Università degli Studi di Napoli Federico II

Dipartimento di Fisica "E. Pancini"

Dottorato di Ricerca in Fisica

Ciclo XXXV (2019-2022)

Settore Scientifico Disciplinare FIS/08

Tesi di Dottorato

A New Look at the Antikythera Mechanism

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Contents

Introduction

The encrusted fragments of the device known as *Antikythera Mechanism* (hereafter AM) were found by a group of sponge-fishers near the coasts of the island of Antikythera in the summer of 1901. This serendipitous discovery turned out to be one of the most important events for the history of ancient science of the XX century.

The shipwreck that saved the AM occurred around 60 BC, but the estimated date for its construction is around the middle of II cent. BC. In total 82 fragments of the original device survive, ranging in size from a few millimeters to about ten centimeters (Fig. 1). First inspections immediately suggested that they belonged to some mechanical artifact involving gear-work, and the readable text engraved on them clearly indicated a connection with astronomical matters. After the intermittent and turbulent early studies following the discovery, strongly limited by the fragility of the fragments, major steps towards the understanding of the actual functions of the original device were made in the seventies by the historian of technology Derek de Solla Price. He was the first to propose a reconstruction of the internal structure of (the surviving part of) the AM based on radiographs of the fragments, and his *Gears from the Greeks* was a mile-stone in the history of technology.[1] Many of the conjectures there exposed about the functions of the AM were confirmed by later studies, and, in particular, Price was the first to recognize the AM as a *computer*. At the end of the century, X-ray tomography was used in the context of the *Antikythera Mechanism Research Project* to get images of unprecedented accuracy of the inscriptions and the internal structure, and these data form the basis of our current understanding of the AM. Some doubts still remain, but after more than a century of scholarly effort a good agreement has been reached about the general features of the original device, which are the following.[2]

The AM was a portable *astronomical computer* (33 cm in height, 18 cm in breadth and 8 cm in depth), composed by more than thirty bronze gears arranged in different trains and originally enclosed in a case (today lost). All the gear trains, hidden to view, were connected the main wheel (well visible in fragment A, see Fig. 2) and ended on pointers moving along dials. In total there were (at least) twelve pointers, seven on the front side and five on the back side, all moving simultaneously and set in motion by the same input rotation (Fig. 5). This was given by hand through an external knob, connected via a crown gear to the main

[1] De Solla Price 1974.

[2] For more details see Jones 2017; Seiradakis and Edmunds 2018; Bruderer 2020 and bibliography therein.

Figure 1: The 82 surviving fragments of the AM. All of them are today exposed in Athens' National Archeological Museum.

wheel. Outside the dials there were inscriptions giving additional informations (including a *parapegma*, i.e. a calendar of stellar phases) and both sides were protected by two cover plates. These were engraved in the interior side with a sort of user manual, explaining the phenomena shown by the device and the meaning of the different dials.

On the front side, the motion of the seven pointers was intended to simulate the observable longitudinal motions of Sun, Moon and five planets along the zodiac, represented by a scale divided in 360 parts (Fig. 3). Outside of it there was a moveable ring representing a calendar, months being indicated with their Egyptian names. For a long time this calendar was thought to be a 365-days solar year composed by 12 months of 30 days, plus 5 intercalary days, but recent investigations have strongly suggested that it represented a 354-day lunar calendar composed of 12 months of alternate full and hollow months of 30 or 29 days.[3] Either case, this calendar ring had to be adjusted from time to time to account for the gap between astronomical and civil year. So, on the internal zodiac scale one could read the *angular positions* of the seven heavenly bodies, and on the outer calendar scale the *date* on which such angular configuration of heavenly bodies could be observable in the sky. The date was probably indicated by a separate date pointer, that could also serve the purpose to indicate the time elapsed between any two configurations. The phases of the Moon were also displayed via a rotating sphere colored in black and white, and according to recent reconstructions an additional pointer indicated also the position of the Moon's orbit nodes, a very relevant information for eclipse predictions.

On the back side, there were two engraved spiral dials, with two pointers

[3]Budiselic et al. 2020.

Figure 2: The main surviving fragments of the AM. In Fragment A (in the middle) the main wheel is clearly visible (source: Bruderer 2020).

Figure 3: Reconstruction of the front dial of the AM (source: Freeth, Higgon, et al. 2021).

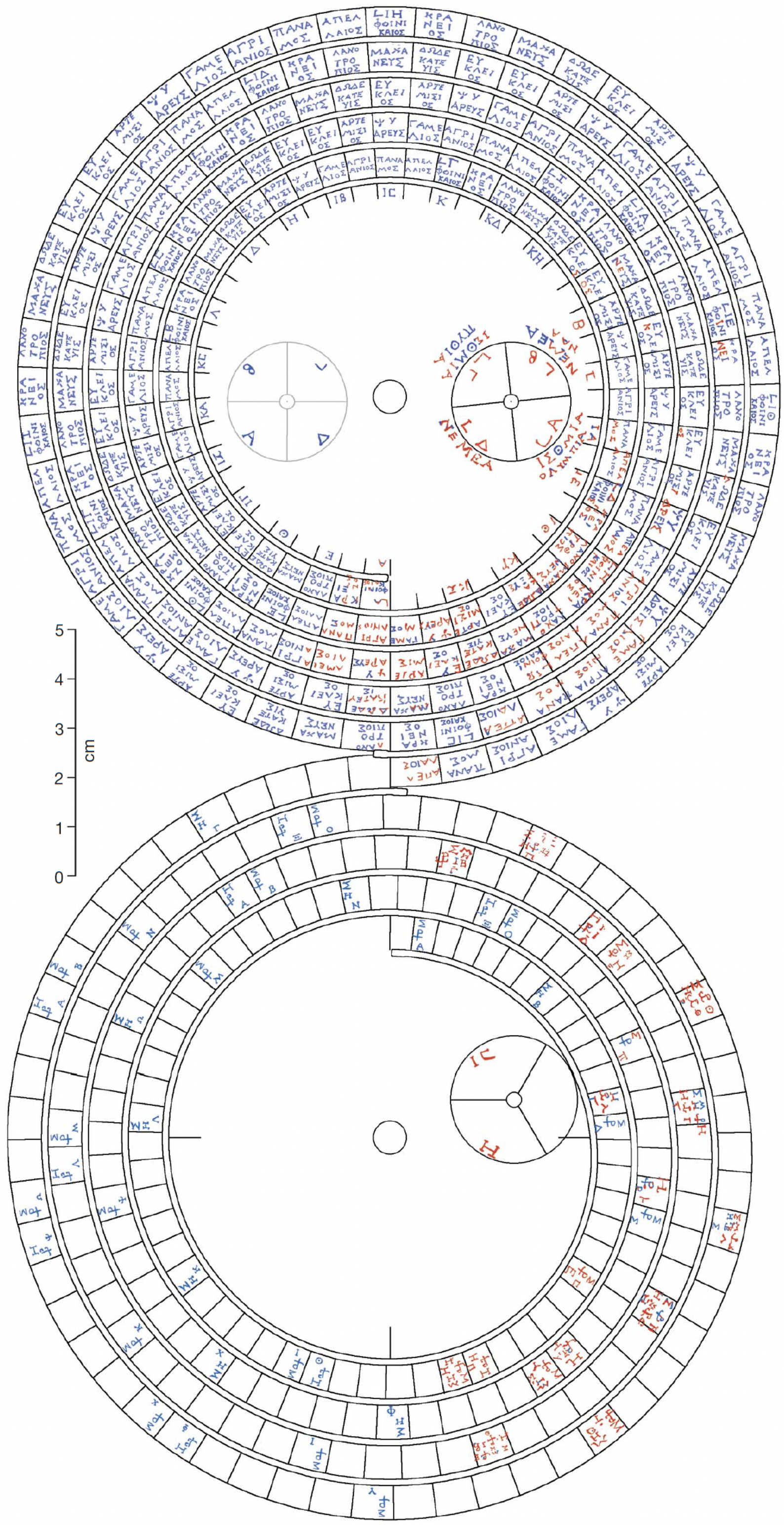

Figure 4: Reconstruction of the back dials of the AM (source: Freeth, Jones, et al. 2008).

moving along them in the outward direction with a mechanism similar to that of modern vinyl players (Fig. 4).

The upper scale was divided in 235 cells representing lunar synodic months and composing one *Metonic Cycle*, approximately corresponding to 19 tropical years. The whole spiral was essentially a lunisolar calendar, not too far in its conception from our own, with the position of the pointer indicating the current month. A fairly sophisticated system indicated if the month was full or hollow, and at the end of a full cycle the pointer had to be reset by hand at the beginning of the spiral. Two smaller dials were placed inside the Meton spiral. One represented the *Callippic Cycle*, a longer 76-years cycle whose purpose was to prescribe a one-day correction to the spiral calendar every four *Metonic Cycles*; the other displayed a four-years cycle of some selected *panhellenic games*, events of great cultural and religious importance for the Greeks.

The lower spiral was divided in 223 cells representing lunar synodic months and composing one *Saros Cycle*, a lunisolar cycle regulating the occurrence of eclipses, one Saros corresponding approximately to the time necessary for the Sun and the Moon to return to the same relative position and in the same points of the ecliptic. On some cells of the spiral there were engraved *glyphs* indicating if in that month a lunar and/or solar eclipse possibility occurred, with referenced informations also about its duration and peak hour. A smaller auxiliary dial inside the Saros spiral represented the *Exeligmos Cycle*, a 54-years cycle that served to correct the indicated hour by 8 or 16 hours, due to the 8-hours gap between a full Saros cycle and the 223 synodic months represented on the dial. It has been suggested that some marks of doubtful interpretation indicated if the eclipse occurred at maximum and minimum distances, two parameters connected to the duration of the event and to the character of the eclipse (total or anular).

The surviving fragments of the AM contain only the gear trains relative to the motions of Sun and Moon (including the calendar and the eclipse-calculator on the back side), whereas all the planetary gears are lost. The surviving *pin-slot device* used to alter the mean motion of the Moon makes quite likely that also in the case of planets variations of velocity were shown. Apart from this clue and a few numbers representing planetary cycles readable in the inscriptions (of unprecedented accuracy and not found in other sources), the reconstruction of the planetary gearings remains for the most part conjectural (Fig. 6).

Another matter of conjecture is that of the authorship. Who *designed* the AM? It is generally accepted that it was the ripe fruit of the Greek tradition of *sphere-making* (in Greek *sphairopoiia*), of which the most famous exponent was Archimedes (III cent. BC), who devoted a full treatise to this art. Indeed, in Cicero there are descriptions of some physical models that Archimedes would have constructed that are quite consistent with the features of the AM, namely the fact that by a single *conversio* it represented the simultaneous motions of Sun, Moon and Planets.[4] It is generally thought that Archimedes' device must have been even more complex than the AM, being *tridimensional* rather than *plane*. For chronological reasons the AM cannot have been built by Archimedes himself, but it is commonly regarded as part of his scientific legacy, probably the result of

[4]The relevant passages of Cicero are quoted in De Solla Price 1974.

Figure 5: Expanded view of Freeth's reconstruction of the AM (Freeth, Higgon, et al. 2021) (source: Bruderer 2020).

a further refinement of his work.

Many contextual elements suggest that Hipparchus (II cent. BC), the last and maximum astronomer of the Hellenistic age, could have been involved in the design of the AM. Among these, the connections of the device with the island of Rhodes, an important center for mechanical studies where Hipparchus was active in the last part of his life, i.e. right at the estimated time of construction of the AM.[5]

Most scholars agree on the idea that the AM was intended for teaching purposes and not as an instrument for professional astronomers. This of course says nothing about its accuracy (provided, of course, that teaching is regarded as a serious activity), which in fact is limited in principle only by technical reasons, e.g. the small size of the components and of the scale-divisions. In this regard, it is worth noting that the AM is, by construction, *scalable at pleasure*, since only the proportions between the sizes of moving parts are relevant for its proper working. This kind of *symmetria*, as we'll see, was indeed regarded by Philo (III cent. BC) as a general principle of machine-design.

Since it is unreasonable to think that such a complex device as the AM could

[5]We find unlikely that the stoic philosopher Posidonius (I cent. BC) designed or built some device similar to the AM, as has been inferred by some scholars on the basis of Cicero's testimony, for the simple reason that Posidonius was neither a mathematician nor an engineer. It appears more reasonable to interpret Cicero's passage as an indication that in the I cent. BC some device similar to the AM was still available in Posidonius' school.

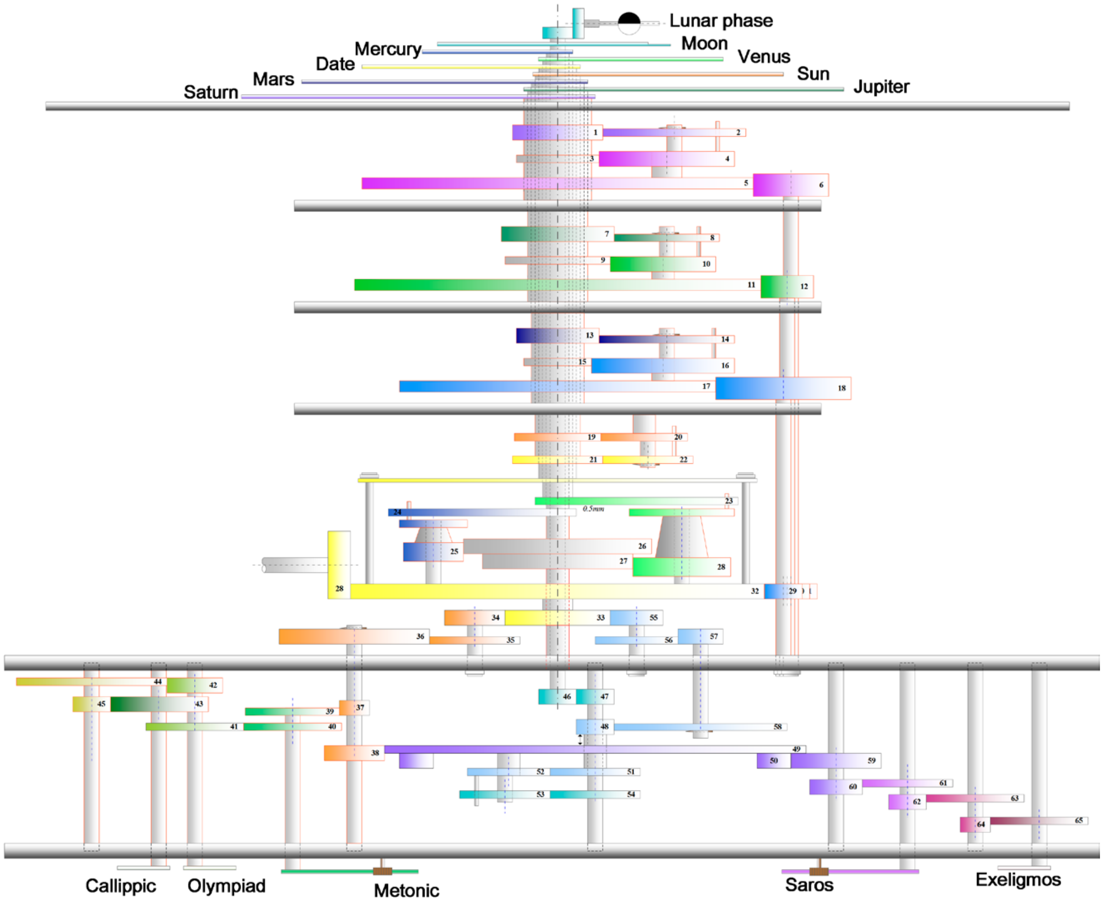

Figure 6: Reconstruction of the mechanical structure of the AM, including the conjectured planetary gear-trains (source: Lin and Yan 2016, p. 56).

be the result of trial-and-error,[6] we feel *compelled* to ask: *on what mathematical theory was based the design of the AM*? The question appears legitimate, since the AM is clearly a brilliant (and unique) specimen of Hellenistic *scientific technology*, an artifact whose design *must* be based on some *theoretical* model of the heavenly motions, that, in some way, matched the possibility of plane mechanization via parallel and multi-axial trains of toothed wheels.

The AM is also, among other things, the only extant astronomical source dating back to the golden age of Hellenistic science. Therefore, its relevance for the reconstruction of the most mature developments of Greek mathematical astronomy is *immense*, its very same survival being a miracle in the general loss of scientific works dating back to the Hellenistic period. Nevertheless, the *theoretical* problem posed by the AM has not received much attention in the existing literature, and indeed the impact of the AM on the current views about Greek astronomy has been practically null. We have therefore the paradoxical circumstance that the current understanding of the AM has rewritten the history of ancient *technology* without even touching upon the history of ancient *science*. How could it be?

For the phenomena shown on the back side the problem, the theoretical problem is indeed relatively simple and the question appears to be settled, albeit there are still controversies on the details of the eclipse scheme.[7] In any case, all the informations shown on the spiral and auxiliary dials pertained only to the proper *synchronization* of the motions of Sun and Moon, and the only astronomical knowledge involved is that of the the appropriate lunisolar cycles. These have been reconstructed from the counting of tooth-numbers and on the basis of the readable inscriptions.[8] It turns out that the astronomical cycles involved in the design of the back-side of the AM are of Babylonian origin and were common knowledge among Greek astronomers way before the middle of II cent. BC.

For the front side the problem is far more complex. Its design involves the theoretical problem of representing via a single rotatory input the simultaneous motions of Sun, Moon and planets *as they are observable from the Earth*. In other terms, the front dial of the AM answered the question: when the Sun is *there* at such a *date*, *where should I look* to see the body x? To answer such a question is equivalent to solve the theoretical problem of *synchronizing all the heavenly motions with one another*, and, then, to represent them in a geocentric reference. Such a task appears very ambitious also by modern standards.

The answer one finds in the existing literature about the AM is that it incorporated some cruder version of Ptolemaic models, i.e. some simple eccentric/epicycle models. Such interpretation fits well with Ptolemy's testimony about the results achieved by his Hellenistic predecessors (namely Hipparchus), and the analogy between the *pin-slot device* found in the Moon gear-train and a suitably calibrated epicycle/eccentric construction has been taken as a confirmation of this view.[9] Therefore, according to this interpretation, the accuracy of the astronomical

[6] In M. Efstathiou et al. 2013 the technical difficulties involved are discussed in an engineering perspective. See also Voulgaris, Mouratidis, and Vossinakis 2019 for a focus on the tools involved in the construction of the parts of the AM.

[7] Iversen and Jones 2019; Voulgaris, Mouratidis, Vossinakis, and Bokovos 2021.

[8] See in particular Freeth, Bitsakis, et al. 2006; Freeth, Jones, et al. 2008; Freeth 2019.

[9] C. Carman, Thorndike, and Evans 2012.

predictions given by the AM was inferior to that of Ptolemaic models, whose predictive power comes mostly from the introduction of the equant point.[10]

This interpretation of the AM is coherent with the general historical narrative which regards the *Almagest* as the culmination point of a continuous, homogeneous and uninterrupted development of Greek mathematical astronomy, going from its origins up to Ptolemy's time. In this view, the *Almagest* would be the only extant astronomical treatise of its kind because it superseded all the preceding works treating the same subjects, just as those of Euclid are the only surviving *Elements* of Greek geometry.[11] Such a *continuist* view seems to be shared by most of the scholars who actively worked on the study of the AM.

A different reconstruction of the general history of Greek astronomy has been proposed by those who emphasized the relevance of the cultural breakdown that occurred in the Mediterranean world at the middle of II cent. BC, when the Romans expanded their dominions to the Hellenistic kingdoms of Greece, Egypt and Mesopotamia. The events connected to such a huge expansion caused the end of the the golden age of Hellenistic science, as clearly shown by the subsequent decline of Alexandria's museum, the main scientific center of antiquity. According to this reconstruction, which was much more fashionable among historians until the beginning of the XX century,[12] Ptolemy *cannot* be considered as a direct successor of scientists like Euclid, Archimedes and Apollonius, but rather as a (very skilled of course) mathematician relying on the works of his predecessors but animated by a very different conception of mathematical sciences (and of astronomy, in particular). Indeed, we think that a direct comparison between Ptolemy's *Mathematical Syntaxis* and any one of Archimedes' extant works is sufficient to convince any scientific reader of the abyss that separates the two authors.

In this latter view, to which we totally adhere, the *Almagest* clearly *cannot* be taken as a paradigm for the reconstruction of the mathematical theory presiding the design of the AM, which is the ripe fruit of a scientific tradition interrupted three centuries before Ptolemy and animated by very different ideas about the character and scope of mathematics and astronomy. Indeed, one may remark that the very same *existence* of a device like the AM is at odds with Ptolemy's claim that, before him, Hipparchus (and, we infer, no one) had *even began* to establish a planetary theory.

In any case, the problem of the relationship between Ptolemy and his Hellenistic sources (notably Hipparchus) is a very complex one, and the origins of Ptolemaic models are, to the least, matter of debate. Historical research and fact-checking on the *Almagest* brought convincing evidence that Ptolemy is not always a reliable source about the results achieved by his predecessors. Well-known examples are the misappropriation of Hipparchus' star catalogue (recently found in a palimpsest), the absence from the *Almagest* of important ideas dating back at least to III cent. BC (like the heliocentric hypothesis, despite traces of it have

[10]For the importance of the equant-point for the effectiveness of Ptolemy's models see Evans 1984.

[11]See for example O. Neugebauer 1969; Toomer 1984; Pedersen 2011.

[12]Among others, we just mention Leopardi 1815; Delambre 1817; Loria 1893; Russo 2004.

been recognized)[13] and, above all, the numerous inconsistencies between Ptolemy's claims and the actual astronomical observations he could have made. These were first noted by Jean-Baptiste Delambre at the beginning of the XIX century,[14] and after him many others have questioned the authenticity of the observations Ptolemy reports in the *Almagest*.[15]

Another cautionary argument to the exclusive use of the Ptolemy's testimony for the reconstruction of Hellenistic astronomy is that he probably had no access to Hipparchus' latest works, that in the three intervening centuries of turbulent events never found their way to the declining Library of Alexandria. On the contrary, there are reasons to believe that valuable informations about Hipparchus' latest works were more easily transmitted in the Latin literature. In this regard, it is important to remember that the research and editorial activity that had made Alexandria's Library the pulsating center of Hellenistic science rapidly ceased after the dramatic events of 146-145 BC, when the Romans put a military at the head of the Museum. On the contrary, Rhodes remained an active scientific center for some time. Overall, the commonplace idea that Ptolemy collected all the relevant results and methods of previous astronomy seems to be the result of a tautology, the *Almagest* itself having been used as the main historical source for the reconstruction itself.[16]

Everything considered, despite Ptolemy's *Almagest* is undoubtedly an important source of informations about the work of previous astronomers, there is nonetheless a considerable risk of being misled if one projects Ptolemy's own conceptions of mathematical astronomy onto Hellenistic mathematicians of III-II cent. BC, namely on those, like Archimedes, who probably played a role in the design of devices like the AM.[17]

If then, as after all seems much more natural, we take the extant Greek mathematical works up to II cent. BC as a leading guide in the necessary guesswork involved in the reconstruction of the theory underlying the AM, we are in a completely different world: geometric algebra, numerical progressions, theory of proportions, exhaustion methods, theory of conic sections and a whole arsenal of mathematical techniques that is *completely absent* from Ptolemy's works becomes available. Obviously, the mathematics we find in the extant works of Euclid, Archimedes and Apollonius must be regarded as only a *lower boundary* to what could have been involved in the the original design of the AM. To this, we

[13] See for example Rawlins 1987 and Neugebauer 1975, p. 146.

[14] Delambre 1817, p. xxvi.

[15] It is worth mentioning Newton 1977, where an extensive study of Ptolemy's reported observations is carried out. The title of the book is sufficient to indicate the level reached by the debate, but, everything considered, the factual results of Newton's analysis seem hardly contestable.

[16] This remark and the previous cautionary argument are in Russo 1994.

[17] Overall, the idea I get from the existing evidence is that Ptolemy heavily relied on the work of Hellenistic mathematicians, but reinterpreting it in light of his own pseudo-aristotelian philosophical views. Elsewhere I set forth the conjecture that the *Almagest* could be the result of a reverse-engineering based on a device not too far from the AM. For more details see Amabile 2020, where it is also attempted an explanation of why a device like the AM could not be considered by Ptolemy's own criteria an actual proof of the existence of a planetary theory, reconciling his primacy claim with his familiarity with *sphairopoiia*.

should add what we are still learning about Babylonian astronomy of the Seleucid period, since the decipherment of cuneiform tablets keeps revealing an unexpected complexity of numerical computational techniques that Hellenistic mathematicians of the II cent. BC had probably incorporated in their astronomical practice.

Also, if we agree that the AM was a follow-up of an Archimedean tradition of *sphere-making*, the theoretical question above asked overlaps with another, that, as far as we know, has never been really addressed: *what kind of theory was exposed in Archimedes' lost treatise on sphere-making?* A conjectural answer to this question, if not of historical value by itself, may well be relevant for the not-less conjectural reconstruction of the planetary gear-work of the AM. Conversely, it seems reasonable that the AM itself can indicate the path for the restoration of the Greek *theory of sphairopoiia*, that, as far as we can tell, died with Archimedes during the siege of Syracuse in 212 BC.

From this perspective, the historical and theoretical question posed by the discovery of the AM becomes much more difficult (and, of course, way more interesting), but it is our conviction that only in this way we can hope to restore the original *form* and *meaning* of such an extraordinary device. The fundamental idea that underlies *all* the present dissertation is that we have still much to learn *about* and *from* the AM.

Inevitably, the route we will follow to give a different look at the AM will be long and a little tortuous, since it involves many ideas that, like the AM itself, are at odds with deep-seated views about Greek science. It is clear that the hermeneutics of a similar artifact touches upon delicate epistemological problems pertaining to mathematical sciences in general, and mathematical astronomy in particular. These must be taken into careful account, so the first thing to do is to put the AM in its own historical, philosophical and scientific context. This is the aim of the first chapter, in which we will dwell at length on the general epistemological framework of Greek mathematics, with a special emphasis on astronomical problems. At the end of it, we will state in general terms our conjecture about what kind of mathematical theory grounded the *conception* and the actual *design* of a device like the AM.

The second chapter is devoted to the work of William Rowan Hamilton, Andrews Professor of Astronomy at the Trinity College of Dublin and Royal Astronomer of Ireland from 1827 to 1865. Such a jump in time and space probably requires a preliminary explanation.

In the history of the complex and varied relationship between modern and ancient science, the XIX century was probably one of the happiest periods. After the recovery of the works of Euclid, Archimedes and Apollonius had formed the basis of the scientific revolution in the early-modern period, during the Age of Enlightment a sort of rebellion against Greek models dominated the scene, fueled by the overwhelming successes of Newtonian science. At the end of the XVIII century, however, a reconciliation began with the old fathers of Western science, that gave his fruits during the XIX century and proved very healthy for the development of modern mathematics, which achieved its full maturity before the "crisis" of the XX century shake again its foundations.

In particular, during the XIX century an unprecedented (and unrepeated) emphasis on the study of Classics in schools and universities across Europe gave to many mathematicians the opportunity to read first-hand the works of their Greek ancestors, and, probably for the first time, to fully understand their scientific meaning.[18] A well-known and very important example is Euclid's complex definition of proportion, that after many criticisms was finally understood and put at the foundations of the theory of real numbers by Dedekind and Weierstrass. Another example is the recovery of the Euclidean axiomatic method, which led, among other things, to the invention of modern non-euclidean geometries (the paradox being only in the name traditionally given to such new systems). Moreover, the *divination* of lost works from authors like Apollonius received new impetus, fueling the development of specific areas of mathematical research (e.g. the study of curves of double curvature). In the same period, the rising of philological studies gave birth to the first accurate editions of fundamental Greek mathematical works such as Heiberg's editions of Archimedes and to the still today reference works of Thomas Little Heath on the history of Greek mathematics.

In the XIX century British science in particular underwent a significant revolution, during which scientists strived to define their own status in the cultural context of the Victorian age. It was this process that ended up with the invention of the very word *scientist*, which took the place of the the old label of *natural philosopher*. Among other things, at the beginning of the century Charles Babbage, John Herschel and George Peacock founded the *Analytical Society* and denounced the bad health of British mathematics. According to them, a too strict adherence to Newtonian, geometrical methods had made British mathematics obsolete with respect to continental mathematics, evolved in accord with the Leibnizian tradition and employing chiefly *symbolical* methods.[19] Thanks to their efforts, Leibniz's *differentials* entered in British universities' curriculums alongside Newton's *fluxions*, but most importantly a general debate unfolded about the relative merits of different methods and approaches in mathematical research and education. A side effect of this debate was a renewed attention on the Euclidean method, as exemplified by the beautiful first colored edition of the first six books of Euclid's *Elements* by Oliver Byrne, published in 1847.[20]

In this general landscape, William Rowan Hamilton occupies a unique place, being an unparalleled prodigy who held with the Classics a direct, constant and fruitful dialogue. As we will see, his general views on the aims and methods of mathematical sciences were strongly influenced by Greek models, an aspect that sometimes put him in plain contrast with some of his contemporaries. As his friend and pupil Peter Guthrie Tait remarked, Hamilton's mathematical works "belong to no particular *school* unless we consider them to form, as they are

[18]It seems worth mentioning here that (after Galileo) William Clifford (1878, p. 15) was the first to rightfully and explicitly ascribe to Archimedes the *definitory* property of *uniform motion*, i.e. the establishment of a *proportionality* between *lengths* and *times*. For a thorough discussion of the role of Classics in the modern history of Western culture, see Russo 2018.

[19]Dubbey 1963.

[20]Byrne 1847, also available online at `https://www.c82.net/euclid/` (last access March 1st 2023). In recent times Byrne's work has been beautifully extended to all the thirteen books of Euclid's *Elements* by *Kronecker Wallis*.

well entitled to do, a school by themselves".[21] Indeed, an essential part of our arguments resides in a proper understanding of Hamilton's *own* views about his work, namely about the character and scope of his beloved *theory of grammarithms* (or, as it is universally known, of *quaternions*), a sort of *aufhebung* between the geometric and symbolical tendencies debated by British mathematicians during the XIX century. The unfortunate fate of this *theory* or *method* or *calculus* (as Hamilton alternatively referred to it) has obscured its original meaning, and we will try to restore it by appealing to its creator's own words.

Despite today his name is familiar to every physicist, Hamilton is one of those giant figures like Newton or Maxwell that are much more often mentioned than actually read. He was, among other things, also a poet and his style of mathematical writing is probably without equals in the history of mathematics. Full of *italics* and CAPITALS, in his written texts Hamilton tried to be as close as possible to the spoken word, but, overall, his pedantry was considered a hard reading already by his Victorian contemporaries, leave alone by modern readers. Yet, the insight that Hamilton's long and often convoluted sentences afford are definitely worth the effort, and we will quote at length from his own writings.[22]

Also, most of Hamilton's works were published in the *Proceedings of the Royal Irish Academy*, at the time a relatively little society with scarce financial resources. As a result, its publications were very little distributed in Europe, and often with a delay of three or four years with respect to the actual communication. In a period of explosion of mathematical researches, often on overlapping domains, this was an important handicap for the diffusion of Hamilton's works, and it is no case that his essays on dynamics, the basis of the well-known *hamiltonian mechanics*, were the only works Hamilton published in the widely distributed *Transactions of the Royal Society of London*.

However, the historian interested in *hamiltonian views* is in a very happy position, since apparently Hamilton spent most of his lifetime in *writing*: he left behind him such a *huge* amount of manuscripts, all preserved in the Library of the Trinity College of Dublin, that one can trace the development of his ideas almost day-by-day.[23] The monumental biography in three volumes written by his friend Robert Perceval Grave is composed for the greatest part of letters written by Hamilton himself, where he dwelled for dozens and even *hundreds* of pages in detailed expositions of his ideas that could never find their way in any published work.[24]

It is a pity that Hamilton's original views are so little known today, since many of the subsequent developments of mathematics and physics find a common root in his forgotten works. This is indeed another of the reasons why we chose to devote such a large amount of space to the exposition of his ideas, a gold mine

[21] Tait 1880.

[22] When not otherwise stated, all the typographical expedients in Hamilton's quotes are from Hamilton himself.

[23] No doubt that Hamilton was fully aware of his genius, and sometimes it almost seems that he wrote for the ease of future historians.

[24] Most of Hamilton's works are collected in LateX format in David Wilkins' personal page (`https://www.maths.tcd.ie/pub/HistMath/People/Hamilton/`, last access February 24th, 2023), a great source about Hamilton's life and works.

from which we are convinced there is still much to learn, today more than ever. After all, this was also Hamilton's wish:

> I hope that it may not be considered as unpardonable vanity or presumption on my part, if, as my own taste has always led me to feel a greater interest in *methods* than in *results*, so it is by METHODS, rather than by *any* THEOREMS, which *can* be separately *quoted*, that I desire and hope to be remembered. Nevertheless it is only human nature, to derive *some* pleasure from being cited, now and then, even about a "Theorem"; especially where the quoter can enrich the subject, by combining it with researches of *his own*.[25]

The third chapter is devoted to the *method of the hodograph*, an offspring of Hamilton's theory of quaternions that led him to a beautiful geometrical formulation of Newton's dynamical theory of gravitation. We will see how such a *method* and the specific *hypothesis* which it naturally *suggests* when applied to astronomical problems were well within the scope of Greek mathematics at the time of the AM, and indeed a natural theoretical framework to ground the art of *sphere-making*. So, after the exposition of the method, we will come back to Hellenistic times and outline more precisely our view about the mathematical theory underlying the design of the AM.

In the concluding remarks, we will summarize the view here proposed. In particular, if our reconstruction is accepted, the AM will appear to be a direct evidence that a theory of heavenly motions mathematically equivalent to Newton's gravitational theory was developed by Hellenistic mathematicians around II cent. BC.

[25] Quoted in Tait 1880.

Chapter 1

What is the Antikythera Mechanism?

1.1 On Hellenistic Mathematics

When one compares modern and ancient mathematics, one of the most striking differences that he/she finds is that this latter was, generally speaking, much more *problem-oriented* than ours. Greek mathematics is no exception to this thumb-rule.

Starting from the stage set by the four Pythagorean sisters - arithmetic, geometry, harmony and astronomy - Greek mathematics gradually evolved, first and foremost, as a collection of *problem-solving* disciplines, *scientific arts* covering a wide range of domains and sharing a unitary methodology. In the Hellenistic period, in particular, the intertwinement between "pure" and "applied" sciences became so tight that most branches of mathematics borrowed their very name from their intended application, like *scenography*, *catoptrics* or *dioptrics*.[1]

Also, there is no doubt that Greek mathematicians experienced the distinction between "abstract" and "practical" problems in a very different way than us. In the domains which we would call "purely" mathematical, research was mostly driven by the search for solutions to problems like the *squaring of the circle*, the *trisection of the angle* or the *doubling of the cube*.[2] Some of these problems had often a mythological origin, like the last mentioned, associated to Apollo's oracle request to his worshippers to double the volume of his altar without altering its shape, or the *isoperimeter problem*, connected to Didon and to the myth of the Minotaur. It seems that still in the III cent. BC celebrations were made in occasion of the finding of new solutions to such problems. Yet, to quote the title of Knorr's classic book, this *ancient tradition of geometrical problems* crossed the boundaries between religious mythology, "pure" mathematics and engineering, since the solution to such abstract (and therefore *general*) mathematical riddles became the key to the solution of entire *classes* of very *practical* problems. Just to make an example, the solution to Apollo's *doubling-problem* became the key ingredient of Philo's solution to the problems of projectile-range posed by catapults-design.

[1]For more details on the Hellenistic scientific method, see Russo 2004, pp. 171–202.
[2]Knorr 1986.

Even arithmetic and geometry, the "purest" among mathematical sciences, have never been for the Greeks entirely *abstract* disciplines, probably because at the time their practical origin - rooted in the operations of *counting* and *measuring* involved in everyday activities - was relatively close and sufficiently clear to their practitioners. Thus arithmetics, the *theory of numbers*, was always accompanied by *logistics*, the art of *computing*, and *geometry*, the *theory of figures*, was never conceived as something distinct from the art of *drawing*.

It has been often noted how the Greeks, in mathematics as elsewhere, were very *visual* thinkers. Martin Heidegger, one of the most insightful interpreters of Greek philosophy, has underlined in particular how the Greek word for *truth* (αλεϑεια) literally translates *not-hiding, not-concealed.*[3] This simple observation is by itself sufficient to understand why the search for *truth* in Greek philosophy has always the character of the *unveiling*: to *find a truth* means in Greek to bring *light* where there was *darkness*, and to *exhibit* what has been found *in plain sight.* Accordingly, in all Greek philosophy the idea of *truth* was always expressed with words pertaining to the domain of *vision*, *false* being often synonymous of *fake appearance.* In the specific domain of Greek mathematics, it seems therefore particularly appropriate to take all the still-in-use locutions as *we see*, *it's evident* or *it's clear* quite literally.[4] For Plato the demonstration (in Greek *apodeixis*) of the properties of a geometrical figure and the connected activity of *drawing* the figure itself worked as a sort of *exhibition to vision* of the general process that leads to the *discovery* of some non-evident *truth.*[5] The proof of a mathematical theorem is evidently the *apotheosis* of this kind of process, and this is probably one of the reasons why Plato gave to the study of geometry such a prominent role.

Still Heidegger has emphasized how also the word *techne* was invested by the Greeks with a strong *epistemological* meaning, always pertaining to the domain of *sight.* Technological artifacts were perceived in fact as an *exhibition to vision* of the *knowledge* required to the artisan to design and build the object in question. In short, as Heidegger put it, for the Greeks that of *techne* was a category of *knowing* and not just of *doing*, and, in spite of later commentators' opinions, for the Greeks the *technological* activity was intertwined with and by no means secondary to the *theoretical* one.[6] To put it with the words of Vico, for the Greeks

[3]See in particular the course he held at Freiburg University in 1931-1932 (Heidegger 1997a).

[4]Indeed, in English *to see* is still used also to mean *to understand.* It is barely necessary to remark that modern cognitive sciences have totally confirmed the relevance of *vision* in the forms of our understanding. Using modern metaphors, it seems that the peculiar hardware and software of our vision process shape a large part of our conscious learning. It would seem advisable that, among other things, mathematical teaching could return to be much more visual than it is today. Important efforts in this direction have been those of Emma Castelnuovo for elementary mathematics and, more recently, those of Tristan Needham for more advanced domains (Needham 1997; Needham 2021).

[5]See for example the famous passage of Socrates and the slave in the *Meno*, about the incommensurability of side and diagonal of a square.

[6]Heidegger 1997b, pp. 38–39. See also Brisson and Pradeau 1998, pp. 53–55: "Dans la philosophie platonicienne, la réflexion sur la technique occupe une place déterminante et constante: la technique est le paradigme du rapport que l'homme entretiens avec tous les objects. [...] Le technicien maîtrise sa technique particulière grâce à la possession d'un certain savoir, d'une certaine science. [...] Et c'est la raison de la relative indistinction, chez Platon, de la technique et de la science: la technique est la science, dans la mesure où elle suppose la

verum and *factum* went hand in hand. Such entanglement between *episteme* and *techne* is indeed one of the hallmarks of Hellenistic mathematics.

This partly explains why the *theoretical* development of Greek mathematics has been so largely shaped by the activity of *drawing*, namely by the use *diagrams* which were always the ground on which mathematicians set their feet to reach the highest peaks of abstraction and generality. It is well known, in particular, the *theoretical* role played in Greek geometry by the elementary operations afforded by the two basic drawing instruments, straight-edge and compass (which, by the way, were not the only tools used in geometrical constructions).

The three fundamental postulates of Euclid's *Elements* are nothing but the abstractions of the three elementary operations one can perform with a straight-edge and a compass - *draw* a line, *produce* a line, *draw* a circle -, and every proposition of the *Elements* is essentially a *list of commands* involving such operations. These commands instruct on how to draw the figure that is the object of the proposition. The *demonstration* of the proposition is either the *exhibition* of a certain property of this figure, or the *proof* that it satisfies a previously required property. Every proposition is thus the *analysis* or *synthesis* of a certain diagram, that in principle may be done in any way, the only constraint being the adherence to the postulates. As Russo has emphasized, it is exactly this *operational* character of the postulates that in Euclid's system of geometry *explicitly* guarantees an unbreakable connection between the *abstract* deductive model and something *tangible* existing in the real world, i.e. a *diagram* drawn according to some pre-established rules. It may be remarked that, in spite of the emphasis that later commentators have put on the difference between *theorems* and *problems*, such difference is a matter of perspective, since *all* the content of *any* proposition of the *Elements* is encoded in *how* the figure is constructed. The proof of a proposition is just the *logical extraction* from the figure of a certain property that, for whatever reason, is regarded as relevant. In this sense, problems were definitely *primary* in Greek mathematics with respect to *theorems*.

There is another reason why problems had a primary role in Greek geometry, i.e. that it was *constructive* in a very strong sense that is rarely adopted in modern mathematics. In Euclid's *Elements* a certain figure literally *does not exist* until it is explicitly *drawn*, i.e. until the *problem* of its construction is solved. It is no case that the first proposition of the *Elements* is a problem, namely that of drawing a regular triangle, i.e. the building block of the whole theoretical architecture. The Euclidean definitions of parallels, squares, tangents etc. are mere conventions, *names* given for convenience, but the very same *existence* of such geometric objects must be *proved by construction* and may never be taken for granted.[7] This point was already emphasized, for example, by Aristotle, who explicitly says that the only requirement of a good definition is that it should be *understandable*, without by this implying that the object defined actually *exists*, a fact that always needs to be *proved*.[8]

connaissance de la manière dont l'activité technique doit convenir à son objet."

[7]The importance of nominalism and linguistic conventionalism in Hellenistic mathematics has been strongly emphasized by Russo.

[8]We may remark that this fact is quite obvious and generally agreed outside of the "mathematical domain". The definition of *unicorn* as "white horse with horn and wings" is very clear

Gemino (I cent. BC), quoted by Proclus, in his classification of Greek mathematics explicitly points out that

> ...it is always the task of geometry, both plane and solid, to construct [figures] or to compare or to divide those that have already been constructed.[9]

Construct, *compare*, and *divide* figures: this is, in short, what Greek geometers did, with a mastery that has no equals in the history of mathematics.

It is also important to remember that for Greek mathematicians drawing was also (but *crucially*) a form of *computing*.[10] After the invention of a general theory of continuous magnitudes (like that exposed in Book V of Euclid's *Elements*) every mathematical problem could be framed in terms of a geometrical diagram, in which the *data* of the problem were represented by the length of some straight lines, and the solution was *constructed* by drawing a line whose length had the required ratio to that of the given lines. In this way, the basic instruments of straight-edge and compass and the constructions made with them became also extremely powerful tools for *analog computing*.[11]

So, in short, in Hellenistic science geometry was, first and foremost, a *theory of diagrams*, which entangled three activities that in modern mathematics are often regarded as independent to each other: drawing, computation and deductive reasoning. As such, it could be put to the service of *any* discipline, abstract or practical, which used diagrams to express the relationship between magnitudes. The more the range of geometry extended, from Euclid's *Elements*, through Archimedes' *Method* up to Apollonius' *Conics*, the more extensive, expressive and powerful such diagrammatic language became for the solution of any kind of problem. Notice, in particular, how far this conception is from the modern idea of geometry as the "science of space".

In their mature form, all Greek mathematics shared a similar conceptual structure, with the drawing and analysis of diagrams occupying a central role in every domain. What changed was the *meaning* attributed to the magnitudes whose reciprocal proportions were encoded in a certain figure, i.e. what diagrams *represented*, which depended of course on the context. This point is emphasized, for example, by Plato, who speaking of the "method of mathematicians" wrote:

> Further you know that they [mathematicians] make use of visible figures and argue about them, but in doing so they are not thinking of these figures but of the things which they represent...[12]

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by itself, but no one would believe that such definition *implies* the existence of similar creatures. Mythological figures like the *minotaur* and the *centaur* could have suggested to the Greeks a similar analogy.

[9] Evans and Berggren 2006, p. 247.

[10] This point is emphasized in Russo 2021, pp. 57–62.

[11] Notice that, as Russo also remarks, the efficiency of geometrical computations remained unsurpassable until the invention of logarithms, whose first tables appeared in 1614. However, during XIX century, there was a renewed interest in graphical methods. See, for example, Clifford 1878, pp. 14–15.

[12] *Republic*, VI, 510C-E. Quoted in Heath 1921a, p. 290.

In Hellenistic mathematical arts the possible meanings and interpretations of geometrical diagrams extended as long as "the method of mathematicians" was applied to many different classes of problems.[13] So, in Euclid's *Optics*, in which it is developed a *theory of vision*, lines represent *visual rays* connecting an observer to an object, and the accompanying propositions pose and solve problems about the *sizes* and *speeds* of such objects as seen by an observer. In Aristarchus' *On the sizes and distances of Sun and Moon* the circles, lines and angles of the diagrams accompanying the propositions represent the relative distances, sizes and orientations of Earth, Sun, Moon and their projected shadows when they are in some definite configurations. In Archimedes' *On Floating Bodies* the paraboloids of revolution whose properties are demonstrated in the related *theorems* are the "abstract" model of a ship's hull, and the theory there exposed has direct bearings on the design of stable ships. We may be confident that, similarly, in Archimedes' lost *Catoptrics* diagrams represented *mirrors* and *visual rays* cutting them. Of course there is no reason to think that the lost treatise *On Sphairopoiia* was methodologically different.

Despite the loss of all the advanced astronomical treatises of the Hellenistic period, it is sure that astronomy was no exception to this epistemological framework. First of all, also astronomers faced very specific *problems*. Today it is too easy to forget this, but it is crucial to keep in mind that, at that time, astronomy was an *extremely* practical discipline devoted, among other things, to the very important offices of *time-reckoning*: calendars, agriculture, and, of course, all the religious and civil affairs depended on the work of astronomers. Also navigation, of course, was strongly dependent on astronomical observations.

It is worth quoting the most relevant passage we have about the difference between *physics* (i.e. *natural philosophy*) and *mathematics* in dealing with astronomical problems. It is a second-hand quote from Gemino, found in Simplicius' commentary to Aristotle (VI cent. CE):

> It is characteristic of physical science to consider what has to do with the substance of heavens and celestial bodies, their powers and quality, their generation and corruption... Astronomy, however, does not concern itself with all that... In many cases astronomers and physicists will set out to demonstrate the same topics, for example the size of the sun or the roundness of the earth, but they don't follow the same route. The latter will deduce whatever it may be from substance or powers, or from optimality arguments, or from generation or transformation, whereas the former will deduce it from appropriate figures or magnitudes or the measurement of motion and corresponding times. The physicist, with an eye towards productive power, often touches on causes, whereas the astronomer, when he is constructing proofs based on what comes from outside, is a poor observer of causes[.] Sometimes [an astronomer] through a hypothesis finds a way to save the phainomena. For example, why do the sun, the moon and the planets appear to move irregularly? If we suppose that their round orbits are eccentric or that these bodies move on epicycles, the apparent irregularities will be

[13] This point is well emphasized in Russo 2004, p. 189. It is noteworthy that also this generalized and extended use of diagrams was fully recovered during the XIX century, among others by James Clerk Maxwell. See, for example, Maxwell 1911.

> saved. One must investigate in how many different ways the phainomena can be represented... [14]

In other words, for the *natural philosopher* (and for Gemino) it is not enough to *save the phenomena*, but it is necessary to find some "causal explanation" or *aitiologia* pertaining to the "nature" of things.[15] This kind of *aitiologia* was the hallmark of Aristotle's *physics*, and indeed Strabo (I cent. BC), criticizing Posidonius for trying to introduce too much *aitiologia* into geography, wrote:

> For him there is too much inquiry into causes, that is, 'Aristotleizing', a thing that our School [i.e. the Stoic School] avoids because of the concealment of the causes.[16]

On the other hand, the importance for the mathematicians to seek for multiple ways in which the same *phainomena* can be represented echoes a similar remark made by Theon of Smyrne (II cent. CE) about Hipparchus' approach.

Like other branches of mathematics, also astronomy worked in strict connection with specific *instruments*. Still Gemino, now quoted by Proclus, describes the subject matter of astronomy as follows:

> There remains astronomy, which treats the cosmic motions, the sizes and shape of the heavenly bodies, their illuminations and their distances from the Earth, and all such questions. [...] Its parts are: gnomonics, which is engaged with the measurement of the hours through the placement of gnomons; meteoroscopy, which discovers the different altitudes and the distances of the stars and teaches many complex matters from astronomical theory; and dioptrics, which examines the positions of the Sun, Moon, and the other stars by means of such instruments.[17]

So, all the three sub-branches of astronomy are identified with the instruments astronomers employ in their inquiry: *gnomonics* with sundials, *meteoroscopy* with the *meteoroskopeion* (a kind of astrolabe)[18] and *dioptrics* with the *dioptra*. To these, we should add the fascinating practice of *sphairopoiia*, which will be the main focus of the following sections.

1.2 On *Sphairopoiia*

Immediately before the passage above quoted, Gemino describes *sphairopoiia* (literally *construction of spheres*) as a sub-branch of mechanics "in imitation of celestial motions, as Archimedes practiced". Today, the word *sphairopoiia*

[14]Quote from Russo 2004, p. 192. The full passage may be read in Evans and Berggren 2006, pp. 252–255.

[15]The reader is pleased to remember this passage whenever in the following we will refer to the distinction between the *physical* and the *mathematical* approaches.

[16]Quote from Evans and Berggren 2006, p. 57, to which we refer the reader for a discussion about *aitiologia* and Gemino's realism.

[17]Evans and Berggren 2006, p. 249.

[18]Evans and Berggren 2006, p. 48.

Figure 1.1: *Erathostenes teaching in Alexandria*, painting by Bernardo Strozzi (1581-1644), today at Montréal Museum of Fine Arts (source: cover of Cassirer 2021).

is generally used by historians to denote the Greek art of building scale-sized objects (or sometimes, by metonymy, the artifacts themselves) exhibiting the configuration and/or the movements of heavenly bodies. The simplest specimens of this mathematical *techne* were *celestial globes* and *armillary spheres*, showing a selection of fixed stars and of the relevant circles of the *celestial sphere*. The AM, as we have seen, is a way more sophisticated device, with a wide range of phenomena represented by multiple pointers moving simultaneously on different dials. In the case of the AM the motion was given by hand, but Pappus (VI cent CE) mentions similar astronomical devices put in motion by water.

In different forms, this art evolved side by side with Greek mathematical astronomy, from the first steps of the Pythagorean school (Archaic period, VI-V cent BC), through its shaping at the time of Plato's Academy (Hellenic period, V-IV cent BC), up to the golden age of Alexandria's Museum (Hellenistic period, III-II cent BC). After II cent. BC there are mentions of *sphairopoiia* as a *received* practice, but we have no evidence that devices such as the AM were ever constructed again in antiquity.[19]

[19]Similar artifacts reappeared in Europe after more than one thousand years (the first being De Dondi's *Astrarium* at the middle of the XIV century), but *sphairopoiia* never attained again the level of the AM, with so many astronomical informations compressed in such a compact manner. Modern *orreries*, dating back to XVII century and owing their name to the English nobleman Charles Boyle, 4th Earl of Orrery, usually represent the solar system in a heliocentric frame of reference and, therefore, have no connections with astronomical observations. Despite the technological complexity, their value is thus more aesthetic than scientific. A XIX century review of this practice is Brewster 1830, and a more recent monograph devoted to it is King and Milburn 1978. In any case, in the present dissertation we won't deal with the complex and

Despite in modern times the construction of mechanical devices incorporating astronomical models has been a charming but mostly secondary activity,[20] the importance of *sphairopoiia* for Greek astronomy can hardly be overestimated. In the hands of Greek mathematicians this art became a powerful tool for the *construction*, *visualization* and *validation* of astronomical models, and the AM shows that in its most mature form it became also a tool for *mechanical computation* specifically adapted to astronomical problems. In the beautiful words of Germaine Aujac:

> Goût pour la spéculation géométrique, référence constante à ce qui est fait de main d' homme, tels sont les deux pôles entre lesquels a jailli l'étincelle et s'est intensifié le courant de la recherche scientifique en Grèce ancienne... faisant appel sans l'ombre d'une hésitation au savoir-faire des artisans, ils ont voulu en fabriquer des répliques sur lesquelles, à l'instar du créateur du monde, ils pouvaient agir et, surtout, qui leur permettaient d'offrir, de leurs conceptions, une vue synthétique. [...] Étudier les propriétés de la sphère en se servant de la géométrie, les vérifier et en chercher de nouvelles applications grâce aux modèles réduits, telles sont les deux démarches qui ont conduit les Grecs à prendre de la terre et du ciel une connaissance qui, aujourd'hui encore, nous confond d'admiration.[21]

We totally agree with Aujac in the view that a proper consideration of *sphairopoiia* sheds a clearer light on the whole evolution of Greek mathematical astronomy. Working like *drawing* as an intermediate *operational* layer between *theory* and *phenomena*, it was this art that guaranteed, among other things, a solid grounding to the theoretical activity of astronomers.

But here we wish to do a step further, and put forward the idea that, in its most mature developments, *sphairopoiia* and theoretical astronomy were actually *the very same thing*, in the sense that mechanical devices were designed to be an *exact* realization of theoretical models, and therefore, like the AM, *theoretical computers* for the occurrence of the phenomena covered by the "abstract" theory. In other words, even if Proclus-Gemino includes *sphairopoiia* in the domain of *mechanics*, we propose to regard it, like *gnomonics* and *dioptrics*, as an essential part of Hellenistic astronomy itself, namely that dealing with *motions* of heavenly bodies. In this regard, we remark that Proclus-Gemino defines astronomy as the science dealing with cosmic *motions*, but then in the description of its different sub-branches the word *motion* does not appear at all. Our conjecture is that such class of problems was the specific object of *sphairopoiia*.

In support of this interpretation, many elements suggest that Geminos' classification should be read *cum grano salis*, being a late systematization of the

important issue of the reception of *sphairopoiia* by modern science. We hope to devote a future work to this topic, and here we will just sketch some conjectures.

[20] As far as I know this practice was not undertaken by most of the major figures of modern science. An important exception is Christiaan Huygens, sometimes named the *Dutch Archimedes*, the first to build a working planetarium in the modern era. It is worth noting that Huygens used the same approximation method to determine the optimal tooth-numbers for gear ratios found in the AM, i.e. the Euclidean algorithm of *antyphairesis* (Freeth, Higgon, et al. 2021), equivalent to *continued fractions*.

[21] Aujac 1993, p. 7.

developments of mathematical sciences in Hellenistic times, made following the Aristotelian lines of old similar classifications.[22] In particular, the main division of Gemino's classification is that between disciplines dealing with "mental things only" and those dealing with "perceptible things":

> But others, such as Geminos, think it proper to divide mathematics according to another scheme. They make one branch concerned with mental things only, and one concerned with perceptible things or touching on them. [...] In the branch engaged with mental things, they place arithmetic and geometry as the two first and most important parts. In the branch that operates with perceptible things they place six parts: mechanics, astronomy, optics, geodesy, canonics, and logistic.[23]

The first thing to remark is that this list is surely incomplete, missing very relevant disciplines such as Erathostenes' geography and Archimedes' hydrostatics. Moreover, in the epistemological framework of Hellenistic mathematics this distinction between "mental" and "touching" things is by itself questionable and indeed quite blurred, even "theoretical" geometry being, as we said, strictly connected to the "practical" activity of drawing figures on various supports. The domain of mechanics, in particular, is later described as follows:

> Besides these sciences there is one called mechanics, which is a part of the study of perceptible and material objects. Under this comes the construction of instruments useful in war, such as the engines for defense that Archimedes is said to have built under the siege of Syracuse, and also the science of wonder-working [*thaumatopoiike*], which works its contrivances by means of air, as both Ctesibius and Hero describe, or by means of weights whose disequilibrium is the cause of motion and whose equilibrium is the cause of rest, as the *Timaeus* has established, or, finally, by means of cords and ropes mimicking the tugs and movements of living creatures. Also under mechanics come the general science of things in equilibrium and the determination of what are called centers of gravity, as well as sphere-making [*sphairopoiia*] in imitation of celestial motions, such as Archimedes practiced, and, generally, all that is concerned with matter in motion.[24]

Immediately after, there is the passage on astronomy quoted above. So, according to Proclus-Geminos mechanics ended up dealing generally with *matter in motion*, and *sphairopoiia* with celestial motions specifically.[25]

Indeed, this overlapping between theoretical, "abstract" models and practical, "concrete" mechanics fits very well with the ambiguous use that Gemino does of the word *sphairopoiia* in the *Introduction to the Phenomena*.[26] Apart from the

[22] For more details on this point and on ancient classifications of mathematical sciences in general see Vitrac 2005a.

[23] Evans and Berggren 2006, p. 246.

[24] Evans and Berggren 2006, p. 249.

[25] A beautiful book on the range and character of ancient mechanics from Mesopotamian civilizations up to the Imperial Period is Di Pasquale 2019.

[26] Such ambiguous use of the word *sphairopoiia* is remarked in Evans and Berggren 2006, pp. 49–58.

literal sense of *sphere-construction*, indicating the *activity* or the *artifact* itself, in different instances Gemino uses *sphairopoiia* in a *theoretical* sense, i.e. to indicate something that is used to *explain* the phenomena observed, or to *deduce* those which cannot be observed, and which in general can be "in accord" or "in contrast" with the actual observations.

For example, speaking about the possible presence of people in the southern hemisphere of the Earth, Gemino writes:

> When we speak of the southern zone and of those dwelling in it, as well as the so called *antipodes* in it, we should be understood in this way: that we have received no account of the southern zone nor whether people live in it, but rather that, because of the whole *sphairopoiia*, and the shape of the Earth, and the path of the Sun between the tropics, there exists a certain other zone, lying toward the south and having the same temperature character as the northern zone in which we live in.[27]

And a little later, speaking of rising and setting stars:

> Because of such *sphairopoiian*, not all the stars both rise and set each night. Rather, certain ones rise and set, certain ones rise but do not set, while some neither rise nor set. [...] And so Krates, speaking in marvels, takes things said by Homer for his own purposes and in archaic fashion, and transfers them to the *sphairopoiia* that accords with reality. For Homer and the ancient poets, nearly all of them, mean to say that the Earth is flat and meets the cosmos; that the Ocean lies around in a circle, occupying the place of the horizon; and that the risings are from the Ocean and the settings are into the Ocean. Consequently, they supposed the *Aithiopians* near the rising and those near the setting to be burned by the Sun. This notion is consistent with their proposed arrangement, but alien to the *sphairopoiia* in accord with nature.[28]

In all these cases, Gemino is dealing with problems of spherical astronomy, but later on, speaking of irregular motions of the planets, he writes:

> As the motion in their case is of such a kind, it is clear that the shift toward the following does not occur by a falling behind, for they would always be falling behind. But, in fact, there is a certain *sphairopoiia* proper for each, in accordance with which they pass sometimes towards the following sometimes toward the preceding, and sometimes standing still. [...] It has resulted, then, from the individual *sphairopoiian* of each [planet] that the shifts are different.[29]

So, all in all, it appears that for Gemino *sphairopoiia* is, to the least, something *more* than the concrete construction of mechanical contrivances, and seems to be consistently used as a byword for *theoretical model* accounting for some astronomical phenomena. It is particularly interesting that Gemino refers to the

[27] Evans and Berggren 2006, p. 214.
[28] Evans and Berggren 2006, p. 215.
[29] Evans and Berggren 2006, p. 199.

sphairopoiia of *each* planet, and to the results that have been deduced from them. Our interpretation of this ambiguous use is that in a late and realist author like Gemino, a full understanding of the complex relationship between *theoretical model* and *physical reality* was already declining and leading to the confusion between the two plans that in later periods will dramatically prevail.

Without dwelling on Ptolemy's conceptions of the relationship between *hypothesis* and *phenomena*, *theory* and *observation*, we limit ourselves to quote the introduction of the *Planetary Hypotheses*, devoted explicitly to the practice of *sphairopoiia*, which clearly exhibits such confusion:

> We have worked out, Syrus, the *hypothesis* of heavenly motions through the books of the Mathematical Syntaxis, demonstrating by arguments, concerning each example, both the logicality and agreement everywhere with the phenomena, with a view to a presentation of uniform and circular motion which necessarily was to arise in things taking part in eternal and orderly motion and that are not capable to undergo increase or decrease in any way. Here we have taken on the task to set out the thing itself briefly, so that it can be more readily comprehended by both ourselves and by those choosing to arrange the models in an instrument, either doing this in a more naked way by restoring each of the motions to its respective epoch by hand, or through a mechanical approach, combining the models with one another and with the motion of the whole. Indeed, this is not the accustomed manner of sphairopoiia; for this [sort of manner], apart from failing to represent the models, emphhpresents the phenomenon only, and not the underlying [emphasis ours], so that the craftsmanship, and not the *hypotheses*, becomes the exhibit. But rather [the manner] where the different motions under our view are arranged together with the anomalies that are apparent to observers and subject to uniform and circular courses, even if it is not possible to intertwine them all in a way that is worthy of the aforementioned, but having to exhibit each separately in this way. Concerning the positions and arrangement of the circles causing the anomalies, we will apply the simpler version in respect to the method of instrument-making, even if some small variations will follow, and moreover we fit the motions to the circles themselves, as if they are freed from the spheres that contain them so that we can gaze upon the visual impact of the models bare and unconcealed.[30]

So, the aim of *sphairopoiia*, for Ptolemy, is not to represent the *phenomena*, but rather the underlying *hypothesis*, that in his astronomy have lost their *theoretical* character and are taken as assertions about the *physical reality* of celestial motions. Clearly a device like the AM wouldn't fit Ptolemy's criteria.[31]

We think that the very same possibility of the *aufhebung* between astronomy and mechanics represented by the art of *sphairopoiia*, and also the origin of the later misunderstanding of this practice, lies in the central role that *circular motion* played for the Greeks as a *theoretical* cornerstone of both disciplines. Such theoretical homogeneity allowed to treat in a unified way astronomical

[30] Hamm 2011, pp. 44–45. As Hamm remarks, Ptolemy's indications are indeed too vague and of no use for the actual design of working planetaria.

[31] For a fuller analysis of Ptolemaic astronomical conceptions see Amabile 2020.

problems, involving the *motions* of stars, Sun, Moon and planets, and mechanical problems, involving the *motions* of the parts of a machine. So, enabling *in principle* to incorporate *exactly* an astronomical theory in a mechanical device, and coherently with the general epistemological framework of Hellenistic mathematics, *sphairopoiia* came to overlap and perhaps even *coincide* with the most mature developments of Greek mathematical astronomy. In these developments, it is likely that Archimedes, mechanician, astronomer and exemplar figure of the Hellenistic intertwinement between *episteme* and *techne*, played a prominent role.

In the next section we will sketch the outlines of this process in the historical evolution of spherical astronomy. Then, we will turn to the more complex problems posed by solar, lunar and planetary models.

1.3 *Sphaerics* and Celestial Globes

It is well known that the roots of Greek mathematical astronomy date back to the works of the Pythagoreans. One of the most celebrated Pythagorean mathematicians is Archytas of Tarentum (IV cent BC), a contemporary and friend of Plato, often remembered for his studies on harmonics.

As we already remarked, astronomy was one of the four fundamental disciplines taught in the Pythagorean school. This is a fragment by Archytas, generally regarded as authentic:

> The mathematicians (τοί τερί τά ματήματα) seem to me to have arrived at correct conclusions, and it is not therefore surprising that they have a true conception of the nature of each individual thing: for, having reached such correct conclusions regarding the nature of the universe, they were bound to see in its true light the nature of particular things as well. Thus they have handed down to us clear knowledge about the speed of stars, their rising and settings, and about geometry, arithmetic, and sphaeric, and last, but not least, about music; for these μαθηήματα seem to be sisters.[32]

The *sisterhood* of such disciplines and thus the homogeneity of their methods was given by their common ground in the notion of *number*. Without dwelling into the details of Pythagorean mathematics, it is nonetheless important to make a little digression on the meaning and extent that such notion had in the Pythagorean school.[33]

For the Pythagoreans *numbers* were, generally speaking, *points having position*. As a consequence, *numbers* and *figures* were not separate entities, but deeply interconnected concepts, every *number* corresponding and being embodied in a definite geometrical *figure*.[34] Accordingly, numbers could be *linear*, *plane* or *solid*. Indeed, according to Anatolius, the Pythagoreans were the first to include under the single word μαθηήματα both arithmetic and geometry, previously regarded as separated disciplines. It has been suggested that such intertwinement came from

[32] Heath 1921a, p. 11.

[33] For more details on Pythagorean arithmetic and geometry see Heath 1921a, pp. 65–117, 141–169.

[34] This link is still present in English, which calls *figure* the *digits* of numbers.

the observation of constellations, whose primary properties are the *number* of stars and the *figure* they form.

Numbers-figures were not seen as static and isolated entities, but *primarily* as *moving* or *changing* entities forming *progressions*, i.e. numerical-geometrical successions unfolding according to a definite *law* or *logos* that determined the reciprocal relationship between the successive terms. The main concepts involved in the study of such progressions were those of *ratio*, *proportion* and *mean*. *Arithmetic*, *geometric* and *harmonic* progressions embodied different possible *laws of succession*. It is thought that the "discovery" of these specific kinds of progressions came from experiments with musical tones, but one cannot exclude a more abstract origin.[35]

It is generally thought that musical experiments were also the origin of the fundamental tenet the Pythagorean natural philosophy that such progressions of numbers-figures are somewhat *printed* in the natural (i.e. *physical*) phenomena, and that therefore could be the key to their decipherment. Iamblichus reports the Pythagorean motto *all things may be compared to numbers*, and a famous fragment from Philolaus says that *all things, at least those we know, contain number; for it is evident that nothing whatever can either be thought or known, without number.* Successions of number-figures were seen as *generated* by some initial *arché* (the *unit*), and by the *logos* (from the verb *legein*, meaning primarily *to collect together, to arrange*) keeping together the terms of the successions and determining the character of the succession itself. Thus, by *analogy*, these successions could represent physical phenomena involving *grow*, *decay* and, of course, *circularity*. This idea of *mimesis* (or, in Plato's language, *methexis*) between progressions of number-figures and the ever-changing phenomena of *physis* was a key, for the Pythagoreans, to the solution of a capital problem of Greek natural philosophy, i.e. that of the *invariance in change*.

In this regard, as Zellini has remarked, an utmost importance must be accorded to another fragment of Philolaus, where it is asserted that *number must be thought according to the nature of the gnomon.* The word *gnomon* here doesn't allude (only?) to the astronomical instrument, but to the number-figure that added or subtracted to another number-figure gives a *similar* one, altering its *magnitude* but leaving unchanged its *shape.* So, *gnomonic constructions* are progressions of numbers-figures in which every term *includes* or *is included* in the preceding one, the whole progression being self-similar. For the Pythagoreans these kind of progressions were, on one hand, compared to the auto-similar and self-generating processes that seem to be at work in many natural phenomena, and, on the other, to the epistemological relationship between *knower* and *known*, one included in the other and reciprocally implied in the act of knowledge. Mathematically speaking, gnomonic constructions and procedures became an essential tool of Greek *geometric algebra* (whose fundamentals were laid down in the Pythagorean school) and the key-ingredient of computational procedures that run through the entire history of mathematics.[36]

[35] A beautiful book on many mathematical topics based on imaginary Pythagorean dialogues motivated by musical problems is Camiz 2019.

[36] See the wonderful Zellini 2007 for more details on the role of gnomonic construction in the history of mathematics. We also can't omit to mention the beautiful insights offered by Simone

In close of this digression, it is important to remark that the subsequent development of Greek mathematics *refined* these Pythagorean conceptions, pruning its more mystical aspects, but without changing some of their essential features. About arithmetics, for example, Gemino wrote (emphasis ours):

> There is a division of arithmetic into the study of linear numbers, plane numbers, and solid numbers: for it examines the classes of number in themselves, as they *proceed from the unit*, and the *generation* of the plane numbers, both similar and dissimilar, and the *progression* to the third dimension.[37]

Turning back to astronomy, Proclus specifies that for the Pythagoreans geometry proper dealt with the study of magnitudes "at rest", while *sphaeric* dealt expressly with magnitudes *in motion*, probably indicating the fact that Pythagorean astronomy dealt primarily with *progressions of number-figures.* Moreover, the term *spheric* indicates that already in the first half of the IV cent. BC such study of the heavenly motions was identified with that of a *turning sphere.* We barely mention that, as well known, some form of heliocentrism was already considered in the context of Pythagorean mathematics.

Archytas, in particular, is also regarded by later sources as one of the founders of the Greek mechanical tradition. According to Diogenes

> ...he was the first to bring mechanics to a system by applying mathematical principles; he also first employed mechanical motion in a geometrical construction, namely, when he tried, by means of a section of a half-cylinder, to find two mean proportionals in order to duplicate the cube.[38]

In other terms, Archytas seems the first to make an explicit use of what much later was called *geometria organica*, a *mechanical* or rather *kinematical* approach to geometry in which *locii* as lines and surfaces are conceived as *traced* or *described* by the *motion* of points and lines. Later extant examples of this approach are Archimedes' *spiral* (defined by two simultaneous motions, one rectilinear and the other circular) and Apollonius' *cones* (defined by the rotation of a triangle around one of its sides). In the lost Apollonius' *Plane Locii* we know from Pappus that geometrical figures were generally classified in *fixed* (εφεκτικοι), *progressing* (διεξοδικοι) and *revolving* (αναστροφικοι), according as they were regarded independently from their mode of generation or as *generated* by a rectilinear or circular motion.[39] It is impossible to trace the development of

Weil about pythagorean mathematics and Greek science in general (S. Weil 1966; S. Weil 2014; S. Weil and A. Weil 2018).

[37] Evans and Berggren 2006, p. 247.

[38] *Vitae Philosophorum*, VIII, 83 (translated by Robert Drew Hicks). As Heath remarks, among the solutions the Greek found to the famous Delian problem Archytas' is "the most remarkable of all, especially when the date is considered (first half of fourth century BC) because it is not a construction in a plane but a bold construction in three dimensions, determining a certain point as the intersection of three surfaces of revolution, (1) a right cone, (2) a cylinder, (3) a *tore* or anchor-ring with inner diameter *nil.* The intersection of the two latter surfaces gives (says Archytas) a certain curve (which is in fact a curve of double curvature), and the point required is found as the point in which the cone meets this curve." (Heath 1921a, pp. 246–247)

[39] Loria 1893, p. 192.

this idea from primary sources, but it seems very likely that such a *kinematical* conception of geometric objects became a standard in Greek geometry, and as such implied in their very same *definitions*. Also Euclid, for example, doesn't define the *sphere* as "the set of points equidistant from a given point", but rather as "the figure *described by the rotation* of a semicircle around its diameter".[40]

The idea of reducing the complexity of astronomical motions to a combination of circular motions has been often ascribed to Plato, that influenced by Pythagoreans and on the basis of his own metaphysical views would have given a sort of "assignment" to astronomers that marked all the subsequent development of this science until Kepler's law of ellipses. Putting aside the fact that Kepler himself introduced ellipses for merely *computational* purposes and referring explicitly to Archimedes, this enduring false myth about Greek astronomy originated from a passage by Simplicius (VI cent. CE), who refers an opinion of the peripatetic philosopher Sosigenes (II cent CE). Nevertheless, no trace of such a "principle" can be found in Plato's works, and Wilbur Knorr (1990) has convincingly demonstrated that the ascription to Plato of such a specific "program" for astronomy is an invention of later commentators, further alimented by the success of Ptolemy's *Almagest*.[41]

A younger contemporary and pupil of both Archyta and Plato was Eudoxus of Cnidus, probably the greatest mathematician of the Hellenic period. Despite only fragments of his works survived, we know that he was an illustrious predecessor of Archimedes in the masterly use of the method of exhaustion,[42] and that the theory of proportion exposed in Book V of Euclid's *Elements* is mainly due to him. Notice that the importance of this theory was immense for Greek mathematicians (as it later was for the modern ones, who started from it to build the concept of *real number*), since it was based on general definitions of *magnitude*, *ratio* and *proportion* framed in such a way to allow an equal treatment of both commensurable and incommensurable quantities, thus solving the difficulties emerged in Pythagorean mathematics. An outcome of this process was the logical subordination of arithmetics to geometry that we see in Euclid's *Elements*, where the theory of numbers exposed in Books VII-IX is built upon the general theory of proportions. This was indeed an inversion of the primacy of arithmetic over geometry generally claimed by Pythagoreans that, incidentally, grounded also arithmetic in the activity of *drawing*.

It was Eudoxus, as far as we know, the first to use a combination of circular motions to account in a relatively simple way for the irregularities observed in

[40]The fact that also Archimedes could have easily defined the *spiral* more generally and without appealing to motion is emphasized in Netz 2017, pp. 33–36.

[41]Of course this doesn't mean that Plato's philosophy had no influence on the development of Greek mathematics (sure it had), but such influence should be assessed much more carefully than sticking to the third-hand testimony of an author who writes almost one thousand years later the things he talks about. In any case, there is no doubt that Plato's influence has been much less important than what asserted by later pseudo-platonic commentators, and it is likely that the opinion exposed by Simplicius gained consensus among historians only because it matches the above alluded narrative of a continuous and uninterrupted line of development going from Plato to Ptolemy.

[42]According to Heath (1921a, p. 328), despite the previous works of Antiphon and Hippocrates, it was Eudoxus who established the method "as part of the regular machinery of geometry."

heavenly motions. His model, exposed in a lost work entitled *On Speeds* (Περί ταχών), was based on a system of concentric spheres for each planet, rotating around different axes and with different velocities, so that a point placed the equator of the external sphere and *carried* by all these *simultaneous* motions periodically traced a spherical lemniscate called *hyppopede*. In this way, identifying the point with the observable position of a planet, the model gave an account of planetary *phainomena*, namely the periodical inversion and restoration of the direction of their motions with respect to the fixed stars.[43]

A long debated issue is whether Eudoxus' spheres should be regarded as real objects existing in physical space or as theoretical entities, i.e. mental models constructed to account for the observed planetary phenomena. The question pertains to the general problem of how mathematical entities were conceived by the Greek mathematicians. Needless to say, the debate is still ongoing today. To the question if mathematical entities are *real* objects or *imaginary* constructions of our minds modern mathematicians hold quite different and often opposite views. It is likely, as Lasserre holds, that also in Plato's Academy different views were represented.[44]

In the specific case of Eudoxus' model, we think that it can be regarded as a masterly application of Archyta's kinematical/mechanical geometry to the solution of a difficult astronomical problem, namely that of planetary stations and retrogradations. Therefore, following Schiaparelli[45] and Heath[46] we find no basis to ascribe any physical reality *at all* to Eudoxus' spheres, *geometrical computational tools* rooted in the specific mathematical techniques mastered by Eudoxus' and his fellows. Eudoxus achieved his goals by the study of the *hyppopede*, and also in this he was a brilliant predecessor of Archimedes, who solved the celebrated problem of *squaring the circle* by rectifying it with the help his kinematically-defined *spiral*. Notice, in particular, that in Archimedes treatise the word for *uniform* is ισοταχέος (literally *with the same speed*), which recalls directly the title of Eudoxus' work Περί ταχών.

The *theoretical* features of Eudoxus' theory by themselves fit the possibility to embed the model of concentric spheres in a real mechanical device. We don't know whether Eudoxus (or someone else) built at the time such a model (probably not, considered the technical difficulties involved), but it is beyond doubt that in Plato's Academy *sphairopoiia* was common practice. Its use as a teaching aid in astronomy is explicitly mentioned in the *Timaeus*, when talking just about planetary motions - so complex that "only few among many" know their periods and "by observation measure their ratios with numbers" - Plato writes:

[43] In modern times, the first to appreciate the full import of Eudoxus' theory was the German chronologist and astronomer Christian Ludwig Ideler around 1830. A beautiful reconstruction of Eudoxus' model and the still-today standard reference about it is Schiaparelli 1874.

[44] Lasserre 1964, pp. 11–42.

[45] Schiaparelli 1874.

[46] According to Heath Eudoxus' model was "on purely mathematical lines", "the first attempt at a purely mathematical theory of astronomy, and "constitutes one of the most remarkable achievements in pure geometry that the whole history of mathematics can show." He also notices that "there is a good deal of similarity of character between Archyta's construction of the curve of double curvature and Eudoxus' construction of the spherical lemniscate by means of revolving spheres." (Heath 1921a, p. 251)

> But the circlings of them and their crossings one of another, and the manner of the returning of their orbits upon themselves and their approximations, and which of the deities meet in their conjunctions and which are in opposition, and how they pass before and behind each other, and at what times they are hidden from us and again reappearing send to them who cannot calculate their motions panics and portents of things to come - to declare all this without visible illustrations of their very movements were labour lost.[47]

It is also noteworthy that the mechanism of nested hemispheres described in the *myth of Er*, put by Plato at the close of the *Republic*, has evident analogies with Eudoxus' planetary model. In support of this interpretation, we suggest that Eudoxus' theory may have represented an example of the "new astronomy" that Plato invokes in this famous passage:

> These decorations in the heaven, since they are embroidered on a visible ceiling, may be believed to be the fairest and most precise of such things; but they fall far short of the true ones, those movements in which the really *fast* and the really *slow*—in true *number* and in all the true *figures*—are moved *with respect to one another* and in their turn move what is contained in them. They, of course, must be grasped *by argument and thought* [lit. *dianoetically*], not sight. [...] Therefore the decoration in the heaven must be used as patterns for the sake of learning these other things, just as if one were to come upon diagrams exceptionally carefully drawn and worked out by Daedalus or some other craftsman or painter. A man experienced in geometry would, on seeing such things, presumably believe that they are fairest in their execution but that it is ridiculous to consider them seriously as though one were to grasp the truth about equals, doubles, or any other proportion in them. [...] Therefore, *by the use of problems, as in geometry, we shall also pursue astronomy*; and we shall let the things in the heaven go, if by really taking part in astronomy we are going to convert the prudence by nature in the soul from uselessness to usefulness.[48]

So, according to Plato, astronomers should not deal *directly* with *visible* bodies, but with the medium of *invisible* entities that "must be grasped by argument and thought" (read: via *consistent theoretical models* formulated in terms of (successions of) *numbers* and *figures*) and their aim should be to seize "the equal, the double and any other proportion" existing between the *motions* - the *fast* and *slow* - of the heavenly bodies. The fact that this new kind of astronomy must be pursued *by the use of problems, as in geometry* is a clear indication that Plato has in mind a mathematical treatment, in which geometrical *diagrams* are set up in such a way to satisfy some previously required properties, and by *analogy* imitate (*mimemata* in the previous passage from the *Timeaus*) the heavenly motions. This is exactly what Eudoxus did with his *kinematical hyppopede*, what probably Archimedes did in his treatise on *sphere-making* and what we see realized in the

[47] *Timaeus*, 40D. Translation by Archer-Hind (1888). It is to be remarked that the role of this dialogue in the history of Western philosophy was immense, being the only one of the Platonic corpus to survive throughout the Middle Age.

[48] *Republic*, 529D. Translation by Bloom (1968) All the emphasis are ours. A review of the proposed interpretations of this passage is in Bulmer-Thomas 1984.

AM. Also, we could add, what in different guises mathematical astronomy has always done ever since.

Whether it was Plato's philosophy that influenced Eudoxus' astronomy, or Eudoxus' astronomy that influenced Plato's philosophy, no one can really say. Or, rather, in such terms the question is ill-posed. The most natural interpretation is that they influenced each other, philosophy and mathematics being clearly in conversation within Plato's Academy. Also, switching from philosophical to strictly mathematical subjects we cannot exclude that the roles of master and pupil could have reversed. Suggestive in this sense are an anonymous scholium to Euclid's Book V (perhaps by Proclus) where Eudoxus is mentioned as "the teacher of Plato",[49] and the fact that Eudoxus visited Plato's Academy at least twice during his life, at some years apart, after having founded his own school. In other words, Eudoxus studied at Plato's Academy, and then returned there as an accomplished mathematician, namely the most revered of his generation.[50]

Eudoxus also composed a *Parapegma* and other works of astronomical content. One of these was the *Phenomena*, partially preserved thanks to the famous poem of Aratus and to Hipparchus' later commentary to it. Here are some passages from it, as quoted by Hipparchus:

> There is a certain star that remains always in the same spot; this star is the pole of the universe.
> Between the Bears is the Tail of the Dragon, the end-star of which is above the head of the Great Bear.
> The Dragon's Head moves where the limits of rising and setting are confounded [i.e. on the arctic circle][51]

Apparently the *Phaenomena* was some sort of astronomical survey, containing among other things a detailed *uranography*, i.e. a description of the relative positions of stars and constellations made with reference to some standard circles cut off from the celestial sphere. Since we know from Hipparchus that this treatise had a content "almost identical" to another work entitled the *Enoptron*, the reference to a sort of *mirror* in the title of this latter suggests that it could have contained a description of a *celestial mirror*, i.e. a celestial globe "mirroring" the celestial sphere. In any case, we know from Cicero that Eudoxus was the first to build a globe with engraved the celestial parallels and the constellations,[52] and it is very reasonable to think that Eudoxus made use of actual globes in his work.

So, to sum up, despite none of his works survives, we know from reliable sources that Eudoxus (1) studied in detail the sphere of the fixed stars, probably making use of celestial globes, and (2) that for each celestial body set up the problem of finding a system of *concentric rotating spheres* generating a spherical curve, the *hyppopede*, that was required to satisfy in its *progressive generation* the properties exhibited to observation by celestial *motions*. As far as we know, he

[49] Heath 1921a, p. 325.

[50] To be clear, a sort of *guest lecturer*. Indeed, Plato's δήμιουργος, the *artisan building the world* mentioned in both the *Timaeus* and the *Republic* seems to me nothing but a *divine mathematician* whose Eudoxus furnished a natural model.

[51] Evans and Berggren 2006, p. 5.

[52] *De Republica*, 22.

was the first to frame the general problems of mathematical astronomy in these terms, and for this reason he may well be regarded as the first well-attested Greek *sphere-maker*.

Eudoxus' approach became the basis for much of the further developments of Greek mathematical astronomy. Along the lines already indicated by the Pythagoreans, astronomy became more and more a byword for *kinematical spherical geometry*, i.e. the theoretical study of a *moving sphere*. In Eudoxus' *Phaenomena* there are still references to the actual astronomical bodies (i.e. stars and constellations), but these progressively disappear in favor of the study of a general *revolving sphere* carrying with it *tracing points* drawing *circles* on its surface. This process can be seen at work in all the surviving Greek astronomical works treating about the *daily motions* of Sun and stars and the connected problems of *rising and setting times*.[53]

The works of Autolycus (360 – c. 290 BC) are the oldest Greek mathematical treatises fully extant today. In his *On the Moving Sphere* the study of the daily motions of stars is carried out in the standard form of mathematical propositions we're still familiar with: the enunciation in general terms of the proposition to be proved, the enunciation of the same proposition with reference to a diagram referenced with letters corresponding to the relevant points, the proof of the proposition, and lastly, in some cases, the conclusion phrased in terms similar to the enunciation. In this treatise Autolycus considers the problems of rising and setting times of stars, but does it by reference to an abstract sphere revolving around one of its diameters and focusing on three kinds of circular sections, corresponding to *meridians*, *parallels* and *oblique* circles like the *ecliptic*. These are some of the propositions demonstrated in the treatise:

> 1. If a sphere *revolves uniformly* around its own axis, all the points on the surface of the sphere which are not on the axis will *describe parallel circles* which have the same poles as the sphere and are also at right angles to the axis. [emphasis ours]
> 2. If the circle in the sphere defining the visible and the invisible portions of the sphere be obliquely inclined to the axis, the circles, the circles which are at right angles to the axis [i.e. the parallels] and cut the defining circle always make both their risings and settings at the same points with respect to the defining circle and further will be also similarly inclined to the that circle.
> 9. If in a sphere a great circle which is obliquely inclined to the axis define the visible and the invisible portions of the sphere, then, of the points which rise at the same time, those towards the visible pole set later and, of those which set at the same time, those towards the visible pole rise earlier.

Contemporary or shortly following Autolykos' *On the Moving Sphere*, Euclid's *Phenomena* treats similar topics (some of their propositions are nearly identical) and also the important problem of determining the length of daylight at a given date and place on the Earth.[54]

[53] This progressive abstraction in the astronomical genre of *Phenomena* is pointed out in Evans and Berggren 2006, pp. 4–8, and was already noticed by Proclus.

[54] See Berggren and Thomas 2000, pp. 1–6 for more details.

This problem had already been treated and partially solved by Babylonian astronomers with arithmetical methods. In particular, they realized that the length of daylight on a given day is the same as the time it takes the semicircle of the ecliptic following the Sun to rise as the point occupied by the Sun moves across the sky from the east to west. Therefore, the total length of daylight was computed (1) by *assigning* rising times to a set of consecutive arcs of the ecliptic, and (2) by adding all the rising times relative to the particular 180° of the ecliptic following the Sun on the given day. Euclid was probably aware of such methods, but sticks to a geometrical approach and invokes arguments based on the symmetry between opposite arcs of the ecliptic. According to Berggrenn and Thomas, one of Euclid's goals in the *Phenomena* was just to *exhibit geometrically* the implicit *symmetry assumptions* underlying Babylonian numerical algorithms.

This is an early example of a general process occurring in Greek astronomy during the Hellenistic period, i.e. the *assimilation* of empirical materials, results and methods employed by Babylonian astronomers. This is a *crucial* feature of Hellenistic mathematics, that was fully understood only in the XX century, after the decipherment of Babylonian astronomical tablets. Since Babylonian astronomy had developed for centuries using sophisticated arithmetical algorithms, specifically adapted to astronomical computations, in the hands of Greek geometers this assimilation took the form of a *geometrical reinterpretation of arithmetical computational methods*.[55] We already mentioned the lunisolar cycles embedded in the back side of the AM, and other well-known examples are the adoption by Greek astronomers of the *zodiac*, the 360 division of the circle (reflecting the Babylonian sexagesimal numerical system) and Hipparchus' use of Babylonian period-relations in the construction of his astronomical models. Despite the loss of all his works, it is generally agreed that Hipparchus played an important part in this assimilation and re-elaboration of Babylonian mathematical methods, also for his geographical area of activity in the eastern part of the Mediterranean.

Theodosius *Spherics* (II cent BC) is an introductory textbook on spherical geometry. Despite no explicit reference at all to celestial bodies is ever made, the three books contain theorems and problems involving circles *cut from* or *drawn on* a sphere that are of immediate application to spherical astronomy. The whole first book is often attributed to Eudoxus, and Theodosius seems to have been a compiler of results known at least since Euclid's time. As Thomas remarks, it is likely that the theory of *spheric sections* exposed in the *Spherics* was preliminary to the way more advanced theory of *conic sections*, such as that exposed in Apollonius' treatise.[56] Here are some of the propositions:

> I.1 - The plane through three points on the surface of a sphere cuts the surface of the sphere in the circumference of a circle. Corollary. If a circle

[55] The origin of this theoretical difference is linked to the different material supports used since millennia for writing in Mesopotamia and in the Mediterranean. On clay tablets arrays of symbols made of strokes accompanied the development of *symbolical* or *algebraical* methods, on papyri the possibility of drawing accurate figures fueled *visual* or *geometrical* approaches. This remark is in Russo 2004.

[56] See Thomas 2018, in which a list of all the definitions and propositions of the *Sphaerics* is given, together with some diagrams drawn using *Mathematica* software. The second book is particularly interesting, since it develops a *theory of tangency* for circles on a sphere.

> is in a sphere, the perpendicular produced from the centre of the sphere to it falls at its centre.
> I.2 - To find the centre of a given sphere. Corollary. If a circle is in a sphere and a perpendicular is erected at its centre, the centre of the sphere is on the perpendicular.
> I.20 - To draw a great circle through two given points on the surface of a sphere.
> I.21 - To find the pole of a given circle in a sphere.
> II.5 - In a sphere, if two circles touch each other, then the great circle drawn through the poles of one and the point of contact goes through the poles of the other.

Even if they don't express with these words, Sidoli and Saito have convincingly argued that Theodosius' *Spherics* was an application to spherical geometry of the *operational* approach we find in Euclid's *Elements*, with the difference that here diagrams are drawn *directly* on a spherical surface:

> ...it is clear that ancient mathematicians were interested in developing mathematical methods that directly modeled the possible operations of actual instruments. Although the diagrams that have been preserved in the manuscript tradition are generally purely schematic, our investigation has shown that the problems in the *Spherics* were written in such a way that they could be carried out on an actual globe and, hence, must have derived from an interest in producing accurate diagrams. Indeed, there is evidence in other mathematical texts that Greek geometers were interested in working with instruments so as to produce metrically accurate diagrams.[57]

Quoting later testimonies corroborating this view, the authors conclude:

> ...the *Spherics* was written for students of spherical astronomy who would have been interested in representing the principal circles of the celestial sphere on a globe. Indeed, a globe inscribed with these lines could well have been produced using the kinds of constructive techniques set fourth in Theodosius's seven problems. Although the text is structured as a purely deductive treatise, it was written by and for individuals who used material objects to aid in their investigations of the mathematical aspects of their cosmos. As an elementary treatise, the Spherics not only develops the basics theorems necessary for understanding the geometry of the sphere, but also sets out a series of problems that would have been useful for anyone solving problems in spherical geometry by drawing diagrams on a real globe.

It is in this context that spherical trigonometry was invented, probably by Hipparchus, who is likely to have used it in his lost work on simultaneous risings. Trigonometrical methods are also used by Ptolemy in the *Almagest* to deal with the same problems and recent scholarship has brought convincing evidence that here as elsewhere Ptolemy drew on the work of Hipparchus. Indeed, the mathematical techniques Ptolemy employs seem to be only a portion of those used for similar problems by his Hellenistic predecessors. From a comparison between

[57]Sidoli and Saito 2009, pp. 606–607.

Ptolemy's treatment of rising times (in Books II and VII of the *Almagest* and in the *Analemma*), and Hipparchus' *Commentary* on Aratus' poem, Sidoli concluded that:

> ...it [is] likely that Ptolemy based his work on material that originated with, or was derived from, Hipparchus's work on spherical astronomy. [...] Whereas Hipparchus seems to have combined the analemma with the trigonometrical methods, Ptolemy took pains to base his spherical astronomy on the trigonometrical methods alone. Even in Almagest VII, where Ptolemy demonstrates how to find the degrees of the ecliptic and equator that rise, culminate and set with a given star, he uses only the trigonometrical theorem In fact, when Ptolemy introduces the mathematical theorems of ancient spherical trigonometry he states that they will allow him "to carry out most demonstrations involving spherical theorems in the simplest and most methodological way possible". By calling his approach simple and methodological, he is likely referring to the fact that it makes use of only one of the two ancient metrical methods on the sphere.[58]

So, to sum up the content of this section, we have sketched how the development of Greek *spherical astronomy* overlaps and actually coincides with the development of the *spherical geometry*. The simplest version of *sphairopoiia*, *celestial globes*, allowed, on one hand, to visualize the *celestial sphere*, i.e. the "abstract" or *theoretical* model used to account for the phenomena; and, on the other, was the very same *instrument* used to frame and solve problems included in this branch of astronomy, perhaps by the drawing of diagrams directly on actual globes.

The point we wish to stress is that the distinction between the "abstract" theory and the "concrete" model simply vanishes: the turning sphere *is* the theory, in the sense that it *is*, by itself, the *model* of a certain set of *phenomena*. A globe with the proper inscribed circles, properly inclined on its support to match the local latitude, and rotating at the proper frequency, would be an *exact* realization of such *theory of the celestial sphere*, an *analog computer* of some well-definite celestial phenomena and the simplest form of *sphairopoiia* in the broader sense we have outlined above.

The originator of this thread, Eudoxus of Cnidus, already tried in his treatise *On Speeds* to extend the same approach to another class of problems, i.e. those involving the motions of Sun, Moon and planets. By the study of a kinematically-generated spherical lemniscate, the *hyppopede*, he was able to account at least in a qualitative way for the periodical occurrence of planetary stations and retrogradations. This was a huge step, but after his immediate successors, Callippus and Aristotle, we have no clue of other attempts to explain planetary motions with tridimensional models. The only exception, in the view we here propose, would be Cicero's testimony on Archimedes' *sphaira*, which is generally believed to have been a tridimensional device.

[58]Sidoli 2004, p. 82.

1.4 Mechanics and Circular Motions

Whereas the *principle of circular motion* in the context of Greek astronomy has been matter of debate, turning to mechanics things are much easier, since circular motion has occupied a central role in it from very remote times, and never ceased to ever since.

The oldest extant Greek work on mechanics is the anonymous collection of *Mechanical Problems*, a list of 35 *how* and *why* questions about motion, weight-lifting, friction and other practical problems involving levers, screws, pulleys and other simple machines. The work survived as part of the pseudo-aristotelian corpus, but apparently no one believes it was written by Aristotle himself. A possible candidate is Strato of Lampsacus, the successor of Aristotle at the head of the Lyceum. Winter (2007) has argued that also Archytas is a possibility. In any case, no scholar has supported a date of composition later than the early III cent. BC.

In spite of its elementary character, the *Mechanical Problems* are extremely interesting for many reasons. The importance of this treatise was huge also for the rising of modern mechanics, being still considered fundamental at the time of Galileo, who lectured upon it in Padua, and Newton, who quotes from it full passages in the *Principia*.[59] In it we find very important results that will prove to be crucial, like the *parallelogram rule* for the composition of motions (see below) and a first attempt to distinguish the concepts of *mass* and *weight*.[60]

This is how mechanics is introduced at the beginning of the work:

> One marvels at things that happen according to nature, to the extent the cause is unknown, and at things happening contrary to nature, done through art for the advantage of humanity. Nature, so far as our benefit is concerned, often works just the opposite to it. For nature always has the same bent, simple, while use gets complex. So whenever it is necessary to do something counter to nature, it presents perplexity on account of the difficulty, and art [*techne*] is required. We call that part of *techne* solving such perplexity a *mechane*. As the poet Antiphon puts it: "We win through *techne* where we are beaten through nature." Such it is where the lesser overcomes the greater, and when things having little impetus move great weights. And we term this entire class of problems *mechanics*.[61]

So, mechanics is explicitly identified with a *class of problems* relating to something that happens *contrary to nature* and therefore *surprisingly*, like those in which a small weight moves a bigger one. Most of these problems deal with moving bodies, and the answers involve geometrical diagrams which are meant to represent *motions* in an abstract way. Most questions ask about the *cause* (*aitia*) of something that is known to happen from experience in dealing with some device (e.g. "through

[59] Rose and Drake 1971.

[60] See Problem 9: "Why is an empty balance beam easier to move than a weighted one? In the same way also a wheel or any such thing, the heavier is harder than the smaller and lighter. This is true not only opposite the weight, but sideways. Opposite to its tilt it is harder to move anything, but there is no tilt sideways." (Winter 2007, p. 17).

[61] Winter 2007, p. 1.

what *aitia* larger balances are more accurate than smaller ones?") and the answer generally lies in some geometrical property of the diagram representing the motion of the device (in this case, the motion of the balance). The author immediately points out this sort of *mixed* character of mechanical problems:

> Mechanics isn't just restricted to physical problems, but is common alike to the theorems of mathematics as well as physics: the *how* is clear through mathematics, the *what* is clear through physics [read *natural philosophy*].

This is indeed, as far as I know, the first explicit mention outside the astronomical and musical domains of the hybrid status of *mathematical physics*, in which problems related to the concrete, *physical* world are formulated and solved in *mathematical* terms.

Despite the hypothetical-deductive method of Euclid is yet to come, in the *Mechanical Problems* we clearly perceive the emergence of the need for a *theory* of mechanics, in which the solutions to *problems* are to be deduced by some assumed *principle*, in the absence of which it is difficult to proceed.[62] Such a guiding principle for mechanics in general is found in the properties of *circular motion*, the origin of the all the *tricks* and the key to their explanation:

> The circle contains the first principle of all such matters. This falls out quite logically: it is nothing absurd for a marvel to stem from something more marvelous still, and most remarkable is for there to be opposites inherent in each other, and the circle is made of opposites. It derives from the moving and the standing, whose nature is opposite each the other. [...] Though one (apparent) absurdity may suffice about the circle, a second is that it moves opposite motions. It moves backwards and forwards at the same time. [...] Therefore, as was said earlier, there is no surprise at its being the first principle of all marvels. Everything about the balance is resolved in the circle; everything about the lever is resolved in the balance, and practically everything about mechanical movement is resolved in the lever.[63]

Aside from these philosophical arguments about the *co-existence of opposites* (commonplace in Greek natural philosophy), from the mathematical point of view the fundamental property of circular motion is the following:

> Further, many of the marvels about the motion of circles derive from the fact that, on any one line drawn from the center, no two points are swept at the same pace as another but always the point further from the motionless end is quicker.

It is important to remark that - coherently with the Greek general practice of considering *figures* - the motion here considered is, primarily, the motion of the *full circle*. This motion entails that of all its parts, and here it is remarked that points

[62] See for example Problem 32, which asks a question involving friction: "Why do objects thrown stop? Is it because the projective force leaves off? Counter-pull? Slope, if it be greater than the throwing force? Or is it foolish to mull such an impasse, absent the principle?" (Winter 2007, p. 34). Significantly, the question is left unanswered.

[63] Winter 2007, p. 2.

internal to a rotating circle at different distances from its center have a different velocity. It is clearly pointed out, in particular, that the points of any rotating radius have speeds ranging from zero to a maximum, and later it is specified more precisely what this means:

> The basis of this is to ask why in the circle is the point standing farther from the center moved faster than the one nearer, when the one nearer is moved by the same force. The "faster" has a double meaning. For if in less time it has crossed equal space we say it is faster; and likewise if in equal time it has crossed more space. *The greater in equal time draws a bigger circle, and the outer is bigger than the inner.* [emphasis ours]

So the essential point of the explanation is that all the points of the radius of a rotating circle move *simultaneously*, and since in the same time the *bigger line* sweeps a *bigger circle* it is said *faster*. It is to be noted that here the author is *not* measuring the velocity of points in terms of ratios between *traveled distances*, but in terms of ratios between *swept areas*. In other terms, *areal* velocity is here considered, and not *linear* velocity. This aspect, that may be surprising for modern readers used to point-mechanics, is indeed a consequence of the fact that the author of the *Mechanical Problems* has in mind real objects like levers, balances and gears, so there is no question that the points on the diagram could be conceived as "disconnected" one another.

If one considers the *diameter* (i.e. the *measure-across*, a term used also for polygons and other figures) of such circles, it is immediate to see that at the two sides of the center points move also in *opposite* directions. This is the principle of gears transmitting motion to each other, as the author immediately points out:

> From the circle going opposite ways at the same time (e.g., one end of the diameter, at A, is moved forward, the other, at *B* is moved backward) some have set up so that from one movement, many circles are in opposite motions, such as they have dedicated in temples, having made the little wheels out of bronze and steel. For if circle CD touches circle AB, CD will be moved backward when the diameter of AB is moved forward, so long as the diameter is moved in place. So the circle CD is moved just the opposite of the circle AB. And that circle again will move its neighboring circle EF just the opposite to itself, and for the same reason. In the same way, if there be more, they will all do this, being moved by one circle alone. So taking this underlying nature of the circle, craftsmen make a machine hiding the cause, so only the marvel of the mechanism is visible while the cause is unseen.

This is the first extant mention of gear-work in the Greek literature, and the evidence that gears were employed at least since the early III cent. BC in fairly complex devices. Such contrivances placed in temples are also ascribed to Archytas by later sources.

In explaining *why* of two concentric circles rotating together the bigger is faster, the author introduces two more important elements pertaining to the theoretical *analysis* and *synthesis* of motions.

The first is what much later was called *parallelogram rule* for the *composition of simultaneous motions*. It is expressed as follows:

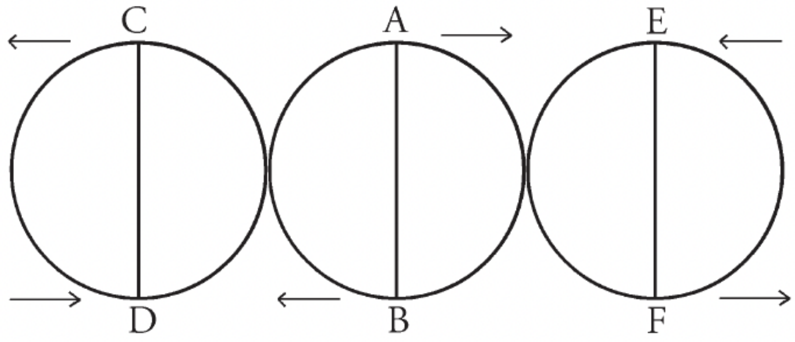

Figure 1.2: Diagram of rotating gears from the *Mechanical Problems* (source: Winter 2007, p. 3).

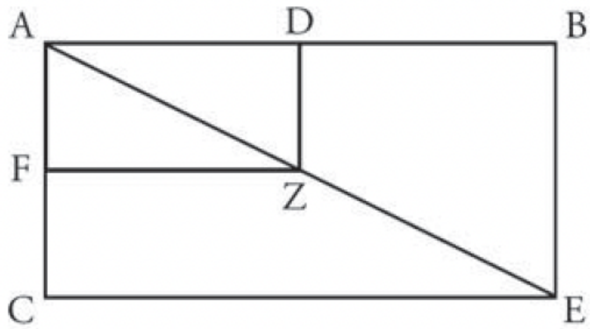

Figure 1.3: Composition of simultaneous motions in the *Mechanical Problems* (source: Winter 2007, p. 4).

> Whenever the moving point is carried in some ratio [*logos*], it is necessarily carried in a straight line, and it becomes the diagonal of the scheme that the lines make which are stretched in that proportion. Let the *logos* the point is carried be the ratio AB has to AC. Let AC be swept toward B. Let AB be swept towards CE. Let A be carried up to D, and the line AB up to F. If the vector ratio is that which AB has to AC, of necessity AD has that ratio to AF. The small four-sider is proportional to the larger since the diagonal is same for both, and the point A will be at Z. The same thing will be seen no matter where the conveying gets stopped: Point A will always be on the diagonal.

This is, incidentally, another example of *kinematical* generation of a figure, in this case a parallelogram, which is conceived as *swept out* by the two sides. The word that Winter translates *vector* in Greek is *phora* (φορα), which means *carriage*, *transport*, and the proposition deals with the *ratios* between simultaneous *transports* or *displacements* or *motions* of points and lines. The accompanying diagram is therefore in all respects a *diagram of motions* or of *velocities*, since it represents two *simultaneous displacements* and their combined result.

The second is a rule for the *decomposition* of circular motion :

> To every line drawing a circle, this happens: it is both conveyed according to nature along the periphery and carried contrary to nature, to the side

> and to the center. Being closer to the anti-pulling center, it is overcome by it more. That the lesser circle is moved more contrary to nature than the larger is clear from the following...

Without entering in the details of the argument, the important point is that also circular motion is conceived as the result of the composition of two elementary and simultaneous motions, one along the *periphery* (i.e. along the tangent) and one towards the center. These, later on, are treated independently and called motion *according-to-nature* (κατα φυσιν) and motion *contrary-to-nature* (παρα φυσιν). The comparison of two distinct circular motions is made in terms of a proportion between the corresponding components:

> There has to be a proportion: the according-to-nature [κατα φυσιν] in the large is to the according-to-nature [κατα φυσιν] in the small as the contrary-to-nature [παρα φυσιν] is to the contrary-to-nature [παρα φυσιν].[64]

So, in short, the author gives some elementary *rules* indicating how rectilinear and circular motions may be analyzed in components. The difference between the two motions is that in circular motion the two components "have no ratio" to one another. Following Russo's interpretation, we take this statement as a reference to Euclid's definition of ratio between two magnitudes (probably introduced by Eudoxus), according to which two magnitudes are said to have a ratio to one another if by adding the small one to itself a sufficient number of times it eventually becomes greater than the big one. So, more precisely, the author is saying that *the motion toward the center is infinitesimal with respect to the motion along the tangent*, a statement that will be made rigorous by Archimedes in the *Spiral*.[65]

To summarize, in the *Mechanical Problems* we find the first extant mathematical treatment of the phenomena of *motion* by the use of *diagrams of velocities* representing *simultaneous displacements* of lines and points, for which are given some elementary rules dictating how these should be theoretical *analyzed* and *constructed*. The properties of circular motions, in particular, are indicated as the cornerstone for the solution of mechanical problems. It is particularly noteworthy that, already in this early work one finds the concept of *areal velocity* alongside with that of *linear* velocity, which indeed is included in the former. In the Greeks perspective this is not so surprising, since dealing with *ratios between simultaneous displacements*, there is little difference, in principle, whether one deals with ratios between *lengths* or *areas*.[66]

During the III cent. BC many of the ideas exposed in the *Mechanical Problems* were fully developed and the mechanics greatly enlarged its scope both practically

[64] Winter 2007, p. 8.

[65] See Russo 2019, pp. 113–116 for more details on this point.

[66] In this regard, it is to be remarked that Kepler introduced the concept of areal velocity in early modern astronomy just recollecting Archimedes' practice to replace infinite sums of "small distances" with infinite sums of "small areas" (*Astronomia Nova*, Chap. XL). Also, Kepler was one of the few who appreciated the often disregarded Book X of Euclid's *Elements* - the longest book of all - where a general theory of commensurability is exposed. Here, magnitudes are classified according to the character of the ratio formed by them and by their *squares*. Of course Kepler read it in light of his own mystic views, but also didn't fail to grasp its theoretical relevance. For more details see Simon 1979, pp. 149–155, 358–366.

and theoretically. In particular, it was turned in a rigorous scientific discipline, methodologically homogeneous to the other branches of Hellenistic mathematics. Despite only a small portion of his mechanical works survive - namely, that dealing with problems of *statics* - it is sure that Archimedes took a strong part in this process, and it is hardly imaginable that he didn't treat also *dynamical* problems involving forces and motions of bodies. From Hero's *Mechanica* we infer, in particular, that in the context of mechanics also the principle of inertia, connected to the phenomena of friction and only adumbrated in the *Mechanical Problems*, received an explicit formulation.

In the Hellenistic period also machine-design came to be based on sound theoretical principles and standardized methods. This aspect is made clear, for example, in Philo's *Belopoeica*, an extant portion of a bigger *Mechanike Syntaxis*. In it Philo discusses many aspects of machine-design, dwelling at long on how only a proper interplay between theory (i.e. mathematics) and experimentation can lead to success (emphasis ours):

> I suppose you are not unaware that the art [technē] contains something unintelligible and baffling to many people; at any rate, many who have undertaken the building of engines of the same size, using the same design, similar wood, and identical metal, without even changing its weight, have made some with long range and powerful impact and others which fall short of the ones mentioned. Asked why this happened, they could not give the reason [*aitia*]. Hence the remark made by Polycletus the sculptor is pertinent to what I am going to say. He maintained that *excellence is achieved gradually through many numbers. Likewise, in this art [technē], since products are brought to completion through many numbers, those who deviate slightly in particular parts produce a large total error at the end.* Therefore, I maintain that we must pay close attention when adapting the design of successful engines to a distinctive construction, especially when one wishes to do this while either increasing or diminishing the scale.[67]

This aspect of *scale* is essential, and throughout the treatise Philo insists on the importance of *symmetria* as a general principle of machine-design. By *symmetria* Philo means the search for a *common measure*, a *unit element*, to which the sizes of all the parts composing the machine must be referred. If the machine is designed in such a way that all the parts are *commensurate* with this single element, the scalability of the device is in principle guaranteed.[68] In the case of catapults, such element is found in the diameter of the hole through which the elastic ropes run through.

There is no doubt that such notion of *symmetria* (literally *same measure*) played a crucial role also in *sphairopoiia*. Symmetria between what? On one hand, between the dimensions of the different parts of the device (e.g. between the gears' tooth-number and radii of), and, on the other, between the actual *time relations* used to model celestial motions. In the AM a clever use of prime-factors between period-ratios, for example, is the key to the optimal use of a small number of gears,

[67]Quoted in Schiefsky 2015, to which the reader is referred for more details on Philo's method in machine-design.

[68]See also Di Pasquale 2019, pp. 110–122 on this point.

some of them being shared by multiple gear-trains.[69] In this perspective, it is sure that the heliocentric hypothesis, indicating the *solar year* as a natural common factor among all the planetary periods, provided essential design indications for sphere-making. Indeed, the whole structure of the AM, centered on the solar wheel, is by itself a strong indication of the heliocentric character of its design, also suggested by Cicero's description of Archimedes' *sphaira*, where it is said that all the motions were obtained "by a single *conversio*". It is no case, after all, that no working planetarium has ever been built in the modern era before Copernicus, albeit the practice of *sphairopoiia* had been resurrected well before.

1.5 Hellenistic Dynamical Astronomy

In 1994 Lucio Russo first proposed a conjectural reconstruction of the astronomical knowledge available at the time of Hipparchus', based on the study of pre-ptolemaic sources and therefore independent from Ptolemy's testimony. The texts Russo analyzes are from literary authors dating back to the period of interruption of astronomical studies occurred between II cent. BC and II cent. CE and reporting scientific ideas from the Hellenistic period. All these authors are extraneous to the scientific method of Greek mathematicians, therefore in some cases a certain hermeneutical effort is required to extract from them relevant informations. However, Russo's study is sound and, in our opinion, his conclusions very convincing. Since the period of Hipparchus is also the period of construction of the AM, Russo's analysis could be very relevant for the not less conjectural reconstruction of the planetary mechanisms of the AM. However, Russo's paper seems to have been ignored by all those who worked on this problem. Here we summarize the most important points of his paper in the perspective of the present dissertation. For more arguments and additional details the reader is referred to Russo's works.[70]

This is a passage from Plutarch's dialogue *De facie quae in orbe lunae apparet*:

> Yet the moon is saved from falling by its very motion and the rapidity of its revolution, just as missiles placed in slings are kept from falling by being whirled around in a circle. For the motion according to nature [κατά φύσιν] governs each thing unless it is diverted by something else. That is why the moon is not governed by its weight, [which is] balanced by the rotatory motion. However, there would be more reason to wonder if she were absolutely unmoved and stationary like the earth.[71]

And, a little later, another speaker of the dialogue says:

> To philosophers one should not listen if they want to repulse paradoxes with paradoxes and in struggling against opinions that are amazing fabricate others that are more strange and amazing, as these people do in introducing their "drift towards the centre". What paradox is not involved in it? ...

[69] See in particular Freeth, Higgon, et al. 2021 on this point.
[70] See, in particular, Russo 2004, pp. 282–319.
[71] *Moralia*, 923C-D.

> Not that incandescent masses of one thousand talents drifted through the depth of the earth, stop if they should reach the centre, though nothing encounter or support them; and if they, drifted downwards with impetus, should go beyond the centre, they turn back and swing ...? Not that pieces of matter cut off from either side of the earth should not be drifted downwards forever but falling upon the earth force their way into it from the outside and conceal themselves about the centre? Not that a turbulent stream of water drifted downwards, if it should reach the centre, a point which they themselves call incorporeal, stops suspended, moves in a circle around it, oscillating in an incessant and perpetual see-saw? ...[72]

In these passages we have a clear exposition of the idea that the motion of the Moon, like missiles in slings, is the result of a twofold effect, namely that of a rectilinear motion "according to nature" and that of a "drift towards the center". We have seen an identical analysis applied to mechanical circular motion in the *Mechanical Problems*, and indeed the word that Russo translates as "drift" is the same φορά used in the *Mechanical Problems* to denote the *transport* or *displacement* of a body due to actual *carrying*, *push* or *pull*. We notice that such homogeneity in the use of technical terms is by itself an indication that the theoretical treatment of *motions* in astronomy and mechanics was the same in the more mature stages of Hellenistic mathematics.

As Russo remarks, Plutarch uses this noun and the corresponding verb φέρω exclusively to describe *varying motions*, i.e. motions of bodies whose *velocity* varies in magnitude and/or direction. The "paradoxes" described in the second excerpt, which are typical solutions to dynamical problems of central force, are clearly different examples of motions resulting from such a "φορά towards the center". Therefore we agree with Russo's conclusion:

> The whole excerpt of Plutarch is therefore consistent, both for the qualitative features of the described motions and for the terminology used, with the possibility that his source might have exposed a dynamics based on the law of inertia and on the idea that what is today called a "force" (in particular gravity) could not uniquely determine the motion, but only the variations of the velocity.[73]

The main adversary in the polemics against "mathematicians" which includes the above passage about the "paradoxes" of the "φορά towards the center" is Hipparchus, and many other elements indicate that he is the main source of the scientific content of Plutarch's dialogue. It is noteworthy that such dialogue was known and well studied in the early modern era by Kepler and Newton, who also reports in the *Principia* the same sling argument in discussing centrifugal forces.

The identification of Hipparchus as the source of Plutarch is confirmed by Simplicius, from which we known that Hipparchus had written a treatise with the title *On bodies drifted downwards by heaviness* (Περί των διά βαρυτητα κατω φερομένον). Simplicius reports that in this treatise it was explained that the motion of a body thrown upwards vertically is, first, upwards with decreasing

[72] *Moralia*, 923F - 924A. It

[73] Russo 1994, p. 216.

velocity, and then downwards with increasing velocity, and that Hipparchus ascribes "the same cause also in the case of bodies let fall from on high". The unification of the explanation of these two motions is possible, as Russo remarks, only in a theory which deals with *accelerations* of bodies, which in the two cases is the same, and not only with their *velocities*, a further confirmation of Russo's interpretation of Plutarch's passages.

In Seneca's *Naturales Quaestionae* we read:

> The five planets force themselves upon our attention. Occurring in one place or another they compel us to be curious. Recently we have begun to understand what their morning and evening risings mean, their positions, the time of their movement straight forwards, why they move backward. Whether Jupiter was rising or whether it was setting or retrograde (for that is the term they have given to it when it recedes) - we learned only a few years ago. People have been found who would say to us: "You are wrong if you judge that any star either stops or alters its orbit. It is not possible for celestial bodies to stand still or turn away. They all move forward. Once they are set in motion they advance. The end of their orbital motion will be the same as their own end. This eternal creation has irrevocable movements. If they stop at any time it means that the bodies which are now maintained by a constancy and equilibrium will fall on each other. What is the reason, then, that some celestial bodies appear to move backward? The encounter with the sun imposes upon them the appearance of slowness, as well as the nature of their paths and their circles which are so placed that at a fixed period they deceive observers. In the same way ships seem to be standing still even though they are moving under full sail".[74]

Here we find, first of all, another allusion to the principle of inertia, according to which planets, once set in motion, always advance "straight forwards". Secondly, Seneca mentions clearly the illusory character of planetary stations and retrogradations, which from time to time "deceive" the observers by the arrangement of the their "circles" ("natura viarum circolorumque sic positorum...") in the same way as "ships seem to be standing still even though they are moving under full sail". In other terms, Seneca is reporting the heliocentric explanation of planetary stations and retrogradation, dating back at least to the III cent. BC and based essentially on the relative character of the observable celestial motions. Moreover, it is explicitly said that the Sun plays a role in the phenomenon ("Solis occursus..."), an element which is hardly understandable in a geocentric setting.

In this regard, it is important to remember that relativity of motion in general and the consequent non-observability of "true" celestial motions had been clearly demonstrated by Euclid in the *Optics* (namely as Theorem 51). It is barely necessary to remark the relevance of optics for astronomical observations, and it is likely that Euclid's theory of vision formed the necessary background of Aristarchus heliocentric model. Thirdly, it is noteworthy that in Seneca's source it was said that celestial bodies. if stopped, would "fall on each other", implying the idea of a *reciprocal attraction* between them.

[74] *Naturales Quaestiones*, VII, xxv, 5-7.

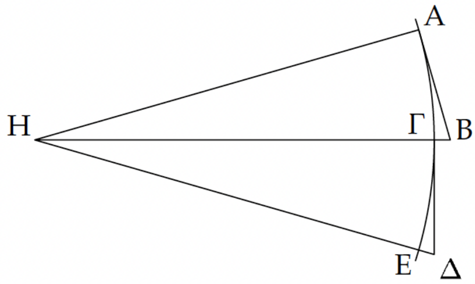

Figure 1.4: Russo's restoration of the the diagram described by Vitruvius' in *De Architectura*, IX, 11-13 (source: Russo 2004, p. 301).

In Lucretius[75] there are similar remarks about the illusory character of planetary phenomena and their connection with relativity of motion, with the important addition that also the immobility of Sun and Moon, suspended in the sky, is indeed a false appearance and due indeed to their very being in motion, an argument we already found in Plutarch's sling analogy.

On of the most impressive results of Russo's paper is the restoration of the meaning of two parallel and otherwise incomprehensible passages from Pliny and Vitruvius mentioning some "triangular rays" emitted by the Sun. These are the following:

> ... [The planets], struck in the aforesaid place, are prevented by a triangular ray of the sun from moving straight forward and they are drawn upwards by [its] burning force.[76]
>
> ... the mighty force of the sun extending its rays in the form of a triangle draws to itself the planets as they follow, and, as it were curbing and restraining those which precede, prevents their onward movement and compels them to return to it ... and to be in a "signum" of the other [out of two] triangle. Perhaps it will be asked why does the sun draw, by these heats, [the planets] in the fifth "signum" away from itself rather than in the second or third, which are nearer. I will therefore explain how this seems to happen. Its rays are spread out in the universe on the lines of a triangle with equal sides. Now [each side] extends neither more nor less than to the fifth "signum" away from its one ...[77]

The word "signum" in Vitruvius' passage is a calque of the Greek σεμειον, the technical term indicating a *point* in Euclid's geometry, and Russo' key insight is that these passages refer to a geometrical construction involving triangles with a vertexes placed in the Sun and two legs connecting it to successive positions of a planet. Russo's conjectural restoration of the diagram described is showed in Fig. 1.4, where a circular motion is analyzed in a series of simultaneous displacements directed along the tangent and towards the Sun. In short, Vitruvius sources applied to the motion of planets around the Sun the same decomposition of a circular motion we have already seen multiple times.

[75] *De Rerum Natura*, IV, 387 - 394.

[76] *Naturalis Historia*, II, 69.

[77] *De Architectura*, IX, i, 11-13

In Seneca we also find the following passages:

> Apollonius [of Myndus] says that the Chaldeans place comets in the category of planets and have determined their orbits.[78]
>
> Apollonius of Myndos has a theory different from Epigenes. He says that a comet is not one body composed of many planets but that many comets are planets. A comet, he says, is not an illusion or fire extending from the edges of two planets but is a celestial body on its own, like the sun and the moon. It has a distinct shape thus: not limited to a disc, but extended and elongated lengthwise. On the other hand its orbit is not clearly visible. A comet cuts through the upper regions of the universe and then finally becomes visible when it reaches the lowest point of its orbit.[79]

These references to comets, and in particular their being compared to planets, is very significant from the theoretical point of view, since the occurrence and *visual* appearances of comets and planets could hardly suggest anything in common between them. Moreover, as Russo remarks, the peculiar character of cometary phenomena makes them a true hallmark of a *dynamical* approach, no theory of comets being conceivable in a purely kinematical approach to heavenly motions.

The same remark applies to Hipparchus' well-known recognition and calculation of the precession of the equinoxes, hardly deducible in purely kinematical approach to the analysis of celestial motions.

Finally, we should add that during the II cent. BC a theory of tides had been probably developed by Seleucus, who may have used it to get a *proof* of the motions of the Earth.[80] This is suggested by the following statement from Plutarch:

> ... Did [Timaeus] put the earth in motion ... and ought the earth, globed about the axis extended through all, be understood to have been devised not as confined and at rest but as revolving and turning, as Aristarchus and Seleucus afterwards maintained that it did, the former stating this as a hypothesis, the latter demonstrating it?[81]

So, to sum up, it seems that at the time of Hipparchus a *heliocentric* and *dynamical* theory of heavenly motions had been developed, similar in its essential features to Newtonian or *classical* dynamics. In particular, it was based on a principle of inertia and on the idea that deviations from such "natural" motion are due to the mutual interactions of celestial bodies among themselves. Such theory was used to account for planetary and cometary motions, and its application to tides would have provided a dynamical justification to Aristarchus' heliocentric hypothesis.

Of course, these ideas are at odds with many commonplace views about Greek astronomy, mainly drawn from Ptolemy's *Almagest*, where no clue of dynamics may be found.

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[78] *Naturales Quaestiones*, VII, iv, 1.

[79] *Naturales Quaestiones*, VII, xvii, 1-2.

[80] A reconstruction of the long-term history of the theory of tides may be found in Russo 2020.

[81] *Moralia*, 1006C.

No doubt that, if Hellenistic mathematicians had developed a dynamical astronomy, it must have been something significantly different from Newton's original theory, which involves many mathematical, philosophical and metaphysical conceptions that are completely extraneous to Greek mathematics. Among these, we just mention the insistence on the notion of *cause*[82] and the connected *absolute* conceptions of times, space and motion.[83] In other terms, If Russo' reconstruction is correct, we would expect from mathematicians like Archimedes, Apollonius and Hipparchus some form of dynamics that is free from such ideas, any conceivable Hellenistic astronomical treatise being necessarily something very different from Newton's *Principia*. Indeed, all the astronomically relevant propositions of the *Principia* are in essence the *analysis* or *construction* of diagrams representing the possible *orbits* of celestial bodies, i.e. the three conic sections. These problems are treated with the method of *first and last ratios*, exposed in the first two books of the *Principia*, which is indeed is another application of the ancient Greeks' kinematical or mechanical approach to the generation of geometrical objects. As Newton wrote in his *De Quadratura Curvarum*:

> Quantitates Mathematicas non ut ex partibus quam minimis constantes, sed ut motu continuo descriptas hic considero. Lineæ describuntur ac describendo generantur non per ap-positionem partium sed per motum continuum punctorum, superficies per motum linearum, solida per motum superficierum, anguli per rotationem laterum, tempora per fluxum continuum, et sic in cæteris. Hæ Geneses in rerum natura locum vere habent et in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectarum immobilium genesin docuerunt rectangulorum.[84]

In short, from the strictly *technical* point of view, Newton's dynamical theory of gravity may be regarded as a set of problems internal to a *kinematical/mechanical* version of Apollonius' theory of conic sections, specifically applied to the motion of celestial bodies (regarded by Newton as parts of a *cosmic machine*).[85] Especially relevant in this regard is Book V of Apollonius' treatise, in which the curvature properties of conics are studied, and his lost works on *On Contacts* (τερί έπαφων), where the general problems of *circles of contacts* were treated.[86]

Recent Newtonian scholarship has emphasized the crucial role that the problem of curvature has played in the shaping of Newton's dynamics, a theoretical ingredient that is much more central than what may appear from the published versions of the *Principia*. Here, methods involving curvature appear only in

[82] See the quote above from Gemino.

[83] As well known, these ideas were abandoned in the subsequent evolution of classical dynamics. One could also mention the *universal* character, a *realistic* view of mathematical objects and the very same metaphor of *physical law* as Newton conceived it (Zilsel 1942). A general and accessible portrait of Newton and his works is Guicciardini 2021.

[84] For more details on Newton's mathematical methods see Guicciardini 2009. For the debts that Newton owes to the ancients, see Russo 2004, pp. 365–378.

[85] See in particular the *Preface* to the *Principia*. It is worth remembering that Newton believed in the myth of the *prisca sapientia*, which to him included also his law of gravitation (ascribed to Pythagora).

[86] Loria 1893, pp. 190–192.

the second edition as "alternate proofs", but from Newton's manuscripts it has emerged that curvature methods were indeed primary in the early formulations of the theory, namely before the correspondence with Hooke suggested to Newton the connection between central forces and Kepler's area law.[87] One of the most important features of these methods is that they don't require to appeal continually to limiting procedures.[88] Brackenridge has analyzed in detail the role and use of different approximation methods in Newton's published and unpublished works, concluding that the real "key" to his dynamics (and therefore to classical dynamics in general) lies in his analysis of *circular motion* and in its use to approximate curves by their osculating circle.[89]

In particular, curvature methods lead systematically to way simpler solutions to the same problems as compared to other methods, and overall today it is clear that the theory exposed in the *Principia* could have been much simpler than it actually was.[90] In fact Newton himself embarked in the project of revisioning it, but in the end such a task proved too big and was never carried to completion. According to Brackenridge, however, there were also other, extra-scientific reasons why Newton didn't adopt curvature methods as basic:

> The disadvantage of the proposed revision is that it appears to restore the devices of uniform circular motion to a prominent position in the analysis of celestial motion. But there has been a fundamental change. For the Scholastics, uniform circular motion occurs in the absence of a force. Newton's circular motion, however, requires a centripetal force that changes with position along an elliptical orbit. But if that difference is not clearly understood, then Newton could be seen as returning to the scholastic tradition and in that sense he would be reactionary. However, Newton may not have been willing to take the risk of being misunderstood. He 'toyed' with the idea and then set it aside. If the diagram for Proposition 6 had been revised [by the addition of the circle of curvature], however, the text would have been much more understandable to nineteenth- and twentieth-century physicists and, perhaps, the *Principia* would have been read as much as it has been revered.[91]

[87] Nauemberg 1994.

[88] See for example Chandrasekhar 1995, p. 78.

[89] For more details see the beautiful Brackenridge 1995, a very clear exposition of the content of the first three sections of Newton's *Principia*.

[90] Indeed, the central role of curvature in Newton's dynamics must have been clear to Huygens, probably one of the few who had the mathematical knowledge necessary to understand and appreciate Newton's *Principia* when it first appeared. Huygens had developed independently a theory of evolutes mathematically equivalent to that exposed by Apollonius in Book V, which could have been used to solve the same problems attacked by Newton in the *Principia*. However, the *Dutch Archimedes*, as Huygens was sometimes called, applied his theory to rectification of curves and to the design of accurate pendulum clocks, and never dealt with dynamical problems such as those addressed by Newton in the *Principia*. Though impressed by Newton's mathematical skills, Huygens was never convinced by his views about absolute space, time and motion, and started a project to re-establish the foundations of mechanics on a relativistic grounding. He died too early to complete such a project, but his beautiful insights may be found in his manuscript notes, first published in Mormino 1993. It is also noteworthy that the approximation of the orbits by osculating circle was explicitly used by Leibniz, who had been Huygens' pupil, in the *Tentamen* (for more details see Aiton 1964).

[91] Brackenridge 1988.

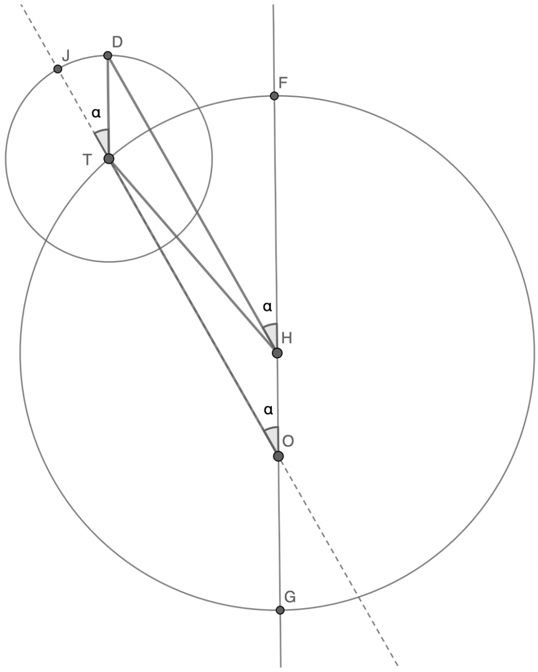

Figure 1.5: Hipparchus' diagram. The motion of D as seen by O is the same as the motion of T as seen by H.

Later on we will see how simple Newton's solution to Kepler's problem may actually be, if one doesn't get so scared about *circular motions*. Quite paradoxically, we will see how in this way Newtonian dynamics may indeed be freed from many of the metaphysical (and *useless*) ideas that obscure its inner simplicity.

1.6 Hipparchus' Diagrams

At some point during the III cent. BC, *plane* astronomical models made their appearance in Greek astronomy, under the well-known form of *eccentric circles* (literally *off-center circles*) and *epicycles* (literally *circles upon circles*). The origin of these models in the context of astronomy is a complete mystery. The earliest extant allusion to such constructions (and also of their equivalence in terms of *saving the phainomena*) is Gemino's above quoted passage about the different approaches of mathematicians and natural philosophers to astronomical problems. We wish to emphasize that, excluding Ptolemy's interpretation and use of such models, all we know about eccentrics and epicycles from other sources is their *name*, which says something only about their structure as *diagrams composed of circles*, and the fact that they were used in some way to account for celestial motions.

Different non-exclusive possibilities have been conjectured. Eccentrics/epicycles could have been the result of stereographic projection of some Eudoxian tridimensional model, or maybe a simplification resulted from Greek absorption of Babylonian astronomy, in which it was common practice to disregard ecliptic latitude altogether. Therefore, plane models would have been a way to match and put to profit Babylonian observations and computational techniques, with

Babylonian numerical period relations translated as relations between the *frequencies of rotation* of eccentrics/epicycles. This seems to have been, in particular, Hipparchus' geometrical translation of Babylonian numerical relations involving planetary periods.

The analogy between eccentrics/epicycles and the *pin-slot device* found in the Moon mechanism of the AM has opened a new and more evidence-based track to the solution of this problem, i.e. that such constructions could have a mechanical origin in the context of *sphairopoiia*. This possibility is explored by J. Evans and C. C. Carman (2000), where the authors conclude:

> Everything considered, it would have been easier to arrive at a mechanical representation of Moon and inner and outer planets based on the pin-and-slot mechanism simply by starting from the phenomena than by starting from epicycle-and-deferent theory.

In this paper it is also remarked that between eccentric/epicycle models and pin-slot device there is only a *quasi-equivalence*, since the two are equivalent only with regard to *observed angular motion*. It seems noteworthy that this is the only relevant magnitude for ancient naked-eye planetary astronomy (and for a device like the AM), putting aside the fact that the variations of distances predicted by (a realistic interpretation of) eccentrics/epicycles are completely wrong and in plain contradictions with observations in the case of the Moon.

Coherently with the general epistemological framework of Hellenistic astronomy, as we said above, we think that also for the problems of Sun, Moon and planets the evolution of *theoretical* models eventually coincided with that of *concrete* models which embodied the theory itself. In other terms, such distinction is simply a *fake* distinction, a sort of modern mirage, eccentrics/epicycles constructions and pin-slot devices being, astronomically speaking, *one and the same thing.* Like with Eudoxus' spheres, we find no reason to ascribe any physical character neither to eccentrics/epicycle constructions (apart from Ptolemy's later view of them), nor to the pin-slot gearing. All these constructions can and should be regarded as mere *tools for computation* internal to a well definite *theory.* Drawn on paper, they are theoretical *diagrams* representing the solution of some astronomical problem; mechanized with gears, they became *analog computers* of such solutions. Since the name usually associated to eccentric/epicycle models are those of Apollonius and Hipparchus, we will call the whole class of eccentric/epicycle/pin-slot constructions *Hipparchus' diagrams.*

Now, the question is: if we reject Ptolemy's interpretation and use of such constructions, what could have been the original *meaning* of Hipparchus' diagrams?

Putting together the pieces laid down in the present chapter, our conjecture is that such diagrams were originally *diagrams of velocities*, resulting from the application of Apollonius' theory of conic sections to the solutions of an astronomical problem, namely that of computing the *succession of relative motions* of two given celestial bodies. Consistently with what is suggested by pre-ptolemaic sources, we think that this solution derived from a *dynamical* theory of celestial motions based on the idea of mutual attraction between bodies, so that these diagrams embodied in their structure the *kinematical relationships* of two given bodies attracting each other, i.e. the *velocities* and *accelerations* of a given body around a given center of attraction. As we will see, it turns out that conic sections and Newton's dynamics are *hidden* in the structure of Hipparchus' diagrams, and therefore in the gear-work of the AM. If our conjecture is correct, in such a framework a device like the AM was an analog computer of the solutions of a well-definite problem internal to such a dynamical theory of celestial motions.

But, before getting there, it will be better to trace the route that led a later follower of the Greeks to free classical dynamics from all the metaphysical dust covering its foundations, and to discover the hidden *symmetry* that, if our interpretation is correct, made the "law of gravitation" easily *visible* to the eye of Greek geometers.

Chapter 2

William Rowan Hamilton. The Classicist Mathematician

William Rowan Hamilton was born at Dominick Street, Dublin, the night between the third and the fourth of August, 1805. At about three years old he was sent to his uncle James Hamilton, curate of Trim, to be raised and educated in a more comfortable environment. James took care personally of William's first instruction, and soon realized that his nephew was a rare prodigy.

Uncle James was a graduate of the Trinity College of Dublin, renewed scholar of the Classics and skilled linguist very passionate about oriental languages. As it happens, the first domains in which William displayed his intellectual talents were reading and languages. When he arrived at Trim he already read the Bible in English, but under the guide of his uncle he immediately started to learn Hebrew. Uncle James had his own teaching methods, consisting in spelling all the monosyllabic words of the dictionary in which the letter *a* occurred, then those with the letter *b*, and so on through all the alphabet; after covering all the monosyllabic words, he moved on to words of two syllables, and so on. As Hankins remarks,[1] from the very beginning Hamilton familiarized with obscure words that most adults had never heard or seen, and probably only his precocity saved him from total confusion. In any case, with him James' method gave his results: after Hebrew Hamilton rapidly absorbed Latin and, above all, Greek, in which he became so proficient to astonish all the learned visitors with his precise and expressive readings of Homer. He soon passed to oriental languages, and at eleven years old wrote a little book on Syriac Grammar. It ends like this:

> Thus have I gone through what is necessary to be known for reading and writing Syriac - the forms of their pronouns, and of their regular nouns and verbs; thus comprising in four chapters the Rudiments of Syriac Grammar. Soon may be expected an account of their irregular and indeclinable words, etc., with a Syntax.

Also in view of a possible diplomatic career in the East India Company, by the age of thirteen Hamilton had already learned a dozen languages including Arabic,

[1] Hankins 1980, p. 13. Hankins' book is an excellent biography of Hamilton, dwelling also on his mathematical works.

Hindu, Sanscrit, Chaldee and Malay, besides of course all the modern European languages. When the Persian Ambassador visited Ireland in 1819, apparently the young William was one of the only two people in the whole country able to write him a welcome letter in his native language.

Another domain in which Hamilton showed an extraordinary ability that never left him throughout his life was *computation*. Since his early childhood he enjoyed making long mental calculations of whatever could occur to him, usually devising his own methods to do it more efficiently. At thirteen years old he confronted Zerah Colburn, an American boy who was famous at the time for his out-of-ordinary computational skills and traveled around Europe exposed by his father. Hamilton lost, but, most importantly, two years later he met Colburn again and didn't miss the occasion to ask him about his methods. Then he tried to figure out *how* and *why* these worked, and dwelled at length comparing Colburn's methods and his owns, in search for better ones. On April 8th, 1820, he wrote to his cousin Arthur:

> I have been considering the methods which Zerah imparted to me of calculating the square and cube roots in particular, and I wish to put this question to him, viz.- Can his method be of any use to discover the nearest square to surd numbers or those which have no exact square root? If not, it will deduct much from its practical utility. As the great use of extracting square root is in operations wherein there will scarcely ever occur an exact square. [...] You may remember my mentioning that he started a new difficulty with respect to his other operation of discovering the factors of high numbers... It is indeed less simple than I at first supposed it to be. [...] I expect to find more and more light on this subject as I continue to consider it...[2]

Hamilton later recalled that it was the first meeting with Colburn to first arise his interest in arithmetic and mathematics in general. However, for some time mathematics was to Hamilton mainly a diversion from the Classics, the main topic of his studies in view of the admission to the Trinity College.

In mathematics Uncle James couldn't help him so much, and Hamilton was almost totally self-taught. His first approaches were Newton's *Arithmetica Universalis* and Clairaut's *Algebra*, which he rapidly mastered. One of his albums dated 1818 contained a "Compendious Treatise on Algebra", in which he summarized his new learning, together with a "Grammar of the Sanscrit Language", an "Arabic Praxis" an "Analyisis of a passage in Syriac".[3] When he discovered Euclid's *Elements* his interest for arithmetic and algebra somewhat declined, but only to be rekindled by the discovery analytic geometry, by which he remained deeply fascinated. This is a passage from a letter dated September 4th, 1822, to his cousin Arthur, in which he summarized the course of his first mathematical studies:

> It is now a good while since I began Euclid. Do you remember when I used to go to breakfast with you, and we read two or three propositions in the morning? I was then so fond of it, that when my uncle wished me to learn Algebra, he said he was afraid I would not like its uphill work after the

[2] Graves 1882, pp. 78–80.
[3] Graves 1882, pp. 54–55.

> smooth and easy path of Geometry. However, I became equally fond of Algebra, though I never mastered some parts of the science. Indeed the resources of Algebra have probably not yet exhausted... Three years ago I read Stack's *Optics.* If you add to what I have mentioned some popular knowledge of Astronomy, you will have the whole of my acquirements in Science, at the beginning of last year. I was lent at that time Brinkley's *Astronomy* and a *Trigonometry*, which I read, but had not time to make myself sufficiently acquainted with them. I bought an Ephemeris, and my favourite amusement was calculating and observing occultations of stars by the moon; eclipses too, but there were not any to observe. But in August, while the King was in Dublin, my uncle gave me Lloyd's *Analytic Geometry.* Ill-omened gift! it was the commencement of my present course of mathematical reading, which has in so great a degree withdrawn my attention, I may say my affection, from the Classics. It prepared the way for Puissant, Garnier, Lagrange. I soon became quite fascinated with it... My next attempt was so much of Newton's *Principia* as is read for the Science Medal. In June I was lent Garnier. At Christmas I was made a present of two *Nautical Almanacks*, which gave me a new impulse to observe the heavenly bodies. In June I was lent Garnier, and some other French mathematical books, which I nearly read through since, though only at stolen intervals from my classical studies with my uncle.

As emerges from this sketch, in these years Hamilton also got into astronomy. He made a great amount of observations, reported them in his journals and wrote letters full of astronomical calculations. His greatest excitement came from eclipses, and in some letters he confessed that waiting for their predicted occurrence he could "think of nothing else". He also started designing his own instruments: he turned a tall tower in a field at Trim in a sun-dial, drawing from it meridian and hour-lines, built a quadrant by himself and also devised a telegraphy system to communicate with his friends at a distance.

It is to be remarked that the acquaintance with languages in general, and with Greek in particular, strongly interacted with his scientific studies. Throughout his youth Hamilton spent a *huge* amount of time reading and, more importantly, *translating texts in different languages*, ancient and modern. It was probably the deep and prolonged study of so many idioms involving different symbols, grammar rules and semantical structures to imprint him that unparalleled attention to the *origin*, *use* and *meaning* of *words* and *symbols* that will become a hallmark of his future mathematical researches. More specifically, his familiarity with Greek gave him insights into scientific terminology that often came directly from a clear understanding of the received *names* used to denote abstract concepts and operations.

How early was his habit of questioning and analyzing the received mathematical terminology is well illustrated by the following remarks, dated 1822 (he was less than seventeen years old), about the meaning of *division* and other arithmetical operations:

> Division, according to the most obvious definition, is the dividing of a quantity into a given number of parts, whence that number is called the Divisor. This kind of Division was probably the first made use of, but is very

limited in extent, not admitting any Divisors but such as are real positive integer numbers; in short such as are the series 1, 2, 3, 4, etc.. The result of this operation always bore the same proportion to the original number that Unity did to the Divisor. By adopting this property as a definition, namely that Division is the finding of a fourth proportional to Divisor, Unity, and Dividend, all sorts of Numerical Divisors were admitted.

> But there is another view of the subject, naturally suggested by the term Quotient; namely, that Division is the finding how often one quantity is contained in another. This is the Definition at present generally adopted. The distinction between it and the former is, that in this the Divisor must be homogeneous to the Dividend; in the former it must be a number. Perhaps the best definition of Division would be "the finding that quantity which multiplied by the Divisor will produce the Dividend".

> Before I quit this subject I may be allowed to remark that all the branches of Arithmetic are applied in a much more extensive manner than was contemplated by the inventors of them. By the introduction of negative and fractional quantities, operations that diminish are included under Addition and Division. As the boundaries of science were extended, new operations were designed by old names. The name of Geometry shows that it was at first confined to what is now only a subordinate part of it, Mensuration: and Calculation itself, the objects of which are so extensive and so wonderful, continues to record by its etymology its humble origin in the rude custom of counting by pebbles.[4]

As we will see, twenty years later exactly the same kind of reflections will form the background of his theory of quaternions.[5]

Indeed, from the very beginning Hamilton showed a great originality and independence of thought, striving to find his own ways to attack mathematical problems without resorting to what others had done before him, and soon confronting on equal grounds with the authors he studied. Needless to say, he could often find new and better solutions to old problems. This is, for example, how he described his first approaches to Euclid's more advanced propositions:

> To return to Euclid: I have since read through the six Books on this plan: when I am walking, or otherwise prevented from graver pursuits, I glance at the title of a proposition and then work it, having resolved not to assist myself by text or figure until I conquer the difficulty by my own resources. In general I find this very easy - sometimes not. Still I have observed my rule. The hardest question I met was Euclid, iv.10: to construct an isosceles triangle having each angle at the base double that at the vertex. I found by Analytic Geometry that the base must be the greater segment of either side, cut in extreme and mean ratio, and then formed a demonstration depending only on the Second Book of Euclid. On referring to his text, I saw that the construction was the same, but the demonstration quite different, being entirely from the Third Book, and therefore less simple than mine.[6]

[4] Graves 1882, pp. 101–102.

[5] It is noteworthy that the introductory chapter of Kendall and Tait (1873) is almost a paraphrase of these remarks.

[6] Graves 1882, p. 140.

A clearer idea of the kind of reflections in which he was engaged while studying Euclid is given by a little dialogue entitled *Waking Dream, or fragment of a dialogue between Pappus and Euclid, in the meads of Asphodel*, composed in 1823, at seventeen years old.[7] In it Pappus asks a series of questions to Euclid about the composition of the *Elements*. This is how it begins:

> *Pappus.* And now that we have discussed these more recent improvements in that science of which you are held the inventor, permit me to inquire how you were enabled to deduce consequences so remote from principles so simple: inform me what it was that first suggested to your mind the consideration of those Theorems which have come down under your name? For so successful have you been in disguising the Analysis which you pursued, that to this day even the learned are doubtful whether your discoveries were made by a gradual process, like that which conducts to truths the minds of other men: or whether they were imparted as an immediate gift from Him who constructed for the Bee its wondrous habitation - of whom it has been justly said, Ὁ Θεός γεωμετρεῖ.
>
> *Euclid.* It was not unintentionally that I adopted, as the medium of communicating to my contemporaries those results at which I arrived, a Synthesis, which presented them under a form the best adapted to excite astonishment, and to disguise the process of discovery. To exoterics the science appeared more interesting as it was more mysterious; and for myself - if the world had known all the fortuitous circumstances to which I owed the perception of so many Theorems, would they have reverenced as they did the Mathematician of Alexandria? The inventor of a curious piece of mechanism does not expose his artifice to the vulgar eye; nor does an architect, when he as erected a magnificent edifice, leave the scaffolding behind. Or think you that the nest of the Phoenix, with its odorous flame, would be regarded with the same veneration, were its place accessible to human foot? Yet now, since here no motive to disguise remains, I am willing, if such be your desire, to reveal the entire process of discovery.

After a question about the origin of Definitions, Postulates and Axioms, Pappus asks about the "intellectual ground" from which Euclid began, and this is the answer:

> *E.* While yet a boy my imagination had been captivated by the 'Eternal and Immutable Ideas' of my illustrious contemporary [*Hamilton's note*: "Plato"]. I sought to discovery what I could fancy to have been in the Divine mind the archetype of Figure: something simple, perfect and *one*. I found it in the equilateral triangle: and from the contemplation of this figure, Geometry as a Science has arisen.
>
> *P.* Was not the Circle at least equally simple?
>
> *E.* You forget that those late discoveries on which our conversation turned not long ago have shown the circle to be the limit of regular rectilinear figures. [*Hamilton's note*: "If you conceive an Equilateral Triangle, a Square, Pentagon, Hexagon etc, inscribed successively in a Circle, you will find

[7] Graves 1882, pp. 662–671. In Greek mythology the Asphodel is one of the rivers of the underworld.

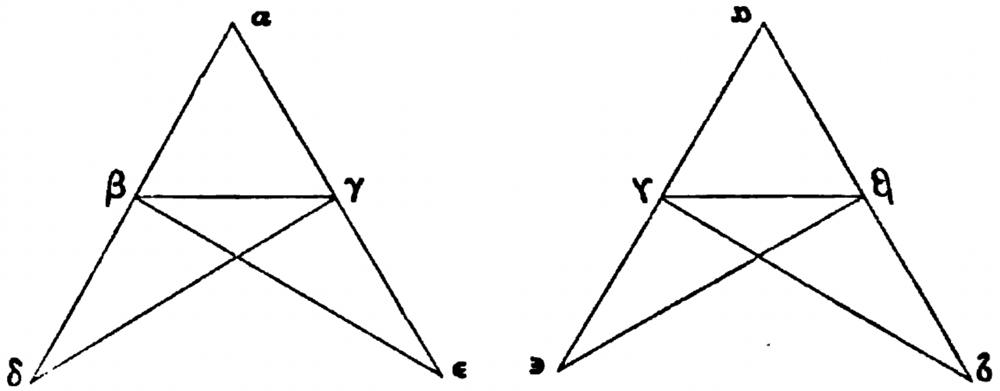

Figure 2.1: Optical origin of Euclidean superposition according to the seventeen-years old Hamilton (source: Graves 1882, p. 668).

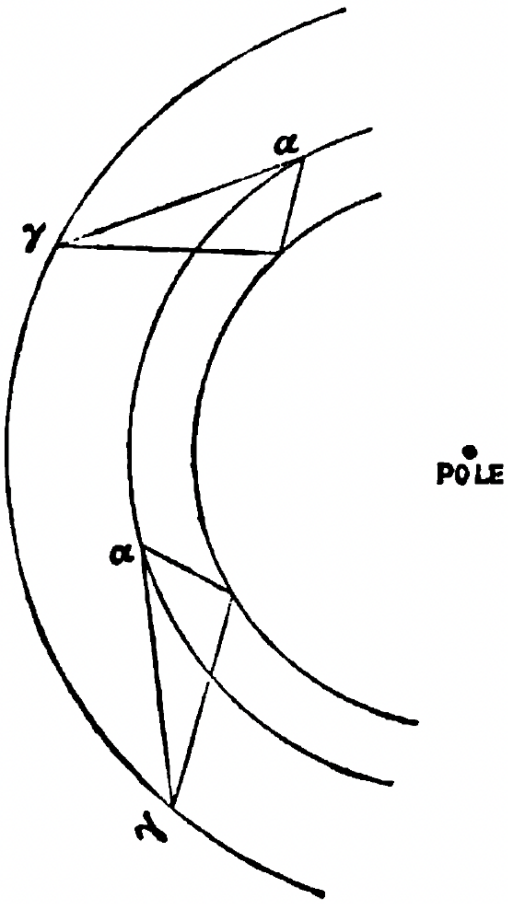

Figure 2.2: Astronomical generalization of Euclidean superposition according to the seventeen-years old Hamilton (source: Graves 1882, p. 669).

> that they go on approaching to it; so that some have called a Circle a Regular Figure of an infinite number of sides.] Of these the simplest is evidently the equilateral triangle. [...] I might mention, as another reason for attending to a rectilinear figure rather than to a curvilinear, the natural bias of the human mind to consider a straight line in some way emblematic of rectitude, and a curve of the contrary; a remark confirmed, I believe, by the etymological analogy of all languages: [*Hamilton's note*: "Curvo dignoscere rectum, atque inter silas Academi quaerere verum."] and which has had so strong an influence on the ideas of those who have inquired into the constitution of Nature, that every curve is thought to be a deviation from a line, and it has been questioned whether curvilinear motion be possible without som external and ever-acting force. [*Hamilton's note*: Newton's Law of Rectilinear Motion was suspected by some among the ancients.] [...] Besides, the idea which I entertained of symmetry, and of the το καλόν, induced me to attend only to regular figures, regarding none else as symmetric and beautiful.

The *Waking Dream* is disseminated with wonderful glimpses into the young Hamilton's ways of thinking. We just quote the explanations of the origin of Propositions 4 and 5 and of the idea of *superposition of triangles*, which is particularly interesting:

> *E.* Chance. Having graved the diagram in this simple form upon a transparent substance, I happened to turn it in such a manner that when placed between my eye and light, the uninscribed surface was next my eye, and the diagram assumed the appearance here delineated [see Fig. 2.1]. You see that the lines themselves appear to preserve the same position; but that the letters are altered in such a manner that β and γ, δ and ϵ, have mutually changed places. [...] Thus you have been admitted to behold my discovery in its embryo state. You will find no difficulty in perceiving how the idea of applying one triangle to another having been once suggested, I was led by my love of generality, and the desire I had to diminish the labour of this demonstration by throwing some of it into a preparatory theorem, to form the 4th Proposition, which was to me the more easy as I had been accustomed to observe the motions of fixed stars round the Pole, revolving as they do in concentric circles, and in such a manner, that if you select any three bright stars α, β, γ, the distance $\alpha\gamma, \beta\gamma$ continue always the same, as also the angle at γ; and, therefore, the triangle alters not in reality, however differently it may be placed to the eye [see Fig. 2.3]. In demonstrating this 4th Proposition separately, the fundamental principle employed therein to prove equality being *conceived coincidence*, I was induced to form the eighth axiom, as also the tenth in words, though I have already mentioned that it occurred to my mind on a former occasion.

And the dialogue concludes thus:

> P. Even for this brief and rapid sketch I thank you. Many however of my most interesting questions remain behind. I wished to have inquired about the origin of several theorems more curious and less likely to have been intuitively perceived; the equality of the three angles of every triangle to

Figure 2.3: Entrance of the Trinity College of Dublin in the late XIX century. Photo by William Mervin Lawrence (source: Wikipedia).

> two right angles, and the 35th Proposition in particular. But see, stalking yonder through the shades, the murderer of Archimedes! Let us disperse in haste, and meet again by Lethe's banks. Bring with you, if you find him, the Samian Sage.[8]

In this very same period Hamilton started to study differential calculus, and in about a year he had absorbed and mastered Laplace's monumental *Mécanique Celeste*. He did it so well, indeed, that he detected a slight error in the deduction of the parallelogram rule.[9] This vaulted him the attention of John Brinkley, Royal Astronomer of Ireland, from whom William received assistance and warm encouragements to pursue his mathematical studies. When he payed him a visit at Dunsink Observatory, he brought with him a certain number of original papers about osculating circles, surfaces of double curvature and other topics that he had developed all by himself.

After the entrance at Trinity College, even if a the great share of his studies was still devoted to the Classics, Hamilton's attention drew more and more on contemporary mathematics. In addition to his normal courses, under the guide of his tutor Charles Boyton Hamilton became familiar also with the mathematical curriculum of the French *École Polytechnique*, at that time by far the most important institution in the world for pure and applied mathematics.

A letter sent to the cousin Arthur on September 28th, 1823, gives a flavor of the kind of student Hamilton was:

> ...My life as a Student has always to me be divided into two principal parts -

[8]The *Samian Sage* is clearly Aristarchus, contemporary of Euclid. The Lethe is another river of the Greek underworld, associated with the ideas of *oblivion*, *forgetfulness* and *concealment* preliminary to the unveiling of *truth*, α-λεϑηια. See above.

[9]Hamilton's analysis and correction of Laplace's error can be found in Graves 1882, pp. 661–662.

> preparation for Entrance; preparation for Fellowship. The part is over, and I think the second has begun. For I consider Academic honors as not only valuable *per se*, but important as steps (*gradus*) to the ultimate rank at which I aim. And were it only for the weight they must give to answering in the Fellowship Hall, I would think them well worth an effort to attain. So you see I am trying to prove that in reading for premiums [i.e. the honors at Term Examination], I am really aiming higher. But besides this, which you may perhaps think a subtlety, whatever study is not given to my immediate course has a tendency to prepare me for remoter objects, and yet at the same time facilitates my intermediate progress. For example, I have found an old *Logic* by Burgersdicius;[10] it is, I believe, read for Fellowship; it is a great deal fuller Murray's [i.e. the text-book of Logic for the Term Examinations], and throws a good deal of light in those parts which he passes rapidly over - for example, the Categories. It tells you, too, what Aristotle said on every part of the subject... A little time, too, is bestowed on Newton's *Algebra*, a subject that is treated of by the great author in the same masterly manner as the *Principia*, and yet in many parts is rendered almost as difficult, by its conciseness and omission of intermediate steps. In Classics I continue the Blank Verse Translation, and Uncle is correcting the Virgil. So much for my studies...[11]

Hamilton's college career was, as expected, more than brilliant: he ranked first at all Term Examinations, won all the medals he could and earned an *optime* in two disciplines - Greek and Physics -, something that apparently had never happened before him.[12] However, it didn't last long: when still an undergraduate, at less than 22 years old, Hamilton was offered the place of Royal Astronomer of Ireland left vacant by Brinkley, and the connected chair of Andrews Professor of Astronomy at the Trinity College. He accepted, and kept this position from 1827 until his death in 1865.

A sort of childish joy continued to animate Hamilton's research. He spent his life at the Dunsink Observatory, often isolated (above all at the end of his career), applying his mathematical prowess to every problem attracted his interest, opening entire new fields of research and giving groundbreaking contributions in any domain he touched. He corresponded with many of the leading scientists of his age (Arthur Cayley, Augustus De Morgan, John Herschel, just to mention some of them), took stimuli from everyone but always followed his own very original views, often at odds with those of his contemporaries. In a letter to Aubrey De Vère dated 1835 he recognized this fact quite plainly:

> I differ from my great contemporaries, my 'brother-band', not in transient or accidental, but in essential and permanent things: in the whole spirit and view with which I study Science.[13]

[10] Franck Pieterszoon Burgersdijk (1590 - 1635), Dutch logician and natural philosopher.

[11] Graves 1882, pp. 148–149.

[12] At the time, an *optime* was a true rarity at Trinity College, meaning that the candidate was out of scale, having displayed a mastery of the subject comparable to that of the examiners.

[13] Graves 1882, p. 319.

Figure 2.4: Photograph of William Rowan Hamilton in his late years (source: Wikipedia).

Despite often misunderstood by his "brother-band", he achieved great honors already in his lifetime, and since the advent of quantum mechanics his name is known to every physicist.

For some time in his youth, Hamilton had been undecided about what career to pursue, if in the Classics, which to him meant also poetry, or in mathematics. In the end, also thanks to the advice of his dear friend William Wordsworth, he chose mathematics, but of course the love for the Classics never left him. He kept writing poems, and we will see the crucial role that Greek models will play throughout his mathematical career. However, Hamilton's ambitions went way beyond the veneration of the Ancients:

> I have been continuing my Classics, as usual, with my uncle. But I fear I shall never be so fond of them as of the Mathematics that I am now reading. [...] Who would not rather have the fame of Archimedes that of his conqueror Marcellus, or than any of those commentators of Classics, whose highest ambition was to be familiar with the thoughts of other men? If indeed I could hope to become myself a Classic, or even to approach in any degree to those great masters of ancient poetry, I would ask no more; but since I have not the presumption to think so, I must enter on that field which is open for me. Mighty minds in all ages have combined to rear upon a lofty eminence the vast and beautiful temple of Science, and inscribed their names upon it in imperishable characters; but the edifice is not completed: it is not yet too late to add another pillar or another ornament. I have

scarcely arrived at its foot, but I may aspire one day to reach its summit.[14]

It is exactly what he did.[15]

2.1 The *Introductory Lecture on Astronomy*

A good starting point to enter Hamilton's more mature conceptions of mathematical sciences is the *Introductory Lecture on Astronomy* he delivered in November 1832, at 27 years old.[16] In this lecture, the first of a full course in Astronomy ranging from the Greeks to Newton's dynamics, Hamilton exposed his general views about the status, epistemology and methodology of pure and applied mathematics. With obvious nuances, he'll remain quite faithful to these ideas throughout his long and varied career, and these views, as we'll see, impressed a peculiar character to much of his mathematical researches.

After a general and highly rhetorical introduction about astronomy, "the parent of all sciences, and the most perfect and beautiful of all" - Hamilton starts his lecture emphasizing that the course he's going to deliver marks the passage from the *sciences of pure reason* - "the logical, the metaphysical, and the mathematical" - to the *physical sciences* - "in which reason is combined with experience" -, astronomy being "a favourable introduction to the rest, and a specimen and type of the whole". However, before entering this new field, Hamilton recalls to his audience "the nature and spirit" of two pure mathematical sciences they're already acquainted with, namely geometry and algebra:

> In all the mathematical sciences we consider and compare relations. The relations of geometry are evidently those of space; the relations of algebra resemble rather those of time. For geometry is the science of figure and extent; algebra, of order and succession. The relations considered in geometry are between points, and lines, and surfaces; the relations of algebra, at least those primary ones, from the comparison of which others of higher kinds are obtained, are relations between successive thoughts, viewed as successive and related states of one more general and regularly changing thought. Thus algebra, it appears, is more refined, more general, than geometry; and has its foundation deeper in the very nature of man; since the ideas of order and succession appear to be less foreign, less separable from us, than those of figure and extent.

Both these pure sciences, algebra and geometry, are called into question in the study of *motion*:

> Motion, although its causes and effects belong to a physical science, yet furnishes, by its conception and by its properties, a remarkable application of each of these two great divisions of the pure mathematics: of geometry, by

[14]Letter to Aunt Mary Hutton, August 26th, 1822 (Graves 1882, pp. 110–111).

[15]The reader who is familiar with Giacomo Leopardi (1798 - 1837) won't fail to notice the deep affinities between him and Hamilton, two intellectual giants who turned out to be the greatest poet and the greatest mathematician of their age.

[16]Hamilton 1833a.

> its connexion with space; of algebra, by its connexion with time. Indeed, the thought of position, whether in space or time, as varied in the conception of motion, is an eminent instance of that passage of one general and regularly changing thought, through successive and related states, which has been spoken of as suggesting to the mind the primary relations of algebra.

So, the observation of motion, although rooted in experience, is for Hamilton the primary source of this general idea of *regularly ordered succession of states.* This idea lies, in Hamilton's view, at the foundation of algebra, in the same way as the idea of *extension* is the intuitive ground for geometry; and the abstract study of motion, involving both the ideas of space and time, is a natural application of both geometry and algebra. It is clear from the context that Hamilton is thinking to astronomy, where, without inquiring about causes and effects, one studies in an abstract way the *regular successions of configurations* of heavenly bodies going through *progressive and related states.*[17]

The ideas of *order*, *number* and *figure*, as deductively developed in the pure sciences of algebra and geometry and as applied to astronomy, compose the "scenery of an inner world" in which we may study *continuous change* in general:

> We may add, that this instance, motion, is also a type of such passage; and that the phrases which originally belong to and betoken motion, are transferred by an expressive figure to every other unbroken transition. For with time and space we connect all continuous change; and by symbols of time and space we reason on and realise progression. Our marks of temporal and local site, our then and there, are at once signs and instruments of that transformation by which thoughts become things, and spirit puts on body, and the act and passion of mind are clothed with an outward existence, and we behold ourselves from afar. The idea of order, with its subordinate ideas of number and of figure [...] appears to be only the development of our original powers, the unfolding of our proper humanity. Foreign, in so far that they touch not the will, nor otherwise than indirectly influence our moral being, they yet compose the scenery of an inner world, which depends not for its existence on the fleeting things of sense, and in which the reason, and even the affections may at times find a home and a refuge.

So, the very same concepts of number and figure are *subordinate* to the ideas of *ordered progression* suggested by astronomical motions, being mere *instruments* through which this human ordering faculty operates. Ultimately, all physical sciences, dealing with *change in space and time*, deal with *motion*, and live for Hamilton between these two worlds of *thoughts* and *things*:

> It has been said, that in all the mathematical sciences we consider and compare relations. But the relations of the pure mathematics are relations between our own thoughts themselves; while the relations of mixed or applied mathematical science are relations between our thoughts and phenomena. To discover laws of nature, which to us are links between reason and experience - to explain appearances, not merely by comparing them with

[17] Compare this idea with that exposed in the passage of the *Waking Dream* above quoted about the origin of the Euclidean idea of *superposition of figures.*

> other appearances, simpler or more familiar, but by showing an analogy between them on the one hand, and our own laws and forms of thought on the other, "darting our being through earth, sea, and air" -[18] such seems to me the great design and office of genuine physical science, in that highest and most philosophical view in which also it is most imaginative.

It is noteworthy that to Hamilton *laws of nature* are neither something purely *descriptive* of the phenomena, nor a pure product of our minds, but rather expression of an *analogy* between the *inner world* of our reason and the *outer world* of appearances, between our mathematical inventions and natural phenomena, the ultimate aim of physical sciences being to build and exhibit a *correspondence* between the two. In this perspective, a physical theory should become at its maturity *co-extensive* with the phenomena it describes. In a draft of the lecture Hamilton is more explicit on this point, giving as chief example Newton's theory of dynamical astronomy:[19]

> And so say I with respect to the observation of phenomena, even when combined with mathematical calculation: that the visible world supposes an invisible world as its interpreter, and that in the application of mathematics themselves there must (if I may venture on the word) be something meta-mathematical. Though the senses may make known the phenomena, and mathematical methods may arrange them, yet the craving of our nature is not satisfied till we trace in them the projection of ourselves, of that which is divine within us; till we perceive an analogy between the laws of outward appearances and our inward laws and forms of thought. [...] This it is, and not the beauty of mathematical reasoning, nor the practical accordance with phenomena, great and important as they are, which gives the highest value and the deepest truth to the dynamical theory of gravitation. Do you think that we *see* the attractions of the Planets? We barely see their orbits. We see, indeed, some brightness afar, some brilliances amid the blue of night. We observe, or rather we *make*, the configurations and arrangements of these *visibles* by mathematical moulds of our own minds; we form them into asterisms and constellations; we give them names; we attribute to each a body and a position. A little while, an hour or two, has passed; we look again, and much is changed, while much remains the same...

He then gives a sketch of the different forms assumed by such "mathematical moulds of our minds" in the history of astronomy, from the revolving celestial *sphere* of the first Greek astronomers, through Ptolemy's *epicycles* and Kepler's *ellipses* until Newton's *dynamical idea* of universal gravitation:

> We have now the idea of a turning sphere, which carries the stars along with it; and this conception, this beginning of astronomical theory, enables us already to draw into our mental view an immense variety of appearances, and to reason, to explain, and to predict them. But there let be a little more of patience and of time, and we find that even this conception is not enough. It will not solve *all* the phenomena, though it solves so *many*.

[18]Quote from Shakespeare.

[19]Graves, 1, p. 67.

> Arcturus and Pleiades may well be represented by it; but Jupiter and Saturn are not. [...] The theory of epicycles did much to satisfy this new want of the intellect... [and] perhaps, if it were properly modified, it might be made commensurate with even modern accuracy... [yet] a single ellipse of Kepler, with his law of equable areas, represented the motion of a planet not only more exactly than any combination of epicycles then known, but more simply and more beautifully by far than any combination of them which could ever be invented. [...] But this was only the opening of a field, the furnishing of an element to Newton. If Kepler had connected *facts*, it was the destiny of Newton to bind together *laws*. While the three great laws of Kepler remained isolated and independent, they seemed to Newton little better than isolated and arbitrary facts [...] all this seemed little to him unless it could be fused by the fire of intellect into one glowing whole: unless all these separate truths could be seen as deductions from one principle, as rays from one common centre. He achieved this fusion, he attained this central point; and he did so by a *dynamical idea* [our emphasis], by an external image of the will, by the principle of universal gravitation. Of the immense extent of this principle it would be hard to give an adequate notion.

For Hamilton the main difference between mathematical and physical sciences is also in the role played by *induction*, admitted in the former only to assist the reasoning, but absolutely central when one deals with empirical matters. So there must be a proper balance, in physical sciences, between *induction* and *deduction*, the paradigm being the route followed by Newton in the creation of his dynamical theory of gravitation.

Dwelling on this point, Hamilton mentions the debt that Newton owes to Bacon, who "more than any other man, of ancient or of modern times, appears to have been penetrated with the desire, and to have conceived and shown the possibility, of uniting the mind to things, say rather of drawing things into the mind". This is instrumental to express a criticism to the inductionist tendency of modern science, especially as it was practiced in Britain:

> For, I cannot suppress my fear, that the signal success, which since the time, and in the country, and by the method of Bacon, has attended the inductive research into the phenomena of the material universe, has injuriously drawn off the intellect from the study of itself and its own nature; and that while we know more than Plato did of the outward and visible world, we know less, far less, of the inward and ideal. But not now will I dwell on this high theme, fearing to desecrate and degrade by feeble and unworthy utterance those deep ideal truths which in the old Athenian days the eloquent philosopher poured forth.[20]

At the end of the lecture, Hamilton emphasizes the role played in physical sciences by *imagination* and *beauty*, elements that make the scientific activity a

[20]In a personal memorandum dating back to 1828 Hamilton remarks that "the Newtonian, no less than the Platonic, Philosophy, appears to me to be a work, a fabric, an architectural edifice. It is only in conformity with vulgar apprehension that Newton's system is stated to be *true*." Similar remarks were made by Leopardi, who referred to Newton's system as a *fantasia*, the same word he used for Plato's theory of *ideas*.

creative *art* analogous in many respects to poetry or sculpture, in which *truth* and *beauty* are strictly connected:

> As to imagination, it results, I think, from the analysis which I have offered of the design and nature of physical science, that into such science generally, and eminently into astronomy, imagination enters as an essential element [...] Be not startled at this, as if in truth there were no beauty, and in beauty no truth; as if [...] no connecting influence could radiate from their common centre. Be not surprised that there should exist an analogy, and that not faint nor distant, between the workings of the poetical and of the scientific imagination...[21]

So, let's summarize in a slightly more modern (and less rhetorical) terms Hamilton's views on mathematical astronomy. To begin with, astronomy is one among other branches of mathematical physics, an essentially creative activity dating back at least to the Greeks, consisting in the construction of *mental models* of natural phenomena and ultimately grounded in the human ability to *order things in time and space*. The *instruments* through which such ordering faculty of our minds operates are *numbers* and *figures*, whose primary conceptions are rooted in the observation of *motion*, namely in the *contemplation* of ordered and regular successions of configurations exhibited to our view by the heavenly motions. The essential point, to Hamilton, is the reduction by a *theory* of a variety of disconnected appearances to few and simple facts, logically connected to each other, bringing unity and order into chaos and thus constituting to our ordering minds a convincing *explanation* of phenomena themselves. The purely deductive sciences of algebra and geometry are the logical development of this natural ordering faculty, and starting from their application to the problem of *motion* they are *metaphorically* extendible to the *abstract* or *mental* or *theoretical* study of every *continuous change* occurring in time and/or space. In this way *all* physical sciences come to be regarded as different applications of one and the same discipline, *mathematics*, and more specifically as the *construction* and *interpretation* of *ordered successions of numbers and/or figures* representing *by analogy* the successive *observable states* of a physical system.

I think it is clear how much Hamilton's conceptions are in direct continuity with those of the beloved Greeks he knew so well, and the homage he pays to Plato only confirms what is otherwise evident.[22] No doubt such a clear and insightful (re-)interpretation of Greek mathematical conceptions came from his familiarity

[21] Such a parallel between science and poetry was not unusual among Victorian Scientists. See Brown 2013 for more details on this very interesting topic.

[22] The reader of Kant may have recognized some traces of the *Critique of the Pure Reason*. Indeed, in the preceding year Hamilton had started studying it (in German, of course), and he greatly appreciated it. As we will see, it played a role in his future researches, but for this lecture Hamilton only borrowed some terminology from Kant, and the whole epistemological framework he outlines is largely independent from its reading. His manuscripts demonstrate the truth of what he later wrote, i.e. that the pleasure he found in Kant came "more from recognition than from learning". After all, Plato is the author most quoted by Kant, and the ease Hamilton found in the reading of Kant's *Kritik* probably came also from his familiarity with Plato's works.

with their language. It is worth quoting a remark he made at seventeen years old on this point:

> In all the Classics, I find that my pleasure in reading them increases with every new perusal. And I think the reason that few people *enjoy* them is this: they do not take the trouble to read them so *often*, that their attention may not be distracted from the beauties of the poetry and the composition in general, by an imperfect knowledge of the meaning of words and sentences. In short, the Classics will not give the degree of pleasure they are calculated to impart, as long as the reader is reminded that they are in a foreign language, by his want of *familiarity* with them. Do you concur in this view on the subject?

In the following sections we will see how Hamilton put to practice his ideas in his works on optics and dynamics. Then we'll turn to his work on the foundations of algebra that led him to the invention of *grammarithms* or *quaternions*, the theory that realized many of his Greek-like conceptions and expressed rigorously the entanglement between algebra and geometry that he regarded as *intrinsic* to the general problem of *motion* and, therefore, of *continuous change in time and space*.

2.2 From *Optics* to *Dynamics*

The formulation of classical mechanics known today as *hamiltonian* grew out directly of Hamilton's researches in optics. His first work in this field was the *Theory of Systems of Rays*, whose roots date back to 1822, when Hamilton devoted great attention to the problem of curvature in space and curved surfaces in general, ending up writing a paper *On Caustics*.[23] The Theory of Rays was developed for the most part when he was still an undergraduate and its first part published in 1828. In it Hamilton develops a very general theory of mathematical optics whose primary objects are *luminous* or *visual rays*, i.e. lines connecting pairs of points in space and passing through whatever combination of mirrors, lenses and media. As far as I know, the only surviving precedent of this kind was Euclid's *Optics*, which however only treated about direct vision.

This is how Hamilton describes the fundamental objects and the general scope of his theory:

> A Ray, in Optics, is to be considered here as a straight or bent or curved line, along which light is propagated; and a System of Rays as a collection or aggregate of such lines, connected by some common bond, some similarity of origin or production, in short some optical unity. Thus the rays which diverge from a luminous point compose one optical system, and, after they have been reflected at a mirror, they compose another. To investigate the geometrical relations of the rays of a system of which we know (as in these simple cases) the origin and history, to inquire how they are disposed among themselves, how they diverge or converge, or are parallel, what

[23] Graves 1882, p. 115.

surfaces or curves they touch or cut, and what angles of section, how they can be combined in partial pencils, and how each ray in particular can be determined and distinguished from every other, is to study that System of Rays. And to generalise this study of one system so as to become able to pass, without change of plan, to the study of other systems, to assign general rules and a general method whereby these separate optical arrangements may be connected and harmonised together, is to form a *Theory of Systems of Rays*.[24]

The key idea of Hamilton's method is quite simple, and consists in studying the dependence between the *directions* of ingoing and outgoing rays (i.e. the six numerical ratios corresponding to the cosine-directors of the rays) and the *positions* of two observers looking at each other from these directions (i.e. the space coordinates of two points belonging to initial and final rays).[25] Hamilton focuses on such *optical unities*, systems composed of *visually connected points*, and develops a general method for the analysis of the geometrical configurations they may assume; every admitted possibility corresponding, in principle, to some realizable optical system and, therefore, to a certain class of optical *phaenomena*. In this way, despite the highly abstract and, therefore, general character, Hamilton's theory bears directly on *vision* and on the *design* of optical instruments, where the only thing one is interested in is how the light path is altered by its passage in the device.[26] Notice that the direction of propagation of the rays is completely irrelevant, exactly as it was irrelevant in Euclid's *Optics*, a fact which caused many misunderstandings in the following centuries.

Treating about *rays*, the conceptions underlying the theory are essentially geometrical, and in fact it is regarded still today as the standard of "geometrical optics". However, this is somewhat paradoxical, since Hamilton explicitly aimed to make available to optics "the powers of the modern mathesis, replacing figures by functions and diagrams by formulae", i.e. to build what he called an *algebraical optics*.[27] In short, he wanted to effect in optics a transformation similar to that brought in early-modern geometry by Descartes. It is worth quoting the masterly description Hamilton gave of the essence of Descartes' contribution in the perspective of his own views about algebra and geometry:

That great and philosophical mathematician conceived the possibility, and employed the plan, of representing or expressing algebraically the position of any point in space by three co-ordinate numbers which answer respectively the questions how far the point is in three rectangular directions (such as

[24]Graves 1882, pp. 228–231.

[25]Today we would probably speak of emitter and receiver of a light signal. This idea of optical unity, i.e. of couples of observers emitting and receiving light signals, will come back in Einstein's relativity.

[26]This is a typical example of a general feature of Hamilton's mathematical intelligence: the ability to perceive what are, among infinite possibilities, the *most relevant variables to look at* for the description of a certain system. This choice translates into the construction of *functions* connecting these variables, that exhibit with their *form* the relationship between such variables and therefore all the relevant properties of the physical system.

[27]Every time Hamilton calls a certain expression *formula*, I think he wishes to underline the fact that, syntactically speaking, it is an array of *symbols* arranged in a particular *form*.

> north, east and west), from some fixed point or origin selected or assumed for the purpose; the three dimensions of space receiving thus their three algebraical equivalents, their appropriate conceptions and symbols in the general science of progression. A plane or curve surface became thus algebraically defined by the assigning *as its equation* the relation connecting the three co-ordinates of any point upon it, and common to all those points: and a line, straight or curved, was expressed according to the same method, by the assigning two such relations, correspondent to two surfaces of which the line might be regarded as the intersection. In this manner it became possible to conduct general investigations respecting surfaces and curves, and to discover properties common to all, through the medium of general investigations respecting equations between three variable numbers: every geometrical problem could be at least algebraically expressed, if not at once resolved, and every improvement or discovery in Algebra became susceptible of application or interpretation in Geometry. The sciences of Space and Time (to adopt here a view of Algebra which I have elsewhere ventured to propose) became intimately intertwined and indissolubly connected with each other. Henceforth it was almost impossible to improve either science without improving the other also. The problem of drawing tangent to curves led to the discovery of Fluxions or Differentials: those of rectification and quadrature to the invention of Fluents or Integrals: the investigation of curvatures of surfaces required the Calculus of Partial Differentials: the isoperimetrical problems resulted in the formation of the Calculus of Variations. And reciprocally, all these great steps in Algebraic Science had immediately their applications to Geometry, and led to the discovery of new relations between points or lines or surfaces. But even if the applications of the method had not been so manifold and important, there would still have been derivable a high intellectual pleasure from the contemplation of it *as* a method.[28]

Now it should be clearer why in the *Introductory Lecture on Astronomy* Hamilton remarked that algebra is a "more general and refined" science than geometry. In Cartesian geometry using one single *algebraic formula* (i.e. one single *equation* between arrays of *symbols*) one may make assertions and deductions about entire *classes* of lines or surfaces, with no need to distinguish among different cases as is often necessary in geometrical constructions. Similarly, by differential *calculus* one may find the tangents, normals etc. to *any* curve by an *algorithm* that is independent from the considered curve. Therefore, if algebra is taken as an *autonomous* discipline, geometry may be ultimately regarded as an *application* of it, one of possibly many ways to *interpret* its symbolical expressions, giving them a definite *meaning* that appeals to the operations of *drawing figures*.[29]

In the case of optics, commonly regarded as a geometry-based science, Hamilton was able to express its fundamental problem into *one single algebraical equation*, whose solution is a function, called *characteristic function*, relating the directions of propagation of ingoing and outgoing rays with the positions of the two extreme

[28] Although not mentioned here, an important source of inspiration for Hamilton was Monge's *analytic geometry*, perhaps even more than Descartes' original system. (Hankins 1980, pp. 64–67)

[29] Or, in the terms of modern logic, a *semantic* to an otherwise purely *syntactical* system, geometrical *constructions* becoming a *model* for algebraical *formulas*.

points connected by these rays. As in Descartes' algebraical geometry all the properties of a line or surface are encoded in its equation, so in Hamilton's algebraic optics all the properties of a *system of rays* are encoded in its characteristic function. Given the features of the optical system, the characteristic function is uniquely determined, and if the form of the characteristic function is known, the paths of light rays through the optical system may be deduced by simple differentiation, or, in Hamilton's words, by the *unfolding of one radical or central relation.*

From a methodological point of view, Hamilton aims to turn mathematical optics into a rigorously *deductive* science. In a paper preliminary to the extension of his optical method to dynamics, Hamilton sketches the history of *theoretical* optics, starting from the "law of seeing in straight lines", known and used since ancient times to explain the appearances (Hamilton explicitly mentions its connections with astronomy). After quoting Newton's famous passage about induction and deduction at the end of the *Optiks*, he complaints about how relatively little the deductive side of this science had been developed as compared to the inductive in two millennia:

> It is, however, remarkable that, while the laws of this science admit of being stated in at least as purely mathematical a form as any other physical results, their mathematical consequences have been far less fully traced than the consequences of many other laws; and that while modern experiments have added so much to the inductive progress of optics, the deductive has profited so little in proportion from the power of the modern algebra. [...] It is better to ascend to the source of the imperfection, the want of a general method, a presiding idea, to guide and assist the deduction.[30]

Following Lagrange's example of *Méchanique Analytique*, the *presiding idea* Hamilton chooses to guide the deductive reasoning, the one and only *principle* on which the whole theory is built upon, is the celebrated *Principle of Least Action*, rephrased more generally as *Law of Stationary Action.* Hamilton attributed a paramount importance to this principle, lying at the foundations of both his optics and dynamics.

Mathematically speaking the Principle of Least Action is a *variational principle*, a postulate requiring that a quantity depending on the *whole* evolution of a physical system, called *action*, must assume a minimum value along the actual motions of the system, if compared to all other neighboring possibilities. Such a requirement is translated into an equation asserting the invariance of the value of a certain sum or path-integral connecting two points, when the path is changed in a smooth manner. Hamilton usually wrote such a variational principle in the form

$$\delta V = \delta \int vds = 0, \tag{2.1}$$

where δ denotes the change of path, v is a magnitude varying along of the path and ds is an element of the path. The product vds is the *action* expended by the system along the element ds, and V is the total action expended by the system in the whole path. Thus the Principle of Least Action operates as a *selection rule*

[30]Hamilton 1833b.

among all conceivable paths, that singles out the one along which the total expense of action is minimum. In the more general form of the principle employed by Hamilton the requirement is that such an expense must have a *stationary property* with respect to its close possibilities, so its value on the actual motion may also be a local maximum. In all the applications of this principle Hamilton always treated the action as a function of the path and/or of its endpoints, sometimes with the addition of some other numerical parameters.

In optics, v is the ratio between the velocity of light in vacuum c and its velocity in the medium $u = ds/dt$, a number called *index of refraction* of the medium. So in this case $vds = cdt$ and the action expended in a portion ds of the path equals the distance traveled by light in the corresponding time. The Principle of Least Action then reads

$$\delta V = c\,\delta \int dt = 0, \tag{2.2}$$

i.e. it reduces to Fermat's Principle of Least Time. As Hamilton remarks sketching the history of this principle, such an idea has its roots just in optics, and also this one was already suggested in antiquity by the observation that light "employs the direct, and, therefore, the *shortest* course to pass from one point to another". Following Laplace's account, he attributes to Ptolemy the observation that such a minimum property is valid also in the case of reflection on a plane mirror, since "the bent line formed by the incident and reflected rays is shorter than any other bent line, having the same extremities, and having its point of bending on the mirror". In the works of Fermat, Maupertuis, Euler and Lagrange this idea was extended to mechanics and progressively evolved until it reached "the last step in the ascending scale of induction", i.e. Hamilton's Law of Stationary Action. This is "the highest and most general axiom (in the Baconian sense) to which optical induction has attained", the maximum generalization achieved starting from the empirical laws of visual communication, and thus a proper starting point for setting out the deductive development of an optical theory.

Hamilton's characteristic optical function V is nothing but the total action integral evaluated between two fixed initial and final points of a light ray (proportional to the total time taken by light to go from one point to the other), expressed as a function of the cosine directors of the initial and final rays and of the color of light. The *equation of the characteristic function* is the analytic development of the formula $\delta V = 0$, i.e. the constraint on the actual light/visual ray expressed by the principle of stationary action.

It is important to remark that Hamilton explicitly rejected all of the physical or metaphysical meanings sometimes attributed to such a variational principle on the basis of some preconceived "economy of nature":

> But although the law of least action has thus attained a rank among the highest theorems of physics, yet its pretensions to a cosmological necessity, on the ground of economy in the universe, are now generally rejected. And the rejection appears just, for this, among other reasons, that the quantity pretended to be economised is in fact often lavishly expended. In optics, for example, though the sum of the incident and reflected portions of the path

> of light, in a single ordinary reflexion at a plane, is always the shortest of any, yet in reflexion at a curved mirror this economy is often violated. If an eye be placed in the interior but not at the centre of a reflecting hollow sphere, it may see itself reflected in two opposite points, of which one indeed is the nearest to it, but the other on the contrary is the furthest; so that of the two different paths of light, corresponding to these two opposite points, the one indeed is the shortest, but the other is the longest of any. In mathematical language, the integral called action, instead of being always a minimum, is often a maximum; and often it is neither the one nor the other: though it has always a certain stationary property [...]. We cannot, therefore, suppose the economy of this quantity to have been designed in the divine idea of the universe: though a simplicity of some high kind may be believed to be included in that idea. And though we may retain the name of *action* to denote the stationary integral to which it has become appropriated - which we may do without adopting either the metaphysical or (in optics) the physical opinions that first suggested the name - yet we ought not (I think) to retain the epithet least: but rather to adopt the alteration proposed above, and to speak, in mechanics and in optics, of the Law of Stationary Action.

Hamilton's optical theory is the deductive development of the consequences of this law, when applied to systems of rays as above defined. One of the key results Hamilton deduces is a generalized version of Malus' theorem, asserting the existence of families of surfaces cut perpendicularly by all the rays of a given system, whatever may be the number of successive reflections or refractions. These surfaces are called *surfaces of constant action*, since in going from one surface of the family to another all the rays expend the same amount of action. In the case of light rays starting from the same point, all of them reach these surfaces expending in the same amount of time. For example, in the simple case of straight rays diverging from a point source in a homogeneous isotropic medium, which is easy to visualize, these surfaces are concentric spheres described by the equation $V = const$.

One of the most relevant features of Hamilton's optics is its independency from any hypothesis about the *nature* of light. At his time the debate between undulatory and corpuscular models was at its apex, and Hamilton's theory simply bypasses the problem. This is a consequence of the fact that the Law of Stationary Action says something only about the *motion* of light, its propagation in *time* and *space*, but nothing about its composition or mechanism of propagation (e.g. the actual vibrations of a possible medium). Therefore, whatever consequence may be drawn from this principle, it will be only about *motion*: if one assumes a corpuscular model, rays represent motions of *particles*; if one assumes a wave model, the same rays represent motions of *wave-fronts*, represented by the surfaces of constant action.[31]

The most important result of Hamilton's theory was the theoretical prediction of *conical refraction*, a phenomenon occurring in biaxial crystals, i.e. crystals in which the speed of light depends on its direction of propagation. On applying

[31] After meeting with Faraday, with which Hamilton got along very well, he reported in a letter that they agreed on the idea that light, after all, is just "the motion of a motion".

his theory to such crystals Hamilton found that if a single ray of light entered or emerged from them in a certain direction, then that ray of light would be refracted into a cone of rays. The actual existence of conical refraction was experimentally confirmed by Humprey Lloyd on December 14th, 1832, making the young Hamilton instantly famous in Britain and abroad.[32]

A few weeks later Hamilton was already extending his method of the characteristic function to mechanics. This is not so surprising, after all, neither from the historical point of view nor from the conceptual one. Describing the optical system in terms of a quantity that is *expended* or *consumed* by light during its propagation (be it called *action* or *time*), Hamilton adopted at the very outset a *dynamical* approach, whose connections with the motion of matter are almost immediate. In fact Hamilton started to apply his method to mechanics as early as 1826, when, preparing for college examinations, he had already determined the explicit form of the characteristic function for the problem of projectile motions, and in the extensive table of contents of the *Theory of System of Rays* he had announced his intention to apply it to a system of mutually attracting points. Now, after a good amount of practice on optical problems, Hamilton proceeded quite fast in generalizing the method to a general dynamical system.[33]

Hamilton's dynamics is based on a slightly different variational principle. If the Law of *Stationary* Action is "the last step in the scale of induction", the first step in the "descending scale of deduction" is what Hamilton calls the Law of *Varying* Action. Mathematically speaking, the difference is that now also the endpoints of the path along which the action integral is evaluated are made to vary. Hamilton changed the adjective from *stationary* to *varying* also to emphasize the fact that, in dynamics much more than in optics, the principle is intended to express the *law of variation* of the action, i.e. the equation describing the *evolution* of the system of bodies, and not a *state* equation as is the case in the optical theory.[34]

What exactly does Hamilton mean by the word *dynamics*? The first paper *On a General Method in Dynamics* opens with some introductory remarks that

[32]As far as I know, this is the first instance of theoretical prediction of the very same *existence* of a *new* phenomenon, never observed before. It has been sometimes compared to the theoretical discovery of Neptune, made in 1846. However this latter, as astonishing as it was, was still an *inference* made to account for the *observed* motions of the other planets. This seems to me an important difference.

[33]In a letter to Lloyd dated February 9th, 1833, he wrote that he had found the form of a new function describing elliptical motion and expressed his intention to apply it to the problem of planetary perturbations; meanwhile, he had already worked out in his new approach the equations of motion for a system of any number of mutually attracting points. The first published paper in which Hamilton gave a sketch of the extension of his method to dynamics, applying it to dynamical astronomy, was published in October 1833. The two essays *On a General Method in Dynamics*, where the method is fully exposed, were complete by the fall of 1834 (Hankins 1980, p. 173).

[34]In optics this necessity arise if one considers non-homogeneous media, where the index of refraction changes continuously in space, the velocity of light varies from point to point and rays are curved. This is the case, for example, of the atmosphere of the Earth, a very important problem for its connections with astronomical observations. Hamilton developed the theory including these systems in the Third Supplement of his Theory of Rays, where the most general version of his optical theory is exposed, but this was never published. From this theory, the transition to dynamics was just a change in the *semantics* of the formulas involved.

clarify this point:

> The theoretical development of the laws of motion of bodies is a problem of such interest and importance, that it has engaged the attention of all the most eminent mathematicians, since the invention of dynamics as a mathematical science by Galileo, and especially since the wonderful extension which was given to that science by Newton. Among the successors of those illustrious men, Lagrange has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem. But the science of force, or of power acting by law in space and time, has undergone already another revolution, and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginably conditions, which Lagrange so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods. And the method proposed in the present essay, for the deductive study of the motions of attracting or repelling systems, will perhaps be received with indulgence, as an attempt to assist in carrying forward so high an inquiry.[35]

So dynamics is to Hamilton the *science of power acting by law in space and time*. Such a science was invented by Galileo (in studying falling bodies), extended by Newton (to astronomical phenomena, by the help of his fluxions) and put into a deductive form by Lagrange (who deduced Newton's theory from the Principle of Least Action). The further step of progress, according to Hamilton, consists in freeing the theory from any consideration about the *mechanism* involved, "resolving all connexions and actions of *bodies* into attractions and repulsions of *points*". In other words, Hamilton aims to expunge *natural philosophy* from Newton's *dynamics* and turn it into a full-fledged mathematical theory. Even if he makes appeal to the *physical* concepts of *power*, *action* and *forces* (coherently with the etimology of the greek δυναμισ), the only magnitudes his theory really deals with are the *effects* of these *actions*, i.e. the *reciprocal motions of bodies*. As Hamiltonian optics is about *motion of light*, modeled by *systems of rays* independently of any hypothesis about their "nature" and "mechanism of propagation", so Hamiltonian dynamics is about *motions of bodies*, modeled as *moving points* independently

[35] These words are echoed by Maxwell, who in the preface of *Matter and Motion* wrote: "Physical Science, which up to the end of the eighteenth century had been fully occupied in forming a conception of natural phenomena as the result of forces acting between one body and another, has now fairly entered on the next stage of progress—that in which the energy of a material system is conceived as determined by the configuration and motion of that system, and in which the ideas of configuration, motion, and force are generalised to the utmost extent warranted by their physical definitions. To become acquainted with these fundamental ideas, to examine them under all their aspects, and habitually to guide the current of thought along the channels of strict dynamical reasoning, must be the foundation of the training of the student of Physical Science."

of any hypothesis about their "nature" and "mechanism of interaction".[36] In another occasion Hamilton was more explicit about the scope of his dynamics:

> Professor Hamilton is of opinion that the mathematical explanation of all the phaenomena of matter distinct from the phaenomena of life, will ultimately be found to depend on the properties of systems of attracting and repelling points. And he thinks that those who do not adopt this opinion in all its extent, must yet admit the properties of such systems to be more highly important in the present state of science, than any other part of the application of mathematics to physics. He therefore accounts it the capital problem of Dynamics "to determine the $3n$ rectangular coordinates, or other marks of position, of a free system of n attracting or repelling points as functions of the time," involving also 6n initial constants, which depend on the initial circumstances of the motion, and involving besides, n other constants called the masses, which measure, for a standard distance, the attractive or repulsive energies.[37]

As in optics the form of the characteristic function is intended to determine the configuration of a system of rays, so in mechanics it must determine fully the *motions* of points, i.e. their mutual *positions* and *velocities* as functions of the *time*. In the solution of this fundamental problem, Hamilton remarks that Lagrange, Laplace, and others have managed to employ a single function to express the different forces of a system (i.e. the potential energy), and thus to encode in a single and general formula the *problem* of the motion (i.e. the Euler-Lagrange equations). His aim, now, is to express with a general formula all the possible *solutions* to such equations, i.e. the *general form* of possible motions (or, in mathematical terms, the form of the *integrals* of the Euler-Lagrange equations).

So, Hamilton's aim *was not*, in principle, to simplify the solution of dynamical problems. He knew it very well, of course, but didn't care too much about this aspect:

> ...and even if it should be that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation.[38]

However, this is what people expected from a "new" theory, and Hamilton's dynamics was not fully understood by many of his contemporaries. As Cayley

[36]This might seem a paraphrase of Newton's *hypothesis non fingo*, but it's not. Aside from his improper use of the word *hypothesis*, Newton stated that he didn't inquire about the *causes* of forces. Since forces are *defined* by Newton as the "cause of motions", his statement amounts to saying that he didn't inquire about the "causes" of the "causes of motion", which is quite obvious if one doesn't want to go backward indefinitely in an endless chain of "causes". Here, it seems to me, Hamilton is actually removing from dynamics the very same notion of *mechanical cause*, his theory dealing expressly with *change*, *forces* being merely "expressive figures" to speak about such changes (compare in this regard the *Introductory Lecture on Astronomy*). It is worth remarking here that, in the case of Hamilton's dynamics, the label *mechanics* is a plain misnomer.

[37]Hamilton 1834b.

[38]Hamilton 1834a.

remarked in 1857 in his *Report on Theoretical Dynamics*, in response to Jacobi's criticisms, even if Hamilton didn't simplify, in the first place, the problem of *solving* the equations of motion, what he invented is a *theory of the representation of the integral equation assumed to be known.*

The question Hamilton asked (and answered) is, in short, the following: given the *structure* of the dynamical equations, what are the properties that *all* their solutions *must* satisfy? Framing the question in these terms, Hamilton found that the integrals of the differential equations of motion depended on the general properties of a "hitherto unimagined" function, i.e. the characteristic or principal function, such properties being common to *any* solution of *any* dynamical problem that is formulated in terms of a variational principle.[39]

2.3 From *Algebraic Couples* to *Geometric Quaternions*

The achievement that Hamilton regarded as the most important of his mathematical career was the invention of quaternions, new mathematical objects to the study of which he devoted almost exclusively the last twenty years of his life. One of the aims of this section and the following is to convey the reasons why Hamilton gave such a big importance to this theory.

Quaternions grew out of Hamiltom's prolonged effort to clarify (above all to himself) the foundations of algebra as an autonomous science, independently from geometry, by making appeal to the fundamental notion of *time progression*. Hamilton's interest in these topics was sparkled by the results of his friend John Graves, who in 1824 had found that the logarithm of a complex number depends on the value of two independent integers. The interpretation of these results was not clear at all, and therefore their validity questioned. Hamilton struggled to solve the issue, and found that the notion of *progression* could be the key element to settle the debate. The first suggestions of this kind appear in a memorandum dated 1827, where Hamilton wrote that the concept of time "might serve to give greater precision and simplicity to our notion of ratio". In 1830 he started his attempts to define the basic operations of algebra in terms of *time-steps* and in November 1833 the *theory of algebraic couples*, fully confirming Graves' results, was presented to the Royal Irish Academy.

However, on this occasion Hamilton said nothing about the very original conceptions underlying his theory. Foundations of algebra were a hot topic of debate among British mathematicians in the first half of the XIX century, and Hamilton was well aware that his somewhat metaphysical views differed from those of his contemporaries. Therefore, for a long time he avoided any public

[39] This "discovery" is often summarized by saying that Hamilton's theory exhibited the so-called *symplectic structure* of classical dynamics, i.e. the fact that every solution of the dynamical equations is a *symplectomorphism*, a coordinate transformation on the phase space that preserves its *symplectic structure*. In this framework, Hamilton's characteristic function is the infinitesimal generator of a one parameter family of such transformations. For more details on the relevance of such "discovery" for mathematical physics see, for example, Gotay and Isenberg 1992; Guillemin and Sternberg 1984.

statement on the matter, sure as he was of the opposition he would have met. The encouragement he needed came from the reading of Kant's *Critique of Pure Reason*,[40] where the possibility of building a science grounded on the *intuition* or *inner mental form* of time is explicitly suggested.[41] Finally, on June 1st, 1835, Hamilton exposed his views about algebra in the *Preliminary and Elementary Essay on Algebra as the Science of Pure Time.* No new results are given in this work - "the novelty being in the view and method"-, but here Hamilton set forth a new *interpretation* of the known symbols, rules and operations of algebra based on the general notion of *ordered continuous progression*, which formed the basis of the work that would later lead him to the invention of quaternions.

In the preceding months Hamilton had discussed the nature, purpose, and methods of mathematics in general and of algebra in particular in a series of letters to his pupil and friend Lord Adare.[42] The content of these letters formed the bulk material of the *General Introductory Remarks* to the *Essay*, and exhibits quite plainly the Greek filiation of Hamilton's conception of *arithmos.*

The first letter is devoted to a careful distinction between the sciences generally dealing with *numbers*:

> In Arithmetic, properly so-called, *Number* is considered as an answer to the question *How many*, and as constituting a Science of *Multitude*, founded on the relation of *more* and *fewer*, or ultimately of the *many* and the *one.* In a more complex Science, of *Magnitude* and *Measure*, which may perhaps be called *Metrology* (though often classed as a higher part of Arithmetic), Number is the answer to the question *How much*, and the fundamental relation is that of *greater and less*, or of *whole and part.* But in *Algebra* I taught that Number answers the question *How placed in a succession*, the guiding relation being that of *before and after* (or of positive, negative, and zero); and the Science itself being one of *Order* and *Progression*, or, as it might be called concisely, of PURE TIME. To *count*, to *measure*, to *order*, are three different, although connected, acts of thought, and belong to these three different, although closely connected, Sciences, of *Arithmetic, Metrology,* and *Algebra.* Groups *as counted*, magnitudes *as measured*, positions or states *as ordered*; and, therefore, finally the relations of the counted to the counter, of the measured to the measurer, of the ordered to the orderer—such are the ultimate objects of these three acts of thought, and the ultimate or elementary conceptions of these three Sciences.

So, according to Hamilton, mathematics in general is about specific *acts of thought* oriented to *answer* specific questions by means of *numbers.* Algebra, in particular, is first and foremost about *ordered progressions* of *positions* or *states*, rather than with *magnitudes*, and its fundamental problem is the *study and comparison of*

[40] In the Preface to the *Lectures on Quaternions* he acknowledged this explicitly, quoting a full passage of Kant's *Trascendental Aesthetic.* Yet, in various letters, Hamilton noted how Kant conceived the possibility that a science of pure time was possible, but didn't realize that such a science could be already at hand, namely, in the existing but still far-from-rigorous operations of algebra. A summary of Hamilton's view of Kant's first *Kritik* is in Graves 1885, pp. 103–105.

[41] The German word for this is *Auunschaung*, literally *to look through* or *inside.* It is sometimes also translated as *contemplation*, a word that Hamilton often used.

[42] Hamilton 1879.

positions or states of the same or of different progressions, with reference only to their *arrangements* in such progressions. Arithmetic, Metrology and Algebra are viewed by Hamilton as three *languages* whose syntax is made of the same *symbols*, but differing according to the *use* and *meaning* accorded to them. Therefore, they are called respectively a *quotitative*, *quantitative* and *ordinal* language. As he expressed elsewhere, Hamilton wishes to make in Algebra "Position, instead of Quantity, or much more instead of Quotity, the primary and elementary conception from which all others are to be deduced".

In the later *Lectures on Quaternions* Hamilton was even more explicit on this point, and extended the primacy of the mental operation of *ordering* to the whole of mathematics:

> For my own part I cannot conceal that I hold it to be of great and even *fundamental* importance, to regard Pure Mathematics as being *primarily* the science of ORDER (in Time and Space), and not primarily the science of MAGNITUDE: if we would attain to a perfectly clear and thoroughly self-consistent view of this great and widely-stretching region, namely, the mathematical, of human thought and knowledge. In mathematical science the doctrine of magnitude, or of quantity, plays indeed a *very* important part, but *not*, as I conceive, the *most* important one. Its importance is SECONDARY and DERIVATIVE, *not primary* and *original*, according to the view which has long approved itself to my own mind, and in entertaining which I think that I could fortify myself by the sanction of some high authorities: although the opposite view is certainly more commonly received.[43]

In accordance with the etymological meaning of the Greek *arithmos*, the emphasis to the *ordinal* instead of the *cardinal* meaning of numbers and to the idea of *position in a succession* instead of that of *magnitude*, are the only *clear* way, according to Hamilton, to avoid the paradoxes involved in what he calls the *doctrines* of negative and imaginary quantities.[44] If the symbols reasoned upon in algebraic formulae are interpreted as *quantities*, one is inevitably led to meaningless statements such as that a *quantity is less than nothing* or that a *a greater quantity may be subtracted from a less*, and to contradictory assertions like those about the *negative squares of imaginary quantities.* Of course these difficulties are not a problem for many practical purposes, or for those who view algebra "as a mere system of symbols". Therefore Hamilton is careful to distinguish three different tendencies among mathematicians, the *practical*, the *philological*, and the *theoretical*, "according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere*,) is eminently prized and sought for". Hamilton belongs, of course, to the last category, "seeking more a clear and lively intuition, by whatever cost of meditation or mental discipline to be attained, than language, however perfect in its structure, or rules, however

[43]Hamilton 1853, p. 12.

[44]See also Graves (1885, p. 303): "In short, *ordinals* seem to me to have, in thought, priority over *cardinal* numbers, "one", "two", "three" mean, *originally* "first", "second", "third", they are names rather of the counted things, than of the groups containing them".

easy of application", and always trying to "*look beyond or through the signs of the things signified.*"[45]

Overall, according to Hamilton's criteria, if algebra is regarded as dealing primarily with *quantities* it cannot be considered as a *science* (read: *episteme*) at all:

> Yet a natural regret might be felt, if such were the destiny of Algebra; if a study, which is continually engaging mathematicians more and more, and has almost superseded the Study of Geometrical Science, were found at last to be not, in any strict or proper sense, the Study of a Science at all: and if, in thus exchanging the ancient for the modern Mathesis, there were a gain only of Skill or Elegance, at the expense of Contemplation and Intuition.

Needless to say, the model Hamilton follows to turn algebra into a proper science - "strict, pure and independent; deduced by valid reasonings from its own intuitive principles" - is Euclid's system of geometry:

> I habitually desire to find or make in Algebra a system of demonstrations resting at last on intuitions, analogous in some way or other to Geometry, as presented by Euclid - for I own that Geometry itself might be presented in a merely logical or symbolical form, though I for one would not thank him who should so present it.[46] And I persuade myself, with a confidence that has been gradually gaining strength for years, that as Geometry, in the popular mind and mine, rests ultimately on the *Intuition of Space*, so Algebra may be made to rest on the kindred *Intuition of Time...* Pure Time - the before and after; precedence, subsequence, and simultaneity; continuous indefinite progression from the past through the present to the future - this thought, or intuition, or form of the human mind, appears to force itself upon me whenever I seek to *analyse* what I and others *mean*, as the *objects reasoned upon*, in Algebraic Science: though I willingly admit that the Time thus considered is *Pure* (just as *Space* of the Geometers is Pure), and does not depend on any phenomenal marks or measures, nor need not use the notion of Cause and Effect. The *moment* is to Algebra, with me, what the *point* is to Geometry; *transitions*, *intervals*, from one moment to another, are analogous to finite straight lines, while Time itself may be conceived or pictured (as indeed metaphysicians generally admit) under the image of an indefinite straight line; and *number* is the ratio of such transition to another, or the complex relation between them, determined partly by their relative largeness and partly by their relative direction...[47]

In the same days Hamilton took a personal note clarifying what he meant by *moment of pure time*:

[45]As examples of *philological school* Hamilton mentions Woodhouse and Peacock, Ohm and Lagrange; to the *theoretical school* he admits Cauchy and Fourier.

[46]Such a program was actually carried out by David Hilbert and the formalistic school during the XX century. No doubt that Hamilton would have been even more isolated from the rest of the mathematical community, had he lived some decades later. However, it is worth remembering that Hilbert's train quite soon crashed on the wall of Kurt Gödel's *incompleteness theorems.*

[47]Graves 1885, pp. 143–144.

> Though the mind exerts an act (of will) in thinking of a moment, yet the thought once formed becomes an object given: and we may and must treat it as such, and conceive ourselves as determining its position, although we ourselves had generated that position. We Must conceive Progression of a Moment, or a Moment as Progressive, and Time itself as generated thereby...[48]

Hamilton justifies his view of the intertwinement between algebra and time noting that the notion of *ordered progression* had actually always been involved in algebraic procedures. This is evident, he argues, from the crucial role that the idea of *continuous progression* has played in the history of algebraical inventions such as Newton's method of fluxions - which "employs, as its primary conception, the thought of a *flowing point*" - and Napier's logarithms - who grew out "not (as it is commonly said) from the arithmetical properties of powers of numbers, but from the contemplation of a *Continuous Progression*; in describing which, he speaks expressly of Fluxions, Velocities and Times".[49] It is worth quoting the following passage:

> The History of Algebraic Science shows that the most remarkable discoveries in it have been made, either expressly through the medium of that notion of Time, or through the closely connected (and in some sort coincident) notion of Continuous Progression. It is the genius of Algebra to consider what it reasons on as flowing, as it was the genius of Geometry to consider what it reasoned on as fixed. Euclid defined a tangent to a circle, Apollonius conceived a tangent to an ellipse, as an indefinite straight line which had only one point in common with the curve; they looked upon the line and curve not as nascent or growing, but as already constructed and existing in space; they studied them as formed and fixed, they compared the one with the other, and the proved exclusion of any second common point was to them the essential property, the constitutive character of the tangent. The Newtonian Method of Tangents rests on another principle; it regards the curve and line not as already formed and fixed, but rather as nascent, or in process of generation: and employs, as its primary conception, the thought of a flowing point. And, generally, the revolution which Newton made in the higher parts of both pure and applied Algebra, was founded mainly on the notion of fluxion, which involves the notion of Time.

Hamilton also quotes Lagrange's definition of algebra as the *science of functions*, a function being essentially a *law connecting Change with Change* and thus intimately connected to *time*.[50]

Hamilton privately described the *Essay* as his "long-anspired union of Mathematics and Metaphysics" and, in later years, defended the legitimacy of his metaphysical introduction in the draft of a letter to De Morgan, dated 8th December 1851:

[48] Graves 1885, p. 695.

[49] Indeed, in Napier's original work logarithms are defined as numbers that *synchronize* an arithmetical and a geometrical progression.

[50] Today, after the decipherment of cuneiform tablets, we could add the origins themselves of arithmetical algorithms in Babylonian astronomy.

> You would do me no great good by criticising - and I can conceive your feeling a temptation to do so - the metaphysics of the early articles *at present.* I know that I have not done justice even to my *own* views in that direction, much lesst to existing philosophy: *sed liberavi animam meam.* I *could not* bring myself to enter on the subject without some such introductory remarks, and must only hope that while thus imperfectly recording some moods or frames of mind, which have really influenced *myself* in mathematical speculations of the class we are now considering, I shall not have materially embarassed the path of any student who will have even a moderate degree of patience to wait till he sees *how I use* the notions with which I profess to set out. At one time I read a good deal of Kant's works in the German, besides portions of Plato in the Greek...

Indeed, apart from such introductory remarks, no metaphysics is involved in the *Essay*, since, as Hamilton beautifully summarizes:

> There is something mysterious and transcendent involved in the idea of Time; but there is also something definite and clear: and while Metaphysicians meditate on the one, Mathematicians may reason from the other.

To give a hint about the character the work, we briefly sketch how Hamilton interprets in his system the "rule of signs", i.e. the fact that the product of two "negative quantities" gives a "positive quantity", and the so-called "imaginary" unit, i.e. the square root of the "negative" unity.

The whole theory starts from the stipulation that the elementary symbols A, B, C, D etc. denote *dates* or, more abstractly, *moments of a one-dimensional progression* (i.e. extending *backward* and *forward*). The elementary binary symbols $=, >, <$ denote respectively *simultaneity, precedence* and *succession* of dates, so that if a moment A is given, any another moment B may be *simultaneous to* A, *earlier than* A, or *later than* A. Accordingly, the formulas

$$\mathrm{B} = \mathrm{A} \tag{2.3}$$

$$\mathrm{B} < \mathrm{A} \tag{2.4}$$

$$\mathrm{B} > \mathrm{A} \tag{2.5}$$

express the possible relationships of simultaneity, precedence and succession of the moment B relatively to the moment A . So, to be completely consistent with Hamilton's view, such formulas should be read respectively "B IS SIMULTANEOUS TO A", "B PRECEDES A", and "B FOLLOWS A".[51] Pairs of moments form *intervals of time* or *time-steps* and are denoted with small letters a, b, c etc. A time-step is clearly a complex object, involving both *magnitude* and *direction*, this latter being either *forward* or *backward* or, more abstractly, *positive* or *contrapositive.* A key point is that *numbers* in Hamilton's theory are not primitive objects, but are *defined* as *quotients of time-steps*, i.e. as *operators* or *factors* or *multipliers* that *act* on a step in such a way as to *produce* or *mentally generate* the other. Hamilton showed that, with this definition, all the common *rules* of addition, subtraction, multiplication, division, extraction of root etc. on algebraic numbers admit a

[51]Some time later this was called the *topological layer.*

simple interpretation in terms of *operations on time-steps*, altering in general both their direction and magnitude. Therefore numbers come to be naturally either *positive* or *contrapositive*, according as the steps they connect are in the *same* or in *opposite* directions, and the *rule of signs* translates the simple circumstance that, given a time-step, two successive reversals restore its original direction.

Moreover, in Hamilton's view there is no difficulty in considering *couples of moments*, *couples of time-steps* and *couples of numbers*, extending to such *couples* all the elementary operations. In particular, simple *numbers* come to be viewed as *degenerate forms* of *number-couples* with a null *secondary part*, and Hamilton recovers by a different route Cauchy's view of complex numbers as *couples* of real numbers, the "imaginary unit" corresponding to the number-couple $(0, 1)$.[52] By Hamilton's definitions this *secondary unit* is an *operator* acting on step-couples in such a way as to *reverse* the second step and to *transpose* the order the two steps, so that applying it *twice* to the same step-couple one gets the original couple with both steps reversed. So - "by a certain *abstraction* of operators from the operand" - he established the formula

$$(0, 1)^2 = (-1, 0) = -1, \tag{2.6}$$

that in his view gives "a new, and clear *interpretation*" to the formula expressing the root of the imaginary unit

$$(0, 1) = (-1, 0)^{\frac{1}{2}} = (-1)^{\frac{1}{2}} = \sqrt{-1}; \tag{2.7}$$

"without anything obscure, impossible, or *imaginary* being in any way involved in the conception."

An important difference between Hamilton's *theory of couples* and other formally equivalent systems in which a similar meaning had been given of the imaginary unit (for example Warren's 1828 work *On the Geometrical Representation of the Square Root of Negative Quantities*, where the symbol $\sqrt{-1}$ is interpreted as a unit line perpendicular to the unit real line) is that no reference is made to the notion of *space*, but only to the concept of *progression*. However, Hamilton's *theory of couples* can be interpreted geometrically and consistently *applied* to problems of *plane* geometry.

For years Hamilton attempted to extend his theory to *triplets* in view of a possible applications to *space* geometry. *A posteriori*, we know that this was an impossible task, since, to speak in modern terms, what Hamilton looked for is a *tridimensional associative division algebra*. Such a structure cannot be constructed, since, up to an isomorphism, there are only *three* associative division algebras over the reals: real numbers themselves, complex numbers and quaternions, having

[52] In his paper on algebraic couples Hamilton derived in an independent way also the so-called *Cauchy-Riemann equations* and remarked: "The author acknowledges with pleasure that he agrees with M. Cauchy, in considering every (so-called) Imaginary Equation as a symbolic representation of two separate Real Equations: but he differs from that excellent mathematician in his method generally, and especially in not introducing the sign $\sqrt{-1}$ until he has provided for it, by his Theory of Couples, a possible and real meaning, as a symbol of the couple $(0, 1)$."

respectively dimensions 1, 2 and 4.[53] So, on one hand, Hamilton's search for an *autonomous* theory of triplets satisfying the criteria he required was doomed to fail, but, on the other, his perseverance and methodological coherence led almost necessarily to the theory of quaternions (in which a *theory of triplets* is actually included as a particular case).

The "discovery" of quaternions, as Hamilton sometimes referred to it, came with a sudden epiphany on October 16th, 1843. This is how Hamilton remembered that moment in 1865, shortly before dying, in a tender letter to his son Archibald:

> ...I happen to be able to put the finger of memory upon the year and month - October, 1843 - when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association, the desire to discover the laws of the multiplication [of triplets]... regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you *multiply* triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only *add* and *subtract* them." But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an *under-current* of thought was going on in my mind, which gave at last a *result*, whereof it is not too much to say that I felt *at once* the importance. An *electric circuit* seemed to *close*; and a spark flashed forth, the herald (as I *foresaw, immediately*) of many long years to come of definitely directed thought and work, by *myself* if spared, and at all events on the part of *others*, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,
>
> $$i^2 = j^2 = k^2 = ijk = -1$$
>
> which contains the *Solution* of the *Problem*, but of course, as an inscription, has long since mouldered away.[54]

[53]The main difficulty arises in defining the multiplication rules in such a way that the inverse of any element of the algebra different from the null element is unique. For more details see May (1966) and bibliography therein. In any case, even if Hamilton didn't give a formal proof of impossibility, it is likely that he perceived this, as also Gauss did before him. In a letter written to John Graves after the "discovery" of quaternions, later published in the *Philosophical Journal*, in describing some of his unsuccessful attempts to multiply triplets Hamilton wrote: "And here there dawned on me the notion that we *must* [emphasis mine] admit, in some sense, a *fourth dimension* of space FOR THE PURPOSE OF CALCULATING [capital mine] with triplets." It is clear then that, at this stage, for Hamilton such "fourth dimension" is necessary in order to construct an effective *calculus* on triplets. He later found a geometrical justification. A fuller account of Hamilton's attempts to find an autonomous theory of triplets is given by himself in the preface of the *Lectures on Quaternions.*

[54]Graves 1885, pp. 434–435. Seven years before, Hamilton gave a similar account of the event

A month later, on November 13th, Hamilton read at the Academy a paper *On a new system of imaginaries in algebra, connected with a theory of quaternions*, in which quaternions are defined as follows:

> It is known to all students of algebra that an imaginary equation of the form $i^2 = -1$ has been employed so as to conduct to very varied and important results. Sir Wm. Hamilton proposes to consider some of the consequences which result from the following system of imaginary equations, or equations between a *system of three different imaginary quantities*:
>
> $$i^2 = j^2 = k^2 = -1$$
> $$ij = k, \quad jk = i, \quad ki = j;$$
> $$ji = -k, \quad kj = -i, \quad ik = -j;$$
>
> no linear relation between i, j, k being supposed to exist, so that the equation
>
> $$\mathrm{Q} = \mathrm{Q}',$$
>
> in which
>
> $$\mathrm{Q} = w + ix + jy + kz,$$
> $$\mathrm{Q}' = w' + ix' + jy' + kz',$$
>
> and w, x, y, z, w', x', y', z' are real, is equivalent to the four separate equations
>
> $$w = w', \quad x = x', \quad y = y', \quad z = z'.$$
>
> Sir W. Hamilton calls an expression of the form Q a *quaternion*; and the four real quantities w, x, y, z he calls the *constituents* thereof.[55]

Hamilton devoted the rest of his life to the "consideration of the consequences" of these definitions, but probably could never imagine how far such consequences would be extended in the XX century. The literature on quaternions, their applications and subsequent fate is exterminate. Here we must limit ourselves to some general remarks more closely related to the subject of this dissertation.[56]

Hamilton's theory of quaternions was a turning point in the history of modern mathematics, marking the beginning of *abstract algebra*. According to Whittaker the formula $ij = -ji$

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in a letter to Peter Tait: "...I then and there felt the galvanic circuit of thought *close*; and the sparks which fell from it were the *fundamental equations between i, j, k; exactly such* as I have used them ever since. I pulled out on the spot a pocket-book, which still exists, and made an entry, on which, *at the very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a *problem* to have been at that moment *solved* - an intellectual want relieved - which had *haunted* me for at least *fifteen years before*." (Graves 1885, pp. 435–436)

[55] Hamilton 1843.

[56] A short and clear introduction to quaternions is Tait's article for the *Encyclopedia Britannica*, published in Tait (1900b, pp. 445–455), where the history and relation of Hamilton's method to others is also discussed. Probably the best starting point to enter quaternions in a way that is more direct to the point but still faithful to Hamilton views is Kendall and Tait 1873. See Gsponer and Hurni 1993 for a more recent review of the relevance of quaternions for contemporary mathematical physics; Gsponer and Hurni 2008a; Gsponer and Hurni 2008b are extensive bibliographies on quaternions and allied systems.

> was the supreme moment in the history of mathematical symbolism. It began the creative process which yielded not only quaternions, but all the other systems which broke away from old rules - Cayley and Sylvester's matrices, Boole's symbolic logic, Grassman's Ausdehnungslehre, Gibbs' dyadics and the Heisenberg-Dirac algebra of quantum mechanics.[57]

For the first time it became clear that one could *create* from scratch new and coherent number systems (and, more generally, *meaningful systems of symbols*), provided a set of postulates defining the properties of the numbers through the *operations* that may be effected on them. Hamilton was one of the first, in fact, to recognize as fundamental (i.e. as *definitory of the meaning of symbols employed*) the commutative, associative and distributive properties of the basic operations of elementary algebra. One of the great novelties of his system was the renounce to the commutative property of multiplication, until then considered a "natural" or "necessary" property of numbers "themselves", conceived as primitive entities whose properties would be "evident". Indeed, it is hard to think differently if numbers themselves are left undefined or implicitly regarded as representing *quantities.* But, as we saw, a critical analysis of the concept of *ratio* led Hamilton to an explicit *operational* definition of real numbers as *quotients of steps.* When numbers are thus regarded as *operators*, it becomes naturally relevant the order in which the operations encoded in a symbolical formula are effected, and commutativity appears clearly as a *possibility* rather than as a *necessity.* Something similar occurred in the same period with the "discovery" of "non-euclidean" geometries, i.e. with the analogous recognition that the existence and uniqueness of parallel lines is a possibility[58] rather than a logical necessity, so that denying Euclid's parallel postulate leads to different but *coherent* and *meaningful* geometries.

As we already remarked, despite quaternions were introduced as algebraic sets (i.e. as systems of *symbols* and *operations* defining them), at the very outset Hamilton framed the search for triplets in geometrical terms, interpreting *triplets* as *positions in space*, and in view of the possible geometrical applications of such a new method. The first such application was to problems of spherical trigonometry. In the first paper of November 1843 Hamilton already remarked

[57]Whittaker 1944. George Boole, in particular, was strongly influenced by Hamilton's works, and repeatedly looked for his support. However, contrarily to his habits, Hamilton was not very responsive to Boole's requests, a fact that Hankins finds hard to explain. I think a clue is given by Hamilton himself in the letter to De Morgan already quoted above, dated December 8th, 1851, which continues as follows: "...it is one of my hopes to resume, at what may be called leisure hours, some of my old studies of that kind, and to combine them with the reading of some other and more Aristotelian than Platonic works - including the 'Formal Logic?'- although my own temperament of mind is far more Platonic than Aristotelian. Don't be so malicious as to quote the *Malim cum Platone errare*, however applicable you may think it to be; and let me tell you that when I was a boy at College I acquired some undergraduate renown by a short proof (which I have totally forgotten, and which would at all events have bee since superseded by one of Mr Boole's), that in no legitimate syllogism can the conclusion change place with either of the premises." (Graves 1889, p. 296). It seems worth noting here that today we know that also in Hellenistic times Chrysippus (III cent. BC), third master of the Stoic school, had demonstrated theorems internal to what later was called *propositional logic.* For more details see Russo 2004, pp. 218–221.

[58]Namely, that of drawing on a *plane* surface rather than on a *curved* one.

how the multiplication properties of quaternions indicated "a new sort of algorithm, or calculus, for spherical trigonometry", and a letter dated October 24th to John Graves proves that Hamilton *immediately* foresaw such application:

> I think that this *Calculus of Quaternions* will at least be found to assist in discovering many theorems in spherical trigonometry. Some such theorems have been suggested to me by it, which I do not know how to prove otherwise...[59]

Alongside with the first applications, Hamilton's attention drew more and more on the *geometrical interpretation* of the results to which the new *symbolic calculus* led him, and the more he clarified the matter the more he came to conceive quaternions chiefly as *geometric objects* (i.e. objects constructed in tridimensional *space*), as contrasted with the initial purely algebraic definition based on the multiplication rules for the fundamental symbols i, j, k. This might seem paradoxical, but as Hamilton immediately perceived (and *this* was his great intuition) the additional *abstract* "extra-spatial" dimension is the price to pay to treat *all directions of space as equivalent.* Some time later in response to De Morgan's attempts to build a system of triplets different from that included in his theory of quaternions Hamilton emphasized the importance of this point:

> It will surprise me, I confess, if either your theory, or any other person's, of *pure triplets* shall be found to surpass that which I have been led to perceive, as *included* in my theory of quaternions on all, or most, of the three following points:
>
> 1. *Algebraic simplicity*; ...analogy to ordinary algebra, as to the rules of addition and multiplication (the commutative property excepted);
> 2. *Geometric simplicity*; ...ease of construction; the rule of the diagonal; and, above all, *simmetricity of space*, no one direction being eminent;
> 3. *Determinateness of division*; ...a quotient being never indeterminate or impossible unless the constituents of the divisor all vanish.
>
> Of all these assumed requisites, or things aimed at me (and I admit that I aimed at others), what *now* appears to me most my own is the SYMMETRICALNESS OF SPACE in my system. If *you* have succeded in representing this with pure triplets, *eris mihi magnus Apollo.* My *real* is the representative of a sort of *fourth* dimension, inclined equally to all lines in space.[60]

A year after the discovery, the geometrical interpretation of the algebraic theory of quaternions was essentially complete. Hamilton exposed it in the short paper *On Quaternions.*[61] In it, this *real* representative of the *fourth* dimension are interpreted as expressing a *time-relation* between steps, pertaining to a one-dimensional ordering representable *on a scale*, whence the name of *scalar* part of the quaternion; and the three imaginary parts, forming the *vector* part of the quaternion, are interpreted as expressing a *space-relation* between steps,

[59] Graves 1885, p. 442.
[60] Quoted in Hankins, p. 305.
[61] Hamilton 1844.

pertaining to their tridimensional ordering. In other words, since quaternions are conceived to express *relations between successions*, their very same *existence in space* presupposes their existence in *time*.[62]

After the triumph of cartesian geometry in the early modern period, this indifference to the directions of space made quaternions the first *coordinate-free language* in which the choice of a reference system, expressed or implied, is not required at all. The importance of this aspect, that became generally acknowledged after Einstein's *theory of relativity*, was clearly perceived by Peter Guthrie Tait, who, after a period of self-study of the *Lectures on Quaternions*, became the only true pupil of Hamilton with regards to his method of quaternions. Therefore, we turn to him and to his closer fellows to get a clearer view of quaternions *as* a method.[63]

2.4 Hamilton's *Elements*

Tait gave great importance to Hamilton's progressive shift in emphasis (at least in his published works) from the algebraical to the geometrical definition of quaternions. In his opinion, the history of the development of this theory should be divided in two distinct phases: the first, going from their (algebraic) invention in 1843 to the publication of the *Lectures on Quaternions* in 1853, when quaternions were *defined* as an "analytical system of imaginaries", and then *interpreted* as geometrical objects (see Fig. 2.5); and a second one, going from 1853 to 1865, when Hamilton died and left unpublished the huge and ever-almost-complete *Elements of Quaternions*, in which he finally exposed his original (and ultimate) view on his own creation. As Tait put it:

> ...[Hamilton] raised Quaternions from the comparatively low estate of a mere system of *Imaginaries* to the proud position of an Organ of Expression;

[62] In this regard, it seems relevant to remark that in a letter to De Morgan dated May 12th, 1841, Hamilton wrote: "Let me own that I am not prepared to decide, with you, that it is possible for a human mind to 'imagine a given length to be instantaneously generated, no one portion of it coming into the thoughts before or after another,' in opposition to the teaching of Kant, which seems to me confirmed by my own consciousness, that 'we can think to ourselves no line without *drawing* it in thought'." (Graves 1885, p. 342)

[63] In a note at the end of the *Elements of Quaternions* Hamilton explicitly elected Tait his "successor" in the further development of the theory of quaternions, and especially of its applications to physics. After the death of the master Tait strongly advocated the use of quaternions as a general language for physical sciences, and got involved in a quarrel on *Nature* with Gibbs' and the supporters of vector calculus. This was born as a byproduct of quaternion calculus, and indeed the quarrel originated by Tait's remark that Gibbs' vector calculus was a "hermafrodite monster" that borrowed words from Hamilton's system but completely misunderstood the nature of the method. Through Tait, quaternions came under the notice of his lifelong friend James Clerk Maxwell, who also became a supporter of Hamilton's system. To illustrate their utility Maxwell wrote his monumental *Treatise on Electricity and Magnetism* in a double language, using both coordinate methods and quaternions. On the contrary, Lord Kelvin remained faithful to coordinates and decided not to include quaternion methods in the influential *Treatise on Natural Philosophy* he co-authored with Tait. It is quite likely, as Hankins remarks, that the subsequent fate of quaternions was strongly influenced by Kelvin's choice. Conversely, it seems that Tait's strict adherence to quaternion methods condemned his impressive works to a relative oblivion.

LECTURES

ON

QUATERNIONS:

CONTAINING A SYSTEMATIC STATEMENT

OF

A New Mathematical Method;

OF WHICH THE PRINCIPLES WERE COMMUNICATED IN 1843 TO

THE ROYAL IRISH ACADEMY;

AND WHICH HAS SINCE FORMED THE SUBJECT OF SUCCESSIVE COURSES OF LECTURES, DELIVERED IN 1848 AND SUBSEQUENT YEARS,

IN

THE HALLS OF TRINITY COLLEGE, DUBLIN:

WITH NUMEROUS ILLUSTRATIVE DIAGRAMS, AND WITH SOME GEOMETRICAL AND PHYSICAL APPLICATIONS.

BY

SIR WILLIAM ROWAN HAMILTON, LL. D., M. R. I. A.,

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OF THE ROYAL SOCIETIES OF EDINBURGH AND DUBLIN; OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY;
THE NEW YORK HISTORICAL SOCIETY; THE SOCIETY OF NATURAL SCIENCES AT LAUSANNE; AND OF OTHER SCIENTIFIC SOCIETIES IN BRITISH AND FOREIGN COUNTRIES;
ANDREWS' PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF DUBLIN;
AND ROYAL ASTRONOMER OF IRELAND.

Figure 2.5: Frontispiece of the 1853 edition of the *Lectures on Quaternions.* Notice the emphasis on the fact that a *new mathematical method* is exposed, "with some *geometrical* and *physical* applications".

giving simple, comprehensive, and (above all) transparently intelligible, embodiment to the most complicated of *Real* geometrical and physical relations. *From the most intensely artificial of systems arose, as if by magic, an absolutely natural one!*

Considering the care that Hamilton put in the choice of *names*, the title of *Elements* he gave to the crowning work of his mathematical career should be read as an explicit homage to his beloved Euclid. Indeed, according to Tait, the Greek ancestry of quaternions is imprinted in their very name:

> The word quaternion properly means "a set of four". In employing such a word to denote a new mathematical method, Sir W. R. Hamilton was probably influenced by the recollection of its Greek equivalent, the Pythagorean *Tetractys*, the mystic source of all things.[64]

It is noteworthy that for some time Hamilton considered the possibility to call these new objects *grammarithms*, which literally translates *figure-number*, i.e. the fundamental notion of Pythagorean mathematics. Indeed, *mutatis mutandis*, it would be not inappropriate to regard the geometric theory of quaternions as a modern realization of the old Pythagorean idea of *spatial numbers disposed in ordered progressions*.

The fundamental problem posed and solved in Hamilton's *Elements* is the very general one of *multiplication and division of directed lines in space*. In the letter to his son we already quoted above, dated 1865, Hamilton was very explicit:

> No more important, or indeed fundamental question, in the whole Theory of Quaternions, can be proposed than that which thus inquires: "What is such MULTIPLICATION? What are its Rules, its Objects, its Results? What Analogies exist between it and other Operations, which have received the same general Name? And finally, what is (if any) its Utility?"

Indeed, as we already remarked, from the very beginning Hamilton pursued the search for *triplets* following Euclid's steps, i.e. searching for the line which is a *fourth proportional of three perpendicular lines in space*.[65] Hamilton found that a quaternion can be seen, very simply, as the *unique quotient of two directed lines in space* that satisfies the requirement of *isotropy* or *symmetry* between all directions of space. This latter property makes the resulting method a *symbolical calculus uniquely adapted to the symmetries of Euclidean space* or, as Hamilton expressed himself, to the *symmetricalness of space*.

The Euclidean filiation of Hamilton's theory was well understood by Maxwell, who wrote in this regard:

> The fact is, that even in the purely geometrical applications of the Quaternion method we meet with three different kinds of directed quantities: the

[64] Tait 1886.

[65] See, in particular, Hamilton 1844, and compare with Hamilton's juvenile remarks on the meaning of *division* quoted above. Note that in a *plane*, the construction of a line that is the fourth proportional of three given lines is equivalent to the calculation of a ratio. In his search for triplets Hamilton has thus generalized Euclid's procedure to three dimensions.

> vector proper which represents transference from A to B; the area or "aperture," which is always understood to have a positive and a negative aspect, according to the direction in which it is swept out by the generating vector; and the versor which represents turning round an axis. The Quaternion ideas of these three quantities differ from the ideas of the line, the surface, and the angle only by giving more prominence to the fact that each of them has a determinate direction as well as a determinate magnitude. When Euclid tells us to draw the line AB he supposes it to be done by the motion of a point A to B or from B to A. But when the line is once generated he makes no distinction between the results of these two operations, which, on Hamilton's system, are each the opposite of the other. Surfaces also, according to Euclid, are generated by the motion of lines, so that the idea of motion is an old one and we have only to take special note of the direction of the motion in order to raise Euclid's idea to the level of Hamilton's.[66]

And also by Kendall (emphasis ours):

> The fundamental idea on which the science is based is that of motion - of transference. Real motion is indeed not needed, any more than real superposition is needed in Euclid's Geometry. An appeal is made to *mental transference* in the one science, to *mental superposition* in the other. We are then to consider how it is possible to frame a new science which shall spring out of Arithmetic, Algebra, and Geometry, and shall add to them the idea of motion - of transference. It must be confessed that the project we entertain is not a project due to the nineteenth century. The Geometry of Des Cartes [*sic*] was based on something very much resembling the idea of motion, and so far the mere introduction of the idea of transference was not of much value. [...] What the nineteenth century has done, then, is to divorce addition from multiplication in the new form in which the two are presented, and to cause the one, in this new character, to signify *motion forwards and backwards*, the other *motion round and round*.[67]

So, in short, Hamilton's theory of quaternions is a *symbolical* and *kinematical* "upgrade" of Euclidean *space* geometry, where the two *algebraic operations* of *addition* and *multiplication* are made to correspond to the *geometrical* operations of *tracing* and *rotating* a straight line, and where, more generally, *all* the magnitudes involved are conceived as *generated* by the *directed motion* of lower-dimensional ones. In Hamilton's view, every quaternion equation is thus the symbolical expression of the *correspondence* or *equivalence* between two *simultaneous progressions of figure-numbers.*

It is clear how Hamilton recovered and put to profit many of the primal conceptions of Greek mathematics. His calculus, in particular, greatly simplifies the study of surfaces of revolution and their intersections, making easily available to modern, symbolic-minded mathematicians many results that the Greeks achieved by their unparalleled geometrical skills and visualization abilities. Indeed, this is what makes Apollonius' *Conics* so hard to read for modern mathematicians.

[66]Maxwell 1873.

[67]Kendall and Tait 1873, p. 4.

As it turns out, many of Apollonius' theorems are demonstrated by quaternion methods with a few lines of calculation.[68] Still Tait (emphasis ours):

> Perhaps to the student there is no part of elementary mathematics so repulsive as is spherical trigonometry. Also, everything relating to change of systems of axes, as for instance in the kinematics of a rigid system, where we have constantly to consider one set of rotations with regard to axes fixed in space, and another set with regard to axes fixed in the system, is a matter of troublesome complexity by the usual methods. But every quaternion formula is a proposition in spherical (sometimes degrading to plane) trigonometry, and has the full advantage of the symmetry of the method. And one of Hamilton's earliest advances in the study of his system (an advance independently made, only a few months later, by Cayley) was the interpretation of the singular operator $q(.)q^{-1}$, where q is a quaternion. Applied to any directed line, this operator at once turns it, *conically*, through a definite angle, about a definite axis. Thus *rotation is now expressed in symbols at least as simply as it can be exhibited by means of a model.* Had quaternions effected nothing more than this, they would still have inaugurated one of the most necessary, and apparently impracticable, of reforms.[69]

If one recalls the young Hamilton's imaginary dialogues with Euclid, it should be clear why he later accorded such a great importance to his new "METHOD".

Those who familiarized with quaternion methods clearly perceived their eminently geometric power. In response to Cayley's opinion that quaternions are just a compact way of using coordinates - which remained to him "the natural and appropriate basis of the science" -, Tait summarized his view on the matter as follows:

> To me Quaternions are primarily a mode of representation:—immensely superior to, but of essentially the same kind of usefulness as, a diagram or a model. They *are*, virtually, the thing represented: and are thus antecedent to, and independent of, coordinates: giving, in general, all the main relations, in the problem to which they are applied, without the necessity of appealing to coordinates *at all.* Coordinates may, however, easily be read *into* them:—when anything (such as metrical or numerical detail) is to be gained thereby. Quaternions, in a word, *exist* in space, and we have only to recognize them :—but we have to *invent* or *imagine* coordinates of all kinds. The grandest characteristic of Quaternions is their transparent intelligibility.[70]

Notice, in particular, that such a use of coordinates *intrinsically adapted* to the figure at hand is exactly the way in which coordinates are employed by Apollonius in the *Conics.*[71]

[68] See for example the exercises proposed in Kendall and Tait 1873.

[69] Tait 1900b, pp. 453–454.

[70] Tait 1900b, p. 393.

[71] It is noteworthy that Hieronymus Zeuthen (1839-1920) was able to recognize Apollonius' methods as a *geometric algebra* only after its modern analogue was invented, in the same span of years, first by Hamilton and then by others.

On different occasions Tait also emphasized the *pedagogical* value of the geometrical approach to quaternions, as contrasted to the algebraical, in particular with regard to the manifold *physical* applications of the method:

> Keeping always in view, as the great end of every mathematical method, the physical applications, I have endeavoured to treat the subject as much as possible from a geometrical instead of an analytical point of view. Of course, if we premise the properties of i, j, k merely, it is possible to construct from them the whole system; just as we deal with the imaginary of Algebra, or, to take a closer analogy, just as Hamilton himself dealt with Couples, Triads, and Sets. This may be interesting to the pure analyst, but it is repulsive to the physical student, who should be led to look upon i, j, k, from the very first as geometric realities, not as algebraic imaginaries.[72]

As he also said, with quaternions there can be no "shut up your eyes, and write down your equations", since every manipulation of a quaternion formula can and *should* be immediately interpreted in geometrical terms. In the same article quoted above Maxwell expressed his enthusiasm about this feature of Hamilton's method:

> Now Quaternions, or the doctrine of Vectors, is a mathematical method, but it is a method of thinking, and not, at least for the present generation, a method of saving thought. It does not, like some more popular mathematical methods, encourage the hope that mathematicians may give their minds a holiday, by transferring all their work to their pens. It calls upon us at every step to form a mental image of the geometrical features represented by the symbols, so that in studying geometry by this method we have our minds engaged with geometrical ideas, and are not permitted to fancy ourselves geometers when we are only arithmeticians.[73]

According to Tait it was an unfortunate choice on Hamilton's part not to publish earlier his theory in its improved form, without appealing at all to the algebraic definition of quaternions. He explained this fact as follows:

> He had fully recognized, and proved to others, that his i, j, k were mere excrescences and blots on his improved method:—but he unfortunately considered that their continued (if only partial) recognition was indispensable to the reception of his method by a world steeped in Cartesianism! Through the whole compass of each of his tremendous volumes one can find traces of his desire to avoid even an allusion to i, j, k; and, along with them, his sorrowful conviction that, should he do so, he would be left without a single reader. [...] And I further believe that, to this cause alone, Quaternions owe the scant favour with which they have hitherto been regarded.[74]

[72]Tait 1890, p. xi.

[73]Maxwell 1873. Such remarks will be echoed in a different context by Vladimir Arnol'd (1990), another strong advocate of geometrical methods in mathematical physics.

[74]In this paper Tait also copied a letter dated 1859 in which Hamilton asked him to differ the publication of his planned manual on quaternions, claiming precedence for the presentation of the theory in his new form: "Meanwhile I trust that it cannot be offensive to you, if I confess,

Tait's analysis applies identically, I think, to the historical reception of the theory quaternions, often mentioned as a mile-stone in the history of algebra but rarely exposed in its *essential* geometrical meaning.[75] In any case, Tait admits that it was not from Hamilton's published works that he grasped the full import of the theory:

> I do not now think that Hamilton, with the "peculiar turn of mind" of which he speaks, could ever, *in a book*, have conveyed adequately to the world his new conception of the Quaternion. I got it from him by correspondence, and in conversation. When he was pressed to answer a definite question, *and could be kept to it*, he replied in ready and effective terms, and no man could express *viva voce* his opinions on such subjects more clearly and concisely than he could:—but he perpetually planed and repolished his printed work at the risk of attenuating the substance: and he fatigued and often irritated his readers by constant excursions into metaphysics. One of his many letters to me gave, in a few dazzling lines, the whole substance of what afterwards became a Chapter of the *Elements*; and some of his *shorter* papers in the *Proc. R. I. A.* are veritable gems. But these were dashed off at a sitting, and were not planed and repolished.

Faithful to himself and to his official charge as Royal Astronomer of Ireland, the first *physical* domain in which Hamilton applied his new method of *grammarithms* was, of course, astronomy. Indeed, the first systematic exposition of his method, the *Lectures on Quaternions*, were in origin included in his Astronomy course at Trinity College, where the occasion allowed him "select his illustrations from Geometry" and, at the same time, to "clothe them in an Astronomical garb."[76] In the astonishing *Lectures on Quaternions* the deep connection between Hamilton's conception of quaternions and the fundamental problems of mathematical astronomy is made crystal clear.

[...] that in any such future publication on the Quaternions as you do me the honour to meditate, I should prefer the establishment of 'PRINCIPLES' being left, for some time longer,—say even 2 or 3 years,— in my own hands. Open to improvement as my treatment of them confessedly is, I wish that improvement, at least to some extent, to be made and published by myself. Briefly, I should like (I own it) that no book, so much more attractive to the mathematical public than any work of mine, as a book of yours is likely to be, should have the appearance of laying a 'FOUNDATION' [...] But my peculiar turn of mind makes me dissatisfied without seeking to go deeper into the philosophy of the whole subject, although I am conscious that it will be imprudent to attempt to gain any lengthened hearing for my reflections. In fact I hope to get much more rapidly on to rules and operations, in the Manual than in the Lectures ; although I cannot consent to neglect the occasion of developing more fully my *conception* of the MULTIPLICATION OF VECTORS, and of seeking to establish such mult[iplication] as a much *less arbitrary* process, than it may seem to most readers of my former book to be." Following the master's wishes, Tait published his *Elementary Treatise on Quaternions* in 1867, a few months after Hamilton's *Elements*. Hamilton's treatise remained the only exposition of such PRINCIPLES and FOUNDATION of the theory of quaternions. However, the difficulty of the treatise, combined with the subsequent fate of the theory, makes it not unlikely that relatively few people have actually ever read Hamilton's major work.

[75] Of course there are some exceptions. See for example Loria (1939), in which the history of the geometrical representation of magnitudes is outlined, and quaternions are mentioned as the latest development of a thread initiated by Greek geometers.

[76] Hamilton 1853, p. 5.

Outcome of this first application was another startling "discovery", which Hamilton described as follows in a letter to De Vère:

> I have, however, a *new conception* to tell you of, which bears on all the applications of Newton's great Law of Attraction, and which has in the strangest way concealed itself from all our mental eyes, till it was pleased to allow me to fix on it my gaze a little time ago - not, I trust, that I may share the fate of him who beheld the Virgin Huntress in her transparent bath, but was afterwards devoured by his own hounds.[77]

To this *new conception* we devote the following chapter.

[77] Graves 1885, p. 546. The Classic reference is to the myth of Artemis and Atteon.

Chapter 3

The Method of the Hodograph

On December 14th 1846 Hamilton read to the Royal Irish Academy a communication "respecting a new mode of geometrically conceiving, and of expressing in symbolical language, the Newtonian law of attraction, and the mathematical problem of determining the orbits and perturbations of bodies which are governed in their motions by that law".[1] In this short paper Hamilton introduces a new mathematical method through which the solution to many relevant dynamical problems may be greatly simplified. The key-idea of the method simple, and consists in focusing the attention, first and foremost, on the *kinematical relationship* or *relative motion* between the bodies under study, rather than on their *spatial relationship* or *relative position.* This conception is concretely realized by the definition of a new curve, which Hamilton calls *hodograph* (from the Greek οδός, a *way*, and γράφω, *to describe*), intended to describe such kinematical relationship.

Generally speaking, the *hodograph* of a given curve γ is another curve γ' determined by how the former is described by a tracing point. In the applications to mechanics, i.e. if the point is conceived to represent a material body moving along a path in "physical" space, its hodograph is the curve traced by the moving extremities of the succession of its *vectors of velocity*, provided all of them are drawn from a common point as origin. In other terms, the hodograph of a moving body is a *diagram* in an abstract *velocity space* answering how the body *moves* at any given point of its path.

The main result of Hamilton's paper is the so-called *Law of the Circular Hodograph*, a geometrical *characterization* of the *inverse-square law* by which the two-body problem is solved in a very simple and elegant way. In particular, starting from the circular hodograph, *all* the solutions to the dynamical two-body problem may be explicitly *constructed* and represented by a diagram which encodes in its structure all (and only) the *observables* of the problem, namely the position along the path and the corresponding velocity of the moving point.

Despite its beauty and simplicity, the method of the hodograph is not widely known today. At the time of its invention, it was well received by some of Hamilton's closest correspondents, mainly for his high pedagogical value. Thomson and Tait, for example, used hodographic methods in their famous textbook (albeit framing it in coordinate language) to solve specific problems that otherwise would

[1] Hamilton 1847b.

require way more sophisticated mathematical techniques.[2] Also Maxwell used the circular hodograph to solve the Kepler problem in his little gem *Matter and Motion*. However, in the first half of the XX century the hodograph practically disappeared from the literature and was later independently rediscovered in different contexts.[3] Its most famous revival is probably Feynman's *lost lecture* about the motion of planets around the Sun, where the circular hodograph, although not named in this way, is used to deduce the elliptic shape of the orbits.[4]

In the last decades the hodograph has been the subject of many works exploring its features and possible generalizations, but none of these has much in common with Hamilton's original approach.[5] In particular, in all the subsequent revivals of the method the *law of the circular hodograph* is *deduced* from the inverse square law, appearing thus as one among its many *consequences*. This is, of course, legitimate, the two laws being completely equivalent from the mathematical standpoint, but doing so one misses the most interesting aspects of Hamilton's method.

In Hamilton's original paper the law of the circular hodograph is the result of a general analysis that yields a *symmetry criterion* which *uniquely selects* the inverse-square proportion among *all* the conceivable *dynamical laws*. In other words, Hamilton's treatment suggests the possibility and indicates the route to an *a priori* justification of the inverse-square proportion based on the *symmetries* of Euclidean space. The *dynamical* symmetry exhibited by Hamilton's procedure is very remarkable, and, as we read in the letter quoted at the end of last chapter, Hamilton himself was surprised when he realized that such a simple property of Newton's gravitational law went unnoticed for so long.

The reasons of such a "strange" circumstance lie, of course, in the intrinsic features of Hamilton's theory of quaternions, of which the geometrical method of the hodograph was an immediate offspring. In July 1845 Hamilton read to the Royal Irish Academy a communication *On the application of the method of Quaternions to some dynamical questions*. Here we read:

> The author stated that, during a visit which he had lately made to England, Sir John Herschel suggested to him that the internal character (if it may be so called) of the method of quaternions, or of vectors, as applied to algebraical geometry, - that character by which it is independent of any

[2]Among others, Thomson was particularly enthusiastic about the hodograph, and asked Hamilton multiple copies of his paper. He also made the hodograph the subject of examination in his courses. In light of the thesis here exposed, it is noteworthy that Thomson was among the modern pioneers of *analog computing*. Probably the first mechanical device of the modern era comparable in conception and sophistication to the AM was Thomson's *Tidal Clock*.

[3]A notable exception is Sommerfeld 1952, who probably knew about Hamilton's method from his master Felix Klein. Klein was one of the few mathematicians on the continent who was thoroughly familiar with Hamilton's works, which probably also influenced the ideas of his *Erlangen Program* (Hankins 1980, pp. 199–210). It is noteworthy that in F. Klein (1949) it is sketched how Euclid's geometry can be reframed by reference to the concept of *motion*, but Hamilton is never named.

[4]The lecture was later published in D. Goodstein and J. Goodstein 1997. The audio recording is available on Youtube.

[5]See for example Eades 1968; Derbes 2001; Ben Ya'acov 2017 and bibliography therein. I don't understand why Hankins regards the hodograph method as a "mathematical curiosity" that has "nothing to do with the quaternions" (Hankins 1980, p. 326).

> foreign and arbitrary axes of coordinates, - might make it useful in researches respecting the attractions of a system of bodies. A beginning of such a research had been made by Sir William Hamilton in October, 1844, which went so far, but only so far, as the deducing of the constancy of the plane of an orbit, and the equable description of areas, under one common formula... Since the suggestion above acknowledged was made, Sir William Hamilton has proposed to himself to express by an equation, on the principles of the method of vectors, the problem of any number of bodies attracting according to Newton's law...

In the later *Elements* Hamilton remarked that the *law of the circular hodograph* was "virtually contained in a quaternion formula" included in this communication, and indeed there he deduced by quaternion methods many of the results which will be obtained geometrically in the hodograph paper of 1846. Also, at the very beginning of the *Lectures on Quaternions*, Hamilton remarked that "theory of *Hodographs*... had been suggested to me as a geometrical interpretation, or construction, of some integrations of equations in physical astronomy whereto I had been conducted by the Method of Quaternions."[6] This later recollection is confirmed by a letter he sent to John Herschel during 1846, where he remarked:

> ...it is a feature of my method that it *suggests* geometrical demonstrations in a degree which I never experienced while practising the method of coordinates...

It was exactly such *suggestive* character of his method that prompted the recognition of the dynamical symmetry hidden in Newton's gravitational law and expressed by the law of the circular hodograph. The last stimulus Hamilton needed for this came from the theoretical discovery of Neptune, occurred in the September 1846, which rekindled his interest in the problems of planetary perturbations. In the preparation of a supplementary lecture to his course on Astronomy at the Trinity College for this special occasion, the geometrical method of the hodograph came to light.

In the 1846 paper quaternions are named just at the end and throughout employed only in the degenerate form of *vectors*. However, the quaternion origin of the method probably explain why the title of the paper refers to a *symbolical language* and why, despite the geometrical character of the conception exposed, no figure is included.[7] This lack makes it quite difficult to read, so we reconstructed the missing diagrams. In any case, before turning to Hamilton's paper, it will be helpful to sketch Feynman's version of the hodograph method.[8]

3.1 Feynman's *Lost Lecture*

On March 13th, 1964, Richard Feynman gave a guest lecture about the motion of the planets around the Sun to the undergraduate students of the California

[6] Hamilton 1853, p. 3.

[7] At least in in the printed version of the paper. It is sure that Hamilton presented diagrams when he communicated the paper at the Royal Irish Academy.

[8] For a full exposition of the hodograph method by quaternions the reader is referred to Hamilton's *Elements* (Book III, Art. 419) and, for a simpler treatment, to Tait 1890, pp. 279–287.

Institute of Technology. The general aim of the lecture is to prove Kepler's first law in the context of Newton's dynamics.

After an historical introduction and some preliminaries devoted to the properties of the ellipse, the steps of Feynman's proof may be summarized as follows.

From Kepler's second law, i.e. from the observation that the planets describe equal areas in equal time, it is deduced that the force which deviates the planets from their inertial path must be directed to the Sun. In this step Feynman reproduces essentially Newton's proof of the inverse result (i.e. that central forces imply equal areas in equal times) as it appears in the *Principia*.

From Kepler's third law, i.e. from the observation that the square of the periods of revolution is proportional to the cube of the major semiaxis of the orbits, assuming for simplicity that the orbits are circles it is deduced that the intensity of the force must decrease as the square of the distance. Also in this step Feynman reproduces the proof sketched in the introduction to the third edition of the *Principia*, written by Roger Cotes.

From the central inverse-square law it is deduced that the planetary orbits are ellipses with the Sun occupying one of the focii, i.e. Kepler's first law. This is the main focus of the lecture, being according to Feynman the "most dramatic" of Newton's discoveries:

> ...what Newton discovered—and which was the most dramatic of his discoveries—was that the third law [Feynman means the First Law] of Kepler was now a consequence of the other two. Given that the force is toward the Sun, and given that the force varies inversely as the square of the distance, to calculate that subtle combination of variations and velocity to determine the shape of the orbit and to discover that it is an ellipse is Newton's contribution, and therefore he felt that the science was moving forward, because he could understand three things in terms of two.

Feynman confesses that on this point he couldn't follow Newton's proof, "because it involves so many properties of the conic sections",[9] so he "cooked up" an original proof, more *elementary* than Newton's in a very specific sense, that is particularly interesting in the perspective of this dissertation:

> "Elementary" means that very little is required to know ahead of time in order to understand it, except to have an infinite amount of intelligence. It is not necessary to have knowledge but to have intelligence, in order to understand an elementary demonstration. There may be a large number of steps that are very hard to follow, but each step does not require already knowing calculus, already knowing Fourier transforms, and so on. So by an elementary demonstration I mean one that goes back as far as one can with regard to how much has to be learned. [...] Secondly, this demonstration is interesting for another reason—it uses completely geometrical methods.[10]

[9]To be pedantic, Newton never gave in the Principia an explicit proof of this result, corresponding to the solution of the so called *inverse* Kepler problem. In the *Principia* it is the *direct* Kepler problem that is explicitly solved, while the inverse result, that guarantees also the uniqueness of the solution, is stated without proof. This gave rise to a long controversy

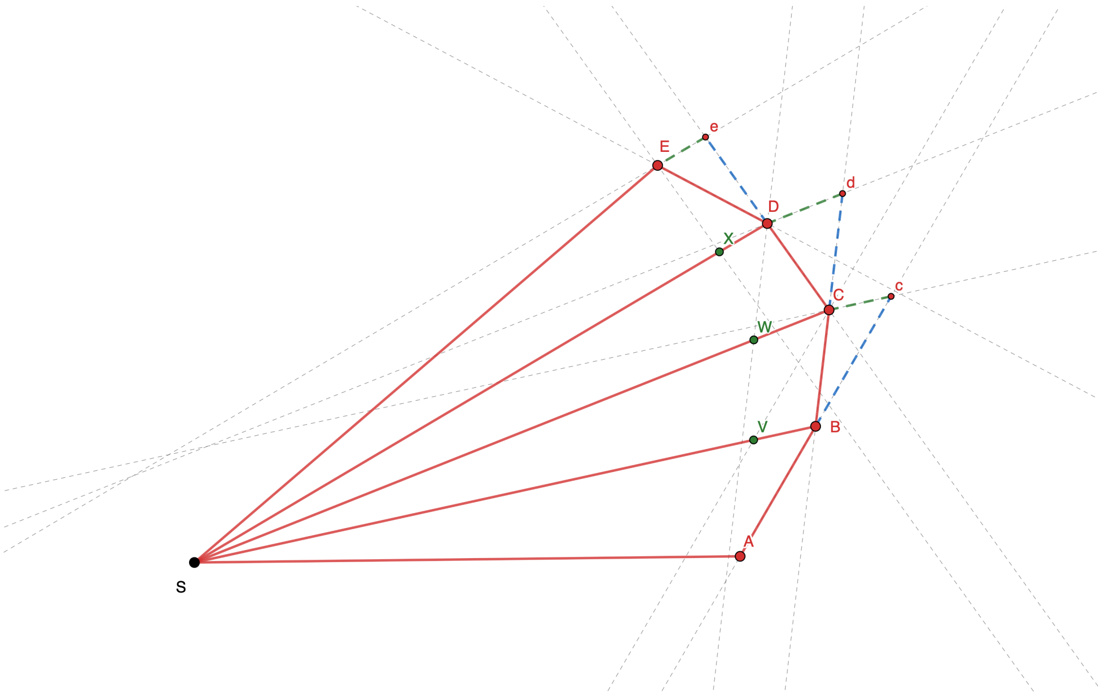

Figure 3.1: Feynman's version of Newton's diagram for *Proposition 1* in the *Principia*.

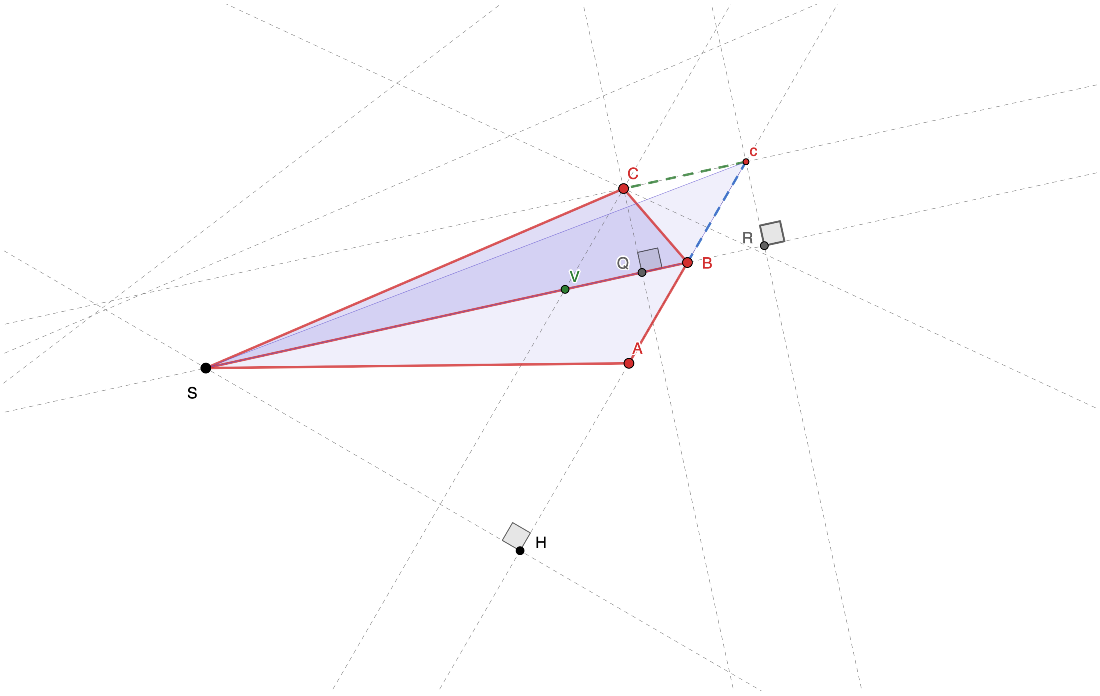

Figure 3.2: Proof that the triangles SAB, SBc and SBC have equal areas. SAB and SBC have the same base because $AB = Bc$ and a common altitude SH, so they have the same area; also SBC has the same area as SBc, because they have the same base SB and are comprised between the same parallel line; therefore, $SAB = SBc = SBC$.

Following Newton's paradigmatic example of the *Principia*, Feynman approximate the orbit whose shape is to be found with a polygon formed by a succession of points A, B, C, D, E, representing the successive positions of a planet, the Sun being placed at the fixed point S, and assuming that the positions of the planet are separated by equal intervals of time (see Fig. 3.1). This assumption, together with Newton's first and second laws of motion and with the hypothesis about the centripetal character of the gravitational force, implies that the successive triangles SAB, SBC, SCD, SDE on the diagram are all in the same plane and have equal areas (see Fig. 3.2). In this approximation the planet moves in each interval at a constant velocity along the lines AB, BC, CD, DE and these triangles have an area equal to that swept by the line joining the Sun and the planet in the corresponding interval of time; therefore the result that the planet describes, in this approximation, equal areas in equal times. Since the successive positions are separated by equal times, the lines AB, BC, CD, DE represent both the mean displacement and the mean velocity of the body in the corresponding intervals of the orbit, and the lines BV, CW, DX represent the changes of velocity impressed by the gravitational force at the points B, C, D. In the limit of smaller and smaller intervals of time the approximate polygonal orbit approaches without limit a curve, and the same lines tend more and more to the instantaneous values of the same magnitudes corresponding to the middle instant of the same intervals.

Next, Feynman departs from Newton and considers a succession of positions of the planet in its orbit, J, K, L, M, N, separated each other by an *equal angle* with respect to the fixed point S (see Fig. 3.3). If we identify this fixed center of attraction with an observer, this is equivalent to choose the *true anomaly* of the orbiting body as the independent variable tracking its successive positions in the orbit. Clearly the true anomaly is not in general proportional to the time, so *equal* angles described by the orbiting planet correspond in general to *unequal* times of description. Nevertheless, these times are in a definite proportion to each other, for the triangles SJK, SKL, SLM, SMN have by construction the angle at the vertex S equal and thus are in the continuous limit more and more similar to each other. So their areas are to each other as the square of any one of their homologous sides,[11] that is as the square of the distance of the planet from the Sun. In other terms, *equal angles of true anomaly are described in times that are among themselves in the same ratio as the squares of the mean distance of the planet from the Sun in those intervals.* Feynman goes quite fast on this important point:

> Now listen: I would point out to you that ... equal angles, which is what I'm aiming for, means that areas are not equal, no, but they are proportional

among later mathematicians and scholars, standing on opposite positions on the matter. For more details see, for example, Arnol'd 1990.

[10] More than twenty years later Feynman saw the fragments of the AM when he visited Athen's National Archeological Museum, and remained deeply fascinated by them. He wrote a letter to his wife about it, which may be read in Feynman 1989, pp. 93–97. I like to think he would be happy to see his lecture employed in the effort to understand the meaning of those fragments he was so curious about.

[11] Euclid, VI.19: "Similar triangles are to one another in the duplicate ratio of their homologous sides."

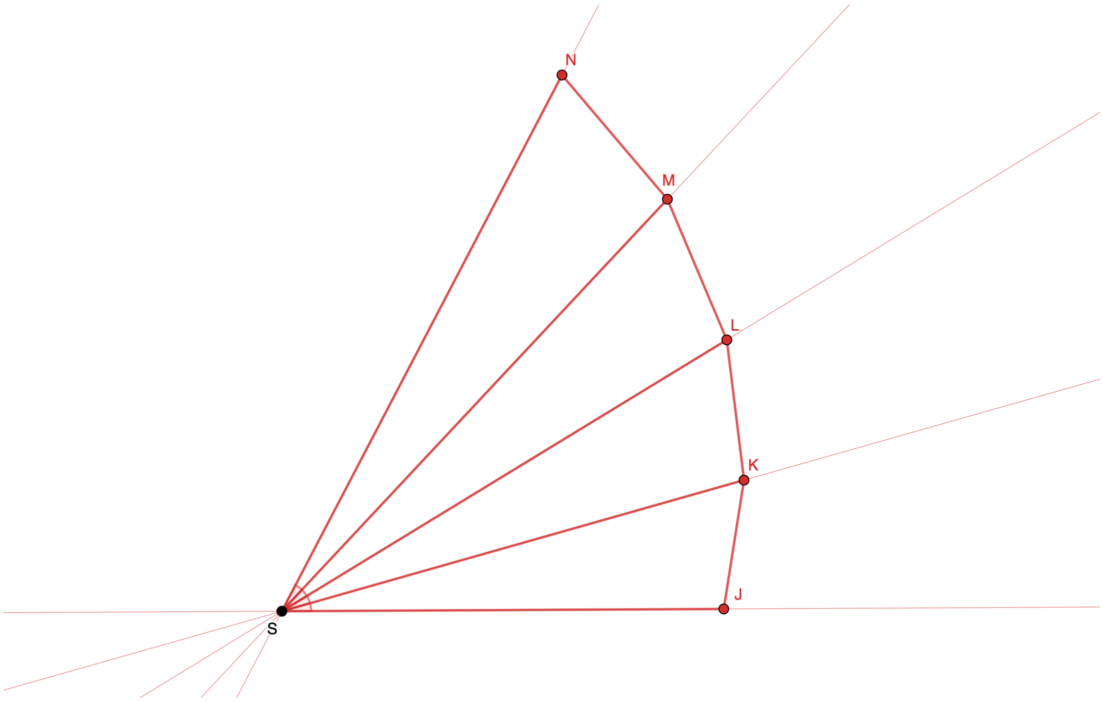

Figure 3.3: Successive positions on the orbit separated by equal angles of true anomaly. The areas of the triangles SJK, SKL, SLM, SMN are unequal and proportional to the square of the mean distance of the planet from the Sun in the corresponding interval of the orbit.

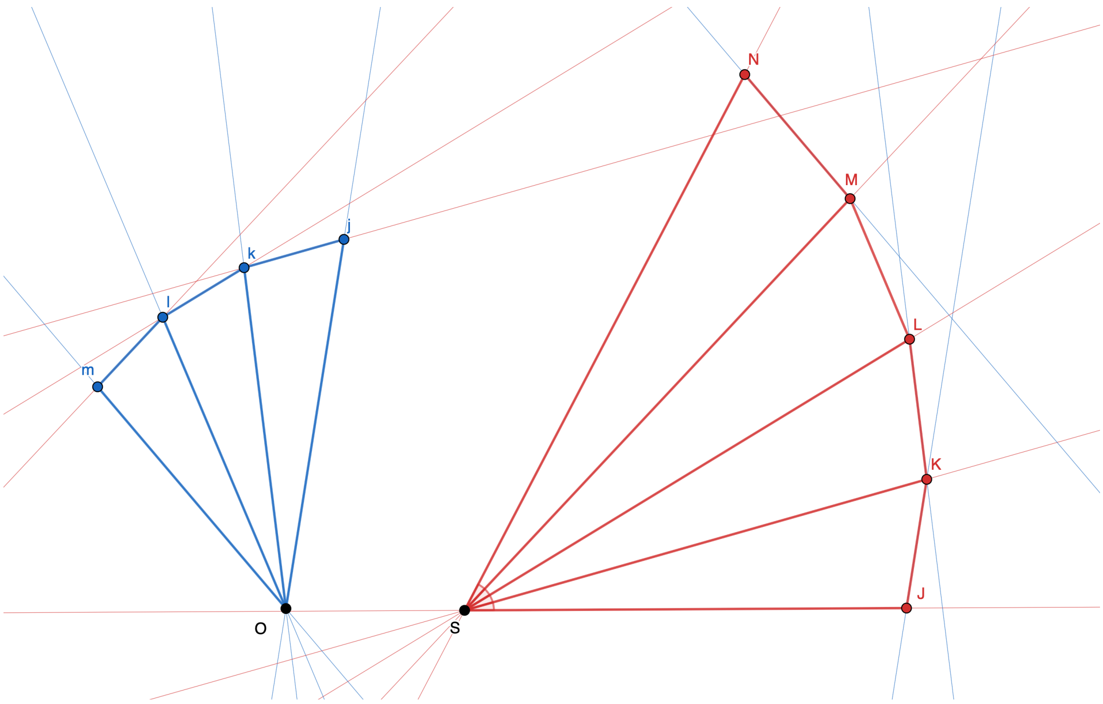

Figure 3.4: Diagrams of position (in red) and velocity (in blue) of the planet in equal angles of true anomaly in the orbit. Oj, Ok, Ol, Om are the mean velocities in the intervals and are parallel to JK, KL, LM, MN; jk, kl, lm are the mean changes of velocities in the intervals, are parallel to SK, SL, SM and have equal lengths. The two diagrams are not in the same scale and their sizes have no connection with each other.

> to the square of the distance from the Sun; for if I have a triangle of a given angle, it is clear that if I make two of them that they are similar; and the proportional area of similar triangles is proportional to the square of their dimensions. Equal angles therefore means—since areas are proportional to time—equal angles therefore means that the times to be swept through these equal angles are proportional to the square of the distance. In other words, these points—J, K, L, and so on—do not represent pictures of the orbit at equal times, no, but they represent pictures of the orbit with successions of times which are proportional to the square of the distance.

Since, by Newton's second law, the mean change of velocity in any interval of the motion is given by the product of the mean force acting in that interval into the time of its action in that interval (in symbols $\Delta v = F\Delta t$),[12] if the force decreases as the square of the distance and in equal intervals of true anomaly the corresponding time of action increases in the same proportion, i.e. if F and Δt vary inversely as each other, their product will be constant and the net effect will be the same in all the intervals. In other terms, *the effect of the gravitational force is such that, whatever may be the distance of the planet from the Sun, in equal intervals of true anomaly equal changes of velocity are produced.* Feynman expresses this important property of the action of the gravitational force in these terms:

> Now, the dynamical law is that there are equal changes in velocity, no—that the changes in velocity vary inversely as the square of the distance from the Sun—that is, the changes of velocity in equal times. Another way of saying the same thing is that equal changes of velocity will occupy times proportional to the square of the distance. It's the same thing. If I take more time, I get more change in the velocity, and, although they are falling off for equal times inversely as the square, if I make my times proportional to the square of the distance, then the changes in velocity will be equal. Or, the dynamical law is: equal changes in velocity occur in times proportional to the square of the distance. But look, equal angles were times proportional to the square of the distance. And so we have the conclusion, from the law of gravitation, that equal changes of velocity will occur in equal angles in the orbit. That's the central core from which all will be deduced—that equal changes in velocity occur when the orbit is moving through equal angles.

In symbols, Feynman's reasoning may be expressed as follows. The dynamical law is given by the proportion $\Delta v_2 : \Delta v_1 = F_2\Delta t_2 : F_1\Delta t_1$, the pedices 1 and 2 indicating any two intervals of the motion. The gravitational force is described by the proportion $F_2 : F_1 = r_1^2 : r_2^2$. If 1 and 2 are equal-angle intervals of the orbit, we saw that $\Delta t_2 : \Delta t_1 = r_2^2 : r_1^2$, therefore in these intervals $F_2 : F_1 = \Delta t_1 : \Delta t_2$ and $\Delta v_2 = \Delta v_1$, i.e. "equal changes in velocity occur when the orbit is moving through equal angles." From the centripetal character of the gravitational force it also follows that the changes in the *directions* of the mean velocity are the same in equal-angle intervals of the orbit.

[12] We assume throughout this analysis that the planet has unit mass.

Next, Feynman proceeds to construct a new diagram representing the succession of the velocities of the planet at the points J, K, L, M, N. This diagram of velocities will be composed of straight lines Oj, Ok, Ol, Om parallel to JK, KL, LM, MN, all drawn from a common origin O, and with a length that must be proportional to the displacement the planet would made in a unit of time starting from J, K, L, M, N if no force acted on it. As Feynman remarks, and this is an important point, now these lengths will *not* be proportional to JK, KL, LM, MN, since these intervals are described in *unequal* times and, therefore, the mean velocities in these times are *not* in the same proportion as the corresponding mean displacements. In symbols, $\Delta r_2 : \Delta r_1 = v_2 \Delta t_2 : v_1 \Delta t_1$, and since in equal-angle intervals $\Delta t_2 \neq \Delta t_1$, then $\Delta r_2 : \Delta r_1 \neq v_2 : v_1$.[13]

The mean changes in velocity occurring in any interval of the motion of the planet are represented on the velocity diagram by the straight lines jk, kl, km connecting the successive velocity points, i.e. the variable extremities of the straight lines representing the velocities - the *vectors of velocity*, as Feynman himself can't resist to say, immediately pointing out, however, that "you're not supposed to know what a vector is in this elementary description". So, the dynamical law expressed by the proposition "equal angles described by the planet in orbit correspond to equal changes of its velocity" translates into a *rule* for the construction of the diagram of velocities, i.e. for the drawing of the succession of points j, k, l, m: every point of the succession is obtained from the preceding by an equal translation, rotated at every step by an equal angle. These equal angles of rotation of the elementary first step corresponds to equal angles of true anomaly described by the body in its orbit around the Sun in the corresponding (unequal) times. If the elementary rotation is expressed as the n-th part of a round angle, the resulting figure will be a regular polygon with n sides.

It is clear that the rule is iterative, so starting from one vector of velocity Oj, the direction of the corresponding vector of position SJ and the relative change of velocity occurring in the first n-th part of true anomaly, the full diagram of velocity may be constructed. In general, the center C of the polygon of velocities will be different from the common origin O of the vectors of velocity, the two points coinciding if and only if the vectors of velocity have all equal length. As the number of divisions of the round angle is taken bigger and bigger, the (irregular) polygon approximating the orbit approaches without limit the curve orbit whose shape we're looking for, while the regular polygon of the velocities approaches without limit a circle of center C, traced by straight lines originating from an eccentric point O.

Feynman now considers this limit and draws the continuous diagram of velocities of a planet orbiting around the Sun (Fig. 3.5).The line Op represents the velocity of the planet when its true anomaly is θ, i.e. the angle that p describes on the circle with respect to its center C, measured starting from the point of

[13] Also on this delicate point Feynman goes over rapidly: "I now draw on this diagram a little line to represent the velocities. Unlike the other diagram, those lines are not the complete line from J to K, for in that diagram those were proportional to the velocities, for the times were equal, and the length divided by equal times represented the velocities. But here I must use some other scale to represent how far the particle would have gone in a given unit of time, rather than in the times which are, in fact, proportional to the square of the distance."

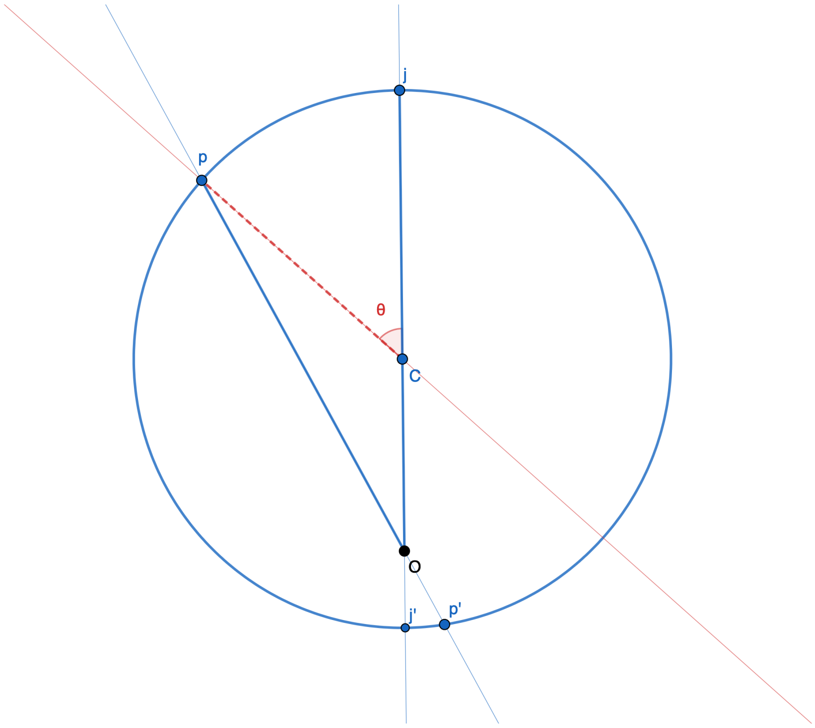

Figure 3.5: Diagram of velocities of a planet orbiting around the Sun. The line Op represents the velocity of the planet when its true anomaly is θ, measured starting from the point of maximum velocity. The orbit is not represented.

maximum velocity j.

To achieve the goal of the lecture, it remains to determine the shape of an orbit corresponding to such an angular distribution of velocities, i.e. the curve having at all his points the tangents parallel to the lines that trace the circle of velocities. Feynman reviews the state of affairs at this point of the proof:

> So here is the problem, here's what we have discovered: that if we draw a circle and take an off-center point, then take an angle in the orbit—any angle you want in the orbit—and draw the corresponding angle inside this constructed circle and draw a line from the eccentric point, then this line will be the direction of the tangent. Because the velocity is evidently the direction of motion at the moment and is in the direction of the tangent to the curve. So our problem is to find the curve such that if we draw a point from an eccentric center, the direction of the tangent of that curve will always be parallel to that when the angle of the curve is given by the angle in the center of that circle.

Now Feynman draws again the diagram of velocities, but rotates it by a right angle, so that the line Op from the eccentric point O to the circumference of the circle is now to be regarded as *perpendicular* to the the tangent at the position on the orbit corresponding to the true anomaly θ. Such a rotation doesn't change anything about the meaning of the diagram, but is instrumental to match the construction of the ellipse that Feynman purposedly set up at the beginning of the lecture (Fig. 3.6). A little before Feynman had closed his preliminaries about the ellipse with these words:

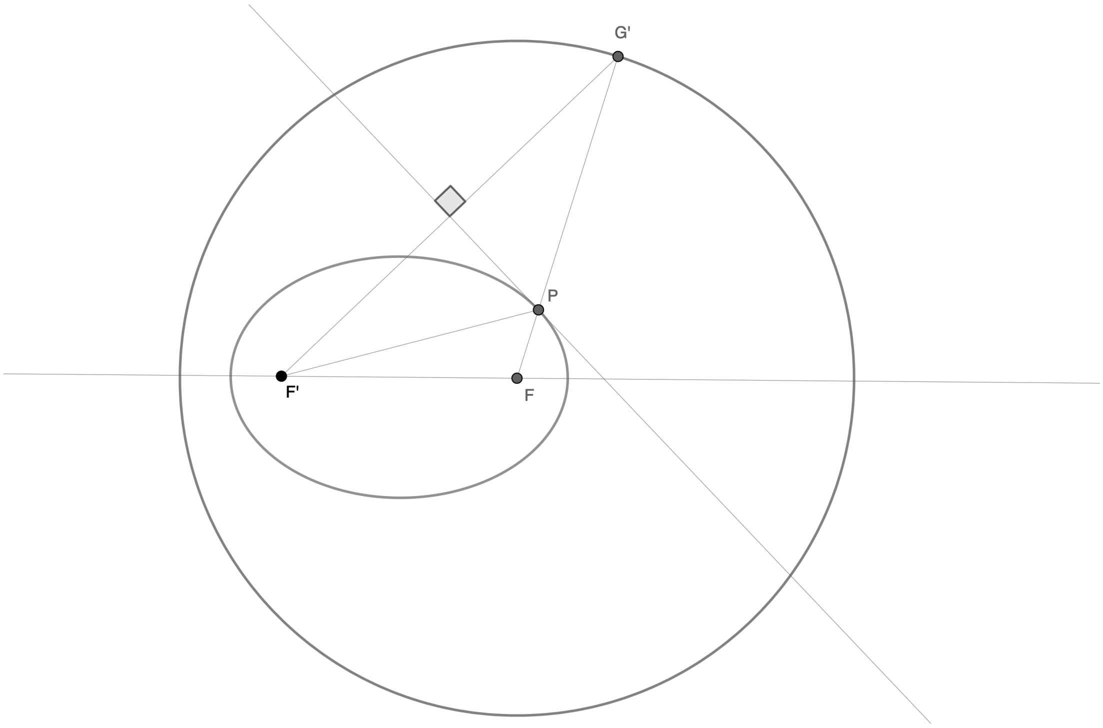

Figure 3.6: Construction of an ellipse by the director circle, showed by Feynman at the beginning of the lecture. As G' describes a circle of center F, the point P describes an ellipse with focii F' and F and major axis equal the half the radius of the circle. The tangent to the ellipse in P is the perpendicular bisector of $F'G'$.

> I just want to summarize that, to remind you of a property of an ellipse, which is this: that as a point G' goes around a circle, a line drawn from an eccentric point to this point G'—this is an off-center point to the point G'—will always be perpendicular to the tangent of the ellipse. Or the other way around: the tangent is always perpendicular to the line—or a line—drawn from an eccentric point. All right, that's all, [and] we'll come back to it and we'll remember...

The circle in Fig. 3.6 is called *director circle* for the ellipse drawn in its interior with the described construction. If one interprets the circle of velocities as a director circle, the ellipse constructed with the same method will have as focii the points O and C. The identity between the constructions shown in Figg. 3.6 and 3.7 proves that the orbit having the correct angular distribution of tangents is, indeed, an ellipse. Therefore, an ellipse constructed as in Fig. 3.7 is a possible orbit according to Newton's dynamics, and the planet in this case describes an ellipse similar to that described inside its diagram of velocities by the point P. The proof is complete: *elementary, but difficult.*

Notice that an identical procedure gives also the *open* possible orbits, namely the parabola and hyperbola, provided the eccentric point origin of the velocity vectors is situated *on* or *outside* the velocity circle.[14]

[14] Indeed, Feynman says that he borrowed the idea of the velocity circle from U. Fano and L. Fano (1959), where it is used to deduce the formula of Rutherford scattering.

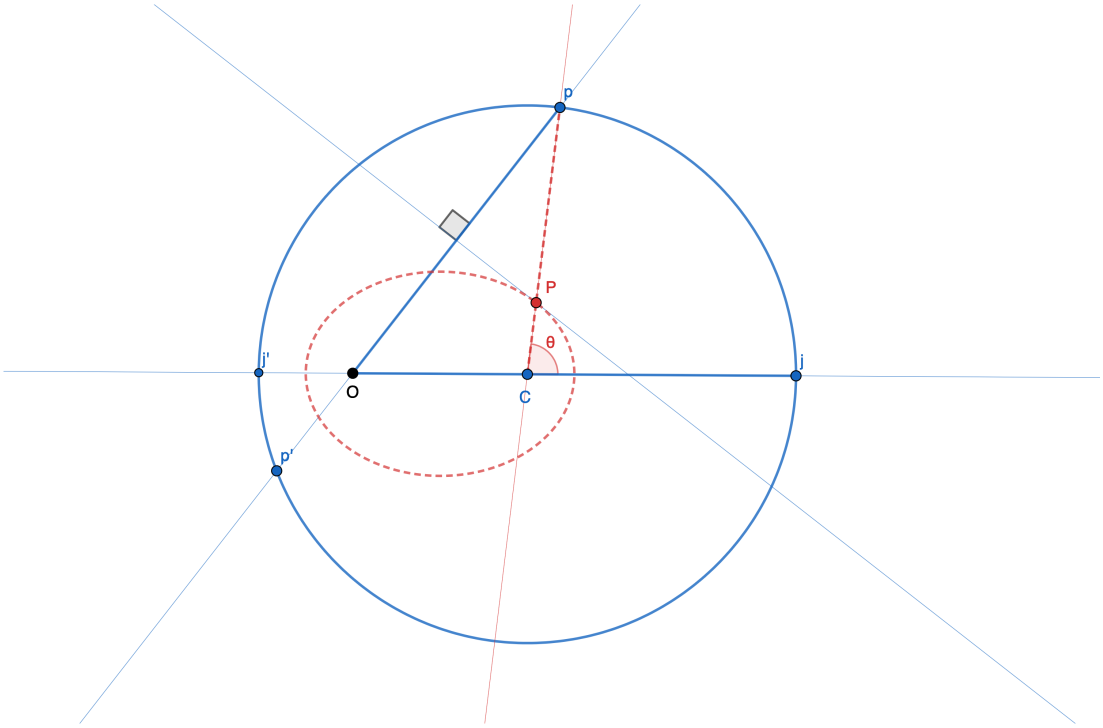

Figure 3.7: Construction of the orbit of a planet corresponding to a circular diagram of velocities of center C and origin O inside the circle. The orbit is the ellipse described by the point P, whose velocity at the true anomaly θ is equal in length and perpendicular to Op.

So, to summarize, Feynman's diagram provides a geometrical construction of the general solution of the two-body problem, i.e. of the whole pattern of relative positions and velocities of the moving body that satisfies the constraints posed by the general laws of dynamics and, in particular, by the inverse-square law of gravity. Every solution is obtained by an identical procedure and is uniquely determined by the relative positions of the centre of the velocity circle and of the eccentric point which is the origin of the straight lines representing in magnitude and direction the motion of the body at each position along its path.

Notice that the diagrams shown in Figg. 3.6 and 3.7 are identical except for colors, to emphasize the fact that the only difference between the two is *semantic*, i.e. they differ only in the *meaning* attributed to the parts of one and the same geometrical construction.[15] This fact is indeed a key feature of Newton's *Principia*, the explicit inspiration of Feynman's lecture. In Newton's original work, apart from the mathematical method he employs, the *essential* point is that lines, angles, areas and other parts of the geometrical diagrams accompanying the propositions (in this case, the construction of an ellipse from its director circle) are given a *dynamical* or *physical* interpretation as representing the *ratios* between the successive positions and velocities of a real body under the assumed conditions (in this case, those of a planet moving under the gravitational pull of the Sun). More generally, cleaning off the metaphysical dust, Feynman's lectures exhibits very

[15]From the audio recording of the lecture it appears that Feynman himself used colors to make diagrams "more interesting".

clearly that Newton's *dynamics* may be regarded as a set of *rules* (i.e. the *laws of motion*) giving instructions on how to *analyze* and *construct* geometrical diagrams that are *meant* to represent *by analogy* the reciprocal motions of bodies.[16]

3.2 The *Law of the Circular Hodograph*

Hamilton's paper on the hodograph method (Hamilton 1847b) begins like this:

> Whatever may be the complication of the accelerating forces which act on any moving body, regarded as a moving point, and, therefore, however complex may be its *orbit*, we may always imagine a succession of straight lines, or vectors, to be drawn from some one point, as from a common origin, in such a manner as to represent, by their directions and lengths, the varying directions and degrees (or quantities) of the velocity of the moving point: and the curve which is the locus of the ends of the straight lines so drawn may be called the *hodograph* of the body, or of its motion, by a combination of the two Greek words, οδος, a *way*, and γραφω, to *write* or *describe*; because the vector of this hodograph, which may also be said to be the *vector of velocity* of the body, and which is always parallel to the tangent at the corresponding point of the orbit, marks out or indicates at once the direction of the momentary path or way in which the body is moving, and the rapidity with which the body, at that moment, is moving in that path or way.

In one single long paragraph Hamilton gives both the geometrical definition and the physical meaning of this new curve, the *hodograph*, intended to describe the *motion* of a point along its path, whatever this may be. Consider a fixed point O and another point P moving relatively to it and tracing a curve $\Gamma(s)$, called the orbit of P relatively to O. Then, by definition, the hodograph of P is another curve, $\gamma(s)$, simultaneously traced by another point p, conceived to be the moving extremity of a straight line op drawn from an arbitrary but fixed origin o. The correspondence between orbit and hodograph of a moving point P is given by the postulate that the line $op(s)$, called the *vector of the hodograph* or *vector of velocity* of P, must be drawn parallel to the tangent at $P = \Gamma(s)$, and with a length proportional to the instantaneous speed of P. Thus the line op represents in direction and magnitude the momentary state of motion of the point P along its orbit.

Notice that Hamilton treats velocity as an *autonomous* and *primary* magnitude, on the same footing as position and not derived from it.[17] Moreover, the hodograph

[16] This is, I think, what Feynman suggests when, after the setting up of the construction shown in Fig. 3.6 says: "That's the ellipse. On the other hand, we have to learn dynamics, we have to put them together. So now we have to explain what dynamics is all about. I want this proposition, that's the *geometry*; now the *mechanics*, what this proposition *means*. What Newton means by this is this: ...", and goes on with the proof of Proposition 1 of the *Principia*, the cornerstone of Newton's dynamics. This seems to imply that, according to Feynman, Newton's *mechanics* provides a *meaning* to well-definite geometrical constructions.

[17] A similar move was made later by Einstein, who set out his relativistic kinematics starting from the *velocity of light* as a *primary* magnitude, and *then* defining *space* and *time* intervals in

is defined kinematically, so that the *vector of the hodograph* (from the latin *vehere*, to *transport* or to *carry*) is a directed straight line that literally *draws* the curve.[18] Since this and the orbit are described by assumption *simultaneously to each other*, there holds a one-to-one correspondence between the succession of position-points forming the orbit and the succession of velocity-points forming the hodograph. Therefore, when in the following we say *motion* we mean the *simultaneous progression* of such a couple of *corresponding points.*

The first property of the hodograph that Hamilton highlights is the fact that it exhibits more directly than the orbit the properties of the forces acting on the moving body:

> This hodographic curve is even more immediately connected than the orbit, with the *forces* which act upon the body, or with the varying resultant of those forces, for the tangent to the hodograph is always parallel to the direction of this resultant; and if the element of the hodograph be divided by the element of the time, the quotient of this division represents (to the usual units) the intensity of the same resultant force; so that the whole accelerating force which acts on the body at any one instant is represented, both in direction and in magnitude, by the element of the hodograph, divided by the element of the time. We have also the general proportion, that *the force is to the velocity*, in any varied motion of a point, *as the element of the hodograph is to the corresponding element of the orbit.*

By definition, in any interval of the motion the mean velocity of p along the hodograph is the same as the mean acceleration of P along the corresponding interval on the orbit, and the same is true in the limit of small intervals (i.e. for the *instantaneous* values of the same magnitudes). Therefore, the principle of dynamics translates into the *hypothesis* that for any interval of motion a definite *proportion* holds between, on one side, the ratio between the *physical* magnitudes describing the motion, i.e. its accelerating force and its velocity, and, on the other, the ratio between the *geometrical* magnitudes that *by analogy* represent them, i.e. the curvilinear *lengths* simultaneously described by the two points p and P. Notice that the passing reference to *time* is just incidental, and that the proportion holds whatever time interval is considered, the only requirement being that it must be *finite*. Moreover, the role of *time* may be played by whatever *continuous parameter* that allows to track the evolution of the binary system.

Next, Hamilton restricts his attention to the case in which the force acting on the body is central, i.e. to the assumption of *rotational symmetry*:

> These general remarks respecting varied motion, under the influence of *any* accelerating forces whatever, having been premised, let it be now supposed that the force is constantly directed towards some one *fixed point* or *centre*, which it will then be natural to choose for the origin of the vectors of the hodograph. The straight lines drawn to the moving body from the centre of

terms of it. I thank Aimeric Colléaux for calling my attention on this important point, many years ago.

[18] It is worth noting here that exactly in *this* sense Winter (2007) translated *vector* the Greek φορά in the *Mechanical Problems*. See above.

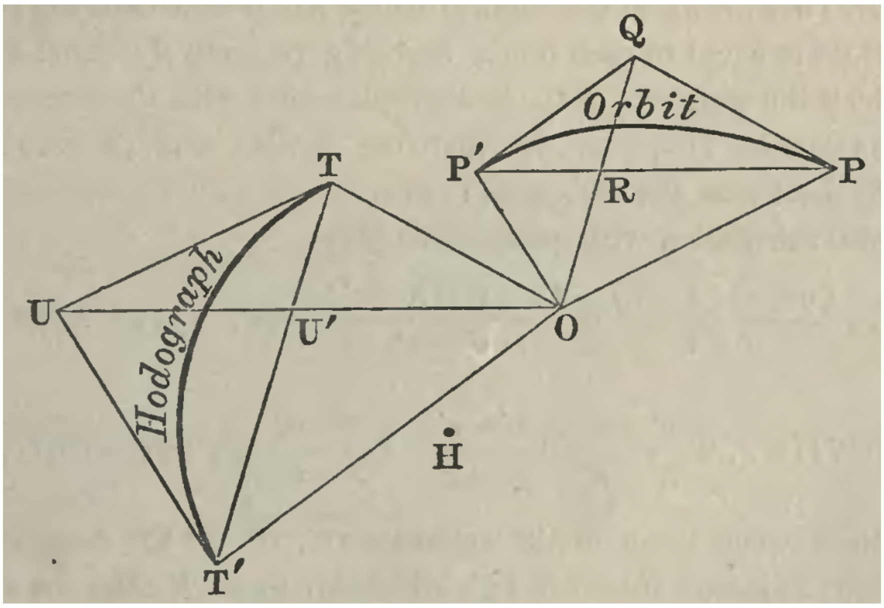

Figure 3.8: Corresponding vectors of orbit and hodograph in the case of central force. O is the fixed center of force, OP and OP' are successive vectors of the orbit, OT and OT' are the corresponding vectors of the hodograph. PP' represents the mean velocity of the moving point, TT' its mean acceleration. H is the center of curvature of the hodograph (source: Hamilton 1866, p. 727).

> force being called, in like manner, the vectors of the orbit, or the *vectors of position* of the body, we see that each such vector of position is now parallel to the tangent of the hodograph drawn at the extremity of the vector of velocity, as the latter vector was seen to be parallel to the tangent of the orbit, drawn at the extremity of the vector of position; so that the two vectors, and the two tangents drawn at their extremities, enclose at each moment a *parallelogram*, of which it is easily seen that the *plane and area are constant*, although its position and its shape are generally variable from one moment to another, in the motion thus performed under the influence of a central force. If, therefore, this constant area be given, and if either the hodograph or the orbit be known, the other of these two curves can be deduced, by a simple and uniform process, of which account the two curves themselves may be called *reciprocal hodographs*.[19]

Hamilton's construction is shown in Fig. 3.8. The hypothesis of central force and Kepler's area law are translate into a *constraint* on the simultaneous descriptions of orbit and hodograph. As Hamilton remarks, the constancy of the plane and area of the parallelogram formed by the corresponding vectors of orbit and hodograph allows in particular to construct one curve given the other and any pair of corresponding vectors, making the two curves *reciprocal* to each other. The

[19] Here the reader familiar with hamiltonian mechanics won't fail to recognize a method that today has become standard, namely that of using the symmetries of the problem at hand and their translation via Nöther theorems into conservation laws to restrict the domain of the phase space accessible to the system. If a sufficient number of symmetries is assumed, the problem may be completely integrated. The Kepler problem, as well known, is *super-integrable*, its symmetries being even more numerous than what is strictly necessary for its solution.

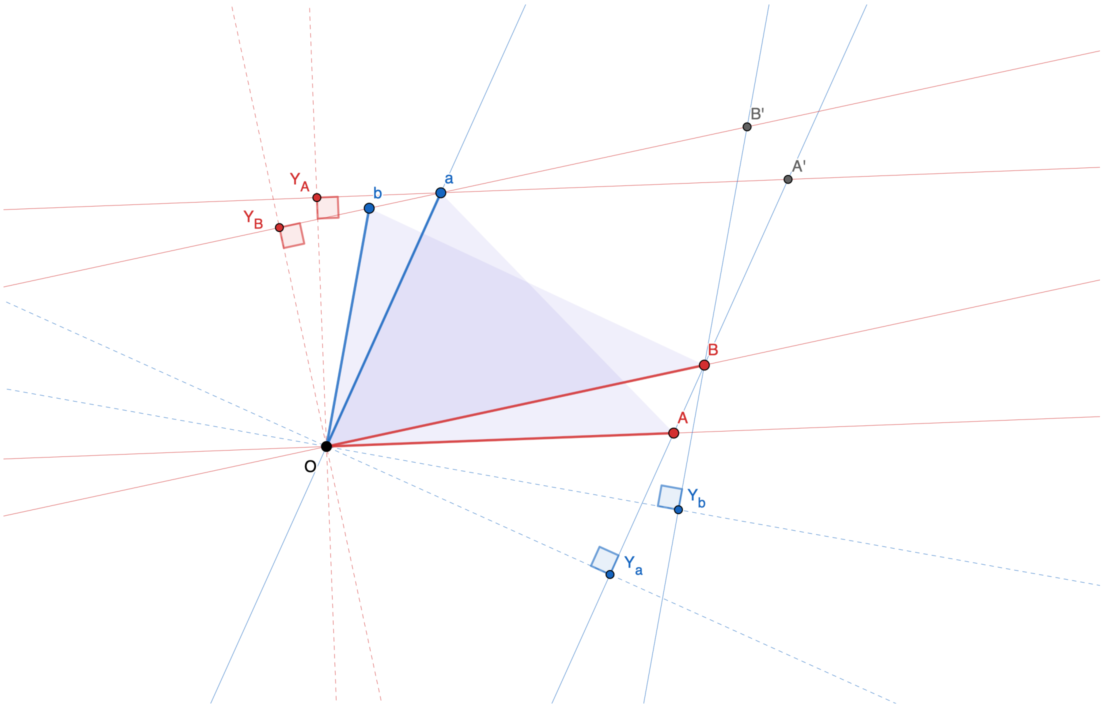

Figure 3.9: Construction of the orbit, given two corresponding vectors and the successive vector of the hodograph. The areas in purple are all equal.

key ingredient of the construction is the same as Newton's and Feynman's proof of the same result, i.e. or the equivalence of parallelograms constructed on the same base and between parallel lines.[20]

To *see* how this comes about, refer to Fig. 3.9. Let's start from the given vectors OA, Oa and Ob, i.e. one vector of the orbit and two successive vectors of the hodograph. Oa is the velocity of the point P when it is at A, and Ob is the velocity it must have at the successive point of the orbit B; ab is the change of velocity that occurs at B. The point B is determined by the dynamical constraint that OB and Ob must enclose a parallelogram of the same area as that enclosed by OA and Oa or, what is the same, that the triangles OAa and OBb must be equal. B is found simply by drawing the parallel to ab through O and intersecting the parallel to Oa through A. The triangles OAa and OBa are equal because they have the same base OA and the same altitude OY_A, and OBa is equal to OBb because they have the same base OB and the same altitude OY_B; therefore, $OAa = OBa = OBb$. Given the successive vectors of the hodograph Oc, Od, the procedure may be iterated and the full orbit constructed starting from the hodograph. An identical procedure allows to construct the hodograph given a succession of points $A, B, C...$ on the orbit (Fig. 3.10).

To summarize, given one of the two *reciprocal hodographs* and any pair of corresponding vectors, the other curve can be drawn constructing a succession of parallelograms equivalent to the one formed by the given pair. Starting from this latter, every parallelogram $OPP'p$ of the succession is obtained from the

[20]This is nothing but Euclid, I.35, the proposition that aroused the special admiration of Pappus in Hamilton's juvenile *Waking Dream*.

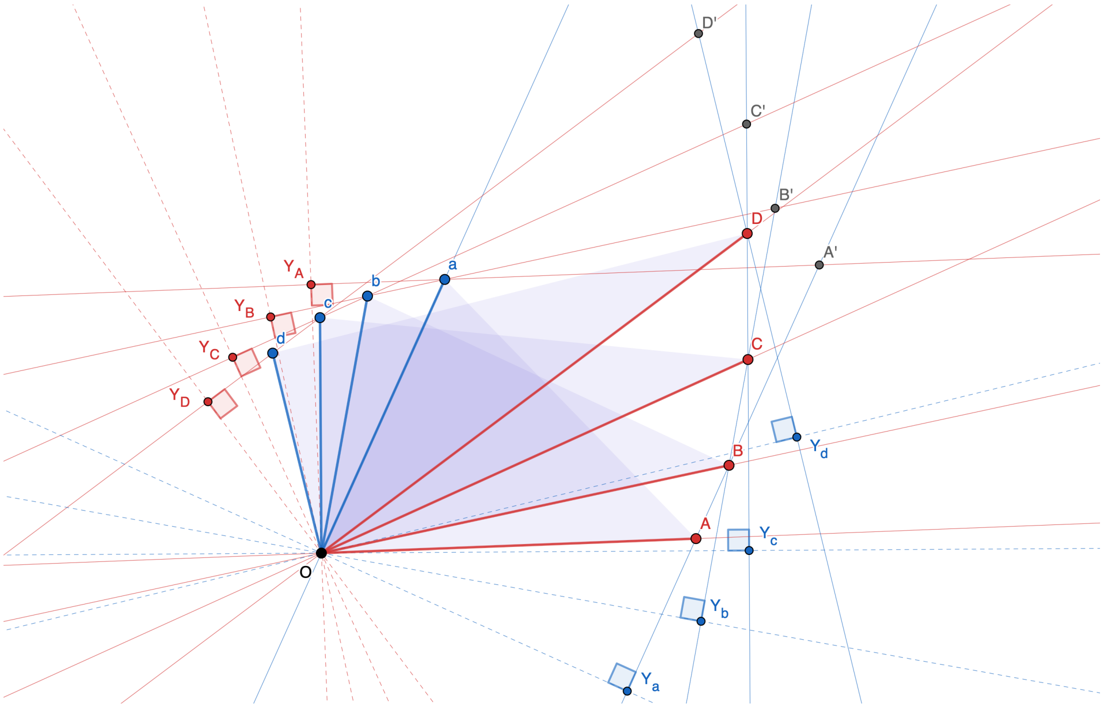

Figure 3.10: Construction of the orbit, given two corresponding vectors and a succession of vectors of the hodograph. The areas in purple are all equal.

preceding keeping the vertex O fixed and sliding alternatively the two adjacent and opposite vertexes P and p along a line parallel to the other side, so that the area is preserved at every step. Notice that the construction is the same, whatever may be the spatial and temporal distribution of the given points on the orbit or on the hodograph, and whichever of the two reciprocal curves is given.

Now it comes the crucial step of Hamilton's analysis. We have alluded to the role that curvature plays in Newton's dynamics, though this is not so apparent from Newton's exposition in the *Principia*. However, this aspect must have been perfectly clear to Hamilton, who indeed devoted much work to the problems of curvature throughout his career.[21] Here, Hamilton chooses to analyze the curvature properties of the hodograph, approximating it with its osculating circle:

> The opposite angles of a parallelogram being equal, it is evident, that if the central force be attractive, any one vector of position is inclined to the next following element of the orbit as the same angle as that at which the corresponding vector of velocity is inclined to the next preceding element of the hodograph. Also, if from either extremity of any small element of the curve, a chord of the circle which osculates to that curve along that element be drawn and bisected, the element subtends, at the middle point of this chord, an angle equal to the angle between the two tangents drawn at the two extremities of the element; that is, here, if the curve be the hodograph,

[21]Significantly, Hamilton's mathematical career opened and closed with problems of curvature. His first paper, as a teenager, was *On Caustics*, and the last one, published posthumous in 1867 in the usual *Proceedings of the Royal Irish Academy*, was *On a New System of Two General Equations of Curvature.*

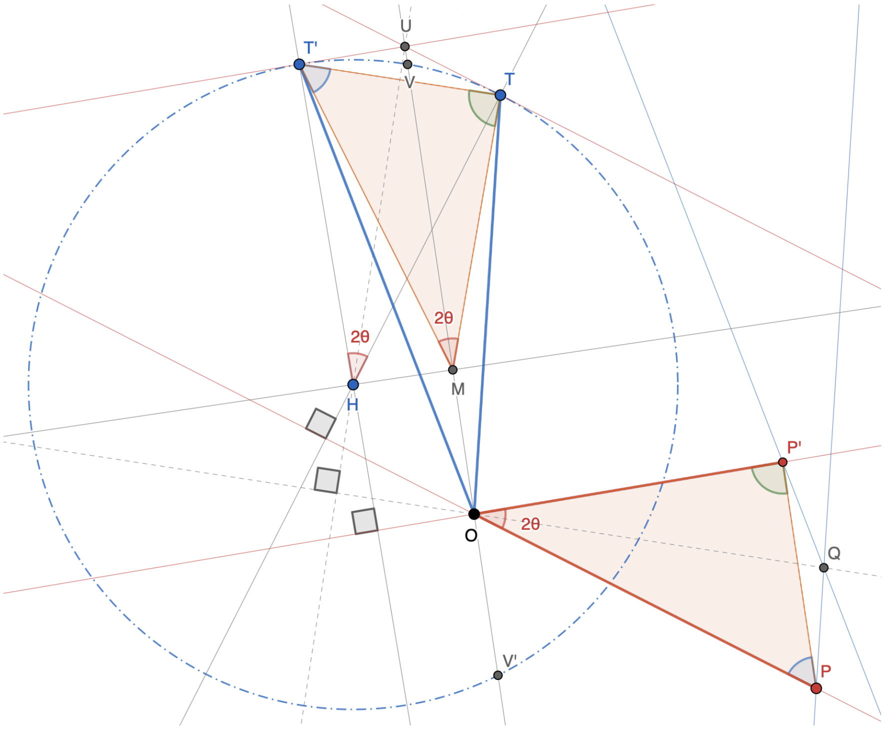

Figure 3.11: Approximation of the hodograph with its circle of curvature. The orange triangles MTT' and OPP' are similar.

> to the angle between the two near vectors of position, which are parallel to the two extreme tangents of its element.

In Hamilton's (and Newton's) kinematical approach to approximate a motion with its osculating circle in a certain interval means to replace it with another motion that is circular and uniform around a point assumed stationary in the considered interval. In other terms, with respect to this point, the momentary *center of curvature*, the approximating motion changes its direction but not its magnitude. Here, Hamilton's wrinkle is to apply such circular approximation to the corresponding motions on the hodograph and on the orbit.

To visualize his construction, refer to Fig. 3.11. The blue circle centered on H osculates the hodograph between the points T and T', separated by an angle 2θ. The point O, which can be anywhere, is the center of force and OT, OT' are the vectors of the hodograph at the extremities of the interval. The hodographic point enters the interval TT' at the point T, with direction perpendicular to HT; after an angle θ its direction is perpendicular to HU; and after an angle 2θ it leaves the circle from T' with direction perpendicular to HT'.

To construct the vectors of the orbit corresponding to such hodographic motion, draw OP perpendicular to HT and with arbitrary length and the perpendicular to HT' through O. Then draw through P the tangent to the orbit, i.e. a line parallel to OT. Draw the perpendicular to HU through O, which intersects this latter tangent in Q. Through this Q draw the parallel to OT'. This meets the perpendicular to HT' in the point P', which will be the point of the orbit corresponding to T'. Next, draw the line through O and U, meeting the osculating

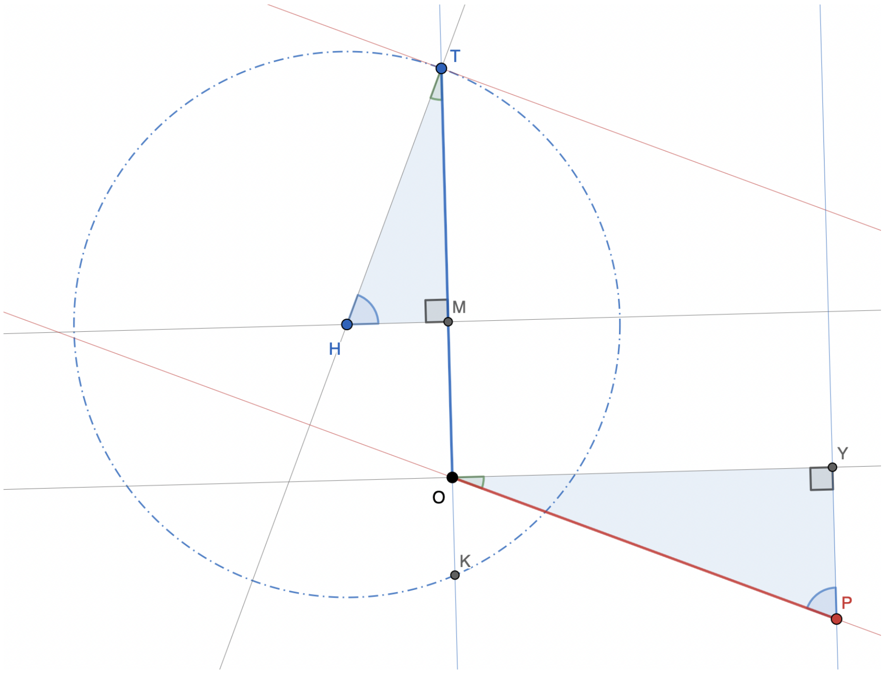

Figure 3.12: The triangles HMT and OPY are similar.

circle in V and V', and bisect it at M. The orange triangle MTT' thus obtained is similar to the triangle OPP', and they remain so also in the limit when T' approaches T:

> We have, therefore, two small and similar triangles, from which results the following proportion, that *the half chord of curvature of the hodograph* (passing through, or tending towards the fixed center of force) [MT] *is to the radius vector of the orbit* [OP] as the element of the hodograph [TT'] is to the element of the orbit [PP'], that is, by what was lately seen, *as the force is to the velocity.*

Moreover, it holds this other general proportion (see Fig. 3.12):

> But also, the radius of curvature of the hodograph [HT] is to the half chord of curvature of the same curve [MT], as the radius vector of the orbit [OP] is to the perpendicular let fall from the fixed center on the tangent to the same orbit [OY];

So, compounding the two proportions:

> therefore, by compounding equal ratios, we obtain this other proportion: *the radius of curvature of the hodograph [HT] is to the radius vector of the orbit [OP], as the rectangle under the same radius vector and the force [OP.TT'] is to the rectangle under the velocity and the perpendicular* [PP'.OY], or to the constant parallelogram under the vectors of position and velocity.

And the final step:

> If, therefore, the law of the inverse square hold good, so that the second and third terms of this last proportion vary inversely as each other, while the fourth term remains unchanged, the first term must be also constant; that is, *with Newton's law of force* (supposed here to act towards a fixed centre) *the curvature of the hodograph is constant:* and, consequently, this curve, having been already seen to be *plane*, is now perceived to be a *circle*, of which the radius is equal to the attracting mass divided by the constant double areal velocity of the orbit. Reciprocally, we see that *no other law* of force would conduct to the same result: so that the Newtonian law may be *characterized* as being the *Law of the Circular Hodograph.*

So, the hodograph is a circle *if and only if* the inverse square law holds. It is noteworthy that Hamilton deduced this result by a geometrical analysis of the *intrinsic* proportions between the two reciprocal curves, i.e. in a way that is independent, in particular, from any consideration of Newtonian *time.*

Immediately Hamilton highlights a fundamental feature of the circular hodograph, which makes it a tool especially useful for astronomical applications:

> The point on the hodograph which is the termination of any one vector of velocity may be called the *hodographic representative* of the moving body, and the foregoing principles show, that with a central force varying as the inverse square of the distance, this representative point describes, in any proposed interval of time, a *circular arc*, which contains the same number of degrees, minutes and seconds, as the angle contemporaneously described round the centre of force by the body itself in its orbit, or by the revolving vector of position; because, whatever that angle may be, an equal angle is described in the same time by the revolving tangent to the hodograph. Thus, with the law of Newton, *the angular motion of a body in its orbit is exactly represented, with all its variations, by the circular motion on the hodograph*; and this remarkable result may be accepted, perhaps, as an additional motive for the use of the new term which it is here proposed to introduce.

In other terms, the circular motion of the hodographic point *extracts* from the motion in space the only component which is available to direct observation from the center of attraction, i.e. the center of the rotational symmetry of the problem, *whatever may be the actual trajectory.*

In the rest of the paper Hamilton deduces other important results from the hodographic representation, and, in particular, the shape of the orbits. Before that, he defines the *vector of eccentricity* as follows:

> Whichever of these situations the centre of force may have [inside, on or outside the circular hodograph], we may call the straight line drawn from it to the centre of the hodograph, the hodographic *vector of eccentricity*; and the *number* which expresses the ratio of the length of this vector to the radius of the hodograph will represent, if the orbit be closed, the ratio of the semidifference to the semisum of the two extreme distances of the body from the centre of force, and may be called generally the *numerical eccentricity* of the hodograph, or of the orbit (without violating the received meaning of the term).

This is instrumental to what comes next:

> Whatever the value of this numerical eccentricity may be, the constant area of the parallelogram under the vectors of position and velocity may always be treated as the sum or difference of two other parallelograms, of which one is equal to the rectangle under the constant radius of the hodographic circle and the varying radius vector of the orbit, while the other is equal to the parallelogram under the vectors of position and eccentricity; and hence it is not difficult to infer that the length of the vector of position, or of the radius vector of the orbit, varies in a constant ratio, expressed by the numerical eccentricity, to the perpendicular let fall from its extremity, that is, from the position of the body, on a constant straight line or *directrix*, which is situated in the plane of the orbit, and is parallel to the hodographic vector of eccentricity. The *orbit*, therefore, whether it be closed or not, is always (with the law of the inverse square) a *conic section*, having the centre of force for a *focus* - a theorem which has indeed been known since the time of Newton, but has not perhaps been proved before from principles so very elementary.

At the end of the paper, Hamilton outlines the application of the method to a binary and multiple system of *mutually* attracting bodies.

Hamilton's construction in the case of a closed orbit is shown in Fig. 3.13, where, inspired by Feynman, we have replaced the two directrix *lines* with one single director *circle*, which corresponds to the hodograph rotated anti-clockwise by a right angle around the eccentric point.

The motion is in the plane of the paper. The hodograph is the blue circle, conceived as drawn by the blue line OT, and the orbit is the red ellipse, conceived drawn by the red line OP. OT and OP are the corresponding vectors of position and velocity, respectively parallel to PQ and TQ and forming the purple parallelogram of constant area representing the double areal velocity of the moving body or angular momentum. The green line OH is the constant vector of eccentricity, perpendicular to the apsidal line of the orbit connecting the points of maximum (minimum) and minimum (maximum) velocity (distance). The angle θ is the true anomaly, here reckoned starting from the point of minimum velocity and maximum distance.

Notice, in particular, the following points:

1. The vector of velocity is decomposed in two parts: one constant in both direction and magnitude (i.e. the constant vector of eccentricity), and the other constant in magnitude but varying direction (i.e. the rotating radius of the hodograph).

2. The line OY is the perpendicular from the center of force to the line tangent the orbit in P. It is equal in length to OT' and, by Euclid I.35, forms with the vector of the hodograph OT a rectangle equivalent to the parallelogram between distance and velocity. By Euclid III.35 all the rectangles formed by the segments in which the chord of the hodograph passing through the eccentric point is cut are equal, so the conservation of angular momentum and the constancy of areal velocity are expressed by the constancy of the product between *opposite velocities*.

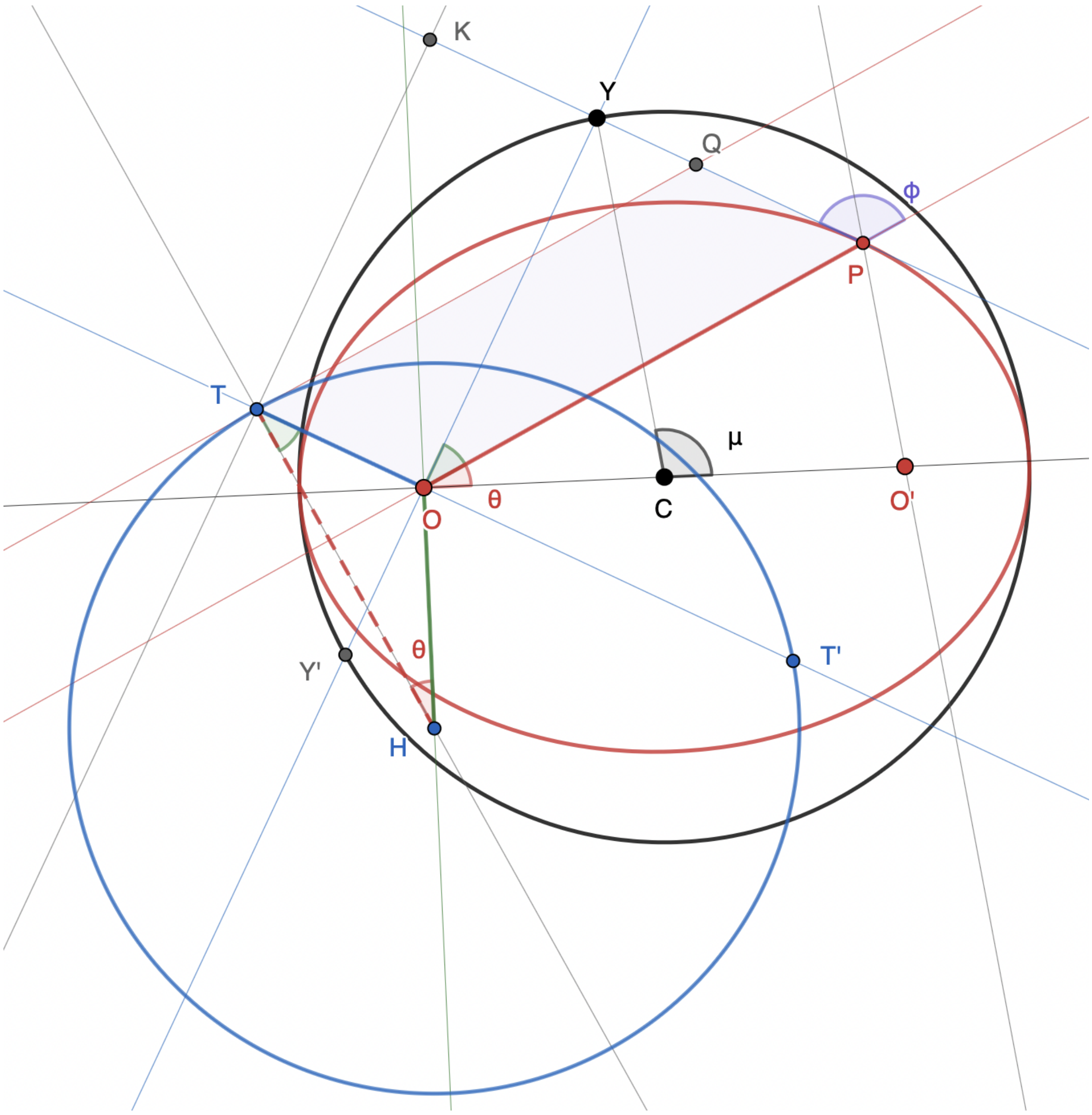

Figure 3.13: Construction of the elliptical orbit from the circular hodograph.

3. The director circle, if regarded as traced by CY, is *also* the zero-eccentricity solution of the problem. In this case, O coalesce with C and the motion is circular with constant speed around this point. Therefore, the angle μ formed by CY with the apsidal line is *also* the so-called *mean anomaly*, and the motion of the point Y is the same as the *mean motion* of the orbiting body. If our interpretation of Hamilton's words is correct, the construction shown in Fig. 3.13 is that alluded to in this passage of the *Elements*:

   > Some not inelegant constructions, deduced from the theory of the hodograph, might be assigned for the case of a *closed orbit*, to represent the *excentric* and *mean anomalies*...[22]

   As a result, if you set in motion the construction moving at a constant angular velocity the point Y, the point P will describe the elliptical orbit in accord with Kepler's area law, and the ratio between the area swept by OP and the area swept by CY will be equal to the ratio between *true* and *mean* time.[23]

So, to summarize, the key idea of Hamilton's hodograph method is to frame the problem of motion in terms of the geometrical analysis of a new curve, the *hodograph*, which is meant to describe the kinematical relationship between two bodies, one of which is regarded as fixed. This curve is put in correspondence with the curve that describes their spatial relationship, the *orbit* or *path* of the moving body, by the hypothesis that the two are traced simultaneously by the moving extremity of two straight lines representing the relative position and velocity of the two bodies in relative motion. In the maximally symmetric case in which the hodograph is a *circle*, the orbit is found to be a conic section having a focus in an eccenter point of the hodograph, and the two curves may be constructed explicitly one from the other starting from any two corresponding values of position and velocity. Moreover, for any considered interval of the motion, the circular arc traced by the hodographic point coincides with the angular displacement of the body on its conical orbit as observed by the other, or, what is the same, the angle at the center of the hodographic circle identifies with the *true anomaly* of the orbiting body. The deviation from uniform circular motion is measured by the eccentricity vector, a straight line parallel to the direction of maximum/minimum velocity, that with its direction and magnitude *characterizes* the particular solution to the two-body problem.

[22] Hamilton 1866, p. 731.

[23] This may be easily done with a software of *dynamic geometry* like GEOGEBRA, which indeed I used extensively in the preparation of this dissertation. All the geometric diagrams, if not otherwise stated, have been made with it.

Chapter 4

A New Look at the Antikythera Mechanism

Consider again Hipparchus' diagram, shown in Fig. 4.1 in the eccentric version, and compare it to Fig. 4.2.

If we interpret the angle at the center of Hipparchus' diagram as the *true anomaly*, and the line from the moving point to the eccenter as the vector of velocity of the moving body, we obtain an exact replica of Hamilton's diagram. In other terms, Hipparchus' diagrams provide, on one hand, a *method of construction* for the ellipse, its tangents and normals (and, in the general case, of all the conic sections) and, on the other, a complete graphical solution to the so called Kepler problem in the context of Newtonian dynamics. Since Hipparchus' and Hamilton's diagram are meant to solve one and the same problem, i.e. that of geometrically representing the observable motions of the heavenly bodies, we find it hard to believe that such a perfect match is the result of a mere coincidence.

If one accepts the idea that the Greeks conceived something like Hamilton's method of the hodograph, the *hypothesis* of a circular hodograph in the context of astronomy would be just natural, and all the rest would follow. In other terms, just like the *law of inverse square* has the status of *hypothesis* in Newton's theory, so the *law of the circular hodograph* could be the fundamental *hypothesis* of the dynamical theory that, if one accepts Russo's reconstruction, Hipparchus developed in his later years. We have already remembered that Hipparchus also solved the problem of falling bodies and projectile motions. It is an easy exercise to see that in this latter case (and, more generally, in presence of a *constant* force) the hodograph turns out to be a *straight line* traced with uniform velocity. In other words, it holds the remarkable circumstance, at least from the Greeks' perspective, that the method of the hodograph leads to the *straight line* and the *circle* as the two *hypotheses* that save the phenomena pertaining to motion under gravity near the surface of the Earth and at astronomical distances from it.

At this point, we can state more fully our conjecture about the theory underlying Hellenistic *sphairopoiia* in general and the AM in particular.

Our proposal is that Hipparchus' diagrams were, in origin, *dynamical* diagrams of successive velocities analogous to those obtained by Hamilton through the hodographic construction of the solution to the Kepler problem to which he was

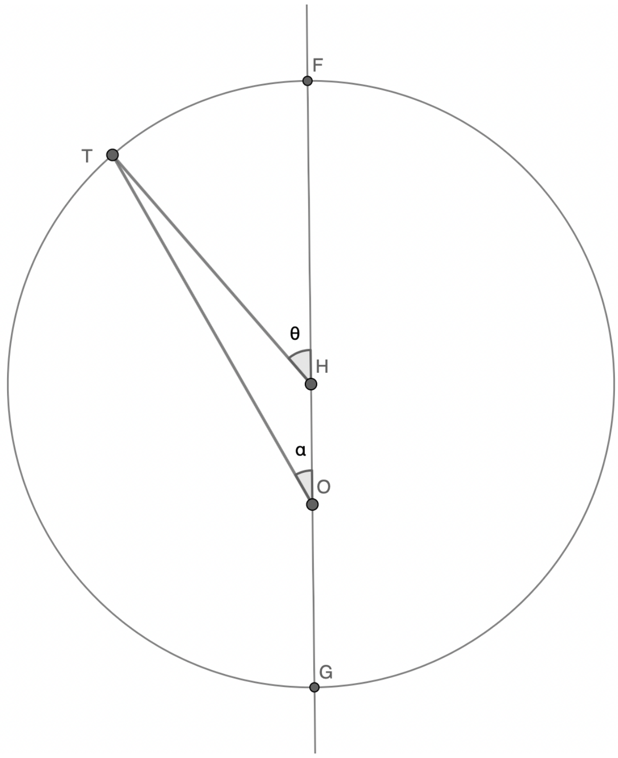

Figure 4.1: Eccentric version of Hipparchus' diagram.

led by the internal character of his method of quaternions. The *hypothesis of the circular hodograph* was the cornerstone of Hellenistic dynamical astronomy, and the theoretical basis for the design of mechanical devices that worked as *hodographic computers* internal to such theory. The AM is a highly refined example of such kind of devices, that put to profit the Babylonian astronomical expertise that during II cent. BC Greek mathematicians (notably Hipparchus') incorporated in their theoretical astronomy.

In light of the historical evidence analyzed in the first chapter, and of the route that led Hamilton to the discovery of such a beautiful and simple *geometrical picture* of the dynamical law of gravitation we outlined in the second chapter, we think that our interpretation fits reasonably well with the existing ancient and modern sources. Before concluding, here are some further arguments.

All the existing sources agree in indicating as the subject matter of astronomy the study of the *motions* of the heavenly bodies, rather than of their *positions*. This is of course a natural consequence of the simple observation that, in the skies, *everything moves at the same time*, and the very same title of Eudoxus seminal work *On Speeds* (Περι ταχεον) is significant in this regard. Eudoxus himself developed a general notion of *magnitude* which naturally applies to *motions* as well as to *lengths*, and we have seen how the *kinematical* approach became a standard in Greek geometry. In short, there is no difficulty in the idea that *diagrams of velocity* could have been considered by Hellenistic mathematicians working on astronomical problems.

In this regard, it is important to remark that relativity of motions (and the connected heliocentric hypothesis) by itself implies the idea that astronomical

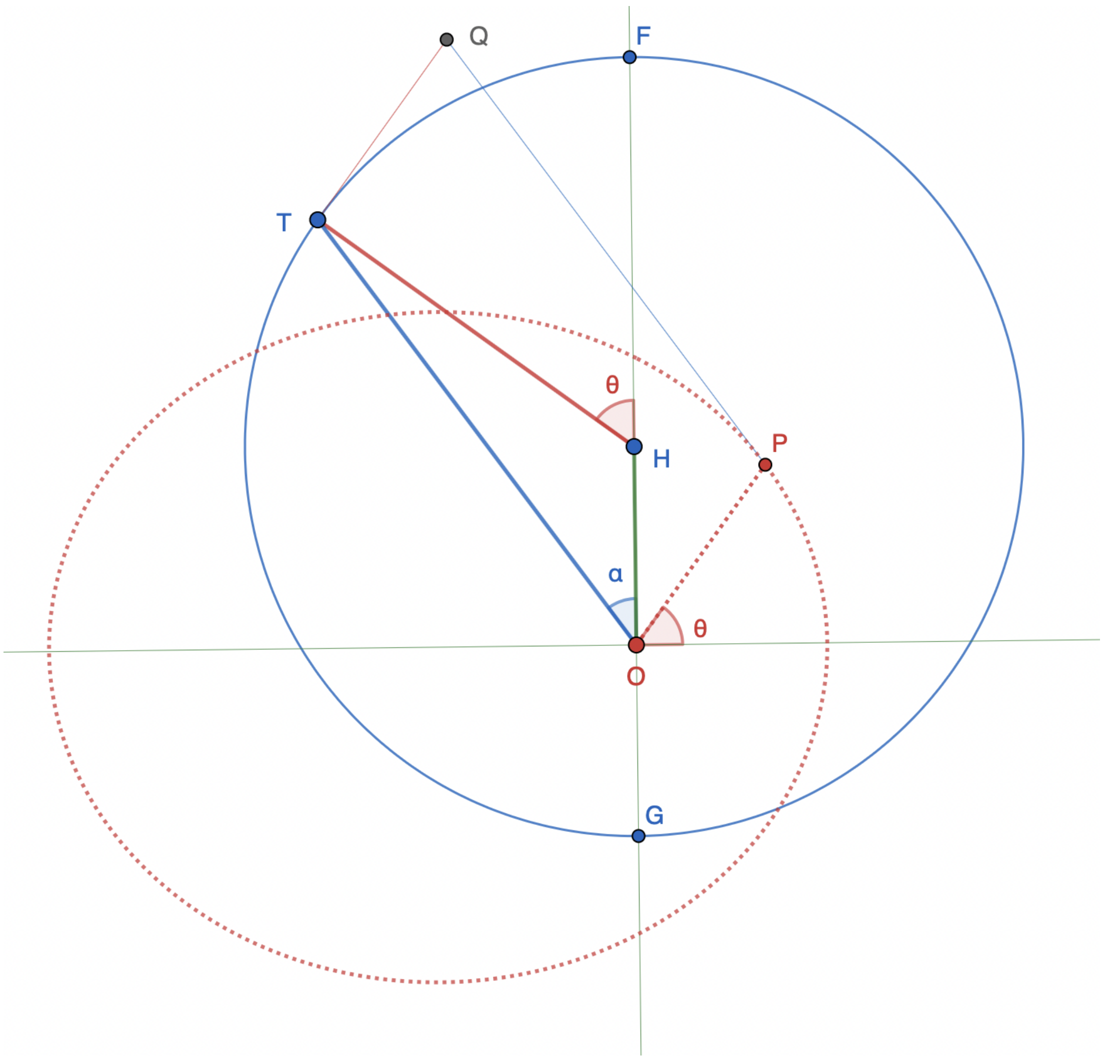

Figure 4.2: Dynamical interpretation of Hipparchus' diagram. OT and OP are corresponding vectors of hodograph and orbit for the Kepler problem. The motion of P observed by O is the same as the motion of T seen by H.

observations *may not* give any information about the *positions in space* of celestial bodies, but only about their *relative velocities.* This observation leads naturally to regard *motions* instead of *positions*, as the primary magnitude to deal with in theoretical astronomy. This is one of the key ideas of Hamilton's method of the hodograph, where, as we already remarked, *relative velocity* is treated autonomously with respect to *relative position.*

It has been often remarked that the concept of *orbit* is extraneous to ancient astronomy, being a modern addition due above all to Kepler's work.[1] In this regard, a noteworthy feature of Hamilton's method is that it exhibits very clearly the fact that, for chiefly astronomical purposes, the *explicit* construction of the orbit is superfluous. From the observations of maximum and minimum velocity and their angular location on the ecliptic one may find the eccentricity vector (in magnitude and direction), and, from this, construct the full diagram of velocities, which completely solves the problem of the observable motions. In particular, we have seen how the preserved area, playing an essential *dynamical* role, is nothing but the rectangle formed by two opposite velocity vectors. So, all in all, the actual *drawing* of the orbit in a diagram solving such astronomical problem is by no means necessary, and, indeed, a useless complication, the net effect of the hodographic construction being the *extraction* of the observable *circular* motion from the *conic* one.

We have also seen how motion was theoretically handled in the context of mechanics in the *Mechanical Problems.* Even if this work was not a rigorous mathematical treatise, the key idea is already clear: *to compare simultaneous motions and study the ratios between the lengths described in the same time.* This is one of the core ideas of Hamilton's method of the hodograph, and indeed a natural approach in astronomy, where the only thing one can do is to compare the simultaneous motions of different bodies, picking one among them and using it as a *clock* to track the motion of all the others. In particular, in the context of mechanics *areal velocity* was considered from the very beginning, a natural choice from the Greek perspective that would later play a pivotal role in modern astronomy thanks to Kepler's "area law".[2]

Notice, in particular, that the construction shown in Fig. 4.2 is scalable at pleasure, one of the fundamental requirements for theory-based machine design according to Philo. Given the eccentricity, an infinite number of *similar* and *confocal* ellipses may be constructed starting from the same hodograph. Among these, there will be the actual orbit, uniquely determined by the length of its major axis.

We already outlined how in Hellenistic astronomy the theoretical treatment of

[1]For more details on the important role of Kepler in the passage from ancient *mathematical astronomy* to modern *physical astronomy* see Stephenson 1987. A beautiful monograph about Kepler is Simon 1979.

[2]It is worth remarking that Kepler always called his famous "laws" simply *ratios* or *proportions*, and often compared the "area law" to the analogous proportion existing between force and arm of a lever (Zilsel 1942, pp. 265–267). For more details on the origin and reception of Kepler's "area law" see Aiton 1969, In Simon 1979, pp. 358–366 it is emphasized how Kepler's *new astronomy* deeply transformed the *meaning* of ancient astronomical terms and concepts, namely that of *mean motion* and, therefore, of *astronomical time.*

motion could well have been one and the same for both mechanics and astronomy, not so differently from what happened in modern times. Since it is sure that in Archimedes' lost mechanical works a rigorous *theory of mechanics* was developed, through the medium of *sphairopoiia* an astronomical theory could well have arisen as an extension/application of such a theory to the reproduction of the observable celestial motions. Such a step was probably made by Archimedes himself, and we think that *this* was, loosely speaking, the subject matter of the lost treatise on *sphairopoiia*: a *mechanical theory of celestial motions*, based on what later Hamilton called *law of the circular hodograph.* Today we know from Archimedes' *Method* that he explicitly used mechanics to *induce* results that he later *deduced* by standard geometrical methods. His *sphairopoiia* would be, in essence, the extension/application of his *mechanical method* to astronomical problems.[3]

Notice that the *Archimedean* development we are outlining - *machines first, celestial bodies later* - would have been just the opposite of that followed in modern times, when Newton's theory, whose main field of application was astronomy, was regarded by Newton and his followers as a *universal* theory of the motion of bodies, provided of the course all the metaphysics involved in the idea of a *mechanical universe.*[4]

As we already remarked, the aim of any astronomical theory and of a device like the AM is, in essence, to *synchronize the heavenly motions with each other*. Archimedes' masterly use of the idea of simultaneous motions describing two connected curves as it appears in the treatise *On Spirals* is a strong indication in this sense. The following *theorem of hodographic isochronism*, communicated by Hamilton to the Royal Irish Academy in March 1847, gives, in our view, a clue about the possible character of Archimedes' mechanical-astronomical theory:

> If two circular hodographs, having a common chord, which passes through or tends towards a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.[5]

Notice that the circles here mentioned are conceived as sections cut off from a sphere, and are in general in different planes.

So, everything considered, apart from Ptolemy's astronomy we find *nothing* in the existing sources which is in *contradiction* with our interpretation of Hipparchus' diagrams as diagrams of velocity. Arguments in its *support* come from the already mentioned Greek assimilation of Babylonian astronomical methods, which, overall,

[3]Compare with the auxiliary role played by *induction* in pure mathematics described by Hamilton in his *Introductory Lecture on Astronomy.* No doubt Hamilton would have been delighted by the discovery of Archimedes' *Method.*

[4]Probably the inverse path would have avoided to the moderns such metaphorical traps and all the painstaking effort that was necessary to get rid of them. It is noteworthy that, despite the huge technological development of mechanics, a rigorous *theory of machines* was developed in modern times only at the end of the XIX century by the German engineer Franz Reuleaux (1876), another passionate reader of the Classics. Reuleaux, among other things, strived to build what he called a *language of invention* for mechanics, intended, in principle, to clear the way for the theoretical *analysis* and *synthesis* of whatever possible *machine.*

[5]Hamilton 1847a.

seems to be the real characteristic element of the more mature stages of Hellenistic astronomy and of Hipparchus' work in particular.

It was common practice of Babylonian astronomers in the Seleucid period to describe the motion of celestial bodies not in terms of *positions* and *time*, but rather in terms of *daily motions*, where one *day* corresponded to the time employed by the Sun to describe one *degree* on the ecliptic. An example is the tablet labeled ACT 190, a listing of Moon daily velocities over a period of 248 days, not attached to any specific date.[6] This was a *template of velocities* that could be used to generate day-by-day positions of the Moon over any desired period. If our conjecture is right, Hipparchus' diagram would be a geometrical translation of this kind of tables, indicating graphically at each angular position the rate of displacement of the considered body.

Moreover, from the decipherment of cuneiform texts reporting calculations relative to the motion of Jupiter, Mathieu Ossendrijver made the groundbreaking discovery that, starting from these observed pairs of time and velocity, the position of Jupiter at any given date was computed by a process equivalent to the so-called *Merton Rule* or *mean-speed theorem*, i.e. by *integration* of the area under a trapezoidal time-velocity diagram. Ossendrijver concludes:

> The Babylonian trapezoid procedures are geometrical in a different sense than the methods of the mentioned Greek astronomers, since the geometrical figures describe configurations not in physical space but in an abstract mathematical space defined by time and velocity (daily displacement).[7]

We find no difficulty in imagining that also Greek mathematicians, and Archimedes in particular, could have made use of such "abstract mathematical spaces defined by time and velocity", computing future celestial positions by *quadrature* of velocity diagrams.

[6] O. Neugebauer 1955, p. 179.

[7] Ossendrijver 2016, p. 484.

Conclusion

Let me summarize in the most direct way the general view of the art of *sphairopoiia* and the new interpretation of the Antikythera Mechanism that, in my opinion, emerges from the previous chapters.

Greek mathematics was a collection of *problem-solving* disciplines sharing a unitary *language* and *method*. In the mature epistemological framework of Hellenistic sciences a central role was played in all domains by geometry and by the connected activity of drawing *diagrams*. These were the main conceptual tool used to frame and solve problems internal to well-definite *theories* or *theoretical models*. These had a twofold aim: to give a unified account of a certain domain of natural phenomena, and to ground the design of technological artifacts. These two moments were not separated but advanced together in a dialectic relationship, technological artifacts becoming also tools for further theoretical investigations.

In the context of astronomy, a central role was played by *sphairopooiia*, the art of building *tangible* and *visible* illustrations of the theoretical models used to account for astronomical phenomena. This practice formed an integral part of the Greek astronomical practice at least since Eudoxus, the greatest mathematician of Plato's time, first in the simple form of *celestial globes* and later in the form of geared mechanisms imitating the observable motions of the heavenly bodies. The prominent role that *circular motion* played in kinematical geometry, mechanics and astronomy became the key ingredient for the blending of these three sciences in the art of *sphairopoiia*.

This process was carried further by Archimedes, the most celebrated mathematician and engineer of the Hellenistic period. His lost treatise *On sphairopoiia* exposed a *theory of sphere-making*, i.e. a theory of celestial motions that *at the same time* grounded the design of mechanical devices intended to IMITATE such motions. Crucial ingredients in this development were Aristarchus' heliocentric hypothesis and the connected idea, already exposed in Euclid's *Optics*, that *in general* observation exhibits to view *nothing but* relative motions. This idea, in particular, was fully restored during the XIX century, and formed the necessary background of Einstein's *theory of relativity.*

The theoretical homogeneity of Hellenistic mechanics and astronomy was granted by the common use of diagrams that encoded the *kinematical relationship* between two given bodies, be these parts of a device or celestial bodies, in which, as usual in Greek mathematics, *circles* played a central role. Motions were represented geometrically as *ratios*, *proportions* and *successions* of the *simultaneous descriptions* of curves and surfaces traced by moving *points* and *lines*. Archimedes' *Spiral* is a beautiful example of this approach, and Archimedes' *Method* gives a clue of how deeply in his lost treatise *On sphairopoiia* mechanics could interact with theoretical astronomy.

Loosely speaking, the astronomical problem as framed and solved by Archimedes was that of *synchronizing all the heavenly motions with the motion of the Sun*. Therefore, in Archimedean *sphairopoiia* theoretical mechanics and astronomy overlapped completely, with *symmetry principles* and *optimality criteria* as benchmarks for theoretical activity, computational procedures and machine-design.

Further developments resulted from the Greek assimilation of computational methods and empirical materials originated in the ancient tradition of Babylonian astronomy, a process that generally took the form of a *geometrical reinterpretation of numerical algorithms*. Already began by Euclid, this work was carried further in the following decades, notably by Hipparchus, generally remembered as the last and greatest astronomer of the Hellenistic period. He probably put to profit the computational algorithms developed by Seleucid astronomers, Apollonius' theory of conic sections and Archimedes' mechanical approach to build a *dynamical* model of celestial motions which gave results equivalent to those of modern dynamics. Our proposed reinterpretation of eccentrics/epicycle diagrams suggests that a central ingredient of Hipparchus' model was the *hypothesis of the circular hodograph*, a geometrical picture of the inverse-square proportion between accelerations and relative positions that, by its very conception, was fully within the range of the mathematical arsenal available at Hipparchus' time.

Plane astronomical models based on the *hypothesis of the circular hodograph* made possible the design of *hodographic computers* that involved eccentric/epicycle constructions and computed graphically the solutions to what later was called *Kepler problem*. The Antikythera Mechanism is an example of such *analog mechanical-astronomical computers*.

After three centuries of scientific decline and the loss of the Hellenistic scientific method, the role and meaning of *sphairopoiia* in the context of mathematical astronomy was no more understandable and the hybrid status of this art deeply misunderstood. The slip consisted, loosely speaking, in taking a *theoretical diagram of relative velocities* as a *realistic picture of relative positions*, which transformed a *dynamical succession of relative velocities* into a *kinematical succession of relative positions*. In other words, hidden behind Ptolemy's metaphysical *principle of circular motions* there is Hipparchus' mathematical *hypothesis of circular hodographs*. However, only a careful analysis of Ptolemy's *Almagest* in the light of this hermeneutical proposal could give a solid ground to such a conjecture about the origin of Ptolemaic models, and must be left to future investigations.

In the following centuries, the art of *sphairopoiia* disappeared, but its memory survived and this practice was resurrected together with the ancient scientific method in the early-modern era. In particular, the first to recompose the dispersed union of kinematical geometry, mechanics and astronomy was Isaac Newton, who elaborated a general *philosophical* system incorporating very different ideas and grounded on the metaphorical notion of *mechanical universe*. Generally speaking, the role of *sphairopoiia* in the transmission and reception of ancient ideas has been immense for the history of Western philosophical, mathematical and cosmological inventions. Also this vast and important topic is open to further investigations in the light of the historical reconstruction here outlined.

Only in the XIX century dynamical astronomy was turned again into the *mathematical* theory it originally was, and this task was accomplished by William Rowan Hamilton, who invented *classical dynamics* with the direct help of the Classics. Hamiltonian dynamics became the ground on which *quantum* dynamics was built, but for multiple reasons the *methods* and *conceptions* that led Hamilton to his theory remained largely unknown to the community of physicists of XX century. It is likely that this fact played some role in the vagueness of the theoretical foundations that, according to many people, one hundred years after their invention still obscure the *meaning* of *quantum theories*, despite their unparalleled empirical success.

Modern technological developments made available different ways to effect the same purposes of ancient *sphairopoiia*, and this art never reached again the level of perfection witnessed by the Antikythera Mechanism.

Clearly there are still many important details to fill in this reconstruction, which, for the moment, is just a new look *at* and *through* the encrusted fragments of the Antikythera device. The aim that I proposed myself in my PhD was to explicitly restore the theory behind the mechanism, a necessary preliminary step to any new and well-grounded reconstruction that is worth of Archimedes' name. However, the task proved too ambitious for the time at my disposal and, for now, must be left unaccomplished. I hope it will be the subject of future works.

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