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Regularity results for solutions to some classes of elliptic problems with sub-quadratic growth

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Contents

Introduction

The aim of this thesis is to describe the regularity results that I obtained during my Ph.D. for the gradient of solutions to some classes of elliptic problems of the Calculus of Variations. Actually, two different families of problems will be faced: unconstrained problems and obstacle problems.
For what concerns unconstrained problems, as it is well known, studying the regularity properties of their solutions means to study the solutions to the corresponding Euler-Lagrange system. The solutions to all the unconstrained problems that will be faced in this thesis are possibly vector-valued functions, i.e., they represent the solutions to a system of partial differential equations.
Solutions to obstacle problems are, instead, scalar-valued functions, that minimize a functional in a class of admissible functions, which values have to be (almost everywhere) greater than those of a map, called *obstacle*, that will be usually denoted with ψ. Studying the regularity properties of solutions to this family of problems means to try to understand how the regularity of the obstacle influences the regularity of the solutions.
We will consider functionals whose energy density satisfies standard p-growth and p-ellipticity conditions with respect to the gradient variable ξ.
More precisely, we will consider local minimizers of functionals of the following form

$$\mathcal{F}\left(w, \Omega\right) = \int_{\Omega} f\left(x, Dw(x)\right) dx, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded open set, and $f : \Omega \times \mathbb{R}^{n \times N} \to \mathbb{R}$, for $N \geq 1$, is a Carathéodory map, such that $\xi \mapsto f(x, \xi)$ is, at least, of class $C^1\left(\mathbb{R}^{n\times N}\right)$.
We shall assume that there exist constants $\ell_1, \ell_2, \nu, L > 0$ and a parameter $\mu \in [0, 1]$ such that the map f satisfies the following p-growth and p-ellipticity conditions

$$\ell_1 \left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}} \leq f(x, \xi) \leq \ell_2 \left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}}, \tag{0.2}$$

$$\langle D_\xi f(x, \xi) - D_\xi f(x, \eta), \xi - \eta\rangle \geq \nu \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{0.3}$$

$$|D_\xi f(x, \xi) - D_\xi f(x, \eta)| \leq L \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{0.4}$$

for any $\xi, \eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$. Our main focus will be on the case $1 < p < 2$.
As it is natural for this kind of problems, the solutions have to belong to the Sobolev space $W^{1,p}$, and since all the results we will describe are local, in the following, with the notations given above, we will always consider functions $w \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$.

So, under this assumptions, when we deal with the regularity of solutions to homogeneous elliptic systems, we are interested in the regularity properties of solutions to the following problem:

$$\min \left\{ \int_\Omega f(x, Dw(x))\, dx : w \in u_0 + W_0^{1,p}\left(\Omega, \mathbb{R}^N\right) \right\},$$

with $u_0 \in W^{1,p}\left(\Omega, \mathbb{R}^N\right)$ a fixed boundary datum.

We also show some higher differentiability results for local minimizers of functionals of the form

$$\mathcal{F}(w, \Omega) = \int_\Omega \left[f(x, Dw(x)) - F(x) \cdot w(x) dx\right], \tag{0.5}$$

where the energy density f still satisfies (0.2)–(0.4).
Local minimizers of functionals of this form are solutions to non-homogeneous elliptic systems and, as we will see in the following, their regularity properties also depend on the assumptions on the datum F.

For what concerns obstacle problems, we are interested in the regularity properties of solutions to problems of the form

$$\min \left\{ \int_\Omega f(x, Dw(x))\, dx : w \in \mathcal{K}_\psi(\Omega) \right\}, \tag{0.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n > 2$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x, \xi)$ is of class $C^1(\mathbb{R}^n)$ for a.e. $x \in \Omega$, $\psi : \Omega \mapsto [-\infty, +\infty)$ belonging to the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the *obstacle*, and

$$\mathcal{K}_\psi(\Omega) = \left\{ w \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}) : w \geq \psi \text{ a.e. in } \Omega \right\}$$

is the class of the admissible functions, with $u_0 \in W^{1,p}(\Omega)$ a fixed boundary datum.
Even when we face this family of problems, we will assume that the function f satisfies growth and ellipticity conditions expressed in (0.2)–(0.4).

It is known that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the obstacle problem (0.6) in $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality

$$\int_\Omega \langle D_\xi f(x, Du(x)), D(\varphi(x) - u(x))\rangle\, dx \geq 0 \qquad \forall \varphi \in \mathcal{K}_\psi(\Omega).$$

As you can notice by (0.1) and (0.5), the energy density of the kind of functionals we will consider is characterized by a dependence on the x-variable and the nature of this dependence is one the key points for the results we will present.
One of the main difficulties of our study relies in the fact that we will always consider situations in which the map $x \mapsto D_\xi f(x, \xi)$ is possibly discontinuous.
Actually, we will assume the existence of a non-negative function $g \in L^q_{\mathrm{loc}}(\Omega)$ for some q, such that, for any $\xi \in \mathbb{R}^{n \times N}$ and for almost every $x, y \in \Omega$, we have

$$|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq (g(x) + g(y))\, |x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}. \tag{0.7}$$

Assuming (0.7) is equivalent to assume that the map $x \mapsto D_\xi f(x, \xi)$ belongs to the Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$ (see [66]).
In Section 4.4, dealing with obstacle problems, we will also consider the weaker situation in which (0.7) is replaced by

$$|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq (g_k(x) + g_k(y))\, |x - y|^\alpha \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{0.8}$$

where $\alpha \in (0,1)$ and $(g_k)_k$ is a sequence of non-negative functions belonging to $L^{\frac{n}{\alpha}}_{\text{loc}}(\Omega)$ such that, for some $q \in [1,+\infty)$ we have

$$\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B_R)} < \infty,$$

for any ball $B_R \Subset \Omega$.
In the same section, we will also consider the case in which (0.8) is replaced by

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le (g(x)+g(y))\,|x-y|^\alpha \left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}, \tag{0.9}$$

where $g \in L^{\frac{n}{\alpha}}_{\text{loc}}(\Omega)$ is a non-negative function.
The two conditions (0.8) and (0.9) represent some fractional-order differentiability properties for the map $x \mapsto D_\xi f(x,\xi)$.
More precisely, inequality (0.8) means that the map $x \mapsto D_\xi f(x,\xi)$ belongs to a Besov-Lipschitz space $B^\alpha_{\frac{n}{\alpha},q,\text{loc}}(\Omega)$ for $1 \le q < \infty$, while (0.9) means that the same map belongs to $B^\alpha_{\frac{n}{\alpha},\infty,\text{loc}}(\Omega)$.
We will prove fractional differentiability results for solutions to some obstacle problems, but this kind of higher differentiability results are available also for solutions to some kinds of elliptic equations (see, for example [2, 30, 31]).

There is a second key point for most of the results contained in this thesis: the value of the growth exponent p.
Indeed, except for the obstacle problem considered in Section 4.2, where we have $p \ge 2$, all the other results deal with variational problems in which assumptions (0.2)–(0.4) hold for $1 < p < 2$.
In case of sub-quadratic growth with respect to the gradient variable, new difficulties arise, expecially when we try to prove higher differentiability properties for solutions, and this issues also affect the results.
This kind of phenomenon was already known for variational problems and elliptic equations with continuous coefficients or, in general, under more regular assumptions on them (see, for example, the pioneering papers [1, 89]), but here we present some results where, for the first time, the difficulties linked to the sub-quadratic growth and ellipticity conditions combine with those that are due to the presence of possibly discontinuous coefficients.

A key tool to study higher differentiability properties of solutions to the wide family of problems described above, is the use of the following auxiliary function of the gradient variable that is usually involved in the study of regularity properties for solutions to p-elliptic problems

$$V_p : \mathbb{R}^k \to \mathbb{R}^k,$$

where

$$V_p(\xi) := \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}} \cdot \xi$$

for any $\xi \in \mathbb{R}^k$ (where $k = n \times N$ in case of variational problems with vector-valued minimizers and $k = n$ if we deal with problems with scalar-valued solutions), and it is naturally linked to the ellipticity condition (0.3).
Moreover, for p-elliptic variational problems, if $u \in W^{1,p}$ is a local minimizer of the functional (0.1), there is always a relation between the higher differentiability properties of u and the properties of the functions $V_p(Du)$.

A key point, as we will see in what follows, is that, when we deal with sub-quadratic p-elliptic problems, this kind of relations are, in some sense, reversed, with respect to what happens for $p \geq 2$.

The regularity properties of minimizers of such integral functionals have been widely investigated in case the energy density $f(x, \xi)$ is continuous as a function of the x-variable, both in the superquadratic and in the sub-quadratic growth case. Actually, the partial continuity of the vectorial minimizers can be obtained with a quantitative modulus of continuity that depends on the modulus of continuity of the coefficients (see for example [1, 49, 58, 89] and the monographs [57, 63] for a more exhaustive treatment). For regularity results under general growth conditions, that of course include the superquadratic and the sub-quadratic ones, we refer to [34, 36, 37, 44, 45, 50, 75, 77, 82].
Recently, there has been an increasing interest in the study of the regularity when the oscillation of $f(x, \xi)$ with respect to the x-variable is controlled through a coefficient that belongs to a suitable Sobolev class of integer or fractional order and the assumptions (0.2)–(0.7) are satisfied with an exponent $p \geq 2$.
Actually, it has been shown that the weak differentiability of the partial map $x \mapsto D_\xi f(x, \xi)$ transfers to the gradient of the minimizers of the functional (0.1) (see [16, 40, 39, 62, 74, 84]) as well as to the gradient of the solutions of non linear elliptic systems (see [3, 26, 25, 28, 60, 76, 85]) and of non linear systems with degenerate ellipticity in case $p \geq 2$ (see [60]).
It is worth mentioning that the continuity of the coefficients is not sufficient to establish the higher differentiability of integer order of the minimizers.

The regularity of minimizers of functionals of the form (0.5) has been object of study under many functional settings (see [87]), and higher integrability (see [19, 71]) and differentiability results of integer and fractional order (see [20, 80, 81]) have been established even under different kind of growth conditions, also for problems with measure data ([24]).
Lipschitz regularity results for solutions of non-homogeneous elliptic problems are proved, for example, in [5, 27, 33].

The interest in the study of the regularity properties of solutions to obstacle problems has been strongly increasing in the last decades as a research topic in Calculus of Variations and Partial Differential Equations.
From the very beginning, obstacle problems were solved applying techniques of functional analysis, and it was clear soon that the regularity properties of the solutions were strictly connected to those of the obstacle.
In the linear setting it was observed that the solutions and the obstacle have the same regularity (see [10, 15, 69]), but this is no longer true in the nonlinear framework for general integrands without any specific structure.
This kind of phenomenon has been studied not only in the case of variational inequalities modelled upon the p-Laplacean energy [22, 23, 83], but also in the case of more general structures [13, 14, 38, 48, 52].
So, recently, there has been an intense research activity concerning the regularity properties of solutions to obstacle problems in the nonlinear setting (see also [11] and the references therein).
In many works about this topic, some extra regularity has been imposed on the obstacle to balance the nonlinearity (see [7, 6, 23, 47, 48, 78]). For example, in some very recent papers, the authors analyzed how an extra differentiability of integer or fractional order of the gradient of the obstacle transfers to the gradient of the solutions (see [42, 43]).
This kind of problem is linked to similar studies about regularity of solutions to partial differential equations, since it has been proved that solutions to the obstacle problem (0.6) are solutions to an equation of the form

$$\mathrm{div} A\left(x, Du\right) = \mathrm{div} A\left(x, D\psi\right). \tag{0.10}$$

It is well known that no extra differentiability properties for the solutions can be expected even if the obstacle ψ is smooth, unless some assumption is given on the x-dependence of the operator A.
As we mentioned above, many recent works deal with regularity properties of solutions to variational problems in which the integrand depends on the x-variable through a function that is possibly discontinuous, such as in the case of Sobolev-type dependence, under quadratic (see [85]), and super-quadratic growth conditions (see [59, 74, 84]).
Therefore, inspired by recent results concerning the higher differentiability of integer ([39, 40, 41, 55, 60, 61, 62, 84]) and fractional ([3, 25]) order for the solutions to elliptic equations or systems, in a number of papers the higher differentiability of the solution of an obstacle problem is proved under a suitable Sobolev assumption on the partial map $x \mapsto A\left(x, \xi\right)$.
More precisely, in [42], the higher differentiability of the solution of an homogeneous obstacle problem with the energy density satisfying p-growth conditions is proved; in [43, 64] the integrand f depends also on the v variable; in [17, 21, 32, 65, 52, 51, 72] the energy density satisfies (p, q)-growth conditions. The non-homogeneous obstacle problem is considered in [79, 12] when the energy density satisfies p-growth conditions, and in [90] under (p, q)-growth conditions.
As we pointed out previously, even for unconstrained problems it is known that the sub-quadratic growth conditions require specific tools and, in general, the expected regularity of the solution, in the case $1 < p < 2$ strongly differs from the case $p \geq 2$ (for a detailed explaination of this phenomenon see [4]).

Many previously known higher differentiability results have been obtained assuming that the map $x \mapsto D_\xi f\left(x, \xi\right)$ belongs to a Sobolev space $W^{1,q}$, with $q \geq n$ but, both in case of equations and in case of obstacle problems, it is possible to weaken these assumptions if we deal with a priori bounded solutions.
Indeed, it is well known that the local boundedness of the solutions to a variational problem is a turning point in the regularity theory. Actually, in [62] it has been proved that, when dealing with bounded solutions to (0.10), higher differentiability properties hold true under weaker assumptions on the partial map $x \mapsto D_\xi f\left(x, \xi\right)$ with respect to $W^{1,n}$. Moreover, in [17], it has been proved that a local boundedness assumption on the obstacle ψ implies a local bound for the solutions to the obstacle problem (0.6).
This kind of properties allow us to prove higher differentiability results, both for bounded solutions to p-elliptic equations with $1 < p < 2$, and for solutions to (0.6) with bounded obstacle, assuming that the partial map $x \mapsto D_\xi f\left(x, \xi\right)$ belongs to a Sobolev class that is not related to the dimension n, but to the growth exponent of the functional.

Let us give a description of the structure of this thesis.

In Chapter 1, we list results and properties of some spaces of functions, that will be useful in the following. The last section of the first chapter is devoted to the properties of the function V_p, with a focus on its relation with differentiability properties of solutions to p-elliptic variational problems when $1 < p < 2$, and the differences with respect to the case $p \geq 2$.

Chapter 2 contains some results about regularity for solutions to unconstrained problems, that is solutions to homogeneous elliptic equations.
As far as we know, no regularity results were available for vectorial minimizers nor to establish their Lipschitz continuity, nor to prove the L^s-integrability of their gradient for every finite $s > 1$, nor to prove higher differentiability, under the so-called sub-quadratic

growth conditions, i.e. when the assumptions (0.2)–(0.7) hold true for $1 < p < 2$ in case of Sobolev coefficients, until the results we will present in Chapter 2 appeared.
In Section 2.1 we give the proof of an a priori estimate for the L^p norm of second-order derivatives of solutions to some p-elliptic problems where $1 < p < 2$, and the solutions are a priori assumed to be in $W^{2,p}$ (see [55]).
It is worth mentioning that, if (0.7) holds, the partial map $x \mapsto D_\xi f(x, \xi)$ needs not to be continuous. Actually, if $q = n$, as in the case, for example, of some results contained in Section 2.2, by the Sobolev embedding theorem, we have that it belongs to the space VMO of function with vanishing mean oscillation (see [63] for the precise definition). The regularity of solutions to PDEs with VMO coefficients goes back to [8, 68, 70].
In Section 2.2, the Lipschitz regularity, the higher integrability for the gradient, and the $W^{2,p}$ regularity results for minimizers of the functional (2.1) are described, whose proofs are contained in [54].
For what concerns higher integrability of the gradient of minimizers, estimate (2.38) of Theorem 2.2.2 can be interpreted as an extension of the result in [70] that concerns the p-Laplace operator to more general operators with sub-quadratic growth.
Moreover, even for problems with sub-quadratic growth conditions, higher integrability for the gradient of minimizers and Lipschitz regularity results have been previously proved, respectively, in [29] and [40], but only for degenerate problems (with ellipticity conditions only at infinity), not for possibly singular ones. Let us recall that degenerate problems are variational problems where the minimal eigenvalue may be zero, while in singular problems the minimal eigenvalue may go to infinity: an example of possibly singular problem is the p-Laplace equation with $1 < p < 2$.

In Chapter 3 we consider a class of non-homogeneous p-elliptic equations for $1 < p < 2$, proving the higher differentiability for their solutions.
Indeed, considering a local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of functional of the form (0.5), where the behavior of the map $x \mapsto D_\xi f(x, \xi)$ is described by (0.7), we focus on the value of r such that, if $F \in L^r_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ we get $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, provided $g \in L^q_{\mathrm{loc}}(\Omega)$ for some suitable q.
The result we prove in Section 3.1 is sharp, in the sense that we identify the largest Lebesgue space L^r to which the datum F has to belong to get higher differentiability of minimizers of the functional (0.5) withouth making stronger assumptions on the solutions themselves.
In Section 3.2 we prove a higher differentiability result for locally bounded minimizers of the functional (0.5).
The results of Chapter 3 will be contained in an upcoming paper, written in collaboration with A. Clop and A. Passarelli di Napoli.

In Chapter 4 we present the regularity results for solutions to some obstacle problems.
The aim of Section 4.2 is to describe a higher diffrentiability result for solutions to a class of obstacle problems contained in [18], where the obstacle is assumed to be bounded, and assumption (0.7) holds with $q = p + 2$.
It is worth to stress that this result is the only one, in this Ph.D. thesis, dealing with problems with super-quadratic growth and ellipticity conditions.
The result of this section is contained in [18], written in collaboration with M. Caselli and R. Giova.
In Section 4.3 and Section 4.4, we move to the study of regularity of solutions to obstacle problems under sub-quadratic growth conditions, extending some results proved in [42] for the super-quadratic growth case (see [53]).
More precisely, in Section 4.3, under assumption (0.7) with $q = n$, we prove an integer order higher differentiability result; then in Section 4.4 we prove two higher differentiability results of fractional order, assuming (0.8) and (0.9) respectively.
In Section 4.5, the result of Section 4.2 is extended to the case $1 < p < 2$, proving a higher

differentiability result for solutions to a class of obstacle problems with sub-quadratic growth conditions, assuming that the obstacle is locally in L^∞.
This result will be contained in an upcoming paper, written in collaboration with R. Giova.

I want to conclude this introduction espressing my deep gratitude to my supervisor, professor Antonia Passarelli di Napoli, for her precious guidance during these three years of studies.
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Chapter 1

Some notations and general properties

This opening chapter is devoted to the description of some general properties of functional spaces and some known result that will be useful in the following chapters.

1.1 Notations and preliminaries

In this section we list the notations that we shall use and recall some tools that will be useful to prove our results.
We shall follow the usual convention and denote by C or c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts.
All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{n\times N}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{n\times N}$ we write $\langle \xi, \eta\rangle := \mathrm{trace}\left(\xi^T\eta\right)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.
When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{n\times N}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column.
For a C^2 function $f\colon \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$, we write

$$D_\xi f(x,\xi)\cdot\eta := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(x,\xi+t\eta) \quad \text{and} \quad \langle D_{\xi\xi} f(x,\xi)\eta,\eta\rangle := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} f(x,\xi+t\eta)$$

for ξ, $\eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$.
With the symbol $B(x,r) = B_r(x) = \{\, y \in \mathbb{R}^n : |y-x| < r \,\}$, we will denote the ball centered at x of radius r and we shall omit the dependence on the center when it is clear from the context. Indeed, since all the results we will discuss are local, proving them, we will denote with B_r any ball $B_r(x_0) \Subset \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $x_0 \in \Omega$. For any set B and any function $u \in L^1(B)$, the notation

$$u_B = \fint_B u(x)dx,$$

stands for the integral mean of u over the set B.
The next lemma can be proved using an iteration technique, and will be very useful in the following, where we will often refer to this as Iteration Lemma.

Lemma 1.1.1 (Iteration Lemma)**.** *Let $h : [\rho, R] \to \mathbb{R}$ be a non-negative bounded function, $0 < \theta < 1$, $A, B \geq 0$ and $\gamma > 0$. Assume that*

$$h(r) \leq \theta h(d) + \frac{A}{(d-r)^\gamma} + B$$

for all $\rho \leq r < d \leq R_0$. *Then*

$$h(\rho) \leq c\left[\frac{A}{(R_0-\rho)^\gamma}+B\right],$$

where $c = c(\theta, \gamma) > 0$.

For the proof we refer to [63, Lemma 6.1].
The following Gagliardo-Nirenberg type inequality is proved, in a more general form, in [86], and can be also found in [46].

Theorem 1.1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $1 \leq q, r \leq \infty$ *and*

$$\frac{1}{s}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{r}\right),$$

then the following implication holds

$$v \in L^q(\Omega) \cap W^{2,r}(\Omega) \implies v \in W^{1,s}(\Omega),$$

with the estimate

$$\|Dv\|_{L^s(\Omega)} \leq c\,\|v\|_{L^q(\Omega)}^{\frac{1}{2}} \cdot \|v\|_{W^{2,r}(\Omega)}^{\frac{1}{2}},$$

for a constant $c(n, q, r, s) > 0$.

The following inequalities are stated in [62]. For the proofs, see [16, Appendix A] and [56, Lemma 3.5] (in case $p(x) \equiv p$, $\forall x$) respectively.

Lemma 1.1.3. *For any* $\phi \in C_0^1(\Omega)$ *with* $\phi \geq 0$*, and any* C^2 *map* $v : \Omega \to \mathbb{R}^N$*, for any* $p > 1$ *and* $\mu \in [0, 1]$*, we have*

$$\begin{aligned}
&\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\,|Dv(x)|^{\frac{m}{m+1}(p+2)}\,dx\\
\leq\;& (p+2)^2\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\,|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\\
&\cdot\left[\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\,|D\phi(x)|^2\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\right.\\
&\left.+n\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p-2}{2}}\left|D^2v(x)\right|^2dx\right)^{\frac{m}{m+1}}\right], \qquad (1.1)
\end{aligned}$$

for any $p \in (1, \infty)$ *and* $m > 1$. *Moreover*

$$\begin{aligned}
&\int_\Omega \phi^2(x)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}|Dv(x)|^2\,dx\\
\leq\;& c\,\|v\|^2_{L^\infty(\mathrm{supp}(\phi))}\int_\Omega \phi^2(x)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p-2}{2}}\left|D^2v(x)\right|^2dx\\
&+c\,\|v\|^2_{L^\infty(\mathrm{supp}(\phi))}\int_\Omega\left(\phi^2(x)+|D\phi(x)|^2\right)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx, \qquad (1.2)
\end{aligned}$$

for a constant $c = c(p)$.

This results will be useful both when dealing with systems of equations and obstacle problems, when the solutions are bounded.
For further needs, we recall the following result, whose proof can be found in [9, Lemma 4.1].

Lemma 1.1.4. *For any $\delta > 0$, $m > 1$ and $\xi, \eta \in \mathbb{R}^k$, let*

$$W(\xi) = (|\xi| - \delta)_+^{2m-1} \frac{\xi}{|\xi|} \qquad \textit{and} \qquad \tilde{W}(\xi) = (|\xi| - \delta)_+^{m} \frac{\xi}{|\xi|}.$$

Then there exists a positive constant $c(m)$ such that

$$\langle W(\xi) - W(\eta), \xi - \eta \rangle \geq c(m) \left| \tilde{W}(\xi) - \tilde{W}(\eta) \right|^2.$$

for any $\eta, \xi \in \mathbb{R}^k$.

1.2 Difference quotients

A key instrument in studying higher differentiability properties of solutions to problems of Calculus of Variations and PDEs is the so called *difference quotients method.*
In this section, we recall the definition and some basic results.

Definition 1.2.1. Given $h \in \mathbb{R}^n$, for every function $F : \mathbb{R}^n \to \mathbb{R}^N$, for any $s = 1, \dots, n$ the finite difference operator in the direction x_s is defined by

$$\tau_{s,h} F(x) = F(x + h e_s) - F(x),$$

where e_s is the unit vector in the direction x_s.

In the following, in order to simplify the notations, we will often omit the vector e_s, denoting

$$\tau_h F(x) = F(x + h) - F(x),$$

where $h \in \mathbb{R}^n$.
We now describe some properties of the operator τ_h, whose proofs can be found, for example, in [63].

Proposition 1.2.2. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{ \, x \in \Omega : d(x, \partial\Omega) > |h| \, \}.$$

Then

(*a*) $\tau_h F \in W^{1,p}\left(\Omega_{|h|}\right)$ *and*

$$D_i(\tau_h F) = \tau_h(D_i F), \qquad \textit{for every } i = 1, \dots, n.$$

(*b*) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then*

$$\int_\Omega F(x) \tau_h G(x) dx = \int_\Omega G(x) \tau_{-h} F(x) dx.$$

(*c*) *We have*

$$\tau_h(FG)(x) = F(x + h) \tau_h G(x) + G(x) \tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 1.2.3. *If* $0 < r < R$, $|h| < \frac{R-r}{2}$, $1 < p < +\infty$, *and* $F, DF \in L^p(B_R)$ *then*

$$\int_{B_r} |\tau_h F(x)|^p \, dx \leq c(n,p)|h|^p \int_{B_R} |DF(x)|^p \, dx.$$

Moreover

$$\int_{B_r} |F(x+h)|^p \, dx \leq \int_{B_R} |F(x)|^p \, dx.$$

The following result is proved in [63].

Lemma 1.2.4. *Let* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ *with* $1 < p < +\infty$. *Suppose that there exist* $r \in (0,R)$ *and* $M > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_r} |\tau_{s,h} F(x)|^p dx \leq M^p |h|^p$$

for $|h| < \frac{R-r}{2}$. *Then* $F \in W^{1,p}\left(B_R, \mathbb{R}^N\right)$.
Moreover

$$\|DF\|_{L^p(B_r)} \leq M,$$

$$\|F\|_{L^{\frac{np}{n-p}}(B_r)} \leq c\left(M + \|F\|_{L^p(B_R)}\right),$$

with $c = c(n,N,p,r,R)$, *and*

$$\frac{\tau_{s,h}F}{|h|} \to D_s F \qquad \text{strongly in } L^p_{\mathrm{loc}}(\Omega), \text{ as } h \to 0,$$

for each $s = 1, \ldots, n$..

Before introducing Besov-Lipschitz spaces, we conclude this section recalling a fractional version of Lemma 1.2.4, whose proof can be found in [74].

Lemma 1.2.5. *Let* $F \in L^2(B_R)$. *Suppose that there exist* $r \in (0,R)$, $\alpha \in (0,1)$ *and* $M > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_r} |\tau_{s,h} F(x)|^2 dx \leq M^2 |h|^{2\alpha},$$

for $|h| < \frac{R-r}{2}$. *Then* $F \in L^{\frac{2n}{n-2\beta}}(B_r)$ *for every* $\beta \in (0,\alpha)$ *and*

$$\|F\|_{L^{\frac{2n}{n-2\beta}}(B_r)} \leq c\left(M + \|F\|_{L^2(B_R)}\right),$$

with $c = c(n,N,p,r,R,\alpha,\beta)$.

1.3 Besov-Lipschitz spaces

The idea of difference quotient can be used also to define some fractional differentiability properties. In this section we introduce the definition of Besov-Lipschitz spaces and list their basic properties.
Let us consider $0 < \alpha < 1$ and $1 \leq p, q < \infty$ and, for a function $v : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}^n$, we denote, as in the previous section, $\tau_h v(x) = v(x+h) - v(x)$. We say that v belongs to the Besov-Lipschitz space $B^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty. \tag{1.3}$$

We define a norm in the space $B^{\alpha}_{p,q}(\mathbb{R}^n)$ setting

$$\|v\|_{B^{\alpha}_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)},$$

and with this norm $B^{\alpha}_{p,q}(\mathbb{R}^n)$ is a Banach space.
Equivalently, we could say that a function $v \in L^p(\mathbb{R}^n)$ belongs to $B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $\frac{\tau_h v}{|h|^{\alpha}} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$. We can also observe that, in (1.3), one can simply integrate for $h \in B(0,\delta)$ for a fixed $\delta > 0$, thus obtaining an equivalent norm, because

$$\left(\int_{\{|h| \geq \delta\}} \left(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \leq c(n,\alpha,p,q,\delta) \|v\|_{L^p(\mathbb{R}^n)}.$$

Moreover, for a function $v \in L^p(\mathbb{R}^n)$, we say that $v \in B^{\alpha}_{p,\infty}(\mathbb{R}^n)$ if

$$[v]_{\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^n)} = \sup_{h \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{1}{p}} < \infty, \tag{1.4}$$

and we define the following norm

$$\|v\|_{B^{\alpha}_{p,\infty}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^n)}.$$

Also in (1.4), the supremum can be taken over the set $\{ |h| \leq \delta \}$ for a fixed $\delta > 0$, and the norm that we obtain is equivalent.
By construction, $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Moreover, the following Sobolev-type embeddings hold for Besov-Lipschitz spaces.

Lemma 1.3.1. *Suppose that* $0 < \alpha < 1$.

(*a*) *If* $1 < p < \frac{n}{\alpha}$ *and* $1 \leq q \leq p^*_{\alpha} = \frac{np}{n - \alpha p}$, *then there is a continuous embedding* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset L^{p^*_{\alpha}}(\mathbb{R}^n)$.

(*b*) *If* $p = \frac{n}{\alpha}$ *and* $1 \leq q \leq \infty$, *then there is a continuous embedding* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$,

where BMO denotes the space of functions with bounded mean oscillations.

The following lemma describes the inclusions between Besov-Lipschitz spaces.

Lemma 1.3.2. *Suppose that* $0 < \beta < \alpha < 1$.

(*a*) *If* $1 < p < \infty$ *and* $1 \leq q \leq r \leq \infty$ *then* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset B^{\alpha}_{p,r}(\mathbb{R}^n)$.

(*b*) *If* $1 < p < \infty$ *and* $1 \leq q, r \leq \infty$ *then* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset B^{\beta}_{p,r}(\mathbb{R}^n)$.

(*c*) *If* $1 \leq q \leq \infty$, *then* $B^{\alpha}_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^{\beta}_{\frac{n}{\beta},q}(\mathbb{R}^n)$.

For the proofs of Lemmas 1.3.1 and 1.3.2 we refer to [67].
We can also define local Besov-Lipschitz spaces as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We say that a function v belongs to $B^{\alpha}_{p,q,\mathrm{loc}}(\Omega)$ if, for any smooth function with compact support in Ω, $\varphi \in C^{\infty}_0(\Omega)$, we have $\varphi v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$. It is easy to extend the embeddings described in Lemma 1.3.1 and 1.3.2 to local Besov spaces. The following Lemma is an easy consequence of the definitions given above and its proof can be found in [3].

Lemma 1.3.3. *A function $v \in L^p_{\mathrm{loc}}(\Omega)$ belongs to the local Besov space $B^{\alpha}_{p,q,\mathrm{loc}}(\Omega)$ if and only if*

$$\left\| \frac{\tau_h v}{|h|^{\alpha}} \right\|_{L^q\left(\frac{dh}{|h|^n};L^p(B)\right)} < \infty$$

for any ball $B \subset 2B \subset \Omega$ with radius r_B. Here the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0, r_B)$ on the h-space.

It is known that Besov-Lipschitz spaces of fractional order $\alpha \in (0,1)$ can be characterized in pointwise terms. Given a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, a *fractional α-Hajłasz gradient* for v is a sequence $(g_k)_k$ of measurable, non-negative functions $g_k : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$ such that the inequality

$$|v(x) - v(y)| \leq (g_k(x) + g_k(y))\, |x-y|^{\alpha}$$

holds for any $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \leq |x-k| \leq 2^{-k+1}$. We say that $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(g_k)_k\|_{\ell^q(L^p)} = \left(\sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < \infty.$$

The following result is proved in [73].

Theorem 1.3.4. *Let $\alpha \in (0,1)$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $v \in L^p(\mathbb{R}^n)$. One has $v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajłasz gradient $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover,*

$$\|v\|_{B^{\alpha}_{p,q}(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{\ell^q(L^p)},$$

where the infimum runs over all the possible α-Hajłasz gradients for v.

1.4 An auxiliary function

Here we define an auxiliary function of the gradient variable that comes out to be very useful to treat regularity properties for solutions to p-elliptic problems.
The function $V_p : \mathbb{R}^k \to \mathbb{R}^k$ is defined as

$$V_p(\xi) := \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi.$$

Of course, when we deal with variational problems whose solutions are scalar-valued functions $v : \mathbb{R}^n \to \mathbb{R}$, such as, fo example, in case of obstacle problems, the map $\xi \mapsto V_p(\xi)$ is defined in $\mathbb{R}^k$ with $k = n$, while, in case of variational problems with vector-valued minimizers $v : \mathbb{R}^n \to \mathbb{R}^N$, we have $k = n \times N$.
As we will see in the following, many higher differentiability properties for solutions to the problems we are considering can be expressed and treated in terms of the function $V_p(Dv)$, where $v \in W^{1,p}$ is a local minimizer of a p-elliptic problem.
A key point that we want to stress is that the role of this function in the study of higher differentiability properties of solutions to elliptic problems changes when we move frome the case $p \geq 2$ to the case $1 < p < 2$. This is the reason why the regularity results available for solutions to super-quadratic problems don't always hold true in the same way in the sub-quadratic case.
The following results are proved in [1], and will be useful to estimate the L^p norm of $D^2 v$, using the L^2 norm of the gradient of $V_p(Dv)$.

Lemma 1.4.1. *For every* $\gamma \in \left(-\frac{1}{2}, 0\right)$ *and* $\mu \geq 0$ *we have*

$$(2\gamma+1)\,|\xi-\eta| \leq \frac{\left|\left(\mu^2+|\xi|^2\right)^{\gamma}\xi-\left(\mu^2+|\eta|^2\right)^{\gamma}\eta\right|}{\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\gamma}} \leq \frac{c(k)}{2\gamma+1}\,|\xi-\eta|\,,$$

for every $\xi, \eta \in \mathbb{R}^k$.

Lemma 1.4.2. *For every* $\gamma \in \left(-\frac{1}{2}, 0\right)$ *we have*

$$c_0(\gamma)\left(1+|\xi|^2+|\eta|^2\right)^{\gamma} \leq \int_0^1 \left(1+|t\xi+(1-t)\eta|^2\right)^{\gamma} dt \leq c_1(\gamma)\left(1+|\xi|^2+|\eta|^2\right)^{\gamma},$$

for every $\xi, \eta \in \mathbb{R}^k$.

Lemma 1.4.3. *Let* $1<p<2$. *There is a constant* $c=c(n,p)>0$ *such that*

$$c^{-1}\,|\xi-\eta| \leq |V_p(\xi)-V_p(\eta)|\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{2-p}{4}} \leq c\,|\xi-\eta|\,, \tag{1.5}$$

for any $\xi, \eta \in \mathbb{R}^k$.

Remark 1.4.4. One can easily check that, if $1<p<\infty$, for a C^2 function g, there exists a constant $C(p)$ such that

$$C^{-1}\left|D^2 g\right|^2\left(\mu^2+|Dg|^2\right)^{\frac{p-2}{2}} \leq |DV_p\left(Dg\right)|^2 \leq C\left|D^2 g\right|^2\left(\mu^2+|Dg|^2\right)^{\frac{p-2}{2}}. \tag{1.6}$$

In what follows, we shall use the following.

Lemma 1.4.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $1<p<2$, *and* $v \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$. *Then the following implication holds*

$$V_p\left(Dv\right) \in W^{1,2}_{\mathrm{loc}}\left(\Omega\right) \implies v \in W^{2,p}_{\mathrm{loc}}\left(\Omega\right), \tag{1.7}$$

and the following estimate

$$\int_{B_r}\left|D^2 v(x)\right|^p dx \leq c\cdot\left[1+\int_{B_R}|DV_p\left(Dv(x)\right)|^2 dx + c\int_{B_R}|Dv(x)|^p dx\right] \tag{1.8}$$

holds for any ball $B_R \Subset \Omega$ *and* $0<r<R$.

Proof. We will prove the existence of the second-order weak derivatives of v and the fact that they are in $L^p_{\mathrm{loc}}\left(\Omega\right)$, by means of the difference quotients method.
Let us consider a ball $B_R \Subset \Omega$ and $0<\frac{R}{2}<r<R$.
For $|h|<\frac{R-r}{2}$, we have $0<\frac{R}{2}<r<r_1 := r+|h|<R-|h| =: r_2<R$, and by (1.5), we get, for any $s=1,\ldots,n$.

$$\int_{B_r}|\tau_{s,h}Dv(x)|^p dx \quad\leq\quad \int_{B_r}|\tau_{s,h}V_p\left(Dv(x)\right)|^p\cdot\left(\mu^2+|Dv(x)|+|Dv\left(x+he_s\right)|\right)^{\frac{p(2-p)}{4}}.$$

By Hölder's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$ and the use of (1.5), we get

$$\int_{B_r}|\tau_{s,h}Dv(x)|^p dx \quad\leq\quad \left(\int_{B_r}|\tau_{s,h}V_p\left(Dv(x)\right)|^2 dx\right)^{\frac{p}{2}}$$

$$\cdot\left(\int_{B_r}\left(\mu^2+|Dv\,(x+he_s)|^2+|Dv\,(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{2-p}{2}},$$

and since $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, by Lemma 1.2.3 and Young's inequality, we have

$$\begin{aligned}\int_{B_r}|\tau_{s,h}Dv(x)|^p\,dx &\le c\left[|h|^2\int_{B_R}|DV_p\,(Dv(x))|^2\,dx\right]^{\frac{p}{2}}\\ &\quad\cdot\left[\int_{B_r}\left(\mu^2+|Dv\,(x+he_s)|^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{2-p}{2}}\\ &\le c|h|^p\left[1+\int_{B_R}|DV_p\,(Dv(x))|^2\,dx+\int_{B_R}|Dv(x)|^p\,dx\right].\end{aligned}$$

Since $v\in W^{1,p}_{\mathrm{loc}}\,(\Omega)$ and $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, then, by Lemma 1.2.4, we get $v\in W^{2,p}_{\mathrm{loc}}\,(\Omega)$, and we have

$$\int_{B_r}\left|D^2v(x)\right|^p dx\le c\left[1+\int_{B_R}|DV_p\,(Dv(x))|^2\,dx+c\int_{B_R}|Dv(x)|^p\,dx\right],$$

that is (1.8). □

Remark 1.4.6. If $\Omega\subset\mathbb{R}^n$ is a bounded open set and $1<p<2$, then one may use Remark 1.4.4 and Lemma 1.4.5 to show that, if $v\in W^{1,p}_{\mathrm{loc}}\,(\Omega)$ and $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, then $v\in W^{2,p}_{\mathrm{loc}}\,(\Omega)$ and (1.6) holds true with v in place of g.

Remark 1.4.7. If $\Omega\subset\mathbb{R}^n$ is a bounded open set and $p\in(1,\infty)$, for any $v\in W^{1,p}_{\mathrm{loc}}\,(\Omega)$ such that $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, if $m>1$ and $v\in L^{2m}_{\mathrm{loc}}\,(\Omega)$, then, thanks to (1.1), $Dv\in L^{\frac{m(p+2)}{m+1}}_{\mathrm{loc}}\,(\Omega)$. Moreover, if $v\in L^{\infty}_{\mathrm{loc}}\,(\Omega)$, thanks to (1.2), we get $Dv\in L^{p+2}_{\mathrm{loc}}\,(\Omega)$.

Remark 1.4.8. For further needs we record the following elementary inequality

$$\left(\mu^2+|\xi|^2\right)^{\frac{p}{2}}\le c(p)\left(\mu^p+|V_p(\xi)|^2\right) \tag{1.9}$$

for every $\xi\in\mathbb{R}^{n\times N}$.
Note that this is obvious if $\mu=0$. In case $\mu>0$, we distinguish two cases.
If $|\xi|\le\mu$ we trivially have

$$\left(\mu^2+|\xi|^2\right)^{\frac{p}{2}}\le 2^{\frac{p}{2}}\mu^p$$

If $|\xi|>\mu$, we have

$$\begin{aligned}\left(\mu^2+|\xi|^2\right)^{\frac{p}{2}} &= \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{2}}\left(\mu^2+|\xi|^2\right)\\ &\le \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{2}}\left(|\xi|^2+|\xi|^2\right)\le 2\left(\mu^2+|\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2\\ &\le 2\,|V_p(\xi)|^2.\end{aligned}$$

Joining two previous inequalities we get (1.9).

Moreover, if $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, by Sobolev's inequality, we have $Dv\in L^{\frac{np}{n-2}}_{\mathrm{loc}}\,(\Omega)=L^{\frac{2^*p}{2}}_{\mathrm{loc}}\,(\Omega)$. Indeed, using (1.9), we get

$$\int_{B_R}|Dv(x)|^{\frac{2^*p}{2}}\,dx=\int_{B_R}\left||Dv(x)|^{\frac{p}{2}-1}Dv(x)\right|^{2^*}dx$$

$$\leq \mu^{\frac{2^* p}{2}} |B_R| + \int_{\{x\in B_R:|Dv|>\mu\}} |V_p(Dv(x))|^{2^*} dx$$
$$\leq \mu^{\frac{2^* p}{2}} |B_R| + \int_{B_R} |V_p(Dv(x))|^{2^*} dx, \qquad (1.10)$$

which is finite by the Sobolev's embedding Theorem, for any ball $B_R \Subset \Omega$.

The following result is a fractional counterpart of Lemma 1.4.5, that will be useful to treat regularity properties of solutions to variational problems with Besov-Lipschitz coefficients (see Section 4.4.)

Lemma 1.4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < 2$, $\alpha \in (0,1)$ and $1 \leq q \leq \infty$. Then, for any function $v \in W^{1,p}_{\text{loc}}(\Omega)$ the following implication holds*

$$V_p(Dv) \in B^{\alpha}_{2,q,\text{loc}}(\Omega) \implies Dv \in B^{\alpha}_{p,q,\text{loc}}(\Omega). \qquad (1.11)$$

Moreover, for any ball $B_R \Subset \Omega$ and $0 < r < R$, the following estimate

$$[Dv]_{\dot{B}^{\alpha}_{p,q}(B_r)} \leq C\left(1 + \|Dv\|_{L^p(B_R)} + \|V_p(Dv)\|_{B^{\alpha}_{2,q}(B_R)}\right)^{\sigma} \qquad (1.12)$$

holds true for $1 \leq q \leq \infty$, where C and σ are positive constants depending on n, p, α and q.

Proof. Let us fix a ball $B_R(x_0) \Subset \Omega$ and $0 < r < R$.
Since $V_p(Dv) \in B^{\alpha}_{2,q,\text{loc}}(\Omega)$, then, by definition, $V_p(Dv) \in L^2_{\text{loc}}(\Omega)$, and so it's easy to check that $Dv \in L^p_{\text{loc}}(\Omega)$.
If we apply Remark 1.4.8, we easily get

$$\int_{B_r} |Dv(x)|^p dx \leq C\left(\int_{B_r} |V_p(Dv(x))|^2 dx + 1\right),$$

where the positive constant C depends on n and p.
Now, let us consider, first, the case $1 \leq q < \infty$.
Using Hölder's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$, Lemmas 1.4.3 and 1.2.3, we have

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}(0)} \left(\int_{B_r} \frac{|\tau_h Dv(x)|^p}{|h|^{p\alpha}} dx\right)^{\frac{q}{p}} \frac{dh}{|h|^n} \\
=\ &\int_{B_{\frac{R}{2}}(0)} \left[\left(\int_{B_r} \frac{|\tau_h Dv(x)|^p}{|h|^{p\alpha}}\right) \cdot \left(\mu^2 + |Dv(x)|^2 + |Dv(x+h)|^2\right)^{\frac{p(p-2)}{4}}\right. \\
&\left.\cdot \left(\mu^2 + |Dv(x)|^2 + |Dv(x+h)|^2\right)^{\frac{p(2-p)}{4}} dx\right]^{\frac{q}{p}} \frac{dh}{|h|^n} \\
\leq\ &\int_{B_{\frac{R}{2}}(0)} \left[\int_{B_r} \frac{|\tau_h Dv(x)|^2}{|h|^{2\alpha}} \cdot \left(\mu^2 + |Dv(x)|^2 + |Dv(x+h)|^2\right)^{\frac{(p-2)}{2}} dx\right]^{\frac{q}{2}} \\
&\cdot \left[\int_{B_r} \left(\mu^2 + |Dv(x)|^2 + |Dv(x+h)|^2\right)^{\frac{p}{2}} dx\right]^{\frac{2-p}{2}\cdot\frac{q}{p}} \frac{dh}{|h|^n} \\
\leq\ &c\left[\int_{B_R} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx\right]^{\frac{2-p}{2}\cdot\frac{q}{p}} \\
&\cdot \left[\int_{B_{\frac{R}{2}}(0)} \left(\int_{B_r} \frac{|\tau_h V_p(Dv(x))|^2}{|h|^{2\alpha}} dx\right)^{\frac{q}{2}} \frac{dh}{|h|^n}\right],
\end{aligned} \qquad (1.13)$$

and the right-hand side of (1.13) is finite since, as we proved above, $Dv \in L^p_{\mathrm{loc}}(\Omega)$, and $V_p(Dv) \in B^{\alpha}_{2,q,\mathrm{loc}}(\Omega)$ by hypothesis.
Let us consider, now, the case $q = \infty$. Arguing as above, we have,

$$\begin{aligned} & \left(\int_{B_r} \frac{|\tau_h Dv(x)|^p}{|h|^{p\alpha}} dx\right)^{\frac{1}{p}} \\ \leq \quad & c\left(\int_{B_R} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}\cdot\frac{1}{p}} \cdot \left(\int_{B_r} \frac{|\tau_h V_p(Dv(x))|^2}{|h|^{2\alpha}} dx\right)^{\frac{1}{2}}, \end{aligned} \tag{1.14}$$

and taking the supremum for $|h| < \frac{R}{2}$, since, by hypothesis, $V_p(Dv) \in B^{\alpha}_{2,\infty,\mathrm{loc}}(\Omega)$, we have $Dv \in B^{\alpha}_{p,\infty,\mathrm{loc}}(\Omega)$.
Recalling the definition of the norms in Besov-Lipschitz spaces, and applying Young's inequality to (1.13) and (1.14), for a suitable choice of C and σ, we conclude with the estimate (1.12) holding true for $1 \leq q \leq \infty$. □

A key point we want to highlight is that, as we mentioned at the beginning of this section, the role of the function V_p with respect to the regularity properties of variatonal problems with standard p-growth conditions and p-ellipticity, changes with the value of the exponent p, in particular when we move from the case $p \geq 2$ to the case $1 < p < 2$.
Indeed, with similar arguments, if $p \geq 2$, it is easy to prove that the reverse of the implications (1.7) and (1.11) hold.

Lemma 1.4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \geq 2$, and $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then the following implication holds*

$$v \in W^{2,p}_{\mathrm{loc}}(\Omega) \implies V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

and the following estimate

$$\int_{B_r} |DV_p(Dv(x))|^2 dx \leq c\left[1 + \int_{B_R} \left|D^2 v(x)\right|^p dx + c\int_{B_R} |Dv(x)|^p dx\right].$$

holds for any ball $B_R \Subset \Omega$ and $0 < r < R$.

This means that, while when $p \geq 2$, if we assume $Dv \in W^{1,p}$, we also have $V_p(Dv) \in W^{1,2}$, this doesn't happen for $1 < p < 2$.
This difference becomes crucial when we have to apply the ellipticity assumption in order to prove regularity results, in particular for what concerns higher differentiability properties, for solutions to p-elliptic problems, and also the results are different because of this reason.
The same phenomenon occurs for Besov-Lipschitz functions.

Lemma 1.4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \geq 2$, $\alpha \in (0,1)$ and $1 \leq q \leq \infty$. Then, for any function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ the following implication holds*

$$Dv \in B^{\alpha}_{p,q,\mathrm{loc}}(\Omega) \implies V_p(Dv) \in B^{\alpha}_{2,q,\mathrm{loc}}(\Omega).$$

Moreover, for any ball $B_R \Subset \Omega$ and $0 < r < R$, the following estimate

$$[V_p(Dv)]_{\dot{B}^{\alpha}_{2,q}(B_r)} \leq C\left(1 + \|Dv\|_{L^p(B_R)} + \|Dv\|_{B^{\alpha}_{p,q}(B_R)}\right)^{\sigma}$$

holds true for $1 \leq q \leq \infty$, where C and σ are positive constants depending on n, p, α and q.

It is obvious that, for $p = 2$, we have $V_p(\xi) = \xi$ for any $\xi \in \mathbb{R}^k$, and all the implications discussed above become equivalences.

Chapter 2

Homogeneous systems

The aim of this chapter is to present some regularity results for solutions to unconstrained variational problems where the energy density of the integral functional satisfies sub-quadratic growth and ellipticity conditions.
Moreover, for what concerns the dependence of the integrand function on the x-variable, we will assume that its derivatives with respect to the gradient variable belong to some suitable Sobolev spaces.
In Section 2.1 a first example of this kind of problem is faced, with the aim to show an estimate for the L^p norm of the second derivatives of solutions to some kind of problem described above, that are a priori assumed to be, locally, in $W^{2,p}$.
This kind of a priori estimates are usually the first step in the study of regularity properties of solutions to variational problems.
In fact, by means of known results available for solutions to problems with stronger properties, which satisfy the a priori assumptions, and using some approximations arguments, it is often possible to prove higher regularity results for solution to less regular problems, starting from a suitable a priori estimate.
This kind of technique is used, in this chapter, in Section 2.2, where higher integrability, differentiability and boundedness results are proved for the gradient of solutions to a class of homogeneous equations.

In this chapter, we will consider integral functionals, whose minimizers are possibly vector-valued functions, of the form

$$\mathcal{F}(w, \Omega) = \int_\Omega f(x, Dw(x))\, dx, \tag{2.1}$$

where, for $n \geq 2$ and $N \geq 1$, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x, \xi)$ is of class $C^1\left(\mathbb{R}^{n\times N}\right)$.
For an exponent $p \in (1, 2)$ and some constants $\ell_1, \ell_2, \nu, L > 0$ and a parameter $\mu \in [0, 1]$ the map f satisfies the following p-growth and p-elipticity conditions:

$$\ell_1\left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}} \leq f(x, \xi) \leq \ell_2\left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}}, \tag{2.2}$$

$$\langle D_\xi f(x, \xi) - D_\xi f(x, \eta), \xi - \eta\rangle \geq \nu\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{2.3}$$

$$|D_\xi f(x, \xi) - D_\xi f(x, \eta)| \leq L\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{2.4}$$

for any $\xi, \eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$.
In order to simplify the presentation, we will also denote

$$A_i^\alpha(x,\xi) := D_{\xi_i^\alpha} f(x,\xi), \qquad \text{for all } \alpha = 1,\dots,N \text{ and } i = 1,\dots,n. \tag{2.5}$$

Assumptions (2.2), (2.3) and (2.4) imply, respectively

$$|A\,(x,\xi)| \le \ell \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{2.6}$$

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle \ge \nu \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{2.7}$$

$$|A(x,\xi) - A(x,\eta)| \le L \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2. \tag{2.8}$$

Let us notice that, if the map $\xi \mapsto f(x,\xi)$ is of class $C^2\left(\mathbb{R}^{n\times N}\right)$, conditions (2.3) and (2.4), for constants $L_1, L_2 > 0$, can be replaced by

$$L_1 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \le \langle D_{\xi\xi} f\,(x,\xi)\,\eta, \eta\rangle \le L_2 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2, \tag{2.9}$$

for almost every x in Ω, and for all ξ, η in $\mathbb{R}^{n\times N}$, i.e., with the notation (2.5),

$$L_1 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \le \langle D_\xi A\,(x,\xi)\,\eta, \eta\rangle \le L_2 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2,$$

for any $\xi, \eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$. The main novelty of the results given here is that difficulties due to the fact that $1 < p < 2$ combine with the presence of Sobolev coefficients for the Euler-Lagrange equations of the functionals. Indeed we assume that the map $x \mapsto D_\xi f(x,\xi)$, for any $\xi \in \mathbb{R}^{n\times N}$, belongs to the Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$, and the results we present also depend on the value of the exponent q.
Let us recall that this condition is equivalent to assume that there exists a non-negative function $\kappa \in L^q_{\mathrm{loc}}(\Omega)$, such that

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le (\kappa(x) + \kappa(y))\,|x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{2.10}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^{n\times N}$, which is also equivalent to say that there exists a non-negative function $\tilde\kappa \in L^q_{\mathrm{loc}}(\Omega)$ such that

$$|D_{x\xi} f(x,\xi)| \le \tilde\kappa(x) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{2.11}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{n\times N}$.
As we will see in the following, (2.10) is useful when we use the difference quotient method. In order to simplify the notations, if we define the function

$$g = \max\{\,\kappa, \tilde\kappa\,\} \qquad \text{a.e. in } \Omega,$$

we have $g \in L^q_{\mathrm{loc}}(\Omega)$ and, in place of (2.10), we can use the condition

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le (g(x) + g(y))\,|x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{2.12}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^{n\times N}$.
Similarly, in place of (2.11), we can use

$$|D_{x\xi} f(x,\xi)| \le g(x) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{2.13}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{n\times N}$.

With the notation (2.5), (2.12) becomes

$$|A(x,\xi)-A(y,\xi)| \le (g(x)+g(y))\,|x-y| \left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}, \tag{2.14}$$

for almost every $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^{n\times N}$. For this point-wise characterization of Sobolev functions we refer to [66].

2.1 An a priori estimate

The aim of this section is to show an a priori estimate for the L^p-norm of second derivatives of local minimizers of the kind of functional described above. This result is contained in [55]. Here we consider the functional (2.1), for which, for $n \ge 2$ and $N \ge 1$, $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x,\xi)$ is of class $C^1\left(\mathbb{R}^{n\times N}\right)$.
For an exponent $p \in (1,2)$, we assume (2.2) and (2.3), for every $\xi, \eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$. For what concerns the dependence of the energy density on the x-variable, we shall assume (2.12).
The result we present here starts the study of the higher differentiability properties of local minimizers of integral functional (2.1) under sub-quadratic growth condition. More precisely, we shall establish the following a priori estimate for the second derivatives of the local minimizers.

Theorem 2.1.1. *Let $u \in W^{2,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ be a local minimizer of the functional* (2.1) *under the assumptions* (2.2), (2.3) *and* (2.12) *for a non-negative function $g \in L^q_{\mathrm{loc}}(\Omega)$, with $q \ge \frac{2n}{p}$, than the following estimate*

$$\left\| D^2u \right\|_{L^p\left(B_{\frac{R}{2}}\right)} \le C\left(\|Du\|_{L^p(B_R)} + \|g\|_{L^q(B_R)} \right) \tag{2.15}$$

holds true for every ball $B_R \Subset \Omega$ with $C = C(\nu, \ell_1, \ell_2, p, n, N, R) > 0$.

The main tool to establish this result is the so called difference quotient method and a double iteration to reabsorb terms with critical summability.

2.1.1 Proof of Theorem 2.1.1

It is well known that every local minimizer of the functional (2.1) is a weak solution $u \in W^{1,p}\left(\Omega, \mathbb{R}^N\right)$ of the corresponding Euler-Lagrange system, which, with the notation (2.5), becomes

$$\mathrm{div} A\,(x, Du) = 0. \tag{2.16}$$

As we mentioned above, with the same notation, assumptions (2.2), (2.3) and (2.12) imply, respectively (2.6), (2.7) and (2.14).

Proof of Theorem 2.1.1. Let us fix a ball $B_R(x_0) = B_R$ of radius $R \in (0, d(x_0, \partial\Omega))$, and consider $\frac{R}{2} \le r < \tilde{s} < t < \tilde{t} < \lambda r < R < 1$, with $1 < \lambda < 2$. Let's test the equation (2.16) with the function $\varphi = \tau_{s,-h}\left(\eta^2 \tau_{s,h} u\right)$, where $\eta \in C_0^\infty(B_t)$ is a cut-off function such that $\eta = 1$ on $B_{\tilde{s}}$, $|D\eta| \le \frac{c}{t-\tilde{s}}$ and $\left|D^2\eta\right| \le \frac{c}{(t-\tilde{s})^2}$.
With this choice of φ, and by Proposition 1.2.2, we get

$$\int_{B_R} \left\langle \tau_{s,h} A\,(x, Du(x))\,, D\left(\eta^2\left(\tau_{s,h} u(x)\right)\right)\right\rangle dx = 0.$$

Since we want to control difference quotients indipendently of the direction, from now on, we simplify the notations dropping the direction s and the corresponding unit vector e_s.
So, recalling the definition of finite difference operator, and the notation
$\tau_h F(x) = F(x+h) - F(x)$ from Section 1.2, we can write previous inequality as follows

$$\begin{aligned}
I_0 &:= \int_{B_R} \left\langle A\left(x+h, Du(x+h)\right) - A\left(x+h, Du(x)\right), \eta^2 D\left(\tau_h u(x)\right)\right\rangle dx \\
&= -\int_{B_R} \left\langle A\left(x+h, Du(x)\right) - A\left(x, Du(x)\right), \eta^2 D\left(\tau_h u(x)\right)\right\rangle dx \\
&\quad -\int_{B_R} \left\langle \tau_h A\left(x, Du(x)\right), 2\eta D\eta \otimes \tau_h u(x)\right\rangle dx \\
&= -\int_{B_R} \left\langle A\left(x+h, Du(x)\right) - A\left(x, Du(x)\right), \eta^2 D\left(\tau_h u(x)\right)\right\rangle dx \\
&\quad -\int_{B_R} \left\langle A\left(x, Du(x)\right), \tau_{-h}\left(2\eta D\eta \otimes \tau_h u(x)\right)\right\rangle dx \\
&= -\int_{B_R} \left\langle A\left(x+h, Du(x)\right) - A\left(x, Du(x)\right), \eta^2 D\left(\tau_h u(x)\right)\right\rangle dx \\
&\quad -\int_{B_R} \left\langle A\left(x, Du(x)\right), \tau_{-h}\left(2\eta D\eta\right) \otimes \tau_h u(x-h)\right\rangle dx \\
&\quad -\int_{B_R} \left\langle A\left(x, Du(x)\right), 2\eta D\eta \otimes \tau_{-h}\left(\tau_h u(x)\right)\right\rangle dx \\
&=: I + II + III.
\end{aligned}$$

Previous equality implies

$$I_0 \le |I| + |II| + |III|. \tag{2.17}$$

In order to estimate the integral I, we use (2.14) and Young's inequality, as follows

$$\begin{aligned}
|I| &\le c|h| \int_{B_R} \eta^2 \left(g(x) + g(x+h)\right) \left(\mu^2 + |Du(x)|^2\right)^{\frac{p-1}{2}} |D\tau_h u(x)|\, dx \\
&\le c|h| \int_{B_R} \eta^2 \left(g(x) + g(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-1}{2}} \\
&\quad \cdot |D\left(\tau_h u(x)\right)|\, dx \\
&\le \varepsilon \int_{B_R} \eta^2 \left|D\left(\tau_h u(x)\right)\right|^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + c_\varepsilon |h|^2 \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx. \quad (2.18)
\end{aligned}$$

Now, we estimate II by (2.6) and the properties of η and Proposition 1.2.2, thus obtaining

$$\begin{aligned}
|II| &\le \frac{c|h|}{(t-\tilde{s})^2} \int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p-1}{2}} |\tau_h u(x-h)|\, dx \\
&\le \frac{c|h|}{(t-\tilde{s})^2} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_t} |\tau_h u(x-h)|^p\, dx\right)^{\frac{1}{p}},
\end{aligned}$$

where, in the last inequality, we used Hölder's inequality. By virtue of the first inequality of Lemma 1.2.3, we obtain

$$|II| \le \frac{c|h|^2}{(t-\tilde{s})^2} \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \tag{2.19}$$

The term III is estimated using (2.6) again, the properties of η and Hölder's inequality as follows

$$\begin{aligned} |III| &\le \frac{c}{t-\tilde{s}} \int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p-1}{2}} |\tau_{-h}\left(\tau_h u(x)\right)| \, dx \\ &\le \frac{c}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_t} |\tau_{-h}\left(\tau_h u(x)\right)|^p \, dx\right)^{\frac{1}{p}} \\ &\le \frac{c|h|}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{t}}} |\tau_h Du(x)|^p \, dx\right)^{\frac{1}{p}}, \end{aligned} \tag{2.20}$$

where, in the last inequality, we used Lemma 1.2.3 and Proposition 1.2.2.
By (2.7), we get

$$|I_0| \ge \nu \int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2 dx. \tag{2.21}$$

Inserting estimates (2.18), (2.19), (2.20) and (2.21) in (2.17), we obtain

$$\begin{aligned} &\nu \int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2 \, dx \\ \le\ & \varepsilon \int_{B_R} \eta^2 \left|D\left(\tau_h u(x)\right)\right|^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\ &+ c_\varepsilon |h|^2 \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx \\ &+ \frac{c|h|^2}{(t-\tilde{s})^2} \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\ &+ \frac{c|h|}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} |\tau_h Du(x)|^p \, dx\right)^{\frac{1}{p}}. \end{aligned}$$

Choosing $\varepsilon = \frac{\nu}{2}$ in the previous estimate, we can reabsorb the first integral in the right-hand side by the left-hand side, thus getting

$$\begin{aligned} &\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2 \, dx \\ \le\ & c|h|^2 \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx \\ &+ \frac{c|h|^2}{(t-\tilde{s})^2} \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\ &+ \frac{c|h|}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} |\tau_h Du(x)|^p \, dx\right)^{\frac{1}{p}}, \end{aligned}$$

with $c = c(\nu, \ell_1, \ell_2, p, n, N)$.
Dividing previous estimate by $|h|^2$ and using Lemma 1.4.1, we have

$$
\begin{aligned}
&\int_{B_R} \eta^2 \frac{|\tau_h V_p(Du(x))|^2}{|h|^2} dx \\
\leq\ & c \int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} \frac{|\tau_h Du(x)|^2}{|h|^2} dx \\
\leq\ & c \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
& + \frac{c}{(t-\tilde{s})^2} \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\
& + \frac{c}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} \frac{|\tau_h Du(x)|^p}{|h|^p} dx\right)^{\frac{1}{p}}. \qquad (2.22)
\end{aligned}
$$

Now, by Hölder's inequality and Lemma 1.4.1, we get

$$
\begin{aligned}
&\int_{B_R} \eta^2 \frac{|\tau_h Du(x)|^p}{|h|^p} dx \\
\leq\ & \int_{B_R} \eta^2 \frac{|\tau_h V_p(Du(x))|^p}{|h|^p} \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p(2-p)}{4}} dx \\
\leq\ & \left(\int_{B_R} \eta^2 \frac{|\tau_h V_p(Du(x))|^2}{|h|^2} dx\right)^{\frac{p}{2}} \\
& \cdot \left(\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}}, \qquad (2.23)
\end{aligned}
$$

and therefore, combining (2.22) and (2.23), we have

$$
\begin{aligned}
&\int_{B_R} \eta^2 \frac{|\tau_h Du(x)|^p}{|h|^p} dx \\
\leq\ & c \Bigg\{ \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
& + \frac{1}{(t-\tilde{s})^2} \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\
& + \frac{1}{t-\tilde{s}} \left(\int_{B_t} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\tilde{t}}} \frac{|\tau_h Du(x)|^p}{|h|^p} dx\right)^{\frac{1}{p}} \Bigg\}^{\frac{p}{2}} \\
& \cdot \left\{\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx\right\}^{\frac{2-p}{2}}.
\end{aligned}
$$

Using Young's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, which is legitimate since $1 < p < 2$, the properties of η, and Lemma 1.2.3, we have

$$
\begin{aligned}
&\int_{B_R} \eta^2 \frac{|\tau_h Du(x)|^p}{|h|^p} dx \\
\leq\ & c \int_{B_R} \eta^2 \left(g^2(x) + g^2(x+h)\right) \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} dx \\
& + c\left(1 + \frac{1}{(t-\tilde{s})^2}\right) \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\right)^{\frac{1}{p}}\\
\le\ & c\int_{B_{\lambda r}}g^2(x)dx+c\int_{B_{\lambda r}}g^2(x)\,|Du(x)|^p\,dx\\
&+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x+h)|^p\,dx+c\int_{B_{\tilde{t}}}g^2(x+h)\,|Du(x)|^p\,dx\\
&+c\left(1+\frac{1}{(t-\tilde{s})^2}\right)\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\
&+\frac{c}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

Using Young's inequality with exponents p and $\frac{p}{p-1}$ to estimate the last integral in the right-hand side, we obtain

$$
\begin{aligned}
&\int_{B_R}\eta^2\frac{|\tau_h Du(x)|^p}{|h|^p}dx\\
\le\ & c\int_{B_{\tilde{t}}}g^2(x)dx+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x)|^p\,dx\\
&+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x+h)|^p\,dx+c\int_{B_{\tilde{t}}}g^2(x+h)\,|Du(x)|^p\,dx\\
&+c\left(1+\frac{1}{(t-\tilde{s})^2}+\frac{1}{(t-\tilde{s})^{\frac{p}{p-1}}}\right)\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\
&+\frac{1}{2}\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx.
\end{aligned}
$$

Recalling that $\eta=1$ on $B_{\tilde{s}}$, we obtain

$$
\begin{aligned}
&\int_{B_{\tilde{s}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\\
\le\ & \frac{1}{2}\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx+c\int_{B_{\tilde{t}}}g^2(x)dx+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x)|^p\,dx\\
&+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x+h)|^p\,dx+c\int_{B_{\tilde{t}}}g^2(x+h)\,|Du(x)|^p\,dx\\
&+c\left(1+\frac{1}{(t-\tilde{s})^2}+\frac{1}{(t-\tilde{s})^{\frac{p}{p-1}}}\right)\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx. \qquad (2.24)
\end{aligned}
$$

Since the previous estimate holds for every $\frac{R}{2}\le r<\tilde{s}<t<\tilde{t}<\lambda r$, and the constant appearing in (2.24) are indipendent of t, we can pass to the limit as $t\to\tilde{t}$, thus getting

$$
\begin{aligned}
&\int_{B_{\tilde{s}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\\
\le\ & \frac{1}{2}\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx+c\int_{B_{\tilde{t}}}g^2(x)dx\\
&+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x)|^p\,dx+c\int_{B_{\tilde{t}}}g^2(x)\,|Du(x+h)|^p\,dx+c\int_{B_{\tilde{t}}}g^2(x+h)\,|Du(x)|^p\,dx
\end{aligned}
$$

$$+c\left(1+\frac{1}{\left(\tilde{t}-\tilde{s}\right)^2}+\frac{1}{\left(\tilde{t}-\tilde{s}\right)^{\frac{p}{p-1}}}\right)\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx. \tag{2.25}$$

To go further, we have to estimate the three integrals in the third line of (2.25).
To this aim, we apply Hölder's inequality with exponents $\left(\frac{q}{2},\frac{q}{q-2}\right)$ and Lemma 1.2.3, thus obtaining

$$\begin{aligned}&\int_{B_{\tilde{t}}} g^2(x)\,|Du(x)|^p\,dx+c\int_{B_{\tilde{t}}} g^2(x)\,|Du(x+h)|^p\,dx+c\int_{B_{\tilde{t}}} g^2(x+h)\,|Du(x)|^p\,dx\\ \le\;& c\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}}\cdot\left(\int_{B_{\lambda r}}|Du(x)|^{\frac{pq}{q-2}}\,dx\right)^{\frac{q-2}{q}},\end{aligned} \tag{2.26}$$

and the second integral in the right-hand side term of (2.26) is finite for $\frac{pq}{q-2}\le\frac{np}{n-p}$, that is $q\ge\frac{2n}{p}$.
So, estimate (2.25) becomes

$$\begin{aligned}&\int_{B_{\tilde{s}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\\ \le\;&\frac{1}{2}\int_{B_{\tilde{t}}}\frac{|\tau_h Du(x)|^p}{|h|^p}dx+c\int_{B_{\tilde{t}}} g^2(x)dx\\ &+c\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}}\cdot\left(\int_{B_{\lambda r}}|Du(x)|^{\frac{pq}{q-2}}\,dx\right)^{\frac{q-2}{q}}\\ &+c\left(1+\frac{1}{\left(\tilde{t}-\tilde{s}\right)^2}+\frac{1}{\left(\tilde{t}-\tilde{s}\right)^{\frac{p}{p-1}}}\right)\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx,\end{aligned}$$

and by virtue of the Iteration Lemma, we have

$$\begin{aligned}&\int_{B_r}\frac{|\tau_h Du(x)|^p}{|h|^p}dx\\ \le\;& c\int_{B_{\lambda r}} g^2(x)dx+c\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}}\cdot\left(\int_{B_{\lambda r}}|Du(x)|^{\frac{pq}{q-2}}\,dx\right)^{\frac{q-2}{q}}\\ &+c\left(1+\frac{1}{r^2(\lambda-1)^2}+\frac{1}{r^{\frac{p}{p-1}}\,(\lambda-1)^{\frac{p}{p-1}}}\right)\cdot\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\end{aligned}$$

and by Lemma 1.2.4, we get

$$\begin{aligned}&\int_{B_r}|D^2u(x)|^p dx\\ \le\;& c\int_{B_{\lambda r}} g^2(x)dx+c\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}}\cdot\left(\int_{B_{\lambda r}}|Du(x)|^{\frac{pq}{q-2}}\,dx\right)^{\frac{q-2}{q}}\\ &+c\left(1+\frac{1}{r^2(\lambda-1)^2}+\frac{1}{r^{\frac{p}{p-1}}\,(\lambda-1)^{\frac{p}{p-1}}}\right)\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx.\end{aligned} \tag{2.27}$$

Now we distinguish between two cases.
Case I $\left(q=\frac{2n}{p}\right)$.
In case $q=\frac{2n}{p}$, the right-hand side of (2.26) becomes

$$c\left(\int_{B_{\lambda r}} g^{\frac{2n}{p}}(x)dx\right)^{\frac{p}{n}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{n}}.$$

Now we observe that, if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, then $Du \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and, by Sobolev's embedding Theorem, $W^{1,p}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{p^*}_{\mathrm{loc}}(\Omega)$, where $p^* = \frac{np}{n-p}$.
So, for a positive constant $c = c(n,p)$, since $\lambda r < 1$, we have

$$\begin{aligned} &\left(\int_{B_{\lambda r}} g^{\frac{2n}{p}}(x)dx\right)^{\frac{p}{n}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{n}} \\ \leq\ & \frac{c}{(\lambda r)^p}\left(\int_{B_{\lambda r}} g^{\frac{2n}{p}}(x)dx\right)^{\frac{p}{n}} \int_{B_{\lambda r}} \left(\left|D^2u(x)\right|^p + |Du(x)|^p\right)dx. \end{aligned} \tag{2.28}$$

Since $g \in L^{\frac{2n}{p}}_{\mathrm{loc}}(\Omega)$, by the absolute continuity of the integral, there exists $R_0 > 0$ such that, for every $R < R_0$, we have

$$c\left(\int_{B_R} g^{\frac{2n}{p}}(x)dx\right)^{\frac{p}{n}} < \frac{1}{2}. \tag{2.29}$$

For this choice of R, joining (2.27), (2.28), (2.29), we get:

$$\begin{aligned} &\int_{B_r} \left|D^2u(x)\right|^p dx \\ \leq\ & c\int_{B_{\lambda r}} g^2(x)dx + \frac{1}{2}\int_{B_{\lambda r}} \left|D^2u(x)\right|^p dx \\ &+c\left(1 + \frac{1}{r^2(\lambda-1)^2} + \frac{1}{r^{\frac{p}{p-1}}(\lambda-1)^{\frac{p}{p-1}}}\right) \cdot \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \end{aligned} \tag{2.30}$$

Since (2.30) holds for all $\frac{R}{2} \leq r < \lambda r < R$ for $\lambda \in (1,2)$, setting $R_0 = R$, $\gamma = \frac{p}{p-1}$ and

$$h(r) = \int_{B_r} \left|D^2u(x)\right|^p dx,$$

by Lemma 1.1.1, we have

$$\left\|D^2u\right\|_{L^p\left(B_{\frac{R}{2}}\right)} \leq c(\nu, \ell_1, \ell_2, p, n, N, R)\left(\|Du\|_{L^p(B_R)} + \|g\|_{L^q(B_R)}\right). \tag{2.31}$$

A standard covering argument yields the conclusion, with (2.15).
Case II $\left(q > \frac{2n}{p}\right)$.
For $q > \frac{2n}{p}$ we have $p < \frac{pq}{q-2} < \frac{np}{n-p}$, and setting $\theta := \frac{2n}{pq} < 1$, we have

$$\frac{q-2}{pq} = \frac{1-\theta}{p} + \frac{\theta(n-p)}{np},$$

and we can use the interpolation inequality to estimate the last integral in (2.26) as follows

$$\left(\int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}}\,dx\right)^{\frac{q-2}{q}}$$

$$\leq \left[\left(\int_{B_{\lambda r}} |Du(x)|^p\,dx\right)^{\frac{1-\theta}{p}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{\theta(n-p)}{np}}\right]^p,$$

and recalling the definition of θ, the right-hand side of (2.26) can be controlled by

$$c\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^p\,dx\right)^{\frac{pq-2n}{pq}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{2(n-p)}{pq}}.$$

Then, by Sobolev's embedding Theorem, since $\lambda r < 1$, we have

$$\begin{aligned}
&\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^p\,dx\right)^{\frac{pq-2n}{pq}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{2(n-p)}{pq}}\\
\leq\ & \frac{c}{(\lambda r)^p}\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^p dx\right)^{\frac{pq-2n}{pq}} \cdot \left(\int_{B_{\lambda r}} \left(\left|Du^2(x)\right|^p + |Du(x)|^p\right)dx\right)^{\frac{2n}{pq}}.
\end{aligned}$$

Now, since $q > \frac{2n}{p}$, we can use Young's inequality with exponents $\left(\frac{pq}{pq-2n}, \frac{pq}{2n}\right)$, thus getting, for every $\varepsilon > 0$,

$$\begin{aligned}
&\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^p\,dx\right)^{\frac{pq-2n}{pq}} \cdot \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{2(n-p)}{pq}}\\
\leq\ & \frac{c}{(\lambda r)^p}\left(\int_{B_{\lambda r}} g^q(x)dx\right)^{\frac{2}{q}} \cdot \left[c_\varepsilon \int_{B_{\lambda r}} |Du(x)|^p dx + \varepsilon \int_{B_{\lambda r}} \left(\left|D^2u(x)\right|^p + |Du(x)|^p\right)dx\right].
\end{aligned}$$

Now we choose ε such that

$$\varepsilon \cdot \left[c\left(\int_{B_R} g^q(x)dx\right)^{\frac{2}{q}}\right] < \frac{1}{2}, \tag{2.32}$$

so that we can obtain the estimate (2.30) again. The use of Iteration Lemma implies (2.31) also in this case, and so we get (2.15) again.

We remark that, differently from the previous case, when $q > \frac{2n}{p}$, we don't need to use a covering argument to conclude. In fact, in (2.32), we just choose a suitable value of ε, which depends on the norm of g in $L^q(B_R)$, while the radius of the ball on which the integral in the left-hand side is taken does not depend on the L^q-norm of g: here, differently from (2.29), we don't use the absolute continuity of the integral. □

2.2 Higher regularity results for solutions to a class of homogeneous systems

In this section we present the regularity results contained in [54].
Let us consider the functional

$$\mathcal{F}(w,\Omega) = \int_\Omega f(x, Dw(x))\,dx, \tag{2.33}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n > 2$, $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a Carathéodory map, with $N \geq 2$, such that $\xi \mapsto f(x,\xi)$ is of class $C^2\left(\mathbb{R}^{n\times N}\right)$ for a.e. $x \in \Omega$, and for an exponent $p \in (1,2)$, assumptions (2.2) and (2.9) hold.
For what concerns the dependence of the energy density on the x-variable, we shall assume that the map $x \mapsto D_\xi f(x,\xi)$ belongs to the Sobolev space $W^{1,q}\left(\Omega \times \mathbb{R}^{n\times N}\right)$, for some $q \geq n$. This is equivalent to assume that there exists a non-negative function $g \in L^q_{\text{loc}}(\Omega)$ such that (2.13) holds.
In order to avoid the irregularity phenomena that are peculiar of the vectorial minimizers (see [35], [88]), we shall assume that

$$f(x,\xi) = k(x,|\xi|) \tag{2.34}$$

with

$$k(x,\cdot) \in C^2(\mathbb{R}) \text{ if } \mu > 0 \qquad \text{or} \qquad k(x,\cdot) \in C^2(\mathbb{R}\setminus\{0\}) \text{ if } \mu = 0, \tag{2.35}$$

for almost every $x \in \Omega$.
It is worth noticing that (2.9), (2.34) and (2.35) imply (2.2).
The results we describe in this section involve both higher differentiability and higher integrability properties of local minimizers of the functional (2.33).
For what concerns higher differentiability, we want to prove that, assuming $g \in L^q_{\text{loc}}(\Omega)$, with $q \geq n$, any local minimizer $u \in W^{1,p}_{\text{loc}}(\Omega)$ of the functional (2.33) is higher differentiable, that is $u \in W^{2,p}_{\text{loc}}(\Omega)$. Moreover, if $q > n$, we establish the Lipschitz continuity of the local minimizers, and for $q = n$ we prove that the gradient of u is in $L^s_{\text{loc}}(\Omega)$ for any $s \in (1,\infty)$.
More precisely, our main results are the following.

Theorem 2.2.1. *Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a local minimizer of the functional* (2.33)*, under the assumptions* (2.9)*,* (2.34) *and* (2.35)*.*
Moreover, let us assume that, for $q > n$, there exists a non-negative function $g \in L^q_{\text{loc}}(\Omega)$, such that (2.13) *holds.*
Then $u \in W^{2,p}_{\text{loc}}(\Omega)$ and $Du \in L^\infty_{\text{loc}}(\Omega)$.
Moreover, there exist two constants $c_1, c_2 > 0$, depending on $n, N, p, q, L_1, L_2, R, \|g\|_{L^q(B_R)}$, such that the following estimates hold:

$$\|Du\|_{L^\infty\left(B_{\frac{R}{2}}\right)} \leq c_1 \left(\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}, \tag{2.36}$$

and

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \leq c_2 \cdot \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.37}$$

for every ball B_R such that $B_R \Subset \Omega$, with $R < 1$.

In the critical case $q = n$, we have the following.

Theorem 2.2.2. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of the functional* (2.33)*, under the assumptions* (2.9)*,* (2.34) *and* (2.35)*.*
Moreover, let us assume that, for $q = n$*, there exists a non-negative function* $g \in L^q_{\mathrm{loc}}(\Omega)$*, such that* (2.13) *holds.*
Then, for any $1 < s < \infty$*,* $Du \in L^s_{\mathrm{loc}}(\Omega)$*, and there exists a constant* $c_1 > 0$*, depending on* $n, N, p, s, L_1, L_2, R, \|g\|_{L^n(B_R)}$*, such that, for every ball* $B_R \Subset \Omega$ *with* $R < 1$*, the following estimate holds*

$$\left(\int_{B_{\frac{R}{2}}} |Du(x)|^s \, dx\right)^{\frac{1}{s}} \le c_1 \cdot \left(\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}. \tag{2.38}$$

Moreover, $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$*, and there exists a constant* $c_2 = c_2\left(n, N, p, L_1, L_2, \|g\|_{L^n(B_R)}\right) > 0$ *such that*

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \le c_2 \cdot \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \tag{2.39}$$

The proofs of our results are achieved combining suitable a priori estimates with an approximation argument. First of all, making the a priori assumptions that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $V_p(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we will use Moser's iteration technique (see, for example, [29]) to find an a priori estimate for the L^∞-norm of Du in case $q > n$, and an a priori estimate for the L^s-norm of Du for any $1 < s < \infty$ if $q = n$. We will also find an a priori estimate for the L^p-norm of the second derivatives of u that improves that established in [55].
After that, by approximation, we will use these a priori estimates to prove that a minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is actually in $W^{2,p}_{\mathrm{loc}}(\Omega)$ and, if $q > n$, then $Du \in L^\infty_{\mathrm{loc}}(\Omega)$, while, if $q = n$, $Du \in L^s_{\mathrm{loc}}(\Omega)$ for all $1 < s < \infty$.
Let us notice that, here, we are making some stronger assumptions about the dependence of f on the ξ-variable, if we compare them to those of the previous section, where we assume that $\xi \mapsto f(x, \xi)$ is of class $C^1\left(\mathbb{R}^{n\times N}\right)$ and, instead of (2.9), we just assumed (2.3).
On the other hand, for what concerns the dependence of the coefficients on the x-variable, in the previous section, instead of (2.13) with $q \ge n$, we assumed (2.12) with $q \ge \frac{2n}{p} > n$ (since $1 < p < 2$). The possibility to weaken this condition is due to the fact that the a priori assumption $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is stronger than $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, by virtue of Lemma 1.4.5.

2.2.1 A priori estimates

Our first step is to prove the a priori estimates. More precisely, making a distinction between the cases $q > n$ and $q = n$ in the assumption (2.13), we want to prove the following claims.

Lemma 2.2.3. *Let* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of the functional* (2.33) *such that* $V_p(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$*, under the assumptions* (2.9)*,* (2.34) *and* (2.35)*.*
Moreover, let us assume that, for $q > n$*, there exists a non-negative function* $g \in L^q_{\mathrm{loc}}(\Omega)$*, such that* (2.13) *holds.*
Then there exist two constants $c_1, c_2 > 0$*, depending on* $n, N, p, q, L_1, L_2, R, \|g\|_{L^q(B_R)}$*, such that the following estimates:*

$$\|Du\|_{L^\infty\left(B_{\frac{R}{2}}\right)} \le c_1 \left(\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}, \tag{2.40}$$

and

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \le c_2 \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.41}$$

hold for every ball B_R such that $B_R \Subset \Omega$ with $R < 1$.

Lemma 2.2.4. *Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (2.33) such that $V_p(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$, under the assumptions (2.9), (2.34) and (2.35). Moreover, let us assume that, for $q = n$, there exists a non-negative function $g \in L^q_{\mathrm{loc}}(\Omega)$, such that (2.13) holds.*
Then, for any $1 < s < \infty$ there is a constant $c_1 > 0$, depending on $n, N, p, s, L_1, L_2, R, \|g\|_{L^n(B_R)}$, and there exists $R_s \in (0,1)$ depending on s, p, n, g such that, for every ball $B_R \Subset \Omega$, with $0 < R \le R_s$, the following estimate holds

$$\left(\int_{B_{\frac{R}{2}}} |Du(x)|^s dx\right)^{\frac{1}{s}} \le c_1 \left(\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}. \tag{2.42}$$

Moreover, there exist a constant $c_2 = c_2(n, N, p, s, L_1, L_2, R, \|g\|_{L^n(B_R)}) > 0$ and $R_0 \in (0,1)$ depending on p, n, g such that

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \le c_2 \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.43}$$

for $0 < R \le R_0$ and $B_R \Subset \Omega$.

Proof of Lemma 2.2.3. Our starting point is the the Second Variation of the functional $\mathcal{F}$. Let us consider a test function $\varphi = D\psi$, with $\psi \in C^\infty_0(\Omega)$, and put φ in the Euler-Lagrange equation of $\mathcal{F}$, so we have

$$\int_\Omega \left\langle D_\xi f(x, Du(x)), D^2\psi(x)\right\rangle dx = 0,$$

and an integration by parts yields

$$\int_\Omega \left\langle D_x\left(D_\xi f(x, Du(x))\right), D\psi(x)\right\rangle = 0,$$

i.e.

$$\int_\Omega \left\langle D_{x\xi} f(x, Du(x)) + D_{\xi\xi} f(x, Du(x))\, D^2u(x), D\psi(x)\right\rangle = 0. \tag{2.44}$$

Now, for a ball $B_R \Subset \Omega$ and $0 < r < R < 1$, we choose a cut-off function $\eta \in C^\infty_0(B_R)$ such as $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r, and $|D\eta| \le \frac{c}{R-r}$ for a constant $c = c(n) > 0$.
The the a priori assumptions $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $V_p(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$ allow us to consider, for $\gamma \ge 0$, the test function $\psi = \eta^2\left(\mu^2 + |Du|^2\right)^{\frac{\gamma}{2}} Du$ in equation (2.44). Computing the derivatives of ψ, we get

$$\begin{aligned} D\psi &= 2\eta\left(\mu^2 + |Du|^2\right)^{\frac{\gamma}{2}} D\eta \otimes Du + \frac{\gamma}{2}\eta^2\left(\mu^2 + |Du|^2\right)^{\frac{\gamma-2}{2}} D\left(|Du|^2\right) \otimes Du \\ &\quad + \eta^2\left(\mu^2 + |Du|^2\right)^{\frac{\gamma}{2}} D^2u, \end{aligned}$$

and the equation (2.44) becomes

$$\begin{aligned}
0 &= 2\int_{B_R}\left\langle D_{x\xi}f\left(x,Du(x)\right),\eta\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}D\eta\otimes Du(x)\right\rangle dx\\
&\quad+\frac{\gamma}{2}\int_{B_R}\left\langle D_{x\xi}f\left(x,Du(x)\right),\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma-2}{2}}D\left(|Du(x)|^2\right)\otimes Du(x)\right\rangle dx\\
&\quad+\int_{B_R}\left\langle D_{x\xi}f\left(x,Du(x)\right),\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}D^2u(x)\right\rangle dx\\
&\quad+2\int_{B_R}\left\langle D_{\xi\xi}f\left(x,Du(x)\right)D^2u(x),\eta\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}D\eta\otimes Du(x)\right\rangle dx\\
&\quad+\frac{\gamma}{2}\int_{B_R}\left\langle D_{\xi\xi}f\left(x,Du(x)\right)D^2u(x),\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma-2}{2}}D\left(|Du(x)|^2\right)\otimes Du(x)\right\rangle dx\\
&\quad+\int_{B_R}\left\langle D_{\xi\xi}f\left(x,Du(x)\right)D^2u(x),\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}D^2u(x)\right\rangle dx\\
&=: I+II+III+IV+V+I_0. \qquad (2.45)
\end{aligned}$$

The integral V is non-negative by the assumption $f(x,\xi)=k(x,|\xi|)$. Actually, it suffices to calculate

$$D_{\xi_i^\alpha\xi_j^\beta}f(x,\xi)=D_{tt}k\left(x,|\xi|\right)\frac{\xi_i^\alpha\xi_j^\beta}{|\xi|^2}+D_tk\left(x,|\xi|\right)\left(\frac{\delta^{\alpha\beta}\delta_{ij}}{|\xi|}-\frac{\xi_i^\alpha\xi_j^\beta}{|\xi|^3}\right)$$

and use the definition of the scalar product to deduce that

$$V\geq 0.$$

So, from (2.45), we get

$$I_0\leq I_0+V\leq |I|+|II|+|III|+|IV|. \qquad (2.46)$$

In the following, we will often use the trivial inequality

$$|\xi|\leq\left(\mu^2+|\xi|^2\right)^{\frac{1}{2}},\qquad \forall\xi\in\mathbb{R}^{n\times N}. \qquad (2.47)$$

By the left inequality in the hypothesis (2.9), we get

$$\begin{aligned}
|I_0| &\geq c\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\left|D^2u(x)\right|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}dx\\
&= c\int_{B_R}\eta^2\left|D^2u(x)\right|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-2}{2}}dx. \qquad (2.48)
\end{aligned}$$

To estimate the term I, we use (2.13) and (2.47), thus getting

$$\begin{aligned}
|I| &\leq 2\int_{B_R}\eta\,|D\eta|\,g(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|Du(x)|\left(\mu^2+|Du(x)|^2\right)^{\frac{\gamma}{2}}dx\\
&\leq 2\int_{B_R}\eta\,|D\eta|\,g(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \qquad (2.49)
\end{aligned}$$

Applying Young's inequality in the right-hand side of (2.49), we get

$$|I| \leq 2\int_{B_R}\eta\,|D\eta|\,g(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}$$

$$\begin{aligned} &\leq c\int_{B_R}\eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx \\ &\quad +c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \end{aligned} \tag{2.50}$$

We use again (2.13) and (2.47) to estimate the term II as follows

$$|II|\leq\gamma\int_{B_R}\eta^2 g(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1+\gamma}{2}}\left|D^2u(x)\right|dx.$$

Since $\frac{p+\gamma-1}{2}=\frac{p+\gamma-2}{4}+\frac{p+\gamma}{4}$, by Young's inequality, we get

$$\begin{aligned} |II| &\leq \varepsilon\int_{B_R}\eta^2\left|D^2u(x)\right|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-2}{2}}dx \\ &\quad +c_\varepsilon\gamma^2\int_{B_R}\eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \end{aligned} \tag{2.51}$$

In order to estimate III, we use (2.13) and Young's inequality as before:

$$\begin{aligned} |III| &\leq \int_{B_R}\eta^2 g(x)\left|D^2u(x)\right|\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-1}{2}}dx \\ &\leq \varepsilon\int_{B_R}\eta^2\left|D^2u(x)\right|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-2}{2}} \\ &\quad +c_\varepsilon\int_{B_R}\eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}. \end{aligned} \tag{2.52}$$

We can estimate IV using (2.9) and (2.47) thus getting

$$|IV|\leq 2\int_{B_R}\eta\,|D\eta|\left|D^2u(x)\right|\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-1}{2}}dx.$$

Using Young's inequality, we have

$$\begin{aligned} |IV| &\leq \varepsilon\int_{B_R}\eta^2\left|D^2u(x)\right|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-2}{2}}dx \\ &\quad +c_\varepsilon\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \end{aligned} \tag{2.53}$$

Now, inserting (2.48), (2.50), (2.51), (2.52) and (2.53) in (2.46), and choosing ε sufficiently small to reabsorb the first terms on the right-hand sides of (2.51), (2.52) and (2.53), we get

$$\begin{aligned} &\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-2}{2}}\left|D^2u(x)\right|^2dx \\ \leq\ & c(1+\gamma^2)\int_{B_R}\eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx \\ &+c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \end{aligned} \tag{2.54}$$

Now, we observe that

$$\left(\mu^2+|Du|^2\right)^{\frac{p+\gamma-4}{2}}\left|D\left(|Du|^2\right)\right|^2 \leq 4\left(\mu^2+|Du|^2\right)^{\frac{p+\gamma-4}{2}}|Du|^2\left|D^2u\right|^2$$
$$\leq 4\left(\mu^2+|Du|^2\right)^{\frac{p+\gamma-2}{2}}\left|D^2u\right|^2, \tag{2.55}$$

where we also used (2.47). So, using (2.55) in the left-hand side of (2.54), we get

$$\begin{aligned}
&\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-4}{2}}\cdot\left|D\left(|Du(x)|^2\right)\right|^2dx\\
\leq\ & c(1+\gamma^2)\int_{B_R}\eta^2g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
&+c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx.
\end{aligned}\tag{2.56}$$

One can easily check that, for any $\alpha\in\mathbb{R}$,

$$D\left[\left(\mu^2+|Du|^2\right)^{\frac{\alpha}{2}}\right]=\frac{\alpha}{2}\cdot\left(\mu^2+|Du|^2\right)^{\frac{\alpha-2}{2}}\cdot D\left(|Du|^2\right), \tag{2.57}$$

So, using (2.57) with $\alpha=\frac{p+\gamma}{2}$, we have

$$\begin{aligned}
&\left(\mu^2+|Du|^2\right)^{\frac{p+\gamma-4}{2}}\cdot\left|D\left(|Du|^2\right)\right|^2=\left|\left(\mu^2+|Du|^2\right)^{\frac{1}{2}\cdot\left(\frac{p+\gamma}{2}-2\right)}\cdot\left|D\left(|Du|^2\right)\right|\right|^2\\
=\ &\left|\frac{4}{p+\gamma}\cdot D\left[\left(\mu^2+|Du|^2\right)^{\frac{p+\gamma}{4}}\right]\right|^2.
\end{aligned}\tag{2.58}$$

Using (2.58) in the left-hand side of (2.56), we get

$$\begin{aligned}
&\frac{c}{(p+\gamma)^2}\int_{B_R}\eta^2\left|D\left[\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{4}}\right]\right|^2dx\\
\leq\ &\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma-4}{2}}\left|D\left(|Du(x)|^2\right)\right|^2dx\\
\leq\ & c(1+\gamma^2)\int_{B_R}\eta^2g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
&+c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx.
\end{aligned}\tag{2.59}$$

Before going further, since we want to apply Moser's iteration technique, let's observe that, if $V_p\left(Du\right)=\left(\mu^2+|Du|^2\right)^{\frac{p-2}{4}}Du\in W^{1,2}_{\mathrm{loc}}\left(\Omega\right)$, then $\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\in W^{1,2}_{\mathrm{loc}}\left(\Omega\right)$.
Indeed, for any ball $B_r\Subset\Omega$, if we recall (2.47) and (1.6), we have

$$\begin{aligned}
\int_{B_r}\left|D\left[\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2dx &= \frac{p}{2}\cdot\int_{B_r}\left[\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{4}-1}\cdot|Du(x)|\cdot\left|D^2u(x)\right|\right]^2dx\\
&\leq \frac{p}{2}\cdot\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\cdot\left|D^2u(x)\right|^2dx\\
&\leq \frac{p}{2}\int_{B_r}\left|DV_p\left(Du(x)\right)\right|^2dx,
\end{aligned}$$

and the last integral of previous inequality is finite since $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.
So, if we set

$$G_\gamma = \eta \cdot \left(\mu^2 + |Du|^2\right)^{\frac{p+\gamma}{4}},$$

by our a priori assumptions, for $\gamma = 0$, we have $G_0 \in W^{1,2}_0(B_R)$, and denoting $2^* = \frac{2n}{n-2}$, by Sobolev's inequality we have

$$\left(\int_{B_R} |G_0(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \le c \int_{B_R} |DG_0(x)|^2 dx.$$

So we have

$$\begin{aligned}
&\left(\int_{B_R} \eta^{2^*} \left(\mu^2 + |Du(x)|^2\right)^{2^* \cdot \frac{p}{4}} dx\right)^{\frac{2}{2^*}} \\
\le\ & c\left(\int_{B_R} \left|\eta \left|D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right| + |D\eta| \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right|^2 dx\right) \\
\le\ & c\int_{B_R} \eta^2 \left|D\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right|^2 dx \\
& + c\int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \qquad (2.60)
\end{aligned}$$

What we just pointed out with (2.60) is the possibility to construct the first step of Moser's iteration, which consists in obtaining higher integrability for Du, starting from $\gamma = 0$: this is possible because all the integrals in the right-hand side of (2.60) are finite.
Since the following steps of the iterations are based on the possibility to let γ increase and eventually go to infinity, we rewrite (2.60) as follows

$$\begin{aligned}
&\left(\int_{B_R} \eta^{2^*} \left(\mu^2 + |Du(x)|^2\right)^{2^* \cdot \frac{p+\gamma}{4}} dx\right)^{\frac{2}{2^*}} \\
\le\ & c\left(\int_{B_R} \left|\eta \left|D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{4}}\right]\right| + |D\eta| \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{4}}\right|^2 dx\right) \\
\le\ & c\int_{B_R} \eta^2 \left|D\left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{4}}\right|^2 dx \\
& + c\int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx. \qquad (2.61)
\end{aligned}$$

Joining (2.61) and (2.59), we get

$$\begin{aligned}
&\left(\int_{B_R} \eta^{2^*} \left(\mu^2 + |Du(x)|^2\right)^{2^* \cdot \frac{p+\gamma}{4}} dx\right)^{\frac{2}{2^*}} \\
\le\ & c\,(p+\gamma)^2 \int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma-4}{2}} \left|D\left(|Du(x)|^2\right)\right|^2 dx \\
& + c\int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx \\
\le\ & c\,(p+\gamma)^2 \left[\left(1+\gamma^2\right) \int_{B_R} \eta^2 g^2(x) \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx\right.
\end{aligned}$$

$$\begin{aligned}
&+c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\Bigg]\\
&+c\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
=\;& c\,(p+\gamma)^2\,(1+\gamma^2)\int_{B_R}\eta^2g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
&+c\left[1+(p+\gamma)^2\right]\int_{B_R}|D\eta|^2\cdot\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
\le\;& c\,(p+\gamma)^4\int_{B_R}\eta^2g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
&+c\,(p+\gamma)^2\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \qquad (2.62)
\end{aligned}$$

Now, recalling that $g\in L^q_{\mathrm{loc}}(\Omega)$, with $q>n>2$, we can use Hölder's inequality with exponents $\left(\frac{q}{2},\frac{q}{q-2}\right)$, and we infer

$$\begin{aligned}
&\int_{B_R}\eta^2g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
\le\;&\left(\int_{B_R}g^q(x)dx\right)^{\frac{2}{q}}\left(\int_{B_R}\eta^{\frac{2q}{q-2}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\cdot\frac{q}{q-2}}dx\right)^{\frac{q-2}{q}}. \qquad (2.63)
\end{aligned}$$

Since $q>n$, $1<\frac{q}{q-2}<\frac{n}{n-2}$, and we can apply the Interpolation inequality to estimate last integral in (2.63) with $\theta\in(0,1)$ such that

$$\frac{q-2}{q}=\theta+\frac{(1-\theta)(n-2)}{n}.$$

One can easily check that

$$\theta=\frac{q-n}{q},$$

and so

$$\begin{aligned}
&\left[\int_{B_R}\left(\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}\right)^{\frac{q}{q-2}}dx\right]^{\frac{q-2}{q}}\\
\le\;& c\left(\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\right)^{\theta}\\
&\cdot\left[\int_{B_R}\left(\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}\right)^{\frac{n}{n-2}}dx\right]^{\frac{(1-\theta)(n-2)}{n}},
\end{aligned}$$

that is

$$\left[\int_{B_R}\left(\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}\right)^{\frac{q}{q-2}}dx\right]^{\frac{q-2}{q}}$$

$$
\begin{aligned}
&\le c\left(\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\right)^{\theta}\\
&\quad\cdot\left[\int_{B_R}\left(\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\cdot\frac{2^*}{2}}\right)dx\right]^{\frac{2(1-\theta)}{2^*}}. \qquad (2.64)
\end{aligned}
$$

Using (2.63), (2.64), and Young's inequality with exponents $\left(\frac{1}{\theta},\frac{1}{1-\theta}\right)$, we can estimate the first term in the right-hand side of (2.62) as follows

$$
\begin{aligned}
&c\,(p+\gamma)^4\int_{B_R}\eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\\
&\le c_\varepsilon\left[c\,(p+\gamma)^4\left(\int_{B_R}g^q(x)dx\right)^{\frac{2}{q}}\right]^{\frac{1}{\theta}}\left(\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\right)\\
&\quad+\varepsilon\left(\int_{B_R}\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\cdot\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}, \qquad (2.65)
\end{aligned}
$$

for any $\varepsilon>0$.
Now, plugging (2.65) into (2.62), we get

$$
\begin{aligned}
&\left(\int_{B_R}\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}\\
&\le \varepsilon\left(\int_{B_R}\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}\\
&\quad+c_\varepsilon\left[(p+\gamma)^4\left(\int_{B_R}g^q(x)dx\right)^{\frac{2}{q}}\right]^{\frac{1}{\theta}}\cdot\left(\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\right)\\
&\quad+c\,(p+\gamma)^2\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \qquad (2.66)
\end{aligned}
$$

Now, for a sufficently small value of ε, we can reabsorb the first term of the right-hand side of (2.66), thus getting

$$
\begin{aligned}
&\left(\int_{B_R}\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}\cdot\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}\\
&\le c\left[(p+\gamma)^4\left(\int_{B_R}g^q(x)dx\right)^{\frac{2}{q}}\right]^{\frac{1}{\theta}}\cdot\left(\int_{B_R}\eta^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx\right)\\
&\quad+c\,(p+\gamma)^2\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}}dx. \qquad (2.67)
\end{aligned}
$$

For $\gamma=0$, recalling the explicit expression of θ, (2.67) gives

$$
\begin{aligned}
&\left(\int_{B_R}\eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}\cdot\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}\\
&\le cp^{\frac{4q}{q-n}}\left(\int_{B_R}g^q(x)dx\right)^{\frac{2}{q-n}}\cdot\left(\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)
\end{aligned}
$$

$$+c\left(1+p^2\right)\left(\int_{B_R}|D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right).$$

Recalling the properties of η, since $0<r<R<1$, we can write

$$\left(\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}\cdot\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}}\le\frac{c\cdot p^{\frac{4q}{q-n}}}{(R-r)^2}\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx, \tag{2.68}$$

where $c=c(n,N,p,q,L_1,L_2,\|g\|_{L^q(B_R)})$.
Now we choose $r=\frac{R}{2}$ and set

$$R_0=R,\qquad R_i=r+\frac{R-r}{2^i}=\frac{R}{2}\left(1+\frac{1}{2^i}\right),\qquad\forall i\in\mathbb{N} \tag{2.69}$$

and

$$p_0=p,\qquad p_i=\frac{2^*}{2}\cdot p_{i-1}=\left(\frac{2^*}{2}\right)^i\cdot p_0,\qquad\forall i\in\mathbb{N}. \tag{2.70}$$

Observe that the sequence R_i is strictly decreasing, and p_i is strictly increasing. Moreover, as $i\to\infty$, $R_i\to\frac{R}{2}$ and $p_i\to\infty$.
Starting from (2.68), we can iterate (2.67), and recalling (2.47), we get, for every $i\in\mathbb{N}$,

$$\begin{aligned}
&\left(\int_{B_{R_{i+1}}}|Du(x)|^{p_{i+1}}dx\right)^{\frac{1}{p_{i+1}}}\\
\le\ &\left(\int_{B_{R_{i+1}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p_{i+1}}{2}}dx\right)^{\frac{1}{p_{i+1}}}\\
\le\ &\left[\frac{c\cdot p_i^{\frac{4q}{q-n}}}{(R_i-R_{i+1})^2}\right]^{\frac{1}{p_i}}\left(\int_{B_{R_i}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p_i}{2}}dx\right)^{\frac{1}{p_i}}\\
\le\ &\left(\prod_{k=0}^{i}\left[\frac{cp_k^{\frac{4q}{q-n}}}{(R_k-R_{k+1})^2}\right]^{\frac{1}{p_k}}\right)\cdot\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
=\ &\exp\left\{\sum_{k=0}^{i}\left[\frac{1}{p_k}\cdot\log\left(\frac{c\cdot 2^{k+2}p_k^{\frac{4q}{q-n}}}{R^2}\right)\right]\right\}\cdot\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}
\end{aligned} \tag{2.71}$$

where we used that $\frac{2}{2^*}=\frac{p_i}{p_{i+1}}$, and $R_i-R_{i+1}=\frac{R}{2^{i+2}}$.
Since the series

$$\sum_{k=0}^{\infty}\left\{\frac{1}{p_k}\cdot\log\left(\frac{c\cdot 2^{k+2}p_k^{\frac{4q}{q-n}}}{R^2}\right)\right\}$$

converges, we can pass to the limit as $i\to\infty$ in (2.71), and we get the following estimate

$$\|Du\|_{L^\infty\left(B_{\frac{R}{2}}\right)}\le c_1\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}},$$

where $c=c\left(n,N,p,q,L_1,L_2,R,\|g\|_{L^q(B_R)}\right)$, i.e. (2.40).

Moreover, by (2.59) for $\gamma = 0$, we get

$$\begin{aligned} &\int_{B_R} \eta^2 \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \\ \leq\ & cp^2 \left[\int_{B_R} \eta^2 g^2(x) \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \right.\\ & \left. + c\int_{B_R} |D\eta|^2 \cdot \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right]. \end{aligned} \tag{2.72}$$

Using (2.63) and (2.64) again for $\gamma = 0$, with the same value of θ, (2.72) becomes

$$\begin{aligned} &\int_{B_R} \eta^2 \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \\ \leq\ & cp^2 \left(\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\theta} \\ & \cdot \left(\int_{B_R} \eta^{2^*} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}\cdot\frac{2^*}{2}} dx\right)^{\frac{2(1-\theta)}{2^*}} \cdot \left(\int_{B_R} g^q(x) dx\right)^{\frac{2}{q}} \\ & + cp^2 \int_{B_R} |D\eta|^2 \cdot \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx, \end{aligned}$$

and now we use Young's inequality with exponents $\left(\frac{1}{\theta}, \frac{1}{1-\theta}\right)$, thus obtaining

$$\begin{aligned} &\int_{B_R} \eta^2 \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \\ \leq\ & cp^2 \left[\left(\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right) \left(\int_{B_R} g^q(x) dx\right)^{\frac{2}{q\theta}} \right.\\ & + \left(\int_{B_R} \eta^{2^*} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}\cdot\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}} \\ & \left. + \int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right], \end{aligned}$$

and, by (2.67) with $\gamma = 0$,

$$\begin{aligned} &\int_{B_R} \eta^2 \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \\ \leq\ & cp^{\frac{4q}{q-n}} \left(\int_{B_R} g^q(x) dx\right)^{\frac{2}{q-n}} \left(\int_{B_R} \eta^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right) \\ & + c\left(1 + p^2\right) \left(\int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right), \end{aligned}$$

where we used that $\theta = \frac{q-n}{q}$.
Recalling the properties of η, and choosing $r = \frac{R}{2}$, we obtain the following estimate

$$\int_{B_{\frac{R}{2}}} \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \leq \frac{c}{R^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.73}$$

where $c = c\left(n, N, p, q, L_1, L_2, \|g\|_{L^q(B_R)}\right)$.
Since $1 < p < 2$, we also have, by Hölder's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$,

$$\begin{aligned} & \int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \\ = & \int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p \left(\mu^2+|Du(x)|^2\right)^{p\cdot\frac{p-2}{4}} \left(\mu^2+|Du(x)|^2\right)^{p\cdot\frac{2-p}{4}} dx \\ \le & \left(\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^2 \left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx\right)^{\frac{p}{2}} \\ & \cdot\left(\int_{B_R} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}}. \end{aligned} \tag{2.74}$$

Now we estimate the first integral in the right-hand side of (2.74) using (2.54) and (2.65) with $\gamma = 0$, and (2.68), so we get

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^2 \left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx \le \frac{c}{R^2}\int_{B_R} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.75}$$

and plugging (2.75) into (2.74), using Young's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$ we get

$$\int_{B_{\frac{R}{2}}} \left|D^2u(x)\right|^p dx \le \frac{c}{R^2}\int_{B_R} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.76}$$

i.e. (2.41). □

Proof of Lemma 2.2.4. Notice that, in this case, we are weakening the assumption on g, since $g \in L^n_{\mathrm{loc}}(\Omega)$. Again, we assume that u is a local minimizer of the functional (2.33) such that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.
First of all, we find an estimate for the L^s-norm of Du, for any $1 < s < \infty$, proving (2.42). We can argue exactly as in the proof of previous Lemma, until the estimate (2.62). Next we use Hölder's inequality with exponents $\left(\frac{n}{2}, \frac{n}{n-2}\right)$, as follows

$$\begin{aligned} & \int_{B_R} \eta^2 g^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx \\ \le & \left(\int_{B_R} g^n(x)dx\right)^{\frac{2}{n}} \cdot \left(\int_{B_R} \eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{2^*}{2}\cdot\frac{p+\gamma}{2}} dx\right)^{\frac{2}{2^*}}. \end{aligned} \tag{2.77}$$

Plugging (2.77) into (2.62), we have

$$\begin{aligned} & \left(\int_{B_R} \eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{2^*}{2}\cdot\frac{p+\gamma}{2}} dx\right)^{\frac{2}{2^*}} \\ \le & \; c\,(p+\gamma)^4 \left(\int_{B_R} g^n(x)dx\right)^{\frac{2}{n}} \cdot \left(\int_{B_R} \eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{2^*}{2}\cdot\frac{p+\gamma}{2}} dx\right)^{\frac{2}{2^*}} \\ & + c\,(p+\gamma)^2 \int_{B_R} |D\eta|^2 \left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx. \end{aligned} \tag{2.78}$$

In order to reabsorb the first term on the right-hand side of (2.78), we have to use the absolute continuity of the integral and take $R < R_\gamma$, with R_γ such that

$$\left(\int_{B_{R_\gamma}} g^n(x)dx\right)^{\frac{2}{n}} < \frac{1}{2c\,(p+\gamma)^4}. \tag{2.79}$$

Observe that, if $\gamma \to \infty$, then $R_\gamma \to 0$, and so, even if we can still use Moser's iterative technique, passing to the limit gives no information.
More precisely, if $R < R_\gamma$, plugging (2.79) into (2.78), we can reabsorb the first term of the right-hand side of (2.78) to the left-hand side, thus getting

$$\begin{aligned}
&\left(\int_{B_R} \eta^{2^*}\left(\mu^2+|Du(x)|^2\right)^{\frac{2^*}{2}\cdot\frac{p+\gamma}{2}} dx\right)^{\frac{2}{2^*}} \\
\leq\;& c\,(p+\gamma)^2\int_{B_R} |D\eta|^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p+\gamma}{2}} dx,
\end{aligned} \tag{2.80}$$

and by the properties of η, for $\gamma = 0$ we get

$$\left(\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{2^*}{2}\cdot\frac{p}{2}} dx\right)^{\frac{2}{2^*}} \leq \frac{c\cdot p^2}{(R-r)^2}\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx. \tag{2.81}$$

Now we use again the sequence of exponents p_i of defined by (2.70). Fixing $s > 1$, since p_i is streactly increasing and $p_i \to \infty$, there exists $\bar{i}$ such that $p_{\bar{i}} > s$, and $\tilde{R}$ such that (2.79) holds true with $p_{\bar{i}}$ in place of $p+\gamma$ and $\tilde{R}$ in place of R_γ. Choosing $R < \tilde{R}$ and $r = \frac{R}{2}$, recalling (2.69), we can iterate (2.81), thus getting

$$\begin{aligned}
&\left(\int_{B_{R_{\bar{i}+1}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p_{\bar{i}+1}}{2}} dx\right)^{\frac{1}{p_{\bar{i}+1}}} \\
\leq\;& \left[\frac{c\cdot p_{\bar{i}}^2}{\left(R_{\bar{i}}-R_{\bar{i}+1}\right)^2}\right]^{\frac{1}{p_{\bar{i}}}}\cdot\left(\int_{B_{R_{\bar{i}}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p_{\bar{i}}}{2}} dx\right)^{\frac{1}{p_{\bar{i}}}} \\
\leq\;& \prod_{k=0}^{\bar{i}}\left(\left[\frac{cp_k^2}{(R_k-R_{k+1})^2}\right]^{\frac{1}{p_k}}\right)\cdot\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\
=\;& \exp\left\{\sum_{k=0}^{\bar{i}}\left[\frac{1}{p_k}\cdot\log\left(\frac{c\cdot 2^{k+2}p_k^2}{R^2}\right)\right]\right\}\cdot\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}},
\end{aligned} \tag{2.82}$$

and since s is arbitrary, the inequality (2.82) allows us to estimate the L^s norm of Du for every $1 < s < \infty$. More precisely, for any finite s, there is $\bar{i} \in \mathbb{N}$ such that $p_{\bar{i}} > s$, and for a constant $c_1 = c_1(s,p,n)$, recalling (2.47), we have

$$\begin{aligned}
&\left(\int_{B_{\frac{R}{2}}} |Du(x)|^s\, dx\right)^{\frac{1}{s}} \\
\leq\;& \left(\int_{B_{\frac{R}{2}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{s}{2}} dx\right)^{\frac{1}{s}} \\
\leq\;& \left(\int_{B_{R_{\bar{i}+1}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{s}{2}} dx\right)^{\frac{1}{s}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_1 \left(\int_{B_{R_{\bar{i}+1}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p_{\bar{i}+1}}{2}} dx \right)^{\frac{1}{p_{\bar{i}+1}}} \\
&\leq c_1 \cdot \exp\left\{ \sum_{k=0}^{\bar{i}} \left[\frac{1}{p_k} \cdot \log\left(\frac{c \cdot 2^{k+2} p_k^2}{R^2} \right) \right] \right\} \cdot \left(\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}, \quad (2.83)
\end{aligned}
$$

and we get (2.42).
Let us prove, now, estimate (2.43). Recalling (2.72), using (2.77) with $\gamma = 0$, we get

$$
\begin{aligned}
&\int_{B_R} \eta^2 \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}} \right] \right|^2 dx \\
\leq\ & \frac{cp^2}{4} \left[\left(\int_{B_R} g^n(x) dx \right)^{\frac{2}{n}} \cdot \left(\int_{B_R} \eta^{2^*}(x) \left(\mu^2 + |Du(x)|^2\right)^{\frac{2^*}{2} \cdot \frac{p}{2}} dx \right)^{\frac{2}{2^*}} \right. \\
& \left. + c \int_{B_R} |D\eta|^2 \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \right],
\end{aligned}
$$

and recalling the properties of η, with $r = \frac{R}{2}$, by (2.81), we obtain

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}} \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}} \right] \right|^2 dx \\
\leq\ & \frac{cp^2}{R^2} \left[p^2 \left(\int_{B_R} g^n(x) dx \right)^{\frac{2}{n}} + 1 \right] \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx.
\end{aligned}
$$

therefore, using (2.79), with $\gamma = 0$, we get

$$
\int_{B_{\frac{R}{2}}} \left| D\left[\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{4}} \right] \right|^2 dx \leq \frac{c}{R^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx,
$$

that is the same a priori estimate as (2.73) under weaker assumption on the coefficients. In a way very similar to (2.74), using (2.54), (2.77) and (2.80) with $\gamma = 0$, , we get (2.75) again, and then the same estimate for the L^p-norm of the second derivatives of u, thus getting (2.43). □

2.2.2 The approximation: proofs of Theorem 2.2.1 and Theorem 2.2.2.

The aim of this section is to prove that the a priori estimates proved in the Section 2.2.1 are preserved in passing to the limit in a sequence of minimizers of a suitable approximating problem, and this allows us to prove Theorem 2.2.1 and Theorem 2.2.2.

Proof of Theorem 2.2.1. Let us consider an open set $\Omega' \Subset \Omega$, and a function $\phi \in C_0^\infty(B_1(0))$ such that $0 \leq \phi \leq 1$ and $\int_{B_1(0)} \phi(x) dx = 1$, and a standard family of mollifiers $\{ \phi_\varepsilon \}_\varepsilon$ defined as follows

$$
\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon} \right),
$$

for any $\varepsilon \in (0, d(\Omega', \partial\Omega))$, so that, for each ε, $\phi_\varepsilon \in C_0^\infty(B_\varepsilon(0))$, $0 \leq \phi_\varepsilon \leq 1$, $\int_{B_\varepsilon(0)} \phi_\varepsilon(x) dx = 1$.

It is well known that, for any $h \in L^1_{\text{loc}}(\Omega)$, setting

$$h_\varepsilon(x) = h * \phi_\varepsilon(x) = \int_{B_\varepsilon} \phi_\varepsilon(y) h(x+y) dy = \int_{B_1} \phi(\omega) h(x+\varepsilon\omega) d\omega,$$

we have $h_\varepsilon \in C^\infty(\Omega')$.
Let us fix a ball $B_{\tilde{R}} = B_{\tilde{R}}(x_0) \Subset \Omega'$ and let us consider the functional

$$\mathcal{F}_\varepsilon\left(w, B_{\tilde{R}}\right) = \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Dw(x)\right) dx, \tag{2.84}$$

that is

$$\mathcal{F}_\varepsilon\left(w, B_{\tilde{R}}\right) = \int_{B_{\tilde{R}}} \left(\int_{B_1} f(x+\varepsilon\omega, Dw(x)) \cdot \phi(\omega) d\omega \right) dx.$$

Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a local minimizer of the functional (2.33), and, for each $\varepsilon > 0$, let $u_\varepsilon \in W^{1,p}_{\text{loc}}\left(B_{\tilde{R}}\right)$ be the unique local minimizer of the functional (2.84) such that $u_\varepsilon - u \in W^{1,p}_0\left(B_{\tilde{R}}\right)$.
It's known that $u_\varepsilon \in W^{1,\infty}_{\text{loc}}\left(B_{\tilde{R}}\right)$ and $V_p\left(Du_\varepsilon\right) = \left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{4}} Du_\varepsilon \in W^{1,2}_{\text{loc}}\left(B_{\tilde{R}}\right)$.
It's easy to check that from (2.2), (2.9) and (2.13), the following properties hold for the function f_ε:

$$\ell_1 \left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}} \le f_\varepsilon(x,\xi) \le \ell_2 \left(\mu^2 + |\xi|^2\right)^{\frac{p}{2}}, \tag{2.85}$$

$$|D_{x\xi} f_\varepsilon(x,\xi)| \le g_\varepsilon(x) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{2.86}$$

$$L_1 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \le \langle D_{\xi\xi} f_\varepsilon(x,\xi)\eta, \eta\rangle \le L_2 \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2, \tag{2.87}$$

for all $\xi, \eta \in \mathbb{R}^{N\times n}$, and for almost every $x \in B_{\tilde{R}}$.
By the growth condition (2.85), and the minimality of u_ε, it follows

$$\begin{aligned} \ell_1 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx &\le \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Du_\varepsilon(x)\right) dx \le \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Du(x)\right) dx \\ &\le \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \end{aligned} \tag{2.88}$$

Since $u \in W^{1,p}\left(B_{\tilde{R}}\right)$, the sequence $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{1,p}\left(B_{\tilde{R}}\right)$ and so there exsists a function $v \in W^{1,p}\left(B_{\tilde{R}}\right)$ such that

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly in } W^{1,p}\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0.$$

Since $u_\varepsilon \in W^{1,\infty}_{\text{loc}}\left(B_{\tilde{R}}\right)$ and $V_p\left(Du_\varepsilon\right) \in W^{1,2}_{\text{loc}}\left(B_{\tilde{R}}\right)$, we can use the estimates (2.73) and (2.76), thus getting

$$\int_{B_{\frac{r}{2}}} \left| D\left[\left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{4}} \right] \right|^2 dx \le \frac{c}{r^2} \int_{B_r} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.89}$$

and applying (2.75), we also have

$$\int_{B_{\frac{r}{2}}} \left|D^2 u_\varepsilon(x)\right|^2 \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} dx \le c \int_{B_r} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \tag{2.90}$$

for any ball $B_r \Subset B_{\tilde{R}}$, and, by Lemma 2.2.3,

$$\int_{B_{\frac{r}{2}}} \left|D^2 u_\varepsilon(x)\right|^p dx \le c \int_{B_r} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx, \tag{2.91}$$

with a constant depending on $\|g_\varepsilon\|_{L^q(B_r)}$.
Let's notice that, since

$$g_\varepsilon \to g \qquad \text{strongly in } L^q\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0,$$

we have

$$\|g_\varepsilon\|_{L^q\left(B_{\tilde{R}}\right)} \le M \|g\|_{L^q\left(B_{\tilde{R}}\right)},$$

and so (2.89) and (2.91) hold true with a constant independent of ε.
So, since the ball $B_r \Subset B_{\tilde{R}}$ is arbitrary, the set $\left\{ \left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p}{4}} \right\}_\varepsilon$ is bounded in $W^{1,2}(B_r)$, and $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{2,p}(B_r)$.
Then there exists a function $\tilde{w} \in W^{1,2}(B_r)$, such that

$$\left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p}{4}} \rightharpoonup \tilde{w} \qquad \text{weakly in } W^{1,2}(B_r),$$

so that

$$\left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p}{4}} \to \tilde{w} \qquad \text{strongly in } L^2(B_r)$$

and

$$\left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p}{4}} \to \tilde{w} \qquad \text{almost everywhere in } B_r,$$

as $\varepsilon \to 0$, up to a subsequence.
Since, by (2.91), $\{ u_\varepsilon \}_\varepsilon$, is bounded in $W^{2,p}(B_r)$, we have

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly in } W^{2,p}(B_r)$$

and

$$u_\varepsilon \to v \qquad \text{strongly in } W^{1,p}(B_r)$$

and

$$Du_\varepsilon \to Dv \qquad \text{almost everywhere in } B_r,$$

up to a subsequence, as $\varepsilon \to 0$.
Moreover, since the function $\xi \mapsto \left(\mu^2 + |\xi|^2\right)^{\frac{p}{4}}$ is continuous, we get

$$\tilde{w} = \left(\mu^2 + |Dv|^2\right)^{\frac{p}{4}} \tag{2.92}$$

almost everywhere, and by (2.89), we get

$$\int_{B_{\frac{r}{2}}} \left|D\left[\left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{4}}\right]\right|^2 dx \le \frac{c}{r^2} \int_{B_r} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx.$$

Similarly, since the function $\xi \mapsto V_p(\xi)$ is of class C^1, up to a subsequence, we have

$$\left|D^2 u_\varepsilon\right|^2 \left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}} \to \left|D^2 v\right|^2 \left(\mu^2 + |Dv|^2\right)^{\frac{p-2}{2}} \qquad \text{almost everywhere in } B_r,$$

as $\varepsilon \to 0$ and, by the dominate convergence theorem, we can pass to the limit in the left-hand side of (2.90), thus getting

$$\int_{B_{\frac{r}{2}}} \left|D^2v(x)\right|^2 \left(\mu^2+|Dv(x)|^2\right)^{\frac{p-2}{2}} dx \le c\int_{B_r} \left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} dx,$$

and applying Young's inequality with exponets $\left(\frac{2}{p}, \frac{2}{2-p}\right)$ as we did in (2.74), we infer

$$\int_{B_{\frac{r}{2}}} \left|D^2v(x)\right|^p dx \le c\int_{B_r} \left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} dx. \tag{2.93}$$

Now we want to prove that $u=v$ almost everywhere in $B_{\tilde{R}}$. Using the minimizing property of u for $\mathcal{F}$, Fatou's Lemma, the lower semi-continuity of $\mathcal{F}_\varepsilon$ (due to the convexity of f_ε), and the fact that u_ε is the minimizer of $\mathcal{F}_\varepsilon$ with boundary value u on $B_{\tilde{R}}$, we have

$$\begin{aligned}\int_{B_{\tilde{R}}} f\left(x, Du(x)\right) dx &\le \int_{B_{\tilde{R}}} f\left(x, Dv(x)\right) dx\\ &\le \liminf_{\varepsilon} \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Du_\varepsilon(x)\right) dx\\ &\le \liminf_{\varepsilon} \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Du(x)\right) dx\\ &= \int_{B_{\tilde{R}}} f\left(x, Du(x)\right) dx.\end{aligned} \tag{2.94}$$

So all the terms of (2.94) are equal, and in particular

$$\int_{B_{\tilde{R}}} f\left(x, Du(x)\right) dx = \int_{B_{\tilde{R}}} f\left(x, Dv(x)\right) dx.$$

By virtue of the strict convexity of the functional (2.33), the local minimizer with boundary value u, is unique, so $u=v$ almost everywhere in $B_{\tilde{R}}$, and therefore $u\in W^{2,p}_{\mathrm{loc}}\left(B_{\tilde{R}}\right)$.
By (2.93), we also obtain the following estimate

$$\int_{B_{\frac{r}{2}}} \left|D^2u(x)\right|^p dx \le c\int_{B_r} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx,$$

for any ball $B_r \Subset B_{\tilde{R}}$ and, by a standard covering argument, we get (2.37).
Now, applying (2.40) to u_ε and recalling (2.88), we get

$$\begin{aligned}\left\|Du_\varepsilon\right\|_{L^\infty\left(B_{\frac{r}{2}}\right)} &\le c\left(\int_{B_r} \left(\mu^2+|Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}\\ &\le c\left(\int_{B_{\tilde{R}}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}},\end{aligned}$$

for any ball $B_r \Subset B_{\tilde{R}}$, and so, since the ball B_r is arbitrary, there exists a function $\bar{w}\in W^{1,\infty}\left(B_r\right)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} \bar{w}$ in $W^{1,\infty}\left(B_r\right)$. So, up to a subsequence, $u_\varepsilon \to \bar{w}$ in $L^\infty\left(B_r\right)$, by which it follows that $\bar{w}=u$. By the weakly-* lower semicontinuity of the map $\xi \mapsto \left(\mu^2+|\xi|^2\right)^{\frac{p}{4}}$ with respect to the L^∞-norm, we get

$$\left\|Du\right\|_{L^\infty\left(B_{\frac{r}{2}}\right)} \le \liminf_{\varepsilon} \left\|Du_\varepsilon\right\|_{L^\infty\left(B_{\frac{r}{2}}\right)}$$

$$\begin{aligned}
&\le c\cdot\liminf_{\varepsilon}\left(\int_{B_r}\left(\mu^2+|Du_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\le c\left(\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}, \qquad (2.95)
\end{aligned}$$

so, using a covering argument, we have $Du \in L^{\infty}_{\text{loc}}(\Omega)$, with the estimate (2.36). □

Proof of Theorem 2.2.2. In order to prove Theorem 2.2.2, let us observe that, by the same arguments given above, we immediately obtain that $u \in W^{2,p}_{\text{loc}}(\Omega)$, with the estimate (2.39). To prove the remaining part of the theorem, for $1 < s < \infty$, using (2.83) and (2.88), we have

$$\begin{aligned}
\left(\int_{B_r}|Du_\varepsilon(x)|^s\right)^{\frac{1}{s}}) &\le c\left(\int_{B_r}\left(\mu^2+|Du_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\le c\left(\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}},
\end{aligned}$$

for any ball $B_r \Subset B_{\tilde{R}}$. Arguing similarly to how we did for (2.95), we get

$$\begin{aligned}
\left(\int_{B_{\frac{r}{2}}}|Du(x)|^s\,dx\right)^{\frac{1}{s}} &\le \liminf_{\varepsilon}\left(\int_{B_{\frac{r}{2}}}|Du_\varepsilon(x)|^s\,dx\right)^{\frac{1}{s}}\\
&\le c\cdot\liminf_{\varepsilon}\left(\int_{B_{\frac{r}{2}}}\left(\mu^2+|Du_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\le c\cdot\left(\int_{B_{\frac{r}{2}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}.
\end{aligned}$$

So, by means of a covering argument, we get $Du \in L^s_{\text{loc}}(\Omega)$, and estimate (2.38) holds, for every $s \in (1,\infty)$. □

Chapter 3

Non-homogeneous systems

In this chapter we discuss some higher differentiability results for local minimizers of functional of the form

$$\mathcal{F}(w,\Omega)=\int_{\Omega}\left[f\left(x,Dw(x)\right)-F(x)\cdot w(x)\right]dx, \tag{3.1}$$

where, for $n>2$, $\Omega\subset\mathbb{R}^n$, is a bounded open set, for $N\geq 1$, $F\in L^r_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)$ for some $r\in(2,n)$ and $f:\Omega\times\mathbb{R}^{n\times N}\to[0,+\infty)$ is a Carathéodory function such that $\xi\mapsto f(x,\xi)$ is $C^2\left(\mathbb{R}^{n\times N}\right)$ for a.e. $x\in\Omega$ and, for an exponent $p\in(1,2)$, and a parameter $\mu\in[0,1]$, (2.2)–(2.4) hold, which, with the notation (2.5), imply (2.6)–(2.8).
The Euler-Langrange system of the functional (3.1) is non-homogeneous, due to the presence of the datum F. The aim of this chapter is to describe the Lebesgue L^r space to which F has to belong, in order to get higher differentiability for the solutions, provided the map $x\mapsto D_\xi f\left(x,\xi\right)$ belongs to a suitable Sobolev space $W^{1,q}$, i.e., we shall assume that there exists a non-negative function $g\in L^q_{\mathrm{loc}}\left(\Omega\right)$ such that (2.12) holds for some q, which is equivalent to (2.14).
More precisely, in Section 3.1 we give a sharp result for the value of r, assuming $q=n$.
In Section 3.1.2, we also provide a counterexample that allows us to understand that we cannot weaken the assumption on F in the scale of Lebesgue spaces, and this is due only to the sub-quadratic growth condition of the energy density f with respect to the gradient variable, and not to the regularity of the coefficients.
In Section 3.2 we give a similar result for a priori bounded minimizers of the functional (3.1), showing that, as often happens to solutions to variational problems, assuming the a priori local boundedness of minimizers allows us to get higher regularity properties, weakening the assumptions on the datum and on the coefficients (we will face this kind of phenomenon also in Chapter 4, in case of obstacle problems: see Sections 4.2 and 4.5).
All the results we describe in this chapter are contained in an upcoming paper, written in cooperation with A. Clop and A. Passarelli di Napoli.

3.1 A sharp higher differentiability result for solutions to some non-homogeneous systems

In this section, we prove the following result.

Theorem 3.1.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, and $1<p<2$.*
Let $u\in W^{1,p}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)$ be a local minimizer of the functional (3.1)*, under the assumptions* (2.2)*–*(2.4) *and* (2.12)*, with*

$$F\in L^{\frac{np}{n(p-1)+2-p}}_{\mathrm{loc}}\left(\Omega\right)\qquad\text{and}\qquad g\in L^n_{\mathrm{loc}}\left(\Omega\right).$$

Then $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$*, and the estimate*

$$\int_{B_{\frac{R}{2}}} |DV_p(Du(x))|^2\,dx \leq \frac{c}{R^{\beta(n,p)}}\left[\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx + \int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx + \int_{B_R} g^n(x)dx + |B_R|\right] \quad (3.2)$$

holds true for any ball $B_R \Subset \Omega$*.*

Let us notice that it is easy to check that

$$2 < \frac{np}{n(p-1)+2-p} < n,$$

for any $n > 2$ and $1 < p < 2$.
Moreover

$$\frac{np}{n(p-1)+2-p} < \frac{p}{p-1} \iff 1 < p < 2.$$

3.1.1 Proof of Theorem 3.1.1

We prove Theorem 3.1.1, dividing the proof into two steps. The first step consists in proving an estimate using the a priori assumption $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.
In the second step, we use an approximation argument, considering a regularized version of the functional to whose minimizer we can apply the a priori estimate. Then we conclude by proving that such estimate is preserved in passing to the limit.
Before entering into the details of the proof, we want to stress that the necessity to use an approximation procedure is due to the assumptions on the function g and on the datum F. If we had $F \in L^\infty_{\mathrm{loc}}(\Omega)$ and $g \in L^\infty_{\mathrm{loc}}(\Omega)$, it would be sufficient to apply the difference quotient method to get $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (see, for example, [1] and [89]).

Proof of Theorem 3.1.1. **Step 1: the a priori estimate.**
Our first step consists in proving that, if $u \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ is a local minimizer of $\mathcal{F}$ such that

$$V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

estimate (3.2) holds.
Since $u \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ is a local minimizer of $\mathcal{F}$, it solves the corresponding Euler-Lagrange system, that is, with the notation (2.5), for any $\varphi \in C^\infty_0\left(\Omega, \mathbb{R}^N\right)$, we have

$$\int_\Omega \langle A(x, Du(x)), D\varphi(x)\rangle dx = \int_\Omega F(x)\cdot\varphi(x). \quad (3.3)$$

Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} \leq r < \tilde{s} < t < \tilde{t} < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C^\infty_0(B_t)$ such that $\eta \equiv 1$ on $B_{\tilde{s}}$, $|D\eta| \leq \frac{c}{t-\tilde{s}}$ and $|D^2\eta| \leq \frac{c}{(t-\tilde{s})^2}$. From now on, with no loss of generality, we suppose $R < 1$. For $|h|$ sufficiently small, we can choose, for any $s = 1, \ldots, n$

$$\varphi = \tau_{s,-h}\left(\eta^2\tau_{s,h}u\right)$$

as a test function in (3.3) and, by Proposition 1.2.2, we get

$$\int_\Omega \left\langle \tau_{s,h}A(x, Du(x)), D\left(\eta^2(x)\tau_{s,h}u(x)\right)\right\rangle dx$$

$$= \int_\Omega F(x) \cdot \tau_{s,-h} \left(\eta^2(x)\tau_{s,h}u(x)\right) dx,$$

that is

$$\begin{aligned} I &:= \int_\Omega \left\langle A\left(x+he_s, Du\left(x+he_s\right)\right) - A\left(x+he_s, Du(x)\right), \eta^2(x)\tau_{s,h}Du(x)\right\rangle dx \\ &= -\int_\Omega \left\langle A\left(x+he_s, Du(x)\right) - A\left(x, Du(x)\right), \eta^2(x)\tau_{s,h}Du(x)\right\rangle dx \\ &\quad -2\int_\Omega \left\langle A\left(x+he_s, Du\left(x+he_s\right)\right) - A\left(x, Du(x)\right), \eta(x)D\eta(x)\otimes\tau_{s,h}u(x)\right\rangle dx \\ &\quad +\int_\Omega F(x)\cdot\tau_{s,-h}\left(\eta^2(x)\tau_{s,h}u(x)\right) dx \\ &=: -II - III + IV. \end{aligned}$$

Therefore

$$I \leq |II| + |III| + |IV|. \tag{3.4}$$

By (2.7), we obtain

$$I \geq \nu \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2\, dx. \tag{3.5}$$

For what concerns the term II, by (2.14) and Young's inequality with exponents $(2,2)$, for any $\varepsilon > 0$, we have

$$\begin{aligned} |II| &\leq |h|\int_\Omega \eta^2(x)\left(g(x)+g\left(x+he_s\right)\right)\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p-1}{2}} |\tau_{s,h}Du(x)|\, dx \\ &\leq \varepsilon\int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2\, dx \\ &\quad + c_\varepsilon |h|^2 \int_\Omega \eta^2(x)\left(g(x)+g\left(x+he_s\right)\right)^2\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p}{2}} dx. \end{aligned}$$

Now, by the assumption $g \in L^n_{\mathrm{loc}}(\Omega)$, we can use Hölder's inequality with exponents $\left(\frac{n}{2}, \frac{n}{n-2}\right)$ and by the properties of η and Lemma 1.2.3, we get

$$\begin{aligned} |II| &\leq \varepsilon\int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2\, dx \\ &\quad + c_\varepsilon |h|^2\left(\int_{B_t}\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{np}{2(n-2)}} dx\right)^{\frac{n-2}{n}} \\ &\quad \cdot\left(\int_{B_t}\left(g(x)+g\left(x+he_s\right)\right)^n dx\right)^{\frac{2}{n}} \\ &\leq \varepsilon\int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du\left(x+he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2\, dx \\ &\quad + c_\varepsilon |h|^2\left(\int_{B_{\lambda r}}\left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx\right)^{\frac{n-2}{n}} \cdot \left(\int_{B_{\lambda r}} g^n(x)dx\right)^{\frac{2}{n}}. \end{aligned} \tag{3.6}$$

Let us consider, now, the term III. We have

$$\begin{aligned} III &= 2\int_{\Omega}\left\langle \tau_{s,h}A\left(x,Du(x)\right),\eta(x)D\eta(x)\otimes\tau_{s,h}u(x)\right\rangle dx \\ &= 2\int_{\Omega}\left\langle A\left(x,Du(x)\right),\tau_{s,-h}\left[\eta(x)D\eta(x)\otimes\tau_{s,h}u(x)\right]\right\rangle dx, \end{aligned}$$

so, by (2.6), we deduce that

$$|III| \leq c\int_{\Omega}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\left|\tau_{s,-h}\left[\eta(x)D\eta(x)\otimes\tau_{s,h}u(x)\right]\right| dx \tag{3.7}$$

and since, for any $x\in\operatorname{supp}(\eta)$ such that $x+he_s, x-he_s\in\operatorname{supp}(\eta)$, recalling the properties of η, we have

$$\begin{aligned} \left|\tau_{s,-h}\left[\eta(x)D\eta(x)\otimes\tau_{s,h}u(x)\right]\right| &\leq \left|\tau_{s,-h}\eta(x)\cdot D\eta\left(x-he_s\right)\otimes\tau_{s,h}u\left(x-he_s\right)\right| \\ &\quad+\left|\eta(x)\tau_{s,-h}D\eta(x)\otimes\tau_{s,h}u\left(x-he_s\right)\right| \\ &\quad+\left|\eta(x)D\eta(x)\otimes\tau_{s,-h}\tau_{s,h}u(x)\right| \\ &\leq \frac{c\,|h|}{(t-\tilde{s})^2}\left|\tau_{s,h}u\left(x-he_s\right)\right| \\ &\quad+\frac{c\,|h|}{t-\tilde{s}}\eta(x)\left|\tau_{s,-h}\tau_{s,h}u(x)\right|. \end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7), we get

$$\begin{aligned} |III| &\leq \frac{c\,|h|}{(t-\tilde{s})^2}\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\left|\tau_{s,h}u\left(x-he_s\right)\right| dx \\ &\quad+\frac{c\,|h|}{t-\tilde{s}}\int_{\Omega}\eta(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\left|\tau_{s,-h}\tau_{s,h}u(x)\right| dx, \end{aligned} \tag{3.9}$$

and by Hölder's inequality with exponents $\left(p,\frac{p}{p-1}\right)$ and the properties of η, (3.9) becomes

$$\begin{aligned} |III| &\leq \frac{c\,|h|}{(t-\tilde{s})^2}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}\cdot\left(\int_{B_t}\left|\tau_{s,h}u\left(x-he_s\right)\right|^p dx\right)^{\frac{1}{p}} \\ &\quad+\frac{c\,|h|}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+\left|Du\left(x+he_s\right)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}} \\ &\quad\cdot\left(\int_{B_t}\left|\tau_{s,-h}\tau_{s,h}u(x)\right|^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

Now, by virtue of Lemma 1.2.3, and using (1.5), we get

$$\begin{aligned} |III| &\leq \frac{c\,|h|^2}{(t-\tilde{s})^2}\int_{B_{\tilde{t}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx \\ &\quad+\frac{c\,|h|^2}{t-\tilde{s}}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+\left|Du\left(x+he_s\right)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}} \end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}Du(x)|^p\,dx\right)^{\frac1p}\\
\le\;& \frac{c\,|h|^2}{(t-\tilde s)^2}\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2\right)^{\frac p2}dx\\
&+\frac{c\,|h|^2}{t-\tilde s}\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du\,(x+he_s)|^2\right)^{\frac p2}dx\right)^{\frac{p-1}{p}}\\
&\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}V_p\,(Du(x))|^p\cdot\left(\mu^2+|Du(x)|^2+|Du\,(x+he_s)|^2\right)^{\frac{p(2-p)}{4}}dx\right)^{\frac1p}\\
\le\;& \frac{c\,|h|^2}{(t-\tilde s)^2}\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2\right)^{\frac p2}dx+\frac{c\,|h|^2}{t-\tilde s}\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac p2}dx\right)^{\frac12}\\
&\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}V_p\,(Du(x))|^2\,dx\right)^{\frac12},
\end{aligned}
\tag{3.10}
$$

where, in the last line, we used Hölder's inequality with exponents $\left(\frac2p,\frac{2}{2-p}\right)$.
Now, using Young's inequality with exponents $(2,2)$ and since $t-\tilde s<1$ and $t<\tilde t<\lambda r<R$, (3.10) gives

$$
|III|\le\sigma\int_{B_{\tilde t}}|\tau_{s,h}V_p\,(Du(x))|^2\,dx+\frac{c_\sigma\,|h|^2}{(t-\tilde s)^2}\int_{B_R}\left(\mu^2+|Du\,(x)|^2\right)^{\frac p2}dx, \tag{3.11}
$$

for any $\sigma>0$. For what concerns the term IV, by virtue of Proposition 1.2.2, we have

$$
\begin{aligned}
IV\;=\;&\int_\Omega\eta^2\,(x)\,F(x)\tau_{s,-h}\,(\tau_{s,h}u(x))\,dx\\
&+\int_\Omega[\eta\,(x-he_s)+\eta(x)]\,F(x)\tau_{s,-h}\eta(x)\tau_{s,h}u\,(x-he_s)\,dx\\
=:\;&J_1+J_2,
\end{aligned}
\tag{3.12}
$$

which yields

$$
|IV|\le|J_1|+|J_2| \tag{3.13}
$$

In order to estimate the term J_1, let us recall that, by virtue of the a priori assumption $V_p\,(Du)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$ and Sobolev's embedding theorem, we have $Du\in L^{\frac{np}{n-2}}_{\mathrm{loc}}\,(\Omega)$, which implies $Du\in L^{\frac{np}{n-2+p}}_{\mathrm{loc}}\,(\Omega)$. So, using Hölder's inequality with exponents $\left(\frac{np}{n(p-1)+2-p},\frac{np}{n-2+p}\right)$, the properties of η and Lemma 1.2.3, we get

$$
\begin{aligned}
|J_1|\;\le\;&\left(\int_{B_t}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx\right)^{\frac{n(p-1)+2-p}{np}}\cdot\left(\int_{B_t}|\tau_{s,-h}\tau_{s,h}u(x)|^{\frac{np}{n-2+p}}dx\right)^{\frac{n-2+p}{np}}\\
\le\;&|h|\left(\int_{B_t}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx\right)^{\frac{n(p-1)+2-p}{np}}\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}Du(x)|^{\frac{np}{n-2+p}}dx\right)^{\frac{n-2+p}{np}}
\end{aligned}
\tag{3.14}
$$

To go further, let us consider the second integral in (3.14). Using (1.5), we get

$$
\int_{B_{\tilde t}}|\tau_{s,h}Du(x)|^{\frac{np}{n-2+p}}dx\;\le\;\int_{B_{\tilde t}}|\tau_{s,h}V_p\,(Du(x))|^{\frac{np}{n-2+p}}
$$

$$\cdot \left(\mu^2+|Du(x)|^2+|Du\left(x+he_s\right)|^2\right)^{\frac{2-p}{4}\cdot\frac{np}{n-2+p}}dx,$$

and, as long as $1<p<2$, we can use Hölder's inequality with exponents $\left(\frac{2(n-2+p)}{np},\frac{2(n-2+p)}{(n-2)(2-p)}\right)$, thus getting

$$\begin{aligned}\int_{B_{\tilde t}}|\tau_{s,h}Du(x)|^{\frac{np}{n-2+p}}dx &\le \left(\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2+|Du\left(x+he_s\right)|^2\right)^{\frac{np}{2(n-2)}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\\ &\quad\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}V_p\left(Du(x)\right)|^2dx\right)^{\frac{np}{2(n-2+p)}}. \qquad (3.15)\end{aligned}$$

Inserting (3.15) into (3.14), and using Young's inequality with exponents $\left(2,\frac{np}{n(p-1)+2-p},\frac{2np}{(n-2)(2-p)}\right)$, we obtain

$$\begin{aligned}|J_1| &\le |h|\left(\int_{B_t}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx\right)^{\frac{n(p-1)+2-p}{np}}\cdot\left(\int_{B_{\tilde t}}|\tau_{s,h}V_p\left(Du(x)\right)|^2dx\right)^{\frac12}\\ &\quad\cdot\left(\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2+|Du\left(x+he_s\right)|^2\right)^{\frac{np}{2(n-2)}}dx\right)^{\frac{(n-2)(2-p)}{2np}}\\ &\le c_\sigma|h|^2\int_{B_t}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx\\ &\quad+\sigma|h|^2\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2+|Du\left(x+he_s\right)|^2\right)^{\frac{np}{2(n-2)}}dx\\ &\quad+\sigma\int_{B_{\tilde t}}|\tau_{s,h}V_p\left(Du(x)\right)|^2dx. \qquad (3.16)\end{aligned}$$

Recalling that $t<\tilde t<\lambda r<R$ and by virtue of Lemma 1.2.3, (3.16) implies

$$\begin{aligned}|J_1| &\le c_\sigma|h|^2\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}dx\\ &\quad+\sigma|h|^2\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2)}}dx\\ &\quad+\sigma\int_{B_{\tilde t}}|\tau_{s,h}V_p\left(Du(x)\right)|^2dx. \qquad (3.17)\end{aligned}$$

For what concerns the term J_2, by virtue of the properties of η, we have

$$\begin{aligned}|J_2| &\le c\int_{B_t}|F(x)|\,|\tau_{s,-h}\eta(x)|\,|\tau_{s,h}u\left(x-he_s\right)|dx\\ &\le |h|\,\|D\eta\|_{L^\infty(B_t)}\int_{B_t}|F(x)|\,|\tau_{s,h}u\left(x-he_s\right)|dx\\ &\le \frac{c\,|h|}{t-\tilde s}\int_{B_t}|F(x)|\,|\tau_{s,h}u\left(x-he_s\right)|dx. \qquad (3.18)\end{aligned}$$

Now, if we apply Hölder's and Young's inequality in (3.18) with exponents $\left(\frac{np}{n(p-1)+2},\frac{np}{n-2}\right)$, we get

$$|J_2| \le \frac{c\,|h|}{t-\tilde s}\left(\int_{B_t}|F(x)|^{\frac{np}{n(p-1)+2}}dx\right)^{\frac{n(p-1)+2}{np}}\cdot\left(\int_{B_t}|\tau_{s,h}u\left(x-he_s\right)|^{\frac{np}{n-2}}dx\right)^{\frac{n-2}{np}}$$

$$\leq \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_t} |F(x)|^{\frac{np}{n(p-1)+2}} dx + \sigma |h|^2 \int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-2}} dx, \tag{3.19}$$

where we also used Lemma 1.2.3, since $Du \in L^{\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$.
By virtue of (3.17) and (3.19), (3.13) gives

$$\begin{aligned} |IV| \leq\ & c_\sigma |h|^2 \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_t} |F(x)|^{\frac{np}{n(p-1)+2}} dx \\ & +2\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx + \sigma \int_{B_{\tilde{t}}} |\tau_{s,h} V_p(Du(x))|^2 dx. \end{aligned} \tag{3.20}$$

Inserting (3.5), (3.6), (3.11) and (3.20) into (3.4), and choosing $\varepsilon < \frac{\nu}{2}$, we get

$$\begin{aligned} & c(\nu) \int_\Omega \eta^2(x) \left(\mu^2 + |Du(x)|^2 + |Du(x+he_s)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h} Du(x)|^2 dx \\ \leq\ & c|h|^2 \left(\int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx\right)^{\frac{n-2}{n}} \cdot \left(\int_{B_{\lambda r}} g^n(x) dx\right)^{\frac{2}{n}} \\ & +2\sigma \int_{B_{\tilde{t}}} |\tau_{s,h} V_p(Du(x))|^2 dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\ & +c_\sigma |h|^2 \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_t} |F(x)|^{\frac{np}{n(p-1)+2}} dx \\ & +2\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx, \end{aligned}$$

for $\sigma > 0$ that will be chosen later.
So, by (1.5) and the properties of η, we have

$$\begin{aligned} & \int_{B_{\tilde{s}}} |\tau_{s,h} V_p(Du(x))|^2 dx \\ \leq\ & 3\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx + c_\sigma \int_{B_{\lambda r}} g^n(x) dx \\ & +2\sigma \int_{B_{\tilde{t}}} |\tau_{s,h} V_p(Du(x))|^2 dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\ & +c_\sigma |h|^2 \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_t} |F(x)|^{\frac{np}{n(p-1)+2}} dx, \end{aligned}$$

where we also used Young's inequality with exponents $\left(\frac{n}{2}, \frac{n}{n-2}\right)$.
Now, Lemma 1.2.3 implies

$$\begin{aligned} & \int_{B_{\tilde{s}}} |\tau_{s,h} V_p(Du(x))|^2 dx \\ \leq\ & 3\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx + c \cdot \sigma |h|^2 \int_{B_{\lambda r}} |DV_p(Du(x))|^2 dx \\ & + \frac{c_\sigma |h|^2}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma \int_{B_{\lambda r}} g^n(x) dx \end{aligned}$$

$$+c_\sigma |h|^2 \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}}\, dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_t} |F(x)|^{\frac{np}{n(p-1)+2}}\, dx. \quad (3.21)$$

Since (3.21) holds for any $s = 1, \dots, n$ and, by virtue of the a priori assumption, $V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, by Lemma 1.2.4, we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\, dx \\
\leq\ & c\cdot\sigma \int_{B_{\lambda r}} |DV_p(Du(x))|^2 + 3\sigma \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx \\
&+ \frac{c_\sigma}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma \int_{B_{\lambda r}} g^n(x) dx \\
&+ c_\sigma \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}}\, dx + \frac{c_\sigma}{(t-\tilde{s})^{\frac{np}{n(p-1)+2}}} \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2}}\, dx,
\end{aligned}$$

and since $t - \tilde{s} < 1$, setting

$$\beta(n,p) = \max\left\{2, \frac{np}{n(p-1)+2}\right\},$$

we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\, dx \\
\leq\ & c\cdot\sigma \int_{B_{\lambda r}} |DV_p(Du(x))|^2 + 3\sigma \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx \\
&+ \frac{c_\sigma}{(t-\tilde{s})^{\beta(n,p)}} \left[\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_{\lambda r}} g^n(x) dx \right. \\
&\left. + \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}}\, dx + \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2}}\, dx \right]. \quad (3.22)
\end{aligned}$$

Now let us notice that, since $\frac{np}{n(p-1)+2} < \frac{np}{n(p-1)+2-p}$, we have

$$\int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2}}\, dx \leq |B_R|^{\frac{p}{n(p-1)+2}} \left(\int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}}\, dx\right)^{\frac{n(p-1)+2-p}{n(p-1)+2}},$$

and using Young's inequality with exponents $\left(\frac{n(p-1)+2}{n(p-1)+2-p}, \frac{n(p-1)+2}{p}\right)$, we get

$$\int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2}}\, dx \leq c \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}}\, dx + c\,|B_R|. \quad (3.23)$$

Plugging (3.23) into (3.22), we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\, dx \\
\leq\ & c\cdot\sigma \int_{B_{\lambda r}} |DV_p(Du(x))|^2 + 3\sigma \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{np}{2(n-2)}} dx \\
&+ \frac{c_\sigma}{(t-\tilde{s})^{\beta(n,p)}} \left[\int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_{\lambda r}} g^n(x) dx \right.
\end{aligned}$$

$$+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+|B_R|\Bigg]. \tag{3.24}$$

Moreover, applying Sobolev's inequality to the function $V_p(Du)$ and recalling (1.10), for a positive constant $c=c(n,p)$ we get

$$\begin{aligned}\int_{B_{\lambda r}}|Du(x)|^{\frac{np}{n-2}}\,dx &\leq c\int_{B_{\lambda r}}|V_p(Du(x))|^{\frac{2n}{n-2}}\,dx+c\mu^{\frac{np}{n-2}}|B_R|\\ &\leq c\int_{B_{\lambda r}}|DV_p(Du(x))|^2\,dx+c\int_{B_{\lambda r}}|V_p(Du(x))|^2\,dx\\ &\quad +c\,|B_R|\\ &\leq c\int_{B_{\lambda r}}|DV_p(Du(x))|^2\,dx+c\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\ &\quad +c\,|B_R|, \end{aligned}\tag{3.25}$$

where we also used the fact that $\mu\in[0,1]$.
Now, plugging (3.25) into (3.24), and recalling that $t-\tilde{s}<1$ and $\lambda r<R$, we get

$$\begin{aligned}\int_{B_{\tilde{s}}}|DV_p(Du(x))|^2\,dx &\leq c\cdot\sigma\int_{B_{\lambda r}}|DV_p(Du(x))|^2\\ &\quad+\frac{c_\sigma}{(t-\tilde{s})^{\beta(n,p)}}\Bigg[\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\ &\quad+\int_{B_R}g^n(x)dx+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+|B_R|\Bigg],\end{aligned}$$

and choosing $\sigma>0$ such that

$$c\cdot\sigma=\frac{1}{2},$$

we get

$$\begin{aligned}\int_{B_{\tilde{s}}}|DV_p(Du(x))|^2\,dx &\leq \frac{1}{2}\int_{B_{\lambda r}}|DV_p(Du(x))|^2\\ &\quad+\frac{c}{(t-\tilde{s})^{\beta(n,p)}}\Bigg[\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\ &\quad+\int_{B_R}g^n(x)dx+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+|B_R|\Bigg].\end{aligned}\tag{3.26}$$

Since (3.26) holds for any $\frac{R}{2}\leq r<\tilde{s}<t<\lambda r<R$, with $1<\lambda<2$, and the constant c depends on n,N,p,L,ν,ℓ but is independent of the radii, we can take the limit as $\tilde{s}\to r$ and $t\to\lambda r$, thus getting

$$\begin{aligned}\int_{B_r}|DV_p(Du(x))|^2\,dx &\leq \frac{1}{2}\int_{B_{\lambda r}}|DV_p(Du(x))|^2\,dx\\ &\quad+\frac{c}{r^{\beta(n,p)}(\lambda-1)^{\beta(n,p)}}\Bigg[\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\ &\quad+\int_{B_R}g^n(x)dx+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+|B_R|\Bigg].\end{aligned}$$

Now, if we set

$$h(r)=\int_{B_r}|DV_p\left(Du(x)\right)|^2\,dx,$$

$$A=c\left[\int_{B_R}\left(\mu^2+|Du\left(x\right)|^2\right)^{\frac{p}{2}}dx+\int_{B_R}g^n(x)dx+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+|B_R|\right]$$

and

$$B=0,$$

and apply Lemma 1.1.1 with

$$\theta=\frac{1}{2}\qquad\text{and}\qquad\gamma=\beta\left(n,p\right),$$

we get

$$\begin{aligned}\int_{B_{\frac{R}{2}}}|DV_p\left(Du(x)\right)|^2\,dx \quad&\le\quad \frac{c}{R^{\beta(n,p)}}\left[\int_{B_R}\left(\mu^2+|Du\left(x\right)|^2\right)^{\frac{p}{2}}dx\right.\\ &\left.+\int_{B_R}|F(x)|^{\frac{np}{n(p-1)+2-p}}\,dx+\int_{B_R}g^n(x)dx+|B_R|\right],\end{aligned} \tag{3.27}$$

that is the desired a priori estimate.

Step 2: the approximation.
Now we want to complete the proof of Theorem 3.1.1, using the a priori estimate (3.27), and a classical approximation argument.
Let us consider an open set $\Omega'\Subset\Omega$, and a function $\phi\in C_0^\infty(B_1(0))$ such that $0\le\phi\le1$ and $\int_{B_1(0)}\phi(x)dx=1$, and a standard family of mollifiers $\{\,\phi_\varepsilon\,\}_\varepsilon$ defined as follows

$$\phi_\varepsilon(x)=\frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right),$$

for any $\varepsilon\in(0,d\left(\Omega',\partial\Omega\right))$, so that, for each ε, $\phi_\varepsilon\in C_0^\infty\left(B_\varepsilon(0)\right)$, $0\le\phi_\varepsilon\le1$, $\int_{B_\varepsilon(0)}\phi_\varepsilon(x)dx=1$.
It is well known that, for any $h\in L^1_{\text{loc}}\left(\Omega\right)$, setting

$$h_\varepsilon(x)=h*\phi_\varepsilon(x)=\int_{B_\varepsilon}\phi_\varepsilon(y)h(x+y)dy=\int_{B_1}\phi(\omega)h(x+\varepsilon\omega)d\omega,$$

we have $h_\varepsilon\in C^\infty\left(\Omega'\right)$.
Let us fix a ball $B_{\tilde R}=B_{\tilde R}\left(x_0\right)\Subset\Omega'$, with $\tilde R<1$ and, for each $\varepsilon\in(0,d\left(\Omega',\partial\Omega\right))$, let us consider the functional

$$\mathcal{F}_\varepsilon\left(w,B_{\tilde R}\right)=\int_{B_{\tilde R}}\left[f_\varepsilon\left(x,Dw(x)\right)-F_\varepsilon(x)\cdot w(x)\right]dx,$$

where

$$f_\varepsilon(x,\xi)=\int_{B_1}\phi(\omega)f\left(x+\varepsilon\omega,\xi\right)d\omega \tag{3.28}$$

and

$$F_\varepsilon=F*\phi_\varepsilon. \tag{3.29}$$

Let us recall that

$$\int_{B_{\tilde{R}}} f_\varepsilon(x,\xi)\,dx \to \int_{B_{\tilde{R}}} f(x,\xi)\,dx, \qquad \text{as } \varepsilon \to 0 \tag{3.30}$$

for any $\xi \in \mathbb{R}^{n\times N}$.
Moreover, since $F \in L_{\text{loc}}^{\frac{np}{n(p-1)+2-p}}(\Omega)$, we have

$$F_\varepsilon \to F \qquad \text{strongly in } L^{\frac{np}{n(p-1)+2-p}}\left(B_{\tilde{R}}\right), \tag{3.31}$$

and since $\frac{np}{n(p-1)+2-p} > \frac{np}{n(p-1)+p} = \frac{p^*}{p^*-1}$, we also have

$$F_\varepsilon \to F \qquad \text{strongly in } L^{\frac{p^*}{p^*-1}}\left(B_{\tilde{R}}\right), \tag{3.32}$$

as $\varepsilon \to 0$.
It is easy to check that (2.2)–(2.4) and (2.12) imply

$$\ell_1\left(\mu^2+|\xi|^2\right)^{\frac{p}{2}} \le f_\varepsilon(x,\xi) \le \ell_2\left(\mu^2+|\xi|^2\right)^{\frac{p}{2}}, \tag{3.33}$$

$$\langle D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(x,\eta), \xi-\eta\rangle \ge \nu\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|^2, \tag{3.34}$$

$$|D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(x,\eta)| \le L\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|, \tag{3.35}$$

$$|D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(y,\xi)| \le (g_\varepsilon(x)+g_\varepsilon(y))\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}|x-y|, \tag{3.36}$$

for a.e. $x,y \in B_{\tilde{R}}$ and every $\xi,\eta \in \mathbb{R}^{n\times N}$, where

$$g_\varepsilon = g * \phi_\varepsilon. \tag{3.37}$$

Since $g \in L^n_{\text{loc}}(\Omega)$, we have

$$g_\varepsilon \to g \qquad \text{strongly in } L^n\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0. \tag{3.38}$$

For each ε, let $u_\varepsilon \in u + W_0^{1,p}\left(B_{\tilde{R}}\right)$ be the solution to

$$\min\left\{\mathcal{F}_\varepsilon\left(w, B_{\tilde{R}}\right) : w \in u + W_0^{1,p}\left(B_{\tilde{R}}\right)\right\},$$

where $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a local minimizer of (3.1).
By virtue of the minimality of u_ε, we have

$$\int_{B_{\tilde{R}}}\left[f_\varepsilon(x,Du_\varepsilon(x)) - F_\varepsilon(x)\cdot u_\varepsilon(x)\right]dx \le \int_{B_{\tilde{R}}}\left[f_\varepsilon(x,Du(x)) - F_\varepsilon(x)\cdot u(x)\right]dx,$$

which means

$$\int_{B_{\tilde{R}}} f_\varepsilon(x,Du_\varepsilon(x))\,dx \le \int_{B_{\tilde{R}}}\left[f_\varepsilon(x,Du(x)) + F_\varepsilon(x)\cdot(u_\varepsilon(x)-u(x))\right]dx,$$

and by (3.33) we get

$$\begin{aligned}\ell_1\int_{B_{\tilde{R}}}\left(\mu^2+|Du_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx &\le \int_{B_{\tilde{R}}} f_\varepsilon(x,Du_\varepsilon(x))\,dx\\ &\le \int_{B_{\tilde{R}}}\left[f_\varepsilon(x,Du(x)) + F_\varepsilon(x)\cdot(u_\varepsilon(x)-u(x))\right]dx\end{aligned}$$

$$\begin{aligned}
&\leq \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \\
&\quad + \int_{B_{\tilde{R}}} |F_\varepsilon(x)| \, |u_\varepsilon(x) - u(x)| \, dx. \qquad (3.39)
\end{aligned}$$

If we use Hölder's and Young's inequalities with exponents $\left(p^*, \frac{p^*}{p^*-1}\right)$ in (3.39) and apply Sobolev's inequality to the function $u_\varepsilon - u \in W_0^{1,p}\left(B_{\tilde{R}}\right)$, for any $\sigma > 0$, we get

$$\begin{aligned}
& \ell_1 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \\
\leq\ & \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma \int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p^*}{p^*-1}} dx \\
& + \sigma \int_{B_{\tilde{R}}} |u_\varepsilon(x) - u(x)|^{p^*} dx \\
\leq\ & \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma \int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p^*}{p^*-1}} dx \\
& + \sigma \left(\int_{B_{\tilde{R}}} |Du_\varepsilon(x) - Du(x)|^p \, dx \right) \\
\leq\ & c_\sigma \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma \int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p^*}{p^*-1}} dx \\
& + \sigma \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \right). \qquad (3.40)
\end{aligned}$$

Now, if we choose $\sigma < \frac{\ell_1}{2}$ in (3.40), we have

$$\begin{aligned}
& \ell_1 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \\
\leq\ & c \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c \int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p^*}{p^*-1}} dx. \qquad (3.41)
\end{aligned}$$

By virtue of (3.32), (3.41) implies that $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{1,p}_{\text{loc}}\left(B_{\tilde{R}}\right)$. Therefore there exists $v \in W^{1,p}\left(B_{\tilde{R}}\right)$ such that

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly in } W^{1,p}\left(B_{\tilde{R}}\right),$$

$$u_\varepsilon \to v \qquad \text{strongly in } L^p\left(B_{\tilde{R}}\right),$$

and

$$u_\varepsilon \to v \qquad \text{almost everywhere in } B_{\tilde{R}},$$

up to a subsequence, as $\varepsilon \to 0$.
On the other hand, since $V_p\left(Du_\varepsilon\right) \in W^{1,2}_{\text{loc}}\left(B_{\tilde{R}}\right)$, we are legitimated to apply estimate (3.27), thus getting

$$\begin{aligned}
\int_{B_{\frac{r}{2}}} \left|DV_p\left(Du_\varepsilon(x)\right)\right|^2 dx \ \leq\ & \frac{c}{r^{\beta(n,p)}} \left[\int_{B_r} \left(\mu^2 + \left|Du_\varepsilon\left(x\right)\right|^2\right)^{\frac{p}{2}} dx \right. \\
& \left. + \int_{B_r} |F_\varepsilon(x)|^{\frac{np}{n(p-1)+2-p}} dx + c \int_{B_r} g_\varepsilon^n(x) dx + |B_r| \right], \qquad (3.42)
\end{aligned}$$

for any ball $B_r \Subset B_{\tilde{R}}$.
By virtue of (3.31), (3.32), (3.38) and (3.41), the right-hand side of (3.42) can be bounded independently of ε. For this reason, recalling Lemma 1.4.5, we also infer that, for each ε, $u_\varepsilon \in W^{2,p}_{\mathrm{loc}}(B_{\tilde{R}})$, and that $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{2,p}_{\mathrm{loc}}(B_r)$.
Hence

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly in } W^{2,p}(B_r),$$

$$u_\varepsilon \to v \qquad \text{strongly in } W^{1,p}(B_r), \tag{3.43}$$

and

$$Du_\varepsilon \to Dv \qquad \text{almost everywhere in } B_r, \tag{3.44}$$

up to a subsequence, as $\varepsilon \to 0$.
Moreover, by the continuity of $\xi \mapsto DV_p(\xi)$ and (3.44), we get $DV_p(Du_\varepsilon) \to DV_p(Dv)$ almost everywhere, and since the right-hand side of (3.42) can be bounded independently of ε, by Fatou's Lemma, passing to the limit as $\varepsilon \to 0$ in (3.42), by (3.31), (3.38) and (3.43), we get

$$\begin{aligned} &\int_{B_{\frac{r}{2}}} |DV_p(Dv(x))|^2 \, dx \\ \leq\ & \frac{c}{r^{\beta(n,p)}} \left[\int_{B_r} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_R} |F(x)|^{\frac{np}{n(p-1)+2-p}} dx \right. \\ & \left. + c\int_{B_r} g^n(x) dx + |B_r|^{\frac{n(p-1)+2}{np}} \right]. \end{aligned} \tag{3.45}$$

Our final step consists in proving that $u = v$ a.e. in $B_{\tilde{R}}$.
First, let us observe that, using Hölder's inequality with exponents $\left(p^*, \frac{p^*}{p^*-1}\right)$, we get

$$\begin{aligned} &\left| \int_{B_{\tilde{R}}} [F_\varepsilon(x) \cdot v(x) - F(x) \cdot v(x)] \, dx \right| \\ \leq\ & \int_{B_{\tilde{R}}} |F_\varepsilon(x) - F(x)| \, |u(x)| \, dx \\ \leq\ & \left(\int_{B_{\tilde{R}}} |u(x)|^{p^*} dx \right)^{\frac{1}{p^*}} \cdot \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x) - F(x)|^{\frac{p^*}{p^*-1}} dx \right)^{\frac{p^*-1}{p^*}}, \end{aligned}$$

that, thanks to (3.32), implies

$$\lim_{\varepsilon \to 0} \int_{B_{\tilde{R}}} F_\varepsilon(x) \cdot v(x) dx = \int_{B_{\tilde{R}}} F(x) \cdot v(x) dx.$$

and recalling (3.30), we obtain

$$\lim_{\varepsilon \to 0} \int_{B_{\tilde{R}}} [f_\varepsilon(x, Du(x)) - F_\varepsilon(x) \cdot u(x)] \, dx = \int_{B_{\tilde{R}}} [f(x, Du(x)) - F(x) \cdot u(x)] \, dx. \tag{3.46}$$

The minimality of u, Fatou's Lemma, the lower semicontinuity of $\mathcal{F}_\varepsilon$ and the minimality of u_ε imply

$$\int_{B_{\tilde{R}}} [f(x, Du(x)) - F(x) \cdot u(x)] \, dx$$

$$
\begin{aligned}
&\leq \int_{B_{\tilde{R}}} [f(x, Dv(x)) - F(x) \cdot v(x)]\, dx \\
&\leq \liminf_{\varepsilon \to 0} \int_{B_{\tilde{R}}} [f(x, Du_\varepsilon(x)) - F(x) \cdot u_\varepsilon(x)]\, dx \\
&\leq \liminf_{\varepsilon \to 0} \int_{B_{\tilde{R}}} [f_\varepsilon(x, Du_\varepsilon(x)) - F_\varepsilon(x) \cdot u_\varepsilon(x)]\, dx \\
&\leq \liminf_{\varepsilon \to 0} \int_{B_{\tilde{R}}} [f_\varepsilon(x, Du(x)) - F_\varepsilon(x) \cdot u(x)]\, dx \\
&= \int_{B_{\tilde{R}}} [f(x, Du(x)) - F(x) \cdot u(x)]\, dx,
\end{aligned}
$$

where the last equivalence follows by (3.46). Therefore, all the previous inequalities hold as equalities and $\mathcal{F}(Du, B_{\tilde{R}}) = \mathcal{F}(Dv, B_{\tilde{R}})$. The strict convexity of the functional yields that $u = v$ a.e. in $B_{\tilde{R}}$, and since the map $\xi \mapsto V_p(\xi)$ is of class C^1, we also have $DV_p(Du) = DV_p(Dv)$ almost everywhere in $B_{\tilde{R}}$, and by (3.45), using a standard covering argument, we can conclude with estimate (3.2). □

Thanks to Lemma 1.4.5, it is easy to prove the following consequence of Theorem 3.1.1.

Corollary 3.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $1 < p < 2$. Let $u \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ be a local minimizer of the functional* (3.1)*, under the assumptions* (2.2)*–*(2.4) *and* (2.12)*, with*

$$
F \in L^{\frac{np}{n(p-1)+2-p}}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad g \in L^n_{\mathrm{loc}}(\Omega).
$$

Then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$.

3.1.2 A Counterexample

The aim of this section is to show that we cannot weaken the assumption $F \in L^{\frac{np}{n(p-1)+2-p}}_{\mathrm{loc}}(\Omega)$ in the scale of Lebesgue spaces.
Our example also shows that this phenomenon is independent of the presence of the coefficients, but it depends only on the sub-quadratic growth of the energy density.
For $\alpha \in \mathbb{R}$, let us set

$$
\beta := (\alpha - 1)(p - 1) - 1,
$$

and consider the functional

$$
\mathcal{F}_\alpha(w, \Omega) = \int_\Omega \left[|Dw(x)|^p - \alpha (n + \beta) |\alpha|^{p-2} |x|^\beta w(x)\right] dx,
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set containing the origin, $1 < p < 2$, $u : \mathbb{R}^n \to \mathbb{R}$.
Using the classical notation for the p-Laplacian

$$
\Delta_p w = \operatorname{div}\left(|Dw|^{p-2} \cdot Dw\right),
$$

a local minimizer of this functional is a weak solution to the p-Poisson equation

$$
\Delta_p w = F_\alpha, \tag{3.47}
$$

with

$$
F_\alpha = \alpha (n + \beta) |\alpha|^{p-2} |x|^\beta.
$$

Before going further, let us notice that (3.47) is an autonomous equation, whose solution are scalar functions, so the problem we're dealing with is much less general with respect to the assumption we considered in order to prove our result.
It is easy to check that, for any $\alpha \in \mathbb{R}$, the function

$$u_\alpha(x) = |x|^\alpha$$

is a solution to (3.47).
Indeed, since, for each $i = 1, \dots, n$, we have

$$D_{x_i} u_\alpha(x) = \alpha \, |x|^{\alpha-2} \, x_i,$$

we get

$$|Du_\alpha(x)| = |\alpha| \, |x|^{\alpha-1} .$$

So, for every $i = 1, \dots, n$, since $\beta = (\alpha-1)(p-1) - 1$, we get

$$|Du_\alpha(x)|^{p-2} \, D_{x_i} u_\alpha(x) = \alpha \, |\alpha|^{p-2} \, |x|^{(p-1)\cdot(\alpha-1)-1} \, x_i = \alpha \, |\alpha|^{p-2} \, |x|^\beta \, x_i$$

and

$$\frac{\partial}{\partial x_i} \left(|Du_\alpha(x)|^{p-2} \, D_{x_i} u_\alpha(x) \right) = \alpha \, |\alpha|^{p-2} \, |x|^\beta \left(1 + \frac{\beta x_i^2}{|x|^2} \right),$$

so

$$\begin{aligned} \Delta_p u_\alpha(x) &= \operatorname{div} \left(|Du_\alpha(x)|^{p-2} \cdot Du_\alpha(x) \right) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|Du_\alpha(x)|^{p-2} \, D_{x_i} u_\alpha(x) \right) \\ &= \alpha \, |\alpha|^{p-2} \, |x|^\beta \sum_{i=1}^n \left(1 + \frac{\beta x_i^2}{|x|^2} \right) = \alpha \, |\alpha|^{p-2} \, (n+\beta) \, |x|^\beta \\ &= F_\alpha(x) \end{aligned}$$

Moreover, for further needs, we observe that

$$\left| D^2 u_\alpha(x) \right| = c(\alpha) \cdot |x|^{\alpha-2} ,$$

for a constant $c(\alpha) \geq 0$. Choosing

$$\alpha - 1 = \frac{2-n}{p}$$

we have

$$F_\alpha \sim |x|^{\frac{(2-n)(p-1)}{p} - 1}$$

and

$$|F_\alpha|^{\frac{np}{n(p-1)+2-p}} \sim |x|^{-n} .$$

Therefore with such a choice of α, F_α doesn't belong to $L^{\frac{np}{n(p-1)+2-p}}(B_1(0))$. With the same choice of α we have

$$|Du_\alpha(x)|^{p-2} \cdot \left| D^2 u_\alpha(x) \right|^2 = c(n,p) \cdot |x|^{p\cdot(\alpha-1)-2} = c(n,p) \cdot |x|^{2-n-2} = c(n,p) \cdot |x|^{-n} ,$$

that doesn't belong to $L^1(B_1(0))$. Therefore we cannot weaken the assumption on datum F in the scale of Lebesgue spaces and obtain the same regularity for the second derivatives of the solution u.
Note that $F_\alpha \in L^{\frac{np}{n(p-1)+2-p} - \varepsilon}(B_1(0))$, for every $\varepsilon > 0$.

3.2 A higher differentiability result for bounded solutions to some non-homogeneous systems

This section is devoted to the proof of an higher differentiability result for a priori bounded minimizers of functional (3.1).
The claim of our result is the following.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (3.1) *under assumptions* (2.2)–(2.4) *and* (2.12) *for* $1 < p < 2$, *with*

$$F \in L^{\frac{p+2}{p}}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad g \in L^{p+2}_{\mathrm{loc}}(\Omega).$$

Then $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the estimate

$$\int_{B_{\frac{R}{2}}} |DV_p(Du(x))|^2 \, dx \leq \frac{c\,\|u\|_{L^{\infty}(B_{4R})}}{R^{\frac{p+2}{p}}} \left[\int_{B_{4R}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_R} g^{p+2}(x)dx + \int_{B_R} |F(x)|^{\frac{p+2}{p}} dx + |B_R| + 1\right] \tag{3.48}$$

holds for any ball $B_{4R} \Subset \Omega$.

It is worth noticing that assuming $F \in L^{\frac{p+2}{p}}_{\mathrm{loc}}(\Omega)$ is weaker than assuming $F \in L^{\frac{np}{n(p-1)+2-p}}_{\mathrm{loc}}(\Omega)$, if and only if

$$\frac{p+2}{p} < \frac{np}{n(p-1)+2-p},$$

and since $1 < p < 2$, this is equivalent to

$$n > p + 2,$$

so, for $n \geq 4$, the result we prove for a priori bounded minimizers improves the one we proved in Section 3.1.
Moreover, for any $n > 2$ and $1 < p < 2$, we have

$$2 < \frac{p+2}{p} < n.$$

3.2.1 A preliminary higher differentiability result

In order to prove Theorem 3.2.1, we need an auxiliary result, concerning the regularity of local minimizers of functionals of the form

$$\mathcal{F}_m(w, \Omega) = \int_{\Omega} \left[f(x, Dw(x)) - F(x) \cdot w(x) + (|w(x)| - a)^{2m}_{+}\right] dx, \tag{3.49}$$

where $a > 0$, $m > 1$, and the function f still satisfies (2.2)–(2.4) and (2.12).
It is clear from the definition and by our assumptions, that the functional in (3.49) admits minimizers in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{2m}_{\mathrm{loc}}(\Omega)$.
We want to prove the following higher differentiability result for local minimizers of the functional $\mathcal{F}_m$.

Theorem 3.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $m > 1$, $a > 0$ and $1 < p < 2$. Let $v \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right) \cap L^{2m}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ be a local minimizer of the functional (3.49), under the assumptions (2.2)–(2.4) and (2.12), with*

$$F \in L^{\frac{2m(p+2)}{2mp+p-2}}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad g \in L^{\frac{2m(p+2)}{2m-p}}_{\mathrm{loc}}(\Omega).$$

Then $V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and the estimate

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |DV_p(Dv(x))|^2 dx \\
\leq\ & \frac{c}{R^{\frac{p+2}{p}}}\left[\int_{B_R}\left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx \right.\\
&+ \left(\int_{B_{4R}} |v(x)|^{2m} dx\right)^{\frac{1}{m+1}} \cdot \left(\int_{B_{4R}}\left(\mu^2 + |Dv(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{m}{m+1}} \\
&\left. + \int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x) dx + \int_{B_R} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} dx + |B_R| + 1\right],
\end{aligned} \tag{3.50}$$

holds true for any ball $B_{4R} \Subset \Omega$.

For further needs, we notice that

$$\frac{2m(p+2)}{2mp+p-2} > \frac{m(p+2)}{mp+m-1}$$

for any $m > 1$ as long as $1 < p < 2$, since it is equivalent to

$$2mp + 2m - 2 > 2mp + p - 2$$

i.e.

$$2m > p.$$

Let us also notice that

$$\frac{2m(p+2)}{2mp+p-2} > \frac{p+2}{p},$$

and

$$\frac{2m(p+2)}{2m-p} > p+2$$

for any $m > 1$ and $p \in (1, 2)$.

Proof of Thorem 3.2.2. **Step 1: the a priori estimate.**
Our first step consists in proving that, if $v \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right) \cap L^{2m}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ is a local minimizer of $\mathcal{F}_m$ such that

$$V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

estimate (3.50) holds.
Since $v \in W^{1,p}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right) \cap L^{2m}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^N\right)$ is a local minimizer of $\mathcal{F}_m$, it is a weak solution of the corresponding Euler-Lagrange system, that is, with the notation (2.5), for any $\varphi \in C^\infty_0\left(\Omega, \mathbb{R}^N\right)$, we have

$$\int_\Omega \langle A(x, Dv(x)), D\varphi(x)\rangle dx = \int_\Omega \left[F(x) - 2m\left(|v(x)| - a\right)_+^{2m-1} \cdot \frac{v(x)}{|v(x)|}\right] \varphi(x). \tag{3.51}$$

Let us fix a ball $B_{4R} \Subset \Omega$ and arbitrary radii $\frac{R}{2} \leq r < \tilde{s} < t < \tilde{t} < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C^\infty_0(B_t)$ such that $\eta \equiv 1$ on $B_{\tilde{s}}$, $|D\eta| \leq \frac{c}{t-\tilde{s}}$ and

$|D^2\eta| \leq \frac{c}{(t-\tilde{s})^2}$. From now on, with no loss of generality, we suppose $R < \frac{1}{4}$. For $|h|$ sufficiently small, we can choose, for any $s = 1, \ldots, n$

$$\varphi = \tau_{s,-h}\left(\eta^2 \tau_{s,h} v\right)$$

as a test function for the equation (3.51), and recalling Proposition 1.2.2, we get

$$\begin{aligned}
& \int_\Omega \left\langle \tau_{s,h} A\left(x, Dv(x)\right), D\left(\eta^2(x)\tau_{s,h}v(x)\right)\right\rangle dx \\
= & \int_\Omega F(x) \cdot \tau_{s,-h}\left(\eta^2(x)\tau_{s,h}v(x)\right) dx \\
& -2m \int_\Omega \tau_{s,h}\left[\left(|v(x)| - a\right)_+^{2m-1} \cdot \frac{v(x)}{|v(x)|}\right] \cdot \eta^2(x)\tau_{s,h}v(x)dx,
\end{aligned}$$

that is

$$\begin{aligned}
I + II \quad := \quad & \int_\Omega \left\langle A\left(x + he_s, Dv\left(x + he_s\right)\right) - A\left(x + he_s, Dv(x)\right), \eta^2(x)\tau_{s,h}Dv(x)\right\rangle dx \\
& +2m \int_\Omega \tau_{s,h}\left[\left(|v(x)| - a\right)_+^{2m-1} \cdot \frac{v(x)}{|v(x)|}\right] \cdot \eta^2(x)\tau_{s,h}v(x)dx \\
= \quad & -\int_\Omega \left\langle A\left(x + he_s, Dv(x)\right) - A\left(x, Dv(x)\right), \eta^2(x)\tau_{s,h}Dv(x)\right\rangle dx \\
& -2\int_\Omega \left\langle \tau_{s,h}\left[A\left(x, Dv(x)\right)\right], \eta(x)D\eta(x) \otimes \tau_{s,h}v(x)\right\rangle dx \\
& + \int_\Omega F(x) \cdot \tau_{s,-h}\left(\eta^2(x)\tau_{s,h}v(x)\right) dx \\
=: \quad & -III - IV + V.
\end{aligned}$$

So we have

$$I + II \leq |III| + |IV| + |V|. \tag{3.52}$$

By virtue of Lemma 1.1.4, we have

$$II \geq c(m)\int_\Omega \eta^2(x)\left|\left(|v(x + he_s)| - a\right)_+^m \cdot \frac{v\left(x + he_s\right)}{\left|v\left(x + he_s\right)\right|} - \left(|v(x)| - a\right)_+^m \cdot \frac{v(x)}{|v(x)|}\right|^2 dx \geq 0,$$

so (3.52) becomes

$$I \leq |III| + |IV| + |V|. \tag{3.53}$$

By (2.7), we get

$$I \geq \nu \int_\Omega \eta^2(x)\left(\mu^2 + |Dv(x)|^2 + \left|Dv\left(x + he_s\right)\right|^2\right)^{\frac{p-2}{2}} \left|\tau_{s,h}Dv(x)\right|^2 dx. \tag{3.54}$$

For what concerns the term III, by (2.14) and using Young's inequality with exponents $(2, 2)$, for any $\varepsilon > 0$, we have

$$\begin{aligned}
|III| \quad \leq \quad & |h|\int_\Omega \eta^2(x)\left(g(x) + g\left(x + he_s\right)\right)\left(\mu^2 + |Dv(x)|^2 + \left|Dv\left(x + he_s\right)\right|^2\right)^{\frac{p-1}{2}} \left|\tau_{s,h}Dv(x)\right| dx \\
\leq \quad & \varepsilon \int_\Omega \eta^2(x)\left(\mu^2 + |Dv(x)|^2 + \left|Dv\left(x + he_s\right)\right|^2\right)^{\frac{p-2}{2}} \left|\tau_{s,h}Dv(x)\right|^2 dx
\end{aligned}$$

$$+c_\varepsilon |h|^2 \int_\Omega \eta^2(x) \left(g(x) + g\left(x + he_s\right)\right)^2 \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x + he_s\right)|^2\right)^{\frac{p}{2}} dx.$$

By Hölder's inequality with exponents $\left(\frac{m(p+2)}{p(m+1)}, \frac{m(p+2)}{2m-p}\right)$, the properties of η and Lemma 1.2.3, we get

$$\begin{aligned}
|III| \quad &\leq \quad \varepsilon \int_\Omega \eta^2(x) \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x + he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h} Dv(x)|^2 \, dx \\
&\quad +c_\varepsilon |h|^2 \left(\int_{B_t} \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x + he_s\right)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx\right)^{\frac{m(p+2)}{m+1}} \\
&\quad \cdot \left(\int_{B_t} \left(g(x) + g\left(x + he_s\right)\right)^{\frac{2m(p+2)}{2m-p}} dx\right)^{\frac{2m-p}{m(p+2)}} \\
&\leq \quad \varepsilon \int_\Omega \eta^2(x) \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x + he_s\right)|^2\right)^{\frac{p-2}{2}} |\tau_{s,h} Dv(x)|^2 \, dx \\
&\quad +c_\varepsilon |h|^2 \left(\int_{B_{\tilde t}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx\right)^{\frac{p(m+1)}{m(p+2)}} \cdot \left(\int_{B_{\lambda r}} g^{\frac{2m(p+2)}{2m-p}}(x) dx\right)^{\frac{2m-p}{m(p+2)}} \qquad (3.55)
\end{aligned}$$

Let us consider, now, the term IV. We have

$$\begin{aligned}
IV \quad &= \quad 2 \int_\Omega \left\langle \tau_{s,h} A\left(x, Dv(x)\right), \eta(x) D\eta(x) \otimes \tau_{s,h} v(x) \right\rangle dx \\
&= \quad 2 \int_\Omega \left\langle A\left(x, Dv(x)\right), \tau_{s,-h}\left[\eta(x) D\eta(x) \otimes \tau_{s,h} v(x)\right] \right\rangle dx,
\end{aligned}$$

so, by (2.6), we get

$$|IV| \quad \leq \quad c \int_\Omega \left(\mu^2 + |Dv(x)|^2\right)^{\frac{p-1}{2}} \left|\tau_{s,-h}\left[\eta(x) D\eta(x) \otimes \tau_{s,h} v(x)\right]\right| dx.$$

We can treat this term as we did after (3.7) in the proof of Theorem 3.1.1, using (3.8) with v in place of u, thus getting

$$|IV| \leq \sigma \int_{B_{\tilde t}} |\tau_{s,h} V_p\left(Dv(x)\right)|^2 dx + \frac{c_\sigma |h|^2}{(t - \tilde s)^2} \int_{B_R} \left(\mu^2 + |Dv\left(x\right)|^2\right)^{\frac{p}{2}} dx, \qquad (3.56)$$

for $\sigma > 0$ that will be chosen later.
In order to estimate the term V, arguing as we did in (3.12), we have

$$\begin{aligned}
V \quad &= \quad \int_\Omega \eta^2\left(x\right) F(x) \tau_{s,-h}\left(\tau_{s,h} v(x)\right) dx \\
&\quad + \int_\Omega \left[\eta\left(x - he_s\right) + \eta(x)\right] F(x) \tau_{s,-h} \eta(x) \tau_{s,h} v\left(x - he_s\right) dx \\
&=: \quad J_1 + J_2,
\end{aligned}$$

which implies

$$|V| \leq |J_1| + |J_2| \qquad (3.57)$$

Let us consider the term J_1. By virtue of the properties of η and using Hölder's inequality with exponents $\left(\frac{2m(p+2)}{2mp+p-2}, \frac{2m(p+2)}{4m+2-p}\right)$, we have

$$\begin{aligned}
|J_1| &\leq \int_{B_t} |F(x)|\left|\tau_{s,-h}\left(\tau_{s,h}v(x)\right)\right| dx \\
&\leq \left(\int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\, dx\right)^{\frac{2mp+p-2}{2m(p+2)}} \cdot \left(\int_{B_t} \left|\tau_{s,-h}\left(\tau_{s,h}v(x)\right)\right|^{\frac{2m(p+2)}{4m+2-p}}\, dx\right)^{\frac{4m+2-p}{2m(p+2)}} \\
&\leq |h| \left(\int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\, dx\right)^{\frac{2mp+p-2}{2m(p+2)}} \cdot \left(\int_{B_{\tilde{t}}} |\tau_{s,h}Dv(x)|^{\frac{2m(p+2)}{4m+2-p}}\, dx\right)^{\frac{4m+2-p}{2m(p+2)}}, \qquad (3.58)
\end{aligned}$$

where, in the last line, we applied Lemma 1.2.3 since, by virtue of the a priori assumption $V_p\left(Dv\right) \in W^{1,2}_{\mathrm{loc}}\left(\Omega\right)$ and recalling Remark 1.4.7, we have $Dv \in L^{\frac{m(p+2)}{m+1}}_{\mathrm{loc}}\left(\Omega\right)$, which implies $Dv \in L^{\frac{2m(p+2)}{4m+2-p}}_{\mathrm{loc}}\left(\Omega\right)$ since, for any $m > 1$ and $1 < p < 2$, we have $\frac{2m(p+2)}{4m+2-p} < \frac{m(p+2)}{m+1}$.
Let us consider the second integral in (3.58). By virtue of (1.5), and using Hölder's inequality with exponents $\left(\frac{4m+2-p}{m(p+2)}, \frac{4m+2-p}{(2-p)(m+1)}\right)$, we have

$$\begin{aligned}
\int_{B_{\tilde{t}}} |\tau_{s,h}Dv(x)|^{\frac{2m(p+2)}{4m+2-p}}\, dx &\leq \int_{B_{\tilde{t}}} \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x+he_s\right)|^2\right)^{\frac{2-p}{4}\cdot\frac{2m(p+2)}{4m+2-p}} \\
&\quad \cdot \left|\tau_{s,h}V_p\left(Dv(x)\right)\right|^{\frac{2m(p+2)}{4m+2-p}}\, dx \\
&\leq \left(\int_{B_{\tilde{t}}} \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x+he_s\right)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\, dx\right)^{\frac{(2-p)(m+1)}{4m+2-p}} \\
&\quad \cdot \left(\int_{B_{\tilde{t}}} \left|\tau_{s,h}V_p\left(Dv(x)\right)\right|^2\, dx\right)^{\frac{m(p+2)}{4m+2-p}}. \qquad (3.59)
\end{aligned}$$

Inserting (3.59) into (3.58), we get

$$\begin{aligned}
|J_1| &\leq |h| \left(\int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\, dx\right)^{\frac{2mp+p-2}{2m(p+2)}} \\
&\quad \cdot \left(\int_{B_{\tilde{t}}} \left(\mu^2 + |Dv(x)|^2 + |Dv\left(x+he_s\right)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\, dx\right)^{\frac{(2-p)(m+1)}{2m(p+2)}} \\
&\quad \cdot \left(\int_{B_{\tilde{t}}} \left|\tau_{s,h}V_p\left(Dv(x)\right)\right|^2\, dx\right)^{\frac{1}{2}},
\end{aligned}$$

Using Lemma 1.2.3 and Young's inequality with exponents $\left(\frac{2m(p+2)}{2mp+p-2}, \frac{2m(p+2)}{(2-p)(m+1)}, 2\right)$, we get

$$\begin{aligned}
|J_1| &\leq c_\sigma |h|^2 \int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\, dx + \sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\, dx \\
&\quad +\sigma \int_{B_{\tilde{t}}} \left|\tau_{s,h}V_p\left(Dv(x)\right)\right|^2\, dx, \qquad (3.60)
\end{aligned}$$

for any $\sigma > 0$.
For what concerns the term J_2, by the properties of η, as in (3.18), we have

$$\begin{aligned}|J_2| &\leq \int_{B_t} |F(x)|\,|\tau_{s,-h}\eta(x)|\,|\tau_{s,h}v\,(x-he_s)|\,dx\\ &\leq \frac{c\,|h|}{t-\tilde{s}}\int_{B_t} |F(x)|\,|\tau_{s,h}v\,(x-he_s)|\,dx.\end{aligned}$$

Now, if we apply Hölder's inequality with exponents $\left(\frac{m(p+2)}{mp+m-1}, \frac{m(p+2)}{m+1}\right)$, we get

$$\begin{aligned}|J_2| &\leq \frac{c\,|h|}{t-\tilde{s}}\left(\int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}}\,dx\right)^{\frac{mp+m-1}{m(p+2)}}\\ &\quad\cdot\left(\int_{B_t} |\tau_{s,h}v\,(x-he_s)|^{\frac{m(p+2)}{m+1}}\,dx\right)^{\frac{m+1}{m(p+2)}}\\ &\leq \frac{c\,|h|^2}{t-\tilde{s}}\left(\int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}}\,dx\right)^{\frac{mp+m-1}{m(p+2)}}\\ &\quad\cdot\left(\int_{B_{\lambda r}} |Dv\,(x)|^{\frac{m(p+2)}{m+1}}\,dx\right)^{\frac{m+1}{m(p+2)}},\end{aligned} \tag{3.61}$$

where we also used Lemma 1.2.3, since $Dv \in L^{\frac{m(p+2)}{m+1}}_{\mathrm{loc}}(\Omega)$.
By virtue of (3.60) and (3.61), (3.57) gives

$$\begin{aligned}|V| &\leq c_\sigma\,|h|^2\int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx + \sigma\,|h|^2\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\,dx\\ &\quad+\sigma\int_{B_{\tilde{t}}} |\tau_{s,h}V_p\,(Dv(x))|^2\,dx\\ &\quad+\frac{c\,|h|^2}{t-\tilde{s}}\left(\int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}}\,dx\right)^{\frac{mp+m-1}{m(p+2)}}\\ &\quad\cdot\left(\int_{B_{\lambda r}} |Dv\,(x)|^{\frac{m(p+2)}{m+1}}\,dx\right)^{\frac{m+1}{m(p+2)}}\\ &\leq 2\sigma\,|h|^2\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\,dx + \sigma\int_{B_{\tilde{t}}} |\tau_{s,h}V_p\,(Dv(x))|^2\,dx\\ &\quad+c_\sigma\,|h|^2\int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx + \frac{c_\sigma\,|h|^2}{(t-\tilde{s})^{\frac{m(p+2)}{mp+m-1}}}\int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}}\,dx,\end{aligned} \tag{3.62}$$

where we also used Young's inequality with exponents $\left(\frac{m(p+2)}{mp+m-1}, \frac{m(p+2)}{m+1}\right)$.
Plugging (3.54), (3.55), (3.56) and (3.62) into (3.53), and choosing $\varepsilon < \frac{\nu}{2}$, we get

$$\begin{aligned}&\int_\Omega \eta^2(x)\,|\tau_{s,h}Dv\,(x)|^2\left(\mu^2+|Dv\,(x+he_s)|^2+|Dv\,(x)|^2\right)^{\frac{p-2}{2}}\,dx\\ &\leq c\,|h|^2\left(\int_{B_{\tilde{t}}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}\,dx\right)^{\frac{p(m+1)}{m(p+2)}}\cdot\left(\int_{B_{\lambda r}} g^{\frac{2m(p+2)}{2m-p}}(x)dx\right)^{\frac{2m-p}{m(p+2)}}\\ &\quad+2\sigma\int_{B_{\tilde{t}}} |\tau_{s,h}V_p\,(Dv(x))|^2\,dx + \frac{c_\sigma\,|h|^2}{(t-\tilde{s})^2}\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}}\,dx\end{aligned}$$

$$+c_\sigma |h|^2 \int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{m(p+2)}{mp+m-1}}} \int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}} dx$$
$$+2\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx$$

which, by virtue of Lemma 1.4.3, and using Young's inequality with exponents $\left(\frac{m(p+2)}{p(m+1)}, \frac{m(p+2)}{2m-p}\right)$ implies

$$\begin{aligned}
&\int_\Omega \eta^2(x) \left|\tau_{s,h} Dv\,(x)\right|^2 \left(\mu^2 + |Dv\,(x+he_s)|^2 + |Dv\,(x)|^2\right)^{\frac{p-2}{2}} dx \\
\le\ & 2\sigma \int_{B_{\tilde{t}}} \left|\tau_{s,h} V_p\,(Dv(x))\right|^2 dx + 3\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx \\
&+ \frac{c_\sigma |h|^2}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Dv\,(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma |h|^2 \int_{B_{\lambda r}} g^{\frac{2m(p+2)}{2m-p}}(x) dx \\
&+ c_\sigma |h|^2 \int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{m(p+2)}{mp+m-1}}} \int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}} dx. \qquad (3.63)
\end{aligned}$$

Applying Lemma 1.2.3, (3.63) becomes

$$\begin{aligned}
&\int_\Omega \eta^2(x) \left|\tau_{s,h} V_p\,(Dv(x))\right|^2 dx \\
\le\ & 3\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx + c \cdot \sigma |h|^2 \int_{B_{\lambda r}} \left|DV_p\,(Dv(x))\right|^2 dx \\
&+ \frac{c_\sigma |h|^2}{(t-\tilde{s})^2} \int_{B_R} \left(\mu^2 + |Dv\,(x)|^2\right)^{\frac{p}{2}} dx + c_\sigma |h|^2 \int_{B_{\lambda r}} g^{\frac{2m(p+2)}{2m-p}}(x) dx \\
&+ c_\sigma |h|^2 \int_{B_t} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} dx + \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{m(p+2)}{mp+m-1}}} \int_{B_t} |F(x)|^{\frac{m(p+2)}{mp+m-1}} dx. \qquad (3.64)
\end{aligned}$$

Let us observe that, for any $m > 1$ and $1 < p < 2$, we have

$$\frac{m\,(p+2)}{mp+m-1} \le \frac{p+2}{p},$$

hence

$$\max\left\{\, 2, \frac{m\,(p+2)}{mp+m-1} \,\right\} \le \max\left\{\, 2, \frac{p+2}{p} \,\right\} = \frac{p+2}{p}.$$

Hence, since $t - \tilde{s} < 1$, by (3.64) we deduce

$$\begin{aligned}
&\int_\Omega \eta^2(x) \left|\tau_{s,h} V_p\,(Dv(x))\right|^2 dx \\
\le\ & 3\sigma |h|^2 \int_{B_{\lambda r}} \left(\mu^2 + |Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx + c \cdot \sigma |h|^2 \int_{B_{\lambda r}} \left|DV_p\,(Dv(x))\right|^2 dx \\
&+ \frac{c_\sigma |h|^2}{(t-\tilde{s})^{\frac{p+2}{p}}} \left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x) dx + \int_{B_R} \left(\mu^2 + |Dv\,(x)|^2\right)^{\frac{p}{2}} dx \right. \\
&\left. + \int_{B_R} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} dx + \int_{B_R} |F(x)|^{\frac{m(p+2)}{mp+m-1}} dx \right]. \qquad (3.65)
\end{aligned}$$

Let us notice that, since $\frac{m(p+2)}{mp+m-1} < \frac{2m(p+2)}{2mp+p-2}$, we have $L_{\mathrm{loc}}^{\frac{2m(p+2)}{2mp+p-2}}(\Omega) \hookrightarrow L_{\mathrm{loc}}^{\frac{m(p+2)}{mp+m-1}}(\Omega)$, and using Young's inequality with exponents $\left(\frac{2(mp+m-1)}{2mp+p-2}, \frac{2(mp+m-1)}{2m-p}\right)$, we have

$$\int_{B_R} |F(x)|^{\frac{m(p+2)}{mp+m-1}}\,dx \le c\,|B_R| + c\int_{B_R} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx. \tag{3.66}$$

So, plugging (3.66) into (3.65), we get

$$\begin{aligned}
&\int_\Omega \eta^2(x)\,|\tau_{s,h}V_p\,(Dv(x))|^2\,dx\\
\le\;& 3\sigma\,|h|^2\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}dx + c\cdot\sigma\,|h|^2\int_{B_{\lambda r}}|DV_p\,(Dv(x))|^2\,dx\\
&+\frac{c_\sigma\,|h|^2}{(t-\tilde{s})^{\frac{p+2}{p}}}\left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}}dx\right.\\
&\left.+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|\right].
\end{aligned} \tag{3.67}$$

Since, by our a priori assumption, $V_p\,(Dv)\in W^{1,2}_{\mathrm{loc}}\,(\Omega)$, and (3.67) holds for any $s=1,\dots,n$, Lemma 1.2.4 implies

$$\begin{aligned}
&\int_\Omega \eta^2(x)\,|DV_p\,(Dv(x))|^2\,dx\\
\le\;& 3\sigma\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}dx + c\cdot\sigma\int_{B_{\lambda r}}|DV_p\,(Dv(x))|^2\,dx\\
&+\frac{c_\sigma}{(t-\tilde{s})^{\frac{p+2}{p}}}\left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}}dx\right.\\
&\left.+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|\right],
\end{aligned}$$

and by the properties of η, we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}}|DV_p\,(Dv(x))|^2\,dx\\
\le\;& c\cdot\sigma\int_{B_{\lambda r}}|DV_p\,(Dv(x))|^2\,dx\\
&+\frac{c_\sigma}{(t-\tilde{s})^{\frac{p+2}{p}}}\left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}}dx\right.\\
&\left.+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|\right]\\
&+3\sigma\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}dx.
\end{aligned} \tag{3.68}$$

Let us remind that, since we choosed $B_{4R}\Subset\Omega$, $\frac{R}{2}\le r<\tilde{s}<t<\tilde{t}<\lambda r<R$, with $1<\lambda<2$ and $R<\frac{1}{4}$, we also have $\lambda r<\lambda\tilde{s}<\lambda t<\lambda^2 r<4r<4R<1$.
Choosing a cut-off function $\phi\in C^\infty_0\,(B_{\lambda t})$ such that $0\le\phi\le 1$, $\phi\equiv 1$ on $B_{\lambda\tilde{s}}$ and $|D\phi|\le\frac{c}{\lambda(t-\tilde{s})}$, we have

$$\int_{B_{\lambda r}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}}dx\le|B_R|+\int_{B_{\lambda t}}\phi^{\frac{m}{m+1}(p+2)}\,|Dv(x)|^{\frac{m}{m+1}(p+2)}\,dx,$$

where we also used that $\mu\in[0,1]$ and $\lambda r<R$. Therefore, applying (1.1), we get

$$
\begin{aligned}
&\int_{B_{\lambda r}} \left(\mu^2+|Dv(x)|^2\right)^{\frac{m(p+2)}{2(m+1)}} dx \\
\le\ & (p+2)^2 \left(\int_{B_{\lambda t}} \phi^{\frac{m}{m+1}(p+2)} |v(x)|^{2m}\, dx\right)^{\frac{1}{m+1}} \\
&\cdot\left[\left(\int_{B_{\lambda t}} \phi^{\frac{m}{m+1}(p+2)} |D\phi|^2 \left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{m}{m+1}}\right. \\
&\left.+n\left(\int_{B_{\lambda t}} \phi^{\frac{m}{m+1}(p+2)} \left(\mu^2+|Dv(x)|^2\right)^{\frac{p-2}{2}} \left|D^2v(x)\right|^2 dx\right)^{\frac{m}{m+1}}\right]+|B_R| \\
\le\ & c\,(n,p)\left(\int_{B_{4R}} |v(x)|^{2m}\, dx\right)^{\frac{1}{m+1}} \\
&\cdot\left[\frac{1}{\lambda^2\,(t-\tilde{s})^2}\left(\int_{B_{4R}} \left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{m}{m+1}}\right. \\
&\left.+\left(\int_{B_{\lambda^2 r}} |DV_p\,(Dv(x))|^2\, dx\right)^{\frac{m}{m+1}}\right]+|B_R|\,,
\end{aligned}
\tag{3.69}
$$

where we used the properties of ϕ, (1.6), and the fact that $\lambda t<\lambda^2 r<4R$.
The elementary inequality

$$
b^{\frac{m}{m+1}} \le b+1, \qquad \text{for any } m>1 \text{ and } b\ge 0,
$$

implies

$$
\left(\int_{B_{\lambda^2 r}} |DV_p\,(Dv(x))|^2\, dx\right)^{\frac{m}{m+1}} \le \int_{B_{\lambda^2 r}} |DV_p\,(Dv(x))|^2\, dx+1. \tag{3.70}
$$

Now, if we recall that $1<\lambda<2$, $t-\tilde{s}<\lambda\,(t-\tilde{s})<1$ and $\frac{p+2}{p}\ge 2$, thanks to (3.69) and (3.70), (3.68) implies

$$
\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p\,(Dv(x))|^2\, dx \\
\le\ & c\cdot\sigma\left(\int_{B_{4R}} |v(x)|^{2m}\, dx\right)^{\frac{1}{m+1}} \cdot \int_{B_{\lambda^2 r}} |DV_p\,(Dv(x))|^2\, dx \\
&+\frac{\lambda^2+1}{\lambda^2}\cdot\frac{c_\sigma}{(t-\tilde{s})^{\frac{p+2}{p}}}\left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}} dx\right. \\
&+\left(\int_{B_{4R}} |v(x)|^{2m}\, dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{m}{m+1}} \\
&\left.+\int_{B_R} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\, dx+|B_R|+1\right] \\
\le\ & c\cdot\sigma\left(\int_{B_{4R}} |v(x)|^{2m}\, dx\right)^{\frac{1}{m+1}} \cdot \int_{B_{\lambda^2 r}} |DV_p\,(Dv(x))|^2\, dx \\
&+\frac{c_\sigma}{(t-\tilde{s})^{\frac{p+2}{p}}}\left[\int_{B_R} g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\,(x)|^2\right)^{\frac{p}{2}} dx\right.
\end{aligned}
$$

$$+\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}$$
$$+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\Bigg],$$

which, if we choose $\sigma>0$ such that

$$c\cdot\sigma\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}<\frac{1}{2},$$

becomes

$$\begin{aligned}\int_{B_{\tilde s}}|DV_p\left(Dv(x)\right)|^2\,dx \;\le\;& \frac{1}{2}\int_{B_{\lambda^2 r}}|DV_p\left(Dv(x)\right)|^2\,dx\\ &+\frac{c}{(t-\tilde s)^{\frac{p+2}{p}}}\Bigg[\int_{B_R}g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}\left(\mu^2+|Dv\left(x\right)|^2\right)^{\frac{p}{2}}dx\\ &+\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\\ &+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\Bigg],\end{aligned}\tag{3.71}$$

Since (3.71) holds for any $\frac{R}{2}\le r<\tilde s<t<\tilde t<\lambda r<R$, with $1<\lambda<2$, with a constant c depending on n,N,p,L,ν,ℓ, but is independent of the radii, passing to the limit as $\tilde s\to r$ and $t\to\lambda r$, we get

$$\begin{aligned}\int_{B_r}|DV_p\left(Dv(x)\right)|^2\,dx \;\le\;& \frac{1}{2}\int_{B_{\lambda^2 r}}|DV_p\left(Dv(x)\right)|^2\,dx\\ &+\frac{c}{r^{\frac{p+2}{p}}(\lambda-1)^{\frac{p+2}{p}}}\Bigg[\int_{B_R}\left(\mu^2+|Dv\left(x\right)|^2\right)^{\frac{p}{2}}dx\\ &+\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\\ &+\int_{B_R}g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\Bigg],\end{aligned}$$

and since $1<\lambda<2$, we have

$$\begin{aligned}\int_{B_r}|DV_p\left(Dv(x)\right)|^2\,dx \;\le\;& \frac{1}{2}\int_{B_{\lambda^2 r}}|DV_p\left(Dv(x)\right)|^2\,dx\\ &+\frac{c}{r^{\frac{p+2}{p}}(\lambda^2-1)^{\frac{p+2}{p}}}\Bigg[\int_{B_R}\left(\mu^2+|Dv\left(x\right)|^2\right)^{\frac{p}{2}}dx\\ &+\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\\ &+\int_{B_R}g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\Bigg]\end{aligned}\tag{3.72}$$

Now, if we set

$$h(r)=\int_{B_r}|DV_p\left(Dv(x)\right)|^2\,dx,$$

$$\begin{aligned} A = & \left[\int_{B_R}\left(\mu^2+|Dv\left(x\right)|^2\right)^{\frac{p}{2}}dx\right. \\ & +\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}} \\ & \left.+\int_{B_R}g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\right], \end{aligned}$$

and

$$B=0$$

since (3.72) holds for any $\lambda\in(1,2)$, we can apply Lemma 1.1.1 with

$$\theta=\frac{1}{2}\qquad\text{and}\qquad\gamma=\frac{p+2}{p},$$

thus getting

$$\begin{aligned} & \int_{B_{\frac{R}{2}}}\left|DV_p\left(Dv(x)\right)\right|^2dx \\ \leq & \frac{c}{R^{\frac{p+2}{p}}}\left[\int_{B_R}\left(\mu^2+|Dv\left(x\right)|^2\right)^{\frac{p}{2}}dx\right. \\ & +\left(\int_{B_{4R}}|v(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4R}}\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}} \\ & \left.+\int_{B_R}g^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_R}|F(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_R|+1\right], \end{aligned} \tag{3.73}$$

which is the desired a priori estimate.

Step 2: the approximation.
As we did in the second step of the proof of Theorem 3.1.1, let us consider an open set $\Omega'\Subset\Omega$ and, for any $\varepsilon\in(0,d\left(\Omega',\partial\Omega\right))$, a standard family of mollifiers $\{\,\phi_\varepsilon\,\}_\varepsilon$.
Let us consider a ball $B_{\tilde{R}}=B_{\tilde{R}}\left(x_0\right)\Subset\Omega'$ with $\tilde{R}<1$ and, for each ε, the functional

$$\mathcal{F}_{m,\varepsilon}\left(w,B_{\tilde{R}}\right)=\int_{B_{\tilde{R}}}\left[f_\varepsilon\left(x,Dw(x)\right)-F_\varepsilon(x)\cdot w(x)+\left(|w(x)|-a\right)_+^{2m}\right]dx, \tag{3.74}$$

where f_ε is defined as in (3.28) and F_ε is defined as in (3.29).
With this choices, we have

$$\int_{B_{\tilde{R}}}f_\varepsilon\left(x,\xi\right)dx\to\int_{B_{\tilde{R}}}f\left(x,\xi\right)dx,\qquad\text{as }\varepsilon\to0 \tag{3.75}$$

for any $\xi\in\mathbb{R}^{n\times N}$.
Moreover, since $F\in L_{\text{loc}}^{\frac{2m(p+2)}{2mp+p-2}}\left(\Omega\right)$, then

$$F_\varepsilon\to F\qquad\text{strongly in }L^{\frac{2m(p+2)}{2mp+p-2}}\left(B_{\tilde{R}}\right),\ \text{as }\varepsilon\to0. \tag{3.76}$$

Let us observe that

$$\frac{2m\left(p+2\right)}{2mp+p-2}\geq\frac{2m}{2m-1}$$

if and only if

$$(2m-1)(p+2) \geq 2mp+p-2,$$

i.e.

$$2m \geq p,$$

which is true for any $m>1$, as long as $1<p<2$.
For this reason, $F \in L_{\mathrm{loc}}^{\frac{2m}{2m-1}}(B_{\tilde{R}})$, and we also have

$$F_\varepsilon \to F \qquad \text{strongly in } L^{\frac{2m}{2m-1}}\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0. \tag{3.77}$$

Again, as in the proof of Theorem 3.1.1, thanks to (2.2)–(2.4) and (2.12), for any $\varepsilon \in (0, d(\Omega', \partial\Omega))$, we have the validity of (3.33)–(3.36), where g_ε is defined in (3.37).
In this case, since $g \in L_{\mathrm{loc}}^{\frac{2m(p+2)}{2mp-p}}(\Omega)$, we have

$$g_\varepsilon \to g \qquad \text{strongly in } L^{\frac{2m(p+2)}{2mp-p}}\left(B_{\tilde{R}}\right) \text{ as } \varepsilon \to 0. \tag{3.78}$$

Let $v_\varepsilon \in \left(v+W_0^{1,p}(B_{\tilde{R}})\right) \cap L^{2m}(B_{\tilde{R}})$ be the solution to

$$\min\left\{ \mathcal{F}_{m,\varepsilon}(w, B_{\tilde{R}}) : w \in \left(v+W_0^{1,p}(B_{\tilde{R}})\right) \cap L^{2m}(B_{\tilde{R}}) \right\},$$

where $v \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap L_{\mathrm{loc}}^{2m}(\Omega)$ is a local minimizer of (3.49).
By virtue of the minimality of v_ε, we have

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left[f_\varepsilon(x, Dv_\varepsilon(x)) + (|v_\varepsilon(x)|-a)_+^{2m}\right] dx \\
\leq\ & \int_{B_{\tilde{R}}} \left[f_\varepsilon(x, Dv(x)) + F_\varepsilon(x)\cdot(v_\varepsilon(x)-v(x)) + (|v(x)|-a)_+^{2m}\right] dx \\
\leq\ & \int_{B_{\tilde{R}}} \left[f_\varepsilon(x, Dv(x)) + |F_\varepsilon(x)|\cdot|v_\varepsilon(x)-v(x)| + (|v(x)|-a)_+^{2m}\right] dx.
\end{aligned} \tag{3.79}$$

Now, using Hölder's and Young's inequalities with exponents $\left(2m, \frac{2m}{2m-1}\right)$, we get

$$\begin{aligned}
&\int_{B_{\tilde{R}}} |F_\varepsilon(x)|\cdot|v_\varepsilon(x)-v(x)|\, dx \\
\leq\ & \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,|v_\varepsilon(x)|\, dx + \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,|v(x)|\, dx \\
=\ & \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,(|v_\varepsilon(x)|-a)\, dx + \int_{B_{\tilde{R}}} a\,|F_\varepsilon(x)|\, dx + \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,|v(x)|\, dx \\
=\ & \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} |F_\varepsilon(x)|\,(|v_\varepsilon(x)|-a)\, dx + \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|<a\,\}} |F_\varepsilon(x)|\,(|v_\varepsilon(x)|-a)\, dx \\
&+ \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,(|v(x)|+a)\, dx \\
\leq\ & \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} |F_\varepsilon(x)|\,(|v_\varepsilon(x)|-a)_+\, dx + \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,(|v(x)|+a)\, dx \\
\leq\ & \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,(|v_\varepsilon(x)|-a)_+\, dx + \int_{B_{\tilde{R}}} |F_\varepsilon(x)|\,(|v(x)|+a)\, dx \\
\leq\ & \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\, dx\right)^{\frac{2m-1}{2m}} \cdot \left(\int_{B_{\tilde{R}}} (|v_\varepsilon(x)|-a)_+^{2m}\, dx\right)^{\frac{1}{2m}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx\right)^{\frac{2m-1}{2m}} \cdot \left(\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx\right)^{\frac{1}{2m}}\\
\leq\ & c_\sigma \int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx + \sigma \int_{B_{\tilde{R}}} (|v_\varepsilon(x)|-a)_+^{2m}\,dx\\
&+c\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx, \qquad (3.80)
\end{aligned}
$$

for $\sigma > 0$ that will be chosen later.
Plugging (3.80) into (3.79), and choosing a sufficiently small σ, we get

$$
\begin{aligned}
&\int_{B_{\tilde{R}}} \left[f_\varepsilon\left(x, Dv_\varepsilon(x)\right) + c\left(|v_\varepsilon(x)|-a\right)_+^{2m}\right] dx\\
\leq\ & \int_{B_{\tilde{R}}} \left[f_\varepsilon\left(x, Dv(x)\right) + c\left(|v(x)|-a\right)_+^{2m}\right] dx\\
&+c\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx + c\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx. \qquad (3.81)
\end{aligned}
$$

Using the right-hand side inequality in (3.33) in (3.81), we have

$$
\begin{aligned}
&\int_{B_{\tilde{R}}} \left[f_\varepsilon\left(x, Dv_\varepsilon(x)\right) + c\left(|v_\varepsilon(x)|-a\right)_+^{2m}\right] dx\\
\leq\ & \ell_2\int_{B_{\tilde{R}}} \left[\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} + c\left(|v(x)|-a\right)_+^{2m}\right] dx\\
&+c\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx + c\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx. \qquad (3.82)
\end{aligned}
$$

Now, by the left-hand side inequality in (3.33), we get

$$
\begin{aligned}
\ell_1\int_{B_{\tilde{R}}} \left(\mu^2+|Dv_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\ \leq\ & \int_{B_{\tilde{R}}} f_\varepsilon\left(x, Dv_\varepsilon(x)\right) dx\\
\leq\ & \int_{B_{\tilde{R}}} \left[f_\varepsilon\left(x, Dv_\varepsilon(x)\right) + \left(|v_\varepsilon(x)|-a\right)_+^{2m}\right] dx\\
\leq\ & \ell_2\int_{B_{\tilde{R}}} \left[\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} + \left(|v(x)|-a\right)_+^{2m}\right] dx\\
&+c\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx\\
&+c\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx, \qquad (3.83)
\end{aligned}
$$

and this, by (3.77), means that $\{\, v_\varepsilon \,\}_\varepsilon$ is bounded in $W^{1,p}\left(B_{\tilde{R}}\right)$, independently of ε, so there exists a function $\tilde{v} \in W^{1,p}\left(B_{\tilde{R}}\right)$ such that, up to a subsequence, we have

$$
v_\varepsilon \rightharpoonup \tilde{v} \qquad \text{weakly in } W^{1,p}\left(B_{\tilde{R}}\right),
$$

$$
v_\varepsilon \to \tilde{v} \qquad \text{strongly in } L^p\left(B_{\tilde{R}}\right),
$$

and

$$v_\varepsilon \to \tilde{v} \qquad \text{almost everywhere in } B_{\tilde{R}},$$

as $\varepsilon \to 0$.
Moreover, we have

$$\begin{aligned}
\int_{B_{\tilde{R}}} |v_\varepsilon(x)|^{2m}\,dx &\leq \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|<a\,\}} |v_\varepsilon(x)|^{2m}\,dx + \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} |v_\varepsilon(x)|^{2m}\,dx \\
&\leq \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|<a\,\}} |v_\varepsilon(x)|^{2m}\,dx + \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} (||v_\varepsilon(x)|-a|+a)^{2m}\,dx \\
&\leq \int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|<a\,\}} a^{2m}dx + c\int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} (|v_\varepsilon(x)|-a)^{2m}\,dx \\
&\quad +c\int_{B_{\tilde{R}}\cap\{\,|v_\varepsilon|\geq a\,\}} a^{2m}dx \\
&\leq c\int_{B_{\tilde{R}}} a^{2m}dx + c\int_{B_{\tilde{R}}} (|v_\varepsilon(x)|-a)_+^{2m}\,dx, \qquad (3.84)
\end{aligned}$$

and since (3.82) implies

$$\begin{aligned}
&\int_{B_{\tilde{R}}} (|v_\varepsilon(x)|-a)_+^{2m}\,dx \\
\leq\ & \ell_2\int_{B_{\tilde{R}}} \left[\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}} + (|v(x)|-a)_+^{2m}\right] dx \\
&+c\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m}{2m-1}}\,dx + c\int_{B_{\tilde{R}}} (|v(x)|+a)^{2m}\,dx, \qquad (3.85)
\end{aligned}$$

by (3.77), and plugging (3.85) into (3.84), using dominate convergence theorem, we have

$$v_\varepsilon \to \tilde{v} \qquad \text{strongly in } L^{2m}\left(B_{\tilde{R}}\right),\ \text{as } \varepsilon \to 0. \qquad (3.86)$$

Since v_ε is a local minimizer of the functional (3.74) and $g_\varepsilon, f_\varepsilon \in C^\infty\left(B_{\tilde{R}}\right)$, we have

$$V_p\left(Dv_\varepsilon\right) \in W^{1,2}_{\text{loc}}\left(B_{\tilde{R}}\right),$$

and we can apply estimate (3.73), thus getting

$$\begin{aligned}
\int_{B_{\frac{r}{2}}} |DV_p\left(Dv_\varepsilon(x)\right)|^2\,dx \ \leq\ & \frac{c}{r^{\frac{p+2}{p}}}\left[\int_{B_r}\left(\mu^2+|Dv_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx\right. \\
&+\left(\int_{B_{4r}}|v_\varepsilon(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4r}}\left(\mu^2+|Dv_\varepsilon(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}} \\
&\left.+\int_{B_r} g_\varepsilon^{\frac{2m(p+2)}{2m-p}}(x)dx + \int_{B_r}|F_\varepsilon(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx + |B_r| + 1\right], \qquad (3.87)
\end{aligned}$$

for any ball $B_{4r} \Subset B_{\tilde{R}}$.
Applying Lemma 1.4.5, by (1.8) and (3.87), recalling (3.76), (3.78), (3.83), (3.84) and (3.85), and by a covering argument, we infer that v_ε is bounded in $W^{2,p}\left(B_{4r}\right)$, which implies

$$v_\varepsilon \to \tilde{v} \qquad \text{strongly in } W^{1,p}\left(B_{4r}\right) \qquad (3.88)$$

and

$$v_\varepsilon \to \tilde{v} \qquad \text{almost everywhere in } B_{4r},$$

up to a subsequence, as $\varepsilon \to 0$.
By virtue of the continuity of the function $\xi \mapsto DV_p(\xi)$, we also have

$$DV_p(Dv_\varepsilon) \to DV_p(D\tilde{v}) \qquad \text{almost everywhere in } B_{4r}, \text{ as } \varepsilon \to 0.$$

For what we discussed above, and recalling (3.76), (3.78), (3.86) and (3.88), thanks to the dominate convergence theorem, we can pass to the limit in (3.87) as $\varepsilon \to 0$, thus getting

$$\begin{aligned} & \int_{B_{\frac{r}{2}}} |DV_p(D\tilde{v}(x))|^2 \, dx \\ \le\ & \frac{c}{r^{\frac{p+2}{p}}} \left[\int_{B_r} \left(\mu^2 + |D\tilde{v}(x)|^2\right)^{\frac{p}{2}} dx \right. \\ & + \left(\int_{B_{4r}} |\tilde{v}(x)|^{2m} \, dx \right)^{\frac{1}{m+1}} \cdot \left(\int_{B_{4r}} \left(\mu^2 + |D\tilde{v}(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{m}{m+1}} \\ & \left. + \int_{B_r} g^{\frac{2m(p+2)}{2m-p}}(x) dx + \int_{B_r} |F(x)|^{\frac{2m(p+2)}{2mp+p-2}} \, dx + |B_r| + 1 \right]. \end{aligned} \tag{3.89}$$

Our next aim is to prove that $\tilde{v} = v$ a.e. in $B_{\tilde{R}}$.
First, let us observe that, using Hölder's inequality with exponents $\left(2m, \frac{2m}{2m-1}\right)$, we get

$$\begin{aligned} & \left| \int_{B_{\tilde{R}}} [F_\varepsilon(x) \cdot v(x) - F(x) \cdot v(x)] \, dx \right| \\ \le\ & \int_{B_{\tilde{R}}} |F_\varepsilon(x) - F(x)| \cdot |v(x)| \, dx \\ \le\ & \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x) - F(x)|^{\frac{2m}{2m-1}} \, dx \right)^{\frac{2m-1}{2m}} \cdot \left(\int_{B_{\tilde{R}}} |v(x)|^{2m} \, dx \right)^{\frac{1}{2m}}, \end{aligned}$$

that, recalling (3.77), implies

$$\lim_{\varepsilon \to 0} \int_{B_{\tilde{R}}} F_\varepsilon(x) \cdot v(x) dx = \int_{B_{\tilde{R}}} F(x) \cdot v(x) dx,$$

and by (3.75), we get

$$\lim_{\varepsilon \to 0} \int_{B_{\tilde{R}}} [f_\varepsilon(x, Du(x)) - F_\varepsilon(x) \cdot u(x)] \, dx = \int_{B_{\tilde{R}}} [f(x, Du(x)) - F(x) \cdot u(x)] \, dx. \tag{3.90}$$

The minimality of v, Fatou's Lemma, the lower semicontinuity of $\mathcal{F}_{m,\varepsilon}$ and the minimality of v_ε imply

$$\begin{aligned} & \int_{B_{\tilde{R}}} \left[f(x, Dv(x)) - F(x) \cdot v(x) + (|v(x)| - a)_+^{2m} \right] dx \\ \le\ & \int_{B_{\tilde{R}}} \left[f(x, D\tilde{v}(x)) - F(x) \cdot \tilde{v}(x) + (|\tilde{v}(x)| - a)_+^{2m} \right] dx \\ \le\ & \liminf_{\varepsilon \to 0} \int_{B_{\tilde{R}}} \left[f_\varepsilon(x, D\tilde{v}(x)) - F_\varepsilon(x) \cdot \tilde{v}(x) + (|\tilde{v}(x)| - a)_+^{2m} \right] dx \end{aligned}$$

$$\begin{aligned}
&\leq \liminf_{\varepsilon\to 0}\int_{B_{\tilde R}}\left[f_\varepsilon\left(x, Dv_\varepsilon(x)\right)-F_\varepsilon(x)\cdot v_\varepsilon(x)+\left(|v_\varepsilon(x)|-a\right)_+^{2m}\right]dx\\
&\leq \liminf_{\varepsilon\to 0}\int_{B_{\tilde R}}\left[f_\varepsilon\left(x, Dv(x)\right)-F_\varepsilon(x)\cdot v(x)+\left(|v(x)|-a\right)_+^{2m}\right]dx\\
&= \int_{B_{\tilde R}}\left[f\left(x, Dv(x)\right)-F(x)\cdot v(x)+\left(|v(x)|-a\right)_+^{2m}\right]dx,
\end{aligned}$$

where the last inequality is a consequence of (3.75) and (3.90).
Therefore $\mathcal{F}_m\left(D\tilde v, B_{\tilde R}\right)=\mathcal{F}_m\left(Dv, B_{\tilde R}\right)$ and the strict convexity of the functional yields that $\tilde v = v$ a.e. in $B_{\tilde R}$. So (3.89) and a covering argument yield (3.50). □

We conclude this section with some consequences of Theorem 3.2.2, which follow by Lemma 1.4.5 and Remark 1.4.7.

Corollary 3.2.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $m>1$, $a>0$ and $1<p<2$. Let $v\in W^{1,p}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)\cap L^{2m}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)$ be a local minimizer of the functional* (3.49)*, under the assumptions* (2.2)*–*(2.4) *and* (2.12)*, with*

$$F\in L^{\frac{2m(p+2)}{2mp+p-2}}_{\mathrm{loc}}\left(\Omega\right)\qquad \text{and}\qquad g\in L^{\frac{2m(p+2)}{2m-p}}_{\mathrm{loc}}\left(\Omega\right).$$

Then $v\in W^{2,p}_{\mathrm{loc}}\left(\Omega\right)$ and $Dv\in L^{\frac{m(p+2)}{m+1}}_{\mathrm{loc}}\left(\Omega\right)$.

3.2.2 Proof of Theorem 3.2.1

The aim of this section is to prove Theorem 3.2.1.
As we will see below, the proof is achieved by using an approximation argument which is based on the possibility to apply Theorem 3.2.2 and pass to the limit as $m\to\infty$.

Proof of Theorem 3.2.1. Arguing as in the second step of the proof of Theorem 3.2.2, let us consider an open set $\Omega'\Subset\Omega$ and, for any $\varepsilon\in\left(0, d\left(\Omega',\partial\Omega\right)\right)$, a standard family of mollifiers $\left\{\phi_\varepsilon\right\}_\varepsilon$.
Let $u\in W^{1,p}_{\mathrm{loc}}\left(\Omega\right)\cap L^\infty_{\mathrm{loc}}\left(\Omega\right)$ be a local minimizer of the functional (3.1), and let us consider a ball $B_{\tilde R}=B_{\tilde R}\left(x_0\right)\Subset\Omega'$, with $\tilde R<1$.
For each ε and any $m>1$, let us consider the functional $\mathcal{F}_{m,\varepsilon}$, defined by (3.74), where f_ε and F_ε are defined by (3.28) and (3.29) respectively, and we fix

$$a=\|u\|_{L^\infty\left(B_{\tilde R}\right)}. \tag{3.91}$$

With these choices, we have (3.75) again, and since $F\in L^{\frac{p+2}{p}}_{\mathrm{loc}}\left(\Omega\right)$, we have

$$F_\varepsilon\to F\qquad \text{strongly in } L^{\frac{p+2}{p}}\left(B_{\tilde R}\right),\qquad \text{as } \varepsilon\to 0. \tag{3.92}$$

Again, thanks to (2.2)–(2.4) and (2.12), for any ε, we have (3.33)–(3.36), where g_ε is defined like in (3.37).
In this case, since $g\in L^{p+2}_{\mathrm{loc}}\left(\Omega\right)$, we have

$$g_\varepsilon\to g\qquad \text{strongly in } L^{p+2}\left(B_{\tilde R}\right),\ \text{as } \varepsilon\to 0. \tag{3.93}$$

Let us observe that $F_\varepsilon\in L^{\frac{2m(p+2)}{2mp+p-2}}\left(B_{\tilde R}\right)$ for any $m>1$, and since

$$\frac{2m\left(p+2\right)}{2mp+p-2}\searrow\frac{p+2}{p},\qquad \text{as } m\to\infty,$$

we have

$$\lim_{m\to\infty}\left(\int_{B_{\tilde R}}|F_\varepsilon(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx\right)^{\frac{2mp+p-2}{2m(p+2)}}=\left(\int_{B_{\tilde R}}|F_\varepsilon(x)|^{\frac{p+2}{p}}\,dx\right)^{\frac{p}{p+2}}, \tag{3.94}$$

for any ε.

Similarly, then $g_\varepsilon\in L^{\frac{2m(p+2)}{2m-p}}\left(B_{\tilde R}\right)$ for any $m>1$ and for any ε, and we have

$$\lim_{m\to\infty}\left(\int_{B_{\tilde R}}|g_\varepsilon(x)|^{\frac{2m(p+2)}{2m-p}}\,dx\right)^{\frac{2m-p}{2m(p+2)}}=\left(\int_{B_{\tilde R}}|g_\varepsilon(x)|^{p+2}\,dx\right)^{\frac{1}{p+2}}, \tag{3.95}$$

for each ε.

Now, for each ε, and for each $m>1$, let $u_{m,\varepsilon}\in\left(u+W_0^{1,p}\left(B_{\tilde R}\right)\right)\cap L^{2m}\left(B_{\tilde R}\right)$ be the solution to

$$\min\left\{\mathcal{F}_{m,\varepsilon}\left(w,B_{\tilde R}\right): w\in\left(u+W_0^{1,p}\left(B_{\tilde R}\right)\right)\cap L^{2m}\left(B_{\tilde R}\right)\right\}.$$

By virtue of the minimality of $u_{m,\varepsilon}$, recalling (3.91), we have

$$\begin{aligned}&\int_{B_{\tilde R}}\left[f_\varepsilon\left(x,Du_{m,\varepsilon}(x)\right)+\left(|u_{m,\varepsilon}(x)|-a\right)_+^{2m}\right]dx\\ \le\;&\int_{B_{\tilde R}}\left[f_\varepsilon\left(x,Du(x)\right)+F_\varepsilon(x)\cdot\left(u_{m,\varepsilon}(x)-u(x)\right)+\left(|u(x)|-a\right)_+^{2m}\right]dx\\ \le\;&\int_{B_{\tilde R}}\left[f_\varepsilon\left(x,Du(x)\right)+|F_\varepsilon(x)|\cdot|u_{m,\varepsilon}(x)-u(x)|\right]dx.\end{aligned} \tag{3.96}$$

Arguing as we did in (3.80) and exploiting (3.91), we get

$$\begin{aligned}&\int_{B_{\tilde R}}|F_\varepsilon(x)|\cdot|u_{m,\varepsilon}(x)-u(x)|\,dx\\ \le\;&\int_{B_{\tilde R}}|F_\varepsilon(x)|\left(|u_{m,\varepsilon}(x)|-a\right)_+dx+2a\int_{B_{\tilde R}}|F_\varepsilon(x)|\,dx\\ \le\;&\left(\int_{B_{\tilde R}}|F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}}\,dx\right)^{\frac{p(2m-1)}{2m(p+2)}}\cdot\left(\int_{B_{\tilde R}}\left(|u_{m,\varepsilon}(x)|-a\right)_+^{\frac{2m(p+2)}{4m+p}}\,dx\right)^{\frac{4m+p}{2m(p+2)}}\\ &+2a\int_{B_{\tilde R}}|F_\varepsilon(x)|\,dx,\end{aligned} \tag{3.97}$$

where, in the last line, we used Hölder's inequality with exponents $\left(\frac{2m(p+2)}{p(2m-1)},\frac{2m(p+2)}{4m+p}\right)$. Let us notice that all the integrals in the last line of (3.97) are finite, since $F_\varepsilon\in C^\infty\left(B_{\tilde R}\right)$ and $\frac{2m(p+2)}{4m+p}<2m$ for any $m>1$ as long as $1<p<2$.

So, since $u_{m,\varepsilon}\in L^{2m}\left(B_{\tilde R}\right)\hookrightarrow L^{\frac{2m(p+2)}{4m+p}}\left(B_{\tilde R}\right)$, using Young's inequality with exponents $\left(2m,\frac{2m}{2m-1}\right)$, we have

$$\begin{aligned}&\int_{B_{\tilde R}}|F_\varepsilon(x)|\cdot|u_{m,\varepsilon}(x)-u(x)|\,dx\\ \le\;&c\left(\int_{B_{\tilde R}}|F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}}\,dx\right)^{\frac{p(2m-1)}{2m(p+2)}}\cdot\left(\int_{B_{\tilde R}}\left(|u_{m,\varepsilon}(x)|-a\right)_+^{2m}\,dx\right)^{\frac{1}{2m}}\\ &+2a\int_{B_{\tilde R}}|F_\varepsilon(x)|\,dx\end{aligned}$$

$$\leq c_\sigma \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}} \, dx \right)^{\frac{p}{p+2}} + \sigma \int_{B_{\tilde{R}}} (|u_{m,\varepsilon}(x)| - a)_+^{2m} \, dx$$
$$+2a \int_{B_{\tilde{R}}} |F_\varepsilon(x)| \, dx, \tag{3.98}$$

for any $\sigma > 0$.
Plugging (3.98) into (3.96), choosing a sufficiently small value of σ and recalling (3.33), we get

$$\int_{B_{\tilde{R}}} \left[\left(\mu^2 + |Du_{m,\varepsilon}(x)|^2\right)^{\frac{p}{2}} + (|u_{m,\varepsilon}(x)| - a)_+^{2m} \right] dx$$
$$\leq \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}} \, dx \right)^{\frac{p}{p+2}}. \tag{3.99}$$

Now let us notice that, since

$$\frac{2m\,(p+2)}{p\,(2m-1)} \geq \frac{p+2}{p}$$

for any $m > 1$, and

$$\frac{2m\,(p+2)}{p\,(2m-1)} \searrow \frac{p+2}{p}, \qquad \text{as } m \to \infty,$$

we have

$$\lim_{m\to\infty} \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}} \, dx \right)^{\frac{p(2m-1)}{2m(p+2)}} = \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p+2}{p}} \, dx \right)^{\frac{p}{p+2}},$$

and so

$$\lim_{m\to\infty} \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}} \, dx \right)^{\frac{p}{p+2}} = \lim_{m\to\infty} \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}} \, dx \right)^{\frac{p(2m-1)}{2m(p+2)} \cdot \frac{2m}{(2m-1)}}$$
$$= \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p+2}{p}} \, dx \right)^{\frac{p}{p+2}}, \tag{3.100}$$

for any ε.
Hence, for any ε, the second integral in the right-hand side of (3.99) can be bounded independently of m.
This implies that, for each ε, $\{ u_{m,\varepsilon} \}_m$ is bounded in $W^{1,p}(B_{\tilde{R}})$, and so there exists a family of functions $\{ u_\varepsilon \}_\varepsilon \subset W^{1,p}(B_{\tilde{R}})$ such that

$$u_{m,\varepsilon} \rightharpoonup u_\varepsilon \qquad \text{weakly in } W^{1,p}(B_{\tilde{R}}),$$

and so

$$u_{m,\varepsilon} \to u_\varepsilon \qquad \text{strongly in } L^p(B_{\tilde{R}}),$$

and

$$u_{m,\varepsilon} \to u_\varepsilon \qquad \text{almost everywhere in } B_{\tilde{R}},$$

as $m \to \infty$, up to a subsequence.
In particular, by (3.99), (3.92) and (3.100), the set of functions $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{1,p}(B_{\tilde{R}})$, and so there exists a function $v \in W^{1,p}(B_{\tilde{R}})$ such that

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly in } W^{1,p}(B_{\tilde{R}}), \text{ as } \varepsilon \to 0.$$

So we have

$$u_\varepsilon \to v \qquad \text{strongly in } L^p(B_{\tilde{R}})$$

and

$$u_\varepsilon \to v \qquad \text{almost everywhere in } B_{\tilde{R}},$$

up to a subsequence, as $\varepsilon \to 0$.
On the other hand, (3.99) implies

$$\begin{aligned} &\int_{B_{\tilde{R}}} (|u_{m,\varepsilon}(x)| - a)_+^{2m}\, dx \\ \leq\ & \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}}\, dx \right)^{\frac{p}{p+2}}, \end{aligned} \tag{3.101}$$

and this bound is independent of m by virtue of (3.100).
One can easily check that, for any $m > 1$, we have

$$\int_{B_{\tilde{R}}} |u_{m,\varepsilon}(x)|^{2m}\, dx \leq \int_{B_{\tilde{R}}} (|u_{m,\varepsilon}(x)| - a)_+^{2m}\, dx + c \int_{B_{\tilde{R}}} a^{2m} dx,$$

and so, by virtue of (3.101), for any $m > 1$, we get

$$\begin{aligned} \int_{B_{\tilde{R}}} |u_{m,\varepsilon}(x)|^{2m}\, dx \ \leq\ & \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \\ & + c \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}}\, dx \right)^{\frac{p}{p+2}} + c \int_{B_{\tilde{R}}} a^{2m} dx \end{aligned}$$

and

$$\begin{aligned} &\left(\int_{B_{\tilde{R}}} |u_{m,\varepsilon}(x)|^{2m}\, dx \right)^{\frac{1}{2m}} \\ =\ & c \left[\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx \right]^{\frac{1}{2m}} + c \left[\int_{B_{\tilde{R}}} a^{2m} dx \right]^{\frac{1}{2m}} \\ & + c \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{2m(p+2)}{p(2m-1)}}\, dx \right)^{\frac{p(2m-1)}{2m(p+2)} \cdot \frac{1}{2m-1}}. \end{aligned} \tag{3.102}$$

Now, if we pass to the lim sup as $m \to \infty$ at both sides of (3.102), recalling (3.91) and (3.100), we get

$$\limsup_{m \to \infty} \left(\int_{B_{\tilde{R}}} |u_{m,\varepsilon}(x)|^{2m}\, dx \right)^{\frac{1}{2m}} \leq c\, \|u\|_{L^\infty(B_{\tilde{R}})},$$

and similarly, for any ball $B_{4r} \Subset B_{\tilde{R}}$, we have

$$\limsup_{m\to\infty}\left(\int_{B_{4r}}|u_{m,\varepsilon}(x)|^{2m}\,dx\right)^{\frac{1}{2m}}\le c\,\|u\|_{L^\infty(B_{4r})},$$

which implies

$$\limsup_{m\to\infty}\left(\int_{B_{4r}}|u_{m,\varepsilon}(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\le c\,\|u\|^2_{L^\infty(B_{4r})}. \tag{3.103}$$

Since, for any $m>1$ and for any ε, $u_{m,\varepsilon}\in\left(u+W^{1,p}_0(B_{\tilde{R}})\right)\cap L^{2m}(B_{\tilde{R}})$ is a minimizer of a functional of the form (3.49), which satisfies (3.33)–(3.36), $g_\varepsilon\in L^{\frac{2m(p+2)}{2m-p}}(B_{\tilde{R}})$ and $f_\varepsilon\in L^{\frac{2m(p+2)}{2mp+p-2}}(B_{\tilde{R}})$, we can apply Theorem 3.2.2, and by (3.50), we get

$$\begin{aligned}
&\int_{B_{\frac{r}{2}}}|DV_p(Du_{m,\varepsilon}(x))|^2\,dx\\
\le\;&\frac{c}{r^{\frac{p+2}{p}}}\left[\int_{B_r}\left(\mu^2+|Du_{m,\varepsilon}(x)|^2\right)^{\frac{p}{2}}dx\right.\\
&+\left(\int_{B_{4r}}|u_{m,\varepsilon}(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4r}}\left(\mu^2+|Du_{m,\varepsilon}(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\\
&\left.+\int_{B_r}g_\varepsilon^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_r}|F_\varepsilon(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_r|+1\right],
\end{aligned}\tag{3.104}$$

for any ball $B_{4r}\Subset B_{\tilde{R}}$.
Moreover, we can use Lemma 1.4.5 and (1.8), thus getting

$$\begin{aligned}
&\int_{B_{\frac{r}{2}}}\left|D^2u_{m,\varepsilon}(x)\right|^p dx\\
\le\;&\frac{c}{r^{\frac{p+2}{p}}}\left[\int_{B_r}\left(\mu^2+|Du_{m,\varepsilon}(x)|^2\right)^{\frac{p}{2}}dx\right.\\
&+\left(\int_{B_{4r}}|u_{m,\varepsilon}(x)|^{2m}\,dx\right)^{\frac{1}{m+1}}\cdot\left(\int_{B_{4r}}\left(\mu^2+|Du_{m,\varepsilon}(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{m}{m+1}}\\
&\left.+\int_{B_r}g_\varepsilon^{\frac{2m(p+2)}{2m-p}}(x)dx+\int_{B_r}|F_\varepsilon(x)|^{\frac{2m(p+2)}{2mp+p-2}}\,dx+|B_r|+1\right].
\end{aligned}\tag{3.105}$$

By virtue of (3.94), (3.95), (3.99), (3.100) and (3.103), all the integrals in the right-hand side of (3.105) are bounded independently of m: for this reason, for each ε, the set of functions $\{u_{m,\varepsilon}\}_m$ is bounded in $W^{2,p}\left(B_{\frac{r}{2}}\right)$, and since the ball B_{4r} is arbitrary, a covering argument implies

$$u_{m,\varepsilon}\to u_\varepsilon\qquad\text{strongly in } W^{1,p}(B_{4r}), \tag{3.106}$$

which gives

$$Du_{m,\varepsilon}\to Du_\varepsilon\qquad\text{almost everywhere in } B_{4r},$$

up to a subsequence, as $m\to\infty$.

So, passing to the limit as $m \to \infty$, recalling (3.99) and (3.100), we also get

$$\begin{aligned} &\int_{B_{2r}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \\ \leq\ & \ell_2 \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + c \left(\int_{B_{\tilde{R}}} |F_\varepsilon(x)|^{\frac{p+2}{p}} dx \right)^{\frac{p}{p+2}}. \end{aligned} \tag{3.107}$$

Therefore, since, by virtue of the continuity of $\xi \mapsto DV_p(\xi)$, we also have

$$DV_p\left(Du_{m,\varepsilon}\right) \to DV_p\left(Du_\varepsilon\right) \qquad \text{almost everywhere in } B_{4r}, \text{ as } m \to \infty,$$

and we can apply Fatou's Lemma passing to the lim sup as $m \to \infty$ in (3.104), using (3.94), (3.95), (3.103) and (3.106), we get

$$\begin{aligned} &\int_{B_{\frac{r}{2}}} \left|DV_p\left(Du_\varepsilon(x)\right)\right|^2 dx \\ \leq\ & \frac{c\left\|u\right\|_{L^\infty(B_{4r})}}{r^{\frac{p+2}{p}}} \left[\int_{B_{4r}} \left(\mu^2 + \left|Du_\varepsilon\left(x\right)\right|^2\right)^{\frac{p}{2}} dx \right. \\ & \left. + \int_{B_r} g_\varepsilon^{p+2}(x) dx + \int_{B_r} |F_\varepsilon(x)|^{\frac{p+2}{p}} dx + |B_r| + 1 \right], \end{aligned} \tag{3.108}$$

where we also used the fact that $r < \tilde{R} < 1$.
Now, since, by virtue of (3.92), (3.93), and (3.107), all the integrals in the right-hand side of (3.108) can be bounded independently of ε, arguing like in the proof of Lemma 1.4.5, it is possible to prove that $\{ u_\varepsilon \}_\varepsilon$ is bounded in $W^{2,p}\left(B_{\frac{r}{2}}\right)$, and since r is arbitrary, a covering argument implies

$$u_\varepsilon \to v \qquad \text{strongly in } W^{1,p}\left(B_{4r}\right),$$

and

$$Du_\varepsilon \to Dv \qquad \text{almost everywhere in } B_{4r},$$

as $\varepsilon \to 0$.
Since, by virtue of the continuity of $\xi \mapsto DV_p(\xi)$, we also have

$$DV_p\left(Du_\varepsilon\right) \to DV_p\left(Dv\right) \qquad \text{almost everywhere in } B_{4r}, \text{ as } \varepsilon \to 0,$$

using Fatou's Lemma in (3.108), we get

$$\begin{aligned} &\int_{B_{\frac{r}{2}}} \left|DV_p\left(Dv(x)\right)\right|^2 dx \\ \leq\ & \frac{c\left\|u\right\|_{L^\infty(B_{4r})}}{r^{\frac{p+2}{p}}} \left[\int_{B_{4r}} \left(\mu^2 + \left|Dv\left(x\right)\right|^2\right)^{\frac{p}{2}} dx \right. \\ & \left. + \int_{B_r} g^{p+2}(x) dx + \int_{B_r} |F(x)|^{\frac{p+2}{p}} dx + |B_r| + 1 \right]. \end{aligned} \tag{3.109}$$

The last step to get the conclusion consists in proving that $u = v$ a.e. on $B_{\tilde{R}}$.
Since we have

$$\begin{aligned} &\left|\int_{B_{\tilde R}}[F_\varepsilon(x)\cdot u(x)-F(x)]\cdot u(x)dx\right| \\ \le\ & \int_{B_{\tilde R}}|F_\varepsilon(x)-F(x)|\cdot|u(x)|\,dx \\ \le\ & \|u\|_{L^\infty(B_{\tilde R})}\int_{B_{\tilde R}}|F_\varepsilon(x)-F(x)|\,dx, \end{aligned}$$

by virtue of (3.92), we have

$$\lim_{\varepsilon\to0}\int_{B_{\tilde R}}F_\varepsilon(x)\cdot u(x)dx=\int_{B_{\tilde R}}F(x)\cdot u(x)dx,$$

and then, by (3.75), we get

$$\lim_{\varepsilon\to0}\int_{B_{\tilde R}}[f_\varepsilon(x,Du(x))-F_\varepsilon(x)\cdot u(x)]\,dx=\int_{B_{\tilde R}}[f(x,Du(x))-F(x)\cdot u(x)]\,dx. \tag{3.110}$$

Using the minimality of u, the lower semicontinuity of the functional $\mathcal{F}$, the minimality of $u_{m,\varepsilon}$ for $\mathcal{F}_{m,\varepsilon}$ and the lower semicontinuity of this functional and recalling (3.91), we get

$$\begin{aligned} &\int_{B_{\tilde R}}[f(x,Du(x))-F(x)\cdot u(x)]\,dx \\ \le\ & \int_{B_{\tilde R}}[f(x,Dv(x))-F(x)\cdot v(x)]\,dx \\ \le\ & \liminf_{\varepsilon\to0}\int_{B_{\tilde R}}[f_\varepsilon(x,Du_\varepsilon(x))-F_\varepsilon(x)\cdot u_\varepsilon(x)]\,dx \\ \le\ & \liminf_{\varepsilon\to0}\liminf_{m\to\infty}\int_{B_{\tilde R}}[f_\varepsilon(x,Du_{m,\varepsilon}(x))-F_\varepsilon(x)\cdot u_{m,\varepsilon}(x)]\,dx \\ \le\ & \liminf_{\varepsilon\to0}\liminf_{m\to\infty}\int_{B_{\tilde R}}\left[f_\varepsilon(x,Du_{m,\varepsilon}(x))-F_\varepsilon(x)\cdot u_{m,\varepsilon}(x)+(|u_{m,\varepsilon}(x)|-a)_+^{2m}\right]dx \\ \le\ & \liminf_{\varepsilon\to0}\int_{B_{\tilde R}}[f_\varepsilon(x,Du(x))-F_\varepsilon(x)\cdot u(x)]\,dx \\ =\ & \int_{B_{\tilde R}}[f(x,Du(x))-F(x)\cdot u(x)]\,dx, \end{aligned} \tag{3.111}$$

where, for the last equality, we used (3.110). Therefore, all the inequalities in (3.111) hold as equalities, and we get

$$\mathcal{F}\left(u,B_{\tilde R}\right)=\mathcal{F}\left(v,B_{\tilde R}\right).$$

So, by virtue of the strict convexity of F with respect to the gradient variable, this implies $u=v$ a.e. on $B_{\tilde R}$. By virtue of (3.109) and a standard covering argument, we get (3.48). □

We conclude this section with some consequences of Theorem 3.2.1, that can be proved applying Lemma 1.4.5 and estimate (1.2) rispectively.

Corollary 3.2.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and $1<p<2$.*
Let $u\in W^{1,p}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)\cap L^\infty_{\mathrm{loc}}\left(\Omega,\mathbb{R}^N\right)$ be a local minimizer of the functional (3.1)*, under the assumptions* (2.2)*–*(2.4) *and* (2.12)*, with*

$$F\in L^{\frac{p+2}{p}}_{\mathrm{loc}}(\Omega)\qquad\text{and}\qquad g\in L^{p+2}_{\mathrm{loc}}(\Omega).$$

Then $u\in W^{2,p}_{\mathrm{loc}}(\Omega)$ and $Du\in L^{p+2}_{\mathrm{loc}}(\Omega)$.

Chapter 4

Obstacle problems

This chapter is devoted to the descriptions of some regularity results for solutions to a class of obstacle problems, that is variational problems whose minimizers are forced to stay inside an admissible class of functions whose value have to stay above a fixed map, called *obstacle*. Of course, in this framework, the local minimizers are scalar-valued functions.
Studying this kind of problems means to figure out how the regularity of the obstale influences the regularity of the solutions.
After giving some preliminary results about regularity properties of solutions to obstacle problems in Section 4.1, in Section 4.2, a first obstacle problem is discussed, that is the only one in this thesis where the integrand of the functional satisfies super-quadratic growth conditions, instead of sub-quadratic ones.
In this kind of problems, the obstacle is assumed to be locally in L^∞. This property transfers to the solutions of the problem and allows to assume weaker Sobolev regularity on the coefficients, if compared with the required regularity in case the obstacle is not assumed to be bounded. As far as we know, this result is new both for the super-quadratic and sub-quadratic growth case. This is the reason why we started from the case $p \geq 2$ and then we moved to the case $1 < p < 2$.
As in the case of the unconstrained problems that have been faced in Chapter 2 and Chapter 3, a key point, here, is the dependence of the energy density of the functional on the x-variable.
For what concerns the case of Sobolev coefficients under sub-quadratic growth conditions, we describe a first higher differentiability result in Section 4.3, where the coefficients are assumed to belong to the critical Sobolev space $W^{1,n}$.
In Section 4.4, assuming sub-quadratic growth conditions again, we consider the case of Besov-Lipschitz coefficient, providing extra fractional differentiability results for the gradient the solutions.
Finally, in Section 4.5, assuming that the obstacle is locally bounded and considering Sobolev coefficients again, we discuss a higher differentiability result that, in some sense, extends to the sub-quadratic case what is proved in Section 4.2.

We are interested in the regularity properties of solutions to problems of the form

$$\min\left\{ \int_\Omega f\left(x, Dw(x)\right) dx : w \in \mathcal{K}_\psi\left(\Omega\right) \right\}, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n > 2$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x, \xi)$ is of class $C^1\left(\mathbb{R}^n\right)$ for a.e. $x \in \Omega$, $\psi : \Omega \mapsto [-\infty, +\infty)$ belonging to the Sobolev class $W^{1,p}_{\mathrm{loc}}\left(\Omega\right)$ is the *obstacle*, and

$$\mathcal{K}_\psi\left(\Omega\right) = \left\{ w \in u_0 + W^{1,p}_0\left(\Omega, \mathbb{R}\right) : w \geq \psi \text{ a.e. in } \Omega \right\}$$

is the class of the admissible functions, with $u_0 \in W^{1,p}\left(\Omega\right)$ a fixed boundary datum.

It is well known that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the obstacle problem (4.1) in $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality

$$\int_\Omega \langle A(x, Du(x)), D(\varphi(x) - u(x))\rangle\, dx \geq 0 \qquad \forall \varphi \in \mathcal{K}_\psi(\Omega), \tag{4.2}$$

where the operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is defined, similarly to how we did in (2.5), as

$$A_i(x, \xi) = D_{\xi_i} f(x, \xi) \qquad \forall i = 1, \ldots, n. \tag{4.3}$$

We assume that A is a p-harmonic type operator, i.e. it satisfies the following p-ellipticity and p-growth conditions with respect to the ξ-variable. There exist positive constants ℓ, ν, L and an exponent $1 < p < +\infty$ and a parameter $0 \leq \mu \leq 1$ such that

$$|A(x, \xi)| \leq \ell \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{4.4}$$

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta\rangle \geq \nu\, |\xi - \eta|^2 \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}, \tag{4.5}$$

$$|A(x, \xi) - A(x, \eta)| \leq L\, |\xi - \eta| \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}, \tag{4.6}$$

for all $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \Omega$.
Let us notice that, if the map $\xi \mapsto f(x, \xi)$ is of class $C^2(\mathbb{R}^n)$ for a.e. $x \in \Omega$, i.e. the map $\xi \mapsto A(x, \xi)$ is of class $C^1(\mathbb{R}^n)$, conditions (4.5) and (4.6), can be replaced, respectively, by

$$\langle D_\xi A(x, \xi)\eta, \eta\rangle \geq \tilde{\nu} \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2,$$

$$|D_\xi A(x, \xi)| \leq \tilde{L} \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}}, \tag{4.7}$$

for any $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

For what concerns the map $x \mapsto D_\xi f(x, \xi)$, in Section 4.2, Section 4.3 and Section 4.5, we shall assume that, for any $\xi \in \mathbb{R}^n$, it belongs to a Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$, and the results we present also depend on the value of the exponent q.
Let us recall that this condition is equivalent to assume that there exists a non-negative function $\kappa \in L^q_{\mathrm{loc}}(\Omega)$, such that

$$|A(x, \xi) - A(y, \xi)| \leq (\kappa(x) + \kappa(y))\, |x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{4.8}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$, which is also equivalent to say that there exists a non-negative function $k \in L^q_{\mathrm{loc}}(\Omega)$ such that

$$|D_x A(x, \xi)| \leq \tilde{\kappa}(x) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{4.9}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$.
As we will see in the following, (4.8) is useful when we use the difference quotient method.
In order to simplify the notations, if we define the function

$$g = \max\{\kappa, \tilde{\kappa}\} \qquad \text{a.e. in } \Omega,$$

we have $g \in L^q_{\mathrm{loc}}(\Omega)$ and, in place of (4.8), we can use the condition

$$|A(x, \xi) - A(y, \xi)| \leq (g(x) + g(y))\, |x - y| \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{4.10}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$.
Similarly, in place of (4.9), we can use

$$|D_x A(x,\xi)| \le g(x)\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{4.11}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$.

For what concerns the case of obstacle problems with Besov-Lipschitz coefficients, see Section 4.4 below.

4.1 Some preliminaries

Here we recall some preliminaries that will be useful for some of the results contained in this chapter, that haven't been mentioned previously.
The following result is proved in [17] for obstacle problems with (p,q)-growth conditions with $2 \le p \le q$, but the same proof works for any $1 < p \le q$, and that suits with our ellipticity and growth assumptions.

Theorem 4.1.1. *Let $u \in K_\psi(\Omega)$ be a solution of* (4.2) *under the assumptions* (4.4) *and* (4.5)*. If the obstacle $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and the following estimate*

$$\|u\|_{L^\infty\left(B_{\frac{R}{2}}\right)} \le \left[\|\psi\|_{L^\infty(B_R)} + \int_{B_R} |u(x)|^{p^*}\,dx\right]^\gamma \tag{4.12}$$

holds for every ball $B_R \Subset \Omega$, for $\gamma(n,p) > 0$ and $c = c(\ell,\nu,p,n)$.

The following result (see [47, 48]) helps us to avoid some difficulties that come out when we use (4.2) in the case of sub-quadratic growth conditions.

Theorem 4.1.2. *A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, with $1 < p < \infty$, is a solution to the problem* (4.1) *if and only if it is a weak solution of the following equation:*

$$\mathrm{div} A(x, Du) = -\mathrm{div} A(x, D\psi)\,\chi_{\{u=\psi\}}. \tag{4.13}$$

4.1.1 VMO coefficients

Here we recall the definition and some properties of VMO functions, since they come into play in the proofs of the results contained in Sections 4.3 and 4.4 below.
This is due to the fact that, if an operator A satisfies (4.4)–(4.6) and (4.10) for a non-negative function $g \in L^n_{\mathrm{loc}}(\Omega)$, or (4.72), or (4.74), then it is locally uniformly in VMO. More precisely, for any a ball $B \Subset \Omega$, let us introduce the operator

$$A_B = \fint_B A(x,\xi)\,dx.$$

One can easily check that $A_B(\xi)$ also satisfies (4.5), (4.6) and (4.4). Setting

$$V(x,B) = \sup_{\xi \ne 0} \frac{|A(x,\xi) - A_B(\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}},$$

we will say that $x \mapsto A(x,\xi)$ is locally uniformly in VMO if, for each compact set $K \subset \Omega$, we have

$$\lim_{R\to 0} \sup_{r<R} \sup_{x_0 \in K} \fint_{B_r(x_0)} V(x,B)\,dx = 0. \tag{4.14}$$

It is known that, if the operator A satisfies (4.4)–(4.6) and (4.10) for a non-negative function $g \in L^n_{\text{loc}}(\Omega)$, then A is locally uniformly in VMO (see [68]), and this will be useful in Section 4.3 to prove Theorem 4.3.1.
The following two results allow us to apply the properties of VMO functions in the proofs of Theorem 4.4.1 and Theorem 4.4.2 respectively, in Section 4.4.
Their proofs, for $p \geq 2$, can be found in [25] (see [25, Lemma 4.1] and [25, Lemma 3.1], respectively), but they works, exactly in the same way, also for $1 < p < 2$.

Lemma 4.1.3. *Let A be such that* (4.4)–(4.6) *and* (4.72) *hold. Then A is locally uniformly in VMO, that is* (4.14) *holds.*

Proof. Given a point $x \in \Omega$, let us denote

$$E_k(x) := \left\{ y \in \Omega : 2^{-k} \leq |x-y| < 2^{-k+1} \right\}.$$

We have

$$\begin{aligned}
\fint_B V(x,B)\,dx &= \fint_B \sup_{\xi \neq 0} \frac{|A(x,\xi) - A_B(\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}} dx \\
&\leq \fint_B \sup_{\xi \neq 0} \fint_B \frac{|A(x,\xi) - A(y,\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}} dydx \\
&= \fint_B \sup_{\xi \neq 0} \frac{1}{|B|} \sum_k \int_{B \cap E_k(x)} \frac{|A(x,\xi) - A(y,\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}} dydx \\
&\leq \frac{1}{|B|^2} \sum_k \int_B \int_{B \cap E_k(x)} |x-y|^\alpha \left(g_k(x) + g_k(y)\right) dydx.
\end{aligned}$$

By Hölder's inequality with exponents $\left(\frac{n}{\alpha}, \frac{n}{n-\alpha}\right)$, the last term of previous inequality can be bounded by

$$\begin{aligned}
&\left(\frac{1}{|B|^2} \sum_k \int_B \int_{B \cap E_k(x)} |x-y|^{\frac{n}{n-\alpha}}\, dydx\right)^{\frac{n-\alpha}{n}} \cdot \left(\frac{1}{|B|^2} \sum_k \int_B \int_{B \cap E_k(x)} \left(g_k(x) + g_k(y)\right)^{\frac{n}{\alpha}}\, dydx\right)^{\frac{\alpha}{n}} \\
= \;& I \cdot II.
\end{aligned}$$

We have

$$I \leq C(n,\alpha)\,|B|^{\frac{\alpha}{n}}.$$

for what concerns the term II, we have

$$\begin{aligned}
II &\leq c \left(\frac{1}{|B|^2} \sum_k |B \cap E_k| \int_B g_k^{\frac{n}{\alpha}}(x)dx\right) \\
&\leq \left(\frac{1}{|B|^2} \sum_k |B \cap E_k|^{\frac{\alpha q}{\alpha q - n}}\right)^{\frac{\alpha q - n}{\alpha q} \cdot \frac{\alpha}{n}} \cdot \left(\frac{1}{|B|^2} \sum_k \left(\int_B g_k^{\frac{n}{\alpha}}(x)dx\right)^{\frac{\alpha q}{n}}\right)^{\frac{n}{\alpha q} \cdot \frac{\alpha}{n}} \\
&= \frac{1}{|B|^{2\left(\frac{\alpha}{n} - \frac{1}{q}\right)}} \left(\sum_k |B \cap E_k|^{\frac{\alpha q}{\alpha q - n}}\right)^{\frac{\alpha q - n}{\alpha q} \cdot \frac{\alpha}{n}} \cdot \frac{1}{|B|^{\frac{2}{q}}} \left(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\right)^{\frac{1}{q}} \\
&\leq \frac{C(n,\alpha,q)\,|B|^{\frac{\alpha}{n}}}{|B|^{2\left(\frac{\alpha}{n} - \frac{1}{q}\right)}} \cdot \frac{1}{|B|^{\frac{2}{q}}} \left(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \quad C(n,\alpha,q)\,|B|^{-\frac{\alpha}{n}}\left(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\right)^{\frac{1}{q}},$$

so we have

$$\fint_B V(x,B)\,dx \le C(n,\alpha,q)\left(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\right)^{\frac{1}{q}}.$$

In order to conclude, we have to prove that

$$\lim_{r\to 0}\sup_{x\in K}\left(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\right)^{\frac{1}{q}} = 0,$$

on every compact set $K \subset \Omega$. To this end, we can fix $r > 0$ small enough, and observe that the function $x \mapsto \|g_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B(x,r))\right)}$ is continuous on the set $\{\, x \in \Omega : d(x,\Omega) > r \,\}$, as a uniformly converging series of continuous functions. As a consequence, there exists a point $x_r \in K$ (at least for small enough $r > 0$) such that

$$\sup_{x\in K}\|g_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B(x,r))\right)} = \|g_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B(x_r,r))\right)}.$$

Now, from $\|g_k\|_{L^{\frac{n}{\alpha}}(B(x,r))} \le \|g_k\|_{L^{\frac{n}{\alpha}}(B(x_r,r))}$, and since this belongs to ℓ^q, we can use dominated convergence to get

$$\lim_{r\to 0}\|g_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B(x_r,r))\right)} = \left(\sum_k \lim_{r\to 0}\left(\int_{B(x_r,r)} g_k^{\frac{n}{\alpha}}(x)dx\right)^{\frac{\alpha q}{n}}\right)^{\frac{1}{q}}.$$

Each of the limits on the term on the right-hand side are equal to 0, since the points x_r cannot escape from the compact set K as $r \to 0$. This completes the proof. □

Lemma 4.1.4. *Let A be such that* (4.4)–(4.6) *and* (4.74) *hold. Then A is locally uniformly in VMO, that is* (4.14) *holds.*

Proof. Let us assume that (4.5), (4.6), (4.4) and (4.74) hold.
Using Hölder's inequality with exponent $\left(\frac{n}{\alpha},\frac{n}{n-\alpha}\right)$, we have

$$\begin{aligned}
\fint_B V(x,B)\,dx &= \fint_B \sup_{\xi\neq 0}\frac{|A(x,\xi)-A_B(\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}dx\\
&\le \fint_B \sup_{\xi\neq 0}\int_B \frac{|A(x,\xi)-A(y,\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}dydx\\
&\le \fint_B \sup_{\xi\neq 0}\fint_B (g(x)+g(y))\,|x-y|^{\alpha}\,dydx\\
&= \fint_B\fint_B (g(x)+g(y))\,|x-y|^{\alpha}\,dydx\\
&\le \left(\fint_B\fint_B (g(x)+g(y))^{\frac{n}{\alpha}}\,dxdy\right)^{\frac{\alpha}{n}}\cdot\left(\fint_B\fint_B |x-y|^{\frac{n\alpha}{n-\alpha}}\,dxdy\right)^{\frac{n-\alpha}{n}}\\
&\le C(\alpha,n)\,|B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B g^{\frac{n}{\alpha}}(x)dx\right),
\end{aligned}$$

thus (4.14) holds. □

The following result is a Calderón-Zygmund type estimate for solutions to obstacle problems with VMO coefficients, and its proof can be found in [11] (in the case $p = p(x)$).

Theorem 4.1.5. *Let $p > 1$, and $q > p$. Assume that (4.4)–(4.6) hold, and that $x \mapsto A(x,\xi)$ is locally uniformly in VMO. Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.1). Then the following implication holds*

$$D\psi \in L^q_{\mathrm{loc}}(\Omega) \implies Du \in L^q_{\mathrm{loc}}(\Omega).$$

Moreover, there exists a constant $C = C(n,\nu,\ell,L,p,q)$ such that the following inequality

$$\fint_{B_R} |Du(x)|^q\, dx \le C\left\{1 + \fint_{B_{2R}} |D\psi(x)|^q\, dx + \left(\fint_{B_{2R}} |Du(x)|^p\, dx\right)^{\frac{q}{p}}\right\} \tag{4.15}$$

holds for any ball B_R such that $B_{2R} \Subset \Omega$.

4.2 A first regularity result for solutions to some obstacle problems

Here we present the result contained in [18].
We are interested in the study of the regularity of the gradient of the solutions to variational obstacle problems of the form (4.1), where $\Omega \subset \mathbb{R}^n$, with $n \ge 2$, is a bounded open set, and the ostacle $\psi : \Omega \to [-\infty,+\infty)$ belongs to the Sobolev class $W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$, and the map $\xi \mapsto f(x,\xi)$ is of class $C^1(\mathbb{R}^n)$ for almost every $x \in \Omega$

With the notation (4.3), we will use assumptions (4.4)–(4.6), and the fact that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to (4.1) if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality (4.2).

For what concerns the dependence of the map A on the x-variable, we assume that it belongs to the Sobolev space $W^{1,p+2}$.
More precisely, recalling the characterization of Sobolev functions proved in [66], we assume that there exists a non-negative function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$ such that (4.10) holds.
Our aim is to establish a higher differentiability result assuming that $D\psi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$.
More precisely, we shall prove the following.

Theorem 4.2.1. *Let $A(x,\xi)$ satisfy the conditions (4.4)–(4.6) and (4.10) with $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$, for an exponent $p \ge 2$, and let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem (4.1). Then, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, the following implication holds*

$$D\psi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega) \implies V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

with the following estimate

$$\begin{aligned} &\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2\, dx \\ \le\ & \frac{c\left(\|\psi\|^2_{L^\infty(B_R)} + \|u\|^2_{L^{p^*}(B_R)}\right)}{R^{\frac{p+2}{2}}} \\ &\cdot \int_{B_R}\left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{2}} + |D\psi(x)|^{\frac{p+2}{2}} + g^{p+2}(x) + |Du(x)|^p\right] dx, \end{aligned} \tag{4.16}$$

for any ball $B_R \Subset \Omega$.

Note that, in the case $p < n-2$, Theorem 4.2.1 improves the results in [42] and [43]. The proof of Theorem 4.2.1 is achieved combining an a priori estimate for the second derivative of the local solutions, obtained using the difference quotient method, with a suitable approximation argument. The local boundedness allows us to use the inequality (1.2), which gives the higher local integrability L^{p+2} of the gradient of the solutions. Such higher integrability is the key tool in order to weaken the assumption on g that in previous results has been assumed at least in L^n.
Moreover, our result is obtained under a weaker assumption also on the gradient of the obstacle. Indeed, previous results assumed $D\psi \in W^{1,p}$ while our assumption is $D\psi \in W^{1,\frac{p+2}{2}}$ with $p > 2$.
Finally, we observe that the assumption of boundedness of the obstacle ψ is needed to get the boundedness of the solution (see Theorem 4.1.1). Therefore if we deal with a priori bounded minimizers, then the result holds without the hypothesis $\psi \in L^\infty$.

4.2.1 Proof of Theorem 4.2.1

The proof of Theorem 4.2.1 will be divided in two steps: in the first one, we will establish the a priori estimate, while in the second one we will conclude through an approximation argument.

Proof of the Theorem 4.2.1. **Step 1: the a priori estimate.** Suppose that u is a local solution to the obstacle problem in $\mathcal{K}_\psi(\Omega)$ such that

$$Du \in W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

By estimate (4.12) and Lemma 1.1.3, we also have $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$. Note that the fact that $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$ implies that the variational inequality (4.2), by a standard density argument, holds true for every $\varphi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$.
In order to choose suitable test functions φ in (4.2) that involve the different quotient of the solution and at the same time belong to the class of the admissible functions $\mathcal{K}_\psi(\Omega)$, we proceed as done in [42].

Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} \le r < s < t < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C^\infty_0(B_t)$ such that $\eta \equiv 1$ on B_s and $|D\eta| \le \frac{c}{t-s}$. From now on, with no loss of generality, we suppose $R < 1$.
Let $v \in W^{1,p}_0(\Omega)$ be such that

$$u - \psi + \tau v \ge 0 \qquad \forall \tau \in [0,1], \tag{4.17}$$

and observe that $\varphi := u + \tau v \in \mathcal{K}_\psi(\Omega)$ for all $\tau \in [0,1]$, since $\varphi = u + \tau v \ge \psi$. For $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right],$$

so we have $v_1 \in W^{1,\frac{p+2}{2}}_0(\Omega)$, and, for any $\tau \in [0,1]$, v_1 satisfies (4.17). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0,1]$

$$\begin{aligned} u(x) - \psi(x) + \tau v_1(x) &= u(x) - \psi(x) + \tau\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right] \\ &= \tau\eta^2(x)(u-\psi)(x+h) + \left(1 - \tau\eta^2(x)\right)(u-\psi)(x) \ge 0, \end{aligned}$$

since $u \in \mathcal{K}_\psi(\Omega)$ and $0 \le \eta \le 1$.
Hence we can use $\varphi = u + \tau v_1$ as a test function in inequality (4.2), thus getting

$$0 \leq \int_\Omega \left\langle A\left(x, Du(x)\right), D\left[\eta^2(x)\left[\left(u-\psi\right)(x+h) - \left(u-\psi\right)(x)\right]\right]\right\rangle dx. \tag{4.18}$$

In a similar way, we define

$$v_2(x) = \eta^2(x-h)\left[\left(u-\psi\right)(x-h) - \left(u-\psi\right)(x)\right],$$

and we have $v_2 \in W_0^{1,\frac{p+2}{2}}(\Omega)$, and (4.17) still is satisfied for any $\tau \in [0,1]$, since

$$\begin{aligned} u(x) - \psi(x) + \tau v_2(x) &= u(x) - \psi(x) + \tau\eta^2(x-h)\left[\left(u-\psi\right)(x-h) - \left(u-\psi\right)(x)\right] \\ &= \tau\eta^2(x)\left(u-\psi\right)(x-h) + \left(1 - \tau\eta^2(x-h)\right)\left(u-\psi\right)(x) \geq 0. \end{aligned}$$

By using in (4.2) as test function $\varphi = u + \tau v_2$, we get

$$0 \leq \int_\Omega \left\langle A\left(x, Du(x)\right), D\left[\eta^2(x-h)\left[\left(u-\psi\right)(x-h) - \left(u-\psi\right)(x)\right]\right]\right\rangle dx,$$

and by means of a change of variable, we obtain

$$0 \leq \int_\Omega \left\langle A\left(x+h, Du(x+h)\right), D\left[\eta^2(x)\left[\left(u-\psi\right)(x) - \left(u-\psi\right)(x+h)\right]\right]\right\rangle dx. \tag{4.19}$$

Now we can add (4.18) and (4.19), thus getting

$$\begin{aligned} 0 \leq & \int_\Omega \left\langle A\left(x, Du(x)\right), D\left[\eta^2\left[\left(u-\psi\right)(x+h) - \left(u-\psi\right)(x)\right]\right]\right\rangle dx \\ & + \int_\Omega \left\langle A\left(x+h, Du(x+h)\right), D\left[\eta^2\left[\left(u-\psi\right)(x) - \left(u-\psi\right)(x+h)\right]\right]\right\rangle dx, \end{aligned}$$

that is

$$0 \leq \int_\Omega \left\langle A\left(x, Du(x)\right) - A\left(x+h, Du(x+h)\right), D\left[\eta^2\left[\left(u-\psi\right)(x+h) - \left(u-\psi\right)(x)\right]\right]\right\rangle dx,$$

which implies

$$\begin{aligned} 0 \geq & \int_\Omega \left\langle A\left(x+h, Du(x+h)\right) - A\left(x, Du(x)\right), \eta^2 D\left[\left(u-\psi\right)(x+h) - \left(u-\psi\right)(x)\right]\right\rangle dx \\ & + \int_\Omega \left\langle A\left(x+h, Du(x+h)\right) - A\left(x, Du(x)\right), 2\eta D\eta\left[\left(u-\psi\right)(x+h) - \left(u-\psi\right)(x)\right]\right\rangle dx. \end{aligned}$$

Previous inequality can be rewritten as follows

$$\begin{aligned} 0 \geq & \int_\Omega \left\langle A\left(x+h, Du(x+h)\right) - A\left(x+h, Du(x)\right), \eta^2\left(Du(x+h) - Du(x)\right)\right\rangle dx \\ & - \int_\Omega \left\langle A\left(x+h, Du(x+h)\right) - A\left(x+h, Du(x)\right), \eta^2\left(D\psi(x+h) - D\psi(x)\right)\right\rangle dx \\ & + \int_\Omega \left\langle A\left(x+h, Du(x+h)\right) - A\left(x+h, Du(x)\right), 2\eta D\eta\tau_h\left(u-\psi\right)(x)\right\rangle dx \\ & + \int_\Omega \left\langle A\left(x+h, Du(x)\right) - A\left(x, Du(x)\right), \eta^2\left(Du(x+h) - Du(x)\right)\right\rangle dx \end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2 \left(D\psi(x+h) - D\psi(x)\right)\right\rangle dx\\
&+\int_\Omega \left\langle A(x+h, Du(x)) - A(x, Du(x)), 2\eta D\eta \tau_h (u-\psi)(x)\right\rangle dx\\
=:\ & I + II + III + IV + V + VI,
\end{aligned}
$$

so we have

$$
I \leq |II| + |III| + |IV| + |V| + |VI|. \tag{4.20}
$$

By the ellipticity assumption (4.5), we get

$$
I \geq \nu \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx. \tag{4.21}
$$

By virtue of assumption (4.6), using Young's inequality with exponents $(2,2)$, and then Hölder's inequality with exponents $\left(\frac{p+2}{4}, \frac{p+2}{p-2}\right)$, by the properties of η, we infer

$$
\begin{aligned}
|II| \leq\ & L \int_\Omega \eta^2 |\tau_h Du(x)| \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} |\tau_h D\psi(x)|\, dx\\
\leq\ & \varepsilon \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx\\
&+ c_\varepsilon \int_\Omega \eta^2 |\tau_h D\psi(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx\\
\leq\ & \varepsilon \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx\\
&+ c_\varepsilon \left(\int_{B_t} |\tau_h D\psi(x)|^{\frac{p+2}{2}} dx\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_{\lambda r}} \left(\mu^{p+2} + |Du(x+h)|^{p+2}\right) dx\right)^{\frac{p-2}{p+2}}\\
\leq\ & \varepsilon \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx\\
&+ c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} \left|D^2\psi(x)\right|^{\frac{p+2}{2}} dx\right)^{\frac{4}{p+2}}\\
&\cdot \left(\int_{B_{\lambda r}} \left(\mu^{p+2} + |Du(x+h)|^{p+2}\right) dx\right)^{\frac{p-2}{p+2}},
\end{aligned} \tag{4.22}
$$

where we used Lemma 1.2.3.

Similarly, by Young's and Hölder's inequalities, by virtue of the properties of η, and Lemma 1.2.3, we can estimate the term III as follows

$$
\begin{aligned}
|III| \leq\ & 2L \int_\Omega \eta |D\eta| |\tau_h Du(x)| \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} |\tau_h (u-\psi)|\, dx\\
\leq\ & \varepsilon \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx\\
&+ \frac{c_\varepsilon(L)}{(t-s)^2} \int_{B_t \setminus B_s} \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} |\tau_h (u-\psi)(x)|^2 dx\\
\leq\ & \varepsilon \int_\Omega \eta^2 |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-2}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}dx\right)^{\frac{4}{p+2}}.
\end{aligned}\tag{4.23}
$$

In order to estimate the term IV, we use assumption (4.10), Young's inequality with exponents $(2,2)$ and the properties of η, thus getting

$$
\begin{aligned}
|IV| &\leq |h|\int_\Omega\eta^2\left(g(x+h)+g(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h Du(x)|\,dx\\
&\leq \varepsilon\int_\Omega\eta^2|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}dx\\
&\quad+c_\varepsilon|h|^2\int_{B_t}\left(g(x+h)+g(x)\right)^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx,
\end{aligned}
$$

and using Hölder's inequality with exponents $\left(\frac{p+2}{2},\frac{p+2}{p}\right)$, and the properties of η, we have

$$
\begin{aligned}
|IV| &\leq \varepsilon\int_\Omega\eta^2|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}dx\\
&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}g^{p+2}(x)dx\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p}{p+2}}.
\end{aligned}\tag{4.24}
$$

In order to estimate the term V, we use condition (4.10) again, then Hölder's inequality with exponents $\left(p+2,\frac{p+2}{p-1},\frac{p+2}{2}\right)$, the properties of η, and the properties of difference quotients of Sobolev functions, so we get

$$
\begin{aligned}
|V| &\leq |h|\int_\Omega\eta^2\left(g(x+h)+g(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h D\psi(x)|\,dx\\
&\leq |h|\left(\int_{B_t}\left(g(x+h)+g(x)\right)^{p+2}dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}|\tau_h D\psi(x)|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}}\\
&\leq |h|^2\left(\int_{B_{\lambda r}}g^{p+2}(x)dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}},
\end{aligned}\tag{4.25}
$$

where we used the assumption $D\psi\in W^{1,\frac{p+2}{2}}$ and first estimate of Lemma 1.2.3.
For what concerns the term VI, using the condition (4.10), the properties of η, Hölder's inequality with exponents $\left(p+2,\frac{p+2}{p-1},\frac{p+2}{2}\right)$, and the properties of difference quotients of Sobolev functions, we have

$$
|VI| \leq 2|h|\int_\Omega\eta|D\eta|\left(g(x+h)+g(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h(u-\psi)(x)|\,dx
$$

$$
\begin{aligned}
&\leq \frac{c|h|}{t-s}\left(\int_{B_t}(g(x+h)+g(x))^{p+2}\,dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}\left|\tau_h\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}}\\
&\leq \frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}g(x)^{p+2}dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}}. \qquad (4.26)
\end{aligned}
$$

Plugging (4.21), (4.22), (4.23), (4.24), (4.25) and (4.26) into (4.20), choosing $\varepsilon=\frac{\nu}{6}$, and reabsorbing the terms with the same integral of the right-hand side to the left-hand, we get

$$
\begin{aligned}
&\nu\int_\Omega \eta^2\left|\tau_h Du(x)\right|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}dx\\
\leq\ & c|h|^2\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-2}{p+2}}\\
&+\frac{c|h|^2}{(t-s)^2}\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-2}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{4}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}g^{p+2}(x)dx\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}g^{p+2}(x)dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}g(x)^{p+2}dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}dx\right)^{\frac{2}{p+2}}. \qquad (4.27)
\end{aligned}
$$

Now we apply Young's inequality with exponents $\left(\frac{p+2}{4},\frac{p+2}{p-2}\right)$ to the first two terms of the right-hand side of (4.27), Young's inequality with exponents $\left(\frac{p+2}{2},\frac{p+2}{p}\right)$ to the third one, and $\left(p+2,\frac{p+2}{p-1},\frac{p+2}{2}\right)$ to the last to terms, and since $u\in\mathcal{K}_\psi\left(\Omega\right)$, we have

$$
\begin{aligned}
&\nu\int_\Omega \eta^2\left|\tau_h Du(x)\right|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}dx\\
\leq\ & \varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx\\
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\cdot\int_{B_{\lambda r}}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx\\
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}g^{p+2}(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx\\
&+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}g^{p+2}(x)dx.
\end{aligned}
$$

Recalling the right-hand side of the inequality (1.5) in Lemma 1.4.3, we get

$$
\begin{aligned}
&\nu\int_\Omega\eta^2\left|\tau_h V_p\left(Du(x)\right)\right|^2dx\\
\le\ &\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\,dx\\
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\cdot\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx\\
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}g^{p+2}(x)dx\\
&+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx\\
&+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}g^{p+2}(x)dx.
\end{aligned}
$$

Now we divide both sides by $|h|^2$ and use the Lemma 1.2.4 so, passing to the limit as $h\to 0$, we get

$$
\begin{aligned}
&\int_\Omega\eta^2\left|DV_p\left(Du(x)\right)\right|^2dx\\
\le\ &4\varepsilon\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\,dx\\
&+c_\varepsilon\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx\\
&+c_\varepsilon\int_{B_{\lambda r}}g^{p+2}(x)dx+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}g^{p+2}(x)dx,
\end{aligned}
$$

which, thanks to the left-hand side of inequality (1.6) gives

$$
\begin{aligned}
&\int_\Omega\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\left|D^2u(x)\right|^2dx\\
\le\ &\int_\Omega\eta^2\left|DV_p\left(Du(x)\right)\right|^2dx\\
\le\ &4\varepsilon\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx+c_\varepsilon\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\,dx\\
&+c_\varepsilon\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,dx\\
&+c_\varepsilon\int_{B_{\lambda r}}g^{p+2}(x)dx+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}g^{p+2}(x)dx.
\end{aligned}
\tag{4.28}
$$

By virtue of Remark 1.4.7, using inequality (1.2) we have

$$\begin{aligned}
&\int_{\Omega}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\,dx\\
\leq\ & c\,\|u\|^2_{L^\infty(\operatorname{supp}(\eta))}\int_{\Omega}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\left|D^2u(x)\right|^2dx\\
&+c\,\|u\|^2_{L^\infty(\operatorname{supp}(\eta))}\int_{\Omega}\left(|\eta|^2+|D\eta|^2\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx.
\end{aligned}$$

Hence, thanks to estimate (4.28), and the properties of η we infer

$$\begin{aligned}
&\int_{\Omega}\eta^2\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\,dx\\
\leq\ & \varepsilon c\,\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\\
&+c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx+c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}dx\\
&+\frac{c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}dx+c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}g^{p+2}(x)dx\\
&+\frac{c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^2}\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\
&+\frac{c_\varepsilon\,\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}g^{p+2}(x)dx.
\end{aligned}$$

Taking into account the properties of η again, since $p\geq 2$ and $t-s<1$, we obtain

$$\begin{aligned}
&\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\,dx\\
\leq\ & \varepsilon c\,\|u\|^2_{L^\infty(B_R)}\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\\
&+c_\varepsilon\,\|u\|^2_{L^\infty(B_R)}\int_{B_R}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx+c_\varepsilon\,\|u\|^2_{L^\infty(B_R)}\int_{B_R}|D\psi(x)|^{\frac{p+2}{2}}dx\\
&+\frac{c_\varepsilon\,\|u\|^2_{L^\infty(B_R)}}{(t-s)^{\frac{p+2}{2}}}\left[\int_{B_R}|D\psi(x)|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx+\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]\\
&+c_\varepsilon\,\|u\|^2_{L^\infty(B_R)}\int_{B_R}g^{p+2}(x)dx,
\end{aligned}$$

and choosing ε such that $\varepsilon\cdot c\,\|u\|^2_{L^\infty(B_R)}\leq\frac{1}{2}$, previous estimate becomes

$$\begin{aligned}
&\int_{B_r}|Du(x)|^{p+2}\,dx\\
\leq\ & \int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\,dx\\
\leq\ & \frac{1}{2}\int_{B_{\lambda r}}|Du(x)|^{p+2}\,dx\\
&+c\,\|u\|^2_{L^\infty(B_R)}\left[\int_{B_R}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}|D\psi(x)|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx\right]
\end{aligned}$$

$$+\frac{c\left\|u\right\|^2_{L^\infty(B_R)}}{(t-s)^{\frac{p+2}{2}}}\left[\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx+\int_{B_R}\left|Du(x)\right|^p dx+c(\mu,p)\left|B_R\right|\right]$$
$$+c(\mu,p)\left|B_R\right|,$$

where $c=c(n,p,\ell,\nu,L,\mu)$ is independent of t and s. Since (4.29) is valid for any $\frac{R}{2}\le r<s<t<\lambda r<R<1$, and the constant $c>0$ is independent on the radii, taking the limit as $s\to r$ and $t\to\lambda r$, we get

$$\begin{aligned}
&\int_{B_r}\left|Du(x)\right|^{p+2}dx\\
\le\ &\frac{1}{2}\int_{B_{\lambda r}}\left|Du(x)\right|^{p+2}dx+c\left\|u\right\|^2_{L^\infty(B_R)}\left[\int_{B_R}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx\right.\\
&\left.+\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx\right]\\
&+\frac{c\left\|u\right\|^2_{L^\infty(B_R)}}{r^{\frac{p+2}{2}}(\lambda-1)^{\frac{p+2}{2}}}\left[\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx\right.\\
&\left.+\int_{B_R}g^{p+2}(x)dx+\int_{B_R}\left|Du(x)\right|^p dx+c(\mu,p)\left|B_R\right|\right]\\
&+c(\mu,p)\left|B_R\right|.
\end{aligned}$$

Now, setting

$$h(r)=\int_{B_r}\left|Du(x)\right|^{p+2}dx,$$

$$A=c\left\|u\right\|^2_{L^\infty(B_R)}\left[\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx+\int_{B_R}\left|Du(x)\right|^p dx+c(\mu,p)\left|B_R\right|\right],$$

and

$$B=c\left\|u\right\|^2_{L^\infty(B_R)}\left[\int_{B_R}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx\right]+c(\mu,p)|B_R|,$$

we can use Iteration Lemma, with

$$\theta=\frac{1}{2}\qquad\text{and}\qquad\gamma=\frac{p+2}{2},$$

thus obtaining

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}}\left|Du(x)\right|^{p+2}dx\\
\le\ &c\left\|u\right\|^2_{L^\infty(B_R)}\left[\int_{B_R}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx\right]\\
&+\frac{c\left\|u\right\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{2}}}\left[\int_{B_R}\left|D\psi(x)\right|^{\frac{p+2}{2}}dx+\int_{B_R}g^{p+2}(x)dx+\int_{B_R}\left|Du(x)\right|^p dx+c(\mu,p)\left|B_R\right|\right]\\
&+c(\mu,p)\left|B_R\right|.
\end{aligned}\tag{4.29}$$

Since $R<1$, estimate (4.29) can be written as follows

$$\begin{aligned}&\int_{B_{\frac{R}{2}}}|Du(x)|^{p+2}\,dx\\ \leq\;& \frac{c\,\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{2}}}\cdot\int_{B_R}\Bigg[1+\left|D^2\psi(x)\right|^{\frac{p+2}{2}}+|D\psi(x)|^{\frac{p+2}{2}}\\ &+g^{p+2}(x)+|Du(x)|^p\Bigg]dx. \qquad (4.30)\end{aligned}$$

Now, we consider the estimate in (4.28) choosing a cut off function $\eta\in C_0^\infty\left(B_{\frac{R}{2}}\right)$ such that $\eta\equiv 1$ on $B_{\frac{R}{4}}$; so that thanks to (4.30), we obtain

$$\begin{aligned}&\int_{B_{\frac{R}{4}}}\left|DV_p\left(Du(x)\right)\right|^2dx\leq\frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{2}}}\\ &\cdot\int_{B_R}\left[1+\left|D^2\psi(x)\right|^{\frac{p+2}{2}}+|D\psi(x)|^{\frac{p+2}{2}}+g^{p+2}(x)+|Du(x)|^p\right]dx.\end{aligned}$$

By virtue of estimate (4.12), we conclude with

$$\begin{aligned}&\int_{B_{\frac{R}{4}}}\left|DV_p\left(Du(x)\right)\right|^2dx\\ \leq\;&\frac{c\left(\|\psi\|^2_{L^\infty(B_R)}+\|u\|^2_{L^{p^*}(B_R)}\right)}{R^{\frac{p+2}{2}}}\\ &\cdot\int_{B_R}\left[1+\left|D^2\psi(x)\right|^{\frac{p+2}{2}}+|D\psi(x)|^{\frac{p+2}{2}}+g^{p+2}(x)+|Du(x)|^p\right]dx. \qquad (4.31)\end{aligned}$$

Step 2: the approximation.
Fix an open set $\Omega'\Subset\Omega$, and for a smooth kernel $\phi\in C_0^\infty\left(B_1(0)\right)$ with $\phi\geq 0$ and $\int_{B_1(0)}\phi=1$, and for any $\varepsilon\in(0,d\left(\Omega',\partial\Omega\right))$, let us consider the corresponding family of mollifiers $\{\,\phi_\varepsilon\,\}_\varepsilon$. Let us set

$$g_\varepsilon=g*\phi_\varepsilon,\qquad\qquad \psi_\varepsilon=\psi*\phi_\varepsilon,$$

$$\mathcal{K}_{\psi_\varepsilon}\left(\Omega\right)=\left\{\,w\in u+W_0^{1,p}\left(\Omega\right):w\geq\psi_\varepsilon\text{ a.e. in }\Omega\,\right\},$$

and

$$A_\varepsilon(x,\xi)=\int_{B_1}\phi(\omega)A\left(x+\varepsilon\omega,\xi\right)d\omega \qquad (4.32)$$

on Ω', for each $\varepsilon\in(0,d\left(\Omega',\partial\Omega\right))$.
Assumptions (4.4)–(4.6) imply

$$\left|A_\varepsilon\left(x,\xi\right)\right|\leq\ell\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}, \qquad (4.33)$$

$$\left\langle A_\varepsilon\left(x,\xi\right)-A_\varepsilon\left(x,\eta\right),\xi-\eta\right\rangle\geq\nu|\eta-\xi|^2\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}, \qquad (4.34)$$

$$|A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta)|\le L|\xi-\eta|\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}. \tag{4.35}$$

By virtue of assumption (4.10), we have

$$|A_\varepsilon(x,\xi)-A_\varepsilon(y,\xi)|\le (g_\varepsilon(x)+g_\varepsilon(y))|x-y|\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}} \tag{4.36}$$

for almost every $x,y\in\Omega$ and for all $\xi,\eta\in\mathbb{R}^n$.
Let u be a solution of the variational inequality (4.2) and let fix a ball $B_{\tilde R}\Subset\Omega'$. Let us denote by $u_\varepsilon\in u+W_0^{1,p}\left(B_{\tilde R}\right)$ the solution to the inequality

$$\int_\Omega\langle A_\varepsilon(x,Dw(x)),D(\varphi-w)(x)\rangle\,dx\ge 0\qquad\forall\varphi\in\mathcal{K}_{\psi_\varepsilon}(\Omega). \tag{4.37}$$

Thanks to [42, Theorem 1.1] we have $V_p(Du_\varepsilon)\in W^{1,2}_{\mathrm{loc}}\left(B_{\tilde R}\right)$ and, since A_ε satisfies conditions (4.33)–(4.36), for a sufficiently small ε, we are legitimated to apply estimate (4.31) to get

$$\begin{aligned}&\int_{B_{\frac{r}{4}}}|DV_p(Du_\varepsilon(x))|^2\,dx\\ \le\ &\frac{c\left(\|\psi_\varepsilon\|^2_{L^\infty(B_r)}+\|u_\varepsilon\|^2_{L^{p^*}(B_r)}\right)}{r^{\frac{p+2}{2}}}\\ &\cdot\int_{B_r}\left[1+\left|D^2\psi_\varepsilon(x)\right|^{\frac{p+2}{2}}+|D\psi_\varepsilon(x)|^{\frac{p+2}{2}}+g_\varepsilon^{p+2}(x)+|Du_\varepsilon(x)|^p\right]dx.\end{aligned} \tag{4.38}$$

for every ball $B_r\Subset B_{\tilde R}$ and for a constant c.
We recall that, since $D\psi\in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$ and $\psi\in L^\infty_{\mathrm{loc}}(\Omega)$, we have

$$\psi_\varepsilon\to\psi\qquad\text{strongly in } L^\infty\left(B_{\tilde R}\right), \tag{4.39}$$

and

$$D^2\psi_\varepsilon\to D^2\psi\qquad\text{strongly in } L^{\frac{p+2}{2}}\left(B_{\tilde R}\right), \tag{4.40}$$

as $\varepsilon\to 0$.
Moreover, applying Lemma 1.1.2, we get $D\psi\in L^{p+2}_{\mathrm{loc}}(\Omega)$, which implies

$$D\psi_\varepsilon\to D\psi\qquad\text{strongly in } L^{p+2}\left(B_{\tilde R}\right)\text{ as }\varepsilon\to 0. \tag{4.41}$$

Since $g\in L^{p+2}_{\mathrm{loc}}(\Omega)$, we have

$$g_\varepsilon\to g\qquad\text{strongly in } L^{p+2}\left(B_{\tilde R}\right),\text{ as }\varepsilon\to 0. \tag{4.42}$$

From (4.33), we have $|A_\varepsilon(x,Du)|\le\ell\left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}}$, and since $A_\varepsilon(x,Du)$ converges to $A(x,Du)$ almost everywhere in $B_{\tilde R}$, by the dominated convergence theorem we have

$$A_\varepsilon(x,Du)\to A(x,Du)\qquad\text{strongly in } L^{\frac{p}{p-1}}\left(B_{\tilde R}\right),\text{ as }\varepsilon\to 0. \tag{4.43}$$

If we consider a cut off function $\eta\in C_0^\infty\left(B_{\tilde R}\right)$ such that $0\le\eta\le 1$, $\eta\equiv 1$ on $B_{\frac{\tilde R}{2}}$ and $|D\eta|\le\frac{c}{\tilde R}$, and choose $\varphi=u_\varepsilon+\eta(\psi-\psi_\varepsilon)$ and $\varphi=u+\eta(\psi_\varepsilon-\psi)$ as test functions in (4.2) and (4.37) respectively, we have

$$\int_{B_{\tilde R}}\langle A(x,Du(x)),(Du_\varepsilon-Du)(x)+D[\eta(\psi-\psi_\varepsilon)(x)]\rangle\,dx\ge 0, \tag{4.44}$$

and

$$\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)), (Du_\varepsilon - Du)(x) + D[\eta(\psi - \psi_\varepsilon)(x)]\rangle \, dx \le 0. \tag{4.45}$$

Using the ellipticity condition (4.34) and (4.44), we have

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2 \, dx \\
\le &\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)) - A_\varepsilon (x, Du(x)), (Du_\varepsilon - Du)(x)\rangle \, dx \\
= &\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)), (Du_\varepsilon - Du)(x)\rangle \, dx \\
&+ \int_{B_{\tilde{R}}} \langle A (x, Du(x)) - A_\varepsilon (x, Du(x)), (Du_\varepsilon - Du)(x)\rangle \, dx \\
&+ \int_{B_{\tilde{R}}} \langle A (x, Du(x)), D[\eta(\psi - \psi_\varepsilon)(x)]\rangle \, dx \\
&- \int_{B_{\tilde{R}}} \langle A (x, Du(x)), (Du_\varepsilon - Du)(x) + D[\eta(\psi - \psi_\varepsilon)(x)]\rangle \, dx,
\end{aligned}$$

and by (4.45), we get

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2 \, dx \\
\le &\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)), D[\eta(\psi_\varepsilon - \psi)(x)]\rangle \, dx \\
&+ \int_{B_{\tilde{R}}} \langle A (x, Du(x)) - A_\varepsilon (x, Du(x)), (Du_\varepsilon - Du)(x)\rangle \, dx \\
&+ \int_{B_{\tilde{R}}} \langle A (x, Du(x)), D[\eta(\psi - \psi_\varepsilon)(x)]\rangle \, dx \\
&- \int_{B_{\tilde{R}}} \langle A (x, Du(x)), (Du_\varepsilon - Du)(x) + D[\eta(\psi - \psi_\varepsilon)(x)]\rangle \, dx \\
= &\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)) - A(x, Du(x)), D[\eta(\psi_\varepsilon - \psi)(x)]\rangle \, dx \\
&+ \int_{B_{\tilde{R}}} \langle A (x, Du(x)) - A_\varepsilon (x, Du(x)), (Du_\varepsilon - Du)(x)\rangle \, dx
\end{aligned} \tag{4.46}$$

Therefore from the inequality (4.46), using Hölder's inequality with exponents $\left(p, \frac{p}{p-1}\right)$ we deduce

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2 \, dx \\
\le &\left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |(Du_\varepsilon - Du)(x)|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{B_{\tilde{R}}} |A(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{B_{\tilde{R}}} |A_\varepsilon(x, Du_\varepsilon(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.47}$$

Since $p \ge 2$, we have

$$\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p \, dx$$

$$\leq \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2 \, dx,$$

and combining previous inequality with (4.47), we deduce

$$\begin{aligned}
&\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p \, dx \\
\leq\ & \left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |(Du_\varepsilon - Du)(x)|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{B_{\tilde{R}}} |A(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{B_{\tilde{R}}} |A_\varepsilon(x, Du_\varepsilon(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}},
\end{aligned}$$

and by Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$, for any $\sigma > 0$ we get

$$\begin{aligned}
&\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p \, dx \\
\leq\ & c_\sigma \int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx + \sigma \int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p \, dx \\
&+ \left(\int_{B_{\tilde{R}}} |A(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{B_{\tilde{R}}} |A_\varepsilon(x, Du_\varepsilon(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}},
\end{aligned} \tag{4.48}$$

and choosing $\sigma = \frac{1}{2}$, we can absorb the second integral in the right-hand side of (4.48), thus getting

$$\begin{aligned}
&\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p \, dx \\
\leq\ & \int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx \\
&+ c\left[\left(\int_{B_{\tilde{R}}} |A(x, Du(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}} + \left(\int_{B_{\tilde{R}}} |A_\varepsilon(x, Du_\varepsilon(x))|^{\frac{p}{p-1}} \, dx\right)^{\frac{p-1}{p}}\right] \\
&\cdot \left(\int_{B_{\tilde{R}}} |D[\eta(x)(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}} \\
\leq\ & \int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx \\
&+ \left[\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right]^{\frac{p-1}{p}} \\
&\cdot \left(\int_{B_{\tilde{R}}} |D[\eta(\psi_\varepsilon - \psi)(x)]|^p \, dx\right)^{\frac{1}{p}} \\
\leq\ & \int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}} \, dx
\end{aligned}$$

$$+c\left[\int_{B_{\tilde{R}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{\tilde{R}}}|(Du-Du_\varepsilon)(x)|^p\,dx\right]^{\frac{p-1}{p}}$$
$$\cdot\left(\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\right)^{\frac{1}{p}}, \tag{4.49}$$

where we used (4.4) and (4.33).
By (4.49), applying Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$, we get

$$\begin{aligned}
&\int_{B_{\tilde{R}}}|(Du-Du_\varepsilon)(x)|^p\,dx\\
\le\ &\int_{B_{\tilde{R}}}|A(x,Du(x))-A_\varepsilon(x,Du(x))|^{\frac{p}{p-1}}\,dx\\
&+c\left[\int_{B_{\tilde{R}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{p-1}{p}}\cdot\left(\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\right)^{\frac{1}{p}}\\
&+c\left[\int_{B_{\tilde{R}}}|(Du-Du_\varepsilon)(x)|^p\,dx\right]^{\frac{p-1}{p}}\cdot\left(\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\right)^{\frac{1}{p}}\\
\le\ &\int_{B_{\tilde{R}}}|A(x,Du(x))-A_\varepsilon(x,Du(x))|^{\frac{p}{p-1}}\,dx\\
&+c\left[\int_{B_{\tilde{R}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{p-1}{p}}\cdot\left(\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\right)^{\frac{1}{p}}\\
&+\sigma\int_{B_{\tilde{R}}}|(Du-Du_\varepsilon)(x)|^p\,dx+c_\sigma\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx,
\end{aligned} \tag{4.50}$$

and if we choose $\sigma>0$ sufficiently small, (4.50) gives

$$\begin{aligned}
&\int_{B_{\tilde{R}}}|(Du-Du_\varepsilon)(x)|^p\,dx\\
\le\ &\int_{B_{\tilde{R}}}|A(x,Du(x))-A_\varepsilon(x,Du(x))|^{\frac{p}{p-1}}\,dx\\
&+c\left[\int_{B_{\tilde{R}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{p-1}{p}}\cdot\left(\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\right)^{\frac{1}{p}}\\
&+c\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx.
\end{aligned} \tag{4.51}$$

Let us notice that, recalling the properties of η, we get

$$\begin{aligned}
&\int_{B_{\tilde{R}}}|D[\eta(\psi_\varepsilon-\psi)(x)]|^p\,dx\\
\le\ &\int_{B_{\tilde{R}}}|D(\psi_\varepsilon-\psi)(x)|^p\,dx+\frac{c}{R}\int_{B_{\tilde{R}}}|(\psi_\varepsilon-\psi)(x)|^p\,dx.
\end{aligned} \tag{4.52}$$

Hence, thanks to (4.52), recalling (4.39), (4.41) and (4.43), the right-hand side of (4.51) vanishes as $\varepsilon\to 0$, and we deduce

$$u_\varepsilon\to u \qquad \text{strongly in } W^{1,p}\left(B_{\tilde{R}}\right),$$

and so

$$u_\varepsilon\to u \qquad \text{strongly in } L^{p^*}\left(B_{\tilde{R}}\right), \tag{4.53}$$

and, for a not relabeled sequence, we also have

$$u_\varepsilon \to u \qquad \text{and} \qquad Du_\varepsilon \to Du$$

almost everywhere in $B_{\tilde{R}}$, as $\varepsilon \to 0$.
Moreover, since the function $\xi \mapsto V_p(\xi)$ is continuous, we also have

$$V_p(Du_\varepsilon) \to V_p(Du) \qquad \text{almost verywhere in } B_r \text{ as } \varepsilon \to 0.$$

For these reasons, we can to pass to the limit in (4.38) and, by virtue of the Fatou's Lemma, recalling (4.39), (4.40), (4.41), (4.42) and (4.53), we get

$$\begin{aligned} &\int_{B_{\frac{r}{4}}} |DV_p(Du(x))|^2\,dx \\ \leq\ & \frac{c\left(\|\psi\|^2_{L^\infty(B_r)} + \|u\|^2_{L^{p^*}(B_r)}\right)}{r^{\frac{p+2}{2}}} \\ &\cdot \int_{B_r}\left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{2}} + |D\psi(x)|^{\frac{p+2}{2}} + g^{p+2}(x) + |Du(x)|^p\right] dx. \end{aligned}$$

Therefore, by means of a covering argument, we conclude, with the estimate (4.16). □

4.3 A higher differentiability result for solutions to some obstacle problems with sub-quadratic growth and Sobolev coefficients

As in the previous section, we are interested in the regularity properties of solutions to problems of the form (4.1), where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n > 2$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x,\xi)$ is of class $C^2(\mathbb{R}^n)$ for a.e. $x \in \Omega$, $\psi : \Omega \mapsto [-\infty, +\infty)$ belonging to the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the obstacle, and

$$\mathcal{K}_\psi(\Omega) = \left\{\, w \in u_0 + W^{1,p}_0(\Omega, \mathbb{R}) : w \geq \psi \text{ a.e. in } \Omega \,\right\}$$

is the class of the admissible functions, with $u_0 \in W^{1,p}(\Omega)$ a fixed boundary datum.
As we noticed before, a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the obstacle problem (4.1) in $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality (4.2) where the operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by (4.3).
Again, we assume that A is a p-harmonic type operator, that is it satisfies p-ellipticity and p-growth conditions with respect to the ξ-variable (4.4)–(4.6) and , for all $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \Omega$.
The main difference with respect to the problem we considered in Section 4.2 is that, here, $1 < p < 2$.
The result we prove in this section is [53, Theorem 1.1], which, in some sense, extends to the sub-quadratic growth case what is stated in [42, Theorem 1.1].
Indeed, we show that an higher differentiability property of integer order of the gradient of the obstacle tranfers to the solution of problem (4.1), provided the partial map $x \mapsto D_\xi f(x,\xi)$ belongs to a suitable Sobolev class, with no loss in the order of differentiation.
More precisely we assume that the map $x \mapsto A(x,\xi)$ belongs to $W^{1,n}_{\mathrm{loc}}(\Omega)$ for every $\xi \in \mathbb{R}^n$ or, equivalently, that there exists a non-negative function $g \in L^n_{\mathrm{loc}}(\Omega)$ such that (4.10) and (4.11) hold.
Note that, since f, as a function of the ξ variable, is of class C^2, then the operator A is of class C^1 with respect to ξ, and (4.6) implies (4.7), for all $\xi \in \mathbb{R}^n \setminus \{\,0\,\}$ and for a.e. $x \in \Omega$.
The result we prove in this section is the following.

Theorem 4.3.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem (4.1) under assumptions (4.4)–(4.6) for $1 < p < 2$. Moreover, let us assume that there exists a function $g \in L^n_{\mathrm{loc}}(\Omega)$ such that (4.10) and (4.11) hold.*
Then the following implication holds:

$$V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega) \implies V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Moreover, for any ball $B_{2R} \Subset \Omega$, the following estimate holds

$$\|DV_p(Du)\|_{L^2\left(B_{\frac{R}{2}}\right)} \le C\left(1 + \|Du\|_{L^p(B_{2R})} + \|V_p(D\psi)\|_{W^{1,2}(B_{2R})} + \|g\|_{L^n(B_R)}\right)^{\sigma}, \tag{4.54}$$

where $C > 0$ depends on n, p, R, ν, L and ℓ and $\sigma > 0$ depends on n and p.

4.3.1 Proof of Theorem 4.3.1

Proof of Theorem 4.3.1. Since the condition $V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ implies the existence of the second order weak derivatives of ψ (see Lemma 1.4.5), and the map $\xi \mapsto f(x,\xi)$ is of class C^2, we are legitimated to apply Theorem 4.1.2, and this gives us the possibility to overcome some technical difficulties that could come out if we start from inequality (4.2), due to the fact that $1 < p < 2$.
Let us recall that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the equation (4.13) if and only if, for any $\varphi \in W^{1,p}_0(\Omega)$,

$$\int_\Omega \langle A(x, Du(x)), D\varphi(x)\rangle\, dx = -\int_\Omega \mathrm{div} A(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\varphi(x) dx. \tag{4.55}$$

Let us fix a ball $B_{2R} \Subset \Omega$ and radii $\frac{R}{2} < r < \frac{3}{4}R < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut-off function $\eta \in C^\infty_0\left(B_{\frac{3}{4}R}\right)$ such that $\eta \equiv 1$ on $B_{\frac{R}{2}}$, $|D\eta| \le \frac{c}{R}$ and $|D^2\eta| \le \frac{c}{R^2}$. From now on, with no loss of generality, we suppose $R < 1$.
Let us consider the test function

$$\varphi = \tau_{-h}\left(\eta^2 \cdot \tau_h u\right).$$

For this choice of φ, using proposition 1.2.2, the left-hand side of (4.55) can be written as follows:

$$\begin{aligned}
&\int_\Omega \left\langle A(x, Du(x)), D\left(\tau_{-h}\left(\eta^2(x)\tau_h u(x)\right)\right)\right\rangle dx \\
= &\int_\Omega \left\langle \tau_h A(x, Du(x)), D\left(\eta^2(x)\tau_h u(x)\right)\right\rangle dx \\
= &\int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), D\left(\eta^2(x)\tau_h u(x)\right)\right\rangle dx \\
= &\int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), \eta^2(x)\tau_h Du(x)\right\rangle dx \\
&+ \int_\Omega \langle A(x+h, Du(x+h)) - A(x, Du(x)), 2\eta(x)D\eta(x)\tau_h u(x)\rangle\, dx \\
= &\int_\Omega \left\langle A(x, Du(x+h)) - A(x, Du(x)), \eta^2(x)\tau_h Du(x)\right\rangle dx \\
&+ \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x+h)), \eta^2(x)\tau_h Du(x)\right\rangle dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \langle A(x+h, Du(x+h)) - A(x, Du(x)), 2\eta(x)D\eta(x)\tau_h u(x)\rangle\, dx \\
:= \quad & I_0 + I + II, \qquad (4.56)
\end{aligned}
$$

where, for the finite differences, we used the simplified notation

$$\tau_h F(x) = F(x+h) - F(x),$$

with $h \in \mathbb{R}^n$, in place of

$$\tau_{s,h} F(x) = F(x + he_s) - F(x),$$

with $h \in \mathbb{R}$ and, in the following, we will specify the direction only if it will be necessary. Since the right-hand side of (4.55) is not zero only where $u = \psi$, using the test function given above, it becomes

$$-\int_\Omega \mathrm{div} A(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\tau_{-h}\left(\eta^2(x)\tau_h\psi(x)\right) dx, \qquad (4.57)$$

and since the map $x \mapsto A(x,\xi)$ belongs to $W^{1,n}_{\mathrm{loc}}(\Omega)$ for any $\xi \in \mathbb{R}^n$, the map $\xi \mapsto A(x,\xi)$ belongs to $C^1(\mathbb{R}^n)$ for a.e. $x \in \Omega$ and $V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we can write (4.57) as follows

$$
\begin{aligned}
&-\int_\Omega \Big\{\Big[A_x(x, D\psi(x)) + A_\xi(x, D\psi(x))\, D^2\psi(x)\Big]\chi_{\{u=\psi\}}(x) \\
&\cdot\tau_{-h}\left(\eta^2(x)\tau_h\psi(x)\right)\Big\}dx \\
= \quad & -\int_\Omega \left\{\Big[A_x(x, D\psi(x)) + A_\xi(x, D\psi(x))\, D^2\psi(x)\Big]\chi_{\{u=\psi\}}(x)\right. \\
&\left.\cdot\tau_{-h}\left(\eta^2(x)\cdot h\int_0^1 D\psi(x+\sigma h)d\sigma\right)\right\}dx \\
= \quad & -\int_\Omega \left\{\Big[A_x(x, D\psi(x)) + A_\xi(x, D\psi(x))\, D^2\psi(x)\Big]\chi_{\{u=\psi\}}(x)\right. \\
&\cdot |h|^2\int_0^1\left[\eta^2(x-\theta h)\int_0^1 D^2\psi(x+\sigma h-\theta h)d\sigma\right. \\
&\left.\left.+2\eta(x-\theta h)D\eta(x-\theta h)\int_0^1 D\psi(x+\sigma h-\theta h)d\sigma\right]d\theta\right\}dx \\
= \quad & -\int_\Omega \left\{\Big[A_x(x, D\psi(x)) + A_\xi(x, D\psi(x))\, D^2\psi(x)\Big]\chi_{\{u=\psi\}}(x)\right. \\
&\cdot\int_0^1\int_0^1 |h|^2\Big[\eta^2(x-\theta h)D^2\psi(x+\sigma h-\theta h) \\
&\left.+2\eta(x-\theta h)D\eta(x-\theta h)D\psi(x+\sigma h-\theta h)\Big]d\sigma d\theta\right\}dx.
\end{aligned}
$$

Therefore, the right-hand side of (4.55) is given by the following expression

$$
\begin{aligned}
&-|h|^2\int_\Omega A_x(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\int_0^1\int_0^1 \eta^2(x-\theta h)D^2\psi(x+\sigma h-\theta h)d\sigma d\theta dx \\
&-2|h|^2\int_\Omega A_x(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\int_0^1\int_0^1 \eta(x-\theta h)D\eta(x-\theta h)D\psi(x+\sigma h-\theta h)d\sigma d\theta dx
\end{aligned}
$$

$$
\begin{aligned}
& -|h|^2 \int_\Omega A_\xi\left(x, D\psi(x)\right) D^2\psi(x)\chi_{\{u=\psi\}}(x) \int_0^1 \int_0^1 \eta^2(x-\theta h) D^2\psi(x+\sigma h-\theta h) d\sigma d\theta dx \\
& -2\,|h|^2 \int_\Omega A_\xi\left(x, D\psi(x)\right) D^2\psi(x)\chi_{\{u=\psi\}}(x) \\
& \cdot \int_0^1 \int_0^1 \eta(x-\theta h) D\eta(x-\theta h) D\psi(x+\sigma h-\theta h) d\sigma d\theta dx \\
=: \quad & -III - IV - V - VI. \qquad (4.58)
\end{aligned}
$$

Inserting (4.56) and (4.58) in (4.55) we get

$$
I_0 = -I - II - III - IV - V - VI,
$$

and so

$$
I_0 \leq |I| + |II| + |III| + |IV| + |V| + |VI|. \qquad (4.59)
$$

By assumption (4.5), we have

$$
I_0 \geq \nu \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx. \qquad (4.60)
$$

Let us consider the term I. By assumption (4.10), and using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{n}{2}, \frac{n}{n-2}\right)$, and the properties of η, we get

$$
\begin{aligned}
|I| \quad \leq \quad & |h| \int_\Omega \left(g(x+h)+g(x)\right)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-1}{2}} \eta^2(x)\,|\tau_h Du(x)|\, dx \\
\leq \quad & \varepsilon \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx \\
& + c_\varepsilon |h|^2 \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p}{2}} \left(g(x+h)+g(x)\right)^2 dx \\
\leq \quad & \varepsilon \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx \\
& + c_\varepsilon |h|^2 \left(\int_{B_{\frac{3}{4}R}} \left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{np}{2(n-2)}} dx\right)^{\frac{n-2}{n}} \\
& \cdot \left(\int_{B_{\lambda r}} g^n(x) dx\right)^{\frac{2}{n}}. \qquad (4.61)
\end{aligned}
$$

For the term II, if we denote again finite differences with respect to a precise direction $s = 1, \dots, n$, with an integration by parts, we have

$$
\begin{aligned}
-II \quad = \quad & -2h_s \int_\Omega \left\langle \int_0^1 \frac{d}{dx_s} A\left(x+\theta h e_s, Du\left(x + h_s\theta e_s\right)\right) d\theta, \eta(x) D\eta(x) \tau_{s,h} u(x)\right\rangle dx \\
= \quad & 2h_s \int_\Omega \left\langle \int_0^1 \left(A\left(x+h_s\theta e_s, Du(x+\theta h e_s)\right)\right) d\theta, \right. \\
& \left. \frac{d}{dx_s}\left(\eta(x) D\eta(x) \tau_{s,h} u(x)\right)\right\rangle dx,
\end{aligned}
$$

where, for each $s = 1, \dots, n$, e_s is the unit vector in the x_s direction, and now $h \in \mathbb{R}$.

So we can estimate II as follows

$$\begin{aligned}
|II| \quad \leq \quad & 2|h| \int_{\Omega} \int_0^1 |A(x+\theta h e_s, Du(x+h_s\theta e_s))| \Big(|D\eta(x)|^2 |\tau_{s,h}u(x)| \\
& + \eta(x) \Big| D^2\eta(x) \Big| |\tau_{s,h}u(x)| \Big) d\theta dx \\
& + 2c|h| \int_{\Omega} \int_0^1 |A(x+h_s\theta e_s, Du(x+\theta h e_s))| \Big(\eta(x) |D\eta(x)| |\tau_{s,h}Du(x)| \Big) d\theta dx \\
\leq \quad & 2c|h| \int_{\Omega} \int_0^1 |A(x+h_s\theta e_s, Du(x+\theta h e_s))| \Big(|D\eta(x)|^2 \\
& + \eta(x) \Big| D^2\eta(x) \Big| \Big) d\theta \, |\tau_{s,h}u(x)| \, dx \\
& + 2c|h| \int_{\Omega} \int_0^1 |A(x+\theta h e_s, Du(x+\theta h e_s))| \, \eta(x) |D\eta(x)| |\tau_{s,h}Du(x)| \, d\theta dx.
\end{aligned}$$

Now, recalling the properties of η, assumption (4.4), and using Hölder's inequality with exponents $\left(p, \frac{p}{p-1}\right)$, Lemma 1.2.3 and Young's inequality with exponents $(2,2)$, we get

$$\begin{aligned}
|II| \quad \leq \quad & 2c|h| \int_{\Omega} \int_0^1 \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{p-1}{2}} \Big(|D\eta(x)|^2 \\
& + \eta(x) \Big| D^2\eta(x) \Big| \Big) d\theta \, |\tau_{s,h}u(x)| \, dx \\
& + 2c|h| \int_{\Omega} \int_0^1 \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{p-1}{2}} \eta(x) |D\eta(x)| |\tau_{s,h}Du(x)| \, d\theta dx \\
= \quad & 2c|h| \int_0^1 \int_{\Omega} \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{p-1}{2}} \Big(|D\eta(x)|^2 \\
& + \eta(x) \Big| D^2\eta(x) \Big| \Big) |\tau_{s,h}u(x)| \, dx d\theta \\
& + 2c|h| \int_0^1 \int_{\Omega} \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{p-1}{2}} \eta(x) |D\eta(x)| |\tau_{s,h}Du(x)| \, dx d\theta \\
\leq \quad & \frac{c|h|}{R^2} \int_0^1 \left(\int_{B_{\frac{3}{4}R}} \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} d\theta \\
& \cdot \left(\int_{B_{\frac{3}{4}R}} |\tau_{s,h}u(x)|^p \, dx \right)^{\frac{1}{p}} \\
& + \varepsilon \int_{\Omega} \eta^2(x) |\tau_{s,h}Du(x)|^2 \left(\mu^2 + |Du(x)|^2 + |Du(x+h e_s)|^2\right)^{\frac{p-2}{2}} dx \\
& + \frac{c_\varepsilon |h|^2}{R^2} \int_0^1 \int_{B_{\frac{3}{4}R}} \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{p-1} \\
& \cdot \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h e_s)|^2\right)^{\frac{2-p}{2}} dx d\theta.
\end{aligned}$$

Now, if we use again the simplified notation for finite diferences, with $h \in \mathbb{R}^n$ in place of he_s where $h \in \mathbb{R}$, by Lemma 1.2.3, we get

$$|II| \quad \leq \quad \frac{c|h|^2}{R^2} \int_0^1 \left(\int_{B_{\frac{3}{4}R}} \left(\mu^2 + |Du(x)|^2 + |Du(x+\theta h)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} d\theta$$

$$
\begin{aligned}
&\cdot\left(\int_{B_{\frac{3}{4}R}}|Du(x)|^p\,dx\right)^{\frac{1}{p}}\\
&+\varepsilon\int_\Omega\eta^2(x)\,|\tau_h Du(x)|^2\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}dx\\
&+\frac{c_\varepsilon\,|h|^2}{R^2}\int_0^1\left[\int_{B_{\frac{3}{4}R}}\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{p-1}{2}}\right.\\
&\left.\cdot\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{2-p}{4}}dx\right]^2d\theta. \qquad (4.62)
\end{aligned}
$$

Let us consider, now, the term III. By (4.11) and the properties of η, we get

$$
\begin{aligned}
|III| \;\le\; & |h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}g(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}}\\
&\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|dxd\sigma d\theta.
\end{aligned}
$$

Using Young's inequality with exponents $(2,2)$, we get

$$
\begin{aligned}
|III| \;\le\; & c|h|^2\int_0^1\int_0^1\left[\int_{B_{\lambda r}}g^2(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p}{2}}dx\right.\\
&+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}\\
&\left.\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|^2dx\right]d\sigma d\theta. \qquad (4.63)
\end{aligned}
$$

Using Young's inequality again, with exponents $\left(\frac{n}{2},\frac{n}{n-2}\right)$ in the first integral of (4.63), we get

$$
\begin{aligned}
|III| \;\le\; & c|h|^2\left[\int_{B_{\lambda r}}g^n(x)dx+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{np}{2(n-2)}}dx\right]\\
&+c\int_0^1\int_0^1\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}\\
&\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|^2dxd\sigma d\theta. \qquad (4.64)
\end{aligned}
$$

We estimate the term IV using (4.11), thus getting

$$
\begin{aligned}
|IV| \;\le\; & 2|h|^2\int_{B_{\lambda r}}g(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}}\\
&\cdot\int_0^1\int_0^1|D\psi(x+\sigma h-\theta h)|\,|D\eta(x-\theta h)|\,d\sigma d\theta dx. \qquad (4.65)
\end{aligned}
$$

Let us consider, now, the term V. By (4.7), we get

$$
|V| \;\le\; |h|^2\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|
$$

$$\cdot \int_0^1 \int_0^1 \left| D^2\psi(x+\sigma h-\theta h) \right| d\sigma d\theta dx. \tag{4.66}$$

In order to estimate the term VI, we recall (4.7) again, thus getting

$$\begin{aligned} |VI| &\leq 2|h|^2 \int_{B_{\lambda r}} \left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}} \left| D^2\psi(x) \right| \\ &\cdot \int_0^1 \int_0^1 |D\eta(x-\theta h)| \, |D\psi(x+\sigma h-\theta h)| \, d\sigma d\theta dx. \end{aligned} \tag{4.67}$$

Now, plugging (4.60), (4.61), (4.62), (4.64), (4.65), (4.66) and (4.67) in (4.59), recalling the properties of η and choosing a sufficiently small value of ε, we get

$$\begin{aligned} &\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2 \, dx \\ \leq\ & c|h|^2 \left(\int_{B_{\frac{3}{4}R}} \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{np}{2(n-2)}} dx\right)^{\frac{n-2}{n}} \cdot \left(\int_{B_{\lambda r}} g^n(x) dx\right)^{\frac{2}{n}} \\ &+\frac{c|h|^2}{R^2} \int_0^1 \left(\int_{B_{\frac{3}{4}R}} \left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} d\theta \\ &\cdot \left(\int_{B_{\frac{3}{4}R}} |Du(x)|^p \, dx\right)^{\frac{1}{p}} \\ &+\frac{c\,|h|^2}{R^2} \int_0^1 \int_{B_{\frac{3}{4}R}} \left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{p-1} \\ &\cdot \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{2-p}{2}} dx d\theta \\ &+c|h|^2 \left[\int_{B_{\lambda r}} g^n(x) dx + \int_{B_{\lambda r}} \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{np}{2(n-2)}} dx\right] \\ &+c|h|^2 \int_0^1 \int_0^1 \int_{B_{\lambda r}} \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}} \left| D^2\psi(x+\sigma h-\theta h) \right|^2 dx d\sigma d\theta \\ &+2|h|^2 \int_0^1 \int_0^1 \int_{B_{\lambda r}} g(x) \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}} \\ &\cdot |D\psi(x+\sigma h-\theta h)| \, |D\eta(x-\theta h)| \, dx d\sigma d\theta \\ &+|h|^2 \int_0^1 \int_0^1 \int_{B_{\lambda r}} \left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}} \left| D^2\psi(x) \right| \\ &\cdot \left| D^2\psi(x+\sigma h-\theta h) \right| dx d\sigma d\theta \\ &+2|h|^2 \int_0^1 \int_0^1 \int_{B_{\lambda r}} \left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}} \left| D^2\psi(x) \right| \\ &\cdot |D\eta(x-\theta h)| \, |D\psi(x+\sigma h-\theta h)| \, dx d\sigma d\theta. \end{aligned} \tag{4.68}$$

By Lemma 1.4.3 and the properties of η, the left-hand side of (4.68) can be bounded from below as follows

$$\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx \geq \int_{B_{\frac{R}{2}}} |\tau_h V_p(Du(x))|^2\,dx. \tag{4.69}$$

So, by (4.69) and (4.68), recalling the properties of η and using Lemma 1.2.3, we get

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |\tau_h V_p(Du(x))|^2\,dx \\
\leq\ & c|h|^2\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2)}}dx\right)^{\frac{n-2}{n}}\cdot\left(\int_{B_R} g^n(x)dx\right)^{\frac{2}{n}} \\
&+\frac{c|h|^2}{R^2}\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx \\
&+c|h|^2\left[\int_{B_R} g^n(x)dx+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{np}{2(n-2)}}dx\right] \\
&+\frac{c|h|^2}{R}\int_{B_R} g(x)\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx \\
&+c|h|^2\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|^2dx \\
&+\frac{c|h|^2}{R}\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-1}{2}}\left|D^2\psi(x)\right|dx.
\end{aligned}$$

Now we apply Hölder's inequality with exponents $\left(n,n,\frac{n}{n-2}\right)$ to the integral of the fifth line, Young's inequality with exponents $(2,2)$ to the last integral, and use Lemma 1.4.3, thus getting

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |\tau_h V_p(Du(x))|^2\,dx \\
\leq\ & c|h|^2\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2)}}dx\right)^{\frac{n-2}{n}}\cdot\left(\int_{B_R} g^n(x)dx\right)^{\frac{2}{n}} \\
&+\frac{c|h|^2}{R^2}\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right) \\
&+c|h|^2\left[\int_{B_R} g^n(x)dx+\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{np}{2(n-2)}}dx\right] \\
&+c|h|^2\left(\int_{B_R} g^n(x)dx\right)^{\frac{1}{n}}\cdot\left(\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{np}{2(n-2)}}dx\right)^{\frac{n-2}{n}} \\
&+\frac{c|h|^2}{R}\left[\int_{B_R}|DV_p(D\psi(x))|^2\,dx+\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]
\end{aligned}$$

for a constant $c=c(n,p,\nu,L,\ell)$. By Young's inequality with exponents $\left(\frac{n}{2},\frac{n}{n-2}\right)$, we get

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |\tau_h V_p(Du(x))|^2\,dx \\
\leq\ & c|h|^2\left[\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2)}}dx+\int_{B_R} g^n(x)dx+\left(\int_{B_R} g^n(x)dx\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{np}{2(n-2)}}dx\Bigg]\\
&+\frac{c|h|^2}{R}\left[\int_{B_R}|DV_p\left(D\psi(x)\right)|^2dx+\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]\\
&+\frac{c|h|^2}{R^2}\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right). \qquad (4.70)
\end{aligned}
$$

Let us observe that, since $V_p\left(D\psi\right)\in W^{1,2}_{\mathrm{loc}}\left(\Omega\right)$, by Sobolev's inequality, $D\psi\in L^{\frac{np}{n-2}}_{\mathrm{loc}}\left(\Omega\right)$. Therefore, applying Theorem 4.1.5 with $q=\frac{np}{n-2}$, we have $Du\in L^{\frac{np}{n-2}}_{\mathrm{loc}}\left(\Omega\right)$, with the following estimate:

$$
⨍_{B_R}|Du(x)|^{\frac{np}{n-2}}dx\le C\left\{1+⨍_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2}}dx+\left(⨍_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2}}\right\}. \qquad (4.71)
$$

Using (4.71), estimate (4.70) becomes

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}}|\tau_hV_p\left(Du(x)\right)|^2dx\\
\le\;& c|h|^2\left\{\int_{B_{2R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{np}{2(n-2)}}dx+\left[\int_{B_{2R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{n}{n-2}}\right.\\
&\left.+\int_{B_R}g^n(x)dx+\left(\int_{B_R}g^n(x)dx\right)^{\frac{1}{2}}dx\right\}\\
&+\frac{c|h|^2}{R}\left[\int_{B_R}|DV_p\left(D\psi(x)\right)|^2dx+\int_{B_R}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]\\
&+\frac{c|h|^2}{R^2}\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right).
\end{aligned}
$$

Applying Sobolev's embedding Theorem to the function $V_p\left(D\psi\right)$, and exploiting the fact that $p<\frac{np}{n-2}$, we get

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}}|\tau_hV_p\left(Du(x)\right)|^2dx\\
\le\;& c|h|^2\left\{\left[\int_{B_{2R}}\left(|V_p\left(D\psi(x)\right)|^2+|DV_p\left(D\psi(x)\right)|^2\right)dx\right]^{\frac{n}{n-2}}\right.\\
&\left.+\left[\int_{B_{2R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right]^{\frac{n}{n-2}}+\int_{B_R}g^n(x)dx+\left(\int_{B_R}g^n(x)dx\right)^{\frac{1}{2}}dx\right\}\\
&+\frac{c|h|^2}{R}\int_{B_R}|DV_p\left(D\psi(x)\right)|^2dx+\frac{c|h|^2}{R^2}\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right).
\end{aligned}
$$

So, applying Lemma 1.2.4, for positive constants C and σ, we get

$$
\|DV_p\left(Du(x)\right)\|_{L^2\left(B_{\frac{R}{2}}\right)}\le C\left(1+\|Du\|_{L^p(B_{2R})}+\|V_p\left(D\psi\right)\|_{W^{1,2}(B_{2R})}+\|g\|_{L^n(B_R)}\right)^{\sigma},
$$

that is (4.54). □

Thanks to Lemma 1.4.5, we get the following consequence of Theorem 4.3.1.

Corollary 4.3.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1) *under assumptions* (4.4)–(4.6) *and let us assume that there exists a function $g \in L^n_{\mathrm{loc}}(\Omega)$ such that* (4.10) *and* (4.11) *hold, for $1 < p < 2$.*
Then the following implication holds:

$$V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega) \implies u \in W^{2,p}_{\mathrm{loc}}(\Omega).$$

4.4 Fractional higher differentiability results for solutions to some obstacle problems with sub-quadratic growth and Besov-Lipschitz coefficients

In this section we give the proof of [53, Theorem 1.2] and [53, Theorem 1.3] which, in some sense, represent the "fractional counterpart" of Theorem 4.3.1.
Indeed, here we assume that the obstacle belongs to a Besov-Lipschitz space, provided we assume a Besov-Lipschitz dependence of the operator A with respect to the x-variable.
First, instead of (4.10) and (4.11), we assume that, given $\alpha \in (0,1)$ and $1 \le q < \infty$ there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{n}{\alpha}}_{\mathrm{loc}}(\Omega)$ such that

$$\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(\Omega')} < \infty,$$

for any open set $\Omega' \Subset \Omega$ and, at the same time,

$$|A(x,\xi) - A(y,\xi)| \le (g_k(x) + g_k(y))\,|x-y|^\alpha \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{4.72}$$

for each $\xi \in \mathbb{R}^n$ and almost every $x, y \in \Omega$ such that $2^{-k}\mathrm{diam}(\Omega) \le |x-y| \le 2^{-k+1}\mathrm{diam}(\Omega)$.
We will shortly write, then, $(g_k)_k \in \ell^q\left(L^{\frac{n}{\alpha}}(\Omega)\right)$. If $A(x,\xi) = \gamma(x)|\xi|^{p-2}\xi$ and $\Omega = \mathbb{R}^n$ then (4.72) says that $\gamma \in B^\alpha_{\frac{n}{\alpha},q}$.
It is worth noticing that, due to the sub-quadratic growth conditions, the Besov regularity of the obstacle transfers to the solution with a small loss in the order of differentiations.

Theorem 4.4.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1), *under the assumptions* (4.4)–(4.6) *and* (4.72), *for $1 < p < 2$. Then the following implication holds*

$$V_p(D\psi) \in B^\alpha_{2,q,\mathrm{loc}}(\Omega) \implies V_p(Du) \in B^{\alpha\beta}_{2,q,\mathrm{loc}}(\Omega)$$

for any $q \le 2^*_\alpha = \frac{2n}{n-2\alpha}$ and $\beta \in (0,1)$.
Moreover, for any ball $B_{4R} \Subset \Omega$, the following estimate holds

$$\begin{aligned} &\left\| \frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}} \right\|_{L^q\left(\frac{dh}{|h|^n}; L^2\left(B_{\frac{R}{2}}\right)\right)} \\ \le\ & C\left(1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B^\alpha_{2,q}(B_{4R})} + \|(g_k)_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)}\right)^\sigma, \end{aligned} \tag{4.73}$$

where $C > 0$ depends on n, p, R, ν, L and ℓ and $\sigma > 0$ depends on n, p, q and α.

This result represents, in some sense, the extention to the case of sub-quadratic growth conditions of [42, Theorem 1.2].
The second result of this section deals with the case of Besov-Lipschitz coefficients in the case $q = \infty$.

In this framework, we have that a fractional differentiability property of the obstacle transfers to the solution with a larger loss on the order of differentiation than the one we have when q is finite. This is due to the fact that the regularity of the type $B^{\alpha}_{p,\infty}$ is the weakest one to assume both on the coefficients and on the gradient of the obstacle in the scale of Besov-Lipschitz spaces (see Lemmas 1.2.5 and 1.3.2 above).
More precisely, the second result we prove in this section is the following.

Theorem 4.4.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1)*, under the assumptions* (4.4)–(4.6) *for $1 < p < 2$. If there exists $\alpha \in (0,1)$ and a function $g \in L^{\frac{n}{\alpha}}_{\mathrm{loc}}(\Omega)$ such that*

$$|A(x,\xi) - A(y,\xi)| \le (g(x) + g(y)) |x-y|^{\alpha} \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{4.74}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$, then, provided $0 < \alpha < \gamma < 1$ the following implication holds

$$V_p(D\psi) \in B^{\gamma}_{2,\infty,\mathrm{loc}}(\Omega) \implies V_p(Du) \in B^{\alpha\beta}_{2,\infty,\mathrm{loc}}(\Omega),$$

for any $\beta \in (0,1)$.
Moreover, for any ball $B_{4R} \Subset \Omega$, the following estimate holds

$$\begin{aligned} & [V_p(Du)]_{\dot{B}^{\alpha\beta}_{2,\infty}\left(B_{\frac{R}{2}}\right)} \\ \le\ & C\left(1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B^{\gamma}_{2,\infty}(B_{4R})} + \|g\|_{L^{\frac{n}{\alpha}}(B_{2R})}\right)^{\sigma}, \end{aligned} \tag{4.75}$$

where $C > 0$ depends on n, p, R, ν, L and ℓ and $\sigma > 0$ depends on n, p, q and α.

This result represents an extension to the case of sub-quadratic growth conditions of [42, Theorem 1.4].

4.4.1 Proof of Theorem 4.4.1

This section is devoted to the proof of Theorem 4.4.1. It is worth noticing that, in this case, our starting point can't be equation (4.13), since our assumption on ψ doesn't allow to calculate the divergence in the right-hand side.
For this reason, here we have to start from (4.2), and apply the same technique we used to prove the result in Section 4.2.

Proof of Theorem 4.4.1. Let us fix a ball $B_{4R} \Subset \Omega$ and radii $\frac{R}{2} < r < \frac{3}{4}R < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut-off function $\eta \in C^{\infty}_0\left(B_{\frac{3}{4}R}\right)$ such that $\eta \equiv 1$ on $B_{\frac{R}{2}}$ and $|D\eta| \le \frac{c}{R}$. From now on, with no loss of generality, we suppose $R < 1$.
Let $v \in W^{1,p}_0(\Omega)$ be such that

$$u - \psi + \tau v \ge 0 \qquad \forall \tau \in [0,1], \tag{4.76}$$

and observe that $\varphi := u + \tau v \in \mathcal{K}_{\psi}(\Omega)$ for all $\tau \in [0,1]$, since $\varphi = u + \tau v \ge \psi$. For $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right],$$

so we have $v_1 \in W^{1,p}_0(\Omega)$, and, for any $\tau \in [0,1]$, v_1 satisfies (4.76). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0,1]$

$$
\begin{aligned}
u(x)-\psi(x)+\tau v_1(x) &= u(x)-\psi(x)+\tau\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\\
&= \tau\eta^2(x)(u-\psi)(x+h)+\Big(1-\tau\eta^2(x)\Big)(u-\psi)(x)\geq 0,
\end{aligned}
$$

since $u\in\mathcal{K}_\psi(\Omega)$ and $0\leq\eta\leq 1$.
So we can use $\varphi=u+\tau v_1$ as a test function in inequality (4.2), thus getting

$$
0\leq\int_\Omega\Big\langle A(x,Du(x)),D\Big[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\Big]\Big\rangle\,dx. \tag{4.77}
$$

In a similar way, we define

$$
v_2(x)=\eta^2(x-h)\left[(u-\psi)(x-h)-(u-\psi)(x)\right],
$$

and we have $v_2\in W_0^{1,p}(\Omega)$, and (4.76) still is satisfied for any $\tau\in[0,1]$, since

$$
\begin{aligned}
u(x)-\psi(x)+\tau v_2(x) &= u(x)-\psi(x)+\tau\eta^2(x-h)\left[(u-\psi)(x-h)-(u-\psi)(x)\right]\\
&= \tau\eta^2(x)(u-\psi)(x-h)+\Big(1-\tau\eta^2(x-h)\Big)(u-\psi)(x)\geq 0.
\end{aligned}
$$

By using in (4.2) as test function $\varphi=u+\tau v_2$, we get

$$
0\leq\int_\Omega\Big\langle A(x,Du(x)),D\Big[\eta^2(x-h)\left[(u-\psi)(x-h)-(u-\psi)(x)\right]\Big]\Big\rangle\,dx,
$$

and by means of a change of variable, we obtain

$$
0\leq\int_\Omega\Big\langle A(x+h,Du(x+h)),D\Big[\eta^2(x)\left[(u-\psi)(x)-(u-\psi)(x+h)\right]\Big]\Big\rangle\,dx. \tag{4.78}
$$

We can add (4.77) and (4.78), thus getting

$$
\begin{aligned}
0 \leq\ & \int_\Omega\Big\langle A(x,Du(x)),D\Big[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\Big]\Big\rangle\,dx\\
&+\int_\Omega\Big\langle A(x+h,Du(x+h)),D\Big[\eta^2(x)\left[(u-\psi)(x)-(u-\psi)(x+h)\right]\Big]\Big\rangle\,dx,
\end{aligned}
$$

that is

$$
0\leq\int_\Omega\Big\langle A(x,Du(x))-A(x+h,Du(x+h)),D\Big[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\Big]\Big\rangle\,dx,
$$

which implies

$$
\begin{aligned}
0 \geq\ & \int_\Omega\Big\langle A(x+h,Du(x+h))-A(x,Du(x)),\eta^2(x)D\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\Big\rangle\,dx\\
&+\int_\Omega\left\langle A(x+h,Du(x+h))-A(x,Du(x)),2\eta(x)D\eta(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\right\rangle\,dx.
\end{aligned}
$$

Previous inequality can be rewritten as follows

$$
0 \geq \int_\Omega\Big\langle A(x+h,Du(x+h))-A(x+h,Du(x)),\eta^2(x)\left(Du(x+h)-Du(x)\right)\Big\rangle\,dx
$$

$$
\begin{aligned}
& -\int_{\Omega}\left\langle A\left(x+h, Du(x+h)\right)-A\left(x+h, Du(x)\right), \eta^2(x)\left(D\psi(x+h)-D\psi(x)\right)\right\rangle dx \\
& +\int_{\Omega}\left\langle A\left(x+h, Du(x+h)\right)-A\left(x+h, Du(x)\right), 2\eta(x)D\eta(x)\tau_h\left(u-\psi\right)(x)\right\rangle dx \\
& +\int_{\Omega}\left\langle A\left(x+h, Du(x)\right)-A\left(x, Du(x)\right), \eta^2(x)\left(Du(x+h)-Du(x)\right)\right\rangle dx \\
& -\int_{\Omega}\left\langle A\left(x+h, Du(x)\right)-A\left(x, Du(x)\right), \eta^2(x)\left(D\psi(x+h)-D\psi(x)\right)\right\rangle dx \\
& +\int_{\Omega}\left\langle A\left(x+h, Du(x)\right)-A\left(x, Du(x)\right), 2\eta(x)D\eta(x)\tau_h\left(u-\psi\right)(x)\right\rangle dx \\
=: \; & I+II+III+IV+V+VI,
\end{aligned}
$$

so we have

$$
I \leq |II|+|III|+|IV|+|V|+|VI|. \tag{4.79}
$$

By (4.5) we have

$$
I \geq \nu \int_{\Omega}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx. \tag{4.80}
$$

Before going further, let us observe that, since $V_p\left(D\psi\right)\in B^{\alpha}_{2,q,\mathrm{loc}}\left(\Omega\right)$ with $q\leq 2^*_{\alpha}$ then, by Lemma 1.3.1, $V_p\left(D\psi\right)\in L^{\frac{2n}{n-2\alpha}}_{\mathrm{loc}}\left(\Omega\right)$, and so $D\psi\in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}\left(\Omega\right)$ and, by Theorem 4.1.5, we also have $Du\in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}\left(\Omega\right)$.
Let us consider the term II. By assumption (4.6) we have

$$
|II| \leq L\int_{\Omega}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|\,|\tau_h D\psi(x)|. \tag{4.81}
$$

Now we set

$$
E_1 := \left\{x\in\Omega : |Du(x)|^2+|Du(x+h)|^2 > |D\psi(x)|^2+|D\psi(x+h)|^2\right\}
$$

and

$$
E_2 := \Omega\setminus E_1,
$$

so (4.81) becomes

$$
\begin{aligned}
|II| \;\leq\; & L\int_{E_1}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|\,|\tau_h D\psi(x)| \\
& +L\int_{E_2}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|\,|\tau_h D\psi(x)| \\
=: \; & II_1+II_2.
\end{aligned} \tag{4.82}
$$

Since $1<p<2$, using Young's inequality with exponents $(2,2)$, the properties of η and Lemma 1.4.3, we get

$$
II_1 \;\leq\; L\int_{E_1}\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{4}}|\tau_h Du(x)|
$$

$$
\begin{aligned}
& \cdot \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{4}} |\tau_h D\psi(x)|\, dx \\
\leq\ & \varepsilon \int_{E_1} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx \\
& + c_\varepsilon \int_{E_1} \eta^2(x) \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h D\psi(x)|^2\, dx \\
\leq\ & \varepsilon \int_{\Omega} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx \\
& + c_\varepsilon \int_{B_{\frac{3}{4}R}} |\tau_h V_p\left(D\psi(x)\right)|^2\, dx. \qquad (4.83)
\end{aligned}
$$

For what concerns the term II_2, using Young's inequality with exponents $(2,2)$, the properties of η, and Lemmas 1.4.3 and 1.2.3, we have

$$
\begin{aligned}
II_2\ \leq\ & L \int_{E_2} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-1}{2}} |\tau_h D\psi(x)| \\
\leq\ & L \int_{E_2} \eta^2(x) \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}} |\tau_h D\psi(x)| \\
\leq\ & c \int_{B_{\frac{3}{4}R}} \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h D\psi(x)|^2\, dx \\
& + c \int_{B_{\frac{3}{4}R}} \left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p}{2}} dx \\
\leq\ & c \int_{B_{\frac{3}{4}R}} |\tau_h V_p\left(D\psi(x)\right)|^2\, dx + c \int_{B_{\lambda r}} \left(\mu^p+|D\psi(x)|^p\right) dx \\
\leq\ & c \int_{B_{\frac{3}{4}R}} |\tau_h V_p\left(D\psi(x)\right)|^2\, dx + cR^{2\alpha} \left[\int_{B_R} \left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right) dx\right]^{\frac{n-2\alpha}{n}}, \qquad (4.84)
\end{aligned}
$$

where we used the fact that $D\psi \in L^{\frac{np}{n-2\alpha}}$, and since $p < \frac{np}{n-2\alpha}$ we have

$$
\int_{B_R} |D\psi(x)|^p\, dx \leq \left(\omega_n R^n\right)^{1-\frac{n-2\alpha}{n}} \left(\int_{B_R} |D\psi(x)|^{\frac{np}{n-2\alpha}}\, dx\right)^{\frac{n-2\alpha}{n}},
$$

where ω_n is the measure of the ball of radius 1 in $\mathbb{R}^n$.
Plugging (4.83) and (4.84) into (4.82), we get the following estimate for the term II:

$$
\begin{aligned}
|II|\ \leq\ & \varepsilon \int_{\Omega} \eta^2(x) \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx \\
& + c_\varepsilon \int_{B_{\frac{3}{4}R}} |\tau_h V_p\left(D\psi(x)\right)|^2\, dx + cR^{2\alpha} \left[\int_{B_R} \left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right) dx\right]^{\frac{n-2\alpha}{n}}. \qquad (4.85)
\end{aligned}
$$

Now we consider the term III. By assumption (4.6), Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$, the fact that $1<p<2$ and the properties of η we have

$$
|III|\ \leq\ L \int_{\Omega} \eta(x)\, |D\eta(x)| \left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|\, |\tau_h\left(u-\psi\right)(x)|\, dx
$$

$$
\begin{aligned}
&\le \varepsilon \int_\Omega \eta^{\frac{p}{p-1}}(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}\cdot\frac{p}{p-1}}|\tau_h Du(x)|^{\frac{p}{p-1}-2}\cdot|\tau_h Du(x)|^2\,dx\\
&\quad+\frac{c_\varepsilon}{R^p}\int_{B_R}|\tau_h(u-\psi)(x)|^p\,dx\\
&\le \varepsilon \int_\Omega \eta^{\frac{p}{p-1}}(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&\quad+\frac{c_\varepsilon}{R^p}\int_{B_R}|\tau_h(u-\psi)(x)|^p\,dx,
\end{aligned}
$$

and using Lemma 1.2.3 we get

$$
\begin{aligned}
|III| &\le \varepsilon \int_\Omega \eta^{\frac{p}{p-1}}(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&\quad+\frac{c_\varepsilon|h|^p}{R^p}\int_{B_{\lambda R}}|D(u-\psi)(x)|^p\,dx\\
&\le \varepsilon \int_\Omega \eta^{\frac{p}{p-1}}(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&\quad+\frac{c_\varepsilon|h|^p}{R^{p-2\alpha}}\left(\int_{B_{2R}}|D(u-\psi)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}},
\end{aligned}
\tag{4.86}
$$

where, in the last line, we used the fact that $D(u-\psi)\in L^{\frac{np}{n-2\alpha}}$, arguing like in (4.84). Let us consider, now, the term IV. By (4.72), Young's inequality with exponents $(2,2)$ and recalling the properties of η we have

$$
\begin{aligned}
|IV| &\le |h|^\alpha \int_\Omega \eta^2(x)\left(g_k(x)+g_k(x+h)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h Du(x)|\,dx\\
&\le \varepsilon \int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&\quad+c_\varepsilon|h|^{2\alpha}\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)^2\\
&\quad\cdot\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}dx,
\end{aligned}
$$

where $2^{-k}\frac{R}{4}\le|h|\le 2^{-k+1}\frac{R}{4}$ for $k\in\mathbb{N}$. Using Hölder's inequality with exponents $\left(\frac{n}{2\alpha},\frac{n}{n-2\alpha}\right)$ and Lemma 1.2.3 we get

$$
\begin{aligned}
|IV| &\le \varepsilon \int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&\quad+c_\varepsilon|h|^{2\alpha}\left(\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{2\alpha}{n}}\\
&\quad\cdot\left(\int_{B_{\frac{3}{4}R}}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{np}{2(n-2\alpha)}}dx\right)^{\frac{n-2\alpha}{n}}\\
&\le \varepsilon \int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+c_\varepsilon|h|^{2\alpha}\left(\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{n-2\alpha}{n}}. \qquad (4.87)
\end{aligned}
$$

Applying estimate (4.15) with $q=\frac{np}{n-2\alpha}$, we have

$$
\fint_{B_R}|Du(x)|^{\frac{np}{n-2\alpha}}dx\le C\left\{1+\fint_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}dx+\left(\fint_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right\}. \qquad (4.88)
$$

Plugging (4.88) into (4.87) we get

$$
\begin{aligned}
|IV| \le\ & \varepsilon\int_\Omega\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_hDu(x)|^2\,dx\\
&+c_\varepsilon|h|^{2\alpha}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}. \qquad (4.89)
\end{aligned}
$$

In order to estimate the term V, we recall the properties of η, consider $2^{-k}\frac{R}{4}\le|h|\le2^{-k+1}\frac{R}{4}$ for $k\in\mathbb{N}$ and use (4.72), Young's inequality with exponents $(2,2)$, and Lemma 1.4.3, thus getting

$$
\begin{aligned}
|V| \le\ & |h|^\alpha\int_\Omega\eta^2(x)\left(g_k(x)+g_k(x+h)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_hD\psi(x)|\,dx\\
\le\ & |h|^\alpha\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_hD\psi(x)|\\
&\cdot\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{4}}\cdot\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{2-p}{4}}dx\\
\le\ & c|h|^{2\alpha}\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)^2\cdot\left(\mu^2+|Du(x)|^2\right)^{p-1}\\
&\cdot\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{2-p}{2}}dx\\
&+c\int_{B_{\frac{3}{4}R}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_hD\psi(x)|^2\,dx\\
\le\ & c|h|^{2\alpha}\int_{B_{\frac{3}{4}R}}\left(g_k(x)+g_k(x+h)\right)^2\cdot\left(\mu^2+|Du(x)|^2\right)^{p-1}\\
&\cdot\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{2-p}{2}}dx+c\int_{B_R}|\tau_hV_p\left(D\psi(x)\right)|^2\,dx. \qquad (4.90)
\end{aligned}
$$

By Hölder's inequality with exponents $\left(\frac{n}{2\alpha},\frac{n}{n-2\alpha}\right)$ and then with exponents $\left(\frac{p}{2(p-1)},\frac{p}{2-p}\right)$, (4.90) gives

$$
\begin{aligned}
|V| \leq\ & c|h|^{2\alpha}\left(\int_{B_{\frac{3}{4}R}}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
& \cdot\left(\int_{B_{\frac{3}{4}R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{n(p-1)}{n-2\alpha}}\cdot\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{n(2-p)}{2(n-2\alpha)}}\,dx\right)^{\frac{n-2\alpha}{n}} \\
& +c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx \\
\leq\ & c|h|^{2\alpha}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
& \cdot\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2\alpha)}}\,dx\right)^{\frac{2(p-1)}{p}\cdot\frac{n-2\alpha}{n}} \\
& \cdot\left(\int_{B_{\frac{3}{4}R}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{np}{2(n-2\alpha)}}\,dx\right)^{\frac{2-p}{p}\cdot\frac{n-2\alpha}{n}} \\
& +c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx \\
\leq\ & c|h|^{2\alpha}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
& \cdot\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{2(p-1)}{p}\cdot\frac{n-2\alpha}{n}} \\
& \cdot\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{2-p}{p}\cdot\frac{n-2\alpha}{n}}+c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx,
\end{aligned}
$$

where we also used Lemma 1.2.3. Using Young's inequality with exponents $\left(\frac{p}{2(p-1)},\frac{p}{2-p}\right)$, we get

$$
\begin{aligned}
|V| \leq\ & c|h|^{2\alpha}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
& \cdot\left[\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{n-2\alpha}{n}}+\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{n-2\alpha}{n}}\right] \\
& +c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx. \qquad (4.91)
\end{aligned}
$$

Using (4.88) to estimate the second integral of the right-hand side of (4.91), we get

$$
\begin{aligned}
|V| \leq\ & c|h|^{2\alpha}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
& \cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}} \\
& +c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx. \qquad (4.92)
\end{aligned}
$$

Now we consider the term VI. Recalling (4.72), taking $2^{-k}\frac{R}{4} \le |h| \le 2^{-k+1}\frac{R}{4}$ for $k \in \mathbb{N}$, the properties of η and using Hölder's inequality with exponents $\left(\frac{n}{2\alpha}, \frac{n}{n-2\alpha}\right)$, and $\left(p, \frac{p}{p-1}\right)$ we get

$$\begin{aligned}
|VI| &\le |h|^\alpha \int_\Omega \eta(x)\,|D\eta(x)|\,(g_k(x)+g_k(x+h))\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}} |\tau_h(u-\psi)(x)|\,dx \\
&\le \frac{c|h|^\alpha}{R}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{2\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
&\quad \cdot \left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{n(p-1)}{2(n-2\alpha)}} |\tau_h(u-\psi(x))|^{\frac{n}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}} \\
&\le \frac{c|h|^\alpha}{R}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{2\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
&\quad \cdot \left[\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2\alpha)}}\,dx\right)^{\frac{p-1}{p}\cdot\frac{n-2\alpha}{n}}\right. \\
&\quad \left.\cdot \left(\int_{B_R}|\tau_h(u-\psi)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np}}\right].
\end{aligned}$$

By virtue of Lemma 1.2.3 we have

$$\begin{aligned}
|VI| &\le \frac{c|h|^{\alpha+1}}{R}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{2\alpha}}\,dx\right)^{\frac{2\alpha}{n}} \\
&\quad \cdot \left[\left(\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{np}{2(n-2\alpha)}}\,dx\right)^{\frac{p-1}{p}\cdot\frac{n-2\alpha}{n}}\right. \\
&\quad \left.\cdot \left(\int_{B_{2R}}|D(u-\psi)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np}}\right] \\
&\le \frac{c|h|^{\alpha+1}}{R^{1-\alpha}}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}} \\
&\quad \cdot \left[\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{p-1}{p}\cdot\frac{n-2\alpha}{n}}\right. \\
&\quad \left.\cdot \left(\int_{B_{\lambda R}}|D(u-\psi)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np}}\right], \qquad (4.93)
\end{aligned}$$

where we used the fact that, for any $k \in \mathbb{N}$, $g_k \in L^{\frac{n}{\alpha}}(\Omega) \hookrightarrow L^{\frac{n}{2\alpha}}(\Omega)$, with the following estimate

$$\|g_k\|_{L^{\frac{n}{2\alpha}}(B_R)} \le cR^\alpha \|g_k\|_{L^{\frac{n}{\alpha}}(B_R)}.$$

Now, by Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$ and (4.88), (4.93) becomes

$$|VI| \le \frac{c|h|^{\alpha+1}}{R^{1-\alpha}}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}$$

$$\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}+\left(\int_{B_{2\lambda R}}|Du(x)|^{p}\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}. \tag{4.94}$$

Plugging (4.80), (4.85), (4.86), (4.89), (4.92) and (4.94) into (4.79), recalling the properties of η and choosing $\varepsilon=\frac{\nu}{6}$, and using Lemma 1.4.3 to estimate the left-hand side, we get

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}}|\tau_h V_p\left(Du(x)\right)|^2\,dx\\
\leq\;& cR^{2\alpha}\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{n-2\alpha}{n}}\\
&+\frac{c|h|^p}{R^{p-2\alpha}}\left(\int_{B_R}|D\left(u-\psi\right)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}\\
&+c|h|^{2\alpha}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^{p}\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}\\
&+c\int_{B_R}|\tau_h V_p\left(D\psi(x)\right)|^2\,dx+\frac{c|h|^{\alpha+1}}{R^{1-\alpha}}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^{p}\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.95}$$

Now, as in Theorem 6.1 in [43] we use a covering argument.
To this aim, we have to introduce some notations.
For any ball $B\Subset\Omega$ of radius ρ, let us denote with $Q_1\left(B\right)$ the largest cube with sides parallel to the coordinate axis concentric with B such that $Q_1\left(B\right)\subset B$, and with $Q_2\left(B\right)$ the smallest cube with sides parallel to the coordinate axis concentric with B such that $B\subset Q_2\left(B\right)$.
So we have

$$Q_1\left(B\right)\subset B\subset Q_2\left(B\right),$$

and

$$|Q_1\left(B\right)|\propto|B|\propto\rho^n.$$

Let us notice that, if we denote $\hat{B}=4B$, we have

$$Q_1\left(B\right)\subset B\Subset 2B\Subset Q_1\left(\hat{B}\right)\subset\hat{B}\Subset Q_2\left(\hat{B}\right).$$

Now let us observe that, for any $\beta\in(0,1)$, if we fix two arbitrary open sets set Ω' and Ω'' such that $\Omega'\Subset\Omega''\Subset\Omega$, for a suffinciently small value of $|h|$, we can find a finite number $K=K(h)\in\mathbb{N}$ of balls of radius $|h|^\beta$, $B_1\left(x_1,|h|^\beta\right)$, …, $B_{K(h)}\left(x_{K(h)},|h|^\beta\right)$, such that the cubes $Q_1\left(B_1\right),\dots,Q_1\left(B_k\right)$ are disjoint and

$$\left|\Omega'\setminus\bigcup_{k=1}^{K}Q_1\left(B_k\right)\right|=0,$$

and such that, if, for any k, we denote $\hat{B}_k=4B_k$, $Q_2\left(\hat{B}_k\right)\Subset\Omega''$. In order to satisfy this properties, the choice of $|h|$ depends on n, β and the distance between the boundary of Ω' and the boundary of Ω''.

It is worth noticing that each of the cubes $Q_2\left(\hat{B}_k\right)$ intersects at most a number of the other cubes $Q_2\left(\hat{B}_j\right)$, with $j \neq k$, that depends on n.
With this construction, for a constant $C = C(n)$ that depends on n but is independent of h, we have

$$|\Omega'| \propto \sum_{k=1}^{K} \left|Q_2\left(\hat{B}_k\right)\right| = C(n)|h|^{n\beta},$$

and the same kind of relation comes out when we consider integrals on this kind of sets.
If we apply this covering argument to the balls of the integrals in (4.95), whose radii are proportional to R, we have $R \propto |h|^{\beta}$, and (4.95) becomes

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |\tau_h V_p\left(Du(x)\right)|^2\, dx \\
\leq\;& c|h|^{p(1-\beta)+2\alpha\beta}\left(\int_{B_R} |D\left(u-\psi\right)(x)|^{\frac{np}{n-2\alpha}}\, dx\right)^{\frac{n-2\alpha}{n}} \\
&+c|h|^{2\alpha\beta}\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right) dx\right]^{\frac{n-2\alpha}{n}} \\
&+c|h|^{2\alpha}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}\, dx\right)^{\frac{2\alpha}{n}} \\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\, dx+\left(\int_{B_{2R}}|Du(x)|^p\, dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}} \\
&+c|h|^{\alpha-\beta+\alpha\beta+1}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}\, dx\right)^{\frac{\alpha}{n}} \\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\, dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\, dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}} \\
&+c\int_{B_R}|\tau_h V_p\left(D\psi(x)\right)|^2\, dx. \qquad (4.96)
\end{aligned}$$

Now, since $\alpha, \beta \in (0,1)$, if we set

$$\begin{aligned}
p_1 &= 2\alpha\beta \in (0,2), \\
p_2 &= p(1-\beta)+2\alpha\beta \in (0,4), \\
p_3 &= 2\alpha \in (0,2), \\
p_4 &= \alpha-\beta+\alpha\beta+1 = (\alpha+1)(1-\beta)+2\alpha\beta \in (0,3)
\end{aligned}$$

we have

$$\min_{i\in\{1,2,3,4\}} p_i = p_1 = 2\alpha\beta.$$

Now let us divide both sides of (4.96) by $|h|^{2\alpha\beta}$.

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} \frac{\left|\tau_h V_p\left(Du(x)\right)\right|^2}{|h|^{2\alpha\beta}} dx \\
\leq\; & c\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{n-2\alpha}{n}} \\
&+c|h|^{p(1-\beta)}\left(\int_{B_R}\left|D\left(u-\psi\right)(x)\right|^{\frac{np}{n-2\alpha}}dx\right)^{\frac{n-2\alpha}{n}} \\
&+c|h|^{2\alpha(1-\beta)}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{2\alpha}{n}} \\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}} \\
&+c\int_{B_R}\frac{\left|\tau_h V_p\left(D\psi(x)\right)\right|^2}{|h|^{2\alpha}}dx \\
&+c|h|^{(\alpha+1)(1-\beta)}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}} \\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}},
\end{aligned}\tag{4.97}$$

where we also used the fact that, if $|h|<1$ and $\beta,\alpha\in(0,1)$, then $|h|^{-2\alpha\beta}\leq|h|^{-2\alpha}$.
In order to conclude, we have to take the L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B\left(0,\frac{R}{4}\right)$ on the h-space, of the L^2 norm of the difference quotient of order $\alpha\beta$ of the function $V_p\left(Du\right)$. Since we have to integrate with respect to the measure $\frac{dh}{|h|^n}$ on the ball $B\left(0,\frac{R}{4}\right)$ and, for each $k\in\mathbb{N}$, the integral in the second-last line of (4.97) is taken for $2^{-k}\frac{R}{4}\leq|h|\leq 2^{-k+1}\frac{R}{4}$, it is useful to notice what follows

$$B\left(0,\frac{R}{4}\right)=\bigcup_{k=1}^{\infty}\left(B\left(0,2^{-k+1}\frac{R}{4}\right)\setminus B\left(0,2^{-k}\frac{R}{4}\right)\right)=:\bigcup_{k=1}^{\infty}E_k,$$

and it is also worth noticing that the choice of the radius $R=|h|^\beta$ is possible for small values of $|h|$, since, for $k\in\mathbb{N}$, $2^{-k}\frac{R}{4}\leq|h|\leq 2^{-k+1}\frac{R}{4}$ if and only if $2^{-\frac{k+2}{1-\beta}}\leq|h|\leq 2^{-\frac{k+1}{1-\beta}}$.
We obtain the following estimate

$$\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{\left|\tau_h V_p\left(Du(x)\right)\right|^2}{|h|^{2\alpha\beta}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n} \\
\leq\; & c\int_{B_{\frac{R}{4}}(0)}\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{q(n-2\alpha)}{2n}}\frac{dh}{|h|^n} \\
&+c\int_{B_{\frac{R}{4}}(0)}|h|^{\frac{qp(1-\beta)}{2}}\frac{dh}{|h|^n}\cdot\left(\int_{B_R}\left|D\left(u-\psi\right)(x)\right|^{\frac{np}{n-2\alpha}}dx\right)^{\frac{q(n-2\alpha)}{2n}} \\
&+c\sum_{k=1}^{\infty}\int_{E_k}|h|^{q\alpha(1-\beta)}\left(\int_{B_R}\left(g_k(x)+g_k(x+h)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{q\alpha}{n}}\frac{dh}{|h|^n}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{q(n-2\alpha)}{2n}}\\
&+c\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h V_p\left(D\psi(x)\right)|^2}{|h|^{2\alpha}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
&+c\sum_{k=1}^{\infty}\int_{E_k}|h|^{\frac{q(\alpha+1)(1-\beta)}{2}}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{q\alpha}{2n}}\frac{dh}{|h|^n}\\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{q(n-2\alpha)}{2n}}. \qquad (4.98)
\end{aligned}
$$

Now, in order to simplify the notations, we set

$$
\tilde{N}=\int_{B_{2\lambda R}}\left(1+|Du(x)|^p+|Du(x)|^{\frac{np}{n-2\alpha}}+|D\psi(x)|^p+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx, \qquad (4.99)
$$

and write (4.98) as follows

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p\left(Du(x)\right)|^2}{|h|^{2\alpha\beta}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\sum_{k=1}^{\infty}\int_{E_k}|h|^{\frac{q(\alpha+1)(1-\beta)}{2}}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{q\alpha}{2n}}\frac{dh}{|h|^n}\\
&+C\sum_{k=1}^{\infty}\int_{E_k}|h|^{q\alpha(1-\beta)}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{q\alpha}{n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h V_p\left(D\psi(x)\right)|^2}{|h|^{2\alpha}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}|h|^{\frac{qp(1-\beta)}{2}}\frac{dh}{|h|^n}, \qquad (4.100)
\end{aligned}
$$

where the constant C now depends on $\nu,\ell,L,n,p,q,\alpha,R,\tilde{N}$.
Applying Young's inequality with exponents $(2,2)$ to the first and the second integral of the right-hand side of (4.100), we get

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p\left(Du(x)\right)|^2}{|h|^{2\alpha\beta}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2q\alpha}{n}}\frac{dh}{|h|^n}\\
&+C\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{q\alpha}{n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h V_p\left(D\psi(x)\right)|^2}{|h|^{2\alpha}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}|h|^{\frac{qp(1-\beta)}{2}}\frac{dh}{|h|^n}+C\int_{B_{\frac{R}{4}}(0)}|h|^{2q\alpha(1-\beta)}\frac{dh}{|h|^n}
\end{aligned}
$$

$$+C\int_{B_{\frac{R}{4}}(0)}|h|^{q(\alpha+1)(1-\beta)}\,\frac{dh}{|h|^n}. \tag{4.101}$$

Now let us observe that, since $\alpha,\beta\in(0,1)$ and $1<p<2$, if we set $p_1=\frac{p(1-\beta)}{2}$, $p_2=2\alpha(1-\beta)$ and $p_3=(\alpha+1)(1-\beta)$ and, for each $i=1,2,3$, $q_i=q\cdot p_i$, we have

$$\kappa:=\min_{i\in\{1,2,3\}}q_i>0$$

and since $|h|<1$ we can write (4.101) as follows

$$\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p(Du(x))|^2}{|h|^{2\beta}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2q\alpha}{n}}\frac{dh}{|h|^n}\\
&+C\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_R}(g_k(x)+g_k(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{q\alpha}{n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h V_p(D\psi(x))|^2}{|h|^{2\alpha}}dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}|h|^{\kappa}\,\frac{dh}{|h|^n}=I_1+I_2+I_3+I_4
\end{aligned}\tag{4.102}$$

Now we notice that

$$I_3\le\|V_p(D\psi)\|_{B^{\alpha}_{2,q}(B_R)}, \tag{4.103}$$

which is finite by hypothesis.
For what concerns the term I_4, by calculating it in polar coordinates, we get

$$I_4=C\int_0^{\frac{R}{4}}\rho^{\kappa-1}d\rho=C(n,p,q,\alpha,R), \tag{4.104}$$

since $\kappa>0$.
Now let us write the integral I_1 in polar coordinates, so $h\in E_k$ if and only if $h=\rho\xi$ for $2^{-k}\frac{R}{4}\le\rho<2^{-k+1}\frac{R}{4}$ and some ξ in the unit sphere S^{n-1} on $\mathbb{R}^n$. Denoting by $d\sigma(\xi)$ the surface measure on S^{n-1}, we have

$$\begin{aligned}
I_1\ =\ & C\sum_{k=1}^{\infty}\int_{r_k}^{r_{k-1}}\int_{S^{n-1}}\left(\int_{B_R}(g_k(x+\rho\xi)-g_k(x))^{\frac{n}{\alpha}}\right)^{\frac{2\alpha q}{n}}d\sigma(\xi)\frac{d\rho}{\rho}\\
\le\ & C\sum_{k=1}^{\infty}\int_{r_k}^{r_{k-1}}\int_{S^{n-1}}\|\tau_{\rho\xi}g_k+g_k\|^{2q}_{L^{\frac{n}{\alpha}}(B_R)}\,d\sigma(\xi)\frac{d\rho}{\rho},
\end{aligned}\tag{4.105}$$

where we set $r_k=2^{-k}\frac{R}{4}$. Let us note that, for each $\xi\in S^{n-1}$ and $r_k\le\rho\le r_{k-1}$,

$$\|\tau_{\rho\xi}g_k+g_k\|_{L^{\frac{n}{\alpha}}(B_R)}\le\|g_k\|_{L^{\frac{n}{\alpha}}\left(B_{R-r_k\xi}\right)}+\|g_k\|_{L^{\frac{n}{\alpha}}(B_R)}\le 2\,\|g_k\|_{L^{\frac{n}{\alpha}}\left(B_{R+\frac{R}{4}}\right)}. \tag{4.106}$$

So, recalling the continuous embedding $\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)\subset\ell^{2q}\left(L^{\frac{n}{\alpha}}(B_{2R})\right)$, by (4.105) and (4.106), we get

$$I_1 \leq C \left\| \{g_k\}_k \right\|^{2q}_{\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)}. \tag{4.107}$$

We can argue in a similiar way to estimate the term I_2, thus getting

$$I_2 \leq C \left\| \{g_k\}_k \right\|^{q}_{\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)}. \tag{4.108}$$

Inserting (4.107), (4.108), (4.103), and (4.104) in (4.102), we have

$$\left\| \frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}} \right\|_{L^q\left(\frac{dh}{|h|^n};L^2\left(B_{\frac{R}{2}}\right)\right)} \leq C\left(1 + \|V_p(D\psi)\|_{B^\alpha_{2,q}(B_R)} + \left\| \{g_k\}_k \right\|^{2q}_{\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)}\right).$$

Recalling explicitly the dependence of the constant C on the value of $\tilde{N}$ given by (4.99), for an exponent $\sigma = \sigma(n,p,q,\alpha) > 0$, using the fact that $\frac{np}{n-2\alpha} > p$ recalling (4.15) and using Lemma 1.3.3 and Lemma 1.3.1, we can conclude with the estimate

$$\begin{aligned} & \left\| \frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}} \right\|_{L^q\left(\frac{dh}{|h|^n};L^2\left(B_{\frac{R}{2}}\right)\right)} \\ \leq\ & C\left(1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B^\alpha_{2,q}(B_{4R})} + \left\| \{g_k\}_k \right\|_{\ell^q\left(L^{\frac{n}{\alpha}}(B_{2R})\right)}\right)^\sigma, \end{aligned}$$

that is (4.73). □

By virtue of Lemma 1.4.9, as a consequence of Theorem 4.4.1, we have the following.

Corollary 4.4.3. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1)*, under the assumptions* (4.4)–(4.6) *and* (4.72)*, for $1 < p < 2$. Then the following implication holds*

$$V_p(D\psi) \in B^\alpha_{2,q,\mathrm{loc}}(\Omega) \implies Du \in B^{\alpha\beta}_{p,q,\mathrm{loc}}(\Omega)$$

for any $q \leq 2^*_\alpha = \frac{2n}{n-2\alpha}$ and $\beta \in (0,1)$.

4.4.2 Proof of Theorem 4.4.2

This section is devoted to the proof of Theorem 4.4.2, which is obtained using the same arguments of the previous section, taking into account that, here, the assumption 4.72 is replaced by the assumption 4.74.

Proof of Theorem 4.4.2. Since, by the hypothesis, $V_p(D\psi) \in B^\gamma_{2,\infty,\mathrm{loc}}(\Omega)$ with $\alpha < \gamma < 1$ then, recalling definition (1.4) and using Lemma 1.2.5, we have $V_p(D\psi) \in L^{\frac{2n}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$, and so $D\psi \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$. This proof goes exactly like the one of Theorem 4.4.1 until we arrive at the estimate (4.79), and the terms I, II and III can be treated in the same way, using (4.80), (4.85) and (4.86) respectively. We just need to use the assumption (4.74) instead of (4.72), in order to estimate the terms IV, V and VI.
For what concerns the term IV, using the assumption (4.74), Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{n}{2\alpha}, \frac{n}{n-2\alpha}\right)$, and Lemma 1.2.3 we get, for $|h| < \frac{R}{4}$,

$$|IV| \leq \varepsilon \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\,dx$$

$$
\begin{aligned}
&+c_\varepsilon |h|^{2\alpha}\left(\int_{B_{\frac{3}{4}R}} (g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left(\int_{B_R}\left(\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\right)dx\right)^{\frac{n-2\alpha}{n}}.
\end{aligned}
$$

and using (4.88), we obtain

$$
\begin{aligned}
|IV| \leq\ & \varepsilon\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx\\
&+c_\varepsilon |h|^{2\alpha}\left(\int_{B_{\frac{3}{4}R}} (g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}. \qquad (4.109)
\end{aligned}
$$

Let us consider, now, the term V to which we apply the assumption (4.74) in place of (4.72), and by the same arguments that we used in the previous section in order to obtain (4.92), we have, for all $h \in B_{\frac{R}{4}}(0)$,

$$
\begin{aligned}
|V| \leq\ & c|h|^{2\alpha}\left(\int_{B_R} (g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}\\
&+c\int_{B_R}|\tau_h V_p(D\psi(x))|^2\,dx. \qquad (4.110)
\end{aligned}
$$

For what concerns the term VI, again, using the assumption (4.74), and the same arguments we used in the previous section in order to get (4.94), for $|h| < \frac{R}{4}$, we obtain

$$
\begin{aligned}
|VI| \leq\ & \frac{c|h|^{\alpha+1}}{R^{1-\alpha}}\left(\int_{B_R} (g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}. \qquad (4.111)
\end{aligned}
$$

Now we plug (4.80), (4.85), (4.86), (4.109), (4.110) and (4.111) into (4.79), choose $\varepsilon = \frac{\nu}{6}$, recall the properties of η and use Lemma 1.4.3 and Lemma 1.2.3, thus getting

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}}|\tau_h V_p(Du(x))|^2\,dx\\
\leq\ & cR^{2\alpha}\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{n-2\alpha}{n}}\\
&+\frac{c|h|^p}{R^{p-2\alpha}}\left(\int_{B_R}|D(u-\psi)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}
\end{aligned}
$$

$$
\begin{aligned}
&+c|h|^{2\alpha}\left(\int_{B_R}(g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}\\
&+c\int_{B_R}|\tau_hV_p\left(D\psi(x)\right)|^2\,dx+\frac{c|h|^{\alpha+1}}{R^{1-\alpha}}\left(\int_{B_R}(g(x)+g(x+h))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}. \qquad (4.112)
\end{aligned}
$$

Now let us notice that, since, for any $\beta\in(0,1)$, $|h|\le\frac{|h|^\beta}{4}$ if and only if $|h|\le 2^{-\frac{2}{1-\beta}}$.
We can recall the same argument that we used in the previous section, after (4.95), so we have $R\propto|h|^\beta$, and dividing both sides of (4.112) by $|h|^{2\alpha\beta}$, we get

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}}\frac{|\tau_hV_p\left(Du(x)\right)|^2}{|h|^{2\alpha\beta}}dx\\
\le\;& c\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{n-2\alpha}{n}}\\
&+c|h|^{p(1-\beta)}\left(\int_{B_R}|D\left(u-\psi\right)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}\\
&+c|h|^{2\alpha(1-\beta)}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)dx\right)^{\frac{2\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}\\
&+c\int_{B_R}\frac{|\tau_hV_p\left(D\psi(x)\right)|^2}{|h|^{2\alpha}}dx+c|h|^{(1-\beta)(\alpha+1)}\left(\int_{B_R}g^{\frac{n}{\alpha}}(x)dx\right)^{\frac{\alpha}{n}}\\
&\cdot\left[1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\frac{n-2\alpha}{n}}, \qquad (4.113)
\end{aligned}
$$

where we also used Lemma 1.2.3 and that, for $|h|<\frac{R}{4}<R<1$, since $\alpha,\beta\in(0,1)$, $|h|^{-2\alpha\beta}<|h|^{-2\alpha}$.
Using Young's inequality with exponents $\left(\frac{n}{2\alpha},\frac{n}{n-2\alpha}\right)$, (4.113) becomes

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}}\frac{|\tau_hV_p\left(Du(x)\right)|^2}{|h|^{2\alpha\beta}}dx\\
\le\;& c\left[\int_{B_R}\left(1+|D\psi(x)|^{\frac{np}{n-2\alpha}}\right)dx\right]^{\frac{n-2\alpha}{n}}+c\int_{B_R}\frac{|\tau_hV_p\left(D\psi(x)\right)|^2}{|h|^{2\alpha}}dx\\
&+c|h|^{p(1-\beta)}\left(\int_{B_R}|D\left(u-\psi\right)(x)|^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}
\end{aligned}
$$

$$
\begin{aligned}
&+c|h|^{2\alpha(1-\beta)}\left[\int_{B_{2R}} g^{\frac{n}{\alpha}}(x)dx+1+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]\\
&+c|h|^{(1-\beta)(\alpha+1)}\left[\left(\int_{B_{2R}} g^{\frac{n}{\alpha}}(x)dx\right)^{\frac{1}{2}}+1+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx\right.\\
&\left.+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]. \qquad (4.114)
\end{aligned}
$$

By Lemma 1.4.9, the hypothesis $V_p\left(D\psi\right)\in B^{\gamma}_{2,\infty,\text{loc}}\left(\Omega\right)$ implies that $D\psi\in B^{\gamma}_{p,\infty,\text{loc}}\left(\Omega\right)$, and since $0<\alpha<\gamma<1$, by Lemma 1.3.2, $V_p\left(D\psi\right)\in B^{\alpha}_{2,\infty,\text{loc}}\left(\Omega\right)$ and $D\psi\in B^{\alpha}_{p,\infty,\text{loc}}\left(\Omega\right)$.
So we can take the supremum for $h\in B_{\frac{R}{4}}(0)$ at the both sides of (4.114), thus getting

$$
\begin{aligned}
&\left[V_p\left(Du\right)\right]_{\dot{B}^{\alpha\beta}_{2,\infty}\left(B_{\frac{R}{2}}\right)}\\
\le\;& C\left[1+\int_{B_{2R}} g^{\frac{n}{\alpha}}(x)dx+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\sigma}\\
&+C\left[V_p\left(D\psi\right)\right]_{\dot{B}^{\alpha}_{2,\infty}(B_R)},
\end{aligned}
$$

where the exponent $\sigma>0$ depends on n,p and α and the constant $C>0$ depends on n,p,α,ν,L, and R.
Recalling the definition of the norm in Besov-Lipschitz spaces and using Lemma 1.3.2, we have

$$
\begin{aligned}
&\left[V_p\left(Du\right)\right]_{\dot{B}^{\alpha\beta}_{2,\infty}\left(B_{\frac{R}{2}}\right)}\\
\le\;& C\left[1+\int_{B_{2R}} g^{\frac{n}{\alpha}}(x)dx+\int_{B_{2\lambda R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\left(\int_{B_{2\lambda R}}|Du(x)|^p\,dx\right)^{\frac{n}{n-2\alpha}}\right]^{\sigma}\\
&+C\left\|V_p\left(D\psi\right)\right\|_{B^{\gamma}_{2,\infty}(B_R)}.
\end{aligned}
$$

Recalling that, for $0<\alpha<\gamma<1$, we have $p<\frac{np}{n-2\alpha}<\frac{np}{n-2\gamma}$, we get

$$
\begin{aligned}
&\left[V_p\left(Du\right)\right]_{\dot{B}^{\alpha\beta}_{2,\infty}\left(B_{\frac{R}{2}}\right)}\\
\le\;& C\Big(1+\|Du\|_{L^p(B_{2\lambda R})}+\|D\psi\|_{L^{\frac{np}{n-2\gamma}}(B_{2\lambda R})}\\
&+\left\|V_p\left(D\psi\right)\right\|_{B^{\gamma}_{2,\infty}(B_R)}+\|g\|_{L^{\frac{n}{\alpha}}(B_{2R})}\Big)^{\sigma},
\end{aligned}
$$

and applying Lemma 1.2.5 to the function $V_p\left(D\psi\right)$, we get

$$
\begin{aligned}
&\left[V_p\left(Du\right)\right]_{\dot{B}^{\alpha\beta}_{2,\infty}\left(B_{\frac{R}{2}}\right)}\\
\le\;& C\left(1+\|Du\|_{L^p(B_{4R})}+\left\|V_p\left(D\psi\right)\right\|_{B^{\gamma}_{2,\infty}(B_{4R})}+\|g\|_{L^{\frac{n}{\alpha}}(B_{2R})}\right)^{\sigma},
\end{aligned}
$$

that is (4.75). □

By virtue of Lemma 1.4.9, we have the following consequence of Theorem 4.4.2.

Corollary 4.4.4. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1)*, under the assumptions* (4.4)–(4.6) *for $1 < p < 2$. If there exists $\alpha \in (0,1)$ and a function $g \in L^{\frac{n}{\alpha}}_{\mathrm{loc}}(\Omega)$ such that* (4.74) *holds, then, provided $0 < \alpha < \gamma < 1$ the following implication holds*

$$V_p(D\psi) \in B^{\gamma}_{2,\infty,\mathrm{loc}}(\Omega) \implies Du \in B^{\alpha\beta}_{p,\infty,\mathrm{loc}}(\Omega),$$

for any $\beta \in (0,1)$.

4.5 Higher differentiability for solutions to problems with bounded obstacle under sub-quadratic growth conditions

In this section we will consider, again, solutions to problems of the form (4.1), under assumptions (4.4)–(4.6), with $1 < p < 2$, where we still use the notation (4.3) and the map $\xi \mapsto A(x,\xi)$ is of class $C^1(\mathbb{R}^n)$.
Under these assumptions, as in Section 4.3, (4.6) implies (4.7), for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and for a.e. $x \in \Omega$.
The result we discuss below can be seen as the sub-quadratic growth version of the result we proved in Section 4.2.
Indeed, in this case we assume that the map $x \mapsto A(x,\xi)$ belongs to $W^{1,p+2}_{\mathrm{loc}}(\Omega)$ for every $\xi \in \mathbb{R}^n$ or, equivalently, that there exists a non-negative function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$ such that (4.10) and (4.11) hold, but now we are dealing with the case $1 < p < 2$.
More precisely, we want to prove the following result.

Theorem 4.5.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1) *under assumptions* (4.4)–(4.6) *and let us assume that there exists a function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$ such that* (4.10) *and* (4.11) *hold, for $1 < p < 2$.*
Then the following implication holds:

$$\psi \in L^{\infty}_{\mathrm{loc}}(\Omega) \text{ and } V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega) \implies V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Moreover, for any ball $B_{8R} \Subset \Omega$, the following estimate holds

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |DV_p(Du(x))|^2\, dx \\
\le\ & \frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})} + \|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{R^2} \\
&\cdot \left[\int_{B_{4R}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_{4R}} g^{p+2}(x)dx \right. \\
&\left. + \int_{B_{4R}} |DV_p(D\psi(x))|^2\, dx + \int_{B_{4R}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p}{2}} dx\right]^{\sigma_2},
\end{aligned} \tag{4.115}$$

where $c > 0$ depends on n, p, ν, L and ℓ and $\sigma_1 > 0$ and $\sigma_2 > 0$ depend on n and p.

Let us notice that here, as in Theorem 4.2.1, the regularity of the coefficients does not depend on the dimension n. Moreover, since we're assuming $1 < p < 2$, if we have $p + 2 < n$ (i.e. if $n \ge 4$), if the obstacle is bounded, we can assume less regularity on the coefficients, if we compare this risult with Theorem 4.3.1, where the obstacle is not assumed to be bounded. The same result can be obtained by removing the assumption of boundedness of the obstacle if we consider a priori bounded minimizers.

4.5.1 Proof of Theorem 4.5.1

Proof of the Theorem 4.5.1. **Step 1: the a priori estimate.**
Let us observe that, if $V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ then, by virtue of Remark 1.4.7 and estimate (1.2), we get $D\psi \in L^{p+2}_{\mathrm{loc}}(\Omega)$.
Suppose that $u \in \mathcal{K}_\psi(\Omega)$ is a solution to the obstacle problem (4.1) such that

$$V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Our first step is to prove the following a priori estimate

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |DV_p(Du(x))|^2 dx \\
\le\ & \frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})} + \|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{R^2} \\
& \cdot\left[\int_{B_{4R}}\left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + \int_{B_{4R}} g^{p+2}(x)dx\right. \\
& \left. + \int_{B_{4R}} |DV_p(D\psi(x))|^2 dx + \int_{B_{4R}}\left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p}{2}} dx\right]^{\sigma_2},
\end{aligned} \tag{4.116}$$

for any ball $B_{8R} \Subset \Omega$.
By estimate (4.12), since $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, we have $u \in L^\infty_{\mathrm{loc}}(\Omega)$.
Recalling Remarks 1.4.6 and 1.4.7, and Lemma 1.1.3, thanks to the a priori assumption $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we have $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$.
In order to apply Theorem 4.1.2, as we did in the proof of Theorem 4.3.1, let us recall that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the equation (4.13) if and only if, for any $\varphi \in W^{1,p}_0(\Omega)$,

$$\int_\Omega \langle A(x, Du(x)), D\varphi(x)\rangle dx = -\int_\Omega \mathrm{div} A(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\varphi(x)dx \tag{4.117}$$

Let us fix a ball $B_{8R} \Subset \Omega$ and arbitrary radii $\frac{R}{2} \le r < \tilde{s} < t < \tilde{t} < \lambda r$, with $1 < \lambda < 2$. Let us consider a cut-off function $\eta \in C^\infty_0(B_t)$ such that $\eta \equiv 1$ on $B_{\tilde{s}}$ and $|D\eta| \le \frac{c}{t-\tilde{s}}$. From now on, with no loss of generality, we suppose $R < \frac{1}{4}$.
For any $s = 1, \dots, n$ and $h \in \mathbb{R}$ with $|h|$ is sufficiently small, let us consider the test function

$$\varphi = \tau_{s,-h}\left(\eta^2 \cdot \tau_{s,h} u\right).$$

For this choice of φ, using Proposition 1.2.2, the left-hand side of (4.117) can be written as follows:

$$\begin{aligned}
&\int_\Omega \left\langle A(x, Du(x)), D\left(\tau_{-h}\left(\eta^2(x)\tau_h u(x)\right)\right)\right\rangle dx \\
=\ & \int_\Omega \left\langle \tau_h A(x, Du(x)), D\left(\eta^2(x)\tau_h u(x)\right)\right\rangle dx \\
=\ & \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), D\left(\eta^2(x)\tau_h u(x)\right)\right\rangle dx \\
=\ & \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), \eta^2(x)\tau_h Du(x)\right\rangle dx \\
& + \int_\Omega \langle A(x+h, Du(x+h)) - A(x, Du(x)), 2\eta(x)D\eta(x)\tau_h u(x)\rangle dx
\end{aligned}$$

$$\begin{aligned}
&= \int_\Omega \left\langle A(x, Du(x+h)) - A(x, Du(x)), \eta^2(x)\tau_h Du(x)\right\rangle dx \\
&\quad + \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x+h)), \eta^2(x)\tau_h Du(x)\right\rangle dx \\
&\quad + \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), 2\eta(x)D\eta(x)\tau_h u(x)\right\rangle dx \\
&:= I_0 + I + II, \qquad (4.118)
\end{aligned}$$

where, for the finite differences, we used the simplified notation

$$\tau_h F(x) = F(x+h) - F(x),$$

with $h \in \mathbb{R}^n$, in place of

$$\tau_{s,h} F(x) = F(x + he_s) - F(x),$$

with $h \in \mathbb{R}$ and, in the following, we will specify the direction only if it will be necessary. Since the right-hand side of (4.117) is not zero only where $u = \psi$, using the test function given above, it becomes

$$-\int_\Omega \mathrm{div} A(x, D\psi(x))\, \chi_{\{u=\psi\}}(x)\tau_{-h}\left(\eta^2(x)\tau_h\psi(x)\right) dx, \qquad (4.119)$$

and since the map $x \mapsto A(x,\xi)$ belongs to $W^{1,p+2}_{\mathrm{loc}}(\Omega)$ for any $\xi \in \mathbb{R}^n$, the map $\xi \mapsto A(x,\xi)$ belongs to $C^1(\mathbb{R}^n)$ for a.e. $x \in \Omega$ and $V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we can argue exactly like in the proof of Theorem 4.3.1, and using the same notations, we can write (4.119) as follows

$$\begin{aligned}
&-\int_\Omega \Big\{\left[A_x(x, D\psi(x)) + A_\xi(x, D\psi(x))\, D^2\psi(x)\right]\chi_{\{u=\psi\}}(x) \\
&\quad \cdot \tau_{-h}\left(\eta^2(x)\tau_h\psi(x)\right)\Big\} dx \\
&= -III - IV - V - VI. \qquad (4.120)
\end{aligned}$$

Inserting (4.118) and (4.120) in (4.117) we get

$$I_0 = -I - II - III - IV - V - VI,$$

and so

$$I_0 \le |I| + |II| + |III| + |IV| + |V| + |VI|. \qquad (4.121)$$

By assumption (4.5), we have

$$I_0 \ge \nu \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx. \qquad (4.122)$$

Let us consider the term I. By assumption (4.10), and using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{p+2}{p}, \frac{p+2}{2}\right)$, and the properties of η, we get

$$\begin{aligned}
|I| &\le \int_\Omega |h|\,(g(x+h) + g(x))\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-1}{2}} \eta^2(x)\,|\tau_h Du(x)|\, dx \\
&\le \varepsilon \int_\Omega \eta^2(x)\left(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} |\tau_h Du(x)|^2\, dx
\end{aligned}$$

$$
\begin{aligned}
& +c_\varepsilon|h|^2\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p}{2}}(g(x+h)+g(x))^2\,dx \\
\leq\ & \varepsilon\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx \\
& +c_\varepsilon|h|^2\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}} \\
& \cdot\left(\int_{B_{\lambda r}} g^{p+2}(x)dx\right)^{\frac{2}{p+2}} \\
\leq\ & \varepsilon\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx \\
& +c_\varepsilon|h|^2\left(\int_{B_{\tilde{t}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}}\cdot\left(\int_{B_{\lambda r}} g^{p+2}(x)dx\right)^{\frac{2}{p+2}}, \qquad (4.123)
\end{aligned}
$$

where we also used Lemma 1.2.3.
Let us consider the term II. If we denote again finite differences with respect to a precise direction $s=1,\dots,n$, with an integration by parts, we have

$$
\begin{aligned}
-II &= -2h\int_\Omega\left\langle\int_0^1\frac{d}{dx_s}A\left(x+\theta he_s,Du(x+\theta he_s)\right)d\theta,\eta(x)D\eta(x)\tau_{s,h}u(x)\right\rangle dx \\
=\ & 2h\int_\Omega\left\langle\int_0^1\left(A\left(x+\theta he_s,Du(x+\theta he_s)\right)\right)d\theta,\frac{d}{dx_s}\left(\eta(x)D\eta(x)\tau_{s,h}u(x)\right)\right\rangle dx,
\end{aligned}
$$

where, for $s=1,\dots,n$, e_s is the unit vector in the x_s direction, and now $h\in\mathbb{R}$.
So we can estimate II as follows

$$
\begin{aligned}
|II| \leq\ & 2|h|\int_\Omega\int_0^1\left|A\left(x+\theta he_s,Du\left(x+\theta he_s\right)\right)\right| \\
& \cdot\left(|D\eta(x)|^2|\tau_{s,h}u(x)|+\eta(x)\left|D^2\eta(x)\right||\tau_{s,h}u(x)|\right)d\theta dx \\
& +2|h|\int_\Omega\int_0^1\left|A\left(x+\theta he_s,Du\left(x+\theta he_s\right)\right)\right| \\
& \cdot\left(\eta(x)\left|D\eta(x)\right||\tau_{s,h}Du(x)|\right)d\theta dx \\
\leq\ & 2|h|\int_\Omega\int_0^1\left|A\left(x+\theta he_s,Du\left(x+\theta he_s\right)\right)\right| \\
& \cdot\left(|D\eta(x)|^2+\eta(x)\left|D^2\eta(x)\right|\right)d\theta\,|\tau_{s,h}u(x)|\,dx \\
& +2|h|\int_\Omega\int_0^1\left|A\left(x+\theta he_s,Du\left(x+h_s\theta e_s\right)\right)\right| \\
& \cdot\eta(x)\left|D\eta(x)\right||\tau_{s,h}Du(x)|\,d\theta dx.
\end{aligned}
$$

Now, recalling the properties of η, assumption (4.4), and using Hölder's inequality with exponents $\left(p,\frac{p}{p-1}\right)$, Lemma 1.2.3 and Young's inequality with exponents $(2,2)$, we get

$$
|II| \leq 2c|h|\int_{B_t}\int_0^1\left(\mu^2+|Du(x)|^2+\left|Du\left(x+\theta he_s\right)\right|^2\right)^{\frac{p-1}{2}}
$$

$$
\begin{aligned}
& \cdot\left(|D\eta(x)|^2+\eta(x)\left|D^2\eta(x)\right|\right) d\theta \left|\tau_{s,h}u(x)\right| dx \\
& +2c|h|\int_\Omega\int_0^1\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{\frac{p-1}{2}} \\
& \cdot\eta(x)\left|D\eta(x)\right|\left|\tau_{s,h}Du(x)\right| d\theta dx \\
= \; & 2c|h|\int_0^1\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{\frac{p-1}{2}} \\
& \cdot\left(|D\eta(x)|^2+\eta(x)\left|D^2\eta(x)\right|\right)\left|\tau_{s,h}u(x)\right| dxd\theta \\
& +2c|h|\int_0^1\int_\Omega\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{\frac{p-1}{2}} \\
& \cdot\eta(x)\left|D\eta(x)\right|\left|\tau_{s,h}Du(x)\right| dxd\theta \\
\leq \; & \frac{c|h|}{(t-\tilde{s})^2}\int_0^1\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} d\theta \\
& \cdot\left(\int_{B_t}\left|\tau_{s,h}u(x)\right|^p dx\right)^{\frac{1}{p}} \\
& +\varepsilon\int_\Omega\eta^2(x)\left|\tau_{s,h}Du(x)\right|^2\left(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\right)^{\frac{p-2}{2}} dx \\
& +\frac{c_\varepsilon\,|h|^2}{(t-\tilde{s})^2}\int_0^1\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{p-1} \\
& \cdot\left(\mu^2+|Du(x)|^2+|Du\left(x+\theta he_s\right)|^2\right)^{\frac{2-p}{2}} dxd\theta.
\end{aligned}
$$

Now, if we use again the simplified notation for finite diferences, with $h \in \mathbb{R}^n$ in place of he_s where $h \in \mathbb{R}$, by Lemma 1.2.3, we get

$$
\begin{aligned}
|II| \; \leq \; & \frac{c|h|^2}{(t-\tilde{s})^2}\int_0^1\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} d\theta \\
& \cdot\left(\int_{B_{\tilde{t}}}|Du(x)|^p dx\right)^{\frac{1}{p}} \\
& +\varepsilon\int_\Omega\eta^2(x)\left|\tau_h Du(x)\right|^2\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
& +\frac{c_\varepsilon\,|h|^2}{(t-\tilde{s})^2}\int_0^1\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{p-1}{2}}\right. \\
& \left.\cdot\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{2-p}{4}} dx\right]^2 d\theta.
\end{aligned} \tag{4.124}
$$

Let us consider, now, the term III. By (4.11) and the properties of η, we get

$$
\begin{aligned}
|III| \; \leq \; & |h|^2\int_0^1\int_0^1\int_{B_{\lambda r}} g(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}} \\
& \cdot\left|D^2\psi(x+\sigma h-\theta h)\right| dxd\sigma d\theta.
\end{aligned}
$$

Using Young's inequality with exponents $(2, 2)$, we get

$$\begin{aligned}|III| &\le c|h|^2\int_0^1\int_0^1\left[\int_{B_{\lambda r}} g^2(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p}{2}}dx\right.\\ &\quad+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}\\ &\quad\left.\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|^2dx\right]d\sigma d\theta. \qquad (4.125)\end{aligned}$$

Again by Young's inequality with exponents $\left(\frac{p+2}{2},\frac{p+2}{p}\right)$ in the first integral of (4.125), we get

$$\begin{aligned}|III| &\le c|h|^2\left[\int_{B_{\lambda r}} g^{p+2}(x)dx+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p+2}{2}}dx\right]\\ &\quad+c|h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}\\ &\quad\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|^2dxd\sigma d\theta. \qquad (4.126)\end{aligned}$$

By (4.11), we can estimate the term IV, thus getting

$$\begin{aligned}|IV| &\le 2|h|^2\int_{B_{\lambda r}} g(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}}\\ &\quad\cdot\int_0^1\int_0^1|D\psi(x+\sigma h-\theta h)|\,|D\eta(x-\theta h)|\,d\sigma d\theta dx. \qquad (4.127)\end{aligned}$$

Let us consider, now, the term V. By assumption (4.7), we get

$$\begin{aligned}|V| &\le |h|^2\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|\\ &\quad\cdot\int_0^1\int_0^1\left|D^2\psi(x+\sigma h-\theta h)\right|d\sigma d\theta dx. \qquad (4.128)\end{aligned}$$

Recalling (4.7) again, we have

$$\begin{aligned}|VI| &\le 2|h|^2\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|\\ &\quad\cdot\int_0^1\int_0^1|D\eta(x-\theta h)|\,|D\psi(x+\sigma h-\theta h)|\,d\sigma d\theta dx. \qquad (4.129)\end{aligned}$$

Now, inserting (4.122), (4.123), (4.124), (4.126), (4.127), (4.128) and (4.129) in (4.121), recalling the properties of η and choosing a sufficiently small value of ε, we get

$$\begin{aligned}&\int_\Omega \eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_h Du(x)|^2dx\\ \le\;& c|h|^2\left(\int_{B_{\tilde t}}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}}\cdot\left(\int_{B_{\lambda r}} g^{p+2}(x)dx\right)^{\frac{2}{p+2}}\\ &+\frac{c|h|^2}{(t-\tilde s)^2}\int_0^1\left(\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-1}{p}}d\theta\end{aligned}$$

$$\cdot\left(\int_{B_{\tilde{t}}}|Du(x)|^p\,dx\right)^{\frac{1}{p}}$$
$$+\frac{c\,|h|^2}{(t-\tilde{s})^2}\int_0^1\int_{B_t}\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{p-1}$$
$$\cdot\left(\mu^2+|Du(x)|^2+|Du(x+\theta h)|^2\right)^{\frac{2-p}{2}}dxd\theta$$
$$+c|h|^2\left[\int_{B_{\lambda r}}g^{p+2}(x)dx+\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p+2}{2}}dx\right]$$
$$+c|h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}$$
$$\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|^2dxd\sigma d\theta$$
$$+2|h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}g(x)\left(\mu^2+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-1}{2}}$$
$$\cdot|D\psi(x+\sigma h-\theta h)|\,|D\eta(x-\theta h)|\,dxd\sigma d\theta$$
$$+|h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|$$
$$\cdot\left|D^2\psi(x+\sigma h-\theta h)\right|dxd\sigma d\theta$$
$$+2|h|^2\int_0^1\int_0^1\int_{B_{\lambda r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|$$
$$\cdot|D\eta(x-\theta h)|\,|D\psi(x+\sigma h-\theta h)|\,dxd\sigma d\theta. \tag{4.130}$$

By Lemma 1.4.3 and the properties of η, the left-hand side of (4.130) can be bounded from below as follows

$$\int_\Omega\eta^2(x)\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\tau_hDu(x)|^2\,dx\geq\int_\Omega\eta^2(x)\,|\tau_hV_p\left(Du(x)\right)|^2\,dx. \tag{4.131}$$

So, by (4.131) and (4.130), recalling the properties of η and using Lemma 1.2.3, we get

$$\int_\Omega\eta^2(x)\,|\tau_hV_p\left(Du(x)\right)|^2\,dx$$
$$\leq c|h|^2\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}}\cdot\left(\int_{B_{2r}}g^{p+2}(x)dx\right)^{\frac{2}{p+2}}$$
$$+\frac{c|h|^2}{(t-\tilde{s})^2}\int_{B_{2r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx$$
$$+c|h|^2\left[\int_{B_{2r}}g^{p+2}(x)dx+\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p+2}{2}}dx\right]$$
$$+\frac{c|h|^2}{t-\tilde{s}}\int_{B_{2r}}g(x)\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx$$
$$+c|h|^2\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-2}{2}}\left|D^2\psi(x)\right|^2dx$$

$$+\frac{c|h|^2}{t-\tilde{s}}\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p-1}{2}}\left|D^2\psi(x)\right|dx.$$

Now we apply Hölder's inequality with exponents $\left(p+2, p+2, \frac{p+2}{p}\right)$ to the integral of the fifth line, Young's inequality with exponents $(2,2)$ to the last integral, thus getting

$$\begin{aligned}
&\int_\Omega \eta^2(x)\left|\tau_h V_p\left(Du(x)\right)\right|^2 dx\\
\leq\ & c|h|^2\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}}\cdot\left(\int_{B_{2r}}g^{p+2}(x)dx\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{(t-\tilde{s})^2}\int_{B_{2r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\\
&+c|h|^2\left[\int_{B_{2r}}g^{p+2}(x)dx+\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p+2}{2}}dx\right]\\
&+\frac{c|h|^2}{t-\tilde{s}}\left(\int_{B_{2r}}g^{p+2}(x)dx\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p+2}{2}}dx\right)^{\frac{p}{p+2}}\\
&+\frac{c|h|^2}{t-\tilde{s}}\left[\int_{B_{2r}}\left|DV_p\left(D\psi(x)\right)\right|^2dx+\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]
\end{aligned}$$

for a constant $c=c(n,p,\nu,L,\ell)$, where we also used (1.6). By Young's inequality with exponents $\left(\frac{p+2}{2},\frac{p+2}{p}\right)$, for some $\varepsilon>0$, we get

$$\begin{aligned}
&\int_\Omega \eta^2(x)\left|\tau_h V_p\left(Du(x)\right)\right|^2 dx\\
\leq\ & c|h|^2\left[\varepsilon\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}dx+c_\varepsilon\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p+2}{2}}dx\right]\\
&+\frac{c|h|^2}{t-\tilde{s}}\left[\int_{B_{2r}}g^{p+2}(x)dx+\left(\int_{B_{2r}}g^{p+2}(x)dx\right)^{\frac{1}{2}}\right]\\
&+\frac{c|h|^2}{t-\tilde{s}}\left[\int_{B_{2r}}\left|DV_p\left(D\psi(x)\right)\right|^2dx+\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]\\
&+\frac{c|h|^2}{(t-\tilde{s})^2}\left(\int_{B_{2r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right),
\end{aligned}$$

and since $\eta\equiv 1$ on $B_{\tilde{s}}$, we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}}\left|\tau_h V_p\left(Du(x)\right)\right|^2 dx\\
\leq\ & c|h|^2\left[\varepsilon\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}dx+c_\varepsilon\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p+2}{2}}dx\right]\\
&+\frac{c|h|^2}{t-\tilde{s}}\left[\int_{B_{2r}}g^{p+2}(x)dx+\left(\int_{B_{2r}}g^{p+2}(x)dx\right)^{\frac{1}{2}}\right]\\
&+\frac{c|h|^2}{t-\tilde{s}}\left[\int_{B_{2r}}\left|DV_p\left(D\psi(x)\right)\right|^2dx+\int_{B_{2r}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]\\
&+\frac{c|h|^2}{(t-\tilde{s})^2}\left(\int_{B_{2r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right).
\end{aligned}$$

Thanks to Lemma 1.2.4, deduce

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2 \, dx \\
\leq\ & c\left[\varepsilon \int_{B_{\lambda r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+2}{2}} dx + c_\varepsilon \int_{B_{2r}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p+2}{2}} dx\right] \\
&+ \frac{c}{t-\tilde{s}} \left[\int_{B_{2r}} g^{p+2}(x) dx + \left(\int_{B_{2r}} g^{p+2}(x) dx\right)^{\frac{1}{2}}\right] \\
&+ \frac{c}{t-\tilde{s}} \left[\int_{B_{2r}} |DV_p(D\psi(x))|^2 \, dx + \int_{B_{2r}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p}{2}} dx\right] \\
&+ \frac{c}{(t-\tilde{s})^2} \left(\int_{B_{2r}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx\right). \qquad (4.132)
\end{aligned}$$

Now, since $\mu \in [0,1]$, we have

$$\begin{aligned}
\left(\mu^2 + |Du|^2\right)^{\frac{p+2}{2}} &= \mu^2 \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} + \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} |Du|^2 \\
&\leq \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} + \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} |Du|^2 \qquad (4.133)
\end{aligned}$$

and, similarly,

$$\left(\mu^2 + |D\psi|^2\right)^{\frac{p+2}{2}} \leq \left(\mu^2 + |D\psi|^2\right)^{\frac{p}{2}} + \left(\mu^2 + |D\psi|^2\right)^{\frac{p}{2}} |D\psi|^2. \qquad (4.134)$$

Since $\tilde{t} < \lambda r < \lambda \tilde{s} < \lambda t < \lambda^2 r < 4R < 1$, if we use (1.2) with $\phi \in C_0^\infty(B_{\lambda t})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\lambda \tilde{s}}$ and $|D\phi| \leq \frac{c}{\lambda(t-\tilde{s})}$, recalling (1.6), thanks to (4.133) we get

$$\begin{aligned}
&\int_{B_{\tilde{t}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p+2}{2}} dx \\
\leq\ & c \|u\|^2_{L^\infty(B_{4R})} \int_{B_{\lambda t}} |DV_p(Du(x))|^2 \, dx \\
&+ \frac{c \|u\|^2_{L^\infty(B_{4R})}}{\lambda^2 (t-\tilde{s})^2} \int_{B_{4R}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx. \qquad (4.135)
\end{aligned}$$

Arguing in the same way, using (1.2) with $\phi \in C_0^\infty(B_{2t})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{2\tilde{s}}$ and $|D\phi| \leq \frac{c}{2(t-\tilde{s})}$, thanks to (4.134), since $r < \tilde{s} < t < R < \frac{1}{4}$, we get

$$\begin{aligned}
&\int_{B_{2r}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p+2}{2}} dx \\
\leq\ & c \|\psi\|^2_{L^\infty(B_{4R})} \int_{B_{4R}} |DV_p(D\psi(x))|^2 \, dx \\
&+ \frac{c \|\psi\|^2_{L^\infty(B_{4R})}}{4 (t-\tilde{s})^2} \int_{B_{4R}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p}{2}} dx. \qquad (4.136)
\end{aligned}$$

Therefore, inserting (4.135) and (4.136) into (4.132), since $1 < \lambda < 2$ and $t - \tilde{s} < 1$, we get

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\,dx \\
\leq\;& c\,\|u\|^2_{L^\infty(B_{4R})}\cdot\varepsilon\int_{B_{\lambda t}} |DV_p(Du(x))|^2\,dx \\
&+\frac{c_\varepsilon\,\|u\|^2_{L^\infty(B_{4R})}}{(t-\tilde{s})^2}\left(\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx\right) \\
&+\frac{c_\varepsilon\,\|\psi\|^2_{L^\infty(B_{4R})}}{(t-\tilde{s})^2}\left[\int_{B_{4R}} g^{p+2}(x)dx+\left(\int_{B_{4R}} g^{p+2}(x)dx\right)^{\frac{1}{2}}\right. \\
&\left.+\int_{B_{4R}} |DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right],
\end{aligned}$$

and recalling (4.12), we have

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\,dx \\
\leq\;& \varepsilon\cdot c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}\int_{B_{\lambda t}} |DV_p(Du(x))|^2\,dx \\
&+\frac{c_\varepsilon\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{(t-\tilde{s})^2} \\
&\cdot\left[\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{4R}} g^{p+2}(x)dx\right. \\
&\left.+\int_{B_{4R}} |DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]^{\sigma_2},
\end{aligned}\tag{4.137}$$

where σ_1 and σ_2 depend on n and p. Now, if we choose $\varepsilon>0$ such that

$$\varepsilon\cdot c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}=\frac{1}{2},$$

(4.137) becomes

$$\begin{aligned}
&\int_{B_{\tilde{s}}} |DV_p(Du(x))|^2\,dx \\
\leq\;& \frac{1}{2}\int_{B_{\lambda t}} |DV_p(Du(x))|^2\,dx \\
&+\frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{(t-\tilde{s})^2} \\
&\cdot\left[\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{4R}} g^{p+2}(x)dx\right. \\
&\left.+\int_{B_{4R}} |DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right]^{\sigma_2},
\end{aligned}\tag{4.138}$$

and since (4.138) holds for any $\frac{R}{2}\leq r<\tilde{s}<t<\lambda r<R$ and for any $\lambda\in(1,2)$ and the constant c is independent of the radii, we can pass to the limit as $\tilde{s}\to r$ and $t\to\lambda r$, thus getting

$$\int_{B_r} |DV_p(Du(x))|^2\,dx$$

$$
\begin{aligned}
\leq\ & \frac{1}{2}\int_{B_{\lambda^2 r}} |DV_p(Du(x))|^2\,dx \\
& +\frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{r^2(\lambda-1)^2} \\
& \cdot\Bigg[\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{4R}} g^{p+2}(x)dx \\
& +\int_{B_{4R}}|DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\Bigg]^{\sigma_2},
\end{aligned}
$$

which also implies

$$
\begin{aligned}
& \int_{B_r} |DV_p(Du(x))|^2\,dx \\
\leq\ & \frac{1}{2}\int_{B_{\lambda^2 r}} |DV_p(Du(x))|^2\,dx \\
& +\frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{r^2(\lambda^2-1)^2} \\
& \cdot\Bigg[\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{4R}} g^{p+2}(x)dx \\
& +\int_{B_{4R}}|DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\Bigg]^{\sigma_2}. \qquad (4.139)
\end{aligned}
$$

Now, setting

$$
h(r)=\int_{B_r}|DV_p(Du(x))|^2\,dx,
$$

$$
\begin{aligned}
A\ =\ & c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1} \\
& \cdot\Bigg[\int_{B_{4R}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+\int_{B_{4R}} g^{p+2}(x)dx \\
& +\int_{B_{4R}}|DV_p(D\psi(x))|^2\,dx+\int_{B_{4R}}\left(\mu^2+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\Bigg]^{\sigma_2}
\end{aligned}
$$

and

$$
B\ =\ 0,
$$

since (4.139) holds for any $1<\lambda<2$, we can apply the Interation Lemma 1.1.1 with

$$
\theta=\frac{1}{2} \qquad \text{and} \qquad \gamma=2,
$$

thus getting

$$
\begin{aligned}
& \int_{B_{\frac{R}{2}}} |DV_p(Du(x))|^2\,dx \\
\leq\ & \frac{c\left(\|\psi\|^2_{L^\infty(B_{8R})}+\|u\|^2_{L^{p^*}(B_{8R})}\right)^{\sigma_1}}{R^2}
\end{aligned}
$$

$$\cdot \left[\int_{B_{4R}} \left(\mu^2 + |Du(x)|^2 \right)^{\frac{p}{2}} dx + \int_{B_{4R}} g^{p+2}(x) dx \right.$$
$$\left. + \int_{B_{4R}} |DV_p(D\psi(x))|^2 dx + \int_{B_{4R}} \left(\mu^2 + |D\psi(x)|^2 \right)^{\frac{p}{2}} dx \right]^{\sigma_2},$$

that is (4.116), where $c > 0$ depends on n, p, ν, L and ℓ, and $\sigma_1, \sigma_2 > 0$ depend on n and p.

Step 2: the approximation.
Fix an open set $\Omega' \Subset \Omega$, and for a smooth kernel $\phi \in C_0^\infty(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)} \phi = 1$, and for any $\varepsilon \in (0, d(\Omega', \partial\Omega))$, let us consider the corresponding family of mollifiers $\{\phi_\varepsilon\}_\varepsilon$, and set

$$g_\varepsilon = g * \phi_\varepsilon,$$

$$\mathcal{K}_{\varepsilon,\psi}(\Omega) = \left\{ w \in u + W_0^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \right\}$$

and

$$A_\varepsilon(x, \xi) = \int_{B_1} \phi(\omega) A(x + \varepsilon\omega, \xi) \, d\omega$$

on Ω', for each $\varepsilon \in (0, d(\Omega', \partial\Omega))$. The assumptions (4.4)–(4.6) imply

$$|A_\varepsilon(x, \xi)| \leq \ell \left(\mu^2 + |\xi|^2 \right)^{\frac{p-1}{2}}, \tag{4.140}$$

$$\langle A_\varepsilon(x, \xi) - A_\varepsilon(x, \eta), \xi - \eta \rangle \geq \nu |\eta - \xi|^2 \left(\mu^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{p-2}{2}}. \tag{4.141}$$

$$|A_\varepsilon(x, \xi) - A_\varepsilon(x, \eta)| \leq L \, |\xi - \eta| \left(\mu^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{p-2}{2}}. \tag{4.142}$$

By virtue of assumption (4.10) we also have

$$|A_\varepsilon(x, \xi) - A_\varepsilon(y, \xi)| \leq (g_\varepsilon(x) + g_\varepsilon(y)) \, |x - y| \left(\mu^2 + |\xi|^2 \right)^{\frac{p-1}{2}} \tag{4.143}$$

for almost every $x, y \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^n$. Let u be a solution of the variational inequality (4.2) and let fix a ball $B_{\tilde{R}} \Subset \Omega'$. Let us denote by $u_\varepsilon \in u + W_0^{1,p}(B_{\tilde{R}})$ the solution to the inequality

$$\int_\Omega \langle A_\varepsilon(x, Dw(x)), D(\varphi - w)(x) \rangle \, dx \geq 0 \qquad \forall \varphi \in \mathcal{K}_{\varepsilon,\psi}(\Omega). \tag{4.144}$$

Thanks to [53, Theorem 1.1] we have $V_p(Du_\varepsilon) \in W^{1,2}_{\mathrm{loc}}(B_{\tilde{R}})$ and, since A_ε satisfies conditions (4.140)–(4.143), for ε sufficiently small, we are legitimated to apply estimate (4.116) thus getting

$$\int_{B_{\frac{r}{2}}} |DV_p(Du_\varepsilon(x))|^2 dx$$
$$\leq \frac{c \left(\|\psi\|^2_{L^\infty(B_{8r})} + \|u_\varepsilon\|^2_{L^{p^*}(B_{8r})} \right)^{\sigma_1}}{r^2}$$
$$\cdot \left[\int_{B_{4r}} \left(\mu^2 + |Du_\varepsilon(x)|^2 \right)^{\frac{p}{2}} dx + \int_{B_{4r}} g_\varepsilon^{p+2}(x) dx \right.$$

$$+\int_{B_{4r}} |DV_p(D\psi(x))|^2\,dx + \int_{B_{4r}} \left(\mu^2 + |D\psi(x)|^2\right)^{\frac{p}{2}} dx\Bigg]^{\sigma_2}, \tag{4.145}$$

for a constant $c = c(n, p, \nu, L, \ell)$, for any ball $B_{8r} \Subset B_{\tilde{R}}$.
Moreover, since and $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$, we have

$$g_\varepsilon \to g \qquad \text{strongly in } L^{p+2}\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0 \tag{4.146}$$

and, up to a subsequence, almost everywhere in $B_{\tilde{R}}$.
Since by (4.140), $|A_\varepsilon(x, Du)| \le \ell\left(\mu^2 + |Du|^2\right)^{\frac{p-1}{2}}$ and since $A_\varepsilon(x, Du)$ converges almost everywhere to $A(x, Du)$, by the dominated convergence Theorem we have

$$A_\varepsilon(x, Du) \to A(x, Du) \qquad \text{strongly in } L^{\frac{p}{p-1}}\left(B_{\tilde{R}}\right), \text{ as } \varepsilon \to 0. \tag{4.147}$$

Using the ellipticity condition (4.141), we have

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2\,dx \\
\le\ &\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)) - A_\varepsilon(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle dx \\
=\ &\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)), (Du_\varepsilon - Du)(x)\rangle dx \\
&- \int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle dx \\
=\ &\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)), (Du_\varepsilon - Du)(x)\rangle dx \\
&- \int_{B_{\tilde{R}}} \langle A(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle dx \\
&- \int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du(x)) - A(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle dx
\end{aligned} \tag{4.148}$$

Using u and u_ε as test functions in (4.144) and (4.2) respectively, we have

$$\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)), (Du_\varepsilon - Du)(x)\rangle dx \le 0 \tag{4.149}$$

and

$$-\int_{B_{\tilde{R}}} \langle A(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle dx \le 0, \tag{4.150}$$

therefore, thanks to (4.149) and (4.150), (4.148) implies

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2\,dx \\
\le\ &- \int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du(x)) - A(x, Du(x)), (Du_\varepsilon - Du)(x)\rangle\,dx \\
\le\ &\left(\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p\,dx\right)^{\frac{1}{p}}
\end{aligned}$$

$$\cdot \left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon (x, Du(x))|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}}. \tag{4.151}$$

Now let us observe that, by (4.149), using Hölder's inequality with exponents $\left(p, \frac{p}{p-1}\right)$ and recalling (4.140), we get

$$\begin{aligned} \int_{B_R} \langle A_\varepsilon (x, Du_\varepsilon(x)), Du_\varepsilon(x) \rangle dx &\leq \int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)), Du(x) \rangle dx \\ &\leq \left(\int_{B_{\tilde{R}}} |A_\varepsilon (x, Du_\varepsilon(x))|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\quad \cdot \left(\int_{B_{\tilde{R}}} |Du(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} \\ &\quad \cdot \left(\int_{B_{\tilde{R}}} |Du(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned} \tag{4.152}$$

Moreover, using Hölder's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$ and thanks to the ellipticity condition (4.141), we have

$$\begin{aligned} \int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p dx &\leq \int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p(p-2)}{4}} \cdot \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p(2-p)}{4}} dx \\ &\leq \left(\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^2 \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} dx \right)^{\frac{p}{2}} \\ &\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{2-p}{2}} \\ &\leq \left(\int_{B_{\tilde{R}}} \langle A_\varepsilon (x, Du_\varepsilon(x)) - A_\varepsilon (x, 0), Du_\varepsilon(x) \rangle dx \right)^{\frac{p}{2}} \\ &\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{2-p}{2}}. \end{aligned} \tag{4.153}$$

We can notice that, since the ellipticity condition implies

$$\langle A_\varepsilon (x, Du_\varepsilon(x)) - A_\varepsilon (x, 0), Du_\varepsilon(x) \rangle \geq 0,$$

and so

$$\langle A_\varepsilon (x, Du_\varepsilon(x)), Du_\varepsilon(x) \rangle \geq \langle A_\varepsilon (x, 0), Du_\varepsilon(x) \rangle,$$

for a.e. $x \in B_{\tilde{R}}$.
Hence, if we denote

$$E_1 := \left\{ x \in B_{\tilde{R}} : \langle A_\varepsilon (x, Du_\varepsilon(x)), Du_\varepsilon(x) \rangle < 0 \right\},$$

and

$$E_2 := \left\{ x \in B_{\tilde{R}} : \langle A_\varepsilon (x, Du_\varepsilon(x)), Du_\varepsilon(x) \rangle \geq 0 \right\},$$

we have

$$|\langle A_\varepsilon (x, Du_\varepsilon(x)), Du_\varepsilon(x) \rangle| \leq |\langle A_\varepsilon (x, 0), Du_\varepsilon(x) \rangle|$$

for a.e. $x \in E_1$, and

$$\langle A_\varepsilon(x, Du_\varepsilon(x)) - A_\varepsilon(x,0), Du_\varepsilon(x)\rangle \le \langle A_\varepsilon(x, Du_\varepsilon(x)), Du_\varepsilon(x)\rangle + |\langle A_\varepsilon(x,0), Du_\varepsilon(x)\rangle|$$

for a.e. $x \in E_2$. Therefore (4.153) implies

$$\begin{aligned}
\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx \le\ & \left(c\int_{E_1} |\langle A_\varepsilon(x,0), Du_\varepsilon(x)\rangle|\,dx\right.\\
&\left. + \int_{E_2} \left(\langle A_\varepsilon(x, Du_\varepsilon(x)), Du_\varepsilon(x)\rangle + |\langle A_\varepsilon(x,0), Du_\varepsilon(x)\rangle|\right) dx\right)^{\frac{p}{2}}\\
&\cdot\left(c\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}}\\
\le\ & \left(\int_{E_1} \mu^{p-1}|Du_\varepsilon(x)|\,dx\right.\\
&\left. + \int_{E_2} \left(\langle A_\varepsilon(x, Du_\varepsilon(x)), Du_\varepsilon(x)\rangle + \mu^{p-1}|Du_\varepsilon(x)|\right) dx\right)^{\frac{p}{2}}\\
&\cdot\left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}}, \qquad (4.154)
\end{aligned}$$

where, in the last line, we used (4.140).
Using Young's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$, by (4.154) we deduce

$$\begin{aligned}
\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx \le\ & c_\sigma \int_{B_{\tilde{R}}} \mu^{p-1}|Du_\varepsilon(x)|\,dx + c_\sigma\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)), Du_\varepsilon(x)\rangle\,dx\\
&+\sigma\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\\
\le\ & c_\sigma\int_{B_{\tilde{R}}} \langle A_\varepsilon(x, Du_\varepsilon(x)), Du_\varepsilon(x)\rangle\,dx\\
&+2\sigma\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx + c_\sigma\left|B_{\tilde{R}}\right|, \qquad (4.155)
\end{aligned}$$

where we also used Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$ and the fact that $\mu \in [0,1]$.
Now, joining (4.152) with (4.155), we get

$$\begin{aligned}
\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx \le\ & c_\sigma\left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{B_{\tilde{R}}} |Du(x)|^p\,dx\right)^{\frac{1}{p}}\\
&+2\sigma\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx + c_\sigma\left|B_{\tilde{R}}\right|\\
\le\ & c_\sigma\int_{B_{\tilde{R}}} |Du(x)|^p\,dx + 3\sigma\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx\\
&+c_\sigma\left|B_{\tilde{R}}\right|, \qquad (4.156)
\end{aligned}$$

where we used Young's inequality with exponents $\left(p, \frac{p}{p-1}\right)$ and the fact that $\mu \in [0,1]$ again.
Choosing $\sigma < \frac{1}{3}$, (4.156) implies

$$\int_{B_{\tilde{R}}} |Du_\varepsilon(x)|^p\,dx \le c\int_{B_{\tilde{R}}} |Du(x)|^p\,dx + c\left|B_{\tilde{R}}\right|. \qquad (4.157)$$

Let us observe that, using Hölder's inequality with exponents $\left(\frac{2}{p}, \frac{2}{2-p}\right)$ recalling (4.157), we have

$$\begin{aligned}
\int_{B_{\tilde{R}}} |(Du_\varepsilon - Du)(x)|^p\, dx &= \int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p(p-2)}{4}} |(Du_\varepsilon - Du)(x)|^p \\
&\quad \cdot \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p(2-p)}{4}} dx \\
&\leq \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2\, dx\right)^{\frac{p}{2}} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}} \\
&\leq c\left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} |(Du_\varepsilon - Du)(x)|^2\, dx\right)^{\frac{p}{2}} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + |B_{\tilde{R}}|\right)^{\frac{2-p}{2}} \\
&\leq c\left[\left(\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p\, dx\right)^{\frac{1}{p}}\right. \\
&\quad \left.\cdot \left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}\right]^{\frac{p}{2}} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + |B_{\tilde{R}}|\right)^{\frac{2-p}{2}} \\
&= c\left(\int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p\, dx\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{2}} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + |B_{\tilde{R}}|\right)^{\frac{2-p}{2}}, \qquad (4.158)
\end{aligned}$$

where we also used (4.151).
By Young's inequality with exponents $(2, 2)$, (4.158) implies

$$\begin{aligned}
\int_{B_{\tilde{R}}} |(Du_\varepsilon - Du)(x)|^p\, dx &\leq \sigma \int_{B_{\tilde{R}}} |(Du - Du_\varepsilon)(x)|^p\, dx \\
&\quad + c_\sigma \left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}}\, dx\right)^{p-1} \\
&\quad \cdot \left(\int_{B_{\tilde{R}}} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} dx + |B_{\tilde{R}}|\right)^{2-p},
\end{aligned}$$

for any $\sigma > 0$, and if we choose $\sigma < \frac{1}{2}$, we have

$$\int_{B_{\tilde{R}}} |(Du_\varepsilon - Du)(x)|^p\, dx \leq c\left(\int_{B_{\tilde{R}}} |A(x, Du(x)) - A_\varepsilon(x, Du(x))|^{\frac{p}{p-1}}\, dx\right)^{p-1}$$

$$\cdot\left(\int_{B_{\tilde{R}}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}dx+|B_{\tilde{R}}|\right)^{2-p}.$$

Hence, by (4.147), we deduce

$$Du_\varepsilon \to Du \qquad \text{strongly in } L^p_{\mathrm{loc}}\left(B_{\tilde{R}}\right),$$

which implies

$$u_\varepsilon \to u \qquad \text{strongly in } L^{p^*}_{\mathrm{loc}}\left(B_{\tilde{R}}\right)$$

and, up to a subsequence

$$u_\varepsilon \to u \qquad \text{almost everywhere in } B_{\tilde{R}},$$

as $\varepsilon \to 0$.
Moreover, by the continuity of the map $\xi \mapsto DV_p(\xi)$, we also get

$$DV_p(Du_\varepsilon) \to DV_p(Du) \qquad \text{a.e. in } B_{\tilde{R}},\ \text{as } \varepsilon \to 0.$$

Therefore, recalling (4.146), if we pass to the limit in (4.145), by Fatou's Lemma and a covering argument, we conclude, proving (4.115). □

As a consequence of Theorem 4.5.1, Lemma 1.4.5 and Remark 1.4.7, we get the following result.

Corollary 4.5.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to the obstacle problem* (4.1) *under assumptions* (4.4)–(4.6) *and let us assume that there exists a function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$ such that* (4.10) *and* (4.11) *hold, for $1 < p < 2$.*
Then the following implication holds:

$$\psi \in L^\infty_{\mathrm{loc}}(\Omega) \text{ and } V_p(D\psi) \in W^{1,2}_{\mathrm{loc}}(\Omega) \implies u \in W^{2,p}_{\mathrm{loc}}(\Omega) \text{ and } Du \in L^{p+2}_{\mathrm{loc}}(\Omega).$$

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