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# Dualities and geometrical aspects of sigma models


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# Declaration of Authorship

I, Francesco BASCONE, declare that this thesis titled *Dualities and geometrical aspects of sigma models* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a PhD at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely based on work that I have done during my doctorate in collaboration with Dr. Franco Pezzella and Prof. Patrizia Vitale.
- I have acknowledged all main sources of help.

Signed:

Date: 08/03/2022

## Details of collaborations and publications

During my PhD I published seven papers in different topics, in collaboration with different groups. Not all of them are pertinent to this thesis, but all of them increased my background and my personal experience, consequently enriching the content of this thesis. The list of publications is outlined below:

**F. Bascone**, F. Pezzella and P. Vitale, "Topological and dynamical aspects of Jacobi sigma models", Symmetry 13.7 (2021), p. 1205; 10.3390/sym13071205

F. Bajardi, **F. Bascone** and S. Capozziello, "Renormalizability of alternative theories of gravity: differences between power counting and entropy argument", Universe 7.5 (2021), p. 148; 10.3390/universe7050148

**F. Bascone**, F. Pezzella and P. Vitale, "Jacobi sigma models", Journal of High Energy Physics 2021, 110 (2021); 10.1007/JHEP03(2021)110

**F. Bascone**, F. Pezzella and P. Vitale, "Poisson-Lie T-duality of WZW model via current algebra deformation", Journal of High Energy Physics 2020, 60 (2020); 10.1007/JHEP09(2020)060

**F. Bascone** and F. Pezzella, "Principal Chiral Model without and with WZ term: Symmetries and Poisson-Lie T-duality", Proceedings of Science (Corfu2019) 134 (2020); 10.22323/1.376.0134

**F. Bascone**, L. Leonforte, D. Valenti, B. Spagnolo, and A. Carollo. "On critical properties of the Berry curvature in the Kitaev honeycomb model". J. Stat. Mech. 1909 (2019), p. 094002; 10.1088/1742-5468/ab35e9.

**F. Bascone**, V. E. Marotta, F. Pezzella and P. Vitale, "T-duality and Doubling of the Isotropic Rigid Rotator", Proceedings of Science (Corfu2018) 123 (2019), 10.22323/1.347.0123

# Sommario

Negli ultimi cinquant'anni anni il concetto di dualità ha assunto un ruolo fondamentale nella fisica teorica. L'estrema importanza delle dualità è dovuta al fatto che queste forniscono strumenti per la costruzione di nuove teorie e permettono ai fisici di fare calcoli in regimi che altrimenti non sarebbero accessibili. Più nello specifico, il concetto di dualità si riferisce all'esistenza di descrizioni diverse, in termini di formalismo matematico, dello stesso fenomeno fisico.

In particolare, le dualità giocano un ruolo chiave nelle teorie di stringa, in quanto permettono di collegare teorie di stringa definite su spazitempo differenti. Un esempio fondamentale è la rete di dualità che collega le cinque teorie di superstringa, permettendo di passare da una all'altra attraverso specifiche mappe di dualità. In questa tesi verrà considerata in particolare la cosiddetta T-dualità, che consente di mappare teorie di stringa definite su varietà con qualche dimensione compatta.

Un altro concetto fondamentale in fisica teorica è quello dei modelli sigma non lineari. Un modello sigma è una teoria di campo in cui i campi sono a valori in una varietà curva, cui ci si riferisce con il nome di spazio target. Tali modelli giocano un ruolo importante in vari settori della fisica teorica, spaziando dalla descrizione di eccitazioni adroniche a bassa energia in quattro dimensioni alla materia condensata e la meccanica statistica oltre che, naturalmente, alle teorie di stringa. Infatti, le teorie di stringa sono descritte da un modello sigma non lineare 2-dimensionale che descrive la superficie spazzata dalla stringa nel suo moto all'interno dello spaziotempo. Tuttavia, una formulazione delle teorie di stringa in cui le simmetrie di dualità siano manifeste è, ad oggi, ancora assente. Una tale formulazione sarebbe utile e interessante per avere più informazioni sulle geometrie di stringa e quindi su aspetti di gravità quantistica.

Questa tesi si basa sull'elaborazione e l'applicazione di un nuovo approccio a quella che viene chiamata Poisson-Lie T-dualità, una dualità tra modelli sigma che generalizza gli altri tipi più comuni di T-dualità di stringa, ovvero quelle Abeliana e non Abeliana. Nello specifico, questo approccio si basa sulla deformazione dell'algebra delle correnti di modelli sigma che hanno gruppi Poisson-Lie duali come spazitempo target. Questo porta alla formulazione di nuove famiglie a più parametri di infiniti modelli duali, tutti legati da particolari trasformazioni di T-dualità. Questo potrebbe portare a nuove teorie ed è particolarmente adatto a una quantizzazione formale nel senso di Drinfeld, ovvero basata su gruppi quantici, con la speranza di poter superare eventualmente i problemi di quantizzazione degli approcci standard.

Al fine di applicare tale formalismo è stato considerato un particolare modello sigma: il modello di Wess-Zumino-Witten (WZW) con la varietà del gruppo di Lie $SU(2)$ come spazio target. Il motivo è che il campo di gioco più naturale in cui inverstigare problemi legati alle simmetrie di dualità è quello della dinamica su varietà di gruppi di Lie, e in particolare il modello WZW descrive stringhe che si propagano su una varietà di gruppo.

E' noto che $SU(2)$ è una componente del gruppo $SL(2,\mathbb{C})$ in una particolare decomposizione di quest'ultimo come Drinfel'd double, insieme al suo partner Poisson-Lie duale $SB(2,\mathbb{C})$, ovvero il gruppo di Lie delle matrici $2 \times 2$ triangolari superiori complesse con determinante unitario e diagonale principale reale. Nella tesi si dimostra che una famiglia a due parametri di modelli duali con spazio delle configurazioni target $SB(2,\mathbb{C})$ può essere ottenuta deformando l'algebra delle correnti del modello originale nell'algebra di Lie di $SL(2,\mathbb{C})$. Definiamo poi una formulazione raddoppiata del modello sul gruppo $SL(2,\mathbb{C})$ da cui poi possono essere ottenute entrambe le famiglie di modelli duali. Inoltre, il formalismo utilizzato sembra essere particolarmente indicato per lo studio di concetti quali la geometria generalizzata, un formalismo che permette di trattare in maniera unificata vettori e 1-forme, ed esplorare le sue relazioni con le simmetrie di Poisson-Lie e con la geometria raddoppiata.

Al fine di generalizzare questo formalismo, abbiamo definito un modello sigma con una varietà di Jacobi come spazio target, e per questa ragione è stato chiamato modello sigma di Jacobi, come generalizzazione del cosiddetto modello sigma di Poisson. Quest'ultimo sembra portare in modo naturale al concetto di dualità di Poisson-Lie quando lo spazio target è un gruppo di Lie, e uno degli obiettivi è includere tali aspetti di dualità nel nuovo modello sigma di Jacobi, generalizzando. Oltre agli aspetti di dualità, abbiamo scoperto che il nostro modello sigma di Jacobi può descrivere nuove soluzioni di stringa (in particolare con flussi) che non è possibile ottenere dal modello sigma di Poisson.

*"Had I been present at the Creation, I would have given some useful hints for the better ordering of the Universe."*

—Alfonso X of Castile

UNIVERSITÀ DEGLI STUDI DI NAPOLI

# *Abstract*

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Doctor of Philosophy

**Dualities and geometrical aspects of sigma models**

by Francesco BASCONE

In theoretical physics, the concept of duality has become the foundation of many important developments in the last 50 years. Dualities are of central importance because they provide powerful tools in the construction of theories, and allow physicists to perform calculations in regimes which would be otherwise inaccessible, giving new insights. More specifically, a duality refers to different descriptions of the same physical phenomenon, just in a different mathematical language.

Dualities play a fundamental role in particular in string theory, relating string theories defined on different backgrounds. A fundamental example is that of the five superstring theories related by duality transformations. We focus in particular on the so-called T-duality, which allows to map string theories defined on target manifolds which have some dimensions compactified.

Another fundamental concept is that of sigma models. In particular, a sigma model is a field theory wherein fields take values in a curved manifold. Nonlinear sigma models play an important role in many sectors of theoretical physics, with applications ranging from the description of low energy hadronic excitations in four dimensions, condensed matter and, in particular, in string theory. Indeed, string theory is encoded in the two-dimensional sigma model describing the area swept out by the string in its motion in the target space and, so far, a formulation of string theory in which those duality symmetries could be manifest is still missing, but it would be interesting to get it in order to have more information on string geometry and, hence, on aspects of string gravity.

This thesis is based on the elaboration and application of a novel approach to the so-called Poisson-Lie T-duality, which is a duality between sigma-models generalizing the most common Abelian T-duality of string theories. In particular, the approach is based on deforming the underlying current algebra structures involved in sigma models having Poisson-Lie dual groups as target spaces. This leads to the formulation of new families of dual models, all related by particular T-duality transformations. This approach is expected to give rise to new theories, and it is particularly suitable for quantization in the sense of Drinfeld, which is based on quantum groups.

We focused on an explicit sigma model: the so-called Wess-Zumino-Witten (WZW) model, with the $SU(2)$ group manifold as target space. This is relevant because the natural framework to investigate such issues in a proper geometric setting is that of dynamics on group manifolds, and the WZW model describes strings propagating on a group manifold.

It is known that $SU(2)$ is a component of $SL(2,\mathbb{C})$ in a Drinfeld double decomposition, together with its Poisson-Lie dual partner $SB(2,\mathbb{C})$, which is the group of complex $2 \times 2$ upper triangular matrices with unit determinant and real diagonal. It is possible to show that a whole two-parameter family of dual models having the latter group as target configuration space can be obtained by deforming the current algebra structure of the original model to $SL(2,\mathbb{C})$. We then define a doubled formulation of model on this group from which both families of dual models can be obtained. Furthermore, the setting under analysis seems to be particularly well suited for Generalized Geometry, which is a way to take into account vector fields and one-forms in a unified fashion, and to study its relations with Poisson-Lie symmetries and Doubled Geometry.

Looking for generalizations of this framework, we have defined a sigma model having a Jacobi manifold as target space, and for this reason we called it Jacobi sigma model, as a generalization of the so-called Poisson sigma model. The Poisson sigma model on Lie groups with Poisson structure seems to lead naturally to the concept of Poisson-Lie duality, and one of the goals of the thesis is to include such duality aspects into the newly defined Jacobi sigma model. Other than duality aspects, we have found our Jacobi sigma model can take new string solutions into account (with fluxes in particular), which is not possible to obtain from the Poisson sigma model.

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*Ai miei nonni*

# 1 Introduction

Non-linear sigma models play an important role in many sectors of theoretical physics, with applications ranging from the description of low energy hadronic excitations in four dimensions [1, 2], to the construction of string backgrounds, like plane waves [3, 4], $AdS$ geometries [5–9] or two-dimensional black hole geometries [10]. Interesting examples of string backgrounds come from WZW models on non-semisimple Lie groups, that we will also consider in our work. In the context of two-dimensional conformal field theories, gauged WZW models with coset target spaces have been investigated since many years (see [11, 12] for early contributions). Recently, non-linear sigma models found new applications in statistical mechanics, describing certain two-dimensional systems at criticality [13], as well as in condensed matter physics, describing transitions for the integer quantum Hall effect [14].

The concept of duality is of central importance in theoretical physics since dualities provide a powerful tool to make problems which are in principle insurmountable, completely within our analytical capabilities. A duality in physics is, roughly speaking, a relation of formal equivalence between different theories. In particular, a physical system may have more than one theoretical formulation and it can be way easier (or even just possible) to work with one instead of the others.

One of the most fundamental dualities in the context of string theory is the so called T-duality [15–17], which is peculiar of strings as extended objects and relates theories defined on different target space backgrounds. The original notion of T-duality emerges in toric compactifications of the target background spacetime. The most basic example is provided by compactification of a spatial dimension on a circle of radius $R$. Here T-duality acts by exchanging momenta $p$ and winding numbers $w$, $p \leftrightarrow w$, while mapping $R \to \frac{\alpha'}{R}$, with $\alpha'$ related to the string fundamental length. This leads to a duality between string theories defined on different backgrounds but yielding the same physics, as it can be easily seen looking at the mass spectrum.

Interestingly, T-duality allows to construct new string backgrounds which could not be obtained otherwise, which are generally referred to as *non-geometric backgrounds* (see for example [18] for a recent review). By non-geometric background it is usually intended a string configuration which cannot be described in terms of Riemannian geometry and T-duality transformations are introduced for gluing coordinate patches, other than the usual diffeomorphisms and $B$-field gauge transformations, as long as it does not exist another T-duality transformation mapping the configuration back into a geometric one. Moreover, T-duality plays an important role, together with S-duality and U-duality, in relating, through a web of dualities, the five superstring theories which in turn appear as low-energy limits of a more general theory, that is, M-theory.

T-duality is certainly to be taken into account when looking at quantum field theory as low-energy limit of the string action. This has suggested since long [17, 19–25] to look for a manifestly T-dual invariant formulation of the Polyakov world-sheet action that has to be based on a doubling of the string coordinates in target space. One relevant objective of this new action would be to obtain new indications for string gravity. This approach leads to Double Field Theory (DFT) with Generalized and Doubled Geometry furnishing the appropriate mathematical framework. In particular, DFT is expected to emerge as a low-energy limit of manifestly T-duality invariant string world-sheet. Then, Doubled Geometry is necessary to accommodate the coordinate doubling in target space. There is a vast literature concerning DFT, including its topological aspects and its description on group manifolds [26–41]. Recently, a global formulation from higher Kaluza-Klein theory has been proposed in [42–45].

The kind of T-duality discussed so far belongs to a particular class, so called *Abelian T-duality*, which is characterized by the fact that the generators of target space duality transformations are Abelian, while generating symmetries of the action only if they are Killing vectors of the metric [46–48]. However, starting from Ref. [49], it was realized that the whole construction could be generalized to include the possibility that one of the two isometry groups be non-Abelian. This is called *non-Abelian*, or, more appropriately, *semi-Abelian* duality. Although interesting, because it enlarges the possible geometries involved, the latter construction is not really symmetric, as a duality would require. In fact, the dual model is typically missing some isometries which are required to go back to the original model by gauging. This means that one can map the original model to the dual one, but then it is not possible to go back anymore. This unsatisfactory feature is overcome with the introduction of *Poisson-Lie T-duality* [50–52] (for some recent work to alternative approaches see [53, 54]). The latter represents a genuine generalization, since it does not require isometries at all, while Abelian and non-Abelian cases can be obtained as particular instances. Recent results on Poisson-Lie T-duality and its relation with para-Hermitian geometry and integrability, as well as low-energy descriptions, can be found in [55–64].

Symmetry under Poisson-Lie duality transformations is based on the concept of Poisson-Lie dual groups and Drinfel'd doubles. A Drinfel'd double is an even-dimensional Lie group $D$ whose Lie algebra $\mathfrak{d}$ can be decomposed into a pair of maximally isotropic subalgebras, $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, with respect to a non-degenerate ad-invariant bilinear form on $\mathfrak{d}$. Lie algebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ are dual as vector spaces, and endowed with compatible Lie structures. Any such triple, $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$, is referred to as a Manin triple. If $D, G, \tilde{G}$ are the corresponding Lie groups, $G, \tilde{G}$ furnish an Iwasawa decomposition of $D$. The simplest example of Drinfel'd double is the cotangent bundle of any $d$-dimensional Lie group $G$, $T^*G \simeq G \ltimes \mathbb{R}^d$, which we shall call the *classical double*, with trivial Lie bracket for the dual algebra $\tilde{\mathfrak{g}} \simeq \mathbb{R}^d$. In general, there may be many decompositions of $\mathfrak{d}$ into maximally isotropic subspaces (not necessarily subalgebras). The set of all such decompositions plays the role of the modular space of field theories mutually connected by a T-duality transformation. In particular, for the Abelian T-duality of the string on a $d$-torus, the Drinfel'd double is $D = U(1)^{2d}$ and its modular space is in one-to-one correspondence with $O(d, d; \mathbb{Z})$ [52], the latter being the pseudo-orthogonal group.

Since Poisson-Lie T-duality generalizes the other kinds of T-duality, one can actually use Drinfel'd doubles to classify T-duality, as it will be shown in Section 3.3.

The appropriate geometric setting to investigate issues related to Poisson-Lie duality is that of dynamics on group manifolds. In this thesis we consider in particular the Isotropic Rigid Rotator as a $(0+1)$-dimensional sigma model on $SU(2)$ [65, 66], as well as its natural generalization, which is the $SU(2)$ Principal Chiral Model (PCM) [67, 68]. However, we will focus in particular on the $SU(2)$ Wess-Zumino-Witten (WZW) model in two space-time dimensions, which is a non-linear sigma model with the group manifold of $SU(2)$ as target space, together with a topological cubic term, which further generalizes the PCM [69]. The WZW model has many interesting applications, both theoretical and phenomenological. Besides its primary phenomenological motivation as an effective model for low energy QCD, it yields, for two-dimensional source spaces, string solutions on group manifolds which represent the appropriate setting to analyse T-duality generalisations. Moreover, the $SL(2,\mathbb{R})$ model has been used to construct bosonic string theories on $AdS_3$ [5–7], while superstring theories on $AdS_3 \times S^3$ geometries can be described by the WZW on $PSU(1,1|2)$ [9]. Strings on black hole geometry can be described by means of coset gauged WZW models [10]. Furthermore, it effectively describes statistical systems at criticality, such as critical antiferromagnetic Potts model [13] or condensed matter systems such as integer quantum Hall [14]. Last but not least, string theory applications with target space a non semisimple group have attracted some interest in the scientific community (see e.g. [3]) and it turns out that the family of dual models that we build is exactly of that type.

Once formulated on Drinfel'd doubles, such models allow for establishing enlightening connections with Generalized Geometry (GG) [70–72], by virtue of the fact that tangent and cotangent vector fields of the group manifold may be respectively related to the span of its Lie algebra and of the dual one. Locally, GG is based on replacing the tangent bundle $TM$ of a manifold $M$ with a kind of Whitney sum $TM \oplus T^*M$, a bundle with the same base space but fibres given by the direct sum of tangent and cotangent spaces, and the Lie brackets on the sections of $TM$ by the so called Courant brackets, involving vector fields and one-forms. Both the brackets and the inner products naturally defined on the generalized bundle are invariant under diffeomorphisms of $M$. More generally, a generalized tangent bundle is a vector bundle $E \to M$ enconded in the exact sequence $0 \to T^*M \to E \to TM \to 0$. This formal setting is certainly relevant in the context of DFT because it takes into account in a unified fashion vector fields, which generate diffeomorphisms for the background metric $G$ field, and one-forms, generating diffeomorphisms for the the background two-form $B$ field. In this framework Doubled Geometry plays a natural role in describing generalized dynamics on the tangent bundle $TD \simeq D \times \mathfrak{d}$, which encodes within a single action dually related models.

It is interesting to note that already in such a simple case as the IRR, many aspects of Poisson-Lie duality can be exploited, and especially some relations with Doubled and Generalized Geometry, although the model is too simple to exhibit manifest invariance. To overcome the inadequacy of the IRR, in [67] the two-dimensional PCM on $SU(2)$ was considered, by means of a one-parameter family of Hamiltonians and Poisson brackets, all equivalent from the point of view of dynamics. Poisson-Lie symmetry and a family of

Poisson-Lie T-dual models were established. Some connections with Born geometry were also made explicit.

In this thesis we describe the WZW model in the Hamiltonian approach with carrier space the cotangent bundle $T^*SU(2)$. One of the two important observations on which the approach relies is the fact that $T^*SU(2)$ is symplectomorphic to $SL(2,\mathbb{C})$, besides being topologically equivalent. The second observation is that $SL(2,\mathbb{C})$ is a Drinfel'd double of the Lie group $SU(2)$ [73–76], which is the main ingredient on which Poisson-Lie T-duality is based.

The core of the approach relies on a deformation of the affine current algebra of the model, which is the semidirect sum of the Kac-Moody algebra associated to $\mathfrak{su}(2)$ and an Abelian algebra $\mathfrak{a}$, to the fully semisimple Kac-Moody algebra $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ [77–79]. By deformation here we mean the usual definition, with a family of some structure which depends on one or more parameters and by sending the parameters to a particular value one recovers the original structure. The deformation of the original current algebra of the model into a non-Abelian one is a crucial step if one observes that $(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{su}(2),\mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple. In particular, $\mathfrak{sb}(2,\mathbb{C})$ is the Lie algebra of the Borel subgroup of $SL(2,\mathbb{C})$ of $2\times 2$ complex valued upper triangular matrices with unit determinant and real diagonal. By $\mathfrak{g}(\mathbb{R})$ we indicate the affine algebra of maps $\mathbb{R}\to\mathfrak{g}$ that are sufficiently fast decreasing at infinity to be square integrable, that is what we will refer to as current algebra. Current algebra deformation is also the essence of a Hamiltonian formulation of the classical world-sheet theory proposed in [80]. After this first deformation resulting in a one-parameter family of Hamiltonian models with algebra of currents homomorphic to $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$, a further deformation is needed in order to make the role of the dual subalgebras involved in the Manin triple decomposition completely symmetric. We show that such a deformation not altering the nature of the current algebra is possible, leaving the dynamics unmodified. In this respect, our findings will differ from existing results, such as $\eta$ or $\lambda$ deformations of non-linear sigma models, which represent true deformations of the dynamics yielding integrable models - recently, relations of these deformed models with Poisson-Lie T-duality have been found and worked out in [55]. We end up with a two-parameter family of models with the group $SL(2,\mathbb{C})$ as target phase space. T-duality transformations are thus realized as $O(3,3)$ rotations in phase space. By performing an exchange of momenta with configuration space fields we obtain a new family of WZW models, with configuration space the group $SB(2,\mathbb{C})$, which is dual to the previous one by construction.

Looking for generalizations of this framework, we have defined a sigma model having a Jacobi manifold as target space, and for this reason we called it Jacobi sigma model. This was done with the aim to build a natural, non-trivial generalization of the well-known Poisson sigma model. The latter is a topological field theory which was first introduced [81, 82] in relation with two-dimensional field theories with non-trivial target space, e.g. gauge and gravity models, as well as gauged WZW models. One of the most interesting features of the Poisson sigma model is its intimate relation with the geometry of the target Poisson manifold. Indeed, it makes it possible to unravel mathematical aspects of such manifolds by employing techniques from field theory. An example of this relation was given by Cattaneo and Felder in [83, 84] where they show that the reduced phase space of the Poisson sigma model is actually the symplectic groupoid integrating the Lie algebroid associated with the Poisson structure of the target Poisson manifold. Moreover, the model made it possible to give an alternative derivation of Kontsevich quantization formula for Poisson manifolds, in terms of the Feynman diagrams coming from the perturbative expansion of the field theory [85]. Analogous questions, such as the geometry of the reduced phase space and the quantization of Jacobi structures, could be addressed once the model is understood.

From a more physical point of view, a further motivation for the introduction of this new model is the possibility to find and analyze new string backgrounds, as well as the possibility of obtaining some useful description of known models within the framework of Jacobi manifolds, just like the Poisson sigma model does.

The Poisson sigma model is described in terms of fields $(X, \eta)$ which are formally associated with a bundle map from the tangent bundle of a source space $\Sigma$, a two-dimensional orientable manifold possibly with boundary, to the cotangent bundle of the target Poisson manifold $M$. In particular, $X$ is the base map, describing the embedding of $\Sigma$ into $M$, while $\eta$ is the fibre map, an auxiliary field which is in particular a one-form on $\Sigma$ with values in the pull-back of the cotangent bundle over $M$. In general it is not possible to integrate out such an auxiliary field, unless the target space is a symplectic manifold. In this case the Poisson bi-vector can be inverted and the equations of motion can be solved for $\eta$. The resulting action is that of a topological A-model [86, 87], i.e., $S = \int_\Sigma X^*(\omega)$, where $\omega = \Pi^{-1}$ is the symplectic form on $M$, $\Pi$ the Poisson bi-vector field (fulfilling the condition of zero Schouten bracket

$[\Pi, \Pi]_S = 0$) and $X^*$ denotes the pull-back map.

Our aim in [88, 89] was to investigate the possibility of relaxing the condition $[\Pi, \Pi]_S = 0$ to a natural generalization represented by a Jacobi structure. The latter is specified by a bivector field $\Lambda$ and a vector field $E$, called Reeb vector field, satisfying the following expression for the Schouten bracket

$$[\Lambda, \Lambda]_S = 2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda]_S = 0. \tag{1.1}$$

The triple $(M, \Lambda, E)$ then defines a Jacobi manifold. The Poisson manifold can be considered as a particular case with identically vanishing Reeb vector field.

Two main families of Jacobi manifolds are represented by contact and locally conformal symplectic manifolds. In this thesis will will consider these kind of manifolds for applications of the model.

From a Jacobi structure one can construct Jacobi brackets on the algebra of functions on $M$ which satisfy the Jacobi identity and the skew-symmetry property just like Poisson brackets. However, it does violate the Leibniz rule,so while the Jacobi bracket still endows the algebra of functions with a Lie algebra structure, it is not a derivation anymore. Thus, the bi-vector field $\Lambda$ may be ascribed to the family of bivector fields violating Jacobi identity, such as "twisted" and "magnetic" Poisson structures (see for example [90, 91]) which recently received some interest in relation with the quantization of higher structures (their Jacobiator being non-trivial) and with the description of non-trivial geometric fluxes in string theory. The violation of Jacobi identity is however under control, because the latter is recovered by the full Jacobi bracket, which is alternatively defined as the most general local bilinear operator on the space of real functions $C^\infty(M, \mathbb{R})$ which is skew-symmetric and satisfies Jacobi identity [92], and this makes its study especially interesting to us.

The Jacobi sigma model generalizes the construction of the Poisson sigma model with field configurations of the model represented by triples $(X, \eta, \lambda)$, where $X : \Sigma \to M$ is the embedding map of the source space into the target Jacobi manifold and $(\eta, \lambda)$ are elements of $\Omega^1(\Sigma, X^*(T^*M \oplus \mathbb{R}))$, being $T^*M \oplus \mathbb{R} = J^1 M$ the vector bundle of 1-jets of real functions on $M$. The result is then a two-dimensional topological non-linear gauge theory which describes strings propagating on a Jacobi manifold. The following main results were achieved in [88, 89], and will be illustrated in the thesis:

- Similarly to the Poisson sigma model, the reduced phase space can be proven to be finite-dimensional, but while for the Poisson case the dimension is 2dim$M$, for the Jacobi sigma model the dimension results to be 2dim$M - 2$.

- The model may be related to a Poisson sigma model with target space $M \times \mathbb{R}$ within a "Poissonization" procedure. The latter approach has been pursued in [93] in relation with non-closed fluxes, and [94] with reference to gauge symmetry.
- The auxiliary fields $(\eta, \lambda)$ can be integrated out, both for contact and locally conformal symplectic manifolds, so to get a model which is solely defined in terms of the field $X$ and its derivatives.
- By including a dynamical term which is proportional to the metric tensor of the target manifold, it is possible to obtain a Polyakov action. The background metric and the $B$-field are expressed in terms of the Jacobi structure. A non-zero three form, $H = dB$, may occur, depending on the details of the model.

The thesis is organized as follows. In Chapter 2 we review the main concepts of Poisson manifolds and Poisson-Lie groups which will be needed in order to discuss Poisson-Lie T-duality issues, as well as to introduce the Poisson sigma model.

In Chapter 3 we review the notion of duality in general, and then we will focus on T-duality. In particular, we will consider Abelian, non-Abelian and Poisson-Lie T-duality. The latter is widely employed in the context of sigma models in its Lagrangian formulation, we shall work out the Hamiltonian counterpart and verify its realization within the model under analysis.

In Chapter 4 the results obtained in [66, 67] are reviewed for the isotropic rigid rotator, thought of as a dynamical model over the group manifold $SU(2)$ with a dual partner defined on the dual group $SB(2,\mathbb{C})$. The two groups appear in the Manin triple decomposition of the Drinfel'd double $SL(2,\mathbb{C})$ whose structure is recalled together with the one of its Lie algebra. We also sketch the main properties of the Principal Chiral Model.

In Chapter 5 the Wess-Zumino-Witten model on the $SU(2)$ group manifold will be introduced with particular emphasis on its Hamiltonian formulation and care will be payed to enlighten the Lie algebraic structure of the Poisson brackets of fields. The main purpose will be to illustrate the one-parameter deformation of the natural current algebra structure of the model to the affine Lie algebra associated to $\mathfrak{sl}(2,\mathbb{C})$ [79]. In particular, we review the derivation of a one-parameter family of models, all equivalent to the $SU(2)$ WZW model, but which clearly do not follow from the standard action principle. A further parameter is introduced in the current algebra in such a way to make the role of the $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ subalgebras symmetric, without modifying the dynamics. This is needed in order to have a manifest Poisson-Lie duality map, which reveals itself to be an $O(3,3)$ rotation in the target phase space $SL(2,\mathbb{C})$. Such a transformation leads to a two-parameter family of models with $SB(2,\mathbb{C})$ as target configuration space, which is dual to the starting family by construction. Independently from the previous Hamiltonian derivation, we introduce a WZW model on $SB(2,\mathbb{C})$ in the Lagrangian approach, together with the corresponding string spacetime background. The model is interesting per se, because it is an instance of a WZW model with non-semisimple Lie group as target space, which exhibits classical conformal invariance. We overcome the intrinsic difficulties deriving from the absence of non-degenerate Cartan-Killing metric. However, the resulting dynamics does not seem to be related by a duality transformation to any of the models belonging to the parametric family described above. We identify the problem as a topological obstruction and we show that in order to establish a connection with any other of the models found, a true deformation of the dynamics is needed, together with a topological modification of the phase space. Finally, having understood what are the basic structures involved in the formulation of both the dually related WZW families, we introduce a generalized doubled WZW action on the Drinfel'd double $SL(2,\mathbb{C})$ with doubled degrees of freedom. Its Hamiltonian description is presented and from it the Hamiltonian descriptions of the two submodels can be obtained by constraining the dynamics to coset spaces $SU(2)$ and $SB(2,\mathbb{C})$.

In the second part of the thesis, in Chapter 6 we present a short summary of the Poisson sigma model.

In Chapter 7 we review the notion of Jacobi manifold and Jacobi structure, and we describe the procedure of Poissonization of a Jacobi manifold $M$ yielding to a higher dimensional manifold $M \times \mathbb{R}$. This construction has played an important role in suggesting the original formulation of the model.

In Chapter 8 the action functional for the Jacobi sigma model is stated. The model in the canonical formulation is constrained, with first class constraints generating gauge transformations. Gauge transformations are implemented by generating functionals $K_{\beta,\lambda_t}$ through Poisson brackets. The constrained phase space is reduced with respect to gauge symmetries and shown to be finite dimensional with dimension equal to $2\dim M - 2$. We also consider the model in both cases of the target space being a contact and locally conformal symplectic manifold in a general fashion. We thus discuss noteworthy examples, such as the manifolds $SU(2)$ and $SU(2) \times S^1$ as instances of contact and LCS target spaces respectively. Finally, we introduce the dynamical model. This consists in supplementing the action of the Jacobi sigma model with a dynamical term which includes a metric tensor both on the source and the target space. It is very much inspired to the dynamical Poisson sigma model discussed in [95], with some interesting differences. The emerging model, besides being non-topological, yields a Polyakov action with background metric $g$ and $B$-field determined by the Jacobi structure involved.

Conclusions and Outlook are reported in the final Chapter 9.

# 2 Poisson-Lie groups and Drinfel'd doubles

In this section we briefly review the mathematical setting of Poisson geometry, and in particular Poisson-Lie groups and Drinfel'd doubles, see [75, 96–99] for details. In particular we will focus on $SL(2,\mathbb{C})$ as a specific example of Drinfel'd double since it plays a major role throughout this thesis. In general, Poisson geometry will prove to be of crucial importance when we will introduce the Poisson sigma model, while the particular framework of Poisson-Lie groups and Drinfel'd doubles will be the background for the discussion on Poisson-Lie duality.

## 2.1 Poisson Geometry

Poisson geometry was born with the introduction of Poisson structures by Lichnerowicz [100] and the study of Weinstein [101]. Poisson geometry is the main ingredient underlying the Hamiltonian formalism of classical mechanics and is now used in many more topics such as noncommutative geometry, topological field theory, integrable systems and more. It is important to remark that not only the importance of Poisson geometry resides on the geometrical formulation of classical mechanics, but also on the quantization approach. In this section we will introduce the concept of Poisson structures and Poisson manifolds which will prove to be of crucial importance for the rest of this thesis. We will focus in particular on the concept of Poisson-Lie groups, which is necessary to understand the concept of Poisson-Lie duality.

### 2.1.1 Poisson manifolds

**Definition 2.1.1.** Let $\mathcal{A}$ be a unital associative and Abelian algebra over the field of real or complex numbers equipped with a bilinear map $\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying the following properties ($\forall f, g, h \in \mathcal{A}$):

- Antisymmetry: $\{f, g\} = -\{g, f\}$,

- Jacobi identity: $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$,
- Leibniz rule: $\{f,gh\} = g\{f,h\} + \{f,g\}h$.

The algebra $\mathcal{A}$ is then called a *Poisson algebra* and the map $\{\cdot,\cdot\}$ is called a *Poisson bracket*.

We are mainly interested in the case in which $\mathcal{A}$ is the algebra of smooth real-valued functions on a manifold $M$, $C^\infty(M)$. An important thing to notice beforehand is that if we fix a function $f \in C^\infty(M)$, the map $g \mapsto \{g,f\}$ is a derivation because of the Leibniz axiom (with the usual product of functions as composition law). This means that with this map we can define a vector field $X_f = \{\cdot,f\}$ that is also smooth since $X_f(g) = \{g,f\} \in C^\infty(M)\ \forall g \in C^\infty(M)$. This leads to the following

**Definition 2.1.2.** For any $f \in C^\infty(M)$, the smooth vector field $X_f \in \mathfrak{X}(M)$ defined by

$$X_f(g) := \{g,f\},\ \forall g \in C^\infty(M) \tag{2.1}$$

is called the *Hamiltonian vector field* generated by the function $f$.

The notion of Hamiltonian vector fields is useful to understand that Poisson brackets can be computed in local coordinates by taking partial derivatives. Indeed, one can note that considering local coordinates $(x^1,\dots,x^m) \in M$, the Hamiltonian vector fields can be written locally as $X_f\big|_p(p) = V_f{}^j(p)\,\frac{\partial}{\partial x^j}\Big|_p$ with respect to the coordinate basis $\{\partial_i := \frac{\partial}{\partial x^i}\}_{i\in\{1,\dots,m\}}$, where $p$ is a point in the local chart $(U,x^i)$ and $V$ represent some smooth coordinate functions $V_f{}^j(p): U \to \mathbb{R}$. Calculating the coordinate functions by using the definition of Hamiltonian vector field in (2.1) we obtain $X_f\big|_p(p) = \{x^j,f\}(p)\,\frac{\partial}{\partial x^j}\Big|_p$. By using the skew-symmetry property to calculate $\{x^j,f\}(p)$ we are then led to the relation $X_f\big|_p = -\{x^i,x^j\}(p)\,\frac{\partial f}{\partial x^i}\Big|_p\frac{\partial g}{\partial x^j}\Big|_p$ and then finally to

$$\{f,g\}(p) = \{x^i,x^j\}(p)\,\frac{\partial f}{\partial x^i}\bigg|_p\frac{\partial g}{\partial x^j}\bigg|_p, \tag{2.2}$$

where the summation convention over repeated indices is used.

Since the Poisson bracket only depends on the partial derivatives of the functions involved, $\{f,g\}$ is then completely determined by the differentials $df$ and $dg$. This means that poisson bracket can be determined by pairing $df \wedge dg$ with a bivector field, so that on a local chart we can write

$$\{f,g\}(p) = \Big\langle\ (df\wedge dg)\big|_p, \{x^i,x^j\}(p)\partial_i\wedge\partial_j\Big\rangle, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $T^*M$ and $TM$.

**Definition 2.1.3.** A *Poisson bivector* (or *Poisson structure*) is a smooth bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ such that locally the following condition (Jacobi identity) holds

$$0 = [\Pi,\Pi]_S^{ijk} = \Pi^{i\ell}\partial_\ell \Pi^{jk} + \text{cycl}(ijk), \tag{2.4}$$

where $[\cdot,\cdot]_S : \Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q-1}(M)$ is the Schouten–Nijenhuis bracket on the algebra of multivector fields on the manifold $M$, namely a skew-symmetric bilinear map $\Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q-1}(M)$ given by

$$\begin{aligned}&\left[A_1 \wedge \cdots \wedge A_p,\, B_1 \wedge \cdots \wedge B_q\right]_S \\ &= \sum (-1)^{t+s} A_1 \wedge \ldots \widehat{A}_s \cdots \wedge A_p \wedge [A_s, B_t] \wedge B_1 \wedge \ldots \widehat{B}_t \ldots \wedge B_q\end{aligned} \tag{2.5}$$

where $A_1, ..., A_p, B_1, ..., B_q$ are vector fields over $M$ and $\widehat{A}$ indicates the omission of the vector field $A$.

Locally, the Poisson bivector can be written as

$$\Pi = \Pi^{ij}\partial_i \wedge \partial_j, \tag{2.6}$$

with $\Pi^{ij} = \{x^i, x^j\}$. The Poisson bracket on $C^\infty(M)$ is then defined as $\{f,g\} = \Pi(df,dg),\ f,g \in C^\infty(M)$.

Note that the relation (2.4) is a direct consequence of Jacobi identity in Definition (2.1.1).

We have seen that we can associate a Poisson bivector to a Poisson bracket. However, it also goes the other way around: if $\Pi$ is a Poisson bivector, then the bracket $\{f,g\} = \big\langle (df \wedge dg), \Pi \big\rangle$ is a Poisson bracket, so that we have bijective correspondence between the set of Poisson structures on a Poisson manifold $M$ and the set of Poisson bivector fields on $M$ which are inverse to each other.

We can now also define equivalently a Poisson manifold $(M,\Pi)$ as a smooth manifold $M$ equipped with a Poisson structure $\Pi$.

It is important to remark that the Poisson structure is in general degenerate. For degenerate Poisson bivectors there are functions whose Hamiltonian vector fields vanish, which are called *Casimir functions*. However, in the case in which it is non-degenerate, the Poisson manifold reduces to the particular case of a symplectic manifold, where the symplectic 2-form $\omega \in \Omega^2(M)$ is given by

$$\omega(X_f, X_g) = \Pi(df,dg). \tag{2.7}$$

A fundamental geometric property of Poisson manifolds is given by the possibility to foliate the manifold with a distribution of symplectic manifolds. To understand this, let us first give the following

**Definition 2.1.4.** A smooth map $\varphi : M \to N$ between Poisson manifolds is called a *Poisson map* if it preserves the Poisson structure [1]

$$\Pi_N = \varphi_* \Pi_M, \tag{2.8}$$

where $\varphi_*$ denotes the push-forward by the map $\varphi$.

One can also have a *anti-Poisson map* by putting a minus sign on one side of the equation (2.8) (as well as in the defining equation in footnote [1]).

**Definition 2.1.5.** Let $S \subset M$ be a submanifold of the Poisson manifold $M$. If the inclusion map $i : S \hookrightarrow M$ is a Poisson map, then $S$ is a *Poisson submanifold* of $M$.

**Definition 2.1.6.** Let $S(M)$ be a set of linear subspaces of the tangent spaces $T_pM$ at each point $p \in M$. If for every $p \in M$ there exists vector fields $V_1, \ldots, V_s \in S(M)$ such that $S_p(M) = \text{span}\{V_1(p), \ldots, V_s(p)\}$, then $S(M)$ is called a *general differentiable distribution*. In particular, the distribution defined by

$$S_p(M) = \{V \in T_pM, p \in M \,|\, \exists f \in C^\infty(M), X_f(p) = V\} \tag{2.9}$$

is called the *characteristic distribution*.

**Definition 2.1.7.** Let $M$ be a differentiable manifold and $E$ a subvectorbundle of $TM$. E is then called (completely) *integrable* if for all $x \in M$ there exists a local submanifold $S \subset M$ such that $TS = TM|_S$.

In general, local submanifolds can be continued to connected maximal integral manifolds which are uniquely determined and regular immersed submanifolds of $M$. In particular, for the case of Poisson manifolds this leads to the following

**Theorem 2.1.1.** The characteristic distribution $S(M)$ of the Poisson manifold $M$ is completely integrable and the Poisson structure induces symplectic structures on the leaves of $S(M)$.

[1] This definition can also be given directly in terms of Poisson bracket as: $\{f, g\}_N(\varphi(x)) = \{f \circ \varphi, g \circ \varphi\}_M(x)$, $\forall x \in M$, $\forall f, g \in C^\infty(N)$.

The leaves of $S(M)$ are then called *symplectic leaves* of the Poisson manifold $M$ and are such that their tangent spaces are spanned by Hamiltonian vector fields. The distribution $S(M)$ is said to be the symplectic foliation of $M$.

Namely, a Poisson structure can be defined by its symplectic foliation and a Poisson manifold can be considered as a disjoint union of symplectic manifolds, which are Poisson submanifolds, called the symplectic leaves. The symplectic form on each leaf is given by a relation like in (2.7) since the Poisson structure restricted to each leaf is non-degenerate.

Furthermore, we give the following definition and proposition which will prove to be useful later in the thesis:

**Definition 2.1.8.** Let $(M, \Pi)$ be a Poisson manifold. If the rank of the Poisson structure $\Pi$ is constant on the symplectic foliation of $M$, then the Poisson manifold is called regular.

**Proposition 2.1.1.** Given $L \subset M$ a symplectic leaf of the Poisson manifold $(M, \Pi)$, a Casimir function $C$ of the Poisson structure $\Pi$ is constant on $L$.

Now let us consider some useful examples of Poisson manifolds and Poisson structures.

**Example 2.1.1.** (Trivial Poisson structure). Any manifold $M$ is a Poisson manifold with the trivial Poisson structure $\Pi = 0$. The identity map is a Poisson map for any Poisson manifold.

**Example 2.1.2.** (Kirillov-Souriau-Kostant structure). Consider as a manifold $M = \mathfrak{g}^*$ the dual of a finite dimensional real Lie algebra $\mathfrak{g}$. By definition, linear functions on the dual $\mathfrak{g}^*$ can be considered as elements of $\mathfrak{g}$, and the Poisson bracket of functions obtained this way is the Lie bracket on $\mathfrak{g}$

$$\{f, g\} = [f, g], \quad f, g \in \mathfrak{g}. \tag{2.10}$$

The Poisson structure obtained this way is called *Kirillov-Souriau-Kostant Poisson structure* (KSK). Poisson structures of this kind are also called *linear Poisson structures* because of the explicit form in terms of local coordinates: $\Pi^{ij} = C^{ij}{}_k x^k$, where the $C^{ij}{}_k$ are the structure constants of the Lie algebra $\mathfrak{g}$.

In this particular case, the symplectic leaves correspond to the coadjoint orbits of any connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Note also that in general the symplectic leaves may have varying dimensions. For example, the origin is always a symplectic leaf since the Poisson structure vanishes there.

**Example 2.1.3.** (Presymplectic structure). Another useful example is that of presymplectic manifolds. In particular, this will be useful in the context of Jacobi sigma models.

A *presymplectic structure* on a smooth manifold $M$ is a closed 2-form $\omega \in \Omega^2(M)$. When $\omega$ is also non-degenerate, then it is a symplectic structure on $M$. A symplectic manifold can be also constructed from a presymplectic one from the quotient of $M$ by the flow of the vector fields in the kernel of $\omega$, if this quotient exists.

In general, it is possible to define a Poisson algebra $\mathcal{A}_\omega$ of functions $f \in C^\infty(M)$ such that the equation

$$\iota_{X_f}\omega = df \tag{2.11}$$

has a solution $X_f \in \mathfrak{X}(M)$. Namely, functions to which it is possible to associate a Hamiltonian vector field. Let $f, g \in \mathcal{A}_\omega$, if $X_f, X_g \in \mathfrak{X}(M)$ are the associated Hamiltonian vector fields, then $fg$ has a associated Hamiltonian vector field $fX_f + gX_g$, hence $\mathcal{A}_\omega \subset C^\infty(M)$. The Poisson bracket is then defined as $\{f, g\} = \omega(X_f, X_g)$ just like in (2.7). In the general presymplectic case the Hamiltonian vector field $X_f$ is not uniquely defined, but since the ambiguities are in the kernel of $\omega$, the Poisson bracket is still well defined.

However, for dynamical systems the degeneracy of the two-form $\omega$ may be a problem in general. In fact, it is well known that a classical physical system can be described with symplectic geometry. In particular, the phase space of a system can be described as a symplectic manifold $(\mathcal{P}, \omega)$. The phase space is the kinematical part of the system, while the dynamical one is encoded into the Hamiltonian, a real-valued function $H \in C^\infty(\mathcal{P})$. The dynamical trajectories of the system in $\mathcal{P}$ are then obtained by solving the Hamilton equation cfr (2.11)

$$\iota_{X_H}\omega = dH, \tag{2.12}$$

where the vector field which solves the equation, $X_H \in \mathfrak{X}(\mathcal{P})$, is the Hamiltonian vector field associated to the function $H$, and the dynamical trajectories in $\mathcal{P}$ are the integral curves of $X_H$. The non-degeneracy of $\omega$ obviously makes the solution for $X_H$ unique. But eventual degeneracy of $\omega$ leading to a presymplectic manifold as a phase space may happen when the phase space is infinite-dimensional or when the system is constrained. In this case, if a solution of (2.11) exists, it will surely not be unique. If solutions exist, then the nonuniqueness is actually characterized by the kernel of $\omega$, but when

(2.11) possesses no globally defined solutions, $M$ or the equations themselves must be modified, or eventually both. To take this situation into account for dynamical systems one can consider the *Presymplectic Constraint Algorithm* (PCA), an algorithm introduced by Gotay and Nester in [102] to account for this problem, providing a recipe for the modification and the resolution of the problem, both in the finite- and infinite-dimensional cases.

## 2.2 Poisson-Lie groups

**Definition 2.2.1.** A *Poisson-Lie group* is a Lie group $G$ which is also a Poisson manifold, with a Poisson structure such that the multiplication $\mu : G \times G \to G$ is a Poisson map if $G \times G$ is equipped with the product Poisson structure.

The condition for $\mu$ being a Poisson map basically means that

$$\begin{aligned} \{f,g\}_G \circ \mu(a,b) &= \{f\circ\mu, g\circ\mu\}_{G\times G}(a,b) \\ &= \{f\circ\mu(\cdot,b), g\circ\mu(\cdot,b)\}(a) + \{f\circ\mu(a,\cdot), g\circ\mu(a,\cdot)\}(b) \\ &= \{f\circ R_b, g\circ R_b\}(a) + \{f\circ L_a, g\circ L_a\}(b), \end{aligned} \tag{2.13}$$

for all $f,g \in C^\infty(G)$, $a,b \in G$ and $L_a$ and $R_a$ denote respectively the left and right translation by the group element $a$. However, compatibility of the Poisson bracket with the group multiplication is required, and this happens if and only if the corresponding Poisson bivector is multiplicative, i.e.:

**Definition 2.2.2.** A Poisson bivector $\Pi$ is called *multiplicative* if it satisfies for all $a,b \in G$

$$\Pi(ab) = \left( dL_a|_b \otimes dL_a|_b \right) \Pi(b) + \left( dR_b|_a \otimes dR_b|_a \right) \Pi(a). \tag{2.14}$$

Let $\mathfrak{g}$ denote the Lie algebra of a Poisson-Lie group $G$, identified with $T_eG$, the tangent space at the group identity $e$. We can use the Poisson bracket defined on the group manifold to introduce a Lie bracket on $\mathfrak{g}^*$, the dual vector space of $\mathfrak{g}$, as follows:

**Definition 2.2.3.** The *induced dual Lie bracket* on $\mathfrak{g}^*$, via the Poisson bracket $\{\cdot,\cdot\}$ on $G$ can be obtained as

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = d\{f_1, f_2\}(e), \tag{2.15}$$

with $f_1, f_2 \in C^\infty(G)$ with the property $df_1(e) = \xi_1$, $df_2(e) = \xi_2$.

It is possible to prove that this induced bracket is indeed a Lie bracket. The compatibility condition between Lie and Poisson structures gives the following relation

$$\begin{aligned}\langle [X,Y],[u,v]^*\rangle + \langle \mathrm{ad}^*_v X, \mathrm{ad}^*_Y v\rangle - \langle \mathrm{ad}^*_v X, \mathrm{ad}^*_Y u\rangle \; - \langle \mathrm{ad}^*_u Y, \mathrm{ad}^*_X v\rangle \\ + \langle \mathrm{ad}^*_v Y, \mathrm{ad}^*_X u\rangle = 0,\end{aligned} \tag{2.16}$$

with $u, v \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$, while $\mathrm{ad}^*_X$ and $\mathrm{ad}^*_u$ respectively denote the coadjoint actions of $\mathfrak{g}$ and $\mathfrak{g}^*$ on each other and $\langle \cdot, \cdot \rangle$ the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. This construction allows to define a Lie bracket on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ as follows:

$$[X+\xi, Y+\zeta] = [X,Y] + [\xi,\zeta]^* - ad^*_X \zeta + ad^*_Y \xi + ad^*_\zeta X - ad^*_\xi Y, \tag{2.17}$$

with $X, Y \in \mathfrak{g}$ and $\xi, \zeta \in \mathfrak{g}^*$.

A Lie algebra with a compatible dual Lie bracket is called a *Lie bialgebra*. If the group $G$ is connected, the compatibility condition is enough to integrate $[\cdot,\cdot]^*$ to a Poisson structure on it, making it Poisson-Lie, and the Poisson structure is unique. Since the role of $\mathfrak{g}$ and $\mathfrak{g}^*$ in (2.16) is symmetric, one has also a Poisson-Lie group $G^*$ with Lie algebra $(\mathfrak{g}^*, [\cdot,\cdot]^*)$ and a Poisson structure whose linearization at $e \in G^*$ gives the bracket $[\cdot,\cdot]$. In this case $G^*$ is said to be the Poisson-Lie dual group of $G$.

The triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ where $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a Lie algebra with bracket given by (2.17) is known as a Manin triple, whereas its exponentiation to a Lie group $D$ is the Drinfel'd double of $G$. More precisely

**Definition 2.2.4.** A *Drinfel'd double* is an even-dimensional Lie group $D$ whose Lie algebra $\mathfrak{d}$ can be decomposed into a pair of maximally isotropic subalgebras[2], $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, with respect to a non-degenerate (ad)-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{d}$.

**Definition 2.2.5.** A *Manin triple* $(\mathfrak{c}, \mathfrak{a}, \mathfrak{b})$ is a Lie algebra with a non-degenerate scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{c}$ such that:

(i) $\langle \cdot, \cdot \rangle$ is invariant under the Lie bracket:
$\langle c_1, [c_2, c_3]\rangle = \langle [c_1, c_2], c_3\rangle, \quad \forall c_1, c_2, c_3 \in \mathfrak{c}$;
(ii) $\mathfrak{a}, \mathfrak{b}$ are maximally isotropic Lie subalgebras with respect to $\langle \cdot, \cdot \rangle$;
(iii) $\mathfrak{a}, \mathfrak{b}$ are complementary (as linear subspaces), i.e. $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$.

[2]An *isotropic subspace* of $\mathfrak{d}$ with respect to $\langle \cdot, \cdot \rangle$ is defined as a subspace $A$ on which the bilinear form vanishes: $\langle a, b \rangle = 0 \ \forall a, b \in A$. An isotropic subspace is said to be *maximal* if it cannot be enlarged while preserving the isotropy property, or, equivalently, if it is not a proper subspace of another isotropic space.

Note that since the bilinear form is non-degenerate by definition, we can identify $\tilde{\mathfrak{g}}$ with the dual vector space $\mathfrak{g}^*$, and the Lie subalgebra structure on $\tilde{\mathfrak{g}}$ then makes $\mathfrak{d}$ into a Lie bialgebra. It is possible to prove that, conversely, every Lie bialgebra defines a Manin triple by identifying $\tilde{\mathfrak{g}} = \mathfrak{g}^*$ and defining the mixed Lie bracket between elements of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ in such a way to make the bilinear form invariant. Indeed, one can prove that if we want to make $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ into a Manin triple, using the natural scalar product on $\mathfrak{d}$, there is only one possibility for the Lie bracket, as explained in the following.

**Lemma 2.2.1.** Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$ and dual Lie bracket $[\cdot,\cdot]_{\mathfrak{g}^*}$. Every Lie bracket on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ such that the natural scalar product is invariant and such that $\mathfrak{g}$, $\mathfrak{g}^*$ are Lie subalgebras is given by:

$$\begin{aligned}
&[x,y]_{\mathfrak{d}} = [x,y]_{\mathfrak{g}} && \forall\, x,y \in \mathfrak{g} \\
&[\alpha,\beta]_{\mathfrak{d}} = [\alpha,\beta]_{\mathfrak{g}^*} && \forall\, \alpha,\beta \in \mathfrak{g}^* \\
&[x,\alpha]_{\mathfrak{d}} = -\operatorname{ad}^*_\alpha x + \operatorname{ad}^*_x \alpha && \forall\, x \in \mathfrak{g}, \alpha \in \mathfrak{g}^*.
\end{aligned} \tag{2.18}$$

In order for the whole algebra to satisfy Jacobi identity the brackets on the two dual spaces have to be compatible. Moreover, this bracket is the unique Lie bracket which makes $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ into a Manin triple.

To make things more explicit, on choosing $T_i$ and $\widetilde{T}^i$ as the generators of the Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively, such that $T_I \equiv (T_i, \widetilde{T}^i)$ are the generators of $\mathfrak{d}$, by the property of isotropy and duality as vector spaces we have

$$\begin{aligned}
\langle T_i, T_j \rangle &= 0 \\
\langle \widetilde{T}^i, \widetilde{T}^j \rangle &= 0 \\
\langle T_i, \widetilde{T}^j \rangle &= \delta_i{}^j,
\end{aligned} \tag{2.19}$$

with $i = 1, 2, \ldots, \dim G$, while the bracket in (2.18), in doubled notation given by $[T_I, T_J] = F_{IJ}{}^K T_K$, can be written explicitly as follows:

$$\begin{aligned}
[T_i, T_j] &= f_{ij}{}^k T_k \\
[\widetilde{T}^i, \widetilde{T}^j] &= g^{ij}{}_k \widetilde{T}^k \\
[T_i, \widetilde{T}^j] &= f_{ki}{}^j \widetilde{T}^k - g_i^{kj} T_k,
\end{aligned} \tag{2.20}$$

with $f_{ij}{}^k$, $g^{ij}{}_k$, $F_{IJ}{}^K$ structure constants for $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ and $\mathfrak{d}$ respectively.

Jacobi identity on $\mathfrak{d}$, or equivalently the compatibility condition, impose the following constraint on structure constants of dual algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$:

$$g^{pk}{}_i f_{qp}{}^j - g^{pj}{}_i f_{qp}^k - g^{pk}{}_q f_{ip}^j + g^{pj}{}_q f_{ip}^k - g^{jk}{}_p f_{qi}^p = 0, \tag{2.21}$$

which is equivalent to Eq. (2.16), obtained as a compatibility condition between Poisson and group structure on a given group G (Poisson-Lie condition).

From previous results some observations follow: the relation is completely symmetric in the structure constants of the dual partners as the entire construction is symmetric, and exchanging the role of the two subalgebras leads exactly to the same structure. This will be important for the formulation of Poisson-Lie duality. It is worth to note that this condition is always satisfied whenever at least one of the two subalgebras is Abelian. This means that if $\mathfrak{d}$ is a Lie algebra of dimension $2d$, we always have at least two Manin triples $(\mathfrak{g}, \mathbb{R}^d)$ and $(\mathbb{R}^d, \mathfrak{g})$, with $\dim \mathfrak{g} = d$.

By exponentiation of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ one gets the dual Poisson-Lie groups $G$ and $\tilde{G}$ such that, in a given local parametrization, $D = G \cdot \tilde{G}$, or by changing parametrization, $D = \tilde{G} \cdot G$. The simplest example is the cotangent bundle of any $d$-dimensional Lie group $G$, $T^*G \simeq G \ltimes \mathbb{R}^d$, which we shall call the classical double, with trivial Lie bracket for the dual algebra $\tilde{\mathfrak{g}} \simeq \mathbb{R}^d$.

The natural symplectic structure on the group manifold of the double $D$ is the so called Semenov-Tian-Shansky structure [103] $\{f, g\}_D$, for $f, g$ functions on $D$. If one considers the functions $f, g$ to be invariant with respect to the action of the group $\tilde{G}$ ($G$) on $D$, they can be basically interpreted as functions on the group manifold of $G$ ($\tilde{G}$), which then inherit the Poisson structure directly from the double.

We finally point out that there may be many decompositions of $\mathfrak{d}$ into maximally isotropic subspaces, which are not necessarily subalgebras: when the whole mathematical setting is applied to sigma models, the set of all such decompositions plays the role of the modular space of sigma models mutually connected by a $O(d, d)$ transformations. In particular, for the manifest Abelian T-duality of the string model on the d-torus, the Drinfel'd double is $D = U(1)^{2d}$ and its modular space is in one-to-one correspondence with $O(d, d; \mathbb{Z})$ [52].

After this brief review of Drinfel'd doubles and Manin triples, for the purposes of this work we will focus on a particular example of Drinfel'd double, $SL(2, \mathbb{C})$.

### 2.2.1 Manin triple decomposition of $SL(2,\mathbb{C})$

As a starting point let us fix the notation. The real form of the $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra is usually represented as:

$$\begin{aligned} \left[e_i, e_j\right] &= i\epsilon_{ij}{}^k e_k \\ \left[b_i, b_j\right] &= -i\epsilon_{ij}{}^k e_k \\ \left[e_i, b_j\right] &= i\epsilon_{ij}{}^k b_k, \end{aligned} \tag{2.22}$$

with $\{e_i\}_{i=1,2,3}$ generators of the $\mathfrak{su}(2)$ subalgebra, $\{b_i\}_{i=1,2,3}$ boosts generators. The linear combinations

$$\hat{e}^i = \delta^{ij}\left(b_j + \epsilon^k{}_{j3} e_k\right), \tag{2.23}$$

are dual to the $e_i$ generators with respect to the Cartan-Killing product naturally defined on $\mathfrak{sl}(2,\mathbb{C})$ as

$$\langle v, w\rangle = 2\left[\mathrm{Im}\,(vw)\right], \ \forall\, v, w \in \mathfrak{sl}(2,\mathbb{C}). \tag{2.24}$$

Indeed, it is easy to show that

$$\left\langle \hat{e}^i, e_j\right\rangle = 2\,\mathrm{Im}\left[\mathrm{Tr}\left(\hat{e}^i e_j\right)\right] = \delta^i_j. \tag{2.25}$$

Moreover, the dual vector space $\mathfrak{su}(2)^*$ spanned by $\{\hat{e}^i\}_{i=1,2,3}$ is the Lie algebra of the Borel subgroup of $SL(2,\mathbb{C})$, so called $SB(2,\mathbb{C})$, of $2\times 2$ upper triangular complex matrices with unit determinant and real diagonal, for which the Lie bracket is defined as follows

$$\left[\hat{e}^i, \hat{e}^j\right] = i f^{ij}{}_k \hat{e}^k \tag{2.26}$$

and

$$\left[\hat{e}^i, e_j\right] = i\epsilon^i_{jk}\hat{e}^k + i e_k f^{ki}, \tag{2.27}$$

with structure constants $f^{ij}{}_k = \epsilon^{ijs}\epsilon_{s3k}$. As a manifold $SB(2,\mathbb{C})$ is non-compact and its Lie algebra is non-semisimple and solvable, which is reflected in the fact that the structure constants $f^{ij}{}_k$ as previously defined are not completely antisymmetric.

It is important to note that the following relations hold

$$\langle e_i, e_j\rangle = \left\langle \hat{e}^i, \hat{e}^j\right\rangle = 0, \tag{2.28}$$

so that both subalgebras $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ are maximal isotropic subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ with respect to $\langle\cdot,\cdot\rangle$. Therefore, $(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{su}(2),\mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple with respect to the natural Cartan-Killing pairing on $\mathfrak{sl}(2,\mathbb{C})$ and $SL(2,\mathbb{C})$ is a Drinfel'd double with respect to this decomposition (polarization): $SL(2,\mathbb{C}) = SU(2)\cdot SB(2,\mathbb{C})$.

Let us observe that the first of the Lie brackets (2.22) together with (2.26) and (2.27) have exactly the form (2.20) and that in doubled notation, $e_I = \begin{pmatrix} e_i \\ \hat{e}^i \end{pmatrix}$, with $e_i \in \mathfrak{su}(2)$ and $\hat{e}^i \in \mathfrak{sb}(2,\mathbb{C})$, the scalar product

$$\langle e_I, e_J\rangle = \eta_{IJ} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix} \tag{2.29}$$

corresponds to an $O(3,3)$ invariant metric.

Other than the natural Cartan-Killing bilinear form there is also another non-degenerate invariant scalar product which can be defined on $\mathfrak{sl}(2,\mathbb{C})$ as:

$$(v,w) = 2\mathrm{Re}\left[\mathrm{Tr}\,(vw)\right], \quad \forall\, v,w \in \mathfrak{sl}(2,\mathbb{C}). \tag{2.30}$$

However, it is easy to check that $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ are no longer isotropic subspaces with respect to this scalar product, it being

$$(e_i,e_j) = \delta_{ij}, \quad (b_i,b_j) = -\delta_{ij}, \quad (e_i,b_j) = 0. \tag{2.31}$$

Note that this does not give rise to a positive-definite metric. However, on denoting by $C_+, C_-$ respectively the two subspaces spanned by $\{e_i\}$ and $\{b_i\}$, the splitting $\mathfrak{sl}(2,\mathbb{C}) = C_+ \oplus C_-$ (which is not a Manin triple polarization by the way, since $C_+$ and $C_-$ do not close as subalgebras) defines a positive definite metric $\mathcal{H}$ on $\mathfrak{sl}(2,\mathbb{C})$ as follows:

$$\mathcal{H} = (\,,\,)_{C_+} - (\,,\,)_{C_-}\,. \tag{2.32}$$

This is a Riemannian metric and we denote it with the symbol $((\,,\,))$. In particular:

$$((e_i,e_j)) \equiv (e_i,e_j)\,, \quad ((b_i,b_j)) \equiv -\,(b_i,b_j)\,, \quad ((e_i,b_j)) \equiv (e_i,b_j) = 0. \tag{2.33}$$

In doubled notation, $e_I = \begin{pmatrix} e_i \\ \hat{e}^i \end{pmatrix}$, this Riemannian product can be written instead as

$$((e_I, e_J)) = \mathcal{H}_{IJ} = \begin{pmatrix} \delta_{ij} & -\delta_{ip}\epsilon^{jp3} \\ -\epsilon^{ip3}\delta_{pj} & \delta^{ij} + \epsilon^{il3}\delta_{\ell k}\epsilon^{jk3} \end{pmatrix}, \tag{2.34}$$

which satisfies the relation $\mathcal{H}\eta\mathcal{H} = \eta$, indicating that $\mathcal{H}$ is a pseudo-orthogonal $O(3,3)$ matrix.

This product can be verified to be equivalent to

$$((u, v)) \equiv 2\mathrm{Re}\left[\mathrm{Tr}\left(u^\dagger v\right)\right], \tag{2.35}$$

and its restriction to the $SB(2,\mathbb{C})$ subalgebra, which will be indicated by $h$, has the following form:

$$h^{ij} = \delta^{ij} + \epsilon^{i\ell 3}\delta_{\ell k}\epsilon^{jk3}. \tag{2.36}$$

It is interesting to notice that the $O(3,3)$ metric in (2.29) and the pseudo-orthogonal metric in (2.34) respectively have the same structure as the $O(d,d)$ invariant metric and the so called generalized metric $\mathcal{H}$ of Double Field Theory [20, 21, 28] (see Sec. 3.2.1).

Finally, let us notice that the most general action functional involving fields valued in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ should contain a combination of the two products, (2.29) and (2.34). This essentially amounts to consider the Hermitian product

$$\mathcal{H}_N = ((u, v))_N \equiv \mathrm{Tr}(u^\dagger v) \tag{2.37}$$

which will prove to be indeed necessary to define a non-vanishing WZ term for the $SB(2,\mathbb{C})$ related model (see Sec. 5.3). When restricting to the $\mathfrak{sb}(2,\mathbb{C})$ subalgebra it acquires the form

$$h_N^{ij} = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \tag{2.38}$$

and obviously satisfies the relation $h_N^{ij} + h_N^{ji} = h^{ij}$. It is possible to check (see Sec. 5.3) that only its real, diagonal part, namely (2.35), contributes when limited to the quadratic term of the WZW action, while only its imaginary, off-diagonal part, namely (2.29), contributes when computing the WZ term, as we will see.

# 3 T-duality

In theoretical physics, in particular in quantum field theory and string theory, the concept of duality is of fundamental importance. A duality can be referred to as a non-trivial equivalence between models, in the sense that they describe the same physics but with a different mathematical formulation. Dualities are of central importance because they allow to perform calculations in regimes which would be otherwise inaccessible, i.e. they allow to perform calculations on a situation in which the calculations are easier to perform. Since the theories are the same from the physical point of view, the calculation is valid also in the other theory where the calculation was too difficult, or even impossible, to perform, giving the possibility to have new important insights.

The first obvious example of duality is that of electromagnetic duality. In fact, by considering Maxwell's equations in the vacuum

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 0\\ \vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\times\vec{B} &= \frac{\partial\vec{E}}{\partial t}\end{aligned} \tag{3.1}$$

it is manifest they are highly symmetric. In particular, they are invariant under (other than Lorentz transformations of course) the so-called *electromagnetic duality*

$$(\vec{E},\vec{B}) \mapsto (\vec{B},-\vec{E}). \tag{3.2}$$

What we call an electric field and what we call a magnetic field is then simply a matter of convention, since every magnetic field has an equivalent description as an electric field, and vice versa. In a manifestly Lorentz invariant description in terms of field strength $F$ it is easy to understand what this duality really is from the geometric perspective. In fact, in this description

Maxwell's equations can be written as

$$\partial_\nu F^{\mu\nu} = 0 \quad \partial_\nu {}^\star F^{\mu\nu} = 0, \tag{3.3}$$

where ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ is the Hodge dual. It is then evident in this formulation that the duality transformation in (3.2) is simply $F \longmapsto F' = \star F$. The action can be written as

$$S_{EM} = -\frac{1}{4}\int F \wedge \star F, \tag{3.4}$$

and under the $F \longmapsto F' = \star F$ transformation the action changes as

$$S'_{EM} = -S_{EM}, \tag{3.5}$$

since in Minkowski space ($(+ - - -)$ signature) $\star^2 = -1$. The action then only changes by a global sign, and so the symmetry transformation (3.2) leaves the equations of motion invariant, or, equivalently, the physics is the same. This is a simple but yet useful example to introduce the concept of T-duality, since the latter can be similarly described as a transformation of fields which does not change the physics, yet changing the action.

Other than the electromagnetic duality which we just considered, there are many kinds of different dualities in physics. Among these we have T-duality, the one we will focus on in this thesis, which is a peculiar stringy feature and that, together with S-duality and U-duality, lies at the heart of relating the five different superstring theories that turn out to be seen as low-energy limits of the so-called M-theory. In particular, S-duality is a kind of generalization of electromagnetic duality. In string theory, S-duality acts by inverting the coupling constant. In particular, it is a strong-weak duality, in the sense that it maps a strongly coupled theory, where perturbative expansion can no longer be trusted, to a weakly coupled theory which can be described using perturbation theory. Strongly coupled systems are in general much more difficult to understand and therefore S-duality can be used to help to understand strongly coupled theories by first dualizing to a weakly coupled theory, and then studying that theory using perturbation theory. U-duality is a duality which combines T- and S- dualities together in a proper way. Another important example of dualities in theoretical physics is AdS/CFT duality [104], but there are in general many more.

## 3.1 Prerequisites

In this section we will introduce notations, conventions and basic results needed for T-duality discussion. We will focus on the worlsheet description of closed bosonic string theory for simplicity, but most of the results carry over superstring theories. The Polyakov action of the bosonic string in a $(G, B, \phi)$ brackground can be written as a non-linear sigma model action described by maps $X : \Sigma \to M$, which are the embedding maps of the two-dimensional Lorentzian worldsheet $(\Sigma, h)$ into the target manifold $(M, G)$, a (pseudo)-Riemannian manifold

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \left[ G_{\mu\nu} dX^\mu \wedge \star dX^\nu + B_{\mu\nu} dX^\mu \wedge dX^\nu + \alpha' \mathrm{R}\phi \star 1 \right], \tag{3.6}$$

with $G_{\mu\nu}$ the background metric of the target space $M$ (which we take to be 26-dimensional Minkowski with $G_{\mu\nu} = \eta_{\mu\nu}$), $B$ describes a constant $B$-field and $\phi$ denotes the dilaton field. The constant $\alpha'$ is related to the string length via the relation $\ell_S = 2\pi\sqrt{\alpha'}$. R is the Ricci scalar of the worldsheet.

It is also useful to write the action in local coordinates by writing the geometric objects in (3.1) in coordinates as follows

$$\begin{aligned} dX^\mu \wedge \star dX^\nu &= \sqrt{h} d^2\sigma\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu, \\ dX^\mu \wedge dX^\nu &= \quad d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu, \\ \star 1 &= \sqrt{h}\, d^2\sigma, \end{aligned} \tag{3.7}$$

where $h$ is the determinant of the worldsheet metric and $(\sigma^0, \sigma^1)$ are the coordinates on $\Sigma$.

In this section we will be mostly interested in worldsheets with cylindrical topology $\Sigma = \mathbb{R} \times S^1$, where the non-compact component is corresponding to the time coordinate $\sigma^0 = t$, while the circle corresponds to the space coordinate $\sigma^1 = \sigma$ with the identification $\sigma \sim \sigma + \ell_s$, and accordingly we impose the periodicity condition of the spatial component on the fields $X(t, \sigma) = X(t, \sigma + \ell_s)$. It is known that by using Weyl invariance of the action one can use a conformal gauge in which the worldsheet metric takes the Minkowski form $h_{\alpha\beta} = \eta_{\alpha\beta}$, and further introducing light-cone coordinates $\sigma^\pm = t \pm \sigma$ the action can be rewritten as

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \left( G_{\mu\nu} + B_{\mu\nu} \right) \partial_+ X^\mu \partial_- X^\nu. \tag{3.8}$$

In this form, the equations of motion have a particularly simple form, and they can be obtained by varying the action as usual, leading to

$$\partial_+\partial_- X^\mu = 0. \tag{3.9}$$

### 3.1.1 Compactification on $S^1$

The plan now is to compactify the closed bosonic string on a circle and explore what T-duality is in this case.

In particular, we compactify the string on a circle of radius $R$ on the last spatial coordinate, which means that the 25th coordinate on the target space $M$ is identified as $X^{25} \sim X^{25} + 2\pi R$. For simplicity, we will also assume that the $B$-field 'has no leg' along the circle direction, namely $B_{\mu 25} = 0$, so we have not to deal with that for the mode description of $X^{25}$.

The mode expansion of the field $X^{25}(t, \sigma)$ can be obtained by solving the e.o.m. (3.9) and imposing the periodic identification on the circle (on the space-time $M$), i.e. $X^{25}(t, \sigma + \pi) = X^{25}(t, \sigma) + 2\pi R w$, with $w$ integer, resulting in

$$X^{25}(\tau, \sigma) = X_R^{25}(t - \sigma) + X_L^{25}(t + \sigma), \tag{3.10}$$

where the right- and left-moving components of the field are given by

$$\begin{aligned} X_R^{25}(t-\sigma) &= x_R^{25} + \frac{2\pi\alpha'}{\ell_s} p_R^{25}(t-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^{25} e^{-\frac{2\pi i}{\ell_s} n(t-\sigma)} \\ X_L^{25}(t+\sigma) &= x_L^{25} + \frac{2\pi\alpha'}{\ell_s} p_L^{25}(t+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \bar{\alpha}_n^{25} e^{-\frac{2\pi i}{\ell_s} n(t+\sigma)}, \end{aligned} \tag{3.11}$$

having defined the centre of mass coordinates $x_R^{25} = \frac{x_0^{25} - c}{2}$, $x_L^{25} = \frac{x_0^{25} + c}{2}$ with $c$ an arbitrary constant, as well as the right and left momenta

$$\begin{aligned} p_R^{25} &= \frac{1}{2}\left(\frac{w}{R} - \frac{kR}{\alpha'}\right), \\ p_L^{25} &= \frac{1}{2}\left(\frac{w}{R} + \frac{kR}{\alpha'}\right), \quad k, w \in \mathbb{Z}. \end{aligned} \tag{3.12}$$

The integer $k$ is the quantized momentum along the compactified direction [1], which is also sometimes called Kaluza-Klein excitation number, while the integer $w$ is called the *winding number*, and it indicated the number of times

[1]The quantization of the momentum eigenvalue along a compactified direction follows from the requirement that the wavefunction must be single-valued, as the latter contains the factor $\exp(ip^{25}x^{25})$.

the string wraps around the circle. Note that having a quantized momentum along compact directions is common to point particles as well, but having non-zero winding number is peculiar to strings, as they can wrap around the compactified direction.

The mass spectrum of the theory can be calculated by using the mass formula together with the level-matching condition, and it is possible to show that the resulting spectrum of excitations is given by

$$M^2 = -p_\mu p^\mu = \frac{k^2}{R^2} + \frac{w^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N_R + N_L - 2), \tag{3.13}$$

where $N_{L,R}$ are the number operators counting string excitations in the corresponding left and right sectors. The first term in (3.13) is the contribution of the momentum to the mass, which is basically a kind of kinetic energy we are familiar with. The second term is the mass contribution that comes from the winding of the string around the circle.

## 3.2 T-duality and introduction to Double Field Theory

The important and peculiar thing to notice from the mass spectrum in (3.13) is its invariance under the following $\mathbb{Z}_2$ action:

$$R \to \frac{\alpha'}{R}, \quad k \leftrightarrow w. \tag{3.14}$$

This symmetry of the spectrum of the bosonic string theory is called *T-duality*. It tells us that compactification on a circle of radius $R$ has the same mass spectrum as a theory which is compactified on a circle of radius $\tilde{R} = \frac{\alpha'}{R}$, as long as momentum and winding modes are exchanged. This is an inherently stringy feature since for point particles there is no winding number. This exchange of momentum and winding is at the core of stringy geometry, which is a term used to emphasize the peculiarity that strings behave differently to particles when it comes to geometry. This is a sign that strings see geometry in a different way, and cannot really distinguish between large and small circles, roughly speaking. Note also that under this action the spectrum is invariant but the momenta are mapped as

$$\left(p_R^{25}, p_L^{25}\right) \quad \longrightarrow \quad \left(-p_R^{25}, +p_L^{25}\right), \tag{3.15}$$

and it is possible to show that also the center of mass coordinates follow the same mapping, but since these momenta and coordinates appear in the full mode expansion, it leads to the mapping

$$\left(X_R^{25}, X_L^{25}\right) \longrightarrow \left(-X_R^{25}, X_L^{25}\right). \tag{3.16}$$

The number operators and the commutation relations of the oscillator modes (when promoted to operators) can be shown to be invariant under this mapping, which is therefore a symmetry of the spectrum. However, note that this $\mathbb{Z}_2$ transformation does change the action of the theory, which is not invariant then. In fact, we have

$$\partial_+ X_L^{25} \partial_- X_R^{25} \longrightarrow -\partial_+ X_L^{25} \partial_- X_R^{25}, \tag{3.17}$$

which means that this is not a symmetry but a duality transformation, similarly to what happened with electromagnetic duality explained at the beginning of this chapter.

A thing worth to notice is that actually the full T-duality group for a circle compactification is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This is due to the fact that $(p_R^{25})^2$ and $(p_L^{25})^2$ are also invariant under

$$R \to \frac{\alpha'}{R}, \quad k \leftrightarrow -w, \tag{3.18}$$

and it extends to the fields $X_{L,R}^{25}$.

Another remark has to be done regarding the mapping of the radius. In fact, the $\mathbb{Z}_2$ transformation in (3.14) there is a fixed point at $R = \sqrt{\alpha'}$, the latter being called the self-dual radius. This may be interpreted as the minimal length scale of the string. However, this only works for the bosonic string.

The T-duality in the case of open strings, for which the worldsheet has a boundary, has the result of transforming a bosonic open string with Neumann boundary conditions to a bosonic open string with Dirichlet boundary conditions, and vice versa, allowing for a physical reasoning for D-branes.

T-duality also allows to build new string backgrounds which could not be addressed otherwise and generally go under the name of *non-geometric backgrounds* (see [18] for a recent review on the subject).

### 3.2.1 Hamiltonian formulation, $T^d$ compactification and Double Field Theory

The analysis which was carried out so far for the $S^1$ compactification can be generalized to toroidal compactifications. On a $d$-torus $T^d$, with constant background provided by the tensor metric field $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$, T-duality is described by $O(d,d;\mathbb{Z})$ transformations. By exchanging momentum and winding modes, it implies that the short distance behavior is governed by the long distance behavior in the dual torus $\tilde{T}^d$.

It has to be observed that, already at the classical level, the indefinite orthogonal group $O(d,d;\mathbb{R})$ naturally appears in the Hamiltonian description of the bosonic string with a world-sheet embedded into a $d$-dimensional target space $M$ together with two peculiar structures, the generalized metric $\mathcal{H}$ and the $O(d,d)$ invariant metric $\eta$ [105].

Consider again the Polyakov action in coordinates

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\sigma\mathrm{d}t \left(h^{\alpha\beta}\sqrt{h}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right), \tag{3.19}$$

and define $\dot{X} := \partial_t X$, $X' := \partial_\sigma X$. The dynamics of the theory is determined by the equations of motion for the coordinates $X^\mu$ accompanied with the constraints (in the conformal gauge):

$$G_{\mu\nu}\left(\dot{X}^\mu\dot{X}^\nu + X'^\mu X'^\nu\right) = 0 \quad ; \quad G_{\mu\nu}\dot{X}^\mu X'^\nu = 0 \tag{3.20}$$

deriving from the vanishing of the energy-momentum tensor $T_{\alpha\beta} \equiv \frac{\delta S}{\delta h^{\alpha\beta}} = 0$.

The Hamiltonian $H$ is given by a Legendre transformation with respect to the canonical momentum $P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\left(G_{\mu\nu}\dot{X}^\nu + B_{\mu\nu}X'^\nu\right)$ and $\dot{X}^\mu$, which leads to

$$H = \frac{1}{4\pi\alpha'}G_{\mu\nu}\left(\dot{X}^\mu\dot{X}^\nu + X'^\mu X'^\nu\right)\Big|_{\dot{X}(P)}. \tag{3.21}$$

By inverting the expression for $P_\mu$ we have

$$\dot{X}^\mu = 2\pi\alpha'\left(G^{-1}\right)^{\mu\nu}P_\nu - \left(G^{-1}\right)^{\mu\rho}B_{\rho\nu}X'^\nu \tag{3.22}$$

so that the Hamiltonian can be written as

$$H = \frac{1}{4\pi\alpha'}\begin{pmatrix} X' \\ 2\pi\alpha' P\end{pmatrix}^T \mathcal{H}(G,B)\begin{pmatrix} X' \\ 2\pi\alpha' P\end{pmatrix}$$

where the *generalized metric* $\mathcal{H}$ is introduced:

$$\mathcal{H}(G,B) \equiv \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{3.23}$$

One can then define *Generalized vector* $A_P \in TM \bigoplus T^*M$ as follows:

$$A_P(X) \quad = \quad X'^{\mu}\frac{\partial}{\partial x^{\mu}} + 2\pi\alpha' P_{\mu} dx^{\mu}, \tag{3.24}$$

so that $H$ results to be proportional to the squared length of the generalized vector $A_P$ as measured by the generalized metric $\mathcal{H}$.

In particular, in terms of $A_P$ the constraints in eq. (3.20) respectively become:

$$A_P^T \mathcal{H} A_P = 0, \quad A_P^T \eta A_P = 0. \tag{3.25}$$

The first sets $H$ to zero while the second one completely determines the dynamics, rewritten in terms of the $O(d,d)$ invariant metric: $\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$. This group is defined by the $d \times d$ matrices $\mathcal{T}$ satisfying the condition $\mathcal{T}^T \eta \mathcal{T} = \eta$. The generalized metric itself is an element of $O(d,d;\mathbb{R})$ since it satisfies $\eta\mathcal{H}\eta = \mathcal{H}^{-1}$, i.e. $\mathcal{H}^T\eta\mathcal{H} = \eta$.

All the admissible generalized vectors satisfying the constraint $A_P^T\eta A_P = 0$ are related by an $O(d,d;\mathbb{R})$ transformation via $A'_P = \mathcal{T} A_P$. Then, for $A'_P$ to solve the first constraint, a compensating transformation $\mathcal{T}^{-1}$ has to be applied to $\mathcal{H}$. , i.e. $\mathcal{H}' = (\mathcal{T}^{-1})^T \mathcal{H} (\mathcal{T}^{-1})$.

The matrix $\mathcal{H}$ and its inverse $\mathcal{H}^{-1}$ can be rewritten in products:

$$\mathcal{H}(G,B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \tag{3.26}$$

$$\mathcal{H}^{-1}(G,B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}. \tag{3.27}$$

This indeed shows that the background $B$ can be created from the $G$-background through a transformation involving $B$, hence named $B$-transformation.

In the case of compactification on the $d$-torus $T^d$ $O(d,d;\mathbb{R}) \to O(d,d;\mathbb{Z})$.

On $T^d$, with directions labeled by $a, b = 0, \ldots d$, where $G_{ab}$ and $B_{ab}$ are constant, with its isometries $U(1)^d$, the e.o.m.'s for the string coordinates are a set of conservation laws on the world-sheet [106]:

$$\partial_\alpha J^\alpha_a = 0 \quad \text{with} \quad J^\alpha_a = h^{\alpha\beta}G_{ab}\partial_\beta X^b + \epsilon^{\alpha\beta}B_{ab}\partial_\beta X^b h \equiv \epsilon^{\alpha\beta}\partial_\beta \tilde{X}_a. \tag{3.28}$$

Through the use of auxiliary fields, one gets the dual Polyakov action $\tilde{S}[\tilde{X};\tilde{G},\tilde{B}]$ on $\tilde{T}^d$ written in terms of the dual string coordinates $\tilde{X}_a$ and connected to $S[X;G,B]$ by $X^a \to \tilde{X}_a$ and suitable transformations of $(G,B) \to (\tilde{G},\tilde{B})$ through the so-called Büscher rules [46, 47]. More specifically, in terms of the coordinates $\tilde{X}_a$'s, one can introduce an action involving auxiliary fields $U^a_\alpha$ [106]:

$$S[U;G,B]{=}\int d^2\sigma \left\{ \left[ h^{\alpha\beta}U^a_\alpha U^b_\beta G_{ab} + \epsilon^{\alpha\beta}U^a_\alpha U^b_\beta B_{ab} \right] + \epsilon^{\alpha\beta}\partial_\alpha \tilde{X}_a U^a_\beta \right\} \tag{3.29}$$

Varying with respect to $\tilde{X}_a$ gives $\epsilon^{\alpha\beta}\partial_\alpha U^a_\beta = 0$, while the $U^a_\alpha$ equation of motion is:

$$h^{\alpha\beta}U^b_\beta G_{ab} + \epsilon^{\alpha\beta}U^b_\beta B_{ab} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}_a = 0. \tag{3.30}$$

This can be used to solve for $U^\alpha_a$ yielding:

$$U^a_\alpha = \left( \epsilon_\alpha{}^\beta \tilde{G}^{ab} + \delta^\beta_\alpha \tilde{B}^{ab} \right) \partial_\beta \tilde{X}_b \tag{3.31}$$

and

$$\tilde{S}[\tilde{X};\tilde{G},\tilde{B}] = \frac{T}{2}\int \left[ \tilde{G}_{ab} d\tilde{X}^a \wedge *d\tilde{X}^b + \tilde{B}_{ab}(X) d\tilde{X}^a \wedge d\tilde{X}^b \right] \tag{3.32}$$

being $\tilde{G} = (G - BG^{-1}B)^{-1}$ and $\tilde{B} = -G^{-1}B\tilde{G}$.

In this case one refers to *Abelian T-duality* for stressing the presence of global Abelian isometries in the target spaces of both the paired sigma models.

For closed strings, toroidal compactification on $T^d$ implies the following periodicity conditions:

$$X^a(t,\sigma) \equiv X^a(t,\sigma+\pi) + 2\pi L^a, \quad L^a = \sum_{i=1}^{d} w^i R_i e^a_i, \tag{3.33}$$

with $w^i$ being the winding numbers and $e^a_i$ vector basis on $T^d$. In this case, $O(d,d;\mathbb{R})$ gets restricted to $O(d,d;\mathbb{Z})$ and this constitutes the T-duality group of the toroidal compactification that provides a symmetry not only of the mass spectrum and the vacuum partition function but also of the scattering amplitudes.

It results very natural to require the string world-sheet sigma model to be, therefore, manifestly $O(d,d)$ invariant (we will omit, in the following, the specification $\mathbb{R}$ or $\mathbb{Z}$) [17, 19–22, 25]. Such formulation is based on a doubling of the string coordinates along the compact directions of the target space, i.e. on the introduction of both the usual coordinates $X^a$ and their respective duals $\tilde{X}_a$. The equations of motion for the doubled coordinates $\mathbb{X}^A \equiv (X^a, \tilde{X}_a)$ $(A = 1, \ldots 2d), (a = 1, \ldots d)$ can be combined into a single $O(d,d)$-invariant equation [17]:

$$\mathcal{H}\partial_\alpha \mathbb{X} = \eta \epsilon_{\alpha\beta} \partial^\beta \mathbb{X}. \tag{3.34}$$

For $G_{ab} = \eta_{ab}$ and $B = 0$, these reproduce the well-known Hodge duality conditions: $\partial_\alpha X^a = \epsilon_{\alpha\beta} \partial^\beta \tilde{X}^a$.

After doubling the coordinates, i.e. putting the coordinates $X^a$ and the dual ones $\tilde{X}_a$ in the *generalized* vector $\mathbb{X}^A$ above introduced, it is natural to replace the standard world-sheet action of string theory based on $G$ and $B$ by an action written in terms of $\eta$ and $\mathcal{H}$ and that could be manifestly invariant under Abelian T-duality.

The action proposed in ref. [21] fulfills this requirement and, for constant backgrounds, highlights the role of the generalized vector $\mathbb{X}$ and of the two metrics:

$$S = -\frac{T}{2} \int dt d\sigma \left[ \partial_t \mathbb{X}^A \partial_\sigma \mathbb{X}^B \eta_{AB} - \partial_\sigma \mathbb{X}^A \partial_\sigma \mathbb{X}^B \mathcal{H}_{AB} \right]. \tag{3.35}$$

This provides a manifest T-dual $O(d,d)$ symmetric formulation that may be considered as a natural generalization, at the string scale, of the usual Polyakov action that can be actually reproduced at compactification radius $R \gg \alpha'$ while its dual can be obtained at $R \ll \alpha'$. The new action, therefore, embodies the core of T-duality on flat compact target spaces: the short distance behavior is governed by the long distance behavior in the dual space. One refers to it as the *doubled world-sheet action* [32–37, 39–41, 107]. From a manifestly T-dual invariant two-dimensional string world-sheet, *Double Field Theory* (DFT) [26] should emerge out as a low-energy limit. DFT developed as a way to encompass the Abelian T-duality in field theory with Doubled Geometry underlying it [70–72]. In DFT, diffeomorphisms rely on an O(d,d) structure defined on the tangent space of a doubled torus. A section condition has then to be imposed for halving the $2d$ coordinates. There is a vast literature concerning DFT [26–41] including topological aspects and its description on group manifolds.

The doubled world-sheet action in ref. (3.35) can be also understood in terms of *Born geometry* that is based on the concept of Born reciprocity principle [108]. Briefly, this latter states that the validity of Quantum Mechanics implies a fundamental symmetry between space and momentum that is broken by General Relativity because it states that spacetime is curved, while energy-momentum space, i.e. the cotangent space, is linear and flat. The simple but radical idea proposed by Max Born, is that in order to unify Quantum Mechanics and General Relativity one should also allow the phase space, and thus momentum space, to carry curvature. The action in (3.35) is defined on the phase space with coordinates $\mathbb{X}^A \equiv (X^a, \tilde{X}_a)$ where the two metrics $\eta$ and $\mathcal{H}_{AB}$ have been introduced. When the corresponding sigma-model can be relaxed away from constant $\eta$ and $\mathcal{H}_{AB}$, this means that not only will space-time become curved, but momentum space as well. This then will lead to an implementation of Born reciprocity.

#### Relation with Generalized Geometry

The Double Field Theory framework has also been considered in connection with *Generalized Geometry* (GG) [70–72], which has consequently arisen as a mean to *geometrize* duality symmetries. GG is based on replacing the tangent bundle $TM$ of a manifold $M$ with a kind of Whitney sum $TM \oplus T^*M$, a bundle with the same base space but fibers given by the direct sum of tangent and cotangent spaces, and the Lie brackets on the sections of $TM$ by the so-called *Courant brackets*, involving vector fields and one-forms. The Courant bracket of two sections of $TM \oplus T^*M$ is defined as

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\left(i_X\eta - i_Y\xi\right), \tag{3.36}$$

where $X, Y$ are tangent vectors and $\xi, \eta$ cotangent vectors, such that $X+\xi$ and $Y+\eta$ are elements of the fibers $T \oplus T^*$. The important point of such bracket is that it commutes with the action of a closed 2-form $B$ [71]. Furthermore, it is easy to note that both this bracket and the inner product naturally defined on the generalized bundle $(x+\xi, X+\xi) = i_X\xi$ are invariant under diffeomorphisms of the underlying manifold $M$. This, together with the fact that a global closed 2-form $B$ will also preserve both the inner product and Courant bracket, means an overall action of the semi-direct product of diffeomorphisms and closed 2-forms. This formal setting has attracted interest in relation to DFT because it takes into account in a unified fashion vector fields, which generate diffeomorphisms for the $G_{ij}$ field, and one-forms, generating diffeomorphisms for the the $B_{ij}$ field. In fact, we have already recovered some of the ingredients along this section while exploring the Hamiltonian formulation in terms of generalized vectors, where the generalized metric also appears in a very natural way.

#### Non-Abelian T-duality

The kind of T-duality discussed so far belongs to a particular class which is characterized by the fact that the generators of target space duality transformations are Abelian, while generating symmetries of the action only if they are Killing vectors of the metric [46–48]. However, starting from Ref. [49], it was realized that the whole construction could be generalized to include the possibility that one of the two isometry groups be non-Abelian. This is called *non-Abelian*, or, more appropriately, *semi-Abelian* duality.

Although interesting, because it enlarges the possible geometries involved, the latter construction is not really symmetric, as a duality would require. In fact, the dual model is typically missing some isometries which are required to go back to the original model by gauging. This means that one can map the original model to the dual one, but then it is not possible to go back anymore (see [53] for a detailed explanation). This unsatisfactory feature is overcome with the introduction of *Poisson-Lie T-duality* [50–52] (for some recent work to alternative approaches see [53, 54]). The latter represents a genuine generalization, since it does not require isometries at all, while Abelian and non-Abelian cases can be obtained as particular instances. Recent results on Poisson-Lie T-duality and its relation with para-Hermitian geometry and integrability, as well as low-energy descriptions, can be found in [55–64].

## 3.3 Poisson-Lie symmetry and duality

In this section we will introduce the concept of Poisson-Lie T-duality as it is usually introduced in the standard literature. The material covered in this section is contained as a review part of the paper [69].

In a field theory context Poisson-Lie symmetry [50–53] is usually introduced as a deformation of standard isometries of two-dimensional non-linear sigma models on pseudo-Riemannian manifolds, in the Lagrangian approach. To make contact with the existing literature let us therefore summarize the main aspects. We use here local, light-cone coordinates, to adhere to common approach.

**Definition 3.3.1.** Let $X^i : \Sigma \to M$, where $(\Sigma, h)$ is a 2-dimensional oriented pseudo-Riemannian manifold, the so called *source space* (the image $X(\Sigma)$ being the worldsheet) with metric $h$ and $(M, g)$ a smooth manifold, the so called *target space* (or background), equipped with a metric $g$ and a 2-form $B$. Let $M$ admit at least a free action of a Lie group $G$ [2]. A 2-dimensional *non-linear sigma model* can be defined by the following action functional:

$$S = \int_\Sigma dz d\bar{z}\, E_{ij} \partial X^i \bar{\partial} X^j, \tag{3.37}$$

with the generalized metric $E_{ij} = g_{ij} + B_{ij}$.

Suppose that the group $G$ acts freely from the right, then the infinitesimal generators of the right action are the left-invariant vector fields $\{V_a\}$, satisfying

$$[V_a, V_b] = f_{ab}{}^c V_c \tag{3.38}$$

with $f_{ab}{}^c$ the structure constants of $\mathfrak{g}$. Under an infinitesimal variation of the fields

$$\delta X^i = V_a^i \epsilon^a, \tag{3.39}$$

with $\epsilon$ the infinitesimal parameters of the transformation, the variation of the action reads as:

$$\delta S = \int_\Sigma dz d\bar{z}\, \mathcal{L}_{V_a} \left(E_{ij}\right) \partial X^i \bar{\partial} X^j \epsilon^a - \int_\Sigma dz d\bar{z} \left[\partial \left(V_a^i E_{ij} \bar{\partial} X^j\right) + \bar{\partial} \left(V_a^i E_{ji} \partial X^j\right)\right] \epsilon^a,$$

where $\mathcal{L}_V$ denotes the Lie derivative along the vector field $V$. Using the fact that

$$dz d\bar{z} \left[\partial \left(V_a^i E_{ij} \bar{\partial} X^j\right) + \bar{\partial} \left(V_a^i E_{ji} \partial X^j\right)\right] = d \left(V_a^i E_{ij} \bar{\partial} X^j d\bar{z} - V_a^i E_{ji} \partial X^j dz\right),$$

we are left with

$$\delta S = \int_\Sigma dz d\bar{z}\, \mathcal{L}_{V_a} \left(E_{ij}\right) \partial X^i \bar{\partial} X^j \epsilon^a - \int_\Sigma d J_a \epsilon^a, \tag{3.40}$$

with

$$J_a = V_a{}^i \left(E_{ij} \bar{\partial} X^j d\bar{z} - E_{ji} \partial X^j dz\right) \tag{3.41}$$

[2] In general the action is only required to be free. If it is also transitive, the model takes the name of Principal Chiral Model and the target space is diffeomorphic with the group itself, as it is the case in this paper.

the Noether one-forms associated with the group transformation. If the action functional is required to be invariant, $\delta S$ has to be zero. Usually a stronger requirement is applied, namely that the target-space geometry be invariant as well. This entails separately the invariance of the metric and of the $B$-field, that is $\mathcal{L}_{V_a}\left(E_{ij}\right)=0$, $\mathcal{L}_{V_a}\left(B_{ij}\right)=0$. The two-form $B$ could be put to zero to start with. Hence, under these assumptions, the symmetry group is a group of isometries and the generators are Killing vector fields. If this is the case, from (3.40) we derive that the Noether one-forms are closed

$$dJ_a=0 \tag{3.42}$$

hence, they are locally exact,

$$J_a=d\widetilde{X}_a. \tag{3.43}$$

In particular, if the symmetry group is Abelian, one can always find a frame where $V_a{}^i=\delta_a{}^i$. In this case Abelian T-duality can be obtained by exchanging $X^i$ with the dual coordinates $\widetilde{X}_i$ and the Bianchi identity $d^2\tilde{X}_i=0$ then leads to the equations of motion for the original theory. Notice that, because of the definition of the one-forms $J_a$, the functions $\widetilde{X}_a$ take value in the tangent space at $M$, namely, they are velocity coordinates. Therefore, the symmetry of the model under target-space duality transformation amounts to the exchange of target space coordinates $X^i$ with velocities $\tilde{X}_i$, the generators of the symmetry are Killing vector fields and T-duality is along directions of isometry.

In the case in which the symmetry group of the action is non-Abelian, but still an isometry, one refers to non-Abelian (or, better, semi-Abelian) T-duality, with Noether currents satisfying Abelian Maurer-Cartan equations.

However, this whole construction can be generalized: suppose the Noether current one-forms are not closed but satisfy instead a Maurer-Cartan equation

$$dJ_a=\frac{1}{2}\tilde{f}_a{}^{bc}J_b\wedge J_c \tag{3.44}$$

being $\tilde{f}_a{}^{bc}$ the structure constants of some Lie algebra $\tilde{\mathfrak{g}}$ not yet specified. Using Eq. (3.44), by imposing invariance of the action, Eq. (3.40) yields

$$\int_\Sigma dzd\bar{z}\,\mathcal{L}_{V_a}\left(E_{ij}\right)\partial X^i\bar{\partial}X^j\epsilon^a=\int_\Sigma\frac{1}{2}\tilde{f}_a{}^{bc}J_b\wedge J_c\,\epsilon^a.$$

From Eq. (3.41), it is straightforward to obtain

$$J_b\wedge J_c=-2V_b^mV_c^lE_{nm}E_{lk}\partial X^n\bar{\partial}X^kdzd\bar{z},$$

and finally

$$\mathcal{L}_{V_a} E_{ij} = -\tilde{f}_a{}^{bc} V_b^k V_c^l E_{ik} E_{lj}. \tag{3.45}$$

The latter relation reveals that in order for the action to be invariant, the generators not only can be non-Abelian, but they do not have to be isometries (one can still find the standard isometry case if the algebra of Noether currents, $\tilde{\mathfrak{g}}$, is Abelian, so that the Lie derivative is again vanishing).

If this is the case, we say that the sigma model is *Poisson-Lie symmetric* (see def. (2.2.1) of Poisson-Lie group). Indeed, from the Lie algebra condition

$$\left[\mathcal{L}_{V_a}, \mathcal{L}_{V_b}\right] E_{ij} = f_{ab}{}^c \mathcal{L}_{V_c} E_{ij} \tag{3.46}$$

the following compatibility condition for the pair structure constants follows:

$$\tilde{f}_a{}^{mc} f_{dm}{}^b - \tilde{f}_a{}^{mb} f_{dm}{}^c - \tilde{f}_d{}^{mc} f_{am}{}^b + \tilde{f}_d{}^{mb} f_{am}{}^c - \tilde{f}_m{}^{bc} f_{da}{}^m = 0, \tag{3.47}$$

which is exactly the compatibility condition in (2.21), in order for two algebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ (which are dual as vector spaces), concur to define a bialgebra, $\mathfrak{d}$, whose underlying vector space is the direct sum of the former. Equivalently, Eq. (3.47) is nothing but the compatibility condition (2.16) between Poisson and group structure of a Poisson-Lie group.

The triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is associated to the starting sigma model but, since the construction of the bialgebra structure is completely symmetric, one can expect to formulate a model associated with the same triple by swapping the role of the subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ and that will be the *Poisson-Lie dual sigma model*, defined by

$$\mathcal{L}_{\tilde{V}_a} \tilde{E}_{ij} = -f_a{}^{bc} \tilde{V}_b^k \tilde{V}_c^\ell \tilde{E}_{ik} \tilde{E}_{\ell j} \tag{3.48}$$

where all fields with $\tilde{}$ refer to the dual model.

On introducing the group $D$ which corresponds to the exponentiation of the bialgebra $\mathfrak{d}$, equivalently we can say that a sigma model is of Poisson-Lie type if the target space is a coset space $D/G$, where $G$ indicates one of its component groups in a chosen polarization. Its dual will be defined on the target coset $D/\tilde{G}$. The group $D$ is the Drinfel'd double and $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is a Manin triple. $G, \tilde{G}$ are dual groups.

Since Poisson-Lie T-duality is a generalization of Abelian and semi-Abelian T-dualities, T-dualities may be classified in terms of the types of Manin triple underlying the sigma model structure:

- Abelian doubles correspond to standard Abelian T-duality. The Drinfel'd double is Abelian, with Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ with the algebra $\mathfrak{g}$

and its dual both Abelian;

- Semi-Abelian doubles (i.e. $\mathfrak{d} = \mathfrak{g} \dot{\oplus} \tilde{\mathfrak{g}}$, with $\mathfrak{g}$ non-Abelian, $\tilde{\mathfrak{g}}$ Abelian and $\dot{\oplus}$ a semi-direct sum) correspond to non-Abelian T-duality between an isometric and a non-isometric sigma model;

- Non-Abelian doubles (all the other possible cases) correspond to Poisson-Lie T-duality. Here no isometries hold for either of the two dual models.

The notion of Poisson-Lie symmetry can also be formulated in the Hamiltonian formalism [67, 69, 109–111]. We may state the following

**Definition 3.3.2.** Let $(M, \omega)$ be a symplectic manifold admitting a right action $M \times G \to M$ of $G$ on $M$, and let $V_a \in \mathfrak{X}(M), a = 1, \ldots d$ be the vectors fields which generate the action, with $d = \dim \mathfrak{g}$. If $\mathcal{L}_{V_a}\omega \neq 0$ but

$$i_{V_a}\omega = \tilde{\theta}^a, \tag{3.49}$$

with $\tilde{\theta}^a$ left(right)-invariant one-forms of the dual group $\tilde{G}$ and $i_V$ the interior derivative along $V$, we say that a dynamical system with phase space $(M, \omega)$ is Poisson-Lie symmetric with respect to $G$ if its Hamiltonian is invariant[3]

$$\mathcal{L}_V H = 0. \tag{3.50}$$

For future convenience, Eq. (3.49) can be equivalently formulated according to

$$\mathcal{L}_{V_a}\omega = -\frac{1}{2} f^a{}_{bc} \tilde{\theta}^b \wedge \tilde{\theta}^c \tag{3.51}$$

which, contracted with dual vector fields, yields $\mathcal{L}_{V_a}\omega(\tilde{X}_b, \tilde{X}_c) = -f^a{}_{bc}$. Finally, let us notice that Poisson-Lie symmetry may be stated in terms of the Poisson bi-vector field $\Pi$ by saying that a dynamical system with target space a Poisson manifold possesses Poisson-Lie symmetry under the action of a Lie group $G$ if the Hamiltonian (or its equations of motion) is invariant and the bivector field $\Pi$ together with the infinitesimal generators of the symmetry implicitly defines one-forms of the dual group according to

$$V_a = \Pi(\tilde{\theta}^a). \tag{3.52}$$

[3]However, it should be sufficient to require that the vector fields $V$ generate symmetries of the equations of motion, not necessarily of the Hamiltonian.

# 4 The Isotropic Rigid Rotator as a toy model

In this chapter we discuss and report the results of papers [65, 66], where a simple mechanical system has been considered: the three-dimensional isotropic rigid rotator (IRR), thought of as a $(0+1)$-field theory described by a sigma model on $SU(2)$.

A remarkable property of this model is that its dynamics exhibits Poisson-Lie symmetries [112, 113] when described in the Hamiltonian approach, by replacing the cotangent space of $SU(2)$ with the group $SL(2,\mathbb{C})$ which plays the role of the alternative phase space of the model. The result is consistent with the two spaces, $T^*SU(2)$ and $SL(2,\mathbb{C})$, being symplectomorphic [113]. Let us remark here that the concept of Poisson-Lie symmetry, which concerns a single dynamical model, can be stated independently and it is indeed a pre-requisite for Poisson-Lie duality, which requires instead two dynamical systems with different carrier spaces to be formulated.

With this distinction in mind, the IRR model was considered under yet another point of view, the goal being to introduce a model on the dual group of $SU(2)$, with the aim of exhibiting a Poisson-Lie dual system. It turned out that the model on the dual group does not describe the same dynamics, though paving the way to a field theory generalization of the whole construction, which describes the Principal Chiral Model on $SU(2)$ and its dual partner as Poisson-Lie duals [67, 68]. The IRR is thus to be conceived as a toy model, where the key features of Poisson-Lie symmetries and Poisson-Lie duality can be clearly understood, although it should be stressed that its dynamics is not invariant under such transformations.

After defining the dual model, a parent action on the Drinfel'd double of $SU(2)$ can be introduced, containing a number of degrees of freedom which is doubled with respect to the original one and from which both this latter and its dual can be recovered by a suitable gauging of the isometries. The geometric structures that appear can then be understood in terms of Generalized Geometry and/or Doubled Geometry.

The material presented in this chapter is taken from the papers [66, 68].

### 4.0.1 The IRR as a $(0+1)$-dimensional sigma model

Let $\varphi : \mathbb{R} \ni t \to g(t) \in SU(2)$ the group-valued dynamical map (in terms of sigma models it is the embedding map). We can build an invariant action on $SU(2)$ by using the Maurer-Cartan left-invariant one-form $g^{-1}dg \in \Omega^1(SU(2)) \otimes \mathfrak{su}(2)$, so that a suitable sigma model action for the IRR model is the following:

$$S_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}\left[\varphi^*\left(g^{-1}dg\right) \wedge *\varphi^*\left(g^{-1}dg\right)\right] = -\frac{1}{4}\int_{\mathbb{R}} dt\, \mathrm{Tr}\left(g^{-1}\dot{g}\right)^2, \quad (4.1)$$

where $*$ is the Hodge star operator on $\mathbb{R}$, defined such that $*dt = 1$ and $\varphi^*$ denotes the pull-back map, so that $\varphi^*\left(g^{-1}dg\right) = g^{-1}\partial_t g\, dt$ defines the pull-back of the Maurer-Cartan one-form on $\mathbb{R}$. In particular, it can be written as $g^{-1}dg = i\alpha^k\sigma_k$, with $\sigma_k$ the Pauli matrices and $\alpha^k$ basic left-invariant one-forms.

The Lagrangian of the model is then $L_0 = -\frac{1}{4}\mathrm{Tr}\left(g^{-1}\dot{g}\right)^2$.

In order to adhere to the notation which is commonly adopted in field theory, we shall identify the dynamical variable $\varphi$ with $g$ for the remainder of this section, as well as simply omit the pull-back notation when there is no chance of confusion.

The group manifold $SU(2)$ can be parametrized by the embedding in the ambient space $\mathbb{R}^4$ as follows:

$$g = 2\left(y^0 e_0 + i y^i e_i\right), \quad g \in SU(2), \quad (4.2)$$

with $\left(y^0\right)^2 + \sum_i \left(y^i\right)^2 = 1$, $e_0 = \mathbb{I}/2$, $e_i = \sigma_i/2$. One has then:

$$y^0 = \langle e_0 | g \rangle = \mathrm{Tr}\left(g e_0\right), y^i = \langle e_i | g \rangle = -i\,\mathrm{Tr}\left(g e_i\right), \quad i = 1,2,3. \quad (4.3)$$

On $SU(2)$ we have $g^{-1} = g^\dagger$, so that

$$\begin{aligned} g^{-1}\dot{g} &= \left(y^0\mathbb{I} - i y^i\sigma_i\right)\left(\dot{y}^0\mathbb{I} + i\dot{y}^j\sigma_j\right) && (4.4)\\ &= i\left(y^0\dot{y}^i - y^i\dot{y}^0 + \epsilon^i{}_{jk}y^j\dot{y}^k\right)\sigma_i + \left(y^0\dot{y}^0 + y^i\dot{y}^i\right), && (4.5)\end{aligned}$$

where we used the well known relation $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ij}{}^k\sigma_k$. Moreover, the last term appearing in the above equation is vanishing since $\left(y^0\dot{y}^0 + y^i\dot{y}^i\right) = \frac{1}{2}\frac{d}{dt}\left(\left(y^0\right)^2 + \sum_i \left(y^i\right)^2\right) = 0$.

By using these relations we are led to

$$g^{-1}\dot{g} = i\left(y^0\dot{y}^i - y^i\dot{y}^0 + \epsilon^i{}_{jk}y^j\dot{y}^k\right)\sigma_i = i\dot{Q}^i\sigma_i, \tag{4.6}$$

where we defined the left generalized velocities

$$\dot{Q}^i \equiv \left(y^0\dot{y}^i - y^i\dot{y}^0 + \epsilon^i{}_{jk}y^j\dot{y}^k\right). \tag{4.7}$$

By using such velocities we can write the Lagrangian in a much more familiar form as

$$L_0 = \frac{1}{2}\dot{Q}^i\dot{Q}^j\delta_{ij}. \tag{4.8}$$

In fact, the latter can be obtained by performing the following computation:

$$\begin{aligned} L_0 &= -\frac{1}{4}\mathrm{Tr}\left(g^{-1}\dot{g}\right)^2 = \\ &-\frac{1}{4}\mathrm{Tr}\left[\left(i\dot{Q}^i\sigma_i\right)\left(i\dot{Q}^j\sigma_j\right)\right] = \frac{1}{4}\mathrm{Tr}\left[\dot{Q}^i\dot{Q}^j\left(\delta_{ij}\mathbb{I} + i\epsilon_{ij}{}^k\sigma_k\right)\right], \end{aligned} \tag{4.9}$$

and using the fact that the $\sigma$ matrices are traceless. From right-invariant one-forms one could define right generalized velocities in an analogous way, they give an alternative set of coordinates over the tangent bundle.

The Euler-Lagrangian equations of motion can be written in its intrinsic formulation [114], especially relevant for non-invariant Lagrangians [1], as:

$$\mathcal{L}_\Gamma\theta_L - dL_0 = 0, \tag{4.10}$$

being $\mathcal{L}_\Gamma$ the Lie derivative along the vector field $\Gamma = \frac{d}{dt}$ and $\theta_L$ the Lagrangian one-form, which is given by

$$\theta_L = \frac{\partial L}{\partial \dot{Q}^j}\alpha^j = \frac{1}{2}\dot{Q}^i\alpha^j\delta_{ij}. \tag{4.11}$$

By projecting (4.10) along the basic left-invariant vector fields $X_i$ (dual to the basic left-invariant one-forms $\alpha^i$), one obtains:

$$\iota_{X_i}\left(\mathcal{L}_\Gamma\theta_L - dL_0\right) = 0. \tag{4.12}$$

[1] This is not the case, but it will be useful for the dual model which we will show to have non-invariant Lagrangian.

Since $\mathcal{L}_\Gamma$ and $\iota_{X_i}$ commute over the Lagrangian one-form [2], one gets:

$$\mathcal{L}_\Gamma\left(\frac{1}{2}\dot{Q}^j\iota_{X_i}\alpha^l\right)\delta_{jl} - \mathcal{L}_{X_i}L_0 = 0, \tag{4.13}$$

where we have used the fact that $\iota_X df = \mathcal{L}_X f$ for $f$ a function. Since $\iota_{X_i}\alpha^l = \delta_i^l$ and $\mathcal{L}_{X_i}L_0 = \frac{1}{2}\dot{Q}^p\dot{Q}^q\epsilon_{ip}{}^k\delta_{qk}$, we are left with the equation of motion

$$\mathcal{L}_\Gamma\dot{Q}^j\delta_{ji} - \dot{Q}^p\dot{Q}^q\epsilon_{ip}{}^k\delta_{qk} = 0, \tag{4.14}$$

but the latter term is vanishing because it is the contraction of a symmetric and of an antisymmetric tensor, hence

$$\ddot{Q}^i = 0, \quad i = 1,2,3. \tag{4.15}$$

Left momenta can be calculated as usual as:

$$I_i = \frac{\partial L_0}{\partial \dot{Q}^i} = \delta_{ij}\dot{Q}^j, \tag{4.16}$$

and cotangent bundle coordinates can then be chosen to be $\left(Q^i, I_i\right)$. An alternative set of fiber coordinates is represented by the right momenta, which are defined analogously in terms of the right generalized velocities.

The Legendre transform from $TSU(2)$ to $T^*SU(2)$ yields the Hamiltonian function:

$$H_0 = \left[I_i\dot{Q}^i - L_0\right]_{\dot{Q}^i=\delta^{ij}I_j} = \delta^{ij}I_iI_j - \frac{1}{2}\delta^{ij}I_jI_k\delta^{lk}\delta_{il} = \frac{1}{2}\delta^{ij}I_iI_j. \tag{4.17}$$

By introducing the dual basis $\left\{e^{i^*}\right\}$ in the cotangent space, such that $\langle e^{i^*}|e_j\rangle = \delta^i_j$, one can consider the form

$$I = -\frac{1}{2}iI_ie^{i^*}. \tag{4.18}$$

In the first order formulation the action results to be

$$S = \int\theta - \int dtH_0, \tag{4.19}$$

where

$$\theta = \langle I|g^{-1}dg\rangle = \langle -\frac{1}{2}iI_ie^{i^*}|2i\alpha^ke_k\rangle = I_i\alpha^k\delta^i_k \tag{4.20}$$

[2] This is general: $\iota_X\mathcal{L} = \iota_X(\iota_Xd + d\,\iota_X) = \iota_Xd\,\iota_X$ since $\iota_X$ is 2-nilpotent, while $\mathcal{L}\iota_X = (\iota_Xd + d\,\iota_X)\,\iota_X = \iota_Xd\,\iota_X$.

is the canonical one-form.

The symplectic form can then be obtained as

$$\omega = d\theta = dI_i \wedge \delta^i_j \alpha^j + I_i \delta^i_j d\alpha^j = dI_i \wedge \delta^i_j \alpha^j + I_i \delta^i_j \epsilon^j{}_{lk} \alpha^l \wedge \alpha^k, \tag{4.21}$$

where we also used the Maurer-Cartan equation $d\alpha^k = \epsilon^k{}_{ij} \alpha^i \wedge \alpha^j$. One can then calculate Poisson brackets by inverting $\omega$. The corresponding bi-vector field $\Lambda$ will be written in terms of the basis vector fields $\partial_{I_j}, X_j$ respectively spanning the fibers and the base manifold of the cotangent bundle. Using the fact that $X_i \left(\alpha^j\right) = \delta^j_i$, $\frac{\partial}{\partial I_i}\left(dI_j\right) = \delta^i_j$, we have

$$\omega^{-1} = \Lambda = a_i{}^j \frac{\partial}{\partial I_i} \wedge X_j + b_{ij} \frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial I_j} + c^{ij} X_i \wedge X_j. \tag{4.22}$$

By imposing the inverse condition one can easily see that $a^j_i = -\delta^j_i$, $b_{ij} = \epsilon_{ij}{}^k I_k$ and $c^{ij} = 0$, so that $\left\{y^i, y^j\right\} = 0$ and $\left\{I_i, I_j\right\} = \epsilon_{ij}{}^k I_k$. In order to calculate the $\left\{y^i, I_j\right\}$ bracket, one has to use the expression of the left invariant vector fields in the chosen $\mathbb{R}^4$ parametrization. To this, by recalling that $\alpha^j = \frac{1}{2}\mathrm{Tr} g^{-1} dg \sigma^j = y^0 dy^j - y^j dy^0 + \epsilon_{lk}{}^j y^l dy^k$, and using the property $X_i \left(\alpha^j\right) = \delta^j_i$ we get:

$$X_j = y^0 \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^0} + \epsilon_{lj}{}^k y^l \frac{\partial}{\partial y^k}. \tag{4.23}$$

By calculating

$$\{I_l, y^m\} = \Lambda(dI_l, dy^m) = -\delta^j_i \frac{\partial I_l}{\partial I_i} X_j(y^m) = -\delta^j_i \delta^i_l X_j(y^m) \tag{4.24}$$

and considering then that

$$X_j(y^m) = y^0 \frac{\partial}{\partial y^j} y^m - y^i \frac{\partial}{\partial y^0} y^m + \epsilon^s{}_{pj} y^p \frac{\partial}{\partial y^s} y^m = y^0 \delta^m_j + \delta^m_s \epsilon^s{}_{pj} y^p, \tag{4.25}$$

we obtain

$$\{I_l, y^m\} = -\delta^j_i \delta^i_l X_j(y^m) = -\delta^j_i \delta^i_l \left(y^0 \delta^m_j + \delta^m_s \epsilon^s{}_{pj} y^p\right) = -y^0 \delta^m_l - \epsilon^m{}_{pj} y^p. \tag{4.26}$$

From the Poisson brackets

$$\left\{y^i, y^j\right\} = 0 \tag{4.27}$$
$$\{I_i, I_j\} = \epsilon_{ij}{}^k I_k \tag{4.28}$$
$$\left\{y^i, I_j\right\} = \delta^i_j y^0 + \epsilon^i{}_{jk} y^k \leftrightarrow \{g, I_j\} = 2ige_j, \tag{4.29}$$

the Hamilton equations of motion can be derived:

$$\dot{I}_i = \{I_i, H\} = 0, \tag{4.30}$$
$$\dot{g} = \{g, H\} = -\delta^{ij} I_i \{I_j, g\} = 2\delta^{ij} I_i ige_j, \tag{4.31}$$

leading to

$$g^{-1}\dot{g} = 2iI_i\delta^{ij}e_j. \tag{4.32}$$

These equations show that the fiber coordinates $I_i$, associated to the angular momentum components, are constants of motion as expected, while $g$ undergoes a uniform precession. In this case, since the Lagrangian and the Hamiltonian are invariant under both left and right action of the $SU(2)$ group, also the right momenta can be seen to be conserved as well, making the model super-integrable. This will not be the case for the dual model, as we show in Sec. 4.0.2.

The fibers of the tangent bundle $TSU(2)$ are, as a vector space, $\mathfrak{su}(2) \simeq \mathbb{R}^3$, being $\dot{Q}^i$ vector fields components, while the fibers of the cotangent bundle $T^*SU(2)$ are isomorphic to the dual Lie algebra $\mathfrak{su}(2)^*$. This, as a vector space, is again $\mathbb{R}^3$, but now $I_i$ are one-form components.

The carrier space of the Hamiltonian dynamics $T^*SU(2)$ is represented, as a group, by the semi-direct product of $SU(2)$ and the Abelian group $\mathbb{R}^3$, i.e. $T^*SU(2) \simeq SU(2) \ltimes \mathbb{R}^3$, with Lie algebra

$$[L_i, L_j] = i\epsilon_{ij}{}^k L_k \tag{4.33}$$
$$[T_i, T_j] = 0 \tag{4.34}$$
$$[L_i, T_j] = i\epsilon_{ij}{}^k T_k, \tag{4.35}$$

being $L_i$ the generators of the $SU(2)$ algebra and $T_i$ the generators of $\mathbb{R}^3$, which behave as vectors under $SU(2)$ rotations as can be seen from the last relation. The linearization of the Poisson structure at the identity of $SU(2)$ provides a Lie algebra structure over the dual algebra $\mathfrak{su}(2)^*$. Thus, the brackets $\{I_i, I_j\} = \epsilon_{ij}{}^k I_k$ are induced by the coadjoint action of the group

$SU(2)$ on its dual algebra, hence the Poisson brackets governing the dynamics of the IRR are the Kirillov-Souriau-Konstant brackets.

It has been shown in [112] that the carrier space of the dynamics of the IRR can be generalized to the semisimple Lie group $SL(2,\mathbb{C})$. This can be realized by replacing the Abelian subgroup $\mathbb{R}^3$ with the non-Abelian group $SB(2,\mathbb{C})$, which, we recall, is the Borel Lie subgroup of $2\times 2$ upper triangular complex matrices with real diagonal and unit determinant. In particular, note that $SU(2)$ and $SB(2,\mathbb{C})$ constitute the pair with respect to which $SL(2,\mathbb{C})$ can be regarded as a Drinfel'd double. This means that the triple $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple with respect to the scalar product $\langle\cdot,\cdot\rangle$ in $\mathfrak{sl}(2,\mathbb{C})$ defined in (2.24), as we have discussed in Sec. 2.2.1.

In the next section we discuss a new model, which will be referred to as dual, in the sense that it is the analogue of the IRR but modeled on the dual group $SB(2,\mathbb{C})$.

### 4.0.2 The dual model

In this section we introduce a dynamical model on the dual group of $SU(2)$, the Borel group $SB(2,\mathbb{C})$, with an action functional that is formally analogous to (4.1).

As carrier space for the dynamics of the dual model in the Lagrangian (respectively Hamiltonian) formulation one can choose the tangent (respectively cotangent) bundle of the group $SB(2,\mathbb{C})$. The latter is the dual Lie group of $SU(2)$ because they are dual partners in a particular polarization realization of $SL(2,\mathbb{C})$ as a Drinfel'd double.

A suitable action for the system is the following:

$$\tilde{S}_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[\tilde{\varphi}^*\left(\tilde{g}^{-1}d\tilde{g}\right)\wedge *\tilde{\varphi}^*\left(\tilde{g}^{-1}d\tilde{g}\right)] = -\frac{1}{4}\int_{\mathbb{R}} dt\, \mathcal{T}r[(\tilde{g}^{-1}\dot{\tilde{g}})(\tilde{g}^{-1}\dot{\tilde{g}})], \tag{4.36}$$

with $\tilde{\varphi} : t \in \mathbb{R} \to \tilde{g} \in SB(2,\mathbb{C})$, the group-valued target space coordinates, and $\tilde{g}^{-1}d\tilde{g} = i\tilde{\alpha}_k\tilde{e}^k \in \Omega^1(SB(2,\mathbb{C}))\otimes\mathfrak{sb}(2,\mathbb{C})$ is the Maurer-Cartan left invariant one-form, with $\tilde{\alpha}_k$ the left-invariant basic one-forms. $*$ is again the Hodge star operator on the source space $\mathbb{R}$ satisfying $*dt = 1$. The symbol $\mathcal{T}r$ is used here to represent a suitable scalar product in the Lie algebra $\mathfrak{sb}(2,\mathbb{C})$. Just as in the previous section, when there is no chance of confusion we will use $\tilde{g}$ for $\tilde{\varphi}$ as well as omit the pull-back notation.

In this case the Lagrangian of the model is given by

$$\tilde{L}_0 = -\frac{1}{4}\mathcal{T}r[(\tilde{g}^{-1}\dot{\tilde{g}})(\tilde{g}^{-1}\dot{\tilde{g}})]. \tag{4.37}$$

In this case the group is not semi-simple, so there is no scalar product which is both non-degenerate and invariant. Therefore, one has two possible different choices: the scalar product defined by the real and/or imaginary part of the trace, given by (2.24) and (2.30) which is $SU(2)$ and $SB(2,\mathbb{C})$ invariant but degenerate, or one could use the scalar product induced by the Riemannian metric $\mathcal{H}$, which, on the algebra $\mathfrak{sb}(2,\mathbb{C})$ takes the form $\left((\tilde{e}^i,\tilde{e}^j)\right) = \delta^{ij} + \epsilon^i{}_{l3}\delta^{lk}\epsilon^j{}_{k3}$, positive definite and non-degenerate. However, this product on $\mathfrak{sl}(2,\mathbb{C})$ is $SU(2)$ invariant but only invariant under left $SB(2,\mathbb{C})$ action. Indeed, by observing that the generators $\tilde{e}^i$ are not Hermitian, (2.34) can be verified to be equivalent to:

$$((u,v)) \equiv 2\mathrm{Re}\left[\mathrm{Tr}\left(u^\dagger v\right)\right], \tag{4.38}$$

so that $((\tilde{g}^{-1}\dot{\tilde{g}},\tilde{g}^{-1}\dot{\tilde{g}})) = 2\mathrm{Re}\left\{\mathrm{Tr}[(\tilde{g}^{-1}\dot{\tilde{g}})^\dagger\tilde{g}^{-1}\dot{\tilde{g}}]\right\}$ which is not invariant under right $SB(2,\mathbb{C})$ action, since $\tilde{g}^{-1} \neq \tilde{g}^\dagger$. We use the latter scalar product: $\mathcal{T}r(ab) \equiv ((a,b))$ to define the model.

Again the group manifold can be embedded in the $\mathbb{R}^4$ ambient space and parametrized so that $\tilde{g} \in SB(2,\mathbb{C})$ can be written as $\tilde{g} = 2\left(u_0\tilde{e}^0 + iu_i\tilde{e}^i\right)$ with $\tilde{e}^0 = \mathbb{I}/2$ and $u_0^2 - u_3^2 = 1$. The latter condition follows from the $\det(\tilde{g}) = 1$ condition. This is easily understood from the explicit form of the generators, as written in (2.23):

$$\tilde{e}^1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}; \quad \tilde{e}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \tilde{e}^3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.39}$$

In order to be consistent we have then:

$$u_i = \frac{1}{4}\left(\left(i\tilde{g},\tilde{e}^i\right)\right),\ i = 1,2, \quad u_3 = \frac{1}{2}\left(\left(i\tilde{g},\tilde{e}^3\right)\right), \quad u_0 = \frac{1}{2}\left(\left(\tilde{g},\tilde{e}^0\right)\right). \tag{4.40}$$

Most of these calculations work in the same way as for the IRR model, with the appropriate differences in the parametrizaton of $\tilde{g} \in SB(2,\mathbb{C})$ and in the scalar product (this time invariant only under left $SB(2,\mathbb{C})$ action) which defines the metric $h^{ij} \equiv \left(\delta^{ij} + \epsilon^i{}_{l3}\epsilon^j{}_{k3}\delta^{lk}\right)$ (cfr. (2.36)), hence we will not go through details.

Since $\tilde{g}^{-1} = 2\left(u_0\tilde{e}^0 - iu_i\tilde{e}^i\right)$ we have

$$\tilde{g}^{-1}\dot{\tilde{g}} = 2i\left(u_0\dot{u}_i - u_i\dot{u}_0 + f_i{}^{jk}u_j\dot{u}_k\right)\tilde{e}^i = 2i\dot{\tilde{Q}}_i\tilde{e}^i, \tag{4.41}$$

where the $f^{ij}{}_k = \epsilon^{ijl}\epsilon_{l3k}$ are the structure constants of $\mathfrak{sb}(2,\mathbb{C})$, so that the Lagrangian can be written as

$$\tilde{L}_0 = h^{ij}\dot{\tilde{Q}}_i\dot{\tilde{Q}}_j, \tag{4.42}$$

having defined

$$\dot{\tilde{Q}}_i \equiv u_0\dot{u}_i - u_i\dot{u}_0 + f_i{}^{jk}u_j\dot{u}_k \tag{4.43}$$

as left generalized velocities.

Following the same approach as with the IRR, the equations of motion of the system can be found to be

$$\mathcal{L}_\Gamma\dot{\tilde{Q}}_j h^{ji} - \dot{\tilde{Q}}_l\dot{\tilde{Q}}_m f_k{}^{il}h^{mk} = 0. \tag{4.44}$$

We can then consider $(\tilde{Q}_i, \dot{\tilde{Q}}_i)$ as tangent bundle coordinates, with $\tilde{Q}_i$ implicitly defined, similarly to the rigid rotator case.

The carrier space of the Hamiltonian dynamics is instead $T^*SB(2,\mathbb{C})$, with coordinates $(\tilde{Q}_i, \tilde{I}^i)$, with $\tilde{I}^i$ the conjugate left momenta defined as usual as

$$\tilde{I}^i = \frac{\partial\tilde{L}_0}{\partial\dot{\tilde{Q}}_i} = h^{ij}\dot{\tilde{Q}}_j. \tag{4.45}$$

To perform the Legendre transform from $TSB(2,\mathbb{C})$ to $T^*SB(2,\mathbb{C})$ we have to invert (4.45), which results in

$$\dot{\tilde{Q}}_i = \tilde{I}^j\left(\delta_{ji} - \frac{1}{2}\epsilon_{jp3}\epsilon_{iq3}\delta^{pq}\right), \tag{4.46}$$

leading to the Hamiltonian

$$\tilde{H}_0 = \frac{1}{2}\left(h^{-1}\right)_{ij}\tilde{I}^i\tilde{I}^j, \tag{4.47}$$

being

$$\left(h^{-1}\right)_{ij} = \left(\delta_{ij} - \frac{1}{2}\epsilon_i p3\epsilon_j{}^{q3}\delta_{pq}\right) \tag{4.48}$$

the inverse of the metric $h^{ij}$ of (2.36). Similarly to what we have done for the rigid rotator, we can introduce the linear combination $\tilde{I} = -i\tilde{I}^i\tilde{e}_i^*$ over the dual basis $\tilde{e}_i^*$, such that $\langle \tilde{e}_j^* | \tilde{e}^i \rangle = \delta_j^i$.

Following the same steps as in the case of the rigid rotator we can find the symplectic form from the first-order action functional, and it reads

$$\tilde{\omega} = d\tilde{\theta} = d\tilde{I}^i \wedge \tilde{\alpha}_i + \tilde{I}^i f_i^{\ jk} \tilde{\alpha}_j \wedge \tilde{\alpha}_k. \tag{4.49}$$

By inverting $\tilde{\omega}$ we find the Poisson algebra

$$\{u_i, u_j\} = 0 \tag{4.50}$$
$$\{\tilde{I}^i, \tilde{I}^j\} = f^{ij}_{\ \ k} \tilde{I}^k \tag{4.51}$$
$$\{u_i, \tilde{I}^j\} = \delta_i^j - f_i^{\ jk} u_k \iff \{\tilde{g}, \tilde{I}^j\} = 2i \tilde{g}\, \tilde{e}^j, \tag{4.52}$$

from which the Hamilton equations of motion can be obtained as follows:

$$\dot{\tilde{I}}^i = \{\tilde{I}^i, \tilde{H}_0\} = f_k^{\ ij} \tilde{I}^k \tilde{I}^l (h^{-1})_{jl}, \tag{4.53}$$
$$\tilde{g}^{-1} \dot{\tilde{g}} = 2i \tilde{e}^i (h^{-1})_{ij} \tilde{I}^j. \tag{4.54}$$

The fact that $\tilde{I}^j$ are not conserved is expected and it expresses the non-invariance of the model under right $SB(2, \mathbb{C})$ action. One can easily check that right momenta, obtained from right-invariant vector fields which generate left action, would result to be constants of motion.

Analogously to the IRR case, we can remark that the fibers of $TSB(2, \mathbb{C})$ can be identified with $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathbb{R}^3$ (as a vector space), as well as the fibers of $T^*SB(2, \mathbb{C})$, identified with the dual algebra $\mathfrak{sb}(2, \mathbb{C})^*$, which is also isomorphic, as vector space, to $\mathbb{R}^3$, but the elements are now components of one-forms. The carrier space of the Hamiltonian dynamics for the dual model $T^*SB(2, \mathbb{C})$ is represented, as a group, by the semi-direct product of $SB(2, \mathbb{C})$ and the Abelian group $\mathbb{R}^3$, i.e. $T^*SB(2, \mathbb{C}) \simeq SB(2, \mathbb{C}) \ltimes \mathbb{R}^3$, and the Lie algebra is a semi-direct sum represented by

$$[B_i, B_j] = i f_{ij}^{\ \ k} B_k \tag{4.55}$$
$$[S_i, S_j] = 0 \tag{4.56}$$
$$[B_i, S_j] = i f_{ij}^{\ \ k} S_k, \tag{4.57}$$

being $B_i$ the generators of the $SB(2, \mathbb{C})$ algebra and $S_i$ the generators of $\mathbb{R}^3$. Again, as before for the IRR, the non-trivial Poisson brackets (4.50) can be understood in terms of the coadjoint action of $SB(2, \mathbb{C})$ on its dual algebra.

### 4.0.3 Doubled IRR action

In the previous sections we have introduced two dynamical models on configuration spaces which are dual Poisson-Lie groups. The Poisson algebras for the respective cotangent bundles, $T^*SU(2)$ and $T^*SB(2,\mathbb{C})$, have both the structure of a semi-direct sum which reflects the semi-direct structure of the Lie algebras $\mathfrak{su}(2)\oplus\mathbb{R}^3$ and $\mathfrak{sb}(2,\mathbb{C})\oplus\mathbb{R}^3$. The two models can be obtained from the same parent action defined on the whole $SL(2,\mathbb{C})$ group which naturally doubles the coordinates as a Drinfel'd double. In this sense they appear as dual.

#### Lagrangian description

In order to unify the two models within a generalized doubled action, whose configuration space has double dimension with respect to the previous ones, let us introduce the configuration space variable $\phi : t \in \mathbb{R} \to \gamma(t) \in SL(2,\mathbb{C})$.

The left-invariant Maurer-Cartan one-form on the group manifold is $\gamma^{-1}d\gamma \in \Omega^1(SL(2,\mathbb{C})) \otimes \mathfrak{sl}(2,\mathbb{C})$ and can be pulled-back to $\mathbb{R}$ yielding

$$\phi^*\left(\gamma^{-1}d\gamma\right) = \gamma^{-1}\dot{\gamma}\,dt \equiv \dot{\mathbf{Q}}^I e_I dt, \tag{4.58}$$

being $\dot{\mathbf{Q}}^I$ the left generalized velocities, which we can decompose as $\dot{\mathbf{Q}}^I \equiv (A^i, B_i)$, resulting in

$$\gamma^{-1}\dot{\gamma}\,dt = \left(A^i e_i + B_i \tilde{e}^i\right) dt. \tag{4.59}$$

Both generalized velocities components are coordinates of the tangent bundle of $SL(2,\mathbb{C})$ but $(A^i, B_i)$ could also alternatively be interpreted in terms of Generalized Geometry as fiber coordinates of the generalized bundle $T \oplus T^*$ with base space $SU(2)$.

The components of the generalized velocity can be obtained by using the scalar product (2.24):

$$A^i = 2\mathrm{Im}\left[\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\tilde{e}^i\right)\right]; \quad B_i = 2\mathrm{Im}\left[\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}e_i\right)\right]. \tag{4.60}$$

The doubled action on $SL(2,\mathbb{C})$ can be introduced at this point using both the scalar products, as follows:

$$S = \frac{1}{2}\int_{\mathbb{R}} \left[k_1 \langle \phi^*\left(\gamma^{-1}d\gamma\right), *\phi^*\left(\gamma^{-1}d\gamma\right)\rangle + k_2\left(\left(\phi^*\left(\gamma^{-1}d\gamma\right), *\phi^*\left(\gamma^{-1}d\gamma\right)\right)\right)\right], \tag{4.61}$$

where $k_1$ and $k_2$ are two real parameters. In terms of generalized velocities, since

$$\langle \phi^* \left(\gamma^{-1} d\gamma\right), *\phi^* \left(\gamma^{-1} d\gamma\right)\rangle = dt\, \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \langle e_I, e_J\rangle = dt\, \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \eta_{IJ} \tag{4.62}$$

and

$$\left(\left(\phi^* \left(\gamma^{-1} d\gamma\right), *\phi^* \left(\gamma^{-1} d\gamma\right)\right)\right) = dt\, \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \left((e_I, e_J)\right) = dt\, \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \mathcal{H}_{IJ}, \tag{4.63}$$

we can write the action (up to an overall constant) explicitly in terms of the Drinfel'd double splitting of $\mathfrak{sl}(2, \mathbb{C})$:

$$S = \frac{1}{2} \int_{\mathbb{R}} dt\, E_{IJ} \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J, \tag{4.64}$$

with $E_{IJ} = k\, \eta_{IJ} + \mathcal{H}_{IJ}$, and we defined $k = \frac{k_1}{k_2}$. We can observe that the matrix $E_{IJ}$ is non-singular provided $k \neq 1$, which is a condition we assume from now on. Explicitly, in terms of fiber coordinates of $TSL(2, \mathbb{C})$ the Lagrangian gets the form:

$$L = \frac{1}{2} \left[\delta_{ij} A^i A^j + \left(k \delta_i^j + \epsilon_i{}^{j3}\right) A^i B_j + \left(k \delta_j^i - \epsilon^i{}_{j3}\right) B_i A^j + h^{ij} B_i B_j\right]. \tag{4.65}$$

The Lagrangian one-form is

$$\theta_L = E_{IJ}\, \dot{\mathbf{Q}}^I \alpha^J, \tag{4.66}$$

so that the equations of motion in the intrinsic formulation can be written as

$$\mathcal{L}_\Gamma \dot{\mathbf{Q}}^I E_{IJ} - \dot{\mathbf{Q}}^P \dot{\mathbf{Q}}^Q C_{IP}{}^K E_{QK} = 0, \tag{4.67}$$

being $C_{IP}{}^K$ the structure constants of the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra.

**Hamiltonian description**

As usual, we can define the left generalized momenta in the doubled description as

$$\mathbf{I}_I = \frac{\partial L}{\partial \dot{\mathbf{Q}}^I} = E_{IJ} \dot{\mathbf{Q}}^J, \tag{4.68}$$

so that the Hamiltonian then reads as:

$$H = \left[\mathbf{I}_I \dot{\mathbf{Q}}^I - L\right]_{|\dot{\mathbf{Q}}^I = (E^{-1})^{IJ} \mathbf{I}_J} = \frac{1}{2} (E^{-1})^{IJ} \mathbf{I}_I \mathbf{I}_J \tag{4.69}$$

with

$$(E^{-1})^{IJ} = \frac{1}{1-k^2} \begin{pmatrix} \delta^{ij} + \epsilon^{il3}\delta_{lk}\epsilon^{jk3} & -\delta_{pj}\left(\epsilon^{ip3} + k\delta^{ip}\right) \\ \delta^{pj}\left(\epsilon_{ip3} - k\delta_{ip}\right) & \delta_{ij} \end{pmatrix}. \tag{4.70}$$

Moreover, we can write the components of $\mathbf{I}_I$ explicitly in terms of the $(A^i, B_i)$ components:

$$\mathbf{I}_I \equiv (I_i, \tilde{I}^i) = \left(\delta_{ij}A^j + \left(k\delta_i^j + \epsilon_i{}^{j3}\right)B_j, \left(k\delta_j^i - \epsilon^i{}_{j3}\right)A^j + \left(\delta^{ij} + \delta^{lk}\epsilon^i{}_{l3}\epsilon^j{}_{k3}\right)B_j\right), \tag{4.71}$$

so that in terms of the components $(I_i, \tilde{I}^i)$ the Hamiltonian can be written as follows:

$$H = \frac{1}{2\left(1-k^2\right)}\left[\left(\delta^{ij} + \epsilon^{il3}\delta_{lk}\epsilon^{jk3}\right)I_iI_j + \delta_{ij}\tilde{I}^i\tilde{I}^j - 2\left(\epsilon^{ip3} + k\delta^{ip}\right)\delta_{pj}I_i\tilde{I}^j\right]. \tag{4.72}$$

We can consider the linear combination $\mathbf{I} = -\frac{i}{2}\mathbf{I}_I e^{I^*} = -\frac{i}{2}\left(I_i e^{i^*} + \tilde{I}_i\tilde{e}_i^*\right)$, such that, using also $\gamma^{-1}d\gamma = 2i\,\alpha^K e_K = i\left(\alpha^k e_k + \beta_k\tilde{e}^k\right)$ we obtain the symplectic form $\omega$ on $T^*SL(2,\mathbb{C})$:

$$\omega = d\theta = dI_i \wedge \alpha^i + \mathbf{I}_I\, C^I{}_{JK}\,\alpha^J \wedge \alpha^K, \tag{4.73}$$

from which the Poisson brackets for the generalized momenta can be obtained:

$$\begin{aligned} &\{I_i, I_j\} = \epsilon_{ij}{}^k I_k \\ &\{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k\tilde{I}^k \\ &\{I_i, \tilde{I}^j\} = \epsilon_{il}{}^j\tilde{I}^l - I_l f_i{}^{lj}, \quad \{\tilde{I}^i, I_j\} = -\epsilon^i{}_{jl}\tilde{I}^l + I_l f^{li}{}_j. \end{aligned} \tag{4.74}$$

Note that the Poisson bracket between momenta and configuration space variables $g, \tilde{g}$ are unchanged with respect to $T^*SU(2)$ and $T^*SB(2,\mathbb{C})$. We can write these brackets in compact (and doubled) form as

$$\{\mathbf{I}_I, \mathbf{I}_J\} = C_{IJ}{}^K\mathbf{I}_K. \tag{4.75}$$

Finally, the Hamilton equations can be derived as follows:

$$\dot{\mathbf{I}}_I = \{\mathbf{I}_I, H\} = (E^{-1})^{JK}\{\mathbf{I}_I, \mathbf{I}_J\}\mathbf{I}_K = (E^{-1})^{JK}C_{IJ}{}^L\,\mathbf{I}_L\mathbf{I}_K, \tag{4.76}$$

which is not zero, consistently with (4.67).

### 4.0.4 Recovering the dual models

The standard dynamics of the isotropic rigid rotator and its dual model can be recovered from the doubled Lagrangian we have introduced. In order to get back one of the two models one has to impose constraints, as is customary in DFT. In particular, one has to gauge either $SU(2)$ or $SB(2,\mathbb{C})$ and integrate out.

For definiteness, we specify and fix a local Iwasawa decomposition for the elements of $SL(2,\mathbb{C})$ as $\gamma = \tilde{g}g$, with $\tilde{g} \in SB(2,\mathbb{C})$ and $g \in SU(2)$. From the action in (4.61) and the properties we have remarked on the two scalar products defined on $SL(2,\mathbb{C})$, it can be seen that the Lagrangian is manifestly globally invariant under both left and right $SU(2)$ actions but only under left $SB(2,\mathbb{C})$ action. Therefore, in order to recover the $TSU(2)$ rotator description this left $SB(2,\mathbb{C})$ invariance has to be promoted to a gauge symmetry and then gauged appropriately. The left $SB(2,\mathbb{C})$ action is given by

$$SB(2,\mathbb{C})_L : \gamma \to \tilde{h}\gamma = \tilde{h}\tilde{g}g, \quad \forall \tilde{h} \in SB(2,\mathbb{C}). \tag{4.77}$$

Promoting this global symmetry to a gauge one, we modify the Maurer-Cartan one-form defining the covariant exterior derivative $D_{\tilde{C}} = d + \tilde{C}$, where $\tilde{C}$ is the gauge connection one-form $\tilde{C} = \tilde{C}_i(t)\tilde{e}^i$, so that

$$\phi^*\left(\gamma^{-1}d\gamma\right) \to \phi^*\left(\gamma^{-1}D_{\tilde{C}}\gamma\right) = \left(\gamma^{-1}\dot{\gamma} + \gamma^{-1}\tilde{C}\gamma\right)dt. \tag{4.78}$$

We can make explicit the doubled notation by performing the following splitting:

$$\gamma^{-1}\dot{\gamma} + \gamma^{-1}\tilde{C}\gamma = U_i\,\tilde{e}^i + W^i e_i, \tag{4.79}$$

whose components can be obtained from

$$U_i = 2\mathrm{Im}\left\{\mathrm{Tr}\left[\left(\gamma^{-1}\dot{\gamma} + \tilde{C}_j\gamma^{-1}\tilde{e}^j\gamma\right)e_i\right]\right\} \tag{4.80}$$

and

$$W^i = 2\mathrm{Im}\left\{\mathrm{Tr}\left[\left(\gamma^{-1}\dot{\gamma} + \tilde{C}_j\gamma^{-1}\tilde{e}^j\gamma\right)\tilde{e}^i\right]\right\}. \tag{4.81}$$

These can be computed explicitly by acting with the coadjoint action of $g$, $\tilde{g}$ on $e_j$, $\tilde{e}^j$ represented by the Lie brackets (2.22) (for the $e_i$), (2.26) and (2.27). However, this is not necessary for our purposes and details can be found in [65]. In terms of these new degrees of freedom we can write down the doubled Lagrangian with the gauge connection added, so the gauged Lagrangian reads

$$L_{\tilde{C}} = \frac{1}{2}\left[\delta_{ij}W^iW^j + 2\left(k\delta_i^j + \epsilon_i{}^{j3}\right)W^iU_j + h^{ij}U_iU_j\right], \tag{4.82}$$

since

$$\begin{aligned} L_{\tilde{C}} = \frac{1}{2}\Big[ & k_1\langle\phi^*\left(\gamma^{-1}D_{\tilde{C}}\gamma\right), *\phi^*\left(\gamma^{-1}D_{\tilde{C}}\gamma\right)\rangle \\ & + k_2\left(\left(\phi^*\left(\gamma^{-1}D_{\tilde{C}}\gamma\right), *\phi^*\left(\gamma^{-1}D_{\tilde{C}}\gamma\right)\right)\right)\Big] = \frac{1}{2}E_{IJ}\dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J, \end{aligned} \tag{4.83}$$

with $\dot{\mathbf{Q}}^I = (W^i, U_i)$. Then, performing the transformation

$$\hat{W}^i = W^i + \left(k\delta^{is} - \epsilon_3{}^{is}\right)U_s, \tag{4.84}$$

we have

$$L_{\tilde{C}} = \frac{1}{2}\left[\delta_{ij}\hat{W}^i\hat{W}^j + \left(1-k^2\right)\delta^{ij}U_iU_j\right]. \tag{4.85}$$

We will use this form for writing the Euclidean partition function of the system

$$\mathcal{Z} = \int \mathcal{D}_g\mathcal{D}_{\tilde{g}}\mathcal{D}_{\tilde{C}}\, e^{-S_{\tilde{C}}} \tag{4.86}$$

and integrate over the gauge connection. In particular, we can trade the integration over $\tilde{C}_i$ with an integration over $U_i$:

$$\mathcal{Z} = \int \mathcal{D}_g\mathcal{D}_{\tilde{g}} \det\left(\frac{\delta\tilde{C}_i}{\delta U_j}\right) e^{-\frac{1}{2}\int_{\mathbb{R}}dt\,\delta_{ij}\hat{W}^i\hat{W}^j}\int \mathcal{D}_U e^{-\frac{1}{2}\int_{\mathbb{R}}dt\,(1-k^2)\delta^{ij}U_iU_j}. \tag{4.87}$$

It is easy to see that the Jacobian determinant of the $\tilde{C} \to U$ transformation is constant since the matrices involved in the gauge transformations are constant, hence it only results in a regularization factor. Using the fact that the Gaussian integral over $U$ is also a constant, the partition function can be finally written in the form

$$\mathcal{Z} \propto \int \mathcal{D}_g\mathcal{D}_{\tilde{g}} e^{-\frac{1}{2}\int_{\mathbb{R}}dt\,\delta_{ij}\hat{W}^i\hat{W}^j}. \tag{4.88}$$

In order to compare with the IRR action (4.1) we can make a step further. It is possible to introduce the endomorphism $E$ of $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{sb}(2,\mathbb{C})$ which preserves the Drinfel'd splitting, defined by the constant matrix

$$E = \begin{pmatrix} \delta^i_j & T^{ij} \\ -(T^{-1})_{ij} & \delta_i^j \end{pmatrix} \tag{4.89}$$

such that we can make the following splitting-preserving change of variables on $\mathfrak{sl}(2,\mathbb{C})$ as a Drinfel'd double:

$$E\begin{pmatrix} W^j \\ U_j \end{pmatrix} = \begin{pmatrix} \hat{W}^j \\ \hat{U}_j \end{pmatrix}. \tag{4.90}$$

In this way, we can write the Maurer-Cartan left-invariant one-forms as

$$\Phi^*\left(g'^{-1}dg'\right) = \hat{W}^i e_i dt, \quad \Phi^*\left(\tilde{g}'^{-1}d\tilde{g}'\right) = \hat{U}_i \tilde{e}^i dt. \tag{4.91}$$

The endomorphism $E$ induces an exponential map $\exp(E) : SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ such that $\gamma = \tilde{g}g$ is mapped into $\gamma' = \tilde{g}'g'$, so that the integration measure can be transformed into $\mathcal{D}_{g'}\mathcal{D}_{\tilde{g}'}$, hence up to a constant factor (the determinant of $\exp(E)$) the partition function can be written as

$$\mathcal{Z} \propto \int \mathcal{D}_{\tilde{g}'} \int \mathcal{D}_{g'} e^{-\frac{1}{2}\int_{\mathbb{R}} \mathrm{Tr}\left[\Phi^*\left(g'^{-1}dg'\right)\wedge *\Phi^*\left(g'^{-1}dg'\right)\right]}. \tag{4.92}$$

Clearly the integration over $\tilde{g}'$ gives another constant, while the other integral is the partition function of the IRR model.

The dual model with carrier space $TSB(2,\mathbb{C})$ can be recovered following exactly the same procedure but gauging this time the global right $SU(2)$ action invariance. The main difference with respect to the previous case is that the gauge connection one-form is now $\mathfrak{su}(2)$-valued, and under the integral it is suitable to be traded for the integration over the dual analogue of $W$ in (4.79). This case has been carried out in detail in [65].

### 4.0.5 Relation with Generalized Geometry

Since we can consider the isomorphism $TSL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ with the fiber

$$\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2) \oplus \mathfrak{sb}(2,\mathbb{C}) \simeq TSU(2) \oplus T^*SU(2), \tag{4.93}$$

one can rewrite the Poisson algebra (4.74) as

$$\{I+\tilde{I}, J+\tilde{J}\} = \{I,J\} - \{J,\tilde{I}\} + \{I,\tilde{J}\} + \{\tilde{I},\tilde{J}\}, \tag{4.94}$$

having defined $I = iI_i e^{i^*}$, $J = iJ_i e^{i^*}$ as one-forms and $\tilde{I} = \tilde{I}^i \tilde{e}^*_i$, $\tilde{J} = \tilde{J}^i \tilde{e}^*_i$ as vector fields. Poisson brackets (4.74) are given by the KSK brackets on the coadjoint orbits of $SL(2,\mathbb{C})$, but in particular, they are induced by the bialgebra structure of $SL(2,\mathbb{C})$ and according to (4.94) they can be identified with the *C-brackets* [115–117] of Generalized geometry [3] for the generalized bundle $T \oplus T^*$, being $\{e^{i^*}\}$ and $\{\tilde{e}^*_i\}$ bases over $T^*$ and over $T$ respectively. Namely, the doubled momenta $(I_i, \tilde{I}^i)$ identify the fiber coordinates of the generalized bundle $T \oplus T^*$ of $SU(2)$.

Furthermore, defining Hamiltonian vector fields in terms of Poisson brackets as usual as

$$X_f = \{\cdot, f\} \tag{4.95}$$

and defining in particular $X_i = \{\cdot, I_i\}$, $\tilde{X}^i = \{\cdot, \tilde{I}^i\}$, one can find, because of the non-trivial Poisson bracket (4.74), and by using Jacobi identity:

$$\begin{aligned}
[X_i, X_j] &= \{\{\cdot, I_j\}, I_i\} - \{\{\cdot, I_i\}, I_j\} = \{\cdot, \{I_i, I_j\}\} = \epsilon_{ij}{}^k X_k, \\
\left[\tilde{X}^i, \tilde{X}^j\right] &= \{\{\cdot, \tilde{I}^j\}, \tilde{I}^i\} - \{\{\cdot, \tilde{I}^i\}, \tilde{I}^j\} = \{\cdot, \{\tilde{I}^i, \tilde{I}^j\}\} = f^{ij}{}_k \tilde{X}^k, \\
\left[X_i, \tilde{X}^j\right] &= \{\{\cdot, \tilde{I}^j\}, I_i\} - \{\{\cdot, I_i\}, \tilde{I}^j\} = \{\cdot, \{I_i, \tilde{I}^j\}\} = -f_i{}^{jk} X_k - \tilde{X}^k \epsilon_{ki}{}^j,
\end{aligned} \tag{4.96}$$

or, in a unified fashion:

$$[X + \tilde{X}, Y + \tilde{Y}] = [X, Y] + [\tilde{X}, \tilde{Y}] + \mathcal{L}_X \tilde{Y} - \mathcal{L}_Y \tilde{X}. \tag{4.97}$$

This shows, remarkably, that the $C$-brackets can be obtained as derived brackets [117] from the canonical Poisson brackets of the dynamics.

It is important at this point to summarize and discuss in what sense the two submodels possess Poisson-Lie symmetries. We have seen in Sec. 3.3 what Poisson-Lie T-duality means and how it is related to the concept of Drinfel'd double. Namely, we have seen that under appropriate conditions the sigma models defined on groups that are dual partners in a Manin triple polarization are indeed dual, in the sense that they describe the same physics even if there is no such manifest symmetry in neither of the two dual actions. Indeed, the two models can be seen to be connected by a canonical transformation on their phase-space variables and classically their dynamics is indistinguishable. However, a parent model can be formulated on the Drinfel'd double group and at this stage Poisson-Lie duality becomes a manifest symmetry and the two submodels can be obtained by gauging conditions. Furthermore, there are two symmetric ways to perform the decomposition: $\gamma = \tilde{g}g$ or $\gamma = g\tilde{g}$. In our simple case, we started from the

[3] C-brackets are mixed brackets between vector fields and forms. They generalize Courant and Dorfmann brackets

action of an isotropic rigid rotator on the group manifold $SU(2)$, and having realized that $SL(2,\mathbb{C})$ can be seen as a Drinfel'd double and in particular $(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{su}(2),\mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple, we built the dual model on $SB(2,\mathbb{C})$, they can then be obtained from the generalized action on $SL(2,\mathbb{C})$ under appropriate gauging. In this sense we can see we have the ingredients under which Poisson-Lie duality relies. It is already enough in principle to state that the model is a Poisson-Lie model.

However, note that in this case the model is too simple to have Poisson-Lie symmetry, indeed, there does not exist a canonical transformation, that is due to the fact the model under analysis is not a genuine field theory. In fact, this can be found instead in its most natural generalization wich is the principal chiral model case [67] that we briefly discuss in the next section. In fact, in the next chapter we will consider a way more general model, which is the PCM with the addition of the so-called Wess-Zumino term. We will analyze only this particular model since we can recover the PCM simply by putting a coupling constant to zero.

## 4.1 The Principal Chiral Model

The Principal Chiral Model (PCM) is a two-dimensional sigma model with target configuration space given by a Lie group $G$. In fact, the field content is simply a field $\phi$ valued in a Lie group. As a source space we will consider the two-dimensional spacetime $\Sigma = \mathbb{R}^{1,1}$ endowed with metric $\mathrm{diag}(1,-1)$. Since the IRR model is too simple to exhibit symmetry under duality transformation, as we have seen in the previous section, it is natural to consider a genuine $(1+1)$-dimensional field theory which resembles the same model. This is the $SU(2)$ Principal Chiral Model which, while being modeled on the IRR system, certainly exhibits interesting properties under duality transformations. More precisely, in [67] it was considered as a starting point an old intuition due to S. G. Rajeev [77, 78] where the $SU(2)$ principal chiral model is shown to exhibit a whole one-parameter family of alternative Hamiltonians and alternative Poisson algebras, all equivalent from the point of view of the dynamics. Then, this intuition was extended with a construction relying on the generalization of the affine algebra of currents, associated with the semi-direct sum $\mathfrak{su}(2)(\mathbb{R})\dot{\oplus}\mathfrak{a}(\mathbb{R})$, being $\mathfrak{a}(\mathbb{R})$ an Abelian Lie algebra, to a fully semi-simple Kac-Moody algebra which is either $\mathfrak{su}(2)(\mathbb{R}) \oplus \mathfrak{su}(2)(\mathbb{R})$ or $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$. Here $\mathfrak{g}(\mathbb{R})$ indicates the affine algebra associated to the Lie algebra $\mathfrak{g}$.

Interestingly, this construction can also be understood in terms of Born Geometry, and by generalizing the Poisson Kac-Moody algebra with the introduction of a second parameter, and performing an $O(3,3)$ transformation over the target phase space, it was showed that a family of sigma models with target configuration space the group manifold of $SB(2,\mathbb{C})$ is obtained. Moreover, the vanishing value of one of the two parameters corresponds to the original $SU(2)$PCM with canonical splitting of its current algebra, whereas the vanishing of the remaining parameter correctly reproduces the dual current algebra $\mathfrak{sb}(2,C)(\mathbb{R})\dot{\oplus}\mathfrak{a}(\mathbb{R})$, but the Hamiltonian exhibits a singular behaviour, which is related to a topological issue that can be understood and solved with a further generalization given by the introduction of a Wess-Zumino term [69], that we will consider in the next chapter.

### 4.1.1 SU(2) Principal Chiral Model

The $SU(2)$ Principal Chiral Model represents a natural generalization to field theory of the dynamics of the IRR, as described in the previous section. Indeed, the action functional is formally the same, while the field variables are defined on two-dimensional spacetime taking values on the group manifold of $SU(2)$. Since we are interested in the more general model with the addition of the Wess-Zumino term, in this section we will simply give a general review of the model and some of its properties and similarities with the IRR. More details can be found in [67, 68].

In the Lagrangian approach the action may be written in terms of fields $\phi : \mathbb{R}^{1,1} \ni (t,\sigma) \to g(t,\sigma) \in SU(2)$ and Lie algebra valued left-invariant one-forms with pull-back to $\mathbb{R}^{1,1}$ given by

$$\phi^*(g^{-1}\mathrm{d}g) = (g^{-1}\partial_t g)\,\mathrm{d}t + (g^{-1}\partial_\sigma g)\,\mathrm{d}\sigma. \tag{4.98}$$

The action of the PCM is given by

$$S = \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{Tr}\,[\phi^*(g^{-1}\mathrm{d}g) \wedge *\phi^*(g^{-1}\mathrm{d}g)], \tag{4.99}$$

where the trace is understood as the Cartan-Killing scalar product in the Lie algebra $\mathfrak{su}(2)$. As usual, $*$ is the Hodge star operator on the source space acting as $*\mathrm{d}t = \mathrm{d}\sigma, *\mathrm{d}\sigma = \mathrm{d}t$ [4].

[4]We adopt the the convention $\epsilon_{01} = 1$.

By writing the pull-back and performing the wedge product explicitly we obtain

$$S = \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}\sigma\ \mathrm{Tr}\left[(g^{-1}\partial_t g)^2 - (g^{-1}\partial_\sigma g)^2\right]. \tag{4.100}$$

A remarkable property of the model is that its Euler-Lagrange equations

$$\partial_t(g^{-1}\partial_t g) - \partial_\sigma(g^{-1}\partial_\sigma g) = 0 \tag{4.101}$$

may be rewritten in terms of an equivalent system of two first order partial differential equations, introducing the so-called *currents*, as it is customary in the framework of integrable systems:

$$A^i = \mathrm{Tr}\,(g^{-1}\partial_t g)e_i, \quad J^i = \mathrm{Tr}\,(g^{-1}\partial_\sigma g)e_i, \tag{4.102}$$

namely, $g^{-1}\partial_t g = 2A^i e_i, g^{-1}\partial_\sigma g = 2J^i e_i$, with $\mathrm{Tr}\,\left(e_i e_j\right) = \frac{1}{2}\delta_{ij}$.

The Lagrangian becomes:

$$L = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\sigma(A^i\delta_{ij}A^j - J^i\delta_{ij}J^j), \tag{4.103}$$

with

$$\partial_t A = \partial_\sigma J, \tag{4.104}$$

$$\partial_t J = \partial_\sigma A - [A, J]. \tag{4.105}$$

The existence of a $g \in SU(2)$ that admits the expression of the currents in the form of eq. (4.102) is guaranteed by eq. (4.105), that can be read as an integrability condition. Moreover, if the usual boundary condition for a physical field is imposed:

$$\lim_{\sigma\to\pm\infty} g(\sigma) = 1, \tag{4.106}$$

one has that $g$ is uniquely determined from eq. (4.102).

The carrier space of the dynamics of our system can be regarded as the tangent bundle of $SU(2)(\mathbb{R})$. Therefore, the tangent bundle description of the dynamics is given in terms of $(J, A)$ with $A$ being left generalized velocities and $J$ left configuration space coordinates.

# 5 Poisson-Lie T-duality of WZW model via current algebra deformation

In this chapter we will further extend the construction of the PCM by introducing a Wess-Zumino term, leading to a WZW model on $SU(2)$ [69] [1]. Our approach will be based on using the Hamiltonian formalism with a pair of currents valued in the target phase space $T^*SU(2)$, which, topologically, is the manifold $S^3 \times \mathbb{R}^3$, while as a group it is the semi-direct product $SU(2) \ltimes \mathbb{R}^3$. An important feature is the fact that, as a symplectic manifold, $T^*SU(2)$ is symplectomorphic to $SL(2,\mathbb{C})$, besides being topologically equivalent. Moreover, both manifolds, $T^*SU(2)$ and $SL(2,\mathbb{C})$ are Drinfel'd doubles of the Lie group $SU(2)$ [73–76], the former being the trivial one, what we call classical double, which can be obtained from the latter via group contraction.

The whole construction relies on a deformation of the affine current algebra of the model, the semidirect sum of the Kac-Moody algebra associated to $\mathfrak{su}(2)$ with an Abelian algebra $\mathfrak{a}$, to the fully semisimple Kac-Moody algebra $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ [77–79]. The latter is a crucial step if one observes that the algebra $\mathfrak{sl}(2,\mathbb{C})$ has a bialgebra structure, with $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ dually related, maximal isotropic subalgebras. By $\mathfrak{g}(\mathbb{R})$ we shall indicate the affine algebra of maps $\mathbb{R} \to \mathfrak{g}$ that are sufficiently fast decreasing at infinity to be square integrable, what we will refer to as current algebra.

Starting from the one-parameter family of Hamiltonian models with algebra of currents homomorphic to $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$, a further deformation is needed in order to make the role of dual subalgebras completely symmetric. We show that such a deformation is possible, which does not alter the nature of the current algebra, nor the dynamics described by the new Hamiltonian.

[1] As was already pointed in the introduction section, and we report here again for highlight, some authors consider the WZW model as the PCM with the addition of the WZ term (and they refer to it as PCM+WZ) when the coupling constants have particular values so that the model describes a conformal field theory. Here for convenience by WZW model we refer in general to the PCM with a WZ term no matter the values of the constants involved.

Our findings will differ from usual deformations of sigma model in the existing literature, for instance $\eta$ or $\lambda$ deformations, since the latter represent true deformations of the dynamics with the aim of preserving integrability. Relations of these deformed models with Poisson-Lie T-duality have been found and worked out in [55].

The result of our approach will be a two-parameter family of models with the group $SL(2,\mathbb{C})$ as target phase space and in which Poisson-Lie T-duality transformations are realized as $O(3,3)$ rotations in phase space, reminiscent of the usual T-duality. By performing an exchange of momenta with configuration space fields (which is again reminiscent of the momenta-winding exchange in standard T-duality) we obtain a new family of dual models with configuration space the group $SB(2,\mathbb{C})$.

The material covered in this chapter is entirely contained in the paper [69].

## 5.1 The WZW model on $SU(2)$

The subject of this section is the Wess-Zumino-Witten model with target space the group manifold of $SU(2)$. First we review the model in the Lagrangian approach and then focus on its Hamiltonian formulation, the latter being more convenient for our purposes.

The main theme of the section is to describe the WZW model with an alternative canonical formulation in terms of a one-parameter current algebra deformation, based on Ref. [79]. Such a richer structure has several interesting consequences; some of them have already been investigated, such as quantization [79] and integrability [118], but in particular it paves the way to target space duality, presented in Section 5.2. We follow the approach of [67] where a similar analysis has been performed for the Principal Chiral Model.

### 5.1.1 Lagrangian formulation

Let $G$ be a semisimple connected Lie group and $\Sigma$ a 2-dimensional oriented (pseudo) Riemannian manifold (we take it with Minkowski signature $(1,-1)$) parametrized by the coordinates $(t,\sigma)$.

The basic invariant objects we need in order to build a group-valued field theory are the left-invariant (or the right-invariant) Maurer-Cartan one-forms, which, if $G$ can be embedded in $GL(n)$, can be written explicitly as $g^{-1}dg \in \Omega^1(G) \otimes \mathfrak{g}$.

Let us denote with $*$ the Hodge star operator on $\Sigma$, acting accordingly to the Minkowski signature as $*dt = d\sigma$, $*d\sigma = dt$.

There is a natural scalar product structure on the Lie algebra of a semisimple Lie group, provided by the Cartan-Killing form and denoted generically with the $\mathrm{Tr}(\cdot,\cdot)$ symbol.

With this notation, we have the following

**Definition 5.1.1.** Let $\varphi : \Sigma \ni (t,\sigma) \to g \in G$ and denote $\varphi^*(g^{-1}dg)$ the pull-back of the Maurer-Cartan left-invariant one-form on $\Sigma$ via $\varphi$. The Wess-Zumino-Witten model is a non-linear sigma model described by the action

$$S = \frac{1}{4\lambda^2}\int_\Sigma \mathrm{Tr}\left[\varphi^*\left(g^{-1}dg\right)\wedge *\varphi^*\left(g^{-1}dg\right)\right] + \kappa S_{WZ}, \tag{5.1}$$

with $S_{WZ}$ the Wess-Zumino term,

$$S_{WZ} = \frac{1}{24\pi}\int_{\mathcal{B}} \mathrm{Tr}\left[\tilde{\varphi}^*\left(\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\right)\right], \tag{5.2}$$

where $\mathcal{B}$ is a 3-manifold whose boundary is the compactification of the original two-dimensional spacetime, while $\tilde{g}$ and $\tilde{\varphi}$ are extensions of previous maps to the 3-manifold $\mathcal{B}$, in the sense that $\tilde{\varphi}|_\Sigma = \varphi$, $\tilde{g}|_\Sigma = g$.

It is always possible to have such an extension since one is dealing with maps $\varphi : S^2 \to G$. The latter are classified by the second homotopy group $\Pi_2(G)$, which is well-known to be trivial for Lie groups. Thus, these maps are homotopically equivalent to the constant map, which can be obviously continued to the interior of the sphere $S^2$. Such an extension is not unique by the way, since there may be many 3-manifolds with the same boundary. However, it is possible to show that the variation of the WZW action remains the same up to a constant term, which is irrelevant classically. For the quantum theory, in order for the partition function to be single-valued, $\kappa$ is taken to be an integer for compact Lie groups (this is the so called level of the theory), while for non-compact Lie groups there is no such a quantization condition.

For future convenience, the action can be written explicitly as

$$\begin{aligned} S = \frac{1}{4\lambda^2}\int_\Sigma d^2\sigma\, \mathrm{Tr}\left(g^{-1}\partial^\mu g g^{-1}\partial_\mu g\right) \\ + \frac{\kappa}{24\pi}\int_{\mathcal{B}} d^3y\, \epsilon^{\alpha\beta\gamma}\mathrm{Tr}\left(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g}\right). \end{aligned} \tag{5.3}$$

Note that although the WZ term is expressed as a three-dimensional integral, since $H \equiv \tilde{g}^{-1}d\tilde{g}^{\wedge 3}$ is a closed 3-form, under the variation $g \to g + \delta g$ (or more precisely $\varphi + \delta\varphi$) it produces a boundary term, which is exactly an integral over $\Sigma$ since the variation of its Lagrangian density can be written as a total derivative. We have indeed

$$\delta S_{WZ} = \int_{\mathcal{B}} \mathcal{L}_{\tilde{V}_a} H = \int_{\mathcal{B}} d i_{\tilde{V}_a} H = \int_{\partial\mathcal{B}} i_{V_a} H, \tag{5.4}$$

with $\partial\mathcal{B} = \Sigma$, $\tilde{V}_a, V_a$ the infinitesimal generators of the variation over $\mathcal{B}$ and $\Sigma$ respectively and $\mathcal{L}_{\tilde{V}_a}$ the Lie derivative along the vector field $\tilde{V}_a$. Then, its contribution to the equations of motion only involves the original fields $\varphi$ on the source space $\Sigma$.

A remarkable property of the model is that its Euler-Lagrange equations may be rewritten as an equivalent system of first order partial differential equations:

$$\partial_t A - \partial_\sigma J = -\frac{\kappa\lambda^2}{4\pi}\,[A, J] \tag{5.5}$$

$$\partial_t J - \partial_\sigma A = -\,[A, J] \tag{5.6}$$

with

$$A = \left(g^{-1}\partial_t g\right)^i e_i = A^i e_i, \tag{5.7}$$

$$J = \left(g^{-1}\partial_\sigma g\right)^i e_i = J^i e_i \tag{5.8}$$

Lie algebra-valued fields (so called currents), $e_i \in \mathfrak{g}$, and the usual physical boundary condition

$$\lim_{|\sigma|\to\infty} g(\sigma) = 1, \tag{5.9}$$

which makes the solution for $g$ unique. This boundary condition has also the purpose to one-point compactify the source space $\Sigma$.

At fixed $t$, the group elements satisfying this boundary condition form an infinite dimensional Lie group: $G(\mathbb{R}) \equiv \mathrm{Map}(\mathbb{R}, G)$, which is given by the smooth maps $g : \mathbb{R} \ni \sigma \to g(\sigma) \in G$ constant at infinity, with standard pointwise multiplication.

The real line may be replaced by any smooth manifold $M$, of dimension $d$, so to have fields in $\mathrm{Map}(M, G)$. The corresponding Lie algebra $\mathfrak{g}(M) \equiv \mathrm{Map}(M, \mathfrak{g})$ of maps $M \to \mathfrak{g}$ that are sufficiently fast decreasing at infinity to be square integrable (this is needed for the finiteness of the energy, as we will see) is the related *current algebra*.

We will stick to the two-dimensional case from now on. Infinitesimal generators of the Lie algebra $\mathfrak{g}(\mathbb{R})$ can be obtained by considering the vector fields which generate the finite-dimensional Lie algebra $\mathfrak{g}$ and replacing ordinary derivatives with functional derivatives:

$$X_i(\sigma) = X_i^{\,a}(\sigma)\frac{\delta}{\delta g^a(\sigma)}, \tag{5.10}$$

with Lie bracket

$$\left[X_i(\sigma), X_j\left(\sigma'\right)\right] = c_{ij}^{\;\;k} X_k(\sigma)\delta^d\left(\sigma-\sigma'\right), \tag{5.11}$$

where $\sigma, \sigma' \in \mathbb{R}$. The latter is $C^\infty(\mathbb{R})$-linear and $\mathfrak{g}(\mathbb{R}) \simeq \mathfrak{g} \otimes C^\infty(\mathbb{R})$.

Let us now consider the target space $G = SU(2)$ and $\mathfrak{su}(2)$ generators $e_i = \sigma_i/2$, with $\sigma_i$ the Pauli matrices, satisfying $[e_i, e_j] = i\epsilon_{ij}^{\;\;k} e_k$ and $\mathrm{Tr}(e_i, e_j) = \frac{1}{2}\delta_{ij}$.

Eq. (5.5) can be easily obtained from the Euler-Lagrange equations for the action (5.1). Eq. (5.6) can be interpreted as an integrability condition for the existence of $g \in SU(2)$ such that $A = g^{-1}\partial_t g$ and $J = g^{-1}\partial_\sigma g$, and it follows from the Maurer-Cartan equation for the $\mathfrak{su}(2)$-valued one-forms $g^{-1}dg$. This can be seen starting from the decomposition of the exterior derivative on the Maurer-Cartan left-invariant one-form:

$$\begin{aligned} d\varphi^*\left(g^{-1}dg\right) &= d\left(g^{-1}\partial_t g\, dt + g^{-1}\partial_\sigma g\, d\sigma\right) \\ &= \left[-\partial_\sigma\left(g^{-1}\partial_t g\right) + \partial_t\left(g^{-1}\partial_\sigma g\right)\right] dt \wedge d\sigma, \end{aligned} \tag{5.12}$$

and since

$$\begin{aligned} d\varphi^*\left(g^{-1}dg\right) &= -\varphi^*(g^{-1}dg) \wedge \varphi^*(g^{-1}dg) \\ &= -\left(g^{-1}\partial_t g g^{-1}\partial_\sigma g - g^{-1}\partial_\sigma g g^{-1}\partial_t g\right) dt \wedge d\sigma \\ &= -\left[g^{-1}\partial_t g, g^{-1}\partial_\sigma g\right] dt \wedge d\sigma, \end{aligned} \tag{5.13}$$

Eq. (5.6) follows.

To summarise, the carrier space of Lagrangian dynamics can be regarded as the tangent bundle $TSU(2)(\mathbb{R}) \simeq (SU(2) \ltimes \mathbb{R}^3)(\mathbb{R})$. It can be described in terms of coordinates $(J^i, A^i)$, with $J^i$ and $A^i$ playing the role of left generalised configuration space coordinates and left generalised velocities respectively. In the next section we will consider the Hamiltonian description, by replacing the generalised velocities $A^i$ with canonical momenta $I_i$ spanning the fibres of the cotangent bundle $T^*SU(2)(\mathbb{R})$.

For future convenience we close this section by introducing the form of the WZ term on $SU(2)$ in terms of the Maurer-Cartan one-form components:

$$S_{WZ} = \frac{1}{24\pi}\int_{\mathcal{B}} d^3y\, \epsilon^{\alpha\beta\gamma}\tilde{A}^i_\alpha \tilde{A}^j_\beta \tilde{A}^k_\gamma \epsilon_{ijk} = \frac{1}{4\pi}\int_{\mathcal{B}} d^3y\, \epsilon^{\alpha\beta\gamma}\tilde{A}_{\alpha 1}\tilde{A}_{\beta 2}\tilde{A}_{\gamma 3}, \tag{5.14}$$

with $\tilde{A}^i_\alpha$ defined from $\tilde{\varphi}^*\left(\tilde{g}^{-1}d\tilde{g}\right) = \tilde{A}^i_\alpha dy^\alpha\, e_i$.

### 5.1.2 Hamiltonian description and deformed $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ current algebra

The Hamiltonian description of the model is the one which mostly lends itself to the introduction of current algebras. The dynamics is described by the following Hamiltonian

$$H = \frac{1}{4\lambda^2}\int_{\mathbb{R}} d\sigma\, \left(\delta^{ij}I_iI_j + \delta_{ij}J^iJ^j\right) = \frac{1}{4\lambda^2}\int_{\mathbb{R}} d\sigma\, I_L(\mathcal{H}_0^{-1})_{LM}I_M \tag{5.15}$$

and equal-time Poisson brackets

$$\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= 2\lambda^2\left[\epsilon_{ij}{}^k I_k(\sigma) + \frac{\kappa\lambda^2}{4\pi}\epsilon_{ijk}J^k(\sigma)\right]\delta(\sigma-\sigma') \\
\{I_i(\sigma), J^j(\sigma')\} &= 2\lambda^2\left[\epsilon_{ki}{}^j J^k(\sigma)\delta(\sigma-\sigma') - \delta_i^j\delta'(\sigma-\sigma')\right] \\
\{J^i(\sigma), J^j(\sigma')\} &= 0
\end{aligned} \tag{5.16}$$

which may be obtained from the action functional. For future reference we have introduced in (5.15) the double notation $I_L = (J^\ell, I_\ell)$ and the diagonal metric

$$\mathcal{H}_0 = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{ij} \end{pmatrix}. \tag{5.17}$$

Momenta $I_i$ are obtained by Legendre transform from the Lagrangian. Configuration space is the space of maps $SU(2)(\mathbb{R}) = \{g : \mathbb{R} \to SU(2)\}$, with boundary condition (5.9), whereas the phase space $\Gamma_1$ is its cotangent bundle. As a manifold this is the product of $SU(2)(\mathbb{R})$ with a vector space, its dual Lie algebra, $\mathfrak{su}(2)^*(\mathbb{R})$, spanned by the currents $I_i$:

$$\Gamma_1 = SU(2)(\mathbb{R}) \times \mathfrak{su}(2)^*(\mathbb{R}). \tag{5.18}$$

Hamilton equations of motion then read as:

$$\partial_t I_j(\sigma) = \partial_\sigma J^k(\sigma)\delta_{kj} + \frac{\kappa\lambda^2}{4\pi}\epsilon_{jk}{}^{\ell} I_\ell(\sigma) J^k(\sigma)\,, \tag{5.19}$$

$$\partial_t J^j(\sigma) = \partial_\sigma I_k(\sigma)\delta^{kj} - \epsilon^{j\ell}{}_k I_\ell(\sigma) J^k(\sigma)\,. \tag{5.20}$$

Remarkably, the Poisson algebra (5.16) is homomorphic to $\mathfrak{c}_1$, the semi-direct sum of the Kac-Moody algebra associated to $SU(2)$ with the Abelian algebra $\mathbb{R}^3(\mathbb{R})$:

$$\mathfrak{c}_1 = \mathfrak{su}(2)(\mathbb{R}) \,\dot{\oplus}\, \mathfrak{a}. \tag{5.21}$$

Therefore, the cotangent bundle $\Gamma_1$ can be alternatively spanned by the conjugate variables $(J^j, I_j)$, with $J^j$ the left configuration space coordinates and $I_j$ the left momenta.

The energy-momentum tensor is traceless and conserved:

$$T_{00} = T_{11} = \frac{1}{4\lambda^2}\mathrm{Tr}(I^2 + J^2); \quad T_{01} = T_{10} = \frac{1}{2\lambda^2}\mathrm{Tr}(IJ), \tag{5.22}$$

so the model is conformally and Poincaré invariant, classically.

It has been shown in Ref. [79] that the current algebra $\mathfrak{c}_1$ may be deformed to a one-parameter family of fully non-Abelian algebras, in such a way that the resulting brackets, together with a one-parameter family of deformed Hamiltonians, lead to an equivalent description of the dynamics. The new Poisson algebra was shown to be homomorphic to either $\mathfrak{so}(4)(\mathbb{R})$ or $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$, depending on the choice of the deformation parameter. In [79] the first possibility was investigated, while from now on we shall choose the second option, for reasons that will be clear in a moment. Accordingly, the cotangent space $\Gamma_1$ shall be replaced by a new one, the set of $SL(2,\mathbb{C})$ valued maps, $\Gamma_2 = SL(2,\mathbb{C})(\mathbb{R})$. We refer to [79] for details about the deformation procedure, while hereafter we shall just state the result with a few steps which will serve our purposes. The new Poisson algebra will be indicated with $\mathfrak{c}_2 = \mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$.

### Deformation to the $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ current algebra

Following the strategy already adopted for the IRR and the PCM, what is interesting for us is the occurrence of the group $SL(2,\mathbb{C})$ as an alternative target phase space for the dynamics of the model. Indeed, $SL(2,\mathbb{C})$ is the Drinfel'd double of $SU(2)$, namely a group which can be locally parametrised as a product of $SU(2)$ with its properly defined dual, $SB(2,\mathbb{C})$. The latter is obtained by exponentiating the Lie algebra structure defined on the dual algebra of $\mathfrak{su}(2)$, under suitable compatibility conditions. Details of the construction are given in Chapter 2. Since the role of the partner groups is symmetric, we are going to see that this shall allow to study Poisson-Lie duality in the appropriate mathematical framework.

Before proceeding further, let us stress here that we are not going to deform the dynamics but only its target phase space description, and in particular its current algebra. This is completely different from the usual deformation approach followed for instance for integrable models. In that case one starts from a given integrable model, and then deforms it while trying to preserve the integrability property, but allowing for a modification of the physical content. In our case no deformation of the dynamics occurs.

Inspired by Wigner-Inonu contraction of semisimple Lie groups, a convenient modification of the Poisson algebra $\mathfrak{c}_1$ which treats $I$ and $J$ on an equal footing is the following:

$$
\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= \xi \left[\epsilon_{ij}{}^k I_k(\sigma) + a\,\epsilon_{ijk} J^k(\sigma)\right] \delta(\sigma-\sigma') \\
\{I_i(\sigma), J^j(\sigma')\} &= \xi \left[\left(\epsilon_{ki}{}^j J^k(\sigma) + b\,\epsilon_i{}^{jk} I_k(\sigma)\right) \delta(\sigma-\sigma') - \gamma\,\delta_i^j \delta'(\sigma-\sigma')\right] \\
\{J^i(\sigma), J^j(\sigma')\} &= \xi \left[\tau^2 \epsilon^{ijk} I_k(\sigma) + \mu\,\epsilon^{ij}{}_k J^k(\sigma)\right] \delta(\sigma-\sigma'),
\end{aligned}
\tag{5.23}
$$

with $a, b, \mu, \xi, \gamma$ real parameters, while $\tau$ can be chosen either real or purely imaginary. Upon imposing that the equations of motion remain unchanged, it can be checked (see [79]) that it is sufficient to rescale the Hamiltonian by an overall factor, depending on $\tau$, according to

$$
H_\tau = \frac{1}{4\lambda^2(1-\tau^2)^2} \int_{\mathbb{R}} d\sigma \left(\delta^{ij} I_i I_j + \delta_{ij} J^i J^j\right) \tag{5.24}
$$

with the parameters obeying the constraints

$$
\begin{aligned}
\xi &= 2\lambda^2 (1-\tau^2) &\quad (5.25)\\
a-b &= \frac{\kappa\lambda^2}{4\pi}(1-\tau^2) &\quad (5.26)\\
\gamma &= (1-\tau^2) &\quad (5.27)
\end{aligned}
$$

while $\mu$ is left arbitrary. In the limit $\tau, b, \mu \to 0$ we recover the standard description.

For real $\tau$ the Poisson algebra (5.23) is isomorphic to $\mathfrak{so}(4)(\mathbb{R})$ [79], while

for imaginary $\tau$ a more convenient choice of coordinates shall be done, which will make it evident the isomorphism with the $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ algebra.
Before doing that, let us shortly address the issue of space-time symmetries of the deformed model. The new formulation is still Poincaré and conformally invariant, although not being derived from the standard action principle. Indeed by following the same approach as in [77, 79] we obtain the new energy-momentum tensor, $\Theta_{\mu\nu}$, by requiring that

$$P = \int_{\mathbb{R}} d\sigma \, \Theta_{01}(\sigma) \tag{5.28}$$

and the Hamiltonian

$$H = \int_{\mathbb{R}} d\sigma \, \Theta_{00}(\sigma) \tag{5.29}$$

generate space-time translations according to

$$\frac{\partial}{\partial\sigma} I_k = \{P, I_k(\sigma)\}, \qquad \frac{\partial}{\partial\sigma} J^k = \{P, J^k(\sigma)\} \tag{5.30}$$

$$\frac{\partial}{\partial t} I_k = \{H, I_k(\sigma)\}, \qquad \frac{\partial}{\partial t} J^k = \{H, J^k(\sigma)\}. \tag{5.31}$$

One finds

$$\Theta_{01} = \Theta_{10} = \frac{1}{4\lambda^2(1-\tau^2)^2}\delta^i_j I_i J^j \tag{5.32}$$

$$\Theta_{00} = \frac{1}{4\lambda^2(1-\tau^2)^2}\left(\delta^{ij} I_i I_j + \delta_{ij} J^i J^j\right) \ . \tag{5.33}$$

To obtain the remaining component of the energy-momentum tensor, we complete the Poincaré algebra by introducing the boost generator, $B$, which has to satisfy the following Poisson brackets

$$\{H, B\} = P, \quad \{P, B\} = H. \tag{5.34}$$

The latter are verified by

$$B = -\frac{1}{2(1-\tau^2)} \int_{\mathbb{R}} d\sigma \, \sigma \, (\delta_{ij} J^i J^j + \delta^{ij} I_i I_j). \tag{5.35}$$

We thus compute the boost transformations of $I$ and $J$, getting

$$\{I_\ell, B\} = 2\lambda^2(1-\tau^2)\left(\sigma\frac{\partial I_\ell}{\partial t} + \delta_{\ell k} J^k\right), \quad \{J^\ell, B\} = 2\lambda^2(1-\tau^2)\left(\sigma\frac{\partial J^\ell}{\partial t} + \delta^{\ell k} I_k\right) \tag{5.36}$$

namely, $I$ and $J$ transform as time and space components of a vector field.

Therefore, the model is Poincaré invariant and the stress-energy tensor has to be conserved. In particular

$$\frac{\partial \Theta_{01}}{\partial t} = \frac{\partial \Theta_{11}}{\partial \sigma} \tag{5.37}$$

which yields $\Theta_{11} = \Theta_{00}$.
Conformal invariance is finally verified by computing the algebra of the energy-momentum tensor, or, equivalently, by checking the classical analogue of the Master Virasoro equation[2](see for example [119]). We do not repeat the calculation, performed in [79], the only difference being the choice of $\tau$ as a real or imaginary parameter.

### New coordinates

It is convenient to introduce the real linear combinations

$$\begin{aligned} S_i(\sigma) &= \frac{1}{\xi(1-a^2\tau^2)}\left[I_i(\sigma) - a\delta_{ik}J^k(\sigma)\right], \\ B^i(\sigma) &= \frac{1}{\xi(1-a^2\tau^2)}\left[J^i(\sigma) - a\tau^2\delta^{ik}I_k(\sigma)\right]. \end{aligned} \tag{5.40}$$

On using the residual freedom for the parameters, we choose $b = \mu = a\tau^2$ and $a = \frac{k\lambda^2}{4\pi}$, so that

$$\{S_i(\sigma), S_j(\sigma')\} = \epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') + C\delta_{ij}\delta'(\sigma-\sigma') \tag{5.41}$$

$$\{B^i(\sigma), B^j(\sigma')\} = \tau^2\epsilon^{ijk} S_k(\sigma)\delta(\sigma-\sigma') + \tau^2 C\delta^{ij}\delta'(\sigma-\sigma') \tag{5.42}$$

$$\{S_i(\sigma), B^j(\sigma')\} = \epsilon_{ki}{}^j B^k(\sigma)\delta(\sigma-\sigma') + C'\delta_i{}^j\delta'(\sigma-\sigma'), \tag{5.43}$$

where we recognise rotations, $S_i$, and boosts, $B^i$, and

$$C = \frac{a}{\lambda^2(1-a^2\tau^2)^2}, \qquad C' = -\frac{(1+a^2\tau^2)}{2\lambda^2(1-a^2\tau^2)^2} \tag{5.44}$$

[2]The classical version of the Master Virasoro equation amounts to the following relations

$$G^{AB} = G^{AC}\Omega_{CD}G^{DB}, \quad \tilde{G}^{AB} = \tilde{G}^{AC}\Omega_{CD}\tilde{G}^{DB} \quad 0 = \tilde{G}^{AC}\Omega_{CD}G^{DB} \tag{5.38}$$

with

$$\Theta = \frac{1}{2}(\Theta_{00}+\Theta_{01}) = G^{AB}M_A M_B \quad \tilde{\Theta} = \frac{1}{2}(\Theta_{00}-\Theta_{01}) = \tilde{G}^{AB}M_A M_B \tag{5.39}$$

$M_A = (J^a, I_a)$ and $\Omega_{AB}$ the matrix of central charges of the current algebra.

the central charges. Note that both transformations and Poisson algebra are consistent and non-singular in the limit $\tau \to 0$. As an intermediate step, it is convenient to write the Hamiltonian in terms of $S$ and $B$. By replacing

$$
\begin{aligned}
I_i(\sigma) &= \xi\left[S_i(\sigma) + a\delta_{ik}B^k(\sigma)\right] \\
J^i(\sigma) &= \xi\left[B^i + a\tau^2\delta^{ik}S_k(\sigma)\right]
\end{aligned}
\tag{5.45}
$$

in Eq. (5.24), it is easy to obtain:

$$
H_\tau = \lambda^2 \int_{\mathbb{R}} d\sigma \Big[ \left(1 + a^2\tau^4\right) \delta^{ij}S_iS_j + \left(1 + a^2\right) \delta_{ij}B^iB^j + 2a\left(1+\tau^2\right) \delta^i{}_j S_iB^j \Big], \tag{5.46}
$$

where we suppressed the $\sigma$-dependence of fields for the sake of notation. The equations of motion in terms of the new generators $S$ and $B$ read as:

$$
\begin{aligned}
\partial_t S_k &= -\frac{a(1-\tau^2)}{1-a^2\tau^2}\partial_\sigma S_k + \frac{1-a^2}{1-a^2\tau^2}\delta_{pk}\partial_\sigma B^p\,, \\
\partial_t B^k &= \frac{1-a^2\tau^4}{1-a^2\tau^2}\delta^{kp}\partial_\sigma S_p + \frac{a(1-\tau^2)}{1-a^2\tau^2}\partial_\sigma B^k - \xi(1-a^2\tau^2)\epsilon^{kp}{}_q S_p B^q\,.
\end{aligned}
\tag{5.47}
$$

Note that now non-diagonal terms appear in the Hamiltonian, which are zero not only for $a = 0$ (i.e. without WZ term), which corresponds to the Principal Chiral Model [67], but also for $\tau = \pm i$, $a \neq 0$.

Our next goal is to make explicit the bialgebra structure of the Poisson algebra (5.41)-(5.43), according to the Manin triple decomposition $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$. To this, another linear transformation of the generators is needed. We leave the $S_i$ unchanged since, according to Eq. (5.41), they already span the $\mathfrak{su}(2)$ algebra and transform the $B^i$ generators as follows:

$$
K^i(\sigma) = B^i(\sigma) \,+\, i\tau\epsilon^{i\ell 3}S_\ell(\sigma)\,. \tag{5.48}
$$

The new generators span the affine algebra associated with the Lie algebra $\mathfrak{sb}(2,\mathbb{C})$, as can be easily checked by computing their Poisson brackets, which read as:

$$
\{K^i(\sigma), K^j(\sigma')\} = i\tau f^{ij}{}_k K^k(\sigma)\delta(\sigma-\sigma') + C\tau^2 h^{ij}\delta'(\sigma-\sigma'), \tag{5.49}
$$

with $f^{ij}{}_k = \epsilon^{ij\ell}\epsilon_{\ell 3k}$ the structure constants of $\mathfrak{sb}(2,\mathbb{C})$ and

$$
h^{ij} = \delta^{ij} + \epsilon^{ip3}\delta_{pq}\epsilon^{jq3} \tag{5.50}
$$

a non-degenerate metric in $\mathfrak{sb}(2,\mathbb{C})$ defined in Eq. (2.36). With similar calculations for the mixed bracket we find:

$$\begin{aligned}\{S_i(\sigma), K^j(\sigma')\} = &\left[\epsilon_{ki}{}^j K^k(\sigma) + i\tau f^{jk}{}_i S_k(\sigma)\right]\delta(\sigma-\sigma') \\ &+ \left(C'\delta_i^j - i\tau C\epsilon_i{}^{j3}\right)\delta'(\sigma-\sigma').\end{aligned} \tag{5.51}$$

The Poisson algebra described by Eqs. (5.41), (5.49), (5.51) is a bialgebra, isomorphic to $\mathfrak{c}_2$, with its maximal subalgebras clearly identified as $\mathfrak{su}(2)(\mathbb{R})$ and $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})$.

By substituting

$$B^i(\sigma) = K^i(\sigma) - i\tau\epsilon^{i\ell 3}S_\ell(\sigma) \tag{5.52}$$

the Hamiltonian is rewritten in terms of the new generators as

$$\begin{aligned}H_\tau = \lambda^2 \int_{\mathbb{R}} d\sigma \Big\{ & S_i S_j\left[(1+a^2\tau^4)\delta^{ij} - \tau^2(1+a^2)\epsilon^{ip3}\delta_{pq}\epsilon^{jq3}\right] \\ + K^i K^j(1+a^2)\delta_{ij} + S_i K^j &\left[2a(1+\tau^2)\delta^{ip} + 2i\tau(1+a^2)\epsilon^{ip3}\right]\delta_{pj}\Big\}.\end{aligned} \tag{5.53}$$

Let us notice that the model remains conformally invariant, because we have only performed linear transformations of the current algebra generators.

In compact form the Hamiltonian reads,

$$H_\tau = \lambda^2 \int_{\mathbb{R}} d\sigma\, S_I\,(\mathcal{M}_\tau)^{IJ}\,S_J, \tag{5.54}$$

where we have introduced the doubled notation $S_I \equiv (K^i, S_i)$, and the generalized metric $\mathcal{M}_\tau(a)$, given by

$$\mathcal{M}_\tau = \begin{pmatrix} (1+a^2\tau^4)\delta^{ij} - \tau^2(1+a^2)\epsilon^{ip3}\delta_{pq}\epsilon^{jq3} & \left[i\tau(1+a^2)\epsilon^{ip3} + a(1+\tau^2)\delta^{ip}\right]\delta_{pj} \\ \delta_{ip}\left[-i\tau(1+a^2)\epsilon^{pj3} + a(1+\tau^2)\delta^{pj}\right] & (1+a^2)\delta_{ij}\end{pmatrix}. \tag{5.55}$$

Let us analyze the latter in more detail, as a function of the parameters $a, \tau$. For $a = 0$ we retrieve the one-parameter family associated to the PCM, studied in [67], and $\mathcal{M}_\tau(a=0)$ can be checked to be an $O(3,3)$ matrix, namely, $\mathcal{M}_\tau^T\eta\mathcal{M}_\tau = \eta$. For $a \neq 0$ the metric is not $O(3,3)$ in general, but it could be for specific values of the parameters; for example, it becomes proportional to an $O(3,3)$ matrix for $\tau = \pm i$. In particular for $\tau = -i$ we find $\mathcal{M}_{-i}(a) = (1+a^2)\,\mathcal{H}^{-1}$, where

$$\mathcal{H}^{-1} = \begin{pmatrix} h^{ij} & +\epsilon^{ip3}\delta_{pj} \\ -\delta_{ip}\epsilon^{pj3} & \delta_{ij}\end{pmatrix} \tag{5.56}$$

is the inverse of the Riemannian metric defined in (2.34) for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Let us summarize the main results of this section. The WZW model with target configuration space $SU(2)$ has been described in terms of a one-parameter family of Hamiltonians $H_\tau$

$$H_\tau = \lambda^2 \int_{\mathbb{R}} d\sigma \, S_I \, (\mathcal{M}_\tau)^{IJ} \, S_J \tag{5.57}$$

with Poisson brackets realizing the non-compact current algebra $\mathfrak{c}_2$

$$\begin{aligned}
\{S_i(\sigma), \, S_j(\sigma') \, \} &= \epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') + C\delta_{ij}\delta'(\sigma-\sigma') \\
\{K^i(\sigma), K^j(\sigma')\} &= i\tau f^{ij}{}_k K^k(\sigma)\delta(\sigma-\sigma') + C\tau^2 h^{ij}\delta'(\sigma-\sigma') \\
\{S_i(\sigma), K^j(\sigma') \, \} &= \left[\epsilon_{ki}{}^j K^k(\sigma) + i\tau f^{jk}{}_i S_k(\sigma)\right]\delta(\sigma-\sigma') \\
&\quad + \left(C'\delta_i^j - i\tau C\epsilon_i{}^{j3}\right)\delta'(\sigma-\sigma')
\end{aligned} \tag{5.58}$$

and

$$K^i \;=\; \frac{1}{\xi(1-a^2\tau^2)}\left((\delta^i_k - i\tau a\epsilon^{i\ell 3}\delta_{\ell k})J^k + (\epsilon^{ik3} - a\tau^2\delta^{ik})I_k\right) \tag{5.59}$$

$$S_i \;=\; = \frac{1}{\xi(1-a^2\tau^2)}\left(I_i - a\delta_{ik}J^k\right). \tag{5.60}$$

The Poisson algebra, thanks to the choice performed for the generators, reveals a bialgebra structure with central terms.

It is interesting to note that the central terms of the brackets for the $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ generators entail the metrics of the respective algebras, obtained directly from the generalized metric on $\mathfrak{sl}(2,\mathbb{C})$ in (2.34).

The alternative canonical formulation which has been presented here has some interesting features in relation with quantization [79] and integrability [118]. In the following we will exploit the bialgebra formulation to analyse the symmetries of the model under Poisson-Lie duality.

### 5.1.3 Poisson-Lie symmetry of the WZW model

Before looking explicitly at the dual models, we address the Poisson-Lie symmetry of the one-parameter family of WZW models described by the Hamiltonian (5.57), and Poisson brackets (5.58), adapting the definition given above to our setting.

Keeping the interpretation of $(K^i, S_i)$ as target phase space coordinates with $K^i$ and $S_i$ respectively base and fibre coordinates, one can associate

Hamiltonian vector fields to $K^i$ by means of

$$X_{K^i} := \{\cdot, K^i\}. \tag{5.61}$$

The fields so defined obey by construction the Lie algebra relations

$$[X_{K^i}, X_{K^j}] = X_{\{K^i, K^j\}} = i\tau f^{ij}{}_k X_{K^k}, \tag{5.62}$$

inherited from the non-trivial Poisson structure (5.58). They span the Lie algebra $\mathfrak{sb}(2,\mathbb{C})$ and, in the limit $\tau \to 0$, reproduce the original Abelian structure of the $\mathfrak{su}(2)$ dual. Because of their definition they satisfy

$$\omega(X_{K^j}, X_{K^k}) = \{K^j, K^k\} = f^{jk}{}_i K^i. \tag{5.63}$$

Moreover, we may define dual one-forms in the standard way, $\alpha_j : \alpha_j(X_{K^k}) = \delta_j^k$, which satisfy the Maurer-Cartan equation

$$d\alpha_i(X_{K^j}, X_{K^k}) = -\alpha_i([X_{K^j}, X_{K^k}]) = -f^{jk}{}_i. \tag{5.64}$$

The latter, being basis one-forms of the dual algebra, can be identified with basis generators of $\mathfrak{su}(2)$, $\alpha_i \to V_i$. By inverting Eq. (5.64) one can check that this is indeed the Poisson-Lie condition stated in Eq. (3.51).

#### $B$ and $\beta$ T-duality transformations

It was already noticed in [67] that the one-parameter family of Principal Chiral Models obtained from deformation of the target phase space could be recognized as a family of Born geometries, generated by $\beta$ T-duality transformations. The situation for the WZW model is more involved.

Starting from the generalized vector $(J^i, I_i)$, which obeys the Poisson algebra (5.16), we have performed a series of transformations, ending up with a new generalized vector, $(K^i, S_i)$, satisfying the Poisson algebra (5.58), while describing the same dynamics. These transformations are therefore symmetries, which can be partially recast in the form of $\beta$-transformations as follows.

According to [120, 121], given a generalized vector field on the target space, $(X, \omega)$, a $\beta$-transformation in the context of Poisson-Lie groups is a T-duality transformation of the algebra of currents $\phi : \mathfrak{d}(\mathbb{R}) \to \mathfrak{d}(\mathbb{R})$, which may be represented as

$$(X, \omega) \to (X + i_\omega \beta, \omega) \quad \beta \in \Gamma(\Lambda^2 TM) \tag{5.65}$$

with $\beta$ a bivector field. As dual to the latter, another $T$-duality transformation is a $B$-transformation, given by

$$(X, \omega) \to (X, \omega + i_X B) \quad B \in \Gamma(\Lambda^2 T^* M) \tag{5.66}$$

with $B$ a two-form. Besides, there are other T-duality transformations, such as factorized transformations, which may be rephrased in the same setting as linear transformations in the bialgebra of generalized vector fields of the target space.
In this perspective let us see how the transformation (5.59) can be reformulated. Differently from the PCM model without WZ term, we need to split the transformation in two steps. We first perform

$$(J^i, I_i) \to (\tilde{J}^i, \tilde{I}_i) = (J^i - a\tau^2 \delta^{ik} I_k,\ I_i - a\delta_{ik} J^k) \tag{5.67}$$

which is a generalized linear transform of the kind

$$(X, \omega) \to (X + C(\omega), \omega + \tilde{C}(X)) \tag{5.68}$$

with $C = -a\tau^2 \delta^{ij} \xi_i \otimes \xi_j$, $\tilde{C} = -a\delta_{ij} \xi^{*i} \otimes \xi^{*j}$, $\{\xi_i\}, \{\xi^{*i}\}$, basis of vector fields and dual one-forms on $M$, and $\tilde{C} = -C^{-1}$ for $i\tau = 1/a$. We thus perform a $\beta$-transformation:

$$(\tilde{J}^i, \tilde{I}_i) \to (\tilde{J}^i + i\tau \epsilon^{i\ell 3} \tilde{I}_\ell, \tilde{I}_i) \equiv (K^i, S_i). \tag{5.69}$$

The generalized metric (5.55) may be obtained from the diagonal metric $\mathcal{H}_0$ applying the above transformations accordingly. Therefore, the one-parameter family of WZW models introduced in previous sections can be regarded as a sequence of $B$- and $\beta$-transformations. For $a = 0$, we resort to the PCM considered in [67], where the one-parameter family is a family of Born geometries related by pure $\beta$-transformations.

## 5.2 Poisson-Lie T-duality

In order to investigate duality transformations within the current algebra which has been obtained at the end of Sec. 5.1.2, we need to make the role of the two subalgebras $\mathfrak{su}(2)(\mathbb{R})$ and $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})$ completely symmetric. To this, we shall introduce a further parameter in the current algebra, so to get a *two-parameter* formulation of the WZW model. As a result, T-duality transformations will be realized as simple $O(3,3)$ rotations in the target phase space $SL(2,\mathbb{C})(\mathbb{R})$ and the two parameters at disposal will allow to consider limiting cases.

### 5.2.1 Two-parameter family of Poisson-Lie dual models

In what follows we slightly modify the current algebra (5.58) by introducing another imaginary parameter, $\alpha$, so to have $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ generators on an equal footing. This will allow to formulate Poisson-Lie duality as a phase space rotation within $SL(2,\mathbb{C})(\mathbb{R})$, namely an $O(3,3)$ transformation, which exchanges configuration space coordinates, $K^i$ with momenta $S_j$. The introduction of the new parameter will make it possible to perform not only the limit $SL(2,\mathbb{C}) \overset{\tau\to 0}{\rightarrow} T^*SU(2)$ but also $SL(2,\mathbb{C}) \overset{\alpha\to 0}{\rightarrow} T^*SB(2,\mathbb{C})$.

To this, let us introduce the two-parameter generaliation of the algebra (5.58) as follows:

$$
\begin{aligned}
\{S_i(\sigma), S_j(\sigma') \} &= i\alpha\epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') - \alpha^2\hat{C}\delta_{ij}\delta'(\sigma-\sigma') \\
\{K^i(\sigma), K^j(\sigma')\} &= i\tau f^{ij}{}_k K^k(\sigma)\delta(\sigma-\sigma') + \tau^2\hat{C}h^{ij}\delta'(\sigma-\sigma') \\
\{S_i(\sigma), K^j(\sigma') \} &= \left[i\alpha\epsilon_{ki}{}^j K^k(\sigma) + i\tau f^{jk}{}_i S_k(\sigma)\right]\delta(\sigma-\sigma') \\
&\quad + (i\alpha\hat{C}'\delta_i^j - i\tau\hat{C}\epsilon_i{}^{j3})\delta'(\sigma-\sigma').
\end{aligned}
\tag{5.70}
$$

It is immediate to check that, in the limit $i\tau \to 0$, the latter reproduces the semi-direct sum $\mathfrak{su}(2)(\mathbb{R})\oplus\mathfrak{a}$, while the limit $i\alpha \to 0$ yields $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})\oplus\mathfrak{a}$. For all non-zero values of the two parameters, the algebra is homomorphic to $\mathfrak{c}_2$, with central extensions. The central charges, $\hat{C}, \hat{C}'$ will be fixed in a while.

By direct calculation one easily verifies that, upon suitably rescaling the fields, one gets back the dynamics of the WZW model, if the Hamiltonian is deformed as follows:

$$
\begin{aligned}
H_{\tau,\alpha} &= \lambda^2 \int_{\mathbb{R}} d\sigma\, S_I\, (\mathcal{M}_{\tau,\alpha})^{IJ}\, S_J \\
&= \lambda^2 \int_{\mathbb{R}} d\sigma \left[S_i(\mathcal{M}_{\tau,\alpha})^{ij}S_j + K^i(\mathcal{M}_{\tau,\alpha})_{ij}K^j + S_i(\mathcal{M}_{\tau,\alpha})^i{}_j K^j + K^i(\mathcal{M}_{\tau,\alpha})_i{}^j S_j\right],
\end{aligned}
\tag{5.71}
$$

with $S_I = (S_i, K^i)$, and

$$\mathcal{M}_{\tau,\alpha} = \begin{pmatrix} \frac{1}{(i\alpha)^2}\left[(1+a^2\bar{\tau}^4)\delta^{ij} - \bar{\tau}^2(1+a^2)\epsilon^{ip3}\delta_{pq}\epsilon^{jq3}\right] & \left[i\bar{\tau}(1+a^2)\epsilon^{ip3} + a(1+\bar{\tau}^2)\delta^{ip}\right]\delta_{pj} \\ \delta_{ip}\left[-i\bar{\tau}(1+a^2)\epsilon^{pj3} + a(1+\bar{\tau}^2)\delta^{pj}\right] & (i\alpha)^2(1+a^2)\delta_{ij} \end{pmatrix} \tag{5.72}$$

where $i\bar{\tau} = i\tau\, i\alpha$. Indeed, by rescaling the fields according to

$$\bar{S}_j = \frac{S_j}{i\alpha}, \quad \bar{K}^j = i\alpha K^j \tag{5.73}$$

the Hamiltonian for the fields $\bar{S}_j, \bar{K}^j$ takes the same form as Eq. (5.57) and the Poisson brackets of the rescaled fields yield back the algebra (5.58) if the central charges are chosen as follows:

$$\hat{C} = \frac{a}{\lambda^2\left(1-a^2\bar{\tau}^2\right)^2}, \quad \hat{C}' = -\frac{1+a^2\bar{\tau}^2}{2\lambda^2\left(1-a^2\bar{\tau}^2\right)^2}. \tag{5.74}$$

This is exactly what we were looking for, since the role of the $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ generators is now completely symmetric. It is easy to check that the two-parameter model indeed reproduces the original WZW dynamics for $i\tau \to 0$, with Poisson algebra $\mathfrak{c}_1 = \mathfrak{su}(2)(\mathbb{R}) \dot{\oplus}\, \mathfrak{a}$. The $i\alpha \to 0$ limit yields the algebra $\mathfrak{c}_3 = \mathfrak{sb}(2,\mathbb{C})(\mathbb{R}) \dot{\oplus}\, \mathfrak{a}$ with central extension, although the Hamiltonian appears to be singular in such a limit. We shall come back to this issue later on. For all other values of $\alpha$ and $\tau$ the algebra is isomorphic to $\mathfrak{c}_2 \simeq \mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$, with $K^i, S_i$ respectively playing the role of configuration space coordinates and momenta.

Since now the role of $S_i$ and $K^i$ is symmetric, if we exchange the momenta $S_i$ with the configuration space fields $K^i$ we obtain a new two-parameter family of models, with the same target phase space, but with the role of coordinates and momenta inverted. The transformation

$$\tilde{K}_i(\sigma) = S_i(\sigma), \quad \tilde{S}^i(\sigma) = K^i(\sigma) \tag{5.75}$$

is an $O(3,3)$ rotation in the target phase space $SL(2,\mathbb{C})$.

Explicitly, under such a rotation we obtain the dual Hamiltonian

$$\tilde{H}_{\tau,\alpha} = \lambda^2 \int_{\mathbb{R}} d\sigma \left[\tilde{K}_i(\mathcal{M}_{\tau,\alpha})^{ij}\tilde{K}_j + \tilde{S}^i(\mathcal{M}_{\tau,\alpha})_{ij}\tilde{S}^j + \tilde{K}_i(\mathcal{M}_{\tau,\alpha})^i{}_j\tilde{S}^j + \tilde{S}^i(\mathcal{M}_{\tau,\alpha})_i{}^j\tilde{K}_j\right], \tag{5.76}$$

and dual Poisson algebra

$$\begin{aligned}\{\tilde{K}_i(\sigma), \tilde{K}_j(\sigma')\} &= i\alpha\epsilon_{ij}{}^k\tilde{K}_k(\sigma)\delta(\sigma-\sigma') - \alpha^2\hat{C}\delta_{ij}\delta'(\sigma-\sigma') \\ \{\tilde{S}^i(\sigma), \tilde{S}^j(\sigma')\} &= i\tau f^{ij}{}_k\tilde{S}^k(\sigma)\delta(\sigma-\sigma') + \tau^2\hat{C}h^{ij}\delta'(\sigma-\sigma') \\ \{\tilde{K}_i(\sigma), \tilde{S}^j(\sigma')\} &= \left[i\alpha\epsilon_{ki}{}^j\tilde{S}^k(\sigma) + i\tau f^{jk}{}_i\tilde{K}_k(\sigma)\right]\delta(\sigma-\sigma') \\ &\quad + (i\alpha\hat{C}'\delta_i^j - i\tau\hat{C}\epsilon_i{}^{j3})\delta'(\sigma-\sigma')\end{aligned} \tag{5.77}$$

which makes it clear that this new two-parameter family of models has target configuration space the group manifold of $SB(2,\mathbb{C})$, spanned by the fields $\tilde{K}_i$, while momenta $\tilde{S}^i$ span the fibres of the target phase space. Hence, this represents by construction a family of dual models.

Note, however, that the limit $i\alpha \to 0$, although giving a well-defined Poisson algebra as a semi-direct sum, does not bring to a well-defined dynamics on $T^*SB(2,\mathbb{C})$, since the Hamiltonian becomes singular. As we shall see in the next section, this seems to be related to the impossibility of obtaining the family of dual Hamiltonians (5.76) from a continuous deformation of a Hamiltonian WZW model on the cotangent space $T^*SB(2,\mathbb{C})$. This obstruction has a topological explanation in the simple fact that $T^*SB(2,\mathbb{C})$, differently from $T^*SU(2)$, is not homeomorphic to $SL(2,\mathbb{C})$. In the next section we shall introduce a WZW model with $SB(2,\mathbb{C})$ as configuration space, in the Lagrangian approach, and shall look for a Hamiltonian description by means of canonical Legendre transform. We shall see that, in order to make contact with one of the dual models described by the two-parameter family (5.76), we need to introduce a true deformation of the dynamics, a topological modification of the phase space and extra terms in the Hamiltonian.

Going back to the Hamiltonian (5.76) we want to show here that, although it has not been obtained from an action principle, nevertheless it is possible to exhibit an action from which it can be derived. Following the standard approach of [77, 122], we shall write the action in the first order formalism. To this, two ingredients are needed: the symplectic form responsible for the current algebra (5.77) and the Hamiltonian (5.76) expressed in terms of the original fields $g \in SU(2)(\mathbb{R})$ and $\ell \in SB(2,\mathbb{C})(\mathbb{R})$. The target phase space $\Gamma_2$ can be identified with $SU(2)(\mathbb{R}) \times SB(2,\mathbb{C})(\mathbb{R})$ as a manifold, and we define

$$-i\alpha\hat{C}g^{-1}\partial_\sigma g = i\delta^{kp}\tilde{K}_p e_k, \quad i\tau\hat{C}\ell^{-1}\partial_\sigma\ell = i(h^{-1})_{kp}\tilde{S}^p\tilde{e}^k, \tag{5.78}$$

with $\hat{e}^k$ the generators of the $\mathfrak{sb}(2,\mathbb{C})$ algebra. It can be shown that the symplectic form which yields the current algebra (5.77) is the following (see Appendix A for more details about this construction):

$$\begin{aligned}\omega = \alpha^2\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg \wedge \partial_\sigma(g^{-1}dg)\right] - \tau^2\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left[\ell^{-1}d\ell \wedge \partial_\sigma(\ell^{-1}d\ell)\right] \\ + i\bar{\tau}\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left\{[g^{-1}dg, \ell^{-1}\partial_\sigma \ell] \wedge (\ell^{-1}d\ell - (\ell^{-1}d\ell)^\dagger)\right\} \\ + i\bar{\tau}\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left\{[\ell^{-1}d\ell, g^{-1}\partial_\sigma g] \wedge (g^{-1}dg - (g^{-1}dg)^\dagger)\right\} \\ + i\alpha\hat{C}'\int_{\mathbb{R}^2} d\sigma\, d\sigma' \partial_\sigma\delta(\sigma-\sigma')\left\{\mathrm{Tr}_\eta\left[g^{-1}dg(\sigma) \wedge \ell^{-1}d\ell(\sigma')\right]\right\} \\ - i\tau\hat{C}\int_{\mathbb{R}^2} d\sigma\, d\sigma' \partial_\sigma\delta(\sigma-\sigma')\left\{\mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg(\sigma) \wedge \ell^{-1}d\ell(\sigma')\right]\right\}.\end{aligned} \tag{5.79}$$

The products denoted by $\mathrm{Tr}_{\mathcal{H}}$ and $\mathrm{Tr}_\eta$ are the two $SL(2,\mathbb{C})$ products defined in (2.34) and (2.29) respectively. The symplectic form is not closed, therefore an action in the first order formalism may be defined according to

$$S_2 = \int \omega - \int H_{|g,\ell}\, dt, \tag{5.80}$$

where $\omega$ has to be integrated on a two-surface and $H_{|g,\ell}$ denotes the Hamiltonian expressed in terms of the original fields $g$ and $\ell$. When the symplectic form is exact, the surface integral reduces to the standard integration of the canonical Lagrangian 1-form along the boundary of the surface. However, this is not the case for our symplectic form and some care is needed. Here one has to consider the closed curve $\gamma$ on $\Gamma_2$, described by functions $g(t,\sigma) : \mathbb{R} \times S^1 \to SU(2)$ and $\ell(t,\sigma) : \mathbb{R} \times S^1 \to SB(2,\mathbb{C})$, as well as the surface $\tilde{\gamma}$ of which it is the boundary: $\partial\tilde{\gamma} = \gamma$. The surface $\tilde{\gamma}$ can then be described by extensions of $g$ and $\ell$ defined such that $\tilde{g}(t,\sigma,y=1) = g(t,\sigma)$, $\tilde{\ell}(t,x,y=1) = \ell(t,\sigma)$ and $\tilde{g}(t,\sigma,y=0) = \tilde{\ell}(t,\sigma,y=0) = 1$, mimicking the Wess-Zumino term construction on a 3-manifold with fields extended from the source space. The action can then be written explicitly with (5.80). Note also that the integration of the first two terms in (5.79) following this recipe results in two WZ terms. The construction is totally symmetric with respect to the exchange of momenta with configuration fields, therefore, the same construction furnishes an action principle for the Hamiltonian (5.71).

Summarizing, we reformulated the WZW model on $SU(2)$ within an alternative canonical picture based on a two-parameter deformation of the current algebra and the Hamiltonian, in which the role of momenta and configuration space fields is made symmetric. By sending to zero either parameter we recover the original current algebra structure $\mathfrak{su}(2)(\mathbb{R})\dot{\oplus}\mathfrak{a}(\mathbb{R})$ or the natural dual one $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})\dot{\oplus}\mathfrak{a}(\mathbb{R})$. By performing an $O(3,3)$ transformation over $SL(2,\mathbb{C})$, which is the deformed target phase space of the system, we obtain a two-parameter family of Hamiltonian models with target configuration space $SB(2,\mathbb{C})$, which represents, by construction, the Poisson-Lie dual family of the $SU(2)$ family we started with.

As a further goal, in parallel to what is done for the $SU(2)$ family, where the limit $i\tau \to 0$ yields back the semi-Abelian model with target phase space $T^*SU(2)$, we would like to perform the limit $i\alpha \to 0$ to recover the semi-Abelian dual model with target phase space $T^*SB(2,\mathbb{C})$. We have seen that, while the current algebra is well-defined in such a limit, yielding the semi-direct sum $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})\dot{\oplus}\mathfrak{a}$, the Hamiltonian is singular. We have argued that this may be related to the different topology of phase spaces $SL(2,\mathbb{C})$ and $T^*SB(2,\mathbb{C})$. This issue will be addressed at the end of Sect. 5.3.2.

## 5.3 Lagrangian WZW model on $SB(2,\mathbb{C})$

In the previous section we have obtained a whole family of dual models having $SB(2,\mathbb{C})$ as target configuration space, which makes it meaningful to look for a dual model having the tangent bundle of $SB(2,\mathbb{C})$ as carrier space from the beginning. However, the latter group algebra is not semi-simple, which means that the Cartan-Killing metric is degenerate. The problem of constructing a WZW model for non-semisimple groups is not new - see for example [3], where the 2-d Poincaré group is considered. In our case, it does not seem to be possible to define any bilinear pairing on $\mathfrak{sb}(2,\mathbb{C})$ which is both non-degenerate and bi-invariant at the same time. As in [67], we could use the induced metric (2.36), $h^{ij} = \delta^{ij} + \epsilon^{i\ell 3}\delta_{\ell k}\epsilon^{jk3}$, which is obtained from restricting the Riemannian metric (2.35) of $\mathfrak{sl}(2,\mathbb{C})$. This product is non-degenerate and positive-definite, and it is not bi-invariant. Indeed, one can check that it is invariant under both left and right $SU(2)$ global action ($\gamma \to g_L \gamma g_R^{-1}$, with $g_L, g_R \in SU(2)$) but only invariant under left $SB(2,\mathbb{C})$ action.

A natural WZW action would then be:

$$S_1 = \frac{1}{n\lambda^2}\int_\Sigma \mathcal{T}r\left[\phi^*(\ell^{-1}d\ell) \overset{\wedge}{,} *\phi^*(\ell^{-1}d\ell)\right] + \frac{\kappa}{m\pi}\int_{\mathcal{B}} \mathcal{T}r\left[\tilde{\phi}^*\left(\tilde{\ell}^{-1}d\tilde{\ell} \overset{\wedge}{,} \tilde{\ell}^{-1}d\tilde{\ell} \wedge \tilde{\ell}^{-1}d\tilde{\ell}\right)\right], \tag{5.81}$$

with $\phi : \Sigma \ni (t,\sigma) \to \ell \in SB(2,\mathbb{C})$, while $\tilde{\phi}$ and $\tilde{\ell}$ are the related extensions to $\mathcal{B}$, and $\mathcal{T}r := ((,))$ as in (2.35). However, it is immediate to check that the WZ term is identically zero for such a product. Indeed, on introducing the notation

$$B_\mu = \ell^{-1}\partial_\mu \ell = B_{\mu j}\tilde{e}^j, \tag{5.82}$$

with $\tilde{e}^j$ indicating the generators of $\mathfrak{sb}(2,\mathbb{C})$ and $\tilde{B}_\mu$ the extension to $\mathcal{B}$, we have

$$\begin{aligned}\left(\left(\tilde{\phi}^* \left[\tilde{\ell}^{-1}d\tilde{\ell} \overset{\wedge}{,} \tilde{\ell}^{-1}d\tilde{\ell} \wedge \tilde{\ell}^{-1}d\tilde{\ell}\right]\right)\right) &= -i\, \mathrm{d}^3y\, \epsilon^{\alpha\beta\gamma}\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k}h^{kp}f^{ij}{}_p \\ &= -2i\, \mathrm{d}^3y\, \epsilon^{\alpha\beta\gamma}\left(\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma 1}\epsilon^{ij2} - \tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{2\gamma}\epsilon^{ij1}\right),\end{aligned}$$

which is obviously vanishing. This means that $h^{ij}$ is not a viable product to define a WZW model on $SB(2,\mathbb{C})$.

Our proposal is then to use the Hermitian product $h_N$ defined in Eq. (2.37). The action of the model will be given by Eq. (5.81) with $\mathcal{T}r(u,v) \to \mathrm{Tr}\,(u^\dagger v)$ and $n, m$, integer coefficients to be determined later. In terms of the latter, the WZ term can be checked to be non-zero and consistent with the equations of motion one expects to obtain. Indeed, on separating the diagonal and off-diagonal part of the product, as

$$h_N^{ij} = \frac{1}{2}h^{ij} + a^{ij}, \tag{5.83}$$

the only contribution to the three-dimensional integral in (5.81) comes from the off-diagonal term, $a^{ij}$, since we just showed that the WZ term vanishes with the metric $h^{ij}$. We have

$$\begin{aligned}\int_{\mathcal{B}} \mathrm{d}^3y\, \epsilon^{\alpha\beta\gamma}\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k}h_N^{kp}f^{ij}{}_p &= \frac{1}{2}\int_{\mathcal{B}} \mathrm{d}^3y\, \epsilon^{\alpha\beta\gamma}\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k}a^{kp}f^{ij}{}_p \\ &= \frac{i}{2}\int_{\mathcal{B}} \mathrm{d}^3y\, \epsilon^{\alpha\beta\gamma}\epsilon^{ijk}\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k}.\end{aligned} \tag{5.84}$$

The latter has the same form as the WZ term on the $SU(2)$ group manifold, which means that the variation is formally the same, leading to the same contribution to the equations of motion but now with $\mathfrak{sb}(2,\mathbb{C})$-valued currents:

$$\delta S_{1,WZ} = \frac{1}{m\pi}\int_\Sigma d^2\sigma\, \epsilon^{ijk}\, B_{0j}B_{1k}\left(\ell^{-1}\delta\ell\right)_i \tag{5.85}$$

with $B_0 = \ell^{-1}\partial_t\ell$, $B_1 = \ell^{-1}\partial_\sigma\ell$.

As for the quadratic term in the action (5.81), on using Eq. (5.83) we have

$$\int_\Sigma \mathrm{d}^2\sigma\, \mathcal{T}r\left[\phi^*(\ell^{-1}d\ell) \overset{\wedge}{,} *\phi^*(\ell^{-1}d\ell)\right] = \int_\Sigma \mathrm{d}^2\sigma\, B_{\mu i}B^\mu_j\, \mathrm{Tr}\, \hat{e}^{i\dagger}\hat{e}^j$$
$$= \frac{1}{2}\int_\Sigma \mathrm{d}^2\sigma\, ((B_\mu, B^\mu)) \tag{5.86}$$

because the off-diagonal contribution proportional to $a^{ij}$ vanishes. Therefore, in absence of the WZ term, the two products yield the same result, up to a numerical factor, and agree with previous findings for the PCM [67]. For the variation of this term with respect to small variations of $\ell$ we will need the following relation:

$$\delta B_\mu = -\ell^{-1}\delta\ell B_\mu + \ell^{-1}\partial_\mu\delta\ell, \tag{5.87}$$

$$\delta((B_\mu, B^\mu)) = ((\delta B_\mu, B^\mu)) + ((B_\mu, \delta B^\mu))$$
$$= 2\left[((B^\mu, [B_\mu, \ell^{-1}\delta\ell])) - ((\partial_\mu B^\mu, \ell^{-1}\delta l)) + \partial_\mu((B^\mu, \ell^{-1}\delta\ell))\right] \tag{5.88}$$

and after integration one obtains

$$\delta S_{1,\mathrm{quad}} = \frac{1}{n\lambda^2}\int_\Sigma d^2\sigma\left[((-\partial^\mu B_\mu, \ell^{-1}\delta\ell)) + ((B^\mu, [B_\mu, \ell^{-1}\delta\ell]))\right].$$

In order to make the comparison with the $SU(2)$ model more transparent we fix $n = 4$ and introduce the notation $\hat{A}_i = B_{0,i}$, $\hat{J}_i = B_{1i}$. We have then

$$\delta S_{1,\mathrm{quad}} = -\frac{1}{4\lambda^2}\int_\Sigma d^2\sigma\left[h^{ij}\left(\partial_t\hat{A}_j - \partial_\sigma\hat{J}_j\right) - f^{pi}{}_q h^{qj}\left(\hat{A}_p\hat{A}_j - \hat{J}_p\hat{J}_j\right)\right](\ell^{-1}\delta\ell)_i. \tag{5.89}$$

By collecting all terms, the resulting equations of motion can then be written as follows:

$$h^{ij}\left(\partial_t\hat{A}_j - \partial_\sigma\hat{J}_j\right) - f^{pi}{}_q h^{qj}\left(\hat{A}_p\hat{A}_j - \hat{J}_p\hat{J}_j\right) = -\frac{4\kappa\lambda^2}{m\pi}\epsilon^{ipj}\,\hat{A}_p\hat{J}_j, \tag{5.90}$$

and we also have the usual integrability condition coming from the Maurer-Cartan equation for the Maurer-Cartan one-forms $\ell^{-1}d\ell$:

$$\partial_t\hat{J} - \partial_\sigma\hat{A} = -[\hat{A}, \hat{J}]. \tag{5.91}$$

Looking at the equations of motion, by analogy with the $SU(2)$ case we will fix $m = 24$.

### 5.3.1 Spacetime geometry

The Lagrangian model which has been derived in the previous section furnishes a possible spacetime background on which strings propagate. Topologically it is the manifold of the group $SB(2,\mathbb{C})$, a noncompact manifold, which can be embedded in $\mathbb{R}^4$ by means of the following parametrization

$$\ell = y_0 \mathbb{1}_2 + 2iy_i \hat{e}^i \tag{5.92}$$

with $y_\mu, \mu = 0,\ldots,3$ global real coordinates, $\tilde{e}^i$ the generators of the group (see def. (2.23)) and the constraint $y_0^2 - y_3^2 = 1$. Its geometry is characterised by a metric tensor and a $\mathbb{B}$-field, which are easier to compute in terms of a local parametrisation. We first rewrite the action as

$$S_1 = \frac{1}{4\lambda^2}\int_\Sigma d^2\sigma B_{\mu i} B^\mu_j h^{ij} + \frac{\kappa}{24\pi}\int_{\mathcal{B}} d^3y\, \epsilon^{\alpha\beta\gamma} \tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k} h_N^{kp} {f^{ij}}_p. \tag{5.93}$$

with $iB_i\hat{e}^i = \ell^{-1}d\ell$ the Maurer-Cartan one-form on the group. We thus parametrise a generic element $\ell \in SB(2,\mathbb{C})$ according to

$$\ell = \begin{pmatrix} \chi & \psi e^{i\theta} \\ 0 & \frac{1}{\chi} \end{pmatrix}, \tag{5.94}$$

with $\chi, \psi \in \mathbb{R}, \chi > 0$ and $\theta \in (0, 2\pi)$. In this way we can write

$$\ell^{-1}d\ell = \begin{pmatrix} \frac{1}{\chi}d\chi & \frac{1}{\chi}e^{i\theta}d\psi + i\frac{\psi}{\chi}e^{i\theta}d\theta + \frac{\psi}{\chi^2}e^{i\theta}d\chi \\ 0 & -\frac{1}{\chi}d\chi \end{pmatrix}. \tag{5.95}$$

Since the generators of the $\mathfrak{sb}(2,\mathbb{C})$ algebra can be written as

$$\hat{e}^k = \frac{1}{2}\delta^{ki}\left(i\sigma_i + \epsilon_{i3}^k\sigma_k\right), \tag{5.96}$$

or, explicitly,

$$\hat{e}^1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \hat{e}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{e}^3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.97}$$

the components of the Maurer-Cartan one-form $\ell^{-1}d\ell$ have the form:

$$B_1 = -\frac{\psi}{\chi^2}\cos\theta d\chi - \frac{1}{\chi}\cos\theta d\psi + \frac{\psi}{\chi}\sin\theta d\theta, \tag{5.98}$$
$$B_2 = \frac{\psi}{\chi^2}\sin\theta d\chi + \frac{\sin\theta}{\chi} d\psi + \frac{\psi}{\chi}\cos\theta d\theta, \tag{5.99}$$
$$B_3 = -\frac{2}{\chi}d\chi. \tag{5.100}$$

On using the explicit expression of the product $h^{ij}$ the quadratic term of the model yields then

$$S_{1quad} = \frac{1}{4\lambda^2}\int_\Sigma d^2\sigma\Bigg[\left(\frac{\psi^2}{2\chi^4}+\frac{4}{\chi^2}\right)\partial_\mu\chi\,\partial^\mu\chi + \frac{1}{2\chi^2}\partial_\mu\psi\,\partial^\mu\psi + \frac{\psi^2}{2\chi^2}\partial_\mu\theta\,\partial^\mu\theta + \frac{\psi}{\chi^3}\partial_\mu\chi\,\partial^\mu\psi\Bigg]. \tag{5.101}$$

Analogously, the WZ term can be calculated in local coordinates to give:

$$\epsilon^{\alpha\beta\gamma}\tilde{B}_{\alpha i}\tilde{B}_{\beta j}\tilde{B}_{\gamma k}h_N^{kp}f^{ij}{}_p = 2\epsilon^{\alpha\beta\gamma}\tilde{B}_{\alpha 1}\tilde{B}_{\beta 2}\tilde{B}_{\gamma 3} = 4\epsilon^{\alpha\beta\gamma}\frac{\tilde{\psi}}{\tilde{\chi}^3}\partial_\alpha\tilde{\chi}\partial_\beta\tilde{\psi}\partial_\gamma\tilde{\theta}$$
$$= -2\epsilon^{\alpha\beta\gamma}\partial_\alpha\left(\frac{\tilde{\psi}}{\tilde{\chi}^2}\partial_\beta\tilde{\psi}\partial_\gamma\tilde{\theta}\right).$$

Hence, by means of Stokes theorem on the latter contribution, the total action can be rewritten as

$$S_1 = \frac{1}{4\lambda^2}\int_\Sigma d^2\sigma\Bigg[\left(\frac{\psi^2}{2\chi^4}+\frac{4}{\chi^2}\right)\partial_\mu\chi\,\partial^\mu\chi + \frac{1}{2\chi^2}\partial_\mu\psi\,\partial^\mu\psi + \frac{\psi^2}{2\chi^2}\partial_\mu\theta\,\partial^\mu\theta + \frac{\psi}{\chi^3}\partial_\mu\chi\,\partial^\mu\psi - \frac{\kappa\lambda^2}{3\pi}\frac{\psi}{\chi^2}\epsilon^{\mu\nu}\partial_\mu\psi\,\partial_\nu\theta\Bigg]. \tag{5.102}$$

By identifying the latter with the Polyakov action

$$\int d^2\sigma\left(G_{ij}\partial_\alpha X^i\partial^\alpha X^j + B_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\right), \tag{5.103}$$

with $X^i \equiv (\chi,\psi,\theta)$, the background spacetime metric and $B$-field read

$$G_{ij} = \frac{1}{4\lambda^2}\begin{pmatrix} \left(\frac{\psi^2}{2\chi^4}+\frac{4}{\chi^2}\right) & \frac{\psi}{2\chi^3} & 0 \\ \frac{\psi}{2\chi^3} & \frac{1}{2\chi^2} & 0 \\ 0 & 0 & \frac{\psi^2}{2\chi^2}\end{pmatrix}, \quad B_{\psi\theta} = -\frac{\kappa}{12\pi}\frac{\psi}{\chi^2}. \tag{5.104}$$

Hence, the spacetime background is a non-compact 3-d Riemannian manifold, embedded in $\mathbb{R}^4$ with the topology of the group manifold of $SB(2,\mathbb{C})$ and its geometry is described by the following above metric and antisymmetric $B$-field. The $B$-field is not closed, thus yielding a 3-form $\mathbb{H}$-flux.

It may be useful to express the metric and the $B$-field in terms of global coordinates in $\mathbb{R}^4$. It can be easily checked that the embedding map reads

$$\psi = 2\sqrt{y_1^2 + y_2^2}, \quad \chi = y_0 - y_3 \quad \theta = -\arctan\frac{y_2}{y_1}. \tag{5.105}$$

Then the metric in (5.104) is obtained by the following Lorentzian metric in $\mathbb{R}^4$

$$\begin{aligned} \mathbb{G}_4 &= \frac{2}{(y_0-y_3)^2}\Big[f_a(-dy_0\otimes dy_0 + dy_3\otimes dy_3) + f_b(dy_1\otimes dy_1 + dy_2\otimes dy_2) \\ &+ f_c d(y_0-y_3)\otimes(y_1dy_1 + y_2dy_2)\Big] \end{aligned} \tag{5.106}$$

with

$$f_a = y_1^2 + y_2^2 + 2(y_0-y_3)^2, \quad f_b = 1, \quad f_c = \frac{2}{(y_0-y_3)}. \tag{5.107}$$

Upon imposing the constraint $(y_0-y_3)(y_0+y_3) = 1$, which characterises the submanifold, we get:

$$\begin{aligned} \mathbb{G}_3 &= \frac{2}{(y_0-y_3)^2}\Big[\frac{f_a}{(y_0-y_e)^2}d(y_0-y_3)\otimes d(y_0-y_3) + f_b(dy_1\otimes dy_1 + dy_2\otimes dy_2) \\ &+ f_c d(y_0-y_3)\otimes(y_1dy_1 + y_2dy_2)\Big]. \end{aligned} \tag{5.108}$$

Analogously, we may write the two-form $B$ in terms of global $\mathbb{R}^4$ coordinates. We obtain

$$B = \frac{\kappa}{3\pi}\frac{1}{(y_0-y_3)^2}dy_1\wedge dy_2 \tag{5.109}$$

with $\mathbb{H}$-flux

$$\mathbb{H} = dB = -\frac{2\kappa}{3\pi}\frac{1}{(y_0-y_3)^3}d(y_0-y_3)\wedge dy_1\wedge dy_2. \tag{5.110}$$

### 5.3.2 Dual Hamiltonian formulation

Let us consider first the situation in which the WZ term is missing ($\kappa = 0$). In this case the equations of motion have the simpler form

$$h^{ij}\left(\partial_t \hat{A}_j - \partial_\sigma \hat{J}_j\right) = f^{pi}{}_q h^{qj}\left(\hat{A}_p \hat{A}_j - \hat{J}_p \hat{J}_j\right), \tag{5.111}$$

$$\partial_t \hat{J} - \partial_\sigma \hat{A} = -[\hat{A}, \hat{J}] \tag{5.112}$$

and we have a clear Lagrangian picture. In particular, we are able to define the left momenta

$$\hat{I}^i = \frac{\delta \mathscr{L}_1}{\delta \hat{A}_i} = \frac{1}{2\lambda^2}\hat{A}_j h^{ij} \tag{5.113}$$

which can be inverted for the generalized velocities to write the Hamiltonian:

$$H_1 = \frac{1}{4\lambda^2}\int_{\mathbb{R}} d\sigma \left(\hat{I}^i \hat{I}^j h_{ij} + \hat{J}_i \hat{J}_j h^{ij}\right)$$

and analogously to the $SU(2)$ case, the pair $(\hat{A}, \hat{J})$ identifies the cotangent bundle of $SB(2,\mathbb{C})$, with $\hat{I}$ fibre coordinates.

Following the usual approach we can then obtain the equal-time Poisson brackets from the action functional:

$$\begin{aligned}
&\{\hat{I}^i(\sigma), \hat{I}^j(\sigma')\} = 2\lambda^2 f^{ij}{}_k \hat{I}^k(\sigma)\delta(\sigma - \sigma') \\
&\{\hat{I}^i(\sigma), \hat{J}_j(\sigma')\} = 2\lambda^2\left[f^{ki}{}_j \hat{J}_k(\sigma)\delta(\sigma - \sigma') - \delta^i_j \delta'(\sigma - \sigma')\right] \\
&\{\hat{J}_i(\sigma), \hat{J}_j(\sigma')\} = 0,
\end{aligned} \tag{5.114}$$

from which, together with the Hamiltonian $H_1$, the equations of motion follow:

$$\partial_t \hat{I}^k(\sigma) = h^{ij}\delta^k{}_j\, \partial_\sigma \hat{J}_i(\sigma) + h_{ij} f^{jk}{}_p\, \hat{I}^i(\sigma)\hat{I}^p(\sigma) + h^{ij} f^{kp}{}_j\, \hat{J}_i(\sigma)\hat{J}_p(\sigma), \tag{5.115}$$

$$\partial_t \hat{J}_k(\sigma) = h_{ij}\left[f^{pi}{}_k\, \hat{I}^j(\sigma)\hat{J}_p(\sigma) + \delta^i{}_k\, \partial_\sigma \hat{I}^j(\sigma)\right]. \tag{5.116}$$

If we now introduce the WZ term contribution, the equations of motion get modified and a new term appears:

$$\begin{aligned}
\partial_t \hat{I}^k(\sigma) =& h^{ik}\, \partial_\sigma \hat{J}_i(\sigma) + h_{ij} f^{jk}{}_p\, \hat{I}^i(\sigma)\hat{I}^p(\sigma) + h^{ij} f^{kp}{}_j\, \hat{J}_i(\sigma)\hat{J}_p(\sigma) \\
&- \frac{\kappa\lambda^2}{3\pi} h_{pi}\epsilon^{kpj}\, \hat{I}^i(\sigma)\hat{J}_j(\sigma),
\end{aligned} \tag{5.117}$$

while the integrability condition does not receive any modification, as it should.

Assuming that the Hamiltonian remains the same after the inclusion of the WZ term, as it is the case for the $SU(2)$ model, we have to find the corresponding Poisson structure leading to the modified equations of motion. By inspection, Hamilton equations for momenta, obtained from the bracket $\{H_1, \hat{I}\}$, only involve the bracket $\{\hat{I}, \hat{I}\}$, therefore we shall just modify the latter, by including a term proportional to $\hat{J}$.

It is straightforward to check that to obtain the right correction to the equations of motion we have to modify the Poisson brackets as follows:

$$\{\hat{I}^i(\sigma), \hat{I}^j(\sigma')\} = 2\lambda^2 \left[ f^{ij}{}_k \hat{I}^k(\sigma)\delta(\sigma - \sigma') - w\,\epsilon^{ijp}\hat{J}_p(\sigma)\delta(\sigma - \sigma') \right], \quad (5.118)$$

and the coefficient $w$ can be determined by direct comparison of Hamilton equation for $I$ with (5.117). We find:

$$w = \frac{2\kappa\lambda^2}{6\pi}.$$

To summarise, the dynamics of the WZW model on $SB(2, \mathbb{C})$ is described by the following Hamiltonian

$$H_1 = \frac{1}{4\lambda^2} \int_{\mathbb{R}} d\sigma \left( \hat{I}^i \hat{I}^j h_{ij} + \hat{J}_i \hat{J}_j h^{ij} \right). \quad (5.119)$$

and Poisson algebra

$$\begin{aligned}
&\{\hat{I}^i(\sigma), \hat{I}^j(\sigma')\} = 2\lambda^2 \left[ f^{ij}{}_k \hat{I}^k(\sigma)\delta(\sigma - \sigma') - \frac{2\kappa\lambda^2}{6\pi}\,\epsilon^{ijp}\hat{J}_p(\sigma)\delta(\sigma - \sigma') \right] \\
&\{\hat{I}^i(\sigma), \hat{J}_j(\sigma')\} = 2\lambda^2 \left[ f^{ki}{}_j \hat{J}_k(\sigma)\delta(\sigma - \sigma') - \delta_j{}^i \delta'(\sigma - \sigma') \right] \\
&\{\hat{J}_i(\sigma), \hat{J}_j(\sigma')\} = 0.
\end{aligned} \quad (5.120)$$

This is the semi-direct sum of an Abelian algebra and a Kac-Moody algebra associated to $SB(2, \mathbb{C})$, with a central extension, just as expected.
Indeed, on defining

$$\hat{S}^i = \hat{I}^i - \frac{w}{2}\epsilon^{ij3}\hat{J}_j \quad (5.121)$$

it is immediate to check that

$$\begin{aligned}
\{\hat{S}^i(\sigma), \hat{S}^j(\sigma')\} &= 2\lambda^2 \left[ f^{ij}{}_k \hat{S}^k(\sigma)\delta(\sigma - \sigma') - w\epsilon^{ij3}\delta'(\sigma - \sigma') \right] \quad (5.122) \\
\{\hat{S}^i(\sigma), \hat{J}_j(\sigma')\} &= 2\lambda^2 \left[ f^{ki}{}_j \hat{J}_k(\sigma) - \delta^i_j \delta'(\sigma - \sigma') \right]. \quad (5.123)
\end{aligned}$$

We have already obtained the same kind of algebra as the $i\alpha \to 0$ limit of the algebra (5.70). The Hamiltonian however is not recovered as a limit of the dual family (5.76), and we have already commented that this should be expected, because $T^*SB(2,\mathbb{C})$, phase space of the former, cannot be obtained by continuous deformation of $SL(2,\mathbb{C})$, phase space of the latter.

There is however another possibility, suggested by the form of the metric (5.72). Within the Hamiltonian picture, we have the freedom of defining another model, taking advantage of the fact the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ has another metric structure, given by (2.29), which is $O(3,3)$ invariant and non-degenerate. This provides a well defined metric for the Lie algebra $\mathfrak{sb}(2,\mathbb{C})\dot{\oplus}\mathbb{R}^3$ as well. Therefore, we may declare the currents $\hat{J}$ to be valued in the Lie algebra $\mathfrak{sb}(2,\mathbb{C})$, $\hat{J} = \hat{J}_i\hat{e}^i$, while the momenta $I$ to be valued in the Abelian algebra $\mathbb{R}^3$, $\hat{I} = \hat{I}^i\hat{t}_i$. The metric (2.29) will give

$$(\hat{I},\hat{I}) = (\hat{J},\hat{J}) = 0, \quad (\hat{I},\hat{J}) = (\hat{J},\hat{I}) = \hat{I}^i\hat{J}_j\delta_i{}^j \tag{5.124}$$

Thus, upon introducing the double field notation $\hat{\mathbf{I}} = (\hat{I},\hat{J})$, the Hamiltonian will be

$$H_2 = \zeta\int_{\mathbb{R}} \mathrm{d}\sigma\, (\hat{\mathbf{I}},\hat{\mathbf{I}}) = 2\zeta\int_{\mathbb{R}} \mathrm{d}\sigma\, \hat{I}^i\hat{J}_i. \tag{5.125}$$

The latter may be obtained from the two-parameter Hamiltonian of the dual family (5.76) in two steps. We first introduce a new Hamiltonian, which is a deformation of (5.76), as

$$\tilde{H}_{\mathrm{def}} = \tilde{H}_{\tau,\alpha} - \lambda^2\int_{\mathbb{R}} d\sigma \tilde{K}_i(\mathcal{M}_{\tau,\alpha})^{ij}\tilde{K}_j \tag{5.126}$$

Then, we perform the limit $i\alpha \to 0$. This yields the wanted result if we suitably choose the parameter as $\zeta = a\lambda^2$:

$$H_2 = \lim_{i\alpha\to 0} \tilde{H}_{\mathrm{def}}. \tag{5.127}$$

On using for the Poisson algebra $\mathfrak{sb}(2,\mathbb{C})\dot{\oplus}\mathbb{R}^3$ the brackets in (5.70) in the limit $i\alpha \to 0$:

$$\begin{aligned}
\{\tilde{S}^i(\sigma),\tilde{S}^j(\sigma')\} &= i\tau f^{ij}{}_k\tilde{S}^k(\sigma)\delta(\sigma-\sigma') + \frac{a\tau^2}{\lambda^2}h^{ij}\delta'(\sigma-\sigma')\\
\{\tilde{K}_i(\sigma),\tilde{S}^j(\sigma')\} &= i\tau f^{jk}{}_i\tilde{K}_k(\sigma)\delta(\sigma-\sigma') - \frac{i\tau a}{\lambda^2}\epsilon_i{}^{j3}\delta'(\sigma-\sigma')\\
\{\tilde{K}_i(\sigma),\tilde{K}_j(\sigma')\} &= 0,
\end{aligned} \tag{5.128}$$

after identifying the currents as $\tilde{S} = \hat{I}$ and $\tilde{K} = \hat{J}$, we finally get the following equations of motion for the model having $SB(2,\mathbb{C})$ as target configuration space.

$$
\begin{aligned}
\dot{\hat{I}}^k &= 2i\tau a^2 \epsilon_p{}^{k3} \partial_\sigma \hat{I}^p - 2a^2\tau^2 h^{pk} \partial_\sigma \hat{J}_p \\
\dot{\hat{J}}_k &= 2i\tau a^2 \epsilon_k{}^{p3} \partial_\sigma \hat{J}_p.
\end{aligned} \tag{5.129}
$$

To summarise, we were not able to recover the 'natural' Hamiltonian model with Hamiltonian $H_1$ from the dual family obtained in Sec. 5.2.1, but we managed to define another model with the same target phase space, $T^*SB(2,\mathbb{C})$, but different metric tensor, which can be related to the dual family of Hamiltonians, (5.76) in the limit $i\alpha \to 0$ once a deformation has been performed. This establishes the wanted connection.

Finally, it is interesting to notice that it would have been impossible to obtain such a connection for the PCM where the WZ term is absent. Indeed, for $a = 0$ the Hamiltonian $H_2$ is identically zero and the equations of motion in (5.129) loose their significance.

To conclude this section, let us shortly address the issue of space-time symmetries. Since the model is Lagrangian, the energy-momentum tensor may be obtained from the action (5.93), yielding

$$
T_{00} = T_{11} = \frac{1}{4\lambda^2}\left(\hat{I}^i h_{ij} \hat{I}^j + \hat{J}_i h^{ij} \hat{J}_j\right); \quad T_{01} = T_{10} = \frac{1}{4\lambda^2}\left(\hat{I}^i \delta_i^j \hat{J}_j\right), \tag{5.130}
$$

which is formally the same as the $SU(2)$ tensor (5.22). However, our product is not bi-invariant, that is, it doesn't satisfy

$$
f^{ab}{}_d\, g^{cd} + f^{ac}{}_d\, g^{bd} = 0, \tag{5.131}
$$

which is a sufficient condition for Poincaré invariance (see for instance [3]). Nonetheless, it is immediate to check that $T_{\mu\nu}$ is conserved and traceless. Moreover, the Master Virasoro equations (5.38) are satisfied as well, so the model is conformally and Poincaré invariant at the classical level.

## 5.4 Double WZW model

So far we have been able to give a description of the $SU(2)$ WZW model current algebra as the affine algebra of $SL(2,\mathbb{C})$ and to construct a map to a family of dual models, with the same current algebra and target phase space, but with momenta and configuration fields exchanged. It is therefore natural to look for an action with manifest $SL(2,\mathbb{C})$ symmetry which could accommodate both models, by doubling the number of degrees of freedom.

Let us consider the $SL(2,\mathbb{C})$-valued field $\Phi : \Sigma \ni (t,\sigma) \to \gamma \in SL(2,\mathbb{C})$ and introduce the $\mathfrak{sl}(2,\mathbb{C})$-valued Maurer-Cartan one-forms $\gamma^{-1}d\gamma$. We postulate the following action functional with $SL(2,\mathbb{C})$ as target configuration space:

$$\begin{aligned} \mathcal{S} = \kappa_1 \int_\Sigma \left(\left(\Phi^*[\gamma^{-1}d\gamma] \overset{\wedge}{,} *\Phi^*[\gamma^{-1}d\gamma]\right)\right)_N \\ + \kappa_2 \int_{\mathcal{B}} \left(\left(\tilde{\Phi}^*\left[\tilde{\gamma}^{-1}d\tilde{\gamma} \overset{\wedge}{,} \tilde{\gamma}^{-1}d\tilde{\gamma} \wedge \tilde{\gamma}^{-1}d\tilde{\gamma}\right]\right)\right)_N \end{aligned} \tag{5.132}$$

with $\kappa_1, \kappa_2$ constants left arbitrary, and the Hermitian product (2.37) is employed.

The equations of motion can be derived by following the same steps as in the previous section for the model on $SB(2,\mathbb{C})$, with the only difference that now the fields are valued in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ with structure constants ${C_{IJ}}^K$. We obtain

$$\begin{aligned} &\mathcal{H}_{(N)LK}\left(\partial_t A^K - \partial_\sigma J^K\right) - {C_{PL}}^Q \mathcal{H}_{(N)QK}\left(A^P A^K - J^P J^K\right) \\ &= -2\frac{\kappa_2}{\kappa_1}\mathcal{H}_{(N)QL}{C_{PS}}^Q A^P J^S, \end{aligned} \tag{5.133}$$

where we denoted by $A^I \equiv (A^i, L_i)$, $J^I \equiv (J^i, M_i)$ the $TSL(2,\mathbb{C})(\mathbb{R})$ coordinates, with double index notation and $\mathcal{H}_{(N)}$ is the Hermitean product defined in (2.37). The generalised doubled action so constructed describes a non-linear sigma model with Wess-Zumino term with target configuration space the group manifold of $SL(2,\mathbb{C})$.

### 5.4.1 Doubled Hamiltonian description

In order to describe the doubled model in the Hamiltonian formalism, let us start by considering only the kinetic term ($\kappa_2 = 0$). In this case the equations of motion can be written as

$$\mathcal{H}_{LK(N)}\left(\partial_t A^K - \partial_\sigma J^K\right) - {C_{PL}}^Q \mathcal{H}_{QK(N)}\left(A^P A^K - J^P J^K\right) = 0, \tag{5.134}$$

and we can define a genuine Lagrangian density as follows:

$$\mathscr{L} = \kappa_1 \mathcal{H}_{LK(N)}\left(A^L A^K - J^L J^K\right), \tag{5.135}$$

from which canonical momenta can be defined

$$I_K \equiv \left(I_k, N^k\right) = \frac{\delta L}{\delta J^K} = 2\kappa_1 \mathcal{H}_{KL(N)} A^L, \tag{5.136}$$

leading to the following Hamiltonian:

$$H = \kappa_1 \int_{\mathbb{R}} d\sigma \left[ \left(\mathcal{H}^{-1}\right)^{LK(N)} I_L I_K + \mathcal{H}_{LK(N)} J^L J^K \right]. \tag{5.137}$$

The equal-time Poisson brackets can then be obtained in the usual way [67], resulting in

$$\begin{aligned} \{I_L(\sigma), I_K(\sigma')\} &= \frac{1}{2\kappa_1} C_{LK}{}^P I_P\, \delta(\sigma-\sigma') \\ \left\{I_L(\sigma), J^K(\sigma')\right\} &= \frac{1}{2\kappa_1}\left[C_{PL}{}^K J^P \delta(\sigma-\sigma') - \delta^K_L \delta'(\sigma-\sigma')\right] \\ \left\{J^L(\sigma), J^K(\sigma')\right\} &= 0, \end{aligned} \tag{5.138}$$

and together with the Hamiltonian, they lead to the equations of motion

$$\begin{aligned} \partial_t I_K(\sigma) = \frac{1}{2\kappa_1}\Bigg[ & \left(\mathcal{H}^{-1}\right)^{LP(N)} C_{LK}{}^Q I_Q(\sigma) I_P(\sigma) - \mathcal{H}_{LP(N)}\, C_{QK}{}^L J^Q(\sigma) J^P(\sigma) \\ & + \mathcal{H}_{KL(N)} \partial_\sigma J^L(\sigma)\Bigg]. \end{aligned} \tag{5.139}$$

Following the same approach we used for the $SB(2,\mathbb{C})$ model case, we can now include the WZ term, resulting in the modification of the equations of motion

$$\begin{aligned} \partial_t I_K(\sigma) = \frac{1}{2\kappa_1}\Bigg[ & \left(\mathcal{H}^{-1}\right)^{LP(N)} C_{LK}{}^Q I_Q(\sigma) I_P(\sigma) - \mathcal{H}_{LP(N)}\, C_{QK}{}^L J^Q(\sigma) J^P(\sigma) \\ & + \mathcal{H}_{KL(N)} \partial_\sigma J^L(\sigma) - 2\frac{\kappa_2}{\kappa_1} \mathcal{H}_{QK(N)} \left(\mathcal{H}^{-1}\right)^{RP(N)} C_{PS}{}^Q I_R J^S \Bigg], \end{aligned} \tag{5.140}$$

which can be obtained from the same Hamiltonian but modifying the Poisson structure as follows:

$$\begin{aligned} \{I_L(\sigma), I_K(\sigma')\} &= \frac{1}{2\kappa_1} C_{LK}{}^P I_P(\sigma)\, \delta(\sigma-\sigma') - \frac{\kappa_2}{\kappa_1^2} \mathcal{H}_{QK(N)} C_{LP}{}^Q J^P(\sigma)\delta(\sigma-\sigma') \\ \left\{I_L(\sigma), J^K(\sigma')\right\} &= \frac{1}{2\kappa_1}\left[C_{PL}{}^K J^P(\sigma) \delta(\sigma-\sigma') - \delta^K_L \delta'(\sigma-\sigma')\right] \\ \left\{J^L(\sigma), J^K(\sigma')\right\} &= 0. \end{aligned} \tag{5.141}$$

Models with target configuration space $SU(2)$ or $SB(2,\mathbb{C})$ could then be obtained by constraining the Hamiltonian (5.137). The Lagrangian approach adopted in [67], which requires to gauge one of the global symmetries of the parent action, presents some difficulties, since minimal coupling is not enough anymore and there may be obstructions to be dealt with. Indeed, although minimal coupling produces a gauge-invariant action, the equations of motion still depend on the extension to the 3-manifold $\mathcal{B}$.

This issue is addressed e.g. in [123–125]. Besides that, another problem, which is specific of the model, might affect the gauging. In fact, in the cited references the gauged action is always formulated for a semisimple group with a Cartan-Killing metric. However, here in order to reproduce the $SB(2,\mathbb{C})$ model we need to work with an Hermitian product. It is not clear how to handle the problem in this case and further investigation is needed.

## 5.5 Some final remarks

Here we summarize the procedure discussed in the rest of this chapter. In particular, we start from a canonical generalization of the Hamiltonian picture associated to the WZW model having $SU(2)$ as target space, which consists in describing the dynamics of the model in terms of a one-parameter family of Hamiltonians and $SL(2,\mathbb{C})$ Kac-Moody algebra of currents. Then, we have highlighted the Drinfel'd double nature of the phase space by introducing a further parameter both in the Hamiltonian and in the Poisson algebra.

The first result has been to show the Poisson-Lie symmetry of the model. Then, by performing a duality transformation in target phase space, we have been able to obtain a two-parameter family of models which are Poisson-Lie dual to the previous ones by construction. The two families share the same target phase space, the group manifold of $SL(2,\mathbb{C})$, but have configuration spaces which are dual to each other, namely $SU(2)$ and its Poisson-Lie dual, $SB(2,\mathbb{C})$. Although they have not been derived from an action principle, it has been shown that it is possible to exhibit an action, by means of an inverse Legendre transform which involves the symplectic form and the Hamiltonian.

As a natural step, we have investigated the possibility of defining a Lagrangian WZW model with target tangent space $TSB(2,\mathbb{C})$. Being the group $SB(2,\mathbb{C})$ not semi-simple, the problem of defining a non-degenerate product on its Lie algebra has been addressed, and a solution has been proposed. We have derived the Hamiltonian description on the cotangent space $T^*SB(2,\mathbb{C})$ and we have shown that, although its current algebra is obtained as the limit

$\alpha \to 0$ of the $SL(2,\mathbb{C})$ Kac-Moody algebra related to the dual family, it is not possible to obtain the Hamiltonian in the same limit, through a continuous deformation of phase spaces. It is however possible to definea new Hamiltonian on $T^*SB(2,\mathbb{C})$ in terms of an alternative $O(3,3)$ metric. Such a model can be related to the dual family of $SL(2,\mathbb{C})$ models by first performing a deformation of the dynamics and then the limit $\alpha \to 0$. It is interesting to notice that such a connection relies on the presence of the WZ term, and the whole construction loses significance if the coefficient in front of this term is zero.

A diagrammatic summary of the different models with corresponding relations between them is depicted in Fig. (5.1), where $Q$, $\Gamma$ and $\mathfrak{c}$ denote the target configuration space, phase space and current algebra respectively.

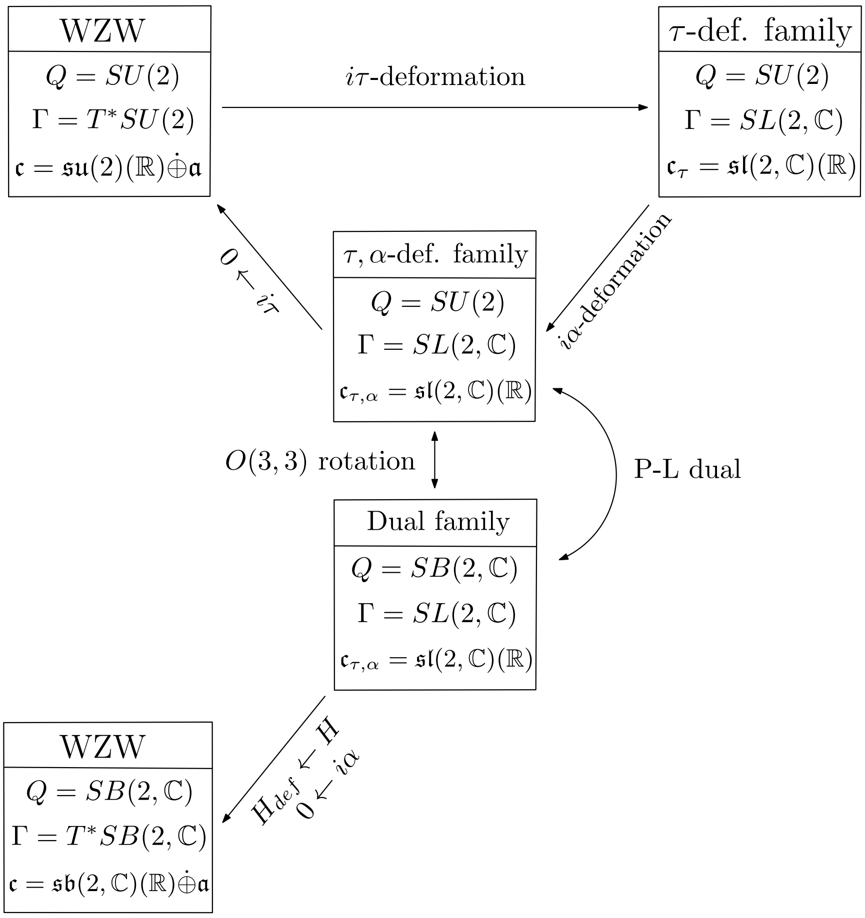


FIGURE 5.1: Diagrammatic summary of the models considered and their relations. $Q$, $\Gamma$ and $\mathfrak{c}$ denote configuration space, phase space and current algebra respectively.

Having introduced a well-defined WZW action on the dual $SB(2,\mathbb{C})$ we have analyzed the geometry of the target space as a string background solution. This is a non-compact Riemannian hypersurface, whose metric is induced by a Lorentzian metric. The $\mathbb{B}$-field and its flux have been calculated as well.

Finally, we have addressed the possibility of making manifest the $SL(2,\mathbb{C})$ symmetry of both families of WZW models, by doubling the degrees of freedom and introducing a parent action with target configuration space the Drinfel'd double $SL(2,\mathbb{C})$. A doubled Hamiltonian formulation has been proposed, such that a restriction to either subgroup, $SU(2)$ or $SB(2,\mathbb{C})$, leads to the Hamiltonian formulation of the two sub-models.

# 6 Poisson sigma model

In this chapter we will introduce the so-called Poisson sigma model, which is a two-dimensional topological field theory with target space a Poisson manifold, first introduced by Ikeda and independently by Schaller and Strobl in the context of two-dimensional gravity [82] and later widely investigated in relation with other models such as two-dimensional Yang–Mills and gravity theories, as well as in relation with deformation quantisation and branes. Related to that, we mention [85] by Cattaneo and Felder where the Poisson sigma model is used to give a physical interpretation of Kontsevich quantisation formula in terms of Feynman diagrams of the perturbative expansion of the model, and the work [83] by the same authors, in which they prove that the reduced phase space of the model is the symplectic groupoid integrating the Lie algebroid associated with the Poisson manifold, inspiring later works on the integrability of Lie algebroids [126]. A brief introduction to the topic can also be found in [127]. Other useful references are [128–130].

## 6.1 Introduction to the Poisson sigma model

Let $\Sigma$ be a two-dimensional oriented manifold, possibly with boundary, and $(M, \Pi)$ a $m$-dimensional Poisson manifold. The topological Poisson sigma model is defined by the bosonic real fields $(X, \eta)$, with $X : \Sigma \to M$ the usual embedding map and $\eta \in \Omega^1(\Sigma, X^*(T^*M))$ a one-form on $\Sigma$ with values in the pull-back of the cotangent bundle over $M$. The action of the model is given by

$$S = \int_\Sigma \left[ \eta_i \wedge dX^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j \right], \quad i, j = 1, \ldots, \dim M \tag{6.1}$$

where $dX \in \Omega^1(\Sigma, X^*(TM))$ and the contraction of covariant and contravariant indices is relative to the pairing between differential forms on $\Sigma$ with values in $X^*(T^*M)$ and $X^*(TM)$, respectively. It is induced by the natural pairing between $T^*M$ and $TM$ and yields a two-form on $\Sigma$. The action is manifestly invariant under diffeomorphisms of $\Sigma$, hence it describes a topological model. In order to make the world-sheet dependence explicit in (6.1), we introduce local coordinates $u^\mu$ ($\mu = 0, 1$) on $\Sigma$ so that $dX^i = \partial_\mu X^i du^\mu$, $\eta_i = \eta_{\mu i} du^\mu$ yielding

$$S = \int_\Sigma d^2u \left[ \epsilon^{\mu\nu} \eta_{\mu i} \partial_\nu X^i + \frac{1}{2} \Pi^{ij}(X) \epsilon^{\mu\nu} \eta_{\mu i} \eta_{\nu j} \right]. \tag{6.2}$$

The variation of the action leads to the following equations of motion in the bulk:

$$dX^i + \Pi^{ij}(X)\eta_j = 0, \tag{6.3}$$

$$d\eta_i + \frac{1}{2}\partial_i \Pi^{jk}(X)\eta_j \wedge \eta_k = 0. \tag{6.4}$$

One thing to notice is that the consistency of the equations of motion requires $\Pi(X)$, as a background field, to satisfy the Jacobi identity. This can be understood by acting on (6.3) with the exterior derivative, then using again (6.3) and finally using (6.4) on the result.

If the manifold $\Sigma$ has a boundary, then it is necessary to impose suitable boundary conditions such that the boundary term $\int_{\partial\Sigma} \delta X^i \eta_i$ vanishes. Many possibilities have been considered [130–133]. Interestingly, these have been associated with different brane solutions when the Poisson sigma model is considered in the framework of topological string theory. Indeed, taking the restriction of the field $X_{|\partial\Sigma} : \partial\Sigma \to N$, for some closed submanifold $N$ (the brane), there may be different conditions for $N$. The one usually chosen (in particular, it was used by Cattaneo and Felder in [83]), is the following:

$$\eta(u)v = 0 \quad \forall\, v \in T(\partial\Sigma), \;\; u \in \partial\Sigma. \tag{6.5}$$

It is important to note that apart from the obvious invariance of the action under diffeomorphisms of the source space $\Sigma$ we have other symmetries. In particular, the infinitesimal transformations

$$\begin{aligned} \delta_\beta X^i &= -\Pi^{ij}(X)\beta_j \\ \delta_\beta \eta_i &= \mathrm{d}\beta_i - \partial_i \Pi^{jk}(X)\eta_j \beta_k \end{aligned}$$

where $\beta = \beta_i(u)\mathrm{d}X^i \in \Gamma(X^* T^* M)$, change the action (6.1) only by a boundary term

$$\delta_\beta S = -\int_\Sigma \mathrm{d}\left(\mathrm{d}X^i \beta_i\right).$$

These give a complete set of local symmetries for the model. In particular, we can easily recover the invariance under diffeomorphisms of $\Sigma$ by a choice of the gauge parameter $\beta$. Given $u^\mu \mapsto u^\mu + \xi^\mu(u)$ the infinitesimal form of the diffeomorphism, if we choose the gauge parameter $\beta_i = \xi^\mu \eta_{i\mu}$ we have

$$\begin{aligned} \delta_\beta X^i &= \mathscr{L}_\xi X^i - \iota_\xi \left( \mathrm{d}X^i + \Pi^{ij}(X)\beta_j \right), \\ \delta_\beta \eta_i &= \mathscr{L}_\xi \eta_i - \iota_\xi \left( \mathrm{d}\eta_i + \frac{1}{2}\partial_i \Pi^{jk}(X)\eta_j \wedge \eta_k \right). \end{aligned} \tag{6.6}$$

It is easy to notice that on-shell (6.6) reduce to the expected transformations for the diffeomorphism invariance.

One could argue that the form of (6.6) is not invariant under a change of coordinates $X'(X)$ of the target space $M$. However, it is possible to show that the form of the transformations in the new coordinates is exactly the same up to the equations of motion, hence the transformations (6.6) are invariant on-shell.

Another important thing to notice is that by applying successive transformations one obtains

$$\begin{aligned} [\delta_\beta, \delta_{\beta'}] X^i &= \delta_{[\beta,\beta']} X^i, \\ [\delta_\beta, \delta_{\beta'}] \eta_i &= \delta_{[\beta,\beta']} \eta_i - \beta_k \beta'_\ell \partial_i \partial_j \Pi^{k\ell} \left( \mathrm{d}X^j - \Pi^{pj} \eta_p \right), \end{aligned} \tag{6.7}$$

where $[\beta, \beta']_i = \beta_j \beta'_k \partial_i \Pi^{jk}(X)$. The equations (6.7) show explicitly that the algebra of gauge transformations is open, i.e. it closes only on-shell. This fact alone makes the quantization of the model more involved and the so-called Batalin-Vilkovisky (BV) formalism [134, 135] has to be used to quantize it. See [136] for an introductory review on the BV formalism.

The Poisson sigma model comprises a variety of models. The most obvious is the one corresponding to a trivial Poisson structure $\Pi = 0$, for which one simply has a BF model with action

$$S = \int_\Sigma \eta_i \wedge dX^i, \tag{6.8}$$

which can be rewritten in the usual BF-theory form $\int XF$ by integrating by parts, with $F = d\eta$.

An interesting non-trivial example is the case corresponding to a linear Poisson structure $\Pi^{ij} = f^{ij}{}_k X^k$, leading to a non-Abelian BF theory with action (in explicit world-sheet dependence)

$$S = \int_\Sigma d^2u \left( \epsilon^{\mu\nu} \eta_{\mu i} \partial_\nu X^i + \frac{1}{2} \epsilon^{\mu\nu} f^{ij}{}_k X^k \eta_{\mu i} \eta_{\nu j} \right), \tag{6.9}$$

or, in the more familiar BF form, by integrating by parts:

$$S = \int_\Sigma X^i F_i, \quad F_i := d\eta_i + \frac{1}{2} f_i{}^{jk} \eta_j \wedge \eta_k. \tag{6.10}$$

In this case, in fact, the Jacobi identity for $\Pi$ makes $M$ the dual of a Lie algebra with structure constants $f^{ij}{}_k$, and $\eta$ takes the role of a one-form connection, while $F$ can be viewed as a curvature two-form. Other cases are two-dimensional Yang-Mills theory, gauged Wess–Zumino–Witten models and two-dimensional gravity models. A useful review where all these models are considered as derived from the Poisson sigma model is [137].

In particular, two-dimensional Yang-Mills theory can be obtained by using a linear Poisson structure $\Pi^{ij}{}_k = f^{ij}{}_k X^k$ and including the function $C(X) = \sum X^i X^i$ as a non-topological term in the action. The latter is the quadratic Casimir function of the Lie algebra $\mathfrak{g}$ whose structure constants are given by $f^{ij}{}_k$, hence its addition does not spoil the gauge invariance of the Poisson sigma model. The resulting first order action would be then the one in (6.10) with the addition of the Casimir function:

$$S_{YM} = \int_\Sigma \left[ X^i F_i + \lambda C(X) \right]. \tag{6.11}$$

By integrating out the field $X$ by using the equations of motion one obtains the usual second order formulation $S_{\mathrm{YM}} = -\frac{1}{4\lambda}\int_\Sigma \mathrm{tr}(F \wedge *F)$.

For what concerns two-dimensional gravity models, most of the two-dimensional ones can be obtained from the Poisson sigma model action. One example we show is two-dimensional $R^2$-gravity as this is particularly simple to work out. In fact, the action

$$S = \frac{1}{4} \int_\Sigma d^2u \sqrt{g} \left( \frac{1}{4} R^2 + 1 \right) \tag{6.12}$$

can be made to the action of a Poisson sigma model by considering Cartan coordinates and identifying the three components of the $\eta$ field as $(\eta_1, \eta_2, \eta_3) = (e^1, e^2, \omega)$, with $e^1, e^2$ the zweibein and $\omega$ denotes the connection one-form, so that the torsion two-form is $De^a = \mathrm{d}e^a + \varepsilon^a_b \omega e^b$, where $\varepsilon = e^1 \wedge e^2$ is the volume form. In particular, it can be shown that in these coordinates the action (6.12) can be written as

$$S = \int_\Sigma \left[ X^a De^a + X^3\, \mathrm{d}\omega + \left( \frac{1}{4} - \left(X^3\right)^2 \right) \varepsilon \right]. \tag{6.13}$$

The action (6.1) can then finally be obtained by integrating by parts, with a particular Poisson structure on $\mathbb{R}^3$ given by

$$\Pi^{12} = \frac{1}{4} - \left(X^3\right)^2, \quad \Pi^{23} = X^1, \quad \Pi^{31} = X^2. \tag{6.14}$$

Note that this Poisson structure is invariant under rotations around the $X^3$-axis.

Furthermore, the Poisson sigma model can ba further generalized by introducing a Wess-Zumino term

$$S = \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j + \int_\mathcal{B} \frac{1}{3!}H_{ijk}(X)dy^i \wedge dy^j \wedge dy^k, \tag{6.15}$$

where $\mathcal{B}$ is a 3-dimensional manifold such that $\Sigma = \partial\mathcal{B}$, and $H$ is a closed three-form on $M$ (obviously pulled back on $\mathcal{B}$), and $y$ are the extensions of the fields $X$ on $\mathcal{B}$, i.e. $y_{|_\Sigma} = X$. The model is then called the WZ-Poisson sigma model, or twisted Poisson sigma model, and was first introduced in [138].

An important remark concerns the auxiliary fields $\eta_i$, which encompass conjugate momenta of the configuration fields $X^i$ and Lagrange multipliers. On using the equations of motion they can be integrated away, resulting in a second order action, only if the target space is a symplectic manifold. In this case, in fact, the Poisson bi-vector can be inverted to a symplectic form $\omega$, and the resulting action is that of the so-called A-model [139, 140], with action

$$S = \int_\Sigma \omega_{ij}\, dX^i \wedge dX^j. \tag{6.16}$$

In the language of strings, this corresponds to a topological action with $B$-field coinciding with the symplectic two-form.

### 6.1.1 Dynamical extension of the Poisson sigma model and Seiberg-Witten limit

An interesting property of the Poisson sigma model is that it can be viewed as a limit of the bosonic string in a massless modes $(G, B)$ background, which can be described by the worldsheet action

$$\mathcal{S}_{string} = \frac{1}{4\pi\alpha'} \int_\Sigma \left[g_{\mu\nu}dX^\mu \wedge \star dX^\nu + B_{\mu\nu}dX^\mu \wedge dX^\nu\right]. \tag{6.17}$$

In particular, it can be shown that this action can be obtained by extending the topological model action with an appropriate dynamical term containing both the metric of the target space $G$ and the metric of the source space $h$ (which is hidden in the Hodge star):

$$S_{dyn} = \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j + \pi\alpha' G^{ij}(X)\eta_i \wedge \star\eta_j, \tag{6.18}$$

where $\star$ is the two-dimensional Hodge star operator acting on forms on $\Sigma$. This is actually a first-order reformulation of the bosonic string action. In fact, the second-order one can be obtained after integrating out the auxiliary field $\eta$.

Now let us consider $\Pi$ a generic bivector. Consider the relation

$$(g+B)^{-1} = G + \frac{\Pi}{2\pi\alpha'} \tag{6.19}$$

holds. It is now possible to show that in the Seiberg-Witten limit [141], which consists in taking the limit $\alpha' \to 0$ while keeping $G$ and $\Pi$ fixed, we have

$$g \sim (2\pi\alpha')^2\, \Pi^{-1} G \left(\Pi^T\right)^{-1}, \quad B \sim 2\pi\alpha' \Pi^{-1}, \tag{6.20}$$

where the bivector $\Pi$ is now assumed to be invertible.

By calculating the beta functions, imposing conformal invariance and taking the limit $\alpha' \to 0$ is possible to prove [142] that the condition $dB = 0$ has to be satisfied, which implies, from (6.20), that $d\Pi^{-1} = 0$, implying $\Pi$ is a Poisson bivector, and then the action

$$S_{string} = \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}\eta_i \wedge \eta_j, \tag{6.21}$$

with $\Pi$ a Poisson structure, hence it is the action of a Poisson sigma model, describes the physical string in the particular limit considered. Note also that in this case being $\Pi$ invertible the target manifold is symplectic.

## 6.2 Hamiltonian description of the Poisson sigma model

We now focus on the Hamiltonian approach. Let us choose the topology of the world-sheet as $\Sigma = \mathbb{R} \times [0,1]$ (we are considering open strings), where we identify the local coordinates $(u_0, u_1)$ with time and space, respectively,

$u_0 = t \in \mathbb{R}$, $u_1 = u \in I = [0,1]$. By further denoting $\beta_i = \eta_{ti}$, $\zeta_i = \eta_{ui}$, $\dot{X} = \partial_t X$ and $X' = \partial_u X$, the first order Lagrangian can be written as

$$L(X,\zeta;\beta) = \int_I du \left[ -\zeta_i \dot{X}^i + \beta_i \left( X'^i + \Pi^{ij}(X)\zeta_j \right) \right], \tag{6.22}$$

from which it is clear that $X$ and $-\zeta$ are canonically conjugate variables, with Poisson brackets

$$\{\zeta_i(u), X^j(v)\} = -\delta_i{}^j \delta(u-v), \tag{6.23}$$

while all the other brackets are vanishing.

Notice that, in this notation, the boundary condition (6.5) means that $\beta_{|\partial I} = 0$, $\beta = \eta_t$ being the component of $\eta$ tangent to the boundary.

Since $\beta$ has no conjugate variable, it has to be considered as a Lagrange multiplier imposing the constraints

$$X'^i + \Pi^{ij}(X)\zeta_j = 0, \tag{6.24}$$

from which it follows that the Hamiltonian

$$H_\beta = -\int_I du\, \beta_i \left[ X'^i + \Pi^{ij}(X)\zeta_j \right] \tag{6.25}$$

is pure constraint and the constraint manifold $\mathcal{C}$ (the space of solutions of (6.24)) can also be understood as the common zero set of the functions $H_\beta$ for all $\beta$ satisfying the boundary condition $\beta(0) = \beta(1) = 0$. This implies that the system is invariant under time-diffeomorphisms. The infinitesimal generators are the Hamiltonian vector fields associated with $H_\beta$ by the canonical Poisson bracket (6.23)

$$\xi_\beta = \{H_\beta, \cdot\} = \int du \left( \dot{X}^i \frac{\delta}{\delta X^i} + \dot{\zeta}_i \frac{\delta}{\delta \zeta_i} \right), \tag{6.26}$$

with

$$\dot{X}^i = -\Pi^{ij}\beta_j, \tag{6.27}$$

$$\dot{\zeta}_i = \partial_u \beta_i - \partial_i \Pi^{jk} \zeta_j \beta_k\,, \tag{6.28}$$

where we identified $\partial_u \beta_i \equiv \beta_i{}'$. The model is also invariant under space-diffeomorphisms $f(u)\partial_u$, the latter being the Hamiltonian vector field associated with $H_\beta$ if one chooses $\beta_j = f(u)\zeta_j$ [84]. However, in order for the algebra of generators to close, one has to extend the dependence of $\beta$ according to $\beta(u) \to \beta(u, X(u))$, with $\beta = \beta_i dX^i$ the associated one-form in local coordinates. Then it is possible to check that

$$\{H_\beta, H_{\tilde{\beta}}\} = H_{[\beta,\tilde{\beta}]} \tag{6.29}$$

with

$$[\beta, \tilde{\beta}] = d\langle\beta, \Pi(\tilde{\beta})\rangle - i_{\Pi(\beta)}d\tilde{\beta} + i_{\Pi(\tilde{\beta})}d\beta \tag{6.30}$$

the Koszul bracket among one-forms on the target manifold $M$, which satisfies the Jacobi identity provided $\Pi$ is a Poisson tensor. $\langle\ ,\ \rangle$ denotes the natural pairing between $T^*M$ and $TM$. Following [83], Eq. (6.30) may be extended to $P_0\Omega^1(M)$, the latter being the algebra of continuous maps $\beta : I \to \Omega^1(M)$, with the property $\beta(0) = \beta(1) = 0$, according to

$$[\beta, \tilde{\beta}](u) = [\beta(u), \tilde{\beta}(u)]. \tag{6.31}$$

Eq. (6.29) shows that the map $\beta \to H_\beta$ is a Lie algebra homomorphism, the Hamiltonian constraints are first class and the Hamiltonian vector fields (6.26) generate gauge transformations. Hence, the reduced phase space of the model is defined in the usual way as the quotient $\mathcal{G} = \mathcal{C}/H$, where $H$ is the group of gauge transformations.

### 6.2.1 Reduced phase space and symplectic groupoids

As we showed in the previous section, the reduced phase space of the model is defined as the quotient $\mathcal{G} = \mathcal{C}/H$, where $\mathcal{C}$ is the constraint manifold, i.e. the manifold defined by the constraints (6.24), and $H$ is the group of gauge transformations

$$\delta_\beta X^i = -\Pi^{ij}\beta_j, \tag{6.32}$$

$$\delta_\beta \zeta_i = \beta_i{}' - \partial_i \Pi^{jk}\zeta_j\beta_k\,. \tag{6.33}$$

One important fact proved by Cattaneo and Felder in [83] about the such phase space is that it is finite-dimensional of dimension 2dim$M$, and it has a groupoid structure, and in particular a symplectic groupoid (being $\mathcal{G}$ a manifold). This is particularly meaningful for the quantization of Poisson manifolds. In fact, the program of quantizing a Poisson manifold $M$ briefly relies on the embedding of $M$ as a Lagrangian submanifold of a symplectic manifold $\mathcal{G}$ in such a way that quantization of $\mathcal{G}$ descends to a quantization of $M$ itself. Here the quantization is used in a general fashion, considering deformation quantization, geometric quantization, etc. The manifold $\mathcal{G}$ is supposed to be a symplectic groupoid. In [83] Cattaneo and Felder show that the program of deformation quantization based on symplectic groupoids may work using topological quantum field theory techniques, and in particular the Poisson sigma model.

To prove that the reduced phase space is indeed finite dimensional, one can start working on the tangent space, so vector fields tangent to the phase space have to satisfy the linearization of the constraint equation (6.24). In particular, one can consider $(X, \zeta)$ as representative of an equivalence class of solutions of (6.24) modulo the gauge transformations in (6.32), (6.33). Consider $(\tilde{X}, \tilde{\zeta})$ as the tangent vector to a point in $\mathcal{C}$, then this has to solve the linearized constraint

$$\tilde{X}'^i + A_j{}^i \tilde{X}^j + \Pi^{ij}\tilde{\zeta}_j = 0, \tag{6.34}$$

with the definition of $A_j{}^i(u) = \partial_j \Pi^{ik}(X(u))\zeta_k(u)$. The formal solution of this equation can be written as

$$\tilde{X}^i(u) = V^i_j(u,0)\tilde{X}^j(0) - \int_0^u V^i_j\left(u,u'\right)\Pi^{jk}\left(X\left(u'\right)\right)\tilde{\zeta}_k\left(u'\right)du', \tag{6.35}$$

where $V(u,u') = \hat{P}\exp\left[-\int_u^{u'} A(v)dv\right]$ is the path-ordered exponential of the matrix $A$, and the solution is unique provided $\tilde{X}(0)$ and $\tilde{\zeta}$ are specified. Considering that $X(0)$ is a set of $m = \dim M$ invariants under the action of $H$ since $\beta_{|_I} = 0$ at the boundary, and that by linearizing (6.33) one can find out another set of $m$ invariants $p_j := \int_0^1 \tilde{\zeta}_i(u)\left(V^{-1}(u)\right)^i_j du$, one can conclude (see [84] in particular for this approach) that every infinitesimal perturbation $(\tilde{X}, \tilde{\zeta})$ determines uniquely $2m$ parameters $(X(0), p)$. Viceversa, given $(X(0), p)$, one can choose $\tilde{\zeta}_i(u) = p_j\left(K^{-1}\right)^j_i$, with $K^i_j = \int_0^1\left(V^{-1}(u)\right)^i_j du$, and then compute $\tilde{X}$ using the solution in (6.35), thus concluding that $\dim\mathcal{G} = 2m$.

#### A brief review of Lie algebroids and Symplectic groupoids

Before considering the groupoid structure of the phase space of the model, we first introduce the main concepts of Lie algebroids and Lie groupoids. In particular, it can be proved [143] that the peculiar algebraic structure of the gauge algebra of the Poisson sigma model is not a Lie algebra but a Lie algebroid over the cotangent bundle $T^*M$. For some part of this section we will closely follow [83].

**Definition 6.2.1.** A *Lie algebroid* $(E, M, \rho, [\cdot,\cdot])$ over a manifold $M$ is a vector bundle $E \to M$ with a Lie algebra structure on the space of the sections $\Gamma(E)$

defined by the Lie bracket $[\cdot,\cdot]$ and a bundle map (called the *anchor map*) $\rho : E \to TM$ satisfying the following conditions:
(i) $[\rho(\alpha), \rho(\beta)] = \rho([\alpha, \beta])$;
(ii) $[\alpha, f\beta] = f[\alpha, \beta] + (\rho(\alpha) f)\beta$;
with $\alpha, \beta \in \Gamma(E), f \in C^\infty(M)$.

An obvious example of Lie algebroid is a Lie algebra, which is obtained by replacing $M$ with a single point and trivial anchor $\rho = 0$, while for $M$ a generic manifold and $\rho = 0$ a bundle of Lie algebras is obtained. The tangent bundle $TM$ is also a Lie algebroid with the usual bracket on vector fields and anchor the identity map. Another particularly relevant example for our context is

**Example 6.2.1.** Let $M$ be a Poisson manifold with Poisson structure $\Pi$. The cotangent bundle $E = T^*M$ is a Lie algebroid with anchor the map given by $\Pi^\#$ [1] and bracket defined on exact forms by the fact that a Poisson bracket on a manifold $M$ defines a Lie bracket on the space of one-forms on $M$, i.e. $[df, dg] = d\{f, g\}$, and then it can be extended to any one-form by Leibniz rule.

Just like Lie groups are the global version (or the integration) of Lie algebras, Lie groupoids are the global version of Lie algebroids. Here we will be only interested in an algebraic definition of groupoids:

**Definition 6.2.2.** A *Lie groupoid* over a manifold $M$ is a manifold $\mathcal{G}$ together with a set of structure maps satisfying certain conditions. In particular, an injection $j : M \hookrightarrow \mathcal{G}$, two surjections $l, r : \mathcal{G} \to M$, as well as a composition law $(g, h) \mapsto g \cdot h$ defined only if $r(g) = l(h)$, with $g, h \in \mathcal{G}$. Denote $\mathcal{G}_{x,y} = l^{-1}(x) \cap r^{-1}(y)$, then the set of axioms these maps have to verify is the following:
(i) $l \circ j = r \circ j = \mathbb{1}_M$;
(ii) If $g \in \mathcal{G}_{x,y}$ and $h \in \mathcal{G}_{y,z}$, then $g \cdot h \in \mathcal{G}_{x,z}$;
(iii) $j(x) \cdot g = g \cdot j(y) = g$, if $g \in \mathcal{G}_{x,y}$;
(iv) For every $g \in \mathcal{G}_{x,y}$ there exists $g^{-1} \in \mathcal{G}_{y,x}$ such that $gg^{-1} = j(x)$;
(v) The composition rule on $\mathcal{G}$ is associative, whenever it is defined.

An example of a Lie groupoid is given by considering the base $M$ as a point and $\mathcal{G}$ a Lie group, with $l, r$ trivial and the injection $j$ maps to the identity of the group. Another useful example is the following:

[1] The map $\Pi^\# : T^*M \to TM$ is defined in the usual way as the map $\omega \mapsto \Pi(\omega, \cdot)$, $\omega \in \Omega^1(M)$

**Example 6.2.2.** ($\mathcal{G} = M \times M$). $M \times M$ can be considered a Lie groupoid by defining the left and right maps $l, r$ as the projections onto the left and right components of the product respectively, and taking $j$ as the diagonal map ($j : X \mapsto (X, X)$). The composition is defined by $(g_1, g_2) \cdot (g_2, g_3) = (g_1, g_3)$ while the inverse is defined by $(g_1, g_2)^{-1} = (g_2, g_1)$.

**Definition 6.2.3.** Let $\mathcal{G}$ be a groupoid equipped with a symplectic structure $\omega$ and $M$ a Poisson manifold. The Lie groupoid $\mathcal{G}$ is called the *symplectic groupoid* of the Poisson manifold $M$ if the following axioms are satisfied:

(i) $j(M)$ is a Lagrangian submanifold [2];

(ii) The map $l$ is a Poisson map and $r$ is a anti-Poisson map;

(iii) Let $\mathcal{G}_0 = \{(g, h) \in \mathcal{G} \mid r(g) = l(h)\}$ and $P : \mathcal{G}_0 \subset \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ the product on $\mathcal{G}_0$. Let us denote the projections onto the first and second factor as $\pi_1, \pi_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$. Then $P^*\omega_{\mathcal{G}} = \pi_1^*\omega_{\mathcal{G}} + \pi_2^*\omega_{\mathcal{G}}$;

(iv) The inverse map is anti-Poisson.

The basic and most used example of a symplectic groupoid is the dual of a Lie algebra:

**Example 6.2.3.** ($M = \mathfrak{g}^*$). Let us consider $M = \mathfrak{g}^*$ the dual of a Lie algebra $\mathfrak{g}$ with Kirillov–Kostant-Souriau Poisson structure. For any Lie group integrating the Lie algebra $\mathfrak{g}$ we can take $j : \mathfrak{g}^* \to T^*G$ as the natural inclusion at the cotangent space at the identity and then it is also natural to define the projections $l, r$ as the left and right (respectively) translations to the cotangent space at the identity. Let us pick two pairs $(g_1, \xi_1)$ and $(g_2, \xi_2)$ such that $r\,(g_1, \xi_1) = l\,(g_2, \xi_2)$. The multiplication law is given by $(g, \xi) = (g_1, \xi_1) \cdot (g_1, \xi_1)$ with $g = g_1 g_2$ and $\xi = (\mathrm{d}R_h(g)^*)^{-1}\xi_1 = \left(\mathrm{d}L_g(h)^*\right)^{-1}\xi_2$ where $L$ and $R$ denotes the left and right translation as usual.

### The symplectic groupoid structure of the phase space

As showed in [83], the groupoid structure of the phase space $\mathcal{G} = \mathcal{C}/H$ is defined via composition of paths. The inclusion map $j : M \hookrightarrow \mathcal{G}$ is defined by taking a point $x \in M$ and sending it to the equivalence class of the constant solution $(X(u) = x, \zeta(u) = 0)$, and the maps $l, r$ give the values of $X$ at the endpoints, which is a good choice since the boundary values are invariant under the action of the gauge group $H$, so these maps descend to $\mathcal{G}$.

It is possible to prove [83] that in each equivalence class $[(X, \zeta)]$ in $\mathcal{G}$ there exists a representative with $\zeta(0) = \zeta(1) = 0$. Then, the composition law

[2] A Lagrangian submanifold is a maximally isotropic submanifold, see Sec. 2.2.

$[(X_3,\zeta_3)] = [(X_1,\zeta_1)] \cdot [(X_2,\zeta_2)]$ is given by choosing such representatives and composing like

$$X_3(\sigma) = \begin{cases} X_1(2u), & 0 \le u \le \frac{1}{2} \\ X_2(2u-1), & \frac{1}{2} \le u \le 1 \end{cases}$$
$$\zeta_3(\sigma) = \begin{cases} 2\zeta_1(2u), & 0 \le u \le \frac{1}{2} \\ 2\zeta_2(2u-1), & \frac{1}{2} \le u \le 1 \end{cases}$$

as long as $X_1(1) = X_2(0)$. Having a representative with $\zeta(0) = \zeta(1) = 0$ is necessary to avoid a singularity at $u = 1/2$. In this way we have met all the conditions to define a Lie groupoid. In fact, we have a partial product $\cdot : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$, where $\mathcal{G} \times_M \mathcal{G}$ is the space of pairs of classes of solutions with the endpoint of the first coinciding with the starting point of the second ($X_1(1) = X_2(0)$). We can see that this product is associative since the two possible ways of combining three solutions are related by a reparametrization of the interval $[0,1]$, which can be obtained by using a gauge transformation. We have already defined inclusion and left/right projections, in fact we know that we have a very peculiar solution: $X(u) \equiv x$ and $\zeta(u) \equiv 0$ playing the role of a unit at $x$ for the product $\cdot$. The last ingredient is the inverse: for every solution $(X,\zeta)$, we have an inverse solution $(\bar{X},\bar{\zeta})$, simply given by the usual inverse for paths: $\bar{X}(u) = X(1-u), \bar{\zeta}(u) = -\zeta(1-u)$.

To see that this is also a symplectic groupoid, it is necessary to look at the boundary values of $X$. We have already seen that these are invariant under gauge transformations since $\beta$ vanishes at the boundary and we can define $x = X(0)$ and $y = X(1)$. Using the Poisson bracket induced by the symplectic structure on $\mathcal{G}$ (see (6.23)), it is possible to obtain

$$\left\{x^i, x^j\right\} = \Pi^{ij}(x), \quad \left\{y^i, y^j\right\} = -\Pi^{ij}(y), \quad \left\{x^i, y^j\right\} = 0. \tag{6.36}$$

The problem is that the boundary functions have no well-defined associated Hamiltonian vector fields. This is due to the fact that the Hamiltonian vanishes for $\beta = 0$. However, one can use a regularized version of boundary values by using the constraint equation (6.24):

$$x^i(u) = X^i(u) + \int_0^u du' \, \Pi^{ij}(X(u'))\eta_j(u'), \tag{6.37}$$

$$y^i(u) = X^i(u) - \int_u^1 du' \, \Pi^{ij}(X(u'))\eta_j(u'), \tag{6.38}$$

which coincides with the original definition on the constraint manifold. Now

Hamiltonian vector fields associated to these functions are well-defined and the Poisson bracket can be computed, resulting in

$$\{x^i, x^j\} = \Pi^{ij}(x), \quad \{y^i, y^j\} = -\Pi^{ij}(y), \quad \{x^i, y^j\} = 0. \tag{6.39}$$

Here we check that we have a "noncommutativity" of the endpoints, and we have some ingredients needed for our definition, in particular it is manifest that the $l$ and $r$ maps previously defined are Poisson and anti-Poisson maps respectively. We also have the fact that $\{(a, b, c) \in G \times_M G \times G : a \cdot b = c\}$ is a Lagrangian submanifold of $G \times G \times \bar{G}$, where $\bar{G}$ is the same as $G$ as a manifold, but has opposite symplectic structure.

# 7 Jacobi geometry

Jacobi manifolds were introduced by Kirillov [144] and Lichnerowicz [100], and have recently seen a rise in popularity, both as a mathematical subject and due to its applications to physics and other sciences. In particular, it has been recently developed for the application on the integrability of Jacobi manifolds by contact groupoids [145], dissipative Liouville's theorem in contact manifolds [146], thermodynamics [147] (see [148] for an application to black holes in AdS spacetimes) and providing a natural framework for the dynamical formulation of mechanical systems subject to time-dependent forces and dissipative effects [149, 150]. It has also been considered for applications in neurosciences [151]. In this section we will introduce the main properties of Jacobi structures and Jacobi manifolds, as well as the concept of contact and locally conformal symplectic manifolds, providing for each simple but useful examples. The concepts introduced in this section will cover the background used for the introduction of the Jacobi sigma model later in this thesis.

The presentation of the material in this section will follow closely the ones in [88, 89] apart from Section 7.1.1 which is not present in the published works.

## 7.1 Jacobi structure and Jacobi manifolds

In this section we review the main properties of Jacobi structures and Jacobi manifolds. This section is mainly based on the papers [92, 152].

Jacobi structures were first introduced indipendently by Kirillov [144] and Lichnerowicz [100] and are defined as the natural generalization of Poisson structures. In fact, a Jacobi bracket can be considered as the most general skew-symmetric local bi-linear differential operator acting on the algebra of functions on a manifold $M$, $C^\infty(M)$, satisfying Jacobi identity. Leibniz rule is then lost, differently from the case of Poisson brackets. A Jacobi structure can be defined by a pair of bi-vector and vector fields on the manifold $M$, satisfying certain conditions:

**Definition 7.1.1.** Let $\Lambda \in \Gamma(\Lambda^2 TM)$ be a bi-vector field and $E \in \mathfrak{X}(M)$ a vector field, called the *Reeb vector field*. The pair $(\Lambda, E)$ is called a *Jacobi structure* if the following relations are satisfied

$$[\Lambda, \Lambda]_S = 2E \wedge \Lambda, \quad [\Lambda, E]_S = \mathcal{L}_E \Lambda = 0, \tag{7.1}$$

where $\mathcal{L}$ denotes the Lie derivative operator and $[,]_S$ denotes the Schouten-Nijenhuis bracket on the algebra of multi-vector fields on the manifold $M$. *Jacobi brackets* are then defined as

$$\{f, g\}_J = \Lambda(df, dg) + f(Eg) - g(Ef). \tag{7.2}$$

It will prove useful to write the explicit expression of these relations in coordinates:

$$\Lambda^{pi}\partial_p \Lambda^{jk} + \mathrm{cycl\,perm}\{ijk\} = E^i \Lambda^{jk} + \mathrm{cycl\,perm}\{ijk\}, \tag{7.3}$$

$$E^k \partial_k \Lambda^{ij} - \Lambda^{kj}\partial_k E^i - \Lambda^{ik}\partial_k E^j = 0. \tag{7.4}$$

Jacobi brackets are linear, skew-symmetric and satisfy Jacobi identity just like Poisson brackets, but in general do not satisfy Leibniz rule, which is instead replaced by the condition

$$\{f, gh\}_J = \{f, g\}_J h + g\{f, h\}_J + gh(Ef). \tag{7.5}$$

This means that the Jacobi brackets still endow the algebra of functions on $M$, $\mathcal{F}(M)$, with the structure of a Lie algebra, but, unlike the Poisson bracket, they are not a derivation of the point-wise product among functions. Furthermore, it also satisfies the condition support $\{f, g\} \subseteq (\text{support } f) \cap (\text{support } g)$. All these properties make the space of functions $C^\infty(M)$, when endowed with a Jacobi bracket, a local Lie algebra [1] and, conversely, a local Lie algebra structure on $C^\infty(M)$ defines a Jacobi bracket on $M$.

Obviously, Jacobi brackets can be considered as a generalization of Poisson brackets since the latter can be obtained from the former if the Reeb vector field identically vanishes, $E = 0$.

**Definition 7.1.2.** A *Jacobi manifold* $(M, \Lambda, E)$ is defined as a smooth manifold equipped with a Jacobi structure. When $E = 0$, $(M, \Lambda)$ is a Poisson manifold.

[1] In the sense of Kirillov [144].

Fundamental examples of Jacobi manifolds are *locally conformal symplectic manifolds* (LCS) and *contact manifolds*. The former ones are even-dimensional manifolds endowed with a non-degenerate two-form $\omega \in \Omega^2(M)$ with the property that for all $x \in M$ there exists an open neighborhood $U \ni x$ and a function $f \in C^\infty(U)$ such that $(U, e^{-f}\omega)$ is a symplectic manifold, i.e. such that $d(e^{-f}\omega) = 0$. The global structure, however, is that of a Jacobi manifold. If $U = M$, the manifold is then called *globally conformal symplectic* (GCS). More explicitly [153], a LCS is characterized by the existence of a non-degenerate two-form $\omega \in \Omega^2(M)$ and a closed exact one-form $\alpha \in \Omega^1(M)$ called *Lee one-form*, such that

$$d\omega + \alpha \wedge \omega = 0. \tag{7.6}$$

Let $E$ and $\Lambda$ be the unique vector and bi-vector fields respectively satisfying

$$\iota_E \omega = -\alpha, \quad \iota_{\Lambda(\gamma)}\omega = -\gamma \quad \forall \gamma \in T^*M, \tag{7.7}$$

then

**Definition 7.1.3.** The triple $(M, \Lambda, E)$ with $\Lambda \in \Gamma^2(TM)$ and $E \in \mathfrak{X}(M)$ satisfying (7.6) and (7.7) is a Jacobi manifold and it is called a locally conformal symplectic manifold.

Note also that

$$\iota_E \alpha = 0, \quad \mathcal{L}_E \alpha = 0, \quad \mathcal{L}_E \omega = 0. \tag{7.8}$$

Of course symplectic manifolds are particular cases of LCS with vanishing Lee one-form, $\alpha = 0$. Using Darboux theorem, around every point there exist local coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ and a local differentiable function $f$ for which it can be written

$$\omega = e^f dx^i \wedge dy_i, \quad \alpha = df = \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y_i} dy_i\right) \tag{7.9}$$

and the Jacobi structure can be written as

$$\Lambda = e^{-f}\left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_i}\right), \quad E = e^{-f}\left(\frac{\partial f}{\partial y_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y_i}\right). \tag{7.10}$$

Contact manifolds are instead odd-dimensional manifolds $(M, \vartheta)$ which are endowed with a contact form (or contact structure), i.e. a one-form $\vartheta$ such that $\vartheta \wedge (d\vartheta)^n$ nowhere vanishes, where $2n+1$ is the dimension of the manifold. The contact form is defined as an equivalence class of one-forms up to multiplication by a non-vanishing function (gauge transformation). It is possible to endow the algebra of functions on a contact manifold with a Lie algebra structure [154], which reads

$$[f,g]\vartheta \wedge (d\vartheta)^n := (n-1)df \wedge dg \wedge \vartheta \wedge (d\vartheta)^{n-1} + (fdg - gdf) \wedge (d\vartheta)^n. \quad (7.11)$$

The latter is local by construction and satisfies Jacobi identity. This one may be taken as a definition of Jacobi bracket, with the usual definition recovered by defining $\Lambda$ and $E$ as follows:

$$\begin{aligned} \iota_E \left(\vartheta \wedge (d\vartheta)^n\right) &= (d\vartheta)^n \\ \iota_\Lambda \left(\vartheta \wedge (d\vartheta)^n\right) &= n\vartheta \wedge (d\vartheta)^{n-1}. \end{aligned} \quad (7.12)$$

The latter conditions imply that

$$\iota_E \vartheta = 1, \quad \iota_E d\vartheta = 0, \quad (7.13)$$

and

$$\iota_\Lambda \vartheta = 0, \quad \iota_\Lambda d\vartheta = 1. \quad (7.14)$$

A contact structure should be thought of as the equation $\vartheta = 0$, which picks out a subspace of the tangent space at each point of the manifold. However, the Frobenius integrability condition for these to fit together as tangent spaces of some submanifold is maximally violated. A submanifold all of whose the tangent vectors will satisfy $\vartheta = 0$ is said to be integral, the largest dimension for such a submanifold being $n$. Such a maximal integral submanifold is called a Legendre submanifold.

There is an analogue of Darboux's theorem which can be considered for contact manifolds, stating that there is a local coordinate system $(z, x^1, \dots, x^n, y_1, \dots, y_n)$ in which the contact form can be written as

$$\theta = dz - y_i dx^i, \quad (7.15)$$

while the Jacobi structure is given by

$$\Lambda = \left(\frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial}\right) \wedge \frac{\partial}{\partial y_i}, \quad E = \frac{\partial}{\partial z}. \quad (7.16)$$

Note that $\mathbb{R}^{2n+1}$ with this choice is the most basic example of a contact manifold.

Other important examples of contact manifolds are three-dimensional semisimple Lie groups, for which the contact form can be taken as one of the basis (left or right) invariant one-forms. Obiously, we are interested in particular to the Lie group $SU(2)$, whose associated sigma models have been widely studied [155, 156], and where there might be some place for eventual application of the duality approach we discussed in the first part of this thesis.

It is important to remark that it is also possible to define Hamiltonian vector fields associated to the Jacobi structure analogously to the case of Poisson structures. Indeed, we can say that for any real-valued function $f \in C^\infty(M)$ one has an associated Hamiltonian vector field $\xi_f$ defined as (see for example [152]):

$$\xi_f = \Lambda(df, \cdot) + fE. \tag{7.17}$$

Note that the Hamiltonian vector field associated with the constant function 1 is just the Reeb vector field $E$ itself. The map $f \to \xi_f$ is homomorphism of Lie algebras, it being $[\xi_f, \xi_g] = \xi_{\{f,g\}_J}$, where the bracket $[\cdot, \cdot]$ is the standard Lie bracket of vector fields.

If one considers, for all $x \in M$, the subspace of $T_xM$ spanned by all the Hamiltonian vector fields at the point $x$, it is possible to show that [153] this defines a generalized foliation, called characteristic foliation, and the Jacobi structure of $M$ induces a Jacobi structure on each leaf of the foliation. For example, if $L$ is the leaf over the point $x \in M$, for $E_x \notin \mathrm{Im}(\Lambda_x)$ it turns out $L$ is a contact manifold, if $E_x \in \mathrm{Im}(\Lambda_x)$ then $L$ is a LCS. In particular, it can be shown [153] that even-dimensional leaves are LCS, while odd-dimensional ones are contact.

In the light of the results of the paper, it is useful to introduce the concept of regularity of Jacobi manifolds:

**Definition 7.1.4.** A Jacobi manifold $(M, \Lambda, E)$ is said to be *regular* if the Reeb vector field $E$ is complete, $E \neq 0$ at every point of $M$ and the foliation defined by $E$ is regular.

The importance of this concept is related to the fact that in this case the space of leaves $\tilde{M} = M/E$ is a differentiable manifold and the canonical projection $\pi : M \to \tilde{M}$ is a fibration. It is also possible to define the bi-vector $\tilde{\Lambda}$ on $\tilde{M}$ as

$$\tilde{\Lambda}(\alpha, \beta) \circ \pi = \Lambda(\pi^*\alpha, \pi^*\beta), \quad \forall\, \alpha, \beta \in \Omega^1(\tilde{M}) \tag{7.18}$$

which makes $(\tilde{M}, \tilde{\Lambda})$ into a Poisson manifold.

An important result concerning the relation between Jacobi and Poisson manifolds is given by the following theorem [100]:

**Theorem 7.1.1.** Given a Jacobi structure $(\Lambda, E)$ on $M$, the product manifold $M \times \mathbb{R}$ carries a Poisson structure with a Poisson bi-vector $P$ defined as

$$P \equiv e^{-\tau}\left(\Lambda + \frac{\partial}{\partial\tau} \wedge E\right), \tag{7.19}$$

where $\tau \in \mathbb{R}$. This procedure is called the *Poissonization* of the Jacobi structure $(\Lambda, E)$ on $M$.

When the Jacobi structure is contact, the Poissonization is a *Symplectification* as the resulting manifold $M \times \mathbb{R}$ is a Symplectic manifold since $P$ is non-degenerate and it defines a symplectic form. Indeed, in such a case, one can define a closed two-form $\omega$ on $M \times \mathbb{R}$ in terms of the contact one-form $\vartheta : \omega = d\left(e^{\tau}\pi^*\vartheta\right) = e^{\tau}\left(d\tau \wedge \pi^*\theta + d\pi^*\theta\right)$, where $\pi : M \times \mathbb{R} \to M$ is the projection map. By using the defining properties of the contact form, it is possible to check that $\omega$ is non-degenerate, hence symplectic.

While this theorem provides a simple recipe to obtain a Poisson structure from a Jacobi structure in a lower dimensional manifold, it is not trivial to construct a Jacobi structure on the Jacobi manifold $M$ itself.

This theorem is particularly useful to obtain results on the Jacobi manifolds by using well known concepts and results from Poisson geometry. For instance, the result $[\xi_f, \xi_g] = \xi_{\{f,g\}_J}$ can be obtained directly using this theorem from the corresponding notion on Poisson manifolds by considering the Poisson-Hamiltonian vector fields corresponding to $P$ and then projecting on $M$, as follows:

$$\xi_f := pr(\xi^P_{e^{\tau}f})|_{\tau=0}, \tag{7.20}$$

where $\xi^P_{e^{\tau}f}$ is the Poisson-Hamiltonian vector field on $M \times \mathbb{R}$ and $pr : TM \times \mathbb{R} \to TM$ is the projection map.

This approach was followed in [88] to give a formulation of the Jacobi sigma model on $M$ starting from the Poisson sigma model on $M \times \mathbb{R}$, considering the immersion $j : M \hookrightarrow M \times \mathbb{R}$ through the identification of $M$ with $M \times \{0\}$.

### 7.1.1 Explicit examples of Jacobi manifolds

We already discussed of $\mathbb{R}^{2n+1}$ with Jacobi structure given by (7.16) as a basic example of contact manifold, as well as of symplectic manifolds as basic examples of LCS manifolds. In this section we will consider $\mathbb{R}^3$ more explicitly, and a few nontrivial examples. In particular, noteworthy examples of contact manifolds are represented by 3-dimensional semi-simple Lie groups, where the contact one-form can be chosen to be one of the basis left-invariant (resp. right-invariant) one-forms on the group manifolds. Non-trivial examples of LCS manifolds may be instead easily constructed by considering the product $M \times S^1$, with $M$ a contact manifold [157].

- Contact manifold examples

**Example 7.1.1.** ($M = \mathbb{R}^3$). $\mathbb{R}^3$ is a contact manifold with contact one-form

$$\vartheta = dx^3 - \frac{1}{2}\left(x^2 dx^1 - x^1 dx^2\right) \tag{7.21}$$

which satisfies $d\vartheta \wedge \vartheta = dx^1 \wedge dx^2 \wedge dx^3$, which is the volume form on $\mathbb{R}^3$. For the applications which will be considered in Sec. 8.2 it is convenient to work with an adapted basis of one-forms

$$\theta^1 = dx^1, \quad \theta^2 = dx^2, \quad \theta^3 = \vartheta = dx^3 - \frac{1}{2}\left(x^2 dx^1 - x^1 dx^2\right) \tag{7.22}$$

and dual vector fields

$$Y_1 = \frac{\partial}{\partial x^1} + \frac{1}{2}x^2\frac{\partial}{\partial x^3}, \quad Y_2 = \frac{\partial}{\partial x^2} - \frac{1}{2}x^1\frac{\partial}{\partial x^3}, \quad Y_3 = \frac{\partial}{\partial x^3}, \tag{7.23}$$

satisfying

$$[Y_1, Y_2] = -Y_3, \quad [Y_a, Y_3] = 0, \quad \theta^i\left(Y_j\right) = \delta^i_j. \tag{7.24}$$

Thus, a Jacobi bracket may be defined through the following structures:

$$\begin{aligned} E &= Y_3 \\ \Lambda &= Y_1 \wedge Y_2, \end{aligned} \tag{7.25}$$

satisfying (7.13) and (7.14).

**Example 7.1.2.** ($M = SU(2)$). It is possible to endow the group manifold of $SU(2)$ with a contact structure [2], provided by one of the left-invariant (resp. right-invariant) one-forms of the group, say $\theta^i$ defined by the Maurer-Cartan one-form $\ell^{-1}d\ell = \theta^i e_i \in \Omega^1(SU(2), \mathfrak{su}(2))$, with $\ell \in SU(2), e_i = i\sigma_i/2$ the Lie algebra generators and $\sigma_i$ the Pauli matrices. Let us choose, to be definite, the contact one-form to be $\vartheta = \theta^3$.

[2] Indeed, this procedure can be adapted to any 3-dimensional semisimple Lie group.

The latter defines a Jacobi bracket according to (7.11), it being

$$\vartheta \wedge d\vartheta = \Omega \tag{7.26}$$

with $\Omega = \theta^1 \wedge \theta^2 \wedge \theta^3$ the volume form on the group manifold. Therefore, the Reeb vector field $E$ and the bivector field $\Lambda$ are easily determined by solving the equations (7.13) and (7.14). In particular, we obtain

$$E = Y_3 \quad \Lambda = Y_1 \wedge Y_2 \tag{7.27}$$

with $Y_i, i = 1, .., 3$ the left-invariant vector fields on the group manifold, which are dual to the one-forms $\theta^i$ by definition. Hence, the Reeb vector field is constant and orthogonal to the distribution spanned by the bivector field $\Lambda$.

- LCS manifold examples
  Examples of LCS manifolds may be built, according to [157], in the following way. The starting point is a contact manifold $\left(M^{2n-1}, \theta\right), n \geq 2$, with contact form $\vartheta$. The manifold $\left(S^1 \times M^{2n-1}, \omega\right)$ is LCS with non degenerate two-form $\omega$ given by

$$\omega = \vartheta \wedge \alpha + d\vartheta \tag{7.28}$$

with $\alpha \in \Omega^1\left(S^1\right)$ the volume form on the circle. Therefore, both $\mathbb{R}^3$ and $S^3$ as contact manifolds are suitable examples of LCS manifolds, when multiplied by the circle $S^1$. In particular, we can consider the product $S^1 \times S^3$, with $S^3$ the contact manifold associated with the group $SU(2)$ previously described. The Jacobi structure $(\Lambda, E)$ can be worked out, yielding

$$\Lambda = \omega^{-1}, \quad E = \Lambda(\alpha) \tag{7.29}$$

which, in local coordinates for the circle $S^1$, with $\alpha = d\phi$ becomes

$$\Lambda = Y_3 \wedge \partial\phi - Y_1 \wedge Y_2, \quad E = -Y_3 \tag{7.30}$$

According to [157], as a generalization of the latter, one can consider principal bundles $\left(P, M^{2n-1}, U(1)\right)$ with basis the contact manifold $M^{2n-1}$ and structure group $U(1)$. $P$ may then be endowed with the LCS structure (7.28), where $\alpha$ is the volume form of the structure group $U(1)$ and $\vartheta$ a $U(1)$ connection. If the curvature of the connection $\psi = d\vartheta$ is such that $\alpha \wedge \vartheta \wedge (\psi)^{n-1} \neq 0$ (namely it defines a volume form on $P$), then $\omega$ is a LCS which is not globally conformal symplectic.

# 8 Jacobi sigma model

In this chapter we will introduce a non-linear sigma model with target space a Jacobi manifold, which we called the *Jacobi sigma model* [88, 89], as a natural generalization of Poisson sigma models. The main motivation for the search for a consistent definition of a sigma model with target space a Jacobi manifold is certainly the fact that it represents a natural, non-trivial generalization of the well-known Poisson sigma model. As we have seen in Chapter 6, one interesting feature of the Poisson sigma model is its intimate relation with the geometry of the target space. An example of this relation is the fact that the reduced phase space of the Poisson sigma model is actually the symplectic groupoid integrating the Lie algebroid associated with the Poisson structure of the target manifold, as shown by Cattaneo and Felder in [83, 84]. Moreover, in [85] it was shown that the Kontsevich quantization formula for Poisson manifolds can be described in terms of the Feynman diagrams from the perturbative expansion of the Poisson sigma model as a field theory. This is important since terms in Kontsevich formula can now be given a physical interpretation. It might be that similar situations can be addressed once the Jacobi sigma model is well understood. Furthermore, another motivation for the introduction of this new model is the perspective of applying techniques from topological quantum field theory to the analysis of new string backgrounds, as well as the possibility of obtaining some useful description of known models within the framework of Jacobi manifolds, as is the case for the Poisson setting.

More specifically, our aim is to investigate the possibility of relaxing the condition $[\Pi, \Pi]_S = 0$ to what is a natural generalization, represented by a Jacobi structure, which is specified not only by a bivector field $\Lambda$ but also by a vector field $E$, the so called Reeb vector field, satisfying

$$[\Lambda, \Lambda]_S = 2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda]_S = 0, \tag{8.1}$$

and then the triple $(M, \Lambda, E)$ defines a Jacobi manifold, as we have seen in Chapter 7. Two main families of Jacobi manifolds, with all other cases being recovered as intermediate situations[1], are represented by contact and locally conformal symplectic manifolds, which we will consider for applications of our model.

Jacobi brackets on the algebra of functions on $M$ can be defined from the Jacobi structure, satisfying the Jacobi identity, but unlike Poisson brackets, violate the Leibniz rule; in other words, the Jacobi bracket still endows the algebra of functions on $M$ with a Lie algebra structure, but it is not a derivation of the point-wise product among functions.

The Jacobi sigma model has the purpose to generalize the Poisson sigma model via the inclusion of an additional field on the source manifold, which is necessary in order to take into account the new background vector field $E$. The field variables of the model are represented by $(X, \eta, \lambda)$, where $X : \Sigma \to M$ is the usual embedding map, while $(\eta, \lambda)$ are put together to give elements of $\Omega^1(\Sigma, X^*(T^*M \oplus \mathbb{R}))$, being $T^*M \oplus \mathbb{R} = J^1M$ the vector bundle of 1-jets of real functions on $M$. The resulting theory is a two-dimensional topological non-linear gauge theory describing strings sweeping a Jacobi manifold. The main results were achieved in [88] and discussed in a more complete and extended way in [89].

The material covered in this chapter is entirely contained in the papers [88] and [89].

## 8.1 The Jacobi sigma model

In this section, we shall analyze the Jacobi sigma model, first introduced in [88] (also see [93]) as a generalization of the Poisson sigma model. Although the defining action functional may be justified in terms of a Poissonization of the target Jacobi manifold and further reduction of the correspondent Poisson sigma model living on $M \times \mathbb{R}$, it has been shown in [89] that an independent formulation can be given. We shall adhere to the latter approach in this thesis. The coordinate-independent formulation can be given by the following

**Definition 8.1.1.** Let $(M, \Lambda, E)$ be a n-dimensional Jacobi manifold. The Jacobi sigma model with source space a two-dimensional manifold $\Sigma$ with

[1] It is possible to show (see for example [152] Thm. 11) that a generic Jacobi manifold admits a foliation by locally conformal symplectic and/or contact leaves. Examples of Jacobi manifolds with "nonpure" characteristic foliation, namely with leaves of odd and even dimension, i.e., contact and l.c.s. leaves may be found in [92].

boundary $\partial\Sigma$ and target space $M$ is defined by the action functional

$$S[X,(\eta,\lambda)] = \int_\Sigma \langle \eta, (dX) \rangle + \frac{1}{2}\langle \eta, (\Lambda \circ X)\eta \rangle + \lambda \wedge (E \circ X)\eta \tag{8.2}$$

with boundary condition $\eta(u)v = 0, u \in \partial\Sigma, v \in T(\partial\Sigma)$.

The field configurations are represented by $X, (\eta, \lambda)$ with $X : \Sigma \to M$ the base map and $(\eta, \lambda) \in \Omega^1(\Sigma, X^*(J^1 M))$, where $J^1 M = T^*M \oplus \mathbb{R}$ is the 1-jet bundle of real functions on $M$.

Sections of the latter are isomorphic as a $C^\infty(M)$-module to the algebra of one-forms [158]

$$\Gamma_0(M) := \{e^\tau(\alpha + f d\tau) | \alpha \in \Omega^1(M), f \in C^\infty(M), \tau \in \mathbb{R}\} \subseteq \Omega^1(M \times \mathbb{R}) \tag{8.3}$$

which is closed with respect to the Koszul bracket of the Poissonized manifold. The map $\langle \ , \ \rangle$ establishes a pairing between differential forms on $\Sigma$ with values in the pull-back $X^*(T^*M)$ and differential forms on $\Sigma$ with values in $X^*(TM)$. It is induced by the natural pairing between $T^*M$ and $TM$ and yields in this case a two-form on $\Sigma$. Then the action may be rewritten as

$$S(X, \eta, \lambda) = \int_\Sigma \left[ \eta_i \wedge dX^i + \frac{1}{2}\Lambda^{ij}(X)\eta_i \wedge \eta_j - E^i(X)\eta_i \wedge \lambda \right]. \tag{8.4}$$

On comparing with the action of the Poisson sigma model (6.1) one important difference is the presence of a new auxiliary field, $\lambda$, which, loosely speaking, is a one-form on the source manifold $\Sigma$ but a scalar on the Jacobi manifold. This is a consequence of the fact that the Jacobi bracket is expressed in terms of a bi-differential operator, not a bivector field. Therefore $\lambda$ is needed in order to take into account the presence of the Reeb vector field $E$.

The variation of the action, together with the boundary condition for $\eta$ in Def. 8.1.1, gives the following equations of motion

$$dX^i + \Lambda^{ij}\eta_j - E^i \lambda = 0, \tag{8.5}$$

$$d\eta_i + \frac{1}{2}\partial_i \Lambda^{jk}\eta_j \wedge \eta_k - \partial_i E^j \eta_j \wedge \lambda = 0, \tag{8.6}$$

$$E^i \eta_i = 0. \tag{8.7}$$

The boundary condition for $\eta$ ensures the vanishing of boundary terms. Consistency of the three yields another dynamical equation. In fact, on applying the exterior derivative to Eq. (8.5) we obtain

$$\partial_k \Lambda^{ij} dX^k \wedge \eta_j + \Lambda^{ij} d\eta_j - \partial_k E^i dX^k \wedge \lambda - E^i d\lambda = 0. \tag{8.8}$$

By substituting Eqs. (8.5)-(8.7) and by using the properties of a Jacobi structure, Eqs. (7.1), we finally get

$$d\lambda = \frac{1}{2} \Lambda^{ij} \eta_i \wedge \eta_j. \tag{8.9}$$

### 8.1.1 Canonical formulation of the model

In this section we will focus on the Hamiltonian formulation of the model, in close analogy with the procedure followed for the Poisson sigma model in Sec. 6.2.

The source manifold is chosen to be $\Sigma = \mathbb{R} \times [0,1]$, with local coordinates $t \in \mathbb{R}$, $u \in [0,1]$. Moreover, by explicitly indicating the time and space components, the one-forms $dX, \eta$ and $\lambda$ shall be locally represented as $dX = \dot{X} dt + X' du$, $\eta = \beta dt + \zeta du$, $\lambda = \lambda_t dt + \lambda_u du$, with $\lambda_t, \lambda_u$ scalar fields, while $\dot{X}, X'$ and $\beta, \zeta$ carrying and extra index on (the pull-back of) the target manifold $M$. Note that the boundary condition in definition 8.1.1 results in $\beta_{\partial\Sigma} = 0$, just like for the Poisson sigma model, while there is no boundary condition for $\lambda$ deriving from the variation of the action. We shall discuss this issue later.

With the notation chosen, the Lagrangian of the model acquires the form

$$L = \int_I du \left[ -\dot{X}^i \zeta_i + \beta_i \left( X'^i + \Lambda^{ij} \zeta_j - E^i \lambda_u \right) + \lambda_t \left( E^i \zeta_i \right) \right], \tag{8.10}$$

with equations of motion

$$\begin{aligned} \dot{X}^i &= -\Lambda^{ij} \beta_j + E^i \lambda_t \\ \dot{\zeta}_i &= \beta_i' - \partial_i \Lambda^{jk} \beta_j \zeta_k - \partial_i E^j \zeta_j \lambda_t + \partial_i E^j \beta_j \lambda_u, \end{aligned} \tag{8.11}$$

$$\begin{aligned} X'^i + \Lambda^{ij} \zeta_j - E^i \lambda_u &= 0 \\ E^i \zeta_i &= 0 \\ E^i \beta_i &= 0. \end{aligned} \tag{8.12}$$

The evolutionary equations are, therefore, represented by Eqs. (8.11), involving time derivatives, while Eqs. (8.12) represent constraints. In the following we perform a detailed analysis of the emergence and nature of constraints in the Hamiltonian approach.

### Dirac analysis of constraints

From the Lagrangian (8.10) the Hamiltonian is seen to be

$$H_0 = -\int_I du\, \beta_i \left( X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u \right) + \lambda_t \left( E^i \zeta_i \right), \tag{8.13}$$

with $\pi_i = \delta L / \delta \dot{X}^i = -\zeta_i$ the conjugate momenta for the field $X^i$, while the conjugate momenta of all other fields are zero. The theory is therefore constrained. We shall perform the analysis à la Dirac, and we provide a very short and brief review of the procedure, but we refer to standard textbooks for a detailed description of the procedure.

**Very brief review on Dirac theory of constraints**

Shortly, we recall that primary constraints are those which emerge from the Lagrangian, without using the equations of motion. They identify a submanifold of the original carrier space of the dynamics, $\mathcal{C}_1 \subset \mathcal{C}_0$. Secondary constraints are all subsequent constraints, obtained by the request that primary constraints be preserved along the motion. They further constrain the motion to some submanifold $\mathcal{C}_2 \subset \mathcal{C}_1$. The process is iterated by imposing conservation of new constraints (tertiary, $\cdots$, n-ary, $\cdots$) at each step, until the true manifold of the motion, $\mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \cdots \subset \mathcal{C}_0$, is found. The term "secondary constraints" is then used for all, except for primary constraints.
Dirac classification of constraints is yet another one, which is specific of the Hamiltonian setting [159]. Here the carrier space of the dynamics is the phase space, endowed with a Poisson bracket. At each step of the reduction from the unconstrained phase space $\mathcal{C}_0$, the so called naive Hamiltonian $H_0$ is replaced by a new one, say $H_i = H_0 + a_\mu \chi_\mu + b_\mu \mathcal{G}_\mu$, with $\{\chi_\mu\}$ the primary constraints, and $\{\mathcal{G}_\mu\}$ the secondary constraints which have emerged up to the step $i$. The parameters $a_\mu, b_\mu$ are also referred to as Lagrange multipliers. The process ends when all constraints, say $\psi_\mu$, are conserved, namely $\dot{\psi}_\mu = \{\psi_\mu, H_n\} \simeq 0$ on the constrained manifold $\mathcal{C}_n$. On considering the Poisson algebra of all constraints, first class (primary and secondary) are those which close a subalgebra, i.e. $\{\psi_\mu, \psi_\nu\} = f^{\kappa}_{\mu\nu}\psi_k \simeq 0$, whereas second class constraints obey $\{\psi_\mu, \psi_\nu\} = c_{\mu\nu}$, with $c_{\mu\nu}$ a non-degenerate matrix (second class constraints are therefore in even number). Because of that, their Lagrange multipliers, say $d_\mu$, may be completely determined according to $d_\mu = -c_{\mu\nu}\{\psi_\nu, H_0\}$ as opposed to first class ones, which are left undetermined, hence, give rise to gauge ambiguities.

We have primary constraints

$$\pi_{\beta_i} = 0, \quad \pi_{\lambda_u} = 0, \quad \pi_{\lambda_t} = 0 \tag{8.14}$$

which have to be added to the Hamiltonian $H_0$.

The unconstrained phase space of the model may be identified as the infinite-dimensional manifold $T^*P(M \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R})$ with $P(N)$ denoting the space of maps from the source space $I = [0,1]$ to some target $N$. The configuration fields will be $X^i : I \to M$, $\beta_i : I \to \mathbb{R}^m, i = 1 \ldots m$, and

$\lambda_t, \lambda_u : I \to \mathbb{R}$. It is possible to read off the non-zero Poisson brackets from the first order action, which yields

$$\{\pi_i(u), X^j(v)\} = \delta_i^j \delta(u-v) \tag{8.15}$$

to which we have to add those related with the extended phase space

$$\begin{aligned}
\{\pi_{\beta_i}(u), \beta_j(v)\} &= \delta_i^j \delta(u-v) && (8.16)\\
\{\pi_{\lambda_t}(u), \lambda_t(v)\} &= \delta(u-v) && (8.17)\\
\{\pi_{\lambda_u}(u), \lambda_u(v)\} &= \delta(u-v). && (8.18)
\end{aligned}$$

By imposing that primary constraints be preserved along the motion, $m+2$ new constraints are obtained

$$\begin{array}{rclcl}
\dot{\pi}_{\beta_i} &=& X'^i + \Lambda^{ij}\zeta_j - E^i\lambda_u &:=& \mathcal{G}_{\beta_i}\\
\dot{\pi}_{\lambda_t} &=& E^i\zeta_i &:=& \mathcal{G}_{\lambda_t}\\
\dot{\pi}_{\lambda_u} &=& E^i\beta_i &:=& \mathcal{G}_{\lambda_u}.
\end{array} \tag{8.19}$$

Hence, the initial Hamiltonian $H_0$ is itself a sum of constraints

$$H_0 = -\int du\ \left[\beta_i \mathcal{G}_{\beta_i} + \lambda_t \mathcal{G}_{\lambda_t}\right]. \tag{8.20}$$

Let us compute their Poisson algebra. For secondary constraints we find

$$\begin{aligned}
\{\mathcal{G}_{\beta_i}(u), \mathcal{G}_{\beta_j}(v)\} &= -\Lambda^{il}\frac{\partial}{\partial X^l(u)}\mathcal{G}_{\beta_j}(v) + \Lambda^{jl}\frac{\partial}{\partial X^l(v)}\mathcal{G}_{\beta_i}(u) && (8.21)\\
\{\mathcal{G}_{\beta_i}(u), \mathcal{G}_{\lambda_u}(v)\} &= -\Lambda^{il}\frac{\partial}{\partial X^l(u)}\mathcal{G}_{\lambda_u}(v) && (8.22)\\
\{\mathcal{G}_{\beta_i}(u), \mathcal{G}_{\lambda_t}(v)\} &= -\Lambda^{il}\frac{\partial}{\partial X^l(u)}\mathcal{G}_{\lambda_t}(v) + E^l\frac{\partial}{\partial X^l(v)}\mathcal{G}_{\beta_i}(u) && (8.23)\\
\{\mathcal{G}_{\lambda_t}(u), \mathcal{G}_{\lambda_u}(v)\} &= -E^l\frac{\partial}{\partial X^l(u)}\mathcal{G}_{\lambda_u}(v). && (8.24)
\end{aligned}$$

Before proceeding further, we assume, without loss of generality, that a basis of vector fields on the target manifold $M$ has been chosen such that the Reeb vector field has non-zero component only along one of the basis elements, say $E^i = \mathcal{E}\delta^{im}$ and we shall indicate with $a = 1, \ldots m-1$ the remaining directions. Thus we compute the remaining brackets, which yield

$$\begin{aligned}
\{\pi_{\beta_i}(u), \mathcal{G}_{\lambda_u}(v)\} &= \mathcal{E}\,\delta^{im}\delta(u-v) && (8.25)\\
\{\pi_{\lambda_u}(u), \mathcal{G}_{\beta_i}(v)\} &= -\mathcal{E}\,\delta^{im}\delta(u-v)) && (8.26)
\end{aligned}$$

with all other brackets strongly zero. By repeated use of Eqs. (7.3), (7.4) in the chosen parameterization for the Reeb vector field as $E^i = \mathcal{E}\delta^{im}$, a tedious but straightforward calculation gives an explicit expression for the Poisson brackets (8.21)-(8.24), which read

$$\begin{aligned}
\{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\beta_b}(v)\} &= \mathcal{G}_{\beta_l}\partial_l\Lambda^{ba} - \mathcal{G}_{\lambda_t}\Lambda^{ba} \simeq 0 && (8.27)\\
\{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\lambda_t}(v)\} &= 0 && (8.28)\\
\{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\beta_m}(v)\} &= \mathcal{G}_{\beta_l}\partial_l\Lambda^{ma} - \mathcal{G}_{\lambda_t}\Lambda^{ma} - \mathcal{E}\Lambda^{ak}\zeta_k \simeq -\mathcal{E}\Lambda^{ak}\zeta_k && (8.29)\\
\{\mathcal{G}_{\beta_m}(u), \mathcal{G}_{\lambda_t}(v)\} &= \mathcal{G}_{\beta_l}\partial_l\mathcal{E} - \mathcal{E}\delta'(u-v) \simeq -\mathcal{E}\delta'(u-v) && (8.30)\\
\{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\lambda_u}(v)\} &= -\mathcal{G}_{\lambda_u}\partial_m\Lambda^{am} \simeq 0 && (8.31)\\
\{\mathcal{G}_{\beta_m}(u), \mathcal{G}_{\lambda_u}(v)\} &= -\beta_m\Lambda^{ml}\partial_l\mathcal{E} && (8.32)\\
\{\mathcal{G}_{\lambda_t}(u), \mathcal{G}_{\lambda_u}(v)\} &= -\mathcal{E}\beta_m\partial_m\mathcal{E} = -\mathcal{G}_{\lambda_u}\partial_m\mathcal{E} \simeq 0. && (8.33)
\end{aligned}$$

The chosen parametrization for the Reeb vector field is particularly useful because it considerably simplifies the classification of constraints as first or second class. By inspecting the rank of the matrix of Poisson brackets, it is easy to verify that the latter is always equal to four. Therefore, we may conclude that four out of $2m+4$ constraints are second class, i.e.,

$$\begin{matrix} \pi_{\lambda_u} & \pi_{\beta_m} \\ \mathcal{G}_{\lambda_u} & \mathcal{G}_{\beta_m} \end{matrix}\ . \tag{8.34}$$

By evaluating the conservation of constraints with respect to the total Hamiltonian

$$H_1 = H_0 + \int du\, [a_i\pi_{\beta_i} + a_t\pi_{\lambda_t} + a_u\pi_{\lambda_u}] \tag{8.35}$$

we may verify that no new constraints arise, but some of the Lagrange multipliers get fixed, namely [2]

$$a_m = \beta_m = 0, \quad a_u = \beta_a\Lambda^{ak}\zeta_k + \partial_u\lambda_t. \tag{8.36}$$

The remaining $2m$ constraints,

$$\begin{matrix} \pi_{\beta_a}, & \mathcal{G}_{\beta_a}, & a = 1, \dots, m-1 \\ \pi_{\lambda_t}, & \mathcal{G}_{\lambda_t} & \end{matrix} \tag{8.37}$$

[2] As for the Lagrange multiplier $a_u$, we notice that its value agrees with the equation of motion for $\lambda_u$ which has been derived in the Lagrangian formalism, (8.9).

are first class, thus generating gauge transformations, with generating functional given by the linear combination

$$K(\beta_a, \lambda_t, a_t, a_{\beta_a}) = \int du\, \lambda_t \mathcal{G}_{\lambda_t} + \beta_a \mathcal{G}_{\beta_a} + a_t \pi_{\lambda_t} + a_{\beta_a}\pi_{\beta_a}, \quad a = 1, \dots, m-1 \tag{8.38}$$

where $\beta_a, \lambda_t, a_t, a_{\beta_a}$ are gauge parameters.

First class constraints have zero Poisson brackets with the total Hamiltonian, with undetermined Lagrange multipliers. Hence, they generate canonical symmetries, that is to say, gauge transformations. One main difference with respect to the Poisson sigma model is that for the latter the whole Hamiltonian is a first class constraint, hence being itself the generating function of gauge transformations. Here instead, the Hamiltonian contains second class constraints as well, which have to be subtracted in order to get the gauge generators.

In order to compute the algebra of gauge generators, $\{K(\beta, \lambda_t, a_t, a_\beta), K(\tilde{\beta}, \tilde{\lambda}_t, \tilde{a}_t, \tilde{a}_\beta)\}$, we notice firstly that primary constraints in (8.38) may be ignored, because their Poisson brackets are strongly zero. Secondly, it is evident from Eq. (8.27) that, similarly to the Poisson sigma model, the algebra will only close on-shell. Therefore, in order to obtain a closed algebra off-shell we allow for the relevant gauge parameters to be functions of the fields. More precisely, given $(\beta_a, \lambda_t) \in C(I \to X^*(T^*M \oplus \mathbb{R}))$ we allow for $\beta_a = \beta_a(u, X(u)), \lambda_t = \lambda_t(u, X(u))$. Thus, we compute

$$\begin{aligned}\{K(\beta, \lambda_t), K(\tilde{\beta}, \tilde{\lambda}_t)\} = \int du du' \Big[ &\{(\beta_a \mathcal{G}_a)(u), (\tilde{\beta}_b \mathcal{G}_b)(u')\} \\ &+ \{(\beta_a \mathcal{G}_a)(u), (\tilde{\lambda}_t \mathcal{G}_t)(u')\} + \{(\lambda_t \mathcal{G}_t)(u), (\tilde{\beta}_b \mathcal{G}_b)(u')\} \\ &+ \{(\lambda_t \mathcal{G}_t)(u), (\tilde{\lambda}_t \mathcal{G}_t)(u')\}\Big].\end{aligned} \tag{8.39}$$

On using (8.15) , where $\zeta_i = -\pi_i$ we find

$$\begin{aligned}\{(\beta_a \mathcal{G}_a)(u), (\tilde{\beta}_b \mathcal{G}_b)(u')\} = \Big[&\mathcal{G}_c \left(\beta_a \tilde{\beta}_b \partial_c \Lambda^{ba} - \beta_a \Lambda^{aj} \partial_j \tilde{\beta}_c + \tilde{\beta}_a \Lambda^{aj} \partial_j \beta_c\right) \\ &- \mathcal{G}_t \beta_a \tilde{\beta}_b \Lambda^{ba}\Big] \delta(u - u')\end{aligned} \tag{8.40}$$

$$\{(\beta_a \mathcal{G}_a)(u), (\tilde{\lambda}_t \mathcal{G}_t)(u')\} = \left(\mathcal{G}_a \mathcal{E} \tilde{\lambda}_t \partial_m \beta_a - \mathcal{G}_t \beta_a \Lambda^{aj} \partial_j \tilde{\lambda}_t\right) \delta(u - u') \tag{8.41}$$

$$\{(\lambda_t \mathcal{G}_t)(u), (\tilde{\beta}_b \mathcal{G}_b)(u')\} = \left(\mathcal{G}_t \tilde{\beta}_b \Lambda^{bj} \partial_j \lambda_t - \mathcal{G}_b \mathcal{E} \lambda_t \partial_m \tilde{\beta}_b\right) \delta(u - u') \tag{8.42}$$

$$\{(\lambda_t \mathcal{G}_t)(u), (\tilde{\lambda}_t \mathcal{G}_t)(u')\} = \mathcal{G}_t \mathcal{E}(\tilde{\lambda}_t \partial_m \lambda_t - \lambda_t \partial_m \tilde{\lambda}_t) \delta(u - u'). \tag{8.43}$$

This yields

$$\{K(\beta,\lambda_t),K(\tilde\beta,\tilde\lambda_t)\}$$
$$= \int du du' \Big[\mathcal{G}_c\Big(\beta_a\tilde\beta_b\partial_c\Lambda^{ba}+\Lambda^{aj}(\tilde\beta_a\partial_j\beta_c-\beta_a\partial_j\tilde\beta_c)+\mathcal{E}\left(\tilde\lambda_t\partial_m\beta_c-\lambda_t\partial_m\tilde\beta_c\right)\Big)$$
$$+\mathcal{G}_t\left(\beta_a\tilde\beta_b\Lambda^{ab}+\Lambda^{aj}(\tilde\beta_a\partial_j\lambda_t-\beta_a\partial_j\tilde\lambda_t)+\mathcal{E}\left(\tilde\lambda_t\partial_m\lambda_t-\tilde\lambda_t\partial_m\tilde\lambda_t\right)\right)\Big]. \tag{8.44}$$

We now observe that a generalization of the Koszul bracket (6.30) is available for Jacobi manifolds, which endows the set of sections of the 1-jet bundle $J^1M$ with a Lie algebra structure [158, 160]. Given $(\alpha,f),(\beta,g)$ sections of $J^1M$, namely $\alpha,\beta\in\Omega^1(M), f,g\in C(M)$, the bracket reads[3]

$$[(\alpha,f),(\beta,g)]$$
$$=\Big(\big(\mathcal{L}_{\sharp_\Lambda\alpha}\beta-\mathcal{L}_{\sharp_\Lambda\beta}\alpha-d(\Lambda(\alpha,\beta)+f\mathcal{L}_E\beta-g\mathcal{L}_E\alpha-\alpha(E)\beta+\beta(E)\alpha\big),$$
$$(\{f,g\}_J-\Lambda(df-\alpha,dg-\beta)\Big) \tag{8.45}$$

where $\sharp_\Lambda\alpha$ denotes the vector field obtained by contracting the bi-vector field $\Lambda$ with the one-form $\alpha$; in local coordinates it reads: $\sharp_\Lambda\alpha=\alpha_i\Lambda^{ij}\partial_j$. The latter satisfies Jacobi identity, provided the manifold is a Jacobi manifold, with $\{f,g\}_J$ the Jacobi bracket. Analogously to the Poisson sigma model, Eq. (8.45) may be extended to maps from the interval $I$ to sections of the 1-jet bundle $(\alpha,f):I\to\Gamma(J^1M)$, with the property $\alpha(0)=\alpha(1)=0$, according to

$$[(\alpha,f),(\beta,g)](u)=[(\alpha,f)(u),(\beta,g)(u)]. \tag{8.46}$$

On computing the bracket (8.45) for $(\beta,\lambda_t),(\tilde\beta,\tilde\lambda_t)$ a lengthy but straightforward calculation yields

$$[(\beta,\lambda_t),(\tilde\beta,\tilde\lambda_t)]=(\underline{\boldsymbol{\beta}},\underline{\boldsymbol{\lambda}}_t) \tag{8.47}$$

with

$$\underline{\boldsymbol{\beta}} = \Big(\Lambda^{ij}(\beta_i\partial_j\tilde\beta_k-\tilde\beta_i\partial_j\beta_k)+\tilde\beta_i\beta_j\partial_k\Lambda^{ij}+\mathcal{E}(\lambda_t\partial_m\tilde\beta_k-\tilde\lambda_t\partial_m\beta_k)$$
$$+\ \mathcal{E}(\tilde\beta_m\beta_k-\beta_m\tilde\beta_k)+(\lambda_t\tilde\beta_m-\tilde\lambda_t\beta_m)\partial_k\mathcal{E}\Big)dX^k \tag{8.48}$$
$$\underline{\boldsymbol{\lambda}}_t = \Lambda^{ij}(\beta_i\partial_j\tilde\lambda_t-\tilde\beta_i\partial_j\lambda_t-\beta_i\tilde\beta_j)+\mathcal{E}(\lambda_t\partial_m\tilde\lambda_t-\tilde\lambda_t\partial_m\lambda_t) \tag{8.49}$$

[3] Vaisman shows in [158] that this is nothing but the Koszul bracket (6.30) defined for the associated "Poissonized" manifold $(M\times\mathbb{R},P)$, with respect to which the algebra of sections of $J^1M$ is closed.

Therefore, by taking into account the second class constraints, which enforce $\beta_m = \tilde{\beta}_m = 0$, the RHS of the Poisson bracket (8.44) may be stated in terms of (8.48),(8.49) to give

$$\{K_{(\beta,\lambda_t)}, K_{(\tilde{\beta},\tilde{\lambda}_t)}\} = -K_{[(\beta,\lambda_t),(\tilde{\beta},\tilde{\lambda}_t)]}. \tag{8.50}$$

Notice that, for $\beta_a = \partial_a \lambda_t$ and analogous expression for $\tilde{\beta}_a$, the latter further reduces to

$$\{K_{(\beta,\lambda_t)}, K_{(\tilde{\beta},\tilde{\lambda}_t)}\} = -K_{(d\{\lambda_t,\tilde{\lambda}_t\}_J, \{\lambda_t,\tilde{\lambda}_t\}_J)} \tag{8.51}$$

which is the particular case considered in [69]. The mapping $f \to e^{\tau}(df + f d\tau)$ with $f \in C^{\infty}(M)$ is a Lie algebra homomorphism from the Jacobi algebra of $M$ to $\Gamma_0(M)$ defined in (8.3).

To summarize, the model exhibits first class constraints, which generate gauge transformations. Differently from the Poisson sigma model, second class constraints are present, which have to be dealt with, before analyzing the algebra of gauge generators. Thanks to the bracket (8.45) the map $(\beta, \lambda_t) \to K(\beta, \lambda_t)$ is a Lie algebra homomorphism. Moreover, because of the homomorphism stated at the end of last paragraph, time-space diffeomorphisms may be explicitly related with the Hamiltonian vector fields associated with $\lambda_t$ (resp. $\lambda_u$) through the Jacobi bracket (see [88] for details).

It is to be noticed that, because of the presence of second class constraints, the Hamiltonian vector fields generating infinitesimal gauge transformations are not directly associated with the Hamiltonian, but rather with the functional $K_{(\beta,\lambda_t)}$. They shall be explicitly worked out in the forthcoming section.

### The reduced phase space

Since the model is gauge invariant under the action of the gauge transformations generated by the flows of the Hamiltonian vector field associated with the functional $K$, we can define the reduced phase space as the quotient space $\mathcal{C}/H$, where $H$ is the gauge group and $\mathcal{C}$ is the constraint manifold. According to Sec. 8.1.1 the former is an infinite-dimensional manifold, which after the imposition of all constraints results to be labelled by $2m$ fields. We choose to parametrize the manifold with $X^i, \zeta_a, \lambda_u$. The quotient manifold $\mathcal{C}/H$ is finite-dimensional, as it follows from the following

**Theorem 8.1.1.** Let $(X^i, \zeta_a, \lambda_u) \in \mathcal{C}$. The subspace of $T_{(X^i,\zeta_a,\lambda_u)}\mathcal{C}$ spanned by the Hamiltonian vector fields $\xi_{\beta,\lambda_t}$ is a closed subspace of codimension $2\dim(M) - 2$.

*Proof.* Let us consider the subspace $\mathcal{S}_{(X^i,\zeta_a,\lambda_u)}$ of $T_{(X^i,\zeta_a,\lambda_u)}\mathcal{C}$ spanned by the Hamiltonian vector fields $\xi_{\beta,\lambda_t}$, associated with the functional $K(\beta,\lambda_t)$, generating infinitesimal gauge transformations. The map $(\beta,\lambda_t) \to \xi_{\beta,\lambda_t}$, explicitly given by

$$\begin{aligned}
\delta_{\xi_K} X^i &:= \{K(\beta,\lambda_t), X^i\} = \Lambda^{ia}\beta_a - \mathcal{E}\delta^i_m\lambda_t && (8.52)\\
\delta_{\xi_K} \zeta_a &:= \{K(\beta,\lambda_t), \zeta_i\} = -(\beta_a)' + \beta_b\partial_a\Lambda^{bk}\zeta_k + \lambda_t\zeta_m\partial_a\mathcal{E} && (8.53)\\
\delta_{\xi_K} \zeta_m &:= \{K(\beta,\lambda_t), \zeta_m\} = \beta_b\partial_m\Lambda^{bk}\zeta_k + \lambda_t\zeta_m\partial_m\mathcal{E} && (8.54)\\
\delta_{\xi_K} \lambda_u &:= \{K(\beta,\lambda_t), \lambda_u\} = 0 && (8.55)
\end{aligned}$$

is linear. However, on the constraint manifold $\mathcal{C}$, the last terms in the r.h.s. of (8.53) and (8.54) vanish because $\zeta_m = 0$, moreover, $\partial_m\Lambda^{bk}\zeta_k = \partial_m\Lambda^{bc}\zeta_c$ which is zero because of eq. (7.1). Therefore, the non-zero components of the map on the constraint manifold are given by

$$\begin{aligned}
\xi_1^i &= \Lambda^{ia}\beta_a - \mathcal{E}\delta^i_m\lambda_t && (8.56)\\
\xi_{2,a} &= -(\beta_a)' + \beta_b\partial_a\Lambda^{bk}\zeta_k. && (8.57)
\end{aligned}$$

The kernel of this linear map is empty, showing that the map is injective. To this, we have to impose that Eqs. (8.56), (8.57) be zero. The second condition yields a homogeneous linear first order ODE with initial condition $\beta(0) = 0$, hence, $\beta$ vanishes identically. The first one is instead an algebraic relation for which, by using the solution $\beta = 0$ we have $\mathcal{E}\delta^i_m\lambda_t = 0$ and since the Reeb vector field is nowhere vanishing it has to be $\lambda_t = 0$. Hence, the map is injective.

Let us, therefore, consider the image space. The tangent vector $(\tilde{X}^i, \tilde{\zeta}_a)$, to a point $(X^i, \zeta_a, \lambda_u) \in \mathcal{C}$ is the solution of the linearized constraint equations

$$\tilde{X}'^i + \left(A_j{}^i - \partial_j\mathcal{E}\delta^{im}\lambda_u\right)\tilde{X}^j + \Lambda^{ib}\tilde{\zeta}_b = 0. \tag{8.58}$$

where we defined $A_j{}^i = \partial_j\Lambda^{ik}\zeta_k$. The tangent field has no component $\tilde{\lambda}_u$ because of the constraint $\mathcal{G}_{\beta_m}$. If $(\tilde{X}, \tilde{\zeta})$ is an Hamiltonian vector field, and thus it is in the image of $\xi$, then it has to be

$$\tilde{X}^i = \Lambda^{ib}\beta_b - \mathcal{E}\delta^{im}\lambda_t, \tag{8.59}$$

$$\tilde{\zeta}_a = -(\beta_a)' + A_a{}^b\beta_b. \tag{8.60}$$

The former have to hold at each $u$, which implies in particular $(\tilde{X}, \tilde{\zeta})$ is in the image of $\xi$ if

$$\tilde{X}^i(0) + \mathcal{E}(X(0))\delta^{im}\lambda_t(0) = 0.$$

If we introduce the matrix $V = \hat{P}\exp[-\int A\, du]$ as the path-ordered exponential of $A$, i.e. the solution of the differential equation

$$\begin{cases} (V_i^j)' = -V_i^k(u) A_k{}^j(u) \\ V_i^j(0) = \delta_i^j, \end{cases} \tag{8.61}$$

then Eq. (8.60) can be rewritten in the form

$$\tilde{\zeta}_a(u) = -(V^{-1}(u))_a^c\, \partial_u[V(u)_c^b \beta_b(u)]. \tag{8.62}$$

From this equation we can define the $m-1$ functions

$$p_a(u) := \int_0^u dv V(v)_a^b \tilde{\zeta}_b(v) = -\int_0^u \partial_v[V(v)_a^b \beta_b(v)], \tag{8.63}$$

from which it follows that

$$\int_I du\, V(u)_a^b \tilde{\zeta}_b(u) = 0.$$

Hence, we conclude that if $(\tilde{X}, \tilde{\zeta})$ is in the image of $\xi$, then we have

$$\tilde{X}^i(0) + \mathcal{E}\delta^{im}(X(0))\lambda_t(0) = 0, \qquad \int_I du\, V(u)_a^b \tilde{\zeta}_b(u) = 0. \tag{8.64}$$

Now, it is important to notice that these conditions yield $2m-2$ invariants and not $2m-1$ as it appears. Indeed, in the chosen parametrization for the Reeb vector field, the first equation in (8.64) amounts to

$$\tilde{X}^a(0) = 0, \quad \tilde{X}^m(0) = -\lambda_t(0). \tag{8.65}$$

However, the second relation is not gauge invariant and does not fix the $m-th$ component of $\tilde{X}$, $\lambda_t(0)$ not being fixed to assume any particular value. Therefore, the first of Eqs. (8.64) yields $m-1$ invariant conditions. The final count of invariant conditions is then $2m-2$.

Vice versa, if we now consider $(\tilde{X}, \tilde{\zeta})$ as a tangent vector at the point $(X, \zeta, \lambda_u) \in \mathcal{C}$ satisfying Eq. (8.64), then we show that this tangent vector is Hamiltonian if we choose $\beta_a = -(V^{-1})_a^b\, p_b = -(V^{-1})_a^b \int_0^u dv\, V(v)_b^c \tilde{\zeta}_c(v)$.

To verify the statement, let us define the vector field

$$Y^i(u) = \Lambda^{ib}(u)\beta_b(u) - \mathcal{E}(u)\delta^{im}\lambda_t(u), \tag{8.66}$$

satisfying the boundary condition $Y^i(0) = -\mathcal{E}(0)\delta^{im}\lambda_t(0)$, with the choice

$$\beta_a = -(V^{-1})_a^b \int_0^u dv\, V(v)_b^c \tilde{\zeta}_c(v). \tag{8.67}$$

We will now check directly that $Y$ satisfies the same ODE as $\tilde{X}$ with the same boundary condition, namely it is a tangent vector field. The derivative of Eq. (8.66) with respect to $u$ yields:

$$Y'^i = -\partial_k\Lambda^{ib}X'^k(V^{-1})_b^c\int_0^u dvV_c^a\tilde{\zeta}_a - \Lambda^{ib}\left[\partial_u(V^{-1})_b^c\int_0^u dvV_c^a\tilde{\zeta}_a + (V^{-1})_b^c V_c^a\tilde{\zeta}_a\right]$$
$$+ \partial_k\mathcal{E}\delta^{im}X'^k\lambda_t + \mathcal{E}\delta^{im}\lambda_t'.$$

By means of Eq. (8.9) with $\dot{\lambda}_u = \{K(\beta,\lambda_t),\lambda_u\} = 0$, namely $\lambda_t' = -\Lambda^{ij}\beta_i\zeta_j$ and the constraint equation $X'^i = -\Lambda^{ib}\zeta_b + \mathcal{E}\delta^{im}\lambda_u$ we arrive at

$$\begin{aligned} Y'^i &= \beta_b\zeta_c\left(\Lambda^{kc}\partial_k\Lambda^{ib} + \Lambda^{ik}\partial_k\Lambda^{cb} - \mathcal{E}\delta^{im}\Lambda^{bc}\right) - \Lambda^{ib}\tilde{\zeta}_b \\ &- \partial_k\mathcal{E}\delta^{im}\Lambda^{kb}\zeta_b\lambda_t - \mathcal{E}\partial_m\left(\Lambda^{ib}\beta_b\lambda_u - \mathcal{E}\delta^{im}\lambda_t\lambda_u\right). \end{aligned}$$

where we have substituted the defining equation for $V$ (8.61) and the explicit form of $\beta$ Eq. (8.67) . Now using the Schouten bracket (7.3) we obtain

$$Y'^i = -\partial_k\Lambda^{ib}\zeta_b Y^k - \Lambda^{ib}\tilde{\zeta}_b + \partial_k\mathcal{E}\delta^{im}\lambda_u Y^k + \left(\partial_m\Lambda^{bi}\mathcal{E} - \partial_k\mathcal{E}\delta^{im}\Lambda^{kb}\right)(\zeta_b\lambda_t + \beta_b\lambda_u)\,.$$

Further implementing $\mathscr{L}_E\Lambda = 0$ we have finally

$$\left(\partial_m\Lambda^{bi}\mathcal{E} - \partial_k\mathcal{E}\delta^{im}\Lambda^{kb}\right)\zeta_b\lambda_t = 0.$$

The same can be shown for the last term proportional to $\lambda_u$, so we have finally that $Y$ satisfies the linearized constraint in Eq. (8.58) with the same boundary condition.

To conclude, we have proven that the image of $\xi$ is the subspace spanned by $\xi_{\beta,\lambda_t}$ modulo the $2m-2$ conditions (8.64), i.e. it is a closed subspace of co-dimension $2m-2$. □

Therefore, similarly to the Poisson sigma model, the constraint manifold quotiented by gauge transformations results to be finite-dimensional, but of dimension equal to $2m-2$, with $m$ the dimension of the target Jacobi manifold.

### 8.1.2 Poissonization

In this section we review the almost [4] one-to-one correspondence between the Jacobi sigma model described in the previous sections and the reduced model which may be obtained on the Jacobi manifold after Poissonization.

The idea in [88] was to formulate a Poisson sigma model with target $(M\times\mathbb{R},P)$ $P$ being the Poisson tensor described in Theorem 7.1.1, and project its dynamics down to $M$. Fig. 8.1 illustrates schematically the procedure.

$$\begin{tikzcd} (M,\Lambda,E) \arrow[r, "\text{Poissonization}"] & \underset{\text{Poisson sigma model}}{(M\times\mathbb{R},P)} \arrow[d, "\pi"] \\ \underset{\text{Jacobi sigma model}}{(M,\Lambda,E)} \arrow[r] & M\text{-projected model} \end{tikzcd}$$

FIGURE 8.1: Diagrammatic summary of the reduction of the dynamics from the Poisson sigma model to the Jacobi sigma model.

For this purpose, let us consider the Poisson sigma model with target space the Poisson manifold $(M\times\mathbb{R},P)$ and Poisson structure $P=e^{-X_0}\left(\Lambda+\frac{\partial}{\partial X_0}\wedge E\right)$ defined in terms of the structures of the embedded Jacobi manifold $(M,\Lambda,E)$ and $X_0\in\mathbb{R}$, according to Theorem 7.1.1. The field configurations are then $(X,\eta)$, with $X^I=(X^i,X^0):\Sigma\to M\times\mathbb{R}$ the usual embedding maps and $\eta\in\Omega^1(\Sigma,X^*(T^*(M\times\mathbb{R})))$, $\eta_I=(\eta_i,\eta_0)$. Capital indices $I=0,\cdots,m$ label coordinates over the Poisson manifold $M\times\mathbb{R}$, while lowercase letters

[4] The reason for the term 'almost' is explained in the last paragraph of this section.

$i = 1, \cdots, m$ shall be reserved to the Jacobi manifold $M$. The Poisson bi-vector field in a coordinate basis $\{\partial/\partial X^I\}$ can be written explicitly as

$$P^{IJ} = e^{-X_0} \begin{pmatrix} & & & -E^1 \\ & \Lambda^{ij} & & \vdots \\ & & & -E^m \\ E^1 & \cdots & E^m & 0 \end{pmatrix}, \tag{8.68}$$

with $P = P^{IJ}\partial_I \wedge \partial_J$ and $E = E^i \partial_i$.

By splitting the equations of motion, (6.3) and (6.4) in terms of target co-ordinates adapted to the product manifold, one obtains:

$$dX^i + e^{-X^0}\left(\Lambda^{ij}\eta_j - E^i\eta_0\right) = 0 \tag{8.69}$$

$$dX^0 + e^{-X^0}E^i\eta_i = 0 \tag{8.70}$$

$$d\eta_i + \frac{1}{2}e^{-X^0}\partial_i\Lambda^{jk}\eta_j \wedge \eta_k + e^{-X^0}\partial_i E^j \eta_0 \wedge \eta_j = 0 \tag{8.71}$$

$$d\eta_0 - \frac{1}{2}e^{-X^0}\Lambda^{jk}\eta_j \wedge \eta_k - e^{-X^0}E^j\eta_0 \wedge \eta_j = 0. \tag{8.72}$$

We now consider the immersion $i : M \hookrightarrow M \times \mathbb{R}$ through the identification of $M$ with $M \times \{0\}$. The reduced dynamics on $M$ is thus obtained by posing $X^0 = 0$. This yields

$$\begin{aligned} &dX^i + \Lambda^{ij}\eta_j - E^i\eta_0 = 0 \\ &E^i\eta_i = 0 \\ &d\eta_i + \frac{1}{2}\partial_i\Lambda^{jk}\eta_j \wedge \eta_k + \partial_i E^j\eta_0 \wedge \eta_j = 0 \\ &d\eta_0 - \frac{1}{2}\Lambda^{jk}\eta_j \wedge \eta_k = 0. \end{aligned} \tag{8.73}$$

On identifying $\eta_0$ with $\pi^*\lambda$, $\pi : M \times \mathbb{R} \to M$ being the projection map, it is immediate to verify that the reduced dynamics coincides with the one obtained from the action functional in Eq. (8.4).

However, it is important to remark that the two models are not completely equivalent. In fact, the reduced sigma model inherits an additional boundary condition for the field $\lambda_t$ which comes from $\eta_{0|\partial\Sigma} = 0$ for the Poissonized sigma model. More precisely, upon performing the splitting of the world-sheet as $\Sigma = \mathbb{R} \times I$ the additional boundary condition requires

$\lambda_{t|\partial I} = 0$ other than $\beta_I = 0$, a condition which is unnecessary for the model described by the action (8.4). This makes a difference in the analysis of the gauge transformations of the two models, although it is always possible to add this condition by hand if one wants to recover a complete equivalence between the two models.

## 8.2 Contact and LCS manifolds

In this section we will consider in some detail two main classes of target spaces for the Jacobi sigma model, that is contact and locally conformal symplectic manifolds. As a first application, we shall show that for both cases an interesting result can be stated, which concerns the possibility of integrating out the auxiliary momenta and obtain a second order formulation of the action functional, solely expressed in terms of the embedding maps $X^i$ and their derivatives. As we already recalled in Sec. 6.1, for the Poisson sigma model this is possible only when the target space is a symplectic manifold. In that case the Poisson bivector can be inverted and the equations of motion can be solved for $\eta$. We shall see in the following that the situation is different for the Jacobi sigma model, both for contact and LCS target.

### 8.2.1 Integration on contact manifolds

Let us start by considering $M$ as a $(2n+1)$-dimensional contact manifold with contact one-form $\vartheta$ satisfying $\vartheta \wedge (d\vartheta)^n \neq 0$ at every point. The Jacobi structure can then be obtained from (7.12), or, equivalently, (7.13)-(7.14).

Let us consider the equations of motion, represented by (8.5)-(8.7), (8.9). Thanks to the relations satisfied by the contact form, Eqs. (7.13), (7.14) , the former can be solved for $\eta$ and $\lambda$. In fact, on multiplying (8.5) by $\vartheta_i$ and summing, we obtain

$$\vartheta_i(dX^i + \Lambda^{ij}\eta_j - E^i\lambda) = \vartheta_i dX^i - \lambda = 0, \tag{8.74}$$

from which

$$\lambda = \vartheta_i dX^i. \tag{8.75}$$

In order to obtain $\eta$ we multiply (8.5) by $(d\vartheta)_{\ell i}$, and sum over $i$. Again, using the properties of the contact form we find

$$d\vartheta_{\ell i}(dX^i + \Lambda^{ij}\eta_j - E^i\lambda) = (d\vartheta)_{\ell i}dX^i + \delta^j_\ell \eta_j = 0, \tag{8.76}$$

from which we obtain

$$\eta_i = (d\vartheta)_{ij} dX^j. \tag{8.77}$$

Thus, we may conclude that the auxiliary fields can be completely integrated out. Substituting (8.75)-(8.77) back into the action (8.4) we find the following second order action

$$S_2 = -\frac{1}{2} \int_\Sigma (d\vartheta)_{ij}\, dX^i \wedge dX^j = -\frac{1}{2} \int_\Sigma X^*(d\vartheta), \tag{8.78}$$

where in the second equality we have restored the pull-back map in order to highlight the geometric content. The exterior derivative of the contact one-form takes the role of the $B$-field, which turns out to be closed for contact manifolds. Despite the analogy with the symplectic case, the latter can only be non-degenerate when appropriately restricted to submanifolds of the target space.

**Topological Jacobi sigma model on $SU(2)$**

As a main example of the model described so far, we consider the target space to be the group manifold of $SU(2)$, bearing in mind that the procedure may be adapted to any three-dimensional semisimple Lie group. The group manifold is diffeomorphic to the sphere $S^3$. The contact one-form may be chosen among the basis left-invariant (resp. right-invariant) one-forms of the group, say $\theta^i$ defined through the Maurer–Cartan one-form $\ell^{-1}d\ell = \theta^i e_i \in \Omega^1(SU(2), \mathfrak{su}(2))$, with $\ell \in SU(2)$, $e_i = i\sigma_i/2$ the Lie algebra generators and $\sigma_i$ the Pauli matrices. Let us choose, to be definite, the contact one form to be $\vartheta = \theta^3$. The latter defines a Jacobi bracket according to Eq. (7.11) it being

$$\vartheta \wedge d\vartheta = \Omega \tag{8.79}$$

with $\Omega = \theta^1 \wedge \theta^2 \wedge \theta^3$ the volume form on the group manifold. Therefore, the Reeb vector field and the bivector field $\Lambda$ are easily determined by solving the equations

$$\iota_E \vartheta = 1, \quad \iota_E d\vartheta = 0, \tag{8.80}$$

$$\iota_\Lambda \vartheta = 0, \quad \iota_\Lambda d\vartheta = 1. \tag{8.81}$$

We obtain

$$E = Y_3 \quad \Lambda = Y_1 \wedge Y_2 \tag{8.82}$$

with $Y_i, i = 1, .., 3$ the left invariant vector fields on the group manifold, which are dual the the one-forms $\theta^i$ by definition. Hence, the Reeb vector field is constant and orthogonal to the distribution spanned by the bivector field $\Lambda$. The action functional of the model is given by

$$S[\phi,(\eta,\lambda)] = \int_\Sigma \langle \eta, \phi^*(g^{-1}\mathbf{d}g)\rangle + \frac{1}{2}\langle \eta, (\Lambda\circ\phi)\eta\rangle + \lambda\wedge(E\circ\phi)\eta \tag{8.83}$$

with field configurations $\phi, (\eta, \lambda)$, $\phi : \Sigma \ni (t, u) \to g \in G$ and $(\eta, \lambda) \in \Omega^1(\Sigma, \phi^*(T^*G \oplus \mathbb{R}))$. We have chosen in this specific example to distinguish the exterior derivative $\mathbf{d}$ on the target manifold from the one on the source, $d$. We recall the boundary condition $\eta(u)v = 0, u \in \partial\Sigma, v \in T(\partial\Sigma)$.

The map $\langle\ ,\ \rangle$ establishes a pairing between differential forms on $\Sigma$ with values in the pull-back $\phi^*(T^*G)$ and differential forms on $\Sigma$ with values in $\phi^*(TG)$.

On identifying the tangent space $TG$ with $G\times\mathfrak{g}$ and $T^*G$ with $G\times\mathfrak{g}^*$ we may write

$$\phi^*(g^{-1}\mathbf{d}g) = (g^{-1}\partial_t g)^i e_i dt + (g^{-1}\partial_u g)^i e_i du = A^i(t,u)e_i dt + J^i(t,u)e_i du \tag{8.84}$$

where we have introduced the notation

$$(g^{-1}\partial_t g)^i = A^i, \quad (g^{-1}\partial_u g)^i = J^i \tag{8.85}$$

with $\{e_i\}$ a basis in the Lie algebra. Analogously

$$\begin{aligned} \eta &= \eta_{tj}e^j dt + \eta_{uj}e^j du := \beta_j e^j dt + \zeta_j e^j du &(8.86)\\ \lambda &= \lambda_t dt + \lambda_u du &(8.87)\end{aligned}$$

with $\eta_{tj} = \beta_j$, $\eta_{uj} = \zeta_j$ and $\{e^i\}$ a dual basis in $\mathfrak{g}^*$. Then the action is rewritten as

$$\begin{aligned} S[g,(\eta,\lambda)] &= \int_\Sigma \eta_i\wedge\phi^*(g^{-1}dg)^i + \frac{1}{2}\Lambda^{ij}\eta_i\eta_j + \lambda\wedge E^i\eta_i \\ &= \int_\Sigma d^2u\left(\beta_i J^i - \zeta_i A^i + \Lambda^{ij}\beta_i\zeta_j + \lambda_t E^j\zeta_j - \lambda_u E^j\beta_j\right) \quad (8.88)\end{aligned}$$

and we have renamed the map $\phi$ with $g$, to simplify the notation.

Let us now derive the equations of motion. By varying the action with respect to the fields $\zeta, \beta, g, \lambda_t, \lambda_u$ we find

$$\begin{aligned} A^j &= -\Lambda^{jl}\beta_l + \lambda_t E^j && (8.89)\\ J^j &= -\Lambda^{jl}\zeta_l + \lambda_u E^j && (8.90)\\ \partial_t \zeta_j &= -(\beta_k J^l - \zeta_k A^l) c^k_{lj} + \partial_u \beta_j && (8.91)\\ E^j \zeta_j &= E^j \beta_j = 0 && (8.92) \end{aligned}$$

where we have used, to derive the third equation,

$$\begin{aligned} (\delta J)^j &= (g^{-1}\partial_u g)^l (g^{-1}\delta g)^k c^j_{lk} + \partial_u (g^{-1}\delta g)^j && (8.93)\\ (\delta A)^j &= (g^{-1}\partial_t g)^l (g^{-1}\delta g)^k c^j_{lk} + \partial_t (g^{-1}\delta g)^j && (8.94) \end{aligned}$$

and $c^j_{lk}$ are the structure constants of the Lie algebra $\mathfrak{su}(2)$. Let us notice that, with the parameterization chosen for the source manifold $\Sigma$, the evolutionary equations are the first and the third one, involving time derivatives, whereas the others are constraints.

In order to make contact with Eqs. (8.5)-(8.7) previously derived for a generic target space, we may write Eqs. (8.89)-(8.92) in compact form

$$\begin{aligned} \phi^*(g^{-1}\mathbf{d}g)^j + \Lambda^{jl}\eta_l - \lambda E^j &= 0 && (8.95)\\ d\eta_j + \eta_k \wedge \phi^*(g^{-1}\mathbf{d}g)^l c^k_{lj} &= 0 && (8.96)\\ E^j \eta_j &= 0. && (8.97) \end{aligned}$$

The first and last one match respectively Eqs. (8.5), (8.7), once we have identified $X^i$ with the local coordinates describing the map $\phi$ in a chart. The second equation needs an intermediate step: we obtain $\phi^*(g^{-1}\mathbf{d}g)^j$ from (8.95) and replace it in (8.96). We find

$$d\eta_j + \eta_k \wedge \left(-\Lambda^{lm}\eta_m + \lambda E^l\right) c^k_{lj} = 0. \qquad (8.98)$$

Then, we observe that

$$\Lambda^{lm} c^k_{lj} = \frac{1}{2}(\mathcal{L}_{Y_j}\Lambda)^{mk} \quad \text{and} \quad E^l c^k_{lj} = -(\mathcal{L}_{Y_j}E)^k \qquad (8.99)$$

so that Eq. (8.98) becomes

$$d\eta_j + \frac{1}{2}(\mathcal{L}_{Y_j}\Lambda)^{km}\eta_k \wedge \eta_m - (\mathcal{L}_{Y_j}E)^k \eta_k \wedge \lambda = 0 \qquad (8.100)$$

and this is exactly Eq. (8.6).

The Lagrangian may also be recast in the following form

$$L[g,\eta,\lambda] = \int_I du \left[-\zeta_i A^i + \beta_i \left(J^i + \Lambda^{ij}\zeta_j - \lambda_u E^i\right) + \lambda_t E^j \zeta_j\right] \tag{8.101}$$

with $A^i$ playing now the role of velocities. The action is already in its first order form, with Hamiltonian

$$H_0 = -\int du \left[\beta_i \left(J^i - \Lambda^{ij}\pi_j - \lambda_u E^i\right) + \lambda_t E^j \zeta_j\right] \tag{8.102}$$

and

$$\pi_i = \frac{\delta L}{\delta A^i} = -\zeta_i \tag{8.103}$$

being the only non-zero momenta, whereas

$$\pi_{\beta_i} = \pi_{\lambda_t} = \pi_{\lambda_u} = 0. \tag{8.104}$$

The latter are primary constraints, which we add to the Hamiltonian to get

$$H_1 = -\int du \left[\beta_i \left(J^i - \Lambda^{ij}\pi_j - \lambda_u E^i\right) + \lambda_t E^j \zeta_j + a_u \pi_{\lambda_u} + a_t \pi_{\lambda_t} + a_{\beta_i}\pi_{\beta_i}\right]. \tag{8.105}$$

In view of performing the Dirac analysis of constraints, the unconstrained phase space of the model may be identified as the infinite-dimensional manifold $T^*(P(G \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}))$, with $PM$ denoting the space of maps from the source space $\Sigma$ to the target manifold $M$. The configuration fields will be $g : \Sigma \to G, \quad \beta_i : \Sigma \to \mathbb{R}^m, i = i \ldots m$ and $\lambda_u, \lambda_t : \Sigma \to \mathbb{R}$ . Then, we read off the non-zero Poisson brackets from the canonical one-form

$$\Theta = \int_I du\ \pi_i \phi^*(g^{-1}\mathbf{d}g)^i \tag{8.106}$$

and its exterior derivative

$$\Omega = d\Theta = \int_I d\pi_i \wedge \phi^*(g^{-1}\mathbf{d}g)^i - \pi_i c^i_{jk}\phi^*(g^{-1}\mathbf{d}g)^j \wedge \phi^*(g^{-1}\mathbf{d}g)^k. \tag{8.107}$$

This yields the non-zero Poisson brackets to be

$$\{\pi_i(u), \pi_j(v)\} = c^k_{ij}\pi_k \delta(u-v) \tag{8.108}$$

$$\{\pi_i(u), g(v)\} = ig\sigma_i \delta(u-v) \tag{8.109}$$

$$\{g(u), \tilde{g}(v)\} = 0 \tag{8.110}$$

(in particular $\{\pi_i, J^j\} = J^k c_{ki}{}^j \delta(u-v) + \delta_i^j \delta'(u-v)$ ), to which we add those on the extended phase space

$$\begin{aligned} \{\pi_{\beta_i}(u), \beta_j(v)\} &= \delta^i_j \delta(u-v) && (8.111)\\ \{\pi_{\lambda_t}(u), \lambda_t(v)\} &= \delta(u-v) && (8.112)\\ \{\pi_{\lambda_u}(u), \lambda_u(v)\} &= \delta(u-v). && (8.113) \end{aligned}$$

Adapting the analysis of constraints to the present case, we find the secondary constraints

$$\begin{aligned} \mathcal{G}_{\lambda_u} &= -\beta_i E^i = -\beta_3 && (8.114)\\ \mathcal{G}_{\lambda_t} &= -\pi_i E^i = -\pi_3 && (8.115)\\ \mathcal{G}_{\beta_i} &= J^i - \Lambda^{ij}\pi_j - \lambda_u \delta^i_3 && (8.116) \end{aligned}$$

whose algebra yields

$$\begin{aligned} \{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\beta_b}(v)\} &= \epsilon_{ab}\mathcal{G}_{\lambda_t}(u)\delta(u-v) && (8.117)\\ \{\mathcal{G}_{\lambda_t}(u), \mathcal{G}_{\beta_a}(v)\} &= \epsilon_{ab}\mathcal{G}_{\beta_b}(u)\delta(u-v) && (8.118)\\ \{\mathcal{G}_{\lambda_t}(u), \mathcal{G}_{\beta_3}(v)\} &= -\delta'(u-v) && (8.119)\\ \{\mathcal{G}_{\beta_a}(u), \mathcal{G}_{\beta_3}(v)\} &= J^a(u)\delta(u-v) && (8.120)\\ \{\mathcal{G}_{\beta_i}(u), \pi_{\lambda_u}(v)\} &= \delta_{i3}\delta(u-v) && (8.121)\\ \{\mathcal{G}_{\lambda_u}(u), \pi_{\beta_i}(v)\} &= \delta_{i3}\delta(u-v) && (8.122) \end{aligned}$$

all others being zero. Therefore, from imposing the conservation of secondary constraints, we obtain

$$\begin{aligned} \dot{\mathcal{G}}_{\lambda_u} &= a_{\beta_3} && (8.123)\\ \dot{\mathcal{G}}_{\beta_a} &= \beta_3 J^a - \epsilon_{ab}(\beta_b \mathcal{G}_{\lambda_t} + \lambda_t \mathcal{G}_{\beta_b}) && (8.124)\\ \dot{\mathcal{G}}_{\beta_3} &= \beta_a J^a - a_u + \partial_u \lambda_t && (8.125) \end{aligned}$$

yielding

$$a_{\beta_3} = \beta_3 = 0; \quad a_u = \beta_a J^a + \partial_u \lambda_t. \tag{8.126}$$

In agreement with the general results of Sec. 8.1.1, we can conclude that, out of the $2n+4 = 10$ constraints of the model, four of them are second class, namely

$$\mathcal{G}_{\lambda_u}, \quad \pi_{\lambda_u}, \quad \mathcal{G}_{\beta_3}, \quad \pi_{\beta_3}. \tag{8.127}$$

The dynamics is retrieved by the total Hamiltonian $H_1$, with canonical Poisson brackets (8.108)-(8.113) and some of the Lagrange multipliers fixed by Eq. (8.126)

$$H_1 = \int du \, \left[ \beta_a \mathcal{G}_{\beta_a} + \lambda_t \mathcal{G}_{\lambda_t} + a_u \pi_{\lambda_u} + a_t \pi_{\lambda_t} + a_{\beta_a} \pi_{\beta_a} \right], \quad a = 1, 2. \tag{8.128}$$

It may be easily verified that the algebra of gauge generators

$$K(\beta, \lambda_t) = \int du \, \left[ \beta_a \mathcal{G}_{\beta_a} + \lambda_t \mathcal{G}_{\lambda_t} + a_t \pi_{\lambda_t} + a_{\beta_a} \pi_{\beta_a} \right], \quad a = 1, 2. \tag{8.129}$$

closes according to Eq. (8.50).

To close this section we apply the results of 8.2.1 to the case of $SU(2)$ for the integration of the fields $\eta$ and $\lambda$. The resulting action is here adapted as

$$S_2 = -\frac{1}{2} \int_\Sigma \langle d\vartheta, (g^{-1}dg) \wedge (g^{-1}dg) \rangle, \tag{8.130}$$

and by writing $d\vartheta$ explicitly we have

$$S_2 = -\frac{1}{2} \int_\Sigma \epsilon_{ab} (g^{-1}dg)^a \wedge (g^{-1}dg)^b = \int_\Sigma d^2u \, \epsilon_{ab} A^a J^b, \tag{8.131}$$

with degenerate $B$-field $B_{ab} = \epsilon_{ab}$, all other components being zero.

### 8.2.2 Integration on locally conformal symplectic manifolds

Let us now consider a $2n$-dimensional locally conformal symplectic manifold $M$ with the non-degenerate two-form $\omega$ and closed one-form $\alpha$ satisfying (7.6), or equivalently, and especially useful for our purposes, (7.7). We have from the latter

$$\Lambda = \omega^{-1}, \quad E^i = (\omega^{-1})^{ij} \alpha_j. \tag{8.132}$$

Therefore, by multiplying (8.5) with $(\omega)_{\ell i}$ we arrive at

$$(\omega)_{\ell i} (dX^i + \Lambda^{ij} \eta_j - E^i \lambda) = \omega_{\ell j} dX^j + \eta_\ell - \alpha_\ell \lambda = 0, \tag{8.133}$$

so that $\eta$ can be written as

$$\eta_\ell = -\omega_{\ell j} dX^j + \alpha_\ell \lambda. \tag{8.134}$$

Note that in this case, differently from contact manifolds, it is not possible to explicitly decouple $\eta$ and $\lambda$. However, on substituting (8.134), together with the second of Eqs. (8.132) into the action functional, after a few simple manipulations it is possible to verify that the terms proportional to $\lambda$ simplify out and we are left with

$$S_2 = \int_\Sigma \omega_{ij}\, dX^i \wedge dX^j = \int_\Sigma X^*(\omega), \tag{8.135}$$

where we have restored the pull-back map in the second equality. Note that this is formally of the same form as (8.78) and of the $A$-model but it differs from both cases. In particular, the role of the $B$-field is represented by the two-form $\omega$ which is non-degenerate and it is not closed since it satisfies (7.6), so in this case there is place for fluxes on the target. Obviously, if $\alpha = 0$ the manifold $M$ becomes a symplectic manifold and the theory reproduces the original $A$-model as a particular case.

The two models considered in this section are new to our knowledge; they cannot be obtained from the Poisson sigma model, unless adding additional degrees of freedom, and fully rely on the underlying Jacobi geometry of the target. The LCS model is especially interesting with respect to its property of being equivalent to a Lagrangian model on the tangent manifold TPM with a two form which is neither degenerate nor closed. In next section we shall see a dynamical generalization of Jacobi sigma models, where this issue will be discussed again.

## 8.3 Dynamical Jacobi

In this section we consider a non-topological extension of the Jacobi sigma model which generalizes the approach proposed in [95] for the Poisson sigma model. As we already briefly discussed in Sec. 6.1, it is possible to add a simple non-topological term to the Poisson sigma model action, which is just a Casimir function on the target manifold, so that it does not spoil the gauge invariance. However, another modification is possible, which might have interesting string applications, in which a dynamical term containing both the metric on $\Sigma$ and on $M$ is considered.

The action for the dynamical model gets modified with respect to the topological action analyzed so far, according to:

$$S(X,\eta,\lambda) = \int_\Sigma \left[\eta_i \wedge dX^i + \frac{1}{2}\Lambda^{ij}(X)\, \eta_i \wedge \eta_j - E^i(X)\, \eta_i \wedge \lambda + \frac{1}{2}(G^{-1})^{ij}(X)\, \eta_i \wedge \star\eta_j\right], \tag{8.136}$$

where the metric on the worldsheet $\Sigma$, $g = diag(1,-1)$, is implemented via the Hodge star operator $\star$, while $G$ is a metric tensor on the target Jacobi manifold $M$.

Since $G$ is non-degenerate by definition, this allows us to integrate the auxiliary fields for a generic Jacobi manifold $M$ so to obtain a Polyakov string action for the embedding maps $X$, as we will see. In fact, the new equations of motion are

$$dX^i + \Lambda^{ij}\eta_j - E^i\lambda + (G^{-1})^{ij} \star \eta_j = 0, \tag{8.137}$$

$$d\eta_i + \frac{1}{2}\partial_i\Lambda^{jk}\eta_j \wedge \eta_k - \partial_i E^j\eta_j \wedge \lambda + \frac{1}{2}\partial_i(G^{-1})^{jk}\eta_j \wedge \star\eta_k = 0, \tag{8.138}$$

$$E^i\eta_i = 0. \tag{8.139}$$

Being $G$ naturally non-degenerate we can solve (8.137) for $\star\eta$,

$$\star\, \eta_j = -G_{ij}\left(dX^i + \Lambda^{ik}\eta_k - E^i\lambda\right) \tag{8.140}$$

and obtain $\eta$ by applying the Hodge star to the latter

$$\eta_p = -(M^{-1})^j{}_p G_{ij}\left(\star dX^i - \Lambda^{ik}G_{\ell k}dX^\ell + \Lambda^{ik}G_{\ell k}E^\ell\lambda - E^i \star \lambda\right), \tag{8.141}$$

with $M^p{}_j = \delta^p{}_j - G_{ji}\Lambda^{ik}G_{k\ell}\Lambda^{\ell p}$ a symmetric matrix, which we may assume to be non-degenerate irrespective of the rank of $\Lambda$. The action becomes then

$$\begin{aligned} S(X,\lambda) = \int_\Sigma \Bigg[&\frac{1}{2}(M^{-1})^p{}_i G_{jp}\, dX^i \wedge \star dX^j - \frac{1}{2}(M^{-1})^p{}_i G_{\ell p}\Lambda^{\ell k}G_{jk}\, dX^i \wedge dX^j \\ &- \frac{1}{2}(M^{-1})^p{}_i G_{\ell p}\Lambda^{\ell k}G_{mk}E^m\lambda \wedge dX^i + \frac{1}{2}(M^{-1})^p{}_i G_{\ell p}E^\ell \star \lambda \wedge dX^i\Bigg]. \end{aligned} \tag{8.142}$$

In order to integrate out the remaining auxiliary field, $\lambda$, we use the inner product on the space of one-forms,

$$\int_\Sigma \star\lambda \wedge dX = -\int \lambda \wedge \star dX, \tag{8.143}$$

so that (8.142) $\lambda$ becomes nothing more than a Lagrange multiplier imposing the geometric constraint

$$(M^{-1})_{i\ell}\left(\Lambda^{\ell k}G_{mk}E^m dX^i + E^\ell \star dX^i\right) = 0. \tag{8.144}$$

This finally leads to the result that the term proportional to $\lambda$ vanishes on-shell and we are left with the second order action

$$S = \int_{\Sigma} \left[ g_{ij} dX^i \wedge \star dX^j + B_{ij} dX^i \wedge dX^j \right] \tag{8.145}$$

where the metric $g$ and the $B$-field are defined in terms of $G$ and $M$ according to:

$$g_{ij} = G_{jp} (M^{-1})^p{}_i, \quad B_{ij} = G_{ik} (M^{-1})^p{}_j G_{p\ell} \Lambda^{\ell k}. \tag{8.146}$$

To summarize, we have obtained a non-linear sigma model action, with target space a Jacobi manifold, represented by Eq. (8.145). The Jacobi bivector field $\Lambda$ enters the definition of the metric and the B-field, while the Reeb vector field $E$ is part of the constraint equation (8.144).

### 8.3.1 Dynamical model on $SU(2)$

To give an example of the Polyakov action obtained in (8.145) we consider again the $SU(2)$ group manifold as target, so to obtain the dynamical completion to the topological model already considered in Sec 8.2.1. In particular, as a metric tensor on the target we introduce the natural Cartan–Killing metric on $SU(2)$: $G_{ij} = \delta_{ij}$. By using $G_{ij} = \delta_{ij}$ and $\Lambda^{ij} = \epsilon^{3ij}$, the metric $g$ and $B$-field are then obtained from (8.146) as

$$g_{ij} = h_{ij} = \delta_{ij} - \frac{1}{2} \epsilon_{ik3} \delta^{kl} \epsilon_{jl3}, \quad B_{ij} = -\frac{1}{2} \epsilon_{3ij}, \tag{8.147}$$

so to have

$$S = \int_{\Sigma} \left[ h_{ij} (g^{-1} dg)^i \wedge \star (g^{-1} dg)^j - \frac{1}{2} \epsilon_{3ij} (g^{-1} dg)^i \wedge (g^{-1} dg)^j \right]. \tag{8.148}$$

From the analysis of the previous section we know that this action has to be complemented with the geometric constraint in Eq. (8.144), i.e. in this case

$$(g^{-1} dg)^3 = 0. \tag{8.149}$$

It is interesting to note the form of the background metric $h$ in (8.147). This metric has been already obtained in the context of Poisson–Lie duality of $SU(2)$ sigma models in Chapter 5 as a non-degenerate metric for the group manifold of $SB(2, \mathbb{C})$, the Borel subgroup of $SL(2, \mathbb{C})$ of upper triangular matrices with complex elements with real diagonal and unit determinant. The latter plays the role of the Poisson-Lie dual of $SU(2)$ in the Manin triple decomposition of the group $SL(2,\mathbb{C})$. Therefore, it is an interesting question to understand the possible relation between the two models. Interestingly, Poisson-Lie groups are discussed in [92] in relation with Jacobi structures. Furthermore, Poisson sigma models have already been analyzed in relation with Poisson-Lie duality [156].

# 9 Conclusion

## 9.1 Conclusions

In this thesis we have reviewed and discussed mathematical and physical aspects of sigma models in the particular framework of geometry and dualities.

Initially, we have discussed the fundamental aspects of Poisson geometry, which is necessary to define the Poisson sigma model, and especially we introduced the concept of Poisson-Lie groups and Drinfel'd doubles, which are of central importance for Poisson-Lie T-duality. Then, we introduced the concept of T-duality from the scratch, giving a general understanding of all the types: Abelian, non-Abelian and Poisson-Lie T-duality. The latter represents a genuine generalization, since it does not require isometries at all, while Abelian and non-Abelian cases can be obtained as particular instances. In the second part of the thesis we introduce the Poisson sigma model, a topological sigma model which was first introduced in relation with two-dimensional field theories with non-trivial target space, e.g. gauge and gravity models, as well as gauged WZW models. One interesting feature of the model is its intimate relation with the geometry of the target space. Indeed, it makes it possible to unravel mathematical aspects of such manifolds by employing techniques from field theory. Then, we reviewed the geometry of a Jacobi manifold, which is a natural generalization of the Poisson manifold (see [161, 162] for another example of generalization), in which the vanishing of the Schouten bracket for the bivector is violated, although in a controlled way, by the introduction of the Reeb vector field. In particular, this violation can be considered as an example of twisted Poisson bracket [138, 163–165] (see also [166] and references therein). Since the Poisson sigma model is intimately related with the geometry of Poisson sigma models, generalizing this model gives the opportunity to extend such geometric aspects of sigma models to the more general Jacobi manifolds.

The bulk of the novel results of this thesis are in Chapters 5 and 8.

In Chapters 5 we focused on duality aspects. In particular, since the appropriate geometric setting to investigate issues related to Poisson-Lie duality is that of dynamics on group manifolds, in this thesis we have considered sigma models on Lie groups. We started by considering the dynamics of the three-dimensional isotropic rigid rotator described as a $(0+1)$-dimensional sigma model with the group manifold of $SU(2)$ as target space. We also considered its dual model on the group $SB(2,\mathbb{C})$, which is Poisson-Lie dual to $SU(2)$, in the spirit of outlining their connection with Poisson-Lie sigma models. We have analyzed the two models from the Poisson-Lie duality point of view and we built a doubled generalized model with $TSL(2,\mathbb{C})$ as carrier space. This was done with the purpose of exploring more deeply the relations between Poisson-Lie symmetries, Double Geometry and Generalized Geometry in a particularly simple system so that the framework could be more easily and explicitly understood. However, the isotropic rigid rotator is just a toy model, representing a sigma model in $(0+1)$ dimensions, which was useful however to pave the way to the analysis of its true field theory generalization, which is the principal chiral model. The latter is a $(1+1)$-dimensional sigma model which,while being modeled on the IRR system, certainly exhibits interesting properties under duality transformations. In this thesis we only gave a general description of the model but we have not explicitly showed its features, which were obtained in [67]. This is because we analyzed a way more general model, which is the so-called Wess-Zumino-Witten model, that we intend as a principal chiral model with the addition of a topological term called the Wess-Zumino term.

Starting from a canonical generalization of the Hamiltonian picture associated to the WZW model with $SU(2)$ target configuration space, which consists in describing the dynamics of the model in terms of a one-parameter family of Hamiltonians and $SL(2,\mathbb{C})$ Kac-Moody algebra of currents, we have highlighted the Drinfel'd double nature of the phase space, by introducing a further parameter both in the Hamiltonian and in the Poisson algebra. Our first result has been to show the Poisson-Lie symmetry of the model. Then, by performing a duality transformation in target phase space, we have been able to obtain a two-parameter family of models which are Poisson-Lie dual to the previous ones by construction. The two families share the same target phase space, the group manifold of $SL(2,\mathbb{C})$, but have configuration spaces which are dual to each other, namely $SU(2)$ and its Poisson-Lie dual, $SB(2,\mathbb{C})$. Although they have not been derived from an action principle, it has been shown that it is possible to exhibit an action, by means of an inverse

Legendre transform which involves the symplectic form and the Hamiltonian. As a natural step, we have investigated the possibility of defining a Lagrangian WZW model with target tangent space $TSB(2,\mathbb{C})$. Being the group $SB(2,\mathbb{C})$ not semi-simple, the problem of defining a non-degenerate product on its Lie algebra has been addressed, and a solution has been proposed. Once accomplished the Lagrangian picture, we have derived the Hamiltonian description on the cotangent space $T^*SB(2,\mathbb{C})$. We have shown that, although its current algebra is obtained as a particular limit ($\alpha \to 0$, with $\alpha$ a deformation parameter) of the $SL(2,\mathbb{C})$ Kac-Moody algebra related to the dual family, it is not possible to obtain the Hamiltonian in the same limit, through a continuous deformation of phase spaces. It is however possible to define on $T^*SB(2,\mathbb{C})$ a new Hamiltonian in terms of an alternative O(3, 3) metric. Such a model can be related to the dual family of $SL(2,\mathbb{C})$ models if one first performs a deformation of the dynamics and then the limit $\alpha \to 0$. It is interesting to notice that such a connection relies on the presence of the WZ term, and the whole construction loses significance if the WZ term is not present. A diagrammatic summary of the different models with corresponding relations is depicted in Fig. 5.1. Having introduced a well-defined WZW action on $SB(2,\mathbb{C})$ we have analyzed the geometry of the target space as a string background solution. This is a non-compact Riemannian hypersurface, whose metric is induced by a Lorentzian metric. The B-field and its flux have been calculated as well. Finally, we have addressed the possibility of making manifest the $SL(2,\mathbb{C})$ symmetry of both families of WZW models, by doubling the degrees of freedom and introducing a parent action with target configuration space the Drinfel'd double $SL(2,\mathbb{C})$. A doubled Hamiltonian formulation has been proposed, such that a restriction to either subgroup, $SU(2)$ or $SB(2,\mathbb{C})$, leads to the Hamiltonian formulation of the two sub-models.

Looking for more geometrical properties of sigma models, as well as further generalization of the framework used for Poisson-Lie T-duality, we have then defined and analyzed a two-dimensional sigma model with target space a Jacobi manifold, as a natural generalization of a Poisson sigma model. It is a two-dimensional topological sigma model, being also a non-linear gauge theory describing strings moving on a Jacobi background. It can be related to a field theory with a higher dimensional target, which is a Poisson sigma model for the 'Poissonized' manifold $M \times \mathbb{R}$.

The so called Poissonization procedure consists in the construction of a homogeneous Poisson structure on $M \times \mathbb{R}$ from the Jacobi structure on the

Jacobi manifold $M$. The two models may be seen to yield the same dynamics, after reduction, provided we impose extra constraints at the boundary. We have analyzed the canonical formulation of the model, which exhibits first and second class constraints, with the former generating gauge transformations. Interestingly, it is possible to establish an homomorphism between the algebra of gauge transformations and the algebra of sections of the 1-jet bundle $J^1M$ , which generalizes an analogous result for the Poisson sigma model, where the role of $J^1M$ is played by $T^*M$. The reduced phase space of the model, which is obtained as the manifold of constraints modulo gauge symmetries, has finite dimension, equal to $2\dim M - 2$. Two main classes of target spaces have been explicitly considered, namely contact and locally conformal symplectic manifolds. We have shown that in both cases the auxiliary fields can be integrated out and a second-order action description in terms of the sole embedding maps can be given. In the case of the Poisson sigma model, this is only possible if the target manifold is symplectic, so that the Poisson bivector can be inverted; in such a case the resulting theory is that of a A-model and the B-field is the symplectic two-form. For the models at hand we obtain different results: on contact manifolds the resulting B-field is the exterior derivative of the contact one-form, which is closed but degenerate, while for the locally conformal symplectic manifolds the B-field is the LCS two-form $\omega$ which is neither degenerate nor closed, allowing for the possibility of generating fluxes without the need to twist the model. A similar situation occurs for dynamical models (cfr. (8.136)). In view of the importance of fluxes in relation with string compactification, the occurrence of two-forms which are not closed in the context of LCS manifolds is, therefore, interesting and needs to be further investigated. The original A-model of string theory is naturally recovered from the locally conformal symplectic case when the one-form is identically vanishing. The group manifold of $SU(2)$ has been considered as an explicit example of contact manifold. As for interesting examples of LCS manifolds, we have shortly reviewed a constructive procedure due to Vaisman and shown that the manifold $SU(2) \times U(1)$ may be endowed with a Jacobi structure. Examples of Jacobi manifolds which are neither contact nor LCS may be found in [92]. They include dual algebras of Poisson–Lie groups, which we think could be of interest in the context of Poisson–Lie T–Duality. We plan to address the problem in the future. Finally, we have reviewed a dynamical extension of the model, which is obtained by adding a metric term to the action functional. On integrating out the auxiliary fields, a Polyakov action is obtained, with a metric and B-field, which are explicitly written in terms of the Jacobi bivector field $\Lambda$. The model is supplemented by a geometric constraint which is related to the Reeb vector field.

## 9.2 Discussion and future perspectives

As for future perspectives for the duality aspect of the work, it would be interesting to quantize the interpolating model, and since it depends on two further parameters, it would be worth looking at conformal invariance in the quantum regime. In this respect however, it should be recalled that finite dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are non-unitary (see [79] for related analysis of the one-parameter family, in the case of $\tau$ real). On the other hand, such an alternative formulation seems to be well suited for a formal quantization in the sense of Drinfel'd [167], and this possibility could be explored in the future.

Furthermore, with respect to the doubled WZW action we presented in Sec. 5.4.1, the usual approach which requires to gauge one of the global symmetries of the parent action to obtain the dual models presents some difficulties, since minimal coupling is not enough anymore and there may be obstructions to be dealt with. Indeed, although minimal coupling produces a gauge-invariant action, the equations of motion still depend on the extension to the 3-manifold $\mathcal{B}$. This issue is addressed e.g. in [123–125], but besides that, another problem, which is specific of the model, might affect the gauging. In fact, in the cited references the gauged action is always formulated for a semisimple group with a Cartan-Killing metric. However, here in order to reproduce the $SB(2,\mathbb{C})$ model we need to work with an Hermitian product. It is not clear how to handle the problem in this case and we plan to further investigate.

Finally, it is interesting to understand if this procedure can be carried out for general groups. In particular, it would be useful and interesting to understand if this might work only for some special examples of Drinfel'd doubles and Manin triple decompositions, and, if this is the case, what are the particular features of these structures.

As a further goal, we believe this work may contribute to the analysis of string theories on AdS geometries, the study of which would be interesting from the AdS/CFT correspondence perspective.

Future directions of research for the Jacobi sigma model include the quantization of the model, its relation with Poisson-Lie symmetry and duality and the groupoid structure of the reduced phase space. In particular, since the algebra of gauge transformations is open, the formalism required to appropriately quantize it is the Batalin-Vilkovisky one.

Since it is known that the reduced phase space of the Poisson sigma model is the symplectic groupoid integrating the Lie algebroid associated to the Poisson structure of the target manifold, it is natural to think that a similar construction can be carried out for the Jacobi sigma model too, giving some geometric relation with the target Jacobi manifold. This is actually a work in progress.

Moreover, the possibility of having non-closed B-fields in the context of LCS manifolds, both for the topological and dynamical models, shall be further investigated, as well as the nature of the new models obtained.

# A Symplectic form of the deformed current algebra

In this section we will briefly sketch the derivation of the symplectic form (5.79) for the two-parameter family of models obtained in Sec. 5.2, as it is presented in [69]. The goal of this construction is to formulate an action principle from which the canonical formalism for the family of models can be obtained, in the first order formulation.

The Poisson algebra we start with is the one in (5.77), which we report for convenience:

$$\begin{aligned}
\{\tilde{K}_i(\sigma), \tilde{K}_j(\sigma')\} &= i\alpha\epsilon_{ij}{}^k \tilde{K}_k(\sigma)\delta(\sigma-\sigma') - \alpha^2 \hat{C}\delta_{ij}\delta'(\sigma-\sigma') \\
\{\tilde{S}^i(\sigma), \tilde{S}^j(\sigma')\} &= i\tau f^{ij}{}_k \tilde{S}^k(\sigma)\delta(\sigma-\sigma') + \tau^2 \hat{C} h^{ij}\delta'(\sigma-\sigma') \\
\{\tilde{K}_i(\sigma), \tilde{S}^j(\sigma')\} &= \left[i\alpha\epsilon_{ki}{}^j \tilde{S}^k(\sigma) + i\tau f^{jk}{}_i \tilde{K}_k(\sigma)\right]\delta(\sigma-\sigma') \\
&\quad + (i\alpha\hat{C}'\delta_i^j - i\tau\hat{C}\epsilon_i{}^{j3})\delta'(\sigma-\sigma').
\end{aligned} \tag{A.1}$$

Let, $X_{\tilde{K}}$ and $X_{\tilde{S}}$ indicate the Hamiltonian vector fields associated with the currents, so that $\omega(X_{\tilde{K}_i}, X_{\tilde{K}_j}) = \{\tilde{K}_i(x), \tilde{K}_j(x)\}$, with analogous expressions for the other brackets. They are left-invariant because so are the currents. On introducing their dual one-forms $\theta^i, \hat{\theta}_i$, with $\theta^i X_{\tilde{K}_j} = \delta^i{}_j$ and $\hat{\theta}_i X_{\tilde{S}^j} = \delta_i{}^j$, the Poisson brackets in (A.1) can be easily obtained from the following symplectic form

$$\begin{aligned}
\omega = \int_{\mathbb{R}^2} d\sigma\, d\sigma' \Big\{ &\theta^i(\sigma)\wedge\theta^j(\sigma')\left[i\alpha\epsilon_{ij}{}^k \tilde{K}_k(\sigma)\delta(\sigma-\sigma') - \alpha^2\hat{C}\delta_{ij}\delta'(\sigma-\sigma')\right] \\
&+ \hat{\theta}_i(\sigma)\wedge\hat{\theta}_j(\sigma')\left[i\tau f^{ij}{}_k \tilde{S}^k(\sigma)\delta(\sigma-\sigma') + \tau^2\hat{C}h^{ij}\delta'(\sigma-\sigma')\right] \\
&+ \theta^i(\sigma)\wedge\hat{\theta}_j(\sigma')\Big[\left(i\alpha\epsilon_{ki}{}^j\tilde{S}^k(\sigma) + i\tau f^{jk}{}_i\tilde{K}_k(\sigma)\right)\delta(\sigma-\sigma') \\
&\qquad + \left(i\alpha\hat{C}'\delta_i^j - i\tau\hat{C}\epsilon_i{}^{j3}\right)\delta'(\sigma-\sigma')\Big]\Big\}.
\end{aligned} \tag{A.2}$$

The latter may be further manipulated and expressed in terms of the original group valued fields $g \in SU(2)$ and $\ell \in SB(2,\mathbb{C})$. This can be obtained by means of the left invariant Maurer-Cartan 1-forms relative to each of the two groups, which read explicitly

$$g^{-1}dg = i\theta^i e_i, \quad \ell^{-1}d\ell = i\hat{\theta}_i\hat{e}^i. \tag{A.3}$$

Hence, by defining

$$-i\alpha\hat{C}g^{-1}\partial_\sigma g = i\delta^{kp}\tilde{K}_p e_k, \quad i\tau\hat{C}\ell^{-1}\partial_\sigma\ell = i(h^{-1})_{kp}\tilde{S}^p\hat{e}^k \tag{A.4}$$

it is possible to show that the symplectic form (A.2) can be written in terms of $g$ and $\ell$ as in (5.79). Since it is not immediate to see that the two expressions are equal, we shall go through the main steps for the first term of (5.79), namely $\int d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg \wedge \partial_\sigma(g^{-1}dg)\right]$, as for the others it works in the same way. The starting point is to decompose the Maurer-Cartan 1-form in its Lie algebra components:

$$\begin{aligned}
&\alpha^2\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg \wedge \partial_\sigma(g^{-1}dg)\right] \\
&= \alpha^2\hat{C}\int_{\mathbb{R}} d\sigma\, \mathrm{Tr}_{\mathcal{H}}\left[-g^{-1}dg \wedge g^{-1}\partial_\sigma g g^{-1}dg + g^{-1}dg \wedge g^{-1}\partial_\sigma dg\right] \\
&= -\alpha\int_{\mathbb{R}} d\sigma \mathrm{Tr}_{\mathcal{H}}\left[\theta^i \wedge \delta^{kp}\tilde{K}_p e_i e_k \theta^j e_j\right] \\
&+ \alpha^2\hat{C}\int_{\mathbb{R}^2} d\sigma d\sigma' \delta(\sigma-\sigma')\mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg(\sigma') \wedge g^{-1}\partial_\sigma dg(\sigma')\right] \\
&= i\alpha\int_{\mathbb{R}^2} d\sigma\, d\sigma' \delta(\sigma-\sigma')\theta^i(\sigma) \wedge \theta^j(\sigma')\epsilon_{ij}{}^k\tilde{K}_k(\sigma) \\
&+ \alpha^2\hat{C}\int_{\mathbb{R}^2} d\sigma d\sigma' \partial_\sigma\left\{\delta(\sigma-\sigma')\mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg(\sigma') \wedge g^{-1}dg(\sigma)\right]\right\} \\
&+ \alpha^2\hat{C}\int_{\mathbb{R}^2} d\sigma d\sigma' \delta'(\sigma-\sigma')\mathrm{Tr}_{\mathcal{H}}\left[g^{-1}dg(\sigma) \wedge g^{-1}dg(\sigma')\right] \\
&= \int_{\mathbb{R}^2} d\sigma\, d\sigma' \theta^i(\sigma) \wedge \theta^j(\sigma')\left[i\alpha\epsilon_{ij}{}^k\tilde{K}_k(\sigma)\delta(\sigma-\sigma') - \alpha^2\hat{C}\delta_{ij}\delta'(\sigma-\sigma')\right],
\end{aligned}$$

which is indeed the first term in (A.2). In the last equation we used the antisymmetry property of the wedge product. Similar calculations can be performed to obtain the remaining terms.

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