c_m = \frac{C_m}{A}$ and the specific

[1]To add, for instance, sodium, potassium and calcium pumps in the model one has to consider some voltage-dependent resistances whose behaviour is described in terms of gating variables. This approach leads to the more complex Hodgkin-Huxley model, see [**51**], of which the Leaky Integrate-and-Fire can be seen as a linear simplification.

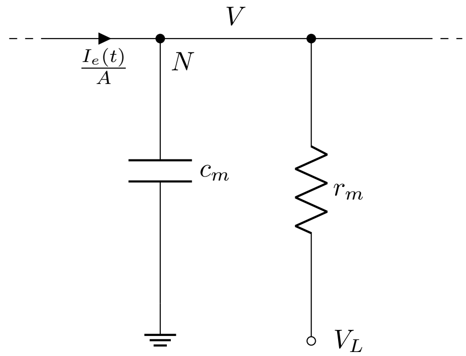

FIGURE 3. Circuit schematization of a section of the neuronal membrane.

membrane resistance as $r_m = AR_m$. By Kirchhoff's Current Law applied to the node N we have

$$c_m \frac{dV}{dt}(t) = -\frac{1}{r_m}(V - V_L) + \frac{I_e(t)}{A}.$$

Now let us observe that $c_m r_m = C_m R_m = \theta$, thus, dividing everything by c_m we have

$$\frac{dV}{dt}(t) = -\frac{1}{\theta}(V - V_L) + \frac{R_m I_e(t)}{\theta}.$$

Finally, defining $I(t) = \frac{R_m}{\theta} I_e(t)$, the membrane potential behaves according to the following differential equation:

$$\frac{dV}{dt}(t) = -\frac{1}{\theta}(V(t) - V_L) + I(t)$$

where V_L is the resting potential and θ is the characteristic time of the membrane, seen as an RC circuit. To represent the action potential, we say that the neuron spikes as $V(t) > V_{th}$ and then the model is reset. For a more detailed study of such a model, we refer to [**144**] and [**51**].

Here we want to focus on the stochastic version of this linear model, that is to say the Stochastic Leaky Integrate-and-Fire (LIF) model. In particular the idea of introducing the noise is due to different reasons:

- The membrane potential is affected by changes in its environment;
- The membrane potential is affected by the physical proximity of other neurons (*ephaptic connections*);
- The membrane potential is affected by the action potential of a large number of other neurons in its local circuit.

All these situations can be easily approximated by the introduction of a Brownian noise in the equation. This leads to the stochastic differential equation

$$dV(t) = \left[-\frac{1}{\theta}(V(t) - V_L) + I(t)\right] dt + \sigma dB(t)$$

where $B(t)$ is a standard Brownian motion and $\sigma > 0$ is the amplitude of the noise: the process $V(t)$ is actually a drifted Ornstein-Uhlenbeck process. Concerning such model, we refer to [**42, 130**]. Although this model is simple, easy to study and works well with different families of neurons, it has been shown to be *too simple* to explain some particular dynamics.

3.6.2. The limits of the LIF model. In [**43**] it has been shown that if the threshold V_{th} is *sufficiently big*, then the behaviour of the inter-spike intervals (ISIs, i.e. the times between two spikes) is approximatively exponential. This approximation is useful when studying some large networks of neurons. However, in [**136**], it has been proved that the model was not consistent for cortical neurons. The solution was found in [**131**], where it has been proved that temporally correlated inputs (and then temporally correlated noise) could lead to a better performance of the model.
However, if we make another jump back in time, we see that the fact that the Ornstein-Uhlenbeck process does not work well for some neurons was already observed in [**124**]. Indeed, the authors notice that not only the exponential distribution does not fit well the behaviour of a particular neuron in the cochlea of the cat, but that the inter-spike intervals seemed to exhibit infinite expectation. This problem was reconsidered in [**69**] where a Cauchy distribution was used to fit the data.
In any case, let us stress out that for the spontaneous activity (i.e. $I(t) \equiv 0$), it is true that in [**43**] one needs the threshold to be big enough to obtain the exponential approximation. However, it has been shown in [**28**] that in any case we obtain finite mean, which is something we want to avoid.

3.6.3. The Semi-Markov LIF model. The idea expressed in [**29**] is the following. Consider the process $V(t)$ defined by means of the classical stochastic LIF model with initial datum V_0 and a driftless Bernstein function $\Phi \in \mathcal{BF}$. Then let us consider the time-changed process $V_\Phi(t) := V(L_\Phi(t))$ where $L_\Phi(t)$ is an inverse-subordinator associated to Φ independent of $V(t)$. What we want to do is to find some hypotheses on Φ in such a way that the process $V_\Phi(t)$ exhibits some quantitative properties that can be useful to describe neurons as the ones found in [**124**].
First of all, let us observe that, without loss of generality, we can set $V_L = 0$. Thus we have

$$V(t) = -\frac{1}{\theta}\int_0^t V(t)dt + \int_0^t I(s)ds + \sigma B(t).$$

On the other hand, we can decompose the process $V(t)$ in two processes:

$$V(t) = \widetilde{V}(t) + J(t)$$

where $\widetilde{V}(t)$ is an Ornstein-Uhlenbeck process with $\widetilde{V}(0) = V_0$ and $J(t) = e^{-\frac{t}{\theta}} \int_0^t I(s)e^{\frac{s}{\theta}}ds$ (which is actually deterministic). Let us in particular stress out that $\widetilde{V}(t)$ represents the *spontaneous activity* part of the process, while $J(t)$ is the integrated stimulus. Thus, taking this in consideration, let us define $\widetilde{V}_\Phi(t) := \widetilde{V}(L_\Phi(t))$ and observe that

$$\mathbb{E}[\widetilde{V}_\Phi(t)] = V_0\, \mathfrak{e}_\Phi\left(t; -\frac{1}{\theta}\right).$$

From now on, let us denote $\lambda = 1/\theta$. Moreover, we directly obtain

$$\mathbb{E}[V_\Phi(t)] = V_0\, \mathfrak{e}_\Phi\left(t; -\lambda\right) + \int_0^{+\infty} J(s) f_\Phi(s;t)ds.$$

Actually, we can show that the mean of V_Φ is solution of a non-local differential equation (see [**29**, Proposition 2]).

PROPOSITION 3.6.1. *Let I be a continuous and bounded function. Then $M_\Phi(t) := \mathbb{E}[V_\Phi(t)]$ is solution of*

$$\begin{cases} \partial^\Phi M_\Phi(t) = -\frac{1}{\theta} M_\Phi(t) + I_\Phi(t), & t > 0 \\ M_\Phi(0) = V_0, \end{cases}$$

where $I_\Phi(t) = \mathbb{E}[I(L_\Phi(t))]$.

PROOF. Let us show that $M_\Phi(t)$ is Laplace transformable in $\mathbb{H}$. To do this, let us observe that being $I(t)$ bounded, we have

$$|J(t)| \le e^{-\frac{t}{\theta}} \int_0^t |I(s)| e^{\frac{s}{\theta}} ds \le \|I\|_{L^\infty(\mathbb{R}^+)}$$

thus also $J(t)$ is bounded. Moreover we have

$$|M_\Phi(t)| \le V_0 \, \mathfrak{e}_\Phi(t; -\lambda) + \int_0^{+\infty} |J(s)| f_\Phi(s;t) ds \le V_0 + \|J\|_{L^\infty(\mathbb{R}^+)} .$$

Thus $M_\Phi(t)$ is bounded and then Laplace transformable in $\mathbb{H}$. Let us denote by $\bar{M}_\Phi(z)$ its Laplace transform. Since it is not difficult to see that, denoting $M(t) = \mathbb{E}[V(t)]$,

$$M_\Phi(t) = \int_0^{+\infty} M(s) f_\Phi(s;t) ds$$

we have that

$$\bar{M}_\Phi(z) = \frac{\Phi(z)}{z} \int_0^{+\infty} M(s) e^{-s\Phi(z)} ds.$$

Now let us recall that $M(t) := \mathbb{E}[V(t)]$ is solution of

$$\begin{cases} \frac{dM}{dt}(t) = -\frac{1}{\theta} M(t) + I(t), & t > 0 \\ M(0) = V_0, \end{cases}$$

thus in particular it is derivable. Hence we have, by integrating by parts,

$$\begin{aligned} \bar{M}_\Phi(z) &= \frac{1}{z} V_0 + \frac{1}{z} \int_0^{+\infty} \frac{dM}{ds}(s) e^{-s\Phi(z)} ds \\ &= \frac{1}{z} V_0 - \frac{1}{z\theta} \int_0^{+\infty} M(s) e^{-s\Phi(z)} ds + \frac{1}{z} \int_0^{+\infty} I(s) e^{-s\Phi(z)} ds. \end{aligned}$$

Now let us multiply everything by $\frac{\Phi(z)}{z}$ to achieve

$$\frac{\Phi(z)}{z} \left(\bar{M}_\Phi(z) - \frac{1}{z} V_0 \right) = -\frac{\Phi(z)}{z^2\theta} \int_0^{+\infty} M(s) e^{-s\Phi(z)} ds + \frac{\Phi(z)}{z^2} \int_0^{+\infty} I(s) e^{-s\Phi(z)} ds. \tag{3.6.1}$$

Let us consider $I_\Phi(t)$ and let us show that it is Laplace transformable. Indeed, it holds

$$|I_\Phi(t)| \le \int_0^{+\infty} |I(s)| f(s;t) ds \le \|I\|_{L^\infty(\mathbb{R}^+)} .$$

Moreover, its Laplace transform is given by

$$\mathcal{L}[I_\Phi](z) = \frac{\Phi(z)}{z} \int_0^{+\infty} I(s) e^{-s\Phi(z)} ds.$$

Thus, taking the inverse Laplace transform in Equation (3.6.1) (recalling that $\frac{\Phi(z)}{z} = \mathcal{L}[\bar{\nu}_\Phi]$), we get

$$\int_0^t \bar{\nu}_\Phi(t-\tau)(M_\Phi(\tau)-V_0)d\tau = -\frac{1}{\theta}\int_0^t M_\Phi(s)ds + \int_0^t I(s)ds.$$

Moreover, observing that $M_\Phi(0) = V_0$ by definition and that the right-hand side is differentiable with continuous derivative, we conclude the proof. □

From last Proposition we deduce that our model is a good candidate to represent a stochastic version of non-local LIF models. In particular for $\Phi(\lambda) = \lambda^\alpha$ as $\alpha \in (0,1)$, $M_\Phi(t)$ solves the equation of a fractional-order LIF model, as described in [**142**].

3.6.4. Correlation structure of the Semi-Markov LIF model. Now let us discuss on the covariance of $V_\Phi(t)$. To study this, we first need to understand what is the covariance of $V(t)$. Actually, since $J(t)$ is a deterministic term, it does not play any role in such covariance, hence $V(t)$ admits the same auto-covariance function of $\widetilde{V}(t)$ and then it is well known that

$$\mathrm{Cov}(V(t),V(s)) = \frac{\sigma^2\theta}{2}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right).$$

Now let us consider the auto-covariance function of $V_\Phi(t)$. Let us define the measure

$$F_\Phi^{(2)}(t,s,A) = \mathbb{P}((L_\Phi(t),L_\Phi(s)) \in A),\ \forall A \in \mathcal{B}(\mathbb{R}^2)$$

and let us observe that

$$\begin{aligned}\mathrm{Cov}(V_\Phi(t),V_\Phi(s)) &= \int_{(\mathbb{R}^+)^2} \mathrm{Cov}(V(u),V(v)) F_\Phi^{(2)}(t,s,dudv)\\ &= \frac{\sigma^2\theta}{2}\left(\int_{(\mathbb{R}^+)^2} e^{-\lambda|u-v|}F_\Phi^{(2)}(t,s,dudv)\right.\\ &\qquad \left. - \int_{(\mathbb{R}^+)^2} e^{-\lambda(u+v)}F_\Phi^{(2)}(t,s,dudv)\right).\end{aligned}$$

The first integral has been already determined in Lemma 2.4.4, thus let us now focus on the second integral, which is actually the bi-variate Laplace transform of $(L_\Phi(t),L_\Phi(s))$.

LEMMA 3.6.2. *For any $\lambda > 0$ and $t \ge s > 0$ it holds*

$$\int_{(\mathbb{R}^+)^2} e^{-\lambda(u+v)}F_\Phi^{(2)}(t,s,dudv) = \mathfrak{e}_\Phi(t;-\lambda) + \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda).$$

PROOF. Let us denote by $G(u,v) = e^{-\lambda(u+v)}$, that is a C^∞ function, and fix $(a,b) \in (\mathbb{R}^+)^2$. By using the bivariate integration by parts formula (see [**70**]) we have

$$\begin{aligned}\int_0^a\int_0^b G(u,v)F_\Phi^{(2)}(t,s,dudv)=&\int_0^a\int_0^b F_\Phi^{(2)}(t,s,[u,a]\times[v,b])G(du,dv)\\&+\int_0^a F_\Phi^{(2)}(t,s,[u,a]\times[0,b])G(du,0)\\&+\int_0^b F_\Phi^{(2)}(t,s,[0,a]\times[v,b])G(0,dv)\\&+F_\Phi^{(2)}(t,s,[0,a]\times[0,b])G(0,0).\end{aligned}$$

Now let us observe that

$$\begin{aligned}G(du,v)&=-\lambda e^{-\lambda(u+v)}du,\\G(u,dv)&=-\lambda e^{-\lambda(u+v)}dv,\\G(du,dv)&=\lambda^2e^{-\lambda(u+v)}dudv,\end{aligned}$$

thus we have

(3.6.2)

$$\begin{aligned}\int_0^a\int_0^b G(u,v)F_\Phi^{(2)}(t,s,dudv)=&\int_0^a\int_0^b \lambda^2F_\Phi^{(2)}(t,s,[u,a]\times[v,b])e^{-\lambda(u+v)}dudv\\&-\int_0^a \lambda F_\Phi^{(2)}(t,s,[u,a]\times[0,b])e^{-\lambda u}du\\&-\int_0^b \lambda F_\Phi^{(2)}(t,s,[0,a]\times[v,b])e^{-\lambda v}dv\\&+F_\Phi^{(2)}(t,s,[0,a]\times[0,b]).\end{aligned}$$

Let us define

$$\begin{aligned}I_1(a,b)&=\int_0^a \lambda F_\Phi^{(2)}(t,s,[u,a]\times[0,b])e^{-\lambda u}du\\I_2(a,b)&=\int_0^b \lambda F_\Phi^{(2)}(t,s,[0,a]\times[v,b])e^{-\lambda v}dv.\end{aligned}$$

First of all, by monotone convergence theorem, we have

$$\begin{aligned}\lim_{a,b\to+\infty}I_1(a,b)&=\int_0^{+\infty}\lambda F_\Phi^{(2)}(t,s,[u,+\infty]\times[0,+\infty])e^{-\lambda u}du\\&=\int_0^{+\infty}\mathbb{P}(L_\Phi(t)\ge u)\lambda e^{-\lambda u}du\\&=1-\int_0^{+\infty}e^{-\lambda u}f_\Phi(u;t)du=1-\mathfrak{e}_\Phi(t;-\lambda).\end{aligned}$$

Analogously we have

$$\lim_{a,b\to+\infty}I_2(a,b)=1-\mathfrak{e}_\Phi(s;-\lambda).$$

Thus, taking the limit as $a, b \to +\infty$ in (3.6.2) we obtain

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} G(u,v)F^{(2)}_\Phi(t,s,dudv)\\ &= \int_0^{+\infty}\int_0^{+\infty} \lambda^2 F^{(2)}_\Phi(t,s,[u,+\infty)\times[v,+\infty))e^{-\lambda(u+v)}dudv\\ &\quad + \mathfrak{e}_\Phi(t;-\lambda)+\mathfrak{e}_\Phi(s;-\lambda)-1.\end{aligned} \tag{3.6.3}$$

Let us denote

$$\begin{aligned}I_4 &= \int_0^{+\infty}\int_0^{+\infty} \lambda^2 F^{(2)}_\Phi(t,s,[u,+\infty)\times[v,+\infty))e^{-\lambda(u+v)}dudv\\ &= \iint_{u<v} \lambda^2 F^{(2)}_\Phi(t,s,[u,+\infty)\times[v,+\infty))e^{-\lambda(u+v)}dudv\\ &+ \iint_{u>v} \lambda^2 F^{(2)}_\Phi(t,s,[u,+\infty)\times[v,+\infty))e^{-\lambda(u+v)}dudv\\ &:= I_5 + I_6.\end{aligned} \tag{3.6.4}$$

Let us first work with I_5. Since $t \ge s$ and $u < v$, we have

$$\begin{aligned}F^{(2)}_\Phi(t,s,[u,+\infty)\times[v,+\infty)) &= \mathbb{P}(L_\Phi(t)\ge u,\ L_\Phi(s)\ge v)\\ &= \mathbb{P}(L_\Phi(s)\ge v)\end{aligned}$$

and then

$$\begin{aligned}I_5 &= \iint_{u<v} \lambda^2\,\mathbb{P}(L_\Phi(s)\ge v)e^{-\lambda(u+v)}dudv\\ &= \int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)(-\lambda)e^{-\lambda v}\left(\int_0^v (-\lambda)e^{-\lambda u}du\right)dv\\ &= \int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)(-\lambda)e^{-2\lambda v}dv - \int_0^{+\infty} \mathbb{P}(L_\Phi(s)\ge v)(-\lambda)e^{-\lambda v}dv\\ &= -\frac{1}{2}+\frac{1}{2}\int_0^{+\infty} e^{-2\lambda v}f_\Phi(v;t)dv + 1 - \int_0^{+\infty} e^{-\lambda v}f_\Phi(v;t)dv\\ &= \frac{1}{2}+\frac{1}{2}\mathfrak{e}_\Phi(s;-2\lambda)-\mathfrak{e}_\Phi(s;-\lambda).\end{aligned} \tag{3.6.5}$$

Now let us consider I_6. By Equation (2.4.10), setting $A(t,s) = \{(x,y)\in\mathbb{R}^2 :\ y \in [0,s],\ x \in [0,t-y]\}$, we have

$$\begin{aligned}I_6 &= \iint_{u>v}\left(\iint_{A(t,s)} g_\Phi(dx;u-v)g_\Phi(dy;v)\right)\lambda^2 e^{-\lambda(u+v)}dudv\\ &= \iint_{A(t,s)}\iint_{u>v}\left(\iint_{u>v} g_\Phi(dx;u-v)g_\Phi(dy;v)\lambda^2 e^{-\lambda(u+v)}dudv\right)\\ &= \iint_{A(t,s)}\int_0^{+\infty} g_\Phi(dy;v)(-\lambda)e^{-\lambda v}\left(\int_v^{+\infty} g_\Phi(dx;u-v)(-\lambda)e^{-\lambda u}du\right)dv.\end{aligned}$$

Consider $u - v = w$ to obtain

$$I_6 = \lambda^2\iint_{A(t,s)}\left(\int_0^{+\infty} g_\Phi(dy;v)e^{-2\lambda v}dv\right)\left(\int_0^{+\infty} g_\Phi(dx;w)e^{-\lambda w}dw\right).$$

Let us consider $\mathbb{P}(\sigma(w) \le x)$. We have

$$\mathbb{P}(\sigma(w) \le x) = \mathbb{P}(w \le L(x)) = 1 - \mathbb{P}(L(x) \le w).$$

Thus, taking the Laplace transform with respect to w we have

$$\mathcal{L}_{w\to\lambda}[\mathbb{P}(\sigma(w) \le x)](\lambda) = \mathcal{L}_{w\to\lambda}[1 - \mathbb{P}(L(x) \le w)](\lambda) = \frac{1}{\lambda} - \frac{\mathfrak{e}_\Phi(x;-\lambda)}{\lambda}.$$

Thus, we get

$$\int_0^{+\infty} g_\Phi(dx;w)e^{-\lambda w}dw = -\frac{\mathfrak{e}_\Phi(dx;-\lambda)}{\lambda}$$

where $\mathfrak{e}_\Phi(dx;-\lambda)$ is well defined as a Radon measure since $\mathfrak{e}_\Phi(x;-\lambda)$ is of bounded variation (being monotone and bounded). We achieve

$$\begin{aligned} I_6 &= \frac{1}{2}\iint_{A(t,s)} \mathfrak{e}_\Phi(dx;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda) \\ &= \frac{1}{2}\int_0^s \left(\int_0^{t-y} \mathfrak{e}_\Phi(dx;-\lambda)\right)\mathfrak{e}_\Phi(dy;-2\lambda) \\ &= \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda) - \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(dy;-2\lambda) \\ &= \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda) - \frac{1}{2}\mathfrak{e}_\Phi(s;-2\lambda) + \frac{1}{2}. \end{aligned} \tag{3.6.6}$$

Substituting Equation (3.6.5) and (3.6.6) in (3.6.4) we have

$$I_4 = 1 - \mathfrak{e}_\Phi(s;-\lambda) + \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda) \tag{3.6.7}$$

and then substituting (3.6.7) in (3.6.3) we conclude the proof. □

By using Lemma 3.6.2 and 2.4.4 we get the following Proposition.

PROPOSITION 3.6.3. *For any $t \ge s > 0$ it holds*

$$\begin{aligned} \mathrm{Cov}(V_\Phi(t), V_\Phi(s)) = \frac{\sigma^2\theta}{2}\Bigg(&\lambda\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)dU_\Phi(y) - 2 \\ &+2\,\mathfrak{e}_\Phi(s;-\lambda) - \frac{1}{2}\int_0^s \mathfrak{e}_\Phi(t-y;-\lambda)\,\mathfrak{e}_\Phi(dy;-2\lambda)\Bigg). \end{aligned}$$

Let us observe that the formula that gives the auto-covariance function is quite complicated, thus we cannot use it directly to obtain some information on the behaviour of the covariance as $t-s$ increases. However, we can still argue on some properties by using the integral representation in terms of the auto-covariance of $V(t)$ (see [**29**, Proposition 4]).

PROPOSITION 3.6.4. *Fix $t > 0$ and define the function*

$$c_\Phi(s;t) := \mathrm{Cov}(V_\Phi(t+s), V_\Phi(t)), \qquad s \ge 0$$

Then $c_\Phi(s;t)$ is decreasing and infinitesimal.

PROOF. Let us first show that $c_\Phi(s;t)$ is decreasing. To do this fix $s_2 > s_1 \ge 0$ and consider the measure

$$F_\Phi^{(3)}(t+s_2, t+s_1, t, A) = \mathbb{P}((L_\Phi(t+s_2), L_\Phi(t+s_1), L_\Phi(t)) \in A), \qquad A \in \mathcal{B}(\mathbb{R}^3).$$

Since $L_\Phi(t)$ is almost surely increasing, the considered measure is concentrated on

$$A = \{(u,v,w) \in \mathbb{R}^3 : \ u \in (0,+\infty), v \in (0,u), \ w \in (0,v)\}. \tag{3.6.8}$$

Let us observe that $s > 0 \mapsto \mathrm{Cov}(V(t+s), V(t))$ is decreasing. Thus, denoting $c(t,s) = \mathrm{Cov}(V(t), V(s))$, we have

$$\begin{aligned} &c_\Phi(s_2;t) - c_\Phi(s_1;t) \\ &\qquad = \int_0^{+\infty}\int_0^u\int_0^v (c(v,w)-c(u,w))F^{(3)}(t+s_2,t+s_1,t,dudvdw) \geq 0, \end{aligned}$$

since $v \leq u$.

Now let us show that $c_\Phi(s;t)$ is infinitesimal. Let us recall that $\lim_{t\to+\infty} c(t,s) = 0$. Moreover there exists a constant $C > 0$ such that $|c(t,s)| \leq C$ for any (t,s). In particular $|c(L_\Phi(t+s), L_\Phi(t))| \leq C$ and then, by dominated convergence theorem (as $\lim_{s\to+\infty} L_\Phi(t+s) = +\infty$) we have

$$\lim_{s\to+\infty} c_\Phi(s;t) = \lim_{s\to+\infty} \mathbb{E}[c(L_\Phi(t+s), L_\Phi(t))] = 0.$$

□

Concerning the variance, we can prove an easier formula.

Proposition 3.6.5. *It holds*

$$\mathrm{Var}[V_\Phi(t)] = \frac{\sigma^2\theta}{2}(1-\mathfrak{e}_\Phi(t;-2\lambda))$$

Proof. Let us recall that $\mathrm{Var}[V(t)] = \frac{\sigma^2\theta}{2}(1-e^{-2\lambda t})$. Thus we have

$$\mathrm{Var}[V_\Phi(t)] = \frac{\sigma^2\theta}{2}\int_0^{+\infty}(1-e^{-2\lambda s})f_\Phi(s;t)ds = \frac{\sigma^2\theta}{2}(1-\mathfrak{e}_\Phi(t;-2\lambda)).$$

□

It is interesting to observe that for big values of $t > 0$ the variance does not go to 0. Indeed it holds $\lim_{t\to+\infty}\mathrm{Var}[V_\Phi(t)] = \frac{\sigma^2\theta}{2}$.

3.6.5. Properties of Inter-Spike Intervals. Now let us focus on the first spiking time and on Inter-Spike Intervals. Let us consider a fixed excitatory input stimuli $I(t) \equiv I_0$ where $I_0 \geq 0$. Moreover, let us suppose, for simplicity, that $V_0 = 0$ and that the process $V(t)$ is reset to 0 after a spike. Then we have that $V(t)$ is a (non-centred) Gauss-Markov process. Moreover, by construction, denoting by T_n the n-th spiking time and with $K_n = T_n - T_{n-1}$ (with $T_{n-1} = 0$) the n-th inter-spike intervals, the family $(K_n)_{n\in\mathbb{N}}$ is constituted of i.i.d. random variables. Let us observe that, by using the comparison results of [**28**], $\mathbb{E}[T_1] < +\infty$ and $\mathbb{P}(T_1 \leq t)$ is rapidly decreasing at 0. $V(t)$ is a diffusion process, hence the survival function of T_1 is solution of a parabolic problem. It is not difficult to see, by typical arguments concerning partial differential equations, that $\mathbb{P}_0(T_1 > t)$ is actually a C^∞ function. With this in mind, one can easily use the results of Section 2.7 to state the following proposition (see [**29**, Proposition 5]).

Proposition 3.6.6. *Denote by $\mathfrak{T}_1$ the first spiking time of $V_\Phi(t)$. Then the following properties hold true:*

- *If Φ is regularly varying at 0^+ with order $\alpha \in [0,1)$, then, as $t \to +\infty$,*
$$\mathbb{P}(\mathfrak{T}_1 > t) \sim \frac{\mathbb{E}[T_1]}{\Gamma(1-\alpha)}\Phi\left(\frac{1}{t}\right);$$
- *If $\bar{\nu}_\Phi(t)$ is absolutely continuous and Φ satisfies Orey's condition, then $\mathfrak{T}_1$ is absolutely continuous with C^∞ density. Moreover, if Φ is regularly varying at $+\infty$, then, for any $\gamma \in \mathbb{R}$, it holds*
$$\lim_{t \to 0^+} \frac{\mathbb{P}(\mathfrak{T}_1 \le t)}{t^\gamma} = 0$$

Now let us observe that, by Lemma 2.7.1, it holds $\mathfrak{T}_n \overset{d}{=} \sigma_\Phi(T_n)$, where $\mathfrak{T}_n$ is the n-th spiking time of $V_\Phi(t)$. Moreover, let us denote by $\mathfrak{K}_n = \mathfrak{T}_n - \mathfrak{T}_{n-1}$ (with $\mathfrak{T}_0 = 0$) the inter-spike intervals of $V_\Phi(t)$. We can prove the following Proposition (see [**29**, Section 5.2]).

Proposition 3.6.7. *The family $(\mathfrak{K}_n)_{n \in \mathbb{N}}$ is constituted of i.i.d. random variables.*

Proof. First let us show that $\mathfrak{K}_n \overset{d}{=} \sigma_\Phi(K_n)$. This will imply that $(\mathfrak{K}_n)_{n\in\mathbb{N}}$ are identically distributed. To do this, let us first observe that $\mathfrak{K}_n \overset{d}{=} \sigma_\Phi(T_n) - \sigma_\Phi(T_{n-1})$. Now let us consider the measure $\mu^{(2)}(A) = \mathbb{P}((T_n, T_{n-1}) \in A)$ for any $A \in \mathcal{B}(\mathbb{R}^2)$. Then $\mu^{(2)}$ is concentrated on
$$A = \{(u,v) \in \mathbb{R}^2 : \ u \in (0,+\infty), \ v \in (0,u)\}$$
being the sequence $(T_n)_{n\in\mathbb{N}}$ strictly increasing by definition. Recalling that $\sigma_\Phi(t)$ is independent of T_n for any $n \in \mathbb{N}$ and that it is a Lévy process, it holds
$$\begin{aligned}\mathbb{P}(\mathfrak{K}_n \le t) &= \mathbb{P}(\sigma_\Phi(T_n) - \sigma_\Phi(T_{n-1}) \le t) \\ &= \int_0^{+\infty}\int_0^u \mathbb{P}(\sigma_\Phi(u) - \sigma_\Phi(v) \le t)\mu^{(2)}(dudv) \\ &= \int_0^{+\infty}\int_0^u \mathbb{P}(\sigma_\Phi(u-v) \le t)\mu^{(2)}(dudv) \\ &= \mathbb{P}(\sigma_\Phi(T_n - T_{n-1}) \le t) = \mathbb{P}(\sigma_\Phi(K_n) \le t),\end{aligned}$$
concluding the first part of the proof.
Now let us show independence. To do this, let us consider $n, m \in \mathbb{N}$ with $m < n$ and let us introduce the measure
$$\mu^{(4)}(A) = \mathbb{P}((T_n, T_{n-1}, T_m, T_{m-1}) \in A), \ A \in \mathcal{B}(\mathbb{R}^4).$$
Since $m < n$, then $m \le n-1$ and we have that the measure is concentrated on
$$A = \{(u,v,w,z) \in \mathbb{R}^4 : \ u \in (0,+\infty), \ v \in (0,u), \ w \in (0,v), \ z \in (0,w)\}.$$
Let us also denote $s(u,t) = \mathbb{P}(\sigma_\Phi(u) \le t)$ and $\eta_n^{(1)}$ the law of K_n. Finally, let us denote by $\eta_{n,m}^{(2)}$ the joint law of K_n and K_m. Since they are independent we have

that $\eta^{(2)}_{n,m} = \eta^{(1)}_n \times \eta^{(1)}_m$. Arguing as before we have, for fixed $t_1, t_2 > 0$,

$$\begin{aligned}
&\mathbb{P}(\mathfrak{K}_n \le t_1, \mathfrak{K}_m \le t_2) = \mathbb{P}(\sigma_\Phi(T_n) - \sigma_\Phi(T_{n-1}) \le t_1, \sigma_\Phi(T_m) - \sigma_\Phi(T_{m-1}) \le t_2) \\
&= \int_0^{+\infty} \int_0^u \int_0^v \int_0^w \mathbb{P}(\sigma_\Phi(u) - \sigma_\Phi(v) \le t_1, \sigma_\Phi(w) - \sigma_\Phi(z) \le t_2) \mu^{(4)}(dudvdwdz) \\
&= \int_0^{+\infty} \int_0^u \int_0^v \int_0^w \mathbb{P}(\sigma_\Phi(u) - \sigma_\Phi(v) \le t_1)\, \mathbb{P}(\sigma_\Phi(w) - \sigma_\Phi(z) \le t_2) \mu^{(4)}(dudvdwdz) \\
&= \int_0^{+\infty} \int_0^u \int_0^v \int_0^w \mathbb{P}(\sigma_\Phi(u-v) \le t_1)\, \mathbb{P}(\sigma_\Phi(w-z) \le t_2) \mu^{(4)}(dudvdwdz) \\
&= \mathbb{E}[s(K_n, t_1) s(K_m, t_2)] \\
&= \int_0^{+\infty} \int_0^{+\infty} \mathbb{P}(\sigma_\Phi(u) \le t_1)\, \mathbb{P}(\sigma_\Phi(v) \le t_1) \eta^{(2)}_{n,m}(dudv) \\
&= \left(\int_0^{+\infty} \mathbb{P}(\sigma_\Phi(u) \le t_1) \eta^{(1)}_n(du) \right) \left(\int_0^{+\infty} \mathbb{P}(\sigma_\Phi(v) \le t_1) \eta^{(1)}_m(dv) \right) \\
&= \mathbb{P}(\sigma_\Phi(K_n) \le t_1)\, \mathbb{P}(\sigma_\Phi(K_m) \le t_2) \\
&= \mathbb{P}(\mathfrak{K}_n \le t_1)\, \mathbb{P}(\mathfrak{K}_m \le t_2),
\end{aligned}$$

concluding the proof. □

Thus we have that $(\mathfrak{K}_n)_{n\in\mathbb{N}}$ are i.i.d random variable that are distributed as $\mathfrak{K}_1 = \mathfrak{T}_1$. Thus, in particular, (3.6.6) still holds for any $\mathfrak{K}_n$. This is true also for spontaneous activity, since it is the case $I(t) \equiv 0$. Thus we can ask if our model satisfies the qualitative observations of [**124**].

3.6.6. The Unit $240-1$. Let us re-consider the problem of the models in [**124**]. The authors focus on some particular neurons of the cochlea of the cat. However, two of them, the Unit $259-2$ and the Unit $240-1$ exhibit some behaviour that was initially inexplicable. Directly citing the paper:

> *The histogram of Unit $259-2$ appears to be unimodal and asymmetric [...] while that of Unit $240-1$ is unimodal and asymmetric, but on a quite different time scale that that of Unit $259-2$. [...] The spike trains of Unit $259-2$ and Unit $240-1$ do not appear to be easily characterizable.*

However, in the same paper, it is shown that Unit $259-2$ sill exhibit exponential decay and the authors suppose that the spike train is still generated by a Poisson process but with some lag time. Concerning Unit $240-1$, the situation is completely different. Indeed:

> *when the histogram of Unit $240-1$ is replotted on a semilogarithmic scale, the decay is clearly seen to be non-exponential.*

In this paper and [**69**], it is shown that Unit $240-1$ exhibit some power law decay, reminding of stable distributions and Mittag-Leffler distributions. In particular in [**69**] the proposed distribution is a Cauchy distribution (that is power-like decaying) since it

> *has essentially the same invariance property as that found for the density of interspike intervals of Unit $240-1$.*

Let us observe that if we consider $\Phi(\lambda) = \lambda^\alpha$, our model recovers the power-like decay observed in [**69**]. Moreover, one expects the distribution function to decay

at 0 quite fast (as we do not expect a neuron to fire *almost instantaneously*), which is something the distribution function of the $\mathfrak{K}_n$ does. In particular, the distribution we obtain (that cannot be characterized explicitly, but of which we know some characteristics such as the asymptotic behaviour) is in agreement with the phenomenological evidence concerning the behaviour of the Unit $240-1$ as described in [**124**] (non-exponential decay and heavy tail) and [**69**] successively (actual power-like behaviour and some property that is similar to stability).
Finally, let us observe that we propose a model for which memory can be also found in the covariance of the process. Indeed it has been shown in [**67**] that such kind of processes exhibit a long-range dependence for $\Phi(\lambda)=\lambda^\alpha$. In the case of spontaneous activity, we can consider the first-order stationary version (that is not second-order stationary, neither in wide sense) and express short-range or long-range dependence with respect to the initial datum, obtaining exactly the same characterization as in Corollary 2.4.6.

3.7. Simulation procedures

As we have shown some applications of the theoretical results given in Chapter 2, it can be useful to present some simulation procedures for the involved stochastic processes. Here we focus on such procedures, outlining some suitable algorithms.

3.7.1. Simulation of a subordinator. Let us consider a driftless Bernstein function $\Phi \in \mathcal{BF}$. The first step, for any kind of simulation procedure we want to show here, is to understand how to simulate a general subordinator. A short description of the algorithm has been given in [**28**]. In particular let us argue as follows. Consider the subordinator $\sigma_\Phi(t)$ and observe that it is a Lévy process, hence its increments are independent and stationary. Let us denote by $\widetilde{\sigma}_\Phi$ the simulated process. In particular let us choose some **nodes** $(y_n)_{n\in\mathbb{N}}$ for which we want to simulate $(\sigma_\Phi(y_n))_{n\in\mathbb{N}}$ and let us call the sequence $(\widetilde{\sigma}_\Phi(y_n))_{n\in\mathbb{N}}$ the **skeleton** of our simulated process. To simplify the discussion let us suppose $y_n-y_{n-1}=\mathfrak{y}$ where $\mathfrak{y}$ is a constant that we will call **increment**.
The idea is the following: we can obtain $\sigma_\Phi(y_n)$ recursively from $\sigma_\Phi(y_{n-1})$ by observing that

$$\sigma_\Phi(y_n)=(\sigma_\Phi(y_n)-\sigma_\Phi(y_{n-1}))+\sigma_\Phi(y_{n-1}),$$

thus, if we have $\widetilde{\sigma}_\Phi(y_{n-1})$, to achieve $\widetilde{\sigma}_\Phi(y_n)$ we only need to simulate $\sigma_\Phi(y_n)-\sigma_\Phi(y_{n-1})$, that is independent of $\sigma_\Phi(y_{n-1})=\sigma_\Phi(y_{n-1})-\sigma_\Phi(0)$. Still by using the fact that $\sigma_\Phi(t)$ is a Lévy process, we have that

$$\sigma_\Phi(y_n)-\sigma_\Phi(y_{n-1})\stackrel{d}{=}\sigma_\Phi(\mathfrak{y}).$$

Thus, if we know how to simulate $\sigma_\Phi(\mathfrak{y})$, we can simulate the whole skeleton by using the recursive formula:

$$\begin{cases}\widetilde{\sigma}_\Phi(0)=0\\ \widetilde{\sigma}_\Phi(t_n)=\widetilde{\sigma}_\Phi(\mathfrak{y})+\widetilde{\sigma}_\Phi(y_{n-1}) & n\in\mathbb{N},\end{cases}$$

where $\widetilde{\sigma}_\Phi(\mathfrak{y})$ is a simulated occurrence of $\sigma_\Phi(\mathfrak{y})$. In conclusion, we only need to simulate $\sigma_\Phi(\mathfrak{y})$. Let us first consider a simple particular case and then we will discuss the general case. Suppose $\Phi(\lambda)=\lambda^\alpha$ for $\alpha\in(0,1)$. σ_α is an α-stable subordinator and we can use the self-similarity property to write $\sigma_\alpha(\mathfrak{y})\stackrel{d}{=}\mathfrak{y}^{\frac{1}{\alpha}}\sigma_\alpha(1)$. In [**104**] a whole chapter is dedicated to the simulation of time-changed processes in such case.

In particular, by using the notation given in [**114**], our $\sigma_\alpha(1)$ is $S(\alpha,1,\gamma(\alpha),0;1)$-distributed, where $\gamma(\alpha)=\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{\alpha}}$. To understand how to simulate $\sigma_\alpha(1)$, we first need to exploit how to simulate a variable $S\sim S(\alpha,1,1,0;1)$. The following equality is a generalization of the Box-Muller algorithm (see [**30, 65, 146**]):

$$S \overset{d}{=} \left(1+\tan^2\left(\frac{\alpha\pi}{2}\right)\right)^{\frac{1}{2\alpha}} \frac{\sin\left(\alpha\left(Y_2+\frac{\pi}{2}\right)\right)}{(\cos(Y_2))^{\frac{1}{\alpha}}}\left(\frac{\cos\left((1-\alpha)Y_2-\frac{\alpha\pi}{2}\right)}{Y_1}\right)^{\frac{1-\alpha}{\alpha}}$$

where $Y_1\sim \mathrm{Exp}(1)$, Y_2 is uniform in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and Y_1 and Y_2 are independent.
By using this equality, one can simulate $S_1\sim S(\alpha,1,1,0;1)$ and set

$$\widetilde{\sigma}_\alpha(1)=\gamma(\alpha)S_1.$$

However, we are interested in the general case, that is actually much more complicated. To simulate $\sigma_\Phi(\mathfrak{y})$ in the general case, we need to combine two different algorithm: a numerical inversion of the Laplace transform, as discussed in [**2**], and an algorithm to simulate random variables from their characteristic functions (see [**123**] and references therein). First of all, let us set $\Psi(\xi)=\Phi(-i\xi)$, $\psi(\xi)=e^{-\mathfrak{y}\,\Psi(\xi)}$ for $\xi\in\mathbb{R}$ and $\varphi(\lambda)=e^{-\mathfrak{y}\,\Phi(\lambda)}$ for $\lambda\in\mathbb{H}$. Thus we can use the following algorithm

(1) By numerical integration set $c=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-\mathfrak{y}\,\Re(\Psi(\xi))}d\xi$;
(2) Setting δ small enough, use the approximation

$$\psi''(\xi)\simeq\frac{\varphi(-i\xi[1+\delta])+\varphi(-i\xi[1-\delta])-2\varphi(-i\xi)}{\delta^2\xi^2}$$

to determine numerically $k=\frac{1}{\pi}\int_{-\infty}^{+\infty}|\psi''(\xi)|d\xi$;
(3) Generate three independent random variables U,V_1,V_2 uniform in $(0,1)$;
(4) Set $x=\sqrt{\frac{k}{c}\frac{V_1}{V_2}}$ and determine numerically

$$f(x)\simeq\frac{h}{\pi}+\frac{2h}{\pi}\sum_{k=1}^{+\infty}e^{-\mathfrak{y}\,\Re(\Phi(ikh))}\cos(\Im(\Phi(ikh)))\cos(kht)$$

for h small enough;
(5) If $V_1\ge V_2$, set $\widetilde{\sigma}_\Phi(\mathfrak{y})=x$ if $cU\le f(x)$, otherwise reject x;
(6) If $V_2>V_1$, set $\widetilde{\sigma}_\Phi(\mathfrak{y})=x$ if $kU\le x^2f(x)$, otherwise reject x.

As we can see, just the simulation of a subordinator requires different approximations and then generates a lot of approximation errors (due to the truncation of the series for Laplace transform inversion, numerical evaluation of the integrals in c and k, numerical approximation of the second derivative of the characteristic function and so on). However, if we want to simulate a time-changed process, in general we need the inverse subordinator, which requires another approximated simulation procedure, and thus leads to much more errors.
Here we want to find some cases in which time-changed processes can be simulated without simulating the inverse subordinator. To do this, we first need to investigate how to simulate Φ-exponential distributions.

3.7.2. Simulations of Φ-exponential variables. Here we want to investigate how to simulate a Φ-exponential random variable. To do this, we first need to show the following Lemma.

LEMMA 3.7.1. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function and $\sigma_\Phi(t)$ the associated subordinator. Let T be an exponential random variable of parameter $\lambda > 0$ independent of $\sigma_\Phi(t)$. Then $\sigma_\Phi(T)$ is a Φ-exponential random variable of parameter λ.*

PROOF. Let us evaluate $\mathbb{P}(\sigma_\Phi(T) > t)$. By using the independence of $\sigma_\Phi(t)$ and T we get

$$\begin{aligned}\mathbb{P}(\sigma_\Phi(T) > t) &= \int_0^{+\infty} \mathbb{P}(\sigma_\Phi(y) > t)\lambda e^{-\lambda y} dy \\ &= \int_0^{+\infty} \mathbb{P}(y > L_\Phi(t))\lambda e^{-\lambda y} dy \\ &= \int_0^{+\infty} e^{-\lambda y} f_\Phi(y;t) dy = \mathfrak{e}_\Phi(t;-\lambda),\end{aligned}$$

concluding the proof. □

As before, let us consider first a simple case. If $\Phi(\lambda) = \lambda^\alpha$ for some $\alpha \in (0,1)$, then it is not difficult to show, by a conditioning argument, that

$$\sigma_\alpha(T) \overset{d}{=} T^{\frac{1}{\alpha}} \sigma_\alpha(1).$$

Thus, simulating a Mittag-Leffler distributed random variable is quite easy: just simulate an exponential random variable T and a stable random variable $\sigma_\alpha(1)$ independent from each other and then multiply them[2].
The general case is instead more difficult. First of all, we need to determine the Laplace transform of $\sigma_\Phi(T)$.

LEMMA 3.7.2. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function and $\sigma_\Phi(t)$ the associated subordinator. Let T be an exponential random variable of parameter $\lambda > 0$ independent of $\sigma_\Phi(t)$. Then*

$$\mathbb{E}[e^{-z\sigma_\Phi(T)}] = \frac{\lambda}{\Phi(z)+\lambda}.$$

PROOF. We have

$$\begin{aligned}\mathbb{E}[e^{-z\sigma_\Phi(T)}] &= \lambda \int_0^{+\infty} \mathbb{E}[e^{-z\sigma_\Phi(y)}] e^{-\lambda y} dy \\ &= \lambda \int_0^{+\infty} e^{-y(\Phi(z)+\lambda)} dy \\ &= \frac{\lambda}{\Phi(z)+\lambda},\end{aligned}$$

concluding the proof. □

Now let us denote $\varphi(z) = \frac{\lambda}{\Phi(z)+\lambda}$ for $z \in \mathbb{H}$ and $\psi(\xi) = \varphi(-i\xi)$ for $\xi \in \mathbb{R}$. Arguing as before, we have the following algorithm to simulate a random variable $\mathfrak{T} \sim \mathrm{Exp}_\Phi(\lambda)$:

(1) By numerical integration set $c = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\psi(\xi)| d\xi$;

[2] There are different representation of Mittag-Leffler random variables in terms of stable variables and exponential variables. For more details, we refer to [**90**]. Moreover, for application of such representations to the simulation of Continuous Time Random Walks, we refer to [**65**].

(2) Setting δ small enough, use the approximation
$$\psi''(\xi) \simeq \frac{\varphi(-i\xi[1+\delta]) + \varphi(-i\xi[1-\delta]) - 2\varphi(-i\xi)}{\delta^2\xi^2}$$
to determine numerically $k = \frac{1}{\pi}\int_{-\infty}^{+\infty}|\psi''(\xi)|d\xi$;
(3) Generate three independent random variables U, V_1, V_2 uniform in $(0,1)$;
(4) Set $x = \sqrt{\frac{k}{c}\frac{V_1}{V_2}}$ and determine numerically
$$f(x) \simeq \frac{h}{\pi} + \frac{2h}{\pi}\sum_{k=1}^{+\infty}\Re(\varphi(ikh))\cos(kht)$$
for h small enough;
(5) If $V_1 \ge V_2$, set $\mathfrak{T} = x$ if $cU \le f(x)$, otherwise reject x;
(6) If $V_2 > V_1$, set $\mathfrak{T} = x$ if $kU \le x^2 f(x)$, otherwise reject x.

3.7.3. Generalization of Gillespie's algorithm. It is well known that a continuous-time Markov chain $M(t)$ can be simulated by simulating separately its jump chain M_n and the sojourn times S_n in each state. Indeed, given a continuous-time Markov chain with transition rate matrix Q on the state space E, we can define the **rate function** $r(s) = -q_{s,s}$ for $s \in E$. Thus, the jump chain M_n admits transition probabilities given by
$$\mathbb{P}(M_n = s_2,\ M_{n-1} = s_1) = p_{s_1,s_2} = \frac{q_{s_1,s_2}}{r(s_1)},$$
and the simulation of the jump chain can be done by simulating at each step n a discrete random variable M_n with probability distribution $(p_s)_{s\in E} = (p_{M_{n-1},s})_{s\in E}$, supposing we already simulated M_{n-1}. Finally, we remain in each state M_n with a sojourn S_n that is exponentially distributed of parameter $r(M_n)$. Thus, we can simulate the two vectors $\vec{M} = (M_n)_{n=0,\dots,N}$ (called the **event vector**) and $\vec{T} = (T_n)_{n=0,\dots,N}$ (called the **calendar vector**), where $T_n = T_{n-1} + S_n$ and $T_0 = 0$, and it not difficult to see that the process $\widetilde{M}(t) = M_n$ as $t \in [T_n, T_{n+1})$ admits the same distribution as $M(t)$.
This simulation algorithm is known as **Gillespie's algorithm** (since it was first presented in [**71**]) and it seems to be based on the Markov property of the process $M(t)$. Actually, it is not difficult to see that the algorithm relies on the Markov property of $M(t)$ only at the jump times T_n, thus the fact that $M(t)$ is Markov is not really needed. Indeed, we can generalize the algorithm to the case of semi-Markov processes.
This generalization has been already applied to different contexts concerning time-changed continuous-time Markov chains with $\Phi(\lambda) = \lambda^\alpha$. For instance, in [**44**] and [**19**] a generalized Gillespie's algorithm has been used to simulate fractional queues, while in [**14, 16**] it has been used to exploit some properties of a time-changed epidemic model. Algorithms for the simulation of non-Markov (precisely semi-Markov) random link activation and deletion with Mittag-Leffler inter-event times are also given in [**68**]. In the same paper, a generalization to the case in which Markov dynamics occur on a non-Markov evolving network is also considered. Here we want to present a general version of the generalized Gillespie's algorithm for time-changed continuous-time Markov chains.
The first thing we have to observe, as we already stated in this Chapter, is that when we apply a time change to a continuous-time Markov chain $M(t)$, then the

jump chain remains untouched. Thus we still know how to simulate the event vector $\vec{M}$. The only problem is the calendar vector. However, let us denote by r the **rate function** of the continuous-time Markov chain $M(t)$ and with $M_\Phi(t)$ the respective time-changed continuous-time Markov chain. Now let us fix a state s and consider a modification of $M(t)$ and $M_\Phi(t)$ such that:

- The process starts from the state s;
- Any other state is absorbing.

Denote by $\bar{M}(t)$ and $\bar{M}_\Phi(t)$ the modified process and observe that $\bar{M}_\Phi(t) = \bar{M}(L_\Phi(t))$. It is not difficult to check that in such case the state probability $\bar{p}(t;s) := \mathbb{P}_s(\bar{M}(t) = s) = \mathbb{P}(S > t)$ where S is the generic sojourn time of $M(t)$ in the state s. Let us denote by S_Φ the sojourn time of $M_\Phi(t)$ in the state s and observe that $\bar{p}_\Phi(t;s) := \mathbb{P}_s(\bar{M}_\Phi(t) = s) = \mathbb{P}(S_\Phi > t)$. However we have

$$\mathbb{P}(S > t) = e^{-r(s)t}$$

and

$$\begin{aligned}\mathbb{P}(S_\Phi > t) &= \mathbb{P}_s(\bar{M}_\Phi(t) = s)\\ &= \int_0^{+\infty} \mathbb{P}_s(\bar{M}(y) = s) f_\Phi(y;t)dy\\ &= \int_0^{+\infty} e^{-r(s)y} f_\Phi(y;t)dy\\ &= \mathfrak{e}_\Phi(t; -r(s)).\end{aligned}$$

In particular we have shown that

LEMMA 3.7.3. *Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function and $M(t)$ a continuous-time Markov chain. Let $L_\Phi(t)$ be an inverse subordinator associated to Φ independent of $M(t)$ and $M_\Phi(t) = M(L_\Phi(t))$. Let r be the rate function of $M(t)$ and $S_\Phi(s)$ the sojourn time of $M_\Phi(t)$ in the state s. Then $S_\Phi(s) \sim \mathrm{Exp}_\Phi(r(s))$.*

By using such lemma, we can give the following **generalized Gillespie's algorithm**:

(1) Initialize M_0 and $T_0 = 0$;
(2) Suppose we have already simulated M_n and T_n:
 (a) Simulate M_{n+1} as a discrete random variable (see [**30**]) with probability distribution $(p_s)_{s\in E} = \left(\frac{q_{M_n,s}}{r(M_n)}\right)_{s\in E}$ where Q is the transition rate matrix of $M(t)$ and r is the rate function of $M(t)$;
 (b) Simulate $S \sim \mathrm{Exp}_\Phi(r(M_n))$ as described in the previous section;
 (c) Set $T_{n+1} = T_n + S$;
(3) Repeat until the stop condition is reached.

3.7.4. Simulating a time-changed Brownian motion. As we stated before, we need to find some simulation procedures that do not rely on the simulation of an inverse subordinator. Concerning time-changed diffusions, like the time-changed Brownian motion, it is not a simple task. In [**104**] it is suggested to consider the couple $(B(t), \sigma_\Phi(t))$ to obtain the points of the graph of $B_\Phi(t)$, where $B(t)$ is a Brownian motion and $B_\Phi(t)$ is the time-changed Brownian motion. In [**28**] we used an approximated simulation procedure to obtain L_Φ. Here we want to show, in the case of the time-changed Brownian motion, a method that does not rely on the simulation of L_Φ. To do this, we first need to recall some technical definitions (see [**148**] for the definition and the subsequent theorems).

DEFINITION 3.7.1. Let us denote by D the space of **cadlag functions**, i.e. functions f defined on some interval of $\mathbb{R}$ that are right-continuous and such that $\lim_{t\to t_0^-} f(t)$ always exists finite. Let $\iota(t) = t$ be the identity map and let us denote by $\|\cdot\|_{L^\infty}$ the supremum norm. Moreover, let Λ be the set of strictly increasing functions. Then we define the J_1 **topology** as the topology induced on D by the metric

$$d_{J_1}(x_1, x_2) = \inf_{\lambda\in\Lambda}(\max\{\|x_1 \circ \lambda - x_2\|_{L^\infty}, \|\lambda - \iota\|_{L^\infty}\}).$$

The metric space (D, d_{J_1}) is called the **Skorohod space**.
Let us denote by D_0 the subspace of D such that $x \in D_0$ if and only if $x(0) \geq 0$. Let $D_\uparrow$ be the subspace of D_0 whose elements are non-decreasing functions and $D_{\uparrow\uparrow}$ the subspace of D_0 whose elements are strictly increasing function. Finally, let us denote by $C_\uparrow = C \cap D_\uparrow$ and $C_{\uparrow\uparrow} = C \cap D_{\uparrow\uparrow}$.

An important thing to remember on (D, d_{J_1}) is that **it is not a topological group**, i.e. **addition is not continuous**. However, in some sense, the composition is a continuous map.

THEOREM 3.7.4. *The map $(x, y) \in D \times D_\uparrow \mapsto x \circ y \in D$ is continuous at $(x, y) \in (C \times D_\uparrow) \cup (D \times C_{\uparrow\uparrow})$.*

Actually, such theorem comes handy when used together with the continuous mapping theorem.

THEOREM 3.7.5 (**Continuous mapping theorem**). *Let X_n be a sequence of random variables in a metric space (S, m) converging towards X in distribution. Let $g : (S, m) \to (S', m')$ where (S', m') is a metric space. Let $\mathrm{Disc}(g)$ be the set of discontinuity points of g. If $\mathbb{P}(X \in \mathrm{Disc}(g)) = 0$, then $g(X_n) \to g(X)$ in (S', m') in distribution.*

Let us recall the following version of Donsker's Functional Central Limit Theorem.

THEOREM 3.7.6 (**Donsker's Theorem**). *Let $N(t)$ be a Poisson process of parameter 1 and $(X_k)_{k\in\mathbb{N}}$ be a family of i.i.d. Gaussian random variables with zero mean and unit variance. Let $X(t) = \sum_{k=1}^{N(t)} X_k$ be the respective compound Poisson process. Moreover let $X^h(t) = hX\left(\frac{t}{h^2}\right)$ for $h \in \mathbb{N}$. Then the sequence $X^h(t)$ converges towards a Brownian motion $B(t)$ in (D, d_{J_1}) in distribution as $h \to 0$.*

By combining the three previous theorem we can easily show the following result.

THEOREM 3.7.7. *Consider $\Phi \in \mathcal{BF}$ a driftless Brownian motion. Let $N(t)$ be a Poisson process of parameter 1 and $(X_k)_{k\in\mathbb{N}}$ be a family of i.i.d. Gaussian random variables with zero mean and unit variance. Let $X(t) = \sum_{k=1}^{N(t)} X_k$ be the respective compound Poisson process. Moreover let $X^h(t) = hX\left(\frac{t}{h^2}\right)$ for $h \in \mathbb{N}$ and consider the time-changed processes $X^h_\Phi(t) := X^h(L_\Phi(t))$ where $L_\Phi(t)$ is an inverse subordinator associated to Φ and independent of $X(t)$. Then the sequence $X^h_\Phi(t)$ converges towards a time-changed Brownian motion $B_\Phi(t)$ in (D, d_{J_1}) in distribution.*

PROOF. Let us consider the sequence (X^h, L_Φ) and observe that it converges towards (B, L_Φ). Now, let us observe that $\mathbb{P}(L_\Phi \not\in D_\uparrow) = 0$ since L_Φ is almost surely increasing and continuous. Moreover, $\mathbb{P}(B \not\in C) = 0$ since the Brownian motion is almost surely continuous. Thus, denoting by g the composition map, it holds $\mathbb{P}((B, L_\Phi) \in \mathrm{Disc}(g)) = 0$. By the continuous mapping theorem, we conclude the proof. $\square$

Now let us show how to use such information to obtain a simulation algorithm for $B_\Phi(t)$. First of all, let us observe that $X^h(t) = \sum_{k=1}^{N\left(\frac{t}{h^2}\right)} hX_k$. It is not difficult to check that $N\left(\frac{t}{h^2}\right) \sim N^h(t)$ where $N^h(t)$ is a Poisson process with parameter $\frac{1}{h^2}$. In particular, by using the consideration we made in the previous section, we actually know that the sojourn times of $N^h_\Phi(t)$ are Φ-exponentials of parameter $\frac{1}{h^2}$. Now let us observe that we can simulate $X^h_\Phi(t)$ by considering two vectors:

- A **state vector** $\vec{Y} = (Y_k)_{k\in\mathbb{N}_0}$ with $Y_0 = 0$;
- A **calendar vector** $\vec{T} = (T_k)_{k\in\mathbb{N}_0}$ with $T_0 = 0$.

Indeed, by definition, there exist two sequences of random variables $\vec{Y}$ and $\vec{T}$ such that $X^h_\Phi(t) = Y_k$ if $t \in [T_k, T_{k+1})$. In particular it holds $Y_{k+1} - Y_k = hX_{k+1}$ where X_{k+1} is a standard normal random variable, and $T_{k+1} - T_k = S_k$ where S_k is the sojourn time of $N^h_\Phi(t)$. Thus, from these observations, we have the following algorithm:

(1) Choose h big enough;
(2) Initialize $Y_0 = T_0 = 0$;
(3) Suppose we have already simulated Y_n and T_n:
   (a) Simulate a standard normal random variable X;
   (b) Set $Y_{n+1} = Y_n + hX$;
   (c) Simulate $S \sim \mathrm{Exp}_\Phi(1/h^2)$;
   (d) Set $T_{n+1} = T_n + S$.
(4) Repeat until the stop condition is reached.

After that, the simulated time-changed Brownian motion is given by $\widetilde{B}_\Phi(t) = Y_k$ for $t \in [T_k, T_{k+1})$.

CHAPTER 4

Non-local operators in space: some results on isotropic Lévy processes and isoperimetric inequalities

Now let us focus on some problems concerning non-locality in space. In particular, we focus on two problems. The first one concerns the asymptotic behaviour of the jump function j_Φ associated with driftless Bernstein functions $\Phi \in \mathcal{BF}$. Let us recall that jump functions are strictly linked with the Lévy measures of subordinated Brownian motions (that are isotropic Lévy processes). Denoting $B^\Phi(t) = B(\sigma_\Phi(t))$ where $B(t)$ is a Brownian motion in $\mathbb{R}^n$ with variance $2t$ and σ_Φ a subordinator independent of it, then, by Bochner subordination, $B^\Phi(t)$ is a Lévy process with Lévy measure $\mu_\Phi(x) = j_\Phi(|x|)dx$. The generator of $B^\Phi(t)$ is given by $-\Phi(-\Delta)$, that is defined via Phillips' formula. However, we have shown in 1.5.3 that $-\Phi(-\Delta)$ can be represented in terms of the jump function j_Φ. Thus, knowing some properties of the jump function j_Φ can be useful to obtain some estimates on the operator $-\Phi(-\Delta)$. This, for instance, can be applied to Schrödinger operators $H_\Phi = \Phi(-\Delta) + V$ with V multiplication operator $Vf \mapsto V(x)f(x)$ where $V : \mathbb{R}^d \to \mathbb{R}$ is a suitable potential. In particular, assuming that H_Φ admits a zero energy eigenvalue φ, i.e. a function $\varphi \in L^2(\mathbb{R}^d)$ with $\varphi \not\equiv 0$ such that $H_\Phi\varphi = 0$, then we have $V(x) = -\frac{1}{\varphi(x)}\Phi(-\Delta)\varphi(x)$. This leads to the usage of asymptotic estimates on j_Φ to deduce some conditions on the decay of the potential $V(x)$, as done in [**24**]. Moreover, the same techniques can be used to study the link between the fractional Laplacian $L_\alpha = (-\Delta)^{\frac{\alpha}{2}}$ and the massive relativistic fractional Laplacian $L_{\alpha,m} = (-\Delta + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$. Indeed, it has been shown in [**128**] that there exists a finite measure $\sigma_{\alpha,m}(x)dx$ such that

$$L_{\alpha,m}f = L_\alpha f - (\sigma_{\alpha,m} - \delta_0) * f.$$

For this reason, we will present also some results on the asymptotic behaviour of the density $\sigma_{\alpha,m}(x)$.

The second problem is related to randomized isoperimetric inequalities. In [**120**] the following isoperimetric inequality has been shown.

THEOREM 4.0.1. *Let $K \subseteq \mathbb{R}^d$ be a compact set and $r \le d$. Let $P_0^K, \cdot, P_r^K$ be $r+1$ random points in K, uniformly distributed and pair-wise independent. Let $K(r) = [P_0^K, \dots, P_r^K]$ be the polytope with vertices $P_0^K, \dots, P_r^K$ and $V_r^K := V_r(K)$ be the r-intrinsic volume of $K(r)$. Let $B \subseteq \mathbb{R}^d$ be any ball such that $|B| = |K|$. Then, for any strictly increasing function Ψ on $\mathbb{R}^+$ it holds*

$$\mathbb{E}[\Psi(V_r^K)] \ge \mathbb{E}[\Psi(V_r^B)].$$

Moreover, if $r < d$, equality holds if and only if K is a ball and, if $r = d$, equality holds if and only if K is an ellipsoid.

In [**63**] the following quantitative version for the previous Theorem, in the case $r = 2$ and $\Psi(x) = x^\beta$, has been shown.

THEOREM 4.0.2. *Let $d \geq 2$ and $\beta > 0$. Then there exists a universal constant $C(d, \beta) > 0$ such that for any measurable set $E \subseteq \mathbb{R}^d$ with $|E| = \omega_d$ it holds*

$$\delta(E) \leq C(d, \beta)\sqrt{D_\beta(E)}$$

where $\delta(E) = \inf_{x \in \mathbb{R}^d} |E \Delta B(x)|$ is Fraenkel asymmetry, $B(x)$ is a ball with radius 1 centred in x, $D_\beta(E) = \mathcal{G}_\beta(E) - \mathcal{G}_\beta(B)$ and $\mathcal{G}_\beta(E) = \mathbb{E}[(V_2^E)^2]$.

In [**17**] we give an alternative proof of the previous Theorem, by means of a Fuglede-type result on nearly spherical sets and some transportation arguments. Let us first state that our methods led us also to the proof of another isoperimetric inequality concerning a mixed energy given by the sum of mean length operator $\mathcal{G}_\beta$ defined before (that is minimized by the ball) and the Riesz potential V_α (that is maximized by the ball), together with a fractional perimeter penalization εP_s.
Here we focus on the main tool we had to introduce to handle the Fugluede-type result. Indeed we had to consider the fractional integral on the sphere $\mathbb{S}^{d-1}$ as defined in [**125**] and a Marchaud-type fractional integral on $\mathbb{S}^{d-1}$. In particular, we give a closed formula for eigenvalues of the two types of integrals and then we show the link between $\mathcal{G}_\beta(B)$ and the first eigenvalue of the Marchaud-type fractional integral.

4.1. Jump functions of the subordinated Brownian motion: general properties

Let us first fix some notation. Let $\Phi \in \mathcal{BF}$ be a driftless Bernstein function whose Lévy measure $\nu_\Phi(dt) = \nu_\Phi(t)dt$ with $\nu_\Phi(t)$ decreasing (but not necessarily completely monotone) and let $\sigma_\Phi(t)$ be the associated subordinator. Let $B(t)$ be a Brownian motion on $\mathbb{R}^d$ (for $d \geq 2$) independent of $\sigma_\Phi(t)$ and let us define the **subordinated Brownian motion** $B^\Phi(t) = B(\sigma_\Phi(t))$. Let us denote by $\mu_\Phi(dx) = j_\Phi(|x|)dx$ the Lévy measure of $B^\Phi(t)$ where $j_\Phi(r)$ is the jump function of $B^\Phi(t)$. We want to exploit the asymptotic behaviour of $j_\Phi(r)$.
Before doing this, let us show some general properties of ν_Φ and j_Φ. First of all, let us show the following technical result (see [**24**, Lemma 2.1]).

LEMMA 4.1.1. *For any $C > 0$ there exists $t_0(C) \in (0, 1)$ such that $\nu_\Phi(t) \leq Ct^{-2}$ for any $t \in (0, t_0(C))$.*

PROOF. Let us argue by contradiction. Suppose there exists $\widetilde{C} > 0$ and a decreasing sequence $(t_n)_{n\geq 1}$ such that $t_1 < 1$, $t_n > 0$ for any $n \in \mathbb{N}$, $t_{n-1} - t_n > \frac{t_{n-1}}{2}$, $t_n \to 0$ and $\nu_\Phi(t_n) > \widetilde{C}t_n^{-2}$. By definition of Lévy measure of a Bernstein function, it holds

$$\int_0^1 t\nu_\Phi(t)dt < +\infty.$$

On the other hand, since ν_Φ is decreasing, we have

$$\int_{t_1}^1 t\nu_\Phi(t)dt \geq t_1\nu_\Phi(1)(1 - t_1)$$

and

$$\int_{t_{n-1}}^{t_n} t\nu_\Phi(t)dt \ge \nu_\Phi(t_n)\frac{(t_n-t_{n-1})(t_n+t_{n-1})}{2} \ge \widetilde{C}\frac{t_{n-1}}{4t_n} \ge \frac{\widetilde{C}}{4}.$$

Thus, we obtain

$$\int_0^1 t\nu_\Phi(t)dt = \int_{t_1}^1 t\nu_\Phi(t)dt + \sum_{n=1}^{+\infty}\int_{t_{n+1}}^{t_n} t\nu_\Phi(t)dt \ge t_1\nu_\Phi(1)(1-t_1) + \widetilde{C}\sum_{n=1}^{+\infty}\frac{1}{4} = +\infty,$$

that is a contradiction. □

Concerning j_Φ, we can show the following result that summarize some of its main properties (see [**24**, Proposition 2.2]).

PROPOSITION 4.1.2. *The following properties hold:*

(1) $j_\Phi(r) < +\infty$ *for any* $r \in \mathbb{R}^+$*;*
(2) $j_\Phi(r)$ *is a decreasing function;*
(3) $j_\Phi(r)$ *is continuous in* $\mathbb{R}^+$*;*
(4) $\lim_{r\to+\infty} j_\Phi(r) = 0$*;*
(5) *It holds*
$$\int_1^{+\infty} r^{d-1} j_\Phi(r)dr < +\infty;$$
(6) *It holds*
$$\int_0^1 r^{d+1} j_\Phi(r)dr < +\infty;$$
(7) *the measure* $\mu_\Phi(dx) = j_\Phi(|x|)dx$ *defined on* $\mathbb{R}^d\setminus\{0\}$ *is a Lévy measure;*
(8) *The function* $x \in \mathbb{R}^d \mapsto j(|x|) \in \mathbb{R}^+$ *belongs to* $L^p(B^c_\varepsilon(0))$ *for any* $\varepsilon > 0$ *and* $p \ge 1$ *(eventually* $p = +\infty$*).*

PROOF. By the definition of jump function given in Equation (1.5.1) we have

$$\begin{aligned} j(r) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\left(\int_0^1 t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}}\nu_\Phi(dt) + \int_1^{+\infty} t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}}\nu_\Phi(dt)\right)\\ &=: \frac{1}{(4\pi)^{\frac{d}{2}}}(I_1(r)+I_2(r)). \end{aligned}$$

Concerning the first integral $I_1(r)$, we have that there exists a constant $C(r)$ such that $t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}} \le C(r)t$ for any $t \in (0,1)$, and then $I_1(r) < +\infty$ by definition of Lévy measure. Concerning the second integral, instead, we have that $I_2(r) < +\infty$ since $t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}} \le 1$ for any $t \ge 1$. This proves property (1). Property (2) follows from the fact that the integrand in (1.5.1) is decreasing and so we have (3) and (4) by monotone convergence. Concerning (5), we have, by Fubini's theorem

$$\int_1^{+\infty} r^{d-1}j_\Phi(r) = \frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^{+\infty} t^{-\frac{d}{2}}\left(\int_1^{+\infty} r^{d-1}e^{-\frac{r^2}{4t}}dr\right)\nu_\Phi(dt).$$

In the inner integral we can consider the change of variables $r = 2\sqrt{st}$ leading to

$$\int_1^{+\infty} r^{d-1}e^{-\frac{r^2}{4t}}dr = 2^{d-1}t^{\frac{d}{2}}\Gamma\left(\frac{d}{2},\frac{1}{4t}\right).$$

Thus we have

$$\int_1^{+\infty} r^{d-1} j_\Phi(r) = \frac{1}{2\pi^{\frac{d}{2}}}\left(\int_0^1 \Gamma\left(\frac{d}{2},\frac{1}{4t}\right)\mu_\Phi(dt) + \int_1^{+\infty}\Gamma\left(\frac{d}{2},\frac{1}{4t}\right)\mu_\Phi(dt)\right).$$

Concerning the second integral, it is finite since $\Gamma\left(\frac{d}{2},\frac{1}{4t}\right) \le \Gamma\left(\frac{d}{2}\right)$ and ν_Φ is a Lévy measure. On the other hand, it is well known that $\Gamma(s,x) \sim e^{-x}x^{s-1}$ as $x \to +\infty$, thus in particular there exists a constant $C > 0$ such that $\Gamma\left(\frac{d}{2},\frac{1}{4t}\right) \le Ct$ for $t \in (0,1)$. Thus, we have that also the first integral is finite since ν_Φ is a Lévy measure.
Now let us prove (6). Arguing as before we have

$$\int_0^1 r^{d+1} j_\Phi(r) = \frac{2}{\pi^{\frac{d}{2}}}\left(\int_0^1 t\gamma\left(\frac{d}{2}+1,\frac{1}{4t}\right)\nu_\Phi(dt) + \int_1^{+\infty} t\gamma\left(\frac{d}{2}+1,\frac{1}{4t}\right)\nu_\Phi(dt)\right),$$

where $\gamma(s,x)$ is the lower incomplete Gamma function defined as

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt.$$

Concerning the first integral, we have $\gamma\left(\frac{d}{2}+1,\frac{1}{4t}\right) \le \Gamma\left(\frac{d}{2}+1\right)$, thus, since ν_Φ is a Lévy measure of a Bernstein function and then t is integrable, it is finite.
Concerning the second integral, let us recall that $\gamma(s,x) \sim sx^s$ as $x \to 0^+$, thus there exists a constant $C > 0$ such that $t\gamma\left(\frac{d}{2}+1,\frac{1}{4t}\right) \le Ct^{-\frac{d}{2}} \le 1$ as $t \ge 1$. Thus, in particular, the second integral is finite since $\nu_\Phi(1,+\infty) < +\infty$.
Property (7) follows from the two previous estimate, since, by coarea formula[1], they imply that

$$\int_{\mathbb{R}^d\setminus\{0\}} (|x|^2 \wedge 1)\mu_\Phi(dx) < +\infty.$$

Finally, concerning property (8), it is immediate for $p = 1$. Concerning $p = +\infty$, it follows from the fact that for $x \in B^c_\varepsilon(0)$ it holds $|x| \ge \varepsilon$ and then $j_\Phi(|x|) \le j_\Phi(\varepsilon)$. Now let us consider $p \in (1,+\infty)$. Since j is decreasing there exists $M > 0$ such that $j_\Phi(|x|) < 1$ for any $x \in B^c_M(0)$. This leads to $j^p_\Phi(|x|) < j_\Phi(|x|)$ for $x \in B^c_M(0)$ and, if $\varepsilon \ge M$, this completes the proof. Otherwise, we only have to observe that

$$\begin{aligned}\int_{B^c_\varepsilon(0)} j^p_\Phi(|x|)dx &= \int_{B_M(0)\setminus B_\varepsilon(0)} j^p_\Phi(|x|)dx + \int_{B^c_M(0)} j^p_\Phi(|x|)dx \\ &\le j^p_\Phi(\varepsilon)(M^p - \varepsilon^p)\omega_d + \nu_\Phi(B^c_M(0)) < +\infty.\end{aligned}$$

□

Now we can focus on the study of the asymptotic behaviour of j_Φ. However, we have to distinguish between two cases:

- The density ν_Φ is regularly varying at 0^+ or at $+\infty$;
- The density ν_Φ exhibit an exponential behaviour at $+\infty$.

[1] Here we actually use only the following corollary of the coarea formula: for any measurable set $E \subseteq \mathbb{R}^d$ and any integrable function f it holds

$$\int_E f(x)dx = \int_0^{+\infty}\int_{E\cap\partial B_r} f(x)d\mathcal{H}^{d-1}(x)dr.$$

For a general formulation of the coarea formula we refer to [**59**].

We will call the first case **regularly varying case**, while the second **exponentially light case**. In particular, let us stress out that the first case is implied by the regular variation of the Bernstein function Φ. In the following sections we will always consider $\Phi \in \mathcal{CBF}$.

4.2. Jump functions of the subordinated Brownian motion: the regularly varying case

Let us first focus on the case in which Φ is regularly varying (at 0 or at ∞). In such case, we already have some asymptotic results on ν_Φ and j_Φ. Indeed, the following result has been shown (see [**141**, Proposition 5.24]).

THEOREM 4.2.1. *Let $\Phi \in \mathcal{CBF}$ be a driftless Bernstein function such that there exists a function $\ell(\lambda)$ slowly varying at infinity (at 0^+) and $\alpha \in (0,2)$ such that*

$$\Phi(\lambda) \sim \lambda^{\frac{\alpha}{2}}\ell(\lambda), \qquad \lambda \to +\infty\,(0^+).$$

Then it holds

$$\nu_\Phi(t) \sim \frac{\frac{\alpha}{2}}{\Gamma\left(1-\frac{\alpha}{2}\right)} t^{-1-\frac{\alpha}{2}}\ell(t^{-1}), \qquad t \to 0^+\,(+\infty).$$

This result can be improved to the case in which we only have an asymptotic bound, as done in [**87**, Theorem 2.10].

THEOREM 4.2.2. *Let $\Phi \in \mathcal{CBF}$ be a driftless Bernstein function such that there exists a function $\ell(\lambda)$ slowly varying at infinity and $\alpha \in (0,2)$ such that, as $\lambda \to +\infty$,*

$$\Phi(\lambda) \asymp \lambda^{\frac{\alpha}{2}}\ell(\lambda).$$

Then it holds

$$\nu_\Phi(t) \asymp t^{-1-\frac{\alpha}{2}}\ell(t^{-1}), \qquad t \to 0^+.$$

Concerning the jump function, the following Theorem holds (see [**87**, Theorem 3.4]).

THEOREM 4.2.3. *Let $\Phi \in \mathcal{CBF}$ be a driftless Bernstein function such that there exists a function $\ell(\lambda)$ slowly varying at infinity and $\alpha \in (0,2)$ such that, as $\lambda \to +\infty$,*

$$\Phi(\lambda) \asymp \lambda^{\frac{\alpha}{2}}\ell(\lambda).$$

Then it holds

$$j_\Phi(|x|) \asymp \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, \ |x| \to 0^+.$$

In [**141**] it is shown that the previous result holds also if we use $\sim$ in place of $\asymp$.
Here we want to focus on the asymptotic behaviour of $j(r)$ as $r \to +\infty$. To do this, we will need to suppose that Φ is regularly varying at 0^+. Moreover, from now on, let us denote $\widetilde{\ell}(t) = \ell(t^{-1})$. We have the following result (see [**24**, Proposition 2.3]).

PROPOSITION 4.2.4. *Let $\Phi \in \mathcal{CBF}$, $\alpha \in (0,2)$ and ℓ a slowly varying function at 0^+ such that $\Phi(\lambda) \sim \lambda^{\frac{\alpha}{2}}\ell(\lambda)$ as $\lambda \to 0^+$. Then it holds*

$$j_\Phi(r) \sim \frac{\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{2-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} r^{-d-\alpha}\,\widetilde{\ell}(r^2), \qquad r \to +\infty.$$

PROOF. By Potter's Theorem we know that there exists a constant M_1 such that

$$\frac{\widetilde{\ell}(t)}{\widetilde{\ell}(s)} \le 2\max\left\{\left(\frac{t}{s}\right)^{\frac{\alpha}{2}}, \left(\frac{t}{s}\right)^{-\frac{\alpha}{2}}\right\}$$

for any $s,t > M_1$. Moreover, by Theorem 4.2.1, we know that there exists $M_2 > 0$ such that

$$\frac{\nu_\Phi(t)}{\frac{\frac{\alpha}{2}}{\Gamma(1-\frac{\alpha}{2})}t^{-1-\frac{\alpha}{2}}\,\widetilde{\ell}(t)} \le 2$$

for any $t > M_2$. Let us set $M = \max\{M_1, M_2\}$. By Lemma 4.1.1 we also know that there exists $t_0 \in (0,1)$ such that for any $t \in (0,t_0)$ it holds $\nu_\Phi(t) \le t^{-2}$. By definition of jump function, we have

$$\frac{j_\Phi(r)}{r^{-d-\alpha}\,\widetilde{\ell}(r^2)} = \frac{1}{(4\pi)^{\frac{d}{2}}}\left(\int_0^{t_0}\frac{t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}}}{r^{-d-\alpha}\,\widetilde{\ell}(r^2)}\nu_\Phi(t)dt + \int_{t_0}^{M}\frac{t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}}}{r^{-d-\alpha}\,\widetilde{\ell}(r^2)}\nu_\Phi(t)dt \right.$$
$$\left. + \int_M^{+\infty}\frac{t^{-\frac{d}{2}}e^{-\frac{r^2}{4t}}}{r^{-d-\alpha}\,\widetilde{\ell}(r^2)}\nu_\Phi(t)dt\right).$$

Now let us use the change of variables $s = \frac{r^2}{4t}$ to obtain

$$\begin{aligned}\frac{j_\Phi(r)}{r^{-d-\alpha}\,\widetilde{\ell}(r^2)} &= \frac{1}{4\pi^{\frac{d}{2}}}\left(\int_{\frac{r^2}{4t_0}}^{+\infty} r^{\alpha+2}\frac{s^{-\frac{d}{2}-2}e^{-s}}{\widetilde{\ell}(r^2)}\nu_\Phi\left(\frac{r^2}{4s}\right)ds\right.\\ &\quad + \int_{\frac{r^2}{4M}}^{\frac{r^2}{4t_0}} r^{\alpha+2}\frac{s^{-\frac{d}{2}-2}e^{-s}}{\widetilde{\ell}(r^2)}\nu_\Phi\left(\frac{r^2}{4s}\right)ds\\ &\quad \left.+ \int_0^{\frac{r^2}{4M}} r^{\alpha+2}\frac{s^{-\frac{d}{2}-2}e^{-s}}{\widetilde{\ell}(r^2)}\nu_\Phi\left(\frac{r^2}{4s}\right)ds\right)\\ &= \frac{1}{4\pi^{\frac{d}{2}}}(I_1(r)+I_2(r)+I_3(r)).\end{aligned}$$

Let us consider $I_3(r)$. We have that, since $s > \frac{r^2}{4t_0}$, it holds $\frac{r^2}{4s} < t_0$ and then $\nu_\Phi\left(\frac{r^2}{4s}\right) \le \frac{16s^2}{r^4}$. This leads to

$$I_3(r) \le \frac{16}{r^{2-\alpha}\,\widetilde{\ell}(r^2)}\int_0^{\frac{r^2}{4M}} s^{-\frac{d}{2}}e^{-s}ds \le \frac{16}{r^{2-\alpha}\,\widetilde{\ell}(r^2)}\Gamma\left(\frac{d}{2}+1\right)$$

and then to $\lim_{r\to+\infty} I_3(r) = 0$.
Now let us consider $I_2(r)$. We have, since ν_Φ is completely monotone,

$$I_2(r) \le \frac{r^{\alpha+2}}{\widetilde{\ell}(r^2)}\nu_\Phi(t_0)\int_{\frac{r^2}{4M}}^{\frac{r^2}{4t_0}} s^{-\frac{d}{2}-2}e^{-s}ds \le \frac{r^{\alpha+2}}{\widetilde{\ell}(r^2)}\nu_\Phi(t_0)\Gamma\left(\frac{d}{2}-1, \frac{r^2}{2M}\right).$$

However, there exists a constant $C > 0$ such that $\Gamma\left(\frac{d}{2}-1, \frac{r^2}{2M}\right) \le Cr^{d-4}e^{-\frac{r^2}{4M}}$ for r sufficiently big. Hence it holds

$$I_2(r) \le \frac{r^{\alpha+d-2}e^{-\frac{r^2}{4M}}}{\widetilde{\ell}(r^2)}\nu_\Phi(t_0).$$

Taking the limit as $r \to +\infty$ we conclude that $\lim_{r\to+\infty} I_2(r) = 0$.
Now we need to find $\lim_{r\to+\infty} I_1(r)$. To do this, let us denote

$$F(r,s) = \frac{\nu_\Phi\left(\frac{r^2}{4s}\right)}{\frac{\frac{\alpha}{2}}{\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{r^2}{4s}\right)^{-1-\frac{\alpha}{2}}\widetilde{\ell}\left(\frac{r^2}{4s}\right)}$$

to obtain

$$I_1(r) = \frac{2^\alpha \alpha}{\Gamma\left(1-\frac{\alpha}{2}\right)}\int_0^{\frac{r^2}{4M}} s^{\frac{d+\alpha}{2}-1}e^{-s}F(r,s)\frac{\widetilde{\ell}\left(\frac{r^2}{4s}\right)}{\widetilde{\ell}(r^2)}ds.$$

By definition of $M > 0$ we have that for $r^2 > M$

$$\frac{\widetilde{\ell}\left(\frac{r^2}{4s}\right)}{\widetilde{\ell}(r^2)} \le 2((4s)^{\frac{\alpha}{2}} + (4s)^{-\frac{\alpha}{2}}).$$

Moreover, $F(r,s) \le 2$, thus we have

$$s^{\frac{d+\alpha}{2}-1}e^{-s}F(r,s)\frac{\widetilde{\ell}\left(\frac{r^2}{4s}\right)}{\widetilde{\ell}(r^2)}\chi_{\left(0,\frac{r^2}{4M}\right)}(s) \le 2^{\alpha+2}s^{\frac{d}{2}+\alpha-1}e^{-s} + 2^{2-\alpha}s^{\frac{d}{2}-1}e^{-s},$$

that is integrable. By dominated convergence theorem, observing that $\lim_{r\to+\infty} F(r,s) = 1$ by Theorem 4.2.1 and $\lim_{r\to+\infty}\frac{\widetilde{\ell}\left(\frac{r^2}{4s}\right)}{\widetilde{\ell}(r^2)} = 1$ by definition of slowly varying functions, we get

$$\lim_{r\to+\infty} I_1(r) = \frac{2^\alpha \alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)},$$

concluding the proof. $\square$

Starting from the asymptotic estimate on $j_\Phi(r)$ we can ask for the asymptotic behaviour of:

- $R \mapsto \mu_\Phi(B_R^c(0))$;
- $R \mapsto \mathcal{J}_\Phi(R) := \int_{B_R(0)} |x|^2 j_\Phi(|x|)dx$;
- $R \mapsto \|j(|x|)\|_{L^p(B_R^c(0))}$.

In particular we can prove the following Corollary (see [**24**, Corollary 2.1]).

Corollary 4.2.5. *Let $\Phi \in \mathcal{CBF}$, $\alpha \in (0,2)$ and ℓ a slowly varying function at 0^+ such that $\Phi(\lambda) \sim \lambda^{\frac{\alpha}{2}}\ell(\lambda)$ as $\lambda \to 0^+$. Then the following properties hold:*

- $$\mu_\Phi(B_R^c(0)) \sim \frac{d\omega_d\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{2-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}R^{-\alpha}\widetilde{\ell}(R^2), \qquad R\to+\infty;$$
- $$\mathcal{J}(R) \sim \frac{\alpha d\omega_d\Gamma\left(\frac{d+\alpha}{2}\right)}{(2-\alpha)2^{2-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}R^{2-\alpha}\widetilde{\ell}(R^2), \qquad R\to+\infty;$$
- *For any $p \in (1,+\infty)$ it holds*

$$\|j_\Phi(|x|)\|_{L^p(B_R^c(0))} \sim \frac{(d\omega_d)^{\frac{1}{p}}\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{2-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)((p-1)d+p\alpha)^{\frac{1}{p}}}R^{-\frac{d}{q}-\alpha}\widetilde{\ell}(R^2), \qquad R\to+\infty$$

where $\frac{1}{q}+\frac{1}{p} = 1$.

PROOF. Let us show only the first property, as the proof of the others is analogous. First of all, let us set

$$A=\frac{\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{2-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}$$

and fix $\varepsilon\in\left(0,\frac{1}{2}\right)$. Thus, by Proposition 4.2.4, we know there exists R_1 such that, for any $r>R_1$, it holds

$$\sqrt{1-\varepsilon}\le\frac{j(r)}{Ar^{-d-\alpha}\,\widetilde{\ell}(r^2)}\le\sqrt{1+\varepsilon}.$$

Moreover, by Karamata's Theorem, we know that there exists $R_2>0$ such that, for any $R>R_2$, it holds

$$\sqrt{1-\varepsilon}\le\frac{\int_R^{+\infty}r^{-1-\alpha}\,\widetilde{\ell}(r^2)dr}{\frac{R^{-\alpha}}{\alpha}\,\widetilde{\ell}(R^2)}\le\sqrt{1+\varepsilon}.$$

Set $R_3=\max\{R_1,R_2\}$ and observe that, by coarea formula,

$$\mu_\Phi(B_R^c(0))=d\omega_d\int_R^{+\infty}r^{d-1}j(r)dr$$

thus, for $R>R_3$ it holds

$$\begin{aligned}1-\varepsilon\le\sqrt{1-\varepsilon}\frac{\int_R^{+\infty}r^{-1-\alpha}\,\widetilde{\ell}(r^2)dr}{\frac{R^{-\alpha}}{\alpha}\,\widetilde{\ell}(R^2)}&\\ &\le\frac{\alpha\mu_\Phi(B_R^c(0))}{Ad\omega_dR^{-\alpha}\,\widetilde{\ell}(R^2)}\\ &\le\sqrt{1+\varepsilon}\frac{\int_R^{+\infty}r^{-1-\alpha}\,\widetilde{\ell}(r^2)dr}{\frac{R^{-\alpha}}{\alpha}\,\widetilde{\ell}(R^2)}\le1+\varepsilon,\end{aligned}$$

concluding the proof since ε is arbitrary. □

4.3. Jump functions of the subordinated Brownian motion: the exponentially light case

Now let us move to the exponentially light case. We say that ν_Φ is **exponentially light** if there exist $\alpha\in(0,2]$ and $\theta,\eta>0$ such that

$$\nu_\Phi(t)\sim\theta t^{-1-\frac{\alpha}{2}}e^{-\eta t},\ t\to+\infty.$$

In this case we will make use of the **modified Bessel function of the third kind**, given by

$$K_\rho(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\rho\int_0^{+\infty}t^{-\rho-1}e^{-t-\frac{z^2}{4t}}dt,\ z>0,\ \rho>-\frac{1}{2}.$$

In particular let us recall the well-known asymptotic formula:

$$K_\rho(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z},\ z\to+\infty.$$

Now that we have set this notation, we can prove the following Proposition (see [**24**, Proposition 2.4]).

PROPOSITION 4.3.1. *Let $\Phi \in \mathcal{CBF}$ be a driftless complete Bernstein function. Suppose there exist $\alpha \in (0,2]$, $\theta, \eta > 0$ such that*

$$\nu_\Phi(t) \sim \theta t^{-1-\frac{\alpha}{2}} e^{-\eta t}, \qquad t \to +\infty.$$

Then it holds

$$j_\Phi(r) \sim \theta \pi^{\frac{1-d}{2}} 2^{\frac{\alpha-1-d}{2}} \eta^{\frac{d+\alpha+2}{4}} r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta} r}, \qquad r \to +\infty.$$

PROOF. Without loss of generality, let us set $\theta = 1$. Fix $\varepsilon \in (0,1)$. Then there exists $t_0 > 0$ such that for any $t > t_0$ it holds

$$(1-\varepsilon) t^{-1-\frac{\alpha}{2}} e^{-\eta t} \le \nu_\Phi(t) \le (1+\varepsilon) t^{-1-\frac{\alpha}{2}} e^{-\eta t}.$$

Now let us split the integral defining j_Φ in t_0 to obtain

$$\begin{aligned} j_\Phi(r) &= \frac{1}{(4\pi)^{\frac{d}{2}}} \left(\int_0^{t_0} t^{-\frac{d}{2}} e^{-\frac{r^2}{2t}} \nu_\Phi(t) dt + \int_{t_0}^{+\infty} t^{-\frac{d}{2}} e^{-\frac{r^2}{2t}} \nu_\Phi(t) dt \right) \\ &=: \frac{1}{(4\pi)^{\frac{d}{2}}} (I_1(r) + I_2(r)). \end{aligned}$$

In particular we have

$$I_2(r) \le 2^d \pi^{\frac{d}{2}} j_\Phi(r) = I_1(r) + I_2(r). \tag{4.3.1}$$

Now let us consider $I_1(r)$. We have

$$\frac{I_1(r)}{r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta} r}} = \int_0^{t_0} t^{-\frac{d}{2}} r^{\frac{d+\alpha+1}{2}} e^{-\frac{r^2}{8t} + \sqrt{\eta} r} e^{-\frac{r^2}{8t}} \nu_\Phi(t) dt.$$

Let us consider the function

$$f_t(r) = r^{\frac{d+\alpha+1}{2}} e^{-\frac{r^2}{8t} + \sqrt{\eta} r}$$

and let us observe that it attains its maximum at

$$r_{\max}(t) = 2\sqrt{\eta} t^2 + 2(\sqrt{\eta} + d + \alpha + 1) t$$

with the value

$$f_t(r_{\max}(t)) = g(t) e^{-\sqrt{\eta}(\sqrt{\eta} t^2 + (\sqrt{\eta} + d + \alpha + 1) t) - \frac{\sqrt{\eta}}{4}(d+\alpha+1)},$$

where g is an increasing function. In particular it holds, for $t \in (0, t_0)$,

$$f_t(r_{\max}(t)) \le g(t_0)$$

and then we have

$$\frac{I_1(r)}{r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta} r}} \le g(t_0) \int_0^{t_0} t^{-\frac{d}{2}} e^{-\frac{r^2}{8t}} \nu_\Phi(t) dt. \tag{4.3.2}$$

Since we want to send $r \to +\infty$, let us suppose $r > 1$ to obtain

$$e^{-\frac{r^2}{8t}} \le e^{-\frac{1}{8t}} \le C(d) t^{\frac{d}{2}+2}$$

as $t \in (0, t_0)$. Thus, we have that the integrand on the right-hand side of (4.3.2) is dominated by a $L^1(\nu_\Phi)$ function and then it holds, by dominated convergence theorem,

$$\lim_{r \to +\infty} \frac{I_1(r)}{r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta} r}} = 0.$$

Now let us consider $I_2(r)$. Since $t > t_0$, we have

$$\begin{aligned} I_2(r) &\le (1+\varepsilon)\int_{t_0}^{+\infty} t^{-1-\frac{d+\alpha}{2}} e^{-\eta t-\frac{r^2}{4t}}\,dt \\ &\le (1+\varepsilon) r^{-\frac{d+\alpha}{2}} 2^{\frac{d+\alpha}{2}} \eta^{\frac{d+\alpha}{4}} \left(\frac{r}{2\sqrt{\eta}}\right)^{\frac{d+\alpha}{2}} \int_{t_0}^{+\infty} t^{-1-\frac{d+\alpha}{2}} e^{-\eta\left(t+\frac{r^2}{4\eta t}\right)}\,dt \\ &= (1+\varepsilon) r^{-\frac{d+\alpha}{2}} 2^{\frac{d+\alpha}{2}} \eta^{\frac{d+\alpha}{4}} K_{\frac{d+\alpha}{2}}(\sqrt{\eta}r), \end{aligned}$$

where we used the integral representation (see [**72**, Formula 8.432.7])

$$K_\rho(xz) = \frac{z^\rho}{2}\int_0^{+\infty} t^{-\rho-1} e^{-\frac{x}{2}\left(t+\frac{z^2}{t}\right)}\,dt.$$

Going back to Equation (4.3.1), we get

$$\begin{aligned} \frac{j_\Phi(r)}{\pi^{\frac{1-d}{2}} 2^{\frac{\alpha-1-d}{2}} \eta^{\frac{d+\alpha+2}{4}} r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta}r}} \le{}& \frac{I_1(r)}{\pi^{\frac{1}{2}} 2^{\frac{\alpha-1-d}{2}} \eta^{\frac{d+\alpha+2}{4}} r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta}r}} \\ &+ (1+\varepsilon)\frac{K_{\frac{d+\alpha}{2}}(\sqrt{\eta}r)}{\pi^{\frac{1}{2}} 2^{-\frac{1}{2}} \eta^{\frac{1}{2}} r^{-\frac{1}{2}} e^{-\sqrt{\eta}r}} \end{aligned}$$

and then, taking the limit superior, we have

$$\limsup_{r\to+\infty} \frac{j_\Phi(r)}{\pi^{\frac{1-d}{2}} 2^{\frac{\alpha-1-d}{2}} \eta^{\frac{d+\alpha+2}{4}} r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta}r}} \le (1+\varepsilon).$$

Now let us consider again $I_2(r)$ and let us control it as

$$\begin{aligned} I_2(r) &\ge (1-\varepsilon)\left(\int_0^{+\infty} t^{-1-\frac{d+\alpha}{2}} e^{-\eta t-\frac{r^2}{4t}}\,dt - \int_0^{t_0} t^{-1-\frac{d+\alpha}{2}} e^{-\eta t-\frac{r^2}{4t}}\,dt\right) \\ &:= (1-\varepsilon)(I_3(r)-I_4(r)). \end{aligned}$$

Concerning $I_3(r)$, we have already shown that

$$I_3(r) = r^{-\frac{d+\alpha}{2}} 2^{\frac{d+\alpha}{2}} \eta^{\frac{d+\alpha}{4}} K_{\frac{d+\alpha}{2}}(\sqrt{\eta}r).$$

On the other hand, we have

$$\begin{aligned} \frac{I_4(r)}{r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta}r}} &= \int_0^{t_0} t^{-1-\frac{d+\alpha}{2}} e^{-\frac{r^2}{8t}-\eta t} f_t(r)dt \\ &\le g(t_0)\int_0^{t_0} t^{-1-\frac{d+\alpha}{2}} e^{-\frac{r^2}{8t}}\,dt. \end{aligned}$$

As before, if $r > 1$ we have for $t \in (0, t_0)$

$$e^{-\frac{r^2}{8t}} \le e^{-\frac{1}{8t}} \le C t^{1+\frac{d+\alpha}{2}}$$

thus, by dominated convergence theorem,

$$\lim_{r\to+\infty} \frac{I_4(r)}{r^{-\frac{d+\alpha+1}{2}} e^{-\sqrt{\eta}r}} = 0.$$

Considering again Equation (4.3.1), we have

$$\frac{j_\Phi(r)}{\pi^{\frac{1-d}{2}}2^{\frac{\alpha-1-d}{2}}\eta^{\frac{d+\alpha+2}{4}}r^{-\frac{d+\alpha+1}{2}}e^{-\sqrt{\eta}r}} \ge (1-\varepsilon)\frac{I_4(r)}{\pi^{\frac{1}{2}}2^{\frac{\alpha-1-d}{2}}\eta^{\frac{d+\alpha+2}{4}}r^{-\frac{d+\alpha+1}{2}}e^{-\sqrt{\eta}r}} + (1-\varepsilon)\frac{K_{\frac{d+\alpha}{2}}(\sqrt{\eta}r)}{\pi^{\frac{1}{2}}2^{-\frac{1}{2}}\eta^{\frac{1}{2}}r^{-\frac{1}{2}}e^{-\sqrt{\eta}r}}$$

and we can take the limit inferior to achieve

$$\liminf_{r\to+\infty}\frac{j_\Phi(r)}{\pi^{\frac{1-d}{2}}2^{\frac{\alpha-1-d}{2}}\eta^{\frac{d+\alpha+2}{4}}r^{-\frac{d+\alpha+1}{2}}e^{-\sqrt{\eta}r}} \ge (1-\varepsilon).$$

Thus, sending $\varepsilon \to 0^+$, we conclude the proof. $\square$

As we did in the regularly varying case, let us investigate the asymptotic behaviour of $R \to \mu_\Phi(B_R^c(0))$ (see [**24**, Corollary 2.2.1]).

Corollary 4.3.2. *Let $\Phi \in \mathcal{CBF}$ be a driftless complete Bernstein function. Suppose there exist $\alpha \in (0,2]$, $\theta, \eta > 0$ such that*

$$\nu_\Phi(t) \sim \theta t^{-1-\frac{\alpha}{2}}e^{-\eta t}, \qquad t \to +\infty.$$

Then it holds

$$\mu_\Phi(B_R^c(0)) \sim \frac{\theta d\omega_d}{\eta^{\frac{d+\alpha}{4}}\pi^{\frac{d-1}{2}}2^{\frac{d+1-\alpha}{2}}}R^{\frac{d-\alpha-3}{2}}e^{-\sqrt{\eta}R}, \qquad R \to +\infty.$$

Proof. Without loss of generality, we can suppose $\theta = 1$. Let us fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ and let us observe, by Proposition 4.3.1, that there exists a constant $R_1 > 0$ such that for any $r > R_1$ it holds

$$\sqrt{1-\varepsilon} \le \frac{j_\Phi(r)}{Ar^{-\frac{d+\alpha+1}{2}}e^{-\sqrt{\eta}r}} \le \sqrt{1+\varepsilon}$$

where

$$A = \pi^{\frac{1-d}{2}}2^{\frac{\alpha-1-d}{2}}\eta^{\frac{d+\alpha+2}{4}}.$$

Now let us suppose $R > R_1$. Then, by coarea formula, we have

$$d\omega_d A\sqrt{1-\varepsilon}\int_R^{+\infty} r^{\frac{d-\alpha-1}{2}-1}e^{-\sqrt{\eta}r}dr \le \mu_\Phi(B_R^c(0)) \le d\omega_d A\sqrt{1+\varepsilon}\int_R^{+\infty} r^{\frac{d-\alpha-1}{2}-1}e^{-\sqrt{\eta}r}dr.$$

By means of the change of variables $w = \sqrt{\eta}r$ we obtain

$$\int_R^{+\infty} r^{\frac{d-\alpha-1}{2}-1}e^{-\sqrt{\eta}r}dr = \eta^{-\frac{d-\alpha-1}{4}}\int_{\sqrt{\eta}R}^{+\infty} w^{\frac{d-\alpha-1}{2}-1}e^{-w}dw = \eta^{-\frac{d-\alpha-1}{4}}\Gamma\left(\frac{d-\alpha-1}{2}, \sqrt{\eta}R\right).$$

Now consider $R_2 > 0$ such that for any $R > R_2$ it holds

$$\begin{aligned}\sqrt{1-\varepsilon}e^{-\sqrt{\eta}R}R^{\frac{d-\alpha-3}{2}}\eta^{\frac{d-\alpha-3}{4}} \\ \le \Gamma\left(\frac{d-\alpha-1}{2}, \sqrt{\eta}R\right) \\ \le \sqrt{1+\varepsilon}e^{-\sqrt{\eta}R}R^{\frac{d-\alpha-3}{2}}\eta^{\frac{d-\alpha-3}{4}}.\end{aligned}$$

Thus we have, for $R > \max\{R_1, R_2\}$

$$d\omega_d A(1-\varepsilon)e^{-\sqrt{\eta}R}R^{\frac{d-\alpha-3}{2}}\eta^{-\frac{1}{2}} \le \mu_\Phi(B^c_R(0)) \le d\omega_d A(1+\varepsilon)e^{-\sqrt{\eta}R}R^{\frac{d-\alpha-3}{2}}\eta^{-\frac{1}{2}},$$

that is to say

$$(1-\varepsilon) \le \frac{\mu_\Phi(B^c_R(0))}{d\omega_d A\eta^{\frac{-d+\alpha-1}{4}}e^{-\sqrt{\eta}R}R^{\frac{d-\alpha-3}{2}}} \le (1+\varepsilon),$$

concluding the proof. □

Concerning the second moment of μ_Φ, this time we have a completely different behaviour (see [**24**, Corollary 2.2]).

COROLLARY 4.3.3. *Let $\Phi \in \mathcal{CBF}$ be a driftless complete Bernstein function. Suppose there exist $\alpha \in (0,2]$, $\theta, \eta > 0$ such that*

$$\nu_\Phi(t) \sim \theta t^{-1-\frac{\alpha}{2}}e^{-\eta t}, \qquad t \to +\infty.$$

Then

$$\int_{\mathbb{R}^d} |h|^2 j_\Phi(|h|)dh < +\infty.$$

PROOF. Let us set, without loss of generality, $\theta = 1$. Let $R_1 > 0$ be such that for any $r > R_1$ it holds

$$\frac{1}{2} \le \frac{j_\Phi(r)}{Ar^{-\frac{d+\alpha+1}{2}}e^{-\sqrt{\eta}r}} \le \frac{3}{2},$$

where

$$A = \pi^{\frac{1-d}{2}}2^{\frac{\alpha-1-d}{2}}\eta^{\frac{d+\alpha+2}{4}}.$$

Now let us split the integral $\int_{\mathbb{R}^d}|h|^2 j_\Phi(|h|)dh$ in

$$\int_{\mathbb{R}^d} |h|^2 j_\Phi(|h|)dh = \int_{B_{R_1}(0)} |h|^2 j_\Phi(|h|)dh + \int_{B^c_{R_1}(0)} |h|^2 j_\Phi(|h|)dh.$$

The first integral is finite since μ_Φ is a Lévy measure. Concerning the second integral, we have

$$\begin{aligned}\int_{B^c_{R_1}(0)} |h|^2 j_\Phi(|h|)dh &\le \frac{3}{2}d\omega_d A\int_{R_1}^{+\infty} r^{\frac{d-\alpha+3}{2}-1}e^{-\sqrt{\eta}r}dr \\ &\le \frac{3}{2}d\omega_d A\eta^{-\frac{d-\alpha+3}{2}}\Gamma\left(\frac{d-\alpha+3}{2}\right) < +\infty,\end{aligned}$$

concluding the proof. □

Let us observe that the difference in the behaviour of the second moment will be one of the main reasons for which, if the zero-energy eigenvalue φ of a non-local Schřodinger operator decays like a power, then the best rate of decay of the potential is $|x|^{-2}$. This does not happen in the regularly varying case, as one obtains different decay rates depending on how fast does φ decay (see [**24**]).

4.4. Ryznar's decomposition for the massive relativistic Laplacian

In Chapter 1 we referred to two particular examples, respectively given by $\Phi_\alpha(\lambda)=\lambda^{\frac{\alpha}{2}}$ and $\Phi_{\alpha,m}(\lambda)=(\lambda+m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}}-m$ with $\alpha\in(0,2)$. The main difference of these two Bernstein functions can be seen in their respective Lévy measures. Indeed Φ_α is regularly varying at 0^+ of index $\frac{\alpha}{2}\in(0,1)$, thus the Lévy density ν_α is regularly varying at infinity. However $\Phi_{\alpha,m}$ is regularly varying at 0^+ of index 1. In particular the Lévy density $\nu_{\alpha,m}$ is exponentially light. As we have seen in the previous Sections, this leads to a completely different behaviour of the respective jump functions j_α and $j_{\alpha,m}$.
However, in [**128**], a link between $L_\alpha=-\Phi_\alpha(-\Delta)$ and $L_{\alpha,m}=-\Phi_{\alpha,m}(-\Delta)$ has been investigated in terms of a finite measure $\sigma_{\alpha,m}(|x|)dx$. We refer to L_α (which is classically called the **fractional Laplacian**) as the **massless relativistic Laplacian** to distinguish it from $L_{\alpha,m}$ which is the **massive relativistic Laplacian**. Now let us show the following result (see [**24**, Proposition 2.7]).

PROPOSITION 4.4.1. *Let $f:\mathbb{R}^d\to\mathbb{R}$ with $f\in L^\infty(\mathbb{R}^d)$. Denote $\mathrm{D}^2_h f(x)=f(x+h)-2f(x)+f(x-h)$. Suppose for any $x\in\mathbb{R}^d$ there exists $L_f(x)$ and $R_f(x)$ such that, for any $h\in B_{R_f(x)}(0)$, it holds $|\mathrm{D}^2_h f(x)|\le L_f(x)|h|^2$. Then it holds*

$$L_{\alpha,m}f(x)=L_\alpha f(x)-\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}^2_h f(x)\sigma_{\alpha,m}(|h|)dh,$$

where

$$\begin{aligned}\sigma_{\alpha,m}(r)&=\frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}\left(\frac{2^{\frac{d+\alpha}{2}-1}\Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha}}-\frac{m^{\frac{d+\alpha}{2\alpha}}K_{\frac{d+\alpha}{2}}(m^{\frac{1}{\alpha}}r)}{r^{\frac{d+\alpha}{2}}}\right)\\&=\frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}r^{d+\alpha}}\int_0^{m^{\frac{1}{\alpha}}r}w^{\frac{d+\alpha}{2}}K_{\frac{d+\alpha}{2}-1}(w)dw.\end{aligned}$$

PROOF. By Proposition 1.5.3, setting $\sigma_{\alpha,m}(r)=j_\alpha(r)-j_{\alpha,m}(r)$, we have

$$L_{\alpha,m}f(x)=\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}^2_h f(x)j_{m,\alpha}(|h|)dh=\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}^2_h f(x)(j_\alpha(|h|)-\sigma_{\alpha,m})dh$$

and

$$L_\alpha f(x)=\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}^2_h f(x)j_\alpha(|h|)dh.$$

Now let us show that the integral $\int_{\mathbb{R}^d}\mathrm{D}^2_h f(x)\sigma_{\alpha,m}(|h|)dh$ is well defined. Fix $x\in\mathbb{R}^d$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\mathrm{D}^2_h f(x)||\sigma_{\alpha,m}(|h|)|dh=&\int_{B_{R_f(x)}(0)}|\mathrm{D}^2_h f(x)||\sigma_{\alpha,m}(|h|)|dh\\&+\int_{B^c_{R_f(x)}(0)}|\mathrm{D}^2_h f(x)||\sigma_{\alpha,m}(|h|)|dh.\end{aligned}$$

Concerning the first integral, we have

$$\begin{aligned}(4.4.1)\quad &\int_{B_{R_f(x)}(0)} |\mathrm{D}_h^2 f(x)||\sigma_{\alpha,m}(|h|)|dh \\ &\qquad \le L_f(x)\int_{B_{R_f(x)}(0)} |h|^2(j_\alpha(|h|)+j_{\alpha,m}(|h|))dh < +\infty,\end{aligned}$$

since $j_\alpha(|h|)$ and $j_{\alpha,m}(|h|)$ are densities of Lévy measures. On the other hand, we have

$$\begin{aligned}&\int_{B_{R^c_f(x)}(0)} |\mathrm{D}_h^2 f(x)||\sigma_{\alpha,m}(|h|)|dh \\ &\qquad \le 4\,\|f\|_{L^\infty}\left(\mu_\alpha(B^c_{R_f(x)}(0))+\mu_{\alpha,m}(B^c_{R_f(x)}(0))\right) < +\infty.\end{aligned}$$

Thus we can conclude that

$$\begin{aligned}L_{\alpha,m}f(x) &= \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{D}_h^2 f(x)(j_\alpha(|h|)-\sigma_{\alpha,m})dh \\ &= L_\alpha f(x) - \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}_h^2 f(x)\sigma_{\alpha,m}(|h|)dh.\end{aligned}$$

Now let us determine $\sigma_{\alpha,m}(r)$. We have, by exploiting the values of ν_α and $\nu_{\alpha,m}$,

$$\sigma_{\alpha,m}(r) = \frac{\alpha 2^{-d}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}\int_0^{+\infty} t^{-1-\frac{d+\alpha}{2}}(1-e^{-m^{\frac{2}{\alpha}}t})e^{-\frac{r^2}{4t}}dt.$$

However, let us observe that

$$\frac{1-e^{-m^{\frac{2}{\alpha}}t}}{t} = \int_0^{m^{\frac{2}{\alpha}}} e^{-tz}dz,$$

thus, by Fubini's theorem, we obtain

$$\begin{aligned}\sigma_{\alpha,m}(r) &= \frac{\alpha 2^{-d}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}\int_0^{m^{\frac{2}{\alpha}}}\int_0^{+\infty} t^{-\frac{d+\alpha}{2}}e^{-\frac{r^2}{4t}-zt}dtdz \\ &= \frac{\alpha 2^{\frac{\alpha-d}{2}}r^{1-\frac{d+\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}\int_0^{m^{\frac{2}{\alpha}}} z^{\frac{d+\alpha}{4}-\frac{1}{2}}\frac{1}{2}\left(\frac{r}{2\sqrt{z}}\right)^{\frac{d+\alpha}{2}-1}\int_0^{+\infty} t^{-\frac{d+\alpha}{2}}e^{-z\left(t+\frac{r^2}{4zt}\right)}dtdz \\ &= \frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{r^{d+\alpha}\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}\int_0^{m^{\frac{2}{\alpha}}r} w^{\frac{d+\alpha}{4}}K_{\frac{d+\alpha}{2}-1}(w)dw.\end{aligned}$$

Finally, [**72**, Formula (5.52)] concludes the proof. □

We refer to the formula $L_{\alpha,m} = L_\alpha - G_{\alpha,m}$, where $G_{\alpha,m}f(x) = \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{D}_h^2 f(x)\sigma_{\alpha,m}(|h|)dh$, as **Ryznar's decomposition** (since it has been first explored in [**128**]).
By using the integral representation we obtained in the previous Proposition, we can show a monotonicity property of $\sigma_{\alpha,m}$ (see [**24**, Corollary 2.4]).

Corollary 4.4.2. *There exists $R > 0$ such that $\sigma_{\alpha,m}$ is decreasing in $(R, +\infty)$.*

PROOF. Let us first observe that being $\sigma_{m,\alpha}(r)$ is in $C^1(0,+\infty)$ and

$$\sigma'_{m,\alpha}(r) = -A(d+\alpha)r^{-d-\alpha-1}\int_0^{m^{\frac{1}{\alpha}}r} w^{\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}-1}(w)dw + Am^{\frac{d+\alpha+2}{\alpha}} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}-1}\left(m^{\frac{1}{\alpha}}r\right),$$

where

$$A = \frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}}.$$

Now, by [**72**, Formula (5.52)], we have

$$\sigma'_{m,\alpha}(r) = A\left(-(d+\alpha)r^{-d-\alpha-1}2^{\frac{\alpha+d+2}{2}}\Gamma\left(\frac{\alpha+d}{2}\right) + m^{\frac{d+\alpha}{2\alpha}} r^{-1-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}\left(m^{\frac{1}{\alpha}}r\right) + m^{\frac{d+\alpha+2}{\alpha}} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}-1}\left(m^{\frac{1}{\alpha}}r\right)\right).$$

By means of the asymptotics of the modified Bessel functions K_ν, we know there exists $R>0$ such that for any $r>R$ it holds

$$m^{\frac{d+\alpha}{2\alpha}} r^{-1-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}\left(m^{\frac{1}{\alpha}}r\right) + m^{\frac{d+\alpha+2}{\alpha}} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}-1}\left(m^{\frac{1}{\alpha}}r\right) \le \frac{d+\alpha}{2} r^{-d-\alpha-1}2^{\frac{\alpha+d-2}{2}}\Gamma\left(\frac{\alpha+d}{2}\right),$$

thus obtaining, for $r>R$,

$$\sigma'_{m,\alpha}(r) \le -A\frac{d+\alpha}{2} r^{-d-\alpha-1}2^{\frac{\alpha+d+2}{2}}\Gamma\left(\frac{\alpha+d}{2}\right) < 0,$$

concluding the proof. $\square$

Now let us introduce the measure $\Sigma_{\alpha,m}(dx) = \sigma_{\alpha,m}(|x|)dx$, which plays the role of μ_Φ in $G_{\alpha,m}$. Concerning the asymptotic behaviour of $\Sigma_{\alpha,m}(B^c_R(0))$ (together with the asymptotic behaviour of the second moment of the measure and the L^p norm of $\sigma_{\alpha,m}$), we can state the following Corollary, whose proof is omitted since it is identical to the one of Corollary 4.2.5 after observing that the Bessel term in $\sigma_{\alpha,m}$ is negligible as $r \to +\infty$.

COROLLARY 4.4.3. *The following properties hold:*

- $$\sigma_{\alpha,m}(r) \sim \frac{2^\alpha\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} r^{-\alpha-d}, \qquad r \to +\infty$$
- $$\Sigma_{\alpha,m}(B^c_R(0)) \sim \frac{d\omega_d 2^\alpha\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} R^{-\alpha}, \qquad R \to +\infty$$
- $$\int_{B_R(0)} |h|^2\sigma_{\alpha,m}(|h|)dh \sim \frac{d\omega_d \alpha 2^\alpha\Gamma\left(\frac{\alpha+d}{2}\right)}{(2-\alpha)\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} R^{2-\alpha}, \qquad R \to +\infty$$
- *For any* $p \in (1,+\infty)$

$$\|\sigma_{\alpha,m}(|h|)\|_{L^p(B^c_R(0))} \sim \left(\frac{d\omega_d}{(p-1)d+p\alpha}\right)^{\frac{1}{p}} \frac{\alpha 2^\alpha\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} R^{-qd-\alpha}, \qquad R \to +\infty$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Moreover, it has been shown in [**128**, Lemma 2] that $\Sigma_{\alpha,m}(\mathbb{R}^d) = m$ as $d \geq 2$. By using this information we easily have the following result.

PROPOSITION 4.4.4. *Let* $f \in L^\infty(\mathbb{R}^d)$ *with* $d \geq 2$. *Then* $G_{\alpha,m} f \in L^\infty(\mathbb{R}^d)$ *is well defined and*

$$\|G_{\alpha,m} f\|_{L^\infty} \leq 2m \|f\|_{L^\infty} .$$

The proof follows just by observing that $|\mathrm{D}_h^2 f(x)| \leq 4 \|f\|_{L^\infty}$.

4.5. Eigenvalues of the fractional integral on the sphere

Now let us move to the second problem in this chapter, that is to say the determination of the eigenvalues of some fractional integrals on the sphere. Let us consider $d \geq 2$ and denote by $\mathbb{S}^d$ the d-dimensional sphere in $\mathbb{R}^{d+1}$. Thus, as done in [**125**], we give the following definition.

DEFINITION 4.5.1. Let $u \in L^\infty(\mathbb{S}^{d-1})$. The **fractional integral** of u of order $\beta + d - 1$ (for $\beta \in (1-d, +\infty)$) is defined as

$$\mathcal{K}_\beta[u](\omega) = c_{d,\beta} \int_{\mathbb{S}^{d-1}} |\omega - \xi|^\beta u(\xi) d\,\mathcal{H}^{d-1}(\xi)$$

where $c_{d,\beta}$ is a normalizing constant and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

Here we will focus on the case $\beta > 0$, since the case $\beta \in (1-d, 0)$ has been already considered in [**61**] and then used in [**66**] and the case $\beta = 0$ is trivial.
To simplify the treatment, since it will not play any particular role, let us suppose $c_{d,\beta} = 2^{-\frac{\beta}{2}}$. Now we can investigate the eigenvalue problem for $\mathcal{K}_\beta$. To do this, we will make use of the Funke-Hecke formula (see, for instance, [**57**]). Indeed we have the following Proposition (see [**17**]).

PROPOSITION 4.5.1. *Let us denote by $\mathcal{S}_k$ the space of the k-th spherical harmonics. For each $k \geq 0$ and $Y_k \in \mathcal{S}_K$ it holds*

$$\mathcal{K}_\beta[Y_k](\omega) = \theta_{k,\beta} Y_k(\omega), \qquad \omega \in \mathbb{S}^{d-1},$$

where

$$\theta_{k,\beta} = (d-1)\omega_{d-1}(-1)^k \frac{2^{\frac{\beta+2d-4}{2}}\Gamma\left(\frac{\beta+d-1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{\beta+2}{2}\right)}{\Gamma\left(\frac{\beta+2}{2}-k\right)\Gamma\left(\frac{\beta+2d-2}{2}+k\right)}.$$

PROOF. Let us first observe that for any $\omega, \xi \in \mathbb{S}^{d-1}$ it holds

$$|\omega - \xi|^\beta = 2^{\frac{\beta}{2}}(1 - \omega \cdot \xi)^{\frac{\beta}{2}}$$

so that for any function $u \in L^\infty(S^{d-1})$ it holds

$$\mathcal{K}_\beta[u](\omega) = \int_{\mathbb{S}^{d-1}} K_\beta(\omega \cdot \xi) u(\xi) d\,\mathcal{H}^{d-1}(\xi) \tag{4.5.1}$$

where the kernel K_β is defined as

$$K_\beta(t) = (1-t)^{\frac{\beta}{2}}.$$

Since we have expressed $\mathcal{K}_\beta$ in terms of an integral kernel that depends only on the scalar product between two points of the sphere, we can use Funke-Hecke formula (see, for instance, [**57**]) to state that the eigenfunctions of $\mathcal{K}_\beta$ are the spherical harmonics and to determine the eigenvalues.
First of all, let us determine $\theta_{0,\beta}$. To do this, we can use [**72**, Formula 7.311.3] to

obtain, recalling that $\mathcal{S}_0 \sim \mathbb{R}$ is the space of constant functions on the sphere,

$$\begin{aligned}\theta_{0,\beta} &= \int_{\mathbb{S}^{d-1}} K_\beta(\omega\cdot\xi)d\mathcal{H}^{N-1}(\xi)\\ &= (d-1)\omega_{d-1}\int_{-1}^{1}(1-t)^{\frac{\beta}{2}}(1-t^2)^{\frac{d-3}{2}}dt\\ &= (d-1)\omega_{d-1}\frac{2^{\frac{\beta+2d-4}{2}}\Gamma\left(\frac{\beta+d-1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{\beta+2d-2}{2}\right)}.\end{aligned}$$

To determine the eigenvalues $\theta_{k,\beta}$ for $k\geq 1$ we have to distinguish between the cases $d=2$ and $d\geq 3$, since, as we will see in the following, we have to refer to different families of orthogonal polynomials.

Let us start with the second one, that is actually simpler. By the Funke-Hecke formula, we know we have to work with a normalization $P_{k,d}(t)$ of the Gegenbauer polynomials $C_k^{\frac{d-2}{2}}(t)$ (see [**72**] for the definition) defined on $t\in[-1,1]$, which constitute a family of orthogonal polynomials with respect to the measure on $[-1,1]$ given by $(1-t^2)^{\frac{d-3}{2}}dt$. In particular, according to [**111**, Page 16], we want $P_{k,d}(1)=1$, while we have (see [**72**, Formula 8.937.4]) $C_k^{\frac{d-2}{2}}(1)=\binom{d+k-3}{k}$. Thus we obtain

$$P_{k,N}(t)=\frac{k!(d-3)!}{(d+k-3)!}C_k^{\frac{d-2}{2}}(t).$$

By using the relation $C_k^{\frac{d-2}{2}}(-t)=(-t)^kC_k^{\frac{d-2}{2}}(t)$ and [**72**, Formula 7.311.3] we achieve

$$\begin{aligned}\theta_{k,\beta} &= (N-1)\omega_{d-1}\frac{k!(d-3)!}{(d+k-3)!}\int_{-1}^{1}(1-t)^{\frac{\beta}{2}}(1-t^2)^{\frac{d-3}{2}}C_k^{\frac{d-2}{2}}(t)dt\\ &= (d-1)\omega_{d-1}(-1)^k\frac{2^{\frac{\beta+2d-4}{2}}\Gamma\left(\frac{\beta+d-1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{\beta+2}{2}\right)}{\Gamma\left(\frac{\beta+2}{2}-k\right)\Gamma\left(\frac{\beta+2d-2}{2}+k\right)},\end{aligned}$$

that is the formula we are searching for.

Concerning the case $d=2$, let us first consider $\beta\neq 2m$ for any $m\in\mathbb{N}$, then we will extend the formula by continuity. First of all, by the Funke-Hecke formula, we know this time we have to work with the Chebyshev polynomials of the first kind $T_k(t)$, which are orthogonal with respect to $\frac{1}{\sqrt{1-t^2}}dt$. By using [**72**, Formula 7.354.6] one achieves

$$\begin{aligned}\theta_{k,\beta} &= 2\pi\int_{-1}^{1}\frac{(1-t)^{\frac{\beta}{2}}}{\sqrt{1+t^2}}T_k(t)dt\\ &= 2\pi\frac{\sqrt{\pi}2^{\frac{\beta}{2}}\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\beta+2}{2}\right)}{}_4F_3\left(-k,k,\frac{\beta+1}{2},\frac{\beta+2}{2};\frac{1}{2},\frac{\beta+2}{2},\frac{\beta+2}{2};1\right)\end{aligned}\tag{4.5.2}$$

where, according to [**72**, Formula 9.14.1], ${}_pF_q$ is the generalized hypergeometric series defined as

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{j=0}^{+\infty}\frac{\prod_{h=1}^{p}(a_h)_j}{\prod_{h=1}^{q}(b_h)_j}\frac{z^k}{k!}$$

and $(a)_j$ is the Pochhammer symbol defined as

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}.$$

By using the definition, it is easy to check that

$${}_4F_3\left(-k,k,\frac{\beta+1}{2},\frac{\beta+2}{2};\frac{1}{2},\frac{\beta+2}{2},\frac{\beta+2}{2};1\right) = {}_3F_2\left(-k,k,\frac{\beta+1}{2};\frac{\beta+2}{2},\frac{1}{2};1\right).$$

Now, by Saalschutz's Theorem (see [**139**, Section 2.3.1]) and by [**139**, Formula 2.3.2.5], we have

$${}_3F_2\left(-k,k,\frac{\beta+1}{2};\frac{\beta+2}{2},\frac{1}{2};1\right) = \frac{\left(\frac{1}{2}\right)_k\left(1+\frac{\beta}{2}-k\right)_k}{\left(\frac{1}{2}-k\right)_k\left(1+\frac{\beta}{2}\right)_k} = \frac{\left(-\frac{\beta}{2}\right)_k}{\left(\frac{\beta+2}{2}\right)_k}.$$

By definition it holds

$$\left(\frac{\beta+2}{2}\right)_k = \frac{\Gamma\left(\frac{\beta+2}{2}+k\right)}{\Gamma\left(\frac{\beta+2}{2}\right)},$$

while, by using Euler's reflection formula, we have

$$\left(-\frac{\beta}{2}\right)_k = \frac{\Gamma\left(k-\frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)} = (-1)^k\frac{\Gamma\left(\frac{\beta+2}{2}\right)}{\Gamma\left(\frac{\beta+2}{2}-k\right)}.$$

Substituting all these equalities back to (4.5.2) we obtain the desired formula. □

In particular let us remark that for $\beta = 2m$ for some $m \in \mathbb{N}$ it holds $\theta_{k,\beta} = 0$ for any $k \geq m+1$. In general we can show that the sequence $(\theta_{k,\beta})_{k\in\mathbb{N}_0}$ is always infinitesimal (see [**17**]). To do this let us first fix some notation:

$$C_\beta = (d-1)\omega_{d-1}2^{\frac{\beta+2d-4}{2}}\Gamma\left(\frac{\beta+d-1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{\beta+2}{2}\right),$$
$$\mu_{k,\beta} = \frac{1}{\Gamma\left(\frac{\beta+2}{2}-k\right)\Gamma\left(\frac{\beta+2d-2}{2}+k\right)}.$$

In this way we have that $\theta_{k,\beta} = (-1)^k C_\beta \mu_{k,\beta}$ and then we only have to show the following result.

PROPOSITION 4.5.2. *It holds* $\lim_{k\to+\infty}\mu_{k,\beta} = 0$

PROOF. By using Euler's reflection formula we have

$$\Gamma\left(\frac{\beta+2}{2}-k\right) = (-1)^{k+1}\frac{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{2-\beta}{2}\right)}{\Gamma\left(k-\frac{\beta}{2}\right)},$$

thus we can rewrite $\mu_{k,\beta}$ as

$$\mu_{k,\beta} = (-1)^{k+1}\frac{\Gamma\left(k-\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{2-\beta}{2}\right)\Gamma\left(\frac{\beta+2d-2}{2}+k\right)}.$$

Taking the absolute value and considering $k > \lfloor \frac{\beta}{2} \rfloor + 3$ we have

$$|\mu_{k,\beta}| = \frac{\Gamma\left(k - \frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\left|\Gamma\left(\frac{2-\beta}{2}\right)\right|\left(\frac{\beta+2d-4}{2}+k\right)\Gamma\left(\frac{\beta+2d-4}{2}+k\right)}.$$

Since Γ is an increasing function in $[2,+\infty)$ and $k - \frac{\beta}{2} < \frac{\beta+2d-4}{2} + k$, we have that

$$|\mu_{k,\beta}| \le \frac{1}{\Gamma\left(\frac{\beta}{2}\right)\left|\Gamma\left(\frac{2-\beta}{2}\right)\right|\left(\frac{\beta+2d-4}{2}+k\right)},$$

concluding the proof. $\square$

Moreover, we can actually exploit a recursive formula for $\mu_{k,\beta}$ (and then for $\theta_{k,\beta}$). Indeed we have the following result.

PROPOSITION 4.5.3. *It holds*

$$\mu_{k,\beta} = \frac{\frac{\beta}{2} - k}{\frac{\beta+2d-2}{2} + k}\mu_{k-1,\beta}.$$

In particular this implies that $|\theta_{k+1,\beta}| \le |\theta_{k,\beta}|$.

We omit the proof since it easily follows from the definition of $\mu_{k,\beta}$.

4.6. Eigenvalues of the Marchaud-type integral on the sphere

Now let us introduce another operator on Banach function spaces on the sphere.

DEFINITION 4.6.1. Let $u \in L^\infty(\mathbb{S}^{d-1})$. The **Marchaud-type fractional integral** of u of order $\beta + d - 1$ (for $\beta \in (d-1,+\infty)$) is defined as

$$\mathcal{I}_\beta[u](\omega) = 2\int_{\mathbb{S}^{d-1}} |\omega - \xi|^\beta (u(\omega) - u(\xi))\, d\,\mathcal{H}^{d-1}(\xi).$$

The name follows from the similarity of this operator with the Marchaud fractional derivative (see [**60**]). As for the fractional integral on the sphere, the eigenvalues of $\mathcal{I}_\beta$ have been already determined in [**61**] for $\beta \in (1-N, 0)$, thus let us focus on $\beta > 0$. First of all, let us observe that $\mathcal{I}_\beta$ can be rewritten in terms of $\mathcal{K}_\beta$ as

$$\mathcal{I}_\beta[u](\omega) = 2^{1+\frac{\beta}{2}}[\theta_{0,\beta}u(\omega) - \mathcal{K}_\beta[u](\omega)], \qquad \omega \in \mathbb{S}^{d-1}.$$

Thus, we obtain the eigenvalues $\lambda_{k,\beta}$ of $\mathcal{I}_\beta$ as

$$\lambda_{k,\beta} = 2^{1+\frac{\beta}{2}}[\theta_{0,\beta} - \theta_{k,\beta}],$$

where the respective eigenfunctions are given by the spherical harmonics $\mathcal{S}_k$. Moreover we have $\lim_{k\to+\infty} \lambda_{k,\beta} = 2^{1+\frac{\beta}{2}}\theta_{0,\beta}$. Finally, we also have the following formula:

$$\lambda_{k,\beta} = \widetilde{C}_\beta\left(1 + (-1)^{k+1}\prod_{j=0}^{k-1}\frac{\beta+2j}{\beta+2d-2+2j}\right)\mu_{0,\beta}$$

where $\widetilde{C}_\beta = 2^{1+\frac{\beta}{2}}C_\beta$.
Now we are interested in the distance between the maximum eigenvalue and the other ones. In particular we have the following result (see [**17**]).

PROPOSITION 4.6.1. *Let $\beta > 0$. Then it holds $\lambda_{1,\beta} = \max_{k\ge 0}\lambda_{k,\beta}$. Moreover, for any $k \ge 2$, it holds*

$$\lambda_{1,\beta} - \lambda_{k,\beta} \ge D_\beta > 0$$

where

$$D_\beta = \frac{\beta\mu_{0,\beta}\widetilde{C}_\beta}{\beta+2d-2}\widetilde{D}_\beta \tag{4.6.1}$$

and

$$\widetilde{D}_\beta = \begin{cases} 1 - \frac{2-\beta}{\beta+2d} & \beta \in (0,2) \\ 1 & \beta \in [2,4] \\ 1 - \frac{(\beta-2)(\beta-4)}{(\beta+2d)(\beta+2d+2)} & \beta > 4. \end{cases}$$

PROOF. Let us first observe that $\lambda_{0,\beta} = 0$ for any $\beta > 0$. Moreover, we have $\mu_{0,\beta} > 0$ and $\widetilde{C}_\beta > 0$, thus

$$\lambda_{1,\beta} = \widetilde{C}_\beta\left(1 + \frac{\beta}{\beta+2d-2}\right)\mu_{0,\beta} > 0.$$

For any $k \ge 2$ we have

$$\begin{aligned} \lambda_{1,\beta} - \lambda_{k,\beta} &= \widetilde{C}_\beta\frac{\beta}{\beta+2d-2}\mu_{0,\beta}\left(1 - (-1)^{k+1}\prod_{j=1}^{k-1}\frac{\beta-2j}{\beta+2d-2+2j}\right) \\ &\ge \widetilde{C}_\beta\frac{\beta}{\beta+2d-2}\mu_0^\beta\left(1 - \prod_{j=1}^{k-1}\frac{|\beta-2j|}{\beta+2d-2+2j}\right) \ge 0, \end{aligned}$$

since $\frac{|\beta-2j|}{\beta+2d-2+2j} \le 1$ for any $j \le k-1$. Thus we get that $\lambda_{1,\beta} = \max_{k\ge 0}\lambda_{k,\beta}$. Now let us show that $\lambda_{1,\beta} - \lambda_{k,\beta} \ge D_\beta > 0$ where D_β is defined in Equation (4.6.1). To do this we have to distinguish between the following five cases:

(*a*) $\beta \in (0,2)$;
(*b*) $\beta = 2$;
(*c*) $\beta \in (2,4)$;
(*d*) $\beta = 4$;
(*e*) $\beta > 4$.

First of all, let us consider case (*a*). If $\beta \in (0,2)$ then we can show that $(\lambda_{k,\beta})_{k\ge 1}$ is a decreasing sequence. Indeed, we have, since $\beta - 2j < 0$,

$$\begin{aligned} \lambda_{k,\beta} - \lambda_{k+1,\beta} = (-1)^{k+1}\widetilde{C}_\beta\frac{\beta}{\beta+2d-2}&\left(\prod_{j=1}^{k-1}\frac{\beta-2j}{\beta+2d-2+2j}\right) \\ &\times\left(1 + \frac{2k-\beta}{\beta+2d-2+2k}\right)\mu_0^\beta \ge 0. \end{aligned}$$

Hence we have that, for $\beta \in (0,2)$, it holds

$$\lambda_{1,\beta} - \lambda_{k,\beta} \ge \lambda_{1,\beta} - \lambda_{2,\beta} = \frac{\beta}{\beta+2d-2}\widetilde{C}_\beta\mu_{0,\beta}\left(1 - \frac{2-\beta}{\beta+2d}\right).$$

Concerning case (*b*), we have $\lambda_{k,\beta} = \widetilde{C}_\beta\mu_{0,\beta}$ for any $k \ge 2$ and then

$$\lambda_{1,\beta} - \lambda_{k,\beta} \ge \lambda_{1,\beta} - \widetilde{C}_\beta\mu_{0,\beta} = \frac{1}{d}\widetilde{C}_\beta\mu_{0,\beta}.$$

Let us now consider $\beta > 2$, which is common for cases (c), (d) and (e). Exploiting $\lambda_{2,\beta}$ we have

$$\lambda_{2,\beta} = \widetilde{C}_\beta \left(1 - \frac{\beta(\beta-2)}{(\beta+2d)(\beta+2d-2)}\right)\mu_{0,\beta} \le \widetilde{C}_\beta \mu_{0,\beta}.$$

Consider case (c) and let us show that the sequence $(\lambda_{k,\beta})_{k\ge 2}$ is increasing. To do this, let us observe that

$$\lambda_{k+1,\beta} - \lambda_{k,\beta} = (-1)^k \widetilde{C}_\beta \frac{\beta(\beta-2)}{(\beta+2d-2)(\beta+2d)} \left(\prod_{j=2}^{k-1} \frac{\beta-2j}{\beta+2d-2+2j}\right)$$
$$\times \left(1 + \frac{2k-\beta}{\beta+2d-2+2k}\right)\mu_0^\beta \ge 0.$$

Let us also recall that $\lambda_{k,\beta} \to \widetilde{C}_\beta \mu_{0,\beta}$ as $k \to +\infty$ to achieve that, for any $k \ge 2$

$$\lambda_{1,\beta} - \lambda_{k,\beta} \ge \lambda_{1,\beta} - \widetilde{C}_\beta \mu_{0,\beta} = \frac{\beta \mu_{0,\beta} \widetilde{C}_\beta}{\beta+2d-2}.$$

Concerning case (d), for any $k \ge 3$ it holds $\lambda_{k,\beta} = \widetilde{C}_\beta \mu_{0,\beta}$ obtaining again the previous estimate.
Finally, for case (e), we have

$$\lambda_{3,\beta} = \widetilde{C}_{n,\beta}\left(1 + \frac{\beta(\beta+2)(\beta+4)}{(\beta+2d-2)(\beta+2d)(\beta+2d+2)}\right)\mu_{0,\beta} > \widetilde{C}_{n,\beta}\mu_{0,\beta}$$

while, with the same strategy as before, we can show that $(\lambda_{k,\beta})_{k\ge 3}$ is a decreasing sequence. Hence, we have, for any $k \ge 2$,

$$\lambda_{1,\beta} - \lambda_{k,\beta} \ge \lambda_{1,\beta} - \lambda_{3,\beta} = \frac{\beta \mu_{0,\beta} \widetilde{C}_\beta}{\beta+2d-2}\left(1 - \frac{(\beta-2)(\beta-4)}{(\beta+2d)(\beta+2d+2)}\right),$$

concluding the proof. □

This property reveals its usefulness as one tries to show a Fuglede-type results for functionals giving the moments of the length of random segments (see [**17**]).

4.7. Moments of the length of random segments in a ball

Now let us exploit the link between the eigenvalues of $\mathcal{I}_\beta$ and the length of random segments with vertices in compact sets. To do this, let us introduce another functional.

Definition 4.7.1. For any $u \in L^\infty(\mathbb{S}^{d-1})$ and $\beta \in (-d, +\infty)$ let us denote

$$[u]_\beta^2 = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\omega - \xi|^\beta |u(\omega) - u(\xi)|^2 d\,\mathcal{H}^{d-1}(\omega) d\,\mathcal{H}^{d-1}(\xi).$$

This is actually a Besov semi-norm on $\mathbb{S}^{d-1}$ if $\beta \in (-d, +\infty)$.
First of all, let us stress out that $[u]_\beta^2$ can be expressed in terms of the integral of the Marchaud-type integral.

Lemma 4.7.1. *Let $u \in L^\infty(\mathbb{S}^{d-1})$ and $\beta \in (1-d, +\infty)$. Then it holds*

$$[u]_\beta^2 = \int_{\mathbb{S}^{d-1}} \mathcal{I}_\beta[u](\omega) u(\omega) d\,\mathcal{H}^{d-1}(\omega). \tag{4.7.1}$$

PROOF. Let us just observe that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}&|\omega-\xi|^{\beta}|u(\omega)-u(\xi)|^2 d\,\mathcal{H}^{d-1}(\omega)d\,\mathcal{H}^{d-1}(\xi)\\ &=\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta}(u(\omega)-u(\xi))u(\omega)d\,\mathcal{H}^{d-1}(\omega)d\,\mathcal{H}^{d-1}(\xi)\\ &+\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta}(u(\xi)-u(\omega))u(\xi)d\,\mathcal{H}^{d-1}(\omega)d\,\mathcal{H}^{d-1}(\xi)\end{aligned}$$

thus Equation (4.7.1) follows from Fubini's theorem and the definition of $\mathcal{I}_\beta[u]$. □

Now let us give some other definitions.

DEFINITION 4.7.2. Let $K \subseteq R^d$ be a compact set. We say that the random variable $X : \Omega \to \mathbb{R}^d$ is **uniform on** K if its law is given by $\frac{1}{|K|}dx$.
We define a **random segment** $[P_0, P_1]$ on K as the segment of vertices P_0, P_1 where P_0, P_1 are random variables that are uniform on K and independent each other. In particular, the length $|P_1 - P_0|$ is a 1-dimensional random variable.
Finally, we define $\mathcal{G}_\beta(K) = \mathbb{E}[|P_1 - P_0|^\beta]$ where $[P_0, P_1]$ is a random segment on K.

Let us recall that it can be shown (see [**120**]) that $\mathcal{G}_\beta(K) \ge \mathcal{G}_\beta(B)$ for any compact set $K \subseteq \mathbb{R}^d$, where B is a ball in $\mathbb{R}^d$ such that $|B| = |K|$. In particular equality holds if and only if K is a ball in $\mathbb{R}^d$. Defining $\mathcal{D}_\beta(K) = \mathcal{G}_\beta(K)-\mathcal{G}_\beta(B)$ the **β-moment deficit** and $\delta(K) = \inf_{x\in\mathbb{R}^N} |K\Delta B_{r_K}(x)|$ the **Fraenkel asymmetry**, where $r_K > 0$ is chosen in such a way that $|K| = |B_{r_k}(x)|$, in [**63**] it has been shown that there exists a constant $C(d, \beta)$ such that

$$\mathcal{D}_\beta(K) \ge C(d, \beta)\delta(K)$$

for any compact set $K \subseteq \mathbb{R}^d$. This means that $\mathcal{D}_\beta(K)$ is in some sense a way to measure the *distance* between the shape of any compact set K and a ball. In [**17**] we provide a different proof that relies on the Marchaud-type fractional integral. Here, we do not want to give the proof of the previous inequality, but a first hint on the link between the Marchaud-type fractional integral and the shape functional $\mathcal{G}_\beta$, via the following result (see [**17**]).

PROPOSITION 4.7.2. *Fix $\beta > 0$. Then it holds*

$$\lambda_{1,\beta} = \frac{(\beta+d)(\beta+2d)}{d}\omega_d\,\mathcal{G}_\beta(B), \tag{4.7.2}$$

where $B \subseteq \mathbb{R}^d$ is the unit ball.

PROOF. Let us first observe that, by definition of uniform distribution one has, for any compact set $K \subseteq \mathbb{R}^d$

$$\mathcal{G}_\beta(K) = \frac{1}{|K|^2}\int_{K^2} |x-y|^\beta dxdy. \tag{4.7.3}$$

In particular, we have

$$\mathcal{G}_\beta(B) = \frac{1}{\omega_d}\int_B |x-y|^\beta dx, \tag{4.7.4}$$

by exploiting the fact that the integral $\int_B |x-y|^\beta dx$ is constant with respect to y (by the fact that B is fixed by rotations and the Lebesgue measure is invariant by isometries).

Now let us consider $\mathcal{S}_1$ the space of first spherical harmonic function. A basis for $\mathcal{S}_1$ is given by the coordinate functions $\omega \mapsto \omega_i$ for $i = 1, \dots, d$. In particular, it holds, by Equation (4.7.1),

$$[\omega_i]_\beta^2 = \lambda_{1,\beta} \int_{\mathbb{S}^{d-1}} \omega_i^2 d\,\mathcal{H}^{d-1}(\omega).$$

On the other hand, by definition,

$$[\omega_i]_\beta^2 = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\omega - \xi|^\beta |\omega_i - \xi_i|^2 d\,\mathcal{H}^{d-1}(\omega) d\,\mathcal{H}^{d-1}(\xi),$$

obtaining the identity

$$\lambda_{1,\beta} \int_{\mathbb{S}^{d-1}} \omega_i^2 d\,\mathcal{H}^{d-1}(\omega) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\omega - \xi|^\beta |\omega_i - \xi_i|^2 d\,\mathcal{H}^{d-1}(\omega) d\,\mathcal{H}^{d-1}(\xi),$$

that holds true for any $i = 1, \dots, d$. Thus, summing over i, we get

$$\lambda_{1,\beta} d\omega_d = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\omega - \xi|^{\beta+2} d\,\mathcal{H}^{d-1}(\omega) d\,\mathcal{H}^{d-1}(\xi).$$

Now let us define the auxiliary function $L : \mathbb{S}^{d-1} \to \mathbb{R}$ as

$$L(\xi) = \int_{\mathbb{S}^{d-1}} |\omega - \xi|^{\beta+2} d\,\mathcal{H}^{d-1}(\omega).$$

Let us also define $\ell(z) = \frac{1}{\beta+2}|z|^{\beta+2}$ so that $\nabla \ell(z) = |z|^\beta z$. This leads to

$$L(\xi) = \int_{\mathbb{S}^{d-1}} \nabla \ell(\omega - \xi) \cdot \omega d\,\mathcal{H}^{d-1}(\omega) - \int_{\mathbb{S}^{d-1}} \nabla \ell(\omega - \xi) \cdot \xi d\,\mathcal{H}^{d-1}(\omega). \tag{4.7.5}$$

Now let us denote by ∇_τ the tangential gradient and with $\frac{\partial}{\partial \nu}$ the normal derivative (with respect to the sphere $\mathbb{S}^{d-1}$). Hence we can split the gradient of ℓ as

$$\nabla \ell(\omega - \xi) = \nabla_\tau \ell(\omega - \xi) + \frac{\partial \ell}{\partial \nu(\omega)}(\omega - \xi)\omega$$

and then, multiplying by $\omega = \nu(\omega)$ on $\mathbb{S}^{d-1}$, we have $\frac{\partial \ell}{\partial \nu}(\omega - \xi) = \nabla \ell(\omega - \xi) \cdot \omega$. Thus, also recalling that $\omega^2 = 1$ and $\nabla(\omega \cdot \xi) = \xi$, we have, by Equation (4.7.5),

$$L(\xi) = \int_{\mathbb{S}^{d-1}} \frac{\partial \ell}{\partial \nu}(\omega - \xi)(1 - \omega \cdot \xi) d\,\mathcal{H}^{d-1}(\omega) - \int_{\mathbb{S}^{d-1}} \nabla_\tau \ell(\omega - \xi) \cdot \nabla_\tau(\omega \cdot \xi) d\,\mathcal{H}^{d-1}(\omega). \tag{4.7.6}$$

Now we need to study the two integrals separately. Let us define the functions $\mathcal{A}, \mathcal{B} : \mathbb{S}^{d-1} \to \mathbb{R}$ as

$$\mathcal{A}(\xi) = \int_{\mathbb{S}^{d-1}} \frac{\partial \ell}{\partial \nu}(\omega - \xi)(1 - \omega \cdot \xi) d\,\mathcal{H}^{d-1}(\omega)$$
$$\mathcal{B}(\xi) = \int_{\mathbb{S}^{d-1}} \nabla_\tau \ell(\omega - \xi) \cdot \nabla_\tau(\omega \cdot \xi) d\,\mathcal{H}^{d-1}(\omega).$$

Concerning $\mathcal{A}$, we have, by divergence theorem,

$$\mathcal{A}(\xi) = \int_B \nabla \ell(\omega - \xi) \cdot \nabla(1 - \omega \cdot \xi) d\omega + \int_B \Delta \ell(\omega - \xi)(1 - \omega \cdot \xi) d\omega.$$

Observing that $\Delta\ell(\omega-\xi)=(\beta+d)|\omega-\xi|^\beta$, $\nabla(1-\omega\cdot\xi)=-\xi$ and recalling that $\nabla\ell(\omega-\xi)=|\omega-\xi|^\beta(\omega-\xi)$, it holds

$$\begin{aligned}\mathcal{A}(\xi)&=\int_B|\omega-\xi|^\beta(1-\omega\cdot\xi)d\omega+(\beta+d)\int_B|\omega-\xi|^\beta(1-\omega\cdot\xi)d\omega\\&=(\beta+d+1)\int_B|\omega-\xi|^\beta(1-\omega\cdot\xi)d\omega.\end{aligned}$$

Concerning $\mathcal{B}$, by integration by parts on $\mathbb{S}^{d-1}$, we have

$$\mathcal{B}(\xi)=-\int_{\mathbb{S}^{d-1}}\ell(\omega-\xi)\Delta_{\mathbb{S}^{d-1}}(\omega\cdot\xi)d\,\mathcal{H}^{d-1}(\omega)$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. In particular, since the first non-trivial eigenvalue of $-\Delta_{\mathbb{S}^{d-1}}$ is $d-1$ and it is achieved for functions in $\mathcal{S}_1$, we have $-\Delta_{\mathbb{S}^{d-1}}(\omega\cdot\xi)=(d-1)\omega\cdot\xi$. Thus, by also using the definition of ℓ,

$$\mathcal{B}(\xi)=\frac{d-1}{\beta+2}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta+2}\omega\cdot\xi d\,\mathcal{H}^{d-1}(\omega).$$

Since $L(\xi)=\mathcal{A}(\xi)+\mathcal{B}(\xi)$, we get

$$L(\xi)=(\beta+d+1)\int_B|\omega-\xi|^\beta(1-\omega\cdot\xi)d\omega-\frac{d-1}{\beta+2}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta+2}\omega\cdot\xi d\,\mathcal{H}^{d-1}(\omega).$$

Integrating both sides on $\mathbb{S}^{d-1}$ and using Fubini's theorem, we have

$$\begin{aligned}d\omega_d\lambda_{1,\beta}=(\beta+d+1)&\int_B\int_{\mathbb{S}^{d-1}}|\omega-\xi|^\beta(1-\omega\cdot\xi)d\,\mathcal{H}^{d-1}(\xi)d\omega\\&-\frac{d-1}{\beta+2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta+2}\omega\cdot\xi d\,\mathcal{H}^{d-1}(\omega)d\,\mathcal{H}^{d-1}(\xi).\end{aligned}$$

Concerning the first integral, we achieve, by using divergence theorem,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^\beta(1-\omega\cdot\xi)d\,\mathcal{H}^{d-1}(\xi)&=\int_{\mathbb{S}^{d-1}}\nabla\ell(\xi-\omega)\cdot\xi d\,\mathcal{H}^{d-1}(\xi)\\&=\int_B\Delta\ell(\xi-\omega)d\xi\\&=(\beta+d)\int_B|\omega-\xi|^\beta d\xi=\omega_d(\beta+d)\,\mathcal{G}_\beta(B)\end{aligned}$$

and then

$$\begin{aligned}\lambda_{1,\beta}=&\frac{\omega_d(\beta+d)(\beta+d+1)}{d}\,\mathcal{G}_\beta(B)\\&-\frac{d-1}{d\omega_d(\beta+2)}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^{\beta+2}\omega\cdot\xi d\,\mathcal{H}^{d-1}(\omega)d\,\mathcal{H}^{d-1}(\xi).\end{aligned}\tag{4.7.7}$$

Now we need to identify the second integral in terms of $\mathcal{G}_\beta(B)$. To do this, let us set $G_1(z)=|z|^\beta$ and, observing that $\nabla G_1(z)=\beta|z|^{\beta-2}z$, we achieve, by divergence theorem,

$$\begin{aligned}\int_B |\omega-\xi|^\beta d\omega &= \frac{1}{\beta}\int_B \nabla G_1(\omega-\xi)\cdot(\omega-\xi)d\omega\\ &= -\frac{1}{\beta}\int_B G_1(\omega-\xi)\mathrm{div}(\omega-\xi)d\omega\\ &\quad+\frac{1}{\beta}\int_{\mathbb{S}^{d-1}} G_1(\omega-\xi)(\omega-\xi)\cdot\omega d\,\mathcal{H}^{d-1}(\omega)\\ &= -\frac{d}{\beta}\int_B |\omega-\xi|^\beta+\frac{1}{\beta}\int_{\mathbb{S}^{d-1}}|\omega-\xi|^\beta(\omega-\xi)\cdot\omega d\,\mathcal{H}^{d-1}(\omega).\end{aligned}$$

Integrating both sides in B, dividing by ω_d^2 and using Fubini's theorem, we have

$$(\beta+d)\,\mathcal{G}_\beta(B)=\frac{1}{\omega_d^2}\int_{\mathbb{S}^{d-1}}\int_B |\xi-\omega|^\beta(\omega-\xi)\cdot\omega d\xi d\,\mathcal{H}^{d-1}(\omega).$$

Setting $G_2(z)=|z|^{\beta+2}$ and arguing as before, by divergence theorem, we get

$$\begin{aligned}\int_B |\xi-\omega|^\beta(\omega-\xi)\cdot\omega d\xi &= -\frac{1}{\beta+2}\int_B \nabla G_2(\xi-\omega)\cdot\omega d\xi\\ &= -\frac{1}{\beta+2}\int_{\mathbb{S}^{d-1}} G_2(\xi-\omega)\xi\cdot\omega d\,\mathcal{H}^{d-1}(\xi)\\ &= -\frac{1}{\beta+2}\int_{\mathbb{S}^{d-1}}|\xi-\omega|^{\beta+2}\xi\cdot\omega d\,\mathcal{H}^{d-1}(\xi).\end{aligned}$$

Hence, we finally obtain

$$-(\beta+2)(\beta+d)\omega_d^2\,\mathcal{G}_\beta(B)=\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|\xi-\omega|^{\beta+2}\xi\cdot\omega d\,\mathcal{H}^{d-1}(\xi)d\,\mathcal{H}^{d-1}(\omega). \tag{4.7.8}$$

Formula (4.7.2) follows by using identity (4.7.8) in Equation (4.7.7). □

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The time is gone,
the song is over,
thought I'd something more to say

Pink Floyd
Time
Lyrics by Roger Waters