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SHARP ESTIMATES FOR EIGENVALUES AND TORSIONAL RIGIDITY OF LINEAR AND NONLINEAR OPERATORS

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Contents

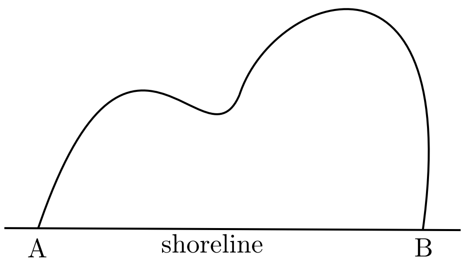
Introduction

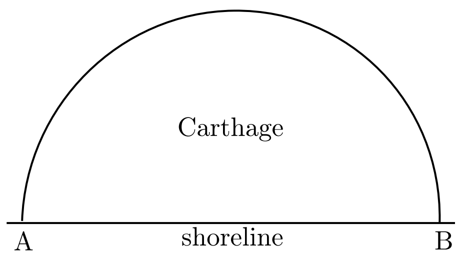
Shape Optimization and Spectral Geometry are fields of mathematics where the main questions are those of maximizing or minimizing, under some geometrical constraints, functionals defined in certain classes of sets. These kind of problems are well known since the ancient times and some of them finds its roots in the verses of Virgil:

> The Kingdom you see is Carthage, the Tyrians, the town of Agenor;
> But the country around is Libya, no folk to meet in war.
> Dido, who left the city of Tyre to escape her brother,
> Rules here - a long and labyrinthine tale of wrong
> Is hers, but I will touch on its salient points in order...
> Dido, in great disquiet, organized her friends for escape.
> They met together, all those who harshly hated the tyrant
> Or keenly feared him: they seized some ships which chanced to be ready...
> They came to this spot, where today you can behold the mighty
> Battlements and rising citadel of New Carthage,
> And purchased a site, which was named "Bull's Hide" after the bargain
> By which they should get as much land as they could enclose with a bull's hide.

Virgil - "The Aeneid"

According to Virgil's epic poem the Aeneid, Dido, the queen of the Phoenician city of Tyre, was forced to leave when her brother usurped the throne and murdered her husband, the king of the reign. With many difficulties, she arrived in Libya, where she bargained with the king of the local tribes to be given as much land as could be enclosed by the hide of a bull. Although this could sound like a unfavorable agreement, the refugee princess managed to find a clever solution: she cut the hide into very thin strips, tied them together into a rope, and looped it around a plot of land by the shoreline in such a way as to maximize the area of her claim.





To this day, the so called Dido's problem consists on finding a curve with fixed endpoints and fixed length that encloses the maximum area between the curve and the line segment between the two endpoints, and belongs to those inequalities that go under the name of isoperimetric inequalities. Namely, if we have a bounded open set $\Omega \in \mathbb{R}^n$, the classical isoperimetric inequalities states that

$$P(\Omega) \geqslant n\omega_n^{\frac{1}{n}}|\Omega|^{1-\frac{1}{n}},$$

where $P(\Omega)$ denotes the Euclidean perimeter of Ω in $\mathbb{R}^n$, $|\Omega|$ is the Lebesgue measure of Ω and ω_n is the Lebesgue measure of the unit ball. In particular the equality case holds if and only if Ω is a ball, up to sets with zero capacity. Even though this problem was well known for thousands of years, the first proofs in two dimensions were given in the nineteenth century by Steiner [127] and Edler [57] and more complete proofs a century later by many other authors [12, 13, 14, 36, 90]. In the three dimensional space, early proofs were given by Tonelli [134], Schmidt [119, 120] and Radò [114]. A rigorous, complete and elegant proof were given only by De Giorgi [46], almost 65 years ago, starting from a general definition of Perimeter.
With regard to Spectral Geometry in particular, the first conjecture goes back to the end of the 19th Century and can be found in the famous book of Lord Rayleigh, *The Theory of Sound* [115]. The author conjectured that, among all planar sets with fixed area, the disk minimizes the first Dirichlet eigenvalue of the Laplace operator. Physically, this eigenvalue represents the principal frequency of a vibrating membrane fixed at its boundary, so that if the conjecture was true, "one could have heard" the shape of the circular drum. This problem was proved 50 years later in two contemporary but independent works, one by Faber [63] and one by Krahn [93], and it was completely solved later with the work of Pólya and Szegö [113]. Let $\Omega \subseteq \mathbb{R}^n$, with $n \geqslant 2$, be an open set with finite Lebesgue measure, the first Dirichlet-Laplacian eigenvalue is the least positive λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits non-trivial solutions in $H_0^1(\Omega)$. The classical result of Faber and Krahn for the first Dirichlet eigenvalue $\lambda_1(\Omega)$ states that, among measurable domains with fixed measure, $\lambda_1(\cdot)$ is minimized by a ball; in other words, the following scaling invariant inequality holds:

$$\lambda_1(\Omega)|\Omega|^{2/n} \geqslant \lambda_1(B)|B|^{2/n}, \tag{2}$$

where by $|\cdot|$ we denote the volume of a measurable set and by B a ball in $\mathbb{R}^n$. Moreover, equality holds in (2) if and only if Ω is equivalent to a ball.
Strictly related to the Dirichlet eigenvalue problem for the Laplacian, there is the torsion problem of elasticity, known as Saint-Venant problem. Adhémar-Jean-Claude Barré de Saint-Venant was a French mathematician and engineer, who devoted his studies to the resistence of materials and to elasticity theory: he conjectured (see [47]) that among all cylindrical bars with constant-shaped cross section and fixed measure, the one that maximized the torsional rigidity was the bar with circular cross section. In n dimensions the torsional rigidity is nothing else that the L^1-norm of the unique and positive solution in $H_0^1(\Omega)$ to the following problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \in \mathbb{R}^n$ is an open set with finite Lebesgue measure. Namely if u_Ω is this unique solution, known as torsion function, the torsional rigidity is defined as

$$T(\Omega) = \int_\Omega u_\Omega \, dx.$$

The conjecture was firstly proved in 1948 by Pólya [111] and successively by Davenport as reported in [113]. Makai [101] in 1966 found another estimate and gave a more general proof of it. Mathematically, the Saint-Venant conjecture is

$$T(\Omega) \leqslant T(B), \tag{4}$$

where B is the ball having the same measure as Ω.
The study of these kind of problems gave rise to a series of other optimization questions, such as other types of boundary conditions of the Laplacian or different operators, linear and non-linear. It is worth mentioning the Laplacian eigenvalue problem with Neumann boundary condition; in this case it makes sense to deal with a maximization problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open and Lipschitz domain; the problem is

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where we denote by $\partial u/\partial \nu$ the outer normal derivative of u on $\partial\Omega$. In this case the first eigenvalue μ_1 is always zero and the associated eigenfunctions are the constant functions. The following inequality was proved by Szegö in the plane [129] and then generalized in higher dimension by Weinberger [140]. The so called Szegö-Weinberger inequality states that the first non-zero Neumann eigenvalue $\mu_2(\Omega)$ is maximized by a ball among domains with fixed measure, that is equivalent to say that the following scaling invariant inequality holds:

$$\mu_2(\Omega)|\Omega|^{2/n} \leqslant \mu_2(B)|B|^{2/n}. \tag{6}$$

The Faber-Krahn, Saint-Venant and Szegö-Weinberger are examples of isoperimetric inequalities. The fact that balls can be characterized as the only sets for which equality holds leads to ask if these inequalities are stable, i.e. if it is possible to improve them by adding a remainder term that measures the deviation of a set Ω from the spherical symmetry. These kind of inequalities are known as quantitative isoperimetric inequalities. Starting from the quantitative isoperimetric inequality proved in [72], several spectral quantitative isoperimetric inequalities were proved, as for example the Faber-Krahn [20] and the Szegö-Weinberger [19] inequalities.

The aim of this thesis is to obtain analogous results in these directions for the eigenvalue problem with different boundary conditions and for some operators of linear and non linear type. In particular, we focus our study on Steklov and Robin boundary conditions, obtaining isoperimetric inequalities as (1) and (5) in particular classes of sets and we obtain a quantitative result in terms of the torsion, perimeter and measure.

In the first part of this thesis we focus on a problem concerning the maximization of the first non-trivial Steklov-Dirichlet eigenvalue on the class of doubly connected domains. But before going on, let us summarise what is a Steklov eigenvalue, introduced by the Russian mathematician V. A. Steklov [128].
Let $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, be a bounded, connected, open set with Lipschitz boundary. A real non-negative number $\sigma \geqslant 0$ is called a Steklov eigenvalue if there exists $u \in H^1(\Omega)$ with $u \neq 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The Steklov eigenvalues can be interpreted as the eigenvalues of the *Dirichlet-to-Neumann operator* $\mathcal{D} : H^{1/2}(\Omega) \to H^{-1/2}(\Omega)$ which maps a function $f \in H^{1/2}(\Omega)$ to $\mathcal{D}f = \frac{\partial Hf}{\partial n}$, where Hf is the harmonic extension of f to Ω. For a survey concerning this topic we refer to [83]. As usual, problem (7) is considered in the weak sense, that is, for every $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \sigma \int_{\partial\Omega} u \, v \, d\mathcal{H}^{n-1}, \tag{8}$$

where $\cdot$ denotes the standard Euclidean scalar product and $\mathcal{H}^{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$. In this framework, since the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ is compact (see [103], Theorem 6.2), it is known that the Steklov spectrum consists of a discrete sequence diverging at infinity

$$0 = \sigma_0(\Omega) \leqslant \sigma_1(\Omega) \leqslant \sigma_2(\Omega) \leqslant \cdots \nearrow +\infty. \tag{9}$$

In particular, the first non-trivial Steklov eigenvalue of Ω has the following variational characterization:

$$\sigma_1(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}} \ : \ v \in H^1(\Omega)\backslash\{0\}, \ \int_{\partial\Omega} v \, d\mathcal{H}^{n-1} = 0 \right\}. \tag{10}$$

If we take $\Omega = B_R(x)$, where $B_R(x)$ is the ball of radius R centered at the point x, then

$$\sigma_1(B_R(x)) = \frac{1}{R}. \tag{11}$$

Moreover, we know that $\sigma_1(B_R(x))$ has multiplicity n and the corresponding eigenfunctions are $u_i(x) = x_i$, with $i = 1, \dots, n$. Let us focus now our attention on shape optimization problems concerning the first non trivial Steklov eigenvalue. In [141] the author considers the problem of maximizing $\sigma_1(\Omega)$ in the plane, keeping the perimeter of Ω fixed. If $\Omega \subseteq \mathbb{R}^2$ is a Lipschitz, simply connected open set, the following inequality, known as Weinstock inequality, is proved

$$\sigma_1(\Omega)P(\Omega) \leqslant \sigma_1(B_R(x))P(B_R(x)), \tag{12}$$

where $P(\Omega)$ denotes the Euclidean perimeter of Ω. In other words, inequality (12) states that, among all planar, simply connected, open sets with prescribed perimeter, $\sigma_1(\Omega)$ is maximum for the disk. Moreover, in [78], it is proved that (12) fails to be true in general in dimension $n > 2$. If we consider indeed the annulus $A_\epsilon = B_1(x)\backslash\overline{B_\epsilon(x)}$, having that $B_R(x)$ is the ball of radius R centered at x, with $\epsilon \approx 0$, that is a simply connected set, the following reverse inequality holds,

$$\sigma_1(A_\epsilon)P(A_\epsilon)^{\frac{1}{n-1}} > \sigma_1(B_R(x))P(B_R(x)))^{\frac{1}{n-1}}.$$

In [27], the authors generalize the Weinstock inequality (12) in any dimension, when restricting to the class of convex sets. More precisely, if $\Omega \subseteq \mathbb{R}^n$ is an open, bounded, convex set, then

$$\sigma_1(\Omega)P(\Omega)^{\frac{1}{n-1}} \leqslant \sigma_1(B_R(x))P(B_R(x))^{\frac{1}{n-1}} \tag{13}$$

and equality holds only if Ω is a ball.

Considering, instead, a volume constraint, in [23] the author proves that the first non-trivial Steklov eigenvalue is maximized by balls, among sets with the same volume. More precisely, if $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is an open bounded set with Lipschitz boundary, then

$$\sigma_1(\Omega)|\Omega|^{\frac{1}{n}} \leqslant \sigma_1(B_R(x))|B_R(x)|^{\frac{1}{n}}, \tag{14}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and equality holds if and only if Ω is a ball. We also observe that (13) and the classical isoperimetric inequality imply (14) for convex sets; so, inequality (14) is weaker than (13) because it contains the volume, but it is more general because it holds without geometric restrictions.

Chapter 2 deals with a different shape optimization problem in domains with a hole, involving the Steklov boundary condition on the outer boundary and Dirichlet or Robin on the inner one. Let $\Omega_0 \subseteq \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, connected set, with Lipschitz boundary such that $B_r \Subset \Omega_0$, where B_r is the open ball of radius $r > 0$ centered at the origin. Let us set $\Omega := \Omega_0 \backslash \overline{B_r}$; then we study the following Steklov-Dirichlet boundary eigenvalue problem for the Laplacian:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial B_r, \\ \dfrac{\partial u}{\partial \nu} = \sigma(\Omega) u & \text{on } \partial \Omega_0. \end{cases} \tag{15}$$

The study of the first eigenvalue of problem (15) leads to the following minimization problem:

$$\sigma_1(\Omega) = \min_{\substack{w \in H^1_{\partial B_r}(\Omega) \\ w \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx}{\displaystyle\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1}}, \tag{16}$$

where $H^1_{\partial B_r}(\Omega)$ is the set of Sobolev functions on Ω that vanish on ∂B_r. Notice also that the value $\sigma_1(\Omega)$ is the optimal constant in the Sobolev-Poincaré trace inequality:

$$\sigma_1(\Omega) ||w||_{L^2(\partial\Omega_0)} \leqslant ||\nabla w||_{H^1_{\partial B_r}(\Omega)}. \tag{17}$$

We treat the following shape optimization issue:

Which sets maximize $\sigma_1(\cdot)$ among sets of the form $\Omega = \Omega_0 \backslash \overline{B_r}$, where Ω_0 contains the fixed ball B_r and Ω has prescribed volume?

In the class of sets of the form $B_R(x_0) \backslash \overline{B}_r$ with $B_R(x_0)$ being a ball containing B_r, the maximizer of σ_1 is the spherical shell, that is the annulus when the balls are concentric (see [68]). This is also proved in [138] and for more general spaces in [125].
We partially solve the problem of the optimality of σ_1, restricting our study to two classes of sets. Firstly we consider the class of nearly spherical sets, that are sets whose boundary can be parametrized on the sphere by means of a Lipschitz function with a small $W^{1,\infty}$-norm; see Definition 1.2 in Chapter 1. In second place, we study the existence of a maximizer and the isoperimetric inequality when Ω is in the class of convex sets.
With regard to the first class, our result is the following and is contained in [106].

Theorem. *Let $\Omega = \Omega_0 \backslash \overline{B_r}$, with Ω_0 a nearly spherical set. Then*

$$\sigma_1(\Omega) \leqslant \sigma_1(A_{r,R}), \tag{18}$$

where $A_{r,R} = B_R \backslash \overline{B_r}$, with $R > r > 0$, is the spherical shell with the same volume as Ω. Moreover the equality in (18) holds if and only if Ω is a spherical shell.

So, we study the optimal shape for $\sigma_1(\Omega)$ when both the volume of the domain and the radius of the internal ball are fixed. We also find some counterexamples showing that when only a volume constraint holds, then σ_1 is not upper bounded, hence we cannot speak about optimality. In order to prove the Theorem, we find $K = K(n, |\Omega|) > 0$, such that

$$\sigma_1(A_{r,R}) \geqslant \sigma_1(\Omega)\left(1 + K(n,|\Omega|)\int_{\mathbb{S}^{n-1}} v^2(\xi)\,d\mathcal{H}^{n-1}\right).$$

In [75] we enlarge the class of nearly spherical sets to the one of convex sets. We prove the existence of a maximizer among convex sets with fixed internal ball and fixed volume. Let $\omega > 0$ and $r > 0$ be fixed, then by $\mathcal{A}_r(\omega)$ we will denote the class of convex sets having measure ω and containing the ball B_r, that is

$$\mathcal{A}_r(\omega) := \left\{D = K\setminus\overline{B}_r,\ \ K \subseteq \mathbb{R}^n \text{ open, bounded, convex}:\ B_r \Subset K,\ |D| = \omega\right\}.$$

The existence theorem is stated as follows

Theorem. *Let $\omega > 0$ and $r > 0$ be fixed. There exists a set $E \in \mathcal{A}_r(\omega)$, such that*

$$\max_{D\in\mathcal{A}_r(\omega)} \sigma_1(D) = \sigma_1(E).$$

In particular this theorem can be easily proved even when fixing the perimeter of Ω_0 instead of the volume. Moreover it can be generalized when we substitute any convex set in place of the ball B_r, fixing its inradius and the measure of Ω_0.
The optimization result in this class is partial and is stated in the following

Theorem. *Let $r > 0$, $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded and convex set, $n \geqslant 2$, such that $B_r \Subset \Omega_0 \subseteq B_{\bar{R}}$, where $B_{\bar{R}}$ is the ball centered at the origin with radius $\bar{R}$ given by*

$$\bar{R} = \begin{cases} re^{\sqrt{2}} & \text{if } n = 2 \\ r\left[\frac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1}\right]^{\frac{1}{n-2}} & \text{if } n \geqslant 3. \end{cases} \tag{19}$$

Then, denoting by $\Omega = \Omega_0\setminus\overline{B}_r$, the following inequality holds

$$\sigma_1(\Omega) \leqslant \sigma_1(A_{r,R}), \tag{20}$$

where $A_{r,R}$ is the spherical shell of radii $r < R$ having the same volume as Ω.

In [76] we replace the Dirichlet boundary condition with the Robin boundary condition. Let $\Omega = \Omega_0\setminus\overline{B}_r$, where B_r is the ball centered at the origin with radius $r > 0$ and $\Omega_0 \subset \mathbb{R}^n$, $n \geqslant 2$, is an open, bounded set with Lipschitz boundary, such that $B_r \Subset \Omega_0$. We deal with the following Steklov-Robin eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega_0 \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B_r, \end{cases}$$

where ν is the outer unit normal to $\partial\Omega$ and $\beta > 0$ is a positive real parameter.
The aim of this paper is to study the first eigenvalue $\sigma_\beta(\Omega)$ of (2.55) defined as

$$\sigma_\beta(\Omega) = \inf_{v\in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx + \beta\int_{\partial B_r} v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} v^2\,d\mathcal{H}^{n-1}}.$$

We prove that $\sigma_\beta(\Omega)$ is actually a minimum, it is simple, and that the corresponding eigenfunctions have constant sign. Hence, also in this case, despite the Steklov condition, $\sigma_\beta(\Omega)$ is formally a Robin type eigenvalue.
When Ω is a spherical shell, that is $\Omega = B_R \backslash \overline{B_r}$, $\sigma_\beta(\Omega)$ and the corresponding eigenfunctions can be explicitly computed.
For sake of simplicity, here we will denote by σ_D the first Steklov-Dirichlet eigenvalue discussed above. We observe that $\sigma_\beta(\Omega)$ depends clearly also on β and we expect that for $\beta \to 0$ and $\beta \to \infty$ then $\sigma_\beta(\Omega)$ goes to 0 and $\sigma_D(\Omega)$, respectively. In order to show this, we will prove some estimates on $\sigma_\beta(\Omega)$ in the spirit of the ones contained in [94] for the first Robin Laplacian eigenvalue (see also [55] for a more general case). More precisely, let us define the following quantities

$$\mu_1(\Omega) := \inf_{\substack{v \in H^1(\Omega)\setminus\{0\} \\ \int_{\partial B_r} v \, d\mathcal{H}^{n-1} = 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\partial\Omega_0} v^2 \, dx},$$

and

$$q_1(\Omega) = \inf_{\substack{\Delta w = 0 \\ w \in H^1(\Omega)\setminus H^1_{\partial B_r}(\Omega)}} \frac{\displaystyle\int_{\partial B_r} w^2 \, \mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} w^2 \, \mathcal{H}^{n-1}},$$

We observe that $\mu_1(\Omega)$ is the first nontrivial Steklov Laplacian eigenvalue in Ω. Then our result is the following

Theorem. *Let $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary and let $\Omega = \Omega_0 \backslash \overline{B_r}$, where B_r is the ball centered at the origin and with radius r such that $B_r \Subset \Omega_0$. Then the following estimates hold*

$$\frac{1}{\sigma_\beta(\Omega)} \leqslant \frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)},$$

and

$$\frac{1}{\sigma_\beta(\Omega)} \leqslant \frac{1}{\sigma_D(\Omega)} + \frac{1}{q_1(\Omega)},$$

where $\sigma_\beta(\Omega)$ is the first Steklov-Robin eigenvalue of Ω, $\sigma_D(\Omega)$ is the first Steklov-Dirichlet eigenvalue, $\mu_1(\Omega)$ and $q_1(\Omega)$ are defined above, respectively.

As a consequence of the above estimates we can obtain the quoted asymptotic behaviour of $\sigma_\beta(\Omega)$ with respect to β in both case, when β either goes to zero or to infinity.

Chapter 3 is devoted to the study of two problems involving a Robin boundary condition: one in the linear and the other one in the non linear case, but both of them gravitate around a result à la Talenti proved in [2].
We start by recalling the Robin eigenvalue problem for the Laplacian. Let Ω be a bounded, open subset of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary; its Robin eigenvalues related to the Laplacian are the real numbers λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

admits non trivial $W^{1,2}(\Omega)$ solutions; α is an arbitrary real constant, which will be referred to as boundary parameter of the Robin problem. We observe that for $\alpha = 0$ we obtain the Neumann problem, for $\alpha = +\infty$ we formally obtain the Dirichlet problem and for $\lambda = 0$ the

Steklov problem; for this reason it can be considered as the most general eigenvalue problem for the Laplace operator. For each fixed Ω and α there is a sequence of eigenvalues

$$\lambda_1(\alpha,\Omega) \leqslant \lambda_2(\alpha,\Omega) \leqslant \cdots \to +\infty$$

which depend on α. In particular, the first non trivial Robin eigenvalue of Ω is characterized by the expression

$$\lambda_1(\alpha,\Omega) = \min_{\substack{u\in W^{1,2}(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} |u|^2 \, d\mathcal{H}^1}{\displaystyle\int_\Omega |u|^2 \, dx}.$$

We refer to [91] for a collection of properties of the Robin Laplacian eigenvalues and the related proofs.

We will always assume that $\alpha > 0$. We have the following Faber-Krahn type inequality, that was proved in [16] in the planar case and was then generalized in [45] in any dimension. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and Lipschitz domain. Then,

$$\lambda_1(\alpha,\Omega) \geqslant \lambda_1(\alpha,B), \tag{22}$$

where B is a ball such that $|B| = |\Omega|$. Equality holds if and only if Ω is a ball. The generalization to the p-Laplacian is given in [44] and in [26]; this result was also shown to hold on general open sets of finite measure, see [28].
As we said before, the authors Alvino, Nitsch and Trombetti studied in [2] a problem that was for the first time introduced by Talenti [130]. He proved, via rearrangements arguments, that the Schwarz symmetrization (see [92]) of the solution to problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

with $f \in L^2(\Omega)$ (non-negative and not identically zero) and Ω an open subset of $\mathbb{R}^n$, is pointwise bounded by the solution to the following symmetrized problem

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{24}$$

with $f^\sharp$ being the Schwarz decreasing rearrangement of f and $\Omega^\sharp$ the ball centered at the origin having the same measure as Ω.

Talenti, with his techniques, gave birth to a series of generalizations and results that still now take his name. For instance see [4, 3, 131, 132] for generalizations to other kind of operators. When we have Robin boundary conditions with positive parameter, problem (23) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

To our knowledge, in literature, there are few comparison results à la Talenti for this kind of problem. A result of this type has been proved only recently in [2], where they have highlighted the importance of the dependence on the dimension of the space. The authors, in fact, managed to compare the Lorentz norm (see [98]) of the solution to problem (25) with that of the solution to the symmetrized problem

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ \dfrac{\partial v}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{26}$$

where the exponents of these norms depend on the dimension of the space.
In particular they proved, for $n \geqslant 2$, that

$$\|u\|_{L^{p,1}(\Omega)} \leqslant \|v\|_{L^{p,1}(\Omega^\sharp)} \quad \text{for all } 0 < p \leqslant \frac{n}{2n-2} \tag{27}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leqslant \|v\|_{L^{2p,2}(\Omega^\sharp)} \quad \text{for all } 0 < p \leqslant \frac{n}{3n-4}, \tag{28}$$

with u solution to (25) and v to (26). Moreover, when $f \equiv 1$ in Ω and $n = 2$, they showed that

$$u^\sharp(x) \leqslant v(x) \qquad x \in \Omega^\sharp, \tag{29}$$

and, for $n \geqslant 3$, that

$$\begin{aligned} \|u\|_{L^{p,1}(\Omega)} &\leqslant \|v\|_{L^{p,1}(\Omega^\sharp)} \\ \|u\|_{L^{2p,2}(\Omega)} &\leqslant \|v\|_{L^{2p,2}(\Omega^\sharp)}, \end{aligned}$$

for all $0 < p \leqslant \frac{n}{n-2}$.
With a different approach [29] proved that

$$\|u\|_{L^1(\Omega)} \leqslant \|v\|_{L^1(\Omega^\sharp)}.$$

A generalization of this result can be found in [5], where the authors consider the p-Laplacian.
The first part of Chapter 3 is dedicated to another nonlinear generalization, involving the anisotropic Laplacian and that can be found in [116]. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, with Lipschitz boundary. Let us consider the following anisotropic problem with Robin boundary conditions

$$\begin{cases} -\operatorname{div}(H(\nabla u)H_\xi(\nabla u)) = f & \text{in } \Omega \\ H(\nabla u)H_\xi(\nabla u)\cdot\nu + \beta H(\nu)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{30}$$

where $f \geqslant 0$ (not identically zero) belongs to $L^2(\Omega)$, H is a sufficiently smooth norm in $\mathbb{R}^n$, ν is the Euclidean outer unit normal to $\partial\Omega$ and $\beta > 0$ is a positive real parameter.
A weak solution to problem (3.1) is a function $u \in H^1(\Omega)$ that satisfies

$$\int_\Omega H(\nabla u)H_\xi(\nabla u)\cdot\nabla\varphi\,dx + \beta\int_{\partial\Omega} H(\nu)u\varphi\,d\mathcal{H}^{n-1} = \int_\Omega f\varphi \qquad \forall\varphi \in H^1(\Omega). \tag{31}$$

We recall that the Wulff Shape centered in $x_0 \in \mathbb{R}^n$ of radius R is defined as follows

$$\mathcal{W}_R(x_0) = \{x \in \mathbb{R}^n : H^\circ(x - x_0) < R\},$$

where H° is the dual norm of H. In particular we will denote by $\mathcal{W}$ the Wulff Shape centered at the origin of radius 1 (for the exact definitions, see (1.4)).

The aim of the work is to estabilish a comparison result with the solution to the following symmetrized problem

$$\begin{cases} -\mathrm{div}(H(\nabla v)H_\xi(\nabla v)) = f^\star & \text{in } \Omega^\star \\ H(\nabla v)H_\xi(\nabla v)\cdot\nu + \beta H(\nu)v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{32}$$

where $f^\star$ is the convex symmetrization of f (see (1.4)) and $\Omega^\star$ is a set homothetic to the Wulff Shape $\mathcal{W}$ such that $|\Omega^\star| = |\Omega|$.
In particular the main theorems are the following

Theorem. *Let be $n \geqslant 2$. If u and v are the solutions to problems* (30) *and* (32) *respectively, then*

$$\|u\|_{L^{p,1}(\Omega)} \leqslant \|v\|_{L^{p,1}(\Omega^\star)} \qquad \textit{for all} \quad 0 < p \leqslant \frac{n}{2n-2} \tag{33}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leqslant \|v\|_{L^{2p,2}(\Omega^\star)} \qquad \textit{for all} \quad 0 < p \leqslant \frac{n}{3n-4}. \tag{34}$$

Theorem. *Let $n = 2$, $f \equiv 1$ in Ω. If u and v are the solutions to problems* (30) *and* (32) *respectively. Then*

$$u^\star(x) \leqslant v(x) \quad x \in \Omega^\star, \tag{35}$$

where $u^\star$ is the convex symmetrization of u.

Theorem. *Let $n \geqslant 3$ and $f \equiv 1$. If u and v are the solutions to problems* (30) *and* (32) *respectively, then*

$$\|u\|_{L^{p,1}(\Omega)} \leqslant \|v\|_{L^{p,1}(\Omega^\star)} \tag{36}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leqslant \|v\|_{L^{2p,2}(\Omega^\star)}, \tag{37}$$

for all $0 < p \leqslant \frac{n}{n-2}$.

The second part of this Chapter, which is contained in [117], is the attempt to solve an open problem left by the authors in [2]. Since the validity of (27),(28), we have that

$$\|u\|_{L^p(\Omega)} \leqslant \|v\|_{L^p(\Omega^\sharp)}, \qquad p = 1, 2. \tag{38}$$

Hence one may ask if (38) is still true for larger values of p in dimension 2 or if it is valid in every dimension and value of p. The authors, though, found counterexamples of the untruthfulness of these questions when $n = 2$ and $p = \infty$, and when $n = 3$ and $p = 2$. This, together with (29), led to the following open problems:

- $u^\sharp \leqslant v$ in $\Omega^\sharp$ for $n \geqslant 3$ and $f \equiv 1$;

- $\|u\|_{L^1(\Omega)} \leqslant \|v\|_{L^1(\Omega^\sharp)}$ for $n \geqslant 3$ and $f \in L^2(\Omega)$.

In [117] we move the first steps in these directions.
In particular let us consider problem (25) with $f \equiv 1$ and $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ a bounded $C^{2,\alpha}$ and simply connected open set.
We set

$$M(\Omega) = \|u\|_{L^\infty(\Omega)},$$

and for every $p \in [1, +\infty)$ we denote the following functional

$$F_p(\Omega) = \int_\Omega |u(x)|^p \, dx = \int_\Omega u^p(x) \, dx = \|u\|^p_{L^p(\Omega)}.$$

We are interested in computing the shape derivative (see [85]) of these two functionals and prove that the ball centered at the origin is a critical shape for them.
Namely, if $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2,\alpha}$ simply connected open set, let us consider a first order perturbation

$$\Omega_t = (\mathbb{1}_{\mathbb{R}^n} + tV)(\Omega),$$

with $\mathbb{1}_{\mathbb{R}^n}$ being the identity function, V a $C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ vector field and t a small real number. We are interested in the study of the limits (if they exist)

$$M'(\Omega, V) = \lim_{t \to 0} \frac{M(\Omega_t) - M(\Omega)}{t} \tag{39}$$

and

$$F_p'(\Omega, V) = \lim_{t \to 0} \frac{F_p(\Omega_t) - F_p(\Omega)}{t}. \tag{40}$$

If we denote by B_R the ball centered at the origin in $\mathbb{R}^n$ with radius $R > 0$, then problem (25) becomes

$$\begin{cases} -\Delta u = 1 & \text{in } B_R \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B_R, \end{cases} \tag{41}$$

where $\nu = \frac{x}{R}$ is the outer unit normal to the boundary. The main theorem is stated as follows

Theorem. *The ball B_R is a critical shape for the functionals $M(\Omega)$ and $F_p(\Omega)$, $p \geqslant 1$, i.e.*

$$M'(B_R, V) = F_p'(B_R, V) = 0,$$

where V is a $C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ vector field volume preserving of the first order and where $M'(\cdot, v)$ and $F_p'(\cdot, v)$ are the shape derivatives of M and F_p respectively.

For the precise definition of vector field volume preserving of the first order see Definition 3.1 in subsection 3.2.2.

Chapter 5 deals with a different problem from the ones discussed above. Indeed in our paper [6] we study a generalization and find a quantitative result for Pólya's inequality, that gives an estimate from below of the torsion of a non-empty open, bounded and convex set, in terms of its perimeter and measure.
In [112] the author proves that, among all bounded, open and convex planar sets, the following inequality holds

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \geqslant \frac{1}{3} \tag{42}$$

and equality is asymptotically achieved by a sequence of thinning rectangles. An upper bound of the same functional is given by Makai in [100] proves that among all bounded, open and convex planar sets, it holds

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \leqslant \frac{2}{3}, \tag{43}$$

which is sharp on a sequence of thinning triangles (for the exact definition of thinning domains see Definition 1.17). Estimates (42) and (43) are generalized to the $p-$Laplacian in [65]. More precisely, the authors prove that, in the class of bounded, open and convex planar sets,

$$\frac{1}{q+1} < \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} < \frac{2^{q+1}}{(q+2)(q+1)} \qquad q = \frac{p}{p-1}, \tag{44}$$

where the lower and the upper bounds hold asymptotically on a sequence of thinning rectangles and on a sequence of thinning isosceles triangles, respectively. In [49] the authors generalize the lower bound (44) in every dimensions, proving that, for bounded, open and convex sets $\Omega \subseteq \mathbb{R}^n$, it holds

$$\frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} > \frac{1}{q+1}, \tag{45}$$

and they extend such result also to the anisotropic case.

We also recall that in [22] the authors consider the functional

$$H_k(\Omega) = \frac{P(\Omega)T^k(\Omega)}{|\Omega|^{\alpha_k}} \qquad \alpha_k = 1 + k + \frac{2k-1}{n}$$

and prove that, among bounded, open and convex sets in $\mathbb{R}^n$, this functional is bounded if and only if $k = 1/2$. More precisely, they prove the following:

$$\frac{1}{\sqrt{3}} \leqslant H_{\frac{1}{2}}(\Omega) \leqslant \frac{2^n n^{3n/2}}{\omega_n} \left(\frac{n}{n+2}\right)^{\frac{1}{2}}, \tag{46}$$

where ω_n is the Lebesgue measure of the unit ball. We note that, in the planar case, the lower bound in (46) coincides with the one given in (42), while the upper bound is strictly larger than the one given in (43). It is conjectured that, in the higher dimensional case, the upper bound is

$$H_{\frac{1}{2}}(\Omega) \leqslant n \left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{2}}.$$

Moreover, we observe that the lower bound in (46) is asymptotically achieved by a sequence of thinning cylinders.

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a non-empty, bounded, open and convex set and let $p \in (1, +\infty)$. We consider the Poisson equation for the $p-$Laplace operator, defined as

$$-\Delta_p u := -\text{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

with Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u(x) = f(d(x,\partial\Omega)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{47}$$

where $f : [0, R_\Omega] \to [0, +\infty[$ is a continuous, non-increasing and not identically zero function, $d(\cdot, \partial\Omega) : \Omega \to [0, +\infty[$ is the distance function from the boundary defined as

$$d(x,\partial\Omega) := \inf_{y\in\partial\Omega} |x - y|$$

and R_Ω is the inradius of Ω, i.e.

$$R_\Omega = \sup_{x\in\Omega} d(x,\partial\Omega).$$

This class of functions, depending only on the distance, are the so called web functions, see as a reference [42]. A function $u \in W_0^{1,p}(\Omega)$ is a weak solution to (4.1) if and only if

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\varphi(x)\,dx = \int_\Omega f(d(x,\partial\Omega))\varphi(x)\,dx \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

The (f,p)-torsional rigidity of Ω, that we denote by $T_{f,p}(\Omega)$, is defined as

$$T_{f,p}(\Omega) = \max_{\substack{\varphi\in W_0^{1,p}(\Omega)\\ \varphi\not\equiv 0}} \frac{\left(\displaystyle\int_\Omega f(d(x,\partial\Omega))\,|\varphi(x)|\,dx\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_\Omega |\nabla\varphi(x)|^p\,dx\right)^{\frac{1}{p-1}}} \tag{48}$$

and, if $u_p \in W_0^{1,p}(\Omega)$ is the unique solution to (4.1), we have

$$T_{f,p}(\Omega) = \int_\Omega f u_p\,dx.$$

For the sake of simplicity, when $f \equiv 1$ in Ω, we set $T_p(\Omega) := T_{1,p}(\Omega)$ and, if we are also in the case $p = 2$, we set $T(\Omega) := T_{1,2}(\Omega)$.
The first result that we prove, following the method of proof used in [112] with the use of web functions as test functions, is a lower bound for the (f,p)-torsional rigidity, which generalizes the lower bound in (44).

Theorem. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$, $n \geqslant 2$, and let $f : [0, R_\Omega] \to [0, +\infty[$ be a continuous and non-increasing function such that $f \not\equiv 0$. Then, it holds*

$$T_{f,p}(\Omega) \geqslant c_p \frac{\mu_f^{q+1}(\Omega)}{f(0)P^q(\Omega)}, \tag{49}$$

where

$$c_p = \frac{p-1}{2p-1}, \qquad q = \frac{p}{p-1},$$

and

$$\mu_f(\Omega) = \int_\Omega f(x)\,dx.$$

Moreover, the equality sign is asymptotically achieved by a sequence of thinning cylinders.

For the definition of thinning cylinder see subsection (1.3.2). We stress that both the estimate and the constant in Theorem 4.1 are independent of the dimension of the space.

In the second part of the present paper, we focus our study on the case $f \equiv 1$ and $n = 2$ and we obtain some quantitative estimates. We define the following functional

$$\mathcal{F}_p(\Omega) = \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} \qquad q = \frac{p}{p-1}, \tag{50}$$

which is scaling invariant, since for every $t > 0$

$$|t\Omega| = t^n\,|\Omega|\,, \qquad P(t\Omega) = t^{n-1}P(\Omega)$$

and

$$T_p(t\Omega) = t^{n+q}T_p(\Omega).$$

We can rewrite inequality (4.3), in the case $f \equiv 1$, as follows

$$\mathcal{F}_p(\Omega) \geqslant c_p.$$

From Theorem 4.1 follows that along a sequence of thinning cylinders $\{\Omega_l\}_{l\in\mathbb{N}}$, we have

$$\mathcal{F}_p(\Omega_l) \xrightarrow{l\to 0} c_p.$$

This leads to the following stability issue: if $\mathcal{F}_p(\Omega)$ is close to c_p, can we say that Ω is close in some sense to a cylinder? The following result gives us information on the nature of the geometry of Ω: when $\mathcal{F}_p(\Omega) - c_p$ is sufficiently small, the set Ω is a thin domain (see definition (1.7)).

The main novelty of the paper consists indeed in the following quantitative results of the Pólya estimates (42) and the Pólya type lower bound in (44) by means of suitable deficits. For completeness, we recall some standard references about isoperimetric quantitative results, see for example [72, 71, 21, 18, 74, 106]. The main difference between these results and ours is that the equality in Pólya's estimates is achieved asymptotically for a sequence of thinning cylinders. Hence, the proof of quantitative result must take into account that we do not have a minimum, as in the classical isoperimetric stability results.

Theorem. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$ and let $f \equiv 1$. Then,*

$$\mathcal{F}_p(\Omega) - c_p \geqslant K(n,p)\left(\frac{w_\Omega}{\operatorname{diam}(\Omega)}\right)^{n-1}, \tag{51}$$

where $K(n,p)$ is a positive constant depending only on p and the dimension of the space n. In particular, in the case $n=2$, the exponent of the quantity $\dfrac{w_\Omega}{\operatorname{diam}(\Omega)}$ is sharp.

We prove a second quantitative result in the case $p = n = 2$.

Theorem. *Let Ω be a non-empty, bounded, open and convex set in $\mathbb{R}^2$, let $f \equiv 1$ and let $p = 2$. Then, there exists a positive constant $\tilde{K}$ such that*

$$\mathcal{F}_2(\Omega) - c_2 = \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \tilde{K}\left(\frac{|\Omega \triangle Q|}{|\Omega|}\right)^3, \tag{52}$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and w_Ω containing Ω.

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Chapter 1

Preliminaries

1.1 Notations

Throughout this thesis, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\cdot$ is the standard Euclidean scalar product for $n \geqslant 2$. Without ambiguity, the same symbol $|\cdot|$ will denote the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$ by $\mathcal{H}^k$, for $k \in [0, n)$, the $k-$dimensional Hausdorff measure in $\mathbb{R}^n$. Moreover, we use the following notation: $B_R(x)$ is the ball of $\mathbb{R}^n$ with radius R and centered at x, B is a generic ball such that $|B| = 1$. Let R_1, R_2 be such that $0 < R_1 < R_2$, the spherical shell will be denoted as follows:

$$A_{R_1,R_2} = \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}.$$

Moreover, we define ω_n as the Lebesgue measure in $\mathbb{R}^n$ of the ball of radius 1, so that $\mathcal{L}^n(B_R(x)) = \omega_n R^n$ and we denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$.

If $\Omega \subseteq \mathbb{R}^n$ has Lipschitz boundary, for $\mathcal{H}^{n-1}-$ almost every $x \in \partial\Omega$, we denote by $\nu_{\partial\Omega}(x)$ the outward unit Euclidean normal to $\partial\Omega$ at x. Sometimes, when there is no possibility of confution, in order to simplify the notation, we will use ν instead of ν_Ω.

1.2 General facts

1.2.1 Basic definitions

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let $E \subseteq \mathbb{R}^n$ be a measurable set. We recall now the definition of the perimeter of E in Ω, that is

$$P(E;\Omega) = \sup\left\{\int_E \text{div}\varphi \, dx : \ \varphi \in C_c^\infty(\Omega;\mathbb{R}^n), \ ||\varphi||_\infty \leqslant 1\right\}.$$

The perimeter of E in $\mathbb{R}^n$ will be denoted by $P(E)$ and, if $P(E) < \infty$, we say that E is a set of finite perimeter. Some references for results relative to the sets of finite perimeter are for example [99, 7]. We observe that a remarkable feature of this definition is that in this way the perimeter is not affected by modifications on sets of measure 0. Moreover, if E has Lipschitz boundary, we have that

$$P(E) = \mathcal{H}^{n-1}(\partial E). \tag{1.1}$$

In order to deduce properties, it is often very useful to approximate sets of finite perimeter with smooth sets. Therefore, we give the following notion of convergence.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set, let $(E_j)_j$ be a sequence of measurable sets in $\mathbb{R}^n$ and let $E \subseteq \mathbb{R}^n$ be a measurable set. We say that $(E_j)_j$ converges in measure in Ω to E, and we write $E_j \to E$, if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$, or in other words, if $\lim_{j\to\infty} |(E_j \Delta E) \cap \Omega| = 0$.

We also recall that the perimeter is lower semicontinuous with respect to the local convergence in measure, that means, if the sequence of sets (E_j) converges in measure in Ω to E, then

$$P(E;\Omega) \leqslant \liminf_{j\to\infty} P(E_j;\Omega).$$

As a consequence of the Rellich-Kondrachov theorem, the following compactness result holds and its proof can be found for instance in [7, Theorem 3.39].

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let $(E_j)_j$ be a sequence of measurable sets of $\mathbb{R}^n$, such that $\sup_j P(E_j;\Omega) < \infty$. Then, there exists a subsequence $(E_{j_k})_k$ converging in measure in Ω to a set E, such that*

$$P(E;\Omega) \leqslant \liminf_{k\to\infty} P(E_{j_k};\Omega).$$

Another useful property concerning the sets of finite perimeter is stated in the next approximation result, see [7, Theorem 3.42].

Proposition 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let E be a set of finite perimeter in Ω. Then, there exists a sequence of smooth, bounded open sets $(E_j)_j$ converging in measure in Ω and such that $\lim_{j\to\infty} P(E_j;\Omega) = P(E;\Omega)$.*

By their respectively definitions, we have that $P(E)$ and $|E|$ satisfy the following scaling properties, for $t > 0$,

$$P(tE) = t^{n-1}P(E), \qquad |tE| = t^n|E|.$$

For completeness we recall the classical isoperimetric inequality, already discussed in the introduction. We refer the reader, for example, to [104, 32, 37, 133] and to the original paper by De Giorgi [46].

Theorem 1.3. *Let $E \subseteq \mathbb{R}^n$, $n \geqslant 2$, a Borel set with finite Lebesgue measure, then*

$$n\omega_n^{1/n}|E|^{(n-1)/n} \leqslant P(E) \tag{1.2}$$

and equality holds if and only if E is a ball.

1.2.2 Coarea Formula and applications

In this subsection we recall the Coarea Formula and some of its consequences, that can be found in [62].

Theorem 1.4 (Coarea Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous function, with $n \geqslant m$. Then for each $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$,*

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\,dy, \tag{1.3}$$

where J denotes the Jacobian.

Theorem 1.5 (Change of variables Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous function, with $n \geqslant m$. Then for each $\mathcal{L}^n$-summable function $g : \mathbb{R}^n \to \mathbb{R}$ we have that $g|_{f^{-1}\{y\}}$ is $\mathcal{H}^{n-m}$-summable for $\mathcal{L}^m$-a.e. y and*

$$\int_{\mathbb{R}^n} gJf\,dx = \int_{\mathbb{R}^m}\int_{f^{-1}\{y\}} g\,d\mathcal{H}^{n-m}\,dy.$$

Applications

As applications to the change of variables formula, we have the integrations over balls and over level sets.

Theorem 1.6 (Integration over balls)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{L}^n$-summable function. Then*

$$\int_{\mathbb{R}^n} g\,dx = \int_0^\infty \int_{\partial B_r} g\,d\mathcal{H}^{n-1}\,dr.$$

In particular for $r > 0$

$$\frac{d}{dr}\left(\int_{B_r} g\,dx\right) = \int_{\partial B_r} g\,d\mathcal{H}^{n-1} \qquad \mathcal{L}^1 - a.e.$$

Theorem 1.7 (Integration over level sets)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous.*

1. *Then*
$$\int_{\mathbb{R}^n} |\nabla f|\,dx = \int_{-\infty}^\infty \mathcal{H}^{n-1}(\{f = t\})dt.$$

2. *Assume also that $ess \inf |\nabla f| > 0$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-summable, then*
$$\int_{\{f>t\}} g\,dx = \int_t^\infty \int_{\{f=s\}} \frac{g}{|\nabla f|}\,d\mathcal{H}^{n-1}\,ds.$$

3. *In particular*
$$\frac{d}{dt}\left(\int_{\{f>t\}} g\,dx\right) = -\int_{\{f=t\}} \frac{g}{|\nabla f|}\,d\mathcal{H}^{n-1} \qquad \mathcal{L}^1 - a.e.$$

1.2.3 Trace and Friedrich inequalities

Here we recall some known inequalities that will be useful in the sequel. Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, then by the classical Sobolev trace inequality (see [60]) we have that

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\|u\|_{H^1(\Omega)}, \tag{1.4}$$

for some positive constant $C > 0$. Moreover the embedding operator of $H^1(\Omega)$ into $L^2(\partial\Omega)$ is compact.

Another important embedding theorem is a consequence of the so-called Friedrich's inequality (see for instance [67, 102] and for a more general case [41]). Let $H^1(\Omega, \partial\Omega)$ the completion of the set of functions in $C^\infty(\Omega) \cap C(\bar{\Omega})$ which have weak gradient in $L^2(\Omega)$, equipped with the following pseudonorm (see [102] for the details)

$$\|u\|_{H^1(\Omega,\partial\Omega)} = \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}.$$

Fridrich's inequality states that

$$\|u\|_{L^2(\Omega)} \leqslant C(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}) \tag{1.5}$$

for some positive constant $C > 0$. Also in this case the embedding operator of $H^1(\Omega, \partial\Omega)$ into $L^2(\Omega)$ is compact (see Corollary 3, p. 392 in [102]).

1.2.4 Gronwall's Lemma

We will state Gronwall lemma (see [11, 82]) that will be useful in Chapter 3.

Lemma 1.8. *(Gronwall) Let $\xi(t)$ be a continuously differentiable function satysfing for some non-negative constant C, the following differential inequality*

$$\tau\xi'(\tau) \leqslant \xi(\tau) + C,$$

for all $\tau \geqslant \tau_0 > 0$. Then we have

$$\xi(\tau) \leqslant \tau\frac{\xi(\tau_0) + C}{\tau_0} + C$$

and

$$\xi'(\tau) \leqslant \frac{\xi(\tau_0) + C}{\tau_0},$$

for all $\tau \geqslant \tau_0$.

1.2.5 Lorentz Spaces

In this paragraph we recall the definition of Lorentz Space (for instance see [98]).
Let $1 \leqslant p, q \leqslant \infty$. A measurable function $f : \Omega \to \mathbb{R}^n$ is in the Lorentz space $L^{p,q}(\Omega)$ if and only if the following norm

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\displaystyle\int_0^{+\infty} (t^{\frac{1}{p}} |\{x \in \Omega : |f(x)| > t\}|)^q \frac{dt}{t} \right)^{\frac{1}{q}} & q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} |\{x \in \Omega : |f(x)| > t\}| & q = \infty \end{cases}$$

is finite.
Let us notice that when $p = q$ then

$$L^{p,q}(\Omega) = L^p(\Omega),$$

i.e. when the exponents are equal, we have the well known Lebesgue Spaces.

1.2.6 Definition of nearly spherical sets and main properties

In this section we give the definition of nearly spherical sets and we recall some of their basic properties (see for instance [18, 69, 70]). The usual definition is the following.

Definition 1.2. Let $n \geqslant 2$. An open, bounded set $E \subseteq \mathbb{R}^n$ with the origin contained in E is said a nearly spherical set parametrized by v if there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that

$$\partial E = \left\{ y \in \mathbb{R}^n \colon y = Rx(1 + v(x)),\, x \in \mathbb{S}^{n-1} \right\}, \tag{1.6}$$

where R is the radius of the ball having the same measure of E and $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant 1/2$.

The perimeter and the volume of a nearly spherical set are given by

$$P(E) = \int_{\mathbb{S}^{n-1}} (1 + v(x))^{n-2} \sqrt{(1 + v(x))^2 + |\nabla_\tau v(x)|^2}\, d\mathcal{H}^{n-1}, \tag{1.7}$$

$$|E| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (1 + v(x))^n \; d\mathcal{H}^{n-1}, \tag{1.8}$$

1.3 Some properties of Convex sets

1.3.1 Hausdorff distance, support function and radial map

We recall here some properties of convex sets that we will use in this thesis. We recall the definition of Hausdorff distance between two non-empty compact sets $E, F \subset \mathbb{R}^n$, that is (see for instance [122])

$$\delta_{\mathcal{H}}(E,F) = \inf\{\varepsilon > 0 \ : \ E \subset F + B_\varepsilon, \ F \subset E + B_\varepsilon\}.$$

Note that, if E, F are both convex sets, then $\delta_{\mathcal{H}}(E,F) = \delta_{\mathcal{H}}(\partial E, \partial F)$.

Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of non-empty compact subsets of $\mathbb{R}^n$, we say that E_k converges to E in the Hausdorff sense and we denote

$$E_k \xrightarrow{\mathcal{H}} E$$

if and only if $\delta_{\mathcal{H}}(E_k, E) \to 0$ as $k \to \infty$. Moreover, we say that $\{E_k\}_{k\in\mathbb{N}}$ converges in measure to E, and we write $E_k \to E$, if $\chi_{E_k} \to \chi_E$ in $L^1(\mathbb{R}^n)$, where χ_E and χ_{E_k} are the characteristic functions of E and E_k respectively.
In what follows we recall some properties of the convex bodies, i.e. compact convex sets without empty interior. We again refer to [122] for further properties and the details.
We give now the definition of support function of a convex set.

Definition 1.3. Let $K \subset \mathbb{R}^n$ be a bounded convex set of $\mathbb{R}^n$. The support function h_K of K is the function $h_K \colon \mathbb{S}^{n-1} \to \mathbb{R}$ defined as follows

$$h_K(x) = \sup_{y\in K}(x,y).$$

It is easy to see that the support function associated to a ball of radius R is constantly equal to R. If the origin belongs to K then h_K is non-negative and $h_K(x) \leqslant \operatorname{diam}(K)$ for every $x \in \mathbb{S}^{n-1}$.

Remark 1.9. Let K, C be two open, convex and bounded sets of $\mathbb{R}^n$; the following relation holds:

$$\delta_{\mathcal{H}}(C,K) = ||h_C - h_K||_{L^\infty(\mathbb{S}^{n-1})}$$

Definition 1.4. Let $K \subset \mathbb{R}^n$ be a bounded convex body such that the origin is an interior point of K. The radial function of K is defined as follows

$$\rho_K(x) = \sup\{\lambda \geqslant 0 \colon \lambda x \in K\}, \quad x \in \mathbb{S}^{n-1}, \tag{1.9}$$

and it is a Lipschitz function. The radial map is the function $r_K \colon \mathbb{S}^{n-1} \to \partial K$ defined as

$$r_K(x) = x\rho_K(x). \tag{1.10}$$

Then we can parametrize the boundary of every convex body containing the origin in this way

$$\partial K = \{x\, \rho_K(x), x \in \mathbb{S}^{n-1}\}. \tag{1.11}$$

Definition 1.5. We will define the minimum and the maximum distance of ∂K from the origin as follows

$$R_m = \min_{\mathbb{S}^{n-1}} \rho_K(x), \qquad R_M = \max_{\mathbb{S}^{n-1}} \rho_K(x). \tag{1.12}$$

Let now $f\colon \partial K \to \mathbb{R}$ be $(n-1)-$integrable. The following formula for the change of variable given by the radial map holds:

$$\int_{\partial K} f\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} f(r_K(x)) \frac{\rho_K(x)}{h_K(\nu_K(r_K(x)))}\, d\mathcal{H}^{n-1}, \tag{1.13}$$

where $\nu_K(r_K(x))$ is the outer unit normal to ∂K at the point $r_K(x) = x\rho_K(x)$. We have (see for example [122])

$$\nu_K(r_K(x)) = \frac{x\rho_K(x) - \nabla_\tau \rho_K(x)}{\sqrt{(\rho_K(x))^2 + |\nabla_\tau \rho_K(x)|^2}},$$

where by $\nabla_\tau \rho_K$ we denote the the component of $\nabla \rho_K$ tangential to $\mathbb{S}^{n-1}$. So, we observe that (1.13) is equivalent to

$$\int_{\partial K} f\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} f(r_K(x))(\rho_K(x))^{n-1} \sqrt{1 + \left(\frac{|\nabla_\tau \rho_K(x)|}{\rho_K(x)}\right)^2}\, d\mathcal{H}^{n-1}.$$

The following result holds (see for instance [38], [89], [122]).

Lemma 1.10. *Let K_n and K be bounded convex bodies containing the origin for any $n \in \mathbb{N}$ and such that $K_n \to K$ in the Hausdorff sense. For any $n \in \{0, 1, 2, ...\}$, let h_{K_n}, ρ_{K_n} be the support function and the radial function K_n, respectively. Then the following statements hold*

(i) Let h_K be the support function of K then

$$\sup_{\theta \in \mathbb{S}^{n-1}} |h_{K_n}(x) - h_K(x)| \to 0.$$

(ii) Let ρ_K the radial function of K then

$$\sup_{x \in \mathbb{S}^{n-1}} |\rho_{K_n}(x) - \rho_K(x)| \to 0.$$

(iii) Let $x \in \partial K$ and $x_n \in \partial K_n$, $n \in \mathbb{N}$, points where $\nu_K(x)$ and $\nu_{K_n}(x_n)$ are well defined and such that

$$\lim_{n \to \infty} x_n = x.$$

Then

$$\lim_{n \to \infty} \nu_{K_n}(x_n) = \nu_K(x).$$

By (1.13), Lemma 1.10 and the Lebesgue's convergence Theorem we immediately get

Theorem 1.11. *Let K_n and K be bounded convex bodies containing the origin for any $n \in \mathbb{N}$ and such that $K_n \to K$ in the Hausdorff sense. Let*

$$f_n\colon \partial K_n \to \mathbb{R}, \quad f\colon \partial K \to \mathbb{R}$$

be $\mathcal{H}^{n-1}$ measurable functions such that

(i) there exists $C > 0$ such that

$$\|f\|_{L^\infty(\partial K)} \leqslant C, \quad \|f_n\|_{L^\infty(\partial K_n)} \leqslant C, \ \forall n \in \mathbb{N}$$

(ii) if $x_n \in \partial K_n$ is such that $\lim_{n\to\infty} x_n = x \in \partial K$, f_n is defined in x_n and

$$\lim_{n \to \infty} f_n(x_n) = f(x).$$

Then

$$\lim_{n \to \infty} \int_{\partial K_n} f_n(x_n)\, d\mathcal{H}^{n-1} = \int_{\partial K} f(x)\, d\mathcal{H}^{n-1}.$$

1.3.2 Inequalities on Convex sets

We conclude this paragraph by recalling some definition concerning convex sets and stating some important inequality that connect different quantities of a convex body in $\mathbb{R}^n$ respectively.
Recall that the diameter $\operatorname{diam}\Omega$ and the inradius R_Ω of Ω are defined as

$$\operatorname{diam}\Omega = \sup_{x,y\in\Omega} |x-y|, \tag{1.14}$$

$$R_\Omega = \sup_{x\in\Omega} \inf_{y\in\partial\Omega} |x-y|. \tag{1.15}$$

By means of the support function of a convex set we can define the minimal width (or thickness) of a convex set as follows

Definition 1.6. Let Ω a bounded, open and convex set of $\mathbb{R}^n$, the width of Ω in the direction $y \in \mathbb{R}$ is defined as

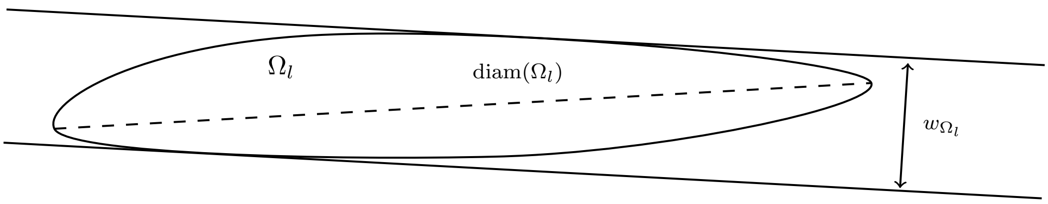
$$\omega_\Omega(y) = h_\Omega(y) + h_\Omega(-y)$$

and the minimal width of Ω as

$$w_\Omega = \min\{\omega_\Omega(y) \mid y \in \mathbb{S}^{n-1}\}.$$

Definition 1.7. Let Ω_l be a sequence of bounded, open and convex sets of $\mathbb{R}^n$. We say that Ω_l is a sequence of thinning domains if

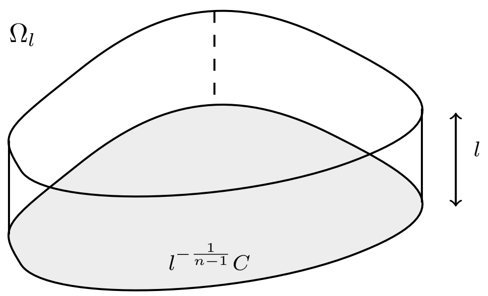
$$\frac{w_{\Omega_l}}{\operatorname{diam}(\Omega_l)} \xrightarrow{l\to 0} 0. \tag{1.16}$$



In particular, if $l > 0$ and C is a bounded, open and convex set of $\mathbb{R}^{n-1}$ with unitary $(n-1)$-dimensional measure, then, if $l \to 0$, the sequence

$$\Omega_l = l^{-\frac{1}{n-1}} C \times \left[-\frac{l}{2}, \frac{l}{2}\right] \tag{1.17}$$

is called a sequence of thinning cylinders. Moreover, in the case $n = 2$, the sequence (1.17) is called sequence of thinning rectangles.



We recall the following estimate, which is proved in [15] in the planar case and is generalized in [17] in every dimensions (see for instance also [54, 122]).

Proposition 1.12. *Let Ω be a bounded, open and convex set of $\mathbb{R}^n$ with non-empty interior. Then,*

$$\frac{1}{n} \leqslant \frac{|\Omega|}{P(\Omega)R_\Omega} < 1. \tag{1.18}$$

The upper bound is sharp on a sequence of thinning cylinders, while the lower bound is sharp, for example, on balls. Moreover, for $n = 2$, any circumscribed polygon, that is a polygon whose incircle touches all the sides, verifies the lower bound with the equality sign.

In the planar case the following inequalities hold true (see as a reference [124, 123, 118]).

Proposition 1.13. *Let Ω be a bounded, open and convex set of $\mathbb{R}^2$. Then,*

$$2 \leqslant \frac{w_\Omega}{R_\Omega} \leqslant 3. \tag{1.19}$$

The upper bound is achieved by equilateral triangles and the lower bound is achieved by disks. Moreover,

$$(w_\Omega - 2R_\Omega)\, P(\Omega) \leqslant \frac{2}{\sqrt{3}} w_\Omega^2, \tag{1.20}$$

with equality holding for equilateral triangles, and

$$|\Omega| \leqslant R_\Omega \left(P(\Omega) - \pi R_\Omega\right), \tag{1.21}$$

with equality holding for the stadii (convex hull of two identical disjoint balls). Eventually,

$$2\operatorname{diam}(\Omega) < P(\Omega) \leqslant \pi \operatorname{diam}(\Omega), \tag{1.22}$$

where the lower bound is asymptotically achieved by a sequence of thinning rectangles and the upper bound by sets of constant width.

Moreover we recall the following inequality (see [59, 80, 121]):

$$P(E)^{n-1} > \omega_{n-1} n^{n-2} \operatorname{diam}(E) |E|^{n-2}. \tag{1.23}$$

1.3.3 Inner parallel sets

Let Ω be a bounded, open and convex set of $\mathbb{R}^n$ with non empty interior. The distance function from the boundary is defined as

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|,$$

and we will denote it by $d(\cdot)$. We remark that the distance function is concave, as a consequence of the convexity of Ω.

The superlevel sets of the distance function

$$\Omega_t = \{x \in \Omega \,:\, d(x) > t\}, \qquad t \in [0, R_\Omega] \tag{1.24}$$

are called *inner parallel sets*, where R_Ω is the inradius and we use the following notations:

$$\mu(t) = |\Omega_t|, \qquad P(t) = P(\Omega_t) \qquad t \in [0, R_\Omega]. \tag{1.25}$$

By coarea formula (1.2.2), recalling that $|\nabla d| = 1$ almost everywhere, we have

$$\mu(t) = \int_{\{d>t\}} dx = \int_{\{d>t\}} \frac{|\nabla d|}{|\nabla d|}\, dx = \int_t^{R_\Omega} \frac{1}{|\nabla d|} \int_{\{d=s\}} d\mathcal{H}^{n-1}\, ds = \int_t^{R_\Omega} P(s)\, ds;$$

hence, the function $\mu(t)$ is absolutely continuous, decreasing and its derivative is $\mu'(t) = -P(t)$ almost everywhere. Moreover, it is possible to prove that the perimeter $P(t)$ is non increasing and absolutely continuous, as a consequence of the concavity of the distance function and the Brunn-Minkowski inequality for the perimeter (see [122] as a reference).

Finally, let us consider the case $n = 2$. For Ω a bounded, open and convex set of $\mathbb{R}^2$ with nonempty interior the Steiner formulas for the inner parallel sets hold (see [8]):

$$P(t) \leqslant P(\Omega) - 2\pi t \qquad \forall t \in [0, R_\Omega], \tag{1.26}$$

$$\mu(t) \geqslant |\Omega| - P(\Omega)t + \pi t^2 \qquad \forall t \in [0, R_\Omega], \tag{1.27}$$

equality holding in both (1.26) and (1.27) for the stadii (see [65]). From (1.26), we have that, if Ω is a convex set, then

$$-P'(t) \geqslant 2\pi, \tag{1.28}$$

with equality if Ω is a ball or a stadium.

1.4 Anisotropy and Convex symmetrization

1.4.1 Anisotropy

What follows can be found in [135]. Let $H : \mathbb{R}^n \longrightarrow [0, +\infty]$, $n \geqslant 2$, be a $C^2(\mathbb{R}^n \backslash \{0\})$ convex function that satisfies the following homogeneity property

$$H(t\xi) = |t| H(\xi) \quad \forall \xi \in \mathbb{R}^n\,,\ \forall t \in \mathbb{R}, \tag{1.29}$$

and such that

$$\gamma |\xi| \leqslant H(\xi) \leqslant \delta |\xi|, \tag{1.30}$$

for some positive constants $\gamma \leqslant \delta$.
These properties guarantee that H is a norm in $\mathbb{R}^n$. Indeed (1.30) guarantees that $H(\xi) = 0$ if and only if $\xi = 0$. It is homogeneous by (1.29) and the triangular inequality follows from the convexity of the function H: if $\xi, \eta \in \mathbb{R}^n$, then

$$\frac{H(x+y)}{2} = H\left(\frac{x}{2} + \frac{y}{2}\right) \leqslant \frac{H(x)}{2} + \frac{H(y)}{2}.$$

Because of (1.29), we can assume that the set

$$K = \{\xi \in \mathbb{R}^n : H(\xi) \leqslant 1\}$$

is such that $|K|$ is equal to the measure ω_n of the unit sphere in $\mathbb{R}^n$. We can define the support function of K as

$$H^\circ(x) = \sup_{\xi \in K} \langle x, \xi \rangle\,, \tag{1.31}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. $H^\circ : \mathbb{R}^n \longrightarrow [0, +\infty]$ is a convex, homogeneous function in the sense of (1.29). Moreover H and H° are polar to each other, in the sense that

$$H^\circ(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)}$$

and

$$H(x) = \sup_{\xi\neq 0} \frac{\langle x, \xi\rangle}{H^\circ(\xi)}.$$

H° is the support function of the set

$$K^\circ = \{x \in \mathbb{R}^n : H^\circ(x) \leqslant 1\}.$$

The set $\mathcal{W} = \{x \in \mathbb{R}^n : H^\circ(x) < 1\}$ is the so-called Wulff shape centered at the origin. We set $k_n = |\mathcal{W}|$. More generally we will denote by $\mathcal{W}_R(x_0)$ the Wulff shape centered in $x_0 \in \mathbb{R}^n$ with measure $k_n R^n$ the set $R\mathcal{W} + x_0$, and $\mathcal{W}_R(0) = \mathcal{W}_R$.
H and H° satisfy the following properties:

$$H_\xi(\xi)\cdot\xi = H(\xi), \quad H^\circ_\xi(\xi)\cdot\xi = H^\circ(\xi), \tag{1.32}$$

$$H(H^\circ_\xi(\xi)) = H^\circ(H_\xi(\xi)) = 1 \quad \forall\xi \in \mathbb{R}^n\backslash\{0\}, \tag{1.33}$$

$$H^\circ(\xi)H_\xi(H^\circ_\xi(\xi)) = H(\xi)H^\circ_\xi(H_\xi(\xi)) = \xi \ \ \forall\xi \in \mathbb{R}^n\backslash\{0\}. \tag{1.34}$$

If $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary and E is an open subset of $\mathbb{R}^n$, we can give a generalized definition of perimeter of E with respect to the anisotropic norm as follows

$$P_H(E,\Omega) = \int_{\partial^* E\cap\Omega} H(\nu)\, d\mathcal{H}^{n-1},$$

where $\partial^* E$ is the reduced boundary of E (for the definition see [62]) and ν is its Euclidean outer normal. Clearly, if E is open, bounded and Lipschitz, then the outer unit normal exists almost everywhere and

$$P_H(E,\mathbb{R}^n) := P_H(E) = \int_{\partial E} H(\nu)\, d\mathcal{H}^{n-1}. \tag{1.35}$$

By (1.30) we have that

$$\gamma P(E) \leqslant P_H(E) \leqslant \delta P(E).$$

In [4, 135] it is shown that if $u \in W^{1,1}(\Omega)$, then for a.e. $t > 0$

$$-\frac{d}{dt}\int_{\{u>t\}} H(\nabla u)\, dx = P_H(\{u > t\},\Omega) = \int_{\partial^*\{u>t\}\cap\Omega} \frac{H(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1}. \tag{1.36}$$

Moreover an isoperimetric inequality for the anisotropic perimeter holds (for instance see [135, 34, 43, 64])

$$P_H(E) \geqslant n k_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}}. \tag{1.37}$$

1.4.2 Convex symmetrization

Let $f : \Omega \longrightarrow [0,+\infty]$ be a measurable function. The decreasing rearrangement f^* of f is defined as follows

$$f^*(s) = \inf\{t \geqslant 0 : |\{x \in \Omega : |f(x)| > t\}| < s\} \quad s \in [0,|\Omega|],$$

which is the generalized inverse function of the distribution function of f. We define the convex symmetrization $f^\star$ of f as

$$f^\star(x) = f^*(k_n H^\circ(x)^n) \quad x \in \Omega^\star.$$

In particular it is well known that the functions f, f^* and $f^\star$ are equimeasurable, i.e.

$$|\{f > t\}| = |\{f^* > t\}| = |\{f^\star > t\}| \quad t \geqslant 0.$$

As a consequence, if $f \in L^p(\Omega)$, $p \geqslant 1$, then $f^* \in L^p([0,|\Omega|])$, $f^\star \in L^p(\Omega^\star)$ and

$$\|f\|_{L^p(\Omega)}\| = \|f^*\|_{L^p([0,|\Omega|])} = \|f^\star\|_{L^p(\Omega^\star)}.$$

Moreover the Hardy-Littlewood inequality holds (see [92])

$$\int_\Omega |f(x)g(x)|\,dx \leqslant \int_0^{|\Omega|} f^*(s)g^*(s)\,ds. \tag{1.38}$$

So, if we consider g as the characteristic function of the set $\{x \in \Omega : u(x) > t\}$, for some measurable function $u : \Omega \to \mathbb{R}$ and $t \geqslant 0$, then we get

$$\int_{\{u>t\}} f(x)\,dx \leqslant \int_0^{\mu(t)} f^*(s)\,ds, \tag{1.39}$$

where, again, $\mu(t)$ is the distribution function of u.

Chapter 2

Results about some eigenvalue problem in annular domains

In this chapter we deal with the study of the first Steklov-Dirichlet eigenvalue for the Laplace operator in a set with an internal spherical obstacle, which can be found in the articles [106, 75]. In the first part we prove, via a stability result, that the spherical shell locally maximizes the first eigenvalue among nearly spherical sets, when both the volume and the internal ball are fixed.
In the second part we work in the class of convex sets. We prove the existence of a maximizer when fixing measure and inradius of the inner ball, generalizing it even when the inner obstacle is any convex set. Moreover we give new bounds for the first non-trivial Steklov-Dirichlet eigenvalue in terms of geometric quantities related to the exterior convex set. Eventually we show a partial result: the maximizer is a spherical shell when our set is contained in a ball centered at the origin and with the radius that depends on the radius of the inner ball and the dimension of the space.

2.1 Introduction to the Steklov-Dirichlet problems and state of art

Let $\Omega_0 \subset \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, connected set with Lipschitz boundary such that $B_{R_1} \Subset \Omega_0$, where B_{R_1} is the open ball of radius $R_1 > 0$ centered at the origin such that its closure is strictly contained in Ω_0 and let us set $\Omega := \Omega_0 \backslash \overline{B_{R_1}}$.
Since we are studying a Steklov eigenvalue problem with a spherical obstacle, we need to introduce the definition of a closed subspace of $H^1(\Omega)$ that incorporates the Dirichlet boundary condition on ∂B_{R_1}. We denote the set of Sobolev functions on Ω that vanish on ∂B_{R_1} by

$$H^1_{\partial B_{R_1}}(\Omega),$$

that is (see [58]) the closure in $H^1(\Omega)$ of the set of test functions

$$C^\infty_{\partial B_{R_1}}(\Omega) := \{u|_\Omega \ : \ u \in C^\infty_0(\mathbb{R}^n), \ \mathrm{spt}(u) \cap \partial B_{R_1} = \emptyset\}.$$

We are dealing with the following boundary eigenvalue problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial B_{R_1} \\ \dfrac{\partial u}{\partial \nu} = \sigma(\Omega)\, u & \text{on } \partial\Omega_0, \end{cases} \tag{2.1}$$

where ν is the outer normal to $\partial\Omega_0$.

Definition 2.1. The real number $\sigma(\Omega)$ and the function $u \in H^1_{\partial B_{R_1}}(\Omega)$ are, respectively, called eigenvalue of (2.1) and eigenfunction associated to $\sigma(\Omega)$, if and only if

$$\int_\Omega \nabla u \nabla \varphi \, dx = \sigma(\Omega) \int_{\partial\Omega_0} u\varphi \, d\mathcal{H}^{n-1}(x)$$

for every $\varphi \in H^1_{\partial B_{R_1}}(\Omega)$.

Furthermore, the first eigenvalue is variationally characterized by

$$\sigma_1(\Omega) = \min_{\substack{v \in H^1_{\partial B_{R_1}}(\Omega) \\ v \neq 0}} J[v], \tag{2.2}$$

where

$$J[v] := \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_{\partial\Omega_0} v^2 \, d\mathcal{H}^{n-1}}. \tag{2.3}$$

We point out that the condition of being orthogonal to constants in $L^2(\partial\Omega)$ is not required, unlike the classical Steklov eigenvalue (when $R_1 = 0$). Notice also that the value $\sigma_1(\Omega)$ is the optimal constant in the Sobolev-Poincaré trace inequality:

$$\sigma_1(\Omega)||v||_{L^2(\partial\Omega_0)} \leqslant ||\nabla v||_{H^1_{\partial B_{R_1}}(\Omega)}.$$

The spectrum of (2.1) is discrete and the sequence of eigenvalues can be ordered (see for instance [1, 56, 105])

$$0 < \sigma_1(\Omega) \leqslant \sigma_2(\Omega) \leqslant \cdots .$$

Furthermore, we will show in section (2.2) that the first eigenvalue is simple and the corresponding eigenfunctions have constant sign (see also [56]).

When $R_1 = 0$, (2.1) is the classical Steklov-Laplacian eigenvalue problem. In this case, Weinstock in [142, 141] proved an isoperimetric inequality for the first non-trivial Steklov eigenvalue in two dimensions. More precisely, he showed that among all simply connected sets of the plane with prescribed perimeter, the disc maximizes the first non-trivial Steklov-Laplacian eigenvalue. In [27] the authors proved that Weinstock inequality holds true in any dimension, provided they restrict to the class of convex sets with fixed perimeter. In [24], it is proved that the ball is still a maximizer for the first non-trivial Steklov eigenvalue among all bounded open sets with Lipschitz boundary of $\mathbb{R}^n$, $n \geqslant 2$, with fixed volume. Stability and instability results are also studied (for instance we refer to [19, 31, 74]).

When we consider a spherical hole with homegeneous Dirichlet boundary condition, that is $R_1 > 0$, the Steklov-Dirichlet eigenvalue problem for the Laplacian (2.1) is substantially different. The study of an eigenvalue problem on sets with a spherical hole is actually a topic of interest and problem (2.1) has been considered by several authors (see for instance [56, 68, 88, 87, 138]). We list here some mixed boundary condition eigenvalue problems on perforated domains: the first eigenvalue of the p-Laplacian with external Robin and internal Neumann boundary condition, when volume and external perimeter are fixed [109, 107]; the first eigenvalue of the p-Laplacian with external Neumann and internal Robin boundary condition, when volume and internal $(n-1)$-quermassintegral [86, 53] are fixed; the problem of optimally insulating a given domain [52]. We

recall that in [9] is studied the Steklov-Dirichlet problem and some properties of the related eigenvalue.
Finally, this kind of estimates has been obtained also for a more general class of equations, involving the so called Finsler operator. We refer the reader, for example, to [48, 51, 50, 77, 108, 110].
When $\Omega_0 = B_{R_2}(x_0)$ is a ball centered at x_0 with radius $R_2 > R_1$, in [68, 138] it is proved that $\sigma_1(\Omega)$ achieves the maximum when Ω is the spherical shell, that is when the two balls are concentric.

2.2 Properties of the Eigenvalues and Eigenfunctions

We give now the definitions and some geometric properties of eigenvalues and eigenfunctions of problem (2.1). For sake of simplicity we define

$$\mathcal{A}_{R_1} := \left\{ \begin{matrix} \Omega = \Omega_0 \backslash \overline{B_{R_1}} \ : \Omega_0 \subset \mathbb{R}^n \text{ open, bounded, connected,} \\ \text{with Lipschitz boundary, s.t.} B_{R_1} \Subset \Omega_0 \end{matrix} \right\}.$$

The following ensures the existence of minimizers of problem (2.2).

Proposition 2.1. *Let $R_1 > 0$ and $\Omega \in \mathcal{A}_{R_1}$, then there exists a function $u \in H^1_{\partial B_{R_1}}(\Omega)$ achieving the minimum in* (2.2) *and satisfying problem* (2.1).

Proof. Let $u_k \in H^1_{\partial B_{R_1}}(\Omega)$ be a minimizing sequence of (2.2) such that $||u_k||_{L^2(\partial\Omega_0)} = 1$. Since the minimum in (2.2) is positive, then there exists a constant $C > 0$ such that $J[u_k] \leqslant C$ for every $k \in \mathbb{N}$ and therefore $||Du_k||_{L^2(\Omega)} \leqslant C$. Moreover, a Poincaré inequality in $H^1_{\partial B_{R_1}}(\Omega)$ holds and this implies that $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^1_{\partial B_{R_1}}(\Omega)$. Therefore, there exist a subsequence, still denoted by u_k, and a function $u \in H^1_{\partial B_{R_1}}(\Omega)$ with $||u||_{L^2(\partial\Omega_0)} = 1$, such that $u_k \to u$ strongly in $L^2(\Omega)$, hence also almost everywhere, and $Du_k \rightharpoonup Du$ weakly in $L^2(\Omega)$. By the compactness of the trace operator (see for example [96, Cor. 18.4]), u_k converges strongly to u in $L^2(\partial\Omega)$ and almost everywhere on $\partial\Omega$ to u. Then, by weak lower semicontinuity we have

$$\lim_{k\to+\infty} J[u_k] \geqslant J[u].$$

Hence, the existence of a minimizer $u \in H^1_{\partial B_{R_1}}(\Omega)$ follows. Moreover, u is harmonic in Ω and so, by strong maximum principle, it has constant sign on Ω. □

Now we state the simplicity of the first eigenvalue of (2.1), following the idea in [61, Section 6.5.1].

Proposition 2.2. *Let $R_1 > 0$ and $\Omega \in \mathcal{A}_{R_1}$, then the first eigenvalue $\sigma_1(\Omega)$ of* (2.1) *is simple, that is all the associated eigenfunctions are scalar multiple of each other.*

Proof. Let $u, \tilde{u}$ be two non trivial weak solutions of the problem (2.1). Since, by Proposition 2.1, we can assume that $\tilde{u}$ is positive in Ω, then it is clear that

$$\int_\Omega \tilde{u}\, dx \neq 0.$$

So, we can find a real constant χ such that

$$\int_\Omega (u - \chi\tilde{u})\, dx = 0. \tag{2.4}$$

Since $u - \chi\tilde{u}$ is still a solution of problem (2.1), then it is also non-negative (or non-positive) in Ω. Therefore, (2.4) implies that $u \equiv \chi\tilde{u}$ in Ω and the simplicity of $\sigma_1(\Omega)$ follows.

□

It is worth noticing that the first nontrivial eigenvalue for the classical Steklov-Laplacian problem (when $R_1 = 0$) on B_{R_2} is $1/R_2$ and the corresponding eigenfunctions are the coordinate axis x_i, for $i = 1, .., n$. This means that the first nontrivial eigenvalue has multiplicity n and this makes a significant difference with problem (2.1), for which we proved that the simplicity holds. On the other hand, it is easy to verify that both have the same scaling property:

$$\sigma_1(t\Omega) = \frac{1}{t}\sigma_1(\Omega), \quad \forall t > 0. \tag{2.5}$$

The first attempts to study the optimal shape of problem (2.1) has been done on spherical shells, i.e. when $\Omega_0 = B_{R_2}$, for $R_2 > R_1 > 0$. We recall from [138], the explicit expression of the first eigenfunction on the spherical shell A_{R_1,R_2}:

$$w(r) = \begin{cases} \ln r - \ln R_1 & \text{for } n = 2 \\ \left(\dfrac{1}{R_1^{n-2}} - \dfrac{1}{r^{n-2}}\right) & \text{for } n \geqslant 3, \end{cases} \tag{2.6}$$

with $r = |x|$. This function is radial, positive, strictly increasing and it is associated to the following eigenvalue:

$$\sigma_1(A_{R_1,R_2}) = \begin{cases} \frac{1}{R_2 \log\left(\frac{R_2}{R_1}\right)} & \text{for } n = 2 \\ \frac{n-2}{R_2\left[\left(\frac{R_2}{R_1}\right)^{n-2} - 1\right]} & \text{for } n \geqslant 3. \end{cases} \tag{2.7}$$

It is worth noting that, since problem (2.1) and the classical Steklov ($R_1 = 0$) have the same scaling property (2.5), then the shape functional $\Omega \to |\Omega|^{\frac{1}{N}}\sigma_1(\Omega)$ is scaling invariant, as in the classical case.

Remark 2.3. We point out that by (2.7), we have that $\sigma_1(A_{R_1,R_2})$ is increasing with respect to the radius of the inner ball, R_1, that is

$$\sigma_1(A_{R_1,R_2}) < \sigma_1(A_{r_1,R_2}), \quad \text{if } r_1 > R_1$$

Moreover it holds

$$\lim_{R_1 \to 0} \sigma_1(A_{R_1,R_2}) = 0, \tag{2.8}$$

that is $\sigma_1(A_{R_1,R_2})$ tends to the first trivial Steklov eigenvalue of the Laplacian for R_1 which goes to zero. Finally we stress that an easy computation gives that $\sigma_1(A_{R_1,R_2})$ is decreasing with respect to the external radius R_2, that is

$$\sigma_1(A_{R_1,R_2}) < \sigma_1(A_{r_1,\bar{R}}), \quad \text{if } \bar{R} < R_2.$$

2.2.1 Upper bounds for $\sigma_1(\Omega)$

We show an upper bound for σ_1 depending only by the dimension n, by the measure of Ω and by the radius of the internal ball R_1.

Proposition 2.4. *Let $R_1 > 0$ and $\Omega \Subset \mathcal{A}_{R_1}$, then*

$$\sigma_1(\Omega) \leqslant \frac{2}{n\omega_n^{\frac{1}{n}}\left(\left(\frac{|\Omega|}{2\omega_n}+R_1^n\right)^{1/n}-R_1\right)^2}|\Omega|^{1/n}. \tag{2.9}$$

Proof. Let $\bar{R} > 0$ be such that $|A_{R_1,\bar{R}}| = |\Omega|/2$, then $\bar{R}$ depends only by the dimension n, the measure $|\Omega|$ and R_1, that is

$$\bar{R} = \left(\frac{|\Omega|}{2\omega_n}+R_1^n\right)^{1/n}.$$

Consider the function

$$\varphi(x) = \begin{cases} |x| - R_1 & \text{if } R_1 \leqslant |x| \leqslant \bar{R}; \\ \bar{R} - R_1 & \text{if } |x| \geqslant \bar{R}. \end{cases} \tag{2.10}$$

We distinguish now two cases. Firstly, we assume that $B_{\bar{R}} \Subset \Omega_0$, i.e. $d := \operatorname{dist}(\partial B_{\bar{R}}, \partial\Omega_0) > 0$. By using (2.10) as test function in the Rayleigh quotient (2.3) and by the isoperimetric inequality, we obtain

$$\sigma_1(\Omega) \leqslant \frac{|\Omega|}{\left(\bar{R}-R_1\right)^2 P(\Omega_0)} \leqslant \frac{1}{n\omega_n^{\frac{1}{n}}\left(\bar{R}-R_1\right)^2}|\Omega|^{\frac{1}{n}}. \tag{2.11}$$

We consider now the case $d = 0$, that is when the ball $B_{\bar{R}}$ is not strictly contained in Ω_0. Therefore, we divide the boundary of Ω_0 in the two sets $\partial^{int}\Omega_0$ and $\partial^{ext}\Omega_0$ that live, respectively, inside and outside of $B_{\bar{R}}$. Using the test function (2.10) in the Rayleigh quotient (2.3), we have

$$\sigma_1(\Omega) \leqslant \frac{|\Omega|}{\int_{\partial\Omega_0}|\varphi|^2\, d\mathcal{H}^{n-1}} \leqslant \frac{|\Omega|}{(\bar{R}-R_1)^2\int_{\partial^{ext}\Omega_0} 1\, d\mathcal{H}^{n-1}}. \tag{2.12}$$

We recall that a relative isoperimetric inequality with supporting set $B_{\bar{R}}$ holds (see as reference e.g. [33, 40, 39]):

$$\mathcal{H}^{n-1}(\partial^{ext}\Omega_0) \geqslant n\left(\frac{\omega_n}{2}\right)^{1/n}\left(\frac{|\Omega_0|}{2}\right)^{1-\frac{1}{n}}. \tag{2.13}$$

By using (2.13) in (2.12), we have

$$\sigma_1(\Omega) \leqslant \frac{2}{n\omega_n^{\frac{1}{n}}(\bar{R}-R_1)^2}|\Omega|^{\frac{1}{n}}. \tag{2.14}$$

The conclusion follows by observing that the upper bound (2.14) is greater than (2.11). □

Obviously $\sigma_1(\Omega)$ is bounded also when we fix the perimeter of Ω, that is equivalent to fix the perimeter of Ω_0, instead of the volume. Indeed by (2.9) and the isoperimetric inequality, we can deduce the following upper bound

$$\sigma_1(\Omega) \leqslant \frac{2V^{\frac{1}{n}}(\Omega)}{n\omega_n^{\frac{1}{n}}\left(\left(\frac{|\Omega|}{2\omega_n}+R_1^n\right)^{\frac{1}{n}}-R_1\right)^2} \leqslant C(n)\frac{P^{\frac{1}{n-1}}(\Omega_0)}{R_1^2}, \tag{2.15}$$

where $C(n)$ is a positive constants that depends only on the dimension n.

An upper bound for $\sigma_1(\Omega)$ for not spherical holes

We are able to prove an upper bound for $\sigma_1(\Omega)$ even in the case of a not spherical hole.
Let $K \subset \mathbb{R}^n$ be a convex set with non-empty interior such that $K \Subset \Omega_0$ and let $\Omega_K = \Omega_0 \backslash \overline{K}$. In this case, according to [58], the natural space of functions that we have to consider are $C^\infty_{\partial K}(\Omega_K)$ and $H^1_{\partial K}(\Omega_K)$. In particular the classical arguments of Calculus of Variations apply, as in Proposition (2.1), and $\sigma_1(\Omega_K)$ is well defined. Let us now assume that the volume $|\Omega| = \omega$ and the inradius $\rho(K) = \overline{r}$ of K are fixed. Let us consider $A_{\overline{r},\overline{R}}$ the spherical shell with radii $\overline{r}$ and $\overline{R}$, where $\overline{R}$ is such that $|A_{\overline{r},\overline{R}}| = |\Omega|/2$. So, we have

$$\bar{R} = \left(\frac{|\Omega|}{2\omega_n} + \bar{r}^n \right)^{1/n}. \tag{2.16}$$

We also consider the following test function $\varphi : \mathbb{R}^n \backslash K \to [0, \infty)$:

$$\varphi(x) = \begin{cases} d_K(x) & \text{if } 0 \leqslant d_K(x) \leqslant \overline{R} \\ \overline{R} & \text{if } d_K(x) \geqslant \overline{R} \end{cases}, \tag{2.17}$$

where

$$d_K(x) := \inf_{y \in \partial K} ||x - y||.$$

and we denote by K_t the set

$$K_t = \{x \in \mathbb{R}^n \backslash \overline{K} \mid d_K(x) < t\}. \tag{2.18}$$

We have now to distinguish two cases. If $K_{\bar{R}} \Subset \Omega$, then, using the test function (2.17) in the variational characterization, we have

$$\begin{aligned} \sigma_1(\Omega) \leqslant \frac{\displaystyle\int_{K_{\overline{R}}} |\nabla d_K(x)|^2 \, dx}{\displaystyle\int_{\partial\Omega_0} d_K^2(x) \, d\mathcal{H}^{n-1}} &= \frac{|K_{\overline{R}}|}{\overline{R}^2 P(\Omega_0)} \leqslant \frac{|\Omega|}{\overline{R}^2 n\omega_n \omega_n^{1/n} |\Omega|^{1-1/n}} \\ &= \frac{|\Omega|^{1/n}}{n\omega_n^{1/n} \left(\frac{|\Omega|}{2\omega_n} + \overline{r}^n \right)^{2/n}} = C(n, \overline{r}, |\Omega|), \end{aligned}$$

where we have used the fact that $|\nabla d_K(x)| = 1$ a.e., the classical isoperimetric inequality and (2.16).
Finally, let us consider the case when $K_{\overline{R}} \nsubseteq \Omega$. We will use the following notations: $\partial^i\Omega_0 = \partial\Omega_0 \cap K_{\overline{R}}$ and $\partial^e\Omega_0 = \partial\Omega_0 \backslash \partial^i\Omega_0$. Using as before the test function (2.17), we have

$$\begin{aligned} \sigma_1(\Omega) &\leqslant \frac{\displaystyle\int_{K_{\overline{R}} \cap \Omega} |\nabla d_K(x)|^2 dx}{\displaystyle\int_{\partial^i\Omega_0} d_K^2(x) d\mathcal{H}^{n-1} + \int_{\partial^e\Omega_0} \bar{R}^2 d\mathcal{H}^{n-1}} \leqslant \frac{|K_{\bar{R}} \cap \Omega|}{\bar{R}^2 |\partial^e\Omega_0|} \\ &\leqslant \frac{2|\Omega|}{\bar{R}^2 n\omega_n^{1/n} |\Omega|^{1-1/n}} = 2C(n, \bar{r}, |\Omega|), \end{aligned} \tag{2.19}$$

where we have used the relative isoperimetric inequality (2.13).

2.2.2 Volume constraint on the spherical shells

We remark that, when a volume constraint for Ω holds, then the upper bound is still finite, when $R_1 \to 0$. On the other hand, when $R_1 \to \infty$, the first eigenvalue cannot be upper bounded. This, together with other examples that we are going to illustrate, motivates the study the optimality of σ_1 when another constraint holds, besides the volume one.

Let us consider the spherical shell A_{R_1,R_2} with the volume constraint:

$$|A_{R_1,R_2}| = \omega_n(R_2^n - R_1^n) = \omega.$$

We show that both in bidimensional case and in higher dimension, σ_1 is not upper bounded in the class of spherical shells of fixed volume.
Let $n = 2$, then $R_2 = \left(R_1^2 + \frac{\omega}{\pi}\right)^{\frac{1}{2}}$ and, by (2.7), we have

$$\sigma_1(A_{R_1,R_2}) = \frac{1}{\left(R_1^2 + \frac{\omega}{\pi}\right)^{\frac{1}{2}} \log\left(1 + \frac{\omega}{\pi R_1^2}\right)^{\frac{1}{2}}} = \frac{2}{R_1\left(1 + \frac{\omega}{\pi R_1^2}\right)^{\frac{1}{2}} \log\left(1 + \frac{\omega}{\pi R_1^2}\right)}.$$

Hence, for R_1 big enough,

$$\sigma_1(A_{R_1,R_2}) \approx \frac{2}{R_1\left(1 + \frac{\omega}{2\pi R_1^2}\right)\frac{\omega}{\pi R_1^2}} = \frac{2\pi R_1}{\omega\left(1 + \frac{\omega}{2\pi R_1^2}\right)}$$

and so

$$\lim_{R_1 \to +\infty} \sigma_1(A_{R_1,R_2}) = +\infty.$$

Let $n \geqslant 3$, then, $R_2 = \left(R_1^n + \frac{\omega}{\omega_n}\right)^{\frac{1}{n}}$ and

$$\sigma_1(A_{R_1,R_2}) = \frac{n-2}{R_1\left(1 + \frac{\omega}{\omega_n R_1^n}\right)^{\frac{1}{n}}\left[\left(1 + \frac{\omega}{\omega_n R_1^n}\right)^{1-\frac{2}{n}} - 1\right]} =$$

$$= \frac{n-2}{R_1\left[\left(1 + \frac{\omega}{\omega_n R_1^n}\right)^{1-\frac{1}{n}} - \left(1 + \frac{\omega}{\omega_n R_1^n}\right)^{\frac{1}{n}}\right]}.$$

Again, if R_1 is big

$$\sigma_1(A_{R_1,R_2}) \approx \frac{n-2}{R_1\left[1 + \left(1 - \frac{1}{n}\right)\frac{\omega}{\omega_n R_1^n} - 1 - \frac{1}{n}\frac{\omega}{\omega_n R_1^n}\right]} = \frac{n\omega_n}{\omega} R_1^{n-1}$$

and hence again

$$\lim_{R_1 \to +\infty} \sigma_1(A_{R_1,R_2}) = +\infty. \tag{2.20}$$

Further, it is clear that, in any dimension, we have

$$\lim_{R_1 \to 0^+} \sigma_1(A_{R_1,R_2}) = 0. \tag{2.21}$$

The limiting results (2.20) and (2.21) motivate the fact that it is not enough to fix the volume to study the first eigenvalue σ_1. Indeed, when R_1 is too big, it is not possible to find an upper bound, and, on the other hand, when R_1 is too small, the eigenvalue is trivial. We remark that, in the class of sets of the form $B_{R_2}(x_0)\backslash\overline{B}_{R_1}$, with $B_{R_2}(x_0)$ being a ball containing B_{R_1}, the maximizer of σ_1 is the spherical shell (see [68]).

Spherical shell with fixed difference between radii.

It is clear now that we cannot study the shape optimization for σ_1 when only a volume constraint holds. On the other hand, it could be interesting to understand if we can study the shape optimization for double connected domains, when only one geometric quantity is fixed. Here, for example, we briefly study the behavior of the spherical shell when the distance between the radii is fixed. Let d be a positive real number such that

$$R_2 - R_1 = d,$$

so that $R_2 = R_1 + d$ and $\frac{R_2}{R_1} = 1 + \frac{d}{R_1}$.
If $n = 2$, then for R_1 big enough, we have

$$\sigma_1(A_{R_1,R_2}) = \frac{1}{(R_1+d)\log\left(1+\frac{d}{R_1}\right)} \approx \frac{R_1}{R_1 d + d^2}$$

and, hence,

$$\lim_{R_1\to+\infty} \sigma_1(A_{R_1,R_2}) = \frac{1}{d}.$$

If $n \geqslant 3$, we have

$$\begin{aligned}\sigma_1(A_{R_1,R_2}) &= \frac{n-2}{(R_1+d)\left[\left(1+\frac{d}{R_1}\right)^{n-2} - 1\right]} \\ &\approx \frac{n-2}{(R_1+d)\left[1+(n-2)\frac{d}{R_1} - 1\right]} = \frac{R_1}{R_1 d + d^2}\end{aligned}$$

and, hence,

$$\lim_{R_1\to+\infty} \sigma_1(A_{R_1,R_2}) = \frac{1}{d}.$$

Furthermore, in any dimensions, we have

$$\lim_{R_1\to 0^+} \sigma_1(A_{R_1,R_2}) = 0.$$

The case of R_1 small is again trivial. On the other hand, σ_1 is upper bounded for any value of R_2 by the reciprocal of the difference between the radii d. The fact that an uniform upper bound holds for spherical shells when only the difference between the radii is fixed, suggests that it could be interesting to study the shapes minizing σ_1 in the class of double connected sets when only the width is fixed.

2.3 Steklov-Dirichlet type problem on a perforated domain for nearly spherical sets

In this Section we prove that the spherical shell is a local maximizer for the first eigenvalue of (2.1) among nearly spherical sets with fixed volume, containing B_{R_1}, for a fixed value $R_1 > 0$.

2.3.1 Main result

We recall that, if Ω_0 is a nearly spherical set, as in Definition 1.2, Chapter 1, its volume is given by

$$|\Omega_0| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (1+v(\xi))^n \; d\mathcal{H}^{n-1}.$$

The class of nearly spherical sets has a peculiar importance in shape optimization theory, in particular for stability results for spectral inequalities. We are considering sets $\Omega = \Omega_0 \backslash \overline{B}_{R_1}$ beloging to $\mathcal{A}_{R_1}$ with $R_1 > 0$, with Ω_0 nearly spherical, and the main result is the following.

Theorem 2.5. *Let $n \geqslant 2$, $R_1 > 0$, $\omega > 0$ and let $R_2 > R_1$ be such that $|A_{R_1,R_2}| = \omega$. There exists $\varepsilon = \varepsilon(n, R_1, \omega) > 0$ such that, for any $\Omega = \Omega_0 \backslash \overline{B_{R_1}}$ belonging to $\mathcal{A}_{R_1}$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $|\Omega| = \omega$, then*

$$\sigma_1(\Omega) \leqslant \sigma_1(A_{R_1,R_2}). \tag{2.22}$$

Moreover the equality in (2.22) *holds if and only if Ω is a spherical shell.*

Let us remark that, in order to have $B_{R_1} \Subset \Omega_0$, we need to require that $\varepsilon \leqslant 1 - R_1/R_2$ to verify that $|y| \geqslant R_1$, that is $R_2(1 + v(\xi)) \geqslant R_1$. Moreover, we observe that, since all the quantities involved are translation invariant, the result in Theorem 2.5 holds also among nearly spherical sets with fixed volume and containing a fixed internal ball.

Recalling the explicit expression (2.6) of the first eigenfunction w on the spherical shell A_{R_1,R_2}, we define the *weighted volume* and the *weighted perimeter* as:

$$\overline{V}(\Omega) := \int_\Omega |\nabla z|^2 \, dx,$$
$$\overline{P}(\Omega) := \int_{\partial\Omega_0} z^2 \, dx.$$

Furthermore, to simplify the notations, we set, for $n = 2$,

$$h_{R_2}(t) = (\ln(tR_2) - \ln R_1)^2, \tag{2.23}$$

$$f_{R_2}(t) = \frac{h'_{R_2}(t)}{2R_2} = \frac{\sqrt{h_{R_2}(t)}}{(tR_2)} \tag{2.24}$$

and for $n \geqslant 3$

$$h_{R_2}(t) = \left(\frac{1}{R_1^{n-2}} - \frac{1}{(tR_2)^{n-2}}\right)^2, \tag{2.25}$$

$$f_{R_2}(t) = \frac{h'_{R_2}(t)}{2R_2} = \frac{n-2}{(tR_2)^{n-1}}\left(\frac{1}{R_1^{n-2}} - \frac{1}{(tR_2)^{n-2}}\right), \tag{2.26}$$

where R_2 is the radius of the ball with the same volume of Ω_0 and $t \geqslant \frac{R_1}{R}{}_2$. Now, we write the Raylegh quotient (2.3) using the parametrization in (1.6).

Lemma 2.6. *Let $n \geqslant 2$, $R_1 > 0$, $\omega > 0$ and let $R_2 > R_1$ be such that $|A_{R_1,R_2}| = \omega$. For any $0 < \varepsilon < 1 - R_1/R_2$ and for any $\Omega = \Omega_0 \backslash \overline{B_{R_1}}$ belonging to $\mathcal{A}_{R_1}$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $|\Omega| = \omega$, then*

$$\sigma_1(\Omega) \leqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} = \frac{\displaystyle\int_{\mathbb{S}^{n-1}} f_{R_2}(1+v(\xi))(1+v(\xi))^{n-1} \, d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{S}^{n-1}} h_{R_2}(1+v(\xi))(1+v(\xi))^{n-1}\sqrt{1+\frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}} \, d\mathcal{H}^{n-1}}. \tag{2.27}$$

Moreover if $\Omega = A_{R_1,R_2}$*, then equality holds in* (2.27) *and* $\sigma_1(A_{R_1,R_2}) = \dfrac{f_{R_2}(1)}{h_{R_2}(1)}$.

Proof. From the variational characterization (2.2) of $\sigma_1(\Omega)$, we have

$$\sigma_1(\Omega) \leqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} = \frac{\displaystyle\int_\Omega |\nabla z|^2\,dx}{\displaystyle\int_{\partial\Omega_0} z^2\,d\mathcal{H}^{n-1}} = \frac{\displaystyle\int_{\partial\Omega_0} \frac{\partial z}{\partial\nu} z\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} z^2\,d\mathcal{H}^{n-1}}.$$

The conclusion follows using the change of variables in (1.6). □

We recall the following result, whose proof can be found in [69].

Lemma 2.7. *Let* $n \geqslant 2$ *and* $R_2 > 0$*. There exists a constant* $C = C(n) > 0$ *such that for any* $0 < \varepsilon < 1$ *and for any* v *parametrizing a nearly spherical set* Ω_0 *such that* $||v||_{W^{1,\infty}} \leqslant \varepsilon$ *and* $|\Omega_0| = |B_{R_2}|$*, then*

$$\left|(1+v)^{n-1} - \left(1 + (n-1)v + (n-1)(n-2)\frac{v^2}{2}\right)\right| \leqslant C\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1},$$

$$1 + \frac{|\nabla v|^2}{2} - \sqrt{1 + \frac{|\nabla v|^2}{(1+v)^2}} \leqslant C\varepsilon\left(v^2 + |\nabla v|^2\right) \ \text{ on } \mathbb{S}^{n-1},$$

$$\left|\int_{\mathbb{S}^{n-1}} v(\xi)\,d\mathcal{H}^{n-1} + \frac{n-1}{2}\int_{\mathbb{S}^{n-1}} v^2(\xi)\,d\mathcal{H}^{n-1}\right| \leqslant C\varepsilon\|v\|^2_{L^2}.$$

As a consequence of the analyticity of h_{R_2} and f_{R_2}, defined in (2.23)-(2.24)-(2.25)-(2.26), the following Lemma holds.

Lemma 2.8. *Let* $n \geqslant 2$ *and* $0 < R_1 < R_2$*. There exists* $K = K(n, R_1, R_2) > 0$ *such that for any* $0 < \varepsilon < 1$ *and for any* v *parametrizing a nearly spherical set* Ω_0 *such that* $||v||_{W^{1,\infty}} \leqslant \varepsilon$ *and* $|\Omega_0| = |B_{R_2}|$*, then*

$$\left|h_{R_2}(1+v) - h_{R_2}(1) - h'_{R_2}(1)v - h''_{R_2}(1)\frac{v^2}{2}\right| \leqslant K\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1},$$

$$\left|f_{R_2}(1+v) - f_{R_2}(1) - f'_{R_2}(1)v - f''_{R_2}(1)\frac{v^2}{2}\right| \leqslant K\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1}.$$

Furthermore, this Poincaré inequality holds.

Lemma 2.9. *(Poincaré inequality) Let* $n \geqslant 2$ *and* $R_2 > 0$*, then there exists a positive constant* $C = C(n)$ *such that for any* $0 < \varepsilon < 1$ *and for any function* v *parametrizing a nearly spherical set* Ω_0 *such that* $||v||_{W^{1,\infty}} \leqslant \varepsilon$ *and* $|\Omega_0| = |B_{R_2}|$*, then*

$$\|\nabla v\|^2_{L^2} \geqslant (n-1)(1 - C\varepsilon)\|v\|^2_{L^2}.$$

Proof. The function $v \in L^2(\mathbb{S}^{n-1})$ admits a harmonic expansion (see e.g. [81, Chap. 3]), in the sense that there exists a family of n-dimensional spherical harmonics $\{H_j(\xi)\}_{j\in\mathbb{N}}$ such that

$$v(\xi) = \sum_{j=0}^{+\infty} c_j H_j(\xi), \quad \xi \in \mathcal{S}^{n-1} \quad \text{with} \quad \|H_j\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

where

$$c_j = \langle v, H_j \rangle_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} v(\xi) H_j(\xi) d\mathcal{H}^{n-1}.$$

and H_j satisfying

$$\Delta_{\mathbb{S}^{n-1}} H_j = j(j+n-2) H_j, \quad \forall\, j \in \mathbb{N},$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator. Furthermore the following identities hold true

$$||v||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{j=0}^{\infty} c_j^2, \tag{2.28}$$

$$||\nabla v||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{j=1}^{\infty} j(j+n-2) c_j^2. \tag{2.29}$$

Since $H_0 = (n\omega_n)^{-\frac{1}{2}}$, we have

$$|c_0| = (n\omega_n)^{-\frac{1}{2}} \left| \int_{\mathbb{S}^{n-1}} v(\xi) d\mathcal{H}^{n-1} \right| \leqslant$$
$$(n\omega_n)^{-\frac{1}{2}} \left| \int_{\mathbb{S}^{n-1}} v^2(\xi) d\mathcal{H}^{n-1} \right| \left(\frac{n-1}{2} + C\varepsilon \right) = C\varepsilon \|v\|_{L^2},$$

where the constant C has been renamed. Using this estimate, by (2.28) and (2.29), we have

$$\|v\|_{L^2} = \sum_{j=0}^{\infty} c_j^2 = c_0^2 + \sum_{j=1}^{\infty} c_j^2 \leqslant C\varepsilon \|v\|^2_{L^2} + \sum_{j=1}^{\infty} c_j^2,$$

and

$$\|\nabla v\|_{L^2} = \sum_{j=1}^{\infty} j(j+n-2) c_j^2 \geqslant (n-1) \sum_{j=1}^{\infty} c_j^2 \geqslant (n-1)(1-C\varepsilon)\|v\|^2_{L^2},$$

which concludes the proof. □

Now we give a key estimate for the main Theorem.

Proposition 2.10. *Let $n \geqslant 2$, $R_1 > 0$, $\omega > 0$ and let $R_2 > R_1$ be such that $|A_{R_1,R_2}| = \omega$. There exist two positive constants $K > 0$ and $0 \leqslant \varepsilon_0 < 1 - R_1/R_2$, depending on n, R_1 and ω only, such that for any $0 < \varepsilon < \varepsilon_0$, for any $\Omega = \Omega_0 \backslash \overline{B_{R_1}}$ belonging to $\mathcal{A}_{R_1}$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $|\Omega| = \omega$, then*

$$\frac{\overline{V}(\Omega^\sharp)\overline{P}(\Omega) - \overline{P}(\Omega^\sharp)\overline{V}(\Omega)}{n\omega_n} =$$
$$= f_{R_2}(1) \int_{\mathbb{S}^{n-1}} h_{R_2}(1+v(\xi))(1+v(\xi))^{n-1} \sqrt{1 + \frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}}\, d\mathcal{H}^{n-1} \tag{2.30}$$
$$-h_{R_2}(1) \int_{\mathbb{S}^{n-1}} f_{R_2}(1+v(\xi))(1+v(\xi))^{n-1}\, d\mathcal{H}^{n-1} \geqslant K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}.$$

Proof. Using Lemmata 2.7, 2.8, 2.9, we have

$$
\begin{aligned}
&f_{R_2}(1)\int_{\mathbb{S}^{n-1}} h_{R_2}(1+v(\xi))(1+v(\xi))^{n-1}\sqrt{1+\frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}}\,d\mathcal{H}^{n-1}\\
&\quad - h_{R_2}(1)\int_{\mathbb{S}^{n-1}} f_{R_2}(1+v(\xi))(1+v(\xi))^{n-1}\,d\mathcal{H}^{n-1}\\
&\geqslant \int_{\mathbb{S}^{n-1}} v\left(f_{R_2}(1)h'_{R_2}(1)-f'_{R_2}(1)h_{R_2}(1)\right)\,d\mathcal{H}^{n-1}\\
&\quad+\int_{\mathbb{S}^{n-1}}\frac{v^2}{2}[f_{R_2}(1)h''_{R_2}(1)-f''_{R_2}(1)h_{R_2}(1)+2(n-1)(f_{R_2}(1)h'_{R_2}(1)-f'_{R_2}(1)h_{R_2}(1))]\,d\mathcal{H}^{n-1}\\
&\quad+\int_{\mathbb{S}^{n-1}} f_{R_2}(1)h_{R_2}(1)\frac{|\nabla v|^2}{2}d\mathcal{H}^{n-1}-\varepsilon K_1\|\nabla v\|^2_{L^2},
\end{aligned}
\tag{2.31}
$$

where K_1 is a positive constant. Let us set

$$
\begin{aligned}
Q_1(t) &:= f_{R_2}(t)h'_{R_2}(t)-f'_{R_2}(t)h_{R_2}(t),\\
Q_2(t) &:= f_{R_2}(t)h''_{R_2}(t)-f''_{R_2}(t)h_{R_2}(t),\\
Q_3(t) &:= f_{R_2}(t)h_{R_2}(t),
\end{aligned}
$$

In order to show (2.30), we need to prove

1. $Q_1(1)>0$,
2. $Q_3(1)>0$,
3. $(n-1)\,[Q_1(1)+Q_3(1)]+Q_2(1)>0$.

Indeed, when (1), (2), (3) hold, then, by using Lemmata 2.7 and 2.9, the last term in (2.31) can be estimated as

$$
\begin{aligned}
&Q_1(1)\int_{\mathbb{S}^{n-1}} v\,d\mathcal{H}^{n-1}+(2(n-1)Q_1(1)+Q_2(1))\int_{\mathbb{S}^{n-1}}\frac{v^2}{2}\,d\mathcal{H}^{n-1}\\
&\qquad\qquad+Q_3(1)\int_{\mathbb{S}^{n-1}}\frac{|\nabla v|^2}{2}d\mathcal{H}^{n-1}-\varepsilon K_1\|\nabla v\|^2_{L^2}\\
&\geqslant -\frac{n-1}{2}Q_1(1)\int_{\mathbb{S}^{n-1}} v^2\,d\mathcal{H}^{n-1}-\varepsilon K_2\|v\|^2_{L^2}+\left((n-1)Q_1(1)+\frac{Q_2(1)}{2}\right)\int_{\mathbb{S}^{n-1}} v^2\,d\mathcal{H}^{n-1}\\
&\qquad\qquad+\frac{n-1}{2}Q_3(1)\int_{\mathbb{S}^{n-1}} v^2-\varepsilon K_3\|v\|^2_{L^2}-\varepsilon K_1\|\nabla v\|^2_{L^2}\\
&=\frac{1}{2}\left\{(n-1)[Q_1(1)+Q_3(1)]+Q_2(1)\right\}\|v\|^2_{L^2}\\
&\qquad\qquad-\varepsilon K_2\|v\|^2_{L^2}-\varepsilon K_3\|v\|^2_{L^2}-\varepsilon K_1\|\nabla v\|^2_{L^2}\\
&\geqslant K\|v\|^2_{L^2}-\varepsilon K_4\|v\|^2_{W^{1,2}(\mathbb{S}^{n-1})},
\end{aligned}
$$

where we denoted $K=\frac{1}{2}\left\{(n-1)\,[Q_1(1)+Q_3(1)]+Q_2(1)\right\}>0$ and $K_4=\max\{K_1,K_2,K_3\}$. The proof concludes by choosing ε small enough.

It remains to prove (1), (2), (3) by distinguishing the bidimensional from the higher dimensional case. We note that

$$
Q_1(t)=f^2_{R_2}(t)\left[\frac{h_{R_2}(t)}{f_{R_2}(t)}\right]'=2Rf^2_{R_2}(t)\left[\frac{h_{R_2}(t)}{h'_{R_2}(t)}\right]',
\tag{2.32}
$$

and

$$Q_2(t) = Q_1'(t) = \left[f_{R_2}^2(t)\right]' \left[\frac{h_{R_2}(t)}{f_{R_2}(t)}\right]' + f_{R_2}^2(t) \left[\frac{h_{R_2}(t)}{f_{R_2}(t)}\right]''. \tag{2.33}$$

Case 1. Let be $n = 2$. We observe that

$$\frac{h_{R_2}(t)}{f_{R_2}(t)} = R_2 t(\ln(tR_2) - \ln R_1),$$

is positive and strictly increasing, since it is a product of two strictly increasing positive functions. Hence $Q_1(t) > 0$ and in particular

$$Q_1(1) = \frac{h_{R_2}(1)}{R_2}\left(\sqrt{h_{R_2}(1)} + 1\right) > 0.$$

Moreover, it is clear that

$$Q_3(1) = \frac{h_{R_2}(1)\sqrt{h_{R_2}(1)}}{R_2} > 0.$$

Let us now calculate all the terms in (2.33) and evaluate them for $t = 1$. We have

$$\begin{aligned}
\left[\frac{h_{R_2}(t)}{f_{R_2}(t)}\right]'_{t=1} &= R_2\left(\sqrt{h_{R_2}(t)} + 1\right)_{t=1} = R_2\left(\sqrt{h_{R_2}(1)} + 1\right) > 0,\\
\left[\frac{h_{R_2}(t)}{f_{R_2}(t)}\right]''_{t=1} &= \left(\frac{R_2}{t}\right)_{t=1} = R_2 > 0
\end{aligned}$$

and

$$\begin{aligned}
f_{R_2}^2(1) &= \frac{h_{R_2}(1)}{R_2^2} > 0,\\
\left[f_{R_2}^2(t)\right]'_{t=1} &= \left[\frac{2R_2}{(tR_2)^3}\left(\sqrt{h_{R_2}(t)} - h_{R_2}(t)\right)\right] = \frac{2}{R_2^2}\left(\sqrt{h_{R_2}(1)} - h_{R_2}(1)\right).
\end{aligned}$$

Summing up, estimate (3) follows by

$$\begin{aligned}
Q_1(1)+Q_3(1) + Q_2(1) = {}& \frac{h_{R_2}(1)\sqrt{h_{R_2}(1)}}{R_2} + \frac{h_{R_2}(1)}{R_2} + \frac{h_{R_2}(1)\sqrt{h_{R_2}(1)}}{R_2} +\\
& 2\frac{\sqrt{h_{R_2}(1)}}{R_2} - 2\frac{h_{R_2}(1)\sqrt{h_{R_2}(1)}}{R_2} + \frac{h_{R_2}(1)}{R_2} = \frac{2}{R_2}(h_{R_2}(1) + \sqrt{h_{R_2}(1)}) > 0.
\end{aligned}$$

Case 2. For $n \geqslant 3$, from (2.32) we have

$$\frac{h_{R_2}(t)}{h'_{R_2}(t)} = \frac{(tR_2)^{n-1}}{2(n-2)R_2}\left(\frac{1}{R_1^{n-2}} - \frac{1}{(tR_2)^{n-2}}\right),$$

that is a strictly increasing function, since it is product of two strictly increasing and positive functions. Hence $Q_1(t) > 0$ and, in particular

$$Q_1(1) = \frac{(n-1)(n-2)}{R_2^{n-1}} h_{R_2}(1)\sqrt{h_{R_2}(1)} + \frac{2(n-2)^2}{R_2^{2n-3}} h_{R_2}(1) > 0.$$

Moreover, it is easily seen that

$$Q_3(1) = \frac{n-2}{R_2^{n-1}} h_{R_2}(1)\sqrt{h_{R_2}(1)} > 0.$$

Eventually, we have

$$\begin{aligned} Q_2(1) = & \frac{(n-2)^3}{R_2^{3n-3}}\sqrt{h_{R_2}(1)} - \frac{(n-1)^2(n-2)}{R_2^{n-1}} h_{R_2}(1)\sqrt{h_{R_2}(1)} \\ & + \frac{(n-1)(n-2)^2}{R_2^{n}} h_{R_2}(1)\sqrt{h_{R_2}(1)} + \frac{(n-1)(n-2)^2}{R_2^{2n-2}} h_{R_2}(1), \end{aligned}$$

and therefore, it follows that $(n-1)\left[Q_1(1) + Q_3(1)\right] + Q_2(1) > 0$. □

We use the previous result to give a stability result in a quantitative form.

Theorem 2.11. *Let $n \geqslant 2$, $R_1 > 0$, $\omega > 0$ and let $R_2 > R_1$ be such that $|A_{R_1,R_2}| = \omega$. There exist two positive constants $K > 0$ and $0 \leqslant \varepsilon_0 < 1 - R_1/R_2$, depending on n, R_1 and ω only, such that for any $0 < \varepsilon < \varepsilon_0$, for any $\Omega = \Omega_0 \backslash \overline{B_{R_1}}$ belonging to $\mathcal{A}_{R_1}$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$, and $|\Omega| = \omega$, then*

$$\sigma_1(A_{R_1,R_2}) \geqslant \sigma_1(\Omega)\left(1 + K(n, R_1, \omega)\int_{\mathbb{S}^{n-1}} v^2(\xi)\, d\mathcal{H}^{n-1}\right).$$

Proof. From Proposition 2.10 we know that there exists $K > 0$ such that

$$\overline{P}(A_{R_1,R_2})\overline{P}(\Omega)\left(\frac{\overline{V}(A_{R_1,R_2})}{\overline{P}(A_{R_1,R_2})} - \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\right) \geqslant n\omega_n K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}.$$

Then, we have

$$\begin{aligned} \sigma_1(A_{R_1,R_2}) &= \frac{\overline{V}(A_{R_1,R_2})}{\overline{P}(A_{R_1,R_2})} \geqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} + \frac{n\omega_n K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{P(A_{R_1,R_2})P(\Omega)} \\ &= \frac{|\Omega|}{P(\Omega)}\left(1 + \frac{n\omega_n K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{P(A_{R_1,R_2})|\Omega|}\right) \\ &= \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\left(1 + \frac{K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{h_{R_2}(1)\int_{\mathcal{S}^{n-1}} f_{R_2}(1+v(\xi))(1+v(\xi))^{n-1}\, d\mathcal{H}^{n-1}}\right) \\ &\geqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\left(1 + \frac{K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{n\omega_n 2^{n-1} h_{R_2}(1) f_{R_2}(2)}\right) \geqslant \sigma_1(\Omega)\left(1 + K\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}\right), \end{aligned}$$

where the second inequality follows by the fact that $\|v\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant \varepsilon < 1$ and by the monotonicity of $f_{R_2}(\cdot)$. □

Eventually, the main result (Theorem 2.5) easily follows by Theorem 2.11. Moreover, if $\Omega = A_{R_1,R_2}$, then the function v parametrizing the outer boundary is constantly equal to zero and equality in (2.22) holds.

2.4 Steklov-Dirichlet problem in the convex case

The aim of this subsection is twofold. First, we prove the existence of a maximum for $\sigma_1(\Omega)$ in the class of sets $\Omega = \Omega_0\setminus\overline{B}_{R_1}$, where $\Omega_0 \subset \mathbb{R}^n$, $n \geqslant 2$, is an open bounded and convex set containing B_{R_1}, when R_1 and the measure of Ω are fixed. Actually, we prove this existence result also when the hole is not spherical, but it is an open, convex set $K \Subset \Omega_0$ with non-empty interior. Our second aim is to find the shape of the maximum when the hole is spherical.
In particular we prove that the spherical shell is a maximizer for a suitable class of annular sets. More precisely our main result is the following.

Theorem 2.12. *Let $R_1 > 0$, $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded and convex set, $n \geqslant 2$, such that $B_{R_1} \Subset \Omega_0 \subseteq B_{\bar{R}}$, where $B_{\bar{R}}$ is the ball centered at the origin with radius $\bar{R}$ given by*

$$\bar{R} = \begin{cases} R_1 e^{\sqrt{2}} & \text{if } n = 2 \\ R_1 \left[\frac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1} \right]^{\frac{1}{n-2}} & \text{if } n \geqslant 3. \end{cases} \tag{2.34}$$

Then, denoting by $\Omega = \Omega_0\setminus\overline{B}_{R_1}$, the following inequality holds

$$\sigma_1(\Omega) \leqslant \sigma_1(A_{R_1,R_2}), \tag{2.35}$$

where A_{R_1,R_2} is the spherical shell of radii $R_1 < R_2$ having the same volume as Ω.

We observe that the convexity assumption is not just technical but it is natural when dealing with Steklov-Dirichlet eigenvalues (see [66]).

2.4.1 Upper and lower bounds for $\sigma_1(\Omega)$ and existence result

In this Section we prove an upper and lower bound for $\sigma_1(\Omega)$ in terms of R_i and R_e, that are the minimal and maximal distance from the origin of the outer boundary as defined in (1.12). Then, we prove an existence results for a maximizer among convex sets with fixed inner ball and fixed volume and we also generalize it in the case of a suitable not spherical hole.

Estimates in terms of R_i and R_e

The proof follows the same idea used in [95] for the planar case and in [73, 137] for any dimension to obtain a lower bounds for the first Steklov Laplacian eigenvalue.

Theorem 2.13. *Let $R_1 > 0$ and $\Omega_0 \subset \mathbb{R}^n$ be an open bounded connected set with Lipschitz boundary such that $B_{R_1} \Subset \Omega_0$ and let $\Omega = \Omega_0\setminus\overline{B}_{R_1}$.*

$$\frac{1}{\max_{\mathbb{S}^{n-1}}\left(\sqrt{1+\frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right)} \left(\frac{1}{R_M}\right)^{n-1} \sigma\left(A_{R_1,R_i}\right) \leqslant \sigma_1(\Omega) \leqslant \left(\frac{1}{R_m}\right)^{n-1} \sigma(A_{R_1,R_e}), \tag{2.36}$$

where R_m and R_M are defined in (1.12), ρ_0 is the radial function of Ω_0 defined in (1.9), A_{R_1,R_i} is the spherical shell with radii R_1 and R_i.
Moreover, the equality case holds if and only if Ω is a ball B_R centered at the origin of radius $R > 0$.

Proof. Let $u \in H^1_{\partial B_{R_1}}(\Omega)$. By using spherical coordinates and the notation introduced in Section 2

$$\partial\Omega_0 = \{x\,\rho_0(x), x \in \mathbb{S}^{n-1}\}.$$

In this case we have

$$\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} u^2 \sqrt{1 + \left(\frac{|\nabla_\tau \rho_0(x)|}{\rho_0(x)}\right)^2} (\rho_0(x))^{n-1} \, d\mathcal{H}^{n-1}. \tag{2.37}$$

Then we get

$$(R_m)^{n-1} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \leqslant \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} \leqslant (R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left(\sqrt{1 + \frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right) \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1}. \tag{2.38}$$

We can parametrize Ω as

$$\Omega = \{s \in \mathbb{R}^n \colon s = x\,r,\ x \in \mathbb{S}^{n-1},\ R_1 \leqslant r \leqslant \rho_0(x)\}$$

by using spherical coordinates, where we denote by $R(y) = \rho_0(x(y))$, where $x\colon y \in U \subset \mathbb{R}^{n-1} \to x(y) \in \mathbb{S}^{n-1}$, is a standard parametrization of the boundary of the unit ball in $\mathbb{R}^n$. Then we get

$$\int_\Omega |\nabla u|^2 \, ds = \int_U \int_{R_1}^{R(y)} \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\tau u|^2\right\} r^{n-1}\sqrt{\tilde{g}}\, dr\, dy, \tag{2.39}$$

where $\sqrt{\tilde{g}}$ is the determinant of the matrix $\tilde{g}_{ij}$, that is the standard metric on $\mathbb{S}^{n-1}$ and $\nabla_\tau u$ is the component of ∇u tangential to $\mathbb{S}^{n-1}$. Then we get

$$\int_U \int_{R_1}^{R_m} \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\tau u|^2\right\} r^{n-1}\sqrt{\tilde{g}}\, dr\, dy, \leqslant \int_\Omega |\nabla u|^2 \, ds \leqslant$$
$$\leqslant \int_U \int_{R_1}^{R_M} \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\tau u|^2\right\} r^{n-1}\sqrt{\tilde{g}}\, dr\, dy, \tag{2.40}$$

Using (2.79) and (2.81), we get

$$\frac{\displaystyle\int_U \int_{R_1}^{R_m} \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\tau u|^2\right\} r^{n-1}\sqrt{\tilde{g}}\, dr\, dy,}{\displaystyle(R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left(\sqrt{1 + \frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right) \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1}} \leqslant \frac{\displaystyle\int_\Omega |\nabla u|^2 \, ds}{\displaystyle\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1}} \leqslant$$
$$\leqslant \frac{\displaystyle\int_U \int_{R_1}^{R_M} \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\tau u|^2\right\} r^{n-1}\sqrt{\tilde{g}}\, dr\, dy,}{\displaystyle(R_m)^{n-1} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1}}. \tag{2.41}$$

The left hand side of (2.77) follows by choosing as u the first positive eigenfunction corresponding to $\sigma_1(\Omega)$, meanwhile the right hand side follows by choosing as u the first positive eigenfunction corresponding to $\sigma_1(A_{R_1,R_M})$.
Finally, we stress that the equality case implies that all the inequalities become equalities. Then we have that $\nabla_\tau \rho_0 = 0$ and $\rho_0(x) = R$, with $R > R_1$ constant. $\square$

Remark 2.14. We observe that the lower bound in (2.77) gives that $\sigma_1(\Omega) > 0$ being $R_1 > 0$ fixed. Moreover, (2.77) also implies a continuity results: $\sigma_1(\Omega) \to 0$ as $R_1 \to 0$. It is worth noticing that the estimate (2.77) also holds when Ω_0 is starshaped.

The existence result

Inequality (2.9) ensures that the Steklov-Dirichlet eigenvalue $\sigma_1(\Omega)$, defined in (2.2), is bounded from above if the volume of Ω is fixed. In this section we prove the existence of a maximizer among convex sets with fixed internal ball and fixed volume. Let $\omega > 0$ and $R_1 > 0$ be fixed, then by $\mathcal{A}_{R_1}(\omega)$ we will denote the class of convex sets having measure ω and containing the ball B_{R_1}, that is

$$\mathcal{A}_{R_1}(\omega) := \left\{D = K\backslash\overline{B}_{R_1},\ \ K \subseteq \mathbb{R}^n \text{ open, bounded, convex} : \ B_{R_1} \Subset K,\ |D| = \omega\right\}.$$

The main theorem of this section is the following existence result.

Theorem 2.15. *Let $\omega > 0$ and $R_1 > 0$ be fixed. There exists a set $E \in \mathcal{A}_{R_1}(\omega)$, such that*

$$\max_{D\in\mathcal{A}_{R_1}(\omega)} \sigma_1(D) = \sigma_1(E).$$

Proof. The upper bound (2.9) implies that there exists $M > 0$ such that

$$\sup_{D\in\mathcal{A}_{R_1}(\omega)} \sigma_1(D) = M < +\infty.$$

Hence, there exists a sequence $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}_{R_1}(\omega)$ such that

$$\lim_{k\to\infty} \sigma_1(E_k) = M.$$

In order to show the desired result, we need to prove the existence of a set $E \in \mathcal{A}_{R_1}(\omega)$ such that $E_k \xrightarrow{\mathcal{H}} E$ with $\sigma_1(E) = M$.

Firstly we prove that, up to a subsequence, $\{E_k\}_{k\in\mathbb{N}}$ converges to a certain $E \in \mathcal{A}_{R_1}(\omega)$ in the Hausdorff metric.

Being $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}_{R_1}(\omega)$ then, for every $k \in \mathbb{N}$ there exists a convex set $E_{0,k}$, such that $B_{R_1} \Subset E_{0,k}$,

$$E_k = E_{0,k}\backslash\overline{B}_{R_1}$$

and

$$\omega_0 := |E_{0,k}| = \omega + \omega_n R_1^n.$$

By the Blaschke selection Theorem and the continuity of the volume functional with respect to the Hausdorff measure (see [122] as a reference), it is enough to show that $\{E_{0,k}\}_{k\in\mathbb{N}}$ is equibounded.

We proceed by contradiction assuming that

$$\lim_{k\to+\infty} \operatorname{diam}(E_{0,k}) = +\infty. \tag{2.42}$$

Inequality in Proposition 1.12 gives

$$R_{E_{0,k}} \leqslant \frac{n|E_{0,k}|}{P(E_{0,k})}, \tag{2.43}$$

where $R_{E_{0,k}}$ is the inradius of $E_{0,k}$ defined in (1.15).

The assumption (2.42) and the inequality (1.23) imply that the right-hand side in (2.43) tends to 0 as $k \to +\infty$, being $|E_{0,k}|$ fixed. Therefore, by (2.43), we have

$$\lim_{k\to+\infty} R_{E_{0,k}} = 0,$$

which is in contradiction with

$$0 < R_1 < R_{E_{0,k}}.$$

Hence, the equiboundeness is proved and then $\{E_k\}_{k\in\mathbb{N}}$ converges up to a subsequence to a set $E \in \mathcal{A}_{R_1}(\omega)$ in the Hausdorff metric. Hence, by the definition of $\mathcal{A}_{R_1}(\omega)$, there exists an open bounded convex set E_0 such that $E = E_0 \backslash \overline{B}_{R_1}$.

In order to complete the proof, we will prove that

$$M = \lim_k \sigma_1(E_k) \leqslant \sigma_1(E). \tag{2.44}$$

Let $u \in H^1_{\partial B_{R_1}}(E)$ be the first positive eigenfunction associated to $\sigma_1(E)$, such that

$$\int_{\partial E_0} u^2 \, d\mathcal{H}^{n-1} = 1.$$

Hence, we have

$$\sigma_1(E) = \int_E |\nabla u|^2 \, dx.$$

By the extension theorem (see for instance [35, 126] for Lipschitz domains), we can extend u in $\mathbb{R}^n$ obtaining a function $\tilde{u} \in H^1_{\partial B_{R_1}}(\mathbb{R}^n)$ such that $\tilde{u} = u$, a.e. in E, and

$$\|\tilde{u}\|_{H^1_{\partial B_{R_1}}(\mathbb{R}^n)} \leqslant c(n)\|u\|_{H^1_{\partial B_{R_1}}(E)},$$

for some positive constant $c = c(n)$. For every $k \in \mathbb{N}$ we define u_k as the restriction of $\tilde{u}$ in E_k. Using u_k as a test function for $\sigma_1(E_k)$, we have

$$\sigma_1(E_k) \leqslant \frac{\displaystyle\int_{E_k} |\nabla \tilde{u}|^2 \, dx}{\displaystyle\int_{\partial E_{0,k}} \tilde{u}^2 \, d\mathcal{H}^{n-1}}. \tag{2.45}$$

In order to get (2.44), we prove that the right-hand side in (2.45) converges to $\sigma_1(E)$. We observe that

$$\int_{E_k} |\nabla \tilde{u}|^2 \, dx - \int_E |\nabla \tilde{u}|^2 \, dx = \int_{\mathbb{R}^n} (\chi_{E_k} - \chi_E) \, |\nabla \tilde{u}|^2 \, dx \to 0, \tag{2.46}$$

since $E_k \to E$ in the Hausdorff metric and by the dominated convergence theorem.

In order to conclude the proof we have to prove

$$\int_{\partial E_{0,k}} \tilde{u}^2 \, d\mathcal{H}^{n-1} \to \int_{\partial E_0} u^2 \, d\mathcal{H}^{n-1} = 1. \tag{2.47}$$

The equiboundedness of the sequence $\{E_{0,k}\}_{k\in\mathbb{N}}$ guarantees the existence of a ball B_R centered at the origin with radius $R > 0$ such that $E_{0,k} \subset B_R$, for every $k \in \mathbb{N}$. Extending $\tilde{u}$ to zero in B_{R_1} and by using an approximation argument, we can suppose that $\tilde{u} \in C^\infty(B_R)$. Then (2.47) follows by Theorem 1.11.

Finally, passing to the limit in (2.45), by (2.46) and (2.47), we get (2.44), that is

$$M \leqslant \sigma_1(E)$$

and, consequently, we can conclude that

$$\sigma_1(E) = M,$$

obtaining the desired claim. □

Remark 2.16. We observe that the above existence result holds even when we consider $\Omega_K = \Omega_0 \backslash K$, where K is a convex set with not empty interior strictly contained in Ω_0. Indeed, by using the upper bound (2.19), the proof can be done following line by line the one just discussed in the case of a spherical hole.

2.4.2 Proof of the main result

In this section we give the proof of the main result. The idea is to take as test function in the quotient (2.3) the eigenfunction of the spherical shell with the same measure as Ω. Before giving the proof, we need a preliminary result.

Lemma 2.17. *Let $R_1 > 0$ and let f be the function defined in $]0, +\infty[$ as*

$$f(t) = \begin{cases} \log^2\left(\frac{\sqrt{t}}{R_1}\right)\sqrt{t} & n = 2 \\ \left(\frac{1}{R_1^{n-2}} - \frac{1}{t^{\frac{n-2}{n}}}\right)^2 t^{\frac{n-1}{n}} & n \geqslant 3. \end{cases}$$

Then, f is convex for every $\alpha_-(n)R_1^n \leqslant t \leqslant \alpha_+(n)R_1^n$, where

$$\alpha_\pm(n) = \begin{cases} e^{\pm 2\sqrt{2}} & n = 2 \\ \left[\frac{(n-1)\pm(n-2)\sqrt{2(n-1)}}{n-1}\right]^{\frac{n}{n-2}} & n \geqslant 3. \end{cases}$$

Proof. Let us begin with the bidimensional case. After an easy computation one can see that

$$f''(t) = \frac{2 - \log^2(\sqrt{t}/R_1)}{4t\sqrt{t}},$$

which gives immediately the conclusion.

Now let us consider $n \geqslant 3$. After some computations the second derivative of the function is the following

$$f''(t) = t^{\frac{3}{n}-3}\left[\frac{R_1^{4-2n}}{n}\left(\frac{1}{n} - 1\right)t^{2-\frac{4}{n}} + \frac{2R_1^{2-n}}{n}\left(1 - \frac{1}{n}\right)t^{1-\frac{2}{n}} + \left(\frac{3}{n} - 2\right)\left(\frac{3}{n} - 1\right)\right].$$

If we call $y = t^{1-\frac{2}{n}}$, the previous function is non-negative if and only if

$$g(y) = \frac{R_1^{4-2n}}{n}\left(\frac{1}{n} - 1\right)y^2 + \frac{2R_1^{2-n}}{n}\left(1 - \frac{1}{n}\right)y + \left(\frac{3}{n} - 2\right)\left(\frac{3}{n} - 1\right) \geqslant 0.$$

It is not difficult to check that the zeros of $g(y)$ are

$$y_\pm = R_1^{n-2}\,\frac{n - 1 \pm (n-2)\sqrt{2(n-1)}}{n-1}.$$

Being $y_- = 0$ for $n = 3$ and $y_- < 0$ for every $n \geqslant 4$ it must be $y_- \leqslant y \leqslant y_+$, which concludes the proof. □

Now we can prove the main result.

Proof of the Theorem 2.12. Let us consider the fundamental solution w, given in (2.6), as a test function in (2.2). Then,

$$\sigma_1(\Omega) \leqslant \frac{\displaystyle\int_\Omega |\nabla w|^2\, dx}{\displaystyle\int_{\partial\Omega_0} w^2\, d\mathcal{H}^{n-1}}.$$

In order to prove the result we will show that

$$\frac{\displaystyle\int_\Omega |\nabla w|^2\, dx}{\displaystyle\int_{\partial\Omega_0} w^2\, d\mathcal{H}^{n-1}} \leqslant \frac{\displaystyle\int_{A_{R_1,R_2}} |\nabla w|^2\, dx}{\displaystyle\int_{\partial B_{R_2}} w^2\, d\mathcal{H}^{n-1}} = \sigma_1(A_{R_1,R_2}). \tag{2.48}$$

Since $|\nabla w|^2$ is a non-negative radially symmetric decreasing function for any $n \geqslant 2$, it coincides with its Schwarz symmetrization. Hence by the Hardy-Littlewood inequality [92, Th. 1.2.2], we have

$$\begin{aligned}\int_\Omega |\nabla w|^2\; dx &= \int_{\Omega_0} |\nabla w|^2\; dx - \int_{B_{R_1}} |\nabla w|^2\; dx\\ &\leqslant \int_{B_{R_2}} |\nabla w|^2\; dx - \int_{B_{R_1}} |\nabla w|^2\; dx = \int_{A_{R_1,R_2}} |\nabla w|^2\; dx.\end{aligned} \tag{2.49}$$

Hence, it remains to prove the following inequality

$$\int_{\partial\Omega_0} w^2\, d\mathcal{H}^{n-1} \geqslant \int_{\partial B_{R_2}} w^2\, d\mathcal{H}^{n-1}. \tag{2.50}$$

Let ρ_0 be the radial function of Ω_0 defined in (1.9). By (1.11), $\partial\Omega_0$ can be represented as follows

$$\partial\Omega_0 = \{x\,\rho_0(x), x \in \mathbb{S}^{n-1}\},$$

with $R_1 < \rho_0(\theta) \leqslant \bar{R}$ and $\bar{R}$ defined in (2.34).
Firstly, let us consider the case $n = 2$. If we denote by $z(\theta) = R^2(\theta) = \rho_0^2(x(\theta))$, being $|\Omega_0| = |B_{R_2}|$, it holds

$$R_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} z(\theta) d\theta}. \tag{2.51}$$

Moreover, we get

$$\begin{aligned}\int_{\partial\Omega_0} w^2\, ds &= \int_{\partial\Omega_0} (\log(|x|) - \log R_1)^2\; ds = \int_0^{2\pi} \log^2\left(\frac{R(\theta)}{R_1}\right) R(\theta)\sqrt{1+\left(\frac{R'(\theta)}{R(\theta)}\right)^2}\; d\theta\\ &\geqslant \int_0^{2\pi} \log^2\left(\frac{R(\theta)}{R_1}\right) R(\theta)\; d\theta = \int_0^{2\pi} \log^2\left(\frac{\sqrt{z(\theta)}}{R_1}\right)\sqrt{z(\theta)}\; d\theta\\ &\geqslant 2\pi \log^2\left(\frac{1}{R_1}\sqrt{\frac{\int_0^{2\pi} z(\theta) d\theta}{2\pi}}\right)\sqrt{\frac{\int_0^{2\pi} z(\theta) d\theta}{2\pi}} =\\ &= 2\pi R_2 \log^2\left(\frac{R_2}{R_1}\right) = \int_{\partial B_{R_2}} w^2\, ds,\end{aligned} \tag{2.52}$$

where, since $\rho_0(x) \leqslant \bar{R}$, the last inequality follows by Lemma 2.17 and by Jensen's inequality. This conclude the proof of (2.50) in the bidimensional case.

Now, let us consider the case $n \geqslant 3$ and we proceed in a similar way.

Moreover since

$$|\Omega_0| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_0^n(x)\, d\mathcal{H}^{n-1}$$

and being $|\Omega_0| = |B_{R_2}|$, it holds

$$R_2 = \left(\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} z(x)\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n}}, \tag{2.53}$$

where $z(x) = \rho_0^n(x)$. Then, we have

$$\begin{aligned}
\int_{\partial\Omega_0} w^2\, d\mathcal{H}^{n-1} &= \int_{\partial\Omega_0}\left(\frac{1}{R_1^{n-2}} - \frac{1}{|x|^{n-2}}\right)^2 d\mathcal{H}^{n-1}\\
&= \int_{\mathbb{S}^{n-1}}\left(\frac{1}{R_1^{n-2}} - \frac{1}{(\rho_0(x))^{n-2}}\right)^2 (\rho_0(x))^{n-1}\sqrt{1+\left(\frac{\nabla_\tau\rho_0(x)}{\rho_0(x)}\right)^2}\, d\mathcal{H}^{n-1}\\
&\geqslant \int_{\mathbb{S}^{n-1}}\left(\frac{1}{R_1^{n-2}} - \frac{1}{(z(x))^{\frac{n-2}{n}}}\right)^2 (z(x))^{\frac{n-1}{n}}\, d\mathcal{H}^{n-1}\\
&\geqslant n\omega_n\left[\frac{1}{R_1^{n-2}} - \frac{n\omega_n}{\left(\int_{\mathbb{S}^{n-1}} z(x)\, d\mathcal{H}^{n-1}\right)^{\frac{n-2}{n}}}\right]^2 \left(\frac{\int_{\mathbb{S}^{n-1}} z(x)\, d\mathcal{H}^{n-1}}{n\omega_n}\right)^{\frac{n-1}{n}}\\
&= n\omega_n\left(\frac{1}{R_1^{n-2}} - \frac{1}{R_2^{n-2}}\right)^2 R_2^{n-1} = \int_{\partial B_{R_2}} w^2\, d\mathcal{H}^{n-1}.
\end{aligned}$$

where last inequality follows by Lemma 2.17 and by Jensen's inequality, being $\rho_0(x) \leqslant \bar{R}$. This gives (2.50) for $n \geqslant 3$ and concludes the proof. □

2.4.3 Some remarks about the perimeter constraint

The estimate (2.15) states that the first Steklov-Dirichlet eigenvalue is bounded from above also when we keep the outer perimeter and the radius of the inner ball fixed. So, it is natural to investigate if there exists a set which maximizes $\sigma_1(\Omega)$ in the following class

$$\mathcal{B}_{R_1}(\kappa) := \left\{D = K\backslash\overline{B}_{R_1}\ ,\ K \subset \mathbb{R}^n,\ \text{open, convex}:\ B_{R_1} \Subset K,\ P(K) = \kappa\right\},$$

where $R_1 > 0$ and $\kappa > n\omega_n R_1^{n-1}$. Arguing as Theorem 2.15, we obtain the following existence result under a perimeter constraint.

Theorem 2.18. *Let $\kappa > n\omega_n R_1^{n-1}$ be fixed. There exists a set $\Omega \in \mathcal{B}_{R_1}(\kappa)$ such that*

$$\sup_{D\in\mathcal{B}_{R_1}(\kappa)} \sigma_1(D) = \sigma_1(\Omega).$$

Remark 2.19. We stress that inequality (2.49) continues to hold true even if we fix the perimeter of Ω_0. Indeed the isoperimetric inequality ensures that the ball B_{R_2} centered at the origin and having the same measure than Ω_0 is contained in the ball centered at the origin and having the same perimeter than Ω_0.

On the other hand we cannot prove, instead, the inequality (2.50) under the perimeter constraint in order to obtain that the spherical shell is still a maximum for $\sigma_1(\Omega)$. Indeed, if we proceed as in the proof of Theorem 2.12, for instance in the planar case, equation (2.51) has to be replaced by the following inequality:

$$2\pi R_2 = P(B_{R_2}) = P(\Omega_0) = \int_0^{2\pi} R(\theta)\sqrt{1+\left(\frac{R'(\theta)}{R(\theta)}\right)^2}\,d\theta \geqslant \int_0^{2\pi} R(\theta)\,d\theta, \tag{2.54}$$

where $R(\theta) = \rho_0(x(\theta))$ Then, in the last step of (2.52), after using Jensen's inequality, we do not obtain the first Steklov-Dirichlet eigenvalue of the spherical shell, since (2.54) is not an equality.

In support of this fact, we give the following numerical counterexample obtained by using ™WxMaxima. We consider $R_1 = 10^{-5}$ and Ω_0 an ellipse with the same perimeter as $A_{R_1,1}$. Let a and b the semi-axes of the ellipse. In order to compute the integral over the ellipse, we used the formula $P(\Omega_0) = 2\pi\sqrt{\frac{a^2+b^2}{2}}$, which is an approximation by excess for the perimeter of the ellipse. Here we have chosen $b = 1.1$. We obtain

$$D(A_{R_1,1}) \approx 832,820208 > 828,919156 \approx D(\Omega_0),$$

where $D(\Omega_0) = \int_{\partial\Omega_0} w^2 ds$ and w is the fundamental solution defined in (2.6).

This means that we cannot study separately the numerator and denominator terms to obtain inequality (2.35) under perimeter constraint.

2.5 The Steklov-Robin eigenvalue problem

Let $\Omega = \Omega_0\backslash\overline{B_r}$. Here $\Omega_0 \subset \mathbb{R}^n$, $n \geqslant 2$, is an open, bounded, connected set with Lipschitz boundary and B_r is the ball of radius $r > 0$ centered at the origin such that $B_r \Subset \Omega_0$. As we said in the introduction we deal with the following Steklov-Robin eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega_0 \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B_r, \end{cases} \tag{2.55}$$

where ν is the outer unit normal to $\partial\Omega$ and $\beta > 0$ is a positive real parameter.

Definition 2.2. A real number $\sigma(\Omega)$ and a function $u \in H^1(\Omega)$ are, respectively, called eigenvalue of (2.55) and associated eigenfunction to $\sigma(\Omega)$, if and only if

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle\,dx + \beta\int_{\partial B_r} u\varphi\,d\mathcal{H}^{n-1} = \sigma(\Omega)\int_{\partial\Omega_0} u\varphi\,d\mathcal{H}^{n-1}$$

for every $\varphi \in H^1(\Omega)$.

We study the first eigenvalue $\sigma_\beta(\Omega)$ of (2.55) defined as (see Section 3 for the details)

$$\sigma_\beta(\Omega) = \inf_{v\in H^1(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx + \beta\int_{\partial B_r} v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} v^2\,d\mathcal{H}^{n-1}}. \tag{2.56}$$

We prove that $\sigma_\beta(\Omega)$ is a minimum, it is simple, and that the corresponding eigenfunctions have constant sign.
Let us define the following quantities

$$\mu_1(\Omega) := \inf_{\substack{v\in H^1(\Omega)\setminus\{0\}\\ \int_{\partial B_r} v\, d\mathcal{H}^{n-1}=0}} \frac{\int_\Omega |\nabla v|^2\, dx}{\int_{\partial\Omega_0} v^2\, dx}, \tag{2.57}$$

and

$$q_1(\Omega) = \inf_{\substack{\Delta w=0\\ w\in H^1(\Omega)\setminus H^1_{\partial B_r}(\Omega)}} \frac{\displaystyle\int_{\partial B_r} w^2\, \mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} w^2\, \mathcal{H}^{n-1}}, \tag{2.58}$$

We observe that $\mu_1(\Omega)$ is the first nontrivial Steklov Laplacian eigenvalue in Ω. Then our result is the following

Theorem 2.20. *Let $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary and let $\Omega = \Omega_0\setminus\overline{B_r}$, where B_r is the ball centered at the origin and with radius r such that $B_r \Subset \Omega_0$. Then the following estimates hold*

$$\frac{1}{\sigma_\beta(\Omega)} \leqslant \frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}, \tag{2.59}$$

and

$$\frac{1}{\sigma_\beta(\Omega)} \leqslant \frac{1}{\sigma_D(\Omega)} + \frac{1}{q_1(\Omega)}, \tag{2.60}$$

where $\sigma_\beta(\Omega)$ is the first Steklov-Robin eigenvalue of Ω defined in (2.56)*, $\sigma_D(\Omega)$ is the first Steklov-Dirichlet eigenvalue defined in* (2.2)*, $\mu_1(\Omega)$ and $q_1(\Omega)$ are defined in* (2.57) *and* (2.58)*, respectively.*

As a consequence of the above estimates we can obtain the quoted asymptotic behaviour of $\sigma_\beta(\Omega)$ with respect to β in both case, when β either goes to zero or to infinity.

2.5.1 Existence and basic properties of $\sigma_\beta(\Omega)$

In this subsection we define and study the main properties of the first Steklov-Robin Laplacian eigenvalue in $\Omega = \Omega_0\setminus\overline{B}_r$. Let $\sigma_\beta(\Omega)$ be the following quantity

$$\sigma_\beta(\Omega) = \inf_{\substack{v\in H^1(\Omega)\\ v\not\equiv 0}} J[v], \tag{2.61}$$

where

$$J[v] = \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \beta\int_{\partial B_r} v^2\, d\mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} v^2\, d\mathcal{H}^{n-1}} \tag{2.62}$$

and β is a positive parameter.
We observe that by (2.61) we immediately get

$$\sigma_\beta(\Omega) \leqslant \sigma_D(\Omega), \tag{2.63}$$

where $\sigma_D(\Omega)$ is the first Steklov-Dirichlet eigenvalue defined in (2.2). In the next result we prove that $\sigma_\beta(\Omega)$ is the first eigenvalue of problem (2.55) and we show some basic properties of $\sigma_\beta(\Omega)$ and its corresponding eigenfunctions.

Theorem 2.21. *Let $n \geqslant 2$ and $\Omega = \Omega_0 \backslash \overline{B}_r$, where Ω_0 is an open, bounded and connected set with Lipschitz boundary in $\mathbb{R}^n$ and B_r a ball centered at the origin of radius $r > 0$ such that $B_r \Subset \Omega_0$. Then $\sigma_\beta(\Omega)$ is actually a minimum, that is*

$$\sigma_\beta(\Omega) = \min_{\substack{v \in H^1(\Omega) \\ v \not\equiv 0}} J[v], \tag{2.64}$$

where $J[v]$ is defined in (2.62). Moreover $\sigma_\beta(\Omega)$ is the first eigenvalue of (2.55), it is strictly positive and any minimizer has constant sign.

Proof. Let us notice that the Rayleigh quotient $J[w]$ defined in the previous proposition is always non-negative and 0-homogeneous. Let us consider a minimizing normalized sequence $\{u_n\}_{n \in \mathbb{N}}$ such that $\|u_n\|_{L^2(\partial\Omega_0)} = 1$, i.e. $\lim_{n\to\infty} J[u_n] = \sigma_\beta(\Omega)$. By (2.63), $J[u_n] \leqslant \sigma_D(\Omega)$, then by Friedrich's inequality (1.5), $\|u_n\|_{L^2(\Omega)}$ is uniformly bounded from above. By the compactness of the embedding $H^1(\Omega, \partial\Omega) \subset L^2(\Omega)$, there exists a subsequence, still denoted by u_n, and a function $u \in H^1(\Omega)$ with $||u||_{L^2(\partial\Omega_0)} = 1$, such that $u_n \to u$ strongly in $L^2(\Omega)$, hence also almost everywhere, and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$. Moreover, by the compactness of the trace embedding theorem (1.4), u_n converges strongly to u also in $L^2(\partial\Omega)$ and almost everywhere on $\partial\Omega$ to u. Then, by weak lower semicontinuity we have

$$\lim_{n\to+\infty} J[u_n] \geqslant J[u].$$

Hence the existence of a minimizer $u \in H^1(\Omega)$ follows.

It is obvious the fact that $\sigma_\beta(\Omega) \geqslant 0$. By contradiction let us suppose that $\sigma_\beta(\Omega) = 0$. This means that

$$\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1} = 0.$$

It follows that $\|\nabla u\|_{L^2(\Omega)}$ and $\|u\|_{L^2(\partial B_r)}$ are both zero. From the first we have that u is constant a.e. in Ω and then it must be $u \equiv 0$ in Ω, which is an absurd. Therefore $\sigma_\beta(\Omega) > 0$.
By classical arguments of Calculus of Variation it is easy to prove that (2.55) is the Euler-Lagrange equation corresponding to (2.64). Here we write down the proof for completeness. Let $u \in H^1(\Omega)$ be minimum of the Rayleigh quotient (2.62) and let $m \in \mathbb{R}$ its value, i.e. $J[u] = m$. Let us now consider the first variation of $J[\cdot]$. If $v \in H^1(\Omega)$, we define the following function

$$f(\varepsilon) = J[u + \varepsilon v].$$

It is clear that $f(0) = m$ and in particular we have that

$$\begin{aligned} \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} \cdot f'(0) = 2\Big(\int_\Omega \langle \nabla u, \nabla v \rangle \, dx + \beta \int_{\partial B_r} uv \, d\mathcal{H}^{n-1} \Big) \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} \\ - \Big(\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1} \Big) \int_{\partial\Omega_0} uv \, d\mathcal{H}^{n-1} = 0 \end{aligned}$$

if and only if

$$\frac{\int_\Omega \langle \nabla u, \nabla v \rangle \, dx + \beta \int_{\partial B_r} uv \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega_0} uv \, d\mathcal{H}^{n-1}} = \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1}} = m.$$

Since the relation written above is valid for every $v \in H^1(\Omega)$, the proposition is proved by definition of weak solution.

In particular it follows that $\sigma_\beta(\Omega)$ is the smallest eigenvalue of problem (2.55). Indeed let us suppose that v is another eigenfunction of (2.55) with corresponding eigenvalue $\tilde{\sigma}$. Then an integration by parts gives

$$\sigma_\beta(\Omega) \leqslant \frac{\int_\Omega |\nabla v|^2 \, dx + \beta \int_{\partial B_r} v^2 \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega_0} v^2 \, d\mathcal{H}^{n-1}} = \frac{\int_{\partial\Omega} \frac{\partial u}{\partial \nu} u \, d\mathcal{H}^{n-1} + \beta \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1}} = \tilde{\sigma}.$$

It only remains to show that any minimizer has constant sign. If u be an eigenfunction corresponding to $\sigma_\beta(\Omega)$, then $J[u] = J[|u|]$. This implies that $u = |u|$ on Ω and therefore $u \geqslant 0$ on Ω. By Harnack inequality (see [136, Thm 1.1]), u is strictly positive on Ω.

□

Next propositions concern the simplicity of $\sigma_\beta(\Omega)$ and sign properties of the corresponding eigenfunctions.

Proposition 2.22. *$\sigma_\beta(\Omega)$ is simple, which means that there exists a unique corresponding eigenfunction up to multiplicative constants.*

Proof. Let us now suppose that v is another eigenfunction corresponding to $\sigma_\beta(\Omega)$. Since $v > 0$ in Ω, it follows

$$\int_\Omega v \, dx \neq 0.$$

Hence there exists a positive number $\lambda > 0$, such that

$$\int_\Omega (u - \lambda v) \, dx = 0.$$

Since $u - \lambda v$ is another eigenfunction corresponding to the same eigenvalue, it is necessary that $u = \lambda v$ in Ω, which proves the simplicity. □

Proposition 2.23. *Let $n \geqslant 2$ and $\Omega = \Omega_0 \backslash \overline{B_r}$, where Ω_0 is an open, bounded and connected set with Lipschitz boundary in $\mathbb{R}^n$ and B_r a ball centered at the origin of radius $r > 0$ such that $B_r \Subset \Omega_0$. Any nonnegative function $v \in H^1(\Omega)$ that satisfies in the sense of definition* (2.2)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega_0 \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B_R, \end{cases} \tag{2.65}$$

is a first eigenfunction of (2.65)*, that is $\sigma = \sigma_\beta(\Omega)$, and $v = u$ (up to multiplicative constants), where u is the eigenfunction corresponding to the first eigenvalue $\sigma_\beta(\Omega)$.*

Proof. Since u is a positive eigenfunction corresponding to $\sigma_\beta(\Omega)$, it satisfies

$$\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1} = \sigma_\beta(\Omega) \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1}. \tag{2.66}$$

While if we consider as a test function for v, the function $u^2/(v + \varepsilon)$, for some $\varepsilon > 0$, we get

$$\int_\Omega \left[\frac{2u\langle \nabla u, \nabla v \rangle}{v + \varepsilon} - \frac{u^2 |\nabla v|^2}{(v + \varepsilon)^2} \right] dx + \beta \int_{\partial B_r} \frac{v}{v + \varepsilon} u^2 \, d\mathcal{H}^{n-1} = \sigma \int_{\partial\Omega_0} \frac{v}{v + \varepsilon} u^2 \, d\mathcal{H}^{n-1}. \tag{2.67}$$

If we subtract (2.66) by (2.67), since $v/(v+\varepsilon)<1$, we get

$$0 \leqslant \int_\Omega \left|\nabla u - \frac{u\nabla v}{v+\varepsilon}\right|^2 dx = \int_\Omega \left[|\nabla u|^2 - \frac{2u\langle \nabla u, \nabla v\rangle}{v+\varepsilon} + \frac{u^2|\nabla v|^2}{(v+\varepsilon)^2}\right] dx$$
$$\leqslant \int_{\partial\Omega_0} \left[\sigma_\beta(\Omega) - \sigma\frac{v}{v+\varepsilon}\right] u^2 \, d\mathcal{H}^{n-1}.$$

Passing to the limit as $\varepsilon \to 0$, we get

$$[\sigma_\beta(\Omega) - \sigma] \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} \geqslant 0.$$

Since $\sigma_\beta(\Omega)$ is the smallest eigenvalue, the only possibility is that $\sigma = \sigma_\beta(\Omega)$ and by the simplicity of the first eigenvalue, it must be $v = u$ up to multiplicative constants. □

2.5.2 The first Steklov-Robin eigenvalue in the spherical shell

Let us consider now $A_{r,R} = B_R \backslash \overline{B_r}$, where B_R and B_r are balls centered at the origin with radii $R > r > 0$. Let $\beta > 0$ be a positive real parameter and let us consider the Steklov-Robin eigenvalue problem for the Laplacian in the spherical shell

$$\begin{cases} \Delta u = 0 & \text{in } A_{r,R} \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B_r \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial B_R, \end{cases} \tag{2.68}$$

where ν is the outer unit normal to $\partial A_{r,R}$.
We are going to compute the solutions to problem (2.68).

Theorem 2.24. *The first Steklov-Robin eigenvalue of the problem* (2.68) *is*

$$\sigma_\beta(A_{r,R}) \begin{cases} \dfrac{1}{\dfrac{R}{\beta r} + R\log\dfrac{R}{r}} & n = 2 \\ \dfrac{n-2}{\dfrac{n-2}{\beta}\left(\dfrac{R}{r}\right)^{n-2} + R\left[\left(\dfrac{R}{r}\right)^{n-2} - 1\right]} & n \geqslant 3, \end{cases} \tag{2.69}$$

and the corresponding eigenfunctions are the following

$$u(x) = \begin{cases} \log\dfrac{|x|}{r} + \dfrac{1}{\beta r} & n = 2 \\ \dfrac{1}{r^{n-2}} - \dfrac{1}{|x|^{n-2}} + \dfrac{n-2}{\beta}\dfrac{1}{Rr^{n-2}} & n \geqslant 3. \end{cases} \tag{2.70}$$

Proof. Since the radial symmetry of the problem and the rotational invariance of the Laplacian, we look forward to a solution which is of the type $u(x) = v(|x|) = v(s)$, where $s = |x|$. Computing the Laplacian of v we get

$$v'' + \frac{n-1}{s}v' = 0,$$

which is equivalent to

$$(s^{n-1}v')' = 0.$$

So integrating twice we get

$$v(s) = \begin{cases} c_1 \log s + c_2 & n = 2 \\ \dfrac{c_1}{s^{n-2}} + c_2 & n \geqslant 3. \end{cases} \tag{2.71}$$

We are going to find the solution to (2.68) by using the boundary conditions on B_r and B_R. Let us begin by the bidimensional case. Using the boundary condition we get the following system in the unknown variables c_1 and c_2

$$\begin{cases} -\frac{c_1}{r} + \beta(c_1 \log r + c_2) = 0 \\ \frac{c_1}{R} - \sigma(c_1 \log R + c_2) = 0. \end{cases} \tag{2.72}$$

Since this is a homogeneous system, the only way not to have $c_1 = c_2 = 0$ is that

$$\det \begin{pmatrix} -1/r + \beta \log r & \beta \\ 1/R - \sigma \log R & -\sigma \end{pmatrix} = 0.$$

From this we get that

$$\sigma_\beta(A_{r,R}) = \frac{1}{\frac{R}{\beta r} + R \log \frac{R}{r}}.$$

Since this choice of σ, c_1 and c_2 must be linearly dependents. Hence if we chose $c_1 = 1$, by using the second equation in (2.72) we have that

$$c_2 = \frac{1}{\sigma R} - \log R = \frac{1}{\beta r} + \log \frac{R}{r} - \log R = \frac{1}{\beta r} - \log r.$$

Hence inserting c_1 and c_2 in (2.71), we have

$$u(x) = \log \frac{|x|}{r} + \frac{1}{\beta r}.$$

In higher dimensions the system becomes

$$\begin{cases} \frac{n-2}{r^{n-1}} c_1 + \beta(\frac{c_1}{r^{n-2}} + c_2) = 0 \\ -\frac{n-2}{R^{n-1}} c_1 - \sigma(\frac{c_1}{R^{n-2}} + c_2) = 0. \end{cases} \tag{2.73}$$

Proceding in the same way as before, we find that

$$\sigma = \sigma_\beta(A_{r,R}) = \frac{n-2}{\dfrac{n-2}{\beta}\left(\dfrac{R}{r}\right)^{n-2} + R\left[\left(\dfrac{R}{r}\right)^{n-2} - 1\right]}.$$

Hence chosing $c_1 = -1$

$$c_2 = \frac{1}{R^{n-2}} + \frac{n-2}{\sigma R^{n-1}} = \frac{1}{r^{n-2}} + \frac{n-2}{\beta} \frac{1}{R r^{n-2}},$$

and

$$u(x) = \frac{1}{r^{n-2}} - \frac{1}{|x|^{n-2}} + \frac{n-2}{\beta} \frac{1}{R r^{n-2}}.$$

Eventually, in any dimension, with these choices of the constants c_1, c_2, the corresponding eigenfunctions do not change sign and so they must be eigenfunctions corresponding to the first Steklov-Robin eigenvalue $\sigma_\beta(A_{r,R})$. $\square$

By the explicit form of $\sigma_\beta(A_{r,R})$ and the corresponding eigenfunctions in (2.69)-(2.70), we deduce the following properties when we let vary β or the radii of the spherical shell.

- $\lim_{\beta\to 0}\sigma_\beta(A_{r,R}) = 0$, and in particular
$$\lim_{\beta\to 0}\frac{\sigma_\beta(A_{r,R})}{\beta} = \frac{P(B_R)}{P(B_r)}. \tag{2.74}$$
- Recalling the explicit value of the first Steklov-Dirichlet eigenvalue of spherical shells (see [139, 75, 106]), we have
$$\lim_{\beta\to\infty}\sigma_\beta(A_{r,R}) = \sigma_D(A_{r,R}). \tag{2.75}$$
- Finally we have
$$\lim_{r\to 0}\sigma_\beta(A_{r,R}) = \lim_{R\to 0}\sigma_\beta(A_{r,R}) = 0, \tag{2.76}$$

We will see in the next section that all of these behaviours will persist in the case of a generic Ω.

2.5.3 Asymptotic estimates of $\sigma_\beta(\Omega)$ with respect to β and r

In this section we will study the behaviour of $\sigma_\beta(\Omega)$ when β and r vary.

Behaviour with respect to the inner radius

We will prove that (2.76) continues to hold for a general annular domain Ω by proving some suitable estimates in terms of the radius of the hole. Indeed let us consider the spherical shell A_{r,R_m}, where R_m is defined in (1.12), which is contained in Ω. If we choose as a test function in the variational characterization of $\sigma_\beta(\Omega)$
$$\varphi = \begin{cases} v(|x|) & \text{in } A_{r,R_m} \\ v(R_m) & \text{in } \Omega\backslash\overline{A_{r,R_m}}, \end{cases}$$
where v is the first eigenfunction in A_{r,R_m}, then
$$\sigma_\beta(\Omega) \leqslant \sigma_\beta(A_{r,R_m}).$$
As a consequence when $r\to 0$,
$$\sigma_\beta(\Omega)\to 0.$$
A natural question, now, is asking if we have a lower bound in terms of the first Steklov-Robin eigenvalue of an opportune spherical shell. In the following result we prove that for starshaped set is possible to have an optimal lower bound $\sigma_\beta(\Omega)$.

Theorem 2.25. *Let $r>0$ and $\Omega_0\subset\mathbb{R}^n$ be an open, bounded starshaped set such that $B_r \Subset \Omega_0$ and let $\Omega = \Omega_0\backslash\overline{B_r}$. Then, it holds*
$$\sigma_\beta(\Omega) \geqslant \frac{\sigma_\beta\left(A_{r,R_m}\right)}{R_M^{n-1}\max_{\mathbb{S}^{n-1}}\left(\sqrt{1+\frac{|\nabla_\tau\rho_0|^2}{\rho_0^2}}\right)}, \tag{2.77}$$
where R_m and R_M are defined in (1.12), ρ_0 is the radial function of Ω_0 defined in (1.9) and A_{r,R_m} is the spherical shell with radii r and R_m.
Moreover, the equality case holds if and only if Ω_0 is also a ball B_R centered at the origin of radius $R>0$.

Proof. We will follow an idea used in [95] for the planar case and in [73, 137] for any dimension. Let $u \in H^1(\Omega)$ By using spherical coordinates and the notation introduced in the preliminaries:

$$\partial\Omega_0 = \{x\,\rho_0(x), x \in \mathbb{S}^{n-1}\}$$

and

$$\partial B_r = \{xr,\, x \in \mathbb{S}^{n-1}\}.$$

The integrals over the boundaries $\partial\Omega_0$ and ∂B_R of u^2 become

$$\int_{\partial\Omega_0} u^2\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} u^2 \sqrt{1 + \left(\frac{|\nabla_\tau \rho_0|}{\rho_0}\right)^2}\, (\rho_0)^{n-1}\, d\mathcal{H}^{n-1},$$

$$\int_{\partial B_R} u^2\, d\mathcal{H}^{n-1} = R^{n-1} \int_{\mathbb{S}^{n-1}} u^2\, \mathcal{H}^{n-1}. \tag{2.78}$$

In particular we have

$$\int_{\partial\Omega_0} u^2\, d\mathcal{H}^{n-1} \leqslant (R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left(\sqrt{1 + \frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right) \int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1}. \tag{2.79}$$

We can parametrize

$$\Omega = \{s \in \mathbb{R}^n \colon s = x\,\tilde{r},\, x \in \mathbb{S}^{n-1},\, \tilde{r} \leqslant \tilde{r} \leqslant \rho_0(x)\},$$

by using spherical coordinates, where we denote by $R(y) = \rho_0(x(y))$, and $x\colon y \in U \subset \mathbb{R}^{n-1} \to x(y) \in \mathbb{S}^{n-1}$ is a standard parametrization of the boundary of the unit ball in $\mathbb{R}^n$. Then we get

$$\int_\Omega |\nabla u|^2\, ds = \int_U \int_r^{R(y)} \left\{\left(\frac{\partial u}{\partial \tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2}|\nabla_\tau u|^2\right\} \tilde{r}^{n-1}\sqrt{\tilde{g}}\, d\tilde{r}\, dy, \tag{2.80}$$

where $\sqrt{\tilde{g}}$ is the determinant of the matrix $\tilde{g}_{ij}$, that is the standard metric on $\mathbb{S}^{n-1}$ and $\nabla_\tau u$ is the component of ∇u tangential to $\mathbb{S}^{n-1}$. Therefore

$$\int_\Omega |\nabla u|^2\, ds \geqslant \int_U \int_r^{R_m} \left\{\left(\frac{\partial u}{\partial \tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2}|\nabla_\tau u|^2\right\} \tilde{r}^{n-1}\sqrt{\tilde{g}}\, d\tilde{r}\, dy, \tag{2.81}$$

Combining (2.79), (2.78) and (2.81) and recalling (2.64), we get

$$\sigma_\beta(\Omega) \geqslant \frac{\displaystyle\int_U \int_r^{R_m} \left\{\left(\frac{\partial u}{\partial \tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2}|\nabla_\tau u|^2\right\} \tilde{r}^{n-1}\sqrt{\tilde{g}}\, d\tilde{r}\, dy + \beta r^{n-1} \int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1}}{\displaystyle(R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left(\sqrt{1 + \frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right) \int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1}} \geqslant$$

$$\frac{\sigma_\beta(A_{r,R_m})}{\displaystyle(R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left(\sqrt{1 + \frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right)}. \tag{2.82}$$

Finally, we stress that the equality case implies that all the inequalities become equalities. So, we have that $\nabla_\tau \rho_0 = 0$ and $\rho_0(x) = \overline{R}$, with $\overline{R} > R$ constant. □

Theorem 2.25 tells us that as long Ω_0 is an open, bounded starshaped set, and $r > 0$, then $\sigma_\beta(\Omega)$ remains away from zero. Is this property still true for any open, bounded set in $\mathbb{R}^n$ with Lipschitz boundary? In general the answer is no, as showed in the following bidimensional counterexample, that is contained in [79], that can be easily generalized in any dimension.

Counterexample 2.26. *Let us consider a sequence of open, bounded and connected sets* $\{\Omega_\varepsilon\} \subset \mathbb{R}^2$ *as follows*

$$\Omega_\varepsilon = B(x_0) \cup R_\varepsilon \cup (B(x_1) \backslash \overline{B_r(x_1)}).$$

Here

$$R_\varepsilon = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \times \left(-\frac{\varepsilon^3}{2}, \frac{\varepsilon^3}{2} \right)$$

is a rectangle centered at the origin with sides of length ε *and* ε^3 *respectively,* $B(x_1)$*,* $B(x_2)$ *are two dimensional balls of radius* 1 *centered at the points* x_1 *and* x_2*, chosen such that the rectangle* R_ε *is well glued and eventually* $B_r(x_2)$ *is a concentric ball in* $B(x_2)$ *of radius* $0 < r < 1$*.*

Let us consider the following function

$$u(x, y) = \begin{cases} \sin\left(\dfrac{2\pi x}{\varepsilon}\right) & \text{in } R_\varepsilon \\ 0 & \text{elsewhere,} \end{cases}$$

which is a continuous test function for the first Steklov-Robin eigenvalue.
Let us evaluate u in the numerator and denominator the Rayleigh quotient. The denominator becomes

$$\int_{\partial\Omega_\varepsilon} u^2 \, d\mathcal{H}^{n-1} = \int_{\partial R_\varepsilon} u^2 \, d\mathcal{H}^{n-1} = 2\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \sin^2\left(\frac{2\pi x}{\varepsilon}\right) dx$$

$$= 4\int_0^{\frac{\varepsilon}{2}} \sin^2\left(\frac{4\pi x}{\varepsilon}\right) dx = 4\int_0^{\frac{\varepsilon}{2}} \frac{1-\cos\left(\dfrac{4\pi x}{\varepsilon}\right)}{2} \, dx = \varepsilon$$

Since

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{2\pi}{\varepsilon}\right)^2 \cos^2\left(\frac{2\pi x}{\varepsilon}\right),$$

we have that the numerator is

$$\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx \, dy = \int_{R_\varepsilon} |\nabla u|^2 \, dx \, dy = \left(\frac{2\pi}{\varepsilon}\right)^2 \int_{-\frac{\varepsilon^3}{2}}^{\frac{\varepsilon^3}{2}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \cos^2\left(\frac{4\pi x}{\varepsilon}\right) dx \, dy$$

$$= 2\left(\frac{2\pi}{\varepsilon}\right)^2 \varepsilon^3 \int_0^{\frac{\varepsilon}{2}} \frac{1+\cos\left(\dfrac{4\pi x}{\varepsilon}\right)}{2} \, dx = 2\pi^2\varepsilon^2.$$

In this way, since u is zero on $\partial B_r(x_2)$*, we get*

$$\sigma_\beta(\Omega_\varepsilon) \leqslant \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx}{\displaystyle\int_{\partial\Omega_\varepsilon} u^2 \, d\mathcal{H}^{n-1}} = 2\pi^2\varepsilon,$$

and

$$\sigma_\beta(\Omega_\varepsilon) \to 0$$

as $\varepsilon \to 0$. *We stress the fact that the same proof can be exhibited even in the Steklov-Dirichlet case.*

This counterexample gives us two information. The first is the one we already mentioned: if Ω_0 is not starshaped, the first eigenvalue could be arbitrarily close to zero. The second one is that when Ω is not connected, then $\sigma_\beta(\Omega)$ could be zero, even though $r > 0$.

Behaviour with respect to β

In this section we will give the proof of the Theorem 2.20.

Proof of Theorem 2.20. Firstly we prove inequality (2.59). We observe that the claim is well posed since, by proceeding analogously as in the existence theorem, $\mu_1(\Omega)$ is positive. For any $w \in H^1(\Omega)$, for simplicity, we will use the following notation

$$D(w) := \int_\Omega |\nabla w|^2\, dx. \tag{2.83}$$

Let u be a positive eigenfunction corresponding to $\sigma_\beta(\Omega)$. By the Minkowski inequality and the definition of $\mu_1(\Omega)$ we have

$$\sqrt{\int_{\partial\Omega_0} u^2\, d\mathcal{H}^{n-1}} \leqslant \sqrt{\int_{\partial\Omega_0} (u-c)^2\, d\mathcal{H}^{n-1}} + \sqrt{c^2 P(\Omega_0)} \leqslant \sqrt{\frac{D(u)}{\mu_1(\Omega)}} + \sqrt{c^2 P(\Omega_0)},$$

where c is

$$c = \frac{1}{P(B_r)} \int_{\partial B_r} u\, d\mathcal{H}^{n-1}, \tag{2.84}$$

Squaring and using the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\int_{\partial\Omega_0} u^2 d\mathcal{H}^{n-1} &\leqslant \frac{D(u)}{\mu(r,\Omega_0)} + c^2 P(\Omega_0) + 2\sqrt{\frac{D(u)c^2P(\Omega_0)}{\mu(r,\Omega_0)}} \\
&= D(u)\left(\frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}\right) + c^2\beta P(B_r)\left(\frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}\right) \\
&= \left(\frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}\right)(D(u) + c^2\beta P(B_r)).
\end{aligned} \tag{2.85}$$

By (2.84) and Hölder inequality, we get

$$\begin{aligned}
\int_{\partial\Omega_0} u^2 d\mathcal{H}^{n-1} &\leqslant \left(\frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}\right)\left(D(u) + \beta\int_{\partial B_r} u^2\, d\mathcal{H}^{n-1}\right) \\
&= \left(\frac{1}{\mu_1(\Omega)} + \frac{P(\Omega_0)}{\beta P(B_r)}\right)\left(\sigma_\beta(\Omega)\int_{\partial\Omega_0} u^2 d\mathcal{H}^{n-1}\right),
\end{aligned} \tag{2.86}$$

which gives (2.57).
Now we prove inequality (2.60). Let u be the eigenfunction corresponding to $\sigma_\beta(\Omega)$, solution to problem (2.55). Let us observe that $u = v + h$, where v and h solve the following problems

$$\begin{cases} \Delta v = 0 & \text{in}\,\Omega \\ v = 0 & \text{on}\,\partial B_r \\ \frac{\partial v}{\partial\nu} = \frac{\partial u}{\partial\nu} & \text{on}\,\partial\Omega_0, \end{cases} \qquad \begin{cases} \Delta h = 0 & \text{in}\,\Omega \\ h = u & \text{on}\,\partial B_r \\ \frac{\partial h}{\partial\nu} = 0 & \text{on}\,\partial\Omega_0. \end{cases}$$

It is easy to check that

$$\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\nabla h|^2 \, dx. \tag{2.87}$$

Then, proceeding as the proof of inequality (2.57), by applying Minkowski inequality and using (2.87), (2.67) and (2.58), we get

$$\begin{aligned}\sqrt{\int_{\partial\Omega_0} u^2 \, \mathcal{H}^{n-1}} &\leqslant \sqrt{\int_{\partial\Omega_0} v^2 \, \mathcal{H}^{n-1}} + \sqrt{\int_{\partial\Omega_0} h^2 \, \mathcal{H}^{n-1}} \\ &\leqslant \sqrt{\frac{1}{\sigma_D(\Omega)} D(u)} + \sqrt{\frac{1}{q_1(\Omega)} \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1}}.\end{aligned}$$

Squaring both sides and applying the arithmetic-geometric mean inequality we have

$$\begin{aligned}\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} &\leqslant \frac{D(u)}{\sigma_D(\Omega)} + \frac{1}{q_1(\Omega)} \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1} + 2\sqrt{\frac{D(u)}{\sigma_D(\Omega)} \frac{1}{q_1(\Omega)} \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1}} \\ &\leqslant \frac{D(u)}{\sigma_D(\Omega)} + \frac{D(u)}{\beta q_1(\Omega)} + \frac{1}{q_1(\Omega)} \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1} + \frac{\beta}{\sigma_D(\Omega)} \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1} \\ &= \left(\frac{1}{\sigma_D(\Omega)} + \frac{1}{\beta q_1(\Omega)} \right) \left(D(u) + \beta \int_{\partial B_r} u^2 \, \mathcal{H}^{n-1} \right).\end{aligned}$$

This gives (2.60).

□

Remark 2.27. We stress that a rough but meaningful estimate can be obtained choosing as a test function in (2.64) the constant function. In this case we have the following upper bound

$$\sigma_\beta(\Omega) \leqslant \beta \frac{P(B_r)}{P(\Omega_0)}, \tag{2.88}$$

which immediately gives that when $\beta \to 0$, then $\sigma_\beta(\Omega) \to 0$. However inequality (2.57) allows us to show that, as in the radial case, it holds

$$\lim_{\beta \to 0} \frac{\sigma_\beta(\Omega)}{\beta} = \frac{P(B_r)}{P(\Omega_0)}. \tag{2.89}$$

Indeed we have

$$\frac{P(B_r)}{P(\Omega_0)} \geqslant \frac{\sigma_\beta(\Omega)}{\beta} \geqslant \frac{P(B_r)\mu_1(\Omega)}{P(\Omega_0)\beta + P(\Omega_0)\mu_1(\Omega)}, \tag{2.90}$$

where the first inequality follows by (2.88) and the second by using (2.59). Taking in (2.90) the limit for β which goes to zero one get (2.89).

Remark 2.28. We observe that inequality (2.60) gives

$$\frac{1}{\sigma_\beta(\Omega)} - \frac{1}{\sigma_D(\Omega)} \leqslant \frac{1}{\beta q_1(\Omega)}$$

which immediately implies that

$$\lim_{\beta \to \infty} \sigma_\beta(\Omega) = \sigma_D(\Omega).$$

Chapter 3

Some results about the Robin type boundary conditions in the linear and non linear case

In this Chapter we focus our attention on varius problem involving a Robin boundary condition type.
In Section 3.1 we prove a result à la Talenti for the solution to the anisotropic Laplacian with Robin boundary condition. In particular we prove that the solution to the above mentioned problem can be upper bounded by the solution to the symmetrized problem in terms of Lorentz norm, and in more particular cases a pointwise estimate is found. Moreover a Bossel-Daners inequality in the anisotropic case is proved in dimension 2.
In Section 3.2 we consider the Torsion problem with robin boundary condition in the linear case. We compute the shape derivatives of the L^p and L^∞ of the torsion function and prove that the ball is a critical shape for these functionals.

3.1 A comparison result à la Talenti for the anisotropic Laplace eigenvalue problem with Robin boundary condition

3.1.1 Definition of the Robin problem in the anisotropic case

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, with Lipschitz boundary. Let us consider the following anisotropic problem with Robin boundary conditions

$$\begin{cases} -\operatorname{div}(H(\nabla u)H_\xi(\nabla u)) = f & \text{in } \Omega \\ H(\nabla u)H_\xi(\nabla u)\cdot\nu + \beta H(\nu)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f \geqslant 0$ (not identically zero) belongs to $L^2(\Omega)$, H is a sufficiently smooth norm in $\mathbb{R}^n$, ν is the Euclidean outer unit normal to $\partial\Omega$ and $\beta > 0$ is a positive real parameter.

A weak solution to problem (3.1) is a function $u \in H^1(\Omega)$ that satisfies

$$\int_\Omega H(\nabla u)H_\xi(\nabla u)\cdot\nabla\varphi\,dx + \beta\int_{\partial\Omega} H(\nu)u\varphi\,d\mathcal{H}^{n-1} = \int_\Omega f\varphi \qquad \forall\varphi\in H^1(\Omega). \tag{3.2}$$

We recall that the Wulff Shape centered in $x_0 \in \mathbb{R}^n$ of radius R is defined as follows

$$\mathcal{W}_R(x_0) = \{x \in \mathbb{R}^n : H^\circ(x - x_0) < R\},$$

where H° is the dual norm of H. In particular we will denote by $\mathcal{W}$ the Wulff Shape centered at the origin of radius 1 (for the exact definitions, see section (1.4)).

The aim is to estabilish a comparison result with the solution to the following symmetrized problem

$$\begin{cases} -\text{div}(H(\nabla v)H_\xi(\nabla v)) = f^\star & \text{in } \Omega^\star \\ H(\nabla v)H_\xi(\nabla v)\cdot\nu + \beta H(\nu)v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{3.3}$$

where $f^\star$ is the convex symmetrization of f and $\Omega^\star$ is a set homothetic to the Wulff Shape $\mathcal{W}$ such that $|\Omega^\star| = |\Omega|$.
In particular what we are going to prove are the following theorems.

Theorem 3.1. *Let be $n \geqslant 2$. If u and v are the solutions to problems* (30) *and* (32) *respectively, then*

$$\|u\|_{L^{p,1}(\Omega)} \leqslant \|v\|_{L^{p,1}(\Omega^\star)} \qquad \textit{for all} \quad 0 < p \leqslant \frac{n}{2n-2} \tag{3.4}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leqslant \|v\|_{L^{2p,2}(\Omega^\star)} \qquad \textit{for all} \quad 0 < p \leqslant \frac{n}{3n-4}. \tag{3.5}$$

Theorem 3.2. *Let $n = 2$, $f \equiv 1$ in Ω. If u and v are the solutions to problems* (30) *and* (32) *respectively. Then*

$$u^\star(x) \leqslant v(x) \quad x \in \Omega^\star, \tag{3.6}$$

where $u^\star$ is the convex symmetrization of u.

Theorem 3.3. *Let $n \geqslant 3$ and $f \equiv 1$. If u and v are the solutions to problems* (30) *and* (32) *respectively, then*

$$\|u\|_{L^{p,1}(\Omega)} \leqslant \|v\|_{L^{p,1}(\Omega^\star)} \tag{3.7}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leqslant \|v\|_{L^{2p,2}(\Omega^\star)}, \tag{3.8}$$

for all $0 < p \leqslant \frac{n}{n-2}$.

3.1.2 Existence, uniqueness and properties of the solution

In this subsection we want to prove that there exists a unique solution to problem (3.1) exists, which is furthermore unique and non-negative.

Existence

Let us consider the following energy functional

$$E[w] = \frac{1}{2}\int_{\Omega} H^2(\nabla w)\,dx + \frac{\beta}{2}\int_{\partial\Omega} H(\nu)w^2\,d\mathcal{H}^{n-1} - \int_{\Omega} fw\,dx, \quad w \in H^1(\Omega). \tag{3.9}$$

We notice that the Euler-Lagrange equations of this functional is (3.1), hence if $E[\cdot]$ has minima, there exists a solution to the considered problem.
Let us proceed with the classical Calculus of Variation method to prove the existence of a minimum.
1) Lower bound. Let us prove that the functional (3.9) is lower bounded. From (1.30) and the generalized Young's inequality we have

$$\begin{aligned} E[u] &\geqslant \frac{\gamma^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\beta\gamma}{2}\|u\|^2_{L^2(\partial\Omega)} - \frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)} \\ &\geqslant C_1\left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\right) - \frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)} \\ &\geqslant \left(C_2 - \frac{\epsilon}{2}\right)\|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)}. \end{aligned}$$

In the last inequality we have used a Poincaré inequality with trace term (see for instance [25, 30, 96]). Here $C_2 = C_2(\beta,\gamma,\Omega)$. If we choose ϵ small enough, then

$$E[u] \geqslant -\frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)} > -\infty.$$

We have proved in this way that the functional is bounded from below. Let us denote by

$$m := \inf_{w\in H^1(\Omega)} E[w]. \tag{3.10}$$

and let $\{u_k\} \subset H^1(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty} E[u_k] = m.$$

We can suppose that $E[u_k] \leqslant m+1$ for all $k \in \mathbb{N}$.
2) Compactness and lower semicontinuity. Using again (1.30) and the generalized Young's inequality, we have

$$\begin{aligned} m+1 &\geqslant \frac{\gamma^2}{2}\|\nabla u_k\|^2_{L^2(\Omega)} + \left(\frac{\beta\gamma}{2} - \frac{\epsilon}{2}\right)\|u_k\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)} \\ &\geqslant \frac{\gamma^2}{2}\|\nabla u_k\|^2_{L^2(\Omega)} - \frac{\epsilon}{2}\|u_k\|^2_{L^2(\Omega)} + C_3\|u_k\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)}, \end{aligned}$$

Choosing ϵ small enough and calling $C_3 = \min(\frac{\gamma^2}{2}, \frac{\beta\gamma}{2} - \frac{\epsilon}{2})$ then

$$\|u_k\|_{H^1(\Omega)} \leqslant \frac{m+1}{C_3} + \frac{1}{2\epsilon C_3}\|f\|^2_{L^2(\Omega)} < \infty.$$

Hence $\{u_k\}$ is bounded in $H^1(\Omega)$, so there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$ that converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function $u \in H^1(\Omega)$. To simplify the notation let us continue to call the subsequence as $\{u_k\}$.

By the strict convexity of the functions $t \longrightarrow t^2$ and H^2 it is straightforward to prove that the functional $E[u]$ is weakly lower semicontinuous, i.e.

$$\liminf_{k\to\infty} E[u_k] \geqslant E[u] = m.$$

This proves that $E[u] = m$ and u is a minimum.

Uniqueness and non-negativeness

Let us prove now that the minimum of (3.9) is unique. If $u, v \in H^1(\Omega)$, by the strict convexity of the funcion H^2 we know that if $t \in [0,1]$, then

$$H^2(t\nabla u + (1-t)\nabla v) \leqslant tH^2(\nabla u) + (1-t)H^2(\nabla v). \tag{3.11}$$

The equality occurs if and only if $t = 0$ or $t = 1$. Analogously

$$[tu + (1-t)v]^2 \leqslant tu^2 + (1-t)v^2, \quad t \in [0,1]. \tag{3.12}$$

Let $u \in H^1(\Omega)$ be a minimizer of (3.9) and let us suppose that there exists another minimizer $v \in H^1(\Omega)$, such that $u \neq v$. Hence $E[u] = E[v] = m$. Let us denote by $w = u + v$ and choose $t = \frac{1}{2}$, then by (3.11) and (3.12)

$$E[w] < \frac{E[u]}{2} + \frac{E[v]}{2} = m.$$

This fact contraddicts the minimality of u and so the minimum must be unique.

Let us now show the non-negativeness of the solution. Let u be the unique minimum of (3.9), namely $E[u] = m$. If we consider $|u|$, by (1.29), we get

$$H^2(\nabla|u|) = H^2\left(\frac{u}{|u|}\nabla u\right) = H^2(\nabla u).$$

Hence

$$\begin{aligned} E[|u|] &= \frac{1}{2}\int_\Omega H^2(\nabla|u|)\,dx + \frac{\beta}{2}\int_{\partial\Omega} H(\nu)|u|^2\,d\mathcal{H}^{n-1} - \int_\Omega f|u|\,dx \\ &= m + \int_\Omega f(u - |u|)\,dx = m + 2\int_{\{u\leqslant 0\}} fu \ \leqslant m. \end{aligned}$$

By the uniqueness of the minimizer we have $u = |u|$ in Ω. Eventually $u \geqslant 0$ in Ω.

The anisotropic radial case

Let us consider the following one dimensional problem

$$\begin{cases} -\frac{1}{r^{n-1}}(r^{n-1}v'(r))' = f^*(k_n r^n) & r \in (0,R) \\ v'(0) = 0 \\ v'(R) + \beta v(R) = 0, \end{cases} \tag{3.13}$$

Integrating the first equation in (3.13), calling $\tilde{t} = k_n t^n$, we get

$$v'(r) = -\frac{1}{r^{n-1}}\int_0^r t^{n-1} f^*(k_n t^n)\,dt + C_1 = -\frac{1}{nk_n r^{n-1}}\int_0^{k_n r^n} f^*(\tilde{t})\,d\tilde{t} + C_1.$$

Since $v'(0)=0$, $C_1=0$. By denoting $\tilde{s}=k_n s^n$, another integration gives

$$v(r)=-\int_0^r \frac{1}{nk_n s^{n-1}}\int_0^{k_n s^n} f^*(\tilde{t})\,d\tilde{t}\,ds+C_2= \\ -\int_0^{k_n r^n}\frac{1}{n^2 k_n^{\frac{2}{n}}\tilde{s}^{2-\frac{2}{n}}}\int_0^{\tilde{s}} f^*(\tilde{t})\,d\tilde{t}\,d\tilde{s}+C_2.$$

From $v'(R)+\beta v(R)=0$ we compute C_2, hence

$$v(r)=-\int_0^{k_n r^n}\frac{1}{n^2 k_n^{\frac{2}{n}}\tilde{s}^{2-\frac{2}{n}}}\int_0^{\tilde{s}} f^*(\tilde{t})\,d\tilde{t}\,d\tilde{s}+ \\ \int_0^{k_n R^n}\frac{1}{n^2 k_n^{\frac{2}{n}}\tilde{s}^{2-\frac{2}{n}}}\int_0^{\tilde{s}} f^*(\tilde{t})\,d\tilde{t}\,d\tilde{s}+\frac{1}{\beta n k_n R^{n-1}}\int_0^{k_n R^n} f^*(\tilde{t})\,d\tilde{t}.$$

Therefore

$$v(r)=\int_{k_n r^n}^{k_n R^n}\frac{1}{n^2 k_n^{\frac{2}{n}}\tilde{s}^{2-\frac{2}{n}}}\int_0^{\tilde{s}} f^*(\tilde{t})\,d\tilde{t}\,d\tilde{s}+\frac{1}{\beta n k_n R^{n-1}}\int_0^{k_n R^n} f^*(\tilde{t})\,d\tilde{t}. \tag{3.14}$$

Now it is easy to check that the function $v(x)=v(H^\circ(x))$ is a $H^1(\Omega^\star)$ solution to problem (3.3), where $\Omega^\star$ is the Wulff Shape centered at the origin with radius R. In this way, by the uniqueness of the solution, we have shown that the unique solution to problem (3.3) is radially symmetric with respect to anisotropic norm and its value on the boundary is given by

$$v(R)=\frac{1}{\beta n k_n R^{n-1}}\int_0^{k_n R^n} f^*(\tilde{t})\,d\tilde{t}\geqslant 0.$$

Remark 3.4. We stress that if $f\equiv 1$ in $\mathcal{W}_R$, then by (3.14) the solution to problem (3.3) can be written explicitly as follows

$$v(x)=v(H^\circ(x))=\frac{R}{\beta n}+\frac{1}{2n}(R^2-H^\circ(x)^2). \tag{3.15}$$

The solution is a paraboloid with respect to the anisotropic norm. Morover if H is the euclidean norm in $\mathbb{R}^n$, we are back to the classical torsion problem (or Saint Venant problem) with Robin boundary conditions, whose radial solution is a concave paraboloid.

Level sets and distribution functions

If u is a solution to problem (3.1), we define

$$U_t=\{x\in\Omega : u(x)>t\}$$

for a non-negative real number $t\geqslant 0$. It is clear that if $t\leqslant u_{\min}$, then $U_t=\Omega$ and that if $t>u_{\max}$, then $U_t=\varnothing$. With $u_{\min}$ and $u_{\max}$ we have denoted the minimum and the maximum of u in Ω. We will denote by

$$\partial U_t^{\text{int}}=\Omega\cap\partial U_t, \qquad \partial U_t^{\text{ext}}=\partial\Omega\cap\partial U_t \tag{3.16}$$

the interior and exterior boundaries of U_t with respect to Ω, and by

$$\mu(t)=|U_t| \tag{3.17}$$

the distribution function of u.
If v is solution to problem (3.3), for $t \geqslant 0$, we define

$$V_t = \{x \in \Omega^\star : v(x) > t\}, \quad \phi(t) = |V_t|$$

the superlevel sets and the distribution function of v, respectively. Furthermore, for $0 \leqslant t \leqslant v_{\min}$, $V_t = \Omega$, while for $v_{\min} < t < v_{\max}$, the superlevel sets V_t are Wulff shapes homothetic to $\Omega^\star$ and strictly contained in it. Again, $v_{\min}$ and $v_{\max}$ are the minimum and the maximum of v in $\Omega^\star$.

3.1.3 Main results

To prove the main results we will use the Gronwall lemma (see subsection (1.2.4)) and prove two others lemmata that will have a central importance for what will follow.

Lemma 3.5. *Let u and v be the solutions to problems* (3.1) *and* (3.3) *respectively. Then for a.e. $t > 0$ we have*

$$n^2 k_n^{\frac{2}{n}} \phi(t)^{\frac{2n-2}{n}} = \left(-\phi'(t) + \frac{1}{\beta} \int_{\partial U_t^{ext}} \frac{H(\nu)}{u} \, d\mathcal{H}^{n-1} \right) \int_0^{\mu(t)} f^*(s)\, ds \tag{3.18}$$

and

$$n^2 k_n^{\frac{2}{n}} \mu(t)^{\frac{2n-2}{n}} \leqslant \left(-\mu'(t) + \frac{1}{\beta} \int_{\partial U_t^{ext}} \frac{H(\nu)}{u} \, d\mathcal{H}^{n-1} \right) \int_0^{\mu(t)} f^*(s)\, ds. \tag{3.19}$$

Proof. Let $t, h > 0$ and let us consider the following test function in $H^1(\Omega)$

$$\varphi_h(x) = \begin{cases} 0 & u \leqslant t \\ u - t & t < u \leqslant t + h \\ h & u \geqslant t + h. \end{cases}$$

Substituting this in (3.2) we have

$$\int_{U_t \setminus U_{t+h}} H(\nabla u) H_\xi(\nabla u) \cdot \nabla u \, dx + \beta \int_{\partial U_t^{\text{ext}} \setminus \partial U_{t+h}^{\text{ext}}} H(\nu)(u-t) u \, d\mathcal{H}^{n-1}$$
$$+ \beta h \int_{\partial U_{t+h}^{\text{ext}}} H(\nu) u \, d\mathcal{H}^{n-1} = \int_{U_t \setminus U_{t+h}} f(u-t) dx + h \int_{U_{t+h}} f \, dx.$$

Applying (1.32) in the first integral, dividing by h and applying the Coarea Formula (see (1.2.2)), we have for a.e. $t > 0$

$$\frac{1}{h} \int_t^{t+h} \int_{\partial U_\tau^{\text{int}}} \frac{H^2(\nabla u)}{|\nabla u|} \, d\mathcal{H}^{n-1} \, d\tau + \frac{\beta}{h} \int_{\partial U_t^{\text{ext}} \setminus \partial U_{t+h}^{\text{ext}}} H(\nu)(u-t) u \, d\mathcal{H}^{n-1}$$
$$+ \beta \int_{\partial U_{t+h}^{\text{ext}}} H(\nu) u \, d\mathcal{H}^{n-1} = \frac{1}{h} \int_{U_t \setminus U_{t+h}} f(u-t) dx + \int_{U_{t+h}} f \, dx.$$

Passing to the limit for $h \to 0^+$ we have

$$\int_{\partial U_t^{\text{int}}} \frac{H^2(\nabla u)}{|\nabla u|} \, d\mathcal{H}^{n-1} + \beta \int_{\partial U_t^{\text{ext}}} H(\nu) u \, d\mathcal{H}^{n-1} = \int_{U_t} f \, dx.$$

Let us set

$$g(x) = \begin{cases} H(\nabla u) & \text{on } \partial U_t^{\text{int}} \\ \beta u & \text{on } \partial U_t^{\text{ext}}. \end{cases} \tag{3.20}$$

We want to end the proof using the anisotropic version of the isoperimetric inequality and to this aim it is necessary to write properly the anisotropic peremeter of U_t. Because of the regularity of $\partial\Omega$ we know that $\partial U_t^{\text{ext}}$ is sufficiently regular and a normal vector can be defined. Since $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$, $\partial U_t^{\text{int}}$ may not have any good regularity property. By (1.36) we can write for a.e. $t > 0$

$$P_H(U_t) = \int_{\partial U_t^{\text{int}}} \frac{H(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1} + \int_{\partial U_t^{\text{ext}}} H(\nu)\, d\mathcal{H}^{n-1},$$

where ν is the outer unit normal to Ω. If we set

$$h(x) = \begin{cases} \dfrac{H(\nabla u)}{|\nabla u|} & \text{on } \partial U_t^{\text{int}} \\ H(\nu) & \text{on } \partial U_t^{\text{ext}}, \end{cases} \tag{3.21}$$

then

$$P_H(U_t) = \int_{\partial U_t} h(x)\, d\mathcal{H}^{n-1}.$$

Furthermore we note that

$$\int_{\partial U_t} h(x)g(x)\, d\mathcal{H}^{n-1} = \int_{U_t} f\, dx. \tag{3.22}$$

Therefore by Schwarz inequality, (3.22) and Hardy-Littlewood inequality, we have for a.e. $t > 0$

$$\begin{aligned} P_H^2(U_t) &= \left(\int_{\partial U_t} h(x)\, d\mathcal{H}^{n-1}\right)^2 = \left(\int_{\partial U_t} \sqrt{h(x)g(x)}\sqrt{\frac{h(x)}{g(x)}}\, d\mathcal{H}^{n-1}\right)^2 \\ &\leqslant \int_{\partial U_t} h(x)g(x)\, d\mathcal{H}^{n-1} \left(\int_{\partial U_t^{\text{int}}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\, d\mathcal{H}^{n-1}\right) \\ &\leqslant \left(-\mu'(t) + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\, d\mathcal{H}^{n-1}\right) \int_0^{\mu(t)} f^*(s)\, ds. \end{aligned}$$

Hence, by (1.37)

$$n^2 k_n^{\frac{2}{n}} \mu(t)^{\frac{2n-2}{n}} \leqslant \left(-\mu'(t) + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\, d\mathcal{H}^{n-1}\right) \int_0^{\mu(t)} f^*(s)\, ds.$$

If we do the same computations, replacing v with u, all the previous inequalities become equalities and we have (3.18).
In particular, if $f \equiv 1$ in Ω, we have

$$n^2 k_n^{\frac{2}{n}} \mu(t)^{\frac{n-2}{n}} \leqslant -\mu'(t) + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\, d\mathcal{H}^{n-1}, \tag{3.23}$$

and

$$n^2 k_n^{\frac{2}{n}} \phi(t)^{\frac{n-2}{n}} = -\phi'(t) + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\, d\mathcal{H}^{n-1}. \tag{3.24}$$

□

Remark 3.6. Let us notice that $u_{\min} \leqslant v_{\min}$. Indeed, being the level sets of v homothetic to $\Omega^\star$, then, using (3.1), (3.3) and the isoperimetric inequality

$$\begin{aligned} v_{\min} P_H(\Omega^\star) &= \int_{\partial\Omega^\star} H(\nu) v(x)\, d\mathcal{H}^{n-1} = -\frac{1}{\beta}\int_{\partial\Omega^\star} H(\nabla v) H_\xi(\nabla v)\cdot \nu\, d\mathcal{H}^{n-1} \\ &= \frac{1}{\beta}\|f^\star\|_{L^1(\Omega^\star)} = \frac{1}{\beta}\|f\|_{L^1(\Omega)} = -\frac{1}{\beta}\int_{\partial\Omega} H(\nabla u) H_\xi(\nabla u)\cdot\nu\, d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} H(\nu) u(x)\, d\mathcal{H}^{n-1} \geqslant u_{\min} P_H(\Omega) \geqslant u_{\min} P_H(\Omega^\star). \end{aligned} \tag{3.25}$$

As a consequence for all $0 < t < v_{\min}$ we have that

$$\mu(t) \leqslant \phi(t) = |\Omega|. \tag{3.26}$$

Lemma 3.7. *For all $t \geqslant v_{\min}$ we have*

$$\int_0^t \tau \left(\int_{\partial V_\tau \cap \partial\Omega^\star} \frac{H(\nu)}{v(x)}\, d\mathcal{H}^{n-1} \right) d\tau = \frac{1}{2\beta}\int_0^{|\Omega|} f^*(s)\, ds \tag{3.27}$$

and

$$\int_0^t \tau \left(\int_{\partial U_\tau^{ext}} \frac{H(\nu)}{u(x)}\, d\mathcal{H}^{n-1} \right) d\tau \leqslant \frac{1}{2\beta}\int_0^{|\Omega|} f^*(s)\, ds \tag{3.28}$$

Proof. By Fubini's Theorem and using (3.1), we have that

$$\begin{aligned} \int_0^\infty \tau \left(\int_{\partial U_\tau^{\text{ext}}} \frac{H(\nu)}{u(x)}\, d\mathcal{H}^{n-1} \right) d\tau &= \int_{\partial\Omega} \left(\int_0^{u(x)} \frac{H(\nu)}{u(x)} \tau\, d\tau \right) d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} \frac{H(\nu)u(x)}{2}\, d\mathcal{H}^{n-1} = \frac{1}{2\beta}\int_0^{|\Omega|} f^*(s)\, ds. \end{aligned}$$

Analogously,

$$\int_0^\infty \tau \left(\int_{\partial V_\tau \cap \partial\Omega^\star} \frac{H(\nu)}{v(x)}\, d\mathcal{H}^{n-1} \right) d\tau = \frac{1}{2\beta}\int_0^{|\Omega|} f^*(s)\, ds.$$

By monotonicity of the integral we have that for $t \geqslant 0$

$$\int_0^t \tau \left(\int_{\partial U_\tau^{\text{ext}}} \frac{H(\nu)}{u(x)}\, d\mathcal{H}^{n-1} \right) d\tau \leqslant \int_0^\infty \tau \left(\int_{\partial U_\tau^{\text{ext}}} \frac{H(\nu)}{u(x)}\, d\mathcal{H}^{n-1} \right) d\tau$$

and if $t \geqslant v_{\min}$, then $\partial V_t \cap \partial\Omega^\star = \varnothing$. Hence

$$\int_0^t \tau \left(\int_{\partial V_\tau \cap \partial\Omega^\star} \frac{H(\nu)}{v(x)}\, d\mathcal{H}^{n-1} \right) d\tau = \int_0^\infty \tau \left(\int_{\partial V_\tau \cap \partial\Omega^\star} \frac{H(\nu)}{v(x)}\, d\mathcal{H}^{n-1} \right) d\tau,$$

and we have (3.28), (3.27).

□

Proof of Theorem 3.1. Let $0 < p \leqslant \frac{n}{2n-2}$ and let us denote $K_n = n^2 k_n^{\frac{2}{n}}$. Let us multiply (3.19) by $t\mu(t)^\eta$, where $\eta = \frac{1}{p} - \frac{2n-2}{n} \geqslant 0$, and integrate from 0 to $\tau \geqslant v_{\min}$

$$\begin{aligned}\int_0^\tau K_n t\mu(t)^{\frac{1}{p}}\,dt \leqslant& \int_0^\tau -\mu'(t)t\mu(t)^\eta\left(\int_0^{\mu(t)} f^*(s)\,ds\right)dt \;+\\ &\frac{1}{\beta}\int_0^\tau t\mu(t)^\eta\left(\int_{\partial U_t^{\text{ext}}}\frac{H(\nu)}{u(x)}\,d\mathcal{H}^{n-1}\int_0^{\mu(t)} f^*(s)\,ds\right)dt\\ &\leqslant \int_0^\tau -t\mu(t)^\eta\left(\int_0^{\mu(t)} f^*(s)\,ds\right)d\mu(t) + \frac{|\Omega|^\eta}{2\beta^2}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^2,\end{aligned}$$

where we applied Lemma 3.7 and the fact that $\mu(t)$ is a monotone non increasing function. By setting $F(l) = \displaystyle\int_0^l w^\eta \int_0^w f^*(s)\,ds\,dw$ and integrating by parts the first and last members in this chain of inequalities we have

$$\begin{aligned}\tau F(\mu(\tau)) + \tau\int_0^\tau K_n\mu(t)^{\frac{1}{p}}\,dt \leqslant& \int_0^\tau F(\mu(t))\,dt + \int_0^\tau\int_0^t K_n\mu(t)^{\frac{1}{p}}\,dr\,dt\\ &+\frac{|\Omega|^\eta}{2\beta^2}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^2\end{aligned}$$

By applying Lemma 1.8, with

$$\xi(\tau) = \int_0^\tau F(\mu(t))\,dt + \int_0^\tau\int_0^t K_n\mu(t)^{\frac{1}{p}}\,dr\,dt,$$

$C = \dfrac{|\Omega|^\eta}{2\beta^2}\left(\displaystyle\int_0^{|\Omega|} f^*(s)\,ds\right)^2$ and $\tau_0 = v_{\min}$, we have that

$$\begin{aligned}F(\mu(\tau)) + \int_0^\tau K_n\mu(t)^{\frac{1}{p}}\,dt \leqslant& \frac{1}{v_{\min}}\left(\int_0^{v_{\min}} F(\mu(t))\,dt\right.\\ &\left.+\int_0^{v_{\min}}\int_0^t K_n\mu(r)^{\frac{1}{p}}\,dr\,dt + \frac{|\Omega|^\eta}{2\beta^2}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^2\right).\end{aligned} \tag{3.29}$$

Analogously

$$\begin{aligned}F(\phi(\tau)) + \int_0^\tau K_n\phi(t)^{\frac{1}{p}}\,dt =& \frac{1}{v_{\min}}\left(\int_0^{v_{\min}} F(\phi(t))\,dt\right.\\ &\left.+\int_0^{v_{\min}}\int_0^t K_n\phi(r)^{\frac{1}{p}}\,dr\,dt + \frac{|\Omega|^\eta}{2\beta^2}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^2\right).\end{aligned} \tag{3.30}$$

By (3.25) and (3.26), then we can compare directly the righthand sides of (3.29) and (3.30). So

$$F(\mu(\tau)) + \int_0^\tau K_n\mu(t)^{\frac{1}{p}}\,dt \leqslant F(\phi(\tau)) + \int_0^\tau K_n\phi(t)^{\frac{1}{p}}\,dt$$

For $\tau \to +\infty$ we have

$$\int_0^\infty \mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\infty \phi(t)^{\frac{1}{p}}\,dt,$$

which is (3.4).
Now we want to prove (3.5). In order to obtain this result let us pass to the limit as $\tau \to \infty$ in the following inequality:

$$\int_0^\tau K_n t\mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\tau -t\mu(t)^\eta \Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)\,d\mu(t) + \frac{|\Omega|^\eta}{2\beta^2}\Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)^2.$$

After an integration by parts we get

$$\int_0^\infty K_n t\mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\infty F(\mu(t))\,dt + \frac{|\Omega|^\eta}{2\beta^2}\Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)^2.$$

On the other hand

$$\int_0^\infty K_n t\phi(t)^{\frac{1}{p}}\,dt = \int_0^\infty F(\phi(t))\,dt + \frac{|\Omega|^\eta}{2\beta^2}\Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)^2,$$

it remains to show that

$$\int_0^\infty F(\mu(t))\,dt \leqslant \int_0^\infty F(\phi(t))\,dt. \tag{3.31}$$

To this aim we multiply (3.19) by $tF(\mu(t))\mu(t)^{-\frac{2n-2}{n}}$. Since $F(l)l^{-\frac{2n-2}{n}}$ is a non-decreasing function in l, when $0 < p \leqslant \frac{n}{3n-4}$, we can integrate from 0 to $\tau \geqslant v_{\min}$ to obtain

$$\begin{aligned}
\int_0^\tau K_n tF(\mu(t))\,dt &\leqslant \int_0^\tau -t\mu(t)^{-\frac{2n-2}{n}} F(\mu(t))\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)\,d\mu(t)\\
&+ \frac{1}{\beta}\int_0^\tau tF(\mu(t))\mu(t)^{-\frac{2n-2}{n}} \int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\,d\mathcal{H}^{n-1}\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)\,dt\\
&\leqslant \int_0^\tau -t\mu(t)^{-\frac{2n-2}{n}} F(\mu(t))\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)\,d\mu(t)\\
&+ \frac{1}{\beta}F(|\Omega|)|\Omega|^{-\frac{2n-2}{n}}\Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)\int_0^\tau t\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u}\,d\mathcal{H}^{n-1}\,dt\\
&\leqslant \int_0^\tau -t\mu(t)^{-\frac{2n-2}{n}} F(\mu(t))\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)\,d\mu(t)\\
&+ F(|\Omega|)\frac{|\Omega|^{\frac{2n-2}{n}}}{2\beta^2}\Big(\int_0^{|\Omega|} f^*(s)\,ds\Big)^2,
\end{aligned}$$

where, again, we applied (3.7). Now, if we call $C = F(|\Omega|)\dfrac{|\Omega|^{\frac{2n-2}{n}}}{2\beta^2}\Big(\displaystyle\int_0^{|\Omega|} f^*(s)\,ds\Big)^2$ and set $J(l) = \displaystyle\int_0^l w^{-\frac{2n-2}{n}} F(w)\Big(\int_0^w f^*(s)\,ds\Big)\,dw$, integrating by parts the first and last member of the previous chain of inequalities, we have

$$\tau\int_0^\tau K_n F(\mu(t))\,dt + \tau J(\mu(\tau)) \leqslant \int_0^\tau\int_0^r K_n F(\mu(z))\,dz\,dr + \int_0^\tau J(\mu(t))\,dt + C.$$

Setting

$$\xi(\tau) = \int_0^\tau\int_0^r K_n F(\mu(z))\,dz\,dr + \int_0^\tau J(\mu(t))\,dt,$$

and applying 1.8 with $\tau_0 = v_{\min}$ we deduce that

$$\int_0^\tau K_n F(\mu(t))\,dt + J(\mu(\tau)) \leqslant \frac{1}{v_{\min}}\Big(\int_0^{v_{\min}}\int_0^r K_n F(\mu(z))\,dz\,dr + \int_0^{v_{\min}} J(\mu(t))\,dt + C\Big).$$

This inequality holds as an equality when we have ϕ in place of μ, so as before

$$\int_0^\tau K_n F(\mu(t))\,dt + J(\mu(t)) \leqslant \int_0^\tau K_n F(\phi(t))\,dt + J(\phi(t)).$$

For $\tau \to \infty$ we have (3.5), which concludes the proof. □

Proof of Theorem 3.2. Multiplying by $t \geqslant 0$ inequality (3.23) and integrating from 0 to $\tau \geqslant v_{\min}$, we have that

$$2k_2\tau^2 \leqslant \int_0^\tau -\mu'(t)t\,dt + \frac{|\Omega|}{2\beta^2}.$$

Here we applied Lemma 3.7. Analogously for (3.24)

$$2k_2\tau^2 = \int_0^\tau -\phi'(t)t\,dt + \frac{|\Omega|}{2\beta^2}.$$

Then

$$\int_0^\tau t(-d\mu(t)) \geqslant \int_0^\tau t(-d\phi(t)),$$

for every $\tau \geqslant v_{\min}$. Integrating by parts

$$\mu(\tau) \leqslant \phi(\tau) \quad \tau \geqslant v_{\min}. \tag{3.32}$$

Since $u_{\min} \leqslant v_{\min}$, inequality (3.32) holds for $t \geqslant 0$ and the claim is proved. □

Proof of Theorem 3.3. Let $0 < p \leqslant \frac{n}{n-2}$. Let us multiply (3.23) by $t\mu(t)^\eta$, where $\eta = \frac{1}{p} - \frac{n-2}{n} \geqslant 0$, and integrate from 0 to $\tau \geqslant v_{\min}$

$$\begin{aligned}\int_0^\tau K_n t\mu(t)^{\frac{1}{p}}\,dt &\leqslant \int_0^\tau -\mu'(t)t\mu(t)^\eta\,dt + \frac{1}{\beta}\int_0^\tau t\mu(t)^\eta \int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u(x)}\,d\mathcal{H}^{n-1}\\ &\leqslant \int_0^\tau -\mu'(t)t\mu(t)^\eta\,dt + \frac{|\Omega|^\eta}{\beta}\int_0^\tau t\int_{\partial U_t^{\text{ext}}} \frac{H(\nu)}{u(x)}\,d\mathcal{H}^{n-1}\\ &\leqslant \int_0^\tau -\mu'(t)t\mu(t)^\eta\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2} \leqslant \int_0^\tau -t\mu(t)^\eta\,d\mu(t) + \frac{|\Omega|^{\eta+1}}{2\beta^2}\end{aligned}$$

where, again, $K_n = n^2 k_n^{\frac{2}{n}}$, in the third inequality we applied Lemma 3.7, and in the last the fact that $\mu(t)$ is a monotone non increasing function.

By setting $G(l) = \displaystyle\int_0^l w^\eta = \frac{l^{\eta+1}}{\eta+1}$ and integrating by parts the first and last members in this chain of inequalities we have

$$\tau G(\mu(\tau)) + \tau\int_0^\tau K_n t\mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\tau G(\mu(t))\,dt + \int_0^\tau\int_0^t K_n\mu(t)^{\frac{1}{p}}\,dr\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}$$

By applying Lemma 1.8, with

$$\xi(t) = \int_0^\tau G(\mu(t))\,dt + \int_0^\tau \int_0^t K_n \mu(t)^{\frac{1}{p}}\,dr\,dt,$$

$C = \dfrac{|\Omega|^{\eta+1}}{2\beta^2}$ and $\tau_0 = v_{\min}$, we have that

$$\begin{aligned} G(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{1}{p}}\,dt \leqslant \frac{1}{v_{\min}} \bigg(& \int_0^{v_{\min}} G(\mu(t))\,dt \\ & + \int_0^{v_{\min}} \int_0^t K_n \mu(r)^{\frac{1}{p}}\,dr\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2} \bigg). \end{aligned} \tag{3.33}$$

Analogously

$$\begin{aligned} G(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}}\,dt = \frac{1}{v_{\min}} \bigg(& \int_0^{v_{\min}} G(\phi(t))\,dt \\ & + \int_0^{v_{\min}} \int_0^t K_n \phi(r)^{\frac{1}{p}}\,dr\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2} \bigg). \end{aligned} \tag{3.34}$$

By (3.25) and (3.26), we compare directly the righthand sides of (3.33) and (3.34). So

$$G(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{1}{p}}\,dt \leqslant G(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}}\,dt$$

For $\tau \to +\infty$ we have

$$\int_0^\infty \mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\infty \phi(t)^{\frac{1}{p}}\,dt,$$

which is (3.7). Now we want to prove (3.8). In order to obtain this result let us pass to the limit as $\tau \to \infty$ in the following inequality:

$$\int_0^\tau K_n t \mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\tau -t\mu(t)^\eta\,d\mu(t) + \frac{|\Omega|^{\eta+1}}{2\beta^2}.$$

After an integration by parts we get

$$\int_0^\infty K_n t \mu(t)^{\frac{1}{p}}\,dt \leqslant \int_0^\infty G(\mu(t))\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}.$$

On the other hand

$$\int_0^\infty K_n t \phi(t)^{\frac{1}{p}}\,dt = \int_0^\infty G(\phi(t))\,dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}.$$

So we need just to show that

$$\int_0^\infty G(\mu(t))\,dt \leqslant \int_0^\infty G(\phi(t))\,dt. \tag{3.35}$$

To this aim we multiply (3.23) by $tG(\mu(t))\mu(t)^{-\frac{n-2}{n}}$. Since $G(l)l^{-\frac{n-2}{n}}$ is a non decreasing function in l, we can integrate from 0 to $\tau \geqslant v_{\min}$, to obtain

$$\begin{aligned}\int_0^\tau K_n t G(\mu(t))\,dt &\leqslant \int_0^\tau -t\mu(t)^{-\frac{n-2}{n}} G(\mu(t))\,d\mu(t)\\ &\quad+\frac{1}{\beta}\int_0^\tau tG(\mu(t))\mu(t)^{-\frac{n-2}{n}}\int_{\partial U_t^{\text{ext}}}\frac{H(\nu)}{u}\,d\mathcal{H}^{n-1}\,dt\\ &\leqslant \int_0^\tau -t\mu(t)^{-\frac{n-2}{n}} G(\mu(t))\,d\mu(t)+\frac{1}{\beta}G(|\Omega|)|\Omega|^{-\frac{n-2}{n}}\int_0^\tau t\int_{\partial U_t^{\text{ext}}}\frac{H(\nu)}{u}\,d\mathcal{H}^{n-1}\,dt\\ &\leqslant \int_0^\tau -t\mu(t)^{-\frac{n-2}{n}} G(\mu(t))\,d\mu(t)+G(|\Omega|)\frac{|\Omega|^{\frac{2}{n}}}{2\beta^2},\end{aligned}$$

where, again, we applied (3.7). Now, if we call $C = G(|\Omega|)\frac{|\Omega|^{\frac{2}{n}}}{2\beta^2}$ and set $J(l) = \int_0^l w^{-\frac{n-2}{n}} G(w)\,dw$, integrating by parts the first and last member of the previous chain of inequalities, we have

$$\tau\int_0^\tau K_n G(\mu(t))\,dt + \tau J(\mu(\tau)) \leqslant \int_0^\tau\int_0^r K_n G(\mu(z))\,dz\,dr + \int_0^\tau J(\mu(t))\,dt + C.$$

Setting

$$\xi(t) = \int_0^\tau\int_0^r K_n G(\mu(z))\,dz\,dr + \int_0^\tau J(\mu(t))\,dt,$$

and applying (1.8) with $\tau_0 = v_{\min}$ we deduce that

$$\begin{aligned}\int_0^\tau K_n G(\mu(t))\,dt + J(\mu(\tau)) \leqslant \frac{1}{v_{\min}}\bigg(&\int_0^{v_{\min}}\int_0^r K_n G(\mu(z))\,dz\,dr\\ &+\int_0^{v_{\min}} J(\mu(t))\,dt, +C\bigg).\end{aligned}$$

This inequality holds as an equality when we have ϕ in place of μ, so as before

$$\int_0^\tau K_n G(\mu(t))\,dt + J(\mu(t)) \leqslant \int_0^\tau K_n G(\phi(t))\,dt + J(\phi(t)).$$

For $\tau \to \infty$ we have (3.35), which concludes the proof. $\square$

3.1.4 Application to PDE's: Bossel-Daners inequality

Let Ω be a bounded and smooth open set in $\mathbb{R}^n$. Let us denote by ν the outer unit normal to $\partial\Omega$ and let $\beta > 0$ be a positive real number. It is well known that for the following Laplacian eigenvalue problem with Robin boundary conditions

$$\begin{cases} -\Delta u = \lambda(\Omega) u & \text{in } \Omega\\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega,\end{cases}$$

a Faber-Krahn type inequality for the first eigenvalue holds. It is famous under the name of Bossel-Daners inequality and it reads as follows

$$\lambda_{1,\beta}(\Omega) \geqslant \lambda_{1,\beta}(\Omega^\sharp),$$

where $\Omega^\sharp$ is the ball centered at the origin with the same measure as Ω. Equality holds if and only if Ω is a ball.
Let us consider now the anisotropic case. If $f = \lambda(\Omega)u$, then (3.1) can be written in this way

$$\begin{cases} -\mathrm{div}(H(\nabla u)H_\xi(\nabla u)) = \lambda(\Omega)u & \text{in } \Omega \\ H(\nabla u)H_\xi(\nabla u)\cdot\nu + \beta H(\nu)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.36}$$

The variational characterization for the first eigenvalue is

$$\lambda_{1,\beta}(\Omega) = \min_{w\in H^1(\Omega)\setminus\{0\}} J[u], \tag{3.37}$$

where

$$J[u] = \frac{\displaystyle\int_\Omega H^2(\nabla u)\,dx + \beta\int_{\partial\Omega} u^2 H(\nu)\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^2\,dx}. \tag{3.38}$$

In [48] the authors proved a Bossel-Daners type inequality for the anisotropic p-Laplacian problem. Indeed, they proved that

$$\lambda_{1,\beta}(\Omega) \geqslant \lambda_{1,\beta}(\Omega^\star), \tag{3.39}$$

where $\Omega^\star$ is the set homothetic to the Wulff Shape having the same measure as Ω. In particular the equality case holds if and only if Ω is a set homothetic to the Wulff Shape.
In this section we want to give an alternative proof of (3.39) in the planar case, using the results found in the previous section.

Corollary 3.8. *Under the hypothesis of Theorem 3.1 we have that*

$$\lambda_{1,\beta}(\Omega) \geqslant \lambda_{1,\beta}(\Omega^\star).$$

Proof. Let u be the first anisotropic Robin eigenfunction associated to $\lambda_{1,\beta}(\Omega)$. Then u is solution to problem

$$\begin{cases} -\mathrm{div}(H(\nabla u)H_\xi(\nabla u)) = \lambda_{1,\beta}(\Omega)u & \text{in } \Omega \\ H(\nabla u)H_\xi(\nabla u)\cdot\nu + \beta H(\nu)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.40}$$

Let z be the solution to the symmetrized problem

$$\begin{cases} -\mathrm{div}(H(\nabla z)H_\xi(\nabla z)) = \lambda_{1,\beta}(\Omega)u^\star & \text{in } \Omega^\star \\ H(\nabla z)H_\xi(\nabla z)\cdot\nu + \beta H(\nu)z = 0 & \text{on } \partial\Omega^\star. \end{cases} \tag{3.41}$$

By theorem 3.2, we know that

$$\int_\Omega u^2\,dx = \int_{\Omega^\star} (u^\star)^2\,dx \leqslant \int_{\Omega^\star} z^2\,dx.$$

So, by Cauchy-Schwarz inequality we have

$$\int_{\Omega^\star} u^\star z\,dx \leqslant \int_{\Omega^\star} z^2\,dx, \tag{3.42}$$

Eventually if we multiply the first equation in (3.41) by z, integrating on $\Omega^\star$ and applying (3.42) we get

$$\lambda_{1,\beta}(\Omega) = \frac{\displaystyle\int_{\Omega^\star} H^2(\nabla z)\,dx + \beta\int_{\partial\Omega^\star} z^2 H(\nu)\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star} u^\star z\,dx}$$
$$\geqslant \frac{\displaystyle\int_{\Omega^\star} H^2(\nabla z)\,dx + \beta\int_{\partial\Omega^\star} z^2 H(\nu)\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star} z^2\,dx} \geqslant \lambda_{1,\beta}(\Omega^\star).$$

□

3.1.5 Conclusions and open problems

As in typhe euclidean case, we have proved that these comparison results depend on the dimension of the space. In particular if we are in the hypothesis of theorem 3.1, when $n = 2$, then

$$\|u\|_{L^1(\Omega)} \leqslant \|v\|_{L^1(\Omega^\star)},$$

and

$$\|u\|_{L^2(\Omega)} \leqslant \|v\|_{L^2(\Omega^\star)}.$$

Therefore a question arises spontaneously. Is it true that

$$\|u\|_{L^p(\Omega)} \leqslant \|v\|_{L^p(\Omega^\star)} \tag{3.43}$$

for all values of p? In dimension 2 the answer is negative for large values of p. Next example will show that (3.43) is untrue when $p = \infty$ and $n = 2$.

Example 1. *Let Ω be the union of two disjoint bidimensional Wulff shapes $\mathcal{W}$ and $\mathcal{W}_r$, with radii 1 and r respectively. If we choose $\beta = \frac{1}{2}$ and f such that it is constantly 1 in $\mathcal{W}$ and constantly zero in $\mathcal{W}_r$, then the solutions to problem (3.1) and (3.3) can be explicitly computed. In particular it is possible to prove that there exitsts a positive constant c such that*

$$\|u\|_{L^\infty(\Omega)} - \|v\|_{L^\infty(\Omega^\star)} = cr^2 + o(r^2).$$

Proof. Considering problem (3.1), since $f \equiv 1$ in $\mathcal{W}$ and $f \equiv 0$ in $\mathcal{W}_r$, then u must be radial in $\mathcal{W}$ and $u|_{\mathcal{W}_r} = 0$. In particular by considering problem (3.13) with $\beta = 1/2$, it is easy to check that in $\mathcal{W}$, denoting by $t = H^\circ(x)$, the solution is as in (3.15)

$$u(t) = 1 + \frac{1}{4}(1 - t^2). \tag{3.44}$$

Let us now compute the solution to the symmetrized problem (3.3). Since $|\Omega| = |\Omega^\star|$ we have that the radius $r^\star$ of the Wulff Shape $\Omega^\star$ is given by

$$r^\star = \sqrt{1 + r^2}.$$

Looking at (3.13), we have to solve the following problem

$$\begin{cases} (tv'(t))' = -t & (0,1], \\ (tv'(t))' = 0 & [1, r^\star), \\ v'(0) = 0, \\ v'(r^\star) + \frac{1}{2}v(r^\star) = 0. \end{cases}$$

When $t \in (0,1]$, then

$$tv'(t) = -\frac{t^2}{2} + c_1.$$

Since $v'(0) = 0$, $c_1 = 0$. Hence integrating

$$v(t) = -\frac{t^2}{4} + c_2.$$

If $t \in [1, r^\star)$, then

$$tv'(t) = c_3.$$

Imposing the continuity of the derivative in 1 we get $c_3 = -1/2$. An integration gives

$$v(t) = -\frac{1}{2}\log t + c_4.$$

The Robin boundary condition allows us to compute c_4 indeed

$$-\frac{1}{2r^\star} - \frac{1}{4}\log r^\star + \frac{c_4}{2} = 0,$$

and consequently

$$c_4 = \frac{1}{r^\star} + \frac{1}{2}\log r^\star.$$

We have

$$v(t) = -\frac{1}{2}\log t + \frac{1}{r^\star} + \frac{1}{2}\log r^\star.$$

Imposing the continuity in 1 of the solution we can compute c_2

$$c_2 = \frac{1}{4} + \frac{1}{r^\star} + \frac{1}{2}\log r^\star.$$

Hence, if we denote by $c(r^\star) = \dfrac{1}{r^\star} + \dfrac{1}{2}\log r^\star$, the solution to the symmetrized problem is

$$v(t) = \begin{cases} -\dfrac{t^2}{4} + \dfrac{1}{4} + c(r^\star) & [0,1], \\ -\dfrac{1}{2}\log t + c(r^\star) & [1, r^\star]. \end{cases}$$

Let us note that in $(0,1]$ we have

$$v(t) = u(t) - 1 + c(r^\star).$$

Eventually

$$\|v\|_{L^\infty(\Omega^\star)} = \|v\|_{L^\infty(\mathcal{W})} = \|u - 1 + c(r^\star)\|_{L^\infty(\mathcal{W})} = \|u\|_{L^\infty(\Omega)} - 1 + c(r^\star).$$

Expanding $c(r^\star)$ in Taylor series we get

$$c(r^\star) = (1+r^2)^{-\frac{1}{2}} + \frac{1}{2}\log(1+r^2)^{\frac{1}{2}} = 1 - \frac{r^2}{2} + \frac{r^2}{4} + o(r^2) = 1 - \frac{r^2}{4} + o(r^2).$$

In this way we have

$$\|u\|_{L^\infty(\Omega)} - \|v\|_{L^\infty(\Omega^\star)} = \frac{r^2}{4} + o(r^2).$$

□

Now someone could ask if (3.43) can be true when $n \geqslant 3$. Next counterexample will show its untruthfulness when $n = 3$ and $p = 2$.

Example 2. *If we consider Ω, β and f as in the example 1 in the corresponding three-dimensional case, then, in the hypothesis of theorem 3.1, the solutions to problem* (3.1) *and* (3.3) *can be explicitly computed. It is possible to prove that there exists a positive constant d such that*

$$\|u\|_{L^2(\Omega)} - \|v\|_{L^2(\Omega^\star)} = dr^3 + o(r^3).$$

Proof. As in the previous example it easy to compute the solution to problem (3.1). If we denote by $t = H^\circ(x)$, then it is given by

$$u(t) = \begin{cases} \dfrac{2}{3} + \dfrac{1}{6}(1 - t^2) & \text{in } \mathcal{W}, \\ 0 & \text{in } \mathcal{W}_r. \end{cases} . \tag{3.45}$$

From the condition $|\Omega| = |\Omega^\star|$, we find the radius $r^\star$ of the Wulff Shape $\Omega^\star$ which is given by $r^\star = (1 + r^3)^{\frac{1}{3}}$. Following exactly the same computations as in 1, we find the solution to the symmetrized problem, and it is

$$v(t) = \begin{cases} u(t) - \dfrac{1}{3} + q(r^\star) & [0, 1], \\ \dfrac{1}{3t} + q(r^\star) & [1, r^\star], \end{cases} \tag{3.46}$$

where

$$q(r^\star) = \frac{2}{3r^{\star 2}} - \frac{1}{3r^\star}.$$

A Taylor series expansion gives us

$$r^{\star\alpha} = (1 + r^3)^{\frac{\alpha}{3}} = 1 + \frac{\alpha}{3} r^3 + o(r^3), \qquad \alpha \in \mathbb{R};$$

$$q(r^\star) = \frac{2}{3(1 + r^3)^{\frac{2}{3}}} - \frac{1}{3(1 + r^3)^{\frac{1}{3}}} = \frac{1}{3} - \frac{r^3}{3} + o(r^3),$$

so that

$$v(t) = \begin{cases} u(t) - \dfrac{r^3}{3} + o(r^3) & [0, 1], \\ \dfrac{2}{3} + o(r) & [1, r^\star]. \end{cases} \tag{3.47}$$

Therefore, recalling that k_3 is the three dimensional measure of the unitary Wulff Shape $\mathcal{W}$, we get

$$\begin{aligned} \|v\|^2_{L^2(\Omega^\star)} &= \int_{\mathcal{W}} \left(u - \frac{r^3}{3} \right) dx + \frac{4}{9} |\mathcal{W}_{r^\star} \backslash \overline{\mathcal{W}}| + o(r^3) \\ &= \|u\|^2_{L^2(\Omega)} + \frac{2}{3} r^3 \int_{\mathcal{W}} u \, dx + \frac{4}{9} k_3 r^3 + o(r^3) \\ &= \|u\|^2_{L^2(\Omega)} - \frac{22}{45} k_3 r^3 + \frac{4}{9} k_3 r^3 + o(r^3) \\ &= \|u\|^2_{L^2(\Omega)} - \frac{2}{45} k_3 r^3 + o(r^3). \end{aligned}$$

Hence we have proved the desired result, since

$$\|u\|^2_{L^2(\Omega)} - \|v\|^2_{L^2(\Omega^\star)} = \frac{2}{45}k_3 r^3 + o(r^3).$$

□

A problem that is still open is the following

Open Problem 1. *In the hypothesis of theorem 3.1,* (3.43) *is true for* $p = 1$ *and* $n \geqslant 3$*?*

If we now consider the theorem 3.2, we have proved that when $n = 2$ and $f \equiv 1$ in Ω, then

$$u^\star(x) \leqslant v(x) \quad x \in \Omega^\star. \tag{3.48}$$

In doing so, another question arises:

Open Problem 2. *In the hypothesis of theorem 3.2, is* (3.48) *true even when* $n \geqslant 3$*?*

3.2 Shape derivative of the L^p and the L^∞ norms of the Robin Torsion function

3.2.1 An introduction to the problem

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded $C^{2,\alpha}$ and simply connected open set. Let us consider the following torsion problem with Robin boundary conditions

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.49}$$

where ν stands for the outer unit normal to $\partial\Omega$ and $\beta > 0$ is a positive real number. A weak solution to (3.49) is a function $u \in H^1(\Omega)$ which satisfies

$$\int_\Omega \nabla u \nabla \phi \, dx + \beta \int_{\partial\Omega} u\phi \, d\mathcal{H}^{n-1} = \int_\Omega \phi, \qquad \forall \phi \in H^1(\Omega).$$

It is well known that the solution to problem (3.49) is unique and positive whenever $\partial\Omega$ is sufficiently smooth.
Let us recall the functionals we want to study and the the definitions of their shape derivatives. We will denote the L^∞ and L^p shape functionals as follows

$$M(\Omega) = \|u\|_{L^\infty(\Omega)},$$

and for every $p \in [1, +\infty)$

$$F_p(\Omega) = \int_\Omega |u(x)|^p \, dx = \int_\Omega u^p(x) \, dx = \|u\|^p_{L^p(\Omega)},$$

where u is solution to (3.49).
As already said in the introduction, let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ simply connected open set and let us consider a first order perturbation

$$\Omega_t = (\mathbb{1}_{\mathbb{R}^n} + tV)(\Omega),$$

with $\mathbb{1}_{\mathbb{R}^n}$ being the identity function, V a $C^{2,\alpha}(\mathbb{R}^n,\mathbb{R}^n)$ vector field and t a small real number. We want to study the limits (if they exist)

$$M'(\Omega, V) = \lim_{t\to 0} \frac{M(\Omega_t) - M(\Omega)}{t} \tag{3.50}$$

and

$$F_p'(\Omega, V) = \lim_{t\to 0} \frac{F_p(\Omega_t) - F_p(\Omega)}{t}. \tag{3.51}$$

What we want to prove is the following

Theorem 3.9. *The ball B_R is a critical shape for the functionals $M(\Omega)$ and $F_p(\Omega)$, $p \geqslant 1$, i.e.*

$$M'(B_R, V) = F_p'(B_R, V) = 0,$$

where V is a $C^{2,\alpha}(\mathbb{R}^n,\mathbb{R}^n)$ vector field volume preserving of the first order and where $M'(\cdot, v)$ and $F_p'(\cdot, v)$ are the shape derivatives of M and F_p respectively.

3.2.2 Shape derivative: some definitions and computations

Here we give some preliminary definitions and results that the reader can find in [10] and [85]. We point out that in this and next subsection, we will use the Einstein summation convention for the repeated indexes.
Let $\Omega \subset \mathbb{R}^n$ be a bounded and simply connected open set. We consider a family of perturbations $\{\Omega_t\}_t$ of the form

$$\Omega_t = \{y = x + tV(x) : x \in \Omega, t \text{ small enough}\}, \tag{3.52}$$

where V is a $C^{2,\alpha}(\mathbb{R}^n,\mathbb{R}^n)$ vector field.

The Jacobian matrix of the transformation

$$y := y(t, \Omega) = x + tV(x), \quad x \in \Omega,\ t \text{ small} \tag{3.53}$$

is

$$D_y = I + tD_V,$$

where I is the identity matrix and $(D_V)_{ij} = \frac{\partial V_i}{\partial x_j}$. By Jacobi's formula, for small t, the Jacobian determinant is given by

$$J(t) = 1 + t \operatorname{div} V. \tag{3.54}$$

It is clear that for t small enough, $J(t) \approx 1$, so $y(t, \Omega)$ is a diffeomorphism and in this case we can consider its inverse transformation $x(y)$.
In particular we can write the measure of Ω_t in terms of the perturbations defined before

$$|\Omega_t| = \int_\Omega J(t)\, dx = |\Omega| + t \int_\Omega \operatorname{div} V\, dx.$$

Definition 3.1. $y(t, \Omega)$ is said to be *volume preserving of the first order* if

$$\int_\Omega \operatorname{div} V\, dx = 0.$$

Let $w \in H^1(\Omega_t)$ and let us consider the following energy functional

$$\mathcal{E}(\Omega_t, w) = \int_{\Omega_t} |\nabla_y w(y,t)|^2 \, dy - 2 \int_{\Omega_t} w(y,t) \, dy + \beta \int_{\partial\Omega_t} w^2(y,t) \, d\sigma_t, \tag{3.55}$$

where with ∇_y we denoted the gradient operator with respect to y and $d\sigma_t$ is the surface element of Ω_t.
A critical point $u \in H^1(\Omega_t)$ of (3.55) satisfies the Euler-Lagrange equations

$$\begin{cases} \Delta_y u(y,t) + 1 = 0 & \text{in } \Omega_t \\ \dfrac{\partial u}{\partial \nu_t}(y,t) + \beta u(y,t) = 0 & \text{on } \partial\Omega_t, \end{cases} \tag{3.56}$$

where Δ_y is the Laplacian operator with respect to y and ν_t is the outer normal to $\partial\Omega_t$.
We want to transform the integrals in (3.55) in integrals onto Ω and $\partial\Omega$. Indeed by a change of variables, using the inverse function $x(y)$ (which exists for small t), we get

$$\begin{aligned} \mathcal{E}(\Omega, u) = & \int_\Omega \frac{\partial u}{\partial x_i}(x + tV(x), t) \frac{\partial u}{\partial x_j}(x + tV(x), t) \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} J(t) \, dx \\ & - 2 \int_\Omega u(x + tV(x), t) J(t) \, dx + \beta \int_{\partial\Omega} u^2(x + tV(x), t) m(t) \, d\sigma. \end{aligned} \tag{3.57}$$

Here $m(t)$ is the index of deformation when passing from $d\sigma$ to $d\sigma_t$ (with $d\sigma$ being the surface element of Ω). If we define the tangential divergence of the vector field V as follows

$$\operatorname{div}_{\partial\Omega} V := \operatorname{div} V - \nu \cdot D_V \nu, \tag{3.58}$$

then, up to first order terms, $m(t)$ can be approximated by (See [10], section 2.2.2)

$$m(t) = 1 + t \operatorname{div}_{\partial\Omega} V. \tag{3.59}$$

If we denote by

$$\tilde{u}(t) := u(x + tV(x), t) \tag{3.60}$$

and

$$A = (A_{ij}(t)) := \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} J(t), \tag{3.61}$$

we can write (3.57) in a more concise form

$$\mathcal{E}(t) := \int_\Omega \nabla \tilde{u}(t) A \nabla \tilde{u}(t) \, dx - 2 \int_\Omega \tilde{u}(t) J(t) \, dx + \beta \int_{\partial\Omega} \tilde{u}^2(t) m(t) \, d\sigma. \tag{3.62}$$

If we simplify one more time the notations and indicate by

$$L_A = \frac{\partial}{\partial x_j}(A_{ij}(t) \frac{\partial}{\partial x_i}) \tag{3.63}$$

and

$$\partial_{\nu_A} = \nu_i A_{ij}(t) \frac{\partial}{\partial x_j}, \tag{3.64}$$

then the transformed function $\tilde{u}(t)$ solves the Eulero-Lagrange equations

$$\begin{cases} L_A\tilde{u}(t) + J(t) = 0 & \text{in } \Omega \\ \partial_{\nu_A}\tilde{u}(t) + \beta m(t)\tilde{u}(t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.65}$$

A key role will be played by (3.60). If we expand it in a neighbourhood of $t = 0$ we have

$$\tilde{u}(t) = \tilde{u}(0) + t\dot{\tilde{u}}(0) + o(t).$$

We remark that the dot notation stands for the derivative with respect to t. In particular the first two coefficients will be

$$\tilde{u}(0) := u(x) \tag{3.66}$$

and

$$\begin{aligned} \dot{\tilde{u}}(0) = \left[\frac{d}{dt}\tilde{u}(t)\right]_{t=0} &= \left[\frac{\partial\tilde{u}}{\partial t}(t) + V(x)\cdot\nabla\tilde{u}(t)\right]_{t=0} \\ &= \frac{\partial\tilde{u}}{\partial t}(0) + V\cdot\nabla u. \end{aligned} \tag{3.67}$$

Definition 3.2. We will call shape derivative of $\tilde{u}$, and it will be denoted by u', the following function

$$u'(x) := \frac{\partial\tilde{u}}{\partial t}(0). \tag{3.68}$$

Hence we can write (3.67) in this way

$$\dot{\tilde{u}}(0) = u' + V\cdot\nabla u. \tag{3.69}$$

Besides the expansion of $\tilde{u}(t)$, it will be helpful to write the taylor series of some other of the function seen until now. Next Lemma will collect all the necessary coefficients of the expansions just mentioned

Lemma 3.10. *We have that*

$$\begin{aligned} &1)\quad J(0) = 1, &&2)\quad \dot{J}(0) = \operatorname{div} V, \\ &3)\quad m(0) = 1, &&4)\quad \dot{m}(0) = \operatorname{div}_{\partial\Omega} V, \\ &5)\quad \tilde{u}(0) = u(x), &&6)\quad \dot{\tilde{u}}(0) = u' + V\cdot\nabla u, \\ &7)\quad A_{ij}(0) = \delta_{ij}, &&8)\quad \dot{A}_{ij}(0) = \operatorname{div} V\delta_{ij} - \frac{\partial V_i}{\partial x_j} - \frac{\partial V_j}{\partial x_i}. \end{aligned} \tag{3.70}$$

Proof. To compute $1), 2), 3), 4)$ it is sufficient to differentiate (3.54) and (3.59) and evaluate for $t = 0$. 5) and 6) are given by (3.66) and (3.69) respectively.
Some more effort will be needed for the matrix A, defined in (3.61). Remembering that the Jacobian matrix of the transformation $y(t, \Omega)$ is

$$D_y = I + tD_V,$$

if t is small enough, we have

$$\begin{aligned} \frac{\partial x_i}{\partial y_k} &= (D_y^{-1})_{ik} = (I + tD_V)^{-1}_{ik} \\ &= (I - tD_V + o(t))_{ik} = \delta_{ik} - t\frac{\partial V_k}{\partial x_i} + o(t). \end{aligned}$$

This allows us to obtain

$$A_{ij}(0) = \delta_{ik}\delta_{jk} = \delta_{ij}$$

and

$$\begin{aligned}\dot{A}_{ij}(0) &= \left[\frac{\partial x_i}{\partial y_k}\frac{\partial x_j}{\partial y_k}\dot{J}(t) + \left(\frac{d}{dt}\frac{\partial x_i}{\partial y_k}\right)\frac{\partial x_j}{\partial y_k}J(t) + \frac{\partial x_i}{\partial y_k}\left(\frac{d}{dt}\frac{\partial x_j}{\partial y_k}\right)J(t)\right]_{t=0} \\ &= \mathrm{div}V\delta_{ij} - \frac{\partial V_i}{\partial x_j} - \frac{\partial V_j}{\partial x_i}.\end{aligned}$$

□

We want to find the equations that are solved by u' in B_R and on its boundary.

3.2.3 An equation for u' in B_R

Let us consider problem (41). It is well known that it admits a unique and positive solution, given by

$$u(x) = \frac{R}{\beta n} + \frac{1}{2n}(R^2 - |x|^2), \tag{3.71}$$

which is a radial and strictly concave function, whose maximum and minimum are achieved in 0 and on ∂B_R respectevely. More precisely

$$u_{\max} = u(0) = \frac{R}{\beta n} + \frac{R^2}{2n}, \quad u_{\min} = u(R) = \frac{R}{\beta n}. \tag{3.72}$$

In order to prove next proposition, it will be useful to keep in mind the gradient and the Hessian matrix of (3.71). The gradient is

$$\nabla u(x) = -\frac{x}{n}. \tag{3.73}$$

In particular, if $x \in \partial B_R$, being $\nu = \frac{x}{R}$ the outer unit normal to the boundary of B_R, then

$$\nabla u(x) = -\frac{R}{n}\nu, \quad \frac{\partial u}{\partial \nu} = -\frac{R}{n}. \tag{3.74}$$

The Hessian matrix is clearly negative definite and it is given by

$$\mathrm{Hess}_u(x) = -\frac{I}{n}, \tag{3.75}$$

where I is the identity matrix.

Proposition 3.11. *Let V be a $C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ vector field and $R > 0$ a positive real number. The function u' solves the following boundary value problem in the ball with radius R*

$$\begin{cases} \Delta u' = 0 & \text{in } B_R \\ \dfrac{\partial u'}{\partial \nu} + \beta u' = \left(\dfrac{1+\beta R}{n}\right)(V\cdot\nu) & \text{on } \partial B_R, \end{cases} \tag{3.76}$$

where ν is the outer unit normal to ∂B_R.

Proof. If we differentiate the first equation in (3.65) with respect to t and evaluate for $t = 0$, we obtain

$$L_{A(0)}\dot{\tilde{u}}(0) + L_{\dot{A}(0)}\tilde{u}(0) + \dot{J}(0) = 0.$$

It will be helpful to write explicitly L_A and $L_{\dot{A}}$. By applying Lemma 3.10 we have

$$\frac{\partial}{\partial x_j}\left(\delta_{ij}\frac{\partial}{\partial x_i}\right)\left(u' + V_k\frac{\partial u}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial V_k}{\partial x_k}\delta_{ij}\frac{\partial}{\partial x_i} - \frac{\partial V_j}{\partial x_i}\frac{\partial}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\frac{\partial}{\partial x_i}\right)u + \frac{\partial V_k}{\partial x_k} = 0.$$

Hence

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u'}{\partial x_j} + \frac{\partial}{\partial x_j}\left(V_k\frac{\partial u}{\partial x_k}\right)\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial V_k}{\partial x_k}\frac{\partial u}{\partial x_j} - \frac{\partial V_j}{\partial x_i}\frac{\partial u}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\frac{\partial u}{\partial x_i}\right) + \frac{\partial V_k}{\partial x_k} = 0.$$

Renaming the indexes k

$$\begin{aligned}
&\frac{\partial^2 u'}{\partial x_j^2} + \frac{\partial^2 V_i}{\partial x_j^2}\frac{\partial u}{\partial x_i} + 2\frac{\partial V_i}{\partial x_j}\frac{\partial^2 u}{\partial x_j\partial x_i} + V_i\frac{\partial^3 u}{\partial^2 x_j\partial x_i} \\
&\quad + \frac{\partial^2 V_i}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial V_i}{\partial x_i}\frac{\partial^2 u}{\partial x_j^2} - \frac{\partial^2 V_j}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_i} \\
&\quad - \frac{\partial V_j}{\partial x_i}\frac{\partial^2 u}{\partial x_j\partial x_i} - \frac{\partial^2 V_i}{\partial x_j^2}\frac{\partial u}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\frac{\partial^2 u}{\partial x_j\partial x_i} + \frac{\partial V_i}{\partial x_i} = 0.
\end{aligned}$$

Considering that $(\mathrm{Hess}_u(x))_{ij} = \frac{\partial^2 u}{\partial x_j\partial x_i} = 0$ whenever $i \neq j$ and the fact that $\frac{\partial^2 u}{\partial x_j^2} = \Delta u = -1$, we have

$$\frac{\partial u'}{\partial x_j^2} + \frac{\partial^2 V_i}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\partial^2 V_j}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_i} = 0.$$

Eventually, by Schwarz Theorem, we obtain $\Delta u' = 0$ in B_R.
If we now differentiate the boundary conditions in (3.65) and evaluate them for $t = 0$, then

$$\partial_{\nu_{A(0)}}\dot{\tilde{u}}(0) + \partial_{\nu_{\dot{A}(0)}}\tilde{u}(0) + \beta\dot{m}(0)\tilde{u}(0) + \beta m(0)\dot{\tilde{u}}(0) = 0. \tag{3.77}$$

Let us compute every term in the previous equation. Considering the boundary conditions satisfied by u, Lemma (3.10) and (3.74) we get

$$\partial_{\nu_{A(0)}}\dot{\tilde{u}}(0) = \frac{\partial u'}{\partial\nu} + \frac{\partial}{\partial\nu}(V\cdot\nabla u) = \frac{\partial u'}{\partial\nu} - \frac{R}{n}\nabla(V\cdot\nu)\cdot\nu,$$

$$\begin{aligned}
\partial_{\nu_{\dot{A}(0)}}\tilde{u}(0) &= \frac{\partial u}{\partial\nu}\operatorname{div}V - \nu\cdot D_V\nabla u - \nabla u\cdot D_V\nu \\
&= -\frac{R}{n}\operatorname{div}V + \frac{2R}{n}\nu\cdot D_V\nu,
\end{aligned}$$

$$\beta\dot{m}(0)\tilde{u}(0) = \beta u\operatorname{div}_{\partial B_R}V = \frac{R}{n}\operatorname{div}V - \frac{R}{n}\nu\cdot D_V\nu,$$

$$\beta m(0)\dot{\tilde{u}}(0) = \beta u' + \beta V\cdot\nabla u = \beta u' - \frac{R\beta}{n}V\cdot\nu.$$

Substituting in (3.77) we have

$$\frac{\partial u'}{\partial \nu} + \beta u' = \frac{R}{n}\nabla(V\cdot\nu)\cdot\nu - \frac{R}{n}\nu\cdot D_V\nu + \frac{\beta R}{n}V\cdot\nu.$$

Now

$$\frac{R}{n}\nabla(V\cdot\nu) = \frac{R}{n}\nu\cdot D_V\nu + \frac{1}{n}V\cdot\nu.$$

Hence

$$\frac{\partial u'}{\partial \nu} + \beta u' = \frac{1}{n}V\cdot\nu + \frac{\beta R}{n}V\cdot\nu = \left(\frac{1+\beta R}{n}\right)(V\cdot\nu).$$

□

As a consequence of the previous proposition, we deduce that

Corollary 3.12. *If $V \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ is a vector field volume preserving of the first order, then the solution u' to* (3.76) *is a function with zero mean value in B_R, that is*

$$\int_{B_R} u'\,dx = \int_{\partial B_R} u'\,d\sigma = 0.$$

Proof. Let us integrate the first equation in (41)

$$0 = \int_{B_R} \Delta u'\,dx = \int_{B_R} \operatorname{div}(\nabla u')\,dx = \int_{\partial B_R} \frac{\partial u'}{\partial \nu}\,d\sigma. \tag{3.78}$$

Since V is volume preserving of the first order

$$\int_{B_R} \operatorname{div} V\,dx = \int_{\partial B_R} V\cdot\nu\,d\sigma = 0. \tag{3.79}$$

So integrating the second equation on the boundary, by (3.78) and (3.79)

$$\int_{\partial B_R} u'\,d\sigma = \frac{1}{\beta}\left(\frac{1+\beta R}{n}\right)\int_{\partial B_R} V\cdot\nu\,d\sigma = 0.$$

In conclusion, being u' a harmonic function, by the mean value theorem

$$\int_{B_R} u'\,dx = \frac{R}{n}\int_{\partial B_R} u'\,d\sigma = 0.$$

□

3.3 Main results

The shape derivative u', solution to problem (3.76) will play a central role to prove the desired results.

3.3.1 Shape derivative of the L^∞-norm

We prove the next result following the proof that can be found in [84].

Theorem 3.13. *Let B_R be a ball centered at the origin with radius $R > 0$. Then for every $C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ vector field V, the shape derivative of M at B_R in any direction V exists and it is given by*

$$M'(B_R, V) = u'(0),$$

where 0 *is the maximum point of* (3.71) *and u' is the solution to* (3.76).

Proof. Let us perturb the ball B_R as we have seen in the previous section

$$B_{R,t} = \{y = x + tV(x) : x \in B_R,\ t \text{ small enough}\},$$

and consider problem (3.56), with $B_{R,t}$ in place of Ω_t.
Being $x = 0$ the unique maximum point of (3.71), then $\nabla u(0,0) = 0$. On the other hand, by the strict concavity of the torsion function u on B_R, the matrix

$$D_y \nabla u(0,0) = \mathrm{Hess}_u(0)$$

is invertible, since Hess_u is negative definite. Hence by the implicit function theorem, in a neighbourhood of the origin and for t small enough, there exists a unique y_t such that $\nabla u(y_t, t) = 0$. Moreover the function $t \longrightarrow y_t$ is differentiable and y_t must be a maximum, so $M(B_{R,t}) = u(y_t, t)$.
We want to prove that

$$\lim_{t\to 0} \frac{M(B_{R,t}) - M(B_R)}{t} = u'(0),$$

where

$$\begin{aligned}\frac{M(B_{R,t}) - M(B_R)}{t} &= \frac{u(y_t,t) - u(0,0)}{t} \\ &= \frac{u(y_t,t) - u(0,t)}{t} + \frac{u(0,t) - u(0,0)}{t}.\end{aligned}$$

By the differentiability of the map $t \longrightarrow u(\cdot, t)$ and the the fact that $\nabla u(0,0) = 0$, we have

$$\lim_{t\to 0} \frac{u(0,t) - u(0,0)}{t} = \frac{d}{dt}\left[u(0,t)\right]_{t=0} = u'(0) + V(0) \cdot \nabla u(0,0) = u'(0).$$

Furthermore, by the differentiability of $t \longrightarrow y_t$, by Lagrange theorem on the segment $[0, y_t]$, the mean value property of $\nabla u(\cdot, t)$ and the regularity of $u(\cdot, t)$, we get

$$\begin{aligned}\frac{u(y_t,t) - u(0,t)}{t} &= \nabla u(\xi_t, t)\frac{y_t}{t} \\ &= \left(\frac{1}{|B_r(\xi_t)|}\int_{B_r(\xi_t)} \nabla u(y,t)\,dy\right)\frac{y_t}{t},\end{aligned}$$

with ξ_t a suitable point in $[0, y_t]$. Hence

$$\lim_{t\to 0} \frac{u(y_t,t) - u(0,t)}{t} = \nabla u(0,0)\left[\frac{dy_t}{dt}\right]_{t=0} = 0.$$

This conclude the proof. □

Corollary 3.14. *The ball is a critical shape for the functional M, for every $V \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ which is volume preserving of the first order, i.e.*

$$M'(B_R, V) = u'(0) = 0.$$

Proof. As a consequence of Lemma 3.12 and Theorem 3.13, applying the mean value theorem, we have that

$$M'(B_R, V) = u'(0) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u'(x)\, d\sigma = \frac{1}{\omega_n R^n} \int_{B_R} u'\, dx = 0.$$

□

3.3.2 Shape derivative of the L^p-norm

Next theorem will be a straightforward computation of the shape derivative of the functional $F_p(\Omega)$.

Theorem 3.15. *For every $C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ vector field V, the shape derivative of F_p at Ω in any direction V exists and it is given by*

$$F_p'(\Omega, V) = p \int_\Omega u^{p-1} u'\, dx + \int_{\partial\Omega} u^p (V \cdot \nu)\, d\sigma,$$

where u' is the shape derivative of u, solution to problem (3.49)*, and ν is the outer unit normal to $\partial\Omega$.*

Proof. Let u be the solution to the perturbed problem

$$\begin{cases} -\Delta_y u(y,t) = 1 & \text{in } \Omega_t \\ \dfrac{\partial u}{\partial \nu_t}(y,t) + \beta u(y,t) = 0 & \text{on } \partial\Omega_t, \end{cases}$$

where Ω_t is the perturbed domain defined in (3.52). Then

$$F_p(\Omega_t) = \int_{\Omega_t} u^p(y,t)\, dy = \int_\Omega \tilde{u}^p(t) J(t)\, dx,$$

with $\tilde{u}(t) = \tilde{u}(x + tV(x), t)$ and $J(t)$ the Jacobian determinant as in (3.54).

Then it is possible to differentiate under the sign of integral and

$$\begin{aligned} \frac{d}{dt} F_p(\Omega_t) &= \frac{d}{dt} \int_\Omega \tilde{u}^p(t) J(t)\, dx = \int_\Omega \frac{d}{dt}[\tilde{u}^p(t) J(t)]\, dx \\ &= p \int_\Omega \tilde{u}^{p-1}(t) \dot{\tilde{u}}(t) J(t)\, dx + \int_\Omega \tilde{u}^p(t) \dot{J}(t)\, dx. \end{aligned}$$

Evaluating this derivative for $t = 0$

$$\left[\frac{d}{dt} F_p(\Omega_t)\right]_{t=0} = p \int_\Omega \tilde{u}^{p-1}(0) \dot{\tilde{u}}(0) J(0)\, dx + \int_\Omega \tilde{u}^p(0) \dot{J}(0)\, dx.$$

Applying Lemma (3.10), we get

$$
\begin{aligned}
F_p'(\Omega, V) &= p \int_\Omega u^{p-1} u' \, dx + p \int_\Omega u^{p-1} V \cdot \nabla u \, dx + \int_\Omega u^p \operatorname{div} V \, dx \\
&= p \int_\Omega u^{p-1} u' \, dx + p \int_\Omega u^{p-1} V \cdot \nabla u \, dx + \int_{\partial\Omega} u^p (V \cdot \nu) \, d\sigma \\
&- p \int_\Omega u^{p-1} V \cdot \nabla u \, dx = p \int_\Omega u^{p-1} u' \, dx + \int_{\partial\Omega} u^p (V \cdot \nu) \, d\sigma.
\end{aligned}
$$

□

When $\Omega = B_R$, we can use the symmetry properties of (3.71) and the property of u' to be a zero mean function, to prove that

Corollary 3.16. *The ball B_R centered at the origin with radius $R > 0$ is a critical shape for the functional F_p, for every $1 \leqslant p < +\infty$ and every vector field $V \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ which is volume preserving of the first order, i.e.*

$$F_p'(B_R, V) = 0.$$

Proof. By previous theorem we know that

$$F_p'(B_R, V) = p \int_{B_R} u^{p-1} u' \, dx + \int_{\partial B_R} u^p (V \cdot \nu) \, d\sigma.$$

Being u constant on the boundary and V a vector field volume preserving of the first order

$$\int_{\partial B_R} u^p (V \cdot \nu) \, d\sigma = u_{\min}^p \int_{\partial B_R} (V \cdot \nu) \, d\sigma = u_{\min}^p \int_{B_R} \operatorname{div} V = 0,$$

where $u_{\min} = u(R) = \frac{R}{\beta n}$. By corollary 3.14, we know that $u'(0) = 0$ and so by the mean value theorem, we have that

$$\int_{\partial B_r} u'(x) \, d\mathcal{H}^{n-1} = u'(0) n \omega_n r^{n-1} = 0,$$

for every $r \in [0; R]$. Eventually, applying the Coarea Formula

$$
\begin{aligned}
&p \int_{B_R} u^{p-1} u' \, dx = p \int_{B_R} \left(\frac{R}{\beta n} - \frac{1}{2n} (R^2 - |x|^2) \right)^{p-1} u' \, dx \\
&= p \int_0^R \left(\frac{R}{\beta n} - \frac{1}{2n} (R^2 - r^2) \right)^{p-1} \int_{\partial B_r} u'(x) \, d\sigma \, dr = 0.
\end{aligned}
$$

Hence

$$F_p'(B_R, V) = 0.$$

□

Chapter 4

Sharp and quantitative estimates for the $p-$Torsion of convex sets

In this Chapter we consider the $(f,p)-$torsional rigidity for the Poisson problem with Dirichlet boundary conditions, denoted by $T_{f,p}(\Omega)$. Firstly, we prove a Pólya type lower bound for $T_{f,p}(\Omega)$ in any dimension; then, we consider the planar case and we provide two quantitative estimates in the case $f \equiv 1$. The following is contained in [6].

4.1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a non-empty, bounded, open and convex set and let $p \in (1,+\infty)$. We consider the Poisson equation for the $p-$Laplace operator, defined as

$$-\Delta_p u := -\text{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

with Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u(x) = f(d(x,\partial\Omega)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f : [0, R_\Omega] \to [0,+\infty[$ is a continuous, non-increasing and not identically zero function, $d(\cdot,\partial\Omega) : \Omega \to [0,+\infty[$ is the distance function from the boundary defined as

$$d(x,\partial\Omega) := \inf_{y\in\partial\Omega} |x-y|$$

and R_Ω is the inradius of Ω. This class of functions, depending only on the distance, are the so called web functions, see as a reference [42].
A function $u \in W_0^{1,p}(\Omega)$ is a weak solution to (4.1) if and only if

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\varphi(x)\,dx = \int_\Omega f(d(x,\partial\Omega))\varphi(x)\,dx \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

The (f,p)-torsional rigidity of Ω, that we denote by $T_{f,p}(\Omega)$, is defined as

$$T_{f,p}(\Omega) = \max_{\substack{\varphi\in W_0^{1,p}(\Omega)\\ \varphi\not\equiv 0}} \frac{\left(\displaystyle\int_\Omega f(d(x,\partial\Omega))\,|\varphi(x)|\,dx\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_\Omega |\nabla\varphi(x)|^p\,dx\right)^{\frac{1}{p-1}}} \tag{4.2}$$

and, if $u_p \in W_0^{1,p}(\Omega)$ is the unique solution to (4.1), we have

$$T_{f,p}(\Omega) = \int_\Omega f u_p \, dx.$$

For the sake of simplicity, when $f \equiv 1$ in Ω, we set $T_p(\Omega) := T_{1,p}(\Omega)$ and, if we are also in the case $p = 2$, we set $T(\Omega) := T_{1,2}(\Omega)$. We recall that the quantities $T(\Omega)$ and $T_p(\Omega)$ are usually called, respectively, torsional rigidity and $p-$torsional rigidity and so, by analogy, we have chosen the above terminology for $T_{f,p}(\Omega)$.
In section 4.2 we prove the following

Theorem 4.1. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$, $n \geqslant 2$, and let $f : [0, R_\Omega] \to [0, +\infty[$ be a continuous and non-increasing function such that $f \not\equiv 0$. Then, it holds*

$$T_{f,p}(\Omega) \geqslant c_p \frac{\mu_f^{q+1}(\Omega)}{f(0)P^q(\Omega)}, \tag{4.3}$$

where

$$c_p = \frac{p-1}{2p-1}, \qquad q = \frac{p}{p-1},$$

and

$$\mu_f(\Omega) = \int_\Omega f(x) \, dx.$$

Moreover, the equality sign is asymptotically achieved by a sequence of thinning cylinders.

In the second part, we focus our study on the case $f \equiv 1$ and $n = 2$ and we obtain some quantitative estimates. We define the following scaling invariant functional

$$\mathcal{F}_p(\Omega) = \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} \qquad q = \frac{p}{p-1}. \tag{4.4}$$

We can rewrite inequality (4.3), in the case $f \equiv 1$, as follows

$$\mathcal{F}_p(\Omega) \geqslant c_p.$$

From Theorem 4.1 follows that along a sequence of thinning cylinders $\{\Omega_l\}_{l \in \mathbb{N}}$, we have

$$\mathcal{F}_p(\Omega_l) \xrightarrow{l \to 0} c_p.$$

This leads to the following stability issue: if $\mathcal{F}_p(\Omega)$ is close to c_p, can we say that Ω is close in some sense to a cylinder? The following result gives us information on the nature of the geometry of Ω: when $\mathcal{F}_p(\Omega) - c_p$ is sufficiently small, the set Ω is a thin domain, in the sense that the ratio $w_\Omega/\text{diam}(\Omega)$ is small. In section 4.3 we prove the following theorems:

Theorem 4.2. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$ and let $f \equiv 1$. Then,*

$$\mathcal{F}_p(\Omega) - c_p \geqslant K(n,p)\left(\frac{w_\Omega}{\text{diam}(\Omega)}\right)^{n-1}, \tag{4.5}$$

where $K(n,p)$ is a positive constant depending only on p and the dimension of the space n. In particular, in the case $n = 2$, the exponent of the quantity $\dfrac{w_\Omega}{\text{diam}(\Omega)}$ is sharp.

We prove a second quantitative result in the case $p = n = 2$.

Theorem 4.3. *Let Ω be a non-empty, bounded, open and convex set in $\mathbb{R}^2$, let $f \equiv 1$ and let $p = 2$. Then, there exists a positive constant $\tilde{K}$ such that*

$$\mathcal{F}_2(\Omega) - c_2 = \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \tilde{K}\left(\frac{|\Omega \triangle Q|}{|\Omega|}\right)^3, \tag{4.6}$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and w_Ω containing Ω.

4.2 A Pólya type estimate for the (f, p)-torsional rigidity

In this Section we prove Theorem 4.1. Since the proof is quite long, we split it in two parts: firstly we prove inequality (4.3) and, then, we prove its sharpness.

Step 1: proof of inequality (4.3) in Theorem 4.1

Proof. Let us choose in the variational characterization (4.2) $\varphi(x) = g(d(x))$ as a test function, where g is a positive and non-decreasing function in $W^{1,p}([0, R_\Omega])$ such that $g(0) = 0$. Then, by coarea formula (1.3),

$$\int_\Omega f(d(x, \partial\Omega))\varphi(x)\, dx = \int_0^{R_\Omega} f(t)g(t)P(t)\, dt \tag{4.7}$$

and

$$\int_\Omega |\nabla\varphi(x)|^p\ dx = \int_0^{R_\Omega} g'^p(t)P(t)\, dt. \tag{4.8}$$

By (4.2), (4.7) and (4.8) we have

$$T_{f,p}(\Omega) \geqslant \frac{\left(\displaystyle\int_0^{R_\Omega} f(t)g(t)P(t)\, dt\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_0^{R_\Omega} g'^p(t)P(t)\, dt\right)^{\frac{1}{p-1}}}. \tag{4.9}$$

Now, if we define the following measure

$$\mu_f(E) = \int_E f(d(x))\, dx,$$

we have

$$\mu_f(t) := \mu_f(\Omega_t) = \int_{\Omega_t} f(d(x))\, dx = \int_t^{R_\Omega} f(s)P(s)\, ds, \tag{4.10}$$

and, since $f(s)P(s)$ is a decreasing function, we get

$$\mu_f(t) \leqslant (R_\Omega - t)f(t)P(t). \tag{4.11}$$

From (4.10), we have

$$-\mu_f'(t) = f(t)P(t) \qquad \text{a.e. } t \in [0, R_\Omega]. \tag{4.12}$$

Using (4.7), (4.12) and integrating by parts, we obtain

$$\int_0^{R_\Omega} f(t)g(t)P(t)\,dt = -\int_0^{R_\Omega} g(t)\mu_f'(t)\,dt = \int_0^{R_\Omega} g'(t)\mu_f(t)\,dt.$$

Consequently, (4.9) becomes

$$T_{f,p}(\Omega) \geqslant \frac{\left(\displaystyle\int_0^{R_\Omega} g'(t)\mu_f(t)\,dt\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_0^{R_\Omega} g'^p(t)P(t)\,dt\right)^{\frac{1}{p-1}}}.$$

We can choose

$$g(t) = \int_0^t \left(\frac{\mu_f(s)}{P(s)}\right)^{1/(p-1)} ds$$

and we observe that $g \in W^{1,p}([0,R_\Omega])$, since, using (4.11), we have

$$g(t) \leqslant \int_0^{R_\Omega} (R_\Omega - s)^{\frac{1}{p-1}} f(s)^{\frac{1}{p-1}}\,ds \leqslant \|f\|_{L^\infty}^{\frac{1}{p-1}} R_\Omega^{\frac{p}{p-1}} \in L^p([0,R_\Omega]),$$
$$g'(t) \leqslant \|f\|_{L^\infty}^{\frac{1}{p-1}} R_\Omega^{\frac{1}{p-1}} \in L^p([0,R_\Omega]).$$

So, we have

$$T_{f,p}(\Omega) \geqslant \int_0^{R_\Omega} \frac{\mu_f^{\frac{p}{p-1}}(t)}{P^{\frac{1}{p-1}}(t)}\,dt = -\frac{p-1}{2p-1}\int_0^{R_\Omega} \frac{(\mu_f^{\frac{2p-1}{p-1}}(t))'}{f(t)P^{\frac{p}{p-1}}(t)}\,dt. \tag{4.13}$$

Let us set $c_p = (p-1)/(2p-1)$. Since $f(s)$ is a non-negative and non-increasing function, integrating by parts in (4.13), we get

$$\begin{aligned} T_{f,p}(\Omega) &\geqslant -c_p \int_0^{R_\Omega} \frac{(\mu_f^{\frac{2p-1}{p-1}}(t))'}{f(t)P^{\frac{p}{p-1}}(t)}\,dt = -c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f(t)P^{\frac{p}{p-1}}(t)}\Bigg|_0^{R_\Omega} + \\ &- c_p \int_0^{R_\Omega} \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f^2(t)P^{\frac{2p}{p-1}}(t)} \left(f'(t)P^{\frac{p}{p-1}}(t) + \frac{p}{p-1} f(t)P^{\frac{1}{p-1}}(t)P'(t)\right) dt \\ &\geqslant c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(\Omega)}{f(0)P^{\frac{p}{p-1}}(\Omega)} + \frac{c_p}{P^{\frac{p}{p-1}}(\Omega)} \int_0^{R_\Omega} \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f^2(t)}(-f'(t))\,dt, \end{aligned} \tag{4.14}$$

where in the last inequality we use (4.11) and the fact that $P'(t) \leqslant 0$. Now, since $f(s)$ is non-increasing, we obtain the desired estimate

$$T_{f,p}(\Omega) \geqslant c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(\Omega)}{f(0)P^{\frac{p}{p-1}}(\Omega)}. \tag{4.15}$$

$\square$

Step 2: proof of the sharpness of (4.3)

Proof. We prove that inequality (4.3) is sharp and that the optimum is asymptotically achieved by the sequence of thinning cylinders Ω_l with unitary measure, as defined in (1.17), that is

$$\Omega_l = l^{-\frac{1}{n-1}} C \times \left(-\frac{l}{2}, \frac{l}{2}\right)$$

where $C \subseteq \mathbb{R}^{n-1}$ is a bounded, open and convex set with unitary $n-1$-measure. It is easy to verify that, for $n \geqslant 3$,

$$\begin{aligned} P(\Omega_l) &= 2\mathcal{H}^{n-1}(l^{-\frac{1}{n-1}} C) + l\mathcal{H}^{n-2}(\partial(l^{-\frac{1}{n-1}} C)) \\ &= 2l^{-1} + l^{\frac{1}{n-1}} \mathcal{H}^{n-2}(\partial C), \end{aligned} \tag{4.16}$$

and we observe that, in the case $n = 2$, we have that $\mathcal{H}^{n-2}(\partial C) = 2$.

Let u be the solution to the following p-torsion problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega_l \\ u = 0 & \text{on } \partial\Omega_l, \end{cases}$$

such that

$$\int_{\Omega_l} u \, dx = T_p(\Omega_l),$$

and let us consider the following function, depending only on the last component x_n of $x \in \mathbb{R}^n$,

$$v(x) = \frac{p-1}{p} \left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - |x_n|^{\frac{p}{p-1}} \right],$$

satisfying

$$\begin{cases} -\Delta_p v = 1 & \text{in } \Omega_l \\ v \geqslant 0 & \text{on } \partial\Omega_l. \end{cases}$$

The comparison principle, see [97], ensures that $u \leqslant v$ in Ω_l and, as a consequence,

$$\begin{aligned} T_p(\Omega_l) &= \int_{\Omega_l} u \, dx \leqslant \int_{\Omega_l} v \, dx = \\ &= \frac{p-1}{p} \int_{l^{-\frac{1}{n-1}} C} \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - |x_n|^{\frac{p}{p-1}} \right] dx_n \, d\mathcal{H}^{n-1} \\ &= 2\, \frac{p-1}{p} l^{-1} \int_0^{\frac{l}{2}} \left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - x_n^{\frac{p}{p-1}} \right] dx_n \\ &= 2\, \frac{p-1}{p} \left[1 - \frac{p-1}{2p-1} \right] l^{-1} \left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}} = 2c_p l^{-1} \left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}}. \end{aligned} \tag{4.17}$$

By (4.17) and (4.16), we have

$$T_p(\Omega_l) P^{\frac{p}{p-1}}(\Omega_l) \leqslant 2c_p l^{-1} \left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}} \left(2l^{-1} + l^{\frac{1}{n-1}} \mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}} = c_p \left(1 + \frac{l^{\frac{n}{n-1}}}{2} \mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}.$$

Now, since $f(x) \leqslant f(0)$, we have that, for every bounded, open and convex set Ω,

$$T_{f,p}(\Omega) \leqslant f^{\frac{p}{p-1}}(0)T_p(\Omega). \tag{4.18}$$

It follows that

$$\begin{aligned} T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) &\leqslant f^{\frac{p}{p-1}}(0)T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \\ &\leqslant c_p f^{\frac{p}{p-1}}(0)\left(1+\frac{l^{\frac{n}{n-1}}}{2}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}. \end{aligned} \tag{4.19}$$

Moreover we observe that, if f never vanishes, we can use its monotonicity property to bound μ_f from below in the following way:

$$\mu_f(\Omega) = \int_\Omega f(d(x))\,dx \geqslant f(R_\Omega)|\Omega|,$$

obtaining

$$T_{f,p}(\Omega) \geqslant c_p \frac{f^{\frac{2p-1}{p-1}}(R_\Omega)|\Omega|^{\frac{2p-1}{p-1}}}{f(0)P^{\frac{p}{p-1}}(\Omega)}. \tag{4.20}$$

Joining (4.20) and (4.19), we obtain

$$c_p\frac{f^{\frac{2p-1}{p-1}}(R_{\Omega_l})}{f(0)} \leqslant T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \leqslant c_p f^{\frac{p}{p-1}}(0)\left(1+\frac{l^{\frac{n}{n-1}}}{2}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}.$$

Eventually, passing to the limit when $l \to 0$, observing that $\lim_{l\to 0} R_{\Omega_l} = 0$ and that f is continuous, we have

$$T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \longrightarrow c_p f^{\frac{p}{p-1}}(0).$$

□

Remark 4.4. If we assume that $f : [0, R_\Omega] \to [0, +\infty[$ is a function in $L^\infty([0, R_\Omega])$, then, using the variational characterization (4.2) and the result (45) proved in [49], we have

$$T_{f,p}(\Omega) \geqslant \left(\inf_{t\in[0,R_\Omega]} f(t)\right)^{\frac{p}{p-1}} T_p(\Omega) \geqslant \left(\inf_{t\in[0,R_\Omega]} f(t)\right)^{\frac{p}{p-1}} c_p \frac{|\Omega|^{\frac{2p-1}{p-1}}}{P(\Omega)^{\frac{p}{p-1}}} \tag{4.21}$$

and the sharpness of (4.21) can be proved in an analogous way as in (4.3).

4.3 The quantitative results

Proof of Theorem 4.2

Proof. Let us start by proving (4.5) in the case $n = 2$. If $f \equiv 1$, (4.14) becomes

$$T_p(\Omega) \geqslant c_p \frac{|\Omega|^{\frac{2p-1}{p-1}}}{P^{\frac{p}{p-1}}(\Omega)} + c_p\frac{p}{p-1}\int_0^{R_\Omega}\left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}}(-P'(t))\,dt. \tag{4.22}$$

Joining (1.18), (1.19), (1.28) and (4.22), we have that

$$\begin{aligned}\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}-c_p > c_p\frac{p}{p-1}\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{R_\Omega}\left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}}(-P'(t))\,dt\\ \geqslant \frac{\pi}{2^{\frac{p}{p-1}}}\frac{p}{2p-1}\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{R_\Omega}(R_\Omega-t)^{\frac{2p-1}{p-1}}\,dt\\ \geqslant \frac{\pi}{2^{\frac{p}{p-1}}}\frac{(p-1)p}{(3p-2)(2p-1)}\frac{R_\Omega}{P(\Omega)}\left(\frac{R_\Omega P(\Omega)}{|\Omega|}\right)^{\frac{2p-1}{p-1}}\\ \geqslant \frac{\pi}{2^{\frac{p}{p-1}}}\frac{(p-1)p}{(3p-2)(2p-1)}\frac{R_\Omega}{P(\Omega)}.\end{aligned}$$

Hence, by applying (1.22) and (1.19) we get

$$\mathcal{F}_p(\Omega)-c_p\geqslant K(2,p)\frac{w_\Omega}{\operatorname{diam}(\Omega)}, \tag{4.23}$$

where

$$K(2,p):=\frac{(p-1)p}{2^{\frac{p}{p-1}}3(3p-2)(2p-1)}.$$

We now prove that the exponent of this ratio is sharp. In order to do that, since we have proved (4.23), we only need to find a sequence $\{\Omega_l\}_{l\in\mathbb{N}}$ of convex sets with fixed measure such that

$$M\frac{w_{\Omega_l}}{\operatorname{diam}(\Omega_l)}\geqslant \mathcal{F}_p(\Omega_l)-c_p,$$

for some positive constant M. Let $0<l<1$, we consider the following rectangle

$$\Omega_l=\left(-\frac{1}{2l},\frac{1}{2l}\right)\times\left(-\frac{l}{2},\frac{l}{2}\right)$$

and we notice that its inradius and area are $R_{\Omega_l}=\frac{l}{2}$ and $|\Omega_l|=1$. Let u be the unique solution to

$$\begin{cases}-\Delta_p u=1 & \text{in } \Omega_l\\ u=0 & \text{on } \partial\Omega_l\end{cases}$$

and let us consider the following function

$$v(y)=\frac{p-1}{p}\left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}}-|y|^{\frac{p}{p-1}}\right],$$

which solves

$$\begin{cases}-\Delta v=1 & \text{in } \Omega_l\\ v\geqslant 0 & \text{on } \partial\Omega_l.\end{cases}$$

The comparison principle gives $u\leqslant v$ in Ω_l and

$$T_p(\Omega_l)=\int_{\Omega_l}u_p\,dx\leqslant\int_{\Omega_l}v\,dx.$$

Arguing as in (4.17), we have

$$\int_{\Omega_l}v\,dx=c_p\left(\frac{l}{2}\right)^{\frac{p}{p-1}}.$$

On the other hand, the perimeter of the rectangle is given by

$$P(\Omega_l) = \frac{2}{l}\left(1+l^2\right)$$

and its Taylor expansion with respect to $l > 0$ is

$$P^{\frac{p}{p-1}}(\Omega_l) = \left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1+l^2\right)^{\frac{p}{p-1}} = \left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1+\frac{p}{p-1}l^2+o(l^2)\right).$$

Using (1.19) and (1.22), we get

$$\begin{aligned} T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) - c_p &\leqslant c_p\left(\frac{l}{2}\right)^{\frac{p}{p-1}}\left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1+\frac{p}{p-1}l^2+o(l^2)\right) - c_p \\ &\leqslant 2c_p\frac{p}{p-1}l^2 \leqslant 16c_p\frac{p}{p-1}\frac{R_{\Omega_l}}{P(\Omega_l)} \\ &\leqslant 4c_p\frac{p}{p-1}\frac{w_{\Omega_l}}{\mathrm{diam}(\Omega_l)} \end{aligned}$$

and this concludes the proof in dimension $n = 2$.
Let us now prove (4.5) in dimension $n > 2$. If we choose $f \equiv 1$, (4.14) becomes

$$T_p(\Omega) \geqslant c_p\frac{|\Omega|^{\frac{2p-1}{p-1}}}{P^{\frac{p}{p-1}}(\Omega)} + c_p\frac{p}{p-1}\int_0^{R_\Omega}\left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}}(-P'(t))\,dt. \tag{4.24}$$

As a consequence of the Alexandrov-Fenchel inequality and the isoperimetric inequality for the quermassintegrals (see [122]), we have

$$-P'(t) \geqslant n(n-1)\omega_n^{\frac{1}{n-1}}\left(\frac{P(t)}{n}\right)^{\frac{n-2}{n-1}}. \tag{4.25}$$

Hence, combining (4.25) and (4.24), we have

$$\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p \geqslant c(n,p)\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{R_\Omega}\left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}}P(t)^{\frac{n-2}{n-1}}\,dt. \tag{4.26}$$

Moreover, from (1.18), we obtain that

$$P(t) \geqslant n\omega_n(R_\Omega - t)^{n-1}, \tag{4.27}$$

and so, using (4.27) in (4.26), we get

$$\begin{aligned} \frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p &\geqslant c(n,p)\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{R_\Omega}(R_\Omega - t)^{\frac{2p-1}{p-1}+n-2}\,dt \\ &= C(n,p)\left(\frac{R_\Omega P(\Omega)}{|\Omega|}\right)^{\frac{2p-1}{p-1}}\frac{R_\Omega^{n-1}}{P(\Omega)} \end{aligned} \tag{4.28}$$

If we combine (4.28) with (1.18), with the following estimate (that can be found in [15]):

$$R_\Omega \geqslant \begin{cases} w_\Omega\dfrac{\sqrt{n+2}}{2n+2} & n \text{ even} \\[2ex] w_\Omega\dfrac{1}{2\sqrt{n}} & n \text{ odd,} \end{cases}$$

and with

$$P(\Omega) \leqslant n\omega_n \left(\frac{n}{2n+2}\right)^{\frac{n-1}{2}} \operatorname{diam}(\Omega)^{n-1},$$

we finally get

$$\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p \geqslant C(n,p)\left(\frac{w_\Omega}{\operatorname{diam}(\Omega)}\right)^{n-1}.$$

□

Remark 4.5. As far as the sharpness of (4.5) in the case $n > 2$, we conjecture that the sharp exponent is 1 as in the planar case. Indeed, the minimizing sequence $\{\Omega_l\}$ satisfies

$$T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) - c_p \approx C\frac{w_{\Omega_l}}{\operatorname{diam}(\Omega_l)}.$$

Remark 4.6. As already remarked in the introduction, inequality (4.23) gives an information on the set Ω. Indeed, if

$$\mathcal{F}_p(\Omega) - c_p$$

is small, then the ratio between w_Ω and $\operatorname{diam}(\Omega)$ has to be necessarily small, i.e. Ω must be a thin domain. Moreover, inequality (4.23) tells us also that the infimum of $\mathcal{F}_p(\Omega)$ is not achieved among bounded, open and convex sets. Assuming by contradiction that there exists a bounded, open and convex set $\tilde\Omega$ such that

$$\mathcal{F}_p(\tilde\Omega) = c_p,$$

we have that

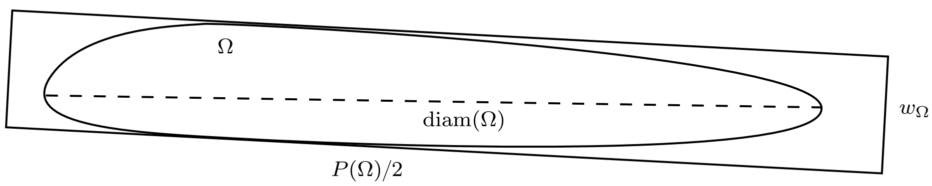
$$\frac{w_{\tilde\Omega}}{\operatorname{diam}(\tilde\Omega)} < \varepsilon \qquad \forall\varepsilon > 0,$$

which is impossible.

Theorem 4.2 only tells us that any minimizing sequence of $\mathcal{F}_p(\cdot)$ is a sequence of thinning domains. On the other hand, Theorem 4.3 gives us more precise information on the geometry of such minimizing sequence.

Proof of Theorem 4.3

Proof. Let Ω be a bounded, open and convex set with nonempty interior in $\mathbb{R}^2$ and let us consider a rectangle Q of sides $P(\Omega)/2$ and w_Ω containing Ω. Such a rectangle exists, since it is enough to choose the shorter side of Q parallel to the direction of w_Ω and to recall the lower bound in (1.22) (see Figure 4.3).



Now, let $\sigma > 0$ be such that

$$\frac{1}{4^3 \cdot 6} - \frac{\pi^2}{2^3 \cdot 3^3} \frac{\sigma^2}{K^2(2)} \geqslant 0; \tag{4.29}$$

$$\frac{1}{3^3 \cdot 6} - \frac{\pi}{48} \frac{\sigma}{K(2))} - \frac{\pi^2}{2^5 \cdot 3} \frac{\sigma^2}{K^2(2)} \geqslant 0; \tag{4.30}$$

$$\frac{\pi}{4} - \frac{\pi}{2\sqrt{3}} \frac{\sigma}{K(2)} \geqslant \frac{4}{3\sqrt{3}}, \tag{4.31}$$

where $K(2)$ is a constant defined in (**??**). If

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \sigma,$$

then, by (1.19) and (1.18), we have

$$\frac{|Q \triangle \Omega|}{|\Omega|} = \left(\frac{P(\Omega) w_\Omega}{2\,|\Omega|} - 1 \right) \leqslant \left(\frac{3}{2} \frac{P(\Omega) R_\Omega}{|\Omega|} - 1 \right) \leqslant 2.$$

So, it follows that

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \frac{\sigma}{2^3} 2^3 \geqslant \frac{\sigma}{2^3} \left(\frac{|Q \triangle \Omega|}{|\Omega|} \right)^3.$$

On the other hand, let us assume that

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} < \sigma. \tag{4.32}$$

By Theorem 4.2, we have that

$$\frac{w_\Omega}{\operatorname{diam}(\Omega)} \leqslant \frac{1}{K(2)} \left[\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \right] < \frac{\sigma}{K(2)}, \tag{4.33}$$

and we observe that, by the choice of σ made in (4.29)-(4.31), a ball cannot satisfy (4.32).

Now, arguing as in (4.13) with $f \equiv 1$ and $p = 2$, we know that

$$T(\Omega) \geqslant \int_0^{R_\Omega} \frac{\mu^2(t)}{P(t)}\, dt. \tag{4.34}$$

We set $\rho = \dfrac{P^2(\Omega)}{4\pi} - |\Omega|$ and $p_R = P(\Omega) - 2\pi R_\Omega$ and we observe that they are both strictly positive by the isoperimetric inequality and the monotonicity of the perimeter, respectively. Using inequalities (1.26) and (1.27) in (4.34), we have that

$$\begin{aligned}
T(\Omega)P^2(\Omega) &\geqslant P^2(\Omega) \int_0^{R_\Omega} \frac{(|\Omega| - P(\Omega)t + \pi t^2)^2}{P(\Omega) - 2\pi t}\, dt \\
&= P^2(\Omega) \int_0^{R_\Omega} \frac{1}{P(\Omega) - 2\pi t} \left(\frac{(P(\Omega) - 2\pi t)^2}{4\pi} - \left(\frac{P^2(\Omega)}{4\pi} - |\Omega| \right) \right)^2 dt \\
&= P^2(\Omega) \int_0^{R_\Omega} \left(\frac{(P(\Omega) - 2\pi t)^3}{(4\pi)^2} - \frac{\rho}{2\pi} (P(\Omega) - 2\pi t) + \frac{\rho^2}{P(\Omega) - 2\pi t} \right) dt \\
&= \frac{P^2(\Omega)}{2\pi} \left(\frac{P^4(\Omega) - p_R^4}{4(4\pi)^2} - \frac{\rho}{4\pi} \left(P^2(\Omega) - p_R^2 \right) - \rho^2 \log \left(1 - \frac{2\pi R_\Omega}{P(\Omega)} \right) \right),
\end{aligned} \tag{4.35}$$

and, using Newton's formula and the Taylor series for the logarithm, we get

$$\begin{aligned}
&P^2(\Omega) - p_R^2 = 4\pi R_\Omega P(\Omega) - 4\pi^2 R_\Omega^2;\\
&P^4(\Omega) - p_R^4 = 8\pi R_\Omega P^3(\Omega) - 24\pi^2 R_\Omega^2 P^2(\Omega) + 32\pi^3 R_\Omega^3 P(\Omega) - 16\pi^4 R_\Omega^4;\\
&-\log\left(1-\frac{2\pi R_\Omega}{P(\Omega)}\right) = \sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{2\pi R_\Omega}{P(\Omega)}\right)^i \geqslant \frac{2\pi R_\Omega}{P(\Omega)} + \frac{2\pi^2 R_\Omega^2}{P^2(\Omega)} + \frac{8}{3}\frac{\pi^3 R_\Omega^3}{P^3(\Omega)} + \frac{4\pi^4 R_\Omega^4}{P^4(\Omega)}.
\end{aligned} \tag{4.36}$$

By (4.36) and (4.35), dividing by $|\Omega|^3$ and subtracting $1/3$, we have

$$\begin{aligned}
\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant &\frac{1}{3}\left(\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right)^3 + \pi\frac{R_\Omega^2}{|\Omega|^2}\left(|\Omega| - \frac{2}{3}P(\Omega)R_\Omega\right)\\
&+ \frac{4}{3}\pi^2\frac{R_\Omega^3}{P(\Omega)\,|\Omega|^2}\left(|\Omega| - \frac{3}{4}P(\Omega)R_\Omega\right).
\end{aligned} \tag{4.37}$$

As an intermediate step we want to prove the following inequality:

$$\begin{aligned}
&\frac{1}{3}\left(\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right)^3 + \pi\frac{R_\Omega^2}{|\Omega|^2}\left(|\Omega| - \frac{2}{3}P(\Omega)R_\Omega\right)\\
&\quad + \frac{4}{3}\pi^2\frac{R_\Omega^3}{P(\Omega)\,|\Omega|^2}\left(|\Omega| - \frac{3}{4}P(\Omega)R_\Omega\right) \geqslant \frac{1}{6}\left(\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right)^3,
\end{aligned} \tag{4.38}$$

that, combined with (4.37), implies

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \frac{1}{6}\left(\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right)^3, \tag{4.39}$$

where we choose the constant $1/6$ as an arbitrary constant less then $1/3$. In particular, (4.38) is equivalent to

$$\frac{1}{6}\big(P(\Omega)R_\Omega - |\Omega|\big)^3 + \pi R_\Omega^2\,|\Omega|\left(|\Omega| - \frac{2}{3}P(\Omega)R_\Omega\right) + \frac{4}{3}\pi^2\frac{R_\Omega^3}{P(\Omega)}\,|\Omega|\left(|\Omega| - \frac{3}{4}P(\Omega)R_\Omega\right) \geqslant 0. \tag{4.40}$$

In order to prove (4.40), we distinguish three cases:

1) if $|\Omega| \geqslant \frac{3}{4}P(\Omega)R_\Omega$, then (4.40) is trivial, since the left hand side is the sum of positive quantities;

2) if $\frac{2}{3}P(\Omega)R_\Omega \leqslant |\Omega| < \frac{3}{4}P(\Omega)R_\Omega$, using (1.19), (1.22), (4.29) and (4.33), we have

$$\begin{aligned}
&\frac{1}{6}\big(P(\Omega)R_\Omega - |\Omega|\big)^3 + \pi R_\Omega^2\,|\Omega|\left(|\Omega| - \frac{2}{3}P(\Omega)R_\Omega\right) + \frac{4}{3}\pi^2\frac{R_\Omega^3}{P(\Omega)}\,|\Omega|\left(|\Omega| - \frac{3}{4}P(\Omega)R_\Omega\right)\\
&\qquad \geqslant P^3(\Omega)R_\Omega^3\left(\frac{1}{4^3\cdot 6} - \frac{2\pi^2}{3^3}\frac{R_\Omega^2}{P^2(\Omega)}\right)\\
&\qquad \geqslant P^3(\Omega)R_\Omega^3\left(\frac{1}{4^3\cdot 6} - \frac{\pi^2}{2^3\cdot 3^3}\frac{w_\Omega^2}{\operatorname{diam}^2(\Omega)}\right)\\
&\qquad \geqslant P^3(\Omega)R_\Omega^3\left(\frac{1}{4^3\cdot 6} - \frac{\pi^2}{2^3\cdot 3^3}\frac{\sigma^2}{K^2(2)}\right) \geqslant 0.
\end{aligned} \tag{4.41}$$

3) if $\frac{1}{2}P(\Omega)R_\Omega \leqslant |\Omega| < \frac{2}{3}P(\Omega)R_\Omega$, arguing as before, we have

$$\begin{aligned}\frac{1}{6}\big(P(\Omega)R_\Omega - |\Omega|\big)^3 + \pi R_\Omega^2 |\Omega| \left(|\Omega| - \frac{2}{3}P(\Omega)R_\Omega\right) + \frac{4}{3}\pi^2 \frac{R_\Omega^3}{P(\Omega)} |\Omega| \left(|\Omega| - \frac{3}{4}P(\Omega)R_\Omega\right)\\ \geqslant P^3(\Omega)R_\Omega^3 \left(\frac{1}{3^3\cdot 6} - \frac{\pi}{48}\frac{w_\Omega}{\mathrm{diam}(\Omega)} - \frac{\pi^2}{2^5\cdot 3}\frac{w_\Omega^2}{\mathrm{diam}^2(\Omega)}\right)\\ \geqslant P^3(\Omega)R_\Omega^3 \left(\frac{1}{3^3\cdot 6} - \frac{\pi}{48}\frac{\sigma}{K(2)} - \frac{\pi^2}{2^5\cdot 3}\frac{\sigma^2}{K^2(2)}\right) \geqslant 0.\end{aligned} \tag{4.42}$$

So, we have proved the intermediate step (4.39). Now, by combining (4.39) and (1.20), we deduce

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \frac{1}{6}\left[\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right]^3 \geqslant \frac{1}{6}\left[\frac{P(\Omega)w_\Omega}{2|\Omega|} - 1 - \frac{1}{\sqrt{3}}\frac{w_\Omega^2}{|\Omega|}\right]^3. \tag{4.43}$$

Using (1.21), (1.20), (4.33) and (4.31), we have

$$\begin{aligned}\frac{P(\Omega)w_\Omega}{2\,|\Omega|} - 1 \geqslant \frac{P(\Omega)R_\Omega}{|\Omega|} - 1 \geqslant \pi\frac{R_\Omega^2}{|\Omega|} \geqslant \frac{\pi}{|\Omega|}\left(\frac{w_\Omega}{2} - \frac{w_\Omega^2}{\sqrt{3}P(\Omega)}\right)^2\\ = \frac{w_\Omega^2}{|\Omega|}\left(\frac{\pi}{4} - \frac{\pi}{\sqrt{3}}\frac{w_\Omega}{P(\Omega)} + \frac{\pi}{3}\frac{w_\Omega^2}{P(\Omega)^2}\right)\\ \geqslant \frac{w_\Omega^2}{|\Omega|}\left(\frac{\pi}{4} - \frac{\pi}{2\sqrt{3}}\frac{w_\Omega}{\mathrm{diam}(\Omega)}\right)\\ \geqslant \frac{w_\Omega^2}{|\Omega|}\left(\frac{\pi}{4} - \frac{\pi}{2\sqrt{3}}\frac{\sigma}{K(2)}\right)\\ \geqslant \frac{4}{3\sqrt{3}}\frac{w_\Omega^2}{|\Omega|}\end{aligned} \tag{4.44}$$

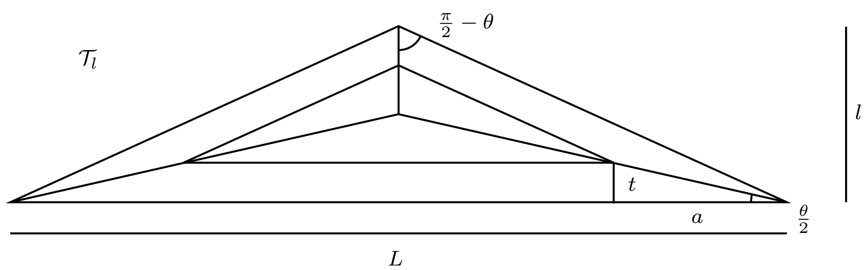
Finally, by combining (4.43) and (4.44), we get the conclusion

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geqslant \frac{1}{6}\left[\frac{P(\Omega)R_\Omega}{|\Omega|} - 1\right]^3 \geqslant \tilde{K}\left[\frac{|Q \triangle \Omega|}{|\Omega|}\right]^3. \tag{4.45}$$

□

The next remark shows that a sequence of thinning triangles is not sharp for (4.5) and this is the reason for which we need Theorem 4.3 to obtain more precise information.

Remark 4.7. Let us consider a sequence of isosceles triangles $\mathcal{T}_l$ of base L and height l such that $|\mathcal{T}_l| = 1$.



If we compute (4.39) on the sequence $\mathcal{T}_l$ and we use (1.18), we get, for every l,

$$\frac{T(\mathcal{T}_l)P^2(\mathcal{T}_l)}{|\mathcal{T}_l|^3} - \frac{1}{3} \geqslant \frac{1}{6}\left(\frac{P(\mathcal{T}_l)R_{\mathcal{T}_l}}{|\mathcal{T}_l|} - 1\right)^3 = \frac{1}{6} \tag{4.46}$$

and, so, the quantity on the left-hand side of (4.46) is bounded away from zero.

Remark 4.8. We point out that

$$\frac{P(\Omega)R_\Omega}{|\Omega|} - 1 \geqslant K\frac{|Q \triangle \Omega|}{|\Omega|},$$

in (4.45) is a quantitative version of the inequality in the right hand side of (1.18).

4.4 Open Problems

We conclude by listing the following open problems:

- We believe that the exponent 3 in the inequality (4.6) is not sharp: we expect it to be 1. We clarify that in Example (3).
- We conjecture that the sharp exponent in (4.5) in the case $n > 2$ is 1 (see Remark 4.5).
- The results contained in Theorem 4.3 could be studied in higher dimension and extended to the (f, p)-torsional rigidity. Our proof cannot be adapted to higher dimension because in dimension $n > 2$ we do not have any more Steiner formulas for inner parallel sets (1.26) and (1.27).

Example 3. *Let* $\Omega_l = \left(-\frac{1}{2l}, \frac{1}{2l}\right) \times \left(-\frac{l}{2}, \frac{l}{2}\right)$ *be a sequence of rectangles of measure* 1*. It is possible to give an explicit upper bound to the functional* $\mathcal{F}_2(\Omega_l)$*. Hence, following the computations in* (4.17)*, we have*

$$\mathcal{F}_2(\Omega_l) - c_2 \leqslant 2l^2.$$

Considering the rectangle Q *with sides* $P(\Omega_l)/2$ *and* w_Ω *containing* Ω_l*, that is*

$$Q = \left(-\frac{1+l^2}{2l}, \frac{1+l^2}{2l}\right) \times \left(-\frac{l}{2}, \frac{l}{2}\right),$$

it is straightfoward to compute

$$|\Omega_l \Delta Q| = 2l^2.$$

Hence, we have

$$2 \geqslant \frac{\mathcal{F}_2(\Omega_l) - c_2}{|\Omega_l \Delta Q|} \geqslant \tilde{K}\ |\Omega_l \Delta Q|^2 = \tilde{K}\, l^4.$$

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