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On the BBDG's decomposition theorem

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Preface

The *decomposition theorem*, which was proved by Beilinson, Bernstein, Deligne (and Gabber) in [BBD82], is a powerful result which describes the derived push-forward $Rf_*IC^\bullet_X$ (see [Ive86, §II.4]), where $f : X \to Y$ is any proper map between complex algebraic varieties and $IC^\bullet_X$ is the intersection cohomology complex of X (see [GM80, §1.3], [GM83, §2.1], [BBD82, §2.1, Remark 2.1.18]), as the direct sum of more elementary objects in the constructible bounded derived category $D^b_c(Y, \mathbb{K})$ of Y, where $\mathbb{K}$ is any field of characteristic 0 (see either [dCM09, §1.5] or [Dim04, §4.1]).

Having said that, this theorem does not provide an explicit description of the direct summands and it may also happen that some of the expected *supports*, that is, the subvarieties of Y on which such direct summands are supported, do not contribute at all to the decomposition. For example, the Ngo's support theorem (see [Ngo10] and [dCM09, §4.6]) is a remarkable result which characterizes the actual supports of the decomposition theorem in certain circumstances. Therefore, it is worth looking for instances in which the description of the direct summands is either more accurate or explicit and examples in which it is possible to improve the statement of the decomposition theorem by characterizing the supports of non-trivial direct summands.

Eligible varieties for the above cause are the Schubert varieties $\mathcal{S}$ contained in a Grassmannian $\mathbb{G}_k(\mathbb{C}^l)$, which are defined as follows. Given a *flag*, i.e. a strictly increasing chain of subspaces $0 \neq F_{j_1} \subset \ldots \subset F_{j_\omega} \subset \mathbb{C}^l$, where the subscripts stand for the dimensions of the corresponding spaces, and an ω-tuple of non-negative integers $(i_1, \ldots, i_\omega)$, a *Schubert variety* $\mathcal{S}$ is

$$\mathcal{S} = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim V \cap F_{j_\alpha} \geq i_\alpha,\ \alpha = 1, \ldots, \omega\}.$$

When $\omega = 1$ and $\mathcal{S}$ is neither empty nor a Grassmannian, the Schubert variety is said to be *special*.

In [DGF19, Theorem 3.1], it has been proved that the decomposition theorem becomes explicit when certain hypotheses are met. In particular, special Schubert varieties $\mathcal{S}$ *with two strata*, that is, the ones which have exactly one Grassmannian as unique Schubert subvariety associated to the same flag of $\mathcal{S}$, have the required properties and are examined in [DGF19, §4].

Special Schubert varieties *with an arbitrary number of strata* have been studied in [Fra20], where the following explicit form of the decomposition theorem is obtained.

Theorem 1. [Fra20, Remark 3.3 and Theorem 3.5]. *Let $0 \neq F_j \subset \mathbb{C}^l$ be a vector space and consider the special Schubert variety $\mathcal{S} = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_j) \geq i\}$ and the projection on the second factor*

$$\pi : \tilde{\mathcal{S}} := \{(Z, V) \in \mathbb{G}_i(F) \times \mathbb{G}_k(\mathbb{C}^l) : Z \subseteq V\} \to \mathcal{S},$$

which is a resolution of singularities (see [Fra20, §2]), i.e. $\tilde{\mathcal{S}}$ is nonsingular and π is birational (in other words, there is an open dense subset U of $\mathcal{S}$ such that the restriction $\pi^{-1}(U) \to U$ is an isomorphism). Assume[1] *that $k \leq j$. Then, if π is small, i.e.,* $\operatorname{codim}\{V \in \mathcal{S} : \dim \pi^{-1}(V) \geq \alpha\} > 2\alpha$ *for any $\alpha > 0$,*

$$R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[\dim \mathcal{S}] \cong IC^\bullet_{\mathcal{S}}$$

and

$$IC^\bullet_{\mathcal{S}}[-\dim \mathcal{S}]|_{\Delta^0_q} \cong \bigoplus_{\alpha \geq 0} H^\alpha(\mathbb{G}_i(\mathbb{C}^{i+q})) \otimes \mathbb{Q}_{\Delta^0_q}[-\alpha],$$

otherwise

$$R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[\dim \mathcal{S}] \cong \bigoplus_{\tau=0}^{k-i} \bigoplus_{\alpha \in \mathbb{Z}} H^{\delta_{0\tau}+\alpha}(\mathbb{G}_{k-i-\tau}(\mathbb{C}^{k-l+j})) \otimes IC^\bullet_{\Delta_\tau}[-\alpha]$$

and

$$IC^\bullet_{\Delta_\tau}[-\dim \Delta_\tau]|_{\Delta^0_q} \cong \bigoplus_{\beta \geq 0} H^\beta(\mathbb{G}_{q-\tau}(\mathbb{C}^{l-j-k+i+q})) \otimes \mathbb{Q}_{\Delta^0_q}[-\beta],$$

where

i) q and τ are non-negative integers such that $q \geq \tau$ and the Schubert varieties $\Delta_q = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_j) \geq i + q\}$ and Δ_τ, defined analogously, are non-empty;

ii) Δ^0_q is the smooth locus of Δ_q (see [Fra20, §2.2]);

iii) $\delta_{0\tau} := 2 \dim \pi^{-1}(\Delta^0_\tau) - (\dim \mathcal{S} - \dim \Delta_\tau)$.

As an application of this fact, two classes of Poincaré polynomial identities are inferred in [CFS21, Theorems 2 and 3]; this is possible because all terms can be described by means of Poincaré polynomials of suitable Grassmannians, for which there is an explicit formula (see [CGM82, p. 329]). In particular, one of the above classes involves some of the so-called Kazhdan-Lusztig polynomials (see [dCM09, §4.4] and [BL00, §6.1]), which, by [KL80, Theorem 4.3], coincide with the Poincaré polynomials of the stalks $\mathcal{H}^\alpha(IC^\bullet_{\mathcal{S}'})_x$ of the cohomology sheaves of the intersection cohomology complexes of the Schubert varieties $\mathcal{S}'$ (not only the ones contained in a Grassmannian).

The conclusive generalization to all Schubert varieties contained in a Grassmannian has been recently achieved in [CFS22]. The reasoning exhibited in [Fra20, §3], which had led to the explicit form of the decomposition theorem stated in Theorem 1, still holds for non-special Schubert varieties.

[1] This is not a restrictive hypothesis; see Sections 2.1.1 and 2.4.1

Theorem 2. [CFS22, Theorem 3.6]. *Let $\mathcal{S}$ be a non-empty Schubert variety. Assume that each of the ω conditions $\dim(V \cap F_{j_\alpha}) \geq i_\alpha$ neither implies another such inequality nor is satisfied by all $V \in \mathbb{G}_k(\mathbb{C}^l)$. Consider the projection on the last factor*

$$\pi : \tilde{\mathcal{S}} := \begin{Bmatrix} (Z_1, \ldots, Z_\omega, V) \in \mathbb{G}_{i_1}(F_{j_1}) \times \ldots \times \mathbb{G}_{i_\omega}(F_{j_\omega}) \times \mathbb{G}_k(\mathbb{C}^l) \\ s.t. \quad Z_1 \subset \ldots \subset Z_\omega \subset V \end{Bmatrix} \to \mathcal{S},$$

which is a resolution of singularities. Then

$$R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[\dim \mathcal{S}] \cong \bigoplus_{\tau} \bigoplus_{\alpha \in \mathbb{Z}} D_{0\tau}^{\delta_{0\tau}+\alpha} \otimes IC^\bullet_{\Delta_\tau}[-\alpha]$$

and

$$IC^\bullet_{\Delta_\tau}[-\dim \Delta_\tau]|_{\Delta_{\tau q}} \cong \bigoplus_{\beta \geq 0} B^\beta_{\tau q} \otimes \mathbb{Q}_{\Delta_{\tau q}}[-\beta],$$

where

i) q and τ are ω-tuples of non-negative integers such that the Schubert varieties Δ_q and Δ_τ, defined respectively by the conditions $\dim(V \cap F_{j_\alpha}) \geq i_\alpha + q_\alpha$ and $\dim(V \cap F_{j_\alpha}) \geq i_\alpha + \tau_\alpha$, are non-empty and such that $\Delta_q \subset \Delta_\tau$;

ii) $\Delta_{\tau q}$ is a suitable smooth open dense subset of Δ_q (see Section 2.2.1);

iii) $\delta_{0\tau} := 2 \dim \pi^{-1}(\Delta_{\tau q}) - (\dim \mathcal{S} - \dim \Delta_\tau)$;

iv) $D^{\delta_{0\tau}+\alpha}_{p\tau}$ are suitable vector spaces symmetric with respect to α, that is, for any $\alpha \geq 0$, $D^{\delta_{0\tau}+\alpha}_{p\tau} \cong D^{\delta_{0\tau}-\alpha}_{p\tau}$;

v) $B^\beta_{\tau q}$ are other suitable vector spaces.

As opposed to the case of special Schubert varieties, an explicit description of the vector spaces $D^{\delta_{0\tau}+\alpha}_{0\tau}$ and $B^\beta_{\tau q}$ is not available, unless the resolution π is small. Indeed, by [GM83, Corollary §6.2], if $\chi : W \to \mathcal{S}$ is a small resolution of singularities, then $IC^\bullet_{\mathcal{S}} \cong R\chi_* \mathbb{Q}_W[\dim \mathcal{S}]$ and, by [BL00, Theorem 9.1.3], the Kazhdan-Lusztig polynomials coincide with the Poincaré polynomials of the fibres $\chi^{-1}(V)$. Anyway, the Poincaré polynomial identities deduced in [CFS21] are, in general, expressions of the following form.

Corollary 1. [CFS22, Corollary 3.12]. *Under the same hypotheses and notations as Theorem 2, if $\Delta_q = \mathcal{S}$, then $a_{00} = g_{00} = b_{00} = 1$; otherwise*

$$a_{0q} = b_{0q} + g_{0q} + \sum_\tau g_{0\tau} b_{\tau q},$$

where

$$a_{0q} := \sum_{\alpha \in \mathbb{Z}} \dim H^\alpha(\pi^{-1}(\Delta^0_q)) \cdot t^\alpha, \qquad b_{\tau q} := \sum_{\alpha \in \mathbb{Z}} \dim B^\alpha_{\tau q} \cdot t^\alpha,$$

$$g_{0\tau} := \sum_{\alpha \in \mathbb{Z}} \dim D^\alpha_{0\tau} \cdot t^{\alpha + 2d_{0\tau}}, \qquad d_{0\tau} := \dim \mathcal{S} - \dim \Delta_\tau - \dim \pi^{-1}(\Delta^0_\tau).$$

The only explicit term in the formula of the preceding corollary is the Poincaré polynomial a_{0q} of the fibre $\pi^{-1}(V)$; moreover, all polynomials $g_{0\tau}$ and $b_{\tau q}$ corresponding to suitable (see item *i)* of Theorem 2) Schubert subvarieties Δ_τ are involved. This suggests that the computation of g_{0q} and b_{0q} has to be performed inductively and this is achieved by means of an algorithm, named *KaLu*, which exploits the explicitness of the polynomial a_{0q} and the fact that the intersection cohomology complexes satisfy the support conditions (see [dCM09, formula 12] and [GM83, p.78, Theorem, item (c)]). Before giving the result concerning the algorithm, it has to be acknowledged that a formula for the computation of this class of Kazhdan-Lusztig polynomials exists in literature; namely, it is [Zel83, Theorem 2]. In a nutshell, an inductive construction of resolution of singularities is given in [Zel83, §3] along with a sufficient condition for the smallness of such maps (see [Zel83, Theorem 1]). Consequently, all Kazhdan-Lusztig polynomials related to Schubert varieties in a Grassmannian are explicit, being the Poincaré polynomial of the fibre of some of these small resolutions as explained after Theorem 2. Nonetheless, these resolution are not that explicit in the sense that, in general, the induction required for their construction makes the description of their fibres, usually highly singular and reducible, not easy. On the contrary, the resolutions π (see Theorem 2) chosen in [CFS22] are explicit and have smooth fibres which are immediate to determine. The drawback of this approach is that the maps π are usually non-small, thus, in this cases, the computation of the Kazhdan-Lusztig polynomials requires the inductive formula described below.

Corollary 2. [CFS22, Corollary 4.1]. *Set*

$$\begin{aligned} U_\beta &: \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] \mapsto \sum_{\alpha\geq\beta} c_\alpha t^\alpha \in \mathbb{Z}[t] \;\; \forall \beta \geq 0, \\ S &: \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] \mapsto c_0 + \sum_{\alpha\geq 1} c_\alpha (t^\alpha + t^{-\alpha}) \in \mathbb{Z}\left[t, t^{-1}\right], \\ \tilde{t}^\beta &: \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] \mapsto \sum_{\alpha\geq 0} c_\alpha t^{\alpha+\beta} \in \mathbb{Z}[t] \;\; \forall \beta \geq 0. \end{aligned}$$

Under the same hypotheses and notations as Theorem 2, if $\Delta_q \subset \mathcal{S}$, then

$$\begin{cases} g_{0q} = \tilde{U}_{0q}(a_{0q} - \sum_\tau g_{0\tau} b_{\tau q}) \\ b_{0q} = a_{0q} - g_{0q} \end{cases}$$

where $\tilde{U}_{0q} := \tilde{t}^{\dim\mathcal{S} - \dim\Delta_q} \circ S \circ \tilde{t}^{-(\dim\mathcal{S} - \dim\Delta_q)} \circ U_{\dim\mathcal{S} - \dim\Delta_q}$.

An implementation of *KaLu* in CoCoa5 [ABR] is available at `http://wpage.unina.it/carmine.sessa2/KaLu`.

KaLu also highlights the existence of supports which do not give any contribution to the decomposition of $R\pi_*\mathbb{Q}_{\tilde{\mathcal{S}}}$. So far, the geometrical reason behind this phenomenon has not been unravelled yet and the question of the characterization of such supports is open.

To conclude, recall that another algorithm for the computation of Kazhdan-Lusztig polynomials of Schubert varieties in Grassmannians can be deduced by a closed formula given in [Zel83], which uses suitable small resolutions. The result of [Zel83] starts from a setting presented in [LS81], where a different method is also introduced, together with a table of the Kazhdan-Lusztig polynomials of a particular case of Schubert varieties in Grassmannians. As far as we know, there has not been an implementation of these methods up to now.

Conversely, implementations for the computation of Kazhdan-Lusztig polynomials of Schubert varieties in flag manifolds, instead of Grassmannians, are available. Among them, it is worth quoting the implementation due to Fokko du Cloux in the program Coxeter 3 [dC05]. Interesting tables of Kazhdan-Lusztig polynomials of Schubert varieties in flag manifolds are published at `https://www.math.ias.edu/~goresky/tables.html`. For some detailed treatments see, for example, [BL00, Bre04, dC96, dC02, L2̈0] and the references therein.

Consider again the decomposition theorem. Another indispensable hypothesis of this result is that sheaves must be taken with coefficients in a field of characteristic 0; therefore, it would be useful to understand the extent to which a decomposition of the derived direct image $Rf_*\mathbb{A}_X$ of a map $f : X \to Y$ exists when $\mathbb{A}$ is not such a field. This problem is tackled with in [GFS22], where the idea of resorting to bivariant theory (see [FH91] and Section 3.1.1) turns out to be effective.

Suppose that X and Y are locally compact Hausdorff spaces embeddable as closed subspaces of $\mathbb{R}^N$ for some N and that f is a proper continuous map of finite cohomological dimension (see Example 3.1.2). The morphism f is associated to the abelian groups $\mathrm{Hom}_{D^b(Y,\mathbb{A})}(Rf_!\mathbb{A}_X, \mathbb{A}_Y[\alpha])$, where $D^b(Y,\mathbb{A})$ denotes the bounded derived category of sheaves of $\mathbb{A}$-modules on Y, $f_!$ is the exceptional direct image of f and $\alpha \in \mathbb{Z}$. The elements θ of these groups are called the *bivariant classes* of f and, as explained in [FH91, pp. 8, 9, 25], they induce Gysin morphisms $\theta_\beta : H^\beta(X) \to H^\beta(Y)$ in singular cohomology. In particular, when θ_0 transforms the unit (with respect to cup product) of $H^0(X)$ into the one of $H^0(Y)$, θ is said to have *degree one* for f.

The existence of a bivariant class of degree one is exactly what characterizes a wide class of proper maps whose derived pushforwards admit a decomposition analogous to the one given by decomposition theorem.

Theorem 3. [GFS22, p. 3, Theorem 1.1]. *Let $f : X \to Y$ be a proper continuous map, with Y path-connected. Let $U \subseteq Y$ be a non-empty open subset such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is a homeomorphism. Set $W = Y \setminus U$ and $\widetilde{W} = f^{-1}(W)$. The following properties are equivalent.*

i) There exists a bivariant class $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one.

ii) There is a cross isomorphism $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$ in $D^b_c(Y,\mathbb{A})$.

iii) There exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y,\mathbb{A})$.

Some explicit isomorphisms in cohomology and Borel-Moore homology (see [BM60], [Ive86, §IX], [Ful98, §19.1]), compatible with the duality morphisms, ensue from the decompositions provided by the above theorem.

Notice, however, that Theorem 3 and its consequences in (co)homology are not completely new; in fact, analogous results had been proved, for example, in [Ful98,

§6.7], [Jou77, §8] and [DGF14, §2]. On the contrary, their application in the study of the relation between the existence of a bivariant class of degree one for f and the property of X and Y of being $\mathbb{A}$-homology manifolds (see [Lef33, p. 487, Definition] and [Wil49, §VIII.1]) does not seem to have been proved elsewhere.

Theorem 4. [GFS22, Theorem 1.2]. *Let $f : X \to Y$ be a projective birational morphism between complex irreducible quasi-projective varieties of the same complex dimension n. Let $U \subseteq Y$ be a non-empty Zariski open subset such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is an isomorphism.*

i) If Y is an $\mathbb{A}$-homology manifold, then

- ▲ *there is a unique bivariant class $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one;*
- ▲ *there exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y,\mathbb{A})$, with $\mathcal{K}^\bullet$ supported on $W := Y \setminus U$;*
- ▲ *if X is an $\mathbb{A}$-homology manifold, $\mathcal{K}^\bullet[n]$ is self-dual.*

ii) If X is an $\mathbb{A}$-homology manifold and there is a bivariant class of degree one $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$, then Y is an $\mathbb{A}$-homology manifold, as well.

As an outcome of this theorem, a simple proof of the fact that the nilpotent cone of any connected reductive complex algebraic group is a homology manifold ensues.

Here is how the thesis is organized. Since decomposition theorem plays a central role, Chapter 1 is meant to settle notations and to make the statement of the theorem intelligible also to readers who are not familiar with the subject. More precisely, the first two sections are devoted to (abelian, derived, triangulated) categories and sheaves, while the third one recalls what perverse sheaves and intersection cohomology are.

Chapter 2 is devoted to the application of decomposition theorem to Schubert varieties contained in a Grassmannian. This part of the dissertation does not respect the chronological order of the results; indeed the general case of any Schubert variety are discussed first, while the results about special Schubert variety, in Section 2.4, are inferred from the general ones. The last section discusses cases in which the Poincaré polynomial expressions are identities and contains several examples of Ferrer's diagrams (for the definition, see Section 2.1.2), which make the study of Schubert varieties easier.

Chapter 3, begins with a preliminary section on bivariant theory and Borel-Moore homology. In Section 3.2, a generalization of the decomposition theorem in circumstances in which this result does not hold is achieved and consequent applications to the study of homology manifolds are obtained in Section 3.3. In particular, nilpotent cones are shown to be homology manifolds in Section 3.4.

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Chapter 1

The decomposition theorem

In this introductory chapter, several definitions about categories and sheaves are recalled. The aim is to provide the reader the minimum information necessary to make the statement of the decomposition theorem 1.3.7 intelligible and this is achieved in three steps. First, the basics of category theory are recalled in Section 1.1. Secondly, the definition and a few properties of sheaves, cohomology with coefficients on sheaves and operations on sheaves are recalled in Section 1.2. Lastly, all notions of the preceding steps are used in Section 1.3 to remind the category in which the decomposition theorem 1.3.7 holds and what intersection cohomology complexes are.

1.1 Categories

The basic definitions of category theory are recalled in this section so as to talk about abelian, derived and triangulated categories, which are fundamental in the study of the (co)homological properties of certain objects. Precisely, (co)homology can be defined in abelian categories because of the existence of kernels and cokernels, while functors which are not exact can be "fixed" if the abelian category is replaced by its derived one. Triangulated categories allow the study (co)homology even when not all morphisms have either kernel or cokernel. This is achieved by suitable axioms and, in particular, by substituting exact sequences for the more general notion of distinguished triangles.

1.1.1 Abelian categories

Definition 1.1.1. A **category** $\mathcal{C}$ is given by the following data.

i) A class $Ob(\mathcal{C})$, which, with abuse of notations, will often be denoted by $\mathcal{C}$, whose elements are called **objects**;

ii) for any pair (A, B) of objects, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, denoted by $\mathrm{Hom}(A, B)$ when no confusion arises, whose elements are called **morphisms**;

iii) an operation, called **composition**, defined for every triple (A, B, C) of objects in $\mathcal{C}$.

$$\circ : (f, g) \in \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \mapsto gf := g \circ f \in \mathrm{Hom}(A, C).$$

The following axioms must be verified:

a. for any object A, $\mathrm{Hom}(A, A) \neq \emptyset$ and it contains the **identity morphism** id_A, also denoted by id when there is no need to highlight A, which has the following property. For any object B, any $f \in \mathrm{Hom}(A, B)$ and any $g \in \mathrm{Hom}(B, A)$, $id_A \circ g = g$ and $f \circ id_A = f$;

b. composition is *associative*, i.e. $h(gf) = (hg)f$ for any $A, B, C, D \in \mathcal{C}$, $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$;

c. $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(C, D) \neq \emptyset$ if and only if $A = C$ and $B = D$.

A morphism f is often denoted by $f : A \to B$ when the underlying category is known. f is said to be an **isomorphism** if there is $g :\in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $gf = id_A$ and $fg = id_B$.

Examples of categories are Sets and Ab, whose objects are sets and abelian groups, respectively, and whose morphisms are functions and group homomorphisms, respectively. Another one is the **opposite category** $\mathcal{C}^{op}$, whose objects are the ones of $\mathcal{C}$ and whose morphisms are the ones of $\mathcal{C}$, but with reversed arrows (e.g. if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then there is $f^{op} \in \mathrm{Hom}_{\mathcal{C}^{op}}(B, A)$). Moreover, $f^{op}g^{op} := (gf)^{op}$ is defined whenever gf makes sense in $\mathcal{C}$.

Given two categories, it is possible to define a "function" between them.

Definition 1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is given by the following data.

i) For any $A \in \mathcal{C}$, $F(A) \in \mathcal{D}$;

ii) for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$;

iii) for any $A \in \mathcal{C}$, $F(id_A) = id_{F(A)}$;

iv) for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F(gf) = F(g)F(f)$.

The notion of **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is *dual* (see [Mac71, §II.1]) to the one of covariant functor; in other words, it is obtained by the definition of covariant functor by "reversing arrows" (e.g. $F(f) : F(B) \to F(A)$ in (ii)).

Let $\mathcal{C}$ be a category and let A be an object. $\mathrm{Hom}(A, -)$ is the covariant functor which associates any $B \in \mathcal{C}$ to $\mathrm{Hom}(A, B)$ and any $f \in \mathrm{Hom}(B, C)$ to the morphism $f_* = \mathrm{Hom}(A, f)$, given by $f_*(g) = fg$ for any $g \in \mathrm{Hom}(A, B)$. Similarly, there is a contravariant functor $\mathrm{Hom}(-, A)$; it associates any object $B \in \mathcal{C}$ to $\mathrm{Hom}(B, A)$ and any morphism $\varphi \in \mathrm{Hom}(B, C)$ to $\varphi^* := \mathrm{Hom}(\varphi, A)$, given by $\varphi^*(\psi) = \psi\varphi$ for any $\psi \in \mathrm{Hom}(C, A)$.

$$\begin{tikzcd}
& A \arrow[dl, "g"'] \arrow[dr, dashed, "f_*g"] & \\
B \arrow[rr, "f"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
& A & \\
B \arrow[ur, dashed, "\varphi^*\psi"] \arrow[rr, "\varphi"'] & & C \arrow[ul, "\psi"']
\end{tikzcd}$$

It is also possible to define "functions" between functors.

Definition 1.1.3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors of the same variance. A **natural transformation** $\tau : F \to G$ is a collection $\{\tau_C : F(C) \to G(C)\}_{C \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that the diagram below commutes for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Moreover, τ is said to be an **isomorphism** if all τ_C are isomorphisms in $\mathcal{D}$.

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\tau_C"] & F(D) \arrow[d, "\tau_D"] \\ G(C) \arrow[r, "G(f)"] & G(D) \end{tikzcd}$$

In order to introduce additive and abelian categories, it is necessary to give further notions which are the solutions to certain universal mapping problems (see [Rot09, Remark p. 217]).

Definition 1.1.4. Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. Their **coproduct** is a triple $(A \sqcup B, \alpha, \beta)$, where $A \sqcup B \in \mathcal{C}$, $\alpha : A \to A \sqcup B$ and $\beta : B \to A \sqcup B$ satisfy the following property. For any $X \in \mathcal{C}$, any $f : A \to X$ and any $g : B \to Y$, there is a unique morphism $\theta : A \sqcup B \to X$ making the diagram on the left commute.

The **product** $(A \sqcap B, \gamma, \delta)$ of A and B, instead, is dual notion of coproduct and can be described by means of the diagram on the right.

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "f"] & \\ A \sqcup B \arrow[rr, dashed, "\exists! \theta"] & & X \\ & B \arrow[ul, "\beta"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} & A & \\ A \sqcap B \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & & X \arrow[ll, dashed, "\exists! \vartheta"] \arrow[ul, "h"'] \arrow[dl, "k"] \\ & B & \end{tikzcd}$$

The product and the coproduct of two objects may either not exist or not coincide. For instance, given $A, B \in$ Sets, $A \sqcup B$ is their disjoint union, whereas $A \sqcap B$ is their cartesian product.

For the next definition, the concept of zero object is required. Given a category $\mathcal{C}$, $I \in \mathcal{C}$ is called an **initial object** if, for any $A \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(I, A) = \{*\}$ is a singleton. Dually, $T \in \mathcal{C}$ is called a **terminal object** if $\mathrm{Hom}_{\mathcal{C}}(A, T) = \{*\}$ for any $A \in \mathcal{C}$. Lastly, a **zero object**, denoted by 0, is both an initial and a terminal object. In this case, the unique morphisms having 0 either as domain or target are denoted by 0, as well. Notice that I, T and 0, when exist, are unique up to a unique morphism (see [Rot09, Lemmas 5.3 and 5.6]), yet, I and T may occur to be different. For instance, $I = \emptyset$ and $T = \{*\}$ in Sets.

Definition 1.1.5. Assume that $\mathcal{C}$ has a zero object and consider two morphisms $f : B \to A$ and $h : A \to B$. A triple $(\ker f, 0, \beta)$, where $\ker f \in \mathcal{C}$, $0 : 0 \to A$ and $\beta : \ker f \to B$ are such that $f\beta = 0$, is called a **kernel** of f if, for any $X \in \mathcal{C}$ and any $\beta' : X \to B$ such that $f\beta' = 0$, there is a unique $\theta : X \to \ker f$ making the diagram on the left commute.

The **cokernel** $(\operatorname{coker} h, 0, \delta)$ of h is, instead, the dual notion of the kernel and can be described by virtue of the diagram on the right.

$$
\begin{tikzcd}
A & B \arrow[l, "f"'] & \\
0 \arrow[u, "0"] & \ker f \arrow[l, "0"] \arrow[u, "\beta"] & \\
& & X \arrow[ul, dashed, "\exists!\theta"] \arrow[uul, bend right, "\beta'"'] \arrow[ull, bend left, "0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "0"] & B \arrow[d, "\delta"] \arrow[ddr, bend left, "\delta'"] & \\
0 \arrow[r, "0"] \arrow[drr, bend right, "0"'] & \operatorname{coker} h \arrow[dr, dashed, "\exists!\vartheta"] & \\
& & X
\end{tikzcd}
$$

In the following, the (co)kernel of a morphism f will be simply denoted by (co)ker$f$.

Kernels and cokernels are examples of *pullbacks* and *pushouts*, respectively, whose definitions are given, for instance, in [Rot09, Definitions pp. 221, 222].

Definition 1.1.6. A category $\mathcal{C}$ is said to be **additive** if the following axioms are satisfied.

A1. For any $A, B \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group and the composition of morphisms is *bi-additive*; in other words, whenever there are morphisms as in the sequence below, $k(f+g) = kf + kg$ and $(f+g)h = fh + gh$.

$$
\begin{tikzcd}
A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "k"] & D
\end{tikzcd}
$$

A2. $\mathcal{C}$ has a zero object;

A3. for any two objects $A, B \in \mathcal{C}$, $A \sqcup B$ and $A \sqcap B$ exist in $\mathcal{C}$.

It is possible to prove that products and coproducts are isomorphic in additive categories (see [Rot09, Lemma 5.87]); for this reason, it makes sense to denote $A \sqcup B \cong A \sqcap B$ by $A \oplus B$ and call them the **direct sum** of A and B.

Definition 1.1.7. An additive category $\mathcal{A}$ is said to be **abelian** if

A4. for any morphism $f : A \to B$, there is a sequence

$$\ker f \xrightarrow{\ i\ } A \xrightarrow{\ \alpha\ } X \xrightarrow{\ \beta\ } B \xrightarrow{\ p\ } \operatorname{coker} f$$

where $\beta\alpha = f$ and $X = \ker p = \operatorname{coker} i$.

In abelian categories, it makes sense to talk about the **image** of a morphism f $\operatorname{im} f := \ker(\operatorname{coker} f)$. This allows to talk about exactness of (possibly infinite) sequences of objects and morphisms

$$\longrightarrow A^{\alpha-1} \xrightarrow{a_{\alpha-1}} A^{\alpha} \xrightarrow{a_{\alpha}} A^{\alpha+1} \longrightarrow$$

Namely, a sequence as the one above is said to be **exact at** A^α if $\ker a_\alpha = \operatorname{im} a_{\alpha-1}$. It is called **exact** if so is it at each A^α. In particular, an exact sequence of the form $0 \to A \to B \to C \to 0$ is called a **short exact sequence**.

Let $\mathcal{A}$ be a category. A (**chain**) **complex** $A^\bullet$ is a sequence of objects A^α, called the **terms** of $A^\bullet$, and morphisms d_A^α, called the **differentials**, such that $d_A^{\alpha+1} d_A^\alpha = 0$ for any $n \in \mathbb{Z}$. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a family of morphisms $\{f_\alpha : A^\alpha \to B^\alpha\}_{\alpha \in \mathbb{Z}}$ in $\mathcal{C}$ making all squares commute in the diagram below.

$$
\begin{array}{cccccccc}
A^\bullet : & \longrightarrow & A^{n-1} & \xrightarrow{d_A^{n-1}} & A^n & \xrightarrow{d_A^n} & A^{n+1} & \longrightarrow \\
 & & \Big\downarrow{\scriptstyle f^{n-1}} & & \Big\downarrow{\scriptstyle f^n} & & \Big\downarrow{\scriptstyle f^{n+1}} & \\
B^\bullet : & \longrightarrow & B^{n-1} & \xrightarrow{d_B^{n-1}} & B^n & \xrightarrow{d_B^n} & B^{n+1} & \longrightarrow
\end{array}
$$

The category $\mathrm{Comp}(\mathcal{A})$ is the one whose objects are the complexes in $\mathcal{A}$ and whose morphisms are the ones just described. The Hom functor on $\mathrm{Comp}(\mathcal{A})$, with $\mathbb{A}$ abelian, has a generalization

$$\mathrm{Hom}^\bullet : \mathrm{Comp}(\mathcal{A})^\circ \times \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(Ab) \tag{1.1.1}$$

which sends any pair $(A^\bullet, B^\bullet)$ of complexes into the complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$, whose terms are

$$\mathrm{Hom}^\alpha(A^\bullet, B^\bullet) := \prod_{\beta \in \mathbb{Z}} \mathrm{Hom}(A^\beta, B^{\alpha+\beta})$$

and whose differentials are given by

$$d^\alpha \varphi = (d_{B^\bullet}^{\alpha+\beta} \varphi^\beta + (-1)^{\alpha+1} \varphi^{\beta+1} d_{A^\bullet}^\beta)_{\beta \in \mathbb{Z}},$$

where $\varphi := (\varphi^\beta)_{\beta \in \mathbb{Z}} \in \mathrm{Hom}^\alpha(A^\bullet, B^\bullet)$.

$\mathcal{A}$ can be thought of as a *full subcategory* of $\mathrm{Comp}(\mathcal{A})$, which means that any object A of $\mathcal{A}$ is also in $\mathrm{Comp}(\mathcal{A})$ (indeed, $A \in \mathcal{A}$ can be identified with the complex having A *concentrated in degree 0*, that is, the complex $A^\bullet$ with $A^0 = A$ and $A^\alpha = 0$ for $\alpha \neq 0$) and, for any $A, B \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(A, B)$ (for a subgategory, the inclusion $\subseteq$ is required; the adjective full means that equality holds). In fact, $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ can be identified with $\{f_\alpha : f_0 = f \text{ and } f_\alpha = 0 \text{ otherwise}\}$. Besides, there are other three important full subcategories of $\mathrm{Comp}(\mathcal{A})$, whose objects are called the **bounded below**, **bounded above** and **bounded** complexes, respectively:

$$
\begin{aligned}
\mathrm{Comp}(\mathcal{A})^+ &:= \{A^\bullet \in \mathrm{Comp}(\mathcal{A}) : \ \exists \beta \in \mathbb{Z} : A^\alpha = 0 \ \forall \alpha \geq \beta\}, \\
\mathrm{Comp}(\mathcal{A})^- &:= \{A^\bullet \in \mathrm{Comp}(\mathcal{A}) : \ \exists \beta \in \mathbb{Z} : A^\alpha = 0 \ \forall \alpha \leq \beta\}, \\
\mathrm{Comp}(\mathcal{A})^b &:= \{A^\bullet \in \mathrm{Comp}(\mathcal{A}) : \ \exists \beta \leq \gamma \in \mathbb{Z} : A^\alpha = 0 \ \forall \alpha : \beta \leq \alpha \leq \gamma\}.
\end{aligned}
$$

If $\mathcal{A}$ is additive or abelian, respectively, so is $\mathrm{Comp}(\mathcal{A})$ (see [Rot09, Proposition 5.100]). In the former case, it makes sense to define, for any $n \in \mathbb{Z}$, the **α-th cohomology functor**

$$H^\alpha : \begin{cases} A^\bullet \in \mathrm{Comp}(\mathcal{A}) \mapsto \ker d^{\alpha+1} / \operatorname{im} d^\alpha \in \mathcal{A} \\ f : A^\bullet \to B^\bullet \mapsto H^\alpha(f) : H^\alpha(A^\bullet) \to H^\alpha(B^\bullet) \end{cases}$$

and a complex $A^\bullet$ is said to be **acyclic at the α-th term** if $H^\alpha(A^\bullet) = 0$; **acyclic** if so is it at all its terms.

1.1.2 Derived categories

Until the end of the subsection, $\mathcal{A}$ shall denote an abelian category and its elements will be denoted by a capital letter and a bullet as superscript (e.g. $A^\bullet$).

The definition of derived categories is reminiscent of the one of the rings of fractions; in that case, a multiplicative subset S of a unitary commutative ring R is taken and the ring $S^{-1}R$, whose elements are fractions with denominators in S, is defined. Hence, the derived category $D(\mathcal{A})$ of $\mathcal{A}$ plays the role of $S^{-1}R$, while the homotopy category $K(\mathcal{A})$, which shall be defined presently, and the set of **quasi-isomorphisms** (i.e., morphisms $f : A^\bullet \to B^\bullet$ such that $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$) represent the ring R and the set S, respectively.

Definition 1.1.8. Let $f, g : A^\bullet \to B^\bullet$ be two morphisms. A **homotopy** between them is a family of morphisms $h = \{h^n : A^n \to B^{n-1}\}_{n\in\mathbb{Z}}$ in $\mathcal{A}$ such that $f^n - g^n = d_B^{n-1}h^n + h^{n+1}d_A^n$ for each $n \in \mathbb{Z}$. In this case, the notation $f \sim g$ is adopted.

$$\begin{tikzcd}
\arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d] & A^n \arrow[r, "d_A^n"] \arrow[dl, "h^n"'] \arrow[d, shift left, "f^n"] \arrow[d, shift right, "g^n"'] & A^{n+1} \arrow[r] \arrow[d] \arrow[dl, "h^{n+1}"'] & {} \\
\arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r] & {}
\end{tikzcd}$$

The **homotopy category** $K(\mathcal{A})$ is nothing but the *quotient* of $\mathrm{Comp}(\mathcal{A})$ with respect to the homotopy relation $\sim$; in other words,

$$\begin{aligned}
&Ob(K(\mathcal{A})) = Ob(\mathrm{Comp}(\mathcal{A})),\\
&\quad [A^\bullet, B^\bullet] := \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim \;\; \forall A^\bullet, B^\bullet.
\end{aligned}$$

Similarly, the homotopy categories $K^*(\mathcal{A})$, where $* = +, -, b$, are defined.

The reason why $K(\mathcal{A})$ is needed to define the derived category is that quasi-isomorphisms $\mathrm{Comp}(\mathcal{A})$ do not form a localizing class (see below) as opposed to their classes in $K(\mathcal{A})$.

Definition 1.1.9. A class of morphisms S in $\mathcal{A}$ is said to be **localizing** if it is closed under composition, contains id_A for any $A \in \mathcal{A}$ and have the following properties.

i) Any two morphisms $f : B \to A$ and $s : C \to A$, with $s \in S$, can be completed to a commutative square whose edge opposite to s is a morphism in S.

$$\begin{tikzcd}
D \arrow[r, dashed] \arrow[d, dashed, "\in S"] & C \arrow[d, "s"] \\
B \arrow[r, "f"] & A
\end{tikzcd}$$

ii) Given $f, g : A \to B$, there is $s \in S$ such that $sf = sg$ if and only if there is $t \in S$ such that $ft = gt$.

Theorem 1.1.10. [GM03, Proposition III.4.2 and Theorem III.4.4]. *The derived category $D(\mathcal{A})$ is canonically isomorphic to the localization of $K(\mathcal{A})$ by quasi-isomorphisms, which form a localizing class.*

For the formal definition of localization, see [GM03, §III.2.2]. Here, the derived category shall be described by saying what its objects, morphisms and composition are.

Definition 1.1.11. The objects of the derived category $D(\mathcal{A})$ of $\mathcal{A}$ are the same as the ones of $\mathrm{Comp}(\mathcal{A})$ and its morphisms $A^\bullet \to B^\bullet$ are classes of (left) **roofs** (see the diagram below on the left), that is, pairs (s, f) with $s \in \mathrm{Hom}_{K(\mathcal{A})}(A_1^\bullet, A^\bullet)$ a quasi-isomorphism and $f \in \mathrm{Hom}_{K(\mathcal{A})}(A_1^\bullet, B^\bullet)$. Two roofs (s, f) and (t, g) are *equivalent* if there is another one (r, e) making the diagram on the right commute in $K(\mathcal{A})$.

$$
\begin{tikzcd}
& A_1^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A_3^\bullet \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "e"] & & \\
& A_1^\bullet \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & A_2^\bullet \arrow[dlll, "t" near end] \arrow[dr, "g"] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}
$$

For any object $A^\bullet$, $id_{A^\bullet}$ is, with abuse of notations, the class of $(id_{A^\bullet}, id_{A^\bullet})$.

Lastly, the composition of two classes roofs (s, f) and (u, h) is the class of (sv, hk), where the roof (v, k) obtained by virtue of Definition 1.1.9 *i)*

$$
\begin{tikzcd}
& & A_2^\bullet \arrow[dl, dashed, "v"'] \arrow[dr, dashed, "k"] & & \\
& A_1^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & & B_1^\bullet \arrow[dl, "u"'] \arrow[dr, "h"] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}
$$

The derived categories $D(\mathcal{A})^*$, where $* = +, -, b$, are defined analogously.

Derived categories are additive (see [GM03, §III.4.5]), yet not abelian, in general.

1.1.3 Triangulated categories

Let $\mathcal{C}$, $\mathcal{D}$ be additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be **additive** if $\mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(D))$ is a homomorphism of abelian groups for any $C, D \in \mathcal{C}$.

Definition 1.1.12. Let $\mathcal{T}$ be an additive category and assume that there is an automorphism

$$
T : \begin{cases} A \in \mathcal{T} \mapsto A[1] \in \mathcal{T} \\ f \in \mathrm{Hom}(A, B) \mapsto f[1] \in \mathrm{Hom}(A[1], B[1]) \end{cases}
$$

called the **translation functor** (be careful; $A[1]$ and $f[1]$ simply stand for the image of A and f by T, respectively). Any sequence of objects and morphisms of any of the two equivalent representations

$$
\begin{tikzcd}
A \arrow[r] & B \arrow[r] & C \arrow[r, "{[1]}"] & A[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr] & \\
C \arrow[ur, "{[1]}"] & & B \arrow[ll]
\end{tikzcd}
$$

is called a **triangle** and a morphism $(\ldots, a, b, c, a[1], \ldots)$ of triangles is any family of morphisms in $\mathcal{T}$ making the diagram below commute.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, dashed, "a"] & B \arrow[r] \arrow[d, dashed, "b"] & C \arrow[r, "{[1]}"] \arrow[d, dashed, "c"] & A[1] \arrow[d, dashed, "{a[1]}"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r, "{[1]}"] & A'[1]
\end{tikzcd}$$

Lastly, an **octahedron** is the diagram consisting of an *upper* and a *lower* cap (left and right diagrams, respectively),

$$\begin{tikzcd}
A' \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] & & C \arrow[ll] \\
& B \arrow[ur] \arrow[dl] \arrow[u, phantom, "\star"] \arrow[d, phantom, "\star"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
C' \arrow[rr, "{[1]}"] & & A \arrow[uu] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[dd, "{[1]}"'] & & C \arrow[ll] \arrow[dl] \\
& B' \arrow[ul] \arrow[dr, "{[1]}"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] \arrow[l, phantom, "\star"] \arrow[r, phantom, "\star"] & \\
C' \arrow[rr, "{[1]}"] \arrow[ur] & & A \arrow[uu]
\end{tikzcd}$$

where the triangles marked by the symbol $\circlearrowleft$ are commutative and the ones with a $\star$ inside are distinguished (see below), such that the following squares commute.

$$\begin{tikzcd}
& B \arrow[dl] \arrow[dr] & \\
C' \arrow[dr] & & C \arrow[dl] \\
& B' &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B' \arrow[dl] \arrow[dr, "{[1]}"] & \\
A' \arrow[dr, "{[1]}"] & & A \arrow[dl] \\
& B &
\end{tikzcd}$$

$\mathcal{T}$ is said to be **triangulated** if it satisfies axioms TR1-TR4 below and is endowed with a translation functor and a class of **distinguished triangles**, characterized by axioms TR1 and TR2.

TR1.
- ▲ For any $A \in \mathcal{T}$, $A \to A \to 0 \to A[1]$, where the first morphism is the identity, is distinguished;
- ▲ triangles isomorphic to distinguished ones are distinguished;
- ▲ any morphism $A \to B$ can be completed to a distinguished triangle $A \to B \to C \to A[1]$ (C is sometimes called the **cone** of $A \to B$).

TR2. $A \to B \to C \to A[1]$ is distinguished if and only if so is $B \to C \to A[1] \to B[1]$.

TR3. Any diagram as the one below on the left, whose rows are distinguished triangles and with squares commutative, can be completed to a morphism of triangles as the one on the right below.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r, "{[1]}"] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r, "{[1]}"] & A'[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r, "{[1]}"] \arrow[d, dashed] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r, "{[1]}"] & A'[1]
\end{tikzcd}$$

TR4. Any upper cap diagram can be completed to an octahedron (equivalently, any lower cap diagram can be completed to an octahedron; see [BBD82, pp. 21, 22]).

The homotopy and the derived categories of an abelian category are examples of triangulated categories, as proved in [GM03, IV.1.9-14 and IV.2].

Here is a hint to remember the construction of the upper cap. Take a commutative triangle of vertices A, B and C and complete $A \to B$ and $B \to C$ to distinguished triangles using TR1. The missing map $A' \to C'$ is just the composition $A' \to B \to C'$ (mind the [1]).

The octahedron diagram can also be depicted in the following way, which highlights the morphisms between the distinguished triangles. Begin with a commutative triangle of vertices A, B, C, as before, and complete all morphisms to distinguished triangles so as to obtain a diagram in which any three consecutive arrows represent a distinguished triangle and all triangles and the square are commutative.

$$\begin{tikzcd}
 & & C' \arrow[dr] & & \\
 & B \arrow[ur] \arrow[dr] & & B' \arrow[ddr] & \\
 & & C \arrow[ur] \arrow[drr] & & \\
A \arrow[uur] \arrow[urr] & & & & A'
\end{tikzcd}$$

Let $\mathcal{T}$ be a triangulated category and denote the image of an object A under the inverse of the translation functor T by $A[-1]$. Then, for any $n \in \mathbb{Z}$,

$$A[n] := \begin{cases} A & \text{if } n = 0, \\ T(A[n-1]) & \text{if } n > 0, \\ T^{-1}(A[n+1]) & \text{if } n < 0. \end{cases}$$

Definition 1.1.13. Let $\mathcal{T}$ be a triangulated category. A pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of two *strictly* ($\neq \mathcal{T}$) full subcategories is called a **t-structure** on $\mathcal{T}$ if

i) $\mathrm{Hom}(A, B) = 0$ for any $A \in \mathcal{D}^{\leq 0}$ and any $B \in \mathcal{D}^{\geq 1}$, where $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n] = T^{-n}(\mathcal{D}^{\geq 0})$ for any $n \in \mathbb{Z}$ (analogously, $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$);

ii) $\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 1}$ and $\mathcal{D}^{\leq 0} \supset \mathcal{D}^{\leq 1}$ ($\subset$ means that the category on the left is a strict subcategory of the one on the right);

iii) for any $X \in \mathcal{T}$, there are $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$ such that $A \to X \to B \to A[1]$ is distinguished.

$\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the **core** of the t-structure.

If $\mathcal{T}$ is a triangulated category endowed with a t-structure, then the core of the t-structure is an abelian category by [BBD82, Theorem 1.3.6]; the objects A, B in Definition 1.1.13 (iii) are unique up to isomorphism by [GM03, IV.4.5] and are denoted by $\tau_{\leq 0}A$ and $\tau_{\geq 1}A$, respectively. In addition, $\tau_{\leq 0}$ and $\tau_{\geq 1}$ are called the **truncation functors** and the composition $H^0 := \tau_{\leq 0} \circ \tau_{\geq 0} \cong \tau_{\leq 0} \circ \tau_{\geq 0}$ (see [BBD82, Proposition 1.3.5]), where $\tau_{\geq 0}A := \tau_{\geq 1}A[-1]$ for any object A, is called the **0-th cohomology functor**. For any $\alpha \in \mathbb{Z}$, the **α-th cohomology functor** H^α is obtained by H^0 by translation; namely, $H^\alpha(A) := H^0(A[\alpha])$ for any object A.

Example 1.1.14. Any abelian category $\mathcal{A}$ is the core of its derived category $D(\mathcal{A})$ with respect to the **natural t-structure** (see [GM03, IV.4.3])

$$\mathcal{D}^{\geq 0} = \{A^\bullet \in D(\mathcal{A}) : H^\alpha(A^\bullet) = 0 \ \forall \alpha < 0\},$$
$$\mathcal{D}^{\leq 0} = \{A^\bullet \in D(\mathcal{A}) : H^\alpha(A^\bullet) = 0 \ \forall \alpha > 0\}.$$

The truncation functor $\tau_{\leq 0}$ is defined on objects by

$$\tau_{\leq 0}(A^\bullet) : \qquad \longrightarrow A^{-2} \longrightarrow A^{-1} \longrightarrow \ker d_A^0 \longrightarrow 0 \longrightarrow$$

and $\tau_{\geq 1} := id_{D(\mathcal{A})}/\tau_{\leq 0}$.

1.2 Sheaves

Here are the basics of sheaf theory. The definition of (pre)sheaves, their stalks and section functors are recalled in Section 1.2.1, along with some indispensable properties. Section 1.2.2 is devoted to (left/right) exact and derived functors, with particular attention to the section functors, which give rise to cohomology with coefficients in a sheaf. Section 1.2.3 concerns operations on functors and duality on complexes of sheaves.

1.2.1 The category of sheaves

Throughout the subsection, $(X, \mathcal{U})$ will be a topological space.

The topology $\mathcal{U}$ can be made into a category, denoted by $\mathcal{U}$, as well, whose objects are the open sets, whose morphisms are the inclusions between open sets and with obvious composition.

Definition 1.2.1. A **presheaf** of abelian groups on X is a contravariant functor

$$\mathcal{P} : \begin{cases} U \in \mathcal{U} \mapsto \mathcal{P}(U) \in \mathrm{Ab} \\ i_{UV} : U \hookrightarrow V \mapsto \rho_{UV} : \mathcal{P}(V) \to \mathcal{P}(U). \end{cases}$$

The elements $s \in \mathcal{P}(V)$ are called **sections** on V, while $s|_U := \rho_{UV}(s)$ is usually called the **restriction** of s to U.

Presheaves of sets, rings, etc. can be defined by replacing the target category Ab suitably.

From the definition, it follows that a *morphism of presheaves $\tau : \mathcal{P} \to \mathcal{P}'$ is a natural transformation.*

Given a presheaf $\mathcal{P}$, it is quite common to think of sections as functions defined on some open set. In many applications, however, some conditions are imposed to them so that they have a "good behaviour"; for instance, sections representing continuous functions are expected to glue together when they agree on the overlaps of their domains and such an extension should be unique.

Definition 1.2.2. A **sheaf** $\mathcal{F}$ of abelian groups on X is a presheaf such that $\mathcal{F}(\emptyset) = 0$ and, if $U \in \mathcal{U} \setminus \{\emptyset\}$ and $\{U_\alpha\}_\alpha$ is an open cover of U, then

i) if $s, t \in \mathcal{F}(U)$ are such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$;

ii) if there are section $s_\alpha \in \mathcal{F}(U_\alpha)$ for all α and $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ whenever $U_\alpha \cap U_\beta \neq \emptyset$, then there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for any α.

Presheaves and sheaves on X form two categories, $\mathrm{pSh}(X)$ and $\mathrm{Sh}(X)$, respectively; the latter a full subcategory of the former. Let $\tau : \mathcal{P} \to \mathcal{P}'$ be a morphism between presheaves on X of abelian groups. The *kernel* $\ker \tau : U \mapsto \ker \tau(U)$ and the *image* $\operatorname{im} \tau : U \mapsto \operatorname{im} \tau(U)$ of τ are presheaves, whereas, if $\mathcal{P}$ and $\mathcal{P}'$ are sheaves, only $\ker \tau$ is always a sheaf. The construction of a sheaf playing the same role as the image of a presheaf map is achieved by means of the notion of *sheafification*.

Definition 1.2.3. Let $(I, \preceq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A **direct system** in $\mathcal{C}$ is a pair $(\{C_i\}_{i \in I}, \{\varphi_{ij} : C_i \to C_j\}_{i \preceq j})$ consisting of a family of objects and a family of morphisms in $\mathcal{C}$ such that $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ whenever $i \preceq j \preceq k$.

The **direct limit** of a direct system is a pair $(\varinjlim C_h, \{\gamma_i : C_i \to \varinjlim C_h\}_{i \in I})$, consisting of an object and a family of morphisms such that $\gamma_j \varphi_{ij} = \gamma_i$ whenever $i \preceq j$, which is the solution of the following universal problem.

For any $X \in \mathcal{C}$ and any family of morphisms $\{f_i : C_i \to X\}_{i \in I}$ such that $f_j \varphi_{ij} = f_i$ whenever $i \preceq j$, there is a unique morphism $\theta : \varinjlim C_h \to X$ making the following diagram commute

$$
\begin{tikzcd}
\varinjlim C_h \arrow[rr, dashed, "\theta"] & & X \\
& C_i \arrow[ul, "\gamma_i"] \arrow[ur, "f_i"'] \arrow[d, "\varphi_{ij}"] & \\
& C_j \arrow[uul, bend left, "\gamma_j"] \arrow[uur, bend right, "f_j"'] &
\end{tikzcd}
$$

Direct systems and limits have their dual notions: inverse systems and limits (see [Rot09, Definitions pp. 230, 231]).

In the categories of abelian groups, (left/right) modules and rings, which will be considered while working with sheaves, the direct limits exist by [Rot09, Proposition 5.23]. Moreover, when I is a **directed set**, that is, when for any $i, j \in I$, there is $k \in I$ such that $i \preceq k$ and $j \preceq k$,

$$\varinjlim C_h = \frac{\sqcup_{i \in I} C_i}{\sim},$$

where $c_i \sim c_j \Leftrightarrow \exists k \in I : i, j \preceq k$ and $\varphi_{ik}(c_i) = \varphi_{jk}(c_j)$ (see [Rot09, Corollary 5.31]). In particular, $[c_i] = 0$ if and only if $\varphi_{ij}(c_i) = 0$ for some $j \succeq i$.

Let $\mathcal{P}$ be a (pre)sheaf of abelian groups on X and let $x \in X$. The family of open sets containing x can be thought of as a directed set with respect to the reversed inclusion $\preceq$, i.e., $V \preceq U$ if and only if $U \subseteq V$. The **stalk of $\mathcal{P}$ at** x is the direct limit

$$\mathcal{P}_x := \varinjlim_{U \ni x} \mathcal{P}(U),$$

which can be thought of as the set of classes of sections on open sets containing x such that $[s] = [t]$ if and only if $s \in \mathcal{P}(U)$, $t \in \mathcal{P}(V)$ and $s|_W = t|_W$ for some $W \subseteq U \cap V$ containing x.

Definition 1.2.4. The **sheafification** $\widetilde{\mathcal{P}}$ of a presheaf $\mathcal{P}$ is the sheaf whose sections on U are the elements $s = (s_x)_{x\in U} \in \prod_{x\in U} \mathcal{P}_x$ of the direct product of the stalks of $\mathcal{P}$ on elements of U such that, for any $y \in U$, there are an open set $V \subseteq U$ containing y and $t \in \mathcal{P}(V)$ such that $s_v = [t] \in \mathcal{P}_v$ for all $v \in V$.

Proposition 1.2 of [Har77, Chapter II] states that $\widetilde{\mathcal{P}}$ is actually a sheaf and that any morphism $\mathcal{P} \to \mathcal{F}$, where $\mathcal{F}$ is a sheaf, factors through $\widetilde{\mathcal{P}}$. Moreover, $\mathcal{P}_x \cong \widetilde{\mathcal{P}}_x$ for any $x \in X$ by construction.

Example 1.2.5. Let A be an abelian group. The **constant presheaf** A_X on X, sometimes denoted also by $\underline{\mathrm{A}}$, is given by

$$A_X(U) := \{f : U \to A : f \text{ is constant}\}.$$

This is not a sheaf (see [Rot09, Example 5.64]), in general, and its sheafification, called the **constant sheaf** on X associated to A and denoted by A_X, as well, is the sheaf of locally constant functions on X.

Definition 1.2.6. Let $\tau : \mathcal{F} \to \mathcal{F}'$ be a morphism between sheaves. The **cokernel sheaf** of τ is the sheafification of $\operatorname{coker}\tau$ and, with abuse of notations, is denoted by $\operatorname{coker}\tau$, too. The **image sheaf** is, instead, $\operatorname{im}\tau = \ker(\operatorname{coker}\tau)$.

Let $\tau :\ \mathcal{F} \to \mathcal{F}'$ be a morphism between (pre)sheaves. There is a well defined morphism $\tau_x :\ \mathcal{F}_x \to \mathcal{F}'_x$ for any $x \in X$; $(\ker\tau)_x = \ker(\tau_x)$ by [Rot09, Proposition 5.80 (iii)] and, consequently, $(\operatorname{im}\tau)_x = \operatorname{im}(\tau_x)$. It follows that *a complex of (pre)sheaves is a short exact sequence if and only if so is the corresponding sequence of stalks at any point* $x \in X$ [Rot09, Theorem 5.85]. This result is fundamental in proving that both $\mathrm{pSh}(X)$ and $\mathrm{Sh}(X)$ *are abelian categories* [Rot09, Theorem 5.91]. Then, in particular, the cohomology functors $\mathcal{H}^n$ are defined.

The **cohomology sheaves** $\mathcal{H}^n(\mathcal{F}^\bullet)$ of a complex of sheaves $\mathcal{F}^\bullet$ can be accurately described by means of the following functor.

Definition 1.2.7. For any open subset $U \subseteq X$,

$$\Gamma(U, -) : \begin{cases} \mathcal{F} \in \mathrm{Sh}(U) \mapsto \mathcal{F}(U) \in \mathrm{Ab} \\ \tau : \mathcal{F} \to \mathcal{F}' \mapsto \tau_U : \mathcal{F}(U) \to \mathcal{F}'(U) \end{cases}$$

is called the **section functor** on U. If $U = X$, $\Gamma(X, -)$ is called the **global section functor**.

Given a sheaf $\mathcal{F}$ and a section $s \in \mathcal{F}(X)$, the **support** $\operatorname{supp} s$ of s is the closure of $\{x \in X : [s] \neq 0 \in \mathcal{F}_x\}$.

If X is a locally compact Hausdorff space, the functor $\Gamma_c(X, -) : \mathrm{Sh} \to \mathrm{Ab}$ defined on objects by $\Gamma_c(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) : \operatorname{supp} s \text{ is compact}\}$ is called the **global section functor with compact support**.

Given a complex of sheaves $\mathcal{F}^\bullet$, $\mathcal{H}^\alpha(\mathcal{F}^\bullet)$ is the sheafification of the presheaf $U \subseteq X$ open $\mapsto H^\alpha(\Gamma(U, \mathcal{F}^\bullet))$ as explained in [Ive86, p. 89]. Similarly, the **compactly supported cohomology** $\mathcal{H}^\alpha_c(\mathcal{F}^\bullet)$ of $\mathcal{F}^\bullet$ is defined.

1.2.2 Cohomology

In algebraic topology, the (co)homology of a topological space X is usually introduced with coefficient in $\mathbb{Z}$ and, later, it is shown how to change the ring of coefficients by means of the universal coefficient theorems. Sheaves permit a further generalization; in fact, it is possible to study *cohomology with coefficients in a sheaf* $\mathcal{F}$, which is the topic of this subsection.

To start with, remember that an additive functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories is said to be **exact** if it preserves the exactness of short exact sequences $0 \to A \to B \to C \to 0$; **left** (**right**, respectively) **exact** if exactness is lost at the term C on the right (A on the left, respectively).

Definition 1.2.8. Let $F : \mathcal{A} \to \mathcal{A}'$ be a left (right, respectively) exact functor between abelian categories. A class of objects R is said to be **adapted to** F if

i) R is closed under finite direct sums;

ii) F transforms acyclic bounded below (above, respectively) complexes of objects of R into acyclic complexes;

iii) *$\mathcal{A}$ has enough objects of R.* In other words, for any object $A \in \mathcal{A}$, there is $B \in R$ and a **monomorphism** $f : A \to B$, i.e. $\ker f = 0$ (an **epimorphism** $g : B \to A$, i.e. $\operatorname{coker} g = 0$, respectively). Moreover, if there is another monomorphism $f' : A' \to B$ (epimorphism $g' : B \to A'$), then A and A' are isomorphic.

If $F : \mathcal{A} \to \mathcal{A}'$ is a right exact functor, R is a class of objects adapted to F if *i)* and the following requirements are met.

ii') F transforms acyclic bounded above complexes of objects of R into acyclic complexes;

iii') *$\mathcal{A}$ has enough objects of R.* In other words, for any object $A \in \mathcal{A}$, there is $B \in R$ and an **epimorphism** $g : B \to A$, i.e. $\operatorname{coker} g = 0$. Moreover, if there is another epimorphism $g' : B \to A'$, then A and A' are isomorphic.

If F is a left exact functor for which there exists an adapted class of objects R, then any complex $A^\bullet \in \mathrm{Comp}(\mathcal{A})$ is quasi-isomorphic to a complex consisting of objects of R and $D^+(\mathcal{A})$ is equivalent to the localization of $K^+(\mathcal{A})$ with respect to the class of quasi-isomorphisms in R (see [GM03, §III.6.4] and its proof). Under these hypotheses, it can be proved the existence of the solution, called the *right derived functor* $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ of F, to a certain universal mapping problem (see [BGK$^+$87, §I.9]). In the following, it will be enough to know that RF is defined on complexes of terms adapted to F by

$$R^\alpha F(A^\bullet) := (RF(A^\bullet))^\alpha = F(A^\alpha) \ \forall \alpha$$

and that this definition extends the "classical" one (see [GM03, §III.6.13] and [Ive86, §I.7]). Precisely, take a **resolution** of objects adapted to F, i.e. a quasi-isomorphism

$$A^\bullet \to A_1^\bullet$$

with $A_1^\bullet$ a complex consisting of terms adapted to F. Then,

$$R^\alpha F(A^\bullet) := H^\alpha(RF(A^\bullet)) = H^\alpha(A_1^\bullet),\ \forall\alpha.$$

As proved in [Rot09, Theorem 6.16], the definition does not depend on the choice of the quasi-isomorphism.

Similarly, if F is a right exact functor and R is an adapted class of objects to F, the *left derived functor* $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ of F exists.

Example 1.2.9. Consider the functors Hom in any abelian category $\mathcal{A}$ and $\otimes$ in the category of (left/right) modules. In general, they are not exact and the corresponding derived functors are denoted by Ext and Tor, respectively. An object $A \in \mathcal{A}$ is said to be **injective** or **projective**, respectively, if $\mathrm{Hom}(A, -)$ or $\mathrm{Hom}(-, A)$ is exact. A left (right, respectively) module M is called **flat** if $- \otimes M$ ($M \otimes -$, respectively) is exact.

Consider again a topological space X. A sheaf $\mathcal{F}$ on X is said to be **flabby** if the restriction $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for any open set U; **soft** if the restriction $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective for any compact subset[1] K. The following implications hold: injective $\Rightarrow$ flabby $\Rightarrow$ soft (see [Ive86, p. 93, Theorem 3.5] and [KS94, p. 104]).

Proposition 1.2.10.

i) The functors $\Gamma(X, -)$ *and* $\Gamma_c(X, -)$ *are left exact (see* [Rot09, Lemma 6.68] *and* [Ive86, p. 147]*);*

ii) the class of flabby sheaves is adapted to $\Gamma(X, -)$ *(see* [Ive86, p. 93, Theorem 3.5] *and* [Rot09, Propositions 6.72 and 6.73]*);*

iii) the class of soft sheaves is adapted to $\Gamma_c(X, -)$ *(see* [Ive86, p. 152, Theorem 2.7]*). If X is also countable at infinity, then the class of soft sheaves is adapted to* $\Gamma(X, -)$ *(see* [KS94, Proposition 2.5.10]*).*

Definition 1.2.11. The n-th **cohomology group** of X **with coefficients in** $\mathcal{F}$ is the n-th right derived functor

$$H^n(X, \mathcal{F}) := H^n(R\Gamma(X, \mathcal{F})).$$

Analogously, the cohomology groups $H_c^n(X, \mathcal{F}) := H^n(R\Gamma_c(X, \mathcal{F}))$ of X **with compact support** are defined.

1.2.3 Operations on sheaves

Given a sheaf on X, it is possible either to construct new sheaves on other topological spaces or to define sections over subspaces which are not open. Here, only the definitions of certain functors will be recalled.

[1] $\mathcal{F}(K)$ requires the definition of pullback of a sheaf, given in §1.2.3.

Let $f : X \to Y$ be a continuous function. The **pullback** $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ and the **pushforward** $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ functors are defined on objects by

$$f^*(\mathcal{G})(U) := \varinjlim_{\substack{V \supseteq f(U) \\ V open}} \mathcal{G}(V), \qquad f_*(\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)),$$

respectively. In particular, when X is a subspace of Y and f is inclusion, $f^*(\mathcal{G})$ is denoted by $\mathcal{G}|_X$ and called the **restriction of $\mathcal{G}$ to X**.

Example 1.2.12. Let $\mathbb{A}$ be a commutative ring and let $\mathrm{Sh}(X, \mathbb{A})$ be the category of sheaves of $\mathbb{A}$-modules, called the **$\mathbb{A}$-sheaves**. An $\mathbb{A}$-sheaf $\mathcal{L}$ is called an **$\mathbb{A}$-local system** if it is **locally constant**; i.e. there are an open cover $\{U_\alpha\}_\alpha$ of X and a family $\{M_\alpha\}_\alpha$ of $\mathbb{A}$-modules such that $\mathcal{L}|_{U_\alpha}$ is the constant sheaf associated to M_α for all α. A local system is called **trivial** if it is the constant sheaf on X.

It is natural to ask how to determine whether a local system $\mathcal{L}$ is trivial. Such a problem can be tackled with by studying the *monodromy representation of* $\mathcal{L}$ (see [BT82, §13]). Indeed, under certain hypotheses on X, the category of $\mathbb{A}$-local systems on X with values in an $\mathbb{A}$-module M is equivalent to the category of representations $\pi_1(X, x) \to \mathrm{Aut}(M)$ of the fundamental group of X on M (see [Dim04, Proposition 2.5.1]); therefore, $\mathcal{L}$ is constant if and only if $\pi_1(X, x) = 0$.

If both X and Y are locally compact Hausdorff, then the **pushforward with proper support** functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is given by

$$f_!(\mathcal{F})(V) := \{s \in f_*(\mathcal{F})(V) : f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to Y \text{ is a proper map}\}.$$

The functors defined so far extend to the categories of complexes of sheaves by applying them term by term; precisely, if $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are complexes of sheaves on X and Y, respectively, then $f^*(\mathcal{G}^\bullet)$, $f_*(\mathcal{F}^\bullet)$ and $f_!(\mathcal{F}^\bullet)$ are the complexes whose α-th terms are $f^*(\mathcal{G}^\alpha)$, $f_*(\mathcal{F}^\alpha)$ and $f_!(\mathcal{F}^\alpha)$, respectively.

When $h : W \hookrightarrow X$ is the inclusion of a **locally closed subspace** (i.e. the intersection of an open and a closed subset), the functors h_* and $h_!$ have several interesting properties. For instance, $h_!$ is exact and so is h_* if W is closed, as opposed to the fact that the functors f_* and $f_!$ are only left exact in general (f^* is, instead, always exact). Moreover, $h_!$ has a *right adjoint* $h^! : \mathrm{Sh}(X) \to \mathrm{Sh}(W)$ (it means that $\mathrm{Hom}(\mathcal{G}, h^!(\mathcal{F})) = \mathrm{Hom}(h_!(\mathcal{G}), \mathcal{F})$ for any $\mathcal{F} \in \mathrm{Sh}(X)$ and any $\mathcal{G} \in \mathrm{Sh}(W)$), given by (see [Ive86, p. 108, Proposition 6.6])

$$h^!(\mathcal{F}) := h^*(\mathcal{F}^W),$$

where

$$\mathcal{F}^W(U) := \{s \in \mathcal{F}(U) : \mathrm{supp}\, s \subseteq W\} \ \forall U \subseteq X \text{ open}.$$

If $\mathbb{A}$ is a Noetherian ring and X and Y are locally compact Hausdorff spaces of **finite homological dimension** n (which means that $H_c^{n+1}(X, \mathcal{F}) = 0$ for any sheaf on X), then the right derived functor $Rf_! : D^+(X, \mathbb{A}) \to D^+(Y, \mathbb{A})$, where $D^+(-, \mathbb{K})$ denotes the derived category of $\mathrm{Sh}(-, \mathbb{A})$, has a right adjoint $f^! :$

$D^+(Y, \mathbb{A}) \to D^+(X, \mathbb{A})$, called the **exceptional inverse image**. For its accurate definition, see the proof of [Ive86, p. 324, Theorem 3.1].

Here are the last two operations. Let $\mathbb{A}$ be a commutative ring. For any pair of $\mathbb{A}$-sheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is the sheaf given on objects by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathrm{Sh}(U,\mathbb{A})}(\mathcal{F}|_U, \mathcal{G}|_U) \ \forall U \subseteq X \text{ open},$$

whereas $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of

$$U \subseteq X \text{ open} \mapsto \mathcal{F}(U) \otimes_{\mathbb{A}} \mathcal{G}(U) \in \mathbb{A}\text{-mod},$$

where $\otimes_{\mathbb{A}}$ stands for the tensor product between $\mathbb{A}$-modules.

The extension of $\mathcal{H}om$ and $\otimes$ is immediate. On the one hand, $\mathcal{H}om^\bullet$ is defined exactly as the $\mathrm{Hom}^\bullet$ functor (1.1.1). On the other hand, $\otimes$ transforms two bounded above complexes of $\mathbb{A}$-sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ into the complex of $\mathbb{A}$-modules $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ whose terms are

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^\alpha := \bigoplus_{\beta \in \mathbb{Z}} \mathcal{F}^\beta \otimes \mathcal{G}^{\alpha-\beta}$$

and whose differentials $d_\otimes^\alpha : (\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^\alpha \to (\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^{\alpha+1}$ are given by

$$d_\otimes^\alpha := (d_{\mathcal{F}^\bullet}^\beta \otimes id_{\mathcal{G}^{\alpha-\beta}} + (-1)^\alpha id_{\mathcal{F}^\beta} \otimes d_{\mathcal{G}^\bullet}^{\alpha-\beta})_{\beta \in \mathbb{Z}}.$$

The derived functors of $\mathcal{H}om^\bullet$ and $\otimes$ are denoted, respectively, by

$$R\mathcal{H}om^\bullet : D^-(X)^\circ \times D^+(X) \to D(X), \qquad \overset{L}{\otimes}: D^-(X) \times D^-(X) \to D^-(X).$$

The functor $R\mathcal{H}om^\bullet$ gives rise to the dual of any complex of sheaves. Let $\mathbb{A}$ be a Noetherian commutative ring and let $\mathbb{K}$ be a field. To start with, remember that the dual of an $\mathbb{A}$-module M is $M^\vee := \mathrm{Hom}_{\mathbb{A}-mod}(M, \mathbb{A})$ and, if $\mathbb{A}$ is replaced by $\mathbb{K}$, the dual of a $\mathbb{K}$-vector space is obtained. For complexes of sheaves, duality is defined either for $\mathbb{A}$-sheaves on a locally compact Hausdorff space X of finite homological dimension or for $\mathbb{K}$-sheaves on locally compact Hausdorff spaces and the role of $\mathbb{A}$ and $\mathbb{K}$ in the case of modules and vector spaces, respectively, is played by a particular complex of sheaves, called the *dualizing complex* ω_X (for its definition, see [Ive86, V.2 and VI.2]).

Definition 1.2.13. The **dual** of $\mathcal{F}^\bullet \in D^b(X, -)$ (here, $-$ can be either $\mathbb{A}$ or $\mathbb{K}$) is the complex of sheaves

$$D\mathcal{F}^\bullet := R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \omega_X)$$

and $D : D^b(X, -) \to D^b(X, -)$ is called the **duality functor**.

Properties concerning duality shall not be listed here, but will be recalled if used.

1.3 Statement of the decomposition theorem

This last section contains the statement of the decomposition theorem 1.3.7, which holds in the derived category of bounded constructible complex of sheaves (see Section 1.3.1) and involves the intersection cohomology complexes, whose definition is recalled in Section 1.3.2.

1.3.1 Perverse sheaves

To begin with, recall the following two definitions. First, remember that an additive functor between triangulated categories is said to be **exact** if it transforms distinguished triangles into distinguished ones. Secondly, a functor $G : \mathcal{C} \to \mathcal{D}$ is said to be **full** if $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(G(A), G(B))$ is surjective for any $A, B \in \mathcal{C}$; **faithful** if such map is injective for any $A, B \in \mathcal{C}$; **fully faithful** if it is both full and faithful.

Let $\mathcal{D}_U$, $\mathcal{D}_F$ and $\mathcal{D}$ be triangulated categories related by two exact functors $i_* : \mathcal{D}_F \to \mathcal{D}$ and $j^* : \mathcal{D} \to \mathcal{D}_U$ such that

i) i_* has a left adjoint i^* and a right adjoint $i^!$;

ii) j^* has a left adjoint $j_!$ and a right adjoint j_*;

iii) $j^*i_* = 0$ and $\mathrm{Hom}_{\mathcal{D}}(j_!B, i_*A) = 0$ and $\mathrm{Hom}_{\mathcal{D}}(i_*A, j_*B) = 0$ for any $A \in \mathcal{D}_F$ and any $B \in \mathcal{D}_U$;

iv) for any $K \in \mathcal{D}$, there are a unique morphism $i_*i^*K \to j_!j^*K[1]$ and a unique morphism $j_*j^*K \to i_*i^!K[1]$, respectively, for which the following triangles are distinguished

$$j_!j^*K \to K \to i_*i^*K \to j_!j^*K[1], \qquad i_*i^!K \to K \to j_*j^*K \to i_*i^!K[1];$$

v) i_*, $j_!$ and j^* are fully faithful.

In the above hypotheses, if $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ and $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ are t-structures on $\mathcal{D}_U$ and $\mathcal{D}_F$, respectively, $\mathcal{D}$ can be endowed with a t-structure obtained by means of the ones of $\mathcal{D}_U$ and $\mathcal{D}_F$.

Theorem 1.3.1. [BBD82, Theorem 1.4.10]. *The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories of $\mathcal{D}$, where*

$$\mathcal{D}^{\leq 0} := \{K \in \mathcal{D} : j^*K \in \mathcal{D}_U^{\leq 0} \text{ and } i^*K \in \mathcal{D}_V^{\leq 0}\},$$
$$\mathcal{D}^{\geq 0} := \{K \in \mathcal{D} : j^*K \in \mathcal{D}_U^{\geq 0} \text{ and } i^!K \in \mathcal{D}_V^{\geq 0}\},$$

*is a t-structure on $\mathcal{D}$ and it is said to be obtained by **gluing** the ones of $\mathcal{D}_U$ and $\mathcal{D}_F$.*

Consider a **stratified topological space** X, i.e. there is a family S_X, called a **stratification** of X, of pairwise disjoint locally closed subsets S whose union is X. Let $\mathcal{O}$ be a sheaf of rings on X and let $\mathfrak{p} : S_X \to \mathbb{Z}$ be a function, called a **perversity**.

Definition 1.3.2. A **sheaf of $\mathcal{O}$-modules** over X is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module for any open subset $U \subseteq X$. The category whose objects are these sheaves is denoted by $\mathrm{Sh}(X, \mathcal{O})$, while its derived category is denoted by $D(X, \mathcal{O})$.

As an application of Theorem 1.3.1, $D(X, \mathcal{O})$ can be endowed with the following t-structure.

Corollary 1.3.3. [BBD82, Corollary 2.1.4]. *The pair $({}^{\mathfrak{p}}\mathcal{D}^{\leq 0}, {}^{\mathfrak{p}}\mathcal{D}^{\geq 0})$ of full subcategories of $D(X, \mathcal{O})$, where*

$$\begin{aligned} {}^{\mathfrak{p}}\mathcal{D}^{\leq 0} &:= \{\mathcal{F}^\bullet \in \mathcal{D}(X, \mathcal{O}) : \mathcal{H}^\alpha \mathcal{F}^\bullet|_S = \mathcal{H}^\alpha i_S^* \mathcal{F}^\bullet = 0,\ \forall \alpha > \mathfrak{p}(S),\ \forall S \in S_X\}, \\ {}^{\mathfrak{p}}\mathcal{D}^{\geq 0} &:= \{\mathcal{F}^\bullet \in \mathcal{D}(X, \mathcal{O}) : \mathcal{H}^\alpha i_S^! \mathcal{F}^\bullet = 0,\ \forall \alpha < \mathfrak{p}(S),\ \forall S \in S_X\}, \end{aligned}$$

is a t-structure on $D(X, \mathcal{O})$, called the ***t-structure of perversity*** *$\mathfrak{p}$. The core of such t-structure is called the the category of* ***$\mathfrak{p}$-perverse sheaves*** *of $\mathcal{O}$-modules on X.*

Since $D(X, \mathcal{O})$ has both the natural and the $\mathfrak{p}$-perversity t-structures, the operations on sheaves and the truncation and homology functors with respect to the latter t-structure are usually denoted with a $\mathfrak{p}$ as a right superscript (e.g. ${}^{\mathfrak{p}}\mathcal{H}^\alpha$).

Now, let X be a complex algebraic variety, let $\mathcal{O} = \mathbb{A}_X$ be the constant sheaf on X over a Noetherian commutative ring $\mathbb{A}$ such that any $\mathbb{A}$-module M has an injective resolution of finite length. By [Ver76, Theorem 2.2], X admits a *Whitney stratification* S_X (for the definition, see also [Mat12, §5]).

A complex $\mathcal{F}^\bullet \in D^+(X, \mathbb{A})$ is said to **constructible** (with respect to S_X) if the stalks of its cohomology sheaves $\mathcal{H}^\alpha(\mathcal{F}^\bullet)$ are finite dimensional $\mathbb{A}$-modules and the restrictions $\mathcal{H}^\alpha(\mathcal{F}^\bullet)|_S$ are locally constant. The **constructible derived category** $D_c(X, \mathbb{A})$ is the full subcategory of $D(X, \mathbb{A})$ consisting of constructible complexes and its full subcategory made of bounded complexes is denoted by $D_c^b(X, \mathbb{A})$.

When $\mathfrak{p}$ is decreasing, the $\mathfrak{p}$-perversity t-structure of $D_c^b(X, \mathbb{A})$ can be described by means of the so-called *support* and *cosupport conditions* as shown in [Dim04, Proposition 5.1.16]. In particular, if $\mathbb{A} := \mathbb{K}$ is a field and $\mathfrak{p}$ is the **middle perversity** function $\mathfrak{p}_{1/2} : S \in S_X \mapsto -\dim_{\mathbb{C}} S$, the cosupport conditions can be described as written below by means of the *Verdier duality* (see [dCM09, §2.3]).

Definition 1.3.4. If $\mathfrak{p} = \mathfrak{p}_{1/2}$, a complex of $\mathbb{K}$-sheaves $\mathcal{F}^\bullet$ is said to satisfy the

- ▲ **support condition** if $\dim \overline{\{x \in X : \mathcal{H}_x^\alpha(\mathcal{F}^\bullet) \neq 0\}} \leq -\alpha,\ \forall \alpha \in \mathbb{Z}$,
- ▲ **cosupport condition** if $\dim \overline{\{x \in X : \mathcal{H}_{c,x}^\alpha(\mathcal{F}^\bullet) \neq 0\}} \leq \alpha\ \forall \alpha \in \mathbb{Z}$,

where $\mathcal{H}_x^\alpha$ (and $\mathcal{H}_{c,x}^\alpha$) denote the stalks of the cohomology (with compact support) sheaves at x.

Proposition 1.3.5. [Dim04, Proposition 5.1.16]. *Let $\mathcal{F}^\bullet \in D_c^b(X, \mathbb{K})$ and let $\mathfrak{p}$ be a decreasing perversity function.*

- ▲ *$\mathcal{F}^\bullet \in {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}$ if and only if it satisfies the support condition;*
- ▲ *$\mathcal{F}^\bullet \in {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}$ if and only if it satisfies the cosupport condition.*

1.3.2 Decomposition theorem

There is one last ingredient needed to state the decomposition theorem; namely the intersection (co)homology complexes.

Intersection homology complexes were defined for the first time in [GM80]. There, topological pseudomanifolds are considered and the *intersection homology complex* $IC^X_\bullet := IC^{\mathfrak{p}}_\bullet(X)$ of X with respect to a sequence of integers $\mathfrak{p} = (0, p_3, \ldots, p_n)$, with $p_{k+1} = p_k$ or $p_{k+1} = p_k + 1$ for any k, called a *perversity*, is defined as a suitable subcomplex of the one of simplicial chains (see [GM80, §1.3]). The corresponding homology groups $IH^X_\alpha := IH^{\mathfrak{p}}_\alpha(X)$ are called the *intersection homology groups*.

Later, in [GM83], intersection cohomology complexes $IC^\bullet_X$ ($IC^{-\alpha}_X := IC^X_\alpha$ for all α) were defined as complexes of sheaves in $D^b(X)$. This point of view enables an axiomatic characterization of intersection cohomology complexes and the proof of the independence of the definition from the choice of a stratification (see [GM83, §4]).

The just mentioned sheaf-theoretical definition of $IC^\bullet_X$ is a particular case of a more general result. To be precise, in the most general setting discussed in Section 1.3.1, it can be shown the existence of a functor $j_{!*}$, sometimes called the *intermediate extension functor*. Its definition and several characterizations are provided in [BBD82, Remark 1.4.14.1 and 1.4.22-26]; however, in the case of a stratified topological space X, the following result holds.

Proposition 1.3.6. [BBD82, Proposition 2.1.11]. *Let S_X be a stratification of X and let $\mathfrak{p}$ be a perversity such that, whenever $S, T \in S_X$ and S is contained in the closure of T, then $\mathfrak{p}(S) \geq \mathfrak{p}(T)$. For any $n \in \mathbb{N}$, let U_n be the union of all strata S such that $\mathfrak{p}(S) \leq n$ and let $j_n : U_{n-1} \hookrightarrow U_n$ be the inclusion. Moreover, let $\mathcal{F}^\bullet$ be a $\mathfrak{p}$-perverse sheaf on U_k for some $k \in \mathbb{N}$, let $m \geq \max\{k, \max\{\mathfrak{p}(S) : S \in S_X\}\}$ be an integer and let $j : U_k \hookrightarrow X = U_m$ be the inclusion. Then*

$$j_{!*}\mathcal{F}^\bullet = \tau_{\leq m-1} R j_{m*}(\ldots(\tau_{\leq k} R j_{k+1*}\mathcal{F}^\bullet)),$$

where each $\tau_{\leq\alpha}$ is the truncation functor with respect to the natural t-structure.

In particular, in the case of sheaves of modules over a regular Noetherian ring $\mathbb{A}$ of finite Krull dimension (see [GM83, §3.0]) the intersection cohomology complex of X with respect to $\mathfrak{p}$ is defined as $IC^\bullet_X := j_{!*}\mathbb{A}_{X\setminus\Sigma}[\dim_{\mathbb{C}} X]$, where Σ is a closed subspace such that $\dim \Sigma \leq \dim X - 2$ and $X \setminus \Sigma$ is a dense manifold of the same dimension as X (see [GM80, §1.1]). When the constant sheaf $\mathbb{A}_{X\setminus\Sigma}$ is replaced by a local system $\mathcal{L}$ on $X \setminus \Sigma$, the **intersection cohomology sheaf** $IC^\bullet_X(\mathcal{L}) := j_{!*}\mathcal{L}[\dim_{\mathbb{C}} X]$ of $\mathcal{L}$ is obtained.

At long last, here is the statement of the *decomposition theorem* [BBD82, Theorem 6.2.5] as maintained in [dCM09, Theorem 1.6.1]. An outline of the several steps of its proof is given, for instance, in [dCM09, §3.1].

Theorem 1.3.7. ***(Decomposition theorem)*** *Let $f : X \to Y$ be a proper map of complex algebraic varieties. There is an isomorphism in the constructible bounded* *derived category* $D^b_c(Y, \mathbb{Q})$

$$Rf_* IC^{\bullet}_X \cong \bigoplus_{\alpha \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X)\,[-\alpha]\,. \tag{1.3.1}$$

Furthermore, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X)$ *are semisimple; i.e. there is a decomposition into finitely many disjoint locally closed and nonsingular subvarieties* $Y = \coprod S_\beta$ *and a canonical decomposition into a direct sum of intersection complexes of semisimple local systems*

$${}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X) \cong \bigoplus_{\beta} IC^{\bullet}_{\overline{S_\beta}}(L_{\alpha, S_\beta}). \tag{1.3.2}$$

The combination of formulae (1.3.1) and (1.3.2) gives

$$Rf_* IC^{\bullet}_X \cong \bigoplus_{\alpha \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X)[-\alpha] \cong \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{\beta} IC^{\bullet}_{\overline{S_\beta}}(L_{\alpha, S_\beta})[-\alpha], \tag{1.3.3}$$

which can be written in the form

$$Rf_* IC^{\bullet}_X \cong \bigoplus_{\alpha \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X)[-\alpha] \cong \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{S} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(Rf_* IC^{\bullet}_X)_S[-\alpha],$$

where S, called a **support** of f, is any $\overline{S_\beta}$ associated to a non-zero local system L_{α, S_β} (see [Max19, Definition 9.3.41]).

Chapter 2

Application of decomposition theorem to Schubert varieties

This chapter is devoted to the application of decomposition theorem to the resolution of singularities π (see Theorem 2) and is organized as follows.

In Section 2.1, notations are settled once and for all and some basic definitions and facts are recalled. In Section 2.2, a more accurate description of the direct summands appearing in the decomposition theorem is provided and, as a consequence, some families of polynomial expressions are inferred in Corollary 2.2.7. Section 2.3 is devoted to the description of the algorithm, *KaLu*, which computes the unknown polynomials involved in the expressions mentioned above and can be used to determine whether a support actually gives a contribution in the decomposition. Some ancillary files are available at `http://wpage.unina.it/carmine.sessa2/KaLu`, along with an implementation of *KaLu* in CoCoA5 [ABR]. The case of special Schubert varieties is treated in Section 2.4; Theorem 2.2.4 is restated coherently and families of polynomial identities are exhibited. The last section consists of several instances in which the polynomial expressions of Corollary 2.2.7 become identities and examples concerning (some of) the properties of Schubert varieties that can be deduced by means of their Ferrer's diagrams (defined in Section 2.1.2).

2.1 Schubert varieties

To begin with, well known facts concerning Grassmannians and Schubert varieties are given so as to settle notations.

Sections 2.1.1 and 2.1.2 are devoted to the description of Schubert varieties and their representation by means of Ferrer's diagrams. In Section 2.1.3, a peculiar class of subvarieties of a given Schubert variety $\mathcal{S}$ is introduced. It is recommended looking at the examples available in Section 2.5.2, which show how most properties of Schubert varieties are conveyed by their Ferrer's diagrams.

2.1.1 Definition and remarks

Hereby, it will be assumed that cohomology groups are with $\mathbb{Q}$-coefficients and that the chosen perversity $\mathfrak{p}$ is the middle one.

Let k be a positive integer and let H be a complex vector space. The *Grassmannian of k-dimensional subspaces of H* shall be denoted by $\mathbb{G}_k(H) := \{V \subseteq H : \dim V = k\}$.

Any complex vector bundle $\psi : E \to B$ (see [GH94, §0.5]) will be simply denoted by E whenever ψ and B are clear in the context. In particular, $\mathcal{G}_h(E) \to B$ stands for the *Grassmannian h-plane bundle of E*, whose fibre at any $b \in B$ is $\mathcal{G}_h(E)_b = \mathbb{G}_h(\psi^{-1}(b))$.

Let ω and l be positive integers with $\omega < l$. A *flag (of length ω)* in $\mathbb{C}^l$ is a finite sequence of vector subspaces $\mathcal{H} : H_1 \subset \ldots \subset H_\omega$ with $H_1 \neq 0$ and $H_\omega \subset \mathbb{C}^l$. It is called *complete* if $\omega = l - 1$ and *partial* otherwise. Another flag $\mathcal{H}' : H'_1 \subset \ldots \subset H'_{\omega'}$ is said to be a *subflag* of $\mathcal{H}$ (in formulae, $\mathcal{H}' \subseteq \mathcal{H}$), if and only if, for each $\alpha \in \{1, \ldots, \omega'\}$, there is $\beta \in \{1, \ldots, \omega\}$ such that $H'_\alpha = H_\beta$.

All flags considered later on shall be assumed to be subflags of a chosen complete flag $\mathcal{F}_c : F_1 \subset \ldots \subset F_{l-1}$ in $\mathbb{C}^l$.

Definition 2.1.1. Given a flag $\mathcal{F} : F_{j_1} \subset \cdots \subset F_{j_\omega}$ of $\mathbb{C}^l$, where $\dim F_{j_\alpha} = j_\alpha$ for every $\alpha \in \{1, \ldots, \omega\}$, and an ω-tuple of non-negative integers $\mathcal{I} = (i_{j_1}, \ldots, i_{j_\omega})$, the **Schubert variety associated to the pair** $(\mathcal{F}, \mathcal{I})$ is the subvariety of $\mathbb{G}_k(\mathbb{C}^l)$ given by

$$\mathcal{S} := \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j_\alpha}) \geq i_{j_\alpha},\ \alpha \in \{1, \ldots, \omega\}\}.$$

Let $\mathcal{S}$ be the Schubert variety associated to $(\mathcal{F}, \mathcal{I})$. From the definition, it immediately follows that $j_1 < \ldots < j_\omega < l$. Notice that $\mathcal{S}$ is neither empty nor contained in a smaller Grassmannian $\mathbb{G}_k(F_{j_\alpha})$ if and only if

$$0 < i_{j_\alpha} < \min\{k, j_\alpha\}\ \forall \alpha \in \{1, \ldots, \omega\}.$$

Moreover, even when $\mathcal{S} \neq \emptyset$, some incidence conditions $\dim(V \cap F_{j_\alpha}) \geq i_{j_\alpha}$ might be superfluous (e.g. some of them may happen to be implied by the others). It is easy to check, by means of the well known Grassmann's formula, that this is not the case if and only if, for every α,

$$i_{j_\alpha} < i_{j_{\alpha+1}}, \qquad i_{j_{\alpha+1}} - i_{j_\alpha} < j_{\alpha+1} - j_\alpha, \qquad k - i_{j_\alpha} < l - j_\alpha.$$

To sum up, given a non-empty Schubert variety $\mathcal{S}$ associated to $(\mathcal{F}, \mathcal{I})$, it is possible to get rid of the redundant conditions so as to describe $\mathcal{S}$ by means of the necessary conditions only. This leads to the following

Definition 2.1.2. Let $\mathcal{S}$ be a non-empty Schubert variety associated to $(\mathcal{F}, \mathcal{I})$. The pair $(\mathcal{F}, \mathcal{I})$ is said to be **essential** if and only if it conveys the minimum information needed to define $\mathcal{S}$; equivalently,

$$\begin{aligned} &0 < i_{j_1} < \ldots < i_{j_\omega} \leq k < l + i_{j_\omega} - j_\omega, \qquad i_{j_\alpha} < j_\alpha\ \forall \alpha \\ &i_{j_{\alpha+1}} - i_{j_\alpha} < j_{\alpha+1} - j_\alpha\ \forall \alpha < \omega. \end{aligned}$$

In particular, $\mathcal{S}$ is called either a **special** or a **single condition** Schubert variety if $\omega = 1$ and $i_{j_\omega} < k$.

Later on, the phrase "the flag of the essential pair of $\mathcal{S}$" will often be shortened to "the essential flag of $\mathcal{S}$" for brevity.

Notice that the property of being essential implies $\omega \leq k$, otherwise $i_{j_\omega} > k$, against the above conditions. Moreover, *each non-empty Schubert variety is uniquely determined by its essential pair.* Nevertheless, when two or more Schubert varieties need comparing, it is more convenient, as shown in the next pages, to describe them by virtue of pairs having the same flag.

Remark 2.1.3. Chosen a non-empty Schubert variety $\mathcal{S}$ and a flag $\mathcal{F}$, the pair $(\mathcal{F}, \mathcal{I})$ associated to $\mathcal{S}$ is unique if

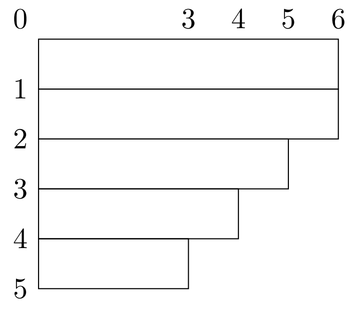
$$\begin{aligned} &0 \leq i_{j_1} \leq \ldots \leq i_{j_\omega} \leq k \leq l + i_{j_\omega} - j_\omega, \qquad i_{j_\alpha} \leq j_\alpha \ \forall \alpha \\ &i_{j_{\alpha+1}} - i_{j_\alpha} \leq j_{\alpha+1} - j_\alpha \ \forall \alpha < \omega. \end{aligned} \tag{2.1.1}$$

Indeed, assume that $\mathcal{S}$ is associated to $(\mathcal{F}, \mathcal{I})$. The condition $\dim(V \cap F_{j_\alpha}) \geq i_{j_\alpha}$ is superfluous if and only if either $i_{j_{\alpha-1}} \geq i_{j_\alpha}$ or $i_{j_{\alpha+1}} - i_{j_\alpha} \geq j_{\alpha+1} - j_\alpha$ or $i_{j_\alpha} \leq k - l + j_\alpha$. To fix ideas, suppose that $i_{j_{\alpha-1}} \geq i_{j_\alpha}$ (a similar argument holds in the other cases). Then, if i_{j_α} is replaced by $i_{j_{\alpha-1}} - \beta$, with $\beta \geq 0$, then the condition $\dim(V \cap F_{j_\alpha}) \geq i_{j_{\alpha-1}} - \beta$ is again redundant. In other words, $\mathcal{S}$ is associated to $(\mathcal{F}, (i_{j_1}, \ldots, i_{j_{\alpha-1}}, i_{j_{\alpha-1}} - \beta, i_{j_{\alpha+1}}, \ldots, i_{j_\omega}))$ for any $\beta \geq 0$.

2.1.2 Ferrer's diagrams

Definition 2.1.4. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a decreasing sequence of k non-negative integers. The **Ferrer's diagram** of λ is the diagram obtained by piling up k rows of length $\lambda_1, \ldots, \lambda_k$, from top to bottom, so that their left edges are aligned.

Example 2.1.5. The Ferrer's diagram of $\lambda = (6, 6, 5, 4, 3)$ is



Given a non-empty Schubert variety $\mathcal{S}$ described by a pair $(\mathcal{F}, \mathcal{I})$, with $\mathcal{F}$ of length ω, it is possible to associate to it the sequence of integers $\lambda^{\mathcal{S}} = (\lambda^{\mathcal{S}}_\alpha)_{\alpha=1,\ldots,k}$ defined as follows:

$$\lambda^{\mathcal{S}}_\alpha = \begin{cases} l - k - j_1 + i_{j_1} & \text{if } \alpha \in \{1, \ldots, i_{j_1}\}, \\ l - k - j_2 + i_{j_2} & \text{if } \alpha \in \{i_{j_1} + 1, \ldots, i_{j_2}\}, \\ \ldots \\ l - k - j_\omega + i_{j_\omega} & \text{if } \alpha \in \{i_{\omega-1} + 1, \ldots, i_{j_\omega}\}, \\ 0 & \text{if } \alpha \in \{i_{j_\omega} + 1, \ldots, k\}. \end{cases}$$

It is worth pointing out that $\lambda^{\mathcal{S}}$ *is independent of the choice of* $(\mathcal{F},\mathcal{I})$, hence it suffices to consider the essential pair of $\mathcal{S}$. Moreover, from the definition of Schubert variety, it follows that the sequence $\lambda^{\mathcal{S}}$ *is decreasing, with each entry non-negative and strictly lower than* $l-k$. Therefore, it makes sense to consider the Ferrer's diagram of $\lambda^{\mathcal{S}}$, which shall be called the *Ferrer's diagram of* $\mathcal{S}$. When $(\mathcal{F},\mathcal{I})$ is essential, $\lambda^{\mathcal{S}}$ contains exactly ω different integers with their repetitions, if any.

Several properties of Schubert varieties are conveyed by their Ferrer's diagrams, as pointed out throughout the chapter. At the moment, just observe that *the (complex) codimension with respect to* $\mathbb{G}_k(\mathbb{C}^l)$ *of the Schubert variety associated to the sequence* $\lambda^{\mathcal{S}} = (\lambda^{\mathcal{S}}_\alpha)_{\alpha=1,\dots,k}$ *equals the area of its Ferrer's diagram* (see [GH94, pp. 194-196]).

2.1.3 Families of subvarieties

From now on, $\mathcal{S}$ *is going to be a non-empty Schubert variety associated to the essential pair* $(\mathcal{F},\mathcal{I})$*, where* $\mathcal{F} : F_{j_1} \subset \ldots \subset F_{j_\omega}$ *and* $\mathcal{I} = (i_{j_1},\dots,i_{j_\omega})$ *with* $i_{j_\omega} < k$.

Definition 2.1.6. An **$\mathcal{S}$-variety** is a non-empty Schubert subvariety of $\mathcal{S}$ whose essential flag is a subflag of $\mathcal{F}$.

Equivalently, setting $\mathcal{F}_p : F_{j_1^p} \subset \ldots \subset F_{j_{\omega_p}^p}$ and $\mathcal{I}_p := (i_{j_1^p},\dots,i_{j^p_{\omega_p}}) + p$, where $p := (p_{j_1^p},\dots,p_{j^p_{\omega_p}})$ is an ω_p-tuple of non-negative integers,

$$\Delta_p := \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j^p_\alpha}) \geq i_{j^p_\alpha} + p_{j^p_\alpha},\ \alpha = 1,\dots,\omega_p\}$$

is the $\mathcal{S}$-variety associated to $(\mathcal{F}_p,\mathcal{I}_p)$ if and only if such pair satisfies conditions (2.1.1).

For instance, $\mathcal{S}$ is the $\mathcal{S}$-variety given by $0 := (0,\dots,0)$.

Remark 2.1.7. With the same notations as Definition 2.1.6, the pair $(\mathcal{F}_p,\mathcal{I}_p)$ satisfies conditions (2.1.1) if and only if

$$\begin{cases} 0 \leq p_{j_1^p} \leq j_1^p - i_{j_1^p} \\ M_\alpha \leq p_{j^p_\alpha} \leq N_\alpha \quad \forall \alpha = 2,\dots,\omega-1 \\ \max\{M_\omega, k-l+j^p_\omega - i_{j^p_\omega}\} \leq p_{j^p_\omega} \leq \min\{N_\omega, k - i_{j^p_\omega}\} \end{cases}$$

where, for all $\alpha \in \{2,\dots,\omega\}$,

$$\begin{aligned} M_\alpha &= \max\{0, i_{j^p_{\alpha-1}} - i_{j^p_\alpha} + p_{j^p_{\alpha-1}}\} \\ N_\alpha &= \min\{j^p_\alpha - i_{j^p_\alpha}, j^p_\alpha - j^p_{\alpha-1} - i_{j^p_\alpha} + i_{j^p_{\alpha-1}} + p_{j^p_{\alpha-1}}\}. \end{aligned}$$

In Section 2.1.2, a way to represent $\mathcal{S}$ by means of its Ferrer's diagram was described. Needless to say, all $\mathcal{S}$-varieties Δ_p can be depicted in the same way and *their associated sequences shall be denoted by* λ^p.

If Δ_p and Δ_q are $\mathcal{S}$-varieties, Δ_q is said to be a Δ_p-variety if it has the properties written in Definition 2.1.6 with $\mathcal{S}$ replaced by Δ_p. In this case, q is said to be p-**admissible**.

Now, assume that Δ_p and Δ_q are associated to $(\mathcal{F}_p, \mathcal{I}_p)$ and $(\mathcal{F}_q, \mathcal{I}_q)$, respectively. If $\mathcal{F}_p = \mathcal{F}_q = \mathcal{F}$, it is straightforward to see that $\Delta_q \subseteq \Delta_p$ if and only if $q_{j^q_\alpha} \geq p_{j^p_\alpha}$ for any $\alpha \in \{1, \dots, \omega\}$. In the general case, the pairs $(\mathcal{F}_p, \mathcal{I}_p)$ and $(\mathcal{F}_q, \mathcal{I}_q)$ can be changed by adding redundant conditions so that $\mathcal{F}_p = \mathcal{F}_q = \mathcal{F}$ again (see Example 2.5.9). In a similar fashion, it is possible to compare any pair of Schubert varieties (e.g. by describing them with respect to $\mathcal{F}_c$).

Notation 2.1.8. Given two $\mathcal{S}$-varieties Δ_p, Δ_q associated to pairs whose flags are $\mathcal{F}$, set $p \leq q \Leftrightarrow \Delta_q \subseteq \Delta_p$. If $\Delta_q \subseteq \Delta_p$, put $d(p,q) := \sum_\alpha q_{j^q_\alpha} - p_{j^p_\alpha}$ and call it the *distance* (with respect to $\mathcal{F}$) between either p and q or Δ_p and Δ_q.

In terms of Ferrer's diagrams, $\Delta_q \subseteq \Delta_p$ *if and only if the Ferrer's diagram of* Δ_p *is contained in the one of* Δ_q (see [Man01, Proposition 3.2.3 (4)]). *When* $\mathcal{S}$ *is a special Schubert variety*, p and q are integers and, as such, comparable. Consequently, in this case, *the set of all* $\mathcal{S}$*-varieties is totally ordered by inclusion.* On the contrary, when $\omega > 1$, Δ_p and Δ_q are unlikely to be comparable with respect to the inclusion relation (see Examples 2.5.9 and 2.5.10).

Remark 2.1.9. If Δ_p is an $\mathcal{S}$-variety, then the families of Δ_p-varieties and $\mathcal{S}$-varieties contained in Δ_p do not coincide, unless the essential flag of Δ_p is $\mathcal{F}$. Indeed, the notion of Δ_p-variety is stronger (see Example 2.5.10).

Later, for any chosen Schubert variety $\mathcal{S}$, the attention will be focused on the family of $\mathcal{S}$-varieties. However, several results (see Section 2.2.2) provide useful information on Schubert varieties $\mathcal{S}' \subset \mathcal{S}$ whose essential flags are not subflags of $\mathcal{F}$ if $\mathcal{S}'$ is replaced with the $\mathcal{S}$-variety $\mathcal{S}'_{\mathcal{F}}$ having the conditions of $\mathcal{S}'$ corresponding to vector spaces of $\mathcal{F}$. Namely, assume that $\mathcal{S}'$ is associated to the pair $(\mathcal{F}_c, \mathcal{I}'_c = (i'_1, \dots, i'_{l-1}))$. Then

$$\mathcal{S}'_{\mathcal{F}} := \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j_\alpha}) \geq i'_{j_\alpha},\ \alpha = 1, \dots, \omega\}.$$

Observe that $\mathcal{S}'_{\mathcal{F}} = \mathcal{S}'$ if and only if $\mathcal{S}'$ is an $\mathcal{S}$-variety.

In the particular case of two $\mathcal{S}$-varieties Δ_p and $\mathcal{S}' := \Delta_q$ associated to the pairs $(\mathcal{F}_p, \mathcal{I}_p)$ and $(\mathcal{F}_c, \mathcal{I}_q)$, respectively, such that $(\mathcal{F}_p, \mathcal{I}_p)$ is essential, the following notation is adopted:

$$\Delta_{q^p} := \mathcal{S}'_{\mathcal{F}} = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j^p_\alpha}) \geq i_{j^p_\alpha} + q_{j^p_\alpha},\ \alpha = 1, \dots, \omega\}.$$

If Δ_q is not a Δ_p-variety, the distance between Δ_{q^p} and Δ_p is strictly lower than the one between Δ_q and Δ_p, since the extra indispensable conditions of Δ_q (with respect to Δ_p) are redundant in Δ_{q^p} (otherwise they would be necessary also for Δ_{q^p}, contradicting the fact that the corresponding vector spaces do not belong to $\mathcal{F}_p$). See Example 2.5.11.

2.2 Schubert varieties and decomposition theorem

Here, a class of resolution of singularities $\pi_0 : \tilde{\mathcal{S}} \to \mathcal{S}$ is defined (see Section 2.2.1). One of the main reasons why this particular family has been chosen is that it is always possible control the fibres of these maps, which is fundamental in what follows. Decomposition theorem is applied to them so as to obtain information on the involved direct summands (see Theorem 2.2.4) and, in Section 2.2.3, certain classes of polynomial expressions.

2.2.1 A family of resolution of singularities

Let $H_1 \subset \ldots \subset H_n$ be complex vector spaces and let $k_1, \ldots, k_n$ be positive integers such that $k_\alpha < \dim H_\alpha$ for any $\alpha = 1, \ldots, n$. Put

$$\mathbb{F}(k_1, \ldots, k_n; H_1, \ldots, H_n) := \left\{ \begin{matrix} (K_1, \ldots, K_n) \in \mathbb{G}_{k_1}(H_1) \times \ldots \times \mathbb{G}_{k_n}(H_n) \\ \text{s.t.} \qquad K_1 \subset \ldots \subset K_n \end{matrix} \right\}.$$

Proposition 2.2.1. [CFS22, Proposition 3.1]. *$\mathbb{F}(k_1, \ldots, k_n; H_1, \ldots, H_n)$ is smooth.*

Proof. If $n = 1$, $\mathbb{F}(k_1; H_1) = \mathbb{G}_{k_1}(H_1)$ is smooth.

Let $n \geq 2$. There is a chain of projections

$$\begin{tikzcd} \mathbb{F}(k_1, \ldots, k_n; H_1, \ldots, H_n) \arrow[r] & \mathbb{F}(k_1, \ldots, k_{n-1}; H_1, \ldots, H_{n-1}) \arrow[r] & \ldots \arrow[dll, hook, rounded corners] \\ \mathbb{F}(k_1, k_2; H_1, H_2) \arrow[r] & \mathbb{G}_{k_1}(H_1) & \end{tikzcd}$$

and each $\mathbb{F}(k_1, \ldots, k_\alpha; H_1, \ldots, H_\alpha)$ is the Grassmannian bundle of a vector bundle over the space $\mathbb{F}(k_1, \ldots, k_{\alpha-1}; H_1, \ldots, H_{\alpha-1})$. In fact, for any $2 \leq \alpha \leq n$, there is an exact sequence of vector bundles

$$\begin{tikzcd} 0 \arrow[r] & S_{\mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1})} \arrow[r] \arrow[dr] & H_\alpha \arrow[r] \arrow[d] & Q_{\alpha-1} \arrow[r] \arrow[dl] & 0 \\ & & \mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1}) & & \end{tikzcd}$$

where $S_{\mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1})}$ and H_α are the tautological and trivial bundle over $\mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1})$, respectively, while $Q_{\alpha-1} = \operatorname{coker}(S_{\mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1})} \to H_\alpha)$.

Let $\psi_{j_{\alpha-1}} : \mathbb{F}(k_1, \ldots, k_{\alpha-1}; H_1, \ldots, H_{\alpha-1}) \to \mathbb{G}_{k_{\alpha-1}}(H_{\alpha-1})$ be the projection map and let $\psi^*_{j_{\alpha-1}}$ be its pullback. Then

$$\mathbb{F}(k_1, \ldots, k_\alpha; H_1, \ldots, H_\alpha) \cong \mathcal{G}_{k_\alpha - k_{\alpha-1}}(\psi^*_{j_{\alpha-1}} Q_{\alpha-1}). \qquad \square$$

The result above is going to be applied to certain maps defined on Schubert varieties. Up to some amendments, the next results are, respectively, [CFS22, Proposition 3.2 and Corollary 3.3].

Proposition 2.2.2. *The smooth locus of $\mathcal{S}$ is*

$$\mathcal{S}^\circ = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j_\alpha}) = i_{j_\alpha},\ \alpha = 1, \ldots, \omega\}.$$

Proof. The aim is to show that $\mathcal{S}^\circ = \mathcal{S} \setminus \operatorname{Sing} \mathcal{S}$.

The singular locus of $\mathcal{S}$ coincides with the union of all $\mathcal{S}$-varieties whose distance from $\mathcal{S}$ is 1 (see [Man01, Example 3.4.3, Theorem 3.4.4]). As a consequence, the smooth locus of $\mathcal{S}$ is

$$\begin{aligned}
&\mathcal{S}\setminus(\Delta_{(1,0,\dots,0)} \cup \ldots \cup \Delta_{(0,\dots,0,1)}) \\
&\quad = \mathcal{S}\setminus\Delta_{(1,0,\dots,0)} \cap \ldots \cap \mathcal{S}\setminus\Delta_{(0,\dots,0,1)} \\
&\quad = \{V \in \Delta_p : \dim(V \cap F_{j_1}) = i_{j_1}\} \cap \ldots \cap \{V \in \Delta_p : \dim(V \cap F_{j_\omega}) = i_{j_\omega}\} \\
&\quad = \{V \in \mathcal{S} : \dim(V \cap F_{j_\alpha}) = i_{j_\alpha},\ \alpha = 1, \dots, \omega\} = \mathcal{S}^\circ. \qquad \square
\end{aligned}$$

Set

$$\begin{aligned}
\tilde{\mathcal{S}} &:= \mathbb{F}(i_{j_1}, \dots, i_{j_\omega}, k; F_{j_1}, \dots, F_{j_\omega}, \mathbb{C}^l) \\
&= \left\{ \begin{matrix} (Z_1, \dots, Z_\omega, V) \in \mathbb{G}_{i_{j_1}}(F_{j_1}) \times \ldots \times \mathbb{G}_{i_{j_\omega}}(F_{j_\omega}) \times \mathbb{G}_k(\mathbb{C}^l) \\ \text{s.t.} \quad Z_1 \subset \ldots \subset Z_\omega \subset V \end{matrix} \right\}.
\end{aligned}$$

Corollary 2.2.3. *$\tilde{\mathcal{S}}$ is smooth and the projection*

$$\pi_0 : (Z_1, \dots, Z_\omega, V) \in \tilde{\mathcal{S}} \mapsto V \in \mathcal{S}$$

is a resolution of singularities.

Proof. Smoothness is a consequence of Lemma 2.2.1.

If $V \in \mathcal{S}^\circ$, then $\dim(V \cap F_{j_\alpha}) = i_{j_\alpha}$ for all $\alpha = 1, \dots, \omega$ and, consequently,

$$\pi_0^{-1}(V) \cong \{(V \cap F_{j_1}, \dots, V \cap F_{j_\omega})\}$$

gives the inverse map of π_0 on the open set $\mathcal{S}^\circ$. $\square$

Let $\mathcal{S}' \subseteq \mathcal{S}$ be a Schubert variety. The **Schubert cell** $\Omega_{\mathcal{S}'}$ of $\mathcal{S}'$ is the set whose elements are the vector spaces $V \in \mathbb{G}_k(\mathbb{C}^l)$ such that

$$\dim(V \cap F_\beta) = \begin{cases} 0 & \text{if } \beta \le l - k - \lambda_1^{\mathcal{S}'} \\ 1 & \text{if } l - k + 1 - \lambda_1^{\mathcal{S}'} \le \beta \le l - k + 1 - \lambda_2^{\mathcal{S}'} \\ \dots & \\ k-1 & \text{if } l - k + (k-1) - \lambda_{k-1}^{\mathcal{S}'} \le \beta \le l - k + (k-1) - \lambda_k^{\mathcal{S}'} \\ k & \text{if } \beta \ge l - \lambda_k^{\mathcal{S}'}. \end{cases}$$

Let $\mathcal{F}'$ be the essential flag of $\mathcal{S}'$ and suppose that such variety is associated to the pair $(\mathcal{F}_c, \mathcal{I}'_c)$, as well. Consider the subset $\Delta_{\mathcal{S}\mathcal{S}'}$ of $\mathcal{S}'$ whose elements satisfy the conditions corresponding to vector spaces of the essential flags of both $\mathcal{S}'$ and $\mathcal{S}$ with an equality. Namely,

$$\Delta_{\mathcal{S}\mathcal{S}'} := \left\{ \begin{matrix} V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j_\alpha}) = i'_{j_\alpha},\ \alpha = 1, \dots, \omega \\ \text{and} \quad \dim(V \cap F_{j'_\alpha}) = i'_{j'_\alpha},\ \alpha = 1, \dots, \omega' \end{matrix} \right\}.$$

By construction, $\Omega_{\mathcal{S}'} \subseteq \Delta_{\mathcal{S}\mathcal{S}'} \subseteq \mathcal{S}'^{\circ}$. When $\mathcal{S}' := \Delta_q$ is an $\mathcal{S}$-variety, the following notation is adopted:

$$\Delta_{0q} := \Delta_{\mathcal{S}\mathcal{S}'} = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F_{j_\alpha}) = i_{j_\alpha} + q_{j_\alpha},\ \alpha = 1, \dots, \omega\}.$$

It follows from the definition that, if $\mathcal{S}' \subseteq \mathcal{S}$ is a Schubert variety and $\Delta_q := \mathcal{S}'_{\mathcal{F}}$, then $\Delta_{\mathcal{S}\mathcal{S}'} \subseteq \Delta_{0q}$. Furthermore, $\mathcal{S}$ is the disjoint union of the sets Δ_{0q} with q 0-admissible.

For any $\mathcal{S}$-variety Δ_q, put

$$\tilde{\Delta}_{0q} := \pi_0^{-1}(\Delta_{0q}) = \left\{ \begin{matrix} (Z_1, \dots, Z_\omega, V) \in \tilde{\mathcal{S}} \\ \text{s.t.} \quad \dim(V \cap F_{j_\alpha}) = i_{j_\alpha} + q_{j_\alpha},\ \alpha = 1, \dots, \omega \end{matrix} \right\}.$$

The restriction of π_0

$$\rho_{0q} : (Z_1, \dots, Z_\omega, V) \in \tilde{\Delta}_{0q} \mapsto V \in \Delta_{0q}$$

is a smooth and proper fibration with fibres

$$\begin{aligned} F_{0q} := \rho_{0q}^{-1}(V) &\cong \left\{ \begin{matrix} (Z_1, \dots, Z_\omega) \in \mathbb{G}_{i_{j_1}}(V \cap F_{j_1}) \times \dots \times \mathbb{G}_{i_{j_\omega}}(V \cap F_{j_\omega}) \\ \text{s.t.} \quad \dim(V \cap F_{j_\alpha}) = i_{j_\alpha} + q_{j_\alpha},\ \alpha = 1, \dots, \omega \end{matrix} \right\} \\ &\cong \mathbb{F}(i_{j_1}, \dots, i_{j_\omega}; \mathbb{C}^{i_{j_1} + q_{j_1}}, \dots, \mathbb{C}^{i_{j_\omega} + q_{j_\omega}}), \end{aligned}$$

whose dimensions are

$$\dim F_{0q} = i_{j_1} \cdot q_{j_1} + \sum_{\alpha=2}^{\omega} q_{j_\alpha}(i_{j_\alpha} - i_{j_{\alpha-1}}).$$

All spaces and maps defined up to now fit in a cartesian square (see either Example 3.1.1 or [Ive86, Definition 5.1, p. 34]) like the one on the right, whose horizontal arrows are inclusions. In particular, i'_{0q} is the restriction of the inclusion $i_{0q} : \Delta_q \hookrightarrow \mathcal{S}$.

$$\begin{tikzcd} \tilde{\Delta}_{0q} \arrow[r, hook, "j_{0q}"] \arrow[d, "\rho_{0q}"'] & \tilde{\mathcal{S}} \arrow[d, "\pi_0"] \\ \Delta_{0q} \arrow[r, hook, "i'_{0q}"'] & \mathcal{S} \end{tikzcd} \tag{2.2.1}$$

2.2.2 Application of the decomposition theorem

Let Δ_q be an $\mathcal{S}$-variety. Put

$$\begin{aligned} &m_q := \dim \Delta_q, \qquad k_{0q} := \dim F_{0q}, \qquad d_{0q} := m_0 - m_q - k_{0q}, \\ &\delta_{0q} := k_{0q} - d_{0q}, \qquad A_{0q}^{\alpha} := H^{\alpha}(F_{0q}). \end{aligned}$$

Proper base change [Ive86, p. 322, 2.6] applied to the square (2.2.1), Deligne's theorem on smooth morphisms [dCM09, Theorem 5.2.2] and the global invariant cycle theorem [dCM09, Theorem 1.2.4] (see also [Fra20, formula (15) and Remark 3.1]) give

$$R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]\,|_{\Delta_{0q}} \cong R\rho_{0q*}\mathbb{Q}_{\tilde{\Delta}_{0q}}[m_0] \cong \bigoplus_{\alpha=0}^{2k_{0q}} A_{0q}^{\alpha} \otimes \mathbb{Q}_{\Delta_{0q}}[m_0 - \alpha] \tag{2.2.2}$$

and, for any $\alpha \in \mathbb{Z}$,

$${}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}|_{\Delta_{0q}}) \cong A_{0q}^{\alpha - m_q} \otimes \mathbb{Q}_{\Delta_{0q}}[m_q]. \tag{2.2.3}$$

Theorem 2.2.4.

i) Up to identifying $IC^\bullet_{\Delta_q}$ with its derived direct image $Ri_{0h*}IC^\bullet_{\Delta_q}$,

$$ {}^{\mathfrak{p}}\mathcal{H}^\alpha(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]) \cong \bigoplus_{\substack{q\geq 0\\ 0-adm.}} D_{0q}^{\delta_{0q}+\alpha} \otimes IC^\bullet_{\Delta_q} $$

for suitable vector spaces such that $D_{0q}^{\delta_{0q}-\alpha} \cong D_{0q}^{\delta_{0q}+\alpha}$ $\forall \alpha \geq 0$. In particular, the family of supports of π_0 coincides with the one of $\mathcal{S}$-varieties.

ii) Given a Schubert variety $\mathcal{S}' \subseteq \mathcal{S}$,

$$ IC^\bullet_{\mathcal{S}}[-m_0]|_{\Delta_{\mathcal{S}\mathcal{S}'}} \cong \bigoplus_{\alpha\geq 0} B^\alpha_{0q} \otimes \mathbb{Q}_{\Delta_{\mathcal{S}\mathcal{S}'}}[-\alpha] \tag{2.2.4} $$

for suitable vector spaces B^α_{0q}, where q is the vector such that $\Delta_q := \mathcal{S}'_{\mathcal{F}}$.

Proof. The proof, inspired by [BM83, Theorem p. 49], is an improvement and a correction of the one of [CFS22, Theorem 3.6].

To start with, notice two facts. Firstly,

$$ {}^{\mathfrak{p}}\mathcal{H}^\alpha(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]) \cong {}^{\mathfrak{p}}\mathcal{H}^{-\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]) \ \forall \alpha \geq 0 $$

by virtue of the relative hard Lefschetz theorem [dCM09, Theorem 1.6.3]. In particular, this implies that $D_{0q}^{\delta_{0q}-\alpha} \cong D_{0q}^{\delta_{0q}+\alpha}$ for any $\alpha \geq 0$. Secondly, if $\mathcal{S}' \subset \mathcal{S}$ is not an $\mathcal{S}$-variety, it suffices to consider $\Delta_q := \mathcal{S}'_{\mathcal{F}}$ (which is an $\mathcal{S}$-variety) because $\Delta_{\mathcal{S}\mathcal{S}'} \subseteq \Delta_{0q}$ and, thus, if item *ii)* holds for $\mathcal{S}$-varieties (Δ_q, in particular), then

$$ IC^\bullet_{\mathcal{S}}[-m_0]|_{\Delta_{\mathcal{S}\mathcal{S}'}} \cong IC^\bullet_{\mathcal{S}}[-m_0]|_{\Delta_{0q}}|_{\Delta_{\mathcal{S}\mathcal{S}'}} \cong \bigoplus_{\alpha\geq 0} B^\alpha_{0q} \otimes \mathbb{Q}_{\Delta_{\mathcal{S}\mathcal{S}'}}[-\alpha]. \tag{2.2.5} $$

In other words, *it suffices to prove isomorphism* (2.2.4) *only for $\mathcal{S}$-varieties.*

Set, for any $\mu \geq 0$,

$$ \mathcal{C}_\mu := \bigcup_q \Delta_q, $$

where q runs through the set of 0-admissible vectors such that $d(0,q) = \mu$, so as to obtain a strictly decreasing finite sequence of closed subsets of $\mathcal{S}$

$$ \mathcal{S} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots $$

Claim: if S is a support such that $S \subseteq \mathcal{C}_\mu$ and $\mathcal{S} \nsubseteq \mathcal{C}_{\mu+1}$, then there is a 0*-admissible q such that $d(0,q) = \mu$, $S \subseteq \Delta_q$ and $S \cap \Delta_{0q} \neq \emptyset$.*

In fact, on the one hand, $S \subseteq \Delta_q$ because S is irreducible by definition (remember that so are Schubert varieties, as well). On the other hand, if Δ_q is assumed to be associated to the pair with flag $\mathcal{F}$, then ($S \subseteq \Delta_q$ is used here)

$$ \begin{aligned} S \cap \Delta_{0q} = \emptyset &\Rightarrow (V \in S \Rightarrow \exists \alpha \in \{1, \dots, \omega\} : \dim(V \cap F_{j_\alpha}) > i_{j_\alpha} + q_{j_\alpha}) \\ &\Leftrightarrow \left(V \in S \Rightarrow \exists \epsilon = (\epsilon_1, \dots, \epsilon_\omega) \in \mathbb{N}^\omega : \sum_\alpha \epsilon_\alpha = 1 \wedge \Delta_{q+\epsilon} \ni V \right) \\ &\Leftrightarrow S \subseteq \bigcup_{\substack{\epsilon\in\mathbb{N}^\omega:\\ \sum_\alpha \epsilon_\alpha = 1}} \Delta_{q+\varepsilon}. \end{aligned} $$

For any such ε, $d(q+\varepsilon,0)=d(0,q)+1=\mu+1$, hence $S\cap\Delta_{0q}=\emptyset$ would imply $S\subseteq\mathcal{C}_{\mu+1}$.

For any $\mu\geq 0$ and any $\mathcal{S}$-variety Δ_q with $d(0,q)=\mu$, put

$$\mathscr{S}:=\{S : S \text{ is a support such that } S\subseteq\Delta_q \wedge S\cap\Delta_{0q}\neq\emptyset\}.$$

By induction on μ and for any $\mathcal{S}$-variety Δ_q with $d(0,q)=\mu$ at a time, it shall be proved that

i.1) $\mathscr{S}=\{\Delta_q\}$, which means that the supports of π_0 are exactly the $\mathcal{S}$-varieties;

i.2) for any $\alpha\geq 0$,

$$ {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_{\Delta_q}|_{\Delta_{0q}}\cong D_{0q}^{\delta_{0q}+\alpha}\otimes\mathbb{Q}_{\Delta_{0q}}[m_q] \tag{2.2.6}$$

ii)

$$IC^{\bullet}_{\mathcal{S}}[-m_0]|_{\Delta_{0q}}\cong\bigoplus_{\alpha\geq 0}B^{\alpha}_{0q}\otimes\mathbb{Q}_{\Delta_{0q}}[-\alpha].$$

The *base step* is straightforward (remember that $\mathbb{Q}_{\mathcal{S}^\circ}[m_0]\cong IC^{\bullet}_{\mathcal{S}}|_{\mathcal{S}^\circ}$ by [GM83, Theorem p. 78, (a)]).

Inductive step.

i) Decomposition theorem 1.3.7 gives

$$\begin{aligned} R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]|_{\Delta_{0q}} &\cong\bigoplus_{\alpha\in\mathbb{Z}}{}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])|_{\Delta_{0q}}[-\alpha]\\ &\cong\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{S\in\mathscr{S}}{}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_S|_{\Delta_{0q}}[-\alpha]\\ &\bigoplus_{\alpha\in\mathbb{Z}}{}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_{\mathcal{S}}|_{\Delta_{0q}}[-\alpha]\\ &\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{\substack{0<\tau<q\\ 0-adm.}}{}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_{\Delta_\tau}|_{\Delta_{0q}}[-\alpha]. \end{aligned} \tag{2.2.7}$$

By inductive hypothesis, for any 0-admissible τ with $0<\tau<q$,

$$\begin{aligned} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_{\Delta_\tau}|_{\Delta_{0q}} &\cong D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes Ri_{0\tau*}IC^{\bullet}_{\Delta_\tau}|_{\Delta_{0q}}\\ &\cong D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes i'^{*}_{qq}\circ i^{*}_{0q}\circ Ri_{0\tau*}IC^{\bullet}_{\Delta_\tau}\\ &\cong D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes i'^{*}_{qq}\circ i^{*}_{\tau q}\circ i^{*}_{0\tau}\circ Ri_{0\tau*}IC^{\bullet}_{\Delta_\tau}\\ &\cong D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes IC^{\bullet}_{\Delta_\tau}|_{\Delta_{0q}}\\ &\cong\bigoplus_{\beta\geq 0}D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes\left(B^{\beta}_{\tau q}\otimes\mathbb{Q}_{\Delta_{0q}}[m_\tau-\beta]\right), \end{aligned}$$

where functoriality and exactness of the pullback have been used, along with the fact that $i^{*}_{0\tau}\circ i_{0\tau*}=id$ by [Ive86, p. 110]. Substitute this in isomorphism (2.2.7) and combine it with formula (2.2.2) so as to obtain

$$
\begin{aligned}
\bigoplus_{\alpha=0}^{2k_{0q}} A_{0q}^{\alpha+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}[-\alpha]\cong & \bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{S\in\mathscr{S}} {}^{\mathfrak{p}}\mathcal{H}^{\alpha}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_S|_{\Delta_{0q}}[-\alpha] \\
& \bigoplus_{\alpha\in\mathbb{Z}} R^{\alpha}(IC^{\bullet}_{\mathcal{S}}|_{\Delta_{0q}})[-\alpha] \\
& \bigoplus_{\substack{0<\tau<q\\ 0-adm.}}\bigoplus_{\substack{\alpha\in\mathbb{Z}\\ \beta\geq 0}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}[-\alpha-\beta].
\end{aligned}
$$

It follows that, for every fixed $\gamma\in\mathbb{Z}$,

$$
\begin{aligned}
A_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong & \bigoplus_{S\in\mathscr{S}} {}^{\mathfrak{p}}\mathcal{H}^{\gamma+m_q}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_S|_{\Delta_{0q}} \\
& \bigoplus R^{\gamma}(IC^{\bullet}_{\mathcal{S}}|_{\Delta_{0q}}) \\
& \bigoplus_{\alpha+\beta=\gamma}\bigoplus_{\substack{0<\tau<q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}.
\end{aligned} \tag{2.2.8}
$$

Remember that it is enough to prove equation (2.2.6) for non-negative exponents; that is, for any $\gamma\geq -m_q$. For such integers, $R^{\gamma}(IC^{\bullet}_{\mathcal{S}}|_{\Delta_{0q}})=0$ because the intersection cohomology complexes satisfy the support conditions and, as a consequence, isomorphism (2.2.8) becomes

$$
\begin{aligned}
A_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong & \bigoplus_{S\in\mathscr{S}} {}^{\mathfrak{p}}\mathcal{H}^{\gamma+m_q}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_S|_{\Delta_{0q}} \\
& \bigoplus_{\alpha+\beta=\gamma}\bigoplus_{\substack{0<\tau<q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}.
\end{aligned}
$$

The category of perverse sheaves on Δ_{0q} is semisimple, thus

$$
\bigoplus_{S\in\mathscr{S}} {}^{\mathfrak{p}}\mathcal{H}^{\gamma+m_q}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])_S|_{\Delta_{0q}}
$$

has $S=\Delta_q$ as unique direct summand. Furthermore, it is a trivial local system on Δ_{0q} (see [Dim04, Proposition 2.5.1]); consequently, there are suitable vector spaces for which (2.2.6) holds.

ii) Now that *i)* has been proved, formula (2.2.8), which holds by induction, can be written as follows:

$$
\begin{aligned}
A_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong & D_{0q}^{\delta_{0q}+\gamma+m_q}\otimes\mathbb{Q}_{\Delta_{0q}}\oplus R^{\gamma}(IC^{\bullet}_{\mathcal{S}}|_{\Delta_{0q}}) \\
& \bigoplus_{\alpha+\beta=\gamma}\bigoplus_{\substack{0<\tau<q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}.
\end{aligned}
$$

In particular, $R^{\gamma}(IC^{\bullet}_{\mathcal{S}}|_{\Delta_{0q}})$ is a trivial local system on Δ_{0q} and the assertion follows from [dCM09, Remark 1.5.1]. □

2.2.3 Polynomial expressions

Given a topological space X, the *Poincaré polynomial of* X shall be denoted by

$$H_X := \sum_{\alpha \geq 0} \dim H^\alpha(X) \cdot t^\alpha.$$

In particular, when $X = \mathbb{G}_k(\mathbb{C}^l)$, it is known that (see [CGM82, §5.2])

$$H_{\mathbb{G}_k(\mathbb{C}^l)} = \frac{P_l}{P_k P_{l-k}},$$

where

$$P_\alpha := \begin{cases} 0 & \text{if } \alpha < 0 \\ 1 & \text{if } \alpha = 0 \\ h_0 \dots h_{\alpha-1} & \text{if } \alpha > 0 \end{cases} \qquad h_\beta := \sum_{\alpha=0}^{\beta} t^{2\alpha} \ \forall \beta \in \mathbb{Z}.$$

Let $\mathcal{S}' \subseteq \mathcal{S}$ be a Schubert variety, set $\Delta_q := \mathcal{S}'_{\mathcal{F}}$ and

$$a_{0q} := \sum_{\alpha \in \mathbb{Z}} \dim A^\alpha_{0q} t^\alpha, \qquad b_{0q} := \sum_{\alpha \in \mathbb{Z}} \dim B^\alpha_{0q} t^\alpha.$$

Here, a_{0q} is the Poincaré polynomial of the fibre F_{0q}, while, due to the fact that $\Omega_{\Delta_q} \subseteq \Delta_{0q}$, b_{0q} is one of the so-called Kazhdan-Lusztig polynomials, which have been named after the mathematicians who defined them for the first time in [KL79] and who related them to intersection cohomology groups in [KL80] (see also [BL00, §6] and [dCM09, §4.4]).

Notice that a_{0q} coincides with the Poincaré polynomial of the fibre of π_0 at any point of $\mathcal{S}'$ because such resolution takes into account only the conditions corresponding to the vector spaces of $\mathcal{F}$. Analogously, b_{0q} coincides with the Kazhdan-Lusztig polynomial of the pair $(\mathcal{S}, \Omega_{\mathcal{S}'})$ owing to formula (2.2.5). In other words, *it is not restrictive to work only with $\mathcal{S}$-varieties in the study of the Poincaré polynomials of the fibres of π_0 and the Kazhdan-Lusztig polynomials.*

Now, assume that $\mathcal{S}' = \Delta_q$ (i.e. it is an $\mathcal{S}$-variety) and put

$$f_{0q} := \sum_{\alpha \in \mathbb{Z}} \dim D^\alpha_{0q} t^\alpha \qquad g_{0q} := f_{0q} t^{2d_{0q}} = \sum_{\alpha \in \mathbb{Z}} \dim D^\alpha_{0q} t^{\alpha + 2d_{0q}}.$$

Remark 2.2.5 (Decomposition of $R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}$). All ingredients to describe $R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}$ as the direct sum of more elementary objects are available. Indeed, the combination of decomposition theorem 1.3.7 with Theorem 2.2.4 gives

$$\begin{aligned} R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}} &\cong \bigoplus_{\substack{\alpha \in \mathbb{Z} \\ 0-adm.\, q}} D^{\delta_{0q}+\alpha}_{0q} \otimes IC^\bullet_{\Delta_q}[-\dim \mathcal{S} - \alpha] \\ &\cong \bigoplus_{\substack{\alpha \in \mathbb{Z} \\ 0-adm.\, q}} IC^\bullet_{\Delta_q}[-\dim \mathcal{S} - \alpha]^{\oplus \dim D^{\delta_{0q}+\alpha}_{0q}}, \end{aligned}$$

Among the above polynomials, the ones *always* explicit are the a_{0q}.

Proposition 2.2.6. [CFS22, Proposition 3.11].

$$a_{0q} = \frac{P_{i_{j_1}+q_{j_1}}}{P_{i_{j_1}}P_{q_{j_1}}} \cdot \prod_{\alpha=2}^{\omega} \frac{P_{i_{j_\alpha}+q_{j_\alpha}-i_{j_{\alpha-1}}}}{P_{i_{j_\alpha}-i_{j_{\alpha-1}}}P_{q_{j_\alpha}}}.$$

Proof. Recall that $a_{0q} = \sum_\alpha \dim H^\alpha(F_{0q}) = H_{F_{0q}}$. For any $\alpha \in \{1, \ldots, \omega\}$, the projection

$$\begin{gathered} \mathbb{F}(i_{j_1}, \ldots, i_{j_\alpha}; \mathbb{C}^{i_{j_1}+q_{j_1}}, \ldots, \mathbb{C}^{i_{j_\alpha}+q_{j_\alpha}}) \\ \downarrow \\ \mathbb{F}(i_{j_1}, \ldots, i_{j_{\alpha-1}}; \mathbb{C}^{i_{j_1}+q_{j_1}}, \ldots, \mathbb{C}^{i_{j_{\alpha-1}}+q_{j_{\alpha-1}}}) \end{gathered}$$

is a fibration with fibres

$$\mathbb{G}_{i_{j_\alpha}-i_{j_{\alpha-1}}}(\mathbb{C}^{i_{j_\alpha}+q_{j_\alpha}-i_{j_{\alpha-1}}}).$$

By Leray-Hirsch theorem [Hat02, Theorem 4D.1],

$$\begin{aligned} H^\bullet(F_{0q}) \cong{}& H^\bullet(\mathbb{G}_{i_{j_1}}(\mathbb{C}^{i_{j_1}+q_{j_1}})) \otimes H^\bullet(\mathbb{G}_{i_{j_2}-i_{j_1}}(\mathbb{C}^{i_{j_2}+q_{j_2}-i_{j_1}})) \\ &\otimes \ldots \otimes H^\bullet(\mathbb{G}_{i_{j_\omega}-i_{\omega-1}}(\mathbb{C}^{i_{j_\omega}+q_{j_\omega}-i_{j_{\omega-1}}})). \end{aligned}$$

The assertion follows by taking the Poincaré polynomials. □

A family of (Poincaré) polynomial expressions is going to be exhibited as a consequence of Theorem 2.2.4. Notice that, if Δ_τ and Δ_σ are $\mathcal{S}$-variety, it is possible to define polynomials $a_{\tau\sigma}$ and $b_{\tau\sigma}$ by taking $\mathcal{S} := \Delta_\tau$ and $\Delta_q := \Delta_\sigma$ in the definitions of a_{0q} and b_{0q}, respectively. Likewise, $f_{\tau\sigma}$ and $g_{\tau\sigma}$ can be defined when, in addition, Δ_σ is a Δ_τ-variety.

Corollary 2.2.7. [CFS22, Corollary 3.12]. *Let Δ_q be an $\mathcal{S}$-variety. If $\Delta_q = \mathcal{S}$,*

$$a_{00} = g_{00} = b_{00} = 1,$$

otherwise

$$a_{0q} = b_{0q} + g_{0q} + \sum_{\substack{0<\tau<q \\ 0-adm.}} g_{0\tau} b_{\tau q}.$$

Proof. Assume $q \neq 0$. Theorem 2.2.4 asserts that

$${}^{\mathfrak{p}}\mathcal{H}^\alpha(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0])|_{\Delta_{0q}} \cong \bigoplus_{\substack{0\le\tau\le q \\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha} \otimes IC^\bullet_{\Delta_\tau}|_{\Delta_{0q}}$$

and

$$IC^\bullet_{\Delta_\tau}|_{\Delta_{0q}} \cong \bigoplus_{\beta\in\mathbb{Z}} B_{\tau q}^\beta \otimes \mathbb{Q}_{\Delta_{0q}}[m_\tau - \beta] \cong \bigoplus_{\beta\in\mathbb{Z}} B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_{0q}}[-\beta].$$

Combining these results with formula (2.2.3), it follows

$$\bigoplus_{\alpha\in\mathbb{Z}} A_{0q}^{\alpha+m_0-m_q}\otimes\mathbb{Q}_{\Delta_{0q}}[m_q-\alpha]$$
$$\cong\bigoplus_{\alpha\in\mathbb{Z}}\left(\bigoplus_{\substack{0\le\tau\le q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes\bigoplus_{\beta\in\mathbb{Z}} B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}[-\beta]\right)[-\alpha],$$

which can be written as

$$\bigoplus_{\alpha\ge -m_0} A_{0q}^{\alpha+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}[-\alpha]\cong\bigoplus_{\alpha,\beta\ge -m_0}\ \bigoplus_{\substack{0\le\tau\le q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}[-\alpha-\beta].$$

For any $\gamma\ge -m_0$,

$$A_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong\bigoplus_{\alpha+\beta=\gamma}\ \bigoplus_{\substack{0\le\tau\le q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}. \tag{2.2.9}$$

When $\tau=0$, $\delta_{00}=0$ and

$$D_{00}^{\alpha}=\begin{cases}\mathbb{Q} & \text{if } \alpha=0,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore

$$\bigoplus_{\alpha+\beta=\gamma} D_{00}^{\delta_{00}+\alpha}\otimes B_{0q}^{\beta+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong B_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}.$$

On the other hand, when $\tau=q$,

$$B_{qq}^{\beta+m_q}=\begin{cases}\mathbb{Q} & \text{if } \beta=-m_q,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore

$$\bigoplus_{\alpha+\beta=\gamma} D_{0q}^{\delta_{0q}+\alpha}\otimes B_{qq}^{\beta+m_q}\otimes\mathbb{Q}_{\Delta_{0q}}\cong D_{0q}^{\delta_{0q}+\gamma+m_q}\otimes\mathbb{Q}_{\Delta_{0q}}.$$

Taking into account these facts, isomorphism (2.2.9) becomes

$$A_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\cong B_{0q}^{\gamma+m_0}\otimes\mathbb{Q}_{\Delta_{0q}}\oplus D_{0q}^{\delta_{0q}+\gamma+m_q}\otimes\mathbb{Q}_{\Delta_{0q}}$$
$$\bigoplus_{\alpha+\beta=\gamma}\ \bigoplus_{\substack{0<\tau<q\\ 0-adm.}} D_{0\tau}^{\delta_{0\tau}+\alpha}\otimes B_{\tau q}^{\beta+m_\tau}\otimes\mathbb{Q}_{\Delta_{0q}}.$$

Put $s=\gamma+m_0$ (≥ 0) and use the equality $m_0-m_\tau-\delta_{0\tau}=2d_{0\tau}$ so as to have

$$A_{0q}^{s}\otimes\mathbb{Q}_{\Delta_{0q}}\cong B_{0q}^{s}\otimes\mathbb{Q}_{\Delta_{0q}}\oplus D_{0q}^{s-2d_{0q}}\otimes\mathbb{Q}_{\Delta_{0q}}$$
$$\bigoplus_{\alpha+\beta=s}\ \bigoplus_{\substack{0<\tau<q\\ 0-adm.}} D_{0\tau}^{\alpha-2d_{0\tau}}\otimes B_{\tau q}^{\beta}\otimes\mathbb{Q}_{\Delta_{0q}}.$$

From this formula it can be inferred that, for any $s \geq 0$,

$$\dim A_{0q}^{s} = \dim B_{0q}^{s} + \dim D_{0q}^{s-2d_{0q}} + \sum_{\alpha+\beta=s} \sum_{\substack{0<\tau<q \\ 0-adm.}} \dim D_{0\tau}^{\alpha-2d_{0\tau}} \dim B_{\tau q}^{\beta},$$

If both sides are formally multiplied by t^s and the sum over s is taken, the desired expression is achieved:

$$\begin{aligned}
a_{0q} &= \sum_{s\geq 0} \dim A_{0q}^{s} t^{s} \\
&= \sum_{s\geq 0} \dim B_{0q}^{s} t^{s} + \sum_{s\geq 0} (\dim D_{0q}^{s-2d_{0q}} t^{s-2d_{0q}}) t^{2d_{0q}} \\
&\quad + \sum_{\substack{0<\tau<q \\ 0-adm.}} \left(\sum_{\alpha\geq 0} \dim D_{0\tau}^{\alpha-2d_{0\tau}} t^{\alpha-2d_{0\tau}} \right) \left(\sum_{\beta\geq 0} \dim B_{\tau q}^{\beta} t^{\beta} \right) t^{2d_{0\tau}} \\
&= b_{0q} + f_{0q} t^{2d_{0q}} + \sum_{\substack{0<\tau<q \\ 0-adm.}} f_{0\tau} b_{\tau q} t^{2d_{0\tau}} \\
&= b_{0q} + g_{0q} + \sum_{\substack{0<\tau<q \\ 0-adm.}} g_{0\tau} b_{\tau q}.
\end{aligned}$$

Now, consider the case $q = 0$. First, $a_{00} = 1$ either by Proposition 2.2.6 or by the fact that $\pi_0 : \pi_0^{-1}(\mathcal{S}^\circ) \to \mathcal{S}^\circ$ is an isomorphism. Secondly, $g_{00} = 1$, being $D_{00}^{\alpha} = \mathbb{Q}$ for $\alpha = 0$ and 0 otherwise (as remarked in the preceding case). Lastly, $b_{00} = 1$, as well, because all but the first coefficients of the Kazhdan-Lusztig polynomial are 0, being $IC_{\mathcal{S}}^{\bullet}|_{\mathcal{S}^\circ}[-m_0] \cong \mathbb{Q}_{\mathcal{S}^\circ}$ by [GM83, Theorem p. 78, (a)]. $\square$

2.3 Computation of certain Poincaré polynomials

In this section, the theoretical results seen up to now are used so as to obtain an iterative algorithm, named *KaLu*, for the computation of the polynomials g_{0q} and b_{0q} (see Section 2.3.1). In Section 2.3.2, the fact that not all $\mathcal{S}$-varieties contribute to the decomposition of $R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}$ is highlighted.

2.3.1 KaLu, the iterative algorithm

From now on, all $\mathcal{S}$-varieties are supposed to be described with respect to the essential flag $\mathcal{F}$ of $\mathcal{S}$.

The polynomial expressions attained in Corollary 2.2.7 shall be written in the form

$$g_{0q} + b_{0q} = R_{0q}, \tag{2.3.1}$$

where

$$R_{0q} = a_{0q} - \sum_{\substack{0<\tau<q \\ 0-adm.}} g_{0\tau} b_{\tau q}, \tag{2.3.2}$$

for any $\mathcal{S}$-variety Δ_q. The fact that the Poincaré polynomials a_{0q} are explicit is fundamental in the achievement of *KaLu*, described below. By the way, the following functions need introducing:

$$\begin{aligned} U_\beta : \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] &\mapsto \sum_{\alpha\geq\beta} c_\alpha t^\alpha \in \mathbb{Z}[t] \ \ \forall\beta\geq 0, \\ Sym : \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] &\mapsto c_0 + \sum_{\alpha\geq 1} c_\alpha (t^\alpha + t^{-\alpha}) \in \mathbb{Z}\left[t, t^{-1}\right], \\ \tilde{t}^\beta : \sum_{\alpha\geq 0} c_\alpha t^\alpha \in \mathbb{Z}[t] &\mapsto \sum_{\alpha\geq 0} c_\alpha t^{\alpha+\beta} \in \mathbb{Z}[t] \ \ \forall\beta\geq 0. \end{aligned}$$

Corollary 2.3.1. [CFS22, Corollary 4.1]. *If $\Delta_q \neq \mathcal{S}$,*

$$\begin{cases} g_{0q} = \tilde{U}_{0q}(R_{0q}) \\ b_{0q} = R_{0q} - g_{0q} \end{cases}$$

where $\tilde{U}_{0q} := \tilde{t}^{m_{0q}} \circ Sym \circ \tilde{t}^{-m_{0q}} \circ U_{m_{0q}}$ and $m_{0q} := m_0 - m_q$.

Proof. Theorem 2.2.4 states that the vector spaces D^α_{0q} are symmetric with respect to the exponent δ_{0q}; that is, the polynomials f_{0q} are symmetric with respect to the degree δ_{0q}. Since $g_{0q} = f_{0q} t^{2d_{0q}}$, these polynomials are symmetric, as well, but with respect to the degree $m_{0q} := 2d_{0q} + \delta_{0q} = m_0 - m_q$. On the other hand, the vector spaces B^α_{0q} were obtained by studying the intersection cohomology complexes $IC^\bullet_{\mathcal{S}}$ locally; that is, their restrictions to the locally closed subsets Δ_{0q}. Being $IC^\bullet_{\mathcal{S}}$ a perverse sheaf, it satisfies, in particular, the support conditions, thus $B^\alpha_{0q} = 0$ for any $\alpha \geq m_{0q}$.

Assume that $g_{0\tau}$ and $b_{\tau q}$ are known for any $0 < \tau < q$; in other words, suppose that R_{0q} is known (when $\sum_{\alpha=1}^{\omega} q_{j_\alpha} = 1$, $R_{0q} = a_{0q}$ is given by Proposition 2.2.6). The polynomial g_{0q} can be obtained by R_{0q} by deleting all terms of degree $< m_{0q}$ and by making the new polynomial symmetric with respect to the term of degree m_{0q}. Formally, $g_{0q} = \tilde{U}_{0q}(R_{0q})$; indeed:

i) the function $U_{m_{0q}}$ deletes the terms of degree $< m_{0q}$ of R_{0q};

ii) the function $\tilde{t}^{-m_{0q}}$ is just multiplication by $t^{-m_{0q}}$;

iii) the function Sym makes the obtained polynomial symmetric with respect to the term of degree 0;

iv) the function $\tilde{t}^{m_{0q}}$ shifts the polynomial so as to make it symmetric with respect to the term of degree m_{0q}.

Finally, b_{0q} is obtained by equation (2.3.1); namely, $b_{0q} = R_{0q} - g_{0q}$. □

The number m_{0q}, which is nothing but the codimension of Δ_q in $\mathcal{S}$, plays an important role in *KaLu* because of Corollary 2.3.1 and is easily calculated if interpreted by means of Ferrer's diagrams. In fact, from the remark at the end of Section 2.1.2, it follows that m_{0q} *is exactly the area of the region between the Ferrer's diagrams of* $\mathcal{S}$ *and* Δ_q. In formulas,

$$m_{0q} = m_0 - m_q = \sum_{\alpha=1}^{k} (\lambda_\alpha^q - \lambda_\alpha^{\mathcal{S}}).$$

Thanks to Proposition 2.2.6 and Corollaries 2.2.7 and 2.3.1, the algorithm *KaLu* (see Algorithm 1) for the computation of the Kazhdan-Lusztig polynomials is obtained. An implementation of *KaLu* in CoCoA5 is available at `http://wpage.unina.it/carmine.sessa2/KaLu`.

Algorithm 1 Algorithm for the computation of the polynomial b_{0q}, where $\Delta_q := \mathcal{S}'_{\mathcal{F}}$ and $\mathcal{S}' \subseteq \mathcal{S}$. It also computes all the polynomials $b_{\tau\eta^\tau}$, with both τ and η 0-admissible and $\tau < \eta$, and $g_{\tau\eta}$ if η is τ-admissible, as well.

KaLu($\mathcal{I}, \mathcal{J}, k, l, \mathcal{I}', \mathcal{J}'$)

Input: $\mathcal{I}, \mathcal{J}, \mathcal{I}', \mathcal{J}'$ vectors of integers of the same length and k, l integers such that $\mathcal{I}, \mathcal{J}, k, l$ and $\mathcal{I}', \mathcal{J}', k, l$ satisfy conditions (2.1.1) (hence, determine Schubert varieties $\mathcal{S}$, $\mathcal{S}'$) and $\mathcal{S} \subseteq \mathcal{S}'$.

Output: The polynomial b_{0q}.

if $\mathcal{S}' = \mathcal{S}$ **then**
  $b_{0q} := 1$;
else
  $q :=$ vector such that $\Delta_q = \mathcal{S}_{\mathcal{F}'}$;
  $T := [0] \cup [\tau : \tau$ is 0-admissible and $\tau < q] \cup [q]$;
  for $\mu = 1, \dots, d(0, q)$ **do**
    for $(\tau, \sigma) \in T \times T$ such that $\tau < \sigma$ and $d(\sigma - \tau) = \mu$ **do**
      if σ is τ-admissible **then**
        $R_{\tau\sigma} := a_{\tau\sigma} - \sum_{\substack{\tau<\eta<\sigma \\ \tau-adm.}} g_{\tau\eta} b_{\eta q}$;
        $g_{\tau\sigma} := \tilde{U}_{\tau\sigma}(R_{\tau\sigma})$;
        $b_{\tau\sigma} := R_{\tau\sigma} - g_{\tau\sigma}$;
      else
        $a_{\tau\sigma} := a_{\tau\sigma^\tau}$; $b_{\tau\sigma} := b_{\tau\sigma^\tau}$;
      end if;
    end for
  end for
end if
return b_{0q}

Proposition 2.3.2. [CFS22, Proposition 4.2]. *Let $\mathcal{S}' \subseteq \mathcal{S}$ be given by means of the integers k, l and the vectors of integers $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{I}'$, $\mathcal{J}'$, respectively, standing* *for the conditions and the dimensions of the vector spaces of the flags representing such varieties. KaLu*$(\mathcal{I}, \mathcal{J}, k, l, \mathcal{I}', \mathcal{J}')$ *returns* b_{0q}*, where* q *is such that* $\Delta_q := \mathcal{S}'_{\mathcal{F}}$*. In addition, whenever* τ *and* η *are* 0*-admissible and such that* $\tau < \eta$*, it computes the polynomials* $b_{\tau\eta^\tau}$ *and, if* η *is* τ*-admissible,* $g_{\tau\eta}$*.*

Proof. This algorithm deals with a finite number of objects that are described by a finite number of data each, hence the termination follows straightforwardly. For the correctness, the command lines have to be analysed.

Impose that all Schubert varieties involved in the computation are represented with respect to the essential pair $(\mathcal{F}, \mathcal{I})$ and satisfy the relations of Remark 2.1.7. Let ω be the length of $\mathcal{I}$.

If $\mathcal{S}' \neq \mathcal{S}$, the algorithm considers $\Delta_q := \mathcal{S}'_{\mathcal{F}}$ and computes the list T of all the ω-tuples $\tau < q$ that are 0-admissible (line 6). Observe that if τ belongs to T and another ω-tuple σ is τ-admissible, then σ is 0-admissible, too, and, consequently, belongs to T (the vice versa does not always hold).

Then, for every μ between 1 and $d(0, q)$, the algorithm considers all pairs (τ, σ) of elements in T such that $d(\sigma, \tau) = \mu$ (lines 7-8). If σ is τ-admissible, then $R_{\tau\sigma}$, $g_{\tau\sigma}$ and hence $b_{\tau\sigma}$ are computed by the formulae of Corollary 2.3.1, being the explicit computation of $a_{\tau\sigma}$ possible thanks to Proposition 2.2.6 (lines 10-12). Note that the algorithm must consider the values of μ in increasing order (line 7) in order to apply formula (2.3.2).

If σ is not τ-admissible, the algorithm considers Δ_{σ^τ} in place of Δ_σ. In this case, only the polynomials $a_{\tau\sigma} = a_{\tau\sigma^\tau}$ and $b_{\tau\sigma} = b_{\tau\sigma^\tau}$ are needed (line 15), where equalities hold as observed in Section 2.2.3. As pointed out at the end of Section 2.1.3, the distance between Δ_{σ^τ} and Δ_τ is lower than the one between Δ_σ and Δ_τ, so $a_{\tau\sigma^\tau}$ and $b_{\tau\sigma^\tau}$ have already been computed.

When μ reaches the value $d(0, q)$, the pair $(0, q)$ is finally considered and b_{0q} can be computed. Indeed, at that moment, all the necessary data to apply formula (2.3.2) to this pair have been obtained and stored. □

2.3.2 Relevant varieties

Here is again the definition of relevant variety, given with the notations introduced so far.

Definition 2.3.3. An $\mathcal{S}$-variety $\Delta_q \neq \mathcal{S}$ is said to be π_0**-relevant** if and only if $m_{0q} \leq 2 \dim F_{0q}$.

According to Theorem 2.2.4, the set of supports of π_0 coincides with the one of $\mathcal{S}$-varieties; nonetheless, it does not mean that all such varieties Δ_q give a contribution in the decomposition of $R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}$. For instance, if q is such that $m_{0q} > 2 \dim F_{0q}$, Δ_q does not provide any contribution in the decomposition. It might seem reasonable to expect that the converse occurs when Δ_q is π_0-relevant, yet, *KaLu* shows that this is not always the case. In Table 2.1 there are some examples of π_0-relevant varieties whose contribution in the decomposition is null; i.e. $g_{0q} = 0$. Richer lists are available in the ancillary files at `http://wpage.unina.it/carmine.sessa2/KaLu/Tests_Relevant_Varieties`.

ω	$I = [i_{j_1}, \ldots, i_\nu]$	k	$J = [j_1, \ldots, j_\nu]$	l	$q = [q_1, \ldots, q_\nu]$
2	[3, 4]	5	[6, 8]	11	[2, 1]
3	[3, 4, 5]	6	[7, 9, 11]	13	[1, 0, 1] [1, 2, 1] [3, 2, 1]
4	[3, 4, 5, 6]	7	[8, 10, 12, 14]	16	[1, 0, 1, 0] [1, 0, 0, 1] [1, 1, 0, 1] [1, 0, 1, 1] [2, 1, 0, 1] [1, 2, 1, 0] [1, 2, 1, 1] [1, 1, 2, 1] [3, 2, 1, 0] [2, 1, 2, 1] [1, 2, 2, 1] [3, 2, 1, 1] [3, 2, 2, 1] [2, 3, 2, 1] [3, 3, 2, 1]

Table 2.1: Here is, for $\omega = 2, 3, 4$, a set of input I, k, J, l for which there are π_0-relevant varieties such that $g_{0q} = 0$.

At the moment, the explanation of the geometrical reason behind this phenomenon is an open problem; nevertheless, it is immediate to see that $KaLu$ gives $g_{0q} = 0$ if $m_{0q} > \deg R_{0q}$ because g_{0q} is obtained by symmetrizing the polynomial R_{0q} with respect to the degree m_{0q}. It would be also interesting to understand if there exists a characterization of the π_0-relevant varieties which actually contribute to the decomposition.

2.4 The case of Special Schubert varieties

In this section, all results proved so far are restated for special Schubert varieties. The reason why it is worth spending time on these varieties is that the decomposition theorem becomes explicit for them (see Theorem 2.4.3) and, consequently, the corresponding polynomial expressions become identities (see Corollary 2.4.4).

In Section 2.4.1, several notations are set in order to restate Theorem 2.2.4 for special Schubert varieties. As an application of this result, two classes of polynomial identities are obtained. The former, in Section 2.4.2, is the subclass of the polynomial expressions seen in Section 2.2.3 related to special Schubert varieties. The latter, in Section 2.4.3, was proved only in this special case and involves the intersection cohomology groups (see Definition 2.4.5). The section is concluded with a partial verification of the last class of identities by means of algebraic manipulation only.

2.4.1 Restatement of Theorem 2.2.4

Definition 2.4.1. Let F be a j-dimensional vector subspace of $\mathbb{C}^l$ and let i and k be non-negative integers such that[1] $0 < i < k \leq j < l$ and $k - i < l - j$. The **special Schubert variety associated to** $(\mathcal{F} : F, \mathcal{I} := (i))$ is the subvariety of $\mathbb{G}_k(\mathbb{C}^l)$ given by

$$\mathcal{S} := \{V \in \mathbb{G}_k(\mathbb{C}^l) \,:\, \dim(V \cap F) \geq i\}.$$

In this case, all $\mathcal{S}$-varieties are either the Grassmannian $\Delta_{k-i} := \mathbb{G}_k(F)$ or special Schubert varieties

$$\Delta_p := \{V \in \mathbb{G}_k(\mathbb{C}^l) \,:\, \dim(V \cap F) \geq i + p\},$$

with $p \in \{0, \dots, k - i - 1\}$, associated to the pairs $(\mathcal{F}, \mathcal{I}_p := (i + p))$, which are always essential.

From now on, assume that a special Schubert variety $\mathcal{S}$, associated to the essential pair $(\mathcal{F}, \mathcal{I})$, has been chosen. For the purposes of this section, it will be enough to consider $\mathcal{S}$-varieties only.

There are two particular resolution of singularities for $\mathcal{S}$:

$$\pi_0 : \tilde{\mathcal{S}} := \{(Z, V) \in \mathbb{G}_i(F) \times \mathbb{G}_k(\mathbb{C}^l) \,:\, Z \subset V\} \mapsto V \in \mathcal{S},$$
$$\xi_0 : \mathcal{D}_{\mathcal{S}} := \{(V, U) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j-i}(\mathbb{C}^l) \,:\, V \subset U\} \mapsto V \in \mathcal{S}.$$

[1] The assumption $k \leq j$ is imposed only to make the polynomial identities easier to handle with. Indeed, in the general case, there is no condition $k \leq j_\alpha$.

The map π_0 has already been introduced in Section 2.2.1, while the ξ_0 will be described for any Schubert variety in Section 2.5.1. Their fibres and the corresponding Poincaré polynomials are easy to write; indeed, if Δ_q is an $\mathcal{S}$-variety and $V \in \Delta_{0q}$, then

$$\begin{aligned}
&F_{0q} := \pi_0^{-1}(V) \cong \mathbb{G}_i(V \cap F) \cong \mathbb{G}_i(\mathbb{C}^{i+q}),\\
&G_{0q} := \xi_0^{-1}(V) \cong \{U \in \mathbb{G}_{k+j-i}(\mathbb{C}^l) \,:\, V + F \subseteq U\} \cong \mathbb{G}_q(\mathbb{C}^{l-k-j+i+q})
\end{aligned}$$

and (see §2.2.3 and 2.5.1)

$$a_{0q} := H_{F_{0q}} = \frac{P_{i+q}}{P_i P_q}, \qquad H_{G_{0q}} = \frac{P_{l-k-j+i+q}}{P_{l-k-j+i} P_q}.$$

Furthermore, as a consequence of Proposition 2.5.6 either π_0 or ξ_0 or both are small. Namely,

Corollary 2.4.2. *ξ_0 is small if and only if $l - j \leq k$. Analogously, π_0 is small if and only if $l - j \geq k$. In particular, both resolutions are small if equality holds.*

Before the adaptation of Theorem 2.2.4 to special Schubert varieties, which merges [Fra20, Remark 3.3 and Theorem 3.5], recall the notations given in §2.2.2:

$$\begin{aligned}
&m_q := \dim \Delta_q = (l-k)(k-i-q) + (j-i-q)(i+q),\\
&k_{0q} := \dim F_{0q} = q \cdot i,\\
&d_{0q} := m_0 - m_q - k_{0q} = q(l-k-j+i-q),\\
&\delta_{0q} := k_{0q} - d_{0q} = -q(l-k-j-q).
\end{aligned}$$

Theorem 2.4.3. *If $l - j \geq k$,*

i) $R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0] \cong IC^\bullet_{\mathcal{S}}$;

ii) for any $\mathcal{S}$-variety Δ_q,

$$IC^\bullet_{\mathcal{S}}[-m_0]|_{\Delta_q^0} \cong R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}|_{\Delta_q^0} \cong \bigoplus_{\alpha \geq 0} H^\alpha(\mathbb{G}_i(\mathbb{C}^{i+q})) \otimes \mathbb{Q}_{\Delta_q^0}[-\alpha].$$

Instead, if $l - j < k$,

i') for any $\alpha \in \mathbb{Z}$,

$${}^{\mathfrak{p}}\mathcal{H}^\alpha(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0]) \cong \bigoplus_{q=0}^{k-i} H^{\delta_{0q}+\alpha}(\mathbb{G}_q(\mathbb{C}^{k-l+j})) \otimes IC^\bullet_{\Delta_q};$$

ii') for any $\mathcal{S}$-variety Δ_q,

$$IC^\bullet_{\mathcal{S}}[-m_0]|_{\Delta_q^0} \cong \bigoplus_{\alpha \geq 0} H^\alpha(\mathbb{G}_q(\mathbb{C}^{l-j-k+i+q})) \otimes \mathbb{Q}_{\Delta_q^0}[-\alpha].$$

For the original proof of the theorem, see [Fra20, Theorem 3.5].

2.4.2 Local polynomial identities

Let Δ_q be an $\mathcal{S}$-variety. In Section 2.2.3, the polynomials

$$g_{0q} := \sum_{\alpha\in\mathbb{Z}} \dim D^\alpha_{0q} t^{\alpha+2d_{0q}}, \qquad b_{0q} := \sum_{\alpha\in\mathbb{Z}} \dim B^\alpha_{0q} t^\alpha$$

were defined. In the case of special Schubert varieties, they are explicit because the vector spaces D^α_{0q} and B^α_{0q} are known; indeed, if $l-j\geq k$,

$$D^\alpha_{0q} = \begin{cases} 0 & \text{if } q\neq 0 \text{ or } q=0 \text{ and } \alpha\neq 0, \\ \mathbb{Q} & \text{if } q=0 \text{ and } \alpha=0. \end{cases} \qquad B^\alpha_{0q} = H^\alpha(\mathbb{G}_i(\mathbb{C}^{i+q}))\ \forall\alpha$$

whereas, if $l-j<k$,

$$D^\alpha_{0q} = H^{\delta_{0q}+\alpha}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\ \forall\alpha, \qquad B^\alpha_{0q} = H^\alpha(\mathbb{G}_q(\mathbb{C}^{l-j-k+i+q}))\ \forall\alpha.$$

As a consequence, the polynomial expressions of Corollary 2.2.7, which had been proven for special Schubert varieties beforehand (see [CFS21, Theorem 2]), take the following form.

Corollary 2.4.4 (Local polynomial identities for special Schubert varieties). *If $\Delta_q = \mathcal{S}$,*

$$a_{00} = g_{00} = b_{00} = 1,$$

otherwise,

$$a_{0q} = \begin{cases} b_{0q} & \text{if } l-j\geq k \\ b_{0q} + g_{0q} + \sum_{\tau=1}^{q-1} g_{0\tau}b_{\tau q} & \text{if } l-j<k \end{cases}$$

In particular, if $\Delta_q\neq\mathcal{S}$ and $l-j<k$, then

$$\frac{P_{i+q}}{P_iP_q} = \sum_{\tau=1}^{q-1}\left(\frac{P_{k-l+j}}{P_\tau P_{k-l+j-\tau}}\cdot\frac{P_{l-j-k+i+q}}{P_{q-\tau}P_{l-j-k+i+\tau}}\cdot t^{2d_{0\tau}}\right) \tag{2.4.1}$$

$$+\frac{P_{k-l+j}}{P_qP_{k-l+j-q}}\cdot t^{2d_{0q}} + \frac{P_{l-j-k+i+q}}{P_qP_{l-j-k+i}}. \tag{2.4.2}$$

Proof. For the original proof, see [CFS21, Theorem 2].

Since Corollary 2.2.7 holds for all Schubert varieties in a Grassmannian, the only thing which needs proving is formula 2.4.1. Since

$$b_{0q} = H_{G_{0q}} = \frac{P_{l-k-j+i+q}}{P_{l-k-j+i}P_q},$$

it suffices to substitute all terms in the equality

$$a_{0q} = b_{0q} + g_{0q} + \sum_{\tau=1}^{q-1} g_{0\tau}b_{\tau q}. \qquad \square$$

2.4.3 Global polynomial identities

Another class of polynomial identities was obtained for special Schubert variety in [CFS21, Theorem 3] as an application of [Fra20, Theorem 3.6]. In order to understand the formulas, it is better to recall the definition of hypercohomology and set some notations.

Definition 2.4.5. Let $\mathcal{F}^\bullet$ be a complex of sheaves on a topological space X and let $\mathcal{F}^\bullet \to I^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. For any $\alpha \in \mathbb{Z}$, the α-th **hypercohomology group of $\mathcal{F}^\bullet$** is

$$\mathbb{H}^\alpha(X, \mathcal{F}^\bullet) := H^\alpha(X, I^\bullet) = H^\alpha(\Gamma(X, I^\bullet)).$$

When $\mathcal{F}^\bullet := IC_X^\bullet[-\dim X]$, the definition of the α-th intersection cohomology group of X

$$IH^\alpha(X) := \mathbb{H}^\alpha(X, IC_X^\bullet[-\dim X])$$

is recovered (see [GM83, §2.1]) and the **Poincaré polynomial of the intersection cohomology groups of X** is

$$IH_X := \sum_{\alpha \in \mathbb{Z}} \dim IH^\alpha(X) \cdot t^\alpha.$$

Corollary 2.4.6. *Let $\mathcal{S}$ be a special Schubert variety. If $l - j \geq k$, then*

$$IH_{\mathcal{S}} = H_{\tilde{\mathcal{S}}};$$

otherwise, if $l - j < k$, then

$$H_{\tilde{\mathcal{S}}} = IH_{\mathcal{S}} + \sum_{q=1}^{k-i} H_{\mathbb{G}_q(\mathbb{C}^{k-l+j})} \cdot IH_{\Delta_q} \cdot t^{2d_{0q}}, \tag{2.4.3}$$

which can be written in terms of Poincaré polynomials as follows:

$$\begin{aligned}\frac{P_j P_{l-i}}{P_i P_{j-i} P_{k-i} P_{l-k}} =& \frac{P_{l-j} P_{k+j-i}}{P_{k-i} P_{l-j-k+i} P_k P_{j-i}} + \\ &+ \sum_{q=1}^{\min\{k-i, k-l+j\}} \frac{P_{k-l+j} P_{l-j} P_{k+j-i-q}}{P_q P_{k-l+j-q} P_{k-i-q} P_{l-j-k+i+q} P_k P_{j-i-q}} t^{2d_{0q}}.\end{aligned}$$

Proof. For the original proof, see [CFS21, Theorem 3].

If $l - j \geq k$, it suffices to apply hypercohomology to the isomorphism in Theorem 2.4.3 *i)*. Indeed, for any $\alpha \in \mathbb{Z}$,

$$\begin{aligned}IH^\alpha(\mathcal{S}) &= \mathbb{H}^\alpha(\mathcal{S}, IC_{\mathcal{S}}^\bullet[-m_p]) \cong \mathbb{H}^\alpha(\mathcal{S}, R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}) \\ &= H^\alpha(\Gamma(\mathcal{S}, \pi_{0*}J^\bullet)) = H^\alpha(\Gamma(\tilde{\mathcal{S}}, J^\bullet)) = H^\alpha(\tilde{\mathcal{S}}, \mathbb{Q}_{\tilde{\mathcal{S}}}) \cong H^\alpha(\tilde{\mathcal{S}}).\end{aligned}$$

Suppose $l-j<k$. Combine the decomposition theorem 1.3.7 with Theorem 2.4.3 *i')* so as to obtain

$$R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}[m_0] \cong \bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{q=0}^{k-i} H^{\delta_{0q}+\alpha}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\otimes IC^{\bullet}_{\Delta_q}[-\alpha],$$

that is (remember that $m_0-m_q-\delta_{0q}=2d_{0q}$),

$$\begin{aligned} R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}} &\cong IC^{\bullet}_{\mathcal{S}}[-m_0]\bigoplus_{\alpha\in\mathbb{Z}}\oplus\bigoplus_{q=1}^{k-i} H^{\delta_{0q}+\alpha}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\otimes IC^{\bullet}_{\Delta_q}[-m_0-\alpha] \\ &\cong IC^{\bullet}_{\mathcal{S}}[-m_0]\oplus\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{q=1}^{k-i} H^{\alpha-2d_{0q}}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\otimes IC^{\bullet}_{\Delta_q}[-m_q-\alpha]. \end{aligned}$$

Apply hypercohomology to both sides; for any $\beta\in\mathbb{Z}$,

$$\begin{aligned} \mathbb{H}^{\beta}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}) &\cong \mathbb{H}^{\beta}(IC^{\bullet}_{\mathcal{S}}[-m_0]) \\ &\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{q=1}^{k-1} H^{\alpha-2d_{0q}}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\otimes \mathbb{H}^{\beta-\alpha}(IC^{\bullet}_{\Delta_q}[-m_q]). \end{aligned} \tag{2.4.4}$$

By Definition 2.4.5,

$$IH^{\beta}(\mathcal{S})=\mathbb{H}^{\beta}(IC^{\bullet}_{\mathcal{S}}[-m_0]),$$

while

$$\begin{aligned} \mathbb{H}^{\beta}(Ri_{0q*}IC^{\bullet}_{\Delta_q}) &= H^{\beta}(\Gamma(\mathcal{S},i_{0q*}I^{\bullet}))= \\ &= H^{\beta}(\Gamma(i_{0q}^{-1}(\mathcal{S}),I^{\bullet}))=H^{\beta}(\Gamma(\Delta_q,I^{\bullet}))=\mathbb{H}^{\beta}(IC^{\bullet}_{\Delta_q}), \end{aligned}$$

where $IC^{\bullet}_{\Delta_q}\to I^{\bullet}$ is an injective resolution of $IC^{\bullet}_{\Delta_q}$. Similarly,

$$\begin{aligned} \mathbb{H}^{\beta}(R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}) &= H^{\beta}(\Gamma(\mathcal{S},R\pi_{0*}\mathbb{Q}_{\tilde{\mathcal{S}}}))=H^{\beta}(\Gamma(\mathcal{S},\pi_{0*}J^{\bullet}))= \\ &= H^{\beta}(\Gamma(\pi_0^{-1}(\mathcal{S}),J^{\bullet}))=H^{\beta}(\Gamma(\tilde{\mathcal{S}},J^{\bullet}))= \\ &= H^{\beta}(\tilde{\mathcal{S}},\mathbb{Q}_{\tilde{\mathcal{S}}})\cong H^{\beta}(\tilde{\mathcal{S}}), \end{aligned}$$

where $\mathbb{Q}_{\tilde{\mathcal{S}}}\to J^{\bullet}$ is an injective resolution of $\mathbb{Q}_{\tilde{\mathcal{S}}}$. Substitute in formula 2.4.4 in order to obtain

$$H^{\beta}(\tilde{\mathcal{S}})\cong IH^{\beta}(\mathcal{S})\oplus\bigoplus_{\alpha\in\mathbb{Z}}\left(\bigoplus_{q=1}^{k-i} H^{\alpha-2d_{0q}}(\mathbb{G}_q(\mathbb{C}^{k-l+j}))\otimes IH^{\beta-\alpha}(\Delta_q)\right).$$

Formula (2.4.3) is attained by taking the dimensions of the vector spaces appearing in the last isomorphism, by multiplying both sides of the new equality by t^{β} and by taking the sum over β.

The last step is writing all the terms by means of the Poincaré polynomials of suitable Grassmannians. The Leray-Hirsch theorem (see [Voi02, Theorem 7.33]) implies that $\tilde{\mathcal{S}}$ has the same Poincaré polynomial as $\mathbb{G}_i(F) \times \mathbb{G}_{k-i}(\mathbb{C}^{l-i})$. Thus, the left-hand side is

$$H_{\tilde{\mathcal{S}}} = H_{\mathbb{G}_i(F)\times\mathbb{G}_{k-i}(\mathbb{C}^{l-i})} = H_{\mathbb{G}_i(F)} \cdot H_{\mathbb{G}_{k-i}(\mathbb{C}^{l-i})}.$$

In addition,

$$\begin{aligned} IH^\alpha(\mathcal{S}) &= \mathbb{H}^\alpha(\mathcal{S}, IC^\bullet_{\mathcal{S}}[-m_0]) = \mathbb{H}^\alpha(\mathcal{S}, R\xi_{0*}\mathbb{Q}_{\mathcal{D}_{\mathcal{S}}}) = \\ &= H^\alpha(\mathcal{D}_{\mathcal{S}}) \cong \bigoplus_{\beta\in\mathbb{Z}} H^\beta(\mathbb{G}_{k-i}(\mathbb{C}^{l-j})) \otimes H^{\alpha-\beta}(\mathbb{G}_k(\mathbb{C}^{k+j-i})), \end{aligned}$$

where $R\xi_{0*}\mathbb{Q}_{\mathcal{D}_{\mathcal{S}}} \cong IC^\bullet_{\mathcal{S}}[-m_0]$ is due to [GM83, Corollary §6.2], since $l - j < k$ is equivalent to the smallness of ξ_0, and the last isomorphism is a combination of Leray-Hirsch theorem with Künneth formula. Then,

$$IH_{\mathcal{S}} = H_{\mathbb{G}_{k-i}(\mathbb{C}^{l-j})\times\mathbb{G}_k(\mathbb{C}^{k+j-i})} = H_{\mathbb{G}_{k-i}(\mathbb{C}^{l-j})} \cdot H_{\mathbb{G}_k(\mathbb{C}^{k+j-i})}.$$

Now,

$$H_{\tilde{\mathcal{S}}} = \frac{P_j}{P_iP_{j-i}} \cdot \frac{P_{l-i}}{P_{k-i}P_{l-k}}, \qquad IH_{\mathcal{S}} = \frac{P_{l-j}}{P_{k-i}P_{l-j-k+i}} \cdot \frac{P_{k+j-i}}{P_kP_{j-i}}$$

and

$$H_{\mathbb{G}_q(\mathbb{C}^{k-l+j})} = \frac{P_{k-l+j}}{P_qP_{k-l+j-q}},$$

hence formula (2.4.3) becomes

$$\begin{aligned} \frac{P_jP_{l-i}}{P_iP_{j-i}P_{k-i}P_{l-k}} =& \frac{P_{l-j}P_{k+j-i}}{P_{k-i}P_{l-j-k+i}P_kP_{j-i}} + \\ &+ \sum_{q=1}^{\min\{k-i,k-l+j\}} \frac{P_{k-l+j}P_{l-j}P_{k+j-i-q}}{P_qP_{k-l+j-q}P_{k-i-q}P_{l-j-k+i+q}P_kP_{j-i-q}} t^{2d_{0q}}. \end{aligned}$$

□

An explicit inductive algorithm for the computation of Poincaré polynomials of the intersection cohomology of special Schubert varieties was obtained in passing. Indeed, when $\mathcal{S} = \mathbb{G}_k(\mathbb{C}^l)$, $IH_{\mathcal{S}} = H_{\tilde{\mathcal{S}}}$ is explicit, whereas, when $p < k - i$, $IH_{\mathcal{S}}$ is obtained inductively by formula (2.4.3). This algorithm can be described by the following equality:

$$\begin{bmatrix} IH_{\mathcal{S}} \\ IH_{\Delta_1} \\ IH_{\Delta_2} \\ \vdots \\ IH_{\Delta_{k-i-1}} \\ IH_{\Delta_{k-i}} \end{bmatrix} = \begin{bmatrix} 1 & g_{k-i+1,k-i} & g_{k-i+1,k-i-1} & \dots & g_{k-i+1,1} \\ 0 & 1 & g_{k-i,k-i-1} & \dots & g_{k-i,1} \\ 0 & 0 & 1 & \dots & g_{k-i-1,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_{21} \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} H_{\mathcal{S}} \\ H_{\Delta_1} \\ H_{\Delta_2} \\ \vdots \\ H_{\Delta_{k-i-1}} \\ H_{\Delta_{k-i}} \end{bmatrix}$$

$$= \sum_{\alpha=0}^{k-i} (-1)^\alpha \begin{bmatrix} 0 & g_{k-i+1,k-i} & g_{k-i+1,k-i-1} & \dots & g_{k-i+1,1} \\ 0 & 0 & g_{k-i,k-i-1} & \dots & g_{k-i,1} \\ 0 & 0 & 0 & \dots & g_{k-i-1,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_{21} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}^{\alpha} \begin{bmatrix} H_{\mathcal{S}} \\ H_{\Delta_1} \\ H_{\Delta_2} \\ \vdots \\ H_{\Delta_{k-i-1}} \\ H_{\Delta_{k-i}} \end{bmatrix}$$

2.4.4 Global identities from a symbolic point of view

The polynomial identity of Corollary 2.4.6 symbolically makes sense, i.e. the denominators do not vanish, under the weaker assumptions $0 \leq i \leq k \leq j$ and $0 \leq k-i \leq l-j \leq k$, which are equivalent to $0 \leq i \leq j$ and $0 \leq k-i \leq l-j \leq k \leq j$.

Set $r := k - i$ and $c := l - j$. In the few further cases $r = 0$, $c = r + i$, $i = 0$ or $i = j$, the polynomial identity trivially holds. For the remaining case $c = r$, there are some experimental evidences verifying the polynomial identity for the 4-tuples (i, j, c, r), as $c = r$, i and j vary, respectively, through $[2, 20]$, $[1, 20]$ and $[r + i, 40]$, by direct computations performed in CoCoA5 (see [ABR]).

A proof of formula (2.4.3) is going to be provided by a mere algebraic manipulation when $2 = \min\{k - i, k - c\}$, which is the first case with a significant geometric meaning.

By direct computations, the validity of formula (2.4.3) has been verified for all 4-tuples (i, j, c, r) as i, r, j and c run through $[1, 20]$, $[2, 20]$, $[r+i, 40]$ and $[r+1, r+i-1]$, respectively. At `http://wpage.unina.it/cioffifr/PolynomialIdentity`, some CoCoA5 functions which perform such computations are available.

Case $2 = k - i \leq k - c$ With the same notation as Corollary 2.4.6, formula (2.4.3) becomes, when $p = 0$,

$$\frac{P_j P_{j+c-i}}{P_i P_{j-i} P_2 P_{j+c-i-2}} = \frac{P_{i-c+2} P_c P_{j+1}}{P_{i-c+1} P_{c-1} P_{i+2} P_{j-i-1}} + \frac{P_c P_{j+2}}{P_2 P_{c-2} P_{i+2} P_{j-i}} \cdot t^{2(c-1)} + \frac{P_{i-c+2} P_j}{P_2 P_{i-c} P_{i+2} P_{j-i-2}} \cdot t^{4c}$$

where only the parameters i, j, c appear. Note that the above formula does make sense for every $j \geq i+2$ and $c \geq 2$. Put

$$H := \frac{P_j P_{j+c-i}}{P_i P_{j-i} P_2 P_{j+c-i-2}} \qquad F := \frac{P_c P_{j+2}}{P_2 P_{c-2} P_{i+2} P_{j-i}}$$
$$F_1 := \frac{P_{i-c+2} P_c P_{j+1}}{P_{i-c+1} P_{c-1} P_{i+2} P_{j-i-1}} \cdot t^{2(c-1)} \qquad F_2 := \frac{P_{i-c+2} P_j}{P_2 P_{i-c} P_{i+2} P_{j-i-2}} \cdot t^{4c}.$$

Since $P_\alpha / P_{\alpha-\beta} = h_{\alpha-1} \ldots h_{\alpha-\beta}$ for every $\alpha > \beta \geq 0$,

$$F = \frac{h_{c-2} h_{c-1} P_{j+2}}{h_1 P_{i+2} P_{j-i}}$$

and

$$H = \frac{h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1}}{h_j h_{j+1} h_{c-2} h_{c-1}} \cdot F \qquad F_1 = \frac{h_1 h_{i-c+1} h_{j-i-1}}{h_{j+1} h_{c-2}} \cdot t^{2(c-1)} \cdot F$$
$$F_2 = \frac{h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1}}{h_{c-2} h_{c-1} h_j h_{j+1}} \cdot t^{4c} \cdot F$$

Hence, letting

$$\begin{aligned} F(i,j,c) := & \frac{h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1}}{h_j h_{j+1} h_{c-2} h_{c-1}} - \frac{h_1 h_{i-c+1} h_{j-i-1}}{h_{j+1} h_{c-2}} \cdot t^{2(c-1)} \\ & - \frac{h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1}}{h_{c-2} h_{c-1} h_j h_{j+1}} \cdot t^{4c}, \end{aligned}$$

the identity at the beginning of the paragraph holds if and only if $F(i,j,c) = 1$. Observe that $F(i,j,c)$ does make sense for every $c \geq 2$ and for every positive integers i, j.

Caution. In the following, the equality $t^{2\alpha} h_\beta = h_{\alpha+\beta} - h_{\alpha-1}$, which holds for every $\alpha, \beta \geq 0$, shall be tacitly applied.

Equality $F(i,j,c) = 1$ shall be proved in two steps.

Step 1. $F(i,j,3) = 1$ for any i and j with $j \geq i$. First of all, notice that

$$\begin{aligned} F(i,j,3) = & \frac{h_i h_{i+1} h_{j-i+1} h_{j-i+2}}{h_j h_{j+1} P_3} - \frac{h_{i-2} h_{j-i-1}}{h_{j+1}} \cdot t^4 \\ & - \frac{h_{j-i-2} h_{j-i-1} h_{i-2} h_{i-3}}{P_3 h_j h_{j+1}} \cdot t^{12}. \end{aligned}$$

Proceed by induction on i. The base step is $i = 2$, although $i \geq c = 3$ is assumed. In this case,

$$\begin{aligned} F(2,j,3) &= \frac{h_2 h_3 h_j h_{j-1}}{h_j h_{j+1} h_1 h_2} - \frac{h_{j-3} h_1}{h_{j+1} h_1} \cdot t^4 = \frac{h_3 h_{j-1}}{h_{j+1} h_1} - \frac{(h_{j-1} - h_1) h_1}{h_1 h_{j+1}} \\ &= \frac{(h_3 - h_1) h_{j-1} + h_1^2}{h_1 h_{j+1}} = \frac{t^4 (1 + t^2) h_{j-1} + h_1^2}{h_1 h_{j+1}} \\ &= \frac{h_{j+1} - h_1 + h_1}{h_{j+1}} = 1. \end{aligned}$$

Now, suppose $F(i-1,j,3)=1$ and prove that $F(i,j,3)-F(i-1,j,3)$ is null, for every $i>2$.

$$\begin{aligned}F(i,j,3)-F(i-1,j,3)=&\frac{h_ih_{j-i+2}(h_{i+1}h_{j-i+1}-h_{i-1}h_{j-i+3})}{h_jh_{j+1}P_3}\\&-\frac{h_jP_3(h_{i-2}h_{j-i-1}-h_{i-3}h_{j-i})t^4}{h_jh_{j+1}P_3}\\&-\frac{h_{j-i-1}h_{i-3}(h_{j-i-2}h_{i-2}-h_{j-i}h_{i-4})t^{12}}{h_jh_{j+1}P_3}.\end{aligned}$$

Therefore, $F(i,j,3)-F(i-1,j,3)=0$ if and only if $G_0+G_1+G_2=0$, where

$$\begin{aligned}G_0&:=h_ih_{j-i+2}(h_{i+1}h_{j-i+1}-h_{i-1}h_{j-i+3}),\\G_1&:=-t^4h_jP_3(h_{i-2}h_{j-i-1}-h_{i-3}h_{j-i}),\\G_2&:=-t^{12}h_{j-i-1}h_{i-3}(h_{j-i-2}h_{i-2}-h_{j-i}h_{i-4}).\end{aligned}$$

Being $h_{i+1}=t^4h_{i-1}+h_1$ and $h_{j-i+3}=t^4h_{j-i+1}+h_1$,

$$G_0=h_ih_{j-i+2}h_1(h_{j-i+1}-h_{i-1}).$$

Moreover $h_{i-2}=t^2h_{i-3}+h_0$, $h_{j-i}=t^2h_{j-i-1}+h_0$, therefore $t^4h_{i-3}=h_{i-1}-h_1$, $t^4h_{j-i-1}=h_{j-i+1}-h_1$ and

$$G_1=h_jP_3(h_{i-1}-h_{j-i+1}).$$

Lastly, notice that $h_{j-i}=t^4h_{j-i-2}+h_1$ and $h_{i-2}=t^4h_{i-4}+h_1$, thus

$$G_2=t^6h_{j-i-1}h_{i-3}h_1(h_{i-1}-h_{j-i+1}).$$

In conclusion,

$$G_0+G_1+G_2=h_1(h_{i-1}-h_{j-i+1})(h_ih_{j-i+2}-h_jh_2-t^6h_{j-i+1}h_{i-3})=0.$$

Step 2. $F(i,j,c)=1$ for any i, j, c with $j\geq i$.

Proceed by induction on c. The base step is $c=3$, but $F(i,j,3)=1$ has just been shown. Hence, suppose $F(i,j,c-1)=1$ for $c>3$ and show that $F(i,j,c)-F(i,j,c-1)=0$.

$$\begin{aligned}&F(i,j,c)-F(i,j,c-1)=\\&\quad\frac{h_{j+c-i-2}h_{j+c-i-1}h_ih_{i+1}}{h_jh_{j+1}h_{c-2}h_{c-1}}-\frac{h_{j+c-i-3}h_{j+c-i-2}h_ih_{i+1}}{h_jh_{j+1}h_{c-3}h_{c-2}}\\&\quad-\frac{h_1h_{i-c+1}h_{j-i-1}}{h_{j+1}h_{c-2}}\cdot t^{2(c-1)}+\frac{h_1h_{i-c+2}h_{j-i-1}}{h_{j+1}h_{c-3}}\cdot t^{2(c-2)}\\&\quad-\frac{h_{i-c}h_{i-c+1}h_{j-i-2}h_{j-i-1}}{h_{c-2}h_{c-1}h_jh_{j+1}}\cdot t^{4c}+\frac{h_{i-c+1}h_{i-c+2}h_{j-i-2}h_{j-i-1}}{h_{c-3}h_{c-2}h_jh_{j+1}}\cdot t^{4(c-1)}.\end{aligned}$$

Therefore, $F(i,j,c) - F(i,j,c-1) = 0$ if and only if $Q_0 + Q_1 + Q_2 = 0$, where

$$\begin{aligned}
Q_0 &:= h_{j+c-i-2}h_{j+c-i-1}h_i h_{i+1}h_{c-3} - h_{j+c-i-3}h_{j+c-i-2}h_i h_{i+1}h_{c-1},\\
Q_1 &:= -t^{2(c-1)}h_1 h_{i-c+1}h_{j-i-1}h_j h_{c-1}h_{c-3}\\
&\qquad + t^{2(c-2)}h_1 h_{i-c+2}h_{j-i-1}h_j h_{c-1}h_{c-2},\\
Q_2 &:= -t^{4c}h_{i-c}h_{i-c+1}h_{j-i-2}h_{j-i-1}h_{c-3}\\
&\qquad + t^{4(c-1)}h_{i-c+1}h_{i-c+2}h_{j-i-2}h_{j-i-1}h_{c-1}.
\end{aligned}$$

Being $h_{j+c-i-1} = t^4 h_{j+c-i-3} + h_1$ and $t^4 h_{c-3} = h_{c-1} - h_1$, then

$$Q_0 = h_{j+c-i-2}h_1 h_i h_{i+1}(h_{c-3} - h_{j+c-i-3}) = -t^{2(c-2)}h_{j+c-i-2}h_1 h_i h_{i+1}h_{j-i-1}.$$

Moreover, $t^{2(c-1)}h_{i-c+1} = h_i - h_{c-2}$ and $t^{2(c-2)}h_{i-c+2} = h_i - h_{c-3}$, therefore

$$Q_1 = t^{2(c-2)}h_1 h_i h_j h_{c-1}h_{j-i-1}.$$

Lastly, $t^4 h_{i-c} = h_{i-c+2} - h_1$, thus

$$Q_2 = t^{4(c-1)}h_{i-c+1}h_{j-i-2}h_{j-i-1}((h_{i-c+2} - h_1)h_{c-3} + h_{i-c+2}h_{c-1}).$$

Note that $Q_0 + Q_1 = 0 = Q_2$ if either $j = i$ or $j = i+1$. So, the less obvious case $j > i+1$ is left. Observe that

$$\begin{aligned}
Q_1 + Q_0 &= t^{2(c-2)}h_1 h_i h_{j-i-1}(h_j h_{c-1} - h_{i+1}h_{j+c-i-2})\\
&= t^{2(c-2)}h_1 h_i h_{j-i-1}h_{j-i-2}(h_{c-1} - h_{i+1})\\
&= -t^{2(c-2)}h_1 h_i h_{j-i-1}h_{j-i-2}t^{2c}h_{i-c+1}.
\end{aligned}$$

The proof is over because

$$\begin{aligned}
&Q_1 + Q_0 + Q_2\\
&\quad = t^{4(c-1)}h_{i-c+1}h_{j-i-2}h_{j-i-1}(-h_1 h_i + (h_{i-c+2} - h_1)h_{c-3} + h_{i-c+2}h_{c-1})\\
&\quad = t^{4(c-1)}h_{i-c+1}h_{j-i-2}h_{j-i-1}h_{i-c+2}(t^{2(c-1)} + t^{2(c-2)} - h_1 t^{2(c-2)}) = 0.
\end{aligned}$$

Case 2 = *k* - *c* < *k* - *i* In this case $r + i = k = c + 2$, $l = j + c$, $c = r + i - 2$ and, for $p = 0$, the global polynomial identity becomes

$$\begin{aligned}
\frac{P_j P_{j+r-2}}{P_i P_{j-i} P_r P_{j-2}} &= \frac{P_{r+i-2}}{P_r P_{i-2}}\frac{P_{r+j}}{P_{r+i}P_{j-i}} + \frac{P_2 P_{r+i-2}P_{r+j-1}}{P_{r-1}P_{i-1}P_{r+i}P_{j-i-1}} \cdot t^{2(i-1)}\\
&\quad + \frac{P_2 P_{r+i-2}P_{r+j-2}}{P_2 P_{r-2}P_i P_{r+i}P_{j-i-2}} \cdot t^{4i},
\end{aligned}$$

where only the parameters i, j, r appear. Note that this formula does make sense for every $r \geq 2$ and $2 \leq i \leq j-2$. Let

$$K := \frac{P_j P_{j+r-2}}{P_i P_{j-i} P_r P_{j-2}} \qquad E := \frac{P_{r+i-2}}{P_r P_{i-2}}\frac{P_{r+j}}{P_{r+i}P_{j-i}}$$

$$E_1 := \frac{P_2 P_{r+i-2}P_{r+j-1}}{P_{r-1}P_{i-1}P_{r+i}P_{j-i-1}} \cdot t^{2(i-1)} \qquad E_2 := \frac{P_2 P_{r+i-2}P_{r+j-2}}{P_2 P_{r-2}P_i P_{r+i}P_{j-i-2}} \cdot t^{4i}.$$

As in the previous case,

$$\begin{aligned}
K &= \frac{h_{j-1}h_{j-2}h_{r+i-1}h_{r+i-2}}{h_{i-1}h_{i-2}h_{r+j-1}h_{r+j-2}} \cdot E\\
E_1 &= \frac{h_{r-1}h_1h_{j-i-1}h_{i-1}h_{r+j-2}}{h_{r+j-1}h_{i-2}h_{i-1}h_{r+j-2}} \cdot t^{2(i-1)} \cdot E\\
E_2 &= \frac{h_{r-2}h_{r-1}h_{j-i-2}h_{j-i-1}}{h_{r+j-2}h_{r+j-1}h_{i-2}h_{i-1}} \cdot t^{4i} \cdot E.
\end{aligned}$$

Hence, letting

$$\begin{aligned}
E'(i,j,r) &:= \frac{h_{j-1}h_{j-2}h_{r+i-1}h_{r+i-2}}{h_{i-1}h_{i-2}h_{r+j-1}h_{r+j-2}}\\
&\quad - \frac{h_{r-1}h_1h_{j-i-1}h_{i-1}h_{r+j-2}}{h_{r+j-1}h_{i-2}h_{i-1}h_{r+j-2}} \cdot t^{2(i-1)} - \frac{h_{r-2}h_{r-1}h_{j-i-2}h_{j-i-1}}{h_{r+j-2}h_{r+j-1}h_{i-2}h_{i-1}} \cdot t^{4i},
\end{aligned}$$

the formula at the beginning of the paragraph holds if and only if $E'(i,j,r) = 1$. Observe that $E'(i,j,r)$ does make sense for all integers $j \geq i \geq 2$ and $r \geq 0$. It is straightforward to check that $E'(i,j,0) = 1$, so assume $r > 0$ and prove that $E'(i,j,r) - E'(i,j,r-1) = 0$. Arguing as in the case $k - i = 2$, put

$$\begin{aligned}
H_0 &:= h_{r+j-3}h_{j-1}h_{j-2}h_{r+i-1}h_{r+i-2} - h_{r+j-1}h_{j-1}h_{j-2}h_{r+i-2}h_{r+i-3}\\
H_1 &:= t^{2(i-1)}h_{r+j-1}h_{r-2}h_1h_{j-i-1}h_{i-1}h_{r+j-3}\\
&\quad - t^{2(i-1)}h_{r+j-3}h_{r-1}h_1h_{j-i-1}h_{i-1}h_{r+j-2}\\
H_2 &:= t^{4i}h_{r+j-1}h_{r-3}h_{r-2}h_{j-i-2}h_{j-i-1} - t^{4i}h_{r+j-3}h_{r-2}h_{r-1}h_{j-i-2}h_{j-i-1}
\end{aligned}$$

so that the thesis becomes $H_0 + H_1 + H_2 = 0$.

Apply the following replacements to H_0, in the given order: $h_{r+i-1} = t^4h_{r+i-3} + h_1$; $h_{r+j-1} = t^4h_{r+j-3} + h_1$; $h_{r+j-2} - h_{r+i-2} = t^{2(r+i-2)}h_{j-i-1}$. Hence,

$$H_0 = h_{j-1}h_{j-2}h_{r+i-2}h_1t^{2(r+i-2)}h_{j-i-1}.$$

Perform the following substitutions in H_1, in the given order. First, $h_{r+j-1} = t^2h_{r+j-2} + h_0$ and $h_{r-1} = t^2h_{r-2} + h_0$, then $h_{r-2} - h_{r+j-2} = -t^{2(r-1)}h_{j-1}$. Thus,

$$H_1 = t^{2(i-1)}h_{r+j-3}h_{j-i-1}h_{i-1}(-t^{2(r-1)}h_{j-1})h_1.$$

Do the following changes in H_2, in the given order. First, $h_{r+j-1} = t^4h_{r+j-3} + h_1$ and $h_{r-1} = t^4h_{r-3} + h_1$, then $h_{r-3} - h_{r+j-3} = -t^{2(r-2)}h_{j-1}$. Consequently,

$$H_2 = -t^{2i}t^{2(r+i-2)}h_{r-2}h_{j-i-2}h_{j-i-1}h_1h_{j-1}.$$

Summing up,

$$\begin{aligned}
&H_0 + H_1 + H_2 =\\
&\qquad t^{2(r+i-2)}h_1h_{j-1}h_{j-i-1}(h_{j-2}h_{r+i-2} - h_{r+j-3}h_{i-1} - t^{2i}h_{r-2}h_{j-i-2}).
\end{aligned}$$

Being $h_{j-2} = t^{2(j-i-1)}h_{i-1} + h_{j-i-2}$ and $h_{r+j-3} = t^{2(j-i-1)}h_{r+i-2} + h_{j-i-1}$,

$$H_0 + H_1 + H_2 = t^{2(r+i-2)}h_1h_{j-1}h_{j-i-1}h_{j-i-2}(h_{r+i-2} - h_{i-1} - t^{2i}h_{r-2} = 0.$$

2.5 Examples

In Section 2.5.1, some cases in which the Kazhdan-Lusztig polynomials are explicit and immediate to determine are exhibited, while, in Section 2.5.2, there are a few examples of Ferrer's diagrams, each of which stresses out certain properties of Schubert varieties.

2.5.1 Smallness and polynomial identities

Definition 2.5.1. A resolution of singularities $\chi : X \to Y$, is said to be **small** if and only if

$$\operatorname{codim}\{y \in Y \,:\, \dim \chi^{-1}(y) \geq \alpha\} > 2\alpha \quad \forall \alpha > 0.$$

The smallness of the resolution χ implies that $IC^\bullet_Y \cong R\chi_*\mathbb{Q}_X[\dim Y]$ (see [GM83, Corollary, §6.2]). In particular, when $Y := \mathcal{S}$ is a Schubert variety, the previous isomorphism gives

$$\mathcal{H}^\alpha(IC^\bullet_{\mathcal{S}})_V \cong H^{\alpha+\dim \mathcal{S}}(\chi^{-1}(V)) \quad \forall V \in \mathcal{S};$$

and, if $V \in \Delta^0_q$, with Δ_q an $\mathcal{S}$-variety, the Kazhdan-Lusztig polynomial corresponding to $\mathcal{S}$ and Δ_q coincides with the Poincaré polynomial of the fibre $\chi^{-1}(V)$ (see [dCM09, Theorem 4.4.7], [BL00, Theorem 9.1.3]).

For this reason, it would be useful to have small resolutions so as to speed up the computation of the Kazhdan-Lustig polynomials. The problem of finding such maps was successfully tackled with in [Zel83] by Zelevinskii, who proved how to construct small resolutions by iteration on the length of the essential flag $\mathcal{F}$ of a Schubert variety (actually, on the number of corners in its Ferrer's diagram). Nevertheless, only the maps π_0, defined in Corollary 2.2.3, along with a new class of resolutions ξ_0 shall be considered here because they have an explicit description and, as it will be soon proved, there is a straightforward way to determine whether they are small.

Let $\mathcal{S}$ be a Schubert variety associated to the essential pair $(\mathcal{F}, \mathcal{I})$.

Proposition 2.5.2.

$$\mathcal{D}_{\mathcal{S}} := \left\{ \begin{matrix} (V, U_1, \ldots, U_\omega) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j_1-i_1}(\mathbb{C}^l) \times \ldots \times \mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l) \\ s.t. \quad U_1 \subset \ldots \subset U_\omega \wedge U_\alpha \supseteq V + F_{j_\alpha}, \quad \alpha = 1, \ldots, \omega \end{matrix} \right\}$$

is a smooth variety.

Proof. Consider the trivial bundle $\mathbb{C}^l$ over $\mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l)$ and its pullback with respect to the inclusion $\mathbb{G}_{k-i_\omega}(\mathbb{C}^l/F_{j_\omega}) \hookrightarrow \mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l)$:

$$\begin{array}{ccc} S_\omega & & \mathbb{C}^l \\ \downarrow & & \downarrow \\ \mathbb{G}_{k-i_\omega}(\mathbb{C}^l/F_{j_\omega}) & \hookrightarrow & \mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l) \end{array}$$

If $\omega = 1$, $\mathcal{D}_{\mathcal{S}}$ coincides with the Grassmannian k-plane bundle of S_ω. If $\omega > 1$, take the Grassmannian $(k + j_{\omega-1} - i_{\omega-1})$-plane bundle of S_ω

$$\mathcal{G}_{k+j_{\omega-1}-i_{\omega-1}}(S_\omega) \cong \left\{ \begin{matrix} (U_{\omega-1}, U_\omega) \in \mathbb{G}_{k+j_{\omega-1}-i_{\omega-1}}(\mathbb{C}^l) \times \mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l) \\ \text{s.t.} \quad U_{\omega-1} \subset U_\omega \ \wedge \ F_{j_\omega} \subset U_\omega \end{matrix} \right\}$$

and the pullback

$$\begin{array}{ccc} S_{\omega-1} & & \mathcal{G}_{k+j_{\omega-1}-i_{\omega-1}}(S_\omega) \\ \downarrow & & \downarrow \\ \mathbb{G}_{k-i_{\omega-1}}(\mathbb{C}^l/F_{\omega-1}) & \hookrightarrow & \mathbb{G}_{k+j_{\omega-1}-i_{\omega-1}}(\mathbb{C}^l). \end{array}$$

Again, if $\omega = 2$, $\mathcal{D}_{\mathcal{S}}$ is the Grassmannian k-plane bundle of $S_{\omega-1}$, otherwise, iterate the process. □

Corollary 2.5.3. *The projection on the first factor*

$$\xi_0 : (V, U_1, \dots, U_\omega) \in \mathcal{D}_{\mathcal{S}} \to V \in \mathcal{S}$$

is a resolution of singularities.

The proof of this result is similar to the one of Corollary 2.2.3, hence it will be omitted.

Let $V \in \mathcal{S}$. There is an admissible ω-tuple q such that $V \in \Delta_{0q}$, so the fibre of ξ_0 at V is

$$G_{0q} := \xi_0^{-1}(V) \cong \left\{ \begin{matrix} (U_1, \dots, U_\omega) \in \mathbb{G}_{k+j_1-i_1}(\mathbb{C}^l) \times \dots \times \mathbb{G}_{k+j_\omega-i_\omega}(\mathbb{C}^l) \\ \text{s.t.} \quad U_1 \subset \dots \subset U_\omega \ \wedge \ U_\alpha \supseteq V + F_{j_\alpha}, \ \alpha = 1, \dots, \omega \end{matrix} \right\}.$$

Its dimension is

$$\dim G_{0q} = q_{j_\omega} \lambda^{\mathcal{S}}_{i_\omega} + \sum_{\alpha=1}^{\omega-1} q_{j_\alpha} (\lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}})$$

and its Poincaré polynomial is

$$\begin{aligned} H_{G_{0q}} &= H_{\mathbb{G}_{q_{j_\omega}}(\mathbb{C}^{l-k-j_\omega+i_\omega+q_{j_\omega}})} \cdot \prod_{\alpha=1}^{\omega-1} H_{\mathbb{G}_{q_{j_\alpha}}(\mathbb{C}^{j_{\alpha+1}-i_{\alpha+1}-j_\alpha+i_\alpha+q_{j_\alpha}})} \\ &= \frac{P_{l-k-j_\omega+i_\omega+q_{j_\omega}}}{P_{q_{j_\omega}} P_{l-k-j_\omega+i_\omega}} \cdot \prod_{\alpha=1}^{\omega-1} \frac{P_{j_{\alpha+1}-i_{\alpha+1}-j_\alpha+i_\alpha+q_{j_\alpha}}}{P_{q_{j_\alpha}} P_{j_{\alpha+1}-i_{\alpha+1}-j_\alpha+i_\alpha}}. \end{aligned}$$

The following fact ensues from the definition of smallness.

Remark 2.5.4. ξ_0 is small if and only if $m_{0q} = m_0 - m_q > 2 \dim G_{0q}$ for all $\mathcal{S}$-varieties Δ_q. Similarly, π_0 is small if and only if $m_{0q} = m_0 - m_q > 2 \dim F_{0q}$ for all $\mathcal{S}$-varieties Δ_q.

Remark 2.5.4 permits checking the smallness property by means of the Ferrer's diagrams, since all numbers m_q, $\dim F_{0q}$ and $\dim G_{0q}$ have a suitable representation (see also Example 2.5.13). Nonetheless, it is not convenient to check either $m_{0q} > 2\dim G_{0q}$ or $m_{0q} > 2\dim F_{0q}$ *for all* 0-admissible q; yet, the combination of Remark 2.5.4 with the next lemma yields an easy-to-compute smallness characterization.

Put $i_0 = \lambda^{\mathcal{S}}_{i_{\omega+1}} = 0$.

Lemma 2.5.5. [Zel83, p. 144]. *For any $\mathcal{S}$-variety Δ_q,*

$$m_{0q} = \sum_{\alpha=1}^{\omega} q_{j_\alpha}(i_\alpha - i_{\alpha-1} + \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}) + B(q), \tag{2.5.1}$$

where the form

$$B(q) = q^2_{j_\omega} + \sum_{\alpha=1}^{\omega-1} q^2_{j_\alpha} - q_{j_\alpha} q_{j_{\alpha+1}}$$

is positive definite (see also Example 2.5.13).

Proposition 2.5.6. *ξ_0 is small if and only if $i_\alpha - i_{\alpha-1} \geq \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}$ for any $\alpha \in \{1, \dots, \omega\}$. Analogously, π_p is small if and only if $i_\alpha - i_{\alpha-1} \leq \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}$ for any $\alpha \in \{1, \dots, \omega\}$. In particular, both resolutions are small if equality holds.*

Proof. We are going to prove the statement for ξ_0 only.

$\Rightarrow$ Suppose that ξ_0 is small. By Remark 2.5.4, for any 0-admissible q,

$$\begin{aligned} 0 &< m_{0q} - 2\dim G_{0q} \\ &= q^2_{j_\omega} + \sum_{\alpha=1}^{\omega-1} q^2_{j_\alpha} - q_{j_\alpha} q_{j_{\alpha+1}} + \sum_{\alpha=1}^{\omega} q_{j_\alpha}(i_\alpha - i_{\alpha-1}) - \sum_{\alpha=1}^{\omega} q_{j_\alpha}(\lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}), \end{aligned}$$

where Lemma 2.5.5 has been used for the equality. This relation holds, in particular, for all

$$q \in \{e_1, \dots, e_\omega\} = \{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$$

(they are all 0-admissible because $\mathcal{S}$ has ω essential conditions). Hence, for any $\alpha \in \{1, \dots, \omega\}$,

$$0 < m_{0e_\alpha} - 2\dim G_{0e_\alpha} = 1 + i_\alpha - i_{\alpha-1} - (\lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}});$$

i.e.

$$i_\alpha - i_{\alpha-1} \geq \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}.$$

$\Leftarrow$ Assume that the inequality holds. For any p-admissible q,

$$\begin{aligned} m_{0q} &= \sum_{\alpha=1}^{\omega} q_{j_\alpha}(i_\alpha - i_{\alpha-1} + \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}) + B(q) && \text{formula (2.5.1)} \\ &= \dim G_{0q} + \dim F_{0q} + B(q) \\ &\geq 2\dim G_{0q} + B(q) && \text{hypothesis} \\ &> 2\dim G_{0q}. && B(q) \text{ is positive definite} \end{aligned}$$

Remark 2.5.4 guarantees that ξ_0 is small. ∎

In terms of Ferrer's diagrams, ξ_0 is small if and only if the α-th vertical line of the diagram of $\mathcal{S}$ is longer than its α-th horizontal line for all α, whereas π_0 is small if and only if the converse is true. If equality holds for all α, both ξ_0 and π_0 are small.

Corollary 2.5.7 (Polynomial identities)**.** *If $i_\alpha - i_{\alpha-1} \geq \lambda^{\mathcal{S}}_{i_\alpha} - \lambda^{\mathcal{S}}_{i_{\alpha+1}}$ for any $\alpha \in \{1, \ldots, \omega\}$ (respectively, $\leq$), then the Poincaré polynomial $H_{G_{0q}}$ (respectively, $H_{F_{0q}}$) of the fibre equals b_{0q} for any 0-admissible q.*

It is worth stressing out that, for instance,

- ▲ π_0 is small as opposed to ξ_0 if
  $i_1 = 1$, $i_2 = 2$, $k = 3$, $j_1 = 4$, $j_2 = 6$ and $l = 9$;
- ▲ ξ_0 is small as opposed to π_0 if
  $i_1 = 1$, $i_2 = 3$, $k = 4$, $j_1 = 5$, $j_2 = 8$ and $l = 10$;
- ▲ both π_0 and ξ_0 are small if
  $i_1 = 1$, $i_2 = 2$, $k = 3$, $j_1 = 4$, $j_2 = 6$ and $l = 8$;
- ▲ neither of π_0 and ξ_0 is small if
  $i_1 = 2$, $i_2 = 3$, $k = 4$, $j_1 = 5$, $j_2 = 7$ and $l = 10$.

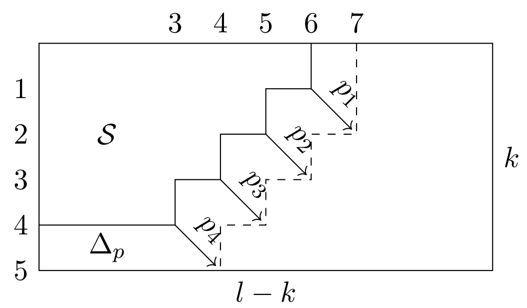
2.5.2 Ferrer's diagrams

In order to make symbols easier to read in the diagrams, notations shall be simplified. Instead of the essential pair $(\mathcal{F}, \mathcal{I})$ of a Schubert variety $\mathcal{S}$, the components $i_1, \ldots, i_\omega$ of $\mathcal{I}$ and the dimensions $j_1, \ldots, j_\omega$ of the vector spaces of $\mathcal{F}$ will be given. $\mathcal{S}$-varieties Δ_p, instead, shall be specified by ω-tuples of non-negative integers $p = (p_1, \ldots, p_\omega)$ and, only in few cases, the pair(s) to which they are associated will be given explicitly.

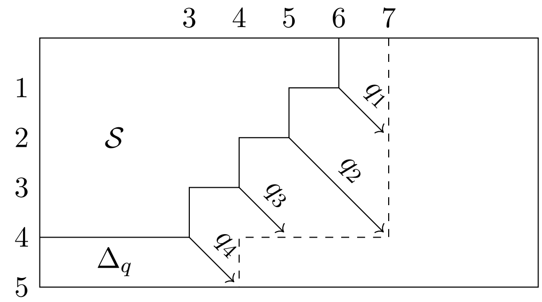
Example 2.5.8. Let $\mathcal{S}$ be the Schubert variety given by

$$\begin{array}{lllll} i_1 = 1, & i_2 = 2, & i_3 = 3, & i_4 = 4, & k = 5, \\ j_1 = 5, & j_2 = 7, & j_3 = 9, & j_4 = 11, & l = 15. \end{array}$$

The sequence associated to $\mathcal{S}$ is $\lambda^{\mathcal{S}} = (6, 5, 4, 3, 0)$, which is shown in the picture below. If $p = (1, 1, 1, 1)$ is taken, the $\mathcal{S}$-variety Δ_p, represented by the dashed diagram below, is associated to $\lambda^p = (7, 7, 6, 5, 4)$.



Now, take $q = (1,2,1,1)$. The first and third conditions are unnecessary since $j_2 - i_2 - q_2 = j_1 - i_1 - q_1 = 3$ and $i_2 + q_2 = i_3 + q_3 = 4$. In other words, the essential pair of Δ_q is $(\mathcal{F}_q, \mathcal{I}_q)$ with $\mathcal{F}_q : F_{j_2} \subset F_{j_4}$, $\mathcal{I}_q = (i_2 + q_2, i_4 + q_4) = (4,5)$. Below, Δ_q is depicted by the dashed diagram.

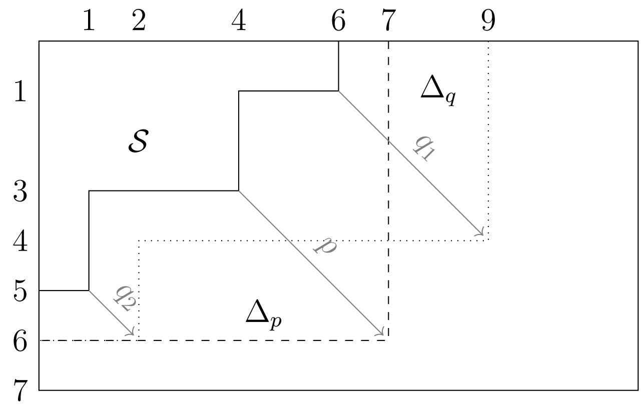


Two important facts are deduced from the pictures. First, *the number of corners of a Schubert variety equals the number of its essential conditions.* Secondly, *when an $\mathcal{S}$-variety Δ_p is associated to the pair with flag $\mathcal{F}$, the components of the ω-tuple p can be interpreted as its distance from $\mathcal{S}$. In particular, the terms corresponding to the essential conditions measure the distance between the corners of $\mathcal{S}$ and Δ_p.*

Example 2.5.9. Let $\mathcal{S}$ be the Schubert variety given by

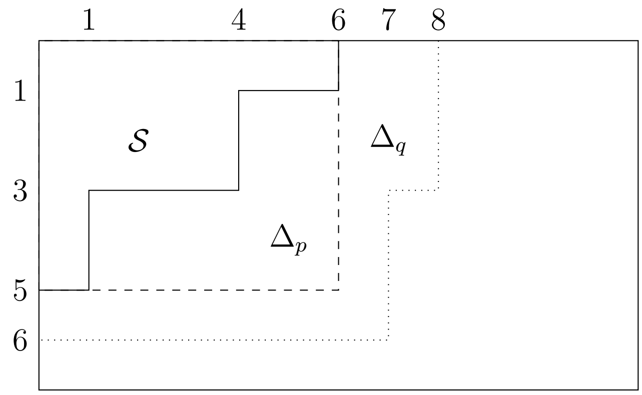
$$\begin{array}{llll} i_1 = 1, & i_2 = 3, & i_3 = 5, & k = 7, \\ j_1 = 8, & j_2 = 12, & j_3 = 17, & l = 20. \end{array}$$

Consider the $\mathcal{S}$-varieties Δ_p (dashed) and Δ_q (dotted), with $p = 3$ and $q = (3,1)$, associated to their essential pairs $(\mathcal{F}_p : F_{j_2}, \mathcal{I}_p = i_2 + p = 6)$ and $(\mathcal{F}_q : F_{j_1} \subset F_{j_3}, \mathcal{I}_q = (i_1 + q_1, i_3 + q_2) = (4,6))$. Neither of their diagrams contains the other, therefore Δ_p and Δ_q are not comparable.



The essential pair of an $\mathcal{S}$-variety is the minimum data needed to draw the Ferrer's diagram. Anyway, if Δ_p and Δ_q were described by means of $\mathcal{F}$, then the associated sequences would be $\mathcal{I}_p = (2,6,6)$, $\mathcal{I}_q = (4,4,6)$ and $p = (1,3,1)$, $q = (3,1,1)$. As you can see, $p_1 < q_1$ and $p_2 > q_2$, which confirms the fact that the studied varieties are not comparable.

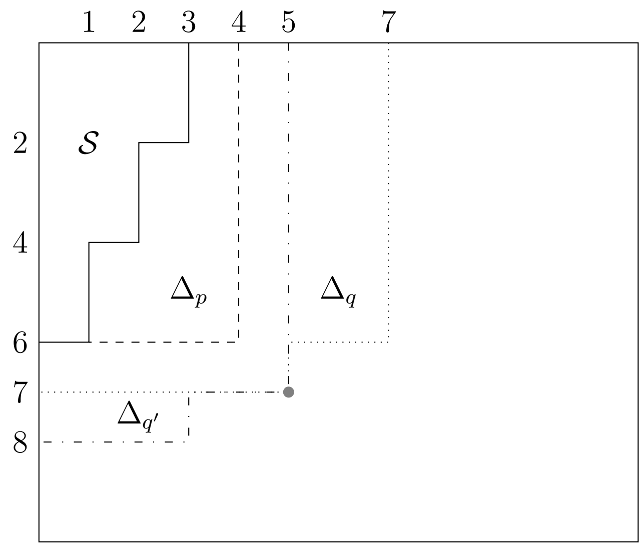
Example 2.5.10. Let $\mathcal{S}$ be the Schubert variety given in Example 2.5.9 and take $p = (0, 2, 0)$. Δ_p (dashed) is a special Schubert variety and Δ_q (dotted), with $q = (2, 3, 1)$, is an $\mathcal{S}$-variety contained in Δ_p which is not a Δ_p-variety ($\mathcal{F}_q \nsubseteq \mathcal{F}_p$, where $\mathcal{F}_p$ and $\mathcal{F}_q$ denote the essential flags of Δ_p and Δ_q, respectively).



Example 2.5.11. Let $\mathcal{S}$ be the Schubert variety given by

$$\begin{aligned} i_1 &= 2, & i_2 &= 4, & i_3 &= 6, & k &= 10, \\ j_1 &= 11, & j_2 &= 14, & j_3 &= 17, & l &= 22 \end{aligned}$$

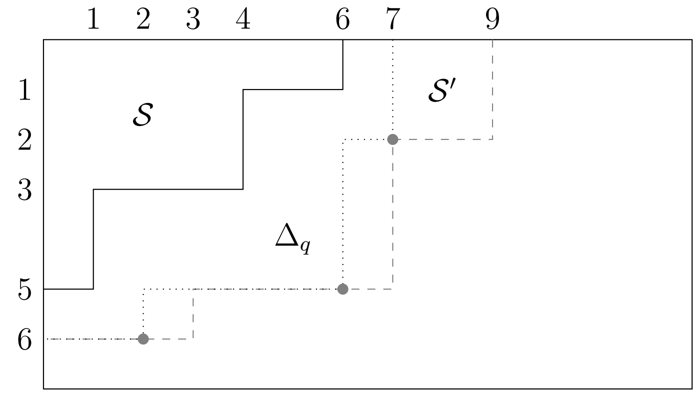
and put $p = (1, 2, 0)$, $q = (4, 3, 1)$, $q' = (2, 3, 2)$. Δ_p, Δ_q and $\Delta_{q'}$ are represented by the dashed, dotted and dashed-dotted diagrams, respectively, and the grey circle highlights the common corner of Δ_q and $\Delta_{q'}$. Δ_p is a special Schubert variety, while Δ_q and $\Delta_{q'}$ have two indispensable conditions, but with respect to different flags. Δ_{q^p} and $\Delta_{q'^p}$ coincide with the special Schubert variety $\Delta_{(2,3,1)}$, whose only corner is represented by the grey circle. It is straightforward to infer from the diagram that $\Delta_q, \Delta_{q'} \neq \Delta_{q^p}$ and that the distance between Δ_p and $\Delta_{(2,3,1)}$ is strictly lower than the one between Δ_p and either Δ_q or $\Delta_{q'}$.



Example 2.5.12. Let $\mathcal{S}$ be the Schubert variety of Example 2.5.9 and let $\mathcal{S}'$ be the Schubert variety given by

$$\begin{array}{llll} i'_1 = 2, & i'_2 = 5, & i'_3 = 6, & k = 7 \\ j'_1 = 6, & j'_2 = 11, & j'_3 = 16, & l\ = 20 \end{array}$$

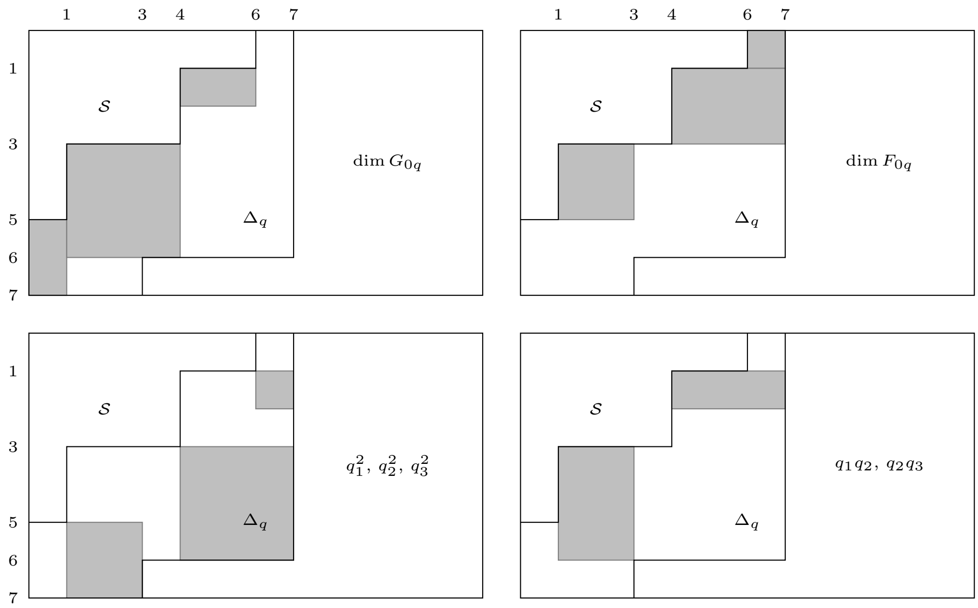
The Ferrer's diagrams of $\mathcal{S}'$ and $\Delta_q := \mathcal{S}'_{\mathcal{F}}$ are, respectively, the dashed and dotted ones below. The diagram of Δ_q is easy to obtain because it is the one whose corners are the intersections (the grey dots in the picture below) of the lines of slope -1 through the corners of $\mathcal{S}$ and the diagram of $\mathcal{S}'$.



Example 2.5.13. Let $\mathcal{S}$ be the Schubert variety in Example 2.5.9.

$$\begin{array}{llll} i_1\ = 1, & i_2 = 3, & i_3 = 5, & k = 7, \\ j_1 = 8, & j_2 = 12, & j_3 = 17, & l\ = 20. \end{array}$$

Take Δ_q with $q = (1, 3, 2)$. $\operatorname{codim}_{\mathcal{S}} \Delta_q$ is easily seen to be given by Formula (2.5.1). The grey rectangles in the pictures below represent the fibre of ξ_0 (on the top left) at any point of Δ_q; the one of π_0 (on the top right); the value q_α^2 (on the bottom left); the quantity $q_\alpha q_{\alpha+1}$ (on the bottom right). Remember that the sum of q_1^2, q_2^2, q_3^2, $q_1 q_2$ and $q_2 q_3$ is the definite positive form $B(q)$ (see Lemma 2.5.5).



Chapter 3

Decomposition theorem and bivariant theory

In this final chapter, it is shown how to obtain a decomposition of the derived direct image $Rf_*\mathbb{A}_X$, where f and $\mathbb{A}$ are, respectively, a suitable map and ring, analogous to the ones given by the decomposition theorem in circumstances in which the hypotheses of this result are not met. Namely, it is proved that the possibility to have such decompositions is equivalent to the existence of a bivariant class of degree one (defined in Section 3.2.1). Moreover, it is also shown that such classes play an important role in discernment of $\mathbb{A}$-homology manifolds (see Section 3.3.4).

In Section 3.1, the axioms of bivariant theory are recalled along with the definition of Borel-Moore homology. Bivariant classes of degree one are introduced in Section 3.2, which is devoted to the proof of a partial generalization of the decomposition theorem; namely, Theorem 3.2.8. Section 3.3 consists of applications of the just mentioned result. Several consequences in cohomology and Borel-Moore homology are inferred so as to describe the duality morphism (3.3.5) as maintained in Corollary 3.3.4 and Theorem 3.3.6, which provides a description of the relation between $\mathbb{A}$-homology manifolds and the existence of a bivariant class of degree one for a suitable morphism, is proved. Lastly, as a consequence of Theorem 3.3.6, Nilpotent cones are proved to be homology manifolds in Section 3.4.

3.1 Preliminaries

This preliminary section is meant to settle notations and is a reminder of bivariant theory and Borel-Moore homology.

3.1.1 Bivariant theory

Let $\mathcal{C}$ be a category endowed with a class of morphisms, called the **confined maps** and a class of commutative squares,

$$\begin{tikzcd} A_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & A \arrow[d, "f"] \\ B_1 \arrow[r, "g"] & B \end{tikzcd} \tag{3.1.1}$$

called the **independent squares**, satisfying the following axioms:

i) all identity morphisms are confined and the composition of confined maps is confined;

ii) for any $A, B \in \mathcal{C}$ and any $f : A \to B$, the square on the right is independent;

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "f"] & A \arrow[d, "f"] \\ B \arrow[r, "id_B"] & B \end{tikzcd}$$

iii) In the two diagrams below, if the inner squares are independent, so is the outer square.

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A \arrow[d] \\ B_2 \arrow[r] & B_1 \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r] \arrow[d] & A \arrow[d] \\ B_1 \arrow[r] \arrow[d] & B \arrow[d] \\ C_1 \arrow[r] & C \end{tikzcd}$$

iv) in any independent square as (3.1.1), if f (g, respectively) is confined, so is f_1 (g_1, respectively).

Observe that, if a square is independent, its transpose (e.g. change B_1 with A and vice versa in (3.1.1)) may not be independent.

Example 3.1.1. Consider the category of topological spaces embeddable as closed subspaces of $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and continuous maps. A possible choice of confined morphisms and independent squares is given, respectively, by proper maps and **fibre** (also called **cartesian**) **squares**, i.e. the ones as (3.1.1) for which A_1 is homeomorphic to $\{(b, a) \in B_1 \oplus A : g(b) = f(a)\}$ (notice that this is the categorical pullback of f and g).

Example 3.1.2. In the category of locally compact Hausdorff spaces and continuous maps $f : X \to Y$ of **finite cohomological dimension**, that is, the ones for which there is $\alpha \in \mathbb{N}$ such that $R^\beta f_! \mathcal{F} = 0$ for any sheaf of abelian groups $\mathcal{F}$ and any $\beta > \alpha$, classes of confined morphisms and independent squares can be, respectively, proper maps and fibre squares.

A **bivariant theory** $\mathbb{B}$ on a category $\mathcal{C}$ as above consists of the following data.

i) For any morphisms $f : A \to B$, there is a graded abelian group

$$\mathbb{B}(A \xrightarrow{f} B) := \bigoplus_{\alpha \in \mathbb{Z}} \mathbb{B}^\alpha(A \xrightarrow{f} B),$$

also denoted by either $\mathbb{B}(A \to B)$ or $\mathbb{B}(f)$. In diagrams, the elements of $\mathbb{B}(A \to B)$ shall be denoted by underlined symbols so as to distinguish them from the maps' names.

ii) Given $f : A \to B$ and $g : B \to C$, there is a **product**

$$\cdot : (a, b) \in \mathbb{B}^\alpha(f) \times \mathbb{B}^\beta(g) \mapsto a \cdot b \in \mathbb{B}^{\alpha+\beta}(gf).$$

$$\begin{tikzcd} & B \arrow[dr, "\underline{b}"] & \\ A \arrow[ur, "\underline{a}"] \arrow[rr, "\underline{a\cdot b}"] & & C \end{tikzcd}$$

iii) Given $f : A \to B$ and $g : B \to C$ with f confined, there is a **pushforward**

$$f_* : a \in \mathbb{B}^\alpha(gf) \mapsto f_*a \in \mathbb{B}^\alpha(g).$$

$$\begin{tikzcd} & B \arrow[dr, "\underline{f_*a}"] & \\ A \arrow[ur, "f"] \arrow[rr, "\underline{a}"] & & C \end{tikzcd}$$

iv) For any independent square as (3.1.1), there is a **pull-back**

$$g^* : a \in \mathbb{B}^\alpha(f) \mapsto g^*a \in \mathbb{B}^\alpha(f_1).$$

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\underline{g^*a}"'] & A \arrow[d, "\underline{a}"] \\ B_1 \arrow[r, "g"] & B \end{tikzcd}$$

v) $\mathbb{B}$ has **units**; in other words, for any $A \in \mathcal{C}$, there is $1_A \in \mathbb{B}^0(id_A)$ such that $a \cdot 1_A = a$ and $1_A \cdot b = b$ whenever product makes sense and $h^*(1_A) = 1_B$ for any $B \in \mathcal{C}$ and any morphism $h : B \to A$.

In addition, the above three operations must satisfy the following axioms. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be morphisms and let $a \in \mathbb{B}^\alpha(f)$, $b \in \mathbb{B}^\beta(g)$ and $c \in \mathbb{B}^\gamma(h)$.

▲ *Associativity of product.* $(a \cdot b) \cdot c = a \cdot (b \cdot c) \in \mathbb{B}^{\alpha+\beta+\gamma}(hgf)$, where

$$\begin{tikzcd} A \arrow[r, "f", "\underline{a}"'] & B \arrow[r, "g", "\underline{b}"'] & C \arrow[r, "h", "\underline{c}"'] & D \end{tikzcd}$$

▲ *Functoriality of pushforward.* $(gf)_*a = g_*(f_*a) \in \mathbb{B}^\alpha(h)$, where f, g are confined and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rrr, bend right, "\underline{a}"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

▲ *Functoriality of pullback.* $(gh)^*a = h^*(g^*a) \in \mathbb{B}^\alpha(f_2)$, where the diagram on the right has independent squares.

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d, "f_2"] & A_1 \arrow[r] \arrow[d] & A \arrow[d, "\underline{a}"] \\ B_2 \arrow[r, "h"] & B_1 \arrow[r, "g"] & B \end{tikzcd}$$

▲ *Product and pushforward commute.* $f_*(a \cdot b) = (f_*a) \cdot b \in \mathbb{B}^{\alpha+\beta}(hg)$, where f is confined and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "\underline{a}"'] & B \arrow[r, "g"] & C \arrow[r, "h", "\underline{b}"'] & D \end{tikzcd}$$

▲ *Product and pullback commute.*
$h^*(a \cdot b) = (h_1^* a) \cdot (h^* b) \in \mathbb{B}^{\alpha+\beta}(g_1 f_1)$, where the diagram on the right has independent squares.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A \arrow[d, "\underline{a}"] \\ B_1 \arrow[r, "h_1"] \arrow[d, "g_1"] & B \arrow[d, "\underline{b}"] \\ C_1 \arrow[r, "h"] & C \end{tikzcd}$$

▲ *Pushforward and pullback commute.*
$f_{1*}(h^* a) = h^*(f_* a) \in \mathbb{B}^{\alpha}(g_1)$, where f is confined and the diagram on the right has independent squares.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A \arrow[d, "f"] \arrow[dd, bend left, "\underline{a}"] \\ B_1 \arrow[r] \arrow[d, "g_1"] & B \arrow[d, "\underline{b}"] \\ C_1 \arrow[r, "h"] & C \end{tikzcd}$$

▲ *Projection formula.*
$h_{1*}((h^* a) \cdot b) = a \cdot (h_* b) \in \mathbb{B}^{\alpha+\beta}(g f_1)$, where g is confined and the square on the right is independent.

$$\begin{tikzcd} A_2 \arrow[r, "h_1"] \arrow[d] & A_1 \arrow[d, "f_1", "\underline{a}"'] & \\ B_2 \arrow[r, "h"] \arrow[rr, bend right, "\underline{b}"'] & B_1 \arrow[r, "g"] & B \end{tikzcd}$$

In the definition of a bivariant theory, it has been assumed that $\mathbb{B}(A \to B)$ is a graded abelian group; yet, it is possible to define bivariant theories for which such objects belong to any arbitrary category (see [FH91, p. 22, Remark]). If $\mathbb{B}(f)$ is not graded the superscript of $\mathbb{B}^\alpha$ is dropped in the axioms.

Example 3.1.3. Consider the category of Example 3.1.1 and a cohomology theory (see [Rot88, pp. 230, 231]) with values in a commutative ring $\mathbb{A}$. For any $f : X \to Y$, there is a morphism $\phi : X \to \mathbb{R}^N$ such that $(f, \phi) : X \to Y \times \mathbb{R}^N$ is a closed embedding. Set $X_\phi := \operatorname{im}(f, \phi)$.

The **topological bivariant homology theory** $\mathbb{B}_{Top}$ is the one that associates any f to

$$\mathbb{B}^\alpha_{Top}(f) := H^{\alpha+N}(Y \times \mathbb{R}^N, (Y \times \mathbb{R}^N) \setminus X_\phi)$$

and whose products, pushforwards and pullbacks are the ones described in [FH91, Part I, §3.1.6-8]. The definition is independent from the choice of ϕ, as proved in [FH91, Part I, §3.1.5, Remark].

Example 3.1.4. Consider the category of Example 3.1.2 and let $\mathbb{A}$ be a Noetherian ring. Remember that (see [Dim04, §1.4])

$$\operatorname{Ext}^\alpha(-,-) := R^\alpha \operatorname{Hom}^\bullet_{D^b(-,\mathbb{A})}(-,-) = \operatorname{Hom}_{D^b(-,\mathbb{A})}(-,-[\alpha]).$$

The **sheaf-theoretical bivariant holomogy theory** $\mathbb{B}_{Sh}$ is the one that associates any morphism $f : X \to Y$ to

$$\begin{aligned}
\mathbb{B}^{\alpha}_{Sh}(f) &:= \mathrm{Ext}^{\alpha}(Rf_!\mathbb{A}_X, \mathbb{A}_Y) \\
&\cong \mathrm{Ext}^{\alpha}(\mathbb{A}_X, f^!\mathbb{A}_Y) \qquad \text{[Dim04, Theorem 3.2.3]} \\
&\cong H^{\alpha}(X, f^!\mathbb{A}_Y) \qquad \text{[Dim04, Remark 2.1.2]}
\end{aligned}$$

and which has the following operations. Let $h : Y \to Z$ be a morphism and consider the fibre square

$$\begin{tikzcd}
X_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & X \arrow[d, "f"] \\
Y_1 \arrow[r, "g"] & Y
\end{tikzcd}$$

- ▲ *Product.* It is the composition

$$\begin{array}{c}
\mathrm{Ext}^{\alpha}(Rf_!\mathbb{A}_X, \mathbb{A}_Y) \otimes \mathrm{Ext}^{\beta}(Rh_!\mathbb{A}_Y, \mathbb{A}_Z) \\
\Big\downarrow {\scriptstyle Rh_! \otimes id_{\mathbb{A}_Z}} \\
\mathrm{Ext}^{\alpha}(Rh_!Rf_!\mathbb{A}_X, Rh_!\mathbb{A}_Y) \otimes \mathrm{Ext}^{\beta}(Rh_!\mathbb{A}_Y, \mathbb{A}_Z) \\
\Big\downarrow \\
\mathrm{Ext}^{\alpha+\beta}(Rh_!Rf_!\mathbb{A}_X, \mathbb{A}_Z)
\end{array}$$

  where the second map is induced by composition.

- ▲ *Pushforward.* It is induced by the map $\mathbb{A}_Y \to Rf_!\mathbb{A}_X$ adjoint to the isomorphism $f^*\mathbb{A}_Y \cong \mathbb{A}_X$ ($f_* = f_!$ because f is proper).

$$\mathrm{Ext}^{\alpha}(Rg_!Rf_!\mathbb{A}_X, \mathbb{A}_Z) \to \mathrm{Ext}^{\alpha}(Rg_!\mathbb{A}_Y, \mathbb{A}_Z).$$

- ▲ *Pullback.* It is the composite

$$\begin{aligned}
\mathrm{Ext}^{\alpha}(Rf_!\mathbb{A}_X, \mathbb{A}_Y) &\xrightarrow{g^*} \mathrm{Ext}^{\alpha}(g^*Rf_!\mathbb{A}_X, g^*\mathbb{A}_Y) \\
&\cong \mathrm{Ext}^{\alpha}(Rf_{1!}\mathbb{A}_{X_1}, \mathbb{A}_{Y_1}) \qquad \text{[Ive86, p. 322, Base change].}
\end{aligned}$$

Let $\mathbb{B}$ be a bivariant theory on a category having final object pt. The **associated contravariant** $\mathbb{B}^*$ and **covariant groups** $\mathbb{B}_*$ are, respectively,

$$\mathbb{B}^{\alpha}(A) := \mathbb{B}^{\alpha}(id_A), \qquad \mathbb{B}_{\alpha}(A) := \mathbb{B}^{-\alpha}(A \to pt).$$

The operations given by the axioms induce cup, cap, cross, external and slant products; besides, any morphism $f : A \to B$ induces a *pullback* $f^*_{\mathbb{B}} : \mathbb{B}^{\alpha}(B) \to \mathbb{B}^{\alpha}(A)$ and, if f is also confined, a *pushforward* $f^{\mathbb{B}}_* : \mathbb{B}^{\alpha}(A) \to \mathbb{B}^{\alpha}(B)$ as explained in [FH91, pp. 23-25]. Furthermore, such operations permit the construction of the **Gysin maps** associated to any $\theta \in \mathbb{B}^{\alpha}(f)$; namely

$$\theta^* : a \in \mathbb{B}_{\beta}(B) \mapsto \theta \cdot a \in \mathbb{B}_{\beta-\alpha}(A)$$

and, if f is confined,

$$\theta_* : a \in \mathbb{B}^{\beta}(A) \mapsto f_*(a \cdot \theta) \in \mathbb{B}^{\alpha+\beta}(B).$$

Gysin maps can be also described by means of the diagrams below. For their main properties, see [FH91, p. 26].

$$\begin{tikzcd} A \arrow[rr, "f", "\underline{\theta}"'] \arrow[dr, "\underline{\theta^* a}"'] & & B \arrow[dl, "\underline{a}"] \\ & pt & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\underline{a}"] \arrow[dr, "\underline{a \cdot \theta}"] & B \arrow[d, "\underline{\theta_* a}"] \\ A \arrow[r, "f", "\underline{\theta}"'] & B \end{tikzcd}$$

Example 3.1.5. The associated contravariant functor to $\mathbb{B}_{Top}$ is the cohomology theory itself; namely, $\mathbb{B}^{\alpha}_{Top}(id_X) = H^{\alpha}(X)$. Instead, the associated covariant functor to $\mathbb{B}_{Top}$ is given by $\mathbb{B}_{Top,\alpha}(X) = H^{N-\alpha}(\mathbb{R}^N, \mathbb{R}^N \setminus X)$.

Let $f : A \to B$ be a morphism. An element $\theta \in \mathbb{B}(f)$ is called a **strong orientation for** f if, for any morphism $h : W \to X$, product by θ

$$\cdot\theta : a \in \mathbb{B}(h) \mapsto a \cdot \theta \in \mathbb{B}(fh)$$

is an isomorphism. In such case, f is said to be **strongly orientable** and θ is said to have either **dimension** $-d$ or **codimension** d if $\theta \in \mathbb{B}^d(f)$.

The product of strong orientations is a strong orientation. Moreover, if a strong orientation exists, it is unique up to multiplication by unit; in other words, if θ and θ' are strong orientations for f, there is a unit $u \in \mathbb{B}(id_A)$ such that $\theta' = u \cdot \theta$ (see [FH91, p. 27]).

Let S be a class of maps in $\mathcal{C}$ closed under composition and containing all identity morphisms. A **canonical orientation** for S is a correspondence that associates each $f : A \to B$ in S to an element $\theta(f) \in \mathbb{B}(f)$ (denoted by $\theta_{\mathbb{B}}(f)$ if $\mathbb{B}$ has to be highlighted), called a **canonical orientation** for f, so that

$$\begin{aligned} &\theta(gf) = \theta(f) \cdot \theta(g) \ \forall f : A \to B, g : B \to C \text{ in } S, \\ &\theta(id_A) = 1_A \ \forall A \in \mathcal{C}. \end{aligned}$$

The Gysin maps induced by $\theta(f)$ are denoted by $f^!_{\mathbb{B}} := \theta(f)^*$ and $f^{\mathbb{B}}_! := \theta(f)_*$. By definition, the functions $f \mapsto f^!_{\mathbb{B}}$ and $f \mapsto f^{\mathbb{B}}_!$ are functors.

Example 3.1.6. In $\mathbb{B}_{Sh}$, a morphism $f : X \to Y$ has a strong orientation in $\mathbb{B}^{-\alpha}_{Sh}(f)$ if and only if $f^!\mathbb{A}_Y$ is quasi-isomorphic to $\mathbb{A}_X[\alpha]$ by [FH91, p. 85, Proposition]. As a consequence, $\mathbb{B}_{Sh}$ coincides with $\mathbb{B}_{Top}$ (with coefficients in $\mathbb{A}$) on the subcategory of topological spaces embeddable in Euclidean spaces (see [FH91, p. 86, Corollary] and [BSY07, Theorem 3.3]).

Let $\mathbb{B}$ and $\bar{\mathbb{B}}$ be two bivariant theories with underlying categories $\mathcal{C}$ and $\bar{\mathcal{C}}$, respectively. Assume there is a functor $\mathcal{C} \to \bar{\mathcal{C}}$ which sends the final object of $\mathcal{C}$ into the one of $\bar{\mathcal{C}}$ and preserves confined maps and independent squares. The images of objects A and morphisms f in $\mathcal{C}$ shall be denoted by $\bar{A}$ and $\bar{f}$, respectively.

A **Grothendieck transformation** is a collection of morphisms

$$t : \mathbb{B} \to \bar{\mathbb{B}}, \qquad t := (t_f : \mathbb{B}(f) \to \bar{\mathbb{B}}(\bar{f}))_f,$$

where f varies through the class of morphisms in $\mathcal{C}$, which commutes with products, pushforwards and pullbacks.

Mind that t may not preserve degrees. From the definition, it follows that a Grothendieck transformation commutes with all products defined on a bivariant theory and it induces natural transformations

$$t^{Gr} : \mathbb{B}^* \to \bar{\mathbb{B}}^*, \qquad t_{Gr} : \mathbb{B}_* \to \bar{\mathbb{B}}_*.$$

If $\theta \in \mathbb{B}(A \to B)$, then $t(\theta) \in \bar{\mathbb{B}}(\bar{A} \to \bar{B})$ and

$$(t(\theta))_* t^{Gr} = t^{Gr}\theta_* : \mathbb{B}^*(A) \to \bar{\mathbb{B}}^*(\bar{B}), \qquad (t(\theta))^* t_{Gr} = t_{Gr}\theta^* : \mathbb{B}_*(A) \to \bar{\mathbb{B}}_*(\bar{B});$$

in other words, t^{Gr} and t_{Gr} commute with Gysin maps.

Lastly, let $f : A \to B$ be a morphism having a canonical orientation $\theta_{\mathbb{B}}(f)$ and assume that $\bar{f}$ has a canonical orientation $\theta_{\bar{\mathbb{B}}}(\bar{f})$. A formula of the form

$$t(\theta_{\mathbb{B}}(f)) = u_f \cdot \theta_{\bar{\mathbb{B}}}(\bar{f}), \quad u_f \in \bar{\mathbb{B}}^*(\bar{A})$$

is called a **Riemann-Roch formula**.

Notice that, if $\theta_{\bar{\mathbb{B}}}(\bar{f})$ is a strong orientation, u_f is unique. In addition, there are two commutative diagrams (see [FH91, p. 11, 31]).

$$\begin{array}{ccc} \mathbb{B}^*(A) & \xrightarrow{t^{Gr}} & \bar{\mathbb{B}}^*(\bar{A}) \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle \bar{f}_!(-\cdot u_f)} \\ \mathbb{B}^*(B) & \xrightarrow{t^{Gr}} & \bar{\mathbb{B}}^*(\bar{B}) \end{array} \qquad \begin{array}{ccc} \mathbb{B}_*(B) & \xrightarrow{t_{Gr}} & \bar{\mathbb{B}}_*(\bar{B}) \\ \downarrow{\scriptstyle f^!} & & \downarrow{\scriptstyle u_f\cdot(\bar{f}^!(-))} \\ \mathbb{B}_*(A) & \xrightarrow{t_{Gr}} & \bar{\mathbb{B}}_*(\bar{A}) \end{array}$$

3.1.2 Borel-Moore homology

Borel-Moore homology was introduced in [BM60] by Borel and Moore in order to obtain a Poincaré duality theorem for generalized (also known as (co)homology) manifolds, which had been defined for the first time by Lefschetz in [Lef33] and Čech in [Č33] independently.

Let X be a locally compact Hausdorff space and let $\mathbb{A}$ be a commutative Noetherian ring.

Definition 3.1.7. The **Borel-Moore homology** groups of X are

$$H_\alpha^{BM}(X, \mathbb{A}) := H^{-\alpha}(\mathrm{Hom}^\bullet_{\mathbb{A}-mod}(R\Gamma_c(X, \mathbb{A}_X), A^\bullet)),$$

where $\mathbb{A} \to A^\bullet$ is an injective resolution of $\mathbb{A}$ in the category of $\mathbb{A}$-modules.

The definition of Borel-Moore homology given above is the one of [Ive86, §IX.1]; nevertheless, there are others equivalent to it. If X has finite homological dimension, then (see [Ive86, p. 380, formula 4.1])

$$H_\alpha^{BM}(X, \mathbb{A}) \cong H^{-\alpha}(\Gamma(X, \omega_X)),$$

where ω_X is the dualizing complex of X. If, in addition, X is an oriented n-manifold, then (see [Ive86, p. 249, Definition 8.1 and p. 481, Theorem 4.7], [Bre97, Theorem 12.1] with ϕ the family of all closed subsets of X)

$$H_\alpha^{BM}(X, \mathbb{A}) \cong H^{n-\alpha}(X, X \setminus Z; \mathbb{A}_X) \cong H^{n-\alpha}(X, X \setminus Z; \mathbb{A})$$

for any closed subspace Z, where the term on the right is the singular cohomology group of the pair (X, W) with coefficients in $\mathbb{A}$, where $W := X \setminus Z$, and the last isomorphism is a combination of the following facts:

i) for any **paracompact** space (i.e. any open covering has a **locally finite** refinement, which means that any point has a neighbourhood which intersects at most a finite number of elements of the refinement), Čech (see [Rot09, §6.3.1]) and sheaf cohomology coincide by [God73, Theorem 5.1.10];

ii) for any **locally contractible** (that is, for any point x and any open set U containing it, there is an open contractible set V such that $x \in V \subseteq U$) paracompact Hausdorff space, Čech and singular cohomology coincide by [Spa66, p. 334, Corollary and p. 340, Corollary];

iii) application of five lemma to the commutative diagram whose rows are the long exact sequences in singular and sheaf cohomology of the pair (X, W) (see [Bre97, p. 84, formula 22] and [Rot88, Theorem 12.9])

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, W; \mathbb{A}_X) \arrow[r] \arrow[d] & H^0(X, \mathbb{A}_X) \arrow[r] \arrow[d, "\cong"] & H^0(W, \mathbb{A}_Z) \arrow[r] \arrow[d, "\cong"] & H^1(X, W; \mathbb{A}_X) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & H^0(X, W; \mathbb{A}) \arrow[r] & H^0(X; \mathbb{A}) \arrow[r] & H^0(W; \mathbb{A}) \arrow[r] & H^1(X, W; \mathbb{A}) \arrow[r] & {}
\end{tikzcd}$$

In particular, if X is a locally compact Hausdorff space embeddable as a closed subspace of an oriented manifold M, then

$$H_\alpha^{BM}(X, \mathbb{A}) \cong H^{\dim M - \alpha}(M, M \setminus X; \mathbb{A}).$$

3.2 Bivariant classes of degree one and pushforwards

The section begins with the definition of bivariant classes of degree one. After the preliminary lemmas of Section 3.2.2, the generalization of the decomposition theorem, that is, Theorem 3.2.8, is provided.

3.2.1 Bivariant classes of degree one

Until the end of the chapter, $\mathbb{A}$ will be a Noetherian unitary commutative ring, unless otherwise stated; every topological space shall be assumed to be a locally compact Hausdorff space embeddable as a closed subspace of $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and all morphisms will be supposed to be proper continuous maps of finite cohomological dimension. All singular cohomology and Borel-Moore homology groups will be taken with coefficients in $\mathbb{A}$ and denoted by $H^\alpha(X)$ and $H_\alpha^{BM}(X)$, respectively.

Let $f : X \to Y$ be a proper continuous map, let

$$\theta \in H^0(f) := \mathrm{Hom}_{D_c^b(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$$

be a bivariant class and let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced Gysin map. θ is said to **have degree one** (for the map f) if $\theta_0(1_X) = 1_Y \in H^0(Y)$.

Clearly, if $\theta_0(1_X) = d \cdot 1_Y$ with d a unit in $\mathbb{A}$, then $d^{-1} \cdot \theta$ is a bivariant class of degree one. Here are other elementary facts concerning such classes.

Remark 3.2.1. [GFS22, Remark 2.1 (ii)]. *The pullback θ' of a bivariant class $\theta \in H^0(f)$ of degree one under the inclusion of any subspace is of degree one. Conversely, if Y is path-connected and θ' is of degree one, then so is θ.*

Indeed, let $i : Z \hookrightarrow Y$ be the inclusion of a non-empty subspace of Y, let $g : f^{-1}(Z) \to Z$ be the restriction of f to $f^{-1}(Z)$ and set $\theta' = i^*(\theta) \in H^0(g)$. By [FM81, p. 26, (G2) (ii)], $i^*\theta_0(1_X) = \theta'_0 j^*(1_X)$, where $j : f^{-1}(Z) \hookrightarrow X$ is inclusion.

Remark 3.2.2. [GFS22, Remark 2.1 (iii)]. *Assume that $f : X \to Y$ is a birational projective locally complete intersection morphism between complex irreducible quasi-projective varieties. If $\theta \in H^0(f)$ is the orientation class of f (see [FM81, §4.1.3, Part I, and p. 131] and [Ful98, §19.2]), then θ has degree one.*

In fact, let U be a non-empty Zariski open subset of Y, such that $f^{-1}(U) \cong U$. Let θ' be the restriction of θ on $f^{-1}(U) \to U$. θ' is the orientation class of $f^{-1}(U) \to U$ by [Ful98, p. 379, Lemma 19.2 (a)], therefore θ' has degree one and the preceding remark implies that θ has degree one, as well.

Remark 3.2.3. [GFS22, Remark 2.1 (vi)]. Let $f : X \to Y$ be a projective map between irreducible quasi-projective varieties with $\dim X = n$ and $\dim Y = m$ and assume the existence of a bivariant class θ of degree one. For any α, $f_*\theta^* = id_{H_\alpha^{BM}(Y)}$ by [DGF17, Remark 2.5], hence the pushforward f_* induces an inclusion $H_\alpha^{BM}(Y) \subseteq H_\alpha^{BM}(X)$ and $m \leq n$. Moreover, f is surjective, otherwise the pushforward $f_* : H_{2m}^{BM}(X) \to H_{2m}^{BM}(Y)$ would vanish. Since the restriction of θ to some special fibre is a bivariant class of degree one, it may happen that $m < n$. If $n = m$, f is birational.

The following result will be used frequently later on.

Lemma 3.2.4. [GFS22, Remark 2.1 (i)]. *θ has degree one if and only if θ is a section of the pullback $f : \mathbb{A}_Y \to Rf_*\mathbb{A}_X$ (see [Voi07, §4.3.1]); in other words,*

$$\theta_0(1_X) = 1_Y \Leftrightarrow \theta f^* = id_{\mathbb{A}_Y}.$$

Proof. $\Rightarrow$ For any α and $y \in H^\alpha(Y)$, [FM81, p. 26, (G4) (i)] and [Spa66, p. 251, 9] imply

$$\theta_*(f^*(y)) = \theta_*(f^*(y) \cup 1_X) = y \cup \theta_*(1_X) = y \cup 1_Y = y.$$

Being y arbitrary, this means that θf^* induced the identity $\theta_* f^* = id_{H^\alpha(Y)}$ for any α. On the other hand, $\theta f^* \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(\mathbb{A}_Y, \mathbb{A}_Y) \cong H^0(Y)$, hence $\theta f^* = id_{\mathbb{A}_Y}$.

$\Leftarrow$ If $\theta f^* = id_{\mathbb{A}_Y}$, the composite

$$H^0(Y) \xrightarrow{f^*} H^0(X) \xrightarrow{\theta_0} H^0(Y)$$

is the identity of $H^0(Y)$. Since $f^*(1_Y) = 1_X$, $\theta_0(1_X) = 1_Y$. □

3.2.2 Results on triangulated categories and sheaves

Lemma 3.2.5. [GFS22, Lemma 3.1]. *Let $\mathcal{T}$ be a triangulated category and let $a \in \mathrm{Hom}_\mathcal{T}(A, B)$. Assume that there exists $b \in \mathrm{Hom}_\mathcal{T}(B, A)$ such that $ba = id_A$. Then $B \cong A \oplus C$ for some $C \in \mathcal{T}$.*

Proof. Axiom TR1 implies that $a : A \to B$ can be completed to a distinguished triangle $A \to B \to C$. Thus, the combination of the hypothesis $ba = id_A$ with axioms TR1 and TR3 gives a morphism of distinguished triangles

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "id"] & B \arrow[r] \arrow[d, "b"] & C \arrow[d] \\ A \arrow[r, "id"] & A \arrow[r] & 0 \end{tikzcd}$$

Axiom TR2 provides the following morphism of distinguished triangles

$$\begin{tikzcd} C \arrow[r, "b"] \arrow[d] & A[1] \arrow[r] \arrow[d, "id"] & B[1] \arrow[d, "b[1]"] \\ 0 \arrow[r] & A[1] \arrow[r] & A[1] \end{tikzcd}$$

from which $b = 0$ follows. As a consequence, $A \to B \to C$ splits by [Huy06, Exercise 1.38]. □

Lemma 3.2.6. [GFS22, Lemma 3.4]. *Consider the following morphisms of distinguished triangles in a triangulated category $\mathcal{T}$.*

$$\begin{tikzcd} A \arrow[r] & B_1 \arrow[r] & C_1 \\ A \arrow[r] \arrow[u, "id_A"] & B \arrow[r] \arrow[u, "b"] & C \arrow[u, "c"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "id_A"] & B_1 \arrow[r] \arrow[d, "b'"] & C_1 \arrow[d, "c'"] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

If $b'b = id_B$, $c'c = id_C$ and $\mathrm{Hom}_\mathcal{T}(A, C_1[-1]) = 0$, there is a "cross" isomorphism

$$B_1 \oplus C \cong B \oplus C_1.$$

Proof. Consider the following commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d] \\
B \arrow[r, "b"] \arrow[d] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B[1] \arrow[d] \\
C \arrow[r, "c"] \arrow[d] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C[1] \arrow[d] \\
A[1] \arrow[r, "{id_A[1]}"] & A[1] \arrow[r] & 0 \arrow[r] & A[2]
\end{tikzcd} \tag{3.2.1}$$

where the first and second columns are the ones given in the hypothesis, and the fourth column is obtained by the first one by means of TR2. The first row, which gives the fourth one by means of TR2, is distinguished by TR1. The second and third rows are given by completion of b and c, respectively, by means of TR1. Lastly, the arrows in the third column are given by TR3. Observe that the third column, a priori, is not a distinguished triangle.

Since $b'b = id_B$ and $c'c = id_C$, Lemma 3.2.5 guarantees that $B_1 \cong B \oplus B_2$ and that $C_1 \cong C \oplus C_2$. Therefore, it suffices to prove that $B_2 \cong C_2$.

Consider the upper cap diagram obtained by the commutative triangle of vertices A, B, B_1 (see below on the left), which can be completed to an octahedron by TR4, whose lower cap is the one drawn below on the right.

$$\begin{tikzcd}
B_2 \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & B_1 \arrow[ll] \\
& B \arrow[ur] \arrow[dl] \arrow[u, phantom, "\star"] \arrow[d, phantom, "\star"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
C \arrow[rr, "{[1]}"] & & A \arrow[ul] \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[dd, "{[1]}"'] & & B_1 \arrow[ll] \arrow[dl] \\
& C_1 \arrow[ul] \arrow[dr, "{[1]}"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] \arrow[l, phantom, "\star"] \arrow[r, phantom, "\star"] & \\
C \arrow[ur, "\varphi"] \arrow[rr, "{[1]}"] & & A \arrow[uu]
\end{tikzcd}$$

In particular, $C \xrightarrow{\varphi} C_1 \to B_2 \to C[1]$ is a distinguished triangle. In addition,

$$\begin{tikzcd}
& & C \arrow[dr, "\varphi"] & & \\
& B \arrow[ur] \arrow[dr, "b"] & & C_1 \arrow[ddr] & \\
& & B_1 \arrow[ur] \arrow[drr] & & \\
A \arrow[uur] \arrow[urr] & & & & B_2
\end{tikzcd}$$

the octahedron shows the existence of a morphism of distinguished triangles

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "\varphi"] & A[1] \arrow[d, "id"] \\
B_1 \arrow[r] & C_1 \arrow[r] & A[1]
\end{tikzcd}$$

This morphism also appears in Diagram (3.2.1), yet with φ instead of c. Nevertheless, [GM03, Corollary IV.1.5] implies that $\varphi = c$. As a consequence, the diagram below, whose rows are distinguished triangles and whose squares commute, can be completed to a morphism of distinguished triangles by TR3.

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "id_C"] & C_1 \arrow[r] \arrow[d, "id_{C_1}"] & B_2 \arrow[r] \arrow[d, dashed] & C[1] \arrow[d, "id"] \\
C \arrow[r, "c"] & C_1 \arrow[r] & C_2 \arrow[r] & C[1]
\end{tikzcd}$$

The dashed vertical arrow is unique because of [GM03, loc. cit.] and is an isomorphism by [GM03, Corollary IV.1.4 (a)]. □

Let $f : X \to Y$ be a proper continuous map and assume the existence of an open subset $U \subseteq Y$ such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is an isomorphism. Set $W := Y \setminus U$ and $\widetilde{W} := X \setminus V$ and let $g : \widetilde{W} \to W$ be the restriction of f to $\widetilde{W}$. Then, there is a commutative diagram

$$\begin{tikzcd}
\widetilde{W} \arrow[r, hook, "i_{\widetilde{W}}"] \arrow[d, "g"] & X \arrow[d, "f"] & V \arrow[l, hook', "j_V"'] \arrow[d, "h"] \\
W \arrow[r, hook, "i_W"] & Y & U \arrow[l, hook', "j_U"']
\end{tikzcd} \tag{3.2.2}$$

whose horizontal arrows are inclusions.

Lemma 3.2.7. [GFS22, Lemma 3.3]. *$f_* j_{V!} = j_{U!} h_!$. Furthermore, f_* induces an exact equivalence between*

$$\mathrm{Sh}_V(X, \mathbb{A}) := \{\mathcal{F} \in \mathrm{Sh}(X, \mathbb{A}) : \mathcal{F}_x = 0 \ \forall x \in \widetilde{W}\}$$

and

$$\mathrm{Sh}_U(Y, \mathbb{A}) := \{\mathcal{G} \in \mathrm{Sh}(Y, \mathbb{A}) : \mathcal{G}_y = 0 \ \forall y \in W\},$$

whose inverse is induced by its pullback f^*.

Proof.

▲ $f_* j_{V!} = j_{U!} h_!$.

Indeed, $f_* = f_!$ for f is proper, while the commutativity of the diagram and [Dim04, Proposition 2.3.23] imply

$$f_! j_{V!} = (f j_V)_! = (j_U h)_! = j_{U!} h_!.$$

▲ f_* induces an equivalence between $\mathrm{Sh}_V(X, \mathbb{A})$ and $\mathrm{Sh}_U(Y, \mathbb{A})$.

By [Ive86, Proposition II.6.4], $j_{U!}$ and $j_{V!}$ induce equivalence of categories

$$\tilde{j}_{U!} : \mathrm{Sh}(U, \mathbb{A}) \to \mathrm{Sh}_U(Y, \mathbb{A}) \qquad \tilde{j}_{V!} : \mathrm{Sh}(V, \mathbb{A}) \to \mathrm{Sh}_V(X, \mathbb{A}).$$

On the other hand, h is an isomorphism, thus $h_! : \mathrm{Sh}(V, \mathbb{A}) \to \mathrm{Sh}(U, \mathbb{A})$ is an equivalence, as well. Then, the desired equivalence of categories is

$$\tilde{f}_! := (\tilde{j}_{V!})^{-1} \circ h_! \circ \tilde{j}_{U!}.$$

▲ $\tilde{f}_!$ is exact.

This follows from the definition, since $h_!$ $\tilde{j}_{U!}$ are exact by [Ive86, p. 106, II.6.3] and $(\tilde{j}_{V!})^{-1}$ is induced by j_V^* (see [Ive86, Proposition II.6.4]), which is exact.

□

3.2.3 Decomposition of the (derived) pushforward

Theorem 3.2.8. [GFS22, Theorem 1.1]. *Let $f : X \to Y$ be a proper continuous map, with Y path-connected. Let $U \subseteq Y$ be a non-empty open subset such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is a homeomorphism. Set $W = Y \setminus U$, $\widetilde{W} = f^{-1}(W)$ and consider the commutative diagram (3.2.2). The following properties are equivalent:*

i) there exists a bivariant class $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one;

ii) there is a cross isomorphism $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$ in $D^b_c(Y, \mathbb{A})$;

iii) there exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y, \mathbb{A})$.

Proof. *i)* ⇒ *iii)* By Remark 3.2.4, $\theta f^* = id_{\mathbb{A}_Y}$. The desired decomposition is achieved by taking $\mathcal{T} = D^b_c(Y, \mathbb{A})$, $A = \mathbb{A}_Y$ and $B = Rf_*\mathbb{A}_X$ in Lemma 3.2.5.

iii) ⇒ *i)* The projection on the first summand $\eta : Rf_*\mathbb{A}_X \to \mathbb{A}_Y$ induces a bivariant class. Let U' be a path-connected component of U. Since the restriction $\eta' := \eta|_{U'}$ is an automorphism of $\mathbb{A}_{U'}$, it follows that $\eta'_0(1_{U'}) = d \cdot 1_{U'} \in H^0(U')$ for some unit $d \in \mathbb{A}$. Hence, $d^{-1} \cdot \eta$ is a bivariant class of degree one.

ii) ⇒ *i)* The proof is the same as *iii)* ⇒ *i)*; it suffices to take the projection $Rf_*\mathbb{A}_X \to \mathbb{A}_Y$.

i) ⇒ *ii)* To start with, recall that

$$0 \to j_{U!}\mathbb{A}_U \to \mathbb{A}_Y \to i_{V*}\mathbb{A}_V \to 0$$

is exact (see [Dim04, Remark 2.4.5]). It follows that there is a morphism of distinguished triangles in $D^b_c(Y, \mathbb{A})$ [Dim04, p. 46]

$$\begin{tikzcd} Rf_*(j_{V!}\mathbb{A}_V) \arrow[r, "a"] & Rf_*\mathbb{A}_X \arrow[r] & Rf_*i_{\widetilde{W}*}\mathbb{A}_{\widetilde{W}} \\ j_{U!}\mathbb{A}_U \arrow[r, "b"] \arrow[u, "\cong"] & \mathbb{A}_Y \arrow[r] \arrow[u, "f^*"] & i_{W*}\mathbb{A}_W \arrow[u, "g^*"] \end{tikzcd} \tag{3.2.3}$$

Lemma 3.2.7 ensures that the left vertical map is actually an isomorphism in $D^b_c(Y, \mathbb{A})$.

On the other hand, there is another morphism of distinguished triangles

$$\begin{tikzcd} Rf_*(j_{V!}\mathbb{A}_U) \arrow[r, "a"] \arrow[d, "\cong", "\varphi"'] & Rf_*\mathbb{A}_X \arrow[r] \arrow[d, "\theta"] & Rf_*i_{\widetilde{W}*}\mathbb{A}_{\widetilde{W}} \arrow[d, "\eta"] \\ j_{U!}\mathbb{A}_U \arrow[r, "b"] & \mathbb{A}_Y \arrow[r] & i_{W*}\mathbb{A}_W \end{tikzcd} \tag{3.2.4}$$

In fact, the left square commutes, for

$$\theta a = \theta(f^* b\varphi) = (\theta f^*) b\varphi = id_{\mathbb{A}_Y} b\varphi = b\varphi,$$

and $\eta : Rf_* i_{\widetilde{W}*}\mathbb{A}_{\widetilde{W}} \to i_{W*}\mathbb{A}_W$, unique by [GM03, Corollary IV.1.5] (bear in mind that $Rf_*(j_{V!}\mathbb{A}_U)$ is isomorphic to $j_{U!}\mathbb{A}_U$), is given by TR3.

Since $\theta f^* = id_{\mathbb{A}_Y}$, the combination of (3.2.3) with (3.2.4) gives the morphism of distinguished triangles

$$\begin{tikzcd} j_{U!}\mathbb{A}_U \arrow[r] \arrow[d, "id"] & \mathbb{A}_Y \arrow[r] \arrow[d, "id"] & i_{W*}\mathbb{A}_W \arrow[d, "\eta g^*"] \\ j_{U!}\mathbb{A}_U \arrow[r] & \mathbb{A}_Y \arrow[r] & i_{W*}\mathbb{A}_W \end{tikzcd}$$

Corollaries IV.1.4 (a) and IV.1.5 of [GM03] guarantee that $\eta g^* = id$, thus the hypotheses of Lemma 3.2.6, which provides the wanted result, are met (take the numbered diagrams).

□

Notice that in the proof of $i) \Rightarrow iii)$ the existence of U is unnecessary.

Remark 3.2.9. [GFS22, remark 3.6]. Bivariant theory provides a pullback morphism (see [FM81, p. 19, (3)])

$$\eta_1 := i_W^*(\theta) : Rf_* i_{\widetilde{W}}^* \mathbb{A}_{\widetilde{W}} \to i_{W*}\mathbb{A}_W$$

of degree one and, consequently, $\eta_1 g^* = id$. However, it is not known if $\eta = \eta_1$.

3.3 Bivariant classes of degree one and homology manifolds

This section is an application of Theorem 3.2.8 to the study of homology manifolds. To begin with, the decomposition of item *ii)* of such theorem implies the existence of isomorphisms between certain cohomology and Borel-Moore homology groups, explicated in Sections 3.3.1 and 3.3.2, respectively. Consequently, in Section 3.3.3, a suitable description of the duality morphism $\mathbb{D}_X$ (see (3.3.5)) induced by X is achieved. Theorem 3.3.6, which gives a relation between the existence of a bivariant class of degree one for a morphism $f : X \to Y$ and the property of X and Y of being $\mathbb{A}$-homology manifolds, is finally proved in Section 3.3.4. Examples and consequences of such theorem are exhibited in Section 3.3.5.

3.3.1 Decompositions in cohomology

Throughout the section, the hypotheses of Theorem 3.2.8 are supposed to hold and the same notations are adopted. Moreover, θ shall denote a bivariant class of degree one for f.

By taking hypercohomology and hypercohomology with compact support in the isomorphism $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$, it follows, for any α (see [Dim04, Corollaries 2.3.4 and 2.3.24]),

$$H^\alpha(X) \oplus H^\alpha(W) \cong H^\alpha(\widetilde{W}) \oplus H^\alpha(Y)$$
$$H^{BM}_\alpha(X) \oplus H^{BM}_\alpha(W) \cong H^{BM}_\alpha(\widetilde{W}) \oplus H^{BM}_\alpha(Y).$$

The aim of this and the following subsection is to make theses isomorphisms explicit.

Taking hypercohomology, morphisms (3.2.3) and (3.2.4) induce the following commutative diagram with exact rows (see [Dim04, Remark 2.4.5]).

$$\begin{tikzcd}
{} \arrow[r] & H^\alpha(Y,W) \arrow[r] \arrow[d, "\cong"] & H^\alpha(Y) \arrow[r, "i^*"] \arrow[d, "f^*"] & H^\alpha(W) \arrow[r] \arrow[d, "g^*"] & H^{\alpha+1}(Y,W) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H^\alpha(X,\widetilde{W}) \arrow[r] \arrow[d, "\cong"] & H^\alpha(X) \arrow[r, "j^*"] \arrow[d, "\theta_*"] & H^\alpha(\widetilde{W}) \arrow[r] \arrow[d, "\eta_*"] & H^{\alpha+1}(X,\widetilde{W}) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H^\alpha(Y,W) \arrow[r] & H^\alpha(Y) \arrow[r, "i^*"] & H^\alpha(W) \arrow[r] & H^{\alpha+1}(Y,W) \arrow[r] & {}
\end{tikzcd}$$

From the fact that $\theta_* f^* = id_{H^\alpha(Y)}$ and $\eta_* g^* = id_{H^\alpha(W)}$ for any α, a diagram chase shows the exactness of the sequence (compare with [Ful98, p. 114, 115, Proposition 6.7 (e)])

$$0 \longrightarrow H^\alpha(X) \xrightarrow{\ \nu\ } H^\alpha(\widetilde{W}) \oplus H^\alpha(Y) \xrightarrow{\ \varrho\ } H^\alpha(W) \longrightarrow 0,$$

where

$$\nu(x) := (j^*(x), -\theta_*(x)), \qquad \varrho(\tilde{w}, y) := \eta_*(\tilde{w}) + i^*(y).$$

Moreover, such sequence has a right section, that is, a right inverse of ϱ,

$$w \in H^\alpha(W) \mapsto (g^*(w), 0) \in H^\alpha(\widetilde{W}) \oplus H^\alpha(Y).$$

Therefore, the sequence is split and there is an isomorphism (see [Rot09, Proposition 2.28])

$$\varphi^* : (x, w) \in H^\alpha(X) \oplus H^\alpha(W) \mapsto (j^*(x) + g^*(w), -\theta_*(x)) \in H^\alpha(\widetilde{W}) \oplus H^\alpha(Y)$$

that can be represented as (compare with [Jou77, p. 328])

$$\begin{bmatrix} \tilde{w} \\ y \end{bmatrix} = \begin{bmatrix} j^* & g^* \\ -\theta_* & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ w \end{bmatrix}$$

Since $\varphi^*(-f^*y, i^*y) = (0, y)$, the inverse map $(\varphi^*)^{-1}$ can be represented as

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} \lambda_* & -f^* \\ \mu_* & i^* \end{bmatrix} \cdot \begin{bmatrix} \tilde{w} \\ y \end{bmatrix},$$

where $\lambda_* : H^\alpha(\widetilde{W}) \to H^\alpha(X)$ and $\mu_* : H^\alpha(\widetilde{W}) \to H^\alpha(W)$ are uniquely determined by the condition that the two matrices above are inverse to one other

$$\begin{cases} \lambda_* j^* + f^* \theta_* = id_{H^\alpha(X)} \ \forall \alpha \\ \lambda_* g^* = 0 \\ \mu_* j^* - i^* \theta_* = 0 \\ \mu_* g^* = id_{H^\alpha(W)} \ \forall \alpha, \end{cases} \tag{3.3.1}$$

which, in turn, are equivalent to the equations

$$\begin{cases} j^* \lambda_* + g^* \mu_* = id_{H^\alpha(\widetilde{W})} \ \forall \alpha \\ \theta_* \lambda_* = 0 \\ j^* f^* = g^* i^* \\ \theta_* f^* = id_{H^\alpha(Y)} \ \forall \alpha. \end{cases} \tag{3.3.2}$$

In particular, $\eta_* = \mu_*$ follows.

The isomorphisms $H^\alpha(X) \oplus H^\alpha(W) \cong H^\alpha(\widetilde{W}) \oplus H^\alpha(Y)$ can be finally described explicitly. Beforehand, observe that, since $\eta_* g^* = id_{H^\alpha(W)}$ for all α, $H^\alpha(W)$ can be thought of, via g^*, as a direct summand of $H^\alpha(\widetilde{W})$ for every integer α.

Proposition 3.3.1. [GFS22, Lemma 4.3]. *For every $k \in \mathbb{Z}$,*

$$x \in H^\alpha(X) \mapsto (\theta_* x, j^* x) \in H^\alpha(Y) \oplus \frac{H^\alpha(\widetilde{W})}{H^\alpha(W)}$$

is an isomorphism with inverse

$$(y, \widetilde{w}) \in H^\alpha(Y) \oplus \frac{H^\alpha(\widetilde{W})}{H^\alpha(W)} \mapsto f^*(y) + \lambda_* \widetilde{w} \in H^\alpha(X).$$

Proof. Firstly, the function

$$x \in H^\alpha(X) \mapsto (\theta_* x, x - f^* \theta_* x) \in H^\alpha(Y) \oplus \ker \theta_*$$

is an isomorphism; in fact,

- ▲ if x is such that $(\theta_* x, x - f^* \theta_* x) = 0$, then the first component is $\theta_* x = 0$ and, consequently, the second one gives $x = 0$.
- ▲ Let $(y, z) \in H^\alpha \oplus \ker \theta_*$. $\theta_* f^* = id_{H^\alpha(Y)}$, thus $\theta_* f^* y = y$ and (y, z) is the image of $f^* y + z$.

Secondly, equations (3.3.1) and (3.3.2) show that j^* induces an isomorphism $\ker \theta_* \to \ker \eta_*$, whose inverse acts as λ_*. Indeed,

- ▲ the map is injective because, for all $x \in \ker \theta_*$, $\lambda_* j^* x = \lambda_* j^* x + f^* \theta_* x = x$;
- ▲ it is injective because, for any $y \in \ker \eta_*$, $j^* \lambda_* y = j^* \lambda_* y + g^* \eta_* y = y$.

Lastly, the equality $\eta_* g^* = id_{H^\alpha(W)}$ implies $\ker \eta_* \cong \operatorname{coker} g^*$ and, consequently, there is an isomorphism

$$\widetilde{w} \in \ker \eta_* \mapsto \widetilde{w} \in \frac{H^\alpha(\widetilde{W})}{H^\alpha(W)}.$$

□

3.3.2 Decompositions in Borel-Moore homology

Consider again the morphisms (3.2.3) and (3.2.4). The following commutative diagram with exact rows is obtained by taking hypercohomology with compact support.

$$\begin{tikzcd}
\longrightarrow H^{BM}_{\alpha+1}(U) \arrow[r] \arrow[d, "\cong"] & H^{BM}_{\alpha}(W) \arrow[r, "i_*"] \arrow[d, "\eta^*"] & H^{BM}_{\alpha}(Y) \arrow[r] \arrow[d, "\theta^*"] & H^{BM}_{\alpha}(U) \longrightarrow \arrow[d, "\cong"] \\
\longrightarrow H^{BM}_{\alpha+1}(V) \arrow[r] \arrow[d, "\cong"] & H^{BM}_{\alpha}(\widetilde{W}) \arrow[r, "j_*"] \arrow[d, "g_*"] & H^{BM}_{\alpha}(X) \arrow[r] \arrow[d, "f_*"] & H^{BM}_{\alpha}(V) \longrightarrow \arrow[d, "\cong"] \\
\longrightarrow H^{BM}_{\alpha+1}(U) \arrow[r] & H^{BM}_{\alpha}(W) \arrow[r, "i_*"] & H^{BM}_{\alpha}(Y) \arrow[r] & H^{BM}_{\alpha}(U) \longrightarrow
\end{tikzcd}$$

From the fact that $f_*\theta^* = id_{H^{BM}_\alpha(Y)}$ and $g_*\eta^* = id_{H^{BM}_\alpha(W)}$ for any α, a diagram chase shows the exactness of the sequence (compare with [DGF14, pp. 264-266, Proposition 2.5])

$$0 \longrightarrow H^{BM}_{\alpha}(W) \xrightarrow{\sigma} H^{BM}_{\alpha}(\widetilde{W}) \oplus H^{BM}_{\alpha}(Y) \xrightarrow{\upsilon} H^{BM}_{\alpha}(X) \longrightarrow 0,$$

where

$$\sigma(w) := (\eta^*(w), -i_*(w)), \qquad \upsilon(\widetilde{w}, y) := j_*(\widetilde{w}) + \theta^*(y)$$

Moreover, such sequence has a left section, that is, a left inverse of σ,

$$(\tilde{w}, y) \in H^{BM}_{\alpha}(\widetilde{W}) \oplus H_{\alpha}(Y) \mapsto g_*(\tilde{w}) \in H^{BM}_{\alpha}(W).$$

Therefore, the sequence is split and there is an isomorphism

$$\varphi_* : H^{BM}_{\alpha}(\widetilde{W}) \oplus H^{BM}_{\alpha}(Y) \to H^{BM}_{\alpha}(X) \oplus H^{BM}_{\alpha}(W),$$

given by $\varphi_*(\tilde{w}, y) = (j_*(\tilde{w}) + \theta^*(y), g_*(\tilde{w}))$, that can be represented as

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} j^* & \theta^* \\ g_* & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{w} \\ y \end{bmatrix}$$

Since $\varphi_*(\eta^* w, -i_* w) = (0, w)$, the inverse map $(\varphi_*)^{-1}$ can be represented as

$$\begin{bmatrix} \tilde{w} \\ y \end{bmatrix} = \begin{bmatrix} \lambda^* & \eta^* \\ \mu^* & -i_* \end{bmatrix} \cdot \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\lambda^* : H^{BM}_{\alpha}(\widetilde{W}) \to H^{BM}_{\alpha}(X)$ and $\mu^* : H^{BM}_{\alpha}(\widetilde{W}) \to H^{BM}_{\alpha}(W)$ are uniquely determined by the condition that the two matrices above are inverse to one other

$$\begin{cases} j_*\lambda^* + \theta^*\mu^* = id_{H^{BM}_\alpha(X)} \ \forall\alpha \\ g_*\lambda^* = 0 \\ j_*\eta^* - \theta^* i_* = 0 \\ g_*\eta^* = id_{H^{BM}_\alpha(W)} \ \forall\alpha, \end{cases} \tag{3.3.3}$$

which, in turn, are equivalent to the equations

$$\begin{cases} \lambda^* j_* + \eta^* g_* = id_{H_\alpha^{BM}(\widetilde{W})} \ \forall \alpha \\ \lambda^* \theta^* = 0 \\ \mu^* j_* = i_* g_* \\ \mu^* \theta^* = id_{H_\alpha^{BM}(Y)} \ \forall \alpha. \end{cases} \tag{3.3.4}$$

In particular, $\mu^* = f_*$ follows.

The isomorphisms $H_\alpha^{BM}(X) \oplus H_\alpha^{BM}(W) \cong H_\alpha^{BM}(\widetilde{W}) \oplus H_\alpha^{BM}(Y)$ can be finally described explicitly. Beforehand, observe that, since $g_*\eta^* = id_{H_\alpha^{BM}(W)}$ for all α, $H_\alpha(W)$ can be thought of, via η^*, as a direct summand of $H_\alpha(\widetilde{W})$ for every integer α.

Proposition 3.3.2. [GFS22, Lemma 4.5]. *For every $k \in \mathbb{Z}$,*

$$x \in H_\alpha^{BM}(X) \mapsto (f_*x, \lambda^*x) \in H_\alpha^{BM}(Y) \oplus \frac{H_\alpha^{BM}(\widetilde{W})}{H_\alpha^{BM}(W)}$$

is an isomorphism with inverse

$$(y, \widetilde{w}) \in H_\alpha^{BM}(Y) \oplus \frac{H_\alpha^{BM}(\widetilde{W})}{H_\alpha^{BM}(W)} \mapsto \theta^*(y) + j_*\lambda^* j_* \widetilde{w} \in H_\alpha^{BM}(X).$$

Proof. Firstly, the function

$$x \in H_\alpha^{BM}(X) \mapsto (f_*x, x - \theta^* f_* x) \in H_\alpha^{BM}(Y) \oplus \ker f_*$$

is an isomorphism; in fact,

- ▲ if x is such that $(f_*x, x - \theta^* f_* x) = 0$, then the first component is $f_*x = 0$ and, consequently, the second one gives $x = 0$.
- ▲ Let $(y, z) \in H_\alpha^{BM}(Y) \oplus \ker f_*$. $f_*\theta^* = id_{H^\alpha(Y)}$, thus $f_*\theta^* y = y$ and (y, z) is the image of $\theta^* y + z$.

Secondly, equations (3.3.3) and (3.3.4) show that λ^* induces an isomorphism $\ker f_* \to \ker g_*$, whose inverse acts as j_*. Indeed,

- ▲ λ^* is injective because, for all $x \in \ker f_*$, $j_*\lambda^* x = j_*\lambda^* x + \theta^* f_* x = x$;
- ▲ it is surjective because, for any $y \in \ker g_*$, $\lambda^* j_* y = \lambda^* j_* y + \eta^* g_* y = y$.

Lastly, the equality $g_*\eta^* = id_{H_\alpha^{BM}(W)}$ implies $\ker g_* \cong \operatorname{coker} \eta^*$ and, consequently, there is an isomorphism

$$\widetilde{w} \in \ker g_* \mapsto \widetilde{w} \in \frac{H_\alpha^{BM}(\widetilde{W})}{H_\alpha^{BM}(W)}.$$

□

Remark 3.3.3. [GFS22, Remark 4.2]. Let $\eta_1 := i^*(\theta)$ be the pullback of θ on W. By [FM81, p. 26, (G2)], $(\eta_1)_* j^* - i^*\theta_* = 0$ and $(\eta_1)_* g^* = id_{H^\alpha(W)}$ for all α, which proves that $\eta_* = (\eta_1)_*$. Similarly, $\eta^* = (\eta_1)^*$.

3.3.3 The duality morphism

In this subsection, f will also be supposed to be surjective and X and Y shall be assumed to be open subsets of complex quasi-projective varieties of the same complex dimension n.

Let $[X] \in H_{2n}^{BM}(X)$ be the fundamental class of X (see [GH94, pp. 60, 61]) and set

$$\mathbb{D}_X : x \in H^{\alpha}(X) \mapsto x \cap [X] \in H_{2n-\alpha}^{BM}(X). \tag{3.3.5}$$

When X is a compact complex variety, this map is called the **duality morphism** (see [McC77, p. 150]). If, in addition, X is smooth, then $\mathbb{D}_X$ is the *Poincaré Duality isomorphism*.

In view of Propositions 3.3.1 and 3.3.2, $\mathbb{D}_X$ identifies with the map

$$\mathbb{D}_X : H^{\alpha}(Y) \oplus \frac{H^{\alpha}(\widetilde{W})}{H^{\alpha}(W)} \to H_{2n-k}^{BM}(Y) \oplus \frac{H_{2n-k}^{BM}(\widetilde{W})}{H_{2n-k}^{BM}(W)}$$

given by

$$\mathbb{D}_X(y, \tilde{w}) = (f_*((f^*y + \lambda_*\tilde{w}) \cap [X]), \lambda^*((f^*y + \lambda_*\tilde{w}) \cap [X]))$$

and induces two projections

$$P_1 : y \in H^{\alpha}(Y) \mapsto f_*(f^*y \cap [X]) \in H_{2n-k}^{BM}(Y),$$
$$P_2 : \tilde{w} \in \frac{H^{\alpha}(\widetilde{W})}{H^{\alpha}(W)} \mapsto \lambda^*(\lambda_*\tilde{w} \cap [X]) \in \frac{H_{2n-k}^{BM}(\widetilde{W})}{H_{2n-k}^{BM}(W)}.$$

Observe that $f_*(f^*y \cap [X]) = y \cap [Y]$ by the projection formula at [FH91, p. 24]; therefore, $P_1 = \mathbb{D}_Y$, i.e. P_1 is nothing but the duality morphism on Y.

Corollary 3.3.4. [GFS22, Corollary 5.1]. $\mathbb{D}_X = \mathbb{D}_Y \oplus P_2$.

Proof. Notice that $\theta^*([Y]) = [X]$, i.e. the Gysin map sends the fundamental class of Y to the fundamental class of X. In fact,

$$\theta^*([Y]) = \theta^* f_*[X] = [X] - (j_*\lambda^*)([X]) = [X]$$

follows from equations (3.3.3) (remember that $f_* = \mu^*$) and the fact that $\lambda^*[X] \in H_{2n}^{BM}(\widetilde{W})$ with $H_{2n}^{BM}(\widetilde{W}) = 0$ for dimensional reasons.

The decomposition of the statement holds if the claims

i) $f_*(\lambda_*\tilde{w} \cap [X]) = 0$ for every $\tilde{w} \in H^{\alpha}(\widetilde{W})/H^{\alpha}(W)$,

ii) $\lambda^*(f^*y \cap [X]) = 0$ for every $y \in H^{\alpha}(Y)$,

are true.

i) formula (G_4) (ii) at [FM81, p. 26] implies

$$f_*(\lambda_*\tilde{w} \cap [X]) = f_*(\lambda_*\tilde{w} \cap \theta^*[Y]) = (\theta_*\lambda_*\tilde{w}) \cap [Y].$$

On the other hand, $\theta_*\lambda_* = 0$ (see formula (3.3.2)), so $f_*(\lambda_*\tilde{w} \cap [X]) = 0$.

ii) By [FM81, p. 26, (G_4) (iii)],

$$\lambda^*(f^*y \cap [X]) = \lambda^*(f^*y \cap \theta^*[Y]) = \lambda^*(\theta^*(y \cap [Y])).$$

On the other hand, $\lambda^*\theta^* = 0$ (see formula (3.3.4)), thus $\lambda^*(f^*y \cap [X]) = 0$.

□

3.3.4 Homology manifolds

Definition 3.3.5. A complex irreducible quasi-projective variety X of complex dimension n is called an **$\mathbb{A}$-homology manifold** if, for any $x \in X$,

$$H_\alpha(X, X \setminus \{x\}) = \begin{cases} 0 & \text{if } \alpha \neq 2n, \\ \mathbb{A} & \text{if } \alpha = 2n. \end{cases}$$

Notice that, as proved in [BSY07, proof of Theorem 3.7], X is an $\mathbb{A}$-homology manifold if and only if $\mathbb{A}_X[n] \cong IC^\bullet_X$.

Theorem 3.3.6. [GFS22, Theorem 1.2]. *Let $f : X \to Y$ be a projective birational morphism between complex irreducible quasi-projective varieties of the same complex dimension n. Let $U \subseteq Y$ be a non-empty Zariski open subset such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is an isomorphism.*

i) *If Y is an $\mathbb{A}$-homology manifold, then*

- ▲ *there is a unique bivariant class $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one;*
- ▲ *there exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y, \mathbb{A})$, with $\mathcal{K}^\bullet$ supported on $W := Y \setminus U$;*
- ▲ *if X is an $\mathbb{A}$-homology manifold, $\mathcal{K}^\bullet[n]$ is self-dual.*

ii) *If X is an $\mathbb{A}$-homology manifold and there is a bivariant class of degree one $\theta \in \mathrm{Hom}_{D^b_c(Y,\mathbb{A})}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$, then Y is an $\mathbb{A}$-homology manifold, as well.*

Proof.

i) By [BSY07, Theorem 3.7], the fundamental class

$$[Y] \in H^{BM}_{2n}(Y) \cong H^{-2n}(Y \to pt)$$

of Y is a strong orientation. Therefore,

$$\begin{aligned} &\mathrm{Hom}_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y) := H^0(f) \overset{\cdot[Y]}{\cong} H^{-2n}(X \to pt) \\ &\cong H^{BM}_{2n}(X) \cong H^0(X). \end{aligned}$$

Since f is birational, the bivariant class corresponding to $1_X \in H^0(X)$ is of degree one for f and is unique (compare with Remark 3.2.2).

By Theorem 3.2.8, there exists a decomposition

$$Rf_*\mathbb{A}_X[n] \cong \mathbb{A}_Y[n] \oplus \mathcal{K}^\bullet[n], \tag{3.3.6}$$

with $\mathcal{K}^\bullet$ supported on W. Taking the Verdier dual,

$$D(Rf_*\mathbb{A}_X[n]) \cong D(\mathbb{A}_Y[n]) \oplus D(\mathcal{K}^\bullet[n]). \tag{3.3.7}$$

Let $[X] \in H_{2n}^{BM}(X)$ be the fundamental class of X. Since

$$H_{2n}^{BM}(X) \cong H^{-2n}(X \to pt) \cong \mathrm{Hom}_{D_c^b(X)}(\mathbb{A}_X[n], D(\mathbb{A}_X[n])),$$

$[X]$ corresponds to a morphism

$$\mathbb{A}_X[n] \to D(\mathbb{A}_X[n]), \tag{3.3.8}$$

whose induced map in hypercohomology is the duality morphism (3.3.5).

Assume that X is an $\mathbb{A}$-homology manifold. Morphism (3.3.8) is an isomorphism by [BSY07, proof of Theorem 3.7] and, since

$$D(Rf_*\mathbb{A}_X[n]) \cong Rf_*D(A_X[n])$$

by [Dim04, Proposition 3.3.7 (ii)], it induces an isomorphism

$$Rf_*\mathbb{A}_X[n] \to D(Rf_*\mathbb{A}_X[n]),$$

which, in turn, via the decompositions (3.3.6) and (3.3.7), induces two projections

$$\mathbb{A}_Y[n] \to D(\mathbb{A}_Y[n]) \qquad \mathcal{K}^\bullet[n] \to D(\mathcal{K}^\bullet[n]).$$

Corollary 3.3.4 implies that the maps induced in hypercohomology by the morphism $\mathcal{K}^\bullet[n] \to D(\mathcal{K}^\bullet[n])$ are isomorphisms. This holds true when restricting to any open subset of Y; therefore $\mathcal{K}^\bullet[n] \cong D(\mathcal{K}^\bullet[n])$, i.e. $\mathcal{K}^\bullet[n]$ is self-dual.

ii) Arguing as before, Corollary 3.3.4 implies that isomorphism (3.3.8) induces an isomorphism $\mathbb{A}_Y[n] \cong D(\mathbb{A}_Y[n])$. This is equivalent to say that Y is an $\mathbb{A}$-homology manifold by [BSY07, loc. cit.].

□

When $\mathbb{A} := \mathbb{K}$ is a field and both X and Y are $\mathbb{K}$-homology manifolds of the same complex dimension n,

$$\dim_{\mathbb{K}} H^{n+\alpha}(\widetilde{W}) - \dim_{\mathbb{K}} H_{n-\alpha}^{BR}(\widetilde{W}) = \dim_{\mathbb{K}} H^{n+\alpha}(W) - \dim_{\mathbb{K}} H_{n-\alpha}^{BR}(W) \tag{3.3.9}$$

for any $\alpha \in \mathbb{Z}$, where $\widetilde{W} := X \setminus V$. Indeed,

$$\dim_{\mathbb{K}} H^{\alpha+n}(X) - \dim_{\mathbb{K}} H^{\alpha+n}(Y) = \dim_{\mathbb{K}} \mathbb{H}^\alpha(\mathcal{K}^\bullet)$$

follows by taking hypercohomology in $Rf_*\mathbb{K}_X \cong \mathbb{K}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y,\mathbb{K})$. On the other hand, $\mathcal{K}^\bullet$ is self dual, thus

$$\dim_{\mathbb{K}} H^{BM}_{\alpha+n}(X) - \dim_{\mathbb{K}} H^{BM}_{\alpha+n}(Y) = \dim_{\mathbb{K}} \mathbb{H}^\alpha(\mathcal{K}^\bullet)$$

is inferred by taking Verdier dual and hypercohomology in $Rf_*\mathbb{K}_X \cong \mathbb{K}_Y \oplus \mathcal{K}^\bullet$ in $D^b_c(Y,\mathbb{K})$. The combination of these equalities gives

$$\dim_{\mathbb{K}} H^{\alpha+n}(X) - \dim_{\mathbb{K}} H^{\alpha+n}(Y) = \dim_{\mathbb{K}} H^{\alpha+n}(X) - \dim_{\mathbb{K}} H^{\alpha+n}(Y)$$

and formula (3.3.9) is a consequence of Propositions 3.3.1 and 3.3.2.

When $\mathbb{A} := \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, item *ii)* of Theorem 3.3.6 can be proved by means of decomposition theorem; indeed, it guarantees that

$$Rf_*\mathbb{A}_X[n] \cong IC^\bullet_Y \oplus \mathcal{G},$$

where $\mathcal{G}$ is a sheaf supported on W. On the other hand, Theorem 3.2.8 gives

$$Rf_*\mathbb{A}_X[n] \cong \mathbb{A}_Y[n] \oplus \mathcal{A}[n],$$

hence there is a non-zero endomorphism $IC^\bullet_Y \to \mathbb{A}_Y[n] \to IC^\bullet_Y$, which is an isomorphism by Schur's Lemma (see [FH91, Lemma 1.7]), since $IC^\bullet_Y$ a simple object of the core of $D^b_c(Y)$ (see [BBD82, Corollary 1.4.25]). On the other hand, the composition $\mathbb{A}_Y[n] \to IC^\bullet_Y \to \mathbb{A}_Y[n]$ is an automorphism, because $\mathrm{Hom}_{D^b_c(Y)}(\mathbb{A}_Y, \mathbb{A}_Y) \cong H^0(Y)$. In conclusion, $IC^\bullet_Y \cong \mathbb{A}_Y[n]$.

3.3.5 Examples and implications

A first class of examples is given by resolutions of singularities of $\mathbb{A}$-homology manifold Y. Indeed, the hypotheses of Theorem 3.3.6 are met, hence, any such morphism has a unique bivariant class of degree one.

Example 3.3.7. [GFS22, Remark 2.1 (v)]. Let $f : X \to Y$ be a projective map between complex irreducible quasi-projective varieties of the same complex dimension n. Assume that Y is smooth (or, more generally, an $\mathbb{A}$-homology manifold). In this case (compare with [FM81, p. 34, §3.1.4], [Ful96, p. 217, Lemma 2] and the proof of Theorem 3.3.6) $H^0(f) \cong H^{BM}_{2n}(X) \cong H^0(X)$. By Remark 3.2.4, if there exists a bivariant class of degree one for f, then, for every α, $H^\alpha(Y)$ is contained in $H^\alpha(X)$ via pullback.

If $\mathbb{A} = \mathbb{Z}$ and $\dim_{\mathbb{Z}} H^\alpha(Y) > \dim_{\mathbb{Z}} H^\alpha(X)$ for some α, then $H^0(f) \neq 0$, yet $\theta_0 = 0$, for every bivariant class θ. However, *if, in addition, f is birational, then the bivariant class θ corresponding to $1_X \in H^0(X)$ is a bivariant class of degree one.* In fact, if U is a Zariski open subset of Y such that $f^{-1}(U) \cong U$, the restriction of θ to $f^{-1}(U) \to U$ has degree one.

The next example shows the existence of a projective birational map $f : X \to Y$ without bivariant classes of degree one such that $H^0(f) \neq 0$.

Example 3.3.8. [GFS22, Remark 6.1 (iii)]. Let $\mathbb{A} := \mathbb{Q}$, let $C \subset \mathbb{P}^3$ be a projective nonsingular curve of genus $g \geq 1$, let $Y \subset \mathbb{P}^4$ be the cone over C and let $f : X \to Y$ be the blowing-up of Y at the vertex $y \in Y$. By the decomposition theorem,

$$Rf_*\mathbb{Q}_X \cong \mathbb{Q}_y[-2] \oplus IC_Y^\bullet[-2].$$

On the other hand, [Ive86, p. 128, 9.13] and [Dim04, Remark 2.4.5 (i)] give

$$\mathrm{Hom}_{D_c^b(Y)}(\mathbb{Q}_y, \mathbb{Q}_Y[2]) \cong H^2(Y, Y \setminus \{y\}) \cong H^1(L),$$

where L is the link of Y at the vertex y (see [Dim04, Example 2.3.18]). The Hopf fibration $L \to C$ induces a Gysin sequence

$$0 \to H^1(C) \to H^1(L) \to H^0(C) \to H^2(C) \to \dots$$

from which $\dim_{\mathbb{Q}} H^1(L) = \dim_{\mathbb{Q}} H^1(C) = 2g \geq 2$ follows. Then

$$H^0(X \to Y) \cong \mathrm{Hom}_{D_c^b(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y) \neq 0$$

and Y is not a homology manifold. In particular, since X is smooth, Theorem 3.3.6 implies that f cannot have a bivariant class of degree one.

Here are two results concerning strong orientations.

Corollary 3.3.9. [GFS22, Corollary 6.2]. *Let $f : X \to Y$ be a projective birational morphism between complex irreducible quasi-projective varieties of the same complex dimension n. Let $\theta \in H^0(f)$ be a bivariant class.*

i) If θ is a strong orientation for f, then θ is of degree one for f, up to multiplication by a unit.

ii) If X is an $\mathbb{A}$-homology manifold and θ is of degree one for f, then θ is a strong orientation for f.

Proof. *i)* Let $U \subset Y$ be a Zariski non-empty open subset of Y such that the restriction $h : V \to U$ of f to $V := f^{-1}(U)$ is an isomorphism. The product by θ gives an isomorphism

$$H^0(V \to X) \xrightarrow{\cdot\theta} H^0(U \to Y).$$

On the other hand, the last formula in [BSY07, p. 803] and [Ive86, p. 109, Proposition II.6.9] imply

$$H^0(V \to X) \cong H^0(V) \qquad H^0(U \to Y) \cong H^0(U).$$

Therefore, θ induces an isomorphism $H^0(V) \to H^0(U)$. It follows that, up to multiplication by a unit, θ is a bivariant class of degree one.

ii) By Theorem 3.3.6, Y is an $\mathbb{A}$-homology manifold, as well, and θ corresponds to 1_X in the isomorphism $H^0(X \to Y) \cong H^0(X)$. Since X and Y are $\mathbb{A}$-manifolds,

$$f^!(\mathbb{A}_Y) = D(f^*(D(\mathbb{A}_Y))) = D(f^*(\mathbb{A}_Y[2n])) = D(\mathbb{A}_X[2n]) = \mathbb{A}_X;$$

therefore, θ corresponds to an isomorphism in

$$\mathrm{Hom}_{D^b_c}(\mathbb{A}_X, f^!\mathbb{A}_Y) \cong \mathrm{Hom}_{D^b_c}(\mathbb{A}_X, \mathbb{A}_X) \cong H^0(X).$$

By [FM81, §7.3.2, proof of Proposition], θ is a strong orientation for f.

□

Proposition 3.3.10. [GFS22, Proposition 6.3]. *Let $f : X \to Y$ be a projective birational morphism between complex irreducible quasi-projective varieties of the same complex dimension n. Let $\theta \in H^0(f)$ be a bivariant class. If θ is a strong orientation for f and Y is an $\mathbb{A}$-homology manifold, then X is an $\mathbb{A}$-homology manifold, as well.*

Proof. Y is an $\mathbb{A}$-homology manifold, thus

$$f^!(\mathbb{A}_Y) = D(f^*(D(\mathbb{A}_Y))) = D(f^*(\mathbb{A}_Y[2n])) = D(\mathbb{A}_X[2n]).$$

On the other hand, if θ is a strong orientation, then [FM81, loc. cit.] gives $f^!(\mathbb{A}_Y) \cong \mathbb{A}_X$ and, consequently, $D(\mathbb{A}_X[2n]) \cong \mathbb{A}_X$. This means that $\mathbb{A}_X[n]$ is self-dual, i.e. X is an $\mathbb{A}$-homology manifold by [BSY07, proof of Theorem 3.7]. □

Remark 3.3.11. [GFS22, Remark 6.4]. Let $f : X \to Y$ be a birational projective locally complete intersection morphism between complex irreducible quasi-projective algebraic varieties. Let $\theta \in H^0(f)$ be the orientation class of f. Then θ has degree one by Remark 3.2.2, but the previous proposition implies that θ cannot be, in general, a strong orientation.

3.4 Nilpotent cones are homology manifolds

Here, Theorem 3.3.6 is resort to so as to provide a short proof of a generalization of a well-known fact (see [BM83, §2.3, Theorem]); namely, that nilpotent cones are homology manifolds (see Section 3.4.2). In order to do that, several definitions are recalled in Section 3.4.1.

3.4.1 Lie groups and algebras

Definition 3.4.1. A **smooth Lie group** G is a set which is both a group and a smooth manifold, such that the multiplication and inverse operations are smooth.

Likewise, *complex* and *algebraic Lie groups* are defined by requiring G to be a complex manifold and the operations to be holomorphic or, respectively, G an algebraic variety and the operations regular maps.

Example 3.4.2. The following are Lie groups. For more details, see [FH91, §7.2].

- ▲ The *general linear groups* $\mathrm{GL}_n \mathbb{R}$ and $\mathrm{GL}_n \mathbb{C}$;
- ▲ the *special linear groups* $\mathrm{SL}_n \mathbb{R}$ and $\mathrm{SL}_n \mathbb{C}$, made of matrices with determinant 1;
- ▲ the *orthogonal groups* $\mathrm{O}_n\mathbb{R}$ and $\mathrm{O}_n\mathbb{C}$, whose automorphisms preserve a symmetric positive-definite bilinear form;
- ▲ the *special orthogonal groups* $\mathrm{SO}_n \mathbb{R}$ and $\mathrm{SO}_n \mathbb{C}$, consisting of the elements of $\mathrm{O}_n\mathbb{R}$ and $\mathrm{O}_n\mathbb{C}$, respectively, with determinant 1;
- ▲ the *symplectic groups* $\mathrm{Sp}_{2n} \mathbb{R}$ and $\mathrm{Sp}_{2n} \mathbb{C}$ whose automorphisms preserve a skew-symmetric non-degenerate bilinear form;
- ▲ the *unitary group* $\mathrm{U}_n\mathbb{C}$, made of automorphisms preserving a positive-definite Hermitian inner product;
- ▲ the *special unitary group* $\mathrm{SU}_n \mathbb{C}$, consisting of the elements of $\mathrm{U}_n\mathbb{C}$ with determinant 1.

Any Lie group G can be associated to its *Lie algebra*, as explained, for instance, in [FH91, §8.1], which is its tangent space $\mathfrak{g} := \operatorname{Lie} G = T_eG$ at the identity element e endowed with a certain map $[-,-]$. The general definition is the following.

Definition 3.4.3. A **Lie algebra** $\mathfrak{g}$ is a vector space endowed with a skew-symmetric bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the **Jacoby identity**:

$$[X,[Y,Z]] + [Y,[X,Z]] + [Z,[X,Y]] = 0 \quad \forall X, Y, Z \in \mathfrak{g}.$$

Example 3.4.4. Let $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form, let $M_n(\mathbb{R})$ and $\mathrm{Tr}(X)$ denote, respectively, the set of all real $n \times n$ matrices and the trace of any such matrix X. Then, the Lie algebras of $\mathrm{SL}_n \mathbb{R}$ and $\mathrm{O}_n\mathbb{R}$ are, respectively,

$$\begin{aligned} \mathfrak{sl}_n\mathbb{R} &= \{X \in M_n(\mathbb{R}) : \mathrm{Tr}(X) = 0\}, \\ \mathfrak{o}_n\mathbb{R} &= \{X \in M_n(\mathbb{R}) : Q(X(v), w) + Q(v, X(w)) = 0 \ \forall v, w \in \mathbb{R}^n\}. \end{aligned}$$

The same formula as the one defining $\mathfrak{o}_n\mathbb{R}$ gives the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$ of $\mathrm{Sp}_{2n} \mathbb{R}$.

If $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is a Hermitian inner product, the Lie algebra of $\mathrm{U}_n\mathbb{C}$ is

$$\mathfrak{u}_n := \{X \in M_n(\mathbb{C}) : H(X(v), w) + H(v, X(w)) = 0 \ \forall v, w \in \mathbb{C}^n\}.$$

For more details, see [FH91, §8.2].

Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called an **ideal** if

$$[X,Y] \in \mathfrak{h} \ \forall X \in \mathfrak{h}, \quad \forall Y \in \mathfrak{g}.$$

The Lie algebra $\mathfrak{g}$ is called **solvable** if there is $\alpha > 0$ such that $\mathscr{D}_\alpha\mathfrak{g} = 0$, where

$$\mathscr{D}_\alpha\mathfrak{g} := \begin{cases} [\mathfrak{g}, \mathfrak{g}] & \text{if } \alpha = 1, \\ [\mathscr{D}_{\alpha-1}\mathfrak{g}, \mathscr{D}_{\alpha-1}\mathfrak{g}] & \text{if } \alpha > 1. \end{cases}$$

The sum of all solvable ideals is called the **radical** $\mathrm{Rad}(\mathfrak{g})$ of $\mathfrak{g}$.

Definition 3.4.5. The Lie algebra $\mathfrak{g}$ is said to be **reductive** if its **centre**

$$Z(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0, \ \forall Y \in \mathfrak{g}\}$$

coincides with $\mathrm{Rad}(\mathfrak{g})$. A Lie group is called **reductive** if so is its Lie algebra.

3.4.2 The nilpotent cone

Theorem 3.4.6. [DGFS22, Theorem 2.1]. *Let $\pi' : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be a projective morphism between complex quasi-projective nonsingular varieties of the same dimension. Assume that π' is generically finite of degree δ. Let $\mathcal{N} \subset \mathfrak{g}$ be a closed irreducible subvariety. Consider the induced fibre square:*

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi"] & \widetilde{\mathfrak{g}} \arrow[d, "\pi'"] \\ \mathcal{N} \arrow[r, hook, "i"] & \mathfrak{g} \end{tikzcd}$$

where $\widetilde{\mathcal{N}} := \mathcal{N} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. If $\widetilde{\mathcal{N}}$ is irreducible and nonsingular and π is birational, then $\mathcal{N}$ is an $\mathbb{A}$-homology manifold for every Noetherian commutative ring with identity $\mathbb{A}$ for which δ is a unit.

Proof. Since $\pi' : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a projective morphism between complex quasi-projective nonsingular varieties of the same dimension, it is a local complete intersection morphism of relative codimension 0 [FM81, p. 130]. Let

$$\theta' \in H^0(\pi') \cong \mathrm{Hom}_{D^b_c(\mathfrak{g})}(R\pi'_* \mathbb{A}_{\widetilde{\mathfrak{g}}}, \mathbb{A}_{\mathfrak{g}})$$

be the orientation class of π' [FM81, p. 131]. Let $\theta'_0 : H^0(\widetilde{\mathfrak{g}}) \to H^0(\mathfrak{g})$ be the induced Gysin map. It is clear that $\theta'_0(1_{\widetilde{\mathfrak{g}}}) = \delta \cdot 1_{\mathfrak{g}} \in H^0(\mathfrak{g})$, where δ is the degree of π'. Therefore, if the pull-back of θ' is denoted by

$$\theta := i^* \theta' \in H^0(\pi) \cong \mathrm{Hom}_{D^b_c(\mathcal{N})}(R\pi_* \mathbb{A}_{\widetilde{\mathcal{N}}}, \mathbb{A}_{\mathcal{N}}),$$

then $\delta^{-1} \cdot \theta$ is a bivariant class of degree one for π (see Remark 3.2.1). At this point, the claim follows by Theorem 3.3.6. □

Remark 3.4.7. [DGFS22, Remark 2.2]. Observe that, as a scheme, $\widetilde{\mathcal{N}}$ could also be nonreduced, but what matters is that, for the usual topology, it is a nonsingular variety [FM81, p. 32, 3.1.1].

A simple application of Theorem 3.4.6 shall prove that nilpotent cones, defined presently, are homology manifolds.

Definition 3.4.8. Let G be a connected reductive complex algebraic group and let $\mathfrak{g}$ be its Lie algebra. For any $X \in \mathfrak{g}$, set

$$\mathrm{ad}_X : Y \in \mathfrak{g} \mapsto [X, Y] \in \mathfrak{g}.$$

The **nilponent cone** of G is the variety (see [AMTT08, §3.1])

$$\mathcal{N} := \mathcal{N}(G) = \{X \in \mathfrak{g} : \exists \alpha > 0 : \mathrm{ad}^\alpha_X = 0\},$$

where ad^α_X is the composition of ad_X with itself α times.

Given the nilpotent cone $\mathcal{N}$ of G as in the definition, let $\mathcal{B}$ denote the variety of all Borel subgroups (for the definition, see [FH91, §23.3]) of G and set

$$\widetilde{\mathcal{N}} := \{(X, B) \in \mathcal{N} \times \mathcal{B} : X \in \operatorname{Lie} B\}.$$

The **Springer resolution** of $\mathcal{N}$, which is semismall by [Ste76, Theorem 4.6], is the projection

$$\pi : (X, B) \in \widetilde{\mathcal{N}} \mapsto X \in \mathcal{N}$$

and it extends to the so-called **Grothendieck simultaneous resolution** (see [Ste74, p. 131])

$$\pi' : \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

defined by omitting the restriction $X \in \mathcal{N}$ in the definition of π. All these spaces and maps form a fibre square

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi"] & \widetilde{\mathfrak{g}} \arrow[d, "\pi'"] \\ \mathcal{N} \arrow[r, hook, "i"] & \mathfrak{g} \end{tikzcd}$$

therefore the hypotheses of Theorem 3.4.6 are met and

Corollary 3.4.9. [DGFS22, Corollary 2.3]. *The nilpotent cone is a rational homology manifold.*

Remark 3.4.10. [DGFS22, Remark 2.4]. If the Grothendieck simultaneous resolution $\pi' : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ has degree δ, Theorem 3.4.6 implies that *the nilpotent cone $\mathcal{N}$ is an $\mathbb{A}$-homology manifold for every Noetherian commutative ring with identity $\mathbb{A}$ for which δ is a unit.* For instance, for the variety $\mathcal{N}$ of nilpotent matrices in $\mathrm{GL}(n, \mathbb{C})$, $\delta = n!$ and, as a result, $\mathcal{N}$ is also a $\mathbb{Z}_h$-homology manifold for every integer h relatively prime with $n!$ in $\mathbb{Z}$.

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