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Ph.D. Thesis in Physics

Semi-leptonic decays of B into charmed mesons in a quark model

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Contents

Introduction

One of the main problems of particle's physicists is to know the parameters of the Standard Model which cannot be fixed by first principles. Between these, the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix are very interesting because of their connection with the Standard Model description of the observed CP violation effects.
The more simple way to fix some of these parameters, in particular their absolute values, is studying weak semi-leptonic decays of mesons; in fact the probabilities of these transitions are proportional to one of the CKM elements.

To evaluate the corresponding amplitudes we have to estimate the hadronic matrix elements which describe mesons's weak transitions. Generally, taking into account Lorentz invariance and transformation properties under CP of these matrix elements, it is possible to express them in terms of some functions of q^2, said form factors. So the problem is deferred to the evaluation of these new functions.

The Standard Model doesn't give us a method to get their analytical or numerical expressions. So it is necessary to introduce some approximation or models.

The simplest models we can talk about are the constituent quark models. These are inspired to quantum mechanic; they consider a meson as a bound state of its valence quarks. These models are not a direct consequence of QCD, nevertheless, the results they get are very good from a phenomenological point of view.
Moreover, these models have allowed to understand new symmetries of QCD, that is the heavy quark symmetry (HQS). And, as a consequence, the heavy quark effective theory (HQET) has been construct. This theory is the QCD developed as a series of the parameter Λ_{QCD}/m_Q, where m_Q is the mass of an heavy quark.

In this approach, it is possible to fix the form factors normalization in the zero recoil point. So this represents a way to test models predictions and to justify them theoretically.

In our work we have studied a constituent quark model and we have applied it to evaluate the form factors which describe the B decay into charmed meson states.
This thesis is organized as follows: in the first two chapters we review the theoretical information necessary to study the semi-leptonic meson decay, the third chapter is centered on the foundation of the model we use and we give the results on the transitions $B \to D^{(*)} l\nu$, in the fourth chapter we put our attention on the B decays in even parity charmed states, in the fifth chapter we compare our model with HQET prediction, this comparison is completed in the last chapter where we study the processes with a tensor charmed meson in the final state.

Chapter 1

The Standard Model

1.1 Particles Classification

Accelerator particle physics experiments show the existence of hundreds of particles.
One way to classify them is using the interaction they are sensible to. This idea gives rise to the classification below:

- hadrons: particles with strong interactions;
- leptons: particles without strong interactions;
- gauge bosons: particles which mediate the interactions.

While leptons and gauge bosons are believed fundamental particles, hadrons show a composite nature. The building blocks they are made of are called quarks. In terms of the number of quarks into the hadrons, we divide them into mesons (with two valence quarks) and baryons (if the valence quarks are three).

In order to take into account the conservation of some quantum numbers, quarks and leptons can be further classified into families structure.
Leptons are:

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \tag{1.1}$$

and quarks are classified in three families:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \tag{1.2}$$

To describe successfully the interactions between these particles, in particular to take into account the maximal parity violation in the weak interactions, it is useful to consider separately left-handed isospin doublets:

$$L = \begin{pmatrix} \nu_{lL} \\ l_L \end{pmatrix}; \tag{1.3}$$

$$L_1 = \begin{pmatrix} d_L \\ u_L \end{pmatrix}, \ L_2 = \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \ L_3 = \begin{pmatrix} b_L \\ t_L \end{pmatrix} \tag{1.4}$$

and right-handed singlets:

$$\nu_R, \ l_R, \tag{1.5}$$

$$u_R, \ c_R, \ t_R, \ d_R, \ s_R, \ b_R. \tag{1.6}$$

The projector operator on left-handed and right-handed components are:

$$P_L = \frac{1}{2}(1 - \gamma_5) \tag{1.7}$$

$$P_R = \frac{1}{2}(1 + \gamma_5) \tag{1.8}$$

Applied to massless particles, these operators project them on elicity eigenstates.

In order to reflect the symmetries shown by the interactions under question, each particle is characterized by some quantum numbers reported in tab.[1.1].

1.2 The Standard Model

The best theory developed to describe strong and elctroweak interactions between elementary particles is the Standard Model. It is a gauge quantum field theory.
The lagrangian of the Standard Model is made of two parts: one describes electroweak interactions and is the Glashow-Weinberg-Salam model ([1],[2],[3]), and the other is the quantum

	I(isospin)	I_3	Q(electric charge)	Y(hypercharge)	B(barionic n^o)
ν_L	1/2	1/2	0	-1/2	0
ν_R	0	0	0	0	0
l_L	1/2	-1/2	-1	-1/2	0
l_R	0	0	-1	-1	0
u_L, c_L, t_L	1/2	1/2	2/3	1/6	1/3
d_L, s_L, b_L	1/2	-1/2	-1/3	1/6	1/3
u_R, c_R, t_R	0	0	2/3	2/3	1/3
d_R, s_R, b_R	0	0	1/3	-1/3	1/3

Table 1.1: Note that if neutrinos are massless, the right-handed one doesn't exist.

chromo dynamics (QCD) for the strong interactions ([4], [5]).
Because of the low mass of the elementary particles, gravitation doesn't give effects comparable to the other forces; so the Standard Model does not include this interaction.

The symmetries that characterize the Standard Model are: $SU(2)$ of isospin I, $U(1)$ of hypercharge Y and $SU(3)$ of color C.
In particular, the part of the theory that describes the electroweak interaction has to be invariant under $SU(2)$x$U(1)$, while QCD has the symmetry $SU(3)$.
In the following subsections we'll see some details about the Standard Model.

1.2.1 The Glashow-Weinberg-Salam model.

Glashow-Weinberg-Salam have written the lagrangian reported below to describe electroweak interactions is the following:

$$\begin{aligned} L \;=\; & \sum_{k=1}^{3} i(\overline{L}^k \not{D} L^k + \overline{R}^k \not{D} R^k) - \frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)^+ (D^\mu \phi) - \\ & \mu^2 \phi^+ \phi - \lambda(\phi^+\phi)^2 - \sum_{j,k=1}^{3} [g_{kl}(\overline{L}_{lk} \phi R_{lk} + \phi^+ \overline{R}_{lk} L_{lk}) + h.c.] \end{aligned} \tag{1.9}$$

where

$$D_\mu = \partial_\mu - i\frac{g}{2}\sigma_j W^j_\mu - ig' Y B_\mu \tag{1.10}$$

$$W^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + i\epsilon^{jkl} W^k_\mu W^l_\nu \tag{1.11}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.12}$$

and the σ_j (j=1,2,3) are Pauli matrices.

The term

$$L = \sum_{k=1}^{3} i(\overline{L}^k \not{D} L^k + \overline{R}^k \not{D} R^k) \tag{1.13}$$

represents the free matter fields and their interactions with the gauge field, whose self interactions are contained in the term:

$$-\frac{1}{4} W^i_{\mu\nu} W^{\mu\nu}_i - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{1.14}$$

In the Standard Model the lonely way to introduce the field masses without lose the renormalizability of the theory and its gauge invariance is the spontaneous symmetry breaking mechanism [6]. To implement it, we introduce the Higgs field:

$$\phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} \tag{1.15}$$

which in general, after a gauge transformation, can be written as:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \sigma \end{pmatrix} \tag{1.16}$$

The charge conjugate of this field is defined as:

$$\tilde{\phi} = -\imath \sigma_2 \phi^* \tag{1.17}$$

Using the Higgs field we can write the Higgs Lagrangian

$$L_{Higgs} = (D_\mu \phi)^+ (D^\mu \phi) - \mu^2 \phi^+ \phi - \lambda (\phi^+ \phi)^2 \tag{1.18}$$

which gives the masses of the gauge fields and of ϕ, and the Yukawa terms:

$$L_Y = -\sum_{j,k=1}^{3} [g_{lk}(\overline{L}_{lk} \phi R_{lk} + \phi^+ \overline{R}_{lk} L_{lk}) + h.c.] \tag{1.19}$$

to explain the masses of the matter fields respectively.

Developing the various terms of the Higgs lagrangian, we see that the massive fields are $W_{1,2}$

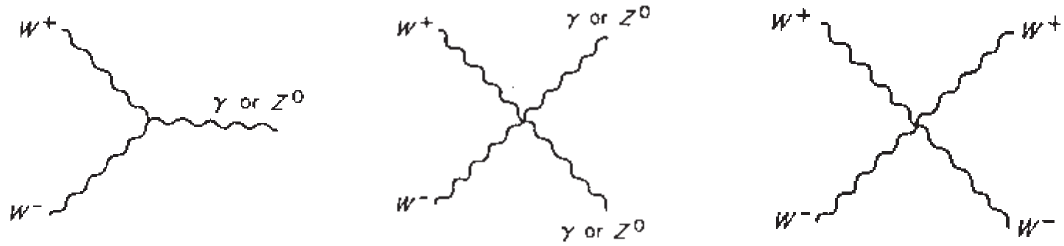


Figure 1.1: Interactions between gauge bosons.

and the combination $gW_3 - g'B$. So we understand that the physical particles are associated to the fields:

$$Z_\mu = W_{3\mu} cos\theta_W - B_\mu sin\theta_W \quad (1.20)$$

$$A_\mu = W_{3\mu} sin\theta_W + B_\mu cos\theta_W \quad (1.21)$$

where θ_W is the Weinberg angle [2].

Moreover we define:

$$W_\mu = \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \quad (1.22)$$

because this field, and its conjugate, are the ones connected to the charged currents.

The lagrangian shows that A_μ is massless, so we can associate it to the photon. This is a constraint on the coupling constants:

$$gsen\theta_w = g'cos\theta_w = e \quad (1.23)$$

because we know that the electromagnetic current and the photon are coupled by the constant e (the electromagnetic electron charge).

The lagrangian we have considered describes the interaction's eigenstates; it is more convenient to rewrite it in terms of the mass eigenstates.
Then, for each state (left-right handed), we introduce a unitary matrix $V_{(L,R)}$ and we define the Cabibbo-Kobayashi-Maskawa (CKM) matrix as:

$$V_{CKM} = V_L^+ V_R \quad (1.24)$$

which is unitary.

If N is the number of families of fermions, the V_{CKM} has $N^2-(2N-1)$ independent parameters. In particular they can be separated into N_A angles and N_{ph} phases:

$$N_A = \frac{1}{2}N(N-1) \quad (1.25)$$

$$N_{ph} = \frac{1}{2}(N-1)(N-2) \quad (1.26)$$

The existence of a phase in the CKM matrix means that CP violation can be described by the Standard Model. So, if $N \geq 3$, CP is not a symmetry of the theory.

With three generations of fundamental fermions, the CKM matrix can be parameterized with four parameters: three Euler angles and one phase.
After the experimental observation that the b quark decays predominantly to the charm ($|V_{cb}| >> |V_{ub}|$), Wolfenstein [7] noticed that $|V_{cb}| \simeq |V_{us}|^2$ and proposed to use $|V_{us}| = \lambda \simeq 0.22$ as an expansion parameter for the elements of the CKM matrix.

$$\begin{aligned} V_{CKM} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &\simeq \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \end{aligned} \quad (1.27)$$

where A is of order unity and ρ and η should be smaller than one.
In this parametrization the role of the phase is played by η. Its non-zero value implies a CP violation.

The unitarity condition of the V_{CKM} can be represented by triangular relations. One of the most interesting for us is the one related to the B decays:

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \quad (1.28)$$

Normalized to $V_{ub}^*V_{cd}$, this relation can be represented as the triangle in fig.[1.2].

Today it is interesting to measure with high precision the elements of the CKM matrix because a test of the triangular relations should assure us about the right number of lepton families in nature.

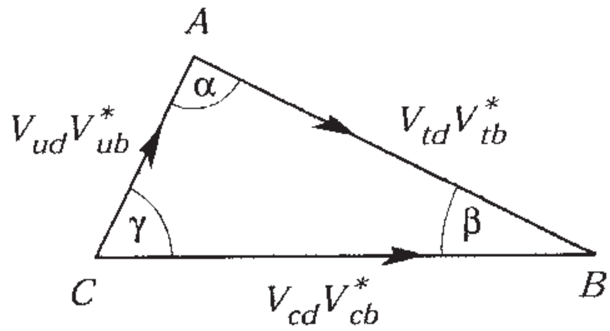


Figure 1.2: Unitarity triangle.

A possible way to measure such quantities is looking at the weak semi-leptonic decays of mesons.

The actual values estimated for the parameters A, λ, η and ρ are in tab. 1.2.

A	$0.818^{+0.007}_{-0.017}$
λ	0.2272 ± 0.0010
ρ	$0.221^{+0.064}_{-0.028}$
η	$0.340^{+0.017}_{-0.045}$

Table 1.2: Parameters of the Cabibbo-Kobayashi-Maskawa matrix [10].

1.2.2 Quantum chromo dynamics.

Quantum chromo dynamics (QCD) is a non-abelian gauge theory (studied the first time by Yang and Mills [9]) which explain strong interactions.

A lot of processes (e-p annihilation into hadrons, π^0 decay,...) show that quark interactions can be described using a new quantum number: the color. In particular each quark can have three different colors; the transformation we do to change the color of a quark can be represented by the group $SU(3)_{color}$.

The Lagrangian of the QCD is the following:

$$\begin{aligned} L &= \bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) \\ &\quad + \frac{1}{2a}(\partial_\mu A^\mu)^2 - i(\partial_\mu \chi^a)^*(D^{\mu\, ab}\chi_b) \end{aligned} \tag{1.29}$$

where ψ are:

$$\psi = \begin{pmatrix} \psi_\alpha \\ \psi_\beta \\ \psi_\gamma \end{pmatrix} \tag{1.30}$$

and each ψ_α is a bi-spinor and m is the mass of the particle associated to the field ψ. Greek letters label different colors.

D_μ is the covariant derivative we need to make the theory invariant under a non abelian gauge transformation, and is:

$$D_\mu = \partial_\mu - i g_s A^c_\mu T_c \qquad c = 1, ...8 \tag{1.31}$$

where T_c are the generators of the $SU(3)_{color}$ group and A^c_μ are the fields which represent the gluons (mediators of the strong interactions).

With this definitions the term

$$\bar{\psi}(i \not{D} - m)\psi \tag{1.32}$$

represents free quarks and their interactions with gluons.

The second term of the Lagrangian

$$-\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) \tag{1.33}$$

takes into account all the dynamic of the gauge fields (see fig.(1.3)). The definition for the tensor $F_{\mu\nu}$ is:

$$F_{\mu\nu} = (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)T_c + i f^c_{ab} A^a_\mu A^b_\nu T_c \tag{1.34}$$

where f^c_{ab} are the structure constants of $SU(3)_{color}$.

The freedom of the gauge transformation lets the field A^a_μ be arbitrary. In order to make the theory consistent it is necessary to fix the gauge, this is done introducing in the Lagrangian the term:

$$+\frac{1}{2a}(\partial_\mu A^\mu)^2 \tag{1.35}$$

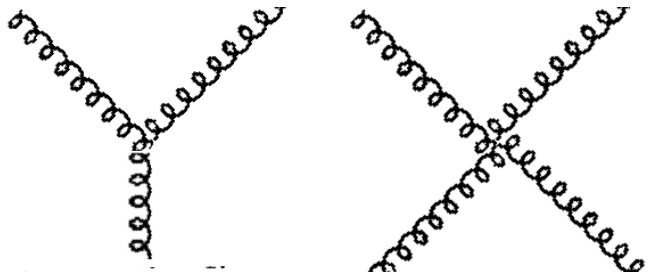


Figure 1.3: Self-interactions between gluons.

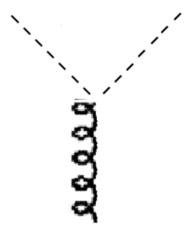


Figure 1.4: Interaction between ghosts (dotted lines) and a gluon.

Finally the fields χ^a are said ghosts. These fields are introduced to have a correct quantization. The term:

$$-i(\partial_\mu \chi^a)^*(D^{\mu\ ab}\chi_b) \tag{1.36}$$

give the interactions between ghosts and gluons (see fig.(1.4)).

1.3 Successes and limits of the Standard Model.

The Standard Model has predicted the existence of some particles and the value of some constants before their experimental observation.
Of particular importance is, for example, the prediction of the ratio of the masses M_Z/M_W that is in perfect agreement with the experimental measurements obtained later.
We can resume the success of the Standard Model saying that it give us a perfect knowledge of the fundamental interactions up to energies of 100 GeV.

Nevertheless, there are some reasons to look at something beyond it and to consider it as an effective theory at energies below of 100 GeV.
The unification of the electromagnetic and weak forces obtained in the Standard Model let us to hope that it exists a theory which unify all forces. This theory would reduce to the Standard Model at low energies.
An other unsatisfactory peculiarity of the Standard Model is that it does not explain the origin of the masses. In fact we can notice that the fermions have very different masses (we move in the range delimited by neutrinos, which should be massless, to the top quark, whose mass is about two hundred GeV) even if they are all fundamental particles, moreover the nature of the mass of the Higgs boson (whose existence is already under investigation) is different from the others: it should be the only particle with a mass which is not generated by a symmetry breaking.

In any case, if after the investigation of phenomena at higher energy, we find that the fundamental interactions are better described by a different theory, we can enjoy that the Standard Model is a good approximation at low energy.

Chapter 2

Potential models and HQET

QCD seems predict the observed confinement: the fact that quarks are confined into hadrons. However, no way to show from first principle this fact is known.

In order to evaluate the transition rates of meson decays, we should write the meson state. The limit of our knowledge of QCD requires some models to describe meson states. A lot of models exist. The simplest of them are the constituent quark potential models.

These models are inspired to quantum mechanic: a meson is considered as a bound state of a fixed number of quarks (valence quarks) interacting instantaneously with a potential.
So we can suppose that the meson state is described by a wave function which depends just by the valence quarks and the bounding potential.
In other words in this models we suppose that a meson is a quantum system in which valence quarks interact by a potential, and we neglect all contributions coming from non-valence quarks and sea gluons in the meson. This approximation is justified by the fact that Fock's state with the minimum number of constituents are prevalent.

The simplest potential models are non-relativistic, but, with a more complicated formalism due to Jaus [19], it is possible to take into account relativistic effects.

From a theoretical point of view potential models are not justified, in fact a correct model should be a relativistic and quantum field theory. Nevertheless they continue to be used for their simplicity and because they work well, in the sense that they predict the decay rates in agreement with experiments. Moreover, they allow to deduce some features of QCD which are not evident from the lagrangian.

In this thesis we'll describe our non-relativistic potential model [11], [12]. But before, it is useful we see briefly the main results and theoretical consequences of the most famous potential model.

2.1 Isgur Scora Grinstein and Wise (ISGW) model

When the ISGW model was published [13], the experimental results about semileptonic B decays were not so accurate as they are now.
At that time, the model [8] used to analyze inclusive semi-leptonic B decays was the parton model. This model considered all quarks as free. It predicted the upper limit of the ratio $|V_{ub}/V_{cb}|^2$ too low respect to the expectation of the Standard Model and of the experimental data; it was difficult to realize if the data, the theory or the model were wrong.
ISGW understood that the problem was the use of the parton model in the region where the lepton energy is at the maximum. In fact this region is populated of low mass hadronic states and quarks cannot be considered free. Then, they implemented the new model we describe below.

The aim of the model is to calculate the hadronic matrix elements. To get this result ISGW give, for each decay, a Lorentz invariant decomposition of the hadronic matrix element defining some relativistic functions we call form factors. For example for the semi-leptonic decay $B \to Dl\nu$, with both mesons pseudo-scalars, they define:

$$< D|j^\mu|B >= f_+(t)(p_B + p_D)^\mu + f_-(t)(p_B - p_D)^\mu \tag{2.1}$$

where the form factors are $f_\pm(t)$, and t is the square of the momentum quarks change between them in the transition, that means:

$$t = (p_B - p_B)^2 \tag{2.2}$$

Even if this definition is covariant, with models it is possible to calculate form factors only in the non-relativistic limit.

The starting point of the model is the hypothesis that it exists a correspondence between this form factors f_i and the ones calculated by the model, $\tilde{f}_i$.
This correspondence is true when the energy between quarks is low or if we are near the rest frame of the two mesons. Nevertheless, we can suppose that the extrapolation of our results in the region $\sqrt{< p^2 >} \simeq < m_u > \simeq \Lambda_{QCD}$ is a good approximation. Some parameters of the model, which have to be fitted, can partially adjust the deficiencies due to this approximation. ISGW consider the extrapolation:

$$f_i(p_X^2) = f_i(0)(1 - (\frac{1}{6}r_i^2 + \frac{a}{m^2})p_X^2) \tag{2.3}$$

where r_i is the radius of the hadron and m the mass of the heavy quark. The term of the order p_X^2 is a relativistic correction.
In this model, to calculate the form factors, the meson state is described by:

$$|\tilde{X}(p_x s_x) >= \sqrt{2\tilde{m}_x} \int d^3p \; \Sigma C^{s_x LS}_{m_L m_S} \; \phi_{xLm_L}(p) \; \chi^{Sm_s}_{s\bar{s}} |q\bar{q} > \tag{2.4}$$

where $\tilde{m}_x = m_q + m_{\bar{q}}$, considering the case we are in the weak binding limit. ϕ is the eigenfunction fixed as a consequence of the choice of the binding potential between quarks. If we use the typical QCD potential

$$V(r) = -\frac{4}{3}\frac{\alpha_S}{r} + br + c \tag{2.5}$$

with $\alpha_S = 0.5$, $b = 0.18 GeV^2$ and $c = -0.84 GeV$, the ϕ are gaussian functions depending by a parameter we can fix with the variational method. This functions are different for each $^{2S+1}J_L$, so the model is able to describe the decay of the B meson in every physical state.

Now it is straightforward to get the expression of the form factors in terms of ϕ.

ISGW model gives a realistic prediction of the ratio $|V_{ub}/V_{cb}|^2$. This is the reason of its immediate success but this is not the lonely goal of the model. The model is very famous also for the discovery of the heavy quark symmetry (HQS) of QCD we describe in the next section.

2.2 Heavy quark symmetry

QCD is difficult to be treated analytically in the non perturbative regime. The best we can do to get predictions from QCD is to use the symmetries of the theory.

ISGW model has put the basis for the discovery of the heavy quark symmetry [14], [15]. Their model shows that if the mass of the quark is $m_Q > \Lambda_{QCD}$ then the physics is insensible to the non-perturbative dynamics. The idea of Isgur and Wise was then to expand the QCD lagrangian respect to the parameter $\frac{\Lambda_{QCD}}{m_Q}$, obtaining an effective theory.

To see the results of such expansion consider a meson Qq with $m_Q >> m_q$, so we can apply the effective theory, and rewrite the Feynman rules of QCD in the proper limit.
First of all we can note that we can express the momentum p_Q as:

$$p_Q^\mu = m_Q v^\mu + k^\mu \tag{2.6}$$

where the momentum k^μ is negligible respect to $m_Q v^\mu$ because we are in the limit $m_Q \to \infty$. As a consequence, because of $p_Q^2 = m_Q^2$, the four-velocity v^μ respects the normalization condition $v^2 = 1$.

The quark propagator in QCD is:

$$\imath \frac{\not{p}_Q + m_Q}{p_Q^2 - m_Q^2} \tag{2.7}$$

With the expansion (2.6) it becomes:

$$\imath \frac{(\not{v} + 1)}{2v{\cdot}k} \tag{2.8}$$

In QCD the interaction vertex Q-gluon is

$$-\imath g \gamma_\mu T^a \tag{2.9}$$

where g is the strong coupling constant and T^a is a generator of the group $SU(3)_{color}$. In the effective theory this vertex becomes:

$$\imath g T^a \frac{(\not{v} + 1) v_\mu}{2(v{\cdot}k)} \tag{2.10}$$

This rules completely define the heavy quark effective theory.

It is useful see how to get the same rules starting from the QCD lagrangian.

$$L = \bar{Q}(\imath \not{D} - m_Q)Q \tag{2.11}$$

We can write the field as

$$Q = e^{-\imath m_Q v \cdot x} h_v^Q \tag{2.12}$$

with

$$\not{v} h_v^Q = h_v^Q \tag{2.13}$$

So the lagrangian is:

$$L_v = \bar{h}^Q \imath v \cdot D h_v^Q \tag{2.14}$$

This lagrangian respects a law of conservation of quarks we can represent by the group U(1).

Moreover we see that gamma-matrices does not appear in the lagrangian, so the spin of quarks cannot change. This is an other symmetry of the theory represented by the group SU(2).

If we have N quarks with the same four-velocity the general lagrangian is:

$$L_v = \Sigma_{j=1}^N \bar{h}^j \imath v \cdot D h_v^j \tag{2.15}$$

The SU(2) symmetry now is a symmetry SU(2N). This group relies also quarks with very different momentum, in fact, even if the velocity is the same, the lagrangian is completely independent by the mass of quarks.

Now we know the lagrangian of the theory, we are ready to calculate the transition matrix element for hadronic transition of the kind $Q_i q \rightarrow Q_j q$.
We can give the definition of form factors $\tilde{f}_i$:

$$\frac{< P_{Q_j}(v')|\bar{h}_{v'}^j \gamma_\mu h_v^i|P_{Q_i}(v) >}{\sqrt{m_{P_{Q_j}} m_{P_{Q_i}}}} = \tilde{f}_+(v+v')_\mu + \tilde{f}_-(v-v')_\mu \tag{2.16}$$

Comparing this relation to the usual definition of form factors in equation (2.1) it is straightforward to obtain the rules which connect the two sets of form factors. For example:

$$f_\pm = \pm\frac{1}{2}\left(\sqrt{\frac{m_{P_{Q_i}}}{m_{P_{Q_j}}}} \pm \sqrt{\frac{m_{P_{Q_j}}}{m_{P_{Q_i}}}}\right)\tilde{f}_+ \pm \frac{1}{2}\left(\sqrt{\frac{m_{P_{Q_j}}}{m_{P_{Q_i}}}} \pm \sqrt{\frac{m_{P_{Q_i}}}{m_{P_{Q_j}}}}\right)\tilde{f}_- \tag{2.17}$$

The form factors $\tilde{f}_i$ are very simples. In fact using the relations:

$$\not{v} h_v = h_v \ , \ \bar{h}_{v'} \ \not{v}' = \bar{h}_{v'} \tag{2.18}$$

and the definition we can obtain

$$\tilde{f}_- = 0 \tag{2.19}$$

Moreover if we put $v = v'$ and $\mu = 0$ the matrix element is associated with a conserved current for the $SU(2)_{flavour}$ symmetry whose value is known. So we realize:

$$\tilde{f}_+(1) = 1 \tag{2.20}$$

It is useful to define a new function $\xi(v{\cdot}v')$, called Isgur and Wise function, which respects the normalization

$$\tilde{\xi}(1) = 1 \tag{2.21}$$

Because of the symmetry $SU(2N)_{spin}$ we can reason in the same way for a transition in an excited state 1^-. If we define the form factors as:

$$\frac{< P^*_{Q_j}(v',\epsilon)|\bar{h}^j_{v'}\gamma_\mu\gamma_5 h^i_v|P_{Q_i}(v) >}{\sqrt{m_{P^*_{Q_j}} m_{P_{Q_i}}}} = \tilde{f}\epsilon^*_\mu + (\epsilon{\cdot}v)(\tilde{a}_+(v+v')_\mu + \tilde{a}_-(v-v')_\mu) \tag{2.22}$$

$$\frac{< P^*_{Q_j}(v',\epsilon)|\bar{h}^j_{v'}\gamma_\mu h^i_v|P_{Q_i}(v) >}{\sqrt{m_{P^*_{Q_j}} m_{P_{Q_i}}}} = i\tilde{g}\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}v'^\lambda v^\sigma \tag{2.23}$$

$$\tag{2.24}$$

we find

$$\tilde{f} = (1 + v{\cdot}v')\xi \tag{2.25}$$

$$\tilde{a}_+ + \tilde{a}_- = 0 \tag{2.26}$$

$$\tilde{a}_+ - \tilde{a}_- = -\xi \tag{2.27}$$

$$\tilde{g} = \xi \tag{2.28}$$

Analogously we can obtain the results of the form factors for every kind of final meson state, we just need to adopt the correct Lorentz decomposition of the hadronic matrix element.

Until now we haven't taken into account the differences between the operators of the effective theory and the ones of QCD, then we cannot compare the form factors of the model with the physical ones. The problem is that at each energy scale there are some graphs that diverges, then the effective theory must be renormalized.
An accurate study [16] shows that all form factors must be multiplied for a factor:

$$C_{ji} = \left(\frac{\alpha_S(m_{Q_i})}{\alpha_S(m_{Q_j})}\right)^{-6/(33-2N)} \left(\frac{\alpha_S(m_{Q_j})}{\alpha_S(\mu)}\right)^{a_L(v{\cdot}v')} \tag{2.29}$$

where

$$a_L(v\cdot v') = \frac{8}{33-2N}(v\cdot v' r(v\cdot v') - 1) \quad (2.30)$$

$$r(v\cdot v') = \frac{1}{\sqrt{(v\cdot v')^2-1}} ln\left(v\cdot v' + \sqrt{(v\cdot v')^2-1}\right) \quad (2.31)$$

and N is the number of flavors we see until the energy μ we are.

In this section we have described the results of the HQS, obtained developing the QCD lagrangian in $1/m_Q$ at the order zero. The heavy quark effective theory (HQET) gives also more accurate results using the expansion at next orders.

2.3 An updated version of the ISGW model (ISGW2)

The discovery of the HQS and the development of the HQET give a method to test potential models. In fact HQET divides the problem of the evaluation of hadronic matrix transition into two factors: one depends by the energy scale and the other is known exactly thanks to the HQS. In this context, a model is associated with a fixed energy scale μ_{qm} where hadronic physics is dominated by valence quark.
Moreover HQET implies that every good model should respect some constraints in the right limit. From this new point of view, the aim of models is to evaluate the deviation of form factors from the HQET results.

In the rest frame of the two mesons involved in the semi-leptonic transition, ISGW model respects the constraints of HQET; but this is not true if we change frame. Moreover this model doesn't impose the relativistic corrections in the right way. This is a first reason to modify the model.
An other reason is that after some years from the born of ISGW model new or more accurate experimental results were obtained, so the models should reproduce them.

In 1995, Isgur and Scora published an updated version of the ISGW model (ISGW2), [17].

We have seen that, for the form factors of the decay $B \to D^{(*)}l\nu$, HQS implies the relations

$$\tilde{f}_+ + \tilde{f}_- = \tilde{f}_+ - \tilde{f}_- = \tilde{g} = \frac{\tilde{f}}{1+w} = \tilde{a}_- - \tilde{a}_+ = \xi(w) \quad (2.32)$$

$$\tilde{a}_+ + \tilde{a}_- = 0 \quad (2.33)$$

where here we define $w = v{\cdot}v'$.
ISGW model respects all this relations a part the one relative to f in which it doesn't have the factor $1+w$. This factor results from relativistic correction to the order v^2/c^2 and is the first modification to introduce in the model.

The currents in the heavy quark limit are not the ones of the complete theory, so the same happens also for the form factors. HQET give us a the rule to evaluate the physical form factor $f_{ji}^{(\alpha)}$, of type α, relative to the quark transition $Q_i \rightarrow Q_j$:

$$f_{ji}^{(\alpha)} = C_{ij}(w)\left(f^{\alpha} + \tilde{\beta}_{ji}^{(\alpha)}(w)\frac{\alpha_s(\mu_{ji})}{\pi}\right)\xi(w) \tag{2.34}$$

where $f^{(\alpha)} = 1$ for $\tilde{f}_+ + \tilde{f}_-$,$\tilde{f}_+ - \tilde{f}_-$, $\tilde{g}$, $\tilde{f}$, $\tilde{a}_- - \tilde{a}_+$ and is 0 for $\tilde{a}_+ + \tilde{a}_-$, while

$$C_{ji} = \left(\frac{\alpha_s(m_i)}{\alpha_s(m_j)}\right)^{-\frac{6}{33-2N_f}}\left(\frac{\alpha_s(m_j)}{\alpha_s(\mu_{qm})}\right)^{\frac{8(wr(w)-1}{33-2N_f'}} \tag{2.35}$$

where $N_f^{(')}$ are the number of flavor the theory is sensible at the scale $m_{i(f)}$ and

$$r = \frac{1}{\sqrt{w^2-1}}ln(w+\sqrt{w^2-1}) \tag{2.36}$$

The functions $\tilde{\beta}$ are the result of radiative corrections evaluated at an intermediate scale between m_i and m_j. Their expressions are quite long, so we don't report them here.

To connect the form factors $\tilde{f}(\tilde{w})$ with the $f(t-t_m)$, relative to the physical transitions, we can start observing the relation between the variables $\tilde{w}$ and $t-t_m$, where the last variable is the difference between the 4-momentum that the quarks change in the decay and the maximum of this value. In the decay $P_Q \rightarrow X_Q l\nu$ we have:

$$\tilde{w} - 1 = \frac{t_m - t}{2\bar{m}_{P_Q}\bar{m}_{X_Q}} \tag{2.37}$$

The form factors in the non relativistic limit and in the approximation of low bounding energy are connected to the real ones by a correction to the order $1/m_Q$. This is, in ISGW2, a factor which takes into account the differences between the physical mass $\bar{m}_H$ of each state of the doublet of spin (indistinguishable by HQS) and the mass $\tilde{m}_H$ (sum of the valence quark) used in the ISGW model.
To evaluate $\bar{m}_H$ for multiplet of spin s_l, Isgur and Scora propose

$$\bar{m}_{s_l} = (\frac{s_l+1}{2s_l+1})m_{j=s_l+1/2} + (\frac{s_l}{2s_l+1})m_{j=s_l-1/2} \tag{2.38}$$

The last fundamental correction to do to the ISGW model comes from the evaluation of relativistic corrections obtained from QCD sum rules. This implies a constraint on the slope of the form factors near $w = 1$:

$$f(t) = f(t_m)\left(1 - \frac{1}{6}r^2(t_m - t) + ...\right) \tag{2.39}$$

with

$$r^2 = \frac{3}{4m_Q m_q} + r_{wf}^2 + \frac{1}{\bar{m}_{P_Q}\bar{m}_{X_q}}\left(\frac{16}{33 - 2N_f'}\right) ln\left(\frac{\alpha_s(\mu_{qm})}{\alpha_s(m_q)}\right) \tag{2.40}$$

and r_{wf} depends by the wave function used in the model.
The result obtained imposing this constraint is the same of the ISGW model without the artificial introduction of the factor κ.

To have a good agreement with experimental results, in ISGW2 model, it is useful to play on the choice of the wave function. The most commonly used are the gaussian and the exponential ones.

For sake of brevity we don't report the complete expression of form factors here.

2.4 Relativistic quark model

In the previous sections we have seen a non-relativistic potential model. This limitation of models imply that form factors can be evaluated correctly just in a point, and their dependence from the momentum transferred has to be imposed as we have seen in the previous cases.

The method to calculate form factors in a covariant way is the light-cone approach. The first author to implement the light-front formalism, due to Dirac [18], to the study of particle decays was Jaus [19]. Then, the method has been often used and a lot of works can be cited [20], [28].
This method allows to write the meson wave function in a covariant way and to construct all spin states using Melosh rotation [21].

Studying particle's decays it is natural to observe that a preferred axis exists. The basis of light-front potential models is an appropriate change of variables to take advantage of this

peculiarity of the problem.
The new variables are defined as:

$$(v_0, v_1, v_2, v_3) \rightarrow (\frac{v_0 + v_3}{\sqrt{2}}, v_1, v_2, \frac{v_0 - v_3}{\sqrt{2}}) \tag{2.41}$$

These new variables have the advantage to transform very simply under a boost along the z-axis, which is chosen as the axis the final meson moves along.

These models give realistic results of the form factors in all the accessible kinematical range.

Chapter 3

A new potential model

The problem of non-relativistic quark models is that they cannot evaluate form factors in all the kinematical accessible range. This limit is due to the fact that the state is not covariant, then we cannot boost it, and that energy is not naturally conserved changing frame.
In our work, using results obtained in [22] and [23], we have developed a model which allows to partially solve this problem imposing the energy conservation in a non conventional way. Then, even if the model we have improved is not relativistic, we'll dare to use it to evaluate form factors at every momentum transferred in the transitions we'll consider.

We can say that our model is an hybrid of the relativistic and non-relativistic ones.

Because of the non relativistic nature of our model, we attend that the decays which involve just heavy mesons are better described then the others. In reality there are other features of our model, we'll see in the next sections, which prevent us to use it when light mesons are involved in the decays.
For this reasons we'll just apply our model to $B_{(s)}$ decays into charmed mesons.

3.1 Feynman rules

To define the model we have to give some Feynman rules to evaluate the matrix element of transition we want to consider.

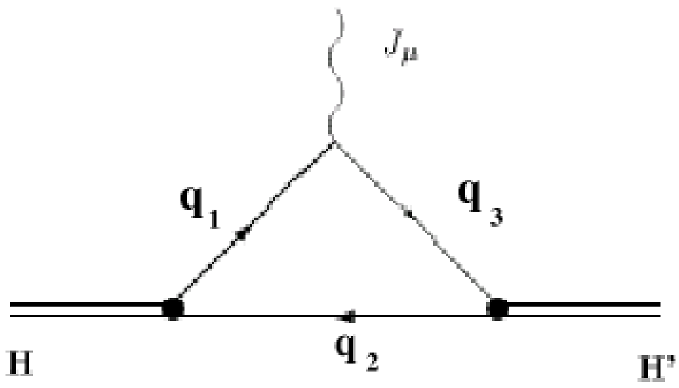


Figure 3.1: $H \to H'l\nu$. In our work we'll consider the case of mesons with a heavy and a light quark. So q_2 represent the light quarks (u, d or s) while q_1 is a beauty and q_3 a charmed quark.

To represent the hadronic part of a semi-leptonic decay of a meson H into H', we refer to the graph in fig.[3.1].

In our work we describe every meson as the Fock state:

$$|H >= \frac{1}{\sqrt{3}}\psi_H \frac{\not{q}_1 + m_1}{2m_1}\Gamma_{J^H}\frac{-\not{q}_2 + m_2}{2m_2} \tag{3.1}$$

where $q_{(1,2)}$ and $m_{(1,2)}$ are respectively the momentum and masses of the two constituent quarks of the meson H of mass M_H, Γ_{J^H} is a matrix that depends from the meson state and ψ_H is the wave function that describes the internal structure of H.
This decomposition is inspired to the transformation properties of the mesons as predicted by HQET and has been written the first time in the paper [23].

We define the state $< H|$ as:

$$< H| = -\gamma_0 |H >^+ \gamma_0 \tag{3.2}$$

The normalization condition is:

$$< H|H >= 2M_H \tag{3.3}$$

realized fixing:

$$\int \frac{d^3k}{(2\pi)^3}|\psi_H(k)|^2 = 2M_H \tag{3.4}$$

where k is the internal momentum of quarks.

At each loop we associate an integration

$$\int_D \frac{d^3k}{(2\pi)^3} \tag{3.5}$$

a trace over Dirac matrices and a color factor 3. In the next section we explain how to fix the region D on which perform the integration.

Finally, at each quark (of mass m_i and energy E_i) involved in the transition, we let correspond a factor

$$\sqrt{\frac{m_i}{E_i}} \tag{3.6}$$

3.2 Energy conservation

From now on we consider only mesons with a heavy Q (b or c quark) and a light q (u, d or s quark) constituent. For what it may concern light quarks, our model will be SU(2) invariant, so we cannot distinguish between u and d quarks.

In non relativistic potential models the interactions between quarks are instantaneous, this means that the time is fixed. As a consequence, also energy, which is the quantum correspondent of the time, is fixed and then, during the process, energy conservation is not respected.

Also our model is based on the approximation of instantaneous interactions, but we impose energy conservation introducing the concept of running mass: we imagine that the interaction between constituent quarks is reflected on the mass of the heavy quark Q.
This mass is fixed so that the condition $E_H = E_q + E_Q$ is respected.
The energies of the constituent quarks are:

$$E_q = \sqrt{m_q^2 + k^2} \tag{3.7}$$

$$E_Q = \sqrt{m_Q^2 + (\vec{p}_H + \vec{k})^2} \tag{3.8}$$

We consider these equations and the conservation of energy as definitions of the function $m_Q(\vec{k})$, while m_q is a parameter of the model.
So we find:

$$m_Q(\vec{k}) = \sqrt{E_H^2 - E_q^2 - (\vec{k} + \vec{p}_H)^2} \tag{3.9}$$

The domain of existence of $m_Q(\vec{k})$ is a limit over $\vec{k}$ and then defines the domain D in equation (3.5).

If we choose spherical coordinates (k, θ, ϕ) for $\vec{k}$ and fix the direction of H as the z axis the domain D is defined by the equations:

$$0 \leq \phi \leq 2\pi \tag{3.10}$$

$$cos\theta \geq \frac{2E_H E_q - m_H^2 - m_q^2}{2kp_H} \tag{3.11}$$

$$k \leq \frac{m_H^2 + m_q^2}{2m_H^2} p_H + \frac{m_H^2 - m_q^2}{2m_H^2} E_H \tag{3.12}$$

3.3 The wave function

A complete theory to describe mesons should be a quantum field theory, because inside hadrons the number of quarks and gluons is not conserved.

In the contest of a potential model we neglect this aspect and imagine that a meson is constituted just by two valence quarks bounded by a potential; so it should exist a wave function that describes the internal structure of the meson and it is the solution of an appropriate equation.

Any model gives us the complete equation to be solved nor the exact potential between quarks, so we cannot calculate exactly this wave function.

There are a lot of different ways to obtain a reasonable approximation of the wave function [24], the final result is that the most used in literature are gaussian or exponential.

In our case, taking into account the right normalization condition, this kind of functions are:

$$\psi_H(k) = 4\pi^{3/4} \sqrt{\frac{m_H}{\omega_H^3}} exp\{-\frac{k^2}{2\omega_H^2}\} \tag{3.13}$$

$$\psi_H(k) = 4\pi \sqrt{\frac{m_H}{\omega_H^3}} exp\{-\frac{k}{\omega_H}\} \tag{3.14}$$

where ω_H is a parameter that can be fitted by the experimental data in the way we'll see after.

We suppose that mesons with the same constituent quarks are characterized by the same value of ω_H, independently from their J^P state.

3.4 Decays $B \to D^{(*)} l\nu$

In this section we give all the ingredients to evaluate the branching fraction of the decay $B \to Dl\nu$ and the partial decay rates of the process $B \to D^* l\nu$, where the $D^{(*)}$ here considered are the pseudoscalar and the vector meson respectively.

This decays are particularly interesting for us because we have experimental results just about this B decays, so we'll use them to fix the free parameters of the model, as we will see in the next section.
To compare the model with the experiments we have to evaluate the matrix transition elements of the decays.
To begin we must fix the vertices of the pseudo-scalar and vector meson.
A pseudo-scalar state ($J^P = 0^-$), because of the negative parity and of its angular momentum, needs a vertex factor of the kind $\Gamma_{0^-} = a\gamma_5$. The coefficient a can be evaluated using the normalization condition of the state.

$$< 0^-|0^- >= \int \frac{d^3k}{(2\pi)^3} |\psi|^2 Tr((\not{q}_2 + m_2)(\not{q}_1 + m_1)) \frac{|a|^2}{4m_1 m_2} = 2M \tag{3.15}$$

then

$$|a| = \sqrt{\frac{m_1 m_2}{q_1 \cdot q_2 + m_1 m_2}} \tag{3.16}$$

To determine the phase of a we evaluate the decay constant f, in the frame where the meson is at rest, and use the condition that it is positive.

$$ifp_\mu =< 0|A_\mu|H >= \int \frac{d^3k}{(2\pi)^3} \sqrt{3}\psi \sqrt{\frac{m_1 m_2}{E_1 E_2}} a \frac{q_{1\mu} m_2 + q_{2\mu} m_1}{m_1 m_2} \tag{3.17}$$

In conclusion we find:

$$\Gamma_{0^-} = -i\gamma_5 \sqrt{\frac{m_1 m_2}{q_1 \cdot q_2 + m_1 m_2}} \tag{3.18}$$

Now we consider the state $J^P = 1^-$. To respect the parity and to take into account its angular momentum, the vertex factor has the form:

$$\Gamma_{1^-} = a \not{\epsilon} + 2b\epsilon \cdot q_2 \tag{3.19}$$

where ϵ^μ is the polarization four vector which satisfies:

$$\epsilon^2 = -1 \ , \ \epsilon \perp p \tag{3.20}$$

The basis of three independent ϵ[1] we use is:

$$\begin{aligned} \epsilon^\mu(1) &= (0,1,0,0) \quad (3.23)\\ \epsilon^\mu(2) &= (0,0,1,0) \quad (3.24)\\ \epsilon^\mu(0) &= \frac{1}{M}(|\vec{p}|,0,0,E) \quad (3.25)\end{aligned}$$

The normalization equation gives a condition that relates the coefficients a and b:

$$\begin{aligned} <1^-|1^-> &= -\int \frac{d^3k}{(2\pi)^3}|\psi|^2\frac{1}{4m_1m_2}(-a^2(-2q_2{\cdot}\epsilon^2+q_1{\cdot}q_2+m_1m_2)\\ &\quad -4\epsilon{\cdot}q_2^2ab(m_1+m_2)+4b^2\epsilon{\cdot}q_2^2(-q_1{\cdot}q_2+m_1m_2))\\ &= 2M \quad (3.26)\end{aligned}$$

that implies, after a sum over ϵ:

$$a^2(q_1{\cdot}q_2+m_1m_2)=m_1m_2 \quad (3.27)$$

$$-2a^2+4ab(m_1+m_2)-4b^2(-q_1{\cdot}q_2+m_1m_2)=0 \quad (3.28)$$

and then

$$\begin{aligned} |a| &= \sqrt{\frac{m_1m_2}{q_1{\cdot}q_2+m_1m_2}} \quad (3.29)\\ |b|_\pm &= \frac{|a|}{m_1+m_2\pm M} \quad (3.30)\end{aligned}$$

For b we chose $|b|_+$ to avoid divergences of the form factors.

To fix the phases of a and b we use the condition that the decay constant is positive. It is defined as:

$$<0|V_\mu|H>=fM\epsilon_\mu \quad (3.31)$$

[1]The reader who would like to compare our model with others should note that our basis is different from the one more often used in literature:

$$\begin{aligned} \epsilon^\mu(\pm) &= \mp\frac{1}{\sqrt{2}}(0,1,\pm\imath,0) \quad (3.21)\\ \epsilon^\mu(0) &= \frac{1}{M}(|\vec{p}|,0,0,E) \quad (3.22)\end{aligned}$$

So we obtain:

$$\Gamma_{1^-} = \sqrt{\frac{m_1 m_2}{q_1{\cdot}q_2 + m_1 m_2}}(\not\epsilon - \frac{\epsilon{\cdot}q_1 - \epsilon{\cdot}q_2}{M + m_1 + m_2}) \tag{3.32}$$

At this point we are ready to evaluate form factors of the decay $B \to D^{(*)} l\nu$. We define them as follows:

$$< D|\gamma_\mu\gamma_5|B > \;=\; f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu \tag{3.33}$$

$$< D^*|\gamma_\mu|B > \;=\; 2g(q^2)\epsilon_{\mu\nu\rho\sigma}\epsilon^\nu p^\rho p'^\sigma \tag{3.34}$$

$$\begin{aligned} < D^*|\gamma_\mu\gamma_5|B > \;=\; & if(q^2)\epsilon_\mu + i(\epsilon{\cdot}p) \\ & ((p+p')_\mu a_+(q^2) + (p-p')_\mu a_-(q^2)) \end{aligned} \tag{3.35}$$

In the rest of our work we'll ever consider the B meson at rest, so the 4-momentum of the particles involved are:

$$p^\mu = (M, 0, 0, 0) \;\; ; \;\; p'^\mu = (E', 0, 0, -|\vec{q}|) \tag{3.36}$$

$$q_2^\mu = (E_2, -\vec{k}) \tag{3.37}$$

$$q_1^\mu = p^\mu - q_2^\mu \;\; ; \;\; q_3^\mu = p'^\mu - q_2^\mu \tag{3.38}$$

Using the Feynman rules given for our model we find:

$$\begin{aligned} & f_+(p+p')_\mu + f_-(p-p')_\mu = \int \frac{d^3k}{(2\pi)^3}\,\psi\psi'\sqrt{\frac{1}{E_1 E_3 (m_2 m_1 + q_1{\cdot}q_2)(m_2 m_3 + q_2{\cdot}q_3)}} \\ & [(m_2 m_3 + q_2{\cdot}q_3)q_{1\mu} + (m_2 m_1 + q_1{\cdot}q_2)q_{3\mu} + (m_3 m_1 - q_1{\cdot}q_3)q_{2\mu}] \end{aligned} \tag{3.39}$$

The extraction of $f_\pm$ from this relation can be done in various ways, for example fixing different values of μ, this allows to write a system of two independent equations, or it is possible to go on in the evaluation multiplying the last expression for two different four vectors. In all cases the final result is:

$$f_+ = \frac{1}{2M_B|\vec{q}|}\int\frac{d^3k}{(2\pi)^3}\,\psi\psi'\sqrt{\frac{1}{E_1E_3(m_2m_1+q_1{\cdot}q_2)(m_2m_3+q_2{\cdot}q_3)}}$$
$$[(c_1+c_3)(|\vec{q}|(M_B-E_2)-(M_B-E_D)kcos\theta)$$
$$+c_2(|\vec{q}|E_2+(M_B-E_D)kcos\theta)] \tag{3.40}$$
$$f_- = -\frac{1}{2E_D|\vec{q}|}\int\frac{d^3k}{(2\pi)^3}\,\psi\psi'\sqrt{\frac{1}{E_1E_3(m_2m_1+q_1{\cdot}q_2)(m_2m_3+q_2{\cdot}q_3)}}$$
$$[c_1(|\vec{q}|(M_B-E_2)-(M_B+E_D)kcos\theta)+c_3(|\vec{q}|(2M_B-E_2+E_D)-(M_B+E_D)kcos\theta)$$
$$+c_2(|\vec{q}|E_2+(M_B+E_D)kcos\theta)] \tag{3.41}$$

where

$$c_1 = (m_2m_3+q_2{\cdot}q_3) \tag{3.42}$$
$$c_3 = (m_2m_1+q_1{\cdot}q_2) \tag{3.43}$$
$$c_2 = (m_3m_1-q_1{\cdot}q_3) \tag{3.44}$$
$$\tag{3.45}$$

As you can see doing reference to the fig.3.1, in the case we are interested, the quantities labelled with 1 and 3 are referred respectively to the b quark and c quark while, with the label 2, we indicate quantities related to the spectator light quark.

In an analogous way it is possible to evaluate the others form factors.

$$g(q^2) = \frac{1}{|\vec{q}|M_B}\int d^3k\,\psi\psi'\sqrt{\frac{1}{E_1E_3(m_2m_1+q_1{\cdot}q_2)(m_2m_3+q_2{\cdot}q_3)}}$$
$$(M_Bm_2|\vec{q}|+(m_3-m_2)M_Bkcos\theta+(m_2-m_1)$$
$$(kcos\theta E_{D^*}-E_2|\vec{q}|)+2ksen\theta cos\phi\epsilon^{(1)}{\cdot}q_2M_B|\vec{q}|\frac{1}{M_{D^*}+m_2+m_3}) \tag{3.46}$$

$$f(q^2) = -\int d^3k\,\psi\psi'\sqrt{\frac{1}{E_1E_3(m_2m_1+q_1{\cdot}q_2)(m_2m_3+q_2{\cdot}q_3)}}$$
$$(c_3^*(\epsilon^{(2)}{\cdot}q_2)^2-c_4) \tag{3.47}$$

$$a_+(q^2) + a_-(q^2) = \frac{1}{(\epsilon^{(3)}\cdot p)^2} \int d^3k \; \psi\psi' \sqrt{\frac{1}{E_1 E_3 (m_2 m_1 + q_1\cdot q_2)(m_2 m_3 + q_2\cdot q_3)}}$$
$$(c_1^{(3)} \epsilon^{(3)}\cdot p + c_3^*((\epsilon^{(3)}\cdot q_2)^2 - (\epsilon^{(2)}\cdot q_2)^2)$$
$$+(m_2 - m_3)(\epsilon^{(3)}\cdot p \epsilon^{(3)}\cdot q_2)) \tag{3.48}$$

$$a_+(q^2) - a_-(q^2) = \frac{1}{(\epsilon^{(3)}\cdot p)|\vec{q}|} \int d^3k \; \psi\psi' \sqrt{\frac{1}{E_1 E_3 (m_2 m_1 + q_1\cdot q_2)(m_2 m_3 + q_2\cdot q_3)}}$$
$$(|\vec{q}| c_2^{(3)} + k cos\theta c_3^{(3)}$$
$$+ \frac{E_{D^*}}{M_{D^*}} c_3^* (\epsilon^{(2)}\cdot q_2)^2) \tag{3.49}$$

where the functions $c_i^\lambda(q^2)$ are given by the following expression:

$$\begin{aligned}
c_1^* &= (m_2 + m_3 + 2\frac{1}{M_{D^*} + m_2 + m_3}(q_2\cdot q_3 - m_2 m_3) \\
c_1(\lambda) &= c_1^* \epsilon^{(\lambda)}\cdot q_2 \\
c_2^* &= -m_1 + m_2 + 2\frac{1}{M_{D^*} + m_2 + m_3}(q_1\cdot q_2 + m_2 m_1) \\
c_2(\lambda) &= c_2^* \epsilon^{(\lambda)}\cdot q_2 - m_2 \epsilon^{(\lambda)}\cdot p \\
c_3^* &= -c_1^* - c_2^* + m_1 + m_3 - 2\frac{1}{M_{D^*} + m_2 + m_3}(q_1\cdot q_3 + m_1 m_3) \\
c_3(\lambda) &= c_3^* \epsilon^{(\lambda)}\cdot q_2 + (m_2 - m_3)\epsilon^{(\lambda)}\cdot p \\
c_4 &= m_2 q_1\cdot q_3 + m_1 q_2\cdot q_3 + m_3 q_1\cdot q_2 + m_2 m_1 m_3
\end{aligned} \tag{3.50}$$

With λ we have indicated the polarization.

We will see that all form factors have the sign expected from heavy quark effective theory. This confirms we have chosen the coefficients a coherently with the definitions used for the decay constant and the form factors.

After the evaluation of the form factors we are ready to get the branching ratio of the decays we are interested in.

For the details regarding the evaluation of the branching ratios see appendix A. Here we report just the useful relations.
If the final meson has zero total angular momentum the differential decay rate is:

$$\frac{d\Gamma}{dq^2} = \frac{G^2|V_{cb}|^2 {f_+}^2(q^2)}{192\pi^3 {M_B}^3}[(q^2 - M_B^2 - M_D)^2 - 4M_B^2 M_D^2]^{\frac{3}{2}} \tag{3.51}$$

where G is the Fermi constant. So the branching ratio is:

$$\Gamma = \int_0^{(M_B-M_D)^2} dq^2 \, \frac{d\Gamma}{dq^2} \tag{3.52}$$

Before to write the differential decay rates in the case of a final state with $J^P = 1^-$, we introduce a different set of form factors: V, A_1, A_2, A_3, A_0, defined as:

$$\begin{aligned} < D^*(p')| : \bar{c}\gamma_\mu(1-\gamma_5)b : |B(p)> \; = \; & \frac{2V(q^2)}{m_B + m_{D^*}} \epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu} p^\alpha p'^\beta \\ & -i\epsilon^*_\mu (m_B + m_{D^*})A_1(q^2) \\ & +i(\epsilon^* {\cdot} q)\frac{(p+p')_\mu}{m_B + m_{D^*}} A_2(q^2) \\ & +i(\epsilon^* {\cdot} q)\frac{2m_{D^*}}{q^2} q_\mu [A_3(q^2) \\ & -A_0(q^2)] \end{aligned} \tag{3.53}$$

with

$$A_3(q^2) = \frac{(m_B + m_{D^*})}{2m_{D^*}} A_1(q^2) - \frac{(m_B - m_{D^*})}{2m_{D^*}} A_2(q^2) \tag{3.54}$$

This form factors are related to the ones calculated before by:

$$V(q^2) = g(q^2)(m_B + m_{D^*}) \tag{3.55}$$

$$A_1(q^2) = \frac{f(q^2)}{m_B + m_{D^*}} \tag{3.56}$$

$$A_2(q^2) = -(m_B + m_{D^*})a_+(q^2) \tag{3.57}$$

$$A_0(q^2) = a_-(q^2)\frac{q^2}{2m_{D^*}} + \frac{f(q^2)}{2m_{D^*}} + \frac{m_B^2 - m_{D^*}^2}{2m_{D^*}} a_+(q^2) \tag{3.58}$$

We introduce this new decomposition at this point because the last form factors are more common in literature and then a comparison of our results with other models will be faster.

With the new definitions the branching ratios can be obtained using the following equations:

$$\frac{d\Gamma}{dq^2} = \frac{d\Gamma_+}{dq^2} + \frac{d\Gamma_-}{dq^2} + \frac{d\Gamma_0}{dq^2} \tag{3.59}$$

where $d\Gamma_i$ are:

$$\frac{d\Gamma_i}{dq^2} = \frac{G^2|V|^2}{(2\pi)^3}\frac{P(q^2)^2}{12M_B^2q^2}|H_i(q^2)|^2\ ,\ i = 0, \pm \tag{3.60}$$

with:

$$H_\pm = -(M_B + M_{D^*})A_1(q^2) \mp \frac{2M_BP}{M_B + M_{D^*}}V(q^2) \tag{3.61}$$

$$H_0 = \frac{1}{2M_{D^*}\sqrt{q^2}}[-(M_B^2 - M_{D^*}^2 - q^2)(M_B + M_{D^*})A_1(q^2) + \frac{4M_B^2P^2}{M_B + M_{D^*}}A_2(q^2)] \tag{3.62}$$

3.5 Parameters of the model

In the rest of our work we want to apply the model described here to semi-leptonic weak decays of the $B_{(s)}$ pseudoscalar meson into $D_{(s)}$ states.

For the B and B_s mesons we have used the masses and lifetimes below[2]:

$$M_B = 5.279GeV\ , \tau_B = 1.60810^{-12}s$$
$$M_{B_s} = 5.370GeV\ , \tau_{B_s} = 1.46110^{-12}s$$

The experimental results for the processes we are studying are the branching ratio of the decays $B^{(-,0)} \to D^{(0,-)}l\nu$ and the partial decay rate of the process $B^{(-,0)} \to D^{(0,-)*}l\nu$ measured in the BABAR experiment [25].
The average of the branching fraction of the decays $B^{(-,0)} \to D^{(0,-)}l\nu$ reported by the PDB (Particle Data Book, [26]) give:

$$Br(B \to Dl\nu) = (2.14 \pm 0.15)\% \tag{3.63}$$

The same average, for the ten points measured of the partial decay rate of the B decay into vector D mesons, gives the results we report in fig.3.3.
We use these experimental values as constraints to fit the free parameters of our model.

[2]Because of the invariance of our model under the symmetry group SU(2), we don't distinguish between neutral and charged mesons, so for each experimental value which characterizes a state of definite charge we'll use an average.

Table 3.1: Values of the parameters of the model using fitting the model on the experimental constraints with the gaussian (gauss.) and exponential (exp.) wave function.

Parameter	gauss.	exp.
m_q(MeV)	0.033946	0.023079
ω_B(MeV)	107.55	101.38
ω_D(MeV)	185.74	51.041
$\lvert V_{cb}\rvert$	0.0429	0.041212

They are $m_{u,d}$ (say m_q), ω_B, ω_D, and the element of the CKM matrix V_{cb}.
The results of the fit are reported in tab.[3.1].

We can observe that our predictions of $|V_{cb}|$ are both in agreement with the PDB value:

$$V_{cb} = (41.6 \pm 0.6)10^{-3} \tag{3.64}$$

In particular the exponential value and the experimental one are compatible within one σ. This is a first goal of our model.

3.6 Numerical results

The values of the form factors we obtain at the extreme points of the kinematical range are reported in tab.[3.2].
If we look at fig. [3.2] we see that the form factors can be described by polar functions:

$$f(q^2) = \frac{f(0)}{1 - a\frac{q^2}{m_B^2}} \tag{3.65}$$

We report the results about the branching fraction of the decays considered in tab.[3.3], they agree with the experimental values [32].

The differential decay rate relative to the decay in D^* is reported in fig.[3.3], we see that there is a very good agreement between our result, even for gaussian than for exponential wave function, and the experimental one.

Table 3.2: Values of the form factors evaluated with exponential(gaussian) wave function. The parameters a and b are the ones useful to describe the behavior of form factors as polar functions of q^2.

	F(0)	$F(q^2_{max})$ (GeV)	a (GeV^{-2})
F_0	0.64(0.61)	0.74(0.71)	0.31(0.31)
F_1	0.64(0.61)	0.98(0.95)	0.16(0.86)
V	0.69(0.67)	0.99(0.97)	0.77(0.81)
A_0	0.67(0.65)	0.97(0.95)	0.81(0.84)
A_1	0.65(0.62)	0.75(0.72)	0.31(0.33)
A_2	0.63(0.59)	0.91(0.85)	0.81(0.81)

Table 3.3: Branching ratios relative to the processes B $\rightarrow$ X l ν evaluated with exponential(gaussian) wave function. For the B meson we have used a mass of 5.279 Gev and a mean life time $\tau = 1.610^{-12}s$; this values are obtained mediating the PDG values for the charged and neutral B meson. The values of the mass of the final state are also obtained with the same kind of average about the D meson.

X	Br(%)	M_X (GeV)
D	2.00(2.01)	1.8693
D^*	0.283(0.277)	2.308

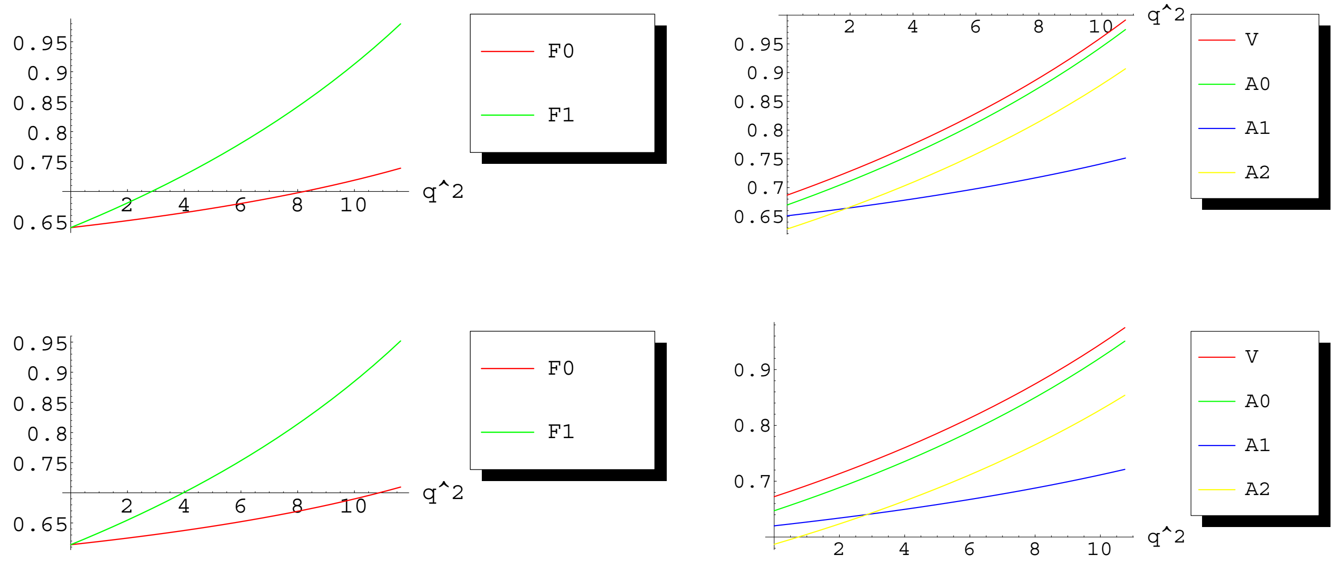


Figure 3.2: Form factors in function of q^2 obtained using the exponential (on the top) wave function, or the gaussian (on the down) one.

We can also give a prediction on the decay of the B_s into $D_s^{(*)}$ states. We just need substitute m_s to m_q in all previous formulas. What we cannot do is to fix m_s because a certain measure of some branching fraction we can calculate doesn't exist. So we can only see what happens at different values of m_s. We find that each branching fraction has a maximum for one value of this mass (we'll investigate more accurately this point in the next paragraph). In tab.[3.4] we just report the results giving to m_s the value which corresponds to the maximum branching ratio we can obtain with the exponential function. We show this particular case because, as an effect of the different phase spaces, we expect that the $\text{Br}(B_s \to D_s l\nu)$ is greater then the $\text{Br}(B \to D l\nu)$.

3.7 Decay constants

To fix the vertices of the mesons we have taken advantage of the heavy quark predictions about decay constants. Once fixed the vertices, the analytical expressions of the correspond-

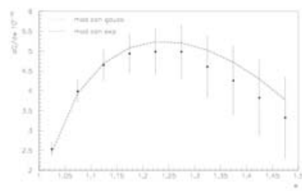


Figure 3.3: The $B \to D^* l\nu$ spectrum. The solid line corresponds to the exponential wave function, while the dashed lines to the results obtained with the gaussian wave function. The data are taken from Ref.[[25]].

Table 3.4: Results relative to the decays $B_s(5.3696GeV) \to Xl\nu$ evaluated with exp.(gauss) wave functions. The value of m_s is fixed to $0.57GeV$; this correspond to the maximum of the $\mathrm{Br}(B_s \to D_s l\nu)$, evaluated using exponential wave function.

X	Br(%)	M_X (GeV)
D_s	2.29(2.21)	1.9685
D_s^*	2.78(2.61)	2.317

ing decay constants are:

$$f(0^-) = \int_D \frac{dkk^2}{2\pi^2}\sqrt{3}\psi\sqrt{\frac{q_1{\cdot}q_2 + m_1 m_2}{E_1 E_2}}\frac{m_1+m_2}{M^2} \tag{3.66}$$

$$f(1^-) = \frac{\sqrt{3}}{M}\int_D \frac{dkk^2}{2\pi^2}\psi(E_1E_2(q_1{\cdot}q_2 + m_1m_2))^{-1/2} \\ (q_1{\cdot}q_2 + m_1m_2 - \frac{2}{3}\frac{k^2M}{M+m_1+m_2}) \tag{3.67}$$

The integration domain is strictly dominated by the difference between meson mass (M in eq.(3.66)) and light mass (m_2). As a consequence of this, for charmed mesons the integration domain is negligible compared to the region in which the integrands in eq.(3.66) give their relevant contribution. So any reliable evaluation of decay constant is possible.

Form factors describing $B \to D$ processes are not too sensible to this problem because the superposition of the two wave functions involved is great enough in the kinematical range allowed for this decay. This means that the new function that derive from this effect allows to take all necessary contributions of the integrand in the domain D.

However, this feature of our model unable us also to evaluate form factors of decays which involve light mesons. For them, in fact, we haven't found any wave function so that their superposition compensate the smallness of the domain D.

Chapter 4

Semi-leptonic weak decays of the B meson

The transitions of the B into charmed even parity mesons were studied in [13] and the heavy quark symmetries was taken into account in [15]. More recently, other quark models [27], also relativistic [28], studied the same topic.

The interest for the charmed states with $J^P = 0^+, 1^+$ was born because the prediction of their masses was believed, by theorists [29], significantly higher than observed [30].

In this chapter we want to study weak semi-leptonic decays of the B into charmed even parity mesons and compare our results with other model calculations.
In the last paragraph we'll give also some predictions about the decays of the B_s into the D_s states.

4.1 Vertex's factors

To fix the factors Γ_{J^P} for each state J^P we proceed as in the cases of the 0^- and 1^- states.

We start with the 0^+ state. Taking into account parity and angular momentum of the state we need the vertex is just a constant, say Z_0. To fix its absolute value we impose the normalization condition:

$$< 0^+|0^+ >= \int \frac{d^3k}{(2\pi)^3}|\psi|^2 Tr((-\not q_2 + m_2)(\not q_1 + m_1))\frac{|Z_0|^2}{4m_1m_2} = 2M \tag{4.1}$$

then

$$|Z_0| = \sqrt{\frac{m_1m_2}{q_1\cdot q_2 - m_1m_2}} \tag{4.2}$$

As in the cases studied before, we choose the phase of Z_0 so that the decay constant f is positive, as predicted by HQET.
We find:

$$ifp_\mu =< 0|V_\mu|H >= \int \frac{d^3k}{(2\pi)^3}\sqrt{3}\psi\sqrt{\frac{m_1m_2}{E_1E_2}}a\frac{q_{1\mu}m_2 - q_{2\mu}m_1}{m_1m_2} \tag{4.3}$$

This quantity is positive if the phase of Z is $-i\frac{\pi}{2}$. Then the vertex factor has to be:

$$\Gamma_{0^+} = -i\sqrt{\frac{m_1m_2}{q_1\cdot q_2 - m_1m_2}} \tag{4.4}$$

A bit more complicated is the case of the state $J^P = 1^+$. In fact, this state can be realized with two different combinations of the spin of the constituent quarks. To distinguish this two states we introduce the formalism $^{2S+1}L_J$, where S is the total spin, L the orbital angular momentum and J the total angular momentum. So we can refer to the states $^1P_1$ and $^3P_1$, which are both pseudo-vector mesons.

In the heavy quark limit the spin of the heavy quark and the total angular momentum j of the light one became the good quantum numbers, so, in this limit, it is more convenient to indicate the states as L^j_J. In this formalism the interesting states are: $P_1^{1/2}$, $P_1^{3/2}$, $P_0^{1/2}$.
$P_0^{1/2}$ represents exactly the scalar meson, while the other two are combinations of the states $^1P_1$ and $^3P_1$.

$$|P_1^{1/2} > = \frac{1}{\sqrt{3}}|^1P_1 > -\sqrt{\frac{2}{3}}|^3P_1 > \tag{4.5}$$

$$|P_1^{3/2} > = \frac{1}{\sqrt{3}}|^3P_1 > +\sqrt{\frac{2}{3}}|^1P_1 > \tag{4.6}$$

Moreover, we see that the states $P_1^{1/2}$ and $P_0^{1/2}$ form a doublet respect the quantum numbers in the limit. Then we can say that in the heavy quark limit they will have the same behavior.

In order to respect this theoretical constraint, we have realized that the two states have to be the same global coefficient in the vertex factors.

So we conclude that the factors of vertices $\Gamma_{^1P_1}$ and $\Gamma_{^3P_1}$ have the general form:

$$\Gamma_{^1P_1} = Z_1(\epsilon\cdot q_1 - \epsilon\cdot q_2)\gamma_5 \tag{4.7}$$

$$\Gamma_{^3P_1} = Z_2(\not{\epsilon} - Z_3\epsilon\cdot(q_1 - q_2))\gamma_5 \tag{4.8}$$

with

$$Z_0 = \frac{1}{\sqrt{3}}Z_1 - \sqrt{\frac{2}{3}}Z_2 \tag{4.9}$$

Z_1 is fixed just by the normalization condition

$$<^1 P_1|^1P_1>=\sum_\lambda \int \frac{d^3k}{(2\pi)^3}|\psi|^2 Tr((\not{q}_2+m_2)(\not{q}_1+m_1))\frac{|Z_1|^2}{4m_1m_2}\epsilon^{(\lambda)}\cdot(q_1-q_2) = 3(2M) \tag{4.10}$$

that implies

$$|Z_1| = \sqrt{\frac{3m_1m_2}{(q_1\cdot q_2 + m_1m_2)((q_1\cdot q_2 + m_2^2)^2 - M^2m_2^2)}\frac{M}{4}} \tag{4.11}$$

To fix the phase of Z_1 we use the condition, derived from HQET, that the decay constant is negative. So we get:

$$\Gamma_{^1P_1} = \epsilon\cdot(q_1 - q_2)\sqrt{\frac{6m_1m_2}{M^2 - (m_1 + m_2)^2}\frac{M}{M^2 - (m_1 - m_2)^2}}\gamma_5 \tag{4.12}$$

and the decay constant is simply:

$$f = -\int \frac{dkk^4}{\pi^2}\psi\sqrt{\frac{2}{(M^2 - (m_1 + m_2)^2)E_1E_2}\frac{m_1 - m_2}{M^2 - (m_1 - m_2)^2}} \tag{4.13}$$

At this point we can fix Z_2 using the condition (4.9). We then have:

$$Z_2 = \sqrt{\frac{3}{2}}\sqrt{\frac{m_1m_2}{q_1\cdot q_2 - m_1m_2}} \tag{4.14}$$

The last factor Z_3 can be deduced using the normalization condition:

$$\begin{aligned} <^3 P_1|^3P_1> &= \int \frac{d^3k}{(2\pi)^3}|\psi|^2\frac{1}{4m_1m_2}(Z_2^2(-2q_2\cdot\epsilon^2 + q_1\cdot q_2 - m_1m_2) \\ &\quad +4\epsilon\cdot q_2^2 a_3b_3(m_1 - m_2) + 4Z_3^2\epsilon\cdot q_2^2(q_1\cdot q_2 + m_1m_2)) \\ &= 2M \end{aligned} \tag{4.15}$$

then

$$Z_3 = \frac{m_1 - m_2}{-M^2 + (m_1 - m_2)^2} \tag{4.16}$$

The final vertex:

$$\Gamma_{^3P_1} = \sqrt{\frac{3}{2}}\sqrt{\frac{m_1 m_2}{q_1 \cdot q_2 - m_1 m_2}}(\not\epsilon - \frac{m_1 - m_2}{-M^2 + (m_1 - m_2)^2}\epsilon \cdot (q_1 - q_2))\gamma_5 \tag{4.17}$$

respects also the expected condition that the decay constant

$$\begin{aligned} f &= \frac{\sqrt{3}}{M}\int \frac{dk k^2}{2\pi^2}\psi(E_1 E_2(q_1 \cdot q_2 - m_1 m_2))^{-1/2} \\ &((q_1 \cdot q_2 - m_1 m_2) + \frac{2}{3}\frac{M k^2}{M + m_1 - m_2})) \end{aligned} \tag{4.18}$$

is positive defined.

4.2 Form factors

At this point we can evaluate the form factors of the weak semi-leptonic decays of a bq pseudo-scalar meson into cq states.

If the final meson is a scalar the definition of the form factors is analogous to the one used in the case of a $0^- \to 0^-$ process. We remember the explicit expression:

$$< P'(p')|j_\mu|P(p) >= F_+(q^2)(p + p')_\mu + F_-(q^2)(p - p')_\mu \tag{4.19}$$

On the contrary of what happens if the final meson is a pseudo-scalar, in the case the final meson is a scalar, only the axial part of the current gives contribution to the form factors. In our model we obtain:

$$\begin{aligned} &F_+(p + p')_\mu + F_-(p - p')_\mu = \int \frac{d^3k}{(2\pi)^3}\psi\psi'\sqrt{\frac{1}{E_1 E_3(m_2 m_1 + q_1 \cdot q_2)(-m_2 m_3 + q_2 \cdot q_3)}} \\ &[(-m_2 m_3 + q_2 \cdot q_3)q_{1\mu} + (m_2 m_1 + q_1 \cdot q_2)q_{3\mu} - (m_3 m_1 + q_1 \cdot q_3)q_{2\mu}] \end{aligned} \tag{4.20}$$

To extract $F_\pm$ separately we can multiply both members of the last equation for a four-vector orthogonal to $(p + p')_\mu$, to evaluate F_-, and to $(p - p')_\mu$ to obtain F_+. The vectors we need are respectively:

$$\chi^-_\mu = (\frac{|\vec{q}|}{M+E'}, 0, 0, 1) \quad (4.21)$$

$$\chi^+_\mu = (\frac{|\vec{q}|}{M-E'}, 0, 0, 1) \quad (4.22)$$

In this way we get the expressions:

$$\begin{aligned} F_+ &= \frac{1}{2M|\vec{q}|}\int \frac{d^3k}{(2\pi)^3}\psi\psi'\frac{1}{m_2}\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\ &[(c_1+c_3)(|\vec{q}|(M-E_2)-(M-E)kcos\theta) \\ &+c_2(|\vec{q}|E_2+(M-E)kcos\theta)] \end{aligned} \quad (4.23)$$

$$\begin{aligned} F_- &= -\frac{1}{2E|\vec{q}|}\int \frac{d^3k}{(2\pi)^3}\psi\psi'\frac{1}{m_2}\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\ &[c_1(|\vec{q}|(M-E_2)-(M+E)kcos\theta)+c_3(|\vec{q}|(2M-E_2+E') \\ &-(M+E)kcos\theta)+c_2(|\vec{q}|E_2+(M+E)kcos\theta)] \end{aligned} \quad (4.24)$$

where

$$c_1 = (-m_2m_3+q_2{\cdot}q_3) \quad (4.25)$$

$$c_3 = (m_2m_1+q_1{\cdot}q_2) \quad (4.26)$$

$$c_2 = (-m_3m_1-q_1{\cdot}q_3) \quad (4.27)$$

and

$$Z_I = \sqrt{\frac{m_1m_2}{q_1{\cdot}q_2+m_1m_2}} \quad (4.28)$$

$$Z_F = \sqrt{\frac{m_3m_2}{q_3{\cdot}q_2-m_3m_2}} \quad (4.29)$$

To describe a decay into a $^3P_1$ meson we define:

$$< P'|\gamma_\mu\gamma_5|P > = -2G'(q^2)\epsilon_{\mu\nu\rho\sigma}\epsilon^\nu p^\rho p'^\sigma \quad (4.30)$$

$$\begin{aligned} < P'|\gamma_\mu|P > &= -i(F'(q^2)\epsilon_\mu+(\epsilon{\cdot}p) \\ &((p+p')_\mu A'_+(q^2)+(p-p')_\mu A'_-(q^2))) \end{aligned} \quad (4.31)$$

With our model we obtain:

$$\begin{aligned}
G'(q^2) &= -\frac{1}{2|\vec{q}|M}\int d^3k\psi\psi'\frac{1}{m_2}\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\
&\quad (Mm_2|\vec{q}| + (m_3+m_2)Mkcos\theta \\
&\quad +(m_2-m_1)(kcos\theta E' - E_2|\vec{q}|) \\
&\quad -2ksen\theta cos\phi\epsilon^{(1)}\cdot q_2 M|\vec{q}|Z'_F) \qquad (4.32) \\
F'(q^2) &= \int d^3k\ \psi\psi'\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\
&\quad (c_3^*(\epsilon^{(2)}\cdot q_2)^2 - c_4) \qquad (4.33) \\
A'_+(q^2)+A'_-(q^2) &= -\frac{1}{(\epsilon^{(3)}\cdot p)^2}\int d^3k\ \psi\psi'\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\
&\quad (c_1^{(3)}\epsilon^{(3)}\cdot p + c_3^*((\epsilon^{(3)}\cdot q_2)^2 - (\epsilon^{(2)}\cdot q_2)^2) \\
&\quad +(m_2-m_3)(\epsilon^{(3)}\cdot p\epsilon^{(3)}\cdot q_2)) \qquad (4.34) \\
A'_+(q^2)-A'_-(q^2) &= -\frac{1}{(\epsilon^{(3)}\cdot p)|\vec{q}|}\int d^3k\ \psi\psi'\sqrt{\frac{m_1m_3}{E_1E_3}}Z_IZ_F \\
&\quad (|\vec{q}|c_2^{(3)} + kcos\theta c_3^{(3)} \\
&\quad +\frac{E_{D^*}}{M_{D^*}}c_3^*(\epsilon^{(2)}\cdot q_2)^2) \qquad (4.35)
\end{aligned}$$

where

$$\begin{aligned}
c_1^* &= (-m_2+m_3+2Z'_F(q_2\cdot q_3+m_2m_3)) \\
c_1(\lambda) &= c_1^*\epsilon^{(\lambda)}\cdot q_2 \\
c_2^* &= -m_1+m_2+2Z'_F(q_1\cdot q_2+m_2m_1) \\
c_2(\lambda) &= c_2^*\epsilon^{(\lambda)}\cdot q_2 + m_2\epsilon^{(\lambda)}\cdot p \\
c_3^* &= -c_1^*-c_2^*-m_1+m_3-2Z'_F(q_1\cdot q_3-m_1m_3) \\
c_3(\lambda) &= c_3^*\epsilon^{(\lambda)}\cdot q_2 + (m_2-m_3)\epsilon^{(\lambda)}\cdot p \\
c_4 &= -m_2q_1\cdot q_3 - m_1q_2\cdot q_3 + m_3q_1\cdot q_2 + m_2m_1m_3 \qquad (4.36)
\end{aligned}$$

$$Z'_F = \frac{m_2-m_3}{M_F^2-(m_3-m_2)^2} \qquad (4.37)$$

In an analogous way, if the final meson of the decay considered is in a ${}^1P_1$ state, we put:

$$
\begin{aligned}
< P'|\gamma_\mu\gamma_5|P > &= -2G(q^2)\epsilon_{\mu\nu\rho\sigma}\epsilon^\nu p^\rho p'^\sigma && (4.38)\\
< P'|\gamma_\mu|P > &= -i(F(q^2)\epsilon_\mu + (\epsilon\cdot p) \\
&\quad ((p+p')_\mu A_+(q^2) + (p-p')_\mu A_-(q^2))) && (4.39)
\end{aligned}
$$

We find that our model predicts that this form factors have the same form of the ones referred to the ${}^3P_1$ state, with the substitutions:

$$
\begin{aligned}
c_1 &= q_2\cdot q_3 + m_2 m_3 \\
c_2 &= -q_1\cdot q_3 + m_1 m_3 \\
c_3 &= q_1\cdot q_2 + m_1 m_2 \\
c_4 &= 0 && (4.40)
\end{aligned}
$$

$$
\begin{aligned}
Z_F &= 1 \\
Z'_F &= -\frac{M_F}{M_F^2-(m_3-m_2)^2}\sqrt{\frac{6}{M_F^2-(m_2+m_3)^2}} && (4.41)
\end{aligned}
$$

and the $G(q^2)$ is formed just by the part proportional to Z'_F.

Also for this form factors we can give an alternative definition, more often used in literature, in the same way used in eq.(3.58). The right sign to use in the definitions of the form factors is fixed by the HQET results.

In tab.[4.1] we report the results relative to the form factors useful to calculate the physically interesting branching fractions.

As we can see in fig.[4.1] our form factors have a polar or a linear behavior in function of the momentum transferred in the transition. In particular we can see that they can be fitted using the functions:

$$
f_i(q^2) = \frac{f_i(0)}{1-a\frac{q^2}{M^2}} \tag{4.42}
$$

or

$$
f_i(q^2) = f_i(0)(1-cq^2) \tag{4.43}
$$

The coefficients a and c are given in table [4.2].

Table 4.1: We report the results of form factors relative to the semi-leptonic decays of a B pseudo-scalar meson into some D exited states. Our results are confronted to the ones of the ISGW2 model [17] and to the ones calculated with a light-front quark model [28].

Form Factor	This work		Ref. [17]		Ref. [28]	
	F(0)	F(q^2_{max})	F(0)	F(q^2_{max})	F(0)	F(q^2_{max})
F_1	-0.32 (-0.30)	-0.35 (-0.33)	-0.18	-0.24	-0.24	-0.34
F_0	-0.32 (-0.30)	-0.025 (-0.036)	-0.18	0.008	-0.24	-0.20
$A_0^{(1/2)}$	-0.25 (-0.23)	-0.31 (-0.28)	-0.18	-0.39	-0.075	-0.083
$A_1^{(1/2)}$	0.096 (0.088)	-0.0018 (-0.0029)	0.070	-0.002	0.073	0.071
$A_2^{(1/2)}$	0.69 (0.63)	0.87 (0.79)	0.49	0.91	0.32	0.56
$V^{(1/2)}$	0.67 (0.61)	0.84 (0.76)	0.44	0.81	0.31	0.55
$A_0^{(3/2)}$	-0.61 (-0.58)	-0.81 (-0.77)	-0.20	-0.46	-0.47	-0.76
$A_1^{(3/2)}$	-0.13 (-0.13)	-0.016 (-0.023)	-0.005	-0.008	-0.20	-0.26
$A_2^{(3/2)}$	0.70 (0.65)	1.27 (1.19)	0.33	0.72	0.25	0.47
$V^{(3/2)}$	-0.81 (-0.77)	-1.09 (-1.03)	-0.44	-0.71	-0.61	-1.24

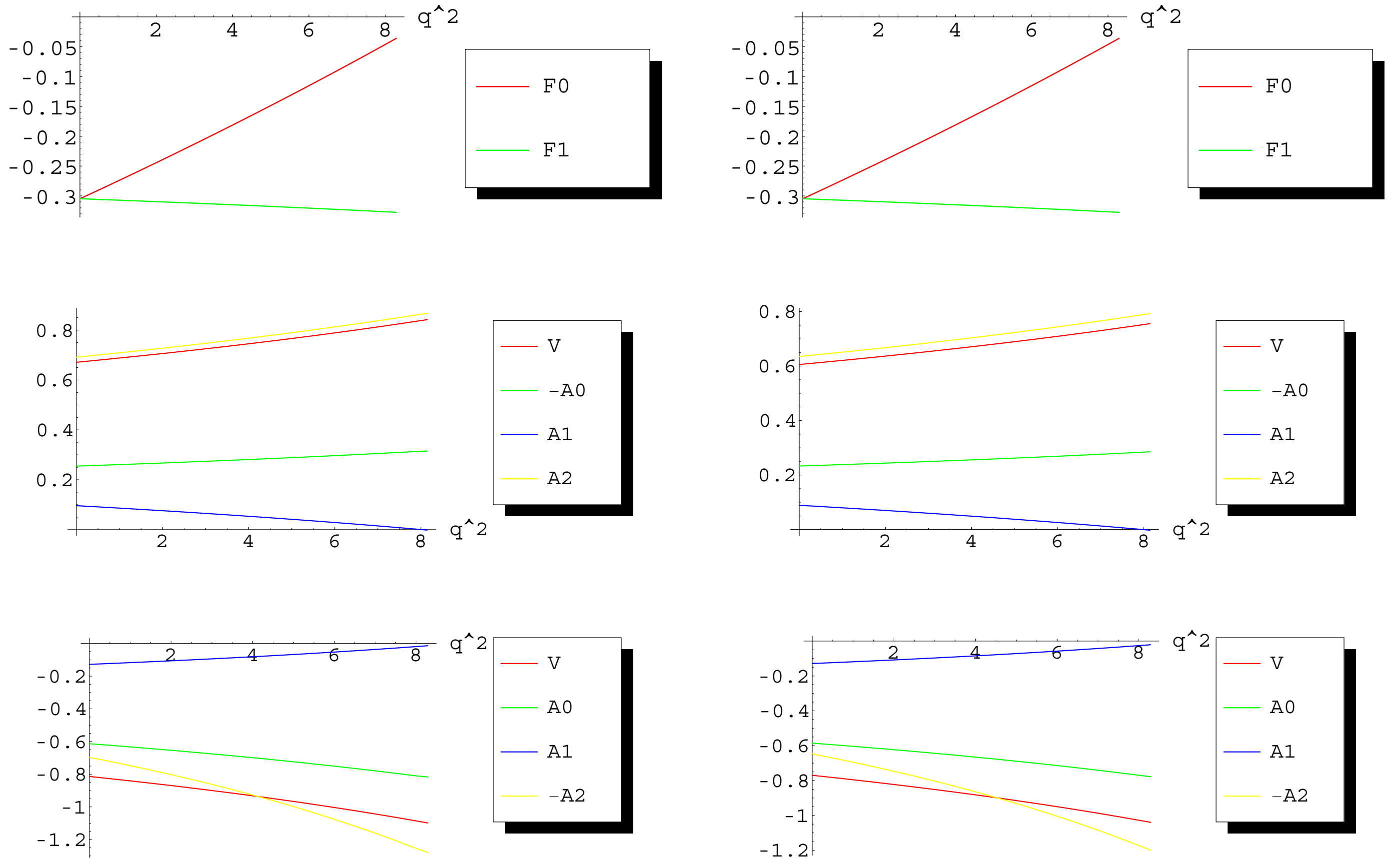


Figure 4.1: Form factors of the physical states , $P_0^{1/2}$, $P_1^{1/2}$, $P_1^{3/2}$, from the top to the bottom, in function of q^2. The different results for each decay are obtained using the exponential (on the left) wave function, or the gaussian one (on the right).

4.3 Numerical results on the branching fractions

At this point we are ready to evaluate the decay rates we are interested in.
Some numerical results we have obtained are reported in tab.[4.3].

To evaluate the branching fractions of the B_s's decays we should fix the mass of the strange quark. This is not possible as for the case of m_q because there is not, until now, an experimental datum on the semi-leptonic decay rate of B_s. What we have studied is the behavior of our predicted branching fractions as a function of the ratio m_s/m_q; in this way we see, fig.[4.2], that they increase with m_s until a pick after which they decrease. Moreover we

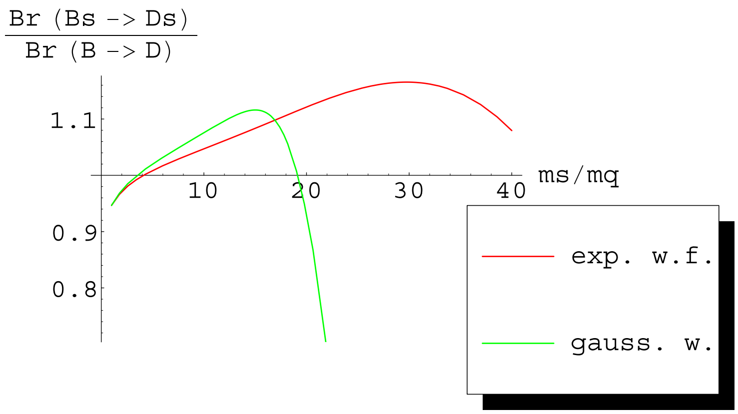


Figure 4.2:

observe that the value of the pick depends on the wave function used.
An important feature of the model is that with $m_q < m_s < \tilde{m}_s$, we can obtain any value between $Br(B \to Dl\nu)$ and the one in the tab.[4.4]. An experimental result would be sufficient to fix the last free parameter of our model.
In the tab.[4.4] we report the results obtained using a value of m_s, say $\tilde{m}_s$, chosen arbitrarily, supposing that the branching fraction of the decay of the B_s into a $D_s(0^-)$ is bigger then the one of the process $B \to D$ because of the enlargement of the phase space.

If we'll have a measure of the $Br(B_s \to D_s l\nu)$ and of the ratio m_s/m_q, all our parameters will be fixed and the agreement between our predictions and experimental values will be significantly dependent by the wave function used. This justifies strongly the necessity of the study of the two cases, initially introduced for the different values of the single form factors they bring to.

From all our results about the branching ratio, we can conclude that the model is in good agreement with experiments for the decays $B \to D^{(*)}l\nu$ and that it predicts that the final state $D_1^{3/2}$ is dominant respect to the $D_1^{1/2}$. This is in agreement with what the Bakamjian-Thomas quark model has demonstrated [33].

Table 4.2: We report the results of the fit of the form factors on the functions of the form in eq.[4.42] and [4.43] for exponential (gaussian) wave function.

Form Factor	F(0)	a	c(GeV^{-2})
F_1	-0.32 (-0.30)	0.263(0.233)	
F_0	-0.32 (-0.30)		-0.109 (-0.103)
$A_0^{(1/2)}$	-0.25 (-0.23)	0.661(0.626)	
$A_1^{(1/2)}$	0.096 (0.088)		-0.12 (-0.12)
$A_2^{(1/2)}$	0.69 (0.63)	0.695(0.680)	
$V^{(1/2)}$	0.67 (0.61)	0.695(0.679)	
$A_0^{(3/2)}$	-0.61 (-0.58)	0.846(0.834)	
$A_1^{(3/2)}$	-0.13 (-0.13)		-0.10 (-0.096)
$A_2^{(3/2)}$	0.70 (0.65)	1.53 (1.54)	
$V^{(3/2)}$	-0.81 (-0.77)	0.872(0.873)	

Table 4.3: Branching fractions of the processes B(5.279 GeV) $\rightarrow$ X l ν obtained using the exponential (gaussian) wave function, in the case of a final lepton with negligible mass (l, μ) and when the decay produces a τ.

X	Br(%)	M_X (GeV)
$D_0 l \bar{\nu}_l$	2.00(2.01)	1.8693
$D^* l \bar{\nu}_l$	5.98(5.99)	2.0067
$D_0^* l \bar{\nu}_l$	0.283(0.277)	2.308
$D_1^{1/2} l \bar{\nu}_l$	0.206(0.185)	2.427
$D_1^{3/2} l \bar{\nu}_l$	0.830(0.827)	2.427
$D_0 \tau \bar{\nu}_\tau$	0.54(0.54)	1.8693
$D^* \tau \bar{\nu}_\tau$	1.61(1.61)	2.0067
$D_0^* \tau \bar{\nu}_\tau$	0.018(0.018)	2.308
$D_1^{1/2} \tau \bar{\nu}_\tau$	0.015(0.013)	2.427
$D_1^{3/2} \tau \bar{\nu}_\tau$	0.057(0.057)	2.427

Table 4.4: Branching fractions of decays B_s(5.3696 GeV) $\rightarrow$ X l ν evaluated using the exponential (gaussian) wave function. The value $m_s = 0.57 GeV$ used is the one for which the rate into the D_{s0} state, evaluated with the exponential wave function, is maximum.

X	Br(%)	M_X (GeV)
D_{s0}	2.29(2.21)	1.9685
D^*_{s0}	2.78(2.61)	2.317
$D^{1/2}_{s1}$	2.78(2.51)	2.457
$D^{3/2}_{s1}$	3.46(3.27)	2.457

Chapter 5

The model in the heavy quark limit

The consistency between quark models and theory can be checked studying the results of the model in the limit of infinite heavy quark mass. We expect that, in this limit, the model agree with the HQS predictions.
Moreover one of the most interesting feature of a model is how much its results deviate from the HQS predictions.

In this chapter we will study these aspects of our model.

5.1 The method

The way to extract the heavy quark dependence from form factors is to expand in power of m_q/m_Q, where m_q is the light spectator quark in the transition and m_Q is one of the two heavy quarks. In this limit we imagine that the mass of each heavy meson corresponds to the mass of the heavy constituent. So it is sensed to use, as small parameter, $z = m_q/M'$, where M' is the mass of the final meson in the process under analysis.

The problem now is that to extract the form factors we have to perform an integration over a domain D which depends by z. Moreover the integral cannot be evaluated analytically.
So the limit is not straightforward.
To do it we introduce a new variable of integration $x = 2\alpha k/M'$, with $0 < \alpha \ll 1$. So the

domain of integration doesn't depend anymore by the mass and the new variable x is small, this allows us to expand the integrand also near $x \simeq 0$; we'll consider truncated series in x truncated to the second order.

We know the results of HQET just near the cinematical point $w = 1$ (with $w = v{\cdot}v'$), that means at the maximum momentum transferred in the transition; so we need just to consider the development of our form factors in this region.
w is related to q^2 as shown by the following relation:

$$w = \frac{M^2 + M'^2 - q^2}{2MM'} \tag{5.1}$$

With $w \simeq 1$ the domain of integration is simplified because the angle θ is free to vary in the range $(-\pi, \pi)$, then this integration is very simple.

Finally every form factor $f_i(q^2)$ takes the form:

$$f_i(q^2) = \int_0^\alpha dx \psi(k(x))\psi'(k(x))\tilde{f}_i(x, z, q^2) \tag{5.2}$$

The integrand $\tilde{f}_i$, after algebraic manipulations, will be a function proportional to x^2 and to the Isgur-Wise function.
Obviously, the results will depend on the wave function, but just by a factor.

5.2 Results and comparison with theory

With the method described in the previous paragraph we find for each form factor the dependence from the mass of the heavy meson near the zero recoil point.

To connect our results with the theoretical prediction we report here the usual definition [14] in HQET of the matrix elements of the transition we are considering:

$$
\begin{aligned}
< D(v')|V_\mu|B(v) > &= \xi(w)(v+v')_\mu & (5.3)\\
< D^*(v',\epsilon)|V_\mu|B(v) > &= -\xi(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}v'^\alpha v^\beta & (5.4)\\
< D^*(v',\epsilon)|A_\mu|B(v) > &= \imath\xi(w)((1+w)\epsilon^*_\mu-(\epsilon^*\cdot v)v'_\mu) & (5.5)\\
< D_0^*(v')|A_\mu|B(v) > &= 2\imath\tau_{1/2}(w)(v-v')_\mu & (5.6)\\
< D_1^{1/2}(v',\epsilon)|V_\mu|B(v) > &= -\imath\tau_{1/2}(w)((1+w)\epsilon^*_\mu+(\epsilon^*\cdot v)v'_\mu) & (5.7)\\
< D_1^{1/2}(v',\epsilon)|A_\mu|B(v) > &= -2\tau_{1/2}(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}v'^\alpha v^\beta & (5.8)\\
< D_1^{3/2}(v',\epsilon)|V_\mu|B(v) > &= \imath\frac{1}{\sqrt{2}}\tau_{3/2}(w)((1-w^2)\epsilon^*_\mu-(\epsilon^*\cdot v)(3v_\mu+(2-w)v'_\mu)) & (5.9)\\
< D_1^{3/2}(v',\epsilon)|A_\mu|B(v) > &= \frac{1}{\sqrt{2}}\tau_{3/2}(w)(1+w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}v'^\alpha v^\beta & (5.10)
\end{aligned}
$$

So the relations between these form factors and the our ones are:

$$
\begin{aligned}
\xi(w) &= \frac{1}{2\sqrt{MM'}}((M+M')f_+(q^2)+(M-M')f_-(q^2))\\
&= -\frac{1}{\sqrt{MM'}}\frac{f(q^2)}{1+w} = -2\sqrt{MM'}g(q^2)\\
&= \sqrt{MM'}(a_+(q^2)-a_-(q^2))\ , & (5.11)\\
0 &= a_+(q^2)+a_-(q^2)\\
&= (M-M')f_+(q^2)+(M+M')f_-(q^2)\ . & (5.12)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{1/2}(w) &= \frac{1}{4\sqrt{MM'}}((M-M')F_+(q^2)+(M+M')F_-(q^2))\\
&= \frac{1}{2\sqrt{MM'}}\frac{f_{1/2}(q^2)}{w-1} = \sqrt{MM'}g_{1/2}(q^2)\\
&= -\frac{\sqrt{MM'}}{2}(a_+^{1/2}(q^2)-a_-^{1/2}(q^2))\ , & (5.13)\\
0 &= a_+^{1/2}(q^2)+a_-^{1/2}(q^2)\\
&= (M+M')F_+(q^2)+(M-M')F_-(q^2)\ . & (5.14)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{3/2}(w) &= -\sqrt{\frac{2}{MM'}}\frac{f_{3/2}(q^2)}{w^2-1} = -\frac{2\sqrt{2}}{w+1}\sqrt{MM'}g_{3/2}(q^2)\\
&= -\frac{\sqrt{2M^3}}{M'}\frac{a_+^{3/2}(q^2)-a_-^{3/2}(q^2)}{w-2}\\
&= -\frac{1}{3}\frac{\sqrt{2M^3}}{M'}(a_+^{3/2}(q^2)+a_-^{3/2}(q^2))\ . & (5.15)
\end{aligned}
$$

In the subsections below we report our form factors in the limit and we'll see that the relations written above are valid in the model.

5.2.1 $B \to D^{(*)} l\nu$

The behavior in the infinite heavy quark limit of our form factors, for the processes $B \to D^{(*)} l\nu$, is summarized by the equations below:

$$f_\pm(q^2)|_{q^2 \simeq q^2_{max}} \simeq \frac{M \pm M'}{2\sqrt{MM'}} \begin{cases} 2\sqrt{2}\left(\frac{\omega\omega'}{\omega^2+\omega'^2}\right)^{3/2} \xi(q^2)\ ; & gaussian\ \ w.f., \\ \frac{\sqrt{\omega^3\omega'^3}}{(\omega+\omega')^3}\ \xi(q^2)\ ; & exponential\ \ w.f. \end{cases} \tag{5.16}$$

$$a_\pm(q^2)|_{q^2 \simeq q^2_{max}} \simeq \mp\frac{1}{2\sqrt{MM'}} \begin{cases} 2\sqrt{2}\left(\frac{\omega\omega'}{\omega^2+\omega'^2}\right)^{3/2} \xi(q^2)\ ; & gaussian\ \ w.f., \\ \frac{\sqrt{\omega^3\omega'^3}}{(\omega+\omega')^3}\ \xi(q^2)\ ; & exponential\ \ w.f. \end{cases} \tag{5.17}$$

$g(q^2)$ has the same behavior of $a_-(q^2)$.

$$f(q^2)|_{q^2 \simeq q^2_{max}} \simeq \sqrt{MM'}(1+w) \begin{cases} 2\sqrt{2}\left(\frac{\omega\omega'}{\omega^2+\omega'^2}\right)^{3/2} \xi(q^2)\ ; & gaussian\ \ w.f., \\ \frac{\sqrt{\omega^3\omega'^3}}{(\omega+\omega')^3}\ \xi(q^2)\ ; & exponential\ \ w.f. \end{cases} \tag{5.18}$$

From this equations we see that each wave function characterizes the expression of the form factor with a fixed factor. For the gaussian and exponential wave functions this factor is:

$$N = \begin{cases} 2\sqrt{2}\left(\frac{\omega\omega'}{\omega^2+\omega'^2}\right)^{3/2}\ ; & gaussian\ \ w.f., \\ \frac{\sqrt{\omega^3\omega'^3}}{(\omega+\omega')^3}\ ; & exponential\ \ w.f. \end{cases} \tag{5.19}$$

The Isgur-Wise function ξ we predict is:

$$\xi(w) = 1 - \frac{11}{12}(w-1) + \frac{77}{96}(w-1)^2 + o((w-1)^3) \tag{5.20}$$

Sum rules give a constraint on the slope [35] (Bjorken sum rule) and the curvature [31] of ξ, they are:

$$\rho^2 = -\xi'(1) \geq \frac{3}{4} \tag{5.21}$$

$$\sigma^2 = \xi''(1) \geq \frac{4}{5}\rho^2\left(1 + \frac{3}{4}\rho^2\right) \tag{5.22}$$

Our model satisfy both constraints, in fact:

$$\rho^2 = \frac{11}{12} \geq \frac{3}{4} \tag{5.23}$$

$$\sigma^2 = \frac{77}{48} \geq \frac{4}{5}\rho^2\left(1+\frac{3}{4}\rho^2\right) = \frac{99}{80} \tag{5.24}$$

5.2.2 $B \to D_0^*(D_1^{1/2,3/2})l\nu$

For the form factors relevant in the $B \to D_0^*(D_1^{1/2,3/2})l\nu$ decays, our model predicts:

$$F_\pm(q^2)|_{q^2\simeq q^2_{max}} \simeq -\frac{M \pm M'}{\sqrt{MM'}} N\tau_{1/2}(w) \tag{5.25}$$

where N is the same of eq.(5.19) and

$$\tau_{1/2}(w) = \frac{1}{3} - \frac{1}{4}(w-1) + \frac{19}{96}(w-1)^2 + o((w-1)^3) \tag{5.26}$$

The form factors $G^{(\prime)}, F^{(\prime)}, A_\pm^{(\prime)}$ are not directly comparable with the ones of the effective theory because they do not refer to the states considered by HQET. To compare our results with the HQET we combine the functions $G^{(\prime)}, F^{(\prime)}, A_\pm^{(\prime)}$ so to obtain the form factors of the B transition into the states $D_1^{1/2,3/2}$, which are the physical states in the heavy quark limit. We use the relations:

$$f_i^{3/2} = \sqrt{\frac{2}{3}} f_1 + \frac{1}{\sqrt{3}} f_i' \tag{5.27}$$

$$f_i^{1/2} = -\sqrt{\frac{2}{3}} f_1' + \frac{1}{\sqrt{3}} f_i \tag{5.28}$$

The heavy quark limit of this form factors are:

$$\left\{\begin{array}{l} g^{1/2}(q^2) \\ f^{1/2}(q^2) \\ a_\pm^{1/2}(q^2) \end{array}\right\} = N \left\{\begin{array}{l} \sqrt{MM'} \\ 2\sqrt{MM'}(w-1) \\ \mp 1/\sqrt{MM'} \end{array}\right\} \tau^{1/2}(w) \tag{5.29}$$

$$\left\{\begin{array}{l} g^{3/2}(q^2) \\ f^{3/2}(q^2) \\ a_+^{3/2}(q^2) \\ a_-^{3/2}(q^2) \end{array}\right\} = N \left\{\begin{array}{l} -\frac{1+w}{2\sqrt{2MM'}} \\ -2\sqrt{\frac{MM'}{2}}(w^2-1) \\ -\sqrt{\frac{M'}{2M^3}}\frac{2(w-2)}{w-3} \\ -\sqrt{\frac{M'}{2M^3}}\frac{6(w-2)}{w+1} \end{array}\right\} \tau^{3/2}(w) \tag{5.30}$$

where:

$$\tau_{3/2}(w) = \frac{5}{6} - \frac{31}{24}(w-1) + \frac{93}{64}(w-1)^2 + o((w-1)^3) \tag{5.31}$$

Our results about the decays into the states $P_{0,1}^{1/2}$ satisfy HQET constraints.
We cannot try any conclusion on the $P_1^{3/2}$ meson state because, until now, we have said nothing about the tensorial state which, in the heavy quark effective theory, should be characterized by the same Isgur-Wise function. We'll see something more in the next chapter.

Uraltsev [34] has derived the relation:

$$\sum_n |\tau_{3/2}^{(n)}(1)|^2 - \sum_n |\tau_{1/2}^{(n)}(1)|^2 = \frac{1}{4} \tag{5.32}$$

where n is the radial excitation.
Moreover, we have an other sum rule, due to Bjorken [35], on the slope of the ξ, which connects the form factors between them:

$$\rho^2 = \frac{1}{4} + \sum_n |\tau_{1/2}^{(n)}(1)|^2 + 2\sum_n |\tau_{3/2}^{(n)}(1)|^2 \tag{5.33}$$

Combining this equation with the eq.[5.32], we get:

$$\rho^2 = \frac{3}{4} + 3\sum_n |\tau_{1/2}^{(n)}(1)|^2 \tag{5.34}$$

In our case $n = 0$ and eq.(5.32) is not an identity because we don't take into account all radial excitations, so it becomes:

$$|\tau_{3/2}(1)|^2 - |\tau_{1/2}(1)|^2 \leq \frac{1}{4} \tag{5.35}$$

Our model violates both sum rules.
We don't know the reason; it is possible that the states with radial excitation bring the necessary terms to get results in agreement with eq.(5.32) and (5.33).

We can compare our predictions of the τ functions with others results in different models (tab.[5.1]). We can observe that our value of $\tau_{3/2}(1)$ is higher than the others, but $\tau_{1/2}(1)$ and all the slopes are similar to these ones. For a discussion on sum rules results and the findings of quark models, see the reference [40].

Table 5.1: The Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ at zero recoil and their slope parameters.

$\tau_{1/2}(1)$	$\rho^2_{1/2}$	$\tau_{3/2}(1)$	$\rho^2_{3/2}$	Ref.
0.33	0.75	0.83	1.29	This work
0.34	0.76	0.59	1.09	[13]
0.22	0.83	0.54	1.5	[36]
0.31	1.18	0.61	1.73	[28]
0.13 ± 0.04	0.50 ± 0.05	0.43 ± 0.09	0.90 ± 0.05	QCDSR [37]
0.35 ± 0.08	2.5 ± 1.0	–	–	QCDSR(NLO)[38]
0.38 ± 0.04		0.53 ± 0.08		Lattice [39]

5.3 Deviations from HQET

To understand the order of the correction to the form factors in the heavy quark limit deriving from the finite value of the mass of the light quark, it is interesting to look at the differences between the form factors of the model evaluated in the limit and the exact ones. We report in fig.5.1 the behaviour of such differences for the form factors f_+, f_- and A_1. This form factors are the more representative because each of them have a different relation with the correspondent Isgur and Wise function.
Generally we see that the corrections for a finite value of m_q are decreasing starting from $q^2 = 0$; in fact we know that the HQET is valid for $q^2 \simeq q^2_{max}$. Moreover we can conclude that the correction is at most of the order of the 10%.

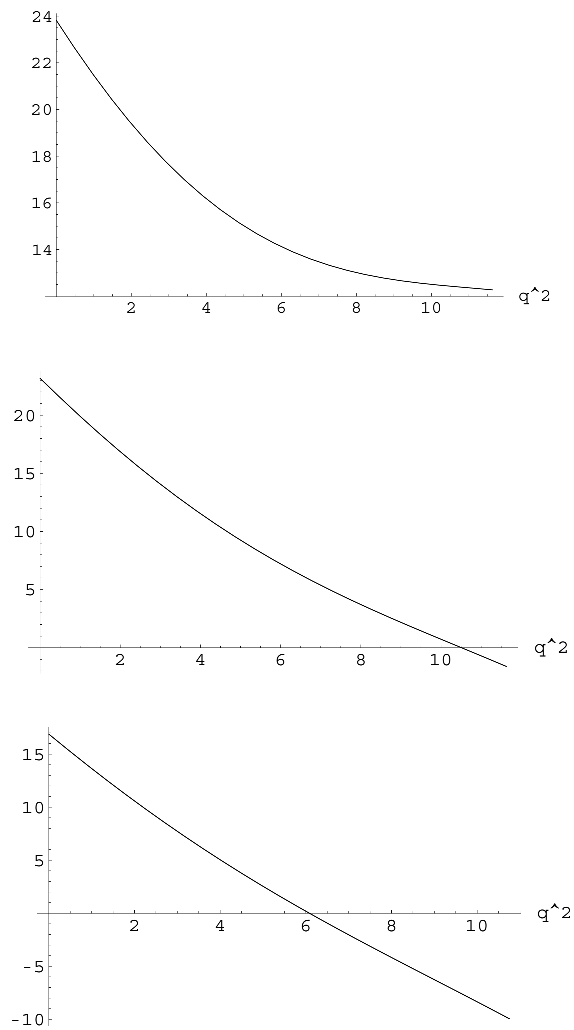


Figure 5.1: In this figures are reported, in function of $q^2(GeV)$, the values of the percentage difference between the exact values of the form factors and the ones in the heavy quark limit. The form factors here considered are respectively, from the top to the bottom, f_+, f_- and A_1. We have considered the results obtained with the exponential wave function.

Chapter 6

Tensorial state

To complete the semi-leptonic decays of the B pseudo-scalar meson into D states without radial excitation, it just fails us the case of a tensor in the final meson. This state is identified by the quantum numbers ${}^3P_2$. We'll indicate the meson under consideration as D_2^*.
The mass of the D_2^* is $2.460 GeV$.

In our model we can evaluate form factors also in this case.

6.1 Representation of the state in the model

As for the other mesons studied here, to represent the tensor meson (${}^{2S+1}L_J ={}^3 P_2$) in our model, first of all we need to fix the vertex factor. To do this we observe that (${}^3P_2$, $P_1^{3/2}$) form a doublet in the heavy quark limit, so they have the same behavior in this limit. To realize this we impose that the global coefficient of the tensorial state is the same of the $P_1^{3/2}$ one. So we have:

$$\Gamma_{{}^3P_2} = \sqrt{\frac{6m_1 m_2}{M^2-(m_1+m_2)^2}}\frac{2M}{M^2-(m_1-m_2)^2}\epsilon^{\mu\nu}(\gamma_\mu + Zq_{2\mu})q_{2\nu} \tag{6.1}$$

The polarization tensors of this state are related to the vector polarization in the way shown below ([41]):

$$\epsilon^{\mu\nu}(\pm 2) = \frac{1}{2}(\epsilon^\mu(1)\epsilon^\nu(1) - \epsilon^\mu(2)\epsilon^\nu(2) \pm \imath\epsilon^\mu(1)\epsilon^\nu(2) \pm \imath\epsilon^\mu(2)\epsilon^\nu(1)) \quad (6.2)$$

$$\epsilon^{\mu\nu}(\pm 1) = \mp\frac{1}{2}(\epsilon^\mu(1)\epsilon^\nu(0) + \epsilon^\mu(0)\epsilon^\nu(1) \pm \imath\epsilon^\mu(0)\epsilon^\nu(2) \pm \imath\epsilon^\mu(2)\epsilon^\nu(0)) \quad (6.3)$$

$$\epsilon^{\mu\nu}(\pm 0) = -\frac{1}{\sqrt{6}}(\epsilon^\mu(1)\epsilon^\nu(2) + \epsilon^\mu(2)\epsilon^\nu(1)) + \sqrt{\frac{2}{3}}\epsilon^\mu(0)\epsilon^\nu(0) \quad (6.4)$$

The factor Z can be fixed by the normalization condition for all polarizations and doing the hypothesis that it is real:

$$2M = \frac{1}{5}\sum_\lambda \int_D \frac{d^3k}{(2\pi)^3}|\psi|^2 \frac{6}{M^2-(m_1+m_2)^2}\frac{4M^2}{(M^2-(m_1-m_2)^2)^2}\epsilon^{*\mu\nu}(\lambda)\epsilon^{\rho\sigma}(\lambda)q_{2\mu}q_{2\rho}$$
$$((2+Z^2(-q_1\cdot q_2+m_1m_2)-2Z(m_1+m_2))q_{2\nu}q_{2\sigma}+g_{\nu\sigma}(q_1\cdot q_2+m_1m_2)) \quad (6.5)$$

and then:

$$Z = \frac{2}{M+m_1+m_2} \quad (6.6)$$

6.2 Form factors

The form factors which describe the transition $B(^1S_0) \to D(^3P_2)$ are:

$$< D|\gamma_\mu|B > = g(q^2)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu\lambda}p_\lambda p^\alpha q^\beta \quad (6.7)$$

$$< D|\gamma_\mu\gamma_5|B > = -\imath\{f(q^2)\epsilon_{\mu\nu}p^\nu + \epsilon_{\alpha\beta}p^\alpha p^\beta[(p+p')_\mu a_+(q^2) + q_\mu a_-(q^2)]\} \quad (6.8)$$

In our model we have:

$$
\begin{aligned}
< D|\gamma_\mu|B > \;=\;& \imath \int_D \frac{d^3k}{(2\pi)^3}\psi\psi' \sqrt{\frac{1}{E_1E_3(q_1{\cdot}q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}} \\
& \frac{2M'}{M'^2-(m_2-m_3)^2}\frac{1}{4}((m_3-m_2)(-k_x^2\frac{ME'}{M'}+ \\
& \epsilon(0){\cdot}q_2k_zM)-(m_1-m_2)(-k_x^2M'+\epsilon(0){\cdot}q_2(k_zE'-|\vec{q}|E_2))+ \\
& |\vec{q}|Mm_2\epsilon(0){\cdot}q_2+\frac{4}{M'+m_3+m_2}k_x^2M|\vec{q}|\epsilon(0){\cdot}q_2) \qquad (6.9) \\
< D|\gamma_\mu\gamma_5|B > \;=\;& -\imath \int_D \frac{d^3k}{(2\pi)^3}\psi\psi' \sqrt{\frac{1}{E_1E_3(q_1{\cdot}q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}} \\
& \frac{2M'}{M'^2-(m_2-m_3)^2}\frac{1}{2}\epsilon^{\alpha\beta}(\lambda)q_{2\beta} \\
& (c_0g_{\mu\alpha}+c_1q_{2\alpha}p_\mu+c_2q_{2\alpha}p'_\mu+c_3p_\alpha p'_\mu+c_4q_{2\alpha}q_{2\mu}+c_5p_\alpha q_{2\mu}) \qquad (6.10)
\end{aligned}
$$

where

$$
\begin{aligned}
c_0 &= m_2q_1{\cdot}q_3+m_1q_2{\cdot}q_3+m_3q_1{\cdot}q_2+m_1m_2m_3 && (6.11) \\
c_1 &= (m_2+m_3+2\frac{q_1{\cdot}q_3+m_1m_3}{M'+m_3+m_2} && (6.12) \\
c_2 &= m_2-m_1-2\frac{q_1{\cdot}q_2+m_1m_2}{M'+m_3+m_2} && (6.13) \\
c_3 &= -m_2 && (6.14) \\
c_4 &= 2(m_1-m_2)+2\frac{q_1{\cdot}q_3+m_1m_3+q_1{\cdot}q_2+m_1m_2-q_3{\cdot}q_2+m_3m_2}{M'+m_3+m_2} && (6.15) \\
c_5 &= m_2-m_3 && (6.16)
\end{aligned}
$$

From these expressions, fixing the polarization and choosing different values of μ it is straightforward to evaluate the explicit expressions of all form factors.

$$
\begin{aligned}
g \;=\;& \imath \int_D \frac{d^3k}{(2\pi)^3}\psi\psi' \sqrt{\frac{1}{E_1E_3(q_1{\cdot}q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}\frac{2M'}{M'^2-(m_2-m_3)^2}} \\
& \frac{1}{8}\frac{M'}{M^2|\vec{q}|^2}((m_3-m_2)(-k_x^2\frac{ME'}{M'}+\epsilon(0){\cdot}q_2k_zM) \\
& -(m_1-m_2)(-k_x^2M'+\epsilon(0){\cdot}q_2(k_zE'-|\vec{q}|E_2)) \\
& +|\vec{q}|Mm_2\epsilon(0){\cdot}q_2+\frac{4}{M'+m_3+m_2}k_x^2M|\vec{q}|\epsilon(0){\cdot}q_2) \qquad (6.17)
\end{aligned}
$$

$$
\begin{aligned}
f &= \frac{1}{\epsilon(0)\cdot p}\int_D \frac{d^3k}{(2\pi)^3}\psi\psi'\sqrt{\frac{1}{E_1E_3(q_1\cdot q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}} \\
&\quad \frac{2M'}{M'^2-(m_2-m_3)^2}\frac{1}{2}\epsilon^{\alpha\beta}(1)q_{2\beta}(c_0g_{1\alpha}-k_x(c_4q_{2\alpha}+c_5p_\alpha)) \qquad (6.18)
\end{aligned}
$$

$$
\begin{aligned}
a_- - a_+ &= f\frac{E'}{M'|q|\epsilon(0)\cdot p}-\frac{1}{|q|(\epsilon(0)\cdot p)^2}\sqrt{\frac{3}{2}}\int_D\frac{d^3k}{(2\pi)^3}\psi\psi' \\
&\quad \sqrt{\frac{1}{E_1E_3(q_1\cdot q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}}\frac{2M'}{M'^2-(m_2-m_3)^2}\frac{1}{2}\epsilon^{\alpha\beta}(0)q_{2\beta} \\
&\quad (c_0g_{3\alpha}-|q|(c_2q_{2\alpha}+c_3p_\alpha)-k_z(c_4q_{2\alpha}+c_5p_\alpha)) \qquad (6.19)
\end{aligned}
$$

$$
\begin{aligned}
a_- + a_+ &= f\frac{|q|}{M'E'\epsilon(0)\cdot p}+\frac{1}{E'(\epsilon(0)\cdot p)^2}\sqrt{\frac{3}{2}}\int_D\frac{d^3k}{(2\pi)^3}\psi\psi' \\
&\quad \sqrt{\frac{1}{E_1E_3(q_1\cdot q_2+m_1m_2)(M'^2-(m_3+m_2)^2)}}\frac{2M'}{M'^2-(m_2-m_3)^2}\frac{1}{2}\epsilon^{\alpha\beta}(0)q_{2\beta} \\
&\quad (c_0g_{3\alpha}+c_1q_{2\alpha}M+E'(c_2q_{2\alpha}+c_3p_\alpha)+E_2(c_4q_{2\alpha}+c_5p_\alpha q_{2\mu})) \qquad (6.20)
\end{aligned}
$$

6.3 Numerical results

In tab.[6.1] we report the numerical results of the form factors of the decay we consider in this chapter. As we can see in fig.[6.1] these form factors show in general a pole in q^2 as in eq.[4.42]; only $A_0(q^2)$ has a different behavior: it is a linear function of q^2, see eq.[4.43]. In tab.[6.2] we write the constants we have introduced to describe the correct function to fit each form factor.
Due to the phase space, the branching ratio of the decay $B \to D_2^* l\nu$ is smaller then the others evaluated until now; the results are in tab.[6.3].

6.4 Limit in the heavy quark effective theory

Using the same method we have applied to the decays studied in the previous chapter, we can obtain the heavy quark limit of the form factors for the process $B \to D_2^* l\nu$.

Table 6.1: We report the results of form factors relative to the semi-leptonic decays of a B pseudo-scalar meson into D_2^*. Our results are confronted to the ones of the ISGW2 model [17] and to the ones calculated with a light-front quark model [28].

Form Factor	This work		Ref. [17]		Ref. [28]	
	F(0)	F(q^2_{max})	F(0)	F(q^2_{max})	F(0)	F(q^2_{max})
A_0	0.017 (0.016)	0.042 (0.041)	0.078	0.093	0.10	0.16
A_1	0.040 (0.037)	0.053 (0.049)	0.077	0.082	0.10	0.14
A_2	0.078 (0.072)	0.121 (0.112)	0.077	0.100	0.10	0.17
V	0.085 (0.082)	0.130 (0.125)	0.085	0.110	0.12	0.19

HQET allows to decompose the matrix transition element of the decays under consideration as:

$$< D_2^*(v',\epsilon)|V_\mu|B(v)> \; = \; \sqrt{3}\tau_{3/2}(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu\gamma}v_\gamma v'^\alpha v^\beta \tag{6.21}$$

$$< D_2^*(v',\epsilon)|A_\mu|B(v)> \; = \; -\imath\sqrt{3}\tau_{3/2}(w)((1+w)\epsilon^*_{\mu\nu}v^\nu - \epsilon^*_{\alpha\beta}v^\alpha v^\beta v'_\mu) \tag{6.22}$$

From heavy quark effective theory the form factors are connected each other to the $\tau_{3/2}(w)$ by the following relations:

$$\begin{aligned}\tau_{3/2}(w) &= 2\sqrt{\frac{M^3M'}{3}}g(q^2) = \sqrt{\frac{M}{3M'}}\frac{f(q^2)}{1+w} \\ &= -\sqrt{\frac{M^3M'}{3}}(a_+(q^2) - a_-(q^2))\end{aligned} \tag{6.23}$$

Table 6.2: We report the results of the fit of the form factors over the functions of the form in eq.[4.42] and [4.43] for exponential (gaussian) wave function.

Form Factor	F(0)	a	c(GeV^{-2})
A_0	0.017 (0.016)		0.16 (0.17)
A_1	0.040 (0.037)	0.87 (0.88)	
A_2	0.078 (0.072)	1.26 (1.28)	
V	0.085 (0.082)	1.24 (1.24)	

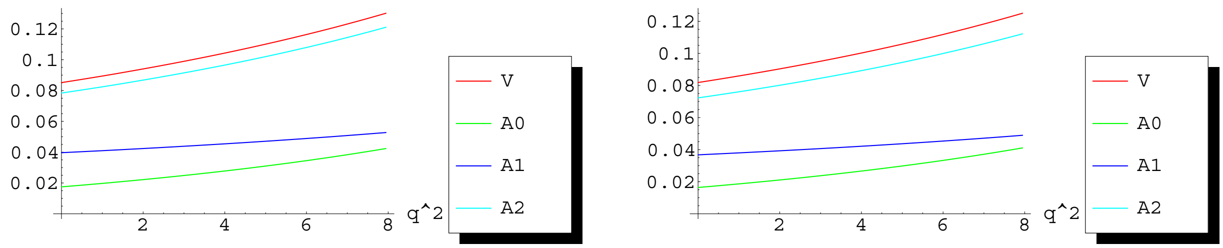


Figure 6.1: On the left we report the plot of the form factors in the case the vertex factor is described by an exponential wave function; on the right there are the results obtained with a gaussian wave function.

and

$$a_+(q^2) + a_-(q^2) = 0 \tag{6.24}$$

So we see that these form factors are related also to the ones obtained for the other decays (see eq.[5.15]).
We find that these relations are valid in our model; the value of the $\tau_{3/2}$ evaluated in this case is exactly the one given in eq.[5.31]. This is a probe of the complete consistency of the model respect to the theory.
On the other hand, this result confirms that the model violates the Bjorken and Uraltsev sum rules. This remain our lonely problem in the heavy quark limit.
In future, we will study radial excitation of charmed mesons in our model, to compare results with the ones by Bjorken and Uraltsev sum rules.

Table 6.3: Branching fractions of the processes $B(5.279GeV) \to D_2^*(2460)l\nu$ obtained using the exponential (gaussian) wave function, for all possible lepton produced.

	Br(10^{-4})
$D_2^* l \bar{\nu}_l$	4.8(4.5)
$D_2^* \tau \bar{\nu}_\tau$	0.39(0.36)

Conclusion

In this thesis we have discussed a constituent quark model adopted to evaluate the form factors of weak semi-leptonic B decays into charmed states.
Constituents quark models are important because they allow to determine the Cabibbo-Kobayashi-Maskawa matrix elements, which are free parameters of the Standard Model. In particular, the decays we are interested in are related to the $|V_{cb}|$ matrix element.

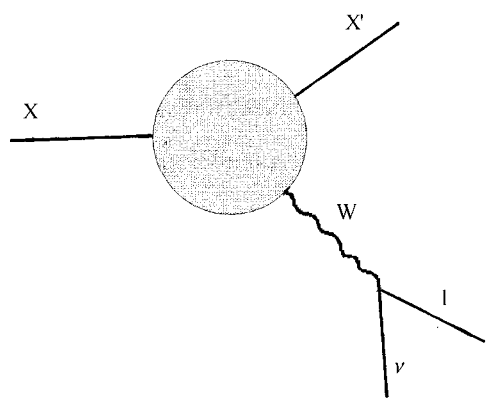
The model, as every kind of constituent quark model, is based on the simplified assumption that mesons are bound states of two valence quarks. This is not a direct consequence of QCD but it is justified by the dominance of the Fock states with the minimum number of constituents.
In our model the meson's state cannot be boosted and so all the calculations should be obtained in a well defined reference frame: the no recoil kinematical point. However, in order to have energy conservation and to obtain the q^2 dependence of the form factors we introduce the concept of running mass. In particular, we consider heavy quarks with a mass which depends by the energy of the meson. So we can compare our results with the ones of relativistic models and we find that the form factors have a comparable behavior.
Moreover, it is possible to perform the heavy quark limit on the form factors; the results are in agreement with the ones dictated by the heavy quark effective theory. Only the sum rules in this limit are not respected by the model, but this problem is probably solvable with future works.

About the numerical results of our model we find a very good agreement with experimental data, when available. In particular, we can see that our estimate of $|V_{cb}|$ agree with the experimental value published on the PDB.

In view of the new experimental results our model can be used to analyze the data.

Appendix A

Branching ratio of the decay X → X'lν



We want to show how evaluate the branching fraction of a semi-leptonic decay of a pseudo-scalar meson X.
We consider the transition $X \rightarrow X'l\nu$ represented in fig.[A].
In the case of low energy respect to the masses involved the propagator of the W boson can be approximated with the Fermi coupling constant:

$$G = 1.166\ 10^{-5}\ GeV^{-2} \tag{A.1}$$

The general form of the transition matrix is then:

$$M = \frac{G}{\sqrt{2}} < l\nu|J_{\mu}^{lept.}|0>< X'|J_{hadr.}^{\mu^+}|X> \tag{A.2}$$

Using the decomposition of the hadronic matrix transition in term of form factors and supposing that the final meson is also in the 0^- state, we have:

$$\begin{aligned} M &= \frac{G}{\sqrt{2}} V(f_+(q^2)(p+p_1)_\mu + f_-(q^2)(p-p_1)_\mu) \\ & \overline{u}(p_2)\gamma^\mu(1-\gamma_5)v(p_3) \end{aligned} \tag{A.3}$$

where

$$q = p - p_1 = p_2 + p_3 \tag{A.4}$$

$$\not{p}_2\overline{u} = m_2\overline{u} \tag{A.5}$$

$$\not{p}_3 v(p_3) = 0 \tag{A.6}$$

If we neglect the masses m_2 and m_3 of the fermions, that is a good approximation if the τ is not involved, then:

$$q_\mu \bar{u}(p_2)\gamma^\mu(1-\gamma_5)v(p_3) = 0 \tag{A.7}$$

Using $p + p_1 = 2p - q$ we obtain:

$$M = 2\frac{G}{\sqrt{2}} V f_+(q^2)\bar{u}(p_2) \not{p}(1-\gamma_5)v(p_3) \tag{A.8}$$

Averaging on the spin of the final states and adding over the initial ones we get:

$$\begin{aligned} \overline{|M|^2} &= 2G^2|V|^2 {f_+}^2(q^2) Tr\{\not{p}_2 \not{p}(1-\gamma_5) \not{p}_3 \not{p}(1-\gamma_5)\} \\ &= 4G^2|V|^2 {f_+}^2(q^2) Tr\{\not{p}_2 \not{p} \not{p}_3 \not{p}(1-\gamma_5)\} \end{aligned} \tag{A.9}$$

where we have used $\{\gamma_5,\gamma_\mu\} = 0$ e $(1-\gamma_5)^2 = 2(1-\gamma_5)$.

γ_5 is defined as

$$\gamma_5 = \frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{A.10}$$

so, taking into account that $p^\mu \neq 0$ only if $\mu = 0$ and that $\epsilon_{\mu\nu\rho\sigma} = 0$ only if two indices are equal, we have:

$$\not{p}_2 \not{p} \not{p}_3 \not{p}\gamma_5 = 0 \tag{A.11}$$

Finally, using the trace theorems, we conclude:

$$\overline{|M|^2} = 32G^2|V|^2 {f_+}^2(q^2)[2(p\cdot p_2)(p\cdot p_3) - p^2(p_2\cdot p_3)] \tag{A.12}$$

Usually it is more convenient to express the amplitude $\overline{|M|^2}$ as a function of three physical invariants, for example q^2, M_X and:

$$m_{13}{}^2 = (p_1 + p_3)^2 = (p - p_2)^2 = M_X^2 - 2M_X E_2 \tag{A.13}$$

To express $\overline{|M|^2}$ in term of this quantities we use the relations below:

$$p_2 \cdot p_3 = \frac{q^2}{2} \tag{A.14}$$

$$p^2 = M_X{}^2 \tag{A.15}$$

$$\begin{aligned} p \cdot p_3 &= (p_2 + p_3 + p_1) \cdot p_3 \\ &= p_1 \cdot p_3 + \frac{q^2}{2} \end{aligned} \tag{A.16}$$

$$\begin{aligned} p \cdot p_2 &= p \cdot (p - p_1 - p_3) \\ &= p^2 - p \cdot p_1 - p \cdot p_3 \end{aligned} \tag{A.17}$$

$$p \cdot p_1 = -\frac{q^2}{2} + \frac{M_X{}^2}{2} + \frac{m_1{}^2}{2} \tag{A.18}$$

$$p_1 \cdot p_3 = \frac{m_{13}{}^2}{2} - \frac{m_1{}^2}{2} \tag{A.19}$$

then

$$\overline{|M|^2} = 32G^2|V|^2 f_+{}^2(q^2)[(m_{13}{}^2 - M_{X'}{}^2)(M_X{}^2 - m_{13}{}^2 - q^2) + M_{X'}{}^2 q^2] \tag{A.20}$$

The differential cross section is generally given by the expression:

$$d\Gamma = \frac{\overline{|M|^2}}{2M_X(2\pi)^5}\delta^{(4)}(p - p_1 - p_2 - p_3)\frac{d^3p_1 d^3p_2 d^3p_3}{8E_1E_2E_3} \tag{A.21}$$

that, after the integration over d^3p_3, becomes:

$$d\Gamma = \frac{\overline{|M|^2}}{2M_X(2\pi)^5}\delta(p - p_1 - p_2 - p_3)\frac{d^3p_1 d^3p_2}{8E_1E_2E_3} \tag{A.22}$$

Now, to integrate over d^3p_2, we define spherical coordinate as in fig.[A]. So we can write:

$$d^3p_2 = dE_2 E_2{}^2 d\phi_2 dcos\theta_2 \tag{A.23}$$

In our case there is not dependence by ϕ_2 of the integrand, so we can immediately perform the integration over this variable:

$$\int_0^{2\pi} d\phi_2 = 2\pi \tag{A.24}$$

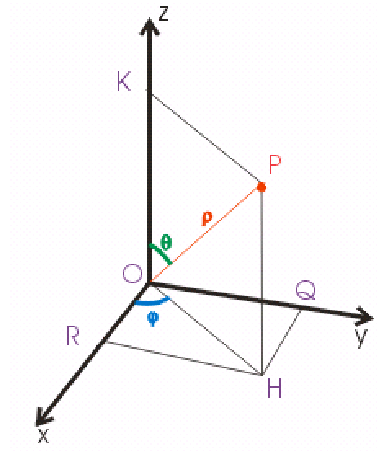


Figure A.1: Cartesian and spherical coordinates of a generic vector $\vec{p}$

To integrate over θ_2 we write δ as a function of it. To do this we use:

$$M_X = E_1 + E_2 + E_3 \Rightarrow E_3 = M_X - E_1 - E_2 \tag{A.25}$$

$$\vec{p_1} + \vec{p_2} + \vec{p_3} = 0 \tag{A.26}$$

$$\vec{p_1} \cdot \vec{p_2} = |\vec{p_1}| E_2 cos\theta_2 \tag{A.27}$$

$$E_3 = \sqrt{{E_1}^2 - {M_{X'}}^2 + {E_2}^2 + 2\sqrt{{E_1}^2 - {M_{X'}}^2} E_2 cos\theta_2} \tag{A.28}$$

The last equations together implie:

$$\begin{aligned} & \int_{-1}^{1} dcos\theta_2 \delta(M_X - E_1 - E_2 - \\ & \sqrt{{E_1}^2 - {M_{X'}}^2 + {E_2}^2 + 2\sqrt{{E_1}^2 - {M_{X'}}^2} E_2 cos\theta_2}) \\ = & \frac{E_3}{2E_2\sqrt{{E_1}^2 - {M_{X'}}^2}} \end{aligned} \tag{A.29}$$

Now we write $d|\vec{p_1}|$ as function of q^2.

$$q^2 = (p - p_1)^2 = {M_X}^2 + {M_{X'}}^2 - 2M_X\sqrt{|\vec{p_1}|^2 + {M_{X'}}^2} \tag{A.30}$$

so

$$\frac{dq^2}{d|\vec{p_1}|} = -\frac{2M_X 2|\vec{p_1}|}{2E_1} \quad \rightarrow \quad \frac{|\vec{p_1}|d|\vec{p_1}|}{2E_1} = -\frac{dq^2}{4M_X} \tag{A.31}$$

and dE_2 as function of m_{13}^2:

$$\frac{d{m_{13}}^2}{dE_2} = -2M_X \;\Rightarrow\; dE_2 = -\frac{d{m_{13}}^2}{2M_X} \tag{A.32}$$

so

$$\frac{d\Gamma}{dq^2} = \frac{\overline{|M|^2}}{32^2 {M_X}^3 \pi^3} d{m_{13}}^2 \tag{A.33}$$

The range of variation of m_{13}^2 is:

$$(E_1^* + E_2^*)^2 - (\sqrt{E_1^{*2} - M_{X'}^2} + E_3^*)^2 < m_{13}^2 < (E_1^* + E_2^*)^2 - (\sqrt{E_1^{*2} - M_{X'}^2} - E_3^*)^2 \tag{A.34}$$

where

$$E_1^* = \frac{q^2 - M_{X'}^2}{2q} \;;\; E_3^* = \frac{M_X^2 - q^2}{2q} \tag{A.35}$$

while q^2 varies in the region:

$$0 < q^2 < (M_X - M_{X'})^2 \tag{A.36}$$

Integrating over $d{m_{13}}^2$:

$$\frac{d\Gamma}{dq^2} = \frac{G^2|V|^2 {f_+}^2(q^2)}{192\pi^3 {M_X}^3}[(q^2 - M_X^2 - M_{X'})^2 - 4M_X^2 M_{X'}^2]^{\frac{3}{2}} \tag{A.37}$$

and then we get the result:

$$\Gamma = \int_0^{(M_X - M_{X'})^2} dq^2 \, \frac{d\Gamma}{dq^2} \tag{A.38}$$

The same result is valid in the case of a scalar final state.

Until now we have supposed null the mass m_l of the lepton, this is valid in any case if the lepton is not the τ. In this last case $m_\tau = 1.778 GeV$ is comparable to the other masses into play, and we cannot neglect it.

In this case the useful result, for (pseudo)-scalar mesons is the following:

$$\Gamma = \int_{m_l}^{(M_X - M_{X'})^2} dq^2 \, \frac{d\Gamma}{dq^2} \tag{A.39}$$

where

$$\frac{d\Gamma}{dq^2} = (1+\frac{m_l^2}{2q^2})\frac{d\Gamma_0}{dq^2} + 3\frac{m_l^2}{2q^2}\frac{d\Gamma_t}{dq^2} \tag{A.40}$$

$$\frac{d\Gamma_i}{dq^2} = \frac{G^2|V|^2}{(2\pi)^3}\frac{P(q^2-m_l^2)^2}{12M_X^2q^2}|H_i(q^2)|^2 \ , \ i=0,t \tag{A.41}$$

$$H_0(q^2) = \frac{2M_XP}{\sqrt{q^2}}F_1(q^2) \tag{A.42}$$

$$H_t(q^2) = \frac{M_X^2-M_{X'}^2}{\sqrt{q^2}}F_0(q^2) \tag{A.43}$$

$$P = \frac{[(q_+^2-q^2)(q_-^2-q^2)]^{1/2}}{2M_X} \tag{A.44}$$

$$q_\pm = (M_X \pm M_{X'})^2 \tag{A.45}$$

Differently, if the final meson has total angular momentum $J=1$, we must take into account the various polarization it can have and sum over them.
In this way the result is that the differential cross section is:

$$\frac{d\Gamma}{dq^2} = \frac{d\Gamma_+}{dq^2} + \frac{d\Gamma_-}{dq^2} + \frac{d\Gamma_0}{dq^2} \tag{A.46}$$

where

$$\frac{d\Gamma_i}{dq^2} = \frac{G^2|V|^2}{(2\pi)^3}\frac{P(q^2-m_l^2)^2}{12M_X^2q^2}|H_i(q^2)|^2 \ , \ i=0,\pm \tag{A.47}$$

with:

$$H_\pm = -(M_X+M_{X'})A_1(q^2) \mp \frac{2M_XP}{M_X+M_{X'}}V(q^2) \tag{A.48}$$

$$H_0 = \frac{1}{2M_{X'}\sqrt{q^2}}\left[-(M_X^2-M_{X'}^2-q^2)(M_X+M_{X'})A_1(q^2) + \frac{4M_X^2P^2}{M_X+M_{X'}}A_2(q^2)\right] \tag{A.49}$$

$$H_t = \frac{M_XP}{M_{X'}\sqrt{q^2}}\left[-(M_X+M_{X'})A_1(q^2) + (M_X-M_{X'})A_2(q^2) + \frac{q^2}{M_X+M_{X'}}A_3(q^2)\right] \tag{A.50}$$

The last case we consider is when the final meson is in a ${}^3P_2$ state. The expressions of the

H_i now became the followings:

$$H_{\pm} = \frac{M_X|\vec{q}|}{\sqrt{2}M_{X'}}\left(f(q^2)) \mp 2M_X|\vec{q}|g(q^2)\right) \quad (A.51)$$

$$H_0 = \sqrt{\frac{1}{6}}\frac{M_X|\vec{q}|}{\sqrt{q^2}M^2_{X'}}\left((M^2_X - M^2_{X'} - q^2)f(q^2) + 4M^2_X|\vec{q}|^2a_+(q^2)\right) \quad (A.52)$$

$$H_t = \sqrt{\frac{2}{3}}\frac{M^2_X|\vec{q}|^2}{\sqrt{q^2}M^2_{X'}}\left(f(q^2) + (|\vec{q}|^2 + E_{X'}q_0 + M_Xq_0)a_+(q^2) + q^2a_-(q^2)\right) \quad (A.53)$$

At the end of all this, to pass from the cross section to the branching ratio, we have to multiply the Γ for:

$$\frac{\tau_X}{\hbar} \quad (A.54)$$

where τ_X is the lifetime of the meson X and $\hbar$ is the Planck constant.

Appendix B

Trace theorems

In this appendix we report some results about the evaluation of the trace of the Dirac matrices we have used in all our work.

$$Tr(I) = 4 \tag{B.1}$$

$$Tr(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu} \tag{B.2}$$

$$Tr(\gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\tau) = 4[g_{\mu\nu} g_{\sigma\tau} - g_{\mu\sigma} g_{\nu\tau} + g_{\mu\tau} g_{\nu\sigma}] \tag{B.3}$$

$$Tr \gamma_5 = 0 \tag{B.4}$$

$$Tr(\gamma_5 \gamma_\mu \gamma_\nu) = 0 \tag{B.5}$$

$$Tr(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\tau) = 4i\epsilon_{\mu\nu\sigma\tau} \tag{B.6}$$

The trace of an odd number of γ matrices is zero.

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