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On spaces with o-O structure and applications

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Contents

Chapter 1

Introduction

In [14] a new function space $\mathcal{B}$ was introduced by J. Bourgain, H. Brezis and P. Mironescu, along with its subspace $\mathcal{B}_0$. This space was introduced in relation to the problem of finding the smallest function space such that an integer-valued function in this space must be constant: it is known that this property holds for the spaces VMO, $W^{1,1}$ and the family of spaces $W^{1/p,p}$ with $1 < p < \infty$, and the authors prove that the space $\mathcal{B}_0$ contains these spaces while still keeping the required property.
This space sparked interesting research: for example a new formula for the perimeter of sets was found [3, 4], using a variant of the $\mathcal{B}$ norm, which led to the discovery of oscillation-type norms for Sobolev spaces and BV spaces [26, 27]; more research inspired by this space can be found in [21, 40, 43, 54, 24].
This pair of spaces also sparked interest in pairs of spaces where one of the spaces has a "big-O" type definition, and the other space is a subspace of the first space whose elements satisfy a corresponding "little-o" property.
In [45], K. M. Perfekt introduced a framework to study these spaces. A pair of Banach spaces (E_0, E) form a little-o/big-O structure if they are of the form

$$E = \left\{ x \in X : \|x\|_E = \sup_{L \in \mathfrak{L}} \|Lx\|_Y < \infty \right\}$$

and

$$E_0 = \left\{ y \in E : \limsup_{L \to \infty} \|Lx\|_Y = 0 \right\}$$

for some Banach spaces X, Y and $\mathfrak{L} \subset \mathcal{L}(X, Y)$ a set of linear maps from X to Y having suitable properties. A more precise definition will be given in Section 3. Under additional assumptions, there exists a canonical isometry between the second dual E_0^{**} of E_0 and E, and they also satisfy additional properties: among others, one can use the functionals L to characterize the distance of an element of E (identified with E_0^{**}) from the subspace E_0, and it is possible to prove that E_0 is an M-embedded space, i.e. the space E^* admits an ℓ^1 decomposition

$$E^* \simeq E_0^* \oplus_{\ell^1} E_0^{\perp},$$

where $E_0^{\perp}$ denotes the set of elements of E^* that annihilate E_0, identified as a subspace of E. One can also prove some properties of the intermediate space $E_* := E_0^*$: for example, the M-embedding property of E_0 implies that E_* is the unique predual of E up to isometry; moreover, we can find a decomposition formula for the elements of E_* in terms of the operators L: we can say that the elements of E_* have atomic decomposition. Additional properties for these spaces can be found in [46, 47, 41].

In this paper we will show examples of little-o/big-O structures, along with some applications. In Section 3.3 we consider a classical example: after recalling the definition of the space BMO of functions of bounded mean oscillation introduced by F. John and L. Nirenberg in [33]:

$$f \in BMO \Leftrightarrow \sup_Q \frac{1}{|Q|} \int_Q \left| f(x) - \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \right| dx < \infty,$$

where the supremum is taken between all cubes having sides parallel to the axes, and its subspace VMO of functions of vanishing mean oscillations introduced by D. Sarason in [50];

$$g \in VMO \Leftrightarrow \lim_{|Q| \to 0} \frac{1}{|Q|} \int_Q \left| g(x) - \left(\frac{1}{|Q|} \int_Q g(y)\, dy \right) \right| dx = 0,$$

we show how the little-o/big-O construction can be applied to this pair and recover classical properties of these spaces, as well as the intermediate space, the (real) Hardy space $\mathcal{H}^1$.

Our next example [8] involves a class of rearrangement-invariant Banach spaces, called Marcinkiewicz (endpoint) spaces. After defining some basic notions on rearrangement-invariant spaces, such as the non-increasing rearrangement f^* of a measurable function f:

$$f^*(t) = \sup_{\mu(E) > t} \operatorname{essinf}_{x \in E} |f(x)|$$

and the maximal (non-increasing) rearrangement function f^{**} of f:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds = \sup_{\mu(E) = t} \frac{1}{t} \int_E |f(x)|,$$

we introduce the spaces M^φ as the space of all functions such that

$$\sup_t \varphi(t) f^{**}(t) < \infty$$

for a function $\varphi : [0, T) \to [0, \infty)$ (with $T \in (0, \infty]$) having the property of being quasi-concave, i.e. it attains the value 0 for $t = 0$ and the functions $\varphi(t)$ and $t/\varphi(t)$ are both non-decreasing for $t > 0$. This class includes positive-valued concave functions φ such that $\varphi(0) = 0$, such as power functions t^α for $\alpha \in [0, 1]$.
The definition of these spaces through the supremum of a quantity makes them a natural candidate for being a big-O space. The little-o space can be proven to be the closure M_b^φ of L^∞ in the Marcinkiewicz space M^φ by proving a distance formula between elements of M^φ and L^∞:

$$\operatorname{dist}_{M^\varphi}(f, L^\infty) = \limsup_{t \to 0^+} \varphi(t) f^{**}(t).$$

We also need to make some assumptions on the function φ so that all the necessary properties of the little-o/big-O construction actually work.

The following topic is Orlicz spaces [6]. These spaces [49, 17] generalize the L^p spaces by replacing the power p in the condition

$$\int |f|^p < \infty$$

with a general function A:

$$\int A(|f|) < \infty.$$

With a small modification on the integral condition to make sure to obtain a linear space - more precisely, one requires the existence of a number $k > 0$ such that $\int A(k|f|)$ is finite - one obtains that the corresponding function space L^A is a (rearrangement invariant) Banach space, provided the function A is an Orlicz function, i.e. it is continuous, increasing and convex. Some classical examples are the exponential spaces EXP_α, which are involved in the classical Sobolev space embedding for the limit case $p = N$ [55, 44], which correspond to $A(t) \simeq e^{t^\alpha}$, the Zygmund spaces $L^p \log^\beta L$, corresponding to $A(t) \simeq t^p \log^\beta t$.
The strategy used in [6] was to reduce this case to the case of a Marcinkiewicz space: it is well known [11] that the exponential space $EXP = EXP_1$ defined on a probability space admits the Marcinkiewicz norm

$$\sup_t (1 + |\log t|)^{-1} f^{**}(t)$$

as a norm which is equivalent to other classical norms for EXP. We therefore studied conditions for an Orlicz space to admit a Marcinkiewicz norm, finding the condition

$$\int A\left(ka\left(\frac{1}{t}\right)\right) < \infty$$

for some $k > 0$, where a is the inverse function of A. We also give examples of Orlicz functions satisfying this condition: examples include nested exponentials

$$A(t) \simeq \exp(\exp(t)), \quad A(t) \simeq \exp(\exp(\exp(t))) \text{ etc.},$$

and in general functions with growth for large t of the form $\exp(\nu(t))$, where ν is a convex function. We also prove that functions with growth of the form $\exp(t)\exp^{\alpha}(\exp(t))$ satisfy the condition if and only if α is greater or equal to the threshold value 1, giving both an affirmative example and a counterexample to the main result.

In chapter 5 we consider spaces defined by giving conditions on the oscillations of its functions. The first example is the space of Lipschitz functions on a compact metric space. Duality and biduality problems about Lipschitz spaces have been studied thoroughly in many papers: in one of the first of these papers, de Leeuw [22] proved that the Hölder space

$$C^{0,\alpha}([0,1]) = Lip_{\alpha}([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \mid \sup_{\substack{x,y \in S \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} < \infty \right\},$$

with $\alpha \in (0,1)$, is the double dual of

$$c^{0,\alpha}([0,1]) = lip_{\alpha}([0,1]) = \left\{ f \in Lip_{\alpha}([0,1]) : \limsup_{|x-y| \to 0} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = 0 \right\}.$$

Hölder spaces can be considered as Lipschitz spaces when we replace the usual euclidean distance $d_{eucl}(x,y) = |x-y|$ with $d_{\alpha}(x,y) = |x-y|^{\alpha}$: this procedure of replacing d with d^{α} is usually called snowflaking.
There is a wide literature on the study of duality and biduality properties of Lipschitz spaces: see for instance [59, 9, 12, 34, 56, 36, 57]. In [5] and [7] we mainly follow the approach found in [30, 31]. Here, we consider the space of finite signed Borel measures $\mathfrak{M}(S)$ of a compact metric space (S,d) endowed with the Kantorovich-Rubinstein norm [37, 38, 39], which is inspired by optimal transport.
In [30] a characterization of compact metric spaces (S,d) such that $(lip(S,d))^{**} \simeq$

$Lip(S,d)$ is shown, by proving that for those spaces $(lip(S,d))^*$ can be identified with the completion of $\mathfrak{M}(S,d)$ with respect to the Kantorovich-Rubinstein norm, for which it is known that $(\mathfrak{M}(S,d))^* \simeq Lip(S,d)$. In [5] we show that the condition is equivalent to the o-O structure of $(lip(S,d), Lip(S,d))$, under the assumption that (S,d) is a doubling compact metric space, and show a "weak" atomic decomposition result for $\mathfrak{M}(S,d)$, while in [7] we study the case of euclidean distance, for which lip is trivial, and consider the problem of atomic decomposition for $\mathfrak{M}(K,d)$ in this case.

Finally, we consider the family of spaces $BMO_{(s)}$, which can be defined through a uniform exponential condition on oscillations:

$$f \in BMO_{(s)} \Leftrightarrow \sup_Q \left\| f - \frac{1}{|Q|}\int_Q f \right\|_{EXP} < \infty$$

and the corresponding vanishing space $VMO_{(s)}$. These spaces were introduced in [52] and later studied in [53, 2, 1] in relation to the family of spaces $X_{(s)}$, a family of spaces defined by atomic decomposition spanning from $L^1_0 = \left\{ f \in L^1 : \int f = 0 \right\}$ to the Hardy space $\mathcal{H}^1$.
Here we will show some properties of these spaces shown in [42], of future publication. In particular, we find the rearrangement-invariant hull of $VMO_{(s)}$, and in particular of VMO, prove that $VMO_{(s)}$ and $BMO_{(s)}$ form a o-O pair and thus a distance formula between elements of $BMO_{(s)}$ and the subspace $VMO_{(s)}$, and a Sobolev type embedding:

$$W^1 L^{N,q} \hookrightarrow VMO_{\left(\frac{q}{q-1}\right)}.$$

Chapter 2

Preliminary notation

- $\mathbb{N}$ denotes the set of all positive integers. $\mathbb{N}_0$ denotes the set of all non-negative integers.
- All vector spaces are considered over the field of real numbers $\mathbb{R}$.
- The letter N will denote the dimension of the euclidean space $\mathbb{R}^N$ we consider.
- The Lebesgue measure of the measurable set $E \subseteq \mathbb{R}^N$ will be denoted by $|E|$.
- For a measure space (Ω, μ) we denote by $\mathcal{M}(\Omega)$ the space of all measurable functions on Ω, by $\mathcal{M}_0(\Omega)$ the subset of all elements of $\mathcal{M}(\Omega)$ that are finite μ-almost everywhere and $\mathcal{M}_+(\Omega)$ the subset of all non-negative elements of $\mathcal{M}_0(\Omega)$.
- All cubes considered in this paper have sides parallel to the axes, i.e. they are of the form

  $$[x_1, x_1 + l] \times \cdots \times [x_N, x_N + l]$$

  for some $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and side length $l > 0$.
- The *characteristic function* χ_E of a set E is the function defined as

  $$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

- A *simple function* s is a finite linear combination of characteristic functions of sets of finite measure: in other words, there exists finitely many disjoint sets $E_1, \dots, E_k$ having finite measure and constants $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$s(x) = \sum_{j=1}^{k} \alpha_i \chi_{E_i}(x).$$

The set of all simple functions of (Ω, μ) is denoted by $\mathcal{S}(\Omega, \mu)$, or just $\mathcal{S}$ when the space is clear from the context.

- When S is a finite measure subset of a measure space (Ω, μ) and f is a function such that the restriction $f|_S$ belongs to $L^1(S)$, we denote by f_S, or by $⨍_S f(x)\,d\mu(x)$, the average of f on the set S:

$$f_S = ⨍_S f(x)\,d\mu(x) := \frac{1}{\mu(S)} \int_S f(x)\,d\mu(x).$$

- When X and Y are two Banach spaces, we will say that X embeds continuously into Y, and write $X \hookrightarrow Y$ if $X \subset Y$ and there exists a constant $C > 0$ such that $\|z\|_Y \leq C\|z\|_X$ for all $z \in X$.

- For $1 \leq p \leq \infty$, we denote by p' its dual exponent, defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

where we use the convention $1/\infty = 0$. If $p < N$, also denote by p^* its Sobolev conjugate exponent (in dimension N)

$$p^* = \frac{Np}{N-p}.$$

- If X is a Banach space and $\mathfrak{F} \subset X^*$, we define the weak topology $\sigma(X, \mathfrak{F})$ induced by $\mathfrak{F}$ to be the weakest topology such that the maps $f : X \to \mathbb{R}$ are continuous for all $f \in \mathfrak{F}$ (where $\mathbb{R}$ is equipped with the euclidean topology).

Chapter 3

Abstract o-O structure and properties

3.1 Basic definitions and properties

Consider a reflexive and separable Banach space X, a Banach space Y and $\mathfrak{L} \subset \mathcal{L}(X,Y)$ a set of bounded linear maps $L : X \to Y$ which is equipped with a topology τ which is Hausdorff, σ-compact, locally compact and such that the maps

$$T_x : L \in \mathfrak{L} \mapsto Lx \in Y$$

are continuous for all $x \in X$. We now define

$$E = E(X, \mathfrak{L}) = \left\{ x \in X : \|x\|_E := \sup_{L \in \mathfrak{L}} \|Lx\|_Y < \infty \right\} \tag{3.1}$$

and

$$E_0 = E_0(X, \mathfrak{L}) = \left\{ x \in E : \limsup_{L \to \infty} \|Lx\|_Y = 0 \right\} \tag{3.2}$$

where $L \to \infty$ is intended in the sense of the one-point compactification of $\mathfrak{L}$, i.e.

$$\limsup_{L \to \infty} \|Lx\|_Y = \sup \lim_{n \to \infty} \|L_n x\|_Y$$

where the supremum is taken among all sequences L_n such that for every compact $K \subset \mathfrak{L}$ we have $L_n \notin K$ for every $n \in \mathbb{N}$ sufficiently large.
One usually wants additional properties from these spaces: for instance, to have a meaningful structure, we assume that E, equipped with the norm $\| \cdot \|_E$, embeds continuously in X; moreover, to make sure the vanishing subspace E_0 is not too small, we make the following *Assumption AP*:

$$\forall x \in E \exists \{y_n\}_{n \in \mathbb{N}} \subset E_0 : y_n \overset{X}{\rightharpoonup} x, \sup_{n \in \mathbb{N}} \|y_n\|_{E_0} \leq \|x\|_E.$$

Theorem 3.1. *[45, 47] Assume that the pair (E_0, E) satisfies Assumption AP. Then there is a continuous embedding $I : X^* \to E_0^*$; moreover, the adjoint $J = I^* : E_0^{**} \to X$ induces an isometry between E_0^{**} and E. Moreover, E_0 is M-embedded, i.e. the decomposition (up to isometry)*

$$E^* = E_0^* \oplus_{\ell^1} E_0^{\perp}$$

holds.

Proposition 3.2. *Assume that the pair (E_0, E) satisfies Assumption AP. Then the following distance formula holds for all $x \in E$:*

$$dist_E(x, E_0) = \limsup_{L \to \infty} \|Lx\|_Y.$$

3.2 Properties of the intermediate space

In a o-O structure (E_0, E), it is also interesting to study properties of the intermediate space $E_* = (E_0)^*$. Combining 3.1 with [32, Proposition III.2.10] we get the following.

Proposition 3.3. *Let (E_0, E) be a o-O pair satisfying assumption AP. Then the space E_* is the unique isometric predual of E.*

Another important property is a decomposition formula for the elements of E_* in terms of simpler objects. We say that the elements of E_* enjoy what is called an *atomic decomposition*. This decomposition follows from the next proposition, which is a particular case of a classical result [23, 35], which is a general result concerning preduals of a Banach space, and can be applied to the pair (E_*, E), assuming $Y = \mathbb{R}$, even if assumption AP does not hold, so that in this case $E_* \neq (E_0)^*$.

Proposition 3.4. *Let E be a Banach space and $\mathfrak{F} = \{f_n\}_{n \in \mathbb{N}} \subset E^*$ be a countable norming set of linear functionals on E (i.e. $\|x\|_E = \sup_{f \in \mathcal{F}} |\langle f, x \rangle_E|$) such that $B_E = \{x \in E : \|x\|_E \leq 1\}$ is $\sigma(E, \mathfrak{F})$-compact.*
Then $E_* := cl_{E^*}(span(\mathfrak{F}))$ is an isometric predual of E, i.e. $(E_*)^* \cong E$; in particular E is the dual of a separable Banach space.

Remark 3.5. When $Y = \mathbb{R}$, the result is applied in the following way: the map

$$V : x \in E \mapsto \{\langle f_n, x \rangle_E\}_{n \in \mathbb{N}} \in \ell^\infty$$

induces an isometry between E and the weak-star closed subset $V(E)$ of ℓ^∞. The space E_* can thus be identified with $\ell^1/V(E)^\perp$, and the element f_n of $\mathfrak{F}$ can be identified with the equivalence class of $\delta_n = \{\delta_{j,n}\}_{j\in\mathbb{N}}$, where

$$\delta_{j,n} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

If Y is not $\mathbb{R}$, a similar construction (see [24, Theorem 3] and [5, Theorem 2.2] for details) can be performed with the space

$$\ell^1(Y^*) = \left\{ z = \{z_n\}_{n\in\mathbb{N}} \in (Y^*)^{\mathbb{N}} : \|z\|_{\ell^1(Y^*)} := \sum_{n\in\mathbb{N}} \|z_n\|_{Y^*} < \infty \right\}$$

replacing ℓ^1.

The weak compactness assumption can be obtained from X by using the following lemma.

Lemma 3.6. *Let X be a reflexive Banach space, $\mathfrak{F} \subset X^*$ and assume that the space E of all elements x of X such that*

$$\|x\|_E := \sup_{f\in\mathfrak{F}} |\langle f, x\rangle_X| < \infty$$

is a Banach space when endowed with the norm $\|\cdot\|_E$ and $E \hookrightarrow X$. Then $\mathbb{B}_E$ is $\sigma(E, \mathfrak{F})$-compact.

Proof. Without loss of generality, we assume that E is densely contained in X. Since $E \hookrightarrow X$, there exists $\lambda > 0$ such that $\mathbb{B}_E \subset \lambda\mathbb{B}_X$, which implies that $\mathbb{B}_E$ has weakly compact closure in X. But $\mathbb{B}_E$ is closed in X: for every net $\{x_\alpha\}_{\alpha\in A} \subset \mathbb{B}_E$ converging to $\tilde{x} \in X$ and for every $f \in \mathfrak{F}$ we have $\langle f, x_\alpha\rangle_X \to \langle f, \tilde{x}\rangle_X$, which implies that $|\langle f, \tilde{x}\rangle_X| \leq 1$, and since $|\langle f, x\rangle_X| < \infty$ if and only if $x \in E$ we have $\tilde{x} \in \mathbb{B}_E$, which implies that $\mathbb{B}_E$ is weakly compact in X, and in particular $\sigma(E, \mathfrak{F})$-compact in E. $\square$

3.3 Space of functions of bounded mean oscillation

As an example, we will now see how this construction can be applied to the classical space BMO of functions of bounded mean oscillation [33]. For simplicity, we will limit ourselves with functions defined on the unit cube Q_0.

Definition 3.7. We say that $f \in L^1_{loc}(Q_0)$ has *bounded mean oscillation*, and we write $f \in BMO(Q_0)$, if the quantity

$$[f]_* = \sup_{Q \subset Q_0} \fint_Q |f(x) - f_Q| \, dx \tag{3.3}$$

is finite, where the supremum is taken among all the subcubes Q of Q_0 with sides parallel to the axes.

Definition 3.8. We say that $g \in BMO(Q_0)$ has *vanishing mean oscillation*, and we write $f \in VMO(Q_0)$, if

$$\limsup_{\varepsilon \to 0} \sup_{\substack{Q \subset Q_0 \\ |Q| \leq \varepsilon}} \fint_Q |f(x) - f_Q| \, dx = 0. \tag{3.4}$$

The level sets of BMO functions satisfy an important inequality.

Proposition 3.9 (John-Nirenberg inequality)**.** *[33] There exist two constants* $c_1, c_2 > 0$*, such that for every* $u \in BMO(Q_0)$ *and every cube* $Q \subset Q_0$ *it holds*

$$|\{x \in Q : |u(x) - u_Q| > \lambda\} \leq c_1 \cdot exp\left(-\frac{c_2 \lambda}{[u]_*}\right) |Q|.$$

Corollary 3.10. *Let* $1 < p < \infty$*. Define*

$$[u]_{BMO_p} := \sup_{Q \subset Q_0} \left(\fint_Q |u(x) - u_Q|^p \, dx \right)^{1/p}.$$

Then $[u]_* \leq [u]_{BMO_p} \leq C_p [u]_*$*, with* $C_p = O(p)$*. In other words,* $[\cdot]_{BMO_p}$ *defines an equivalent norm in* BMO*.*

The spaces BMO and VMO are related to the (real) Hardy space $\mathcal{H}^1$. There are multiple ways to characterize this space [51]. Here we define this space through atomic decomposition.

Definition 3.11. We say that a function $a \in L^p(Q_0)$ is an L^p-atom if

- $\text{supp}(a) \subset Q$ for a cube Q;
- $\int_Q a(x) \, dx = 0$;
- $\|a\|_{L^p(Q)} \leq |Q|^{-1/p}$.

We say $f \in \mathcal{H}^1(Q_0)$ if and only if there exists a sequence of L^2-atoms $\{a_j\}_{j\in\mathbb{N}}$ and real coefficients $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$f(x) = \sum_{j\in\mathbb{N}} \lambda_j a_j(x) \quad \text{a.e.}$$

where the a_j are L^p cubes for a fixed $p > 1$. We also define the norm

$$\|f\|_{\mathcal{H}^1_{atom}} = \inf \sum_{j\in\mathbb{N}} |\lambda_j|$$

where the infimum is taken among all representations of f.

Remark 3.12. The decomposition of elements of $\mathcal{H}^1$ can also be given in terms of L^p-atoms, instead of L^2 atoms, for any fixed $p > 1$.

Proposition 3.13. *[25] The Hardy space $H^1(Q_0)$ can be identified with the dual of $VMO(Q_0)$ via the duality $\langle f, g\rangle = \int_{Q_0} f(x)g(x)\,dx$, and $BMO(Q_0)$ can be identified with the dual of $H^1(Q_0)$ via the same pairing.*

Let us now see how to apply the o-O construction. First, we need to consider the spaces $BMO(Q_0)/\mathbb{R}$ and $VMO(Q_0)/\mathbb{R}$ so that $[\cdot]_*$ is a norm. We now choose $X = L^2(Q_0)/\mathbb{R}$, $Y = L^1(Q_0)/\mathbb{R}$ and the family $\mathfrak{L}$ of operators of the form

$$L_Q : f + \mathbb{R} \in L^2(Q_0)/\mathbb{R} \mapsto \frac{\chi_Q}{|Q|}(f - f_Q)$$

with Q a subcube of Q_0; this family is equipped with the topology induced by the embedding $Q = [x_1 - l/2, x_1 + l/2] \times \cdots \times [x_N - l/2, x_N + l/2] \mapsto (x_1, \ldots, x_n, l) \in \mathbb{R}^{N+1}$. With this construction we easily recover $E = BMO/\mathbb{R}$ and $E_0 = VMO/\mathbb{R}$, and Assumption AP is easily recovered by using regularization.

To recover the atomic decomposition of $\mathcal{H}^1$ we can apply remark 3.5, but we need to make a small alteration to the construction first: instead of $Y = L^1(Q_0)/\mathbb{R}$ we take $Y = L^2(Q_0)/\mathbb{R}$ and alter the operators L_Q appropriately so that we recover $BMO/\mathbb{R}$ with the $[\cdot]_{BMO_2}$ norm. The following distance formula was discovered in [50] up to equivalence. We can use the o-O construction to prove equality.

Corollary 3.14. *Let $u \in BMO$ (modulo constants). Then the following holds:*

$$dist_{BMO}(u, VMO) = dist_{BMO}(u, C^\infty) = \limsup_{|Q|\to 0} \fint_Q |u(x) - u_Q|\,dx.$$

Chapter 4

Rearrangement-invariant spaces

4.1 Definitions

4.1.1 Banach function spaces

Let (Ω, μ) be a measure space.

Definition 4.1. A function $\rho : \mathcal{M}_+(\Omega) \to [0, +\infty]$ is called a *Banach function norm* if it satisfies the following properties for all $f, g \in \mathcal{M}_+(\Omega)$:

i. $\rho(f) = 0 \Leftrightarrow f = 0$ almost everywhere;

ii. $\rho(\lambda f) = \lambda \rho(f)$ for all $\lambda \geq 0$;

iii. $\rho(f + g) \leq \rho(f) + \rho(g)$;

iv. $f \leq g$ almost everywhere $\Rightarrow \rho(f) \leq \rho(g)$ (*lattice property*);

v. $\rho(f_n) \uparrow \rho(f)$ for all sequences $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_+(\Omega)$ such that $f_n \uparrow f$ almost everywhere (*Fatou property*);

vi. $\rho(\chi_F) < \infty$ for all measurable subsets F of Ω, where χ_F denotes the characteristic function of F;

vii. for every measurable subset F of Ω there exists a constant C_F, dependent on F but not the function f, such that

$$\|f\chi_F\|_{L^1} = \int_F f(x)\, d\mu(x) \leq C_F \rho(f).$$

A Banach function space is a vector space X of the form

$$X = X(\Omega, \rho) = \{f \in \mathcal{M}(\Omega) : \rho(|f|) < +\infty\} \tag{4.1}$$

for some Banach function norm ρ, equipped with the norm $\|f\|_X = \rho(|f|)$.

It can be shown [48] that a Banach function space X, endowed with the norm $\|\cdot\|_X$, is a Banach space. Inclusions of Banach function spaces behave in a simple way.

Proposition 4.2. *[11, 48] Let ρ, σ be two Banach function norms, (Ω, μ) a measure space and $X = X(\Omega, \rho)$, $Y = Y(\Omega, \sigma)$ be the Banach function spaces generated by these functions. Then $X \hookrightarrow Y$ if and only if $X \subset Y$. In particular, they coincide as sets if and only if they have equivalent norms.*

If ρ is a Banach function norm, one can define the functional

$$\rho' : f \in \mathcal{M}_+(\Omega) \mapsto \inf \left\{ \int_\Omega fg : g \in \mathcal{M}_+(\Omega), \rho(g) \leq 1 \right\},$$

called the *associate norm*. The associate norm is also a Banach function norm, thus it generates a Banach function space X', called the *associate space* of the Banach function space $X = X(\rho)$ generated by ρ. The associate ρ'' of the associate norm coincides with ρ.

4.1.2 The spaces X_a and X_b

There are two important subsets of a Banach function space X.

Definition 4.3. Let X be a Banach function space. An element f of X is said to have *absolutely continuous norm* if

$$\|f \chi_{E_n}\|_X \to 0$$

for every sequence of subspaces $\{E_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n} \to 0$ μ-almost everywhere.
The subset of all elements of X having absolutely continuous norm is denoted by X_a.

Definition 4.4. Let X be a Banach function space defined on (Ω, μ). The closure in X of the set $\mathcal{S} = \mathcal{S}(\Omega, \mu)$ of all simple functions is denoted by X_b.

Proposition 4.5. *[48] Let X be a Banach function space. The following inclusions hold:*

$$X_a \subseteq X_b \subseteq X.$$

Proposition 4.6. *[48] Let X be a Banach function space. We have that $X_a = X_b$ if and only if $\chi_E \in X_a$ for all sets E with finite measure.*

Proposition 4.7. *[48] Let X be a Banach function space. Then $X' = X^*$ if and only if $X = X_a$. In particular, X is reflexive if and only if $X = X_a$ and $X' = (X_a)'$.*

4.1.3 Rearrangements and rearrangement-invariant Banach function spaces

For a measurable function f, we define its distribution function μ_f as

$$\mu_f : \lambda \in [0, \infty) \to \mu(\{x \in \Omega : |f(x)| > \lambda\})$$

and its non-increasing rearrangement f^* as

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}.$$

Two measurable functions $f, g : \Omega \to \mathbb{R}$ are said to be equimeasurable if $f^* \equiv g^*$. A Banach Function space X is said to be rearrangement-invariant if $f \in X$ if and only if $g \in X$ for all measurable functions g equimeasurable to f.

Definition 4.8. A measure space (Ω, μ) is said to be *resonant* if for every pair of measurable functions $f, g \in \mathcal{M}_0(\Omega, \mu)$ we have

$$\int_0^{\mu(\Omega)} f^*(t) g^*(t)\, dt = \sup_{\tilde{g}} \int_\Omega |f\tilde{g}|\, d\mu,$$

where the supremum is taken between all $\tilde{g} \in \mathcal{M}_0(\Omega, \mu)$ such that $\tilde{g}$ is equimeasurable to g.
A resonant measure space (Ω, μ) is said to be *strongly resonant* if for every $f, g \in \mathcal{M}_0(\Omega, \mu)$ there exists $\tilde{g} \in \mathcal{M}_0(\Omega, \mu)$ equimeasurable with g such that

$$\int_\Omega |f\tilde{g}|\, dx = \int_0^{\mu(\Omega)} f^*(t) g^*(t)\, dt.$$

For these spaces there is a characterization.

Proposition 4.9. *[48] A measure space is resonant if it is either σ-finite and non-atomic or it is completely atomic and all atoms have the same measure. A resonant measure space is strongly resonant if $\mu(\Omega) < \infty$.*

We are mainly going to be concerned with the non-atomic case, as it is the case for open subsets of $\mathbb{R}^N$ when equipped with the Lebesgue measure (or a measure that is absolutely continuous with respect to the Lebesgue measure).

Proposition 4.10. *[48] Let (Ω, μ) be a resonant measure space and let X be a rearrangement-invariant Banach function space defined on (Ω, μ). Then there is a rearrangement-invariant Banach function space $\overline{X}$, called the* (Luxemburg) representation space *of X, defined on the interval $(0, \mu(\Omega))$ such that for every $f \in X$ we have*

$$\|f\|_X = \|f^*\|_{\overline{X}}.$$

An important tool to study rearrangement-invariant Banach function spaces is the fundamental function

Proposition 4.11. *Let X be a rearrangement-invariant function space. Its fundamental function φ_X is defined as*

$$\varphi_X(t) := \|\chi_{E_t}\|_X$$

where E_t is any set of measure t.

Proposition 4.12. *[11, 48] Let X be a r.i. Banach function space defined on a measure space (Ω, μ). The fundamental function φ_X of X and the fundamental function $\varphi_{X'}$ of the associate space X' of X are related in the following way:*

$$\varphi_X(t)\varphi_{X'}(t) = t$$

Definition 4.13. Let I be an interval of the form $[0, T)$ with $T \in (0, \infty]$. A function $\varphi : I \to [0, +\infty)$ is called a *quasi-concave function* if $\varphi(0) = 0$, φ is non-decreasing and the function $t \mapsto \varphi(t)/t$ is non-increasing.

Fundamental functions of r.i. Banach function spaces are closely related to quasi-concave functions:

Proposition 4.14. *[11, 48] Let X be a r.i Banach function space and φ_X be its fundamental function. Then φ_X is a quasi-concave function. Conversely, if φ is a quasi-concave function, there exists a r.i. Banach function space X such that $\varphi = \varphi_X$.*

One could ask what is the link between concave and quasi-concave function. Concerning this question, one can prove the following Lemma.

Lemma 4.15. *[48] Any rearrangement-invariant Banach function space can be equivalently renormed so that its fundamental function is concave.*

4.2 Marcinkiewicz spaces

Definition 4.16. Let (Ω, μ) be a resonant measure space and φ be a quasi-concave function. The *Marcinkiewicz space* $M^\varphi(\Omega, \mu)$ is defined as

$$M^\varphi(\Omega, \mu) = \left\{ u \in \mathcal{M}_0(\Omega, \mu) : \|u\|_{M_\varphi} := \sup_{0<t<\mu(\Omega)} u^{**}(t)\varphi(t) < \infty \right\}.$$

If $\varphi = \varphi_X$ is the fundamental function of a r.i. Banach function space X on Ω, we denote M^{φ_X} by $M(X)$.

Example 4.17.
- The weak Lebesgue spaces $L^{p,\infty}$ are obvious examples of Marcinkiewicz spacesm with $\varphi(t) = \varphi_{L^p}(t) = t^{1/p}$.
- A non-trivial example is the space $EXP(\Omega)$, where Ω has finite measure: this space can be defined (see [11]) as the space of functions f such that the quantity

$$\|f\|_{M(EXP(\Omega))} = \sup_{0<t\leq|\Omega|} \frac{f^{**}(t)}{1-\log\left(\frac{t}{|\Omega|}\right)}$$

  is finite. We will later give another definition of this space.

The following proposition shows that a Marcinkiewicz space is the largest r.i. space among those having a fixed fundamental function.

Proposition 4.18. *Let X be a r.i. Banach function space defined on a measure space (Ω,μ) and φ its fundamental function. Then*

$$\|f\|_{M^{\varphi}(\Omega)} \leq \|f\|_X$$

for every $f \in X$.

For the rest of this section all measure spaces will be implicitly assumed to be finite and non-atomic, unless stated otherwise.
We now need to find a good candidate for a little-o subspace of M^{φ}: this is given by

$$M^{\varphi}_b(\Omega) = \overline{L^{\infty}(\Omega)}^{M^{\varphi}(\Omega)}.$$

Lemma 4.19. *Let φ be a continuous quasi-concave function. Then $M^{\varphi}_a(\Omega) = M^{\varphi}_b(\Omega)$.*

Proof. By [48, theorem 6.3.18] we only need to show that $\chi_E \in M^{\varphi}_a$ for all E having finite measure.
We have that $\chi^{**}_E(t) = \frac{1}{t}\max\{t,|E|\}$, so that

$$\lim_{t\to 0^+} \varphi(t)\chi^{**}_E(t) = \lim_{t\to 0^+} \varphi(t) = 0.$$

Since obviously $\chi_E \in L^1$ we can use [48, theorem 7.10.23] in the case $\lim_{t\to\infty} \dfrac{\varphi(t)}{t} > 0$.
For the case $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$ we have

$$\lim_{t\to\infty} \varphi(t)\chi^{**}_E(t) = \lim_{t\to\infty} \varphi(t)\frac{|E|}{t} = 0$$

so we can use the other part of [48, theorem 7.10.23]. □

By using the fact that $M_a^\varphi(\Omega) = M_b^\varphi(\Omega)$, we can also show a separability result for $M_b^\varphi(\Omega)$.

Lemma 4.20. *Let φ be a continuous quasi-concave function. Then the space $M_b^\varphi(\Omega)$ is separable.*

Proof. By [48, theorem 6.3.16] the space $M_b^\varphi(\Omega)$ is an ideal in $M^\varphi(\Omega)$. We obtain the separability from [48, theorem 6.5.9] since $M_b^\varphi(\Omega)$ contains the simple functions by definition, by lemma 4.19 it coincides with $M_a^\varphi(\Omega)$ and the Lebesgue measure is separable. □

To obtain the o-O structure for the Marcinkiewicz space, we want to know when we can embed M^φ in a reflexive space. We therefore study embedding properties for M^φ. We start with the following simple proposition.

Proposition 4.21. *Let X be a rearrangement invariant space with fundamental function φ. Let $\overline{X}$ be the representation space of X. If $\frac{1}{\varphi} \in \overline{X}$, then X coincides M^φ and the quantity*

$$|f|_{M^\varphi} = \sup_{t\in[0,1]} f^*(t)\varphi(t)$$

is equivalent to $\|\cdot\|_X$. Moreover the converse is true if and only if there exists a constant $M > 0$ such that

$$\int_0^t \frac{ds}{\varphi(s)} \leq \frac{Mt}{\varphi(t)}.$$

Proof. We can write

$$\begin{aligned} \|f\|_X = \|f^*\|_{\overline{X}} = \left\| \frac{f^*\varphi}{\varphi} \right\|_{\overline{X}} \\ &\leq \|1/\varphi\|_{\overline{X}} \|f^*\varphi\|_{L^\infty} \\ &= \|1/\varphi\|_{\overline{X}} |f|_{M^\varphi} \\ &\leq \|1/\varphi\|_{\overline{X}} \|f\|_{M^\varphi} \end{aligned}$$

which together with the inequality $\|f\|_{M^\varphi} \leq \|f\|_X$ proves the first half of the statement. The second half of the proof is just [48, Proposition 7.10.5]. □

We are interested in finding a characterization of this problem.

Lemma 4.22. *[8, Lemma 4] Let $\varphi : [0,1] \to \mathbb{R}$ be a quasi-concave function. Then there is a function $\theta \in L^1(\Omega)$ and a quasi-concave function $\psi : [0,1] \to \mathbb{R}$ such that*

- *$M^{\varphi}(\Omega)$ is equivalent to $M^{\psi}(\Omega)$;*
- *It holds*

$$\theta^{**}(t) = \frac{1}{\psi(t)} \tag{4.2}$$

Proof. Let us first consider the case $\Omega = [0,1]$. The function

$$\eta(t) = \frac{t}{\varphi(t)}$$

is the fundamental function of $(M^{\varphi})'$ by Proposition 4.12. By using lemma 4.15 we know there exists a rearrangement invariant space Y such that its fundamental function η is concave and its norm $\|\cdot\|_Y$ is equivalent to $\|\cdot\|_{(M^{\varphi})'}$, so in particular $Y' = M^{\varphi}$ with equivalent norm. Let us consider ψ the fundamental function of Y'. It is not hard to show that Y' contains every rearrangement invariant space with fundamental function ψ, therefore $Y' = M^{\varphi} = M^{\psi}$. The fundamental function $\overline{\eta}$ of Y' is absolutely continuous in $[\varepsilon, 1]$ for all $\varepsilon \in]0,1[$, so that if we consider its weak derivative θ we can write

$$\overline{\eta}(t) - \overline{\eta}(\varepsilon) = \int_{\varepsilon}^{t} \theta(s)\, ds$$

for all $\varepsilon \in]0,1[$ and for all $t \in]\varepsilon, 1]$, and since $\theta(s) \geq 0$ because $\overline{\eta}$ is increasing we can use the monotone convergence theorem to show that

$$\int_0^t \theta(s)\, ds = \overline{\eta}(t) = \frac{t}{\psi(t)},$$

i.e., since θ is non-decreasing by the convexity of $\overline{\eta}$ and then $\theta^* = \theta$, $\theta^{**}(t) = \frac{1}{\psi(t)}$.
For generic Ω we can build a function $h : \Omega \to \mathbb{R}$ such that $h^* = \theta$ by approximating with simple functions. □

The next result allows us to characterize the inclusion of a Marcinkiewicz space in a Banach function space in terms of the function θ defined before. Moreover, this function can be used to give a control of the norm of the inclusion. Indeed, we have the following Proposition.

Proposition 4.23. *[8, Proposition 5] Let $\varphi : [0,1] \to \mathbb{R}$ be a quasi-concave function and $X(\Omega)$ a rearrangement invariant space. Let θ, ψ be the functions defined in 4.22. Then $M^{\varphi}(\Omega) \hookrightarrow X(\Omega)$ if and only if $\theta \in X(\Omega)$. Moreover, if we denote by $J : M^{\varphi}(\Omega) \to X(\Omega)$ the inclusion, then $\|J\| \leq C \|\theta\|_X$ where C is such that*

$$\varphi \leq C\psi.$$

Proof. First of all, let us suppose that $\theta \in X(\Omega)$. Consider $f \in M^\varphi$ and observe that, by definition of Marcinkiewicz space and the equivalence between M^φ and M^ψ,

$$f^{**}(t) \leq \frac{\|f\|_{M^\psi}}{\varphi(t)} = \|f\|_{M^\psi}\, \theta^{**}(t) = (\|f\|_{M^\psi}\, \theta)^{**}(t).$$

Now, since $\theta \in X(\Omega)$, also $\|f\|_{M^\psi}\, g \in X(\Omega)$ and by Hardy-Lettlewood-Polýa principle, we have

$$\|f\|_X \leq \|\|f\|_{M^\psi}\, \theta\|_X = \|f\|_{M^\psi}\, \|\theta\|_X \leq C\, \|f\|_{M^\varphi}\, \|\theta\|_X\,.$$

Now let us suppose $\theta \notin X(0,1)$. Then let us consider the functions ψ and θ as in 4.22. From $C\psi \leq \varphi$ we have

$$\frac{1}{\varphi} \leq \frac{1}{C_1\psi}$$

hence $\frac{1}{\psi} \notin X(0,1)$. Now, since $\theta^{**} = \frac{1}{\psi}$, we have that $\|\theta^{**}\|_{M^\psi} = 1$ and then, by equivalence, $\theta \in M^\varphi$, concluding the proof. □

This also leads to a characterization of spaces that are equivalent to Marcinkiewicz spaces.

Corollary 4.24. *Let X be a rearrangement invariant space with fundamental function φ and let θ, ψ be the functions defined in 4.22. Then X is equivalent to the Marcinkiewicz space if and only if $\theta \in X$.*

A distance formula between $f \in M^\varphi(\Omega)$ and $L^\infty(\Omega)$ can be proven, generalizing some results in [16]. This in particular gives an alternative description of $M^\varphi_b(\Omega)$.

Proposition 4.25. *[8] Let φ be a quasi-concave function that is continuous at 0. Then for all $f \in M^\varphi(\Omega)$ the following holds:*

$$dist_{M^\varphi}(f, L^\infty(\Omega)) = \limsup_{t\to 0^+}(f^{**}(t)\varphi(t)). \tag{4.3}$$

Proof. In the following f will be a generic function of $M^\varphi(\Omega)$ and g a generic function of L^∞. Let us write $[f]$ for $\limsup_{t\to 0^+}(f^{**}(t)\varphi(t))$. This quantity is easily seen to be subadditive, and one also has that

$$[g] \leq \limsup_{t\to 0^+}(\|g\|_\infty \varphi(t)) = 0; \tag{4.4}$$

as a consequence we obtain that $[f+g] = [f]$.
Now, we have $[f] = [f-g] \leq \|f-g\|_{M^\varphi}$ for all f, g, so that $[f] \leq dist_{M^\varphi}(f, L^\infty)$.

For the other inequality, we have that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that $f^{**}(t)\varphi(t) \leq [f] + \varepsilon$ for all $t \in (0,\delta)$. We define

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } f(x) < f^*(\delta) \\ 0 & \text{if } f(x) \geq f^*(\delta). \end{cases} \tag{4.5}$$

We have that $f_\varepsilon \in L^\infty$ and $(f - f_\varepsilon)^*(t) = \chi_{[0,\delta]}(t) f^*(t)$, so that

$$dist_{M^\varphi}(f, L^\infty) \leq \|f - f_\varepsilon\|_{M^\varphi} = \sup_{0<t<1} ((f - f_\varepsilon)^{**}(t)\varphi(t)) = \sup_{0<t<\delta} (f^{**}(t)\varphi(t)) \leq [f] + \varepsilon, \tag{4.6}$$

where we used the fact that if $f^*(t) = 0$ for $t > t_0$ then $f^{**}(t)\varphi(t) \leq f^{**}(t_0)\varphi(t_0)$, which is easy to prove. As a consequence, we proved the equality of the two quantities. □

Now that we have this formula, we can show the following theorem.

Theorem 4.26. *[8] Let φ be a quasi-concave function that is continuous at 0. Consider the functions g and $\overline{\varphi}$ defined in Lemma 4.22. Suppose there exists a reflexive rearrangement-invariant space $X(\Omega)$ such that $g \in X(0,1)$ and the fundamental function $\varphi_{X'}$ of the associate of X is continuous at* 0. *Then the pair $(M_b^\varphi(\Omega), M^\varphi(\Omega))$ is an o-O pair satisfying assumption AP.*

We are going to use a technical lemma for this theorem. A proof can be found in [8].

Lemma 4.27. *Let (Ω, μ) be a finite measure space. There exists a family of set functions $K = \{\omega : [0, \mu(\Omega)] \to \mathcal{P}(\Omega)\}$ such that:*

- *$\mu(\omega(t)) = t$ for all $\omega \in K$, $t \in [0, \mu(\Omega)]$;*
- *$0 \leq s < t \leq \mu(\Omega) \Rightarrow \omega(s) \subset \omega(t)$ for all $\omega \in K$;*
- *for every measurable subset $E \subset \Omega$ having positive measure there exists $\omega \in K$ such that $\omega(\mu(E)) = E$.*

Proof of Theorem 4.26. Proposition 4.23 insures that $M^\varphi(\Omega)$ embeds continuously into $X(\Omega)$, so we only need to focus on Y and $\mathcal{L}$.
We will use the following representation formula for u^{**}:

$$f^{**}(t) = \frac{1}{t} \sup_{|E|=t} \int_E |f|. \tag{4.7}$$

We can take as space Y the following:

$$Y = \left\{ u := \{u_\omega\}_{\omega \in K} : \|u\|_Y := \sup_{\omega \in K} \|u_\omega\|_{L^1} < \infty \right\},$$

where we take K as in Lemma 4.27. The corresponding space is a Banach space. We then take $\mathcal{L} = \{L_t : t \in (0,1]\}$, where

$$L_t f := \left\{ \frac{1}{t} \chi_{\omega(t)} \varphi(t) f(t) \right\}_{\omega \in K}, \tag{4.8}$$

we have $\|f\|_{M(X(\Omega),\mathcal{L})} = \|f\|_{M^\varphi}$, as required. It remains to check that $\mathcal{L}$, endowed with the induced topology from $[0,1]$, satisfies our hypotheses.
Compactness, σ-compactness and the Hausdorff separation property are immediate from our choice for the topology, while boundedness of the operators is a consequence of the continuous immersion of X in L^1, which is a consequence of the definition of Banach function norm.
We now need to prove the continuity of $T_f : t \mapsto L_t f$ for fixed f. We take $h > 0$ and we consider the upper bound:

$$\begin{aligned}
\|L_{t+h}f - L_t f\|_Y = &\sup_{\omega \in K} \left\| \frac{f\chi_{\omega(t+h)}\varphi(t+h)}{t+h} - \frac{f\chi_{\omega(t)}\varphi(t)}{t} \right\| \leq \\
&\sup_{\omega} \int_{\omega(t)} \left| \frac{\varphi(t+h)}{t+h} - \frac{\varphi(t)}{t} \right| |f(x)|\, dx + \\
&+ \sup_{\omega} \int_{\omega(t+h)\setminus\omega(t)} \left| \frac{\varphi(t+h)}{t+h} \right| |f(x)|\, dx \leq \\
&\frac{h}{t(t+h)} \varphi(t) \|f\|_{L^1} + \frac{\varphi(t+h)}{t+h} \sup_{\omega} \int_{\omega(t+h)\setminus\omega(t)} |f(x)|\, dx \leq \\
&\frac{h}{t^2} \varphi(t) \|f\|_{L^1} + \frac{\varphi(t)}{t} \|f\|_X \varphi_{X'}(h)
\end{aligned}$$

where $\varphi_{X'}$ is continuous at 0 by assumption.
Concerning Assumption AP, let us first observe that since $X(\Omega)$ is a reflexive Banach function space, then $L^\infty(\Omega)$ is dense in $X(\Omega)$ (as a consequence of [48, Corollary 6.4.6]). In particular, for $f \in M^\varphi(\Omega) \subset X(\Omega)$ there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega) \subset M^\varphi_b(\Omega)$ such that $g_n \to f$ in $X(\Omega)$. By the continuous embedding of $X(\Omega)$ in $L^1(\Omega)$, we can suppose, up to a subsequence, that $g_n \to f$ almost everywhere; let us denote by V the (negligible) set where $g_n \nrightarrow f$. Let us define the sequence $h_n = \min\{g_n^+, f^+\}\chi_{\{f\geq 0\}} - \min\{g_n^-, f^-\}\chi_{\{f\leq 0\}}$. It is easy to see that $|h_n| \leq |f|$. Moreover we have

$$|h_n - f| = |g_n - f|\chi_{\{fg_n>0\}\cap\{|g_n|\leq|f|\}} + |f|\chi_{\{fg_n<0\}} \leq |g_n - f| + |f|\chi_{E_n}.$$

where $E_n = \{fg_n < 0\}$. Denote $F_n = \bigcup_{k=n}^{+\infty} E_k$. Now, since $g_n(x) \to f(x)$ for any $x \in \Omega \setminus V$, for any $x \in \Omega \setminus V$ we have that there exists a $\nu_x \in \mathbb{N}$ such that for $n \geq \nu_x$ $x \notin E_n$ and then $x \notin F_n$. Hence $F_n \downarrow F$ where $F \subseteq V$. Thus we have that $E_n \to \emptyset$ almost everywhere. Now, by definition of Banach function norm, we have

$$\|h_n - f\|_X \leq \|g_n - f\|_X + \|f\chi_{E_n}\|_X$$

and, since by [48, Corollary 6.4.6] $X(\Omega) = X_a(\Omega)$, $g_n \to f$ in $X(\Omega)$ and $E_n \to \emptyset$ almost everywhere, we have

$$\limsup_{n\to+\infty} \|h_n - f\|_X \leq \lim_{n\to+\infty} \|g_n - f\|_X + \lim_{n\to+\infty} \|f\chi_{E_n}\|_X = 0$$

hence $h_n \to f$ in $X(\Omega)$. Finally, since we have that $|h_n| \leq |f|$, we also obtain by definition of Banach function norm $\|h_n\|_{M^\varphi(\Omega)} \leq \|f\|_{M^\varphi(\Omega)}$ for any $n \in \mathbb{N}$, completing the proof. □

4.3 Orlicz spaces

We now consider o-O structures for Orlicz spaces [49, 17], which are an important class of rearrangement-invariant Banach spaces generalizing the Lebesgue spaces.

Definition 4.28. A function $A : [0, +\infty[\to [0, \infty[$ is said to be a *Young function* (also Orlicz function or N-function) if it is continuous, strictly increasing and convex on $[0, +\infty)$.
Let (Ω, μ) be a finite measure space. The *Orlicz modular* ρ_A of A is the function

$$\rho_A(f) = \int_E A(f)\, d\mu \quad f \in L^1_{loc}(\Omega)$$

The *Orlicz space* is the space

$$L^A(\Omega) = \left\{ f \in \mathcal{M}_0(\Omega) \mid \exists\lambda > 0 : \rho_A(\lambda^{-1} f) < \infty \right\}.$$

For the rest of this section, without loss of generality, (Ω, μ) will be a probability space, i.e. a measure space such that $\mu(\Omega) = 1$.

Example 4.29.

- The simplest example of Orlicz space is with $A(t) = t^p$ with $1 < p < \infty$: in this case, $L^A(\Omega)$ coincides with the usual $L^p(\Omega)$ space.
- If $|\Omega| < \infty$, another example is with a function $A(t)$ that grows like e^{t^α} asymptotically as $t \to \infty$, with $\alpha > 0$: in this case the corresponding space is the *exponential space* $EXP_\alpha(\Omega)$. One can show that this definition does not depend on the function A.

- If $A(t) = t^p(\log^+ t)^\beta$, with $1 < p < \infty$ and $\beta \in \mathbb{R}$, or $p = 1$ and $\beta \geq 0$, we obtain the *Zygmund space* $L^p \log^\beta L(\Omega)$.

Remark 4.30. Most of the properties of an Orlicz space we are interested in depend only on the behaviour of the behaviour of A at infinity. However some properties, especially concerning the geometry of the space, depend on the choice of equivalent norm (more details on the geometry of Orlicz spaces can be found in [17]).

There are multiple equivalent norms that can be given to an Orlicz space: the usual one is the *Luxemburg norm*:

$$\|f\|_{L^A} = \inf\left\{\lambda > 0 : \rho_A(\lambda^{-1}f) \leq 1\right\}$$

Remark 4.31. We have that $\|f\|_{L^A(\Omega)} \leq 1$ if and only if $\rho_A(f) \leq 1$.

Proposition 4.32. *[48] The space $L^A(\Omega)$, endowed with the Luxemburg norm $\|\cdot\|_{L^A}$, is a rearrangement-invariant Banach function space. If A^{-1} denotes the inverse function of A, the fundamental function of $L^A(\Omega)$ is*

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}\left(\frac{1}{t}\right)}.$$

From an Orlicz function A one can construct a function related to it.

Definition 4.33. Let A be an Orlicz function. Then the function $\widetilde{A}(t) := \inf_{s>0}(st - A(s))$ is called the complementary function of A.

Proposition 4.34. *[48] The complementary function $\widetilde{A}$ of A is an Orlicz function. Moreover, up to equivalence of norms, $L^{\widetilde{A}}(\Omega)$ is the associate space of $L^A(\Omega)$.*

This proposition gives an equivalent norm for L^A.

Definition 4.35. Let A be an Orlicz function and let $\widetilde{A}$ be its complementary function. The *Orlicz norm* of a function $f \in L^A(\Omega)$ is defined as

$$\begin{aligned}\|f\|^o_{L^A} &= \sup\left\{\int_E fg : g \in L^{\widetilde{A}}(\Omega), \|g\|_{L^{\widetilde{A}}} \leq 1\right\} \\ &= \sup\left\{\int_E fg : g \in L^{\widetilde{A}}(\Omega), \rho_{\widetilde{A}}(g) \leq 1\right\}\end{aligned}$$

The following facts are obvious consequences of this definition.

Corollary 4.36. *Let A be an Orlicz function and let $\widetilde{A}$ be its complementary function. The associate space of $L^A(\Omega)$, equipped with the Luxemburg norm, coincides with $L^{\widetilde{A}}(\Omega)$, equipped with the Orlicz norm. In particular, the following generalized Hölder inequality holds:*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^A(\Omega)} \|g\|^o_{L^{\widetilde{A}}(\Omega)}.$$

Corollary 4.37. *Let A be an Orlicz function. The fundamental function of $L^A(\Omega)$, equipped with the Orlicz norm, is*

$$\varphi^o_{L^A}(t) = tA^{-1}\left(\frac{1}{t}\right).$$

Some properties of an Orlicz space depend on the behaviour of its Orlicz function A. Let us define the following.

Definition 4.38. An Orlicz function A satisfies the Δ_2 condition (at infinity), and we write $A \in \Delta_2$ if there exists $T > 0$ such that

$$\sup_{t>T} \frac{A(2t)}{A(t)} < \infty.$$

An Orlicz function A satisfies the ∇_2 condition (we also write $A \in \nabla_2$) if its complementary function $\widetilde{A}$ satisfies the Δ_2 condition.

Proposition 4.39. *[48] Let A be an Orlicz function satisfying Δ_2. Then the dual space of $L^A(\Omega)$ coincides with its associate $L^{\widetilde{A}}(\Omega)$.*
In particular, $L^A(\Omega)$ is reflexive if and only if $A \in \Delta_2 \cap \nabla_2$.

Since we are not interested in the reflexive case, our case of interest is with $A \in \nabla_2 \backslash \Delta_2$. To study what happens in this case, we introduce a subspace of L^A.

Definition 4.40. Let A be an Orlicz function. Then we define the *Morse space* $E^A(\Omega)$ as the set of all functions $g \in L^A(\Omega)$ such that $\rho_A(\alpha g) < \infty$ for all $\alpha > 0$.

Proposition 4.41. *[48] The subspace $E^A(\Omega)$ of $L^A(\Omega)$ coincides with the space $L^A_a(\Omega)$ of all elements of $L^A(\Omega)$ having absolutely continuous norm, and with the closure $L^A_b(\Omega)$ of $L^\infty(\Omega)$ in $L^A(\Omega)$.*

Proposition 4.42. *[48] Let A be an Orlicz function satisfying the Δ_2 condition. Then $E^A(\Omega) = L^A(\Omega)$.*

Proposition 4.43. *[48] Let A be an Orlicz function. Then the dual space of $E^A(\Omega)$, equipped with the Luxemburg norm induced by $L^A(\Omega)$, is $L^{\widetilde{A}}(\Omega)$, equipped with the Orlicz norm.*

We now need an equivalent norm for $L^A(\Omega)$ defined using a supremum. As we saw before, $EXP(\Omega)$ admits an equivalent Marcinkiewicz type norm. Inspired by this fact, in [6] we looked for a criterion that tells us when an Orlicz space admits an equivalent Marcinkiewicz norm.

Proposition 4.44. *[6, Theorem 1] Let A be an Orlicz function and let us denote by $\alpha(t)$ the fundamental function of L^A, i.e.*

$$\alpha(t) = \frac{1}{A^{-1}(1/t)}$$

The following statements are equivalent.

i. $1/\alpha(t) = A^{-1}(1/t) \in L^A(0,1)$, *i.e. there exists $\beta > 0$ such that*

$$\int_0^1 A\left(\beta A^{-1}\left(\frac{1}{t}\right)\right) < \infty;$$

ii. $f \in L^A(\Omega)$ *if and only if* $\|f\|_{M^\alpha} < \infty$, *i.e.* $L^A(\Omega) = M^\alpha(\Omega)$;

iii. $f \in L^A(\Omega)$ *if and only if the quasinorm* $|f|_{M^A} := \sup_{0<t\leq 1} \alpha(t) f^*(t)$ *is finite;*

iv. The quasinorm $|\cdot|_{M^A}$ and the norms $\|\cdot\|_{L^A}$ and $\|\cdot\|_{M^\alpha}$ are equivalent; more precisely there exist constants $c_1, c_2 > 0$ such that

$$c_1|f|_{M^\alpha} \leq c_1\|f\|_{M^\alpha} \leq \|f\|_{L^A} \leq c_2|f|_{M^\alpha} \leq c_2\|f\|_{M^\alpha}$$

for all $f \in L^A(\Omega)$.

Proof. i.⇒ii. Since $\|f\|_{M^\alpha} \leq \|f\|_{L^A}$, we only need to prove that $f \in M^\alpha \Rightarrow f \in L^A$. Let $\lambda > 0$. If we divide by λ and then apply the modular ρ_A to both sides of the inequality

$$f^{**}(t) \leq A^{-1}\left(\frac{1}{t}\right)\|f\|_{M^A}$$

we get

$$\int_\Omega A(\lambda^{-1} f^{**}(t)) \leq \int_\Omega A\left(\lambda^{-1} A^{-1}\left(\frac{1}{t}\right)\|f\|_{M^\alpha}\right)$$

which by i. implies that f^{**}, and therefore f^*, belongs to $L^A(0,1)$, and by the Luxemburg representation theorem $f \in L^A(\Omega)$.
ii.⇒iii. Immediate consequence of the inequality $f^* \leq f^{**}$.
iii.⇒i. Using approximations by simple functions, one obtains that there is a function $f \in L^A(\Omega)$ such that $f^* = \alpha$, which implies i.
iii.⇒iv. The inequality $\|f\|_{M^\alpha} \leq \|f\|_{L^A}$ is well known, so that one can always take $c_1 = 1$. On the other hand, we can write

$$\begin{aligned}\|f\|_{L^A(\Omega)} = \|f^*\|_{L^A(0,1)} &= \left\| \frac{f^*(t)\alpha(t)}{\alpha(t)} \right\|_{L^A(0,1)} \\ &\leq \left\| A^{-1}\left(\frac{1}{t}\right) \right\|_{L^A(0,1)} |f|_{M^\alpha} \\ &\leq \left\| A^{-1}\left(\frac{1}{t}\right) \right\|_{L^A(0,1)} \|f\|_{M^\alpha}\end{aligned}$$

so that we can take $c_2 = \|A^{-1}(1/t)\|_{L^A(0,1)}$.
iv.⇒ii. Trivial. □

Using this criterion, we can just use the results from the previous section to show

Theorem 4.45. *[6] Let A be an Orlicz function satisfying the conditions of Proposition 4.44 and such that*

$$\liminf_{t\to\infty} \frac{A(t)}{t^p} = +\infty$$

for some $p > 1$. Then the pair $(E^A(\Omega), L^A(\Omega))$ forms a o-O pair.

Remark 4.46. The condition that

$$\liminf_{t\to\infty} \frac{A(t)}{t^p} = +\infty$$

is there to exclude some Orlicz functions that exhibit both growth faster than all polinomials and slower than every power t^p with p greater than one: for example, one can construct counterexamples by alternating an exponential behaviour and a linear behaviour. Of course, the theorem still holds if we can find a reflexive space X, not necessarily of the form L^p such that $L^A \hookrightarrow X$.

Corollary 4.47. *Let A be an Orlicz function satisfying the assumptions of Theorem 4.45 and let $\alpha(t) = \frac{1}{\alpha(1/t)}$, where a is the inverse function of A. Then E^A, equipped with the Marcinkiewicz norm $\|\cdot\|_{M^\alpha}$, is M-embedded in L^A.*

Remark 4.48. The property of E^A being M-embedded in L^A depends on the norm chosen: it is known that E^A is M-embedded when you use the Luxemburg norm [49], but this is not the case if you endow it with the Orlicz norm [18].

4.3.1 Examples

In the following, for an Orlicz function A we will denote by a its inverse function and by α the function $\alpha(t) = \frac{1}{a(1/t)}$. Let us also introduce the following growth condition.

Definition 4.49. An Orlicz function A is said to satisfy the Δ^0 condition, and we write $A \in \Delta^0$, if there exists $\lambda > 1$ such that

$$\liminf_{t\to\infty} \frac{A(\lambda t)}{A(t)} = +\infty.$$

It is not hard to show that a Δ_0 function A satisfies

$$\liminf_{t\to\infty} \frac{A(t)}{t^p} = +\infty$$

for all $p > 0$. Let us show a result covering a large class of functions.

Proposition 4.50. *Let A be an Orlicz function with growth of the form $e^{\nu(t)}$, where ν is a convex function. Then A satisfies the Δ^0 condition and $a(1/t) \in L^A(I_0)$. In particular, they satisfy the assumptions of Theorem 4.45*

Proof. For simplicity we will do our calculations with the function $A(t) = e^{\nu(t)} - 1$. The Δ^0 condition easily follows from the inequality

$$A(kt) = e^{\nu(kt)} - 1 \geq e^{k\nu(t)} - 1$$

for any $k > 1$ and $\nu(t) \to +\infty$ as $t \to +\infty$.
Now observe that, since A is an Orlicz function, $\nu(0) = 0$ and ν is strictly increasing. Moreover for any $r > 0$

$$e^{\nu(r/2)} \leq \sqrt{e^{\nu(r)}} \leq \frac{e^{\nu(r)}}{\sqrt{e^{\nu(r)} - 1}},$$

since $\nu(r/2) \leq \nu(r)/2$. Since ν is convex, it is also derivable a.e., so let r be such that $\nu'(r)$ and $\nu'(r/2)$ both exist. For such r we have $\nu(r/2) \leq \nu'(r)$, and if we combine this with (4.3.1) we get

$$\frac{1}{2}\nu'\left(\frac{r}{2}\right) e^{\nu(r/2)} \leq \frac{1}{2}\frac{\nu'(r)e^{\nu(r)}}{\sqrt{e^{\nu(r)} - 1}}.$$

Integrating this inequality in the interval $[0, r]$, applying first the logarithm and then ν^{-1} we obtain

$$\frac{r}{2} \leq \nu^{-1}\left(\log\left(\sqrt{e^{\nu(r)}-1}+1\right)\right).$$

Now, for any $t > 0$, consider $r = a(1/t) = \nu^{-1}(\log(1 + 1/t))$ to obtain

$$\frac{1}{2}a\left(\frac{1}{t}\right) \leq \nu^{-1}\left(\log\left(\sqrt{1+\frac{1}{t}}\right)\right) = a\left(\sqrt{\frac{1}{t}}\right)$$

and then applying A we finally have

$$A\left(\frac{1}{2}a\left(\frac{1}{t}\right)\right) \leq \sqrt{\frac{1}{t}}$$

hence $t \mapsto a(1/t) \in L^A(0, 1)$. □

Observe that $A(t) = e^t - 1$, which is the Orlicz function defining EXP, falls in this case, together with the functions $A(t) = e^{t^\alpha} - 1$ for $\alpha > 1$, as well as functions that grow asymptotically as Ae^{e^t} or more general nested exponentials.
However, Orlicz functions with asymptotic growth $A(t) = \exp\left(\log^{1+\varepsilon}(t)\right)$, with $\varepsilon > 0$, do not fall in the previous cases. For these we can show the following Proposition.

Proposition 4.51. *Let A be the Orlicz function with asymptotic growth $A(t) = \exp\left(\log^{1+\varepsilon}(t)\right)$, with $\varepsilon > 0$. Then A satisfies the Δ^0 condition and $a(1/t) \in L^A(I_0)$.*

Proof. It is easy to prove that A satisfies the Δ^0 conditon. To study this case, one can consider the change of variable $t = 1/s$, by which the condition $a(1/t) \in L^A(I_0)$ is shown to be equivalent to the existence of a $k > 1$ such that

$$\int_0^1 A\left(\frac{1}{k}a\left(\frac{1}{s}\right)\right) ds = \int_1^{+\infty} \frac{1}{t^2} A\left(\frac{1}{k}a(t)\right) dt < +\infty.$$

Let us study the case in which $\varepsilon \in (0, 1)$. We have

$$a(t) \simeq \exp\left(\log^{\frac{1}{1+\varepsilon}}(t)\right),$$

where the symbol $\simeq$ means we are considering functions with the same asymptotic behaviour at $+\infty$, and then, after some calculations

$$A\left(\frac{1}{k}a(t)\right) \simeq t\exp\left(-(1+\varepsilon)\log^{\frac{\varepsilon}{1+\varepsilon}}(t)\right).$$

This asymptotic equivalence follows from Taylor expansion of $t \mapsto (1+t)^{1+\varepsilon}$ near 0 up to the second order term.
Since for t large enough

$$e^{(1+\varepsilon)\log^{\frac{\varepsilon}{1+\varepsilon}} t} \geq \log^p t$$

for any $p > 1$, we obtain

$$\int_1^{+\infty} \frac{1}{t^2} A\left(\frac{1}{k}a(t)\right) dt < +\infty.$$

A similar argument works for $\varepsilon > 1$ by considering a Taylor expansion of higher order. □

Other interesting cases are covered by the following Proposition.

Proposition 4.52. *Let A be a Young function of the form $e^{\nu(t)} - 1$ where ν is a definitely derivable concave function such that*

- *for any $p > 1$*
$$\lim_{t\to+\infty} \frac{\nu(t)}{\log^p t} = 0;$$
- $$\lim_{t\to+\infty} \frac{\nu(t)}{\log t \log\log t} = +\infty;$$
- *there exists a $M_1 > 0$ such that*
$$\int_{M_1}^{+\infty} \frac{1}{t\nu(t)} dt < +\infty.$$

Then $a(1/t) \in L^A(I_0)$.

Proof. To work with this case let us also notice that, because of the regularity properties of A, the condition $a(1/t) \in L^A(I_0)$ is equivalent to the existence of a $k > 1$ such that (by using the change of variables $t = a(1/s)$)

$$\int_{a(1)}^{+\infty} \frac{A'(t)A(t/k)}{A^2(t)} dt = \int_{a(1)}^{+\infty} (\log(A(t)))' \frac{A(t/k)}{A(t)} dt < +\infty. \tag{4.9}$$

Notice that this condition can be verified quite easily since a appears in this formula only as integration extrema. Also, since A is sufficiently regular (in particular definitively derivable), it is also possible to replace A with a (sufficiently regular) function that is asymptotically equivalent to it.

In our case, from Equation (4.9), we only need to show that there exist a $M > 0$ and a $k > 1$ such that

$$\int_M^{+\infty} \nu'(t) e^{\nu(t/k)-\nu(t)} dz < +\infty.$$

To do this, let us observe that, by Lagrange theorem and the fact that ν' is decreasing, we obtain

$$\int_M^{+\infty} \nu'(t) e^{\nu(t/k)-\nu(t)} dz \leq \int_M^{+\infty} e^{\log(\nu'(t))-\nu'(t)\left(1-\frac{1}{k}\right)t} dz.$$

Hence, taking $\alpha = 1 - 1/k$, our aim is to show that there exists a $\alpha \in (0, 1)$ such that

$$\log(\nu'(t)) - \alpha\nu'(t)t \leq -\log t - \log(\nu(t))$$

that is to say

$$\alpha t\nu'(t) \geq \log(t\nu(t)\nu'(t)).$$

Now, if we apply l'Hopital rule to the growth conditions we can infer that, for $t > M$ (with $M > 0$ large enough), $t\nu'(t) \geq 4 \log\log t$ and $\log^2 t \geq t\nu(t)\nu'(t)$, so that

$$\frac{1}{2} t\nu'(t) \geq \log(t\nu(t)\nu'(t))$$

that leads to

$$\int_M^{+\infty} \nu'(t) e^{\nu\left(\frac{t}{k}\right)-\nu(t)} dt \leq \int_{M_2}^{+\infty} e^{\log(\nu'(t))-\nu'(t)\left(1-\frac{1}{k}\right)t} dt \leq \int_{M_2}^{+\infty} \frac{1}{t\nu(t)} dt < +\infty.$$

□

With this result in mind, we can show the following result.

Proposition 4.53. *Let A be an Orlicz function with asymptotic growth $e^{\log(t)\log^p(\log(t))}$, with $p > 0$. Then A satisfies the Δ^0 condition, and $a(1/t) \in L^A(I_0)$ if and only if $p \geq 1$.*

Proof. Let us first observe that for any $k \neq 1$ we have

$$\frac{A(t/k)}{A(t)} \simeq e^{-\log k \log^p \log t}.$$

In particular for $k < 1$ we have $\lim_{t\to+\infty} \frac{A(t/k)}{A(t)} = +\infty$, so the Δ^0 condition is still satisfied.

Now observe that for $p > 1$ we are under the hypotheses of the previous

proposition, hence we only need to check the case in which $p \in (0, 1]$.
In general we have

$$(\log(A(t)))' \simeq \frac{\log^p \log t}{t} \left(1 + \frac{1}{\log \log t}\right)$$

hence it is easy to check that for $p = 1$, $k > e$ and for some $M > 0$

$$\int_M^{+\infty} (\log(A(t)))' \frac{A(t/k)}{A(t)} dt < +\infty.$$

Instead for $p \in (0, 1)$, we know that

$$\lim_{t \to +\infty} \frac{\log \log t}{\log^p \log t} = +\infty$$

hence for any $k > 1$ there exists a $M > 0$ such that for any $t > M$

$$\log \log t \geq \log k \log^p \log t.$$

Hence we have, for $t > M$:

$$\log(A(t))' \frac{A(t/k)}{A(t)} \simeq \frac{\log^p \log t}{t} e^{-\log k \log^p \log t} \geq \frac{\log^p \log t}{t \log t}$$

and finally

$$\int_M^{+\infty} \log(A'(t)) \frac{A(t/k)}{A(t)} dt = +\infty.$$

□

Chapter 5

Oscillation-type spaces

5.1 Lipschitz spaces

An important class of functions is the class of Lipschitz functions [57].

Definition 5.1. Let (S, d) a metric space. We define the space of *Lipschitz functions* on (S, d) as

$$Lip(S,d) = \left\{ f : S \to \mathbb{R} \mid \sup_{\substack{x,y \in S \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} < \infty \right\}.$$

If $f \in Lip(S, d)$, the constant $[f]_{Lip} := L = \sup_{\substack{x,y \in S \\ x \neq y}} \frac{|f(x)-f(y)|}{d(x,y)}$ is called the *Lipschitz constant* of f.

The quantity $[\cdot]_{Lip}$ is a seminorm that can be made into a norm in two main ways:

- the first way is to only consider functions that take the value 0 at a prescribed point, that we denote by 0: this way $[f]_{Lip}$ becomes a norm in the space that we denote by $Lip^0(S, d)$;

- the second way is to consider the norm $\|f\|_{Lip} := \max\{[f]_{Lip}, \|f\|_{L^\infty}\}$: this way, the space $Lip(S, d)$ becomes a Banach space. Note that this case can be reduced to the first one by adding an artificial point having distance 1 from every other point, and imposing the condition $f = 0$ at this point.

We will mainly consider the second approach.
We define the following subspace of $Lip(S,d)$:

$$lip(S,d) = \left\{ u \in Lip(S,d) : \limsup_{\varepsilon\to 0} \sup_{\substack{x,y\in S\\ d(x,y)\le\varepsilon}} \varepsilon^{-1}|u(x)-u(y)| = 0 \right\}.$$

If $S \subset \mathbb{R}^N$ and $d = d_E$ is the Euclidean distance, one finds that $lip(S,d)$ contains only constant functions; however, if we consider $d = d_E^\alpha$ with $\alpha \in (0,1)$ we have that $Lip(S,d_E) \subset lip(S,d_E^\alpha)$.
We denote by $\mathfrak{M}(S,d)$ the space of all signed Borel measures on (S,d) having bounded total variation. This space can be made into a Banach space when equipped with the total variation, which defines a norm on $\mathfrak{M}(S,d)$. This norm, however, does not have any correlation with the distance d: for any two distinct points x,y of S we have $\|\delta_x - \delta_y\| = |\delta_x - \delta_y|(S) = 2$. One can however define another norm on $\mathfrak{M}(S,d)$, inspired by optimal transport theory, that is connected to the distance. If $\mathfrak{M}_0(S,d)$ denotes the elements ν of $\mathfrak{M}(S,d)$ such that $\nu(S) = 0$, we define, for $\nu \in \mathfrak{M}_0(S,d)$, the quantity

$$\|\nu\|_{KR}^0 = \inf\left\{ \int_{S\times S} d(x,y)\, d\Psi(x,y) \right\},$$

where Ψ varies among all finite signed Borel measures on $(S\times S, d\times d)$ such that $\Psi(S\times F) - \Psi(F\times S) = \nu(F)$ for all Borel subsets F of S, and for $\mu \in \mathfrak{M}(S,d)$ the quantity

$$\|\mu\|_{KR} = \inf_{\nu\in\mathfrak{M}_0(S,d)} \left(|\mu-\nu|(S) + \|\nu\|_{KR}^0\right).$$

For example, we have that $\|\delta_x - \delta_y\|_{KR}^0 = d(x,y)$ (while $\|\delta_x - \delta_y\|_{KR} = \min\{2, d(x,y)\}$) and $\|\delta_z\|_{KR} = 1$. The quantity $\|\cdot\|_{KR}$ defines a norm on $\mathfrak{M}(S,d)$, however the space $(\mathfrak{M}(S,d), \|\cdot\|_{KR})$ is not complete: we can consider the completion $\mathfrak{M}^c(S,d)$ with respect to this norm. This norm is connected to Lipschitz spaces through the following results.

Proposition 5.2. *[37, 38, 39] Let (S,d) be a compact metric space. The topological dual of the normed vector space $\mathfrak{M}(S,d)$, endowed with the Kantorovich-Rubinstein norm, coincides with $Lip(S,d)$, via the duality*

$$\langle\cdot,\cdot\rangle_{\mathfrak{M}(S,d)} : (f,\mu) \in Lip(S,d)\times\mathfrak{M}(S,d) \mapsto \int_S f\, d\mu.$$

This identification is an isometry if $Lip(S,d)$ is endowed with the norm $\|\cdot\|_{Lip}$. Moreover, we have $(\mathfrak{M}_0(S,d))^* \simeq Lip^0(S,d)$ (this result is independent from the choice of the distinguished point 0).

Proposition 5.3. *[30] Let (S, d) be a compact metric space. The following are equivalent:*

- *The dual of $lip(S, d)$ can be identified with the completion $\mathfrak{M}^c(S, d)$ of $\mathfrak{M}(S, d)$, via the duality defined for $g \in lip(S, d)$ and $\mu \in \mathfrak{M}(S, d)$ by*

$$\langle \mu, g \rangle_{lip(S,d)} = \int_S g \, d\mu$$

and extended on $\mathfrak{M}^c$ by continuity.

- *(Assumption H) For any $f \in Lip(S, d)$, A a finite subset of K and $C > 1$ real constant, there exists a function $g \in lip(S, d)$ such that $g|_A \equiv f|_A$ and $\|g\| \leq C\|f\|$.*

Remark 5.4. One can show that euclidean distance d_{eucl} on a compact subset K of $\mathbb{R}^N$ does not satisfy Assumption H: indeed, the space $lip(K, d_{eucl})$ becomes trivial in this case. However, if one considers the distance $d_\alpha(x, y) := (d_{eucl}(x, y))^\alpha$, where $\alpha \in (0, 1)$, and the corresponding (Hölder) spaces $C^{0,\alpha}(K) = Lip_\alpha(K) := Lip(K, d_\alpha)$ and $c^{0,\alpha}(K) = lip_\alpha(K) := lip(K, d_\alpha)$ one obtains that Assumption H is satisfied and $(c^{0,\alpha})^{**} \simeq C^{0,\alpha}$.

Assumption H is actually equivalent to an approximation property in $Lip(S, d)$: for any function $f \in Lip(S, d)$ we can find a sequence of functions in a suitable subspace that pointwise converges towards f. This property will take the role of Assumption AP in proving the o-O structure of (lip, Lip).

Proposition 5.5. *[5] Let us suppose Assumption H holds. Let $f \in Lip(S, d)$. There is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset lip(S, d)$ pointwise converging to f and such that* $\sup_{n\in\mathbb{N}} \|f_n\|_1 \leq \|f\|_1$.

Proof. Since S is totally bounded, it can be covered by a finite number of balls of radius 1, so let's call A_0 the set of centers of these balls. Suppose now that we have defined the set A_n and consider the set $K_{n+1} := S \setminus \bigcup_{x\in A_n} B_{2^{-n-1}}(x)$. Since K_{n+1} is a compact and thus totally bounded subset of S, it can be covered by balls of radius 2^{-n-1}, so if we denote by B_{n+1} the corresponding set of centers, we can take $A_{n+1} := A_n \cup B_{n+1}$. This ensures that every point of S has distance less that 2^{-n} from the points in A_n. We also take $C_n := 1 + \frac{1}{n+1}$.
Let g_n be the function from Assumption H obtained by considering $A = A_n$ and $C = C_n$ and define $f_n := \frac{g_n}{C_n} \in lip(S, d)$. We have that $\|f_n\|_\alpha \leq \|f\|_\alpha$, so the only thing that's left to show is the pointwise convergence, which implies weak convergence in X. We notice that, by definition of f_n, it is enough to show that $g_n \to f$ pointwise. If we define $A_\infty := \bigcup_{n\in\mathbb{N}} A_n$ we see that A_∞ is

dense and for all $x \in A_\infty$ the sequence $g_n(x)$ eventually becomes constantly equal to $f(x)$. By using the Lipschitz property we can easily extend the pointwise convergence to the whole S.

□

We now need to find a good candidate for the reflexive space X. This is given by a class of fractional Besov spaces, under additional assumptions on (S, d).

Definition 5.6. We say that a metric space (S, d) has the *doubling condition* if there exists a positive integer C such that any ball B can be covered by at most C balls having half the radius.
A Borel measure μ on a metric space (S, d) is said to have the *doubling condition* if

(i) there exist two balls B_1, B_2 such that $\mu(B_1) > 0$ and $\mu(B_2) < +\infty$;

(ii) there exists a constant $C > 0$ such that

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)) \tag{5.1}$$

for all $x \in S$ and all $r > 0$. The space (S, d, μ) is said to be a doubling metric measure space. A measure μ that satisfies (i) is said to be non-degenerate. Condition (ii) is called doubling condition and the constant C is called doubling constant.

Definition 5.7 ([29])**.** Let (S, d, μ) be a doubling measure space. The Besov space of parameters $s \in (0, 1)$ and $p, q \in [1, \infty)$ on (S, d, μ) is the space

$$\begin{aligned} \mathcal{B}^s_{p,q}(S, d, \mu) = \Big\{ & f : K \to \mathbb{R} : f \in L^p(S, \mu) \text{ and} \\ [f]_{\mathcal{B}^s_{p,q}} := & \left[\int_0^{+\infty} \frac{dr}{r} \left[\int_S \fint_{B_r(x)} \frac{|f(x) - f(y)|^p}{r^{sp}} \, d\mu(y) \, d\mu(x) \right]^{q/p} \right]^{1/q} < +\infty \Big\}. \end{aligned}$$

It is a Banach space when endowed with the norm

$$\|f\|_{\mathcal{B}^s_{p,q}} = \|f\|_{L^p} + [f]_{\mathcal{B}^s_{p,q}}.$$

Lemma 5.8. *Let (K, ρ) be a compact doubling metric space and μ a doubling measure on it. Then there exist two constants $C, Q > 0$ such that*

$$\mu(B_r(x)) \geq Cr^Q.$$

It can be shown that the Besov space $\mathcal{B}^s_{p,p}$ is separable, and it is reflexive when $p > 2$. We also need an embedding result for this space (see e.g. [5]).

Proposition 5.9. *Let (S, d, μ) be a doubling compact metric measure space and $p > \frac{Q}{s}$, with Q as in Lemma 5.8 Then $\mathcal{B}^s_{p,p}(S, d, \mu)$ embeds continuously in L^∞.*

Proposition 5.10. *[5] Consider $\alpha \in (0, 1]$. Then the space $Lip_\alpha(S, d)$ continuously embeds in $\mathcal{B}^s_{p,p}$ for $s \in (0, \alpha)$ and $p \in [1, +\infty)$.*

Proof. To prove that $\mathrm{Lip}_\alpha(S, d)$ embeds continuously in X, it is necessary to assume $s < \alpha$.
Indeed, if so, assume that $C = \|f\|_1$. Then $\|f\|_{L^p} \leq \|f\|_{L^\infty} \leq C$ and

$$\left[\int_0^{+\infty} \frac{dr}{r} \int_K \fint_{B_r(x)} \frac{|f(x) - f(y)|^p}{r^{sp}} \, d\mu(y) \, d\mu(x)\right]^{1/p} \leq$$

$$\leq \left[\int_0^D \frac{1}{r} C^p r^{(\alpha - s)p} dr + \int_D^{+\infty} 2C^p \, \frac{dr}{r^{1+sp}}\right]^{1/p} \leq kC$$

where in the first integral the idea was using the fact that C bounds the Lipschitz constant, while in the second one the fact that C bounds the L^∞ norm of f was used.

□

Now we are ready to prove that (lip, Lip) form an o-O structure.

Theorem 5.11. *[5] Let (S, d, μ) be a doubling compact metric measure space. Then the pair $(lip(S, d), Lip(S, d))$ exhibit a o–O structure if and only if Assumption H holds. Supposing that Assumption H holds, as a consequence we have the following properties:*

- *$(lip(S, d))^{**} \simeq Lip(S, d)$ isometrically;*
- *for $f \in Lip(S, d)$ the following distance formula holds:*

$$dist_{Lip(S,d)}(f, lip(S, d)) = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x, y)}; \tag{5.2}$$

- *$lip(S, d)$ is an M-ideal in $Lip(S, d)$, that is*

$$(Lip(S, d))^* \simeq (lip(S, d))^* \oplus_1 (lip(S, d))^\perp, \tag{5.3}$$

- *$(lip(S, d))^*$ is the strongly unique predual of $Lip(S, d)$.*

Proof. First we choose the reflexive and separable space to be $X = \overline{Lip(S,d)}^{\mathcal{B}^s_{p,p}}$, with $s \in (0,1)$ and $p > \max\{Q/s, 2\}$, with Q taken as in Lemma 5.8. As Banach space Y let us choose $\mathbb{R}\times\mathbb{R}$, endowed with the ℓ^∞ norm, i.e. $\|(x,y)\|_{\mathbb{R}\times\mathbb{R}} = \max\{|x|,|y|\}$.
Our family of operators will be the following:

$$\mathfrak{L} = \left\{ L_{x,y,z} : f \in X \mapsto \left(\frac{f(x)-f(y)}{d(x,y)}, \frac{d(x,y)}{D} f(z) \right) \in \mathbb{R}\times\mathbb{R}, x,y,z \in S, x \neq y \right\},$$

where $D = \operatorname{diam} S = \sup_{x,y\in S} d(x,y)$. It is clear that these operators are linear.
If we set $V := S^2 \backslash \mathrm{Diag}(S^2)$, we can give $\mathcal{L}$ the product topology of $V \times S$, where on V we have the trace topology induced by the topology on S^2. In the following we will identify $\mathcal{L}$ with $W := V \times S$.
Since S is a compact metric space, it is σ-compact, locally compact, Hausdorff and separable and so is also V. These properties easily transfer to $\mathfrak{L}$, being it a product space. In particular an exhaustive sequence S_n of compact subsets of $\mathfrak{L}$ is given by

$$S_n = \left\{ (x,y) \in S^2 : \ d(x,y) \geq \frac{1}{n} \right\}$$

hence taking the limit as $L \to \infty$ is equivalent to taking the limit as $d(x,y) \to 0$. Now we need to show the continuity of the maps $T_f : L \in \mathfrak{L} \mapsto L(f) \in \mathbb{R}\times\mathbb{R}$ for $f \in X$. We notice that it is enough to prove this for $f \in Lip(S,d)$, since we can use a diagonal argument, combined with the boundedness of the operators themselves, to extend this to the whole X.
This is easy because $(x_n, y_n, z_n) \to (x,y,z)$ as n goes to infinity implies $d(x_n,y_n) \to d(x,y)$ and $d(z_n,z) \to 0$, so using the continuity of f and d we easily obtain

$$\max\left\{ \left| \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} - \frac{f(x)-f(y)}{d(x,y)} \right|, \left| \frac{d(x_n,y_n)}{D} f(z_n) - \frac{d(x,y)}{D} f(z) \right| \right\} \to 0$$

proving that T_f is continuous for any $f \in \mathrm{Lip}(S,d)$.
It is easy to observe that

$$\sup_{(x,y,z)\in W} \|L_{x,y,z} f\|_{\mathbb{R}\times\mathbb{R}} = \sup_{(x,y,z)\in W} \max\left\{ \left| \frac{f(x)-f(y)}{d(x,y)} \right|, \frac{d(x,y)}{D} |f(z)| \right\}$$
$$= \max\{[f]_1, \|f\|_\infty\} = \|f\|_1,$$

Concerning the continuity of $L_{x,y,z}$, let us recall, from Proposition 5.9, that there exists a constant C such that $\|f\|_{\mathcal{B}^s_{p,p}} \geq C\|f\|_{L^\infty}$. Hence we have

$$\frac{|f(x)-f(y)|}{d(x,y)\,\|f\|_{\mathcal{B}^s_{p,p}}} \leq \frac{2}{Cd(x,y)}$$

while

$$\frac{d(x,y)|f(z)|}{D\,\|f\|_{\mathcal{B}^s_{p,p}}} \leq \frac{d(x,y)}{CD},$$

thus $L_{x,y,z} : X \to \mathbb{R} \times \mathbb{R}$ is a bounded linear operator.
Finally let us observe that we have shown that, supposed that Assumption H holds, for any $f \in Lip(S,d)$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset lip(S,d)$ such that $f_n \to f$ point-wise and $\sup_{n\in\mathbb{N}} \|f_n\|_1 \leq \|f\|_1$, hence, by Banach-Alaoglu theorem, we can extract a subsequence of f_n that weakly converges to f in X, concluding the proof of one implication.
Concerning the other implication, let us suppose that the *o–O* structure holds. Then we know that $(lip(S,d))^{**} \simeq Lip(S,d)$ isometrically. However, in [30] it is shown that such isometry is equivalent to Assumption H, concluding the proof. □

Remark 5.12. A proof of Theorem 5.11 without the requirement of the doubling property of the space was found in [58], using a different space X.

We can give many examples of metric spaces that satisfy assumption H.

Proposition 5.13. *[30] Let (S,d) be a compact metric space and $\omega : [0,\infty) \to [0,\infty)$ be a quasiconcave function such that $\lim_{t\to 0}\omega(t) = \lim_{t\to 0} t/\omega(t) = 0$. Then $(S,\omega(d))$ is a compact metric space satisfying Assumption H.*

Remark 5.14. If (S,d) is doubling, it is not hard to see that $(S,\omega(d))$ is also doubling.

In particular, we recover the case d_α, corresponding to $\omega(t) = t^\alpha$.

5.1.1 Arens-Eells space and atomic decomposition

We can give a result on an incomplete atomic decomposition of $\mathfrak{M}(S,d)$ (since the space is not complete). This result easily extends to an atomic decomposition of $(\mathfrak{M}(S,d))^c$

Proposition 5.15. *[5] Let (S,d) be a compact metric space satisfying the assumptions of Theorem 5.11 and let $\mu \in \mathfrak{M}(S)$. Then there exist a sequence of atomic measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathfrak{M}(S)$ with $card(supp(\mu_n)) \leq 3$ and a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{R})$ with $\gamma_n \geq 0$ such that*

$$\mu = \sum_{n=1}^{+\infty} \gamma_n \mu_n$$

where the convergence is intended in the Kantorovich-Rubinstein norm. Moreover there is $C > 0$ such that

$$C \sum_{n=1}^{+\infty} \gamma_n \leq \|\mu\|_{KR} \leq \sum_{n=1}^{+\infty} \gamma_n \tag{5.4}$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ be three dense sequences in S such that $x_m \neq y_n$ for all $m, n \in \mathbb{N}$. Applying remark 3.5 to Theorem 5.11 we can obtain $(lip(S,d))^*$ is spanned by the functionals

$$L_{m,n,p,\lambda} : f \in lip(S,d) \mapsto \lambda \frac{f(x_m) - f(y_n)}{d(x_m, y_n)} + (1-\lambda) \frac{f(z_p) d(x_m, y_n)}{\operatorname{diam} S}$$

with $m, n, p \in \mathbb{N}$ and $\lambda \in [0,1] \cap \mathbb{Q}$. These functionals can be identified with

$$\lambda \frac{\delta_{x_m} - \delta_{y_n}}{d(x_m, y_n)} + (1-\lambda) \frac{\delta_{z_p} d(x_m, y_n)}{\operatorname{diam} S}, \tag{5.5}$$

which easily implies our result. □

Since the atomic decomposition does not require the space E_0 to be nontrivial, we can show this result in cases where Assumption H does not hold. In particular, we can show the following results about $Lip(K, d_{eucl})$ and $\mathfrak{M}(K)$, where K is a compact subset of $\mathbb{R}^N$.

Proposition 5.16. *[7] Let $K \subset \mathbb{R}^N$ be a compact set. There exists a constant $C \in (0,1)$ such that for any functional $\mu \in \mathfrak{M}_0(K)^c$ there exists a sequence $(\alpha_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{R})$ such that*

$$\mu = \sum_{j=1}^{+\infty} \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j,$$

where the series converges in KR_0, and

$$C \sum_{j=1}^{+\infty} |\alpha_j| \leq \|\mu\|_{KR_0} \leq \sum_{j=1}^{+\infty} |\alpha_j|, \tag{5.6}$$

where the sequences $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ are defined in Lemma **??**. *Moreover, the sequence of δ-atoms $(\mu_j)_{j\in\mathbb{N}} \subset \mathfrak{M}_0(K)$ defined as*

$$\mu_j = \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|}$$

spans $\mathfrak{M}_0(K)^c$, with $\|\mu_j\|_{KR_0} = 1$ for any $j \in \mathbb{N}$. In particular, the δ-atoms μ_j are dipoles, hence have support of cardinality exactly 2.

Proposition 5.17. *[7] Let $K \subset \mathbb{R}^N$ be a compact set. There exists a constant $C \in (0,1)$ such that for any functional $\mu \in \mathfrak{M}(K)^c$ there exists a sequence $((\alpha_j^1, \alpha_j^2))_{j\in\mathbb{N}} \in \ell^1(\mathbb{R}^2)$ such that*

$$\mu = \sum_{j=1}^{+\infty} \left(\frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j^1 + \delta_{x_j} \alpha_j^2 \right),$$

where the series converges in KR, and

$$C \sum_{j=1}^{+\infty} (|\alpha_j^1| + |\alpha_j^2|) \le \|\mu\|_{KR} \le \sum_{j=1}^{+\infty} (|\alpha_j^1| + |\alpha_j^2|), \tag{5.7}$$

where the sequences $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ are defined in Lemma **??**. *In particular, the sequence of δ-atoms $(\mu_j)_{j\in\mathbb{N}} \subset \mathfrak{M}(K)$ defined as*

$$\mu_j = \begin{cases} \frac{\delta_{x_k} - \delta_{y_k}}{|x_k - y_k|} & j = 2k-1 \\ \delta_{x_k} & j = 2k \end{cases} \tag{5.8}$$

spans $\mathfrak{M}(K)^c$, and $\|\mu_j\|_{KR} \le 1$ for any $j \in \mathbb{N}$.

Remark 5.18. There are other representations of the predual of *Lip*, known in literature as the Lipschitz-free space or the Arens-Eells space. For example, in [13] and in [28] the predual of $Lip^0(S)$ is characterized, for $S \subseteq \mathbb{R}^N$, as the quotient of $L^1(S;\mathbb{R}^N)$ modulo its elements having null distributional divergence. For these spaces, a result of atomic decomposition also holds, with different atoms from $\mathfrak{M}(S)$.

5.2 BMO(s) spaces

Definition 5.19. [52] Let $1 \le s < \infty$. A function $f \in L^1(Q_0)$ belongs to the space $BMO_{(s)}(Q_0)$ if the quantity

$$[f]_s = \sup_{1\le p<\infty} \sup_{Q\subset Q_0} p^{-1/s} \left(\fint_Q |f(x) - f_Q|^p \, dx \right)^{1/p}$$

is finite.

Again, as in the BMO and B cases, the quantity $[f]_s$ is a seminorm, which becomes a norm if we identify functions modulo constants.

Remark 5.20. If $s = 1$, we obtain by Corollary 3.10 that $BMO_{(s)}(Q_0) = BMO(Q_0)$, with equivalence of seminorms. Moreover, if we extend the definition to the case $s = \infty$, we recover that $BMO_{(\infty)}(Q_0) = L^\infty(Q_0)$:

$$\begin{aligned}\sup_{1\le p<\infty}\sup_{Q\subset Q_0}\left(\fint_Q |f(x)-f_Q|^p\,dx\right)^{1/p} &= \lim_{1\le p<\infty}\sup_{Q\subset Q_0} p^{-1/s}\left(\fint_Q |f(x)-f_Q|^p\,dx\right)^{1/p}\\ &= \sup_{Q\subset Q_0}\|f-f_Q\|_{L^\infty(Q)}\end{aligned}$$

Just like for BMO, it is possible to define a vanishing subspace

Definition 5.21. Let $1 \le s < \infty$, $f \in L^1(Q_0)$ and $Q \subset Q_0$ a cube. We define the quantity

$$[f]_{s,Q} := \sup_{1\le p<\infty} p^{-1/s}\left(\fint_Q |f(x)-f_Q|^p\,dx\right)^{1/p}.$$

We say that a function $g \in L^1(Q_0)$ belongs to $VMO_{(s)}(Q_0)$ if

$$\limsup_{\varepsilon\to 0}\sup_{\substack{Q\subset\Omega\\ |Q|\le\varepsilon}}[g]_{s,Q} = 0.$$

Just like for $BMO_{(s)}$, $VMO_{(1)}$ coincides with VMO. However, if we extend the definition to $s = \infty$, the corresponding space is trivial.
The main motivation for the introduction of the spaces $BMO_{(s)}$ in [52] is their relation to the spaces X_s, introduced in the same paper and later expanded upon in [1, 2].

Definition 5.22. Let $1 < q \le 2$ and $1 \le s < \infty$. A (q, s)-atom is an $L^q(Q_0)$ function a such that:

1. $\operatorname{supp}(a) \subseteq Q$ for some cube Q;
2. $\int_Q a(x)\,dx = 0$;
3. $\|u\|_{L^q(Q)} \le (q')^{-1/s}|Q|^{-1/q'}$.

A function $f \in L^1_{loc}(Q_0)$ belongs to the space $X_s(Q_0)$ if there exists a sequence $\lambda = \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1$ and there exist (q_j, s) atoms a_j, with q_j possibly varying with the index j, such that $f(x) = \sum_{j\in\mathbb{N}} \lambda_j a_j(x)$ almost everywhere.

This family spans from the Hardy space $\mathcal{H}^1$, for $s = 1$, to $L^1_0(Q_0) = \left\{f \in L^1(Q_0) : \int_{Q_0} f = 0\right\}$ for $s \to \infty$. The relation between $BMO_{(s)}$ and X_s is the following.

Proposition 5.23. *[52, 2] Let $1 \leq s < \infty$. Then the dual space of X_s is isomorphic to $BMO_{(s)}/\mathbb{R}$, via the duality*

$$\langle f + \mathbb{R}, u \rangle_{X_s} = \int_{Q_0} fu.$$

This result actually holds for $s = \infty$ as well. This implies an atomic decomposition formula for L^1:

Theorem 5.24. *[1] A function $f \in L^1_{loc}(Q_0)$ belongs to $L^1(Q_0)$ if and only if there exists a sequence $\{q_j\}_{j\in\mathbb{N}}$ with $1 < q_j \leq 2$, a sequence $\{a_j\}_{j\in\mathbb{N}}$ of functions such that a_j is a q_j-atom and a sequence $\{\lambda_j\}_{j\in\mathbb{N}_0}$ of real numbers such that*

$$f(x) = \lambda_0 + \sum_{j\in\mathbb{N}} \lambda_j a_j(x)$$

holds almost everywhere.

There are many equivalent seminorms in $BMO_{(s)}$.

Lemma 5.25. *Let (Ω, μ) be a probability space and let $\alpha > 0$. Then the quantity*

$$|f|_\alpha := \sup_{1\leq p<\infty} \frac{\|f\|_{L^p(\Omega)}}{p^\alpha}$$

defines an equivalent norm in $EXP_{1/\alpha}(\Omega)$.

Definition 5.26. We consider the case $\alpha = 1$ for simplicity. The other cases are similar. Let us recall Stirling's formula:

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!} = 1.$$

We are going to use this in the form of the inequality (for $n > 0$)

$$Cn! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n!,$$

where in particular we can take $C = \sqrt{2\pi}e^{-1} \sim 0.922$ Together with the Taylor expansion for the exponential, this gives

$$Ce^x \leq 1 + \sum_{n=1}^{\infty} \frac{(ex)^n}{n^n\sqrt{2\pi n}} \leq e^x$$

and, using dominated convergence and monotone convergence, the inequality

$$C\int_\Omega e^{|f(x)|}\,dx \leq 1 + \sum_{n=1}^{\infty} \frac{e^n}{n^n\sqrt{2\pi n}} \int_\Omega |f(x)|^n\,dx \leq \int_{Q_0} e^{|f(x)|}\,dx$$

for every measurable function f. If $f \in L^1_{loc}(Q_0)$ satisfies $|f|_1 < \infty$, we have that

$$\int_\Omega \frac{1}{n^n} \left(\frac{|f(x)|}{\lambda |f|_1} \right)^n dx \le \lambda^{-n}$$

and therefore

$$\int_\Omega e^{\frac{|f(x)|}{\lambda |f|_1}} \le C^{-1} \left(1 + \sum_{n=1}^{\infty} \frac{e^n}{\sqrt{2\pi n} \lambda^n} \right) \overset{\lambda \to +\infty}{\longrightarrow} C^{-1},$$

so that for some constant $c > 0$ we have $\|f\|_{EXP} \le c|f|_1$. On the other hand, we have

$$2 \ge \int_{Q_0} e^{\frac{|f(x)|}{\|f\|_{EXP}}} \ge 1 + \sum_{n=1}^{\infty} \frac{e^n}{\sqrt{2\pi n}} \int_\Omega \frac{|f(x)|^n}{n^n \|f\|^n_{EXP}} \, dx \ge 1 + \frac{1}{\sqrt{2\pi}} \int_\Omega \frac{1}{n^n} \left(\frac{2|f(x)|}{\|f\|_{EXP}} \right)^n dx$$

and therefore

$$\frac{\|f\|_{L^p}}{p} \le 2 \frac{\|f\|_{L^{\lceil p \rceil}}}{\lceil p \rceil} \le (2\pi)^{\frac{1}{2\lceil p \rceil}} \|f\|_{EXP} \le 2\pi \|f\|_{EXP},$$

where $\lceil p \rceil$ denotes the smallest integer that is larger or equal to p.

Corollary 5.27. *The quantity*

$$\{f\}_s := \sup_{Q \subset Q_0} \|f - f_Q\|_{EXP_s}$$

is equivalent to $[f]_s$.

Corollary 5.27, together with paragraph 4.3, imply that, after some computation, the quantity

$$\sup_{Q \subset Q_0} \sup_{E \subset Q} \frac{\fint_E |u(x) - u_Q| \, dx}{\log^{1/s} \left(\frac{e|Q|}{|E|} \right)}$$

and the quasinorm

$$N_s(u) := \sup_{Q \subset Q_0} \sup_{0 < t \le |Q|} \frac{((u - u_Q)\chi_Q)^*(t)}{\log^{1/s} \left(\frac{e|Q|}{t} \right)} \tag{5.9}$$

are equivalent to $[u]_s$. We can use these quantities to show a version of the John-Nirenberg inequality for $BMO_{(s)}(Q_0)$.

Proposition 5.28. *There exist two constants* c_1, c_2 *such that for every* $u \in BMO_{(s)}(Q_0)$ *and every cube* $Q \subset Q_0$ *we have*

$$|\{x \in Q : |u(x) - u_Q| > \lambda\}| \leq c_1 \exp\left(-c_2 \left(\frac{\lambda}{[u]_s}\right)^s\right) |Q|. \tag{5.10}$$

Proof. By using the quasinorm (5.9), we see that $u \in BMO_{(s)}(Q_0)$ is equivalent to the existence of a constant $M > 0$ such that

$$((u - u_Q)\chi_Q)^*(t) \leq M \log^{1/s}\left(\frac{e|Q|}{t}\right) \tag{5.11}$$

for all cubes $Q \subset Q_0$. If we now fix a cube Q and call $\lambda = ((u - u_Q)\chi_Q)^*(t)$, we get from the properties of the non-increasing rearrangement that $t = |E_\lambda|$, with $E_\lambda := \{x \in Q : |u(x) - u_Q| > \lambda\}$. We can then rewrite (5.11) as

$$\frac{|Q|}{|E_\lambda|} \geq e^{-1} \cdot e^{\frac{\lambda}{M}},$$

which is exactly (5.10) with $c_1 = 1/e, c_2 = 1$. □

An application of the $BMO_{(s)}$ spaces is in Sobolev type embeddings. Let us define the Lorentz spaces.

Definition 5.29. Let $0 < p, q \leq \infty$ and Ω a bounded domain in $\mathbb{R}^N$. The Lorentz space $L^{p,q}(\Omega)$ [11] is defined as the space of measurable functions f such that $\|f\|_{p,q}$ is finite, where

$$\|f\|_{p,q} = \left(\int_0^{|\Omega|} (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^{|\Omega|} \mu_f^{q/p}(\lambda)\, d\,(\lambda^q)\right)^{1/q}$$

if q is finite, while for $q = \infty$ we have

$$\|f\|_{p,\infty} = \sup_{0<t\leq|\Omega|} t^{1/p} f^*(t).$$

It is trivial to see that $L^{p,p} = L^p$ for $0 < p \leq \infty$, and we have the embedding $L^{p,q} \hookrightarrow L^{p,r}$ if $q < r$. Notice that $\|\cdot\|_{L^{p,q}}$ only defines a norm if $1 \leq q \leq p < \infty$ or $p = q = \infty$, but if $1 < p < q \leq \infty$ it is equivalent to a norm.

Definition 5.30. Let $\Omega \subset \mathbb{R}^n$ be an open set and $X \subset L^1(\Omega)$ a Banach space. The Sobolev-type space $W^1X(\Omega)$ is defined as

$$W^1X(\Omega) = \{u \in W^{1,1}(\Omega) \cap X(\Omega) : |Du| \in X\}.$$

This space is a Banach space when endowed with the norm $\|u\|_{W^1X(\Omega)} = \|u\|_X + \|Du\|_X$.

Proposition 5.31. *Let* $1 \le q < \infty$, *and* $u \in W^1L^{N,q}(Q_0)$. *Then* $u \in BMO_{(q')}(Q_0)$ *and* $[u]_{q'} \le C\|Du\|_{L^{N,q}(Q_0)}$, *with* C *depending only on* N *and* q.

Proof. We start by noticing that if $u \in W^1L^{N,q}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a generic domain, then for all $\lambda > 0$ and $x_0 \in \mathbb{R}^N$ it follows from an easy computation that the function $v(x) := u\left(\dfrac{x - x_0}{\lambda}\right)$ satisfies $\|Dv\|_{L^{N,q}(\lambda\Omega + x_0)} = \|Du\|_{L^{N,q}(\Omega)}$, where $\lambda\Omega + x_0 = \{x \in \mathbb{R}^N : x = \lambda y + x_0, \quad y \in \Omega\}$. From [19] we have

$$\int_{Q_0} \exp\left(\alpha \left|\frac{u(x) - u_{Q_0}}{\|Du\|_{L^{N,q}(Q_0)}}\right|^{\frac{q}{q-1}}\right) \le \beta$$

for all $u \in W^1L^{N,q}(Q_0)$ for some constants α, β depending on N and q. Using the scaling and translation properties of $\|Du\|_{L^{N,q}}$ we obtain that

$$\int_{Q} \exp\left(\alpha \left|\frac{u(x) - u_{Q}}{\|Du\|_{L^{N,q}(Q)}}\right|^{\frac{q}{q-1}}\right) \le \beta|Q|$$

for every $u \in W^1L^{N,q}(Q_0)$ and every $Q \subset Q_0$, with the same constants α and β. Considering equivalence of norms, one obtains that $[u]_{q'} \le C\|Du\|_{L^{N,q}(\Omega)}$, so that $u \in BMO_{(q')}$. □

It is actually possible to prove a stronger result.

Lemma 5.32. *Let* $1 < p < \infty$ *and* $1 \le q < \infty$. *Then* $C^\infty(Q_0)$ *is dense in* $W^1L^{p,q}(Q_0)$.

Proof. Let us define, for a function $f \in L^1(Q_0)$, the Hardy-Littlewood maximal operator [11]:

$$Mf(x) = \sup_{\substack{Q \subset Q_0 \\ Q \ni x}} \int_Q |f(x)|\, dx.$$

Using standard interpolation techniques [11], one can prove that M maps $L^{p,q}(Q_0)$ into itself continuously:

$$\|Mf\|_{L^{p,q}} \le C\|f\|_{L^{p,q}}. \tag{5.12}$$

This implies that using regularization by convolution with standard mollifiers one can find for every $f \in W^1L^{p,q}(Q_0)$ a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^{p,q}(Q_0)$ and $f_n \to f$ in $W^1L^{p,q}(\Omega)$ for every $\Omega \subset\subset Q_0$. Using a partition of unity we can find a sequence that converges on the whole Q_0. □

Proposition 5.33. *Let* $1 < q < \infty$ $u \in W^1L^{N,q}(Q_0)$*. Then* $u \in VMO_{(q')}(Q_0)$.

Proof. By Lemma 5.32, we know that $C^\infty(Q_0)$ is dense in $W^1L^{N,q}$: we therefore consider, for $u \in W^1L^{N,q}(Q_0)$, a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(Q_0)$ such that $u_j \to u$ in $W^1L^{N,q}(Q_0)$. Since $[v]_{q'} \leq C\|Dv\|_{L^{N,q}(Q_0)}$, we have that $u_j \to u$ in $BMO_{(s)}(Q_0)$, and since $C^\infty(Q_0) \subset VMO_{(s)}(Q_0)$ we have $u \in VMO_{(s)}(Q_0)$. □

The spaces $BMO_{(s)}$ are not rearrangement invariant; it can be interesting to give a characterization of the rearrangement-invariant hull of these spaces, i.e. the set of all functions (on Q_0) equimeasurable to a function in $BMO_{(s)}$; we are also going to do the same for $VMO_{(s)}$.

Proposition 5.34. *Let* $Q_0 \subset \mathbb{R}^N$ *be the unit cube. For* $f \in L^1_{loc}(\Omega)$ *and* $t \in (0,1]$ *we consider the following quantity:*

$$R_s f(t) := \sup_{1\leq p<\infty} \frac{1}{p^{1/s}} \left(\frac{1}{t} \int_0^t |f^*(\tau) - f^*(t)|^p \, d\tau \right)^{1/p}$$

Let us now define

$$W_{(s)} = \left\{ f \in L^1_{loc}(Q_0) : \sup_{0<t\leq 1} < \infty \right\}.$$

and

$$w_{(s)} = \left\{ g \in W_{(s)}(Q_0) \cap C(Q_0) : \limsup_{t\to 0} R_s g(t) = 0 \right\}.$$

Then $W_{(s)}$ *is the rearrangement invariant hull of* $BMO_{(s)}$*, in the sense that* u *belongs to* $W_{(s)}$ *if and only if it is equimeasurable to a function* $\overline{u} \in BMO_{(s)}$*, and* $w_{(s)}$ *the rearrangement invariant hull of* $VMO_{(s)}$*.*

Remark 5.35. The spaces $W_{(s)}$ and $w_{(s)}$ are not vector spaces.

To prove this result we need a lemma.

Lemma 5.36. *[10, 11] Let* $\mathcal{O}$ *be a relatively open subset of a cube* Q *such that* $2|\mathcal{O}| \leq |Q|$*. Then there exists a family of cubes* $\{Q_j\}_{j\in\mathbb{N}}$ *with pairwise disjoint interiors such that*

- $|\mathcal{O} \cap Q_j| \leq |Q_j|/2 \leq |Q_j \setminus \mathcal{O}|$;
- $\mathcal{O} \subset \bigcup_{j\in\mathbb{N}} Q_j \subset Q$;
- $|\mathcal{O}| \leq \sum_{j\in\mathbb{N}} |Q_j| \leq 2^{N+1}|Q|$.

Proof of Proposition 5.34. A proof that $W_s(Q_0)$ is the rearrangement invariant hull of $BMO_{(s)}(Q_0)$, which generalizes the classical result by Bennett, Sharpley and Devore on BMO and $W = W_1$ (also called weak-L^∞) [10], can be found in [53]. We will therefore focus on w_s and $VMO_{(s)}$.
We claim that the inequality

$$\left(\fint_0^t (f^*(\tau) - f^*(t))^p \, d\tau \right)^{1/p} \leq C \sup_{\substack{Q \subset Q_0 \\ |Q| \leq 4t}} \left(\fint_Q |f(x) - f_Q|^p \, dx \right)^{1/p}$$

holds for all $f \in L^1_{loc}(Q_0)$, $1 \leq p < \infty$ and $t < 1/4$, with a constant $C > 0$ independent from these quantities. Let $f \in L^1_{loc}(Q_0)$ and $1 \leq p < \infty$.Without loss of generality, we can assume $f \geq 0$, since

$$\int_Q ||f| - |f|_Q| \leq \int_Q |f - f_Q|.$$

Consider the set $E = \{x : f(x) > f^*(t)\}$; we have $|E| < t$, so we can consider a relatively open set $\mathcal{O} \supset E$ such that $|\,\mathcal{O}\,| < 2t$. We now consider a sequence of cubes Q_j as in Lemma 5.36 and obtain

$$\left(\int_0^t (f^*(\tau) - f^*(t))^p \, d\tau \right)^{1/p} = \left(\int_E (f(x) - f^*(t))^p \, dx \right)^{1/p}$$
$$= \left(\sum_{j \in \mathbb{N}} \int_{E \cap Q_j} (f(x) - f^*(t))^p \, dx \right)^{1/p} \leq \sum_{j \in \mathbb{N}} \left(\int_{E \cap Q_j} (f(x) - f^*(t))^p \, dx \right)^{1/p}.$$

Let us denote by J_1 the set of indexes j for which $f_{Q_j} > f^*(t)$ and by J_2 the other indexes. We have

$$\begin{aligned} \sum_{j \in J_2} \left(\int_{E \cap Q_j} (f(x) - f^*(t))^p \, dx \right)^{1/p} &\leq \sum_{j \in J_2} \left(\int_{E \cap Q_j} (f(x) - f_{Q_j})^p \, dx \right)^{1/p} \\ &\leq \sum_{j \in J_2} \left(\int_{Q_j} (f(x) - f_{Q_j})^p \, dx \right)^{1/p} \\ &= \sum_{j \in J_2} |Q_j|^{1/p} \left(\fint_{Q_j} |f(x) - f_{Q_j}|^p \, dx \right)^{1/p}, \end{aligned}$$

while the sum on the J_1 indexes can be controlled by

$$\begin{aligned}
&\sum_{j\in J_1}\left(\int_{E\cap Q_j}(f(x)-f_{Q_j})^p\,dx\right)^{1/p} &&+\sum_{j\in J_1}\left(\int_{E\cap Q_j}(f_{Q_j}-f^*(t))^p\,dx\right)^{1/p}\\
\leq&\sum_{j\in J_1}\left(\int_{Q_j}(f(x)-f_{Q_j})^p\,dx\right)^{1/p} &&+\sum_{j\in J_1}\left(\int_{E\cap Q_j}(f_{Q_j}-f^*(t))^p\,dx\right)^{1/p}\\
=&\sum_{j\in J_1}|Q_j|^{1/p}\left(⨍_{Q_j}|f(x)-f_{Q_j}|^p\,dx\right)^{1/p} &&+\sum_{j\in J_1}|E\cap Q_j|^{1/p}(f_{Q_j}-f^*(t)).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\sum_{j\in J_1}|E\cap Q_j|^{1/p}(f_{Q_j}-f^*(t)) &\leq \sum_{j\in J_1}|\mathcal{O}\cap Q_j|^{1/p}(f_{Q_j}-f^*(t))\\
&\leq \sum_{j\in J_1}|\mathcal{O}^c\cap Q_j|^{1/p}(f_{Q_j}-f^*(t))\\
&= \sum_{j\in J_1}\left(\int_{\mathcal{O}^c\cap Q_j}(f_{Q_j}-f^*(t))^p\,dx\right)^{1/p}\\
&\leq \sum_{j\in J_1}\left(\int_{\mathcal{O}^c\cap Q_j}(f_{Q_j}-f(x))^p\,dx\right)^{1/p}\\
&\leq \sum_{j\in J_1}\left(\int_{Q_j}(f_{Q_j}-f(x))^p\,dx\right)^{1/p}\\
&= \sum_{j\in J_1}|Q_j|^{1/p}\left(⨍_{Q_j}|f(x)-f_{Q_j}|^p\,dx\right)^{1/p},
\end{aligned}$$

where we used the fact that so that $|\mathcal{O}\cap Q_j| < |\mathcal{O}^c\cap Q_j|$ and that $f(z)\leq f^*(t)$ on $\mathcal{O}^c$. Putting everything together we obtain

$$\left(\frac{1}{t}\int_0^t(f^*(\tau)-f^*(t))^p\,d\tau\right)^{1/p}\leq 2t^{-1/p}\sum_{j\in\mathbb{N}}|Q_j|^{1/p}\left(⨍_{Q_j}|f(x)-f_{Q_j}|^p\,dx\right)^{1/p},$$

and since $\sum_{j\in\mathbb{N}}|Q_j|^{1/p}\leq\left(\sum_{j\in\mathbb{N}}|Q_j|\right)^{1/p}\leq\left(2^{N+1}|\mathcal{O}|\right)^{1/p}\leq 2^{\frac{N+2}{p}}t^{1/p}$ and

$|Q_j| \leq 4t$ for all $j \in \mathbb{N}$ we get that

$$\left(\frac{1}{t}\int_0^t |f^*(\tau) - f^*(t)|^p \, d\tau\right)^{1/p} \leq 2t^{-1/p} \sum_{j\in\mathbb{N}} |Q_j|^{1/p} \sup_{\substack{Q\subset Q_0 \\ |Q|\leq 4t}} \left(\fint_Q |f(x) - f_Q|^p \, dx\right)^{1/p}$$
$$\leq 2^{\frac{N+2}{p}} \sup_{\substack{Q\subset Q_0 \\ |Q|\leq 4t}} \left(\fint_Q |f(x) - f_Q|^p \, dx\right)^{1/p}$$
$$\leq 2^{N+2} \sup_{\substack{Q\subset Q_0 \\ |Q|\leq 4t}} \left(\fint_Q |f(x) - f_Q|^p \, dx\right)^{1/p}$$

for all $t \leq 1/4$, which proves our claim. In particular, this implies that if $g \in VMO_{(s)}(Q_0)$ then

$$\limsup_{t\to 0} R_s g(t) = 0.$$

To show that $VMO_{(s)}(Q_0) \subseteq w_s(Q_0)$, we need to show that g^* is continuous for all $g \in VMO_{(s)}(Q_0)$. Since g^* is monotone, the only discontinuity it can have is a jump discontinuity, so let us assume $\alpha := \lim_{\tau\to t^+} g^*(\tau) < \lim_{\tau\to t^-} g^*(\tau) =: \beta$ for some $t > 0$. By [20], the truncated function

$$g(x) = \begin{cases} \alpha & \text{if } g(x) < \alpha \\ g(x) & \text{if } \alpha \leq g(x) \leq \beta \\ \beta & \text{if } g(x) > \beta \end{cases}$$

is also in $VMO_{(s)}$ and $g^*(\tau) = \alpha$ if $\tau > t$ and $g^*(\tau) = \beta$ if $\tau < t$, which implies that $g(x) = \alpha + (\beta - \alpha)\chi_E(x)$ almost everywhere, with $E = \{x : g(x) > \beta\}$, but such a function cannot be in $VMO_{(s)}$, so we obtain a contradiction. Therefore, g^* is continuous.

To complete the proof, it is necessary to show that every function g in $w_{(s)}$ is equimeasurable to a function in $VMO_{(s)}$. In fact, $g^* \in VMO_{(s)}(0,1)$: since for a bounded monotone function continuity is equivalent to belonging to $VMO_{(s)}$, it it enough to show the vanishing condition only for intervals approaching zero. Fix $\varepsilon > 0$ and let t be such that

$$\sup_{1\leq p<\infty} \frac{1}{p^{1/s}} \left(\frac{1}{t}\int_0^t |g^*(\tau) - g^*(t)|^p \, d\tau\right)^{1/p} \leq \varepsilon;$$

if $0 < a < b < t$, by [20, Proposition 4.5] we have

$$\begin{aligned}\frac{1}{p^{1/s}}\left(\fint_a^b |g^*(x)-(g^*)_{[a,b]}|^p\,dx\right)^{1/p} &\le \frac{2}{p^{1/s}}\left(\fint_a^b |g^*(x)-g^*(b)|^p\,dx\right)^{1/p}\\ &\le \frac{2}{p^{1/s}}\left(\fint_0^b |g^*(x)-g^*(b)|^p\,dx\right)^{1/p}\\ &\le 2\varepsilon\end{aligned}$$

which concludes the proof. □

To incorporate the pair $(VMO_{(s)}, BMO_{(s)})$ in the o-O framework, we need to use a different seminorm than the usual one.

Proposition 5.37. *Let $1 < s < \infty$, $p_0 \in (1,\infty)$ and $u \in L^1_{loc}(Q_0)$. Define*

$$[f]_{(s;p_0)} = \sup_{p_0 \le p < \infty}\ \sup_{Q \subset Q_0} p^{-1/s}\left(\fint_Q |f(x)-f_Q|^p\,dx\right)^{1/p}.$$

Then

$$[f]_{(s;p_0)} \le [u]_s \le p_0^{1/s}[f]_{(s;p_0)}$$

.

Proof. The first inequality is obvious. Concerning the second one, using the Hölder inequality we get that $\|f\|_{L^p(\Omega)} \le \|f\|_{L^q(\Omega)}$ for every Ω having measure 1. As a consequence, we have

$$\begin{aligned}p^{-1/s}\left(\fint_Q |f-f_Q|^p\right)^{1/p} &\le \left(\frac{p_0}{p}\right)^{1/s}(p_0)^{-1/s}\left(\fint_Q |f-f_Q|^{p_0}\right)^{1/p_0}\\ &\le p_0^{1/s}\left((p_0)^{-1/s}\left(\fint_Q |f-f_Q|^{p_0}\right)^{1/p_0}\right),\end{aligned}$$

which implies $[u]_s \le p_0^{1/s}[u]_{(s;p_0)}$. □

We are now ready to state the main theorem of this section

Theorem 5.38. *[42] Let $1 \le s < \infty$ and $p_0 > 1$. Consider the space $BMO_{(s)}(Q_0)/\mathbb{R}$ endowed with the norm $[\cdot]_{(s;p_0)}$. Then $(VMO_{(s)}/\mathbb{R}, BMO_{(s)}/\mathbb{R})$ is a o-O pair.*

In the proof, we use the following result.

Lemma 5.39. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p \le \infty$ and $u \in L^p(\Omega)$. Then the function $q \in [1,p] \mapsto \|u\|_{L^q(\Omega)}$ is continuous.*

Proof of Theorem 5.38. Write

$$[f]_{(s;p_0)} = \sup_{Q\subset Q_0} \sup_{p_0\le p<\infty} \frac{1}{p^{1/s}} \left(\fint_Q |f(x)-f_Q|^p \, dx \right)^{1/p}$$
$$= \sup_{Q\subset Q_0} \sup_{p_0\le p<\infty} \sup_{\|g\|_{L^{p'}(Q)}\le 1} \frac{1}{p^{1/s}} \fint_Q (f(x)-f_Q)g(x)\, dx.$$

Since $L^{p_0}(Q)$ is dense in $L^q(Q)$ for $1 \le q < p_0$ for all cubes $Q \subset Q_0$, we have

$$\sup_{\|g\|_{L^q(Q)}\le 1} \fint_Q f(x)g(x)\, dx = \sup_{g\in L^{p_0}(Q)\setminus\{0\}} \frac{1}{|Q|\|g\|_{L^q(Q)}} \int_Q f(x)g(x)\, dx,$$

therefore

$$[f]_{(s;p_0)} = \sup_{g\in L^{p_0}(Q)\setminus\{0\}} \frac{C(g)}{|Q|} \int_Q (f(x)-f_Q)g(x)\, dx,$$

where $C(g) = \sup_{1\le q<p_0} \frac{1}{\|g\|_{L^q(Q)}}$.

We now need to introduce, for $\Omega \subset \mathbb{R}^N$ a bounded open set and $1 < p < \infty$, the spaces $L^p_0(\Omega) = \left\{u \in L^p(\Omega) : \int_\Omega u(x)\, dx = 0\right\}$, endowed with the usual L^p norm, and $L^p(\Omega)/\mathbb{R}$, endowed with the quotient norm $\|u + \mathbb{R}\|_{L^p(\Omega)/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|u + c\|_{L^p(\Omega)}$. We have that for all $u + \mathbb{R} \in L^p(\Omega)/\mathbb{R}$ and $\varphi \in L^{p'}_0(\Omega)$ it is possible to consider the duality $\langle \varphi, [u]\rangle_{L^p(\Omega)/\mathbb{R}} = \int_\Omega f(x)g(x)\, dx$, which is independent from the choice of the representative g, and using some standard results on duals of subspaces and quotient spaces [15, Section 11.2] we have $(L^p_0(\Omega))^* \equiv L^{p'}(\Omega)/\mathbb{R}$ and $(L^p(\Omega)/\mathbb{R})^* \equiv L^{p'}_0(\Omega)$ via the previously mentioned duality. Define $\widetilde{\mathfrak{Q}}$ as the set of all cubes contained in Q_0 and denote by B_X the unit ball of a Banach space X. We consider $\widetilde{\mathfrak{F}} = \{\varphi_{Q,h}\}_{Q\in\widetilde{\mathfrak{Q}}, h\in B_{L^{p_0}_0(Q)}}$ defined on $X = L^{p_0}(\Omega)/\mathbb{R}$ in the following way:

$$\langle \varphi_{Q,h}, u\rangle_{L^{p_0}(\Omega)/\mathbb{R}} := C(h) \int_{Q_0} (u(x) - u_Q)h(x)\, dx,$$

where $C(h) = \max_{1<q\le p_0'} \dfrac{1}{|Q|^{1/q}(q')^{1/s}\|h\|_q}$. We will also consider the subset(s) $\mathfrak{F} = \{\varphi_{Q,h}\}_{Q\in\mathfrak{Q}, h\in\mathfrak{S}_Q}$, where $\mathfrak{Q}$ is the set of all rational cubes contained in Q_0 and $\mathfrak{S}_Q$ is any countable dense subset of the unit ball of $L^{p_0}_0(Q)$ with respect to the strong topology. It is not hard to see that $\mathfrak{F} \subset X^*$.

We now need to define a topology on $\widetilde{\mathfrak{F}}$. Identifying a cube with the pair center-sidelength, we can induce a topology on $\widetilde{\mathfrak{Q}}$ from the natural topology of $\mathbb{R}^{N+1}$, and $B_{L^{p_0}_0(Q)}$ with the topology induced by the weak topology in $L^{p_0}(Q)\setminus\mathbb{R}$; finally, by identifying each $L^{p_0}(Q)$ naturally with $L^{p_0}(Q_0)$, we

can 'glue' the various balls in a continuous way. We remark that $\mathfrak{F}$ is countable and dense in $\widetilde{\mathfrak{F}}$.
With our choice of topology we have $\varphi \to \infty$ iff $|Q| \to 0$, so we have that $E_0 = VMO_{(s)}(\Omega)$, so let us show assumption AP. Since $C^\infty(Q_0) \subset VMO_{(s)}(Q_0)$, it is enough to show it for a sequence in C^∞. Reasoning in a similar way as in [24], we take $\rho \in C_c^\infty(\mathbb{R}^N)$, $\operatorname{supp}(\rho) \subset [-1,1]^N$, $\rho \geq 0$, $\int_{[-1,1]^N} \rho = 1$ and for $\delta > 0$ define $\rho_\delta(x) = \delta^{-N}\rho(x/\delta)$. Let $x_0 \in Q_0$ and for $f \in BMO_{(s)}(Q_0)$ define $h = h_\varepsilon(x) = f\left((1-\varepsilon)x + \varepsilon x_0\right)$ and $g = g_\varepsilon = \rho_\delta * h_\varepsilon$, with $\delta = \kappa\varepsilon$ and κ such that the cube centered in x_0 with side length 2κ is entirely contained in Q_0. We have that $g_\varepsilon \in C^\infty(Q_0) \subset VMO_{(s)}(Q_0)$ and that $g_\varepsilon \rightharpoonup u$.
For simplicity, let us denote by $M_p(f,Q)$, with $1 \leq p < \infty$ and Q a cube, the quantity:

$$M_p(f,Q) = \left(\fint_Q |f(x) - f_Q|^p \, dx \right).$$

We have:

$$\begin{aligned}
g(x) - g_Q &= g(x) - \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^N} \rho_\delta(y) h(z-y) \, dy \, dz \\
&= \frac{1}{|Q|} \int_{\mathbb{R}^N} \rho_\delta(y) g(x-y) - \rho_\delta(y) \int_Q h(z-y) \, dz \, dy \\
&= \frac{1}{|Q|} \int_{\mathbb{R}^N} \rho_\delta(y)(g(x-y) - g_{Q-y}) \, dy
\end{aligned}$$

so that

$$\begin{aligned}
M_p(g,Q) &= \fint_Q \left| \int_{[-\delta,\delta]^N} \rho_\delta(y)(h(x-y) - h_{Q-y}) \, dy \right|^p \, dx \\
&= \fint_Q (2\delta)^N \left| \fint_{[-\delta,\delta]^N} \rho_\delta(y)(h(x-y) - h_{Q-y}) \, dy \right|^p \, dx \\
&\leq \fint_Q (2\delta)^N \fint_{[-\delta,\delta]^N} |\rho_\delta(y)(h(x-y) - h_{Q-y})|^p \, dy \, dx \\
&= (2\delta)^{N(p-1)} \int_{\mathbb{R}^N} \rho_\delta(y) M_p(h, Q-y) \, dy \\
&\leq \int_{\mathbb{R}^N} \rho_\delta(y) M_p(h, Q-y) \, dy.
\end{aligned}$$

Now, we have $M_p(h, Q-y) = M_p(f, Q(y,\varepsilon))$, with $Q(y,\varepsilon) = (1-\varepsilon)(Q-y) + \varepsilon x_0 = (1-\varepsilon)Q + \varepsilon(x_0 + (1-\varepsilon)y)$, which by using the assumption on κ and the convexity of Q_0 implies that $Q(y,\varepsilon) \subset Q_0$, so that $M_p(g,Q) \leq M_p(f, Q(y,\varepsilon))$ for all cubes Q and all $p \in [p_0, \infty)$, which gives $[g]_{(s;p_0)} \leq [f]_{(s;p_0)}$. $\square$

Just like in [45], this result implies a distance formula. However, the proof is slightly more involved.

Corollary 5.40. *Let $u \in BMO_{(s)}$ (modulo constants). The following distance formula holds:*

$$dist_{BMO_{(s)}}\big(u, VMO_{(s)}\big) = \inf_{v \in VMO_{(s)}} [u-v]_s = \limsup_{|Q|\to 0} [u]_{s,Q}$$

Proof. We notice that, for any fixed $f \in BMO_{(s)}$, the map $p \in [1,\infty) \mapsto [u]_{(s;p)}$ is continuous. If we denote by $D_{p_0}(f)$ the distance of a function $f \in BMO_{(s)}$ from $VMO_{(s)}$ with respect of the norm $[\cdot]_{(s;p_0)}$, we have by Proposition 3.2 that

$$D_{p_0}(f) = \limsup_{|Q|\to 0} \sup_{p_0 \le p < \infty} \frac{1}{p^{1/s}} \left(\fint_Q |f(x) - f_Q|^p \, dx \right)^{1/p}$$

for all $p_0 > 1$. Since $D_1(f) = \lim_{p_0 \to 1^+} D_{p_0}(f)$, all we need to show is that

$$\limsup_{|Q|\to 0} [f]_{s,Q} = \lim_{p_0 \to 1^+} \left(\limsup_{|Q|\to 0} \sup_{\overline{p} \le p < \infty} \frac{1}{p^{1/s}} \left(\fint_Q |u(x) - u_Q|^p \, dx \right)^{1/p} \right).$$

Using Lemma 5.39 we have that, for a fixed cube Q,

$$[u]_{s,Q} = \lim_{p_0 \to 1^+} \sup_{p_0 \le p < \infty} \frac{1}{p^{1/s}} \left(\fint_Q |u(x) - u_Q|^p \, dx \right)^{1/p},$$

so the claim follows by a diagonal argument, concluding the proof. □

We can also recover the atomic decomposition of the space X_s. Let us show that the conditions of Proposition 3.4 hold: B_E **is** $\sigma(E, \mathfrak{F})$**-compact.** As mentioned before, we just need to apply 3.6, using the fact that $\mathfrak{F} \subset \widetilde{\mathfrak{F}} \subset X^*$.
$\mathfrak{F}$ **is norming.** We use the fact that

$$\|f\|_p = \sup_{\|g\|_{p'} \le 1} \int_{Q_0} f(x) g(x) \, dx$$

and the density of $\mathfrak{S}_Q$ in every $L^q(Q)$ with $1 < q \le 2$:

$$\sup_{h \in \mathfrak{S}_Q} \langle \varphi_{Q,h}, u \rangle \ge \frac{1}{p^{1/s}} \left(\fint_Q |u(x) - u_Q|^p \, dx \right)^{1/p} \qquad \forall p \in [2, +\infty)$$

since $C(h)\|h\|_q \geq \frac{1}{p^{1/s}|Q|^{1/p}}$ by definition of $C(h)$, and we complete by taking the supremum over all q and Q.
For the other inequality, Lemma 5.39 implies that $\frac{1}{|Q|^{1/q}(q')^{1/s}\|h\|_q}$ is continuous with respect to q and

$$\lim_{q\to 1^+} \frac{1}{|Q|^{1/q}(q')^{1/s}\|h\|_q} = 0,$$

so that it attains its maximum at $p' \in (1,2]$, and

$$\langle \varphi_{Q,h}, u\rangle \leq \frac{1}{p^{1/s}} \left(\fint_Q |u(x) - u_Q|^p \, dx \right)^{1/p} \leq [[u]]_s.$$

Let us now show that E_* coincides with the space X_s. An element $\varphi_{Q,h}$ of $\mathfrak{F}$ can be represented, via the usual duality by the function

$$\varphi_{Q,h} = C(h)h(x).$$

It can easily be seen that this function is actually a (q,s)-atom for a suitable q: by definition, it is supported on Q and it has zero average, and by letting q be the exponent where $\frac{1}{|Q|^{1/q}(q')^{1/s}\|h\|_q}$ attains its maximum, we easily obtain that $\|\varphi_{Q,h}\|_{L^q(Q)} \leq (q')^{-1/s}|Q|^{-1/q'}$, and by using Proposition 3.4 we obtain that E_* has an atomic representation. To show that it actually coincides with X_s it is necessary to show that the same formula holds for any choice of atoms, but this can be obtained by varying the sets $\mathfrak{S}_Q$ among all possibilities.

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