UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

Dipartimento di Matematica e Applicazioni "Renato Caccioppoli"

Dottorato di Ricerca in Matematica e Applicazioni

Tesi di Dottorato di
Martina Liccardo

Some Model Theoretic Aspects of Ordered Abelian Groups

Ciclo XXXIV, S.S.D. MAT/01

Abstract

This thesis is concerned with the model theory of ordered abelian groups, in particular, in relation to the properties of eliminating imaginaries and having definable types. The theory of ordered abelian groups was initially studied by Y. Gurevich and P. Schmitt in the second half of the 20th century. They classified up to elementary equivalence the class of ordered abelian groups by a sequence of coloured chains, called spines, arising from every ordered abelian group.

The analysis of imaginaries and the description of definable types are two active research areas in model theory. We investigate the property of elimination of imaginaries for some special cases of ordered abelian groups. We prove that certain Hahn products of ordered abelian groups do not eliminate imaginaries in the pure language of ordered abelian groups. Moreover, we show that, adding finitely many constants to the language of ordered abelian groups, the theories of the finite lexicographic products $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$ have definable Skolem functions. We then study the property for an ordered abelian group to be stably embedded (i.e. to have definable types). We identify a sufficient and necessary condition for certain ordered abelian groups to be stably embedded. These include regular ordered abelian groups, ordered abelian groups with finite regular rank and models of the theory of a maximal ordered abelian group satisfying a condition on the definability of its principal convex subgroups. For the last class of groups G, we establish, in particular, a transfer principle for stable embeddedness from G to the spine of G in the spirit of the work of Gurevich and Schmitt.

Acknowledgements

I would like to express my gratitude to my supervisor Paola D'Aquino for her constant guidance throughout my studies.

I would like to thank Immanuel Halupczok for suggesting the problem of eliminating imaginaries in ordered abelian groups, which is one of the topics discussed in this thesis. Moreover, I owe a special thank to Lorna Gregory and Angus Macintyre for many fruitful discussions and useful remarks.

I wish to thank Martin Hils and Pierre Touchard for sharing with me the interest in stably embedded ordered abelian groups. The collaboration with them has led to Chapter 4 of this thesis. In particular, I would like to express my thanks to Martin Hils for being a so welcoming host during my visit to the University of Münster and for his precious contribution to Chapter 3 by a simple, but important example. I would also like to thank Westfälische Wilhelms-Universität Münster for financial support during my research visit at the Institute of Mathematical Logic and Foundational Research.

I would like to thank the referees of this thesis, Françoise Point and Carlo Toffalori, for their suggestions and remarks which certainly improved the thesis.

Last but not least, I wish to thank all the members of the model-theoretic community that I have met over these years and that have contributed to making my doctoral studies a so valuable experience. I am particularly grateful to Antongiulio Fornasiero and Marcus Tressl for several interesting and stimulating conversations. Special thanks goes to my colleagues Angela Borrata and Anna De Mase for their unwavering support.

Contents

Chapter 1

Introduction

Ordered abelian groups form a class of first order structures widely studied in model theory, and several results, including quantifier elimination results and classifications of the model theoretic complexity, have been obtained (see for example [17], [49], [50], [6], [13] and [10]). They are all unstable structures, so that their model theory is very different and more complex than that of abelian groups, whose theory is stable. In his pioneering paper [16], Gurevich determined how to transfer the elementary properties of an ordered abelian group to simpler and more manageable structures. Indeed, for any natural number $n \geqslant 2$, one can associate with every ordered abelian group G a linear order with unary predicates (coloured chain) $\mathrm{Sp}_n(G)$, called the n-spine of G. Roughly speaking, the collection of the n-spines of G "contains" all the information of the group expressible in the language. This leads to some fundamental transfer principles from G to $\mathrm{Sp}_n(G)$, introduced by Gurevich first and revisited later by Schmitt in [41]. For instance, one can reduce the elementary equivalence of two ordered abelian groups G and H to the elementary equivalence of the corresponding n-spines $\mathrm{Sp}_n(G)$ and $\mathrm{Sp}_n(H)$, for any $n \geqslant 2$. Furthermore, using that the theory of linear orders is decidable ([11]), Gurevich proved the decidability of the theory of ordered abelian groups.

Ordered abelian groups play an important role also in the context of valued fields. Indeed, since the fundamental results obtained by Ax, Kochen and Ershov for henselian valued fields, one can reduce, in many cases, the study of a property for a valued field to the study of the same property for its residue field and value group. Since the value group is an ordered abelian group, this gives an extra motivation for better understanding the model theory of ordered abelian groups. For contributions in this direction see [18], [1] and [47].

In this thesis we will investigate two model-theoretical properties in the class of ordered abelian groups, both somehow related to a definability issue: *elimination of imaginaries* and *stable embeddedness*.

The imaginary elements of a given first order structure $\mathcal{M}$ are equivalence classes of 0-definable equivalence relations. Shelah, in [43], introduced for any structure $\mathcal{M}$ the expanded structure $\mathcal{M}^{\mathrm{eq}}$, in order to make the imaginary elements of $\mathcal{M}$ real elements of the structure. Adding the imaginaries to $\mathcal{M}$ has several advantages. For instance, it allows to consider interpretations as subsets of the structure, and it entails "nice" properties, as the Galois correspondence. In some cases, one can find already in the structure canonical codes which identify the imaginary elements and, in this case, we do not need to expand $\mathcal{M}$ to $\mathcal{M}^{\mathrm{eq}}$. If this occurs, we say that $\mathcal{M}$ eliminates imaginaries. We may say that in a structure eliminating imaginaries, some quotient structures, in general not definable, can essentially be treated as definable. In the literature there are partial results on elimination of imaginaries in ordered abelian groups. Examples of ordered abelian groups that eliminate imaginaries are divisible ordered abelian groups, see [31], and discretely ordered abelian groups elementarily equivalent to $\mathbb{Z}$, see [5] and Appendix A of [9]. But examples of ordered abelian groups that do not have elimination of imaginaries in the language $L_{\mathrm{oag}} = \{0, +, -, <\}$ are not present in the literature. Providing such examples is the first goal of this thesis.

The second property we will examine is related to the definability of types. One of the main results of Shelah's classification theory is the characterization of stable theories in terms of properties for types. In particular, a theory T is stable if and only if all types over any model of T are definable. Then, in unstable theories, one can ask if the definability of types holds for at least some model $\mathcal{M}$. This is equivalent to saying that, for any elementary extension $\mathcal{N}$ and every definable set $\varphi(N)$ with parameters from N, the intersection $M \cap \varphi(N)$ is definable with parameters from M. If this is the case, we say that $\mathcal{M}$ is stably embedded in all elementary extensions or simply it is stably embedded. If the same holds, not for all elementary extensions of $\mathcal{M}$, but for a fixed elementary extension $\mathcal{N}$, we say that $\mathcal{M}$ is stably embbeded in $\mathcal{N}$ or that the pair $(\mathcal{N}, \mathcal{M})$ is stably embedded. In [46], stably embedded pairs of models in a particular class of valued fields were studied. Touchard proved that, in some cases, an elementary pair of valued fields is stably embedded if and only if the extension is separated and the corresponding pairs of value groups and residue fields are stably embedded. This result motivates then an analysis of stably embedded ordered abelian groups, which is the second goal of this thesis. In particular, we aim at characterizing stable embeddedness for ordered abelian groups, establishing a result similar to that obtained for valued fields.

We outline the structure and the main results of the thesis in the following. After collecting some preliminaries on general model theory in the next section, we introduce the class of ordered abelian groups in Chapter 2. In this chapter, we review definitions and results we will need through the thesis. We adapt, in particular, some notions from general valuation theory to the context of ordered abelian groups, and we present the state of art of the model theory of ordered abelian groups, summarizing the most important results that have been achieved in the last years.

In Chapter 3, we prove that elimination of imaginaries fails for the theories of the Hahn products $H_{i<\alpha}\mathbb{Z}$ and $H_{i<\alpha}\mathbb{Z} \times \mathbb{Q}$, with α well-ordered index set. A property related to the elimination of imaginaries is the existence of definable Skolem functions. In particular, definable Skolem functions allow to reduce the goal of finding a code for every imaginary to coding just one dimensional definable sets. We prove that, once we add finitely many judiciously chosen elements as new constants to L_{oag}, the theories of $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$, for any $n \geqslant 1$, have definable Skolem functions. These results are part of the preprint [27].

Chapter 4 is dedicated to the study of the ordered abelian groups which are stably embedded in all elementary extensions. To this purpose, we firstly study stably embedded coloured chains. We show that a coloured chain is stably embedded if and only if all cuts are definable. Then, for certain ordered abelian groups G, we identify some necessary and sufficient conditions for G to be stably embedded. These involve in particular the non-existence of proper immediate extensions and the stable embeddedness of the spine, establishing a transfer principle for stable embeddedness in the same spirit of that of Gurevich and Schmitt for elementary equivalence. Moreover, we exhibit some concrete examples of stably embedded ordered abelian groups, including the Hahn product $G = H_{i<\omega}\mathbb{Z}$. To this end, we deduce from [6] a specific language in which G eliminates quantifiers. Some of the results presented in this chapter have been obtained jointly with M. Hils and P. Touchard and will appear in [19].

1.1 Notation and preliminaries

We recall here some notions and results of model theory that we will use throughout the thesis. For the most basic notions, one may refer to any model theory text, such as [20], [35], [28] or [45].

We use the convention that $0 \in \mathbb{N}$. We write $\mathbb{N}^*$ and $\mathbb{P}$ for the set $\mathbb{N}\backslash\{0\}$ and the set of primes, respectively. We denote the successor of an ordinal α by $\alpha + 1$ and, if a set S has order-type α, the order type of the reverse order of S is denoted by α^*. A tuple of variables $(x_0, \dots, x_n)$ or of elements in an L-structure $(a_0, \dots, a_n)$ is denoted by $\bar{x}$ or $\bar{a}$. Moreover, the notations $|\bar{x}|$ and $|\bar{a}|$ stand for the arity of the tuple. Curvy letters $\mathcal{M}, \mathcal{N}, \mathcal{K}, \dots$ typically denote structures, whereas $M, N, K, \dots$ denote the corresponding underlying sets. If $\mathcal{M}$ is an L-structure, by "definable" we will mean definable in L with parameters in M, and by "0-definable" we will mean definable without parameters.

Elimination of imaginaries

Let $\mathcal{M}$ be an L-structure, L any first order language. Let m be a positive integer and $E(\bar{x}, \bar{y})$ a 0-definable equivalence relation on $\mathcal{M}^m$. The E-equivalence classes of $\mathcal{M}^m$ are called *imaginary elements* of $\mathcal{M}$, and we say that $\mathcal{M}$ eliminates imaginaries if each imaginary can be coded in the structure. More precisely, the notion of elimination of imaginaries is stated in the following definition.

Definition 1.1.1 ([20])**.** An L-structure $\mathcal{M}$ has *elimination of imaginaries* if for any positive integer m, any 0-definable equivalence relation $E(\bar{x}, \bar{y})$ on $\mathcal{M}^m$ and any E-class X, there is an L-formula $\vartheta(\bar{x}, \bar{z})$ such that $X = \vartheta(M^m, \bar{b})$ for some UNIQUE tuple $\bar{b} \subseteq M$. Moreover, such a tuple $\bar{b}$ is called a *canonical parameter* for X.

Let $\mathcal{M}$ be an L-structure. We denote by $\mathrm{Aut}(\mathcal{M})$ the group of automorphisms of $\mathcal{M}$. Moreover, let $S \subseteq \mathrm{Aut}(\mathcal{M})$. By $\mathrm{Fix}(S)$ we mean the set

$$\mathrm{Fix}(S) = \{ a \in M \mid f(a) = a \text{ for all } f \in S \}.$$

Let $A \subseteq M$. By $\mathrm{Aut}(\mathcal{M}/A)$ and $\mathrm{Stab}_{\mathcal{M}}(A)$ we mean the group of all automorphisms of

$\mathcal{M}$ fixing A pointwise and fixing A setwise, respectively. Namely,

$$\mathrm{Aut}(\mathcal{M}/A) = \{ f \in \mathrm{Aut}(\mathcal{M}) \mid f(a) = a \text{ for every } a \in A \}$$
$$\mathrm{Stab}_{\mathcal{M}}(A) = \{ f \in \mathrm{Aut}(\mathcal{M}) \mid f(A) = A \}.$$

Remark 1.1.2. Let $\mathcal{M}$ be an L-structure that admits elimination of imaginaries and let $\bar{b}$ be a canonical parameter for an E-class X, where $E(\bar{x}, \bar{y})$ is a 0-definable equivalence relation on $\mathcal{M}^m$. Then $\bar{b}$ is fixed by the same automorphisms of $\mathcal{M}$ which leave X invariant. Therefore, if $\mathcal{M}$ eliminates imaginaries, for every E-class X there exists a tuple $\bar{b} \subseteq M$ such that, for any automorphism f of $\mathcal{M}$, $f \in \mathrm{Stab}_{\mathcal{M}}(X)$ if and only if $f \in \mathrm{Aut}(\mathcal{M}/\bar{b})$.

Definition 1.1.3. An L-structure $\mathcal{M}$ has *uniform elimination of imaginaries* if for any positive integer m, and any 0-definable equivalence relation $E(\bar{x}, \bar{y})$ on M^m, there is an L-formula $\vartheta(\bar{x}, \bar{z})$ such that for every E-class X, there is a unique tuple $\bar{b} \subseteq M$ such that $X = \vartheta(M^m, \bar{b})$.

In other words, $\mathcal{M}$ uniformly eliminates imaginaries if one can find a formula $\vartheta(\bar{x}, \bar{z})$ as in Definition 1.1.1 depending only on the equivalence relation $E(\bar{x}, \bar{y})$ and not on the equivalence class X. Notice that uniform elimination of imaginaries is preserved under elementary equivalence. Moreover, one can easily see that, in this case, every 0-definable equivalence relation is the fibration of a 0-definable function. In particular, it holds that

Proposition 1.1.4. *$\mathcal{M}$ has uniform elimination of imaginaries if and only if for every 0-definable equivalence relation $E(\bar{x}, \bar{y})$ there is a 0-definable function f_E on $M^{|\bar{x}|}$ such that for all $\bar{b_1}, \bar{b_2}$,*

$$E(\bar{b_1}, \bar{b_2}) \text{ if and only if } f_E(\bar{b_1}) = f_E(\bar{b_2}).$$

We say that a theory T in L has (uniform) elimination of imaginaries if every model of T has (uniform) elimination of imaginaries. The uniformity of elimination of imaginaries, in the sense of Definition 1.1.3, holds under the condition stated in the following well-known result (see, for example, [35, Theorem 16.16]).

Theorem 1.1.5. *Let T be an L-theory. Suppose T has elimination of imaginaries and, in every model of T, there are at least two elements 0-definable. Then T has uniform elimination of imaginaries.*

Note that the existence of at least two 0-definable elements in order to have uniform elimination of imaginaries cannot be avoided. Indeed, both classes of the equivalence relation $(x_1 = x_2 \wedge y_1 = y_2) \vee (x_1 \neq x_2 \wedge y_1 \neq y_2)$ are definable without parameters by the formulas $x_1 = x_2$ and $x_1 \neq x_2$, respectively, but there is no way of using a unique tuple to pick out one of these formulas.

Another form of elimination of imaginaries is the following.

Definition 1.1.6. We say that an L-structure $\mathcal{M}$ has *weak elimination of imaginaries* if for any positive integer m, any 0-definable equivalence relation $E(\bar{x}, \bar{y})$ on $\mathcal{M}^m$ and any E-class X, there are an L-formula $\vartheta(\bar{x}, \bar{z})$ and a FINITE set B of tuples of M such that $X = \vartheta(M^m, \bar{b})$ if and only if $\bar{b} \in B$.
We say that a theory T in L has weak elimination of imaginaries if every model of T has weak elimination of imaginaries.

Fact 1.1.7 ([36])**.** *Let T be an expansion of the theory of linear order. If T weakly eliminates imaginaries, then T eliminates imaginaries.*

Stable embeddedness

Definition 1.1.8. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. $\mathcal{M}$ is said to be *stably embedded* in $\mathcal{N}$ if for every definable set $\varphi(N^m, \bar{a})$, $\bar{a} \subset N$, its trace $\varphi(N^m, \bar{a}) \cap M^m$ is $L(M)$-definable, i.e. there exist an L-formula $\psi(\bar{x}, \bar{z})$ and a tuple $\bar{b}$ of parameters in M such that

$$\varphi(N^m, \bar{a}) \cap M^m = \psi(M^m, \bar{b}). \tag{1.1}$$

Note that $\psi(\bar{x}, \bar{z})$ may depend on the parameters $\bar{a}$. If $\psi(\bar{x}, \bar{z})$ depends only on the formula $\varphi(\bar{x}, \bar{y})$ and not on $\bar{a}$, $\mathcal{M}$ is said to be *uniformly stably embedded* in $\mathcal{N}$.

Definition 1.1.9. We say that $\mathcal{M}$ is *stably embedded* if $\mathcal{M}$ is stably embedded in every elementary extension of $\mathcal{M}$. Similarly, we say that $\mathcal{M}$ is *uniformly stably embedded*.

It is easy to see that the property of being stably embedded in an elementary extension is equivalent to the definability of the types realized in that extension. First of all we recall the notion of definable type over an arbitrary subset of M.

Definition 1.1.10. Let $A \subseteq M$. A type $p(\bar{x}) \in S_n(A)$ is said to be *definable* if for every L-formula $\varphi(\bar{x}, \bar{y})$, there exists an $L(A)$-formula $d_p\varphi(\bar{y})$ such that for all $\bar{a} \subseteq A$

$$p(\bar{x}) \vdash \varphi(\bar{x}, \bar{a}) \text{ if and only if } M \models d_p\varphi(\bar{a}). \tag{1.2}$$

The collection $(d_p\varphi)_\varphi$ is called a *defining scheme* for p.

Example 1.1.11. Every realized type is definable: let $\bar{\alpha} \in M^n$, then $\mathrm{tp}(\bar{\alpha}/M)$ is definable. Indeed, trivially, $(d_p\varphi)_\varphi$ with $d_p\varphi(\bar{y}) = \varphi(\bar{\alpha}, \bar{y})$ is a defining scheme for p.

Example 1.1.12. Let T be a stable theory, then all types over all models of T are definable (Shelah's theorem). In particular, for any formula $\varphi(\bar{x}, \bar{y})$, there is a formula $\psi(\bar{y}, \bar{z})$ such that for any $\mathcal{M} \models T$, for every type $p(\bar{x})$ over M there is $\bar{b} \subset M$ such that

$$d_p\varphi(\bar{y}) = \psi(\bar{y}, \bar{b}). \tag{1.3}$$

In this case, we say that all the types are *uniformly definable*.

One can see immediately the following fact:

Fact 1.1.13. *$\mathcal{M}$ is stably embedded in $\mathcal{N}$ if and only if all n-types over M realized in N are definable, i.e. for every $\bar{\alpha} \subset N$, $p(\bar{x}) = \mathrm{tp}(\bar{\alpha}/M)$ is definable.*

Similarly, $\mathcal{M}$ is uniformly stably embedded in $\mathcal{N}$ if and only if all n-types over M realized in N are uniformly definable ($d\varphi$ does not depend on p, i.e. on $\bar{\alpha} \subset N$).

It is worth mentioning that uniform stable embeddedness of elementary pairs is preserved by elementary extension. Recall that an elementary pair is a pair of L-structures $(\mathcal{N}, \mathcal{M})$ such that $\mathcal{M} \preceq \mathcal{N}$. Let P be a unary predicate, then $(\mathcal{N}, \mathcal{M})$ can be seen as a $L_{\mathrm{P}} = L \cup \{\mathrm{P}\}$-structure by interpreting P as the underlying set of $\mathcal{M}$. One can prove that, if $\mathcal{M}$ is uniformly stably embedded in $\mathcal{N}$, then $\mathcal{M}'$ is uniformly stably embedded in $\mathcal{N}'$ for any elementary extension $(\mathcal{N}', \mathcal{M}')$ of $(\mathcal{N}, \mathcal{M})$ in L_{P}.

Remark 1.1.14. The following are equivalent:

1. $\mathcal{M}$ is (uniformly) stably embedded in every elementary extension,
2. $\mathcal{M}$ is (uniformly) stably embedded in a monster model $\mathcal{U}$ of $Th(\mathcal{M})$.

In order to characterize the definability of types, we now introduce the notion of 'heir' of a type p.

Definition 1.1.15. Let $p \in S_n(M)$, and $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. An extension q of p over N is called an *heir* of p if for every formula $\varphi(\bar{x}, \bar{y})$ with parameters from M and every $\bar{n} \subset N$ such that $q \vdash \varphi(\bar{x}, \bar{n})$, there exists a tuple $\bar{m} \subset M$ with $p \vdash \varphi(\bar{x}, \bar{m})$.

The following result is well-known [26]:

Theorem 1.1.16 (Lascar-Poizat)**.** *Let $p \in S_n(M)$. Then p is definable if and only if p has only one heir over every elementary extension of $\mathcal{M}$.*

Another important result about the definability of types which we recall is Marker-Steinhorn's Theorem. In [29], Marker and Steinhorn showed that, in o-minimal structures, one can reduce the question of definability of types to the question of definability of 1-types:

Theorem 1.1.17 (Marker-Steinhorn)**.** *Let T be an o-minimal theory, and let $\mathcal{M} \preceq \mathcal{N}$ be two models of T. Then, all types over M realized in N are (uniformly) definable if and only if all 1-types over M realized in N are (uniformly) definable.*

Note that since Marker-Steinhorn's theorem, the question whether definability of 1-types implies definability of n-types has been studied and analogue versions of Marker-Steinhorn's result are known in other contexts, see, for example, [7, Theorem 3.3] in the class of algebraically closed valued fields and, more generally, [23, Lemma 6.2.7] and [33, Lemma 2.7].

Relative quantifier elimination

Let us recall here some terminology and basic facts about relative quantifier elimination. The material presented here is taken essentially from [38, Annex A]. In this section, L will denote a many-sorted language and $\{ \Pi, \Sigma \}$ a partition of its sorts. Moreover, we denote by $L\restriction_\Sigma$ the language of all function symbols and relation symbols in L involving only sorts in Σ.

Definition 1.1.18. Let $L^{\Sigma-\mathrm{Mor}}$ be the definitional extension of L obtained by adding a new predicate $P_\varphi(\bar{x})$ for each $L\restriction_\Sigma$ formula $\varphi(\bar{x})$. The Morleyization of T on Σ is the $L^{\Sigma-\mathrm{Mor}}$-theory $T^{\Sigma-\mathrm{Mor}} := T \cup \{ P_\varphi(\bar{x}) \leftrightarrow \varphi(\bar{x}) \mid \varphi(\bar{x})\ L\restriction_\Sigma \text{ formula} \}$.

Definition 1.1.19. Let T be an L-theory. We say that

- T eliminates Π-quantifiers if every L-formula is equivalent modulo T to a formula where quantification only occurs on variables from the sorts in Σ.
- T eliminates quantifiers relatively to Σ if the Morleyzation of T on Σ, $T^{\Sigma-\mathrm{Mor}}$, eliminates quantifiers.

As observed in [38, Annex A], if T eliminates quantifiers relatively to Σ, then T eliminates Π-quantifiers. But the converse is not in general true.

Definition 1.1.20. We say that Σ is *closed* if any relation symbol involving a sort in Σ or any function symbol with a domain involving a sort in Σ only involves sorts in Σ. Equivalently, every symbol involving sorts from both Π and Σ are function symbols of the form $f\colon \prod_i P_i \to S$, where $P_i \in \Pi$ and $S \in \Sigma$.

If Σ is a closed set of sorts, we will denote the set of functions $f\colon \prod_i P_i \to S$ by $\mathcal{F}$.

Fact 1.1.21. *If Σ is a closed set of sorts, then T eliminates Π-quantifiers if and only if T eliminates quantifiers relatively to Σ.*

Fact 1.1.22. *If Σ is a closed set of sorts and T eliminates quantifiers relatively to Σ, then for any $\mathcal{M} \models T$, any $L(M)$-definable subset of $\Sigma(M)$ is defined by a formula of the form*

$$\varphi(\bar{x}, \bar{f}(\bar{a}), \bar{b}),$$

where φ is an $L{\restriction_\Sigma}$-formula, $\bar{a}$ is a tuple from $\Pi(M)$, $\bar{b}$ a tuple from $\Sigma(M)$ and $\bar{f}$ are functions from $\mathcal{F}$.

In particular, we have that if Σ is a closed set of sorts and T eliminates quantifiers relatively to Σ, then any $L(M)$-definable subset of $\Sigma(M)$ is $L{\restriction_\Sigma}(\Sigma(M))$-definable. In that case, we say that Σ is *purely stably embedded*.

Chapter 2

Generalities on ordered abelian groups

2.1 Algebraic properties

In this section we review some preliminaries on ordered abelian groups that we shall need throughout the thesis, and for which we refer mainly to [14].

Basic notions and notations

Let G be an ordered abelian group, that is an abelian group endowed with a linear order which is compatible with the group operation: $a < a'$ if and only if $a + b < a' + b$ for all $a, a', b \in G$. Clearly, such a group is always torsion-free. Conversely, using a compactness argument, one can easily see that any abelian group which is torsion-free can be endowed with a linear order that makes it into an ordered group (see [20, Exercise 6.2.13]). For every $a \in G, a \neq 0$, we say that a is *positive* if $a > 0$; otherwise, we have $a < 0$ and we say that a is *negative*. We say that G is *discrete* if there exists a minimal positive element, and *dense* otherwise.

Notation. By $\operatorname{div}(G)$, we denote the divisible hull of G, that is the minimal (unique up to isomorphism) divisible group that contains G. Moreover, let $a \in G$, and H be any subgroup of G, we write $a \mod H$ for the coset of a in G/H, i.e. $a \mod H$ denotes the set $\{ a + b \mid b \in H \}$.

Convex subgroups

A subgroup C of an ordered abelian group G is *convex* in G if, for every $g, h \in G$ such that $0 < g < h$ and $h \in C$, we also have $g \in C$. It is easy to see that the intersection of convex subgroups is again a convex subgroup. Hence, we can define the convex subgroup generated by a subset X of G, $\langle X \rangle^{\text{conv}}$, as the smallest convex subgroup containing X. In particular, we mean by a *principal convex subgroup C* one generated by a single element $a \in G$, i.e. if $C = \langle a \rangle^{\text{conv}}$ for some $a \in G$. Clearly, all other convex subgroups are unions of principal ones.

Proposition 2.1.1. *(a) The set of all convex subgroups of G is linearly ordered by inclusion.*

(b) If C is a convex subgroup of G, then G/C is an ordered abelian group with respect to the ordering relation defined by, for any $a, b \in G$,

$$a \mod C < b \mod C \text{ if and only if } a < b \text{ and } b - a \notin C.$$

(c) If C is a convex subgroup of G, then C is a pure subgroup of G, i.e. for any $n \in \mathbb{N}$ and $g \in G$, if $ng \in C$, then $g \in C$.

The order-type of the set of all proper convex subgroups of G is called the *rank* of G and it is an invariant under isomorphisms of G.

Definition 2.1.2. We say that G is *archimedean* if $\{0\}$ and G are the only convex subgroups of G.

Proposition 2.1.3. *If G is non trivial, the following conditions are equivalent.*

(i) G is archimedean.

(ii) G has rank one.

(iii) For every pair of positive elements $a, b \in G$ there exists $n \in \mathbb{N}, n > 0$ such that $na > b$.

The following notable result holds.

Theorem 2.1.4 (Hölder's Theorem)**.** *Every archimedean group is isomorphic (as ordered group) to a subgroup of* $\mathbb{R}$*.*

Clearly, the converse is also true, i.e. every subgroup of $\mathbb{R}$ is archimedean.

Cuts in ordered abelian groups

Let X be a totally ordered set. A *cut* in X is a pair (L, R) of subsets of X such that $L \cup R = X$ and $L < R$, i.e. $a < b$ for all $a \in L$, $b \in R$. Trivially, the pairs $(\emptyset, X)$ and $(X, \emptyset)$ are cuts in X and are denoted by $-\infty, +\infty$, respectively. If $Y \subseteq X$, then Y^+ denotes the cut (L, R) with $R = \{ x \in X \mid x > Y \}$ and $L = X \backslash R$. Similarly, Y^- is the cut (L, R) with $L = \{ x \in X \mid x < Y \}$ and $R = X \backslash L$. In particular, by a *principal cut* we mean one of the form $-\infty, +\infty$ or the form a^+, a^- for some $a \in X$.

Let (L, R) be a cut in an ordered abelian group G. For any $g \in G$, we define $g + (L, R)$ as the cut $(g + L, g + R)$. Then, to every cut (L, R) in G, we may associate the following subgroup of G

$$H(L, R) := \{ g \in G \mid g + (L, R) = (L, R) \},$$

which is called the *invariance group* of (L, R). It is straightforward that $H(L, R)$ is a convex subgroup of G. Moreover, for every convex subgroup C and any $g \in G$, we have

$$C = H(g + C^+) = H(g + C^-).$$

When the context is clear, we will denote the cut (L, R) by just the set L as well.

Hahn product and lexicographic sum

Let $(I, <)$ be an ordered set, and for each $i \in I$ let G_i be an ordered abelian group. From the family $\{G_i\}_{i \in I}$, we can construct two ordered abelian groups in the following way. Consider the direct product of the groups G_i, $\prod_{i \in I} G_i$, that is, the group of functions $f : I \to \bigcup_{i \in I} G_i$ such that for all $i \in I$, $f(i) \in G_i$. For every $f \in \prod_{i \in I} G_i$, the *support* of f is the set $\operatorname{supp}(f) := \{ i \in I \mid f(i) \neq 0 \}$. Let H be the following subgroup of $\prod_{i \in I} G_i$:

$$\left\{ f \in \prod_{i \in I} G_i \mid \operatorname{supp}(f) \text{ is a well-ordered subset of } (I, <) \right\},$$

where the empty-set is assumed to be well-ordered. It is easily checked that H is an ordered abelian group with respect the *lexicographic order*, whose positive cone is given by

$$f >_{lex} 0 \iff f(\min(\mathrm{supp}(f))) > 0.$$

The group H endowed with this order is called the *Hahn product* of the family $\{G_i\}_{i\in I}$ and is denoted by $\boldsymbol{H}_{i\in I} G_i$. The ordered subgroup

$$\{ f \in \boldsymbol{H}_{i\in I} G_i \mid \mathrm{supp}(f) \text{ is finite } \}$$

is the *lexicographic sum* of the family $\{G_i\}_{i\in I}$ and is denoted by $\sum_{i\in I} G_i$ or $\coprod_{i\in I} G_i$.

The lexicographic sum and the Hahn product of a family of ordered abelian groups are indistinguishable by first order properties. Indeed, the following holds.

Proposition 2.1.5 ([41, Corollary 6.3])**.** *Let $(I, <)$ be an ordered set and for each $i \in I$ let G_i be an ordered abelian group. The lexicographic sum $\sum_{i\in I} G_i$ is an elementary substructure of the Hahn product $\boldsymbol{H}_{i\in I} G_i$.*

The skeleton and the natural valuation

We introduce a fundamental algebraic invariant of an ordered abelian group.

Definition 2.1.6. Let G be an ordered abelian group. We denote by Γ_G or simply Γ a set indexing the set $\{\langle a\rangle^{\mathrm{conv}}\}_{a\in G}$ of principal convex subgroups of G, and inversely ordered, i.e. we set, for any $\gamma, \delta \in \Gamma_G$,

$$\gamma < \delta \iff C_\delta \subset C_\gamma,$$

where, for any $\gamma \in \Gamma_G$, C_γ denotes the corresponding principal convex subgroup. Then, Γ_G has a maximal element corresponding to $\{0\}$, which we denote by ∞.
For every $\gamma \in \Gamma_G \backslash \{\infty\}$, let V_γ be the union of all convex subgroups strictly contained in C_γ, that is the largest convex subgroup which does not contain a, with $a \in G, a \neq 0$ such that $C_\gamma = \langle a\rangle^{\mathrm{conv}}$. For $\gamma = \infty$, set $V_\gamma = \{0\}$. For any $\gamma \in \Gamma_G$, the quotient $G_\gamma = C_\gamma / V_\gamma$ is an archimedean group, and the pair

$$(\Gamma_G, (G_\gamma)_{\gamma\in\Gamma_G})$$

is called the *skeleton* of G. Moreover, we call the set Γ_G the *archimedean spine* of G, a pair (γ, G_γ) a *bone* of G, and G_γ a *rib* of G, for any $\gamma \in \Gamma_G$.

Example 2.1.7. Excluding the index for the zero set $\{0\}$, the skeleton of the lexicographic sum $G = \sum_{i \in I} H_i$ of the family $\{H_i\}_{i \in I}$ is (isomorphic to) the pair $(I, (H_i)_{i \in I})$, namely, $\Gamma_G \backslash \{\infty\} \cong I$ and $G_i \cong H_i$ for every $i \in I$. The same holds for the skeleton of the Hahn product $G' = \boldsymbol{H}_{i \in I} H_i$.

It follows immediately that

Proposition 2.1.8. *If Γ_G is well-ordered, then all convex subgroups of G are principal.*

One of the deepest results in the theory of ordered abelian groups states that every ordered abelian group lives, as ordered subgroup, in the lexicographically ordered real function space determined by the skeleton of its divisible hull. More precisely:

Theorem 2.1.9 (Hahn Embedding Theorem)**.** *Consider an ordered abelian group G, and let $(\Gamma_G, (G_\gamma)_{\gamma \in \Gamma_G})$ be the skeleton of G. Then G embeds (as an ordered abelian group) into $\boldsymbol{H}_{\gamma \in \Gamma_G} div(G_\gamma)$, where $div(G_\gamma)$ is the divisible hull of G_γ.*

The order structure on G induces a metric structure, coming from the natural valuation, defined as follows.

Definition 2.1.10. The *natural valuation* on G is the map

$$\mathrm{val}\colon G \to \Gamma_G$$

defined by

$$\mathrm{val}(a) = \gamma, \text{where } \langle a \rangle^{\mathrm{conv}} = C_\gamma.$$

For any $a \in G$, we will denote by C_a, V_a, and G_a the corresponding ordered abelian groups $C_{\mathrm{val}(a)}$, $V_{\mathrm{val}(a)}$, $G_{\mathrm{val}(a)}$, respectively.

It is straightforward to show that val satisfies the following properties, for all $a, b \in G$

(i) $\mathrm{val}(a) = \infty \iff a = 0$,

(ii) $\mathrm{val}(a - b) \geqslant \min\{\mathrm{val}(a), \mathrm{val}(b)\}$,

(iii) $\mathrm{val}(na) = \mathrm{val}(a)$ for every integer $n \neq 0$,

(iv) $\mathrm{val}(a) \neq \mathrm{val}(b) \implies \mathrm{val}(a - b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.

Moreover, note that, for every $a \in G$, $a \neq 0$,

$$C_a = \{ g \in G \mid \mathrm{val}(g) \geqslant \mathrm{val}(a) \}, \quad \text{and } V_a = \{ g \in G \mid \mathrm{val}(g) > \mathrm{val}(a) \}.$$

We recall that an abelian group G equipped with a map $v : G \to \Gamma$, with Γ totally ordered set with ∞ as maximal element, satisfying (i)-(ii) is called a *valued group*, and, in this case, v is called a *valuation*. Therefore, we can see any ordered abelian group as a valued group with respect to the natural valuation, and we will refer to the archimedean spine Γ_G of G as the *value set* of G.

Then, the natural valuation allows to establish a relation between the two invariants of an ordered abelian group we have introduced so far: the archimedean spine Γ_G of G and the rank of G. To this end, we recall that an endsegment of Γ_G is a subset of Γ_G that is closed upward. One can easily see that

Proposition 2.1.11. *1. If C is a convex subgroup of G, then the set*

$$\Delta_C = \{ \mathrm{val}(f) \in \Gamma_G \mid f \in C \}$$

is an endsegment of Γ_G.

2. If Δ is an endsegment of Γ_G, then the set

$$C_\Delta = \{ f \in G \mid \mathrm{val}(f) \in \Delta \}$$

is a convex subgroup of G.

Moreover, we have that for any convex subgroup of C, $C_{\Delta_C} = C$ and, for any endsegment Δ of Γ_G, $\Delta_{C_\Delta} = \Delta$. Then, there exists an order-preserving bijection between the set of all endsegments of Γ_G and the set of all convex subgroups of G.

Therefore, the rank of G is isomorphic to the set of endsegments of Γ_G, totally ordered by inclusion. Moreover, notice that

Proposition 2.1.12. *The following are equivalent:*

(i) C is a principal convex subgroup.

(ii) Δ_C^- is a principal cut.

2.2 Kaplansky theory for ordered abelian groups

In this section, we discuss extensions of ordered abelian groups. In particular, we will need the notion of *maximality* from general valuation theory, and the characterization of this notion in terms of sequence completion (Theorem 2.2.7). The relationship between these two properties was originally proved by Kaplansky in the context of valued fields [24]. Then his ideas have been adapted in other contexts; see e.g. [15] for the case of valued vector spaces and [21] for the case of ordered groups, even non-commutative.

Firstly, notice that an embedding of ordered abelian groups induces an embedding between their skeletons (see Definition 2.1.6). Particularly notable is the case where the skeletons are actually equal. Indeed, we have the following definitions.

Definition 2.2.1. Let G be an ordered abelian group. We say that an extension H of G is *immediate* if it preserves the skeleton, i.e. if $\Gamma_G = \Gamma_H$ and for each value $\gamma \in \Gamma_G$, $G_\gamma = H_\gamma$. Moreover, we say that G is *maximal* if it has no proper immediate extension.

Example 2.2.2. Let I be any ordered set and $\{G_i\}_{i\in I}$ a family of ordered abelian groups. Then the Hahn product $G' = \mathbf{H}_{i\in I} G_i$ of the family is an immediate extension of the lexicographic sum $G = \sum_{i\in I} G_i$.

Remark 2.2.3. Not every immediate extension of an ordered abelian group G is isomorphic to G, neither it is an elementary extension of G. Indeed, consider for instance the lexicographic sum $G = \sum_{i<\omega} \mathbb{Q}$, and the element $a \in \mathbf{H}_{i<\omega} \mathbb{Q}$ such that $a(i) = 1$ for every $i < \omega$. Then, the ordered group $G' = \langle G, a\rangle$ generated by G and a is an immediate extension of G, but it is not elementarily equivalent: a is not divisible by 2 (or by any integer).

The notions of skeleton, immediate extension and maximality from general valuation theory (see e.g. [25, Chapter 0]) coincide in this context with the definitions given so far. We now recall the main results in [21]. First of all, we have to show that the definition of immediate extension introduced here is equivalent to that of "c-extension" presented in the paper. In particular, we have the following characterization of immediate extensions.

Fact 2.2.4. *Let G, H be ordered abelian groups such that $G \subset H$. Then the following are equivalent:*

1. *H is an immediate extension of G,*
2. *for every $h \in H \backslash \{0\}$ there exists $g \in G$ such that $\operatorname{val}(h-g) > \operatorname{val}(h)$,*
3. *for every $h \in H \backslash G$, the set of values $\Delta = \{ \operatorname{val}(h-g) \mid g \in G \}$ does not admit a maximal element,*
4. *for every $0 < h \in H$ there exists $g \in G$ such that for all integers n, $n(h-g) < h$.*

Proof. It is trivial to show the equivalences $2 \Leftrightarrow 3$ and $2 \Leftrightarrow 4$. We prove $1 \Leftrightarrow 2$.
$(1 \Rightarrow 2)$ Let H be an immediate extension of G and $h \in H, h \neq 0$. Let $g \in G$. Then, there is some $g' \in G$ such that $\operatorname{val}(h-g) = \operatorname{val}(g')$ and $h - g \mod V_h = g' \mod V_h$. It follows that $\operatorname{val}(h-g-g') > \operatorname{val}(h-g)$. This shows that $\Delta = \{ \operatorname{val}(h-g) \mid g \in G \}$ does not admit a maximal element.
$(1 \Leftarrow 2)$ Let $h \in H, h \neq 0$ and $g \in G$ such that $\operatorname{val}(h-g) > \operatorname{val}(h)$. Then, the bone of h is equal to the the bone of g. Indeed, by (iv), $\operatorname{val}(h) = \operatorname{val}(g)$, and $h \mod V_h = g \mod V_h$. □

In [21], a *c-extension* is defined as an extension H of G satisfying condition 4. in the above Fact, and, thus, we have showed that it is equivalent to an immediate extension. So the notion of *c-closed* ordered abelian group, i.e. an ordered abelian group without proper c-extensions, is equivalent to that of maximal ordered abelian group.

Definition 2.2.5. Let G be a valued group with respect to the valuation v. Consider a sequence $(a_i)_{i \in I}$ of elements in G, where I is a well-ordered set with no maximal element.

- The sequence $(a_i)_{i \in I}$ is called *eventually pseudo-Cauchy* or just *pseudo-Cauchy* if there is $\alpha \in I$ such that for all $\alpha < i < j < k$, $v(a_i - a_j) < v(a_j - a_k)$.
- An element a of G is called a *pseudo-limit* of $(a_i)_{i \in I}$ if there exists $\alpha \in I$ such that for all $\alpha < i < j$, $v(a_i - a) = v(a_i - a_j)$.

Moreover, we say that G is *pseudo-complete* if every pseudo-Cauchy sequence in G admits a pseudo-limit in G.

Note that a pseudo-Cauchy sequence may admit more than one limit. Henceforth, if $(a_i)_{i\in I}$ is a sequence of elements in a valued group G, we will say that an assertion about its elements holds for *eventually all i* if there is some $i_0 \in I$ such that it holds for all a_i with $i \geqslant i_0$.

We can state the following results.

Proposition 2.2.6 ([21, Theorem C1]). *Let G, H be ordered abelian groups such that $G \subset H$. If H is an immediate extension of G and $h \in H \backslash G$, then there is a pseudo-Cauchy sequence of elements of G with pseudo-limit h and with no pseudo-limits in G.*

Theorem 2.2.7 ([21, Theorem C6]). *An ordered abelian group G is maximal if and only if it is pseudo-complete.*

Important examples of maximal ordered abelian groups are the Hahn products of archimedean groups. Indeed,

Proposition 2.2.8 ([21, Lemma C4]). *Let I be an ordered set, and $(G_i)_{i\in I}$ be a collection of archimedian groups. Then the Hahn product $\mathbf{H}_{i\in I} G_i$ is pseudo-complete and, thus, maximal.*

We conclude this section introducing a characterization of pseudo-completeness in terms of another notion of completion: the spherical completeness. Let G be a valued group with respect to the valuation $v\colon G \to \Gamma$. Recall that the (closed) ball around a with radius γ is the set

$$B_\gamma(a) := \{ x \in G \mid v(x-a) \geqslant \gamma \}$$

where $a \in G$ and $\gamma \in \Gamma$.

Definition 2.2.9. We say that G is *spherically complete* if every nested family of balls has non-empty intersection, i.e. if whenever $\{B_{\gamma_i}(a_i)\}_{i\in I}$ is such that for any i, j, either $B_{\gamma_i}(a_i) \subseteq B_{\gamma_j}(a_j)$ or $B_{\gamma_j}(a_j) \subseteq B_{\gamma_i}(a_i)$, then $\bigcap_{i\in I} B_{\gamma_i}(a_i) \neq \varnothing$.

The following result is well-known (see, for example, [30, pag.34] in the more general context of ultra-metric spaces):

Theorem 2.2.10. *A valued group G is spherically complete if and only if it is pseudo-complete.*

In particular, by Theorem 2.2.7, we have

Theorem 2.2.11. *An ordered abelian group G is maximal if and only if it is spherically complete.*

2.3 Model theory of ordered abelian groups

Elementary classes of ordered abelian groups

The model theory of ordered abelian groups is complex and varied. Not all ordered abelian groups satisfy the same model-theoretic properties and, since the sixties, several subclasses of ordered abelian groups with a "good" model theoretic behaviour have been isolated.

The first complete theory of ordered abelian groups to be studied was *Presburger arithmetic*, i.e. the theory of $(\mathbb{Z}, 0, +, <)$. Let L_{oag} denote the language $\{0, +, -, <\}$ of ordered abelian groups. Assume that G is elementarily equivalent to $\mathbb{Z}$ in L_{oag}. The *Presburger language*, denoted by L_{Pres}, is the definitional extension of L_{oag} consisting of the symbols $0, 1, +, -, <$ and a binary relation symbol $\equiv_m$ for each $m \in \mathbb{N}^*$, where $0, +, -, <$ take their obvious interpretation, 1 is interpreted as the minimal positive element and $\equiv_m$ is interpreted as the equivalence relation modulo m defined by

$$a \equiv_m b \text{ if and only if } a - b \in mG.$$

It is well known that

Theorem 2.3.1 (Presburger's Theorem, [37]). *Presburger arithmetic admits quantifier elimination in L_{Pres}.*

Another notable class of ordered abelian groups is given by the models of the theory of divisible ordered abelian groups: DOAG. It is well known that DOAG is complete and eliminate quantifiers in L_{oag} ([39]). Recall that a structure $\mathcal{M} = (M, <, \dots)$ which is totally ordered by $<$ is said to be o-minimal if any definable subset of M is a finite union of points and intervals with endpoints in $M \cup \{\pm\infty\}$. It follows that divisible ordered abelian groups are o-minimal. In particular, these groups are exactly all the o-minimal ordered groups, as the following result shows [32, Theorem 2.1].

Theorem 2.3.2. *Any o-minimal ordered group is abelian and divisible. In particular, it is elementarily equivalent to* $(\mathbb{Q}, 0, +, <)$.

In [40], Robinson and Zakon identified a first elementary class of ordered abelian groups, which includes both the models of Presburger arithmetic and DOAG, the class of regular groups.

Definition 2.3.3.
- Let $n \in \mathbb{N}, n \geqslant 2$. An ordered abelian group G is said to be *n-regular* if any interval with at least n points contains an element divisible by n.
- An ordered abelian group is said to be *regular* if it is n-regular for any $n \in \mathbb{N}, n \geqslant 2$.

Fact 2.3.4. *For an ordered abelian group G, the following are equivalent:*

1. *G is regular;*
2. *G is p-regular for every prime p;*
3. *there exists an archimedean group G' elementarily equivalent to G;*
4. *the only definable convex subgroups of G are $\{0\}$ and G.*

Robinson and Zakon characterized completely all possible completions of the theory of regular groups as well. Indeed, they proved

Theorem 2.3.5.
- *The theory of discrete regular groups is complete, and it is the theory of* $(\mathbb{Z}, 0, +, <)$.
- *If G, H are dense regular groups, then $G \equiv H$ if and only if, for each prime p, G/pG and H/pG are either both infinite or have the same finite cardinality.*

Note that every ordered abelian group can be seen as an L_{Pres}-structure, once 1 is interpreted by 0 if there is no minimal positive element. However, in [50], Weispfenning proved that the regular groups are the only ordered abelian groups that admit quantifier elimination in L_{Pres}:

Theorem 2.3.6. *An ordered abelian group G is regular if and only if it admits elimination of quantifiers in* L_{Pres}.

In [2], Belegradek studied the following ordered abelian groups, which include the regular groups.

Definition 2.3.7. An ordered abelian group G is said to have *finite regular rank* if it has a finite series of convex subgroups

$$(0) = B_0 < B_1 < \cdots < B_d = G$$

with regular quotients B_{i+1}/B_i for any $i < d$.

Note that such a series is not necessarily unique. However, in any ordered abelian group of finite regular rank, there exists a unique finite series of convex subgroups $(0) = B_0 < B_1 < \cdots < B_m = G$ such that

- for each $0 \leqslant i < m$, B_{i+1}/B_i is regular,
- for each $0 < i < m$, B_{i+1}/B_i is not divisible.

Such a natural number m is called the *regular rank* of the group. Moreover, one can prove that such convex subgroups are 0-definable [2, Corollary 3.5].

Fact 2.3.8. *For an ordered abelian group G, the following are equivalent:*

1. *G has finite regular rank;*
2. *G has finitely many definable convex subgroups;*
3. *G is elementarily equivalent to a subgroup of the lexicographically ordered group* $\mathbb{R}^m$*, for some* $m \in \mathbb{N}$.

The class of ordered abelian groups of rank at most m is an elementary class, as it follows from

Theorem 2.3.9 ([2, Theorem 3.8]). *For any positive integer m, the class of ordered abelian groups of rank at most m is first order axiomatizable.*

In [49], Weispfenning obtained a quantifier elimination for the class of ordered abelian groups of finite regular rank, in the language of ordered abelian groups extended with predicates to distinguish the subgroups $C + nG$, where C is a definable convex subgroup and $n \in \mathbb{N}, n \neq 1$, and constants for a representative of the least positive element of B_{i+1}/B_i for any $i < m$ such that B_{i+1}/B_i is discrete.

A further generalization can be obtained extending the notion of regular rank to that of n-regular rank, as follows (see [13]). Let $n \in \mathbb{N}, n \geqslant 2$. We say that an ordered abelian group G has n-regular rank equal to m if there are $B_0^{(n)}, \dots, B_m^{(n)}$ convex subgroups of G such that

- $(0) = B_0^{(n)} < \cdots < B_m^{(n)} = G$,
- for each $0 \leqslant i < m$, $B_{i+1}^{(n)}/B_i^{(n)}$ is n-regular,
- for each $0 < i < m$, $B_{i+1}^{(n)}/B_i^{(n)}$ is not n-divisible.

Let $\mathrm{R}_n(G)$ denote the set of convex subgroups $\left\{ B_0^{(n)}, \dots, B_m^{(n)} \right\}$.

Definition 2.3.10. If G has finite n-regular rank for all $n \in \mathbb{N}, n \geqslant 2$, then the cardinality of $\mathrm{R}(G) := \bigcup_{n \geqslant 2} \mathrm{R}_n(G)$ is either finite or $\aleph_0$, and it is called the *regular rank* of G. In this case, we say that G has *bounded regular rank*.

If G has bounded regular rank, then $R(G)$ is the collection of all proper definable convex subgroups of G and they are all definable without parameters ([13, Proposition 2.3]). Moreover, it holds that

Fact 2.3.11. *If G has finite n-regular rank and $H \equiv G$, then H has the same n-regular rank as G. In particular, the value of the regular rank depends only on the theory of G.*

It follows that the class of ordered abelian groups of bounded regular rank of a fixed value is closed under elementary equivalence. In [13, Theorem 2.4], Farrè generalized Weispfenning quantifier elimination as well, identifying a language to eliminate quantifiers for the more general class of ordered abelian groups of bounded regular rank.

Quantifier elimination and the auxiliary sorts

Since the work of Gurevich [16] and Schmitt [41], it has been clear that linear orders, up to additional unary predicates, are the key to understand ordered abelian groups. We call this kind of structures coloured chain. More, precisely:

Definition 2.3.12. Let λ be a cardinal. A *λ-chain*, or simply a *coloured chain*, $(C, <, (P_i)_{i\in\lambda})$ is a linearly ordered set $(C, <)$ equipped with λ-many unary predicates P_i.

The tool introduced by Gurevich and Schmitt consisted in assigning to every ordered abelian group G countably many coloured chains, called the *n-spines* of G, which could allow to translate formulas about ordered abelian groups into formulas in a simpler language. Indeed, they proved the so-called Transfer Theorem stating that every formula of an ordered abelian group can be translated into a formula in some of its spines, plus a quantifier free formula in a definitional extension of L_{oag} [41, Theorem 4.5]. A direct consequence of this result is that:

Theorem 2.3.13. *Let G and H be two ordered abelian groups and $\mathrm{Sp}_n(G)$ and $\mathrm{Sp}_n(H)$ the n-spines of G and H, respectively. Then, $G \equiv H$ if and only if $\mathrm{Sp}_n(G) \equiv \mathrm{Sp}_n(H)$ for all $n \in \mathbb{N}, n \geqslant 2$.*

Remark 2.3.14. We have already seen an example of a sequence of invariants under elementary equivalence of G in the class of ordered abelian groups of bounded regular rank: the sequence of n-regular ranks (Fact 2.3.11). Actually, as it is proved in [13], the n-spine of G coincides in this case with the set $\mathrm{R}_n(G)$ and so the n-regular rank is nothing more than the cardinality of the n-spine. Indeed, in literature ordered abelian groups of bounded regular rank are also known as ordered abelian groups with finite spines (e.g. in [18]).

The Transfer Theorem has been later revisited by Cluckers and Halupczok in [6]. In this paper, they introduced two languages, denoted by L_{qe} and L_{syn}, which are very close to that of Gurevich and Schmitt, but in line with the modern notion of many-sorted language. Roughly speaking, in order to eliminate quantifiers, we need a (many-sorted) expansion of L_{Pres} that can deal with the quotients G/H, where H is a definable convex subgroup of G.

Let us review the language L_{syn} which we will need throughout the thesis. We begin by describing the set of *auxiliary sorts* of L_{syn}:

$$\mathcal{A} := \{\mathcal{S}_n, \mathcal{T}_n, \mathcal{T}_n^+ \mid n \in \mathbb{N}, n > 0\}.$$

Definition 2.3.15. Fix a natural number $n > 0$.

1. For $a \in G \backslash nG$, let V_a^n be the largest convex subgroup of G such that $a \notin V_a^n + nG$; for $a \in nG$, set $V_a^n = \{0\}$. Define $\mathcal{S}_n := G/\sim$, with $a \sim a'$ if and only if $V_a^n = V_{a'}^n$, and let $\mathfrak{s}_n \colon G \twoheadrightarrow \mathcal{S}_n$ be the canonical map. Denote by $G(\alpha)$ the convex subgroup V_a^n, with $\alpha = \mathfrak{s}_n(a)$.

2. For $b \in G$, set $\hat{V}_b^n = \bigcup_{a \in G, b \notin V_a^n} V_a^n$, where the union over the empty set is declared to be $\{0\}$. Define $\mathcal{T}_n := G/\sim$, with $b \sim b'$ if and only if $\hat{V}_b^n = \hat{V}_{b'}^n$, and let $\mathfrak{t}_n \colon G \twoheadrightarrow \mathcal{T}_n$ be the canonical map. Denote by $G(\alpha)$ the convex subgroup $\hat{V}_b^n$, with $\alpha = \mathfrak{t}_n(b)$.

3. Denote by $\mathcal{T}_n^+$ a copy of $\mathcal{T}_n$, i.e. $\mathcal{T}_n^+ := \{\beta^+\}_{\beta \in \mathcal{T}_n}$. For each $\beta^+ \in \mathcal{T}_n^+$, let $G(\beta^+) = \bigcap_{\alpha \in \mathcal{S}_n, G(\alpha) \supsetneq G(\beta)} G(\alpha)$, where the intersection over the empty set is G. In particular, if $\beta = \mathfrak{t}_n(b)$, we have $G(\beta^+) = \bigcap_{a \in G, b \in V_a^n} V_a^n$.

Some remarks:

- The notation introduced is slightly different from that adopted in [6].

- Note that for any convex subgroup C of G, $a \notin C + nG$ if and only if $C \cap a + nG = \emptyset$. Then, for $a \notin nG$, we could define V_a^n as the largest convex subgroup not intersecting $a + nG$ as well.

- In [6], it is proved that the convex subgroups in each of the three families

$$\{G(\alpha)\}_{\alpha \in \mathcal{S}_n}, \{G(\alpha)\}_{\alpha \in \mathcal{T}_n}, \{G(\alpha)\}_{\alpha \in \mathcal{T}_n^+}$$

  are (uniformly) definable in L_{oag}. It follows that all the auxiliary sorts are imaginary sorts of L_{oag}.

- The convex subgroups considered by Cluckers and Halupczock are the same subgroups introduced by Gurevich and Schmitt for the definition of the n-spine. In particular, the underlying set of the n-spine $\mathrm{Sp}_n(G)$ of G is the set of convex subgroups

$$\{\, V_a^n \mid a \in G, a \notin nG \,\} \cup \{\, \hat{V}_a^n \mid a \in G, a \neq 0 \,\}. \tag{2.1}$$

Adopting the terminology introduced by Gurevich, we will call V_a^n the *n-fundament of a* and the convex subgroups of this form *n-fundamental*. Notice that the collection of the n-spines of G is the model-theoretic counterpart of the archimedean spine Γ_G introduced in Definition 2.1.6. It is a model-theoretic invariant, whereas the archimedean spine is an algebraic invariant.

In order to present the complete definition of L_{syn}, we have to introduce also, for any $\alpha \in \bigcup_{n\in\mathbb{N},n>0} \mathcal{S}_n \dot\cup \mathcal{T}_n \dot\cup \mathcal{T}_n^+$ and $m \in \mathbb{N}, m > 0$, the subgroup

$$G(\alpha)^{[m]} := \bigcap_{H \supsetneq G(\alpha), H \text{ convex subgroup of } G} (H + mG).$$

Notice that it is definable by [6, Lemma 2.4].

Definition 2.3.16. The language L_{syn} consists of the following:

(a) The main sort $(G, 0, +, -, <, (\equiv_m)_{m\in\mathbb{N},m>0})$;

(b) the auxiliary sorts $\mathcal{S}_n, \mathcal{T}_n, \mathcal{T}_n^+$, for each $n \in \mathbb{N}, n > 0$, with the binary relations $\leqslant$ on $(\mathcal{S}_n \dot\cup \mathcal{T}_n \dot\cup \mathcal{T}_n^+) \times (\mathcal{S}_m \dot\cup \mathcal{T}_m \dot\cup \mathcal{T}_m^+)$ (each pair (m, n) giving rise to nine binary relations), defined by $\alpha \leqslant \alpha'$ if and only if $G(\alpha) \subseteq G(\alpha')$;

(c) the canonical maps $\mathfrak{s}_n \colon G \twoheadrightarrow \mathcal{S}_n$ and $\mathrm{t}_n \colon G \twoheadrightarrow \mathcal{T}_n$, for each $n \in \mathbb{N}, n > 0$;

(d) a unary predicate $x =^\bullet k_\bullet$ on G, for each $k \in \mathbb{Z}\setminus\{0\}$, defined by $g =^\bullet k_\bullet$ if and only if there exists a convex subgroup H of G such that G/H is discrete and g mod H is equal to k times the smallest positive element of G/H, for every $g \in G$;

(e) a unary predicate $x \equiv_m^\bullet k_\bullet$ on G, for each $m \in \mathbb{N}, m > 0$ and $k \in \{1, \ldots, m-1\}$, defined by $g \equiv_m^\bullet k_\bullet$ if and only if there exists a convex subgroup H of G such that G/H is discrete and $g \mod H$ is congruent modulo m to k times the smallest positive element of G/H, for every $g \in G$;

(f) a unary predicate $D^{[p^s]}_{p^r}(x)$ on G, for each prime p and each $r, s \in \mathbb{N}\backslash\{0\}$ with $s \geqslant r$, defined by $D^{[p^s]}_{p^r}(g)$ if and only if there exists an $\alpha \in \mathcal{S}_p$ such that $g \in G(\alpha)^{[p^s]} + p^r G$ and $g \notin G(\alpha) + p^r G$, for every $g \in G$;

(g) a unary predicate $\mathrm{discr}(x)$ on the sort $\mathcal{S}_p$, with p prime, defined by $\mathrm{discr}(\alpha)$ if and only if $G/G(\alpha)$ is discrete, for every $\alpha \in \mathcal{S}_p$;

(h) two unary predicates on the sort $\mathcal{S}_p$, with p prime, for each $l, n \in \mathbb{N}\backslash\{0\}$, defining the sets

$$\{\alpha \in \mathcal{S}_p \mid \dim_{\mathbb{F}_p}(G(\alpha)^{[p^n]} + pG)/(G(\alpha)^{[p^{n+1}]} + pG) = l\} \text{ and}$$
$$\{\alpha \in \mathcal{S}_p \mid \dim_{\mathbb{F}_p}(G(\alpha)^{[p^n]} + pG)/(G(\alpha) + pG) = l\},$$

where $\dim_{\mathbb{F}_p}(\bullet)$ denotes the dimension of the group as $\mathbb{F}_p$-vector space.

Fact 2.3.17 ([6, Theorem 1.13]). *In the theory of ordered abelian groups, each L_{syn}-formula is equivalent to an L_{syn}-formula without quantifiers ranging over the main sort G.*

In particular, we have that any L_{syn}-formula $\varphi(\bar{x}, \bar{\alpha})$, with G-variables $\bar{x}$ and $\mathcal{A}$-variables $\bar{\alpha}$, is a boolean combination of formulas of the form:

- $\psi(\bar{x})$, where ψ is quantifier free and it lives purely in the main sort G, and
- $\chi(\bar{x}, \bar{\alpha}) := \xi((\mathfrak{s}_p(\sum_{i<n} z_i x_i), \mathrm{t}_p(\sum_{i<n} z_i x_i))_{p \in \mathbb{P}}, \bar{\alpha})$, where ξ is an $\mathcal{A}$-formula and $z_0, \ldots, z_{n-1} \in \mathbb{Z}$.

The following fact will be useful as well:

Fact 2.3.18 ([6, Lemma 2.12]). *For any $g \in G$, we have the following equivalences.*

1. *$g =^{\bullet} k_{\bullet}$ if and only if $G/G(\mathrm{t}_2(g))$ is discrete and $g \mod G(\mathrm{t}_2(g))$ is equal to k times the smallest positive element of $G/G(\mathrm{t}_2(g))$.*
2. *$g \equiv^{\bullet}_m k_{\bullet}$ if and only if $G/G(\mathfrak{s}_m(g))$ is discrete and $g \mod G(\mathfrak{s}_m(g))$ is congruent modulo m to k times the smallest positive element of $G/G(\mathfrak{s}_m(g))$.*

As remarked in [6], the map t_2 can be replaced by any other map t_p, with $p \in \mathbb{P}$.

Model-theoretic complexity

Shelah's classification theory [43] aims at identifying a series of properties which would determine whether a theory is "tame". These properties are characterized by the absence or presence of different combinatorial configurations and yield a partition of first order theories in various classes. A classification of ordered abelian groups in this context was initiated by Gurevich and Schmitt who proved that no ordered abelian group has the independence property [17]. A more recent development in this direction is the characterization of strongly dependent ordered abelian groups, obtained independently by Halevi and Hasson [18], Dolich and Goodrick [10] and Farré [13]. In particular, the following holds.

Theorem 2.3.19. *For an ordered abelian group G, the following are equivalent:*

1. *G is strongly dependent,*
2. *G has finite dp-rank,*
3. *G has bounded regular rank and the cardinality of G/pG is infinite for only finitely many prime p.*

In [22], the case of dp-rank equal to 1 was also investigated and it was proved that

Theorem 2.3.20. *An ordered abelian group G has dp-rank equal to* 1 *(i.e. it is dp-minimal) if and only if G/pG is finite for every prime p.*

Chapter 3

Ordered abelian groups that do not have elimination of imaginaries

3.1 Hahn products of $\mathbb{Z}$ over a well-ordered set

In this chapter, we aim at investigating elimination of imaginaries for some ordered abelian groups, including the Hahn products of $\mathbb{Z}$ with the usual order over a well-ordered set I with $|I| > 1$. Notice that the case $|I| = 1$ corresponds to the ordered group of integers $\mathbb{Z}$ and it has been already studied in [5]. To this purpose, in this section we are going to observe some basic facts on such groups.

Throughout the chapter, unless otherwise stated, we work in the language of ordered abelian groups L_{oag}. Henceforth, we identify I with the corresponding ordinal α, and we assume that $\alpha > 1$. We denote by Λ and Ψ the groups $H_{i<\alpha}\mathbb{Z}$ and $\sum_{i<\alpha}\mathbb{Z}$, respectively. Note that, since α is well-ordered, the domain of Λ coincides with the direct product $\prod_{i<\alpha}\mathbb{Z}$. Furthermore, we add the symbol ∞ to α and set $i < \infty$ for all $i < \alpha$. Then, we define the following map $v\colon \Lambda \to \alpha \cup \{\infty\}$ where, for any $f \in \Lambda$,

$$v(f) = \begin{cases} \min \operatorname{supp}(f) & \text{if } f \neq 0 \\ \infty & \text{otherwise} \end{cases} \tag{3.1}$$

One can easily prove the following properties:

Fact 3.1.1. *(i) $v(f) = v(-f)$ for any $f \in \Lambda$;*

(ii) $v(f+g) \geqslant \min\{v(f), v(g)\}$ *for any* $f, g \in \Lambda$*;*
(iii) Let $f, g \in \Lambda$ *such that* $0 < g < f$*. Then* $v(f) \leqslant v(g)$.

Recall that the set $\alpha \cup \{\infty\}$ is isomorphic to the archimedean spine of Λ. Hence, by Proposition 2.1.8, it follows that all the convex subgroups of Λ are principal. Moreover, it is easy to see that the map v coincides with the natural valuation introduced in Definition 2.1.10. Therefore, the convex subgroups of Λ of the form

$$\{ f \in \Lambda \mid v(f) \geqslant i \}$$

with $i \in \alpha \cup \{\infty\}$, are exactly all the convex subgroups of Λ. In particular, the following holds.

Proposition 3.1.2. *For every ordinal* $\alpha > 1$, $\Lambda = H_{i<\alpha}\mathbb{Z}$ *is of rank* $(\alpha+1)^*$.
In particular, if α is finite, i.e. $\alpha = n$ *for some* $n \in \mathbb{N}$, $n > 1$, *there are exactly n proper convex subgroups of* Λ.

Note that the same holds also for $\Psi = \sum_{i<\alpha}\mathbb{Z}$ and, in particular, the sets

$$\{ f \in \Psi \mid v(f) \geqslant i \}$$

are exactly all the convex subgroups of Ψ. Throughout this chapter, we will denote by Λ_i and Γ_i the convex subgroups of Λ and Ψ, respectively.

Let $i < \alpha$, $e_{\{i\}}$ stands for the following element of Λ:

$$(e_{\{i\}}(j))_{j<\alpha} \text{ where } e_{\{i\}}(j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ otherwise} \end{cases} \tag{3.2}$$

Note that, for every $i, j < \alpha$ such that $i < j$, $0 < e_{\{j\}} < e_{\{i\}}$. Moreover, the family $\{e_{\{i\}}\}_{i<\alpha}$ generates $\Psi = \sum_{i<\alpha}\mathbb{Z}$ as an abelian group.

Proposition 3.1.3. *If* $\alpha = \beta + 1$ *for some ordinal β, then* $\Lambda = H_{i<\alpha}\mathbb{Z}$ *is discrete and, in particular,* $e_{\{\beta\}}$ *is the minimal positive element of* Λ*. Otherwise, if α is a limit ordinal,* Λ *is dense.*

Proof. Let $\alpha = \beta + 1$ and suppose $f \in \Lambda$ is such that $0 < f \leqslant e_{\{\beta\}}$. Then $v(f) = \beta$ and $0 < f(\beta) \leqslant 1$. Since $f(\beta) \in \mathbb{Z}$, $f(\beta) = 1$ and $f = e_{\{\beta\}}$.

Let α be a limit ordinal and $f \in \Lambda$, $f > 0$. Let $i = v(f)$. Then $f(i) > 0$ and $f(j) = 0$ for every $j < i$. Clearly, $0 < e_{\{i+1\}} < f$. □

Note that the same is true for $\Psi = \sum_{i<\alpha} \mathbb{Z}$: if $\alpha = \beta + 1$, $e_{\{\beta\}}$ is the minimal positive element of $\sum_{i<\alpha} \mathbb{Z}$; otherwise, $\sum_{i<\alpha} \mathbb{Z}$ is dense.

Let us show that all the convex subgroups of Λ are definable. Indeed, one can easily see that they coincide with the convex subgroups introduced in Definition 2.3.15. Let us recall that, for any $n \in \mathbb{N}$, $n > 0$, if $f \in \Lambda \backslash n\Lambda$, V^n_f denotes the largest convex subgroup H of Λ not intersecting $f + n\Lambda$; otherwise $V^n_f = \{0\}$. Firstly, we observe that

Proposition 3.1.4. *The family $\{\Lambda_{i+1}\}_{i+1<\alpha}$ is uniformly definable.*

Proof. Let $i < \alpha$ be such that $i + 1 < \alpha$. Then $\Lambda_{i+1} = V^2_{e_{\{i\}}}$ and, so, is definable. Indeed, for every $f \in e_{\{i\}} + 2\Lambda$, we have $v(f) \leqslant i$. Hence, $V^2_{e_{\{i\}}} = \{ f \in \Lambda \mid v(f) > i \} = \{ f \in \Lambda \mid v(f) \geqslant i + 1 \} = \Lambda_{i+1}$. □

More generally, for every $f \in \Lambda \backslash n\Lambda$, we have

$$V^n_f = \begin{cases} \Lambda_{i+1} & \text{if } i + 1 < \alpha \\ \{0\} & \text{otherwise} \end{cases}$$

where $i = \min \{ j < \alpha \mid f(j) \notin n\mathbb{Z} \}$.

Now we are able to prove

Corollary 3.1.5. *All convex subgroups of Λ are definable.*

Proof. We have only to show the definability of Λ_i for i limit ordinal. Let $i < \alpha$ be a limit ordinal, then

$$\Lambda_i = \bigcap_{\{ j<\alpha \mid j<i \}} \Lambda_j = \bigcap_{\{ j<\alpha \mid j<i \text{ and } j \text{ is not limit } \}} \Lambda_j = \bigcap_{\left\{ g\in\Lambda \mid e_{\{i\}}\in V^n_g \right\}} V^n_g .$$

□

It is clear that the above statements hold also for $\Psi = \sum_{i<\alpha} \mathbb{Z}$.

3.1.1 The group of automorphisms of the lexicographic sum of $\mathbb{Z}$

In this subsection, we will describe the group of automorphisms of $\Psi = \sum_{i<\alpha} \mathbb{Z}$.

First of all, we characterize the automorphisms of Γ when α is finite, i.e. $\alpha = n$ for some natural number $n > 1$. In this case, $\Psi = \sum_{0 \leqslant i \leqslant n-1} \mathbb{Z} = \boldsymbol{H}_{0 \leqslant i \leqslant n-1} \mathbb{Z}$, and we will denote Ψ by its domain $\mathbb{Z}^n$. For any $z \in \mathbb{Z}^n$, let $z(i) = z_i$ for every $0 \leqslant i \leqslant n-1$. Note that $\mathbb{Z}^n$ is a discrete ordered abelian group. Therefore, every element of the convex subgroup

$$\mathbb{Z}^n{}_{n-1} = \{ z \in \mathbb{Z}^n \mid v(z) \geqslant n-1 \} = \{ (0, \dots, 0, m) \mid m \in \mathbb{Z} \}$$

is 0-definable. Moreover, each convex subgroup $\mathbb{Z}^n{}_i$ of $\mathbb{Z}^n$ is 0-definable, as it is proved in the following lemma.

Lemma 3.1.6. *For every* $0 \leqslant i \leqslant n-1$, $\mathbb{Z}^n{}_i = \{ z \in \mathbb{Z}^n \mid v(z) \geqslant i \}$ *is* 0*-definable.*

Proof. For $i = 0$ it is trivial. So, let $0 < i \leqslant n-1$ and fix a prime p. For every $z \in \mathbb{Z}^n$, $z \in \mathbb{Z}^n{}_i$ if and only if the set $\{ w + p\mathbb{Z}^n \mid -z \leqslant w \leqslant z \}$ of the $\equiv_p$ - equivalence classes of elements in the interval $[-z, z]$ has cardinality at most p^{n-i}. Indeed, let $z \in \mathbb{Z}^n_i$. Then $[-z, z] \subseteq (-e_{\{i-1\}}, e_{\{i-1\}})$. Since, for every $0 \leqslant j \leqslant n-1$ the $\equiv_p$ - equivalence classes in $(-e_{\{j\}}, e_{\{j\}})$ are exactly p^{n-1-j}, it follows that $|\{ w + p\mathbb{Z}^n \mid -z \leqslant w \leqslant z \}| \leqslant p^{n-i}$. Conversely, if $z \notin \mathbb{Z}^n_i$, then $v(z) < i$ and $[-z, z]$ contains the interval $[-e_{\{i-1\}}, e_{\{i-1\}}]$. Therefore $|\{ w + p\mathbb{Z}^n \mid -z \leqslant w \leqslant z \}| \geqslant p^{n-i} + 1$.

Clearly, the set of $z \in \mathbb{Z}^n$ such that $|\{ w + p\mathbb{Z}^n \mid -z \leqslant w \leqslant z \}| \leqslant p^{n-i}$ is 0-definable. □

Consider the following upper triangular matrix of size $n \times n$, whose elements are in $\mathbb{Z}$

$$K_n = \begin{pmatrix} 1 & k_{12} & \dots & k_{1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & k_{n-1n} \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{3.3}$$

Note that K_n is invertible and its inverse is an upper triangular matrix on $\mathbb{Z}$ of the same form, with all the entries of the main diagonal equal to 1. Then the function

$$f_{K_n} \colon z \in \mathbb{Z}^n \mapsto zK_n \in \mathbb{Z}^n$$

where zK_n is the matrix product of the row vector $z = (z_1, \ldots, z_n)$ and K_n, is a group automorphism of $\mathbb{Z}^n$. Moreover, it is straightforward to show that f_{K_n} is order-preserving. Therefore, every matrix as in (3.3) determines an automorphism of the ordered group $\mathbb{Z}^n$. We now prove that every automorphism of $\mathbb{Z}^n$ is obtained in this way and it can be represented by a matrix as in (3.3).

Note that, since $\mathbb{Z}^n$ is a free module of rank n, every group automorphism φ of $\mathbb{Z}^n$ is given by an invertible matrix $M \in GL_n(\mathbb{Z})$. In other words, there exists an $n \times n$ matrix M over $\mathbb{Z}$ with $\det(M) = \pm 1$ such that $\varphi(z) = zM$ for every $z \in \mathbb{Z}^n$. Indeed, the matrix

$$M = \begin{pmatrix} \varphi(e_{\{1\}}) \\ \vdots \\ \varphi(e_{\{n\}}) \end{pmatrix} \tag{3.4}$$

represents φ in the above sense. Hence, it suffices to prove that if φ preserves the order on $\mathbb{Z}^n$, then M is of the form (3.3).

Let $0 \leqslant i \leqslant n-1$ and m be any element in $\mathbb{Z}\backslash\{0\}$, consider

$$A_{i,m} = \left\{ z \in \mathbb{Z}^n \mid v(z) = i \text{ and } z_{v(z)} = m \right\}.$$

Note that, for $i = n-1$, $A_{n-1,m}$ is the singleton of $(0, \ldots, 0, m)$. For $i = 0$, $A_{0,m}$ is the set $\{m\} \times \mathbb{Z}^{n-1}$, and, for $0 < i < n-1$, $A_{i,m} = \{0\}^i \times \{m\} \times \mathbb{Z}^{n-(i+1)}$.

Proposition 3.1.7. *For every $0 \leqslant i \leqslant n-1$ and every $m \in \mathbb{Z}\backslash\{0\}$, the set $A_{i,m}$ is 0-definable.*

Proof. For $i = n-1$ it is clear. Let $0 \leqslant i < n-1$ be fixed and consider the set $A_i = \{ z \in \mathbb{Z}^n \mid v(z) = i \}$. By Lemma 3.1.6, since $A_i = {\mathbb{Z}^n}_i \backslash {\mathbb{Z}^n}_{i+1}$, A_i is 0-definable. Let $\alpha(x)$ and $\beta(x)$ be the formulas defining A_i and ${\mathbb{Z}^n}_{i+1}$, respectively. Since $A_{i,m} = \{ z \in \mathbb{Z}^n \mid -z \in A_{i,-m} \}$, it suffices to prove the statement for $m > 0$. If $m = 1$, then $A_{i,1}$ is defined by the formula

$$\psi_1(x) := \alpha(x) \wedge 0 < x \wedge \forall y\big((\alpha(y) \wedge 0 < y \wedge y < x) \implies \beta(y - x)\big).$$

Indeed, it is trivial that any element z in $A_{i,1}$ satisfies $\psi_1(x)$. Conversely, let x be an element of $\mathbb{Z}^n$ satisfying $\psi_1(x)$. Then $v(x) = i$ and $x_{v(x)} > 0$. If $x_{v(x)} > 1$, then

$v(x - e_{\{i\}}) = i$. Since $0 < e_{\{i\}} < x$, we get a contradiction and $x_{v(x)} = 1$. Let $m > 1$ and suppose by induction that $A_{i,m-1}$ is defined by the formula $\psi_{m-1}(x)$. Consider the formula

$$\gamma_{m-1}(x) := \forall y(\psi_{m-1}(y) \implies y < x).$$

Then, $A_{i,m}$ is defined by the formula

$$\psi_m(x) := \alpha(x) \wedge \gamma_{m-1}(x) \wedge \forall y\big(\gamma_{m-1}(y) \wedge y < x) \implies \beta(y - x)\big).$$

Indeed, let x in $\mathbb{Z}^n$ satisfy $\psi_m(x)$. Then $v(x) = i$ and $x_{v(x)} > m - 1$. If $x_{v(x)} > m$, then $v(x - me_{\{i\}}) = i$, and we get a contradiction. Therefore, $x_{v(x)} = m$ and $x \in A_{i,m}$. □

Now we are able to prove the following

Theorem 3.1.8. *Let φ be an automorphism of the lexicographic sum $\mathbb{Z}^n$, where $n > 1$. Then there exists an upper triangular matrix K_n as in (3.3) such that $\varphi = f_{K_n}$.*

Proof. For every $z \in \mathbb{Z}^n$, $\varphi(z) = zM$, where M is the matrix (3.4). By Proposition 3.1.7, φ fixes $A_{i,1}$ setwise for every $0 \leqslant i \leqslant n - 1$. Therefore, for any $0 \leqslant i \leqslant n - 1$, $v(\varphi(e_{\{i\}})) = i$ and $\varphi(e_{\{i\}})_i = 1$, so the result is proved. □

Henceforth we focus on the case $\alpha \geqslant \omega$. Recall that the chain

$$(0) = \Psi_\infty \subset \cdots \subset \Psi_n \subset \cdots \subset \Psi_1 \subset \Gamma_0 = \Psi = \sum_{i<\alpha} \mathbb{Z} \tag{3.5}$$

represents the set of all convex subgroups of Ψ. Note that, contrary to the finite case, it is not a well-ordered set, since it contains an infinite descending chain.

Let φ be an automorphism of Ψ. For every $f \in \Psi$, there exists a finite subset $\{ i_1, \ldots, i_m \}$ of α such that $f = \sum_{j=1}^m f(i_j)e_{\{i_j\}}$. Then, since φ is a group homomorphism, $\varphi(f) = \sum_{j=1}^m f(i_j)\varphi(e_{\{i_j\}})$. For every $i, k < \alpha$, we set

$$m_{ik} = (\varphi(e_{\{i\}}))(k).$$

Then, we have

$$(\varphi(f))(k) = \sum_{i<\alpha} f(i)m_{ik}$$

for every $k < \alpha$. In other words, also in the case of α infinite, we can associate to φ an infinite matrix $M = (m_{ik})_{i,k<\alpha}$ such that $\varphi(f) = fM$ for every $f \in \Psi$. Note that any row of M has only finitely many nonzero elements.

In order to generalize the result of Theorem 3.1.8, we now determine which conditions M has to satisfy, for preserving the order on Ψ. We will prove that also in the infinite case M is upper triangular with all the entries of the main diagonal equal to 1, i.e.

$$m_{ik} = \begin{cases} 0 \text{ if } i > k \\ 1 \text{ if } i = k \end{cases} \tag{3.6}$$

Lemma 3.1.9. *Let φ be an automorphism of Ψ. Then φ fixes Ψ_i setwise for every $0 \leqslant i < \alpha$.*

Proof. Note that, for every $0 \leqslant i < \alpha$, $\varphi(\Psi_i)$ is a convex subgroup of Ψ. It follows that φ induces an order-preserving bijection on the set of all convex subgroups of Ψ $\Psi_i \mapsto \varphi(\Psi_i)$, which will be denoted again by φ. We prove that the automorphism induced by φ on the set of convex subgroups of Ψ is the identity. Clearly, $\varphi(\Psi_0) = \Psi_0$. Let $0 < i < \alpha$ and suppose, by induction, $\varphi(\Psi_j) = \Psi_j$ for every $j < i$. Therefore, $\varphi(\Psi_i) \subseteq \Psi_i$. Let Δ be a convex subgroup of Ψ such that $\varphi(\Delta) = \Psi_i$. Then, $\Delta = \Psi_h$ for some $h \geqslant i$. It follows that $\Psi_h \subseteq \Psi_i$ and, since φ is order-preserving, $\Psi_i \subseteq \varphi(\Psi_i)$. Therefore, $\varphi(\Psi_i) = \Psi_i$, and the statement is proved. □

From Lemma 3.1.9, it follows immediately that

Corollary 3.1.10. *If φ is an automorphism of Ψ, then φ fixes $\{ f \in \Gamma \mid v(f) = i \}$ setwise for every $0 \leqslant i < \alpha$.*

Proof. Let $i < \alpha$ be such that $i + 1 < \alpha$. Then $\{ f \in \Psi \mid v(f) = i \} = \Psi_i \backslash \Psi_{i+1}$ and, by Lemma 3.1.9, it is fixed setwise. Furthermore, if $\alpha = \beta + 1$ for some $\beta < \alpha$, each element of Ψ_β is 0-definable and Ψ_β is fixed pointwise. □

Theorem 3.1.11. *Let φ be an automorphism of Ψ. Then there exists $M = (m_{ik})_{i,k<\alpha}$ with m_{ik}'s as in (3.6) such that $\varphi(f) = fM$ for every $f \in \Psi$.*

Proof. Fix $i < \alpha$ and consider $e_{\{i\}} \in \Psi$. Set $m_{ik} = (\varphi(e_{\{i\}}))(k)$ for every $k < \alpha$. Then by Corollary 3.1.10, $m_{ik} = 0$ for $k < i$ and $m_{ik} \geqslant 1$ for $k = i$. Note that Lemma 3.1.9

implies that each $\Psi_i = \{ f \in \Psi \mid v(f) \geqslant i \}$ is generated by $\{\varphi(e_{\{j\}})\}_{j \geqslant i}$ as an abelian group. Therefore, for every $g \in \Psi_i$ with $v(g) = i$ we have $g(i) = km_{ii}$ for some $k \in \mathbb{Z}$. Therefore $g(i) \equiv 0 \pmod{m_{ii}}$ for every $g \in \Psi_i$. Then $m_{ii} = 1$, and so the statement is proved. □

Summarizing, we have shown that, for any ordinal α, $\alpha > 1$, every automorphism of $\Psi = \sum_{i<\alpha} \mathbb{Z}$ can be represented as a matrix $M = (m_{ik})_{i,k<\alpha}$ with m_{ik}'s as in (3.6).

3.2 Failure of elimination of imaginaries

We now prove that in both cases $\Psi = \sum_{i<\alpha} \mathbb{Z}$ and $\Lambda = H_{i<\alpha} \mathbb{Z}$, there exist some imaginaries of the ordered abelian group that do not admit a code in the group. We have seen that a code for an imaginary element is, in particular, a finite tuple of elements fixed pointwise by the same automorphisms which leave the imaginary invariant. Then, the argument we will use consists in determining, for a fixed E-equivalence class X, with E 0-definable equivalence relation on the group G,

1. a set $S \subseteq \mathrm{Stab}_G(X)$ with $\mathrm{Fix}(S) \subseteq \mathrm{Fix}(\mathrm{Aut}(G))$
2. an automorphism $\varphi \in \mathrm{Aut}(G) \setminus \mathrm{Stab}_G(X)$.

We will first focus on the case of the lexicographic sum. To this purpose, recall that, for any $m \in \mathbb{N}, m > 0$, $\equiv_m$ denotes the binary relation defined by

$$f \equiv_m g \text{ if and only if } f - g \text{ is divisible by } m.$$

Theorem 3.2.1. *Let $\alpha > 1$ be an ordinal. Then $\Psi = \sum_{i<\alpha} \mathbb{Z}$ does not admit elimination of imaginaries in the pure language of ordered abelian groups.*

Proof. Suppose by contradiction that Ψ admits elimination of imaginaries.

Let p be a prime and fix an element $a \in \Psi$ such that $a(0) \notin p\mathbb{Z}$. Let $X = [a]_{\equiv_p}$ be the $\equiv_p$ - equivalence class of a. Since $\equiv_p$ is 0-definable, there exists a canonical parameter $\bar{b}$ for X, let $\bar{b} = (b_j)_{j<\mu}$ for some positive integer μ.

Let φ be an automorphism of Ψ and M the matrix $(m_{ik})_{i,k<\alpha}$ such that $\varphi(f) = fM$ for every $f \in \Psi$. Recall that M is either finite or infinite depending on the cardinality of

α, and that, for every $i, k < \alpha$, $m_{ii} = 1$ and $m_{ik} = 0$ for $k < i$. Suppose $\varphi \in \mathrm{Stab}_\Psi(X)$. Then $\varphi(a) \equiv_p a$, namely, $(\varphi(a))(i) - a(i) \in p\mathbb{Z}$ for every $i < \alpha$. Therefore, from $(\varphi(a))(1) = a(0)m_{01} + a(1)$ and $a(0) \notin p\mathbb{Z}$, it follows that $m_{01} \in p\mathbb{Z}$. Moreover, since $\varphi \in \mathrm{Stab}_\Psi(X)$ then, by Remark 1.1.2, $\varphi \in \mathrm{Aut}(\Psi/\bar{b})$, namely, $\varphi(b_j) = b_j$ for every $j < \mu$. In particular, since for every $0 < k < \alpha$, $(\varphi(b_j))(k) = b_j(k) + \sum_{0 \leqslant i < k}(b_j(i))m_{ik}$ for every $j < \mu$, it follows that

$$\sum_{0 \leqslant i < k} (b_j(i))m_{ik} = 0 \text{ for every } 0 < k < \alpha \text{ and for every } j < \mu. \tag{3.7}$$

Let $h < \alpha$ be such that $h + 1 < \alpha$, and consider the following matrix $\tilde{M}^h = (\tilde{m}^h_{ik})_{i,k<\alpha}$ where

$$\tilde{m}^h_{ik} = \begin{cases} 1 & \text{if } i = k \\ p & \text{if } i = h, k = h+1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, $\tilde{M}^h$ is an upper triangular matrix, with all entries of the diagonal equal to 1. Since for every $f \in \Psi$, $(f\tilde{M}^h)(h+1) = p(f(h)) + f(h+1)$ and $(f\tilde{M}^h)(k) = f(k)$ for every $k < \alpha$, $k \neq h$, $f\tilde{M}^h \in \Gamma$ and $\tilde{M}^h$ induces the function

$$\varphi_{\tilde{M}} \colon f \in \Psi \mapsto f\tilde{M}^h \in \Psi.$$

In particular, $\varphi_{\tilde{M}^h}(f) \equiv_p f$ for every $f \in \Psi$. Moreover, $\varphi_{\tilde{M}^h}$ is a group automorphism and is order-preserving, hence $\varphi_{\tilde{M}^h} \in \mathrm{Stab}_\Psi(X)$. From (3.7) it follows that $p(b_j(h)) = 0$ for every $j < \mu$, and, so, $b_j(h) = 0$ for every $j < \mu$. Now we need to distinguish two cases, α limit ordinal and α successor ordinal.

case α limit Since for every $h < \alpha$, $h + 1 < \alpha$, we obtain $b_j = 0$ for every $j < \mu$.

case $\alpha = \beta + 1$ for some $\beta < \alpha$ Then $b_j(h) = 0$ for every $h < \beta$ and for every $j < \mu$. In particular, for every $j < \mu$, $b_j \in \Psi_\beta$, i.e. $b_j = k_j e_{\{\beta\}}$ for some $k_j \in \mathbb{Z}$. Since $e_{\{\beta\}}$ is the minimal positive element of Ψ, b_j is 0-definable for every $j < \mu$.

Therefore, in both cases, all the automorphisms of Ψ fix $\bar{b}$. Hence, by Remark 1.1.2, any automorphism of Ψ fixes X. This is clearly false. Indeed, for example, the automorphism $\psi \colon \Psi \to \Psi$ defined by $\psi(f) = f\bar{M}$, for every $f \in \Psi$, and $\bar{M} = (\bar{m}_{ik})_{i,k<\alpha}$

with

$$\bar{m}_{ik} = \begin{cases} 1 & \text{if either } i = k \text{ or } i = 0, k = 1 \\ 0 & \text{otherwise} \end{cases}$$

does not fix X since $\bar{m}_{01} \notin p\mathbb{Z}$. The contradiction follows from the existence of a canonical parameter for X, and so Ψ does not admit elimination of imaginaries. □

Using the same argument as in Theorem 3.2.1, we can prove the failure of elimination of imaginaries in L_{oag} also for $\Lambda = \mathop{\boldsymbol{H}}_{i<\alpha} \mathbb{Z}$. Indeed, let p be a prime, $a \in \Lambda$ be such that $a(0) \notin p\mathbb{Z}$ and $X = [a]_{\equiv_p}$. Fix $h < \alpha$, then the automorphism $\varphi_h \colon \Lambda \to \Lambda$ defined by

$$(\varphi_h(f))(j) = \begin{cases} pf(j-1) + f(j) & \text{if } j = h+1 \\ f(j) & \text{otherwise} \end{cases} \tag{3.8}$$

for every $f \in \Lambda$, fixes X setwise. Therefore, if $\bar{b}$ is a canonical parameter for X, we obtain that $\bar{b}$ is a tuple of $ke_{\{\beta\}}$'s, with $k \in \mathbb{Z}$, if $\alpha = \beta + 1$ for some $\beta < \alpha$, and $\bar{b} = \bar{0}$ otherwise. In both cases we have a contradiction. Hence, we have proved

Theorem 3.2.2. *Let $\alpha > 1$ be an ordinal. Then $\Lambda = \mathop{\boldsymbol{H}}_{i<\alpha} \mathbb{Z}$ does not admit elimination of imaginaries in the pure language of ordered abelian groups.*

This argument can be adapted for proving the failure of elimination of imaginaries for other ordered abelian groups, such as $\mathop{\boldsymbol{H}}_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$ and $\sum_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$, with $\alpha > 1$ ordinal. Indeed, in a similar way we can prove

Theorem 3.2.3. *Let $\alpha > 1$ be an ordinal. Then $\mathop{\boldsymbol{H}}_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$ does not admit elimination of imaginaries in the pure language of ordered abelian groups.*

Proof. Let $\Omega = \mathop{\boldsymbol{H}}_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$. Then Ω is the lexicographic product $\mathop{\boldsymbol{H}}_{i<\alpha+1} G_i$, where $G_i = \mathbb{Z}$ for every $i < \alpha$ and $G_\alpha = \mathbb{Q}$. As in the proof of Theorem 3.2.1, let p be a prime, $a \in \Omega$ such that $a(0) \notin pG_0 = p\mathbb{Z}$ and $X = [a]_{\equiv_p}$. Let $\bar{b}$ be a canonical parameter for X, $\bar{b} = (b_j)_{j<\mu}$ for some positive integer μ.

Fix $h < \alpha$, and consider the automorphism $\psi_h \colon \Omega \to \Omega$ defined by

$$(\psi_h(f))(j) = \begin{cases} pf(j-1) + f(j) & \text{if } j = h+1 \\ f(j) & \text{otherwise} \end{cases}$$

for every $f \in \Omega$. Since $\psi_h(f) \equiv_p f$ for every $f \in \Omega$, in particular, $\psi_h \in \mathrm{Stab}_\Omega(X)$. Then, by Remark 1.1.2, $\psi_h \in \mathrm{Aut}(\Gamma/\bar{b})$, i.e. $\psi_h(b_j) = b_j$ for every $j < \mu$. Therefore, $p(b_j(h)) = 0$ for every $j < \mu$, and, so, $b_j(h) = 0$ for every $j < \mu$. From the generality of $h < \alpha$, it follows that $b_j(h) = 0$ for every $h < \alpha$ and every $j < \mu$, namely, $b_j \in \{0\}^\alpha \times \mathbb{Q}$ for every $j < \mu$.

Now consider the function $\varphi\colon \Omega \to \Omega$ defined by

$$(\varphi(f))(j) = \begin{cases} 2f(j) & \text{if } j = \alpha \\ f(j) & \text{otherwise} \end{cases}$$

for every $f \in \Omega$. Recall that $G_\alpha = \mathbb{Q}$. Then, φ is an order-preserving group automorphism and, trivially, $\varphi \in \mathrm{Stab}_\Omega(X)$. Therefore, by Remark 1.1.2, $\varphi(b_j) = b_j$ for every $j < \mu$, and, so, $b_j(\alpha) = 0$ for every $j < \mu$. It follows that $b_j = 0$ for every $j < \mu$ and X is 0-definable. Hence, $\chi(X) = X$ for all automorphisms χ of Ω. This gives a contradiction, since the automorphism $\chi\colon \Omega \to \Omega$ defined by

$$(\chi(f))(j) = \begin{cases} f(0) + f(1) & \text{if } j = 1 \\ f(j) & \text{otherwise} \end{cases}$$

for every $f \in \Omega$, does not fix X. □

Note that the proof of Theorem 3.2.3 also works for $\sum_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$.

The above arguments can be used in order to prove the failure of elimination of imaginaries for many other ordered abelian groups, not only for Hahn products and lexicographic sums. For instance, let p be a prime, $p \neq 2$, and consider $\mathbb{Z}_{(p)}$ as an ordered abelian group with the order induced from the usual order on $\mathbb{Q}$. Note that this is a regular ordered abelian group, different from a divisible ordered abelian group and a model of Presburger arithmetic. Let $X = [1]_{\equiv_p} = 1 + p\mathbb{Z}_{(p)}$ be the $\equiv_p$-equivalence class of 1. Define $\varphi\colon \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$ by $\varphi(w) = (p + 1)w$. Then, φ is bijective and order-preserving, since $p + 1 > 0$. Moreover, $\varphi \in \mathrm{Stab}_{\mathbb{Z}_{(p)}}(X)$, and $\mathrm{Fix}(\varphi) = \{0\}$. Therefore, if $\mathbb{Z}_{(p)}$ eliminates imaginaries (in L_{oag}), X is 0-definable and, in particular, $\mathrm{Stab}_{\mathbb{Z}_{(p)}}(X) = \mathrm{Aut}(\mathbb{Z}_{(p)})$. This is clearly false, since the automorphism $\psi\colon \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$ defined by $\psi(w) = 2w$ does not fix X.

Weak elimination of imaginaries

From Fact 1.1.7, one can deduce that the behaviour of the theories of $\mathcal{H}_{i<\alpha}\mathbb{Z}$ and $\mathcal{H}_{i<\alpha}\mathbb{Z} \times \mathbb{Q}$ in terms of "coding" imaginaries is even worse. Indeed, both $\mathrm{Th}(\mathcal{H}_{i<\alpha}\mathbb{Z})$ and $\mathrm{Th}(\mathcal{H}_{i<\alpha}\mathbb{Z} \times \mathbb{Q})$ do not have even weak elimination of imaginaries. Using a similar argument to that used in the proof of Theorem 3.2.1, we now provide a direct proof of the failure of weak elimination of imaginaries for the theory of $\mathcal{H}_{i<\alpha}\mathbb{Z}$.

Theorem 3.2.4. *Let α be an ordinal, $\alpha > 1$. Then $T = \mathrm{Th}(\mathcal{H}_{i<\alpha}\mathbb{Z})$ does not have weak elimination of imaginaries in the pure language of ordered abelian groups.*

Proof. Suppose for a contradiction that T has weak elimination of imaginaries. For simplicity, consider $\Psi = \sum_{i<\alpha}\mathbb{Z}$. As in the proof of Theorem 3.2.1, let p be a prime and consider $X = [a]_{\equiv_p}$ the $\equiv_p$-equivalence class of $a \in \Psi$ such that $a(0) \notin p\mathbb{Z}$. Then there exist a formula $\vartheta(x, \bar{w})$ and B a finite set of $|\bar{w}|$-tuples such that $X = \vartheta(\Psi, \bar{b})$ if and only if $\bar{b} \in B$.

Let $\mu = |\bar{w}|$ and $\bar{b} = (b_0, \ldots, b_\mu) \in B$. Then $\bar{b}$ is not 0-definable, since X is not 0-definable. In particular, if α is a limit ordinal, then $b_j \neq 0$ for every $j \leqslant \mu$, otherwise, if $\alpha = \beta + 1$ for some $\beta < \alpha$, then $b_j \notin \Psi_\beta$ for every $j \leqslant \mu$. Fix $j \leqslant \mu$ and consider b_j. Without loss of generality, we may assume $j = 0$. Let $h = v(b_0) < \infty$. If $\alpha = \beta + 1$, then $h < \beta$. In any case, since $h + 1 < \alpha$, we can consider the function $\varphi_h \colon \Psi \to \Psi$, defined as in (3.8). Therefore, since $\varphi_h \in \mathrm{Stab}_\Psi(X)$, we have $X = \vartheta(\Psi, \varphi_h(\bar{b}))$ and $\varphi_h(\bar{b}) \in B$. In particular, $\varphi_h(\bar{b}) = (\varphi_h(b_0), \ldots, \varphi_h(b_\mu))$ and, by definition of φ_h, we have $(\varphi_h(b_0))(h + 1) = p(b_0(h)) + b_0(h + 1)$. Then, from $b_0(h) \neq 0$, it follows that $(\varphi_h(b_0))(h + 1) \neq b_0(h + 1)$ and $\varphi_h(b_0) \neq b_0$. Therefore, $\varphi_h(\bar{b}) \neq \bar{b}$. Now consider $\varphi_h^\delta = \underbrace{\varphi_h \circ \cdots \circ \varphi_h}_{\delta \text{ times}}$ for any natural number $\delta > 0$. For every $\delta > 0$, $\varphi_h^\delta \in \mathrm{Stab}_\Psi(X)$ and, then, $\varphi_h^\delta(\bar{b}) \in B$. Clearly, $(\varphi_h^\delta(b_0))(h + 1) = \delta p(b_0(h)) + b_0(h + 1)$. It follows that $\{\varphi_h^\delta(\bar{b})\}_{\delta\in\mathbb{N}}$ is an infinite sequence of pairwise distinct elements of Ψ. This gives a contradiction since B is finite. □

Similarly, we can provide a direct proof of the failure of weak elimination of imaginaries for the theory of $\mathcal{H}_{i<\alpha}\mathbb{Z} \times \mathbb{Q}$.

3.3 Definable Skolem functions

In this section, we prove that the theories of $\Lambda = \mathbb{Z}^n$ and $\Omega = \mathbb{Z}^n \times \mathbb{Q}$, with $n \geqslant 1$, have definable Skolem functions once finitely many new constants are added to the language of ordered abelian groups. Let us recall that

Definition 3.3.1. We say that a theory T in L has definable Skolem functions if for every L-formula $\varphi(\bar{x}, y)$, there is an L-formula $\psi(\bar{x}, y)$ such that

$$T \vdash \forall \bar{x}(\exists y \varphi(\bar{x}, y) \to (\exists! y \psi(\bar{x}, y) \wedge \forall y(\psi(\bar{x}, y) \to \varphi(\bar{x}, y)))). \tag{3.9}$$

First of all, note that the convex subgroup $C = \{0\}^n \times \mathbb{Q}$ of $\mathbb{Z}^n \times \mathbb{Q}$ is 0-definable by the formula $\gamma(x)$ which says that all elements of $[0, |x|]$ are divisible by 2. We highlight

Fact 3.3.2. *Let G be an ordered abelian group.*
(1) If $G \equiv \mathbb{Z}^n$, with $n \geqslant 1$, then G has n 0-definable proper convex subgroups $G^{(0)} = (0) < G^{(1)} < \cdots < G^{(n-1)} < G^{(n)} = G$ such that $G^{(i)}/G^{(i-1)} \equiv \mathbb{Z}$ for every $1 \leqslant i \leqslant n$.
(2) If $G \equiv \mathbb{Z}^n \times \mathbb{Q}$, with $n \geqslant 1$, then G has $n+1$ 0-definable proper convex subgroups $G^{(-1)} = (0) < G^{(0)} < \cdots < G^{(n-1)} < G^{(n)} = G$ such that $G^{(0)} \equiv \mathbb{Q}$ and $G^{(i)}/G^{(i-1)} \equiv \mathbb{Z}$ for every $1 \leqslant i \leqslant n$.

We rely on the following fact from [49]. Consider the first order language $L_{\text{Weis}} = \{0, 1^{(1)}, 1^{(2)}, \ldots, 1^{(n)}, +, -, <, (\equiv_m)_{m>0}\}$, where $1^{(1)}, 1^{(2)}, \ldots, 1^{(n)}$ are new constant symbols. In $\mathbb{Z}^n$ we interpret $1^{(1)}, 1^{(2)}, \ldots, 1^{(n)}$ as $(0, \ldots, 0, 1)$, $(0, \ldots, 0, 1, 0)$, $\ldots$, $(1, 0, \ldots, 0)$, respectively. In $\mathbb{Z}^n \times \mathbb{Q}$ we interpret $1^{(1)}, 1^{(2)}, \ldots, 1^{(n)}$ as $(0, \ldots, 0, 1, 0)$, $(0, \ldots, 0, 1, 0, 0)$, $\ldots$, $(1, 0, \ldots, 0)$, respectively. Namely, we expand L_{oag} by the equivalence relations $\equiv_m$, for each positive integer m, and n constants $1^{(i)}$, $1 \leqslant i \leqslant n$, for a representative of the smallest positive element in each discretely ordered quotient $G^{(i)}/G^{(i-1)}$, where either $G \equiv \mathbb{Z}^n$ or $G \equiv \mathbb{Z}^n \times \mathbb{Q}$ for some $n \geqslant 1$. In [49], Weisfenning proved that

Fact 3.3.3. *Both $\text{Th}_{L_{Weis}}(\mathbb{Z}^n)$ and $\text{Th}_{L_{Weis}}(\mathbb{Z}^n \times \mathbb{Q})$ admit elimination of quantifiers.*

Let G be either a model of $\text{Th}_{L_{\text{Weis}}}(\mathbb{Z}^n)$ or a model of $\text{Th}_{L_{\text{Weis}}}(\mathbb{Z}^n \times \mathbb{Q})$. In particular, every formula $\sigma(x, \bar{a})$ in L_{Weis} with parameters $\bar{a} \subset G$, is equivalent modulo G to a positive boolean combinations of formulas of the following forms

$$(1) \quad kx \equiv_m t(\bar{a}), \text{or } kx \not\equiv_m t(\bar{a}),$$
$$(2) \quad kx = t(\bar{a}), kx < t(\bar{a}), \text{or } t(\bar{a}) < kx,$$

where $k > 0$ is a natural number and $t(\bar{x})$ is a term in L_{Weis}. Let $t(\bar{a}) = g \in G$. Since $[G : mG] < \infty$, every formula in (1) is equivalent to a finite disjuction of formulas of the form $kx \equiv_m g$. If such a formula defines a nonempty set, then there exists $h \in G$ such that $kx \equiv_m g$ is equivalent to $x \equiv_{m'} h$ for some $m' > 0$. So we may assume that all formulas in (1) are of the form $x \equiv_m g$. Moreover, every formula in (2) defines a set which is a finite union of cosets of $2G$ intersected with intervals, see Theorem 12 and Theorem 15 in [4]. Therefore, every definable set in G is a finite union of cosets of subgroups mG intersected with intervals with endpoints in $G \cup \{\pm\infty\}$, for some positive integer m. Hence, G is coset-minimal. We recall that

Definition 3.3.4. A totally ordered group (with possibly extra structure) is *coset-minimal* if every definable set is a finite union of cosets of definable subgroups intersected with intervals.

In [34], the groups elementarily equivalent to either $\mathbb{Q}$, or $\mathbb{Z}^n$, or $\mathbb{Z}^n \times \mathbb{Q}$, for some $n \geqslant 1$, have been characterized as the coset-minimal pure (modulo some constants) groups. Moreover, since any $\equiv_m$-equivalence class is 0-definable, any such group provides an example of a quasi o-minimal structure (see [3]).

We show that the theories of $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$ have definable Skolem functions in L_{Weis} and in a suitable language expanding L_{Weis}, respectively. In [42], using proof-theoretic arguments, Scowcroft identified the following sufficient condition for a model complete theory in order to have definable Skolem functions.

Proposition 3.3.5. *Let L be a first order language with at least one constant symbol and T be a model complete theory in L. Let Σ be a set of $\forall\exists$-axioms for T and Δ be the set of all quantifier free L-formulas $\delta(\bar{u}, \bar{v})$ such that $\forall\bar{u}\exists\bar{v}\delta(\bar{u}, \bar{v}) \in \Sigma$. Suppose for each $\delta(\bar{u}, \bar{v}) \in \Delta$ there is an L-formula $\gamma_\delta(\bar{u}, \bar{v})$ such that $T \vdash \forall\bar{u}\exists!\bar{v}\, \gamma_\delta(\bar{u}, \bar{v})$ and $T \vdash \forall\bar{u}\forall\bar{v}(\gamma_\delta(\bar{u}, \bar{v}) \to \delta(\bar{u}, \bar{v}))$. Then T has definable Skolem functions.*

Note that by Fact 3.3.3 the theories $\text{Th}_{L_{\text{Weis}}}(\mathbb{Z}^n)$ and $\text{Th}_{L_{\text{Weis}}}(\mathbb{Z}^n \times \mathbb{Q})$ are model-complete and, hence, can be axiomatised by $\forall\exists$-sentences. Consider the following sets of sentences introduced in [44, pp. 149-150]:

$$\begin{aligned}
\Sigma_0 =& \forall x \forall y \forall z((x+y)+z = x+(y+z));\\
& \forall x(x+0=x);\\
& \forall x(x-x=0);\\
& \forall x \forall y(x+y=y+x);\\
& \forall x(\neg(x<x));\\
& \forall x \forall y \forall z(x<y<z \rightarrow x<z);\\
& \forall x(x=0 \vee 0<x \vee x<0);\\
& \forall x \forall y(0<x \wedge 0<y \rightarrow 0<x+y).\\
\Sigma_1 =& \; 0 < 2\cdot 1^{(i)} < 1^{(i+1)} \text{ for each } i \text{ such that } 1 \leqslant i \leqslant n-1;\\
& \forall x(2x < 1^{(i)} \vee 1^{(i)} < 2x) \text{ for each } i \text{ such that } 1 \leqslant i \leqslant n;\\
& \forall x(2x < 1^{(i)} \rightarrow mx < 1^{(i)}) \text{ for each } i \text{ such that } 1 \leqslant i \leqslant n \text{ and } m \geqslant 2;\\
& \forall x(x \equiv_m 0 \leftrightarrow \exists y \exists z(-1^{(1)} < 2z < 1^{(1)} \wedge x = my+z)) \text{ for each } m>0;\\
& \forall x(\bigvee_{0\leqslant q_1,\ldots,q_n \leqslant m-1} (x \equiv_m q_1 1^{(1)} + \cdots + q_n 1^{(n)})) \text{ for each } m>1;\\
& \forall x(-1^{(1)} < 2x < 1^{(1)} \rightarrow \exists y(x=my)) \text{ for each } m>1.\\
\Sigma_2 =& \; \forall x(\neg(0<x<1^{(1)})).\\
\Sigma_3 =& \; \exists x(0<x<1^{(1)}).
\end{aligned}$$

It was shown in [44] that the theories $\mathrm{Th}_{L_{\mathrm{Weis}}}(\mathbb{Z}^n)$ and $\mathrm{Th}_{L_{\mathrm{Weis}}}(\mathbb{Z}^n \times \mathbb{Q})$ are axiomatized by $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ and $\Sigma_0 \cup \Sigma_1 \cup \Sigma_3$, respectively. Therefore we are able to use Scowcroft's criterion for the existence of definable Skolem functions in model complete theories. It follows easily that

Theorem 3.3.6. *The theory* $\mathrm{Th}_{L_{Weis}}(\mathbb{Z}^n)$ *has definable Skolem functions.*

Proof. We just need to show the existence of definable Skolem functions for the non-universal axioms:

$$\forall x(x \equiv_m 0 \leftrightarrow \exists y \exists z(-1^{(1)} < 2z < 1^{(1)} \wedge x = my+z)) \text{ for each } m>0, \text{ and}$$
$$\forall x(-1^{(1)} < 2x < 1^{(1)} \rightarrow \exists y(x=my)) \text{ for each } m>1.$$

It is sufficient to note that, in any model G of $\mathrm{Th}_{L_{\mathrm{Weis}}}(\mathbb{Z}^n)$, if $g \in G$ is such that $g = mh$ for some $h \in G$, then h is unique. Indeed, let $g, h, k \in G$ be such that $g = mh + k$ and $-1^{(1)} < 2k < 1^{(1)}$. Trivially, by the interpretation of $1^{(1)}$, the inequalities $-1^{(1)} < 2k < 1^{(1)}$ imply $k = 0$. Hence, k is unique and, so also h in $g = mh$ is unique. □

In order to prove the same result for $\mathbb{Z}^n \times \mathbb{Q}$, for any $n \geqslant 1$, fix an element c in $C = \{0\}^n \times \mathbb{Q}$ such that $c > 0$. Then the quantifier elimination in $\mathrm{Th}_{L_{\mathrm{Weis}}}(\mathbb{Z}^n \times \mathbb{Q})$ is not affected by adding c as a new constant to L_{Weis}. Let $L_{\mathrm{Weis}}(c)$ denote $L_{\mathrm{Weis}} \cup \{c\}$. Now an axiomatization of $\mathrm{Th}_{L_{\mathrm{Weis}}(c)}(\mathbb{Z}^n \times \mathbb{Q})$ is given by $\Sigma_0 \cup \Sigma_1 \cup \Sigma_3$ and the following axiom:

$$0 < 2c < 1^{(1)}.$$

Theorem 3.3.7. *The theory* $\mathrm{Th}_{L_{Weis}(c)}(\mathbb{Z}^n \times \mathbb{Q})$ *has definable Skolem functions.*

Proof. We just need to show the existence of definable Skolem functions for the non-universal axioms:

$$\forall x(x \equiv_m 0 \leftrightarrow \exists y \exists z(-1^{(1)} < 2z < 1^{(1)} \wedge x = my + z)) \text{ for each } m > 0, \text{ and}$$
$$\forall x(-1^{(1)} < 2x < 1^{(1)} \rightarrow \exists y(x = my)) \text{ for each } m > 1, \text{ and}$$
$$\exists x(0 < x < 1^{(1)}).$$

Note that $0 < c < 1^{(1)}$. Then, as in the proof of Theorem 3.3.6, it suffices to note that, in any model G of $\mathrm{Th}_{L_{\mathrm{Weis}}}(\mathbb{Z}^n \times \mathbb{Q})$, if $g \in G$ is such that $g = mh$ for some $h \in G$, then h is unique. Indeed, let $g, h, k \in G$ be such that $g = mh + k$ and $-1^{(1)} < 2k < 1^{(1)}$. In particular, there exists $k' \in G$ such that $k = mk'$, and $g = m(h + k')$. Therefore, we may assume $k = 0$. □

By the characterization of dp-minimal ordered groups in [22] (see Theorem 2.3.20), for any $n \geqslant 1$, the finite lexicographic products $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$ are dp-minimal. The author has recently learned that Vicaria [48] has identified a suitable many-sorted language in which dp-minimal ordered abelian groups eliminates imaginaries. We aimed at identifying a single-sorted language that could suffice for eliminating imaginaries for the theory of the ordered groups $\mathbb{Z}^n$, and $\mathbb{Z}^n \times \mathbb{Q}$, for any $n \geqslant 1$. The languages L_{Weis} and $L_{\mathrm{Weis}}(c)$ seemed to be promising for this goal, since in these languages we can eliminate the $\equiv_p$-equivalence classes and, by Theorem 3.3.6 and Theorem 3.3.7, there are definable Skolem functions. Indeed, it is well known that, provided that one can uniformly associate a canonical parameter to every unary definable set, the existence of definable Skolem functions is a sufficient condition for uniform elimination of imaginaries (see [20, Lemma 4.4.3]).

The following example pointed out to the author by M. Hils shows that unfortunately this is not plausible, and a many-sorted language seems unavoidable. Consider the lexicographic product $G = \mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$ in L_{Weis} (or in some expansion L of L_{Weis} by adding new constants) and suppose G admits elimination of imaginaries. By Fact 3.3.3, we have that every infinite definable set $X \subseteq G^m$ is uncountable. In particular, the set of canonical parameters of cosets $a + \{0\} \times \mathbb{R} \times \mathbb{Z}$, with $a \in G$, is a definable set, since it is the image of a definable map $f\colon \frac{\mathbb{Z}\times\mathbb{R}\times\mathbb{Z}}{\{0\}\times\mathbb{R}\times\mathbb{Z}} \to (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z})^m$ for some $m \in \mathbb{N}$, and, hence it is uncountable. This is clearly false, since $\frac{\mathbb{Z}\times\mathbb{R}\times\mathbb{Z}}{\{0\}\times\mathbb{R}\times\mathbb{Z}} \cong \mathbb{Z}$. The contradiction follows from the existence of a canonical parameter for $a + \{0\} \times \mathbb{R} \times \mathbb{Z}$, and so $\text{Th}_{L_{\text{Weis}}}(\mathbb{Z}^2)$ does not admit elimination of imaginaries. Similar arguments can be used to show that the theory of $\mathbb{Z}^n$, for any $n > 1$, and the theory of $\mathbb{Z}^n \times \mathbb{Q}$, for any $n \geqslant 1$, do not admit elimination of imaginaries in any expansion L of L_{oag} by adding new constants. Moreover, it seems plausible that a much more general statement can be proven along the same lines, namely that whenever G is an ordered abelian group admitting a non-trivial definable convex subgroup, then G does not eliminate imaginaries, even after naming constants.

Remark 3.3.8. It is still unsolved the problem of eliminating imaginaries for the theories of $\mathop{H}_{i<\alpha} \mathbb{Z}$ and $\mathop{H}_{i<\alpha} \mathbb{Z} \times \mathbb{Q}$ with α any ordinal, since these groups do not belong to the class of ordered abelian groups of bounded regular rank considered in [48].

Chapter 4

Stably embedded ordered abelian groups

4.1 Stably embedded coloured chains

We initially study stably embedded models of a theory of linear orders with unary predicates, as these structures appear naturally in the study of ordered abelian groups. In [35, Section 12.6], Poizat has shown that a 1-type p over a coloured chain is definable if and only if its cut is definable. We strengthen this result, generalizing it to every n-type.

For this purpose, we recall the following fact, which is a fundamental tool in the model theory of coloured chains. Let C be a coloured chain, and $a, b \in C$. We say that a sequence $\varphi_1(x), \ldots, \varphi_n(x)$ of formulas is realized between a and b if there are n elements $c_1, \ldots, c_n$, with $a < c_1 < \cdots < c_n < b$, such that each c_i satisfies the corresponding formula φ_i.

Fact 4.1.1 (Rubin's Theorem)**.** *Let C and D be two coloured chains in the same language $\{<, (P_i)_{i \in I}\}$. Two increasing n-tuples, $a_1 < \cdots < a_n$ in C and $b_1 < \cdots < b_n$ in D, have the same type if and only if they satisfy the following conditions:*

- *for every $i \leqslant n$, a_i and b_i have the same type,*
- *for every $i < n$, the same finite sequences of formulas are realized between a_i and a_{i+1} and between b_i and b_{i+1}.*

As a consequence of Rubin's theorem, we can simply describe the types over a coloured chain in the following way. Let C be a coloured chain and $p(x_1, \ldots, x_n)$ a n-type over C. Let p be non-realized in C, since in the case p is realized it is completely determined by a realization of p in C. Moreover, we may assume that no coordinate of $(x_1, \ldots, x_n)$ is realized in C. For any $1 \leqslant i \leqslant n$, the cut determined by p and x_i is defined to be the pair (A_i^p, B_i^p), with

$$A_i^p = \{ c \in C \mid c < x_i \in p \}, \qquad B_i^p = \{ c \in C \mid x_i < c \in p \}.$$

Note that, since for any $1 \leqslant i \leqslant n$ the pair (A_i^p, B_i^p) is a partition of C, for every finite subset $\mathfrak{p}$ of p, $\mathfrak{p}$ is satisfiable either by an element of A_i^p or by an element of B_i^p, for any $1 \leqslant i \leqslant n$. In particular, if the cut (A_i^p, B_i^p) is definable, that is to say if A_i^p is definable, then p is finitely satisfiable on one side. Indeed, we can introduce the following definition.

Definition 4.1.2. Suppose that (A_i^p, B_i^p) is a definable cut, and let $\psi(x_i)$ be a definition of A_i^p. We say that p is *satisfiable on the left at x_i* if $p \vdash \psi(x_i)$; otherwise, we say that p is *satisfiable on the right at x_i*.

Rubin's theorem implies then that a non-realized type $p \in S_n(C)$ is completely determined by its restriction to the empty set of parameters, the sequence of its cuts $(A_i^p, B_i^p)_{1 \leqslant i \leqslant n}$, and its side of satisfiability at x_i for every i such that (A_i^p, B_i^p) is definable.

We deduce from it the following fact:

Proposition 4.1.3. *Let $p \in S_n(C)$ be a non-realized type over a coloured chain C. Then p is definable if and only if the cut (A_i^p, B_i^p) is definable for every $1 \leqslant i \leqslant n$.*

Proof. If p is definable, then trivially (A_i^p, B_i^p) is definable for every $1 \leqslant i \leqslant n$. Conversely, suppose that (A_i^p, B_i^p) is definable for every $1 \leqslant i \leqslant n$. Let D be a very saturated elementary extension of C. By Theorem 1.1.16, it is sufficient to show that p has only one heir over D. In particular, we need to show that if $q \in S_n(D)$ is an heir of p, for each i, $1 \leqslant i \leqslant n$, there is only one possibility for its cut (A_i^q, B_i^q) and its side of satisfiability at x_i. Let $1 \leqslant i \leqslant n$ be fixed, and let $\psi(x_i)$ be a definition of A_i^p. If $q \in S_n(D)$ is an heir of p, then for every $d \in D$, $D \models \psi(d)$ if and only if $q \vdash d < x_i$. Therefore, $A_i^q = \{ d \in D \mid d < x_i \in q \}$ is necessarily the subset of D defined by $\psi(x_i)$ and, hence, we have no choice for the cut determined by q and x_i. The same holds for its side of satisfiability. Suppose, for instance, that p is satisfiable on the left at x_i. Then, p is finitely satisfiable in A_i^p and, so q is finitely satisfiable in A_i^q. □

Therefore, we obtain the following characterization of stably embedded coloured chains.

Corollary 4.1.4. *A coloured chain C is stably embedded if and only if all cuts of C are definable.*

Examples 4.1.5. The following coloured chains are stably embedded:

1. $(\omega, <)$;

2. $(\mathbb{R}, <)$;

3. $(\mathbb{R}, P_{\mathbb{Q}}, <)$, where $P_{\mathbb{Q}}$ stands for a predicate defining $\mathbb{Q}$.

4.2 The case of ordered abelian groups with finite regular rank

In the next sections, we aim at characterizing all stably embedded ordered abelian groups. It is worth analyzing, firstly, the case of ordered abelian groups with finite regular rank, since in this case one can obtain without great effort a very simple characterization. To this purpose, in the following section we are going to investigate stable embeddedness for the subclass of regular ordered abelian groups. A similar study for this class of groups can be also found in [8, Section 4], while the case of models of DOAG and Presburger arithmetic has been already covered in [46].

4.2.1 Regular ordered abelian groups

We begin by observing a fundamental fact that will be useful also later.

Fact 4.2.1. *Let G be any ordered abelian group. If G is stably embedded, then all convex subgroups of G are definable.*

Proof. It is sufficient to show that any convex subgroup determines a cut which is definable if and only if the subgroup is definable. Indeed, clearly, if G is stably embedded, then every cut in G is definable.

Let C be a convex subgroup of G, and consider the cut $C^+ = (L, R)$, with $R = \{ g \in G \mid g > C \}$ and $L = G \backslash R$. Trivially, if C is definable, then C^+ is definable. Conversely, suppose C^+ is definable. Then, so it is the invariance group of C^+, $H(C^+) = \{ g \in G \mid g + C^+ = C^+ \}$. Since C is a convex subgroup, $H(C^+) = C$ and, hence, C is definable. □

In particular, it follows that every regular ordered abelian group which is stably embedded is necessarily archimedean. As a consequence, we will see that both in the class of divisible ordered abelian groups and in the class of $\mathbb{Z}$-groups there is a unique stably embedded model.

Divisible ordered abelian groups

We now show that $(\mathbb{R}, 0, +, <)$ is stably embedded in any elementary extension and it is the unique model of DOAG to be stably embedded.

Since divisible ordered abelian groups are o-minimal, by Theorem 1.1.17 it is sufficient to consider just 1-types. Therefore, let $D \models$ DOAG and $p(x) \in S_1(D)$ be a non-realized type. Then, p is determined by the cut $C^p = \{d \in D \mid p \vdash d < x\}$ and, the definability of p is equivalent to the definability of C^p. Clearly, the only possible cuts in $\mathbb{R}$ are of the form $a^-, a^+, -\infty, +\infty$. Therefore, $(\mathbb{R}, 0, +, <)$ is stably embedded. Moreover, let $D \models$ DOAG be such that every $p(x) \in S_1(D)$ is definable. By Fact 4.2.1, D is a subgroup of $\mathbb{R}$, and so equal to its completion. Hence, we have proved:

Theorem 4.2.2. *Let D be a divisible ordered abelian group. Then, D is stably embedded if and only if $D \cong \mathbb{R}$.*

Note that, since there are only finitely many kinds of types over $\mathbb{R}$, each one corresponding to a kind of cut among $a^-, a^+, -\infty, +\infty$, $\mathbb{R}$ is, in particular, uniformly stably embedded.

Presburger Arithmetic

We now show that $(\mathbb{Z}, 0, +, <)$ is stably embedded in any elementary extension and it is the unique model of Presburger Arithmetic to be stably embedded.

Let T be the theory of $\mathbb{Z}$ in L_{Pres}. Let $\mathcal{M}$ be a model of T, and let $\bar{a} = (a_0, \dots, a_{n-1})$ be a finite tuple of elements in an elementary extension $\mathcal{N}$ of $\mathcal{M}$. Then, by quantifier elimination,

$$\bigcup_{z_0,\dots,z_{n-1}\in\mathbb{Z}} \operatorname{tp}(\sum_{i<n} z_i a_i/M) \vdash \operatorname{tp}(\bar{a}/M).$$

In particular, if all 1-types over M are definable, then all n-types over M are definable, for each n. Hence, as in the case of divisible ordered abelian groups, we can concentrate just on 1-types.

Let $\mathbb{Z} \preceq \mathcal{M}$, then $\mathcal{M} = \mathcal{D} \times \mathbb{Z}$, where $\mathcal{D} \models$ DOAG. In particular, $\mathbb{Z}$ is (isomorphic to) the non-trivial convex subgroup of $\mathcal{M}$ and, it is pure in $\mathcal{M}$. Let $p(x)$ $=\operatorname{tp}(a/\mathbb{Z})$, with $a \in M$. Then, $p(x)$ is determined by the class modulo n of a and by the cut $C_n^p = \{d \in \mathbb{Z} \mid p \vdash d < nx\}$, with n ranging over $\mathbb{N}$. Therefore, the only possible 1-types over $\mathbb{Z}$ are the realized types and the types of the form $-\infty, +\infty$. Moreover, any other model of T is not archimedean and, hence, not stably embedded by Fact 4.2.1. We have proved:

Theorem 4.2.3. *Let G be a $\mathbb{Z}$-group. Then, G is stably embedded if and only if $G \cong \mathbb{Z}$.*

Note that, in particular, all types over $\mathbb{Z}$ are uniformly definable, and then $\mathbb{Z}$ is uniformly stably embedded.

Dense regular ordered abelian groups

It remains the case of $(G, 0, +, <)$ regular ordered abelian group which is neither divisible nor discrete. So, let G be any stably embedded regular ordered abelian group, and assume $G \leqslant \mathbb{R}$ by Fact 4.2.1. Assume, also, that G is dense. By quantifier elimination in L_{Pres}, as in the previous cases, we can use a *Marker-Steinhorn-type argument* and deduce the definability of n-types from the definability of 1-types for all regular ordered abelian groups. Therefore, let $G' \succeq G$ be an elementary extension of G, and $p(x) = \operatorname{tp}(a/G)$ be a non realized type over G, with $a \in G'$. Then, p is determined by the classes modulo n of a and by the cuts $C_n^p = \{d \in \mathbb{Z} \mid p \vdash d < nx\}$, with $n \in \mathbb{N}$.

In particular, p is definable if and only if the type of a over the divisible hull of G is definable. Then, by Theorem 4.2.2 we have that

Theorem 4.2.4. *Let G be any regular ordered abelian group densely ordered. Then G is stably embedded if and only if G is archimedean and $div(G) \cong \mathbb{R}$.*

Uniform stable embeddedness does not hold in general for any dense regular group stably embedded, as the following example shows.

Example 4.2.5. Let $(b_i)_{i\in I}$ be a $\mathbb{Q}$-basis of $\mathbb{R}$ and consider $G = \{\sum_{i\in I} \frac{z_i}{2^n} b_i \mid z_i \in \mathbb{Z}, n \in \mathbb{N}\} \leqslant \mathbb{R}$. Then G is stably embedded, but not uniformly.

Let G' be an elementary extension of G which contains a realization g'_n of (a completion of) the partial type $p_n(x)$ determined by the cut $(\frac{1}{2^n})^+$, for every $n \in \mathbb{N}$. If G is uniformly stably embedded in G', then we have that for every elementary extension $(\tilde{G}', \tilde{G})$ of the pair (G', G), $\tilde{G}$ is stably embedded in $\tilde{G}'$. This is clearly false. Indeed, consider any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and let $g' = (g'_n)_{n\in\mathbb{N}}/\mathcal{U}$ an element of the ultraproduct $\prod_{\mathcal{U}} G'$. Then, the type $p(x) = \text{tp}(g'/\prod_{\mathcal{U}} G)$ is not definable, hence, $\tilde{G} = \prod_{\mathcal{U}} G$ is not stably embedded in $\tilde{G}' = \prod_{\mathcal{U}} G'$.

Note that, by a similar argument, one can show that no archimedean ordered abelian group not isomorphic to $\mathbb{Z}$ or $\mathbb{R}$ is uniformly stably embedded.

4.2.2 Ordered abelian groups with finite regular rank

In this section, we are going to characterize all stably embedded models in the class of ordered abelian groups with finite regular rank. Let G be an ordered abelian group with finitely many definable convex subgroups

$$(0) = \Delta_0 < \cdots < \Delta_i < \cdots < \Delta_n = G.$$

For any $i < n$, we can associate to G the following short exact sequence of ordered abelian groups

$$0 \longrightarrow \Delta_i \xrightarrow{\iota} \Delta_{i+1} \xrightarrow{\nu} \Delta_{i+1}/\Delta_i \longrightarrow 0 \tag{4.1}$$

This simple observation will allow us to deduce some necessary and sufficient conditions for an ordered abelian group with finite regular rank in order to be stably embedded.

Recall that a short exact sequence of abelian groups is a sequence $\mathcal{M} = (A, B, C, \iota, \nu)$, where A, B, C are abelian groups and ι, ν homomorphisms such that

- $\iota \colon A \to B$ is injective
- $\nu \colon B \to C$ is surjective
- $\mathrm{Im}\iota = \ker\nu$

A short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} C \longrightarrow 0$$

is said to be pure if $\iota(A)$ is a pure subgroup of B. In [46], Touchard proved the following characterization for pure short exact sequences of abelian groups:

Proposition 4.2.6. *Let $\mathcal{M} = (A, B, C, \iota, \nu, \dots)$ be a pure short exact sequence of abelian groups, with eventually additional structure on the sort A and on the sort C. Let $\mathcal{N} = (A', B', C', \iota', \nu')$ be an elementary extension of $\mathcal{M}$.*
Then, $\mathcal{M}$ is (uniformly) stably embedded in $\mathcal{N}$ if and only if A is (uniformly) stably embedded in A' and C is stably embedded in C'.

In particular, this proposition can be applied to short exact sequences of ordered abelian groups

$$0 \longrightarrow (A, <) \xrightarrow{\iota} (B, <) \xrightarrow{\nu} (C, <) \longrightarrow 0$$

with A convex subgroup of B and $C \cong B/A$ and, so to the short exact sequences in (4.1) as well. Indeed, we have that A is a pure subgroup of B and, the order on B can be recovered from the orderings on A and C. It follows

Theorem 4.2.7. *Let G be an ordered abelian group with finite regular rank, and let $(0) = \Delta_0 < \cdots < \Delta_i < \cdots < \Delta_n = G$ be all the definable convex subgroups of G. Then, G is (uniformly) stably embedded if and only if Δ_{i+1}/Δ_i is (uniformly) stably embedded for every $i < n$.*

Proof. By induction on n. For $n = 1$, it is trivial. Let $n > 1$ and suppose that the statement holds for any ordered abelian group with n definable convex subgroups. Let

G' be any elementary extension of G. Consider the following short exact sequences

$$\mathcal{M}\colon 0 \longrightarrow \Delta_1 \xrightarrow{\iota} G \xrightarrow{\nu} G/\Delta_1 \longrightarrow 0$$
$$\mathcal{N}\colon 0 \longrightarrow \Delta_1' \xrightarrow{\iota'} G' \xrightarrow{\nu'} G'/\Delta_1' \longrightarrow 0$$

where ι, ι' are the immersion maps and ν, ν' the canonical maps. We have $\mathcal{M} \preceq \mathcal{N}$ and, since Δ_1 is pure in G, $\mathcal{M}$ is pure. Therefore, by Proposition 4.2.6, G is (uniformly) stably embedded in G' if and only if Δ_1 is (uniformly) stably embedded in Δ_1' and G/Δ_1 is (uniformly) stably embedded in G'/Δ_1'. Moreover, by induction hypothesis, G/Δ_1 is (uniformly) stably embedded if and only if $\frac{\Delta_{i+1}/\Delta_1}{\Delta_i/\Delta_1} \simeq \Delta_{i+1}/\Delta_i$ is (uniformly) stably embedded for every $1 \leqslant i < n$. □

Notice that for any $i < n$, the group Δ_{i+1}/Δ_i has no proper nonzero definable subgroups and hence is regular by Fact 2.3.4. Then, by the characterization of stably embedded regular ordered abelian groups in Section 4.2.1, we have

Corollary 4.2.8. *G is stably embedded if and only if Δ_{i+1}/Δ_i is archimedean and either $\Delta_{i+1}/\Delta_i \cong \mathbb{Z}$ or $div(\Delta_{i+1}/\Delta_i) \cong \mathbb{R}$.*

Example 4.2.9. The ordered abelian groups $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{R}$ are uniformly stably embedded, for every $n \in \mathbb{N}$, and they are the unique models of their own theory to be stably embedded.

4.3 Towards a characterization of stable embeddedness

We have seen in the previous section that if an ordered abelian group G is stably embedded, then all its convex subgroups are definable. One can easily see that this is far from being a sufficient condition for stable embeddedness. Indeed, even if all convex subgroups of G are definable, stable embeddedness certainly can fail in two cases:

- if a rib G_γ of G is not stably embedded as an ordered abelian group, for some $\gamma \in \Gamma_G$, and
- if G is not maximal.

By Theorem 4.2.4, a very simple example for the first occurrence is given by the group of the rationals $\mathbb{Q}$ and, more generally, by any archimedean group G, which actually coincides with its unique rib, such that $\operatorname{div}(G)$ is a proper subgroup of $\mathbb{R}$. As we will see in Section 4.3.2, the lexicographic sum $\sum_{i<\omega} \mathbb{Z}$ represents instead an ordered abelian group which is not maximal and not stably embedded.

We are going to prove that, for a certain class of ordered abelian groups, the above conditions, together with the existence of a convex subgroup which is not definable, represent actually the only cases where an ordered abelian group is not stably embedded. In other words, for a class of ordered abelian groups, the maximality of the group, the definability of all its convex subgroups, and the stable embeddedness of all its ribs are sufficient to deduce the stable embeddedness of the group (see Theorem 4.3.23 and Remark 4.3.28.)

In the next sections, we will introduce all the tools we need to attain this goal.

4.3.1 The induced valued group modulo m

Let G be an ordered abelian group. In Section 2.2, we have seen that G is, in particular, a valued group with respect to the natural valuation. Now we show that we can actually associate to G a family of valued groups, one for every natural number m. Indeed, let $m \in \mathbb{N}$. As in Definition 2.3.15, for any $a \in G \backslash mG$, let V_a^m be the largest convex subgroup of G such that $a \notin V_a^m + mG$. For any $a \in mG$, set $V_a^m = \{0\}$.

Definition 4.3.1. We denote by Γ_G^m or simply Γ^m a set indexing the set of the convex subgroups $\{V_g^m\}_{g\in G}$, and inversely ordered. In other words, we set $\{V_g^m\}_{g\in G} = \{V_\gamma^m\}_{\gamma\in\Gamma_G^m}$, and, for any $\gamma, \delta \in \Gamma_G^m$,

$$\gamma < \delta \iff V_\delta^m \subset V_\gamma^m.$$

As usual, let ∞ denote the maximal element of Γ^m, corresponding to $\{0\}$. For every $m \in \mathbb{N}$, we define a map $\operatorname{val}^m : G \to \Gamma^m$, as follows:

$$\operatorname{val}^m(a) = \gamma, \text{ where } V_a^m = V_\gamma^m$$

Notice that, for any $m > 0$, Γ^m is equal to the underlying set of the auxiliary sort S_m introduced in Definition 2.3.15, and val^m corresponds to the canonical map $\mathfrak{s}_m$. In particular, for $m = 0$, Γ^0 is the archimedean spine Γ_G of G and val^0 is the natural valuation on G, and, for $m = 1$, $\Gamma^1 = \{\infty\}$ and val^1 is the trivial map sending every element of G to ∞.

Clearly, for every $m \in \mathbb{N}$, the map val^m induces the valuation map $\hat{\mathrm{val}}^m : G/mG \to \Gamma^m$ defined by $\hat{\mathrm{val}}^m(g \mod mG) = \mathrm{val}^m(g)$, for any $g \in G$. We call the valued group $(G/mG, \hat{\mathrm{val}}^m)$ the *induced valued group modulo m* of G and Γ^m_G the *m-value set* of G. By abuse of notation, we will also write val^m to refer to the induced valuation $\hat{\mathrm{val}}^m$.

The aim of this section is to prove that the property of pseudo-completeness transfers from G to the induced valued group modulo m's. We first show that, in the case of G pseudo-complete ordered abelian group, we can actually identify Γ^m with a subset of $\Gamma = \Gamma_G$ (depending on m), for any $m \in \mathbb{N}$.

Lemma 4.3.2. *Let G be a pseudo-complete ordered abelian group, and $m \in \mathbb{N}$ be a natural number such that $m > 1$. Then, for every $g \in G, g \notin mG$, there exists $g' \in G$ such that $g \equiv_m g'$ and $V^m_g = V_{g'}$.*

Proof. Let $g \in G \backslash mG$ be fixed. It is enough to show that the set of values

$$\Delta := \{ \mathrm{val}(g - ma) \mid a \in G \}$$

has a maximal element. Indeed, let $a' \in G$ be such that $\mathrm{val}(g - ma') = \max \Delta$, then, clearly, $V^m_g = V_{g-ma'} = V^m_{g-ma'}$. Suppose Δ does not admit a maximal element. Let $(a_i)_{i \in I}$ be a sequence such that $\mathrm{val}(g - ma_i)$ is increasing and cofinal in Δ. Then, we have that $(g - ma_i)_{i \in I}$ and $(a_i)_{i \in I}$ are pseudo-Cauchy sequences. Consider a pseudo-limit a of $(a_i)_{i \in I}$. Then, $\mathrm{val}(g - ma) > \mathrm{val}(g - ma_i)$ for all i's, and so we get a contradiction. □

Henceforth, if G is pseudo-complete, then, for any $m \in \mathbb{N}$, we have $\Gamma^m_G \subseteq \Gamma_G$, and the map $\mathrm{val}^m : G \to \Gamma^m_G$ is given by:

$$\mathrm{val}^m(g) = \begin{cases} \min\{\gamma \in \Gamma_G \mid g \notin V_\gamma + mG\} & \text{if } g \notin mG, \\ \infty & \text{otherwise.} \end{cases} \tag{4.2}$$

We observe also the following property that will be used repeatedly from now on.

Remark 4.3.3. Let G be a a peudo-complete ordered abelian group, and $g \in G$. From $V^m_g \subseteq V_g$ it follows $\mathrm{val}^m(g) \geqslant \mathrm{val}(g)$, for any $m \in \mathbb{N}, m > 1$.

We highlight that Γ^m may be a proper subset of Γ. In particular, it holds that

Proposition 4.3.4. *Let G be a pseudo-complete ordered abelian group, and $m \in \mathbb{N}$ be a natural number such that $m > 1$. Then, Γ^m is the set $\{ \gamma \in \Gamma \mid G_\gamma$ is not divisible by $m \}$.*

Proof. Let $\gamma \in \Gamma$ be a value for which the corresponding rib G_γ is not divisible by m. This means that there exists $g \in C_\gamma$ such that for all $g' \in C_\gamma$, $\mathrm{val}(g+mg') = \gamma$ (otherwise, $\mathrm{val}(g+mg') > \gamma$ and, $g+V_\gamma = mg'+V_\gamma$ for some $g' \in C_\gamma$). Thus, V_γ is the largest convex subgroup of G not intersecting $g + mG$ and, hence, $\gamma \in \Gamma^m$. Conversely, if $\gamma \in \Gamma^m$, then there exists g of m-value γ. By Lemma 4.3.2, we may assume that $\mathrm{val}(g) = \gamma$. Then, we have that $g \in C_\gamma + mG$ and $g \notin V_\gamma + mG$, and so G_γ is not divisible by m. □

Example 4.3.5. Consider the following ordered abelian group: $H_{p\in\mathbb{P}} \mathbb{Z}_{(p)}$.
Clearly, for any $p \in \mathbb{P}$, since we have $p\mathbb{Z}_{(p)} \subsetneq \mathbb{Z}_{(p)}$ and $q\mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}$ for all $q \neq p$, $\Gamma^p = \{p\}$. Then, $\Gamma = \mathbb{P} = \bigcup_{p\in\mathbb{P}} \Gamma^p$ and, in particular, all convex subgroups are definable.

Note that, if G is not pseudo-complete, the convex subgroup V^m_g is not necessarily equal to a value V_i for some $i \in \Gamma$. Consider, for instance, $G' := \sum_{r\in\mathbb{Q}} \mathbb{Z}$, an increasing sequence of positive rationals $(r_k)_{k\in\mathbb{N}}$ converging to $\sqrt{2}$ and $a := (a_r)_{r\in\mathbb{Q}} \in H_{r\in\mathbb{Q}} \mathbb{Z}$ with

$$a_r = \begin{cases} m & \text{if } r = r_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be the ordered subgroup generated by $G' \cup \{a\}$ (notice that a is not divisible by m in G). We have that $V^m_a = \sum_{i>\sqrt{2}} \mathbb{Z}$, but $V^m_a \neq V_g$ for any $g \in G$.

Theorem 4.3.6. *Let G be a pseudo-complete ordered abelian group. Then, for every $m \in \mathbb{N}, m > 1$, the induced valued group modulo m $(G/mG, \mathrm{val}^m)$ is pseudo-complete.*

Proof. Note that, by Lemma 4.3.2, for any coset $a \mod mG$ in G/mG, there exists $b_a \in G$ such that $a \equiv_m b_a$ and $\mathrm{val}^m(a) = \mathrm{val}(b_a)$ (if $a \in mG$, we trivially have $b_a = 0$). We show that a pseudo-Cauchy sequence $(a_i \mod mG)_{i\in I}$ in G/mG can be lifted into a pseudo-Cauchy sequence $(a'_i)_{i\in I}$ in G such that $a_i \equiv_m a'_i$ for every $i \in I$. We may assume that $\mathrm{val}^m(a_i - a_j) < \mathrm{val}^m(a_j - a_k)$ for all $i < j < k$. Let $I = \lambda$ for some limit ordinal λ. By transfinite induction, we can define the following sequence in G:

- For $\alpha = 0$, let a'_0 be any element of G such that $a'_0 \equiv_m a_0$ and $\mathrm{val}(a'_0) = \mathrm{val}^m(a_0)$.

- For any $\alpha < \lambda$, let $b_{\alpha+1}$ denote an element in G such that $b_{\alpha+1} \equiv_m a_\alpha - a_{\alpha+1}$ and $\mathrm{val}(b_{\alpha+1}) = \mathrm{val}^m(a_\alpha - a_{\alpha+1})$. We set $a'_{\alpha+1} = a'_\alpha + b_{\alpha+1}$.

- Let $0 < \alpha < \lambda$ be a limit ordinal, and assume $\mathrm{val}(a'_\beta - a'_\gamma) = \mathrm{val}^m(a_\beta - a_\gamma)$, for eventually all $\beta < \gamma < \alpha$, and $a'_\beta \equiv_m a_\beta$ for all $\beta < \alpha$. Then, $(a'_\beta)_{\beta<\alpha}$ is pseudo-Cauchy in G and, in particular, it admits a pseudo-limit c_α in G. Let b_α be such that $b_\alpha \equiv_m a_\alpha - c_\alpha$ and $\mathrm{val}(b_\alpha) = \mathrm{val}^m(a_\alpha - c_\alpha)$. We set $a'_\alpha = c_\alpha + b_\alpha$.

Note that $a_\alpha \equiv_m a'_\alpha$ for any $\alpha < \lambda$. Let us prove the following claim:

Claim 4.3.7. *Let α be a limit ordinal, and suppose $(a'_\beta)_{\beta<\alpha}$ is pseudo-Cauchy. Then, there exists $\beta_0 < \alpha$ such that $\mathrm{val}(a'_\alpha - a'_\beta) = \mathrm{val}(a'_{\beta+1} - a'_\beta)$ for any $\beta \geqslant \beta_0$. In particular, a'_α is a pseudo-limit of $(a'_\beta)_{\beta<\alpha}$.*

Proof of the claim. Clearly, we have that $\mathrm{val}(a'_\alpha - a'_\beta) = \mathrm{val}(c_\alpha + b_\alpha - a'_\beta) \geqslant \min\{\mathrm{val}(c_\alpha - a'_\beta), \mathrm{val}(b_\alpha)\}$. Then, since c_α is a pseudo-limit of $(a'_\beta)_{\beta<\alpha}$, it is sufficient to show that $\mathrm{val}(b_\alpha) > \mathrm{val}(c_\alpha - a'_\beta) = \mathrm{val}(a'_{\beta+1} - a'_\beta)$ for eventually all $\beta < \alpha$. Let $\beta < \alpha$ be fixed sufficiently large. Then, by definition, $\mathrm{val}(b_\alpha) = \mathrm{val}^m(a_\alpha - c_\alpha)$ and, in particular, it holds that

$$\mathrm{val}^m(a_\alpha - c_\alpha) = \mathrm{val}^m(a_\alpha - a_{\beta+1} + a_{\beta+1} - c_\alpha) \geqslant \min\{\mathrm{val}^m(a_\alpha - a_{\beta+1}), \mathrm{val}^m(a_{\beta+1} - c_\alpha)\}.$$

Since the sequence $(a_\beta \mod mG)_{\beta<\alpha}$ is pseudo-Cauchy in G/mG and $a_\beta \equiv_m a'_\beta$ for any $\beta < \alpha$, it follows that

1. $\mathrm{val}^m(a_\alpha - a_{\beta+1}) > \mathrm{val}^m(a_{\beta+1} - a_\beta) = \mathrm{val}(a'_{\beta+1} - a'_\beta)$, and
2. $\mathrm{val}^m(a_{\beta+1} - c_\alpha) = \mathrm{val}^m(a'_{\beta+1} - c_\alpha) \geqslant \mathrm{val}(a'_{\beta+1} - c_\alpha) > \mathrm{val}(a'_{\beta+1} - a'_\beta)$.

Therefore, $\mathrm{val}(b_\alpha) > \mathrm{val}(a'_{\beta+1} - a'_\beta)$, and $\mathrm{val}(a'_\alpha - a'_\beta) = \mathrm{val}(a'_{\beta+1} - a'_\beta)$. □

Then, by transfinite induction, it follows easily that $\mathrm{val}(a'_\beta - a'_\gamma) = \mathrm{val}^m(a_\beta - a_\gamma)$, for eventually all $\beta < \gamma < \lambda$. In particular, $(a'_\beta)_{\beta<\lambda}$ is pseudo-Cauchy, and it admits a pseudo-limit a' in G. Then, $a' \mod mG$ is a pseudo-limit of the sequence $(a_\alpha \mod mG)_{\alpha<\lambda}$ in G/mG. Indeed, for any $\beta < \alpha < \gamma < \lambda$ large enough, we have that $\mathrm{val}^m(a' - a_\alpha) = \mathrm{val}^m(a' - a'_\alpha)$ and

$$\mathrm{val}^m(a' - a'_\alpha) \geqslant \mathrm{val}(a' - a'_\alpha) = \mathrm{val}(a'_\gamma - a'_\alpha) = \mathrm{val}^m(a_\gamma - a_\alpha) > \mathrm{val}^m(a_\alpha - a_\beta).$$

It follows that $\mathrm{val}^m(a' - a_\beta) = \mathrm{val}^m(a_\alpha - a_\beta)$ for eventually all $\beta < \lambda$. Hence, a' mod mG is a pseudo-limit of $(a_\alpha \mod mG)_{\alpha<\lambda}$, and the statement is proved. □

4.3.2 An instructive example

Before analysing the case of a more general class of ordered abelian groups, it is worth presenting a new interesting example of stably embedded ordered abelian group. Consider the Hahn product $G = \mathbf{H}_{i<\omega}\mathbb{Z}$. We prove that G is stably embedded in every elementary extension. Note that

1. G is maximal, by Proposition 2.2.8;
2. for any $\gamma < \omega$, $G_\gamma = \mathbb{Z}$ and, hence, every rib of G is stably embedded by Theorem 4.2.3;
3. the archimedean spine Γ_G of G is the ordered set $\omega \cup \{\infty\}$ and, hence, it is stably embedded, by Corollary 4.1.4.

First of all, we look for a language in which G admits quantifier elimination. We have already mentioned that in [6] Cluckers and Halupczok have introduced two many-sorted languages, L_{qe} and L_{syn}, in which any ordered abelian group eliminates quantifiers from G. To our purposes, it is convenient to consider the language L_{syn}, instead of L_{qe}. Indeed, although it may seem rather technical, this language has better syntactic properties, since the only symbols in L_{syn} connecting the main sort G and the auxiliary sorts in $\mathcal{A}$ are functions from G to a sort in $\mathcal{A}$.

In 3.1 we have seen that all convex subgroups of $G = \mathbf{H}_{i<\omega}\mathbb{Z}$ are unifomly definable, and that for any $n \in \mathbb{N}$, the n-spine of G is isomorphic as ordered set to the archimedean spine $\Gamma_G = \omega \cup \{\infty\}$. It is well known that the theory of ω eliminates quantifiers in the language $\{0, <, s\}$, where $0, <$ are interpreted in the obvious way and s is interpreted by the successor function $s\colon n \mapsto n + 1$ (see e.g. [12, Theorem 32A]). One can easily see that this is equivalent to the following

Fact 4.3.8. *Let T be the theory of $\omega \cup \{\infty\}$ in the language $\{<, 0, \infty, s\}$, where $0, <, \infty$ are interpreted in the obvious way and s is interpreted by the function $s\colon \omega \cup \{\infty\} \to \omega \cup \{\infty\}$ defined by $s(n) = n + 1$ if $n < \omega$ and $s(\infty) = \infty$. Then T admits elimination of quantifiers.*

Then, from Facts 2.3.17 and 2.3.18 one can deduce a specific many-sorted language $\mathcal{L}_H$ in which G eliminates quantifiers.

Proposition 4.3.9. *Let $\mathcal{L}_H$ be the language consisting of*

- *the main sort G with the symbols $+, -, 0, <, \equiv_m$ for each $m \in \mathbb{N}$, interpreted in the obvious way,*
- *an auxiliary sort Γ for Γ_G with the symbols $<, 0, \infty, s$, interpreted as in the above Fact,*
- *a function symbol val^m for each $m \in \mathbb{N}$, interpreted by the map $\mathrm{val}^m : G \to \Gamma$ defined in (4.2),*
- *a unary predicate $x =^\bullet k_\bullet$ on G for each $k \in \mathbb{Z}\backslash\{0\}$, defined by, for any $a \in G$, $a =^\bullet k_\bullet$ if $a \mod V_{\mathrm{val}(a)}$ is equal to k times the minimal positive element of $G/V_{\mathrm{val}(a)}$,*
- *a unary predicate $x \equiv_m^\bullet k_\bullet$ on G for each $m \in \mathbb{N}, m > 0$ and $k \in \{1, \dots, m-1\}$, defined, for any $a \in G$, by $a \equiv_m^\bullet k_\bullet$ if $a \mod V^m_{\mathrm{val}^m(a)}$ is congruent modulo m to k times the minimal positive element of $G/V^m_{\mathrm{val}^m(a)}$.*

Then the theory of $G = \bigoplus_{i<\omega} \mathbb{Z}$ admits quantifier elimination in $\mathcal{L}_H$.

Proof. We compute the auxiliary sorts $\mathcal{S}_n, \mathcal{T}_n, \mathcal{T}_n^+$ and the projection maps $\mathfrak{s}_n, \mathfrak{t}_n$ of L_{syn} for $G = \bigoplus_{i<\omega} \mathbb{Z}$. We have that, for any $n \in \mathbb{N}, n > 0$, the underlying sets of $\mathcal{S}_n, \mathcal{T}_n, \mathcal{T}_n^+$ is the archimedean spine of G, namely, the set $\omega \cup \{\infty\}$. We have already noticed that, for any $n \in \mathbb{N}, n > 0$, $\mathcal{S}_n$ is in order-reversing bijection with Γ^n, and $\mathfrak{s}_n$ corresponds to $\mathrm{val}^n : G \to \Gamma^n$. On the other hand, for $n \neq 1$, we can actually identify Γ^n with $\Gamma_G = \omega \cup \{\infty\}$. Indeed, let $a \in G\backslash nG$, and $j = \min \mathrm{supp}(a \mod nG)$. Then, $V_a^n = \bigoplus_{i>j} \mathbb{Z}$. Therefore, we have

$$\Gamma^n \cong \omega \cup \{\infty\} \quad \text{and} \quad \mathrm{val}^n(a) = \begin{cases} \min \mathrm{supp}(a \mod nG) & \text{if } a \notin nG \\ \infty & \text{otherwise} \end{cases}$$

for any $n \in \mathbb{N}, n > 1$. In particular, for any $b \in G$, the convex subgroups $\bigcup_{a \in G, b \notin V_a^n} V_a^n$ and $\bigcap_{a \in G, b \in V_a^n} V_a^n$ are the largest convex subgroup not containing b and the smallest convex subgroup containing b, respectively. Then, for any $n \in \mathbb{N}, n > 1$, we can identify

$\mathcal{T}_n$ and $\mathcal{T}_n^+$ with $\omega \cup \{\infty\}$, and $\mathrm{t}_n = \mathrm{val}$, where $\mathrm{val}(a) = \min \mathrm{supp}(a)$ if $a \neq 0$ and $\mathrm{val}(0) = \infty$.

For any $n \in \omega \cup \{\infty\}$, we denote by G_n the convex subgroup $\boldsymbol{H}_{n \leqslant i} \mathbb{Z}$. It remains to prove that the symbols in $L_{\mathrm{syn}} \backslash \mathcal{L}_H$ can be defined without quantifiers over G in $\mathcal{L}_H$.

For any $n \in \omega \cup \{\infty\}$, and $m \in \mathbb{N}$, we have

$$G_n^{[m]} = \bigcap_{i<n} G_i + mG = \begin{cases} G_{n-1} + mG & \text{if } n < \omega \\ mG & \text{if } n = \infty \end{cases} \tag{4.3}$$

Moreover, for any $n \in \omega$,

$$g \in G_n + mG \text{ if and only if } \mathrm{val}^m(g) \geqslant n.$$

It follows that the unary predicates $D_{p^r}^{[p^s]}(x)$ on G can be expressed in $\mathcal{L}_H$ without G-quantifiers. The same holds for the unary predicates defined on $\mathcal{S}_p = \Gamma_G$. Indeed, clearly, the set of values n such that G/G_n is discrete is equal to ω. Moreover, from (4.3), it follows that, for any $n < \omega$,

$$(G_n^{[p^s]} + pG)/(G_n + pG) \cong \mathbb{Z}/p\mathbb{Z},$$

and for $n = \infty$, $(G_n^{[p^s]} + pG)/(G_n + pG) = \{0\}$. Therefore, for each $l, s \in \mathbb{N} \backslash \{0\}$, the set

$$\{n \in \Gamma_G \mid \dim_{\mathbb{F}_p} (G_n^{[p^s]} + pG)/(G_n + pG) = l\}$$

is either ω or the empty set. Similarly, one can deduce that the set

$$\{n \in \Gamma_G \mid \dim_{\mathbb{F}_p} (G_n^{[p^s]} + pG)/(G_n^{[p^{s+1}]} + pG) = l\}$$

can be defined without quantifiers. Then, by Facts 2.3.17 and 4.2.1, the theory of G admits quantifier elimination relatively to Γ in $\mathcal{L}_H$. In particular, since Γ eliminates quantifiers by Fact 4.3.8, every $\mathcal{L}_H$-formula is equivalent modulo the theory of G to a $\mathcal{L}_H$-formula without quantifiers. □

As a first consequence of the previous result, we show that the lexicographic sum $\sum_{i<\omega} \mathbb{Z}$ is not stably embedded in $\boldsymbol{H}_{i<\omega} \mathbb{Z}$. Note that $\sum_{i<\omega} \mathbb{Z}$ is not maximal, since

$\mathbf{H}_{i<\omega}\mathbb{Z}$ is an immediate extension.

Example 4.3.10. Let G' denote the lexicographic sum $\sum_{i<\omega}\mathbb{Z}$. Consider $a = (a_i)_{i<\omega} \in \mathbf{H}_{i<\omega}\mathbb{Z}\setminus\sum_{i<\omega}\mathbb{Z}$ with $a_i = 2$ for every $i < \omega$. Let $C = \{g \in G' \mid g < a\}$. We prove that C is not definable with parameters in G'. Assume, by contradiction, that C is definable in G'. Then, by Proposition 4.3.9, the cut determined by C is one of the form

$$g + \Delta^+, \quad g + \Delta^-$$

for some $g \in G'$ and Δ convex subgroup of G'. One can easily see that the invariance group $H(C, G'\setminus C)$ of the cut $(C, G'\setminus C)$ is the zero-set $\{0\}$. Therefore, the only possibilities for $(C, G'\setminus C)$ are the cuts of the form g^+ and g^-. Thus, we get a contradiction, from the fact that for any $g_1, g_2 \in G'$ with $g_1 < a < g_2$, there are g_1', g_2' in G' such that $g_1 < g_1' < a < g_2' < g_2$. Hence, for any $g \in G'$ the cut determined by C cannot be of the form g^- or of the form g^+. Then we have that C is not definable and G' is not stably embedded in $\mathbf{H}_{i<\omega}\mathbb{Z}$.

Remark 4.3.11. Notice that for an arbitrary model G' of $\mathrm{Th}(\mathbf{H}_{i<\omega}\mathbb{Z})$ in $\mathcal{L}_H$, the auxiliary sort $\Gamma(G')$ is not necessarily interpreted, as ordered set, by the archimedean spine $\Gamma_{G'}$ of G'. However, it still represents the "definable" spine in the sense given by model theorists, that is to say a (multi-sorted) linear order of definable convex subgroups (see Definition 2.3.15 and Equation (2.1)). In particular, the function symbol val may be no longer interpreted by the natural valuation. Clearly, its interpretation is still a valuation of ordered abelian groups, i.e. a map $v\colon G' \to \Gamma(G')$ such that, for all $a, b \in G'$,

(i) $v(a) = \infty \iff a = 0$,

(ii) $v(a - b) \geqslant \min\{v(a), v(b)\}$,

(iii) $v(na) = v(a)$ for every integer $n \neq 0$.

Note that from (i) and (ii), it follows that

$$v(a) \neq v(b) \implies v(a - b) = \min\{v(a), v(b)\}.$$

Then, for every $\gamma \in \Gamma(G')$, we can define the ordered abelian group C_γ/V_γ, where

$$C_\gamma = \{\, g \in G' \mid v(g) \geqslant \gamma \,\}, \quad \text{and } V_\gamma = \{\, g \in G' \mid v(g) > \gamma \,\}.$$

By abuse of notation, we will not distinguish between the function symbol val and its interpretation, even if it is not interpreted as the natural valuation. Moreover, we denote the group C_γ/V_γ by G'_γ and we will call it the rib of G' as well. Notice that, by elementary equivalence, for any $\gamma \in \Gamma(G')$, G'_γ is a model of Presburger arithmetic.

The following result is true for any pseudo-complete valued group (G, v) and, so in particular for $G = H_{i<\omega}\mathbb{Z}$ with respect to the natural valuation.

Proposition 4.3.12. *Let $(G_1, v_1), (G_2, v_2)$ be two valued groups such that (G_2, v_2) is an extension of (G_1, v_1), i.e. such that $G_1 \subseteq G_2$ and $v_2(g) = v_1(g)$ for every $g \in G_1$. If G_1 is pseudo-complete, then, for any $a \in G_2$, the set of values*

$$\{ v_2(a - g) \mid g \in G_1 \}$$

admits a maximal element.

Proof. If $a \in G_1$, it is trivial. Then, suppose $a \in G_2 \backslash G_1$. Assume for a contradiction that $\{ v_2(a - g) \mid g \in G_1 \}$ has no maximal element and consider a sequence $(g_i)_{i\in I}$ of elements in G_1 such that $(v_2(a - g_i))_{i\in I}$ is cofinal and strictly increasing in $\{v_2(a - g) \mid g \in G_1\}$. We have that $(g_i)_{i\in I}$ is pseudo-Cauchy. Indeed, from $v_2(a-g_i) > v_2(a-g_j)$ for all $i < j$, it follows that $v_1(g_j - g_i) = v_2(a - g_j)$ depends only on j and increases with j. Thus, since G_1 is pseudo-complete, $(g_i)_{i\in I}$ admits a limit g' in G_1. Then, we show that $v_2(a - g')$ is maximal in the set $\{ v_2(a - g) \mid g \in G_1 \}$. Indeed, for eventually all $i \in I$, $v_2(a-g') \geqslant \min\{v_2(a-g_i), v_1(g'-g_i)\} = v_2(a-g_i)$. Hence, we get a contradiction. □

Therefore, we can introduce the following definition.

Definition 4.3.13. Let G be a pseudo-complete valued group and a be a new element in a proper extension of valued groups of G. We say that $g' \in G$ is a *best approximation* of a in G, if $v(a - g') = \max\{ v(a - g) \mid g \in G \}$.

Note that a best approximation is not in general unique.

Now we are able to prove

Theorem 4.3.14. *Let G be the Hahn product $H_{i<\omega}\mathbb{Z}$. Then, G is stably embedded.*

Proof. It is enough to prove that all types over G are definable in $\mathcal{L}_H$. First of all, we show that every 1-type over G is definable in $\mathcal{L}_H$. Then, consider a proper elementary extension $(G', \Gamma(G'))$ of $(G, \Gamma(G) = \Gamma_G)$ in $\mathcal{L}_H$, and let $a \in G' \backslash G$. Denote by Θ the set

$$\Theta(a) := \{\mathrm{val}^m(na - g) \mid n \in \mathbb{Z}, g \in G, m \in \mathbb{N}\} \subseteq \Gamma(G').$$

Note that the sort Γ is purely stably embedded in the theory of G in $\mathcal{L}_H$. Then, we have

$$\mathrm{tp}_{\mathrm{Th}(\Gamma_G)}(\Theta/\Gamma_G) \vdash \mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(\Theta/G).$$

Therefore, in order to deduce the definability of $\mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(a/G)$ it suffices to show that the types

$$\mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(a/G \cup \Theta) \text{ and } \mathrm{tp}_{\mathrm{Th}(\Gamma_G)}(\Theta/\Gamma_G)$$

are definable. Clearly, since $\Gamma_G = \omega \cup \{\infty\}$ is stably embedded as ordered set, the type $\mathrm{tp}_{\mathrm{Th}(\Gamma_G)}(\Theta/\Gamma_G)$ is definable. Let us prove the definability of $\mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(a/G \cup \Theta)$.

By Proposition 4.3.9, it is sufficient to show the existence of a defining formula with parameters in $G \cup \Theta$ for the sets defined by the following formulas with variables x, x_1, x_2 from the sort G and variable y from the sort Γ and with parameter a:

a) $na - x > 0$,

b) $na - x \equiv_m 0$,

c) $na - x =^\bullet k_\bullet$,

d) $na - x \equiv_m^\bullet k_\bullet$,

e) $s^{l_1}(\mathrm{val}^{m'}(na - x)) \square s^{l_2}(y)$,

f) $s^{l_1}(\mathrm{val}^{m'_1}(na - x_1)) \square s^{l_2}(\mathrm{val}^{m'_2}(n'a - x_2))$,

where $\square \in \{ >, = \}$, $n, n', k \in \mathbb{Z}\backslash\{0\}$, $m, l_1, l_2 \in \mathbb{N}\backslash\{0\}$ and $m', m'_1, m'_2 \in \mathbb{N}$.

By Theorems 2.2.7 and 4.3.6, (G, val) and $(G/mG, \mathrm{val}^m)$ are pseudo-complete valued groups. Therefore, by Proposition 4.3.12, for every $n \in \mathbb{Z}$, one can find a best approximation g_n of na in G and a best approximation $g_n^m \mod mG$ of $na \mod mG'$ in G/mG, for any $m \in \mathbb{N}, m > 1$.

Claim 4.3.15. *If $\beta_n = \mathrm{val}(na - g_n)$ and $\beta_n^m = \mathrm{val}^m(na - g_n^m)$, then, for any $g \in G$, we have*

$$\mathrm{val}(na - g) = \min\{\mathrm{val}(g_n - g), \beta_n\} \tag{4.4}$$

$$\mathrm{val}^m(na - g) = \min\{\mathrm{val}^m(g_n^m - g), \beta_n^m\} \tag{4.5}$$

Proof of the claim. Clearly, $\mathrm{val}(na-g) = \mathrm{val}(na-g_n+g_n-g) \geqslant \min\{\mathrm{val}(g_n-g), \beta_n\}$. If $\mathrm{val}(g_n - g) \neq \beta_n$ then, trivially, $\mathrm{val}(na - g) = \min\{\mathrm{val}(g_n - g), \beta_n\}$. Hence, suppose $\mathrm{val}(g_n - g) = \beta_n$. Then, $\mathrm{val}(na - g) \geqslant \beta_n$. Moreover, by the maximality of β_n in the set $\{ \mathrm{val}(na - g) \mid g \in G \}$, we have $\beta_n \geqslant \mathrm{val}(na - g)$. Therefore, $\mathrm{val}(na - g) = \beta_n = \mathrm{val}(g_n-g)$. Thus, in both cases we have obtained that $\mathrm{val}(na-g) = \min\{\mathrm{val}(g_n-g), \beta_n\}$.

Similarly, we have $\mathrm{val}^m(na - g) = \min\{\mathrm{val}^m(g_n^m - g), \beta_n^m\}$. □

Note that, since $\Gamma(G') \succ \Gamma_G = \omega \cup \{\infty\}$, we have either $\beta_n \in \omega$ or $\beta_n > \omega$ and similarly for β_n^m. In particular, we have $\beta_n^m > \omega$ for any $m \in \mathbb{N}, m > 1$. Indeed, if $\beta_n^m \in \omega$, then $na - g_n^m \mod V^m_{\beta_n^m} \equiv_m g \mod V^m_{\beta_n^m}$ for some $g \in G$ since

$$\left|\frac{G'/V^m_{\beta_n^m}}{mG'/V^m_{\beta_n^m}}\right| = \left|\frac{G/V^m_{\beta_n^m}}{mG/V^m_{\beta_n^m}}\right| < \infty$$

Therefore, $\mathrm{val}^m(na-(g_n^m+g)) > \mathrm{val}^m(na-g_n^m) = \beta_n^m$ and this contradicts the maximality of β_n^m in $\{ \mathrm{val}^m(na - g) \mid g \in G \}$. Moreover, from (4.4) it follows that if $\beta_n > \omega$, then there exists a unique best approximation of na in G. Indeed, if $g' \neq g_n$ is such that $\mathrm{val}(na - g') = \beta_n$, then $\beta_n \leqslant \mathrm{val}(g_n - g') \in \omega$. Similarly, one can show that for any $g \in G$, $\mathrm{val}^m(na - g) = \beta_n^m$ if and only if $g \equiv_m g_n^m$. Then, there exists a unique best approximation of $na \mod mG'$ in G/mG.

By (4.4) and (4.5) we have a definition with parameters in $G \cup \{\beta_n, \beta_n^m\}_{m>1}$ of the sets

$$E := \{(g, \vartheta) \in G \times \Theta \mid s^{l_1}(\mathrm{val}^{m'}(na - g)) \square s^{l_2}(\vartheta)\}$$

and

$$F := \{(g_0, g_1) \in G^2 \mid s^{l_1}(\mathrm{val}^{m'_1}(na - g_0)) \square s^{l_2}(\mathrm{val}^{m'_2}(n'a - g_1))\}.$$

We show that the sets

$$A := \{g \in G \mid na - g > 0\},$$
$$B := \{g \in G \mid na - g \equiv_m 0\},$$
$$C := \{g \in G \mid na - g \equiv_m^\bullet k_\bullet\},$$
$$D := \{g \in G \mid na - g =^\bullet k_\bullet\}$$

are definable with parameters in $G \cup \Theta$. To this purpose, for any $k \in \mathbb{N}, k \neq 0$ and $\vartheta \in \Theta$, we denote by k_ϑ a representative in G' of k times the minimal positive element of G'/V_ϑ.

$A = \{g \in G \mid na - g > 0\}$. Assume, for instance, that $na - g_n > 0$. We distinguish the cases $\beta_n \in \omega$ and $\beta_n > \omega$.

If $\beta_n > \omega$, then, clearly, $g \in A$ if and only if $g_n - g > 0$.

So, let $\beta_n \in \omega$. Then, we have that $g \in A$ if and only if either $g_n - g > 0$ or $\mathrm{val}(g_n - g) \geqslant \beta_n$. It is sufficient to notice that if g is such that $\mathrm{val}(g_n - g) = \beta_n$, then g is a best approximation of na in G and, in particular, $na - g > 0$. Indeed, if $g' \in G$ is such that $\mathrm{val}(na - g') = \beta_n$ and $na - g' < 0 < na - g_n$, then there exists an integer $k > 0$ such that $\mathrm{val}(na - g' + k_{\beta_n}) > \mathrm{val}(na - g_n)$, where $k_{\beta_n} \in G$. By the maximality of $\mathrm{val}(na - g_n)$ in $\{\ \mathrm{val}(na - g) \mid g \in G\ \}$, this is clearly a contradiction. Therefore, in particular, the sign of $na - g_n$ does not depend on the choice of the best approximation.

If $na - g_n < 0$, similarly one can find a definition of A with parameters in $G \cup \{\beta_n\}$. In any case we obtain that A is definable over $G \cup \{\beta_n\}$.

$B = \{g \in G \mid na - g \equiv_m 0\}$. Clearly, $g \in B$ if and only if $\mathrm{val}^m(na - g) = \infty$. Then, by (4.5), B is not empty if and only if $na \equiv_m g_n^m$. In that case, $g \in B$ if and only if $g \equiv_m g_n^m$. In particular, B is definable over G.

$C = \{g \in G \mid na - g \equiv_m^\bullet k_\bullet\}$. Recall that $na - g \equiv_m^\bullet k_\bullet$ if and only if

$$na - g \mod V^m_{\mathrm{val}^m(na-g)} \equiv_m k_{\mathrm{val}^m(na-g)} \mod V^m_{\mathrm{val}^m(na-g)}.$$

Set $\beta := \mathrm{val}^m(na - g) = \min\{\mathrm{val}^m(g_n^m - g), \beta_n^m\}$. If $\mathrm{val}^m(g_n^m - g) < \beta_n^m$, then, from $V^m_\beta \supset V^m_{\beta_n^m}$ it follows that $na - g_n^m \in V^m_\beta + mG'$. Hence, in that case, $g \in C$ if and only if $g_n^m - g \equiv_m^\bullet k_\bullet$. Otherwise, since $\beta_n^m > \omega$, if $\mathrm{val}^m(g_n^m - g) \geqslant \beta_n^m$, then we have necessarily $\mathrm{val}^m(g_n^m - g) = \infty$. In this case, $g_n^m \equiv_m g$ and $g \in C$ if and only if $na - g_n^m \equiv_m^\bullet k_\bullet$, which depends only a. Therefore, we obtain that C is definable over $G \cup \{\beta_n^m\}$.

$D = \{g \in G \mid na - g =^\bullet k_\bullet\}$ Recall that $na - g =^\bullet k_\bullet$ if and only if

$$na - g \mod V_{\mathrm{val}(na-g)} = k_{\mathrm{val}(na-g)} \mod V_{\mathrm{val}(na-g)}.$$

We use a similar argument to that used for C, considering val and g_n instead of val^m and g_n^m. Set $\beta := \mathrm{val}(na - g) = \min\{\mathrm{val}(g_n - g), \beta_n\}$. If $\mathrm{val}(g_n - g) < \beta_n$, then, by (4.4), $\beta = \mathrm{val}(g_n - g)$, and, since $V_\beta \supset V_{\beta_n}$, $na - g_n \in V_\beta$. Hence, in that case, $g \in D$ if and only if $g_n - g =^\bullet k_\bullet$. Suppose $\mathrm{val}(g - g_n) \geqslant \beta_n$. We need to distinguish between the cases

(i) $na - g_n =^\bullet k_\bullet$

(ii) $na - g_n =^\bullet k'_\bullet$ for some $k' \in \mathbb{Z}\backslash\{0\}, k' \neq k$

(iii) $\neg(na - g_n) =^\bullet k_\bullet$ for any $k \in \mathbb{Z}\backslash\{0\}$.

If (i) holds, then, clearly, $g \in D$ if and only if $\beta_n < \mathrm{val}(g_n - g)$. Suppose (ii) and $k' < k$. Then, $g \in D$ if and only if $\beta_n = \mathrm{val}(g_n - g)$ and $g_n - g' =^\bullet (k - k')_\bullet$. Similarly, if $k < k'$, then $g \in C$ if and only if $\beta_n = \mathrm{val}(g_n - g)$ and $g_n - g' =^\bullet (k' - k)_\bullet$. To conclude, assume (iii). Then, there is no $g \in G$ such that $\mathrm{val}(g - g_n) \geqslant \beta_n$ and $g \in D$. In any case, we obtain that D is definable over $G \cup \{\beta_n\}$.

Therefore, $\mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(a/G \cup \Theta)$ is definable, and so it is $\mathrm{tp}_{\mathrm{Th}_{\mathcal{L}_H}(G)}(a/G)$. Now let $\bar{a} = (a_0, \ldots, a_{k-1})$ be any tuple of new elements in G'. The type of $\mathrm{tp}(\bar{a}/G)$ is determined by the following set of formulas:

$$\bigcup_{z_0,\ldots,z_{k-1}\in\mathbb{Z}} \mathrm{tp}(\sum_{i<k} z_i a_i/G) \cup \mathrm{tp}_{\mathrm{Th}(\Gamma_G)}(\Theta(\bar{a})/\Gamma_G),$$

where $\Theta(\bar{a}) = \bigcup_{z_0,\ldots,z_{k-1}\in\mathbb{Z}} \Theta(\sum_{i<k} z_i a_i)$. Then, from the definability of 1-types over G and the stable embeddedness of Γ_G, it follows that $\mathrm{tp}(\bar{a}/G)$ is definable. This concludes our proof. □

4.3.3 Maximal ordered abelian groups with uniformly definable principal convex subgroups

We have already seen that, by Fact 4.2.1, if G is stably embedded, then all convex subgroups of G are definable, and so they are, in particular, all principal convex subgroups of G. We consider the case where all principal convex subgroups are uniformly definable. (We recall that the principal convex subgroups of an ordered abelian group are in general not even definable.)

Proposition 4.3.16. *Let G be such that all principal convex subgroups of G are uniformly definable. Then, $(\Gamma_G, <)$ is interpretable in G. Moreover, the natural valuation* $\mathrm{val}\colon G \to \Gamma_G$ *is definable, once we add a sort Γ for Γ_G.*

Proof. Assume that the family $\{ \langle a \rangle^{\mathrm{conv}} \mid a \in G \}$ of principal convex subgroups of G is uniformly defined by the formula $\varphi(x, \bar{y})$. Without loss of generality, we may also assume that for all $\bar{b} \in G^{|\bar{y}|}$, $\varphi(x, \bar{b})$ is a convex subgroup. Then, the structure $(\Gamma_G, <)$ is interpreted in G as the quotient $(G/\sim, <)$, where the equivalence relation $\sim$ and the ordering relation $<$ are defined as follows: for all $a, a' \in G$,

$$a \sim a' \iff \forall \bar{y}\ (\varphi(a, \bar{y}) \leftrightarrow \varphi(a', \bar{y}))\,,$$

and

$$[a]_\sim < [a']_\sim \iff \forall \bar{y}\ (\varphi(a', \bar{y}) \to \varphi(a, \bar{y})) \wedge \exists \bar{y}\big(\neg\, (\varphi(a, \bar{y}) \to \varphi(a', \bar{y}))\big).$$

Moreover, notice that the natural valuation is the projection map $G \to G/\sim$ and, clearly, its graph is the set

$$\{ (a, a') \in G \times G \mid a \sim a' \}\,.$$

□

We now introduce the following definition:

Definition 4.3.17. In analogy with the terminology adopted by Gurevich for the convex subgroups of the form V_g^n, for any $g \in G, g \neq 0$ we call the largest convex subgroup not containing g, V_g, *the fundament of g* and the convex subgroups of this form *fundamental*.

Then, it is easy to show that the uniform definability of the principal convex subgroups is equivalent to the uniform definability of the fundamental convex subgroups. Indeed, we have

Proposition 4.3.18. *Let G be any ordered abelian group. The following are equivalent:*

1. *all principal convex subgroups of G are uniformly definable,*
2. *all fundamental convex subgroups of G are uniformly definable,*

Proof. $(1 \Rightarrow 2)$. Let $\varphi(x, \bar{y})$ be a formula defining the family $\{\langle a\rangle^{\text{conv}}\}_{a \in G}$ of principal convex subgroups of G. Moreover, assume that every instance of φ defines a convex subgroup whenever it defines a non-empty set. Clearly, the following formula

$$\psi(z, \bar{y}) := \varphi(z, \bar{y}) \wedge \forall \bar{w}\Big(\varphi(z, \bar{w}) \to \big(\forall x(\varphi(x, \bar{y}) \to \varphi(x, \bar{w}))\big)\Big)$$

defines the set of pairs $(a, \bar{b})$ such that $\varphi(G, \bar{b})$ is the convex subgroup generated by a. Let $a \in G$, $a \neq 0$, and let $\bar{b}$ be a tuple of parameters such that $\varphi(G, \bar{b}) = \langle a\rangle^{\text{conv}}$. Note that V_a is the set $\{\, x \in G \mid \langle x\rangle^{\text{conv}} \subsetneq \langle a\rangle^{\text{conv}} \,\}$, then, it is defined by the formula

$$\forall \bar{y}\big(\psi(x, \bar{y}) \to \eta(\bar{y})\big),$$

where $\eta(\bar{y})$ expresses the property that $\varphi(G, \bar{y})$ is strictly contained in $\varphi(G, \bar{b})$. Similarly, we can deduce $(2 \Rightarrow 1)$ from the fact that, for any $a \in G$, $a \neq 0$, $\langle a\rangle^{\text{conv}} = \{\, x \in G \mid V_x \subseteq V_a \,\}$. ☐

We now look for a class of ordered abelian groups which allow us to consider a language for eliminating quantifiers relatively to a single auxiliary sort Γ. To this end, we introduce the following condition for an ordered abelian group G:

- $\star$ There is $n \in \mathbb{N}, n > 1$ such that, for any $a \in G$, $C_a = \langle a\rangle^{\text{conv}} = \bigcap_{i \in \Gamma^n, a \in V_i^n} V_i^n$.

It easily follows from the uniform definability of the family $\{V_g^n\}_{g \in G}$ (see [6, Lemma 2.1]) that an ordered abelian group G satisfying $\star$ has uniformly definable principal convex subgroups. Moreover, in this particular case, Γ is in order-reversing bijection with the underlying set of the auxiliary sort $\mathcal{T}_n$ and $\mathcal{T}_n^+$ (introduced in Definition 2.3.15), for some $n \in \mathbb{N}, n > 1$. If G is also maximal, we can deduce from Facts 2.3.17 and 2.3.18

the following language in which this kind of maximal ordered abelian groups with interpretable archimedean spine eliminate G-quantifiers (see Section 1.1 for definitions of relative quantifier elimination and their equivalence in the case of closed sorts). Note that the ordered abelian group $\prod_{i<\omega}\mathbb{Z}$ analyzed in Section 4.3.2 is an instance of this kind of groups.

Definition 4.3.19. Assume that G is maximal and satisfies $\star$. Let $\mathcal{L}$ be the language consisting of

- the main sort G, with the symbols $+, -, 0, <, \equiv_m$ for each $m \in \mathbb{N}$, interpreted in the obvious way,
- an auxiliary sort Γ for Γ_G, with a binary relation $<$, interpreted by the ordering relation on Γ_G; a unary predicate $C_\varphi(x)$ for each sentence φ in L_{oag}, defined, for any $\gamma \in \Gamma$, by $C_\varphi(\gamma)$ if and only if $G_\gamma \models \varphi$,
- a function symbol val^m for each integer m, interpreted by the map $\mathrm{val}^m : G \to \Gamma^m$ defined in (4.2),
- a unary predicate $x =^\bullet k_\bullet$ on G for each $k \in \mathbb{Z}\backslash\{0\}$, defined by, for any $a \in G$, $a =^\bullet k_\bullet$ if the quotient $G/V_{\mathrm{val}(a)}$ is discrete and $a \mod V_{\mathrm{val}(a)}$ is k times the minimal positive element of $G/V_{\mathrm{val}(a)}$,
- a unary predicate $x \equiv^\bullet_m k_\bullet$ on G for each $m \in \mathbb{N}, m > 0$ and $k \in \{1, \ldots, m-1\}$, defined by, for any $a \in G$, $a \equiv^\bullet_m k_\bullet$ if $G/V^m_{\mathrm{val}^m(a)}$ is discrete and $a \mod V^m_{\mathrm{val}^m(a)}$ is congruent modulo m to k times the minimal positive element of $G/V^m_{\mathrm{val}^m(a)}$.

Note that, since the divisibility by m of the rib G_γ is definable by a first order formula in L_{oag}, the language $\mathcal{L}$ defined above contains a unary predicate for the m-value set Γ^m, for each $m > 1$.

Theorem 4.3.20. *Assume that G is maximal and satisfies $\star$. Then the theory of G eliminates quantifiers relatively to the sort Γ in the language $\mathcal{L}$.*

Proof. We deduce it from the more general Fact 2.3.17. For the notation involved, see Definition 2.3.16. We have to recover G-quantifier freely the language L_{syn} using the language $\mathcal{L}$.

By hypothesis, there exists $n \in \mathbb{N}, n > 1$ such that $\Gamma \cong \mathcal{T}_n$ and the valuation map val can be identified with $\mathfrak{t}_n$. Moreover, by Proposition 4.3.4, for every $m \in \mathbb{N}$, Γ^m corresponds to $\mathcal{S}_m$, and the m-valuation map val^m corresponds to the map $\mathfrak{s}_m$. Therefore, it remains to show that we can interpret the sorts $\mathcal{T}_m, \mathcal{T}_m^+$ and the projection maps $\mathfrak{t}_m$ for all $m > 0$ without using quantifiers over G.

Let $m > 0$ be fixed. We show that $\mathcal{T}_m$ is interpretable in $\mathcal{L}$ without using quantifiers over G. Take $a \in G$ and let $\beta = \mathfrak{t}_m(a)$. The convex subgroup $G(\beta)$ is the union of all principal convex subgroups $G(\alpha)$ which do not contain a, where $\alpha \in \Gamma^m = \mathcal{S}_m$, i.e. it is the set

$$\{x \in G \mid \exists \delta' \in \mathcal{S}_m \ \mathrm{val}(x) \geqslant \delta' > \mathrm{val}(a)\}.$$

It follows that $\mathcal{T}_m$ is interpretable in $\mathcal{T}_n = \Gamma$ as the quotient $\Gamma/\sim_m$ where $\sim_m$ is the equivalence relation defined by $\gamma \sim_m \gamma'$ if and only if

$$\{\delta \in \Gamma \mid \exists \delta' \in S_m \ \delta \geqslant \delta' > \gamma\} = \{\delta \in \Gamma \mid \exists \delta' \in S_m \ \delta \geqslant \delta' > \gamma'\}.$$

We interpret the order $<$ between $\mathcal{T}_m$ and $\mathcal{T}_k$ as the ordering relation defined by $[\gamma]_{\sim_m} \leqslant [\gamma']_{\sim_k}$ if and only if

$$\{\delta \in \Gamma \mid \exists \delta' \in S_m \ \delta \geqslant \delta' > \gamma\} \subseteq \{\delta \in \Gamma \mid \exists \delta' \in S_k \ \delta \geqslant \delta' > \gamma'\}.$$

The projection map $\mathfrak{t}_m : G \to \mathcal{T}^m$ is interpreted by the map $a \in G \mapsto [\mathrm{val}(a)]_{\sim_m}$.

We show that $\mathcal{T}_m^+$ is interpretable in $\mathcal{L}$ without using quantifiers over G as well. Take $a \in G$ and let $\beta = \mathfrak{t}_m^+(a)$. The convex subgroup $G(\beta)$ is the intersection of all principal convex subgroups $G(\alpha)$ which contain a, where $\alpha \in \Gamma^m = \mathcal{S}_m$, i.e. it is the set

$$\{x \in G \mid \forall \gamma \in \mathcal{S}_m \ \gamma \leqslant \mathrm{val}(a) \to \gamma \leqslant \mathrm{val}(x)\}.$$

It follows that $\mathcal{T}_k^+$ can be interpretable as $\mathcal{T}_k$; the ordering $<$ between $\mathcal{T}_m$ and $\mathcal{T}_k^+$ is interpreted by $\mathfrak{t}_m(a) \leqslant \mathfrak{t}_k^+(b)$ if and only if

$$\{\delta \in \Gamma \mid \exists \delta' \in S_m \ \delta \geqslant \delta' \geqslant \mathrm{val}(a)\} \subseteq \{\delta \in \Gamma \mid \forall \delta' \in S_k \ \delta' \leqslant \mathrm{val}(b) \to \delta' \leqslant \delta\}.$$

Finally, we show that the predicates $D_{p^r}^{[p^s]}(x)$ on G are not required, as for all $\alpha \in \bigcup_n \mathcal{S}_n$, $G(\alpha)^{[m]} = C_\alpha + mG$, for some principal convex subgroup C_α. Recall that for $\alpha \in \bigcup_n \mathcal{S}_n$

$$G(\alpha)^{[m]} := \bigcap_{H \supsetneq G(\alpha), H \text{ convex subgroup of } G} (H + mG).$$

This intersection can be restricted to principal convex subgroups. Firstly, note that if $C_\beta := \{x \in G \mid \text{val}(x) \geqslant \beta\}$ is a principal convex subgroup, then $C_\beta + mG$ is the set $\{x \in G \mid \text{val}^m(x) \geqslant \beta\}$. Indeed, the inclusion $C_\beta + mG \supseteq \{x \mid \text{val}^m(x) \geqslant \beta\}$ follows from Lemma 4.3.2, as for all x there is x' such that $x' \equiv_m x$ and $\text{val}^m(x) = \text{val}(x)$. The other inclusion is clear. Then, we have that

$$\begin{aligned} G(\alpha)^{[m]} &= \bigcap_{\beta<\alpha} C_\beta + mG \\ &= \{x \mid \forall \beta < \alpha \ \text{val}^m(x) \geqslant \beta\} \\ &= \{x \mid \text{val}^m(x) \geqslant \alpha'\} = C_{\alpha'} + mG \end{aligned}$$

where α' is the immediate predecessor of α in Γ^m if it exists, or is equal to α otherwise. It follows that the unary predicates $D_{p^r}^{[p^s]}(x)$ on G can be expressed in $\mathcal{L}$ without G-quantifiers. Moreover, all unary predicate symbols on the sort $\mathcal{A}$ of L_{syn} correspond to a predicate C_φ in $\mathcal{L}$ for some sentence φ in L_{oag}. Notice that since we have only quantified over Γ in order to recover the language L_{syn}, all G-quantifier-free formulas in the language L_{syn} are equivalent to a G-quantifier-free formula in the language $\mathcal{L}$. □

Remark 4.3.21. Note that, since the auxiliary sort $(\Gamma, (C_\varphi)_{\varphi \in L_{\text{oag}}}, <)$ is closed, we obtain that it is a pure coloured chain and is stably embedded in G (see Fact 1.1.22).

Henceforward, let G be any ordered abelian group with uniformly definable principal convex subgroups (not necessarily maximal). One can expand G to the many-sorted structure $\mathcal{G} = ((G, 0, +, -, <), (\Gamma, (C_\varphi)_{\varphi \in L_{\text{oag}}}, <), \text{val})$, where the sort Γ is interpreted by the archimedean spine Γ_G and the function symbol val is interpreted by the natural valuation $\text{val}\colon G \to \Gamma_G$. In particular, by Proposition 4.3.16, every formula in G is equivalent to a formula in $\mathcal{G}$. As already noticed in Remark 4.3.11 for the case of $G = \prod_{i<\omega} \mathbb{Z}$, when we consider an arbitrary model $\mathcal{G}' = (G', \Gamma(G'), \text{val})$ of the theory of $\mathcal{G}$, the sort Γ and the function symbol val can no longer be interpreted as the archimedean spine and the natural valuation of G', respectively. However, since val is always interpreted as a valuation of ordered abelian groups, we adopt the same notation, and G'_γ will denote the ordered abelian group C_γ/V_γ as defined in Remark 4.3.11. Notice, in particular, that, if $\mathcal{G}'$ is an elementary extension of $\mathcal{G}$, then

- $\Gamma(G) \preceq \Gamma(G')$ as coloured chains;
- $G_\gamma \preceq G'_\gamma$ as ordered abelian groups, for any $\gamma \in \Gamma(G)$.

Moreover, by elementary equivalence, G'_γ is a regular ordered abelian group for any $\gamma \in \Gamma(G')$. Therefore, we may say that $\Gamma(G')$ and G'_γ, for any $\gamma \in \Gamma(G')$, are the *regular spine* and a *regular rib* of G', respectively.

Let G' be any maximal ordered abelian group with uniformly definable principal convex subgroups. We prove that the non-existence of immediate extensions is actually a necessary condition for any model of the theory of G' in order to be stably embedded.

Proposition 4.3.22. *Let G' be any maximal ordered abelian group with uniformly definable principal convex subgroups, and let G be any model of* $\mathrm{Th}(G')$. *If G is stably embedded, then G is maximal.*

Proof. We expand G to the many-sorted structure $\mathcal{G} = (G, \Gamma, \mathrm{val})$. First of all, note that by Fact 2.2.4, all convex subgroups of G are definable. In particular, for any $\gamma \in \Gamma(G)$, G_γ is archimedean and the interpretation of val coincides with the natural valuation of G. It follows that it suffices to show that (G, val) is pseudo-complete. By contradiction, suppose that G is not pseudo-complete, and so there exists a pseudo-Cauchy sequence $(g_i)_{i\in I}$ of elements of G with no pseudo-limits in G. Let a be a pseudo-limit of $(g_i)_{i\in I}$ in an elementary extension $\hat{G}$ of G. Then, for any $g \in G$, $\mathrm{val}(a-g) \in \Gamma(G)$. Indeed, since by assumption $(g_i)_{i\in I}$ does not have a pseudo-limit in G, there exists $i \in I$ such that $\mathrm{val}(a-g) < \mathrm{val}(a-g_i)$. In particular, we have that $\mathrm{val}(a-g) = \mathrm{val}(g-g_i) \in \Gamma(G)$. Therefore, for any $g \in G$, we can consider the ball $B_g := \{\, h \in G \mid \mathrm{val}(a-h) \geqslant \mathrm{val}(a-g) \,\}$. We show that $\bigcap_{g\in G} B_g \neq \varnothing$.
Since G is stably embedded in $\hat{G}$, the subset

$$\{\, (h,g) \in G^2 \mid v(a-h) \geqslant v(a-g) \,\}$$

of G^2 may be defined with a formula $\varphi(x, y, \bar{c})$, where $\varphi(x, y, \bar{z})$ is a formula without parameters and $\bar{c}$ is a tuple from G. Moreover, we may assume that, for any tuple of parameters $\bar{c}$, the non-empty instances of $\varphi(x, g, \bar{c})$, with $g \in G$, define a nested family of balls. Notice that the property of $\varphi(x, y, \bar{c})$ to define a nested family of balls is first-order expressible. By hypothesis G' is maximal, then, by Theorem 2.2.11,

$$G' \models \forall \bar{z} \exists x \forall y (\exists w \varphi(w, y, z) \rightarrow \varphi(x, y, \bar{z})),$$

and so also

$$G \models \forall \bar{z} \exists x \forall y (\exists w \varphi(w, y, z) \rightarrow \varphi(x, y, \bar{z})),$$

since $G \equiv G'$. In particular, we have that $\bigcap_{g \in G} B_g \neq \emptyset$. Let $b \in \bigcap_{g \in G} B_g$. Clearly, $\mathrm{val}(a - b) > \mathrm{val}(a - g_i)$ for eventually all $i \in I$. Hence, $\mathrm{val}(b - g_i) = \mathrm{val}(a - g_i)$, and b is a pseudo-limit of $(g_i)_{i \in I}$ in G. We get a contradiction. □

Now we are able to prove the main theorem of this chapter.

Theorem 4.3.23. *Let $\mathcal{G} = (G, \Gamma(G), \mathrm{val})$ be any model of the theory of a maximal ordered abelian group G' such that*

$$\star \textit{ there is } n \in \mathbb{N}, n > 1 \textit{ such that, for any } a \in G', C_a = \langle a \rangle^{conv} = \bigcap_{i \in \Gamma^n, a \in V_i^n} V_i^n.$$

Then, G is stably embedded if and only if it is maximal, its regular ribs $(G_\gamma, +, 0, <)$ are stably embedded for all $\gamma \in \Gamma(G)$ and its regular spine $(\Gamma(G), (C_\varphi)_{\varphi \in L_{oag}}, <)$ is stably embedded.

Proof. $(\Rightarrow)$ By Proposition 4.3.22, G is maximal. We show that G_γ is stably embedded, for any $\gamma \in \Gamma(G)$. Let γ be any value in $\Gamma(G)$ and consider a proper elementary extension $\hat{G}$ of G_γ in L_{oag}. We need to show that G_γ is stably embedded in $\hat{G}$. Note that G_γ is a pure ordered abelian group. Moreover, since G_γ is regular, it is sufficient to consider just 1-types (see Section 4.2.1). Clearly, there exists an elementary extension $\mathcal{G}' = (G', \Gamma(G'), \mathrm{val})$ of $\mathcal{G} = (G, \Gamma(G), \mathrm{val})$ such that $G'_\gamma \geq \hat{G} > G_\gamma$. It is enough to show then that G_γ is stably embedded in G'_γ. Let $a \in G' \backslash G$ with $\mathrm{val}(a) = \gamma$. By hypothesis, G is stably embedded in G'. Then, the type of a over G is definable and, thus, so it is the type of $a \mod V_\gamma$ over G_γ. Therefore, G_γ is stably embedded in G'_γ as ordered abelian groups. It follows that G_γ is stably embedded for any $\gamma \in \Gamma(G)$.

Similarly, we show that $\Gamma(G)$ is stably embedded in every elementary extension. Let $\hat{\Gamma}$ be a proper elementary extension of $\Gamma(G)$, and an elementary extension $\mathcal{G}' =$

$(G', \Gamma(G'), \mathrm{val})$ of $\mathcal{G} = (G, \Gamma(G), \mathrm{val})$ such that $\Gamma(G') \geq \hat{\Gamma} > \Gamma(G)$. Let $\gamma \in \Gamma(G')\backslash\Gamma(G)$ and consider an element $a \in G'$ of value γ. Since the type of a over G is definable and $\Gamma(G)$ is purely stably embedded, it follows that the type of $\gamma = \mathrm{val}(a)$ over $\Gamma(G)$ is definable. Therefore, every 1-type over $\Gamma(G)$ is definable. Hence, by Corollary 4.1.4, $\Gamma(G)$ is stably embedded.
$(\Leftarrow)$ First of all, note that, for every $\gamma \in \Gamma(G)$, G_γ is archimedean by Fact 4.2.1. It follows that the interpretation of val coincides with the natural valuation of G, and, in particular, all principal convex subgroups of G are uniformly definable.

We show that every 1-type over G is definable in $\mathcal{L}$. Consider an elementary extension $\mathcal{G}' = (G', \Gamma(G'), \mathrm{val}^m)$ of $\mathcal{G} = (G, \Gamma(G), \mathrm{val}^m)$ in $\mathcal{L}$, and let $a \in G'\backslash G$. In particular, by the maximality of G, G' is not an immediate extension of G. As in the proof of Theorem 4.3.14, denote by Θ the set $\Theta(a) := \{\mathrm{val}^n(a-g) \mid n \in \mathbb{N}, g \in G\}$. We want to show that $\mathrm{tp}(a/G \cup \Theta)$ is definable. Then, since $\Gamma(G)$ is purely stably embedded and $\mathrm{tp}(\bar{\beta}/\Gamma(G))$ is definable for all $\bar{\beta} \in \Theta^{|\bar{\beta}|}$ by hypothesis, we deduce that the type $\mathrm{tp}(a/G)$ of a over G is definable. By Theorem 4.3.20, a formula $\varphi(x, \bar{g}, \bar{\vartheta})$ in the language $\mathcal{L}$ with parameters in $G \cup \Theta$ is a finite Boolean combination of formulas of the form:

a) $nx - g > 0$,

b) $nx - g \equiv_m 0$,

c) $nx - g \equiv_m^\bullet k_\bullet$,

d) $nx - g =^\bullet k_\bullet$,

e) $\psi(\mathrm{val}^{m_0}(n_0x - g_0), \ldots, \mathrm{val}^{m_{h-1}}(n_{h-1}x - g_{h-1}), \vartheta_0, \ldots, \vartheta_{h'-1})$,

where ψ is formula in $(\Gamma, (C_\varphi)_{\varphi \in L_{\mathrm{oag}}}, <)$, $n, n_0, \ldots, n_{h-1}, k \in \mathbb{Z}\backslash\{0\}$, $m_0, \ldots, m_{h-1} \in \mathbb{N}$, $m \in \mathbb{N}\backslash\{0\}$ and $g, g_0, \ldots, g_{h-1} \in G$, $\vartheta_0, \ldots, \vartheta_{h'-1} \in \Theta$. Since (G, val) is pseudo-complete, as in Claim 4.3.15, from Proposition 4.3.12, it follows that for any $n \in \mathbb{Z}$ and $g \in G$ we have that

$$\mathrm{val}(na - g) = \min\{\mathrm{val}(a_n^0 - g), \beta_n^0\} \tag{4.6}$$

$$\mathrm{val}^m(na - g) = \min\{\mathrm{val}^m(a_n^m - g), \beta_n^m\}. \tag{4.7}$$

where a_n^0, a_n^m are a best approximation of na in G and a representative of a best approximation of $na \mod mG$ in G/mG, respectively, and $\beta_n^0 := \mathrm{val}(na - a_n^0)$ and $\beta_n^m :=$

$\mathrm{val}^m(na-a_n^m)$. Therefore, by (4.6) and (4.7) we clearly have a definition with parameters in $G \cup \{\beta_n^0, \beta_n^m\}_{n,m>1}$ of the set

$$E := \{(\bar{g}, \bar{\vartheta}) \in G^h \times \Theta^{h'} \mid \psi(\mathrm{val}^{m_0}(n_0 a - g_0), \ldots, \mathrm{val}^{m_{h_1}}(n_{h-1}a - g_{h-1}), \vartheta_0, \ldots, \vartheta_{h'-1})\}.$$

Claim 4.3.24. *The set*

$$A := \{g \in G \mid na - g > 0\}$$

is definable with parameters in $G \cup \Theta$.

Proof of the claim. Let $g \in G$, and set $\beta := \mathrm{val}(na-g) = \min\{\mathrm{val}(a_n^0 - g), \beta_n^0\}$. Clearly, the sign of $na - g$ is determined by the sign of $na - g \mod V_\beta$. If $\mathrm{val}(a_n^0 - g) < \beta_n^0$, then, in particular, $na - g \mod V_\beta = a_n^0 - g \mod V_{\mathrm{val}(a_n^0 - g)}$. Therefore, in that case, $g \in A$ if and only if $a_n^0 - g > 0$. Then, suppose $\mathrm{val}(a_n^0 - g) \geqslant \beta_n^0$.
If $\beta_n^0 < \mathrm{val}(a_n^0 - g)$, then, the sign of $na - g$ is determined by the sign of $na - a_n^0$. Otherwise, suppose $\mathrm{val}(a_n^0 - g) = \beta_n^0$ for some $g \in G$. Then, we have $\beta = \beta_n^0 \in \Gamma(G)$ and $na - g \mod V_\beta \in G'_\beta$. By hypothesis G_β is stably embedded in G'_β and, in particular, the type of $na - g \mod V_\beta$ over G_β is definable. Hence, there exists a formula with parameters in G_β, and so in G, defining the set of $g \in G$ such that $na - g \mod V_\beta > 0$. Therefore, we obtain that one can find $\psi_1(x), \psi_2(x), \psi_3(x)$ with parameters in G equivalent to the formula $na - g > 0$ for each of the occurrences $\mathrm{val}(a_n^0 - g) < \beta_n^0$, $\mathrm{val}(a_n^0 - g) > \beta_n^0$ and $\mathrm{val}(a_n^0 - g) = \beta_n^0$. Thus A is definable with parameters in $G \cup \{\beta_n^0\}$ by the formula

$$\big(\mathrm{val}(a_n^0 - x) < \beta_n^0 \wedge \psi_1(x)\big) \vee \big(\mathrm{val}(a_n^0 - x) > \beta_n^0 \wedge \psi_2(x)\big) \vee \big(\mathrm{val}(a_n^0 - x) = \beta_n^0 \wedge \psi_3(x)\big).$$

□

Moreover, we observe that the set

$$B := \{g \in G \mid na - g \equiv_m 0\}$$

is definable with parameters in G since $g \in B$ if and only if $a_n^m - g \equiv_m 0$ and $\beta_n^m = \infty$.

Claim 4.3.25. *The set*

$$C := \{g \in G \mid na - g \equiv_m^\bullet k_\bullet\}$$

is definable with parameters in $G \cup \Theta$.

Proof of the claim. Let $g \in G$, and set $\beta := \mathrm{val}^m(na - g) = \min\{\mathrm{val}^m(a_n^m - g), \beta_n^m\}$. Similarly to the proof of the Claim 4.3.24, we show that one can find $\psi_1(x), \psi_2(x), \psi_3(x)$ with parameters in G equivalent to the formula $na - g \equiv_m^\bullet k_\bullet$ for each of the occurrences $\mathrm{val}^m(a_n^m - g) < \beta_n^m$, $\mathrm{val}^m(a_n^m - g) > \beta_n^m$ and $\mathrm{val}^m(a_n^m - g) = \beta_n^m$. Then, C will be definable with parameters in $G \cup \{\beta_n^m\}$ by the following disjunction

$$\big(\mathrm{val}^m(a_n^m - x) < \beta_n^m \wedge \psi_1(x)\big) \vee \big(\mathrm{val}^m(a_n^m - x) > \beta_n^m \wedge \psi_2(x)\big) \vee \big(\mathrm{val}^m(a_n^m - x) = \beta_n^m \wedge \psi_3(x)\big).$$

If $\mathrm{val}^m(a_n^m - g) < \beta_n^m$, then we simply need to observe that $na - g \mod V_\beta^m + mG' = a_n^m - g \mod V^m_{\mathrm{val}^m(a_n^m - g)} + mG'$. In particular, $na - g \equiv_m^\bullet k_\bullet$ if and only if $a_n^m - g \equiv_m^\bullet k_\bullet$, so take $\psi_1(x) := a_n^m - g \equiv_m^\bullet k_\bullet$.
If $\beta_n^m < \mathrm{val}^m(a_n^m - g)$, then we have $na - g \mod V_\beta^m + mG' = na - a_n^m \mod V^m_{\beta_n^m} + mG'$. In particular, $na - g \equiv_\bullet^m k_\bullet$ if and only if $na - a_n^m \equiv_\bullet^m k_\bullet$, which depends only on a. Then, if $na - a_n^m \equiv_m^\bullet k_\bullet$, take $\psi_2(x) := x = x$; otherwise, take $\psi_2(x) := \neg(x = x)$.
Suppose either $\beta_n^m \notin \Gamma(G)$ or $G_{\beta_n^m}$ is not discrete. Then, trivially, there is no $g \in G$ such that $\mathrm{val}^m(a_n^m - g) = \beta_n^m$ and $g \in C$. So, in both cases, take $\psi_3(x) := \neg(x = x)$. Therefore, suppose that $\mathrm{val}^m(a_n^m - g) = \beta_n^m$ and $G_{\beta_n^m}$ is discrete. By (the proof of) Lemma 4.3.2, there exists $a' \in G'$ such that $a' - g \equiv_m na - g$ and $\mathrm{val}^m(a' - g) = \mathrm{val}(a' - g)$. In particular, we have that $\mathrm{val}^m(a' - g) = \mathrm{val}^m(na - g) = \beta$ and, $a' - g \mod V_\beta + mG' = a' - g \mod V_\beta^m + mG' = na - g \mod V_\beta^m + mG'$. Then, $na - g \equiv_\bullet^m k_\bullet$ if and only if $a' - g \equiv_\bullet^m k_\bullet$. Since $a' - g \mod V_\beta \in G'_\beta$ and G_β is stably embedded in G'_β, there is a formula $\psi_3(x)$ with parameters in G_β, and so in G, defining the set of g's such that $a' - g \mod V_\beta^m \equiv_m k_\beta \mod V_\beta^m$, where k_β denotes a representative in G' of k times the minimal positive element of G'/V_β^m. Thus C is definable with parameters in $G \cup \{\beta_n^m\}$. □

Claim 4.3.26. *The set*

$$D := \{g \in G \mid na - g =^\bullet k_\bullet\}$$

is definable with parameters in $G \cup \Theta$.

Proof of the claim. Consider $g \in G$, and set $\beta := \mathrm{val}(na - g) = \min\{\mathrm{val}(a_n^0 - g), \beta_n^0\}$. Similarly to the proof of Claim 4.3.24, we show that one can find $\psi_1(x), \psi_2(x), \psi_3(x)$ with parameters in G equivalent to the formula $na - g =^\bullet k_\bullet$ for each of the occurrences

$\mathrm{val}(a^0_n - g) < \beta^0_n$, $\mathrm{val}(a^0_n - g) > \beta^0_n$ and $\mathrm{val}(a^0_n - g) = \beta^0_n$. Then, D will be definable with parameters in $G \cup \{\beta^0_n\}$ by the following disjunction

$$\big(\mathrm{val}(a^0_n - x) < \beta^0_n \wedge \psi_1(x)\big) \vee \big(\mathrm{val}(a^0_n - x) > \beta^0_n \wedge \psi_2(x)\big) \vee \big(\mathrm{val}(a^0_n - x) = \beta^0_n \wedge \psi_3(x)\big).$$

If $\mathrm{val}(a^0_n - g) < \beta^0_n$, then we simply observe that $na - g \mod V_\beta = a^0_n - g \mod V_{\mathrm{val}(a^0_n - g)}$. In particular, $na - g =^\bullet k_\bullet$ if and only if $a^0_n - g =^\bullet k_\bullet$, so take $\psi_1(x) := a^0_n - g =^\bullet k_\bullet$. If $\beta^0_n < \mathrm{val}(a^0_n - g)$, then we have $na - g \mod V_\beta = na - a^0_n \mod V_{\beta^0_n}$. In particular, $na - g =^\bullet k_\bullet$ if and only if $na - a^0_n =^\bullet k_\bullet$, which depends only on a. Then, if $na - a^0_n =^\bullet k_\bullet$, take $\psi_2(x) := x = x$; otherwise, take $\psi_2(x) := \neg(x = x)$.
Suppose either $\beta^0_n \notin \Gamma(G)$ or $G_{\beta^0_n}$ is not discrete. Then, trivially, there is no $g \in G$ such that $\mathrm{val}(a^0_n - g) = \beta^0_n$ and $g \in D$. So, in both cases, take $\psi_3(x) := \neg(x = x)$. Therefore, suppose that $\mathrm{val}(a^0_n - g) = \beta^0_n$ and $G_{\beta^0_n}$ is discrete. Since G_β is stably embedded in G'_β, there exists a formula $\psi_3(x)$ with parameters in G_β, and so in G, defining the set of g's such that $na - g \mod V_\beta = k_\beta \mod V_\beta$, where k_β denotes a representative in G' of k times the minimal positive element of G'_β. Therefore, we obtain that D is definable with parameters in $G \cup \{\beta^0_n\}$. □

Now let $\bar{a} = (a_0, \ldots, a_{k-1})$ be any tuple of new elements in G'. The type of $\mathrm{tp}(\bar{a}/G)$ is determined by the following set of formulas:

$$\bigcup_{z_0,\ldots,z_{k-1}\in\mathbb{Z}} \mathrm{tp}(\sum_{i<k} z_i a_i/G) \cup \mathrm{tp}_{\mathrm{Th}(\Gamma_G)}(\Theta(\bar{a})/\Gamma_G),$$

where $\Theta(\bar{a}) = \bigcup_{z_0,\ldots,z_{k-1}\in\mathbb{Z}} \Theta(\sum_{i<k} z_i a_i)$. Since each of these types are definable over G, so is $\mathrm{tp}(\bar{a}/G)$. This concludes our proof. □

We have studied stably embedded regular ordered abelian groups and stably embedded coloured chains in Sections 4.2.1 and 4.1, respectively. Therefore, from Theorems 4.2.3 and 4.2.4 and Corollary 4.1.4, it follows that

Corollary 4.3.27. *Let $\mathcal{G} = (G, \Gamma(G), \mathrm{val})$ be any model of the theory of a maximal ordered abelian group G' satisfying*

$$\star \text{ there is } n \in \mathbb{N}, n > 1 \text{ such that, for any } a \in G', C_a = \langle a \rangle^{conv} = \bigcap_{i \in \Gamma^n, a \in V^n_i} V^n_i.$$

Then, G is stably embedded if and only if

1. *G is maximal;*

2. *for every $\gamma \in \Gamma(G)$, either $G_\gamma \cong \mathbb{Z}$, or G_γ is densely ordered and $div(G_\gamma) \cong \mathbb{R}$;*

3. *all cuts of $(\Gamma(G), (C_\varphi)_{\varphi \in L_{oag}}, <)$ are definable.*

Remark 4.3.28. Let G be an ordered abelian group with uniformly definable principal convex subgroups. By Proposition 2.1.11, every cut $\Delta = (\Delta_L, \Delta_R)$ in the archimedean spine Γ_G corresponds to a convex subgroup of G by the bijection

$$F\colon \Delta_R \mapsto C_{\Delta_R} = \{ g \in G \mid \mathrm{val}(g) \in \Delta_R \} .$$

By Proposition 4.3.16 it follows that the definability of all cuts in $(\Gamma_G, (C_\varphi)_{\varphi \in L_{\mathrm{oag}}}, <)$ is equivalent to the definability of all convex subgroups of G.

Remark 4.3.29. We can deduce that the ordered abelian group $G := \mathop{H}_{i<\omega} \mathbb{Z}$ analyzed in Section 4.3.2 is the unique model of its own theory to be stably embedded. Indeed, one can easily see that the lexicographic sum $\sum_{i<\omega} \mathbb{Z}$ is a prime model of $T = \mathrm{Th}(G)$ and, hence, any maximal model of T contains G. It is clear that any proper extension of G is not stably embedded, since, by maximality of G, it is not immediate and no proper extension of a rib $(\mathbb{Z}, +, 0, <)$ nor of the archimedean spine $(\omega, <)$ is stably embedded.

The next step in the study of stable embeddedness for ordered abelian groups could be looking for a similar characterization for stably embedded pairs (G, G') of ordered abelian groups. More precisely, it is natural to ask whether the following is true.

Conjecture 4.3.30. *Consider the complete theory T of a maximal ordered abelian group G such that*

$$\star \text{ there is } n \in \mathbb{N}, n > 1 \text{ such that, for any } a \in G, C_a = \langle a \rangle^{conv} = \bigcap_{i \in \Gamma^n, a \in V_i^n} V_i^n .$$

Consider an extension of models $G_1 \preceq G_2$ of T in $\mathcal{L}$. Then, G_1 is stably embedded in G_2 if and only if the following occurs:

- *G_1 is maximal in G_2: there is no intermediate immediate extensions of G_1 in G_2;*

- *for every $\gamma \in \Gamma(G_1)$, $(G_1)_\gamma$ is stably embedded in $(G_2)_\gamma$;*
- $(\Gamma(G_1), (C_\varphi)_{\varphi \in L_{oag}}, <)$ *is stably embedded in* $(\Gamma(G_2), (C_\varphi)_{\varphi \in L_{oag}}, <)$.

As an application of Theorem 4.3.23, we conclude presenting a new example of stably embedded ordered abelian group:

Example 4.3.31. The ordered abelian group $G = H_{i \in \mathbb{R}} G_i$ where

$$G_i = \begin{cases} \mathbb{Z} & \text{if } i \text{ is rational} \\ \mathbb{R} & \text{if } i \text{ is irrational} \end{cases}$$

is stably embedded in every elementary extension. Indeed, G is maximal by Fact 2.2.8. Moreover, one can easily see that, as ordered sets, $\mathcal{S}_n \cong \mathbb{Q} \cup \{\infty\}$ and $\mathcal{T}_n \cong \mathcal{T}_n^+ \cong \mathbb{R} \cup \{\infty\}$ (see for example [6, Section 4.2]). Therefore, the archimedean spine Γ_G is interpretable in the language of ordered abelian groups, and the natural valuation is given by t_n. Then, the auxiliary sort $(\Gamma, (C_\varphi)_{\varphi \in L_{\mathrm{oag}}}, <)$ is interpreted by the coloured chain $(\mathbb{R}, \mathbb{Q}, <)$. Moreover, the ribs of G are equal to either $\mathbb{Z}$ or $\mathbb{R}$. Therefore, by Corollary 4.3.27 G is stably embedded.

Bibliography

[1] Matthias Aschenbrenner, Artem Chernikov Allen Gehret, and Martin Ziegler. *Distality in Valued Fields and Related Structures*. arXiv Prepint: 2008.09889. 2020.

[2] Oleg Belegradek. “Poly-regular ordered abelian groups”. In: *Logic and Algebra*. Ed. by Yi Zhang. Contemp. Math. 302. Providence, RI: AMS, 2003.

[3] Oleg Belegradek, Ya’Acov Peterzil, and Frank Wagner. “Quasi-O-Minimal Structures”. In: *The Journal of Symbolic Logic* 65.3 (2000), pp. 1115–1132.

[4] Oleg Belegradek, Viktor Verbovskiy, and Frank Wagner. “Coset-minimal groups”. In: *Annals of Pure and Applied Logic* 121 (2003), pp. 113–143.

[5] Raf Cluckers. “Presburger Sets and P-Minimal Fields”. In: *The Journal of Symbolic Logic* 68.1 (2003), pp. 153–162.

[6] Raf Cluckers and Immanuel Halupczok. “Quantifier elimination in ordered abelian groups”. In: *Confluentes Mathematici* 3.4 (2011), pp. 587–615.

[7] Pablo Cubides Kovacsics and Françoise Delon. “Definable types in algebraically closed valued fields”. In: *MLQ. Mathematical Logic Quarterly* 62.1-2 (2016), pp. 35–45.

[8] Pablo Cubides Kovacsics, Martin Hils, and Jinhe Ye. *Beautiful pairs*. arXiv Prepint: 2112.00651. 2021.

[9] Pablo Cubides Kovacsics and Eva Leenknegt. “Integration and cell decomposition in P-minimal structures”. In: *The Journal of Symbolic Logic* 81.3 (2016), pp. 1124–1141.

[10] Alfred Dolich and John Goodrick. "A Characterization of Strongly Dependent Ordered Abelian Groups". In: *Revista Colombiana de Matemáticas* 52 (2017), pp. 139–159.

[11] Andrzej Ehrenfeucht. "Decidability of the theory of linear ordering relation". In: *Notices of the American Mathematical Society* 6 (1959), pp. 268–269.

[12] Herbert B. Enderton. *A mathematical introduction to logic*. Second Ed. Harcourt/Academic Press, Burlington, MA, 2001.

[13] Rafel Farré. *Strong ordered Abelian groups and dp-rank*. arXiv:1706.05471. 2017.

[14] László Fuchs. *Partially ordered algebraic systems*. Pergamon Press, Oxford - London - New York - Paris; Addison-Wesley Publishing Co., 1963.

[15] K. A. H. Gravett. "Valued linear spaces". In: *The Quarterly Journal of Mathematics* 6.1 (1955), pp. 309–315.

[16] Yuri Gurevich. "Elementary properties of ordered abelian groups". In: *Transactions of the American Mathematical Society* 261 (1965), pp. 165–192.

[17] Yuri Gurevich and Peter H. Schmitt. "The theory of ordered abelian groups does not have the independence property". In: *Transactions of the American Mathematical Society* 284.1 (1984), pp. 171–182.

[18] Yatir Halevi and Assaf Hasson. "Strongly Dependent Ordered Abelian Groups and Henselian Fields". In: *Israel Journal of Mathematics* 232 (2019), pp. 719–758.

[19] Martin Hils, Martina Liccardo, and Pierre Touchard. *Stably Embedded Pairs of Ordered Abelian Groups*. In preparation.

[20] Wilfrid Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.

[21] Charles Holland. "Extensions of Ordered Groups and Sequence Completion". In: *Transactions of the American Mathematical Society* 107.1 (1963), pp. 71–82.

[22] Franziska Jahnke, Pierre Simon, and Erik Walsberg. "Dp-minimal valued fields". In: *The Journal of Symbolic Logic* 82.1 (2017), pp. 151–165.

[23] Will Johnson. "Fun with fields". PhD thesis. UC Berkeley, 2016.

[24] Irving Kaplansky. “Maximal fields with valuations”. In: *Duke Mathematical Journal* 9.2 (1942), pp. 303–321.

[25] Salma Kuhlmann. *Ordered Exponential Fields*. AMS Publications, 2000.

[26] Daniel Lascar and Bruno Poizat. “An Introduction to Forking”. In: *The Journal of Symbolic Logic* 44.3 (1979), pp. 330–350.

[27] Martina Liccardo. *Ordered abelian groups that do not have elimination of imaginaries*. arXiv Prepint: 2105.14646. 2021.

[28] Annalisa Marcja and Carlo Toffalori. *A Guide to Classical and Modern Model Theory*. Trends in Logic. Springer, 2003.

[29] David Marker and Charles I. Steinhorn. “Definable types in O-minimal theories”. In: *The Journal of Symbolic Logic* 59.1 (1994), pp. 185–198.

[30] Lawrence Narici, Edward Beckenstein, and George Bachman. *Functional Analysis and Valuation Theory*. Marcel Dekker Inc., 1971.

[31] Anand Pillay. “Some Remarks on Definable Equivalence Relations in O-Minimal Structures”. In: *The Journal of Symbolic Logic* 51.3 (1986), pp. 709–714.

[32] Anand Pillay and Charles I. Steinhorn. “Definable sets in ordered structures I”. In: *Transactions of the American Mathematical Society* 295 (1986), pp. 565–592.

[33] Françoise Point. “Definability of types and VC density in differential topological fields”. In: *Archive for Mathematical Logic* 57 (2018), pp. 809–828.

[34] Françoise Point and Frank O. Wagner. “Essentially Periodic Ordered Groups”. In: *Annals of Pure and Applied Logic* 105.1-3 (2000), pp. 261–291.

[35] Bruno Poizat. *A course in model theory: An introduction to contemporary mathematical logic*. Universitext. Springer-Verlag, New York, 2000.

[36] Bruno Poizat. “Une Théorie de Galois Imaginaire”. In: *The Journal of Symbolic Logic* 48 (1983), pp. 1151–1170.

[37] Mojżesz Presburger and Dale Jabcquette. “On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation”. In: *History and Philosophy of Logic* 12.2 (1991), pp. 225–233.

[38] Silvain Rideau. “Some properties of analytic difference valued fields”. In: *Journal of the Institute of Mathematics of Jussieu* 16.3 (2017), pp. 447–499.

[39] Abraham Robinson. *Complete theories*. North-Holland, Amsterdam, 1956.

[40] Abraham Robinson and Elias Zakon. "Elementary properties of ordered abelian groups". In: *Transactions of the American Mathematical Society* 96 (1960), pp. 222–236.

[41] Peter H. Schmitt. *Model theory of ordered abelian groups*. Habilitationsschrift. 1982.

[42] Philip Scowcroft. "A note on definable Skolem functions". In: *The Journal of Symbolic Logic* 53.3 (1988), pp. 905–911.

[43] Saharon Shelah. *Classification theory and the number of nonisomorphic models*. Second Ed. Vol. 92. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1990.

[44] Hiroshi Tanaka. "Direct products of ordered abelian groups". In: *Scientiae Mathematicae Japonicae Online* (2006), pp. 149–155.

[45] Katrin Tent and Martin Ziegler. *A course in model theory*. Vol. 40. Lecture Notes in Logic. Association for Symbolic Logic; Cambridge University Press, Cambridge, 2012.

[46] Pierre Touchard. *Stably embedded submodels of Henselian valued fields*. arXiv Prepint:2005.02363. 2020.

[47] Mariana Vicaria. *Elimination of imaginaries in multivalued fields*. arXiv Prepint: 2109.08140. 2021.

[48] Mariana Vicaria. *Elimination of imaginaries in ordered abelian groups with bounded regular rank*. arXiv Prepint: 2106.01500. 2021.

[49] Volker Weispfenning. "Elimination of quantifiers for certain ordered and lattice-ordered abelian groups". In: *Bulletin de la Société Mathématique de Belgique, Ser. B* 33 (1981), pp. 131–155.

[50] Volker Weispfenning. "Quantifier eliminable ordered abelian groups". In: *Algebra and Order, Proc. First Symp. Ordered Algebraic Structures, Luminy - Marseilles, 1984, Res. Exp. Math.* 14 (1986), pp. 113–126.