Università degli Studi di Napoli Federico II

Dottorado di Ricerca in Fisica

Scuola Politecnica e delle Scienze di Base
Area Didattica di Scienze Matematiche Fisiche e Naturali

Dipartimento di Fisica "Ettore Pancini"

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Theoretical Physics
Ciclo XXXV

Tutori
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Abstract

We study the $1+1$ (space-time) dimensional extension of the $0+1$ dimensional Sachdev-Ye-Kitaev (SYK) model for N Majorana fermions, with random all-to-all quartic interactions, averaged over disorder. At large effective couplings J and Q, and large N, the conformal symmetry of the effective action emerges, which is not broken spontaneously as in the original $0+1$ d SYK model. Critical two-point correlators are obtained from a coupling expansion of the Schwinger-Dyson equations. For $N = 4$, the model can be mapped onto complex fermions and solved exactly via the bosonization technique. The model separates in two sectors, nicknamed as "pseudo-charge" and "pseudo-spin", with gapped and gapless excitations, respectively. Excitations allow to give an approximate analytic form of the two-point correlators at large distances, which is adopted heuristically to numerically evaluate the Ground State Energy of the large N model. Absolute minimum of the energy is found in a restricted range of parameters.

Contents

Introduction

In recent years, there has been a lot of activity around the Sachdev-Ye-Kitaev (SYK) model [1, 2, 3, 4, 5] due to its interesting properties. The model is a $0 + 1$ dimensional system described by N Majorana fermions with many fermionic degrees of freedom and all-to-all random interactions. The theory is partially solvable in a tractable large N limit which is possible to study through the dynamical mean field theory as well as the so-called tensor models [6]. This allows applications within both condensed matter physics and high energy.

These tensor models are considered as a new class of large N theories, different from vector models [7, 8] or matrix models [9, 10]. For example, the two-point function in tensor Klebanov-Gurau model [11, 12] is dominated by melon diagrams in the large N limit and fixed by the N dependent parameter J, whereas the dominant diagrams for vector and matrix models are bubble and planar diagrams, respectively. However, beyond the melonic dominance, the low degree of symmetry makes the tensor formalism difficult to study. In the SYK model case, this difficulty is mitigated by introducing disorder. Furthermore, the dominance of this simple class of Feynman diagrams and their iterations allows to evaluate all the correlation functions. Indeed, the Out-of-Time-Ordered-Correlator (OTOC) grows exponentially on an inverse time scale which corresponds to a classical Lyapunov exponent λ_L and saturates at times less than the "scrambling time" t_* with $\lambda_L t_* \sim \ln 1/\hbar$ [13]. In this way, the model can be seen as a holographic dual for gravity theories of black holes [14, 15, 16, 17]. The model presents an emergent approximate conformal symmetry at low energies, where the reparametrization symmetry is spontaneously broken down to a $SL(2,\mathbb{R})$ subgroup. Goldstone soft modes appear in the excitation spectrum. These gapless excitations become gapped when the approximate conformal symmetry is explicitly broken by reintroducing the derivative term of the Lagrangian as an ultraviolet correction. This implies that the soft modes acquire a mass, denoted as pseudo-Goldstone in the literature. As a solvable many-body system, the SYK model serves as a building block for constructing a metal and study its properties. Its capability to describe strongly correlated systems opens up the possibility of study "strange metal" phase [18].

Because of these interesting properties, extensions of the model to higher dimensions have been explored. The path to reach this goal includes many different approaches. In particular, in the context of Condensed Matter, some authors have extended the model to higher dimensions by building a chain or a lattice of SYK dots [19, 20, 21, 22] due to the fact that the original $0+1\,d$ SYK model can be seen as mimicking a quantum dot in Condensed Matter Physics. These extensions to higher space dimensionality appear to be a tractable benchmark for a quantum many particle interacting system with non-Fermi-Liquid (NFL) behaviour [23, 24]. There are complex fermion versions of the SYK model [25, 26, 27, 28] dubbed as "strange metal" because there is at least a branch of gapless excitations which are not the quasiparticles of the Fermi Liquid Theory. When dealing with hopping in a spatial lattice at lowest perturbative order [29], in the infrared limit, the response of the fermionic excitations, in the conformal symmetry limit, to an external driving to be specified, gives rise to the celebrated linear temperature dependence of the transported current over a large range of temperatures and to the constancy in temperature of the thermal conductivity [27, 30, 31]. As this is a striking feature of the resistivity which is experimentally found in the normal phase of the High Critical Temperature (HT_c) superconducting materials, these models are extensively studied in that connection [5, 32, 33]. It is interesting that the addition of a kinetic term to the model carries a complex $U(1)$ phase with, to be added to the real fields, which gives rise to bosonic collective and gapped diffusive modes [4, 29].

Other possibility is to extend directly the quantum field theory to $1{+}1d$ by changing the canonical scale dimension of the fermionic fields. By adding an extra spatial dimension, the scale dimension of the coupling constants change as well as the relevance of the interactions [34]. For a q-fermion interaction, the scale dimension of the interaction term is $q/2$, which is marginal if $q = 4$, i.e. the four-body interaction, which is regularly studied in SYK models, is not relevant in the power counting in $1 + 1$ dimensions. Since the $0 + 1\,d$ model has relevant interactions, the above is an important obstacle in the generalization of the SYK model to higher dimensions. However, there are some attempts in the direct dimensional extension in which some features of the model have been shown to be preserved. In the extended $1 + 1$ dimensional chiral SYK model [35], the interactions are exactly marginal, leading to an exact scaling symmetry which makes the model exactly solvable at all energy scales. This allows to compute the OTOC which has a Lyapunov regime and an asymmetric butterfly cone in the large N limit. The model is integrable in the case $N = 4$ by bosonization, where the functional form of the two-point correlators coincides with the ones coming from the large N case. Non-chiral extension have been also considered [36, 37] showing that, statistically, the random couplings are overall marginally irrelevant. An emergent approximate conformal symmetry is present at low energies which, unlike the $0+1\,d$ SYK case, is not broken spontaneously by the conformal correlator. Furthermore, the model requires a regulator to be included already in the action, which breaks conformal invariance explicitly. The non-chiral $1 + 1\,d$ model in [37] can also be seen as a random version of the Thirring model, with random couplings $J_{ij;kl}$ drawn from a Gaussian ensemble. From the Callan-Symanzik equation, a positive β function is obtained, which means a non-renormalizable theory. Nevertheless, the model can be studied as an effective theory below some scale.

As we can see, the study of the extension of the SYK model as a $1 + 1$ dimensional quantum field theory is intricate, specially in the non-chiral case. Even though, it is worthy study some open issues. First, we would like to know if the theory is stable below some scale and find an approximation which makes the model solvable. Non-chiral $1+1$ dimensional models are statistically marginal irrelevant, in the sense that after averaging over disorder and using conformal perturbation theory, the β function is positive. However, there are relevant and irrelevant operators that will grow or decrease as we flow into the infrared, and these can also change as the couplings themselves evolve. As all these contributions are screened by the net effect of the average over disorder, we can think of the model as an effective model with an effective coupling J. Furthermore, the model is not truly conformal symmetric, which implies that we can still look for some approximation to study the model. Secondly, it would be interesting to see if gapped excitations are still present in higher dimensions for the non-chiral $1+1$ d SYK model. In this case, it is known that the model is not strictly Lorentz invariant due to the regulator that must be included. However, we speak of "quasi" Lorentz invariance at IR and an emerging conformal symmetry at large interaction coupling J, as it happens for the $0+1$ d case. In this limit, when an UV cutoff Λ is introduced in real time-space by regularising the singularity at small arguments with a logarithmic factor, the disorder average of the model provides a Schwinger Dyson equation in the $1/N \to 0$ limit that can be solved without breaking the conformal symmetry and the model is found to be still critical. This does not give us any hint about excitations being gapped: they all could be gapless if the system remains critical. Thus, it is necessary to seek for a different approach in order to find solutions. Finally, as the Thirring model is dual to sine-Gordon models via bosonization, it is interesting to search for a dual sine-Gordon version of the non-chiral SYK model in order to obtain its properties and infer from them, the features of the model in the large N limit.

In this thesis we propose an extended non-chiral $1+1$ d Sachdev-Ye-Kitaev model with the disorder average which includes the cross chirality interaction, that is to say, we are not fixed to a diagonal coupling $J_{ij;kl} \sim J\delta_{ij}\delta_{kl}$. In this sense, we can think the model as an extension of the random Thirring model [37]. As the "quasi" conformal invariance solutions do not give any information about excitations, we seek for solutions from the case $N = 4$. In this limit, the model is non symmetrical (non Lorentz invariant) and non traceless (non conformal invariant), however, it can be solved exactly by bosonization. The two-point correlation functions inform us about excitations: they are both gapless and gapped. These two branches exist due to the separation of what we nickname as "pseudo-charge" and "pseudo-spin" sectors. Nonetheless, we should ask if this behavior remains for large N. From the $1+1$ d chiral SYK model [35], it is shown that the two-point correlator for $N = 4$ and large N are equal (beyond the fact that the meaning of the coupling is not strictly the same in both cases) and the model remains critical with gapless excitations. On the other side, from the non-chiral case, the model is still critical in the infrared, as it is shown by the Callan-Symanzik equation and the powerlaw behavior of the correlator. For the strong J coupling limit, it can be seen by variational arguments that the model presents gap excitations. Therefore, we assume heuristically analytic correlators suggested by the $N = 4$ case to evaluate an approximated free energy in the large N case, and find the gap in a range of J values.

This thesis is organized as follows: In Chapter 1 we briefly summarize some aspects of the original $0 + 1$ d SYK model. We present its effective action by averaging over disorder, finding the Schwinger-Dyson (SD) equations and obtaining the correlation functions by considering the emergent approximate conformal invariance. These aspects lay the foundation for calculations in the large N limit.

In Chapter 2 we present a general view of our $1 + 1$ d extended non-chiral model in the large N limit. We explore by dimensional analysis its renormalizability and then we discuss if interactions in our model are relevant, irrelevant or marginal. We complement the analysis by comparing with the original $0 + 1$ dimensional case. A brief analysis is also included for the case $N = 4$.

In Chapter 3 we begin to develop the model and study its properties in the large N limit. We derive the effective action by using the replica trick after averaging over disorder. After we introduce cross chirality bilocal Green's functions, the Schwinger-Dyson equations are represented including also off-diagonal chirality terms, besides the diagonal ones. The solution of SD equations is obtained within the approximate conformal invariance scheme which leads to free-like correlators in the $1/N$ expansion limit. An alternative expansion around the conformal symmetry limit is also investigated giving critical correlators with an strange exponent $\Gamma \neq 1$ which implies Non-Fermi-Liquid behavior.

Chapter 4 is the core of the original work and includes most of the results. An approximate free energy of the non-chiral $1 + 1$ d extended SYK model is plotted as a function of parameters for the interaction. We follow the derivation of the free energy introduced for the $0 + 1$ d SYK model in [2]. The Green's chiral conserving functions are required as well as the off-diagonal non-chiral conserving ones. We use results coming from the Chapter 5 where we derive the nature of the excitation spectrum and some correlators for our $1 + 1$ d SYK extended model for $N = 4$. Bosonization of the $N = 4$ case allows to map the model onto the sine-Gordon field theory model and these results are exact. For a chiral conserving case, the spectrum includes a pseudo-spin branch which is gapless and a gapped pseudo-charge branch. We assume that these features are conserved when $N >> 4$ and that the functional form of the correlators of the $N = 4$ case holds also for the $N >> 4$ extension. We are encouraged to do so, based on what we have discussed above. In the $0 + 1$ d SYK model the conformal symmetry of the $N \to \infty$ limit is spontaneously broken by the ground state as it is revealed by the time correlator in the infrared limit. Then, symmetry is explicitly broken by reintroducing the kinetic term as an ultraviolet correction. This implies that the Goldstone bosons acquire a gap, denoted as pseudo-Goldstone in the literature. At higher space dimension the conformal symmetry is broken by hand, as well as ultraviolet corrections and there is consensus on the fact that gapped excitations persist in all its extensions. The free energy expression also requires the knowledge of the off-diagonal non-chiral conserving Green's function. We prove that they vanish identically in the $N = 4$ limit because they correspond to non number conserving correlators which are absent in the $N = 4$ case. However, we find that by posing the interaction parameters Q and J equal we can circumvent this difficulty because the free energy expression only requires the product $g_{\cap} g_{\cup}$ which is represented heuristically by the number conserving operator $\langle O^{z\dagger}_{TS}(z, z) O^{z}_{TS}(0, 0)\rangle$, in the bosonization approach. In this way, we are able to graph by numerical methods the free energy versus the supposed gap and we find that there is a limited range of values for the J parameter in which the free energy develops an absolute minimum at finite Δ. This findings are the most relevant results of this work and seem to confirm that the main features of the $N = 4$ case are still present when $N >> 4$ at least in the $1/N$ expansion limit.

In Chapter 5 we derive the main features of the model in the $N = 4$ case. At $N = 4$, the model can be bosonized exactly by constructing complex fermions. Within the sine-Gordon scenario, two sector fields arise which we call the pseudo-charge and pseudo-spin, characterized by the renormalized velocities $u_c = u_0\sqrt{1 + J/\pi u_0}$ and $u_s = u_0\sqrt{1 - J/\pi u_0}$, respectively, where u_0 is a velocity scale. The theory is diagonalized in these sectors but it still keeps mixed chiralities. The pseudo-spin interaction factor $\mathcal{K}_s = \left(1 - \frac{J}{\pi u_0}\right)^{-1/2}$ diverges when $J \longrightarrow \pi u_0$ which establish the strong coupling limit of this $N = 4$ model. For the pseudo-charge case, this factor appears to be $\mathcal{K}_c = \left(1 + \frac{J}{\pi u_0}\right)^{-1/2}$. As is well known, the sine-Gordon model is in the critical phase with a powerlaw decay of the two-point correlation function when $\mathcal{K} > 1$, while it has a gapped spectrum with exponentially decaying correlators when $\mathcal{K} < 1$. In this way, the excitation spectrum of the pseudo-charge modes turns out to be gapped, while the pseudo-spin modes remain gapless. Correlators are computed, which are after used in the calculations of the free energy in Chapter 4. The energy-momentum tensor is obtained, which is non traceless. This indicates the fact that the theory appears to be non conformal invariance in the $N = 4$ case. Finally, other physical quantities are obtained, distinguished by the fact that they are renormalized by a factor depending on the interaction.

We close this thesis with a brief summary and discussion.

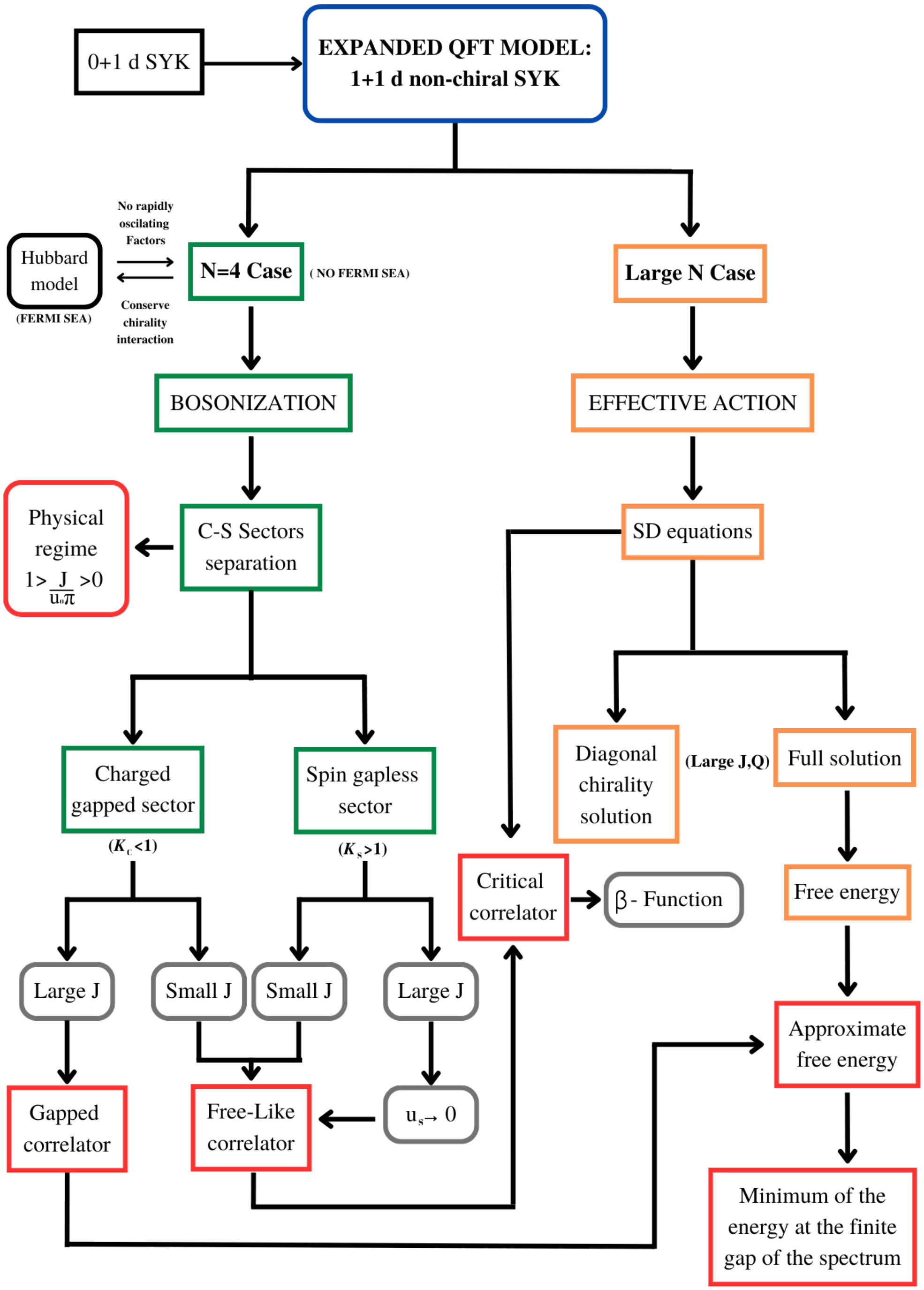


Figure 1: Schematic summary of the thesis. Black squares indicate models that can be related with our non-chiral $1 + 1$ d SYK within some limit. Green squares indicate the path to study the model in the $N = 4$ case. Orange squares indicate the path to study the model in the large $N =$ case. Red squares indicate important outcome or concepts developed, including the novel results. Gray squares indicate minor steps.

–1–
The 0+1 dimensional Sachdev-Ye-Kitaev model

In this initial chapter we describe the original $0+1\,d$ SYK model and explore some aspects that concern us. We present its Hamiltonian and describe the random nature of the coupling constants. In the large N limit, by considering the average over disorder and the replica trick, the effective action is obtained. Self-consistent Schwinger-Dyson equations are derived. At low energies, by dropping the kinetic term, an approximate conformal symmetry emerges and the model can be solve analytically.

1.1 The model

The Sachdev-Ye-Kitaev model is a $0+1$ dimensional quantum system of N Majorana fermions with many degrees of freedom and random all-to-all interactions [1]. The model proposed by Kitaev, as a variant of the original Sachdev and Ye model for pairwise coupled spins [38], consists in a Hamiltonian with $N >> 1$ Majorana sites and four-body interaction (generalized after to q-interacting fields):

$$H = -\frac{1}{4!} \sum_{i<j<k<l} J_{ijkl} \psi_i \psi_j \psi_k \psi_l \tag{1.1}$$

where $\psi_i \psi_j + \psi_j \psi_i = \delta_{ij}$ and all the indices i, j, k, l go from 1 to N. Here disorder effects are weaker than in systems with pairwise interactions. The random couplings J_{ijkl} are time independent and completely antisymmetric. In the large N context, it is not really important the specific distribution and we can assumed a Gaussian distribution with zero mean value and the following variance:

$$\overline{(J_{ijkl})^2} = \frac{3! J^2}{N^3} \tag{1.2}$$

with J being the characteristic energy scale. The model can be generalized to q-interacting fermions, but we are going to focus in the quartic interacting case. The reason for this is that, when we restrict to time-reversal-symmetric interactions, the model with $q = 4$ represents the dominant interactions at low energy [2].

In order to obtain the free correlator, we use the path integral formulation in the free theory:

$$Z_0[J] = \int \mathcal{D}\boldsymbol{\psi} \exp\left[-\int d\tau \sum_i \left(\frac{1}{2}\psi_i \partial_\tau \psi_i + J_i \psi_i\right)\right] \tag{1.3}$$

where a Grassmann source was included and the shorthand notation $\mathcal{D}\boldsymbol{\psi}$ denotes integration measure over all paths. Expanding the fields in terms of fermionic Matsubara frequencies $\omega_n = \frac{n\pi}{\beta}$ with $n \in 2\mathbb{Z}+1$, the path integral becomes:

$$Z_0[J] = \int \mathcal{D}\boldsymbol{\psi} \exp\left[-\frac{1}{\beta} \sum_{i,n\in 2\mathbb{Z}+1} \left(\frac{1}{2}\psi_{i,n} i\omega_{i,n} \psi_{i,-n} + J_{i,n}\psi_{i,-n}\right)\right]. \tag{1.4}$$

It is possible to transform the path integral into a Gaussian integral. This can be done by shifting the fields with $\psi_{i,n} \to \psi_{i,n} - \frac{J_{i,n}}{i\omega_n}$ and completing the square. Then, absorbing the proportionality constant into the measure, we obtain:

$$Z_0[J] = \exp\left[\frac{1}{\beta} \sum_{i,n\in 2\mathbb{Z}+1} \left(\frac{1}{2} J_{i,n} \frac{1}{-i\omega_{i,n}} J_{i,-n}\right)\right], \tag{1.5}$$

where the Euclidean time propagator

$$\Delta(\tau - \tau') = \frac{1}{\beta} \sum_{i,n\in 2\mathbb{Z}+1} \frac{e^{-i\omega_n(\tau-\tau')}}{-i\omega_{i,n}} \tag{1.6}$$

allows us to write the time-ordered Euclidean free propagator for Majorana fermion at zero temperature as:

$$G_0(\tau) = -\frac{1}{2} sgn(\tau). \tag{1.7}$$

From here, it is possible to compute corrections due to the interactions. A diagrammatic expansion is shown in Fig. 1.1, where the gray circle is the two-point function G (dress correlator), the thin solid lines are the free two-point function G_0 (bare correlator), and the dashed lines denote the average over J_{ijkl}.

As an operator, the bare correlator $G_0(\tau) = -\frac{1}{2} sgn(\tau)$ can be expressed as $\hat{G}_0 = (-\partial_\tau)^{-1}$. By expanding it, the diagrammatic expansion in Fig. 1.1 is obtained and can be written in terms of operators as

$$\hat{G} = \hat{G}_0 + \hat{G}_0\, \hat{\Sigma}\, \hat{G}_0 + (\hat{G}_0\, \hat{\Sigma}\, \hat{G}_0)^2 + \cdots \tag{1.8}$$

where the melon diagrams (and all the nested melon that contribute at leading order) are represented by $\hat{\Sigma}$. By factorizing the bare correlator in the right side of Eq. (1.8) (or similarly the thin solid line in Fig. 1.1), the expanded part can be seen as the Neumann series analog to the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ and write:

$$\hat{G} = \hat{G}_0(1 - \hat{\Sigma}\, \hat{G}_0)^{-1} = \left[(\hat{G}_0)^{-1} - \hat{\Sigma}\right]^{-1} = (-\partial_\tau - \hat{\Sigma})^{-1} \tag{1.9}$$

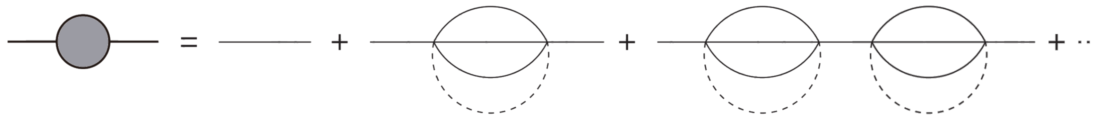


Figure 1.1: Corrections to the two-point function due to quartic interactions. The expansion is dominated by melon diagrams at leading order of $1/N$. Nested melon diagrams can also contribute.

where it was used $\hat{G}_0(\hat{G}_0)^{-1} = 1$. Eq. (1.9) corresponds to one of the Schwinger-Dyson equations (see Eq. 1.17) as we will see in following sections.

1.2 The effective action

In this section we show how the disorder-averaged partition function $\overline{Z}$ can be obtained. This is important for computation of the effective action which leads to the Schwinger-Dyson equations. Furthermore, the free energy can also be obtained by means of disorder-averaged partition function in the replica context [2].

The functional integral of $e^{-\overline{S}[G,\Sigma]}$ over some variables G and Σ gives the disorder-averaged partition function $\overline{Z}$, where $\overline{S}$ is the effective action (strictly speaking, the disorder-averaged action). On the other side, the average value of the free energy F can be obtained from $\beta\,\overline{F} = -\overline{\ln Z}$. However, $\overline{\ln Z}$ is not the same as $\ln \overline{Z}$ by terms of order $O(N^{2-q})$ [1]. If we consider quartic interactions, i.e. $q = 4$, these terms are not too important in the large N limit and the diagrammatic expansion around the saddle point of the effective action reproduces correctly all connected $2n$-point functions. More accurate computations can be done by considering a similar action with M replicas and then make the usual $M \to 0$ trick. In the following, we use the standard replica method to perform the disorder average over random coupling constants assuming that the replica symmetry is unbroken.

In the replica context, the average value of the free energy is given by:

$$\beta\,\overline{F} = -\overline{\ln Z} = -\lim_{M\to 0}\frac{\ln \overline{Z^M}}{M}. \tag{1.10}$$

For each realization of disorder, $Z(J_{ijkl})^M$ is equal to the partition function of M replicas. The exact form of the probability distribution is not really important when N is large, thus we can assume that it is Gaussian. Therefore, we can consider a probability density function of the form $P(K_{abcd}) = \frac{1}{\sqrt{2\pi\overline{(K_{abcd})^2}}}\exp\left(-\frac{1}{2}\sum_{abcd}\frac{1}{\overline{(K_{abcd})^2}}(K_{abcd})^2\right)$. Finally, considering an extended set of Grassmann variables ψ_j^α parametrized by $\tau \in [0,\beta]$, with

$j = 1, ..., N$ and the replica index $\alpha = 1, ..., M$, the partition function becomes:

$$\begin{aligned}
\overline{Z^M} &= \int \mathcal{D}\psi\mathcal{D}\mathbf{J} \exp\left[\sum_\alpha \int_0^\beta d\tau \left(-\frac{1}{2}\sum_j \psi_j^\alpha \partial_\tau \psi_j^\alpha + \sum_{i<j<k<l} \sqrt{\frac{3!J^2}{N^3}} J_{ijkl}\psi_i^\alpha \psi_j^\alpha \psi_k^\alpha \psi_l^\alpha\right)\right] \\
&= \int \mathcal{D}\psi \exp\left[\sum_{\alpha,j} \int d\tau \left(-\frac{1}{2}\psi_j^\alpha \partial_\tau \psi_j^\alpha + \frac{3!J^2}{2N^3}\sum_{i<j<k<l}\left(\sum_\alpha \int d\tau \psi_i^\alpha \psi_j^\alpha \psi_k^\alpha \psi_l^\alpha\right)^2\right)\right] \\
&= \int \mathcal{D}\psi \exp\left[-\frac{1}{2}\sum_{\alpha,j}\int d\tau \psi_j^\alpha \partial_\tau \psi_j^\alpha + \frac{NJ^2}{8}\sum_{\alpha,\beta}\int d\tau d\tau' \left(\frac{1}{N}\sum_j \psi_j^\alpha(\tau)\psi_j^\beta(\tau')\right)^4\right].
\end{aligned} \tag{1.11}$$

The latter can be solved by using the following identity:

$$f(\tilde{x}) = \int_{-\infty}^{+\infty} dx\, f(x)\delta(x-\tilde{x}) = \frac{N}{2\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, f(x) e^{iNy(x-\tilde{x})}. \tag{1.12}$$

If we define $G_{\alpha\beta}(\tau,\tau') = \frac{1}{N}\sum_j \psi_j^\alpha(\tau)\psi_j^\beta(\tau')$, we can identify that $x = -G_{\alpha\beta}(\tau,\tau')$. Additionally, we denote y by $-i\Sigma_{\alpha\beta}(\tau,\tau')$. Integration over ψ fields can be assumed as N identical integrals, in such a way that the partition function becomes:

$$\begin{aligned}
\overline{Z^M} = \int \mathcal{D}\Sigma\mathcal{D}G \Bigg(\int \mathcal{D}\psi \exp\Bigg[&-\frac{1}{2}\sum_\alpha \int d\tau \psi^\alpha \partial_\tau \psi^\alpha \\
&-\frac{1}{2}\sum_{\alpha,\beta}\int d\tau d\tau' \Sigma_{\alpha\beta}(\tau,\tau')\psi^\alpha(\tau)\psi^\beta(\tau') \\
&+\frac{1}{2}\sum_{\alpha,\beta}\int d\tau d\tau' \left(\frac{J^2}{4}G^4_{\alpha\beta}(\tau,\tau') - \Sigma_{\alpha\beta}(\tau,\tau')G_{\alpha\beta}(\tau,\tau')\right)\Bigg]\Bigg)^N.
\end{aligned} \tag{1.13}$$

Now it is easy to perform the integral over the Grassmann variables $\psi^\alpha(\tau)$. This gives the Pfaffian of the operator $-\partial_\tau - \hat{\Sigma}$. Integrals over auxiliary fields Σ and G can be performed by using the saddle point approximation:

$$-\ln \overline{Z^M} = -\ln\left(\int \mathcal{D}\Sigma\mathcal{D}G \exp\left(-\overline{S}^{(M)}[G,\Sigma]\right)\right) \approx max_{(G)}\; min_{(\Sigma)} \overline{S}^{(M)}[G,\Sigma], \tag{1.14}$$

where

$$\overline{S}^{(M)}[G,\Sigma] = N\left[-\ln Pf(-\partial_\tau - \hat{\Sigma}) + \frac{1}{2}\sum_{\alpha,\beta}\int d\tau d\tau' \left(\Sigma_{\alpha\beta}(\tau,\tau')G_{\alpha\beta}(\tau,\tau') - \frac{J^2}{4}G^4_{\alpha\beta}(\tau,\tau')\right)\right] \tag{1.15}$$

is the effective action for M replicas. Considering the following ansatz $\Sigma_{\alpha\beta}(\tau,\tau') = \Sigma(\tau,\tau')\delta_{\alpha\beta}$, that is to say, omitting the off-diagonal in replicas, the limit $M \to 0$ becomes trivial and the effective action is:

$$\overline{S}[G,\Sigma] = N\left[-\ln Pf(-\partial_\tau - \hat{\Sigma}) + \frac{1}{2}\int d\tau d\tau' \left(\Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{4}G^4(\tau,\tau')\right)\right]. \quad (1.16)$$

The replica-diagonal approximation can be justified as follows: on the one hand, the expansion of $\beta\,\overline{F} = -\overline{\ln Z}$ consists of those diagrams that are connected along fermionic lines. On the other hand, the high temperature expansion of $-\ln\overline{Z}$ includes all connected diagrams, i.e. there are leading diagrams proportional to N^{-2}. Therefore, the replica-diagonal approximated free energy differs by $O(N^{-2})$ terms, which can be neglected in the large N limit [1]. A way to obtain this free energy is provided in section 1.4.

As it can be seen, all this process was possible after averaging over disorder and using the replica trick. The saddle point values of G and Σ in the effective action correspond exactly to the Green function and the self-energy in the mean field approximation.

1.3 The Schwinger-Dyson equations

In the large N limit the model is solvable by dynamical mean field theory, making it possible to write the self-consistency Schwinger-Dyson equations for the imaginary time correlator $G(\tau_1,\tau_2) = -\langle \mathbf{T}\psi_i(\tau_1)\psi_j(\tau_2)\rangle$, where $\mathbf{T}$ denotes time ordering. By taking the maximum over G and the minimum over Σ in the effective action (1.16), it is possible to write the SD equations as (see Fig. 1.2):

$$\hat{G} = (-\partial_\tau - \hat{\Sigma})^{-1}, \qquad \Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)^3. \quad (1.17)$$

Both Green function and self-energy are bilocal antisymmetric fields with antiperiodic boundary conditions, parametrized by the inverse temperature β by $\tau \in [0,\beta]$. At low energies, there is an emergent conformal symmetry and the model can be solved analytically.

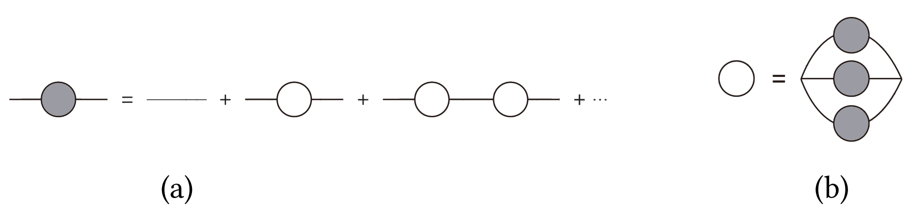


Figure 1.2: Graphical representation of the Schwinger-Dyson equations. The gray circle denote the full two-point function, whereas the white circle represent the self-energy. Equations (a) and (b) correspond to the left and right equations of (1.17), respectively.

1.3.1 Conformal limit

When we consider strong coupling (low energies), the kinetic part of Eq. (1.17) can be ignored and this let us write the unitary condition:

$$\int d\tau' G(\tau,\tau')\Sigma(\tau',\tau'') = -\delta(\tau-\tau''). \tag{1.18}$$

In this form, the Green function and the self-energy are invariant under the reparametrizations:

$$\begin{aligned} G(\tau,\tau') &\rightarrow [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau),f(\tau')), \\ \Sigma(\tau,\tau') &\rightarrow [f'(\tau)f'(\tau')]^{\Delta} \Sigma(f(\tau),f(\tau')) \end{aligned} \tag{1.19}$$

where $\Delta = 1/q$, for general q-interacting case. The Majorana correlator obtains the form:

$$G_c(\tau) = b\frac{sgn(\tau)}{|\tau|^{2\Delta}} \tag{1.20}$$

for some constant b. The last expression solves the Schwinger-Dyson equations provided the scale dimension Δ. Although the effective action and the SD equations are invariant for any reparametrization of conformal transformations, this does not happen for the conformal Green function, which is only invariant if we consider a reparametrization $\in SL(2,\mathbb{R})$. In this way, the strong coupling limit implies that it is possible to approximate the model by neglecting the kinetic part in the Schwinger-Dyson equations, where reparametrization invariance is spontaneously broken by the conformal solution $G_c(\tau)$. In Section 3.2 of Chapter 3, we will see that, for $1+1$ d SYK model case, a regulator has to be already included in the action. This also emerges when Fourier transforming the self-energy appearing in Eq. (3.16) as the integral does not converge and requires the introduction of a regulator. This UV regularization breaks the conformal invariance, which implies that the breaking of the scale invariance is explicit and not spontaneous. This reparametrization invariance can be used to solve the SD equations and find at least the two-point correlator functions.

1.4 Free energy

The free energy can be studied from the functional integral by considering the original partition function. However, as it was observed, the leading large N approximation of the free energy reads:

$$-\beta F/N = \ln Pf(-\partial_\tau - \hat{\Sigma}) - \frac{1}{2}\int d\tau d\tau' \left(\Sigma(\tau,\tau')G_(\tau,\tau') - \frac{J^2}{4}G^4(\tau,\tau')\right). \tag{1.21}$$

As it is shown in [2], it is convenient to take the derivative with respect $J\partial_J$ in order to avoid evaluating the Pfaffian term. Since G and Σ obey the equations of motion, the only contributing term is the derivative of the explicit dependence on J, such that:

$$\begin{aligned} J\partial_J(-\beta F/N) &= \frac{J^2\beta}{4}\int_0^\beta d\tau G^q(\tau) = -\frac{\beta}{4}\partial_\tau G\Big|_{\tau\to 0^+} \\ &= -\beta E. \end{aligned} \tag{1.22}$$

Because the partition function just depends on the combination βJ, we can consider that $J\partial_J$ is the same as $\beta\partial_\beta$. This provides a way to obtain the energy. We adopt this method to obtain the free energy in our extended $1+1$ d non-chiral SYK model, as it is shown in Section 4.1, where diagonal in chirality and cross-chirality (off-diagonal) correlators are required.

With this we have finished our review of the original $0+1$ d SYK model. Even if the theory has other interesting properties such as its relation with chaos and black holes, we are not going to focus on them in this work. This is basically due to the fact that reparametrization invariance in the $1+1$ d case is different from the $0+1$ d case. In the original SYK model, the action was consistently reparametrization invariant and a scale invariant solution can be found. Invariance is seen after spontaneously broken. By including the explicit breaking of reparametrization, the four-point function can be computed, and these are required to study the chaos limit. In the extended case, the cut-off already breaks reparametrization explicitly in the action and solution is not scale invariant. A four-point function is much more complicated to obtain and possibly requires a different approach that we do not study here.

–2–
The non-chiral 1+1 dimensional SYK model: generalities

In this chapter we describe some general aspects of the theory. We introduce the action of the model. The free spectrum is linearized around $k = 0$ and two branches for right and left-movers appears. The generalized interaction includes two different random couplings which control interactions between the same or different chirality branches. We perform dimensional analysis of the action, making the comparison with the original $0 + 1$ d SYK model. For the $N = 4$ case, the relevance of the interaction is studied by analyzing the scaling dimension and the conformal spin parameters.

2.1 The model

We propose an extended $1 + 1$ dimensional non-chiral SYK model described by Majorana fermions with fermionic degrees of freedom labeled by a flavour index that can take N (even) values on each site x. In a general way, we can express the free theory action as

$$S_0 = \frac{i}{2}\sum_{i}^{N}\int d^2x \left(\overline{\psi}_i \gamma^\mu \partial_\mu \psi_i\right), \tag{2.1}$$

where the spinor is $\overline{\psi} = \psi^\dagger \gamma^0$ and γ^μ are the gamma matrices. In the chiral representation [39] the real Majorana fermionic operator of flavor i will be denoted by $\psi_i(x)$ with $\psi_i(x)^T \equiv (\psi_{i-}(x), \psi_{i+}(x))$, where $\psi_{i\pm}(x)$ are eigenstates of σ_z and $\pm$ labels the chirality. The $\psi_i(x)$'s of different flavor or site satisfy the anticommutation relation

$$\left\{\psi_i(x),\ \psi_j(y)\right\} = \delta_{i,j}\ \delta(x - y). \tag{2.2}$$

In Euclidean space, $\psi_{i+}(x, \tau)$ is only a function of the complex coordinate $z = x + i\,\tau$ while $\psi_{i-}(x, \tau)$ is only a function of $\overline{z} = x - i\,\tau$. In the following, x in $\psi_{i\pm}(x)$ will denote both variables (x, τ) if no ambiguity arises.

Following within the chiral representation, in 1 + 1 dimensions the gamma matrices are given by Pauli's matrices as follow:

$$\gamma^0 = \sigma_x \quad ; \quad \gamma^1 = -i\sigma_y \quad ; \quad \gamma^5 = \sigma_z. \tag{2.3}$$

They satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.4}$$

where $\eta^{\mu\nu}$ is the Minkowski metric with signature $(+,-)$. The action for the free massless case can then be expressed by

$$S_0 = \frac{1}{2}\sum_i^N \int d^2x\, \psi_i(x)^T \left(-\partial_\tau + i\sigma_z\, u_0\partial_x\right) \psi_i(x). \tag{2.5}$$

After a Legendre transformation of the free Lagrangian, the free Hamiltonian can be easily obtained to be

$$H = \frac{1}{2}\sum_i^N \int dx\, \left[\psi_{i-}(x)(u_0 i\partial_x)\psi_{i-} + \psi_{i+}(x)(-u_0 i\partial_x)\psi_{i+}\right], \tag{2.6}$$

where it is observed that, after a Fourier transform into k-momentum space, the free spectrum is linearized around $k = 0$ with velocity $\pm u_0$ for $\pm k$.

In the original SYK model, N Majorana fermions have all-to-all fermion random interaction where $J_{i1\ldots iq}$ are a set of random Gaussian couplings. Here, we generalize it by defining

$$\begin{aligned} S_I \;=\; \int d^2x \Bigg[& \frac{1}{2}\sum_{i<j<k<l} J_{ijkl}(\overline{\psi}_i\gamma^5\gamma^\mu\psi_j)(\overline{\psi}_k\gamma^5\gamma_\mu\psi_l) \\ & + \sum_{i<j<k<l} Q_{ijkl}\left(\frac{1}{2}(\overline{\psi}_i\gamma^\mu\psi_j)(\overline{\psi}_k\gamma_\mu\psi_l) + (\overline{\psi}_i\psi_j\overline{\psi}_k\psi_l)\right)\Bigg], \end{aligned} \tag{2.7}$$

where the couplings J_{ijkl} and Q_{ijkl} are real and antisymmetric with respect to any two indices, and the labels $\{i, j, k, l\}$ run from 1 to N. In Section 3.2.1, for the large N case and after averaging over disorder, we will see that the model can be seen as the Random Thirring model [37] when $J = 0$, while if $Q = 0$, the model becomes in two decoupled left/right-mover SYK chiral systems. As for the original SYK model, the specific distribution is not very important in the large N case, therefore we will assume that they obey the random Gaussian distribution P with zero mean and the following variances:

$$\overline{(J_{ijkl})^2} = \frac{3!J^2}{N^3} \quad ; \quad \overline{(Q_{ijkl})^2} = \frac{Q^2}{3N^3}. \tag{2.8}$$

It is important to note that the model is just random with respect to Majorana flavor indices, while the translational invariance is unaffected.

On the other side, for the $N = 4$ case, just one way to order indices is valid and the couplings become fixed. Hence, without loss of generality, we will assume that both J and Q are equal. As we will see in Chapter 5, the model can be bosonized by constructing complex fermions and using a bosonization technique (see Appendix B). Doing that, the model is separated into "pseudo-charge" and "pseudo-spin" sectors, as it is shown in Section 5.2. The interaction has two effects: on one hand, the following part of the interaction

$$\frac{1}{2}\left((\overline{\psi}_i\gamma^5\gamma^\mu\psi_j)(\overline{\psi}_k\gamma^5\gamma_\mu\psi_l) + (\overline{\psi}_i\gamma^\mu\psi_j)(\overline{\psi}_k\gamma_\mu\psi_l)\right)$$

rescales the velocity. On the other hand, the interacting part

$$\overline{\psi}_i\psi_j\overline{\psi}_k\psi_l$$

introduces cosine-like interaction terms in both sectors. These two branches has different behavior on their excitations according to the interaction parameter $\mathcal{K}_{c/s} = 1/\sqrt{1 \pm \frac{J}{\pi u_0}}$ introduced in Eq. (5.34). For pseudo-spin sector it is found that excitations remains gapless, while on the contrary, for pseudo-charge sector it is found that excitations becomes gapped. We will assume that these two branches hold for the $N >> 4$ case, based on the results coming from Section 3.2.3 for small coupling regime (where the system remains critical) , and from variational analysis in the strong coupling regime (where indications of gapped excitations appear).

As a last comment, if we add to the interaction the following term

$$(\overline{\psi}_i\gamma^5\psi_j)(\overline{\psi}_k\gamma^5\psi_l), \tag{2.9}$$

it is possible to obtain a non-symmetrical model having the cosine interacting term in just one of the spin-charge sectors. For instance, in the Random Gross-Neveu like interaction [40, 41]

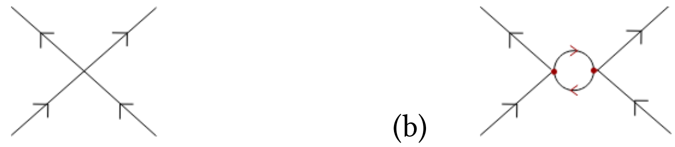
$$\frac{1}{2}\left[(\overline{\psi}_i\psi_j\overline{\psi}_k\psi_l) - (\overline{\psi}_i\gamma^5\psi_j)(\overline{\psi}_k\gamma^5\psi_l)\right] \sim \frac{1}{2\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_s\right) \tag{2.10}$$

the cosine of the charge sector disappears.

2.2 Dimensional analysis

Doing a dimensional extension in a model implies that we have different scaling dimensions for fields and hence, for interactions. The latter invites us to analyze the conditions in which interactions are relevant, irrelevant or marginal. It is also worth asking whether the theory is still renormalizable or not, and see if perturbations can be made.

One of the criteria that gives us information about the convergence of Feynman integrals and, therefore, about renormalizability of the model, is the "power counting". This method is used to classify divergences systematically. It can be related with the superficial degree of divergence, which studies the UV behavior of Feynman diagram



(a) (b)

Figure 2.1: (a) Four-body fermionic interaction represented by one interaction vertex and four external fermionic lines. (b) Four-body fermionic interaction with one internal loop and two interaction vertices.

integrals containing all the contributions with powers of momentum k. In Fig. 2.1 (a), it is observed a $q = 4$ fermionic interaction where, for some coupling constant g, there is a family of diagrams of order g, g^2, $O(g)$ in an expansion to one loop, two loops, three loops, etc. For example, one diagram of order g^2 and one loop of fermions is represented in Fig. 2.1 (b), where there are two interaction vertices and one internal loop. In a general d space-time dimensional system, for a diagram of order n (with n being the number of interaction vertices), E external lines, I internal lines and L loops, we have a k-momentum integral:

$$\int \frac{d^{dL}k}{k^{2I}}. \tag{2.11}$$

The superficial degree of divergence D is defined as [42]:

$$D = dL - 2I + \sum_a D_a \tag{2.12}$$

where D_a is the number of derivatives in the interaction for each vertex a. It is useful to express (2.12) in terms of external lines and vertices. The number of loops is given by:

$$L = I - n + 1 \tag{2.13}$$

while the number of external lines is:

$$E = \sum_a v_a - 2I. \tag{2.14}$$

Here, v_a is the number of lines (bosonic ones or fermionic ones) that connect with the vertex a. Using these expressions, and considering bosonic and fermionic fields, we can express an extended version of Eq. (2.12) as:

$$D = d - \left(\frac{d-2}{2}\right) E_B - \left(\frac{d-1}{2}\right) E_F - \sum_a \left[d - D_a - \left(\frac{d-2}{2}\right) v_a^B - \left(\frac{d-1}{2}\right) v_a^F\right] \tag{2.15}$$

where labels B and F are for bosons and fermions, respectively. It is recognized the vertex's dimension d_a as:

$$d_a = D_a + \left(\frac{d-2}{2}\right) v_a^B + \left(\frac{d-1}{2}\right) v_a^F. \tag{2.16}$$

Thus, according to the dimension of the vertex, the model can be considered

- super-renormalizable, if $d_a < d$
- renormalizable, if $d_a = d$
- non-renormalizable, if $d_a > d$.

The dimension of the vertex a is related to the interacting part of the Lagrandian density. Thus, by performing a dimensional analysis of the action, we can relate d_a with the dimension of the coupling constant:

$$d_a + [g] = d. \tag{2.17}$$

It is not surprising this relation between convergence of Feynman integrals and the dimension of the coupling constants. Indeed, when $d_a < 2$ for all vertices, the coupling constant has a positive dimension. Then, going to higher orders in perturbation theory produces more powers of g and forces the Feynman integrand to vanish faster at large momenta so that the total dimension remains fixed. On the contrary, when $d_a > 2$, it is expected more and more divergent integrals [42]. Accordingly, depending on the coupling scale $[g]$, the model can be (in mass units):

- super renormalizable if $[g] > 0$
- renormalizable if $[g] = 0$
- non-renormalizable if $[g] < 0$.

A theory can be expressed in terms of different kind of interactions. These interactions can be relevant, irrelevant or marginal. Broadly speaking, an interaction becomes relevant if it grows when it is considered larger scales of the theory, having noticeable effects in the IR. On the other side, an irrelevant interaction decrease when it is increased the scale of the model, having negligible effects in the IR. Other cases are said marginal. Now let us consider, for example, the irrelevant terms. As the interaction becomes irrelevant in the IR, in the momentum space it diverges in the UV. If the model is capable to include enough counter-terms to cancel these divergent terms, the theory it is said renormalizable (or super renormalizable). If not, the model is non-renormalizable.

In the following, we are going to analyze the renormalizability of the model and the relevance of the interactions for each $0+1\ d$ and $1+1\ d$ models. We must notice that we have a set of random couplings, which implies that the flows of each realisation can be different. This is specially important when interactions are marginal in the dimensional analysis because some operators will grow and some will decrease as we flow to the IR [37].

2.2.1 Power counting in the 0+1 dimensional SYK model

Let us start with the dimensional analysis of the original SYK model. The dimensionless action is

$$S = \int d\tau \left(\sum_i \psi_i \partial_\tau \psi_i + \sum_{ijkl} J_{ijkl} \psi_i \psi_j \psi_k \psi_l \right). \tag{2.18}$$

For the analysis, we are going to consider the characteristic energy scale J, which is related to the set of coupling constants J_{ijkl} by means of the variance (1.2). Then, doing a dimensional analysis, we have

$$1 = [t][\psi]^2[t]^{-1} = [t][J][\psi]^4 \tag{2.19}$$

$$\rightarrow [\psi] = 1 \quad \rightarrow [J] = [t]^{-1}. \tag{2.20}$$

It is common to make the power counting by putting time as length by multiplying a velocity such as $[t] \rightarrow [L]$ which is the inverse of mass (in natural units) $[M] = [L]^{-1}$. With this, the convention indicates that the field is dimensionless and the scale dimension of the coupling is 1. Therefore, we conclude that the model is super-renormalizable and the interaction term is relevant, being strongly coupled in the IR. The same conclusion can be reached evaluating (2.16) by considering $D_a = 0$, $d = 1$, $v_a^B = 0$ and $v_a^F = 4$, such that $d_a = 0 < d$, the model being super-renormalizable.

2.2.2 Power counting in the 1+1 dimensional extension

Let us see what happens if we extend the theory to $1 + 1$ dimensions. The dimensionless action (omitting some interaction terms for simplicity) is

$$S \sim \int d\tau dx \left(\sum_i \psi_i(\partial_\tau + u_0\partial_x)\psi_i + \sum_{ijkl} J_{ijkl}\psi_i\psi_j\psi_k\psi_l \right). \tag{2.21}$$

In $0 + 1\, d$, the set of random couplings is related to the characteristic energy scale J. In our $1 + 1\, d$ model, the sets of random couplings J_{ijkl} and Q_{ijkl} are related to the characteristic velocity scales J and Q, respectively, by means of the variances (2.8). Repeating the dimensional analysis, we have

$$1 = [t][x][\psi]^2[t]^{-1} = [t][x][J][\psi]^4 \tag{2.22}$$

$$\rightarrow [\psi] = [x]^{-1/2} \quad \rightarrow [J] = [x][t]^{-1}. \tag{2.23}$$

Considering again that $[t] \rightarrow [L]$ and $[M] = [L]^{-1}$, the scale dimension for Majorana fields is $1/2$, while the coupling is now dimensionless. The model is still renormalizable, but interaction is marginal. The same conclusion can be reached evaluating (2.16) by considering $D_a = 0$, $d = 2$, $v_a^B = 0$ and $v_a^F = 4$, such that $d_a = 2 = d$, the model being renormalizable.

Finally we can conclude that including space in the original $0 + 1\, d$ theory, the model remains renormalizable, but interaction becomes marginal when we consider a four-fermion interaction.

2.2.3 Scaling dimension and conformal spin

Most of the work on SYK model has been described by Majorana fermions with all-to-all fermion random interaction. Notwithstanding, in the Chapter 5, we present the model in the bosonization picture for the specific case $N = 4$, where the model can be related to its dual sine-Gordon version, after introducing complex fermions. In this subsection, we explore an alternative approach to analyze the relevance of the interactions in the specific $N = 4$ case. In this scenario, fermionic fields can be replaced by bosonic ones described in general by analytic and anti-analytic fields [43]:

$$c(\beta, z) \sim exp\left[\frac{i}{2}\beta(\phi(z,\bar{z}) + \theta(z,\bar{z}))\right] \quad ; \quad \bar{c}(\bar{\beta},\bar{z}) \sim exp\left[\frac{i}{2}\bar{\beta}(\phi(z,\bar{z}) - \theta(z,\bar{z}))\right], \tag{2.24}$$

where c and $\bar{c}$ are operators related to the bosonic field ϕ and its dual field θ, z and $\bar{z}$ are the complex coordinates, and β and $\bar{\beta}$ are generally different numerical factors. By construction, these dual fields are, in fact, compose by chiral "±" components (see Section 5.1).

If we consider first the free case of this $N = 4$ SYK model in the bosonization picture, we basically have the Gaussian bosonic model, where the pair correlation function is $(z_{12})^{-\beta^2/4\pi}(\bar{z}_{12})^{-\bar{\beta}^2/4\pi}$. By defining the "scaling dimension" $\tilde{d}$ as

$$\tilde{d} \equiv \Delta + \bar{\Delta} = \frac{1}{8\pi}(\beta^2 + \bar{\beta}^2) \tag{2.25}$$

and the "conformal spin" S like

$$S \equiv \Delta - \bar{\Delta} = \frac{1}{8\pi}(\beta^2 - \bar{\beta}^2) \tag{2.26}$$

we can study how perturbations affect our model and whether they are relevant or not. For example, in a d-dimensional system (space and time), a perturbation with scaling dimension $\tilde{d}$ and conformal spin $S = 0$ is said to be relevant if [43]

$$\tilde{d} < d \tag{2.27}$$

irrelevant if

$$\tilde{d} > d \tag{2.28}$$

and marginal if

$$\tilde{d} = d. \tag{2.29}$$

As already mentioned, in the $N = 4$ case cosine-like interaction terms will appear, which are not strange in one-spatial direction systems. This cosine is constructed by exponentials with exponent's prefactors proportional to β and $\bar{\beta}$, which are in general equal but opposite in sign. Keeping this in mind, cosine interaction will be relevant if

$$\beta^2 < 4\pi d. \tag{2.30}$$

For our $1 + 1$ dimensional model $d = 2$. Thus, in the $N = 4$ case, relevant or marginal interactions occur when $\beta^2 < 8\pi$ or $\beta^2 = 8\pi$, respectively.

We will see in Chapter 5 that the model separates in two sectors: the pseudo-charge and pseudo-spin, as it is showed in the Hamiltonian (5.25). In both sectors there is a cosine-like interaction of the form $\cos(\beta\phi) = \cos(\sqrt{8\pi}\phi)$. Therefore, we can again conclude that interactions are marginal. However, by re-scaling the bosonic fields and the conjugate momenta in the Hamiltonian, it is possible to obtain extra information about excitations, as it is observed in Eqs. (5.30), (5.31) and (5.32), where it is found that pseudo-charge has gapped excitations while the pseudo-spin case is gapless. This analysis shows that the case $N = 4$ can help us to explore features of the model.

–3–
Large N limit of the non-chiral 1+1 dimensional SYK model

In this chapter we solve the model for a large number N of left and right-movers Majorana fermions. The $q = 4$ fermionic interactions dominate and are mediated by the independent sets of real random couplings J_{ijkl} and Q_{ijkl} with zero mean and random Gaussian distribution. They are characterized by dimensionless parameters J and Q which determinate interactions between fermions of the same and opposite branches, respectively. Schwinger-Dyson equations are computed after averaging over disorder. An approximate free energy is obtained. Cross interacting correlators are also considered, and it is found that they modify the free energy. In the strong coupling limit an emergent approximate conformal limit appears.

3.1 Effective action

Let us remember our extended $1+1\ d$ SYK model of N (even) $q = 4$ interacting Majorana fermions. The full action is given by:

$$\begin{aligned} S \equiv S_0 + S_I &= \frac{i}{2}\sum_i^N \int d^2x \left(\overline{\psi}_i \gamma^\mu \partial_\mu \psi_i\right) + \int d^2x \left[\frac{1}{2}\sum_{i<j<k<l} J_{ijkl}(\overline{\psi}_i \gamma^5 \gamma^\mu \psi_j)(\overline{\psi}_k \gamma^5 \gamma_\mu \psi_l) \right. \\ &\left. + \sum_{i<j<k<l} Q_{ijkl}\left(\frac{1}{2}(\overline{\psi}_i \gamma^\mu \psi_j)(\overline{\psi}_k \gamma_\mu \psi_l) + (\overline{\psi}_i \psi_j \overline{\psi}_k \psi_l)\right)\right], \end{aligned} \tag{3.1}$$

where spinors and gamma matrices are in the chiral representation [39]. S_0 refers to the free action, while on the contrary, S_I refers to the interacting part of the full action. In Euclidean space, $\psi_{i+}(x,\tau)$ is only a function of the complex coordinate $z = x + i\,\tau$ while $\psi_{i-}(x,\tau)$ is only a function of $\overline{z} = x - i\,\tau$. We maintain the convention from the previous chapters where x in $\psi_{i\pm}(x)$ will denote both variables (x,τ) if no ambiguity arises. As in the original SYK model, N Majorana fermions have all-to-all fermion random interaction where the couplings J_{ijkl} and Q_{ijkl} are a set of real random Gaussian couplings and they are antisymmetric with respect to any two indices. In the large N case, the exact form of the probability distribution is actually unimportant and we will assume that they obey a random Gaussian distribution P with zero mean and the following variance:

$$\overline{(J_{ijkl})^2} = \frac{3!J^2}{N^3} \qquad ; \qquad \overline{(Q_{ijkl})^2} = \frac{Q^2}{3N^3} \tag{3.2}$$

where its randomness is just with respect to Majorana flavor indices, keeping translational invariance unaffected.

In the incoming calculations, we follow the same procedure as in Section 1.2 to obtain the effective action. We use the standard replica method [1] to perform the ensemble average over random coupling constants assuming that the replica symmetry is unbroken. To achieve it we must find the average partition function $\overline{Z^M}$ for the replica integer M and then take the limit $M \to 0$. For each realization of disorder, $Z(J_{ijkl})^M$ is equal to the partition function of M replicas. In this way we can relate $\frac{1}{M}\ln\overline{Z^M}$ with $\overline{\ln Z} = -\beta\overline{F}$, where $\overline{F}$ is the averaged free energy. For a general probability density function $P(K_{abcd})$, it is known that $P(K_{abcd}) = \frac{1}{\sqrt{2\pi\overline{(K_{abcd})^2}}}\exp\left(-\frac{1}{2}\sum_{abcd}\frac{1}{\overline{(K_{abcd})^2}}(K_{abcd})^2\right)$. Then, considering $P(J_{ijkl})$ and $P(Q_{ijkl})$ from our model, averaging disorder, completing the square and doing Gaussian integration over couplings, the partition function

$$\overline{Z} = \int \mathcal{D}\psi \int \mathcal{D}\mathbf{J} \int \mathcal{D}\mathbf{Q}\, P(J_{ijkl})P(Q_{ijkl}) \exp\left[-S_0 + S_I\right] \tag{3.3}$$

becomes:

$$\begin{aligned}
\overline{Z} \;=\; & \int \mathcal{D}\psi \;\, e^{-\frac{i}{2}\sum_i^N \int d^2x\left(\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i\right)} \\
& \times\exp\left\{\frac{N}{8}\sum_{a=\pm}\int d^2x d^2x'\left[J^2\left(\sum_j \frac{\psi_{ja}(x)\psi_{ja}(x')}{N}\right)^4 + J^2\left(\sum_j \frac{\psi_{ja}(x)\psi_{j\overline{a}}(x')}{N}\right)^4\right.\right. \\
& +2Q^2\left(\frac{1}{N}\sum_j \psi_{ja}(x)\psi_{ja}(x')\right)^2\left(\frac{1}{N}\sum_k \psi_{k\overline{a}}(x)\psi_{k\overline{a}}(x')\right)^2 \\
& +2Q^2\left(\frac{1}{N}\sum_j \psi_{ja}(x)\psi_{j\overline{a}}(x')\right)^2\left(\frac{1}{N}\sum_k \psi_{k\overline{a}}(x)\psi_{ka}(x')\right)^2 \\
& \left.\left. +8Q^2\sum_i\frac{\psi_{ia}(x)\psi_{ia}(x')}{N}\sum_j\frac{\psi_{ja}(x)\psi_{j\overline{a}}(x')}{N}\sum_k\frac{\psi_{k\overline{a}}(x)\psi_{k\overline{a}}(x')}{N}\sum_l\frac{\psi_{l\overline{a}}(x)\psi_{la}(x')}{N}\right]\right\}
\end{aligned} \tag{3.4}$$

where, if $a \to \pm$, then $\overline{a} \to \mp$. In the previous calculation, after Gaussian integration, we obtained sums like $\sum_{ijkl}\left(\int d^2x\psi_{i\pm}\psi_{j\pm}\psi_{k\pm}\psi_{l\pm}\right)^2$. Taking care on the antisymmetry of index and using combinatorial properties of the sum $ijkl$, it was possible to write the latter as $\frac{1}{4!}\int d^2xd^2x' \left(\sum_j \psi_{j\pm}(x)\psi_{j\pm}(x')\right)^4$ and similar expressions for the other sums. Now, we use the Hubbard-Stratonovich procedure which requires the introduction of bilocal auxiliary fields g and Σ, with the following definition: $\frac{1}{N}\sum_j \psi_{ja}(x)\,\psi_{ja'}(x') = g_{aa'}(x,x')$, where a is the chirality label. Thus, the partition function is written as:

$$\begin{aligned}\overline{Z} &= \int \partial\Sigma\partial G\Bigg[\int \mathcal{D}\psi\; e^{-\frac{i}{2}\sum_i^N \int d^2x\left(\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i\right)} \times \exp\Bigg\{-\int \frac{d^2xd^2x'}{2}\Bigg(\Sigma_+\psi_+(x)\psi_+(x') \\ &\quad +\Sigma_\cap\psi_-(x)\psi_+(x') + \Sigma_\cup\psi_+(x)\psi_-(x') + \Sigma_-\psi_-(x)\psi_-(x') \\ &\quad -\frac{J^2}{4}\sum_\alpha (g_\alpha(x,x'))^4 - \frac{Q^2}{2}\left(g_+^2g_-^2 + g_\cap^2 g_\cup^2 + 4\,g_+g_\cap g_-g_\cup\right)\Bigg)\Bigg\}\Bigg]^N, \end{aligned} \tag{3.5}$$

where the label α runs over $(+,-,\cap,\cup)$ with the short notation $g_\pm \equiv g_{\pm\pm}$ and $g_\cap \equiv g_{+-},\ g_\cup \equiv g_{-+}$ (and similar ones for Σ auxiliary fields). In the original $0+1\ d$ SYK model, due to the fact that diagonal replicas terms minimized the solution in the saddle-point approximation for effective action, only diagonal replica terms of the partition function were considered (see Section 1.2). In this way, the action is diagonal in all its labels. In our case, we also consider just diagonal replica terms, however, as our model is $1+1\ d$ with chirality label, it is possible for off-diagonal chirality terms have some contribution to the free energy. The integral over fermions $\psi_\pm$ is equal to the Pfaffian of the operator $-(\partial_\tau \pm i\partial_x) - \Sigma_\pm$ while, in the large N limit, the outer integrals $\partial\Sigma\partial G$ can be performed by finding a saddle point [1]. We introduce the $\hat{G}$ and the self energy $\hat{\Sigma}$ as 2×2 matrices in the chirality label:

$$\hat{G} = \begin{pmatrix} g_+ & g_\cap \\ g_\cup & g_- \end{pmatrix}, \qquad \hat{\Sigma} = \begin{pmatrix} \Sigma_+ & \Sigma_\cap \\ \Sigma_\cup & \Sigma_- \end{pmatrix}, \tag{3.6}$$

$$\hat{G}^{-1} = \begin{pmatrix} -(\partial_\tau - i\partial_x) - \Sigma_+ & -\Sigma_\cap \\ -\Sigma_\cup & -(\partial_\tau + i\partial_x) - \Sigma_- \end{pmatrix} \tag{3.7}$$

and the parity transformed (and transposed) $\hat{G}$ function, $\mathcal{P}\hat{G}\mathcal{P}^\dagger = \begin{pmatrix} g_- & g_\cap \\ g_\cup & g_+ \end{pmatrix}$. Integrating the fermion degree of freedom in Eq. (3.5), and considering the previous definition of the matrices, we can write the effective action as:

$$\begin{aligned}-\overline{S}[\hat{\Sigma},\hat{G}] &= N\Bigg[\ln Pf\left[\hat{G}^{-1}\right] - \frac{1}{2}\int d^2xd^2x'\Bigg(\mathbf{Tr}\left[\hat{\Sigma}(x,x')\,\hat{G}(x',x)\right] \\ &\quad -\frac{J^2}{4}\sum_\alpha (g_\alpha(x,x'))^4 - \frac{Q^2}{4}\left(\mathbf{Tr}\left[\left(\mathcal{P}\hat{G}\mathcal{P}^\dagger\,\hat{G}\right)^2\right] - 4\,g_+g_\cap g_-g_\cup\right)\Bigg)\Bigg]. \end{aligned} \tag{3.8}$$

Since the model is translationally invariant in both time and space, its two-point functions g_α and self-energy Σ_α will depend on the difference of the two space-time points, e.g. $g_\alpha(\tau_1, ix_1; \tau_2, ix_2) = g_\alpha((\tau_1 - \tau_2), i(x_1 - x_2))$.

As for the original SYK model, the effective action (3.8) allows us to obtain the Schwinger-Dyson equations using a saddle point approximation.

3.2 The Schwinger-Dyson equations

In the replica-diagonal approximation, the diagrammatic expansion of $-\ln \overline{Z}$ contains all the connected diagrams, whereas $\beta \overline{F} = -\overline{\ln Z}$ just contains those connected along fermionic lines. The difference between both are leading diagrams of order $O(N^{-2})$, with the free energy error being of the same order [1]. Since we are in the large N limit, the integrals in $-\ln \overline{Z} = -\ln\left(\int \mathcal{D}\Sigma\mathcal{D}G \exp\left(-\overline{S}[\Sigma, G]\right)\right)$ can be performed by the saddle point approximation. The maximum for G fields and the minimum over Σ fields give the Schwinger-Dyson equations. In the mean field approximation, these saddle point values of G and Σ are exactly the Green function and self-energy of the model.

3.2.1 First approximation: diagonal solution

As in the $0 + 1$ d SYK model, in the limit of large J, Q, the Schwinger-Dyson equations provide solutions which are invariant under reparametrizations. We prove this here in the simpler setup which drops the off-diagonal terms in the matrices of Eqs.(3.6) and (3.7). This is partially justified since, when focusing in the case $N = 4$, the cross-correlators vanish (see Appendix C, Eq. C.45). In this context, the effective action becomes:

$$\begin{aligned}\overline{S}[\Sigma, G] \;=\; N\sum_{a=\pm}\Bigg[&-\ln Pf\left[-(\partial_\tau + ai\partial_x) - \Sigma_a(x.x')\right] \\ &+\frac{1}{2}\int d^2x d^2x' \left(\Sigma_a(x,x')g_a(x,x') - \frac{J^2}{4}g_a^4(x,x') - \frac{Q^2}{2}g_+^2(x,x')g_-^2(x,x')\right)\Bigg].\end{aligned} \tag{3.9}$$

In the large N limit we resort to the saddle point approximation which gives the Schwinger-Dyson equations:

$$\Sigma_+(x,x') = J^2 g_+^3(x,x') + Q^2 g_+(x,x')g_-^2(x,x') \tag{3.10}$$

$$\Sigma_-(x,x') = J^2 g_-^3(x,x') + Q^2 g_-(x,x')g_+^2(x,x'). \tag{3.11}$$

The approximate solution of the Schwinger-Dyson equations in the large J, Q limit, where it is dropped the inverse free Green function term appearing in Eq.(3.7), is justified as the prefactor of Σ_a given by Eqs. (3.10) and (3.11) includes positive powers of J and Q.

Finally, in the Schwinger-Dyson equations (3.10) and (3.11), it is evident the role that play J and Q. If $J = 0$, we have the Random Thirring model [37], while on the contrary if $Q = 0$, we have two decoupled left/right-mover SYK models [35].

Reparametrization invariance

In original $0+1\ d$ SYK model, the strong coupling limit implies that it is possible to approximate the model by neglecting the kinetic part in the Schwinger-Dyson equations. Written in that way, they are invariant under reparametrizations and a scale invariant solution can be found. Then, it occurs a spontaneous symmetry breaking down to $SL(2,\mathbb{R})$ in the two-point correlator. In $1+1\ d$ SYK models [35, 37] it has seen that UV regularization plays an important role on correlation functions. Regularization breaks conformal invariance and a regulator has to be included in the action, with the breaking of scale invariance being explicit and not spontaneous.

In the large J, Q limit, by dropping the dependence on free propagator, the Schwinger-Dyson equations are invariant under reparametrization. According to [37], it is also possible to include the kinetic part in the equations and still have conformal invariance. However, as it was said, the regulator which is included in the action, breaks the scale invariance and the situation becomes quite different from $0+1\ d$ SYK model. Within this approximation, Eqs. (3.10) and (3.11) are invariant under the conformal transformation $z \longrightarrow f(z)$ and $\overline{z} \longrightarrow \overline{f}(\overline{z})$, which reads:

$$\begin{aligned} g_{\pm}(z,z';\overline{z},\overline{z}') &= [f'(z)f'(z')]^{\Delta_{\pm}}[\overline{f}'(\overline{z})\overline{f}'(\overline{z}')]^{\overline{\Delta}_{\pm}} g_{\pm}(f(z),f(z');\overline{f}(\overline{z}),\overline{f}(\overline{z}')) \\ &\equiv [f^2]^{\Delta_{\pm}}[\overline{f}^2]^{\overline{\Delta}_{\pm}}\widetilde{g_{\pm}} \end{aligned} \quad (3.12)$$

where

$$[f^2] = [f'(z)f'(z')] \quad ; \quad \overline{f}^2 = \overline{f}'(\overline{z})\overline{f}'(\overline{z}') \quad ; \quad \widetilde{g_{\pm}} = g_{\pm}(f(z),f(z');\overline{f}(\overline{z}),\overline{f}(\overline{z}')). \quad (3.13)$$

Up to now we have considered a four fields interacting model. Generalising to a $q-$interaction model, the self-energy Σ_a, according to Eq.(3.11), with the same short-hand notation, transforms as:

$$\begin{aligned} \Sigma_{\pm}(z,z';\overline{z},\overline{z}') &= J^2[f^2]^{\Delta_{\pm}(q-1)}[\overline{f}^2]^{\overline{\Delta}_{\pm}(q-1)}\widetilde{g_{\pm}}^{q-1} \\ &\quad + Q^2[f^2]^{\frac{q}{2}(\Delta_{\pm}+\Delta_{\mp})-\frac{q}{4}\Delta_{\pm}}[\overline{f}^2]^{\frac{q}{2}(\overline{\Delta}_{\pm}+\overline{\Delta}_{\mp})-\frac{q}{4}\overline{\Delta}_{\pm}}\widetilde{g_{\pm}}^{\frac{q}{4}}\widetilde{g_{\mp}}^{\frac{q}{2}}. \end{aligned} \quad (3.14)$$

Under the same approximations, the unitary condition,

$$\int d^2z'\ g_{\pm}(z,z';\overline{z},\overline{z}')\ \Sigma_{\pm}(z',z'';\overline{z}',\overline{z}'') = -\delta(z-z'')\ \delta(\overline{z}-\overline{z}''), \quad (3.15)$$

arises from minimization of the action with respect to $\Sigma_{\pm}$. The unitary condition of Eq.(3.15) implies that $\frac{q}{2}(\Delta_{\pm}+\Delta_{\mp}) = \frac{q}{2}(\overline{\Delta}_{\pm}+\overline{\Delta}_{\mp}) = 1$. Unbroken parity implies that $g_{+}(z,\bar{z}) = g_{-}(\bar{z},z) \equiv g(z,\bar{z})$. Under these assumptions, $\Delta_{+}=0$ and $\overline{\Delta}_{-}=0$, so that we can just redefine $\overline{\Delta}_{+} \longrightarrow \Delta_{-}$ and $\overline{\Delta}_{-} \longrightarrow \Delta_{+}$, and we can conclude that the saddle point and unitary equations are invariant under conformal transformation $z \longrightarrow f(z), \overline{z} \longrightarrow f(\overline{z})$ with $\Delta \equiv \Delta_{-} = \frac{2}{q}$.

Solutions of the Schwinger-Dyson equations in the q=4 case

Reparametrization invariance suggests the following solutions for Eqs.(3.10), (3.11) and (3.15) in the $q = 4$ case:

$$g(z,\bar{z}) = \frac{C'}{z}, \qquad \Sigma(z,\bar{z}) = C'^3\left(\frac{J^2}{z^3} + \frac{Q^2}{z\bar{z}^2}\right). \tag{3.16}$$

Here C' is a constant to be fixed by the unitary condition (3.15). Unlike the $0+1$ d SYK model, where scale invariance breaking was spontaneous, here the scale invariance is an explicitly broken solution starting from the action where a UV regularization has to be introduced all the way down to the IR limit with an UV cutoff Λ. This also emerges when Fourier transforming the self-energy appearing in Eq. (3.16) as the integral does not converge and requires the introduction of a regulator. Following [37], we pose the Ansatz:

$$g(z,\bar{z}) = \frac{C}{z}\ln^{\alpha}(z\bar{z}\Lambda^2), \qquad \Sigma(z,\bar{z}) = C^3\left(\frac{J^2}{z^3} + \frac{Q^2}{z\bar{z}^2}\right)\ln^{3\alpha}(z\bar{z}\Lambda^2), \tag{3.17}$$

where the log-term softens the RG flow on top of some leading power law term. The second of the Eqs. (3.17) derives from the saddle point equation (3.10). In the present form, these equation already solve the Schwinger-Dyson equations. To obtain the constant C from the unitary condition (3.15), we have to express it, along with the Ansatz (3.17), in momentum space:

$$g(p,\bar{p})\ \Sigma(p,\bar{p}) = -1, \tag{3.18}$$

$$g(p,\bar{p}) = C\int \frac{d^2z}{z}\ln(z\bar{z}\Lambda^2)^{\alpha}e^{ipz+i\overline{pz}} \tag{3.19}$$

$$\Sigma(p,\bar{p}) = C^3\left(J^2\int \frac{d^2z}{z^3}\ln(z\bar{z}\Lambda^2)^{3\alpha}e^{ipz+i\overline{pz}} + Q^2\int \frac{d^2z}{z\bar{z}^2}\ln(z\bar{z}\Lambda^2)^{3\alpha}e^{ipz+i\overline{pz}}\right). \tag{3.20}$$

The Fourier transforms in the large J, Q limit are given by:

$$g(p,\bar{p}) = i\pi\frac{C}{\bar{p}}\ln^{\alpha}\left(\frac{\Lambda^2}{|p|^2}\right), \tag{3.21}$$

$$\Sigma(p,\bar{p}) \approx i\pi\ \bar{p}\ C^3\ \frac{(J^2+Q^2)}{3\alpha+1}\ \ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right), \tag{3.22}$$

where $|p|^2 = p\bar{p}$. Computations are developed in Appendix A. From the unitary condition in momentum space given by Eq.(3.18), we get $\alpha = -\frac{1}{4}$ and $4\pi^2C^4\left(J^2+Q^2\right) = 1$.

Inclusion of the cutoff Λ makes the model not strictly Lorentz invariant. Just by considering low energy solution, i.e. by neglecting the kinetic part of Eq. (3.9), we have a Lorentz covariance and conformal symmetry solution. This is a free-like solution except by the inclusion of the regulator in the logarithmic term, as it is shown in Eq. (3.17). Our "quasi" conformal symmetry is not broken by this free-like solution, therefore, not spontaneously breaking of symmetry occurs. Also, by Fourier transform the quasi-conformal correlator, it is possible to solve the quasi-conformal Schwinger-Dyson equations. However, we are still in a quasi-conformal invariant case, in such a way that no information about excitations can be obtained from the quasi-conformal correlator solution.

The logarithmic dependence appearing in Eqs. (3.21) and (3.22) suggests to look for a solution of the unitarity condition in terms of the correlation functions $F_\alpha\left[\ln\left(\frac{\Lambda^2}{|p|^2}\right)\right] \equiv \ln^\alpha\left(\frac{\Lambda^2}{|p|^2}\right)$ by rewriting Eq. (3.15) as:

$$\frac{1}{F_\alpha} = \pi^2(J^2+Q^2)\int^{\ln(\Lambda^2/|p|^2)} d\ln y\, F_\alpha^3[y]. \tag{3.23}$$

From Eq. (3.23) one can get the differential equation $F'_\alpha = \pi^2(J^2+Q^2)F_\alpha^5$ (the prime means derivative), which provides the solution: $F_\alpha = \left[1+4\pi^2(J^2+Q^2)\ln\left(\frac{\Lambda^2}{|p|^2}\right)\right]^{-1/4}$.

Correlation functions involving F_α can be plugged into the Callan-Symanzik equation (with $\tilde{J}^2 = J^2+Q^2$):

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(\tilde{J})\frac{\partial}{\partial\tilde{J}} + 2\,\gamma(\tilde{J})\right]\frac{1}{\left[1+4\pi^2\tilde{J}^2\ln\left(\frac{4\Lambda^2}{|p|^2}\right)\right]^{1/4}} = 0. \tag{3.24}$$

The β-function satisfied by the fermion propagator $g(p,\overline{p})$ can be obtained

$$\beta(\tilde{J}) = 4\pi^2\tilde{J}^3 \quad ; \quad \gamma(\tilde{J}) = \pi^2\tilde{J}^2. \tag{3.25}$$

Since the β-function is positive, the coupling increases with increasing energy scale, and the model becomes strongly coupled at high energy.

We must notice that RG flows have been obtained by considering the average coupling $\tilde{J}$, which is coming from random coupling for realisations i, j, k, l. One can think that some realisations make also the coupling scale invariant or decreasing with increasing energy scale. Then, there are relevant and irrelevant operators that will grow or decrease as we flow into the IR, and these can also change as the couplings themselves evolve. However, all these contributions are screening by the net effect of the average over disorder, and we can think the model as an effective model with an effective coupling $\tilde{J}$. A nice description of this analysis is discussed in [37], where authors studied the flow of ensemble for the random Thirring model, which is also a model with an infinite number of couplings in a given realisation. Despite the above, the model serves as an effective theory below some scale, how it is shown in Chapters 4 and 5, where properties of the model are inferred from the limiting case $N = 4$.

3.2.2 Full effective action: off-diagonal solution

In order to obtain the full solution from the effective action (3.8), we have to include the off-diagonal terms of Eqs.(3.6) and (3.7). These give us the extra terms $\Sigma_\cap g_\cup$ and $\Sigma_\cup g_\cap$

for $\mathbf{Tr}\left[\hat{\Sigma}(x,x')\,\hat{G}(x',x)\right]$ and the extra terms $2g_{\cap}^2 g_{\cup}^2$ and $12\,g_+g_\cap g_- g_\cup$ for $\mathbf{Tr}\left[\left(\mathcal{P}\hat{G}\mathcal{P}^\dagger\,\hat{G}\right)^2\right]$. Collecting these terms, the action becomes:

$$
\begin{aligned}
-\overline{S}[\hat{\Sigma},\hat{G}] \;=\; & N\Bigg[\ln Pf\left[\hat{G}^{-1}\right] - \frac{1}{2}\int d^2x d^2x' \Bigg(\Sigma_+ g_+ + \Sigma_\cap g_\cup + \Sigma_\cup g_\cap + \Sigma_- g_- \\
& -\frac{J^2}{4}\sum_\alpha \left(g_\alpha(x,x')\right)^4 - \frac{Q^2}{2}\left(g_+^2 g_-^2 + g_\cap^2 g_\cup^2 + 4\,g_+ g_\cap g_- g_\cup\right)\Bigg)\Bigg],
\end{aligned}
\tag{3.26}
$$

where the label α runs over $(+,-,\cap,\cup)$. Most of the arguments developed for the diagonal case can be extended and applied to the off-diagonal solution. The Schwinger-Dyson equations are taken from the saddle-point approximation where, from the maximum for g_α we have:

$$
\Sigma_+(z,\overline{z}) = J^2 g_+^3 + Q^2\left[g_-^2 g_+ + 2\,g_\cap g_- g_\cup\right] \tag{3.27}
$$

$$
\Sigma_-(z,\overline{z}) = J^2 g_-^3 + Q^2\left[g_+^2 g_- + 2\,g_\cup g_+ g_\cap\right] \tag{3.28}
$$

$$
\Sigma_\cap(z,\overline{z}) = J^2 g_\cup^3 + Q^2\left[g_\cup^2 g_\cap + 2\,g_- g_\cap g_+\right] \tag{3.29}
$$

$$
\Sigma_\cup(z,\overline{z}) = J^2 g_\cap^3 + Q^2\left[g_\cap^2 g_\cup + 2\,g_+ g_\cup g_-\right]. \tag{3.30}
$$

In the same way, the equations can be solved in the conformal limit, which is suggested by reparametrization arguments. Thus, in real space we assume that $g_+(z,\overline{z}) = \frac{a}{z}\ln^\alpha\left(|z|^2\Lambda^2\right)$, $g_-(z,\overline{z}) = \frac{a}{\overline{z}}\ln^\alpha\left(|z|^2\Lambda^2\right)$ and $g_\cap = g_\cup = \frac{b}{|z|}\ln^\alpha\left(|z|^2\Lambda^2\right)$, obtaining:

$$
\Sigma_+(z,\overline{z}) = \left[J^2\frac{a^3}{z^3} + Q^2\left(\frac{a^3}{\overline{z}|z|^2} + 2\,\frac{ab^2}{\overline{z}|z|^2}\right)\right]\ln^{3\alpha}\left(|z|^2\Lambda^2\right) \tag{3.31}
$$

$$
\Sigma_-(z,\overline{z}) = \left[J^2\frac{a^3}{\overline{z}^3} + Q^2\left(\frac{a^3}{z|z|^2} + 2\,\frac{ab^2}{z|z|^2}\right)\right]\ln^{3\alpha}\left(|z|^2\Lambda^2\right) \tag{3.32}
$$

$$
\Sigma_\cup(z,\overline{z}) = \Sigma_\cap(z,\overline{z}) = \left[J^2\frac{b^3}{|z|^3} + Q^2\left(\frac{b^3}{|z|^3} + 2\,\frac{a^2 b}{|z|^3}\right)\right]\ln^{3\alpha}\left(|z|^2\Lambda^2\right), \tag{3.33}
$$

where $|z|^2 = z\overline{z}$ and a and b are two constant parameters. From the minimization of effective action over self-energy we can obtain the second group of Schwinger-Dyson equations, which are of the form $\hat{G}^{-1} = -\hat{\Sigma}$ in momentum space. Accordingly, we need the matrix equations (3.6) and (3.7) expressed in Fourier space. In conformal limit they take the form:

$$\hat{G}^{-1}(p,\overline{p}) = \frac{1}{g_+g_- - g_\cap g_\cup}\begin{pmatrix} g_- & -g_\cap \\ -g_\cup & g_+ \end{pmatrix}(p,\overline{p}) = \begin{pmatrix} -\Sigma_+ & -\Sigma_\cap \\ -\Sigma_\cup & -\Sigma_- \end{pmatrix}(p,\overline{p}) = -\hat{\Sigma}(p,\overline{p}). \tag{3.34}$$

The latter takes the form of the unitary condition (3.18) when off-diagonal terms are neglected. Equations from diagonal terms are equal. The same occurs from the off-diagonal terms, therefore we have two independent equations to obtain constants a and b. By taking the Ansatz, it is not difficult to find that, after Fourier transform (see Appendix A), we have

$$4\pi^2\left[a^2(J^2+Q^2)+2b^2Q^2\right]\ln^{4\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right) = \frac{1}{(a^2-b^2)} \tag{3.35}$$

and

$$4\pi^2\left[b^2(J^2+Q^2)+2a^2Q^2\right]\ln^{4\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right) = -\frac{1}{(a^2-b^2)} \tag{3.36}$$

which again requires $\alpha = -1/4$ and has the solution $b = \pm ia$. Inserting the solution in any of previous equation we found that:

$$a^4 = \frac{1}{8\pi^2(J^2-Q^2)}. \tag{3.37}$$

This proves that the Schwinger-Dyson equations can be easily solved in the conformal limit. In this limit, the off-diagonal part changes the contribution of the Q interaction in the self-energy. However, as for the diagonal solution, the quasi-conformal correlator does not give us any hint about excitation spectrum. Since we are still in a quasi-conformal invariant case, excitations could be gapless if the system remains critial and/or gapped if not.

3.2.3 Critical correlator at large distances in the conformal symmetry limit

The previous case shows that except for a very soft breaking obtained by the envelope function $\ln\left(\frac{|p|^2}{\Lambda^2}\right)^{\frac{1}{4}}$, the conformal symmetry forces the correlation function $g_+(z,\overline{z}) \propto 1/z$ as in the free $1+1\,d$ case. Here we show that a critical powerlaw decay of the correlators at large distance with non free-like exponent $\Gamma \neq 1$ can also be obtained from the conformal symmetry limit, for intermediate values of the couplings. Actually we will use linearization in approximating the self-energy obtained from the Schwinger-Dyson equations, which will hold for just one single value of the coupling strengths. However, the method could be extended to introduce dependence of Γ on J in some range of J values.

We consider the correlator by keeping just the lowest order of the expansion in inverse powers of z for large z. Introducing the additive correction $\eta(z)$ and the expansion parameter $\lambda << 1$ we have:

$$g(z,\bar{z}) \approx i\, C_\Gamma \frac{r_0}{z}\, [1 + \lambda\, \eta(z)] + ... \tag{3.38}$$

We linearize the saddle point equations for Σ, given in Eq. (3.16), with $\eta(z) = \eta_J(z) + \eta_Q(z,\bar{z})$:

$$\Sigma_J(z,\bar{z}) \approx -i \left(\frac{J}{\pi u_0}\right)^2 \frac{1}{C_\Gamma^3} \left(\frac{r_0}{z}\right)^3 \left(1 + 3\lambda\, \eta_J(z)\right), \tag{3.39}$$

$$\Sigma_Q(z,\bar{z}) \approx -i\, 3\lambda \left(\frac{Q}{\pi u_0}\right)^2 \frac{1}{3C_Q^3} \left(\frac{r_0^3}{z\, |z|^2}\right), \tag{3.40}$$

where we have put $\eta_Q(z,\bar{z})$ just constant in Σ_Q for simplicity. We have dropped the regularizing logarithmic term of Eq. (3.31). C_Γ and C_Q are fixed by the unitarity condition. We invert the Fourier transform of Eq. (3.38) and use the Schwinger-Dyson equation $g^{-1}(p) = g_0^{-1}(p) - \Sigma(p)$, assuming that, as in the conformal symmetry limit, all three quantities have a $\bar{p}-$ dependence, which we drop, obtaining:

$$\begin{aligned} C_\Gamma \times g^{-1}(p) \quad &\to \quad \frac{1}{\frac{1}{i\,\bar{p}} - i\,\lambda \mathcal{FT}\left[(r_0/z)\times\eta\right](p,\bar{p})} \\ &\approx \quad i\bar{p} - i\,\bar{p}^2 \lambda \mathcal{FT}\left[(r_0/z)\times\eta\right](p,\bar{p}) = i\,\bar{p} - C_\Gamma \Sigma(p,\bar{p}), \end{aligned} \tag{3.41}$$

where $\mathcal{FT}$ stands for Fourier transformation. The last equality allows to write down two separate differential equations for $\eta_J(r)$ and $\eta_Q(r)$. The first one is:

$$\partial^2_{z/r_0}\left[\frac{r_0}{z} \times \eta_J(z)\right] \approx \left(\frac{J}{\pi u_0 C_\Gamma}\right)^2 \left(\frac{r_0}{z}\right)^3 \left(\frac{1}{\lambda} + 3\,\eta_J(z)\right). \tag{3.42}$$

Introducing $h_J(z) = \left[\frac{1}{z} \times \eta_J(z)\right]$, the first contribution in Eq. (3.38) can be written as $\tilde{g}(z)/C_\Gamma = \frac{i}{z} + i\,\lambda\, h_J$, where z is in units of r_0. Defining $b = \left(\frac{J}{\pi u_0 C_\Gamma}\right)^2$, we get the simple differential equation

$$\partial_z^2\, h_J(z) - \frac{3b_J}{z^2}\, h_J(z) = \frac{b_J}{\lambda\, z^3}, \tag{3.43}$$

whose solution is

$$h_J(z) = -\frac{1}{z} \frac{1}{\lambda\left[1 + 2\left(1 - \frac{1}{b_J}\right)\right]} + z^{\frac{1}{2}}\left(c_1\, z^{s/2} + c_2\, z^{-s/2}\right), \qquad s^2 = 1 + 12\, b.$$

Putting $c_1 = 0$, we get:

$$\frac{\tilde{g}(z)}{C_\Gamma} = \frac{i}{z} - \frac{i}{z} \frac{1}{\left[1 + 2\left(1 - \frac{1}{b_J}\right)\right]} + \lambda\, c_2\, \frac{i}{z^{\frac{s}{2} - \frac{1}{2}}}. \tag{3.44}$$

If we want that the free-like $1/z$ dependence to disappear in favour of $\frac{1}{z^\Gamma} = 1/\left(z^{\frac{s}{2}-\frac{1}{2}}\right)$, we have to put $b_J = 1$ and $c_2 \propto 1/\lambda$. Also the exponent $\Gamma = (\sqrt{13}-1)/2 \approx 1.3$ is fixed. The value of the exponent has been obtained just for a fixed J, Q pair, due to the linearization procedure adopted in Eqs. (3.39) and (3.40). It is remarkable that it is independent of λ.

We now turn to the second term, Σ_Q, which provides a contribution that does not conserve chirality.

As Σ_Q involves $g_-^2 g_+$ in real space, in place of Eq. (3.42), we have:

$$\partial^2_{z/r_0}\left[\frac{r_0}{z} \times \eta_Q(z,\bar{z})\right] \approx -\left(\frac{Q}{\pi u_0}\right)^2 \frac{C_\Gamma}{C_Q^3}\,\frac{r_0^3}{z\,\bar{z}^2}. \tag{3.45}$$

Defining once more $b_Q = \left(\frac{Q}{\pi u_0 C_Q}\right)^3$, η_Q solves the differential equation

$$\partial_z^2\,\eta_Q(z,\bar{z}) - \frac{2}{z}\partial_z\,\eta_Q(z,\bar{z}) + \frac{2}{z^2}\,\eta_Q(z,\bar{z}) = -b_Q\,\frac{1}{\bar{z}^2}\,e^{3\,h_Q(z,\bar{z})}.$$

The solution for this equation, which adds non chirality to the general solution, is $\eta_Q(z,\bar{z}) = \frac{z^2}{\bar{z}^2}\left[1 - b_Q\ \ln(z\bar{z})\right]$. Adding this contribution to Eq. (3.44) gives finally

$$\frac{g(z,\bar{z})}{C_\Gamma} = \frac{i}{z} + i\,\lambda\,h_J(z) + \frac{i}{z}\,\lambda\,\eta_Q(z,\bar{z}) \approx \left(\frac{i}{z} + i\,\lambda h_J(z)\right)\left[1 + \frac{1}{\frac{i}{z}}\frac{i\,\lambda\,\eta_Q(z,\bar{z})}{z}\right]$$
$$\rightarrow \frac{i}{\left(\frac{z}{r_0}\right)^{\frac{\sqrt{13}}{2}-\frac{1}{2}}}\left[1 + \lambda\,\frac{z^2}{\bar{z}^2}\frac{1}{1 + b_Q\ \ln\left(\frac{z\bar{z}}{r_0^2}\right)}\right]. \tag{3.46}$$

The term in square brackets recalls an analogous term (but in $\mathcal{FT}$), characterizing F_α in Eq. (3.24). However, the linearization of Eqs. (3.39) and (3.40) has the consequence that the non chiral contribution disappears in the limit $\lambda \longrightarrow 0$.

As last comment, correlator $g(z,\bar{z})$ can be Fourier transformed with respect to time and acquires a powerlaw dependence $\sim \omega^{\Gamma-1}$. As the exponent is smaller than unity (in the approximate derivation presented above is $\Gamma - 1 = 0.3$), we find that this correlation function originates from Non-Fermi-Liquid collective excitations.

–4– Approximate Free Energy in the non-chiral 1+1 dimensional extended SYK model

This Chapter is the core of the original work and includes most of the results. It is a direct continuation of the previous Chapter. However, we heavily use results and concepts coming from the Chapter 5. Readers are invited to start reading the next Chapter if they want to go deeper into the topic, or continue with this Chapter first to go directly to the main results.

In the following Chapter 5 we derive the nature of the excitation spectrum and some correlators for the $N = 4$ version of the model which can be bosonized exactly. We find that the pseudo-charge excitations display a gap, while the pseudo-spin ones are gapless. In this Chapter we try to draw an analogy between the $0+1\,d$ SYK model and our extended model by assuming that the features of the spectrum found in the $N = 4$ case still hold when $N >> 4$.

In the $0+1\,d$ SYK model the conformal symmetry of the $N \to \infty$ limit is spontaneously broken by the ground state. Then, symmetry is explicitly broken by reintroduce the derivative term of the free Lagrangian as an ultraviolet correction. This implies that the Goldstone bosons acquire a gap and they are denoted as pseudo-Goldstone in the literature. At higher space dimension the symmetry is broken by ultraviolet corrections and we can guess that gapped excitations are also present in our model. In this Chapter we derive an expression for the free energy of the model in terms of the Green's functions that solve the SD equations and we assume that the correlators of the $N >> 4$ case consist again of two branches, one corresponding to the pseudo-spin, with free-like excitations (Eq.5.77) and renormalized velocity $u_s = u_0\sqrt{1 - J/\pi u_0}$ of the linear spectrum, and another corresponding to the pseudo-charge, with gapped excitations (Eq.5.100) with velocity $u_c = u_0\sqrt{1 + J/\pi u_0}$, that reduce to the free-like excitations when $J \to 0$. While the pseudo-charge excitations allow any large value of J, the velocity vanishes at $J/\pi u_0 = 1$ in the pseudo-spin branch. Therefore, this should be considered as the strong coupling limit.

Evaluation of the free energy requires also that the cross-chirality Green's functions $g_\cap, g_\cup$ are known. We show that the correlators corresponding to these ones in the $N = 4$ model vanish because they are non number conserving (Appendix C.3). A direct evaluation of these correlators would require a precise knowledge of the excitation spectrum and it is out of the present possibilities. However, the correlator corresponding to $g_\cap g_\cup$ (Eq.5.104) is number conserving in the model and can be evaluated in the $N = 4$ limit exactly.

By looking at the free energy, we realize that if the coupling Q is set equal to the coupling J, the single $g_\cap$, $g_\cup$ are not needed in the free energy expression, as they always appear as a product. Consequently we have decided to limit ourselves to the $Q = J$ case. In this way we can surmise the contribution of the off-diagonal chirality correlators to the free energy and give its approximate expression in terms of the velocities u_c, u_s and of a gap Δ for the gapped excitation spectrum branch, which is chosen as a parameter. In this way we are able to plot the free energy vs the gap Δ, given the value of J and of the velocities in the two spectrum branches and look for the minimum of the free energy for a given coupling J. The $J = 0$ limit corresponds to the conformal symmetry at $N \to \infty$ limit which is a free limit. The results at finite J are plotted in Fig. 4.1 and 4.2. They show that a small value of J is unable produce a minimum of the free energy at finite gap, but the minimum at $J = 0$ disappears and the free energy increase with Δ. When J increases the free energy develops a minimum which becomes the absolute minimum of the free energy in a restricted range of values of J. In this range, at least within our approximations, our model confirms the presence of gapped excitations in the spectrum. At higher values of J, $0.7 < J < 0.9$, the minimum is still present but it is metastable. Meanwhile a stable minimum of free energy is lost. Values of $J > 0.9$ make the minimum fully disappear.

We now turn to show these results starting from the derivation of the free energy in the $0 + 1\ d$ SYK model [2].

4.1 Free energy

In the following we are going to derive the free energy as it can be done for the $0 + 1\ d$ case. It is possible to relate the free energy with the effective action by writing the original partition function as a functional integral and then inserting the solutions of Green's function and self-energy in the resulting expression [44, 45]. Due to the fact that G and Σ obey the equations of motion, the only contributing term are the derivatives of the explicit dependence on J and Q, so that we can take derivatives to avoid evaluating the Pfaffian term [2]:

$$\frac{1}{Pf(A)}\frac{\partial Pf(A)}{\partial x} = \frac{1}{2}\,Tr\left(A^{-1}\frac{\partial A}{\partial x}\right).$$

Therefore, for the J part first, we have:

$$\partial_J \ln Pf\left[-(\partial_\tau + ai\partial_x) - \Sigma_a(x.x')\right] = \frac{1}{2}\,Tr\left(G\,\frac{\partial\Sigma}{\partial J}\right); \tag{4.1}$$

$$-\partial_J \frac{1}{2}(\Sigma\, G) = -\frac{1}{2}\left[\Sigma \partial_J G + G \partial_J \Sigma\right]; \tag{4.2}$$

$$\begin{aligned}\partial_J\left[\frac{1}{2}\frac{J^2}{4}\sum_\alpha (g_\alpha(x,x'))^4\right] &= \frac{1}{2}\,2\frac{J}{4}\left(\sum_\alpha g_\alpha(x,x')\right)^4 + \frac{1}{2}\sum_\alpha J^2\,(g_\alpha(x,x'))^3\ \partial_J g_\alpha \\ &= \frac{J}{q}\,G^q + \frac{1}{2}\mathbf{Tr}\left[\Sigma\,\partial_J G\right]\end{aligned} \tag{4.3}$$

where the last term is $\Sigma_+ g_+ + \Sigma_\cap g_\cup + \Sigma_\cup g_\cap + \Sigma_- g_-$ and cancels with the one in the previous line. Therefore, by summing all together, only one term survives from the J contribution:

$$-\beta F^{(J)} = J\,\partial_J\left(-\frac{\beta F^{(J)}}{N}\right) = \frac{(\beta J)^2}{q}\int d^2z\ \sum_\alpha (g_\alpha(z,\bar z))^4 \tag{4.4}$$

Applying the same argument to evaluate the Q^2 contribution to the free energy we obtain the final result:

$$\begin{aligned}-\beta F &= J\,\partial_J\,(-\beta F/N) + Q\,\partial_Q\,(-\beta F/N) \\ &= \int d^2z\left(\frac{(\beta J)^2}{4}\sum_\alpha (g_\alpha(z,\bar z))^4 + \frac{(\beta Q)^2}{2}\left[g_+^2 g_-^2 + g_\cap^2 g_\cup^2 + 4 g_+ g_\cap g_- g_\cup\right](z,\bar z)\right).\end{aligned} \tag{4.5}$$

As we can observed, single cross-chirality correlators $g_\cap$, $g_\cup$ are required. We do not have any information about them. However, the product $g_\cap g_\cup$ can be assumed heuristically from correlators obtained in the $N = 4$ case. This can help us to circumvent the lack of information about off-diagonal correlators, as we will explore in Section 4.3.

4.2 Green's functions

Let us recap the necessary Green's functions to be used in the free energy expression. From the $N = 4$ case (see Chapter 5 for details) we have suggestions for $g_\pm(z,\bar z)$ (see Eq.5.100) and for the product $g_\cap(z,\bar z)\ g_\cup(z,\bar z)$ (see Eq.5.104). Here we emphasize the fact that the capital G will refer to correlators coming from the $N = 4$ case, while g stands for large N case. These quantities are all real and non chiral. However, in principle we have no hint on $g_\cap(z,\bar z)$, which is not expected to be real, but can be reasonably assumed to be complex conjugate of $g_\cup(z,\bar z)$. In fact, as it turned out in Section 3.2.2, $g_{\cap/\cup} \sim b/|z|$ with $b = \pm ia$, and a being a real constant.

Assuming that the functional form of the correlators of the $N = 4$ case holds also for the large N case, we are going to use the average Majorana fermion two-point function

$$G_\pm(r) = \frac{1}{4}\sum_i \langle \psi^i_\pm(r)\psi^i_\pm(0)\rangle \tag{4.6}$$

as an indication for the $N >> 4$ limit. In the $N = 4$ case, it is possible to write the latter as:

$$G_{\pm}(r) = \frac{1}{4} \sum_{\sigma=\uparrow,\downarrow} \langle c_{\sigma\pm}(r) c^{\dagger}_{\sigma\pm}(0) + c^{\dagger}_{\sigma\pm}(r) c_{\sigma\pm}(0) \rangle. \tag{4.7}$$

To reach this result, we have to introduce the complex fermions (5.64) and (5.65). The bosonized version of this two-point function is:

$$G_{\pm}(r) = \pm \frac{i}{2\pi\alpha} (e^{\frac{\pi}{2}\langle \phi_c(r)\phi_c(0) - \phi_c^2(0)\rangle} e^{\frac{\pi}{2}\langle \theta_c(r)\theta_c(0) - \theta_c^2(0)\rangle} e^{\frac{\pi}{2}\langle \phi_s(r)\phi_s(0) - \phi_s^2(0)\rangle} e^{\frac{\pi}{2}\langle \theta_s(r)\theta_s(0) - \theta_s^2(0)\rangle}). \tag{4.8}$$

The explicit form of the latter depends on the strength of the coupling. By considering the strong coupling limit, the Green's chiral conserving function $g_{\pm}(z, \bar{z}) \sim G_{\pm}(r)$ is:

$$\begin{aligned} g_{\pm}(z,\bar{z}) = \pm \tfrac{1}{2\pi\alpha} \left(\tfrac{\alpha}{\sqrt{(u_s\tau+\alpha)^2 + x^2}} \right)^{\frac{1}{4}\left(\mathcal{K}_s + \frac{1}{\mathcal{K}_s}\right)} e^{-\frac{1}{8}\left(\mathcal{K}_c + \frac{1}{\mathcal{K}_c}\right)\left[\int_0^{i\,x+u_c\tau} \frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\, K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right) dz + c.c.\right]} \\ \times \prod\nolimits_{\pm} e^{\frac{\Delta}{8\mathcal{K}_c}\left[e^{-\frac{\Delta u_c \tau \pm i\,x}{L}}\right]} \left[\sqrt{\tfrac{\pm i\,x}{2\pi L}}\, \Gamma\left(\tfrac{\pm i\,x}{L}\right)\ e^{\mp i\,\frac{x}{L}(\gamma+1)} \right]^{\frac{\Delta}{8\mathcal{K}_c} e^{-\frac{\Delta u_c\tau}{L}}} \end{aligned} \tag{4.9}$$

which reproduces the free-like case when $J \longrightarrow 0$. Evaluation of the free energy requires also the off-diagonal Green's functions $g_{\cap}(z, \bar{z})$ and $g_{\cup}(z, \bar{z})$, which are different from zero in the large N limit but they unfortunately vanish in the $N = 4$ model (Appendix C.3). For instance, some of the combinations which mix both left and right-movers Majorana fermions that we can consider for the off-diagonal Green's functions are: the sum on Majorana's flavors,

$$\frac{1}{4} \sum_i \psi^i_{\pm}(r)\psi^i_{\mp}(0) = \frac{1}{4} \left[\psi^1_{\pm}(r)\psi^1_{\mp}(0) + \psi^2_{\pm}(r)\psi^2_{\mp}(0) + \psi^3_{\pm}(r)\psi^3_{\mp}(0) + \psi^4_{\pm}(r)\psi^4_{\mp}(0) \right], \tag{4.10}$$

the next combination that we will call $O^x_{TS}(r)$,

$$O^x_{TS}(r) = \frac{1}{2} \left[(\psi^1_+\psi^1_- - \psi^2_+\psi^2_- + \psi^3_-\psi^3_+ - \psi^4_-\psi^4_+) - i(\psi^1_+\psi^2_- + \psi^2_+\psi^1_- + \psi^3_-\psi^4_+ + \psi^4_-\psi^3_+) \right], \tag{4.11}$$

and other similar combination that we call $O^y_{TS}(r)$,

$$O^y_{TS}(r) = -\frac{1}{2} \left[(\psi^1_+\psi^2_- + \psi^2_+\psi^1_- - \psi^3_-\psi^4_+ - \psi^4_-\psi^3_+) + i(\psi^1_+\psi^1_- - \psi^2_+\psi^2_- - \psi^3_-\psi^3_+ + \psi^4_-\psi^4_+) \right]. \tag{4.12}$$

Introducing again complex fermions (5.64) and (5.65) and its complex conjugates, previous expressions are given by:

$$\frac{1}{4} \sum_i \psi^i_{\pm}(r)\psi^i_{\mp}(0) = \frac{1}{4} \sum_{\sigma=\uparrow,\downarrow} \left[c_{\sigma\pm}(r) c^{\dagger}_{\sigma\mp}(0) + c^{\dagger}_{\sigma\pm}(r) c_{\sigma\mp}(0) \right], \tag{4.13}$$

$$O^x_{TS}(r) = c^{\dagger}_{\uparrow+}(r) c^{\dagger}_{\uparrow-}(r) + c^{\dagger}_{\downarrow-}(r) c^{\dagger}_{\downarrow+}(r), \tag{4.14}$$

$$O_{TS}^{y(r)} = -i\left[c^\dagger_{\uparrow+}(r)c^\dagger_{\uparrow-}(r) - c^\dagger_{\downarrow-}(r)c^\dagger_{\downarrow+}(r)\right]. \tag{4.15}$$

They reflect that, starting from mixed chirality Majorana components, expressions of mixed chirality complex fermions are reached. The first of those, Eq. (4.13), is non-chiral conserving and its correlator in the $N = 4$ case vanishes (see Appendix C, Eq. (C.45)) and is given by:

$$G_{\cap/\cup}(r) = \frac{1}{4}\sum_i \langle \psi^i_\pm(r)\psi^i_\mp(0)\rangle = \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow}\langle c_{\sigma\pm}(r)c^\dagger_{\sigma\mp}(0) + c^\dagger_{\sigma\pm}(r)c_{\sigma\mp}(0)\rangle = 0. \tag{4.16}$$

The other two, Eqs. (4.14) and (4.15), are non number conserving and, therefore, their correlators also vanish in the $N = 4$ case. Accordingly, no information about single $g_{\cap/\cup}(z,\bar{z})$ is suggested from the $N = 4$ case, at least for the cross-chirality expressions considered above.

On the other side, the correlator corresponding to $g_\cap g_\cup$ is number conserving and it does not vanish in the $N = 4$ model. To see this, let's consider again the off-diagonal expression Eq. (4.13) and let us define the four-Majorana component

$$\sum_{i,j}\psi^i_+(r)\psi^i_-(0)\psi^j_-(r)\psi^j_+(0) = \sum_{\sigma,\sigma'}\Big[c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0) + c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0)$$
$$+c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0) + c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0)\Big]. \tag{4.17}$$

The corresponding correlator of the last expression depends on the first and fourth terms of the right side, while the others do not contribute. In this way, we obtain heuristically the correlator corresponding to the product $g_\cap g_\cup$ to be a four-point function that, in the $N = 4$ language, we define as $G_{\cap\cup}$:

$$\begin{aligned}
G_{\cap\cup} &\equiv \sum_{i,j}\langle\psi^i_+(r)\psi^i_-(0)\psi^j_-(r)\psi^j_+(0)\rangle \\
&\sim \sum_{\sigma,\sigma'=\uparrow,\downarrow}\langle c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0) + c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0) \\
&\quad +c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0) + c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0)\rangle \\
&\sim \langle c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c_{\sigma-}(r)c^\dagger_{\sigma+}(0) + c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0)\rangle \neq 0.
\end{aligned} \tag{4.18}$$

These behaviors of the correlators for $g_{\cap/\cup}$ and $g_\cap g_\cup$ corresponds to the ones of the triple-pairing operators $O^x_{TS}(r)$, $O^y_{TS}(r)$ and $O^z_{TS}(r) = O^x_{TS}(r) + iO^y_{TS}(r)$, which describe pairing with zero total momentum and are detailed in Subsection 5.4.4. Specifically, the correlators of the non number conserving operators

$$O^z_{TS}(r) = 2c^\dagger_{\uparrow+}(r)c^\dagger_{\uparrow-}(r) = \frac{1}{\pi\alpha}e^{-i\sqrt{2\pi}\theta_c(r)}e^{-i\sqrt{2\pi}\theta_s(r)} \tag{4.19}$$

and its complex conjugate, vanish, while for the number conserving operator

$$O_{TS}^{z\dagger}(r)O_{TS}^{z}(0) = 4c_{\uparrow+}(r)c_{\uparrow-}^{\dagger}(0)c_{\uparrow-}(r)c_{\uparrow+}^{\dagger}(0) \tag{4.20}$$

does not. As Eq. (4.20) behaves as Eq. (4.18) in the $N = 4$ case, we adopt the correlator $\langle O_{TS}^{z\dagger}(z,z)O_{TS}^{z}(0,0)\rangle$ as an indication on the product $g_{\cap}(z,\bar{z})g_{\cup}(z,\bar{z})$. Its explicit expression is computed in Chapter 5 (Subsection 5.4.4) and allows us to write $g_{\cap}(z,\bar{z})\; g_{\cup}(z,\bar{z}) \sim \langle O_{TS}^{z\dagger}(z,z)O_{TS}^{z}(0,0)\rangle$ in such a way that we have:

$$g_{\cap}g_{\cup}(z,\bar{z}) = \frac{1}{\pi^2\alpha^2}\left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{\mathcal{K}_s}} e^{-\frac{1}{2\mathcal{K}_c}\left[\int_0^{i\,x+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\,K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right)dz+\ln(\alpha^2)+c.c.\right]}$$
$$\times\prod_{\pm} e^{\frac{\Delta}{2\mathcal{K}_c}\left[e^{-\frac{\Delta u_c\tau\pm i\,x}{L}}\right]}\left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)\,e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\frac{\Delta}{2\mathcal{K}_c}e^{-\frac{\Delta u_c\tau}{L}}} \tag{4.21}$$

which reproduces the free-like case when $J \longrightarrow 0$. In this way, we can relate the triple-pairing operators with the correlators needed to compute the free energy, with the single $g_{\cap/\cup}(z,\overline{z})$ being non number conserving, while the product $g_{\cap}g_{\cup}$ is number conserving, in the $N = 4$ case.

4.3 Approximate Free energy in the Q = J case

Since we have no information about single $g_{\cap/\cup}(z,\overline{z})$ in the large N limit (and it is not possible to obtain any suggestion from $N = 4$ model), we restrict the model to the case $J^2 = Q^2$:

$$\beta F = \frac{(\beta J)^2}{4}\int d^2z\left(\sum_{\alpha}\left(g_{\alpha}(z,\bar{z})\right)^4 + 2\left[g_+^2g_-^2 + g_{\cap}^2g_{\cup}^2 + 4\,g_+g_{\cap}g_-g_{\cup}\right](z,\bar{z})\right). \tag{4.22}$$

By doing this, we can rearrange the related terms in order to obtain dependence only in the product $g_{\cap}g_{\cup}$. We proceed as follows:

$$\begin{aligned} g_{\cap}^4 + g_{\cup}^4 + 2g_{\cap}^2g_{\cup}^2 &= \left(g_{\cap}^2 + g_{\cup}^2\right)^2 \equiv \left(g_{\cap}^2 + \overline{g_{\cap}}^2\right)^2 \\ &= 4\left[(\Re e g_{\cap})^2 - (\Im m g_{\cap})^2\right]^2 = 4\left[\Re e(g_{\cap}g_{\cup}) + \Im m(g_{\cap}g_{\cup})\right]^2 \\ &= 4\left|g_{\cap}g_{\cup}\right|^2. \end{aligned} \tag{4.23}$$

The free energy takes the form:

$$\begin{aligned} -\beta F &= (\beta J)^2\int d^2z\,\left\{(g_+g_-)^2 + (g_{\cap}g_{\cup})^2 + 2\,g_+g_{\cap}g_-g_{\cup}\right\} \\ &= \frac{(\beta J)^2}{2}\int d^2z\left(\mathbf{Tr}\,\hat{G}^2\right)^2, \end{aligned} \tag{4.24}$$

with $\hat{G}$ given in Eq. (3.6). This is the extension of the result for the $0+1d$ Sachdev-Ye-Kitaev model [2] to the $1+1\,d$ non chiral case for $J^2 = Q^2$. We now discuss how this expression behaves in presence of the gap Δ, defined in Eq. (5.94) as:

$$\frac{\Delta}{L} = \frac{2}{\alpha}\sqrt{\frac{\mathcal{K}_c^{-2}-1}{\mathcal{K}_c^{2}}}, \tag{4.25}$$

where $\mathcal{K}_c^{-1} = \sqrt{1+\frac{J}{\pi u_0}}$ is a dimensionless factor related to the interaction. Up to now, all ours calculations have been performed analytically. Now, graphs are drawn from numerical solutions of the diagonal correlators $g_{\pm}(z,\bar{z})$, given by Eq. (4.9), and the off-diagonal correlators product $g_{\cap}(z,\bar{z})g_{\cup}(z,\bar{z})$, given by Eq. (4.21). We also have to consider the gap Δ, which is choosen as a parameter.

In Fig. 4.1 it is shown the free energy βF vs the gap Δ for different values of the coupling J normalized to $u_0\pi$, in the physical bound $0 < J/u_0\pi < 1$. In the range $J \leq 0.3$, the model is unable to produce a minimum of the free energy for a finite gap. When J goes to zero, the minimum disappears and the free energy increases with Δ. For increasing values of J, the free energy develops an absolute minimum which confirms, at least within our approximations, the presence of gapped excitations in the model's spectrum in the range $0.4 \leq J \leq 0.6$. At higher values of J, let us say $0.7 \leq J \leq 0.8$, the minimum is still present but becomes metastable, meanwhile a stable minimum of the free energy is lost and the model with a gap no longer holds. Close to the upper limit $J = 1$ of the physical bound, when $J \geq 0.9$, the minimum fully disappears, as it is shown in Fig.4.1(b) and, more in detail, in Fig. 4.2.

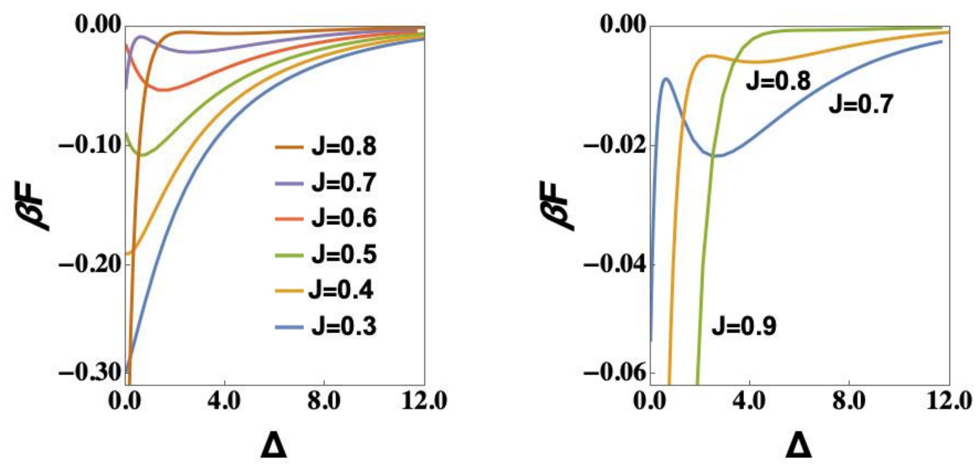


Figure 4.1: Free energy βF vs the gap Δ for different values of J. From (a) it is observed that the model does not develop any minimum when $J = 0.3$ or lower. At $J = 0.4$ an absolute minimum starts. Gapped excitations hold until $J = 0.6$. Figure (b) shows high values of J. The minimum is not longer stable, but it is metastable and well defined at $J = 0.7$. For $J > 0.7$, the minimum becomes shallow and eventually disappears for $J \geq 0.9$.

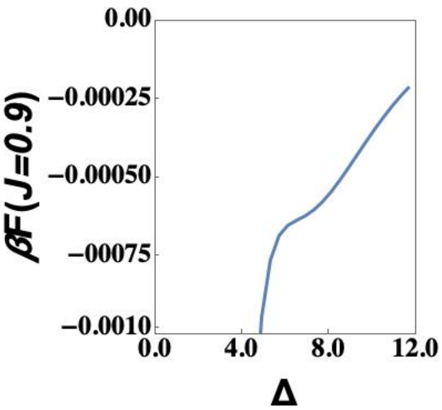


Figure 4.2: Free energy βF vs the gap Δ for $J = 0.9$. In the strong coupling limit, close to the upper limit $J = 1$ of the physical bound, the minimum of the free energy fully disappears.

In Fig. 4.3 are represented the x/L dependent Green's functions used in the code in the gapped regime $J = 0.6$, where x represents the difference of the two space points, due to the translational invariance, and L is the size of the system. Both $g_{++} \equiv g_+$ and $g_{+-}g_{-+} \equiv g_\cap g_\cup$ are taken from the $N = 4$ case in such a way that $g_+ \sim G_{++}$ and $g_\cap g_\cup \sim G_{+-}G_{-+}$ by assuming that the functional form of the correlators are the same for $N >> 4$. Correlators were drawn for the specific time $\tau = 0.01$, and the fixed gap $\Delta = 1.71$. $g_+ = \overline{g_-}$ tends to unity by construction when $x \longrightarrow 0$ and shows a crossover from powerlaw decay to exponential decay $\sim e^{-\frac{\pi x}{8}\frac{\Delta}{\mathcal{K}_c}}$ at large x. The product $g_\cap(z,\overline{z})g_\cup(z,\overline{z})$ has been chopped to unity at small distances and has an exponential decay $\sim e^{-\frac{\pi x}{2}\frac{\Delta}{\mathcal{K}_c}}$ at large distances. This does not affect the qualitative results.

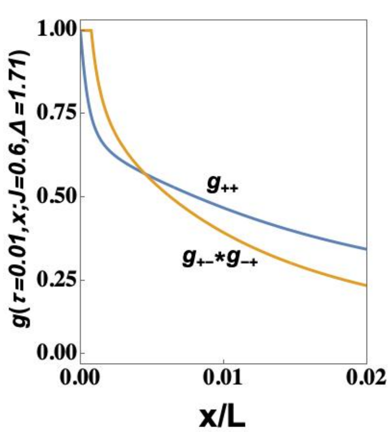
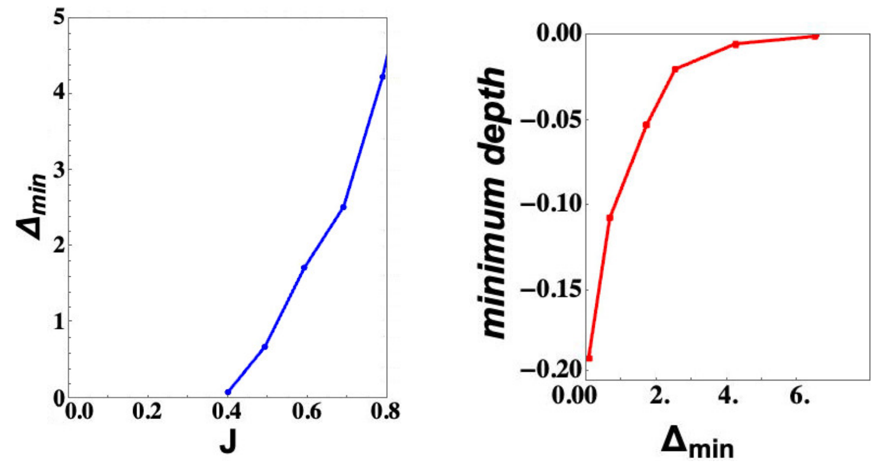


Figure 4.3: Spatial x/L dependent decay of the normalized Green functions $g_{++} \equiv g_+(\tau = 0,01, x; J = 0.60, \Delta = 1.71)$ and $g_{+-} * g_{-+} \equiv g_\cap(\tau = 0,01, x; J = 0.60, \Delta = 1.71)g_\cup(\tau = 0,01, x; J = 0.60, \Delta = 1.71)$ used to compute the free energy. The figure shows them for the specific fixed values $\Delta = 1.71$, $\tau = 0.01$ and $J = 0.6$, in the gapped regime.



(a) (b)

Figure 4.4: (a) Gap of the minimum Δ_{min} vs coupling J. By increasing J also the Δ_{min} increases. (b) Minimum depth of free energy vs gap of the minimum Δ_{min}. The minimum of βF approaches to zero when Δ_{min} increases, i.e. when J increases, becoming shallow.

Finally, from Fig. 4.4 (a) we can realize that, as J increases, so does Δ of the minimum. The divergent tendency for values close to the upper limit $J = 1$ of the physical bound reflects the fact that the minimum becomes shallow for higher values of J, as can be seen in Fig. 4.4(b).

In conclusion, we have drawn an analogy between the $0 + 1$ d SYK model and our extended model by assuming that the features of the spectrum found in the $N = 4$ case still hold when $N >> 4$. We surmise that the spectrum of the $N >> 4$ case includes again two branches as in the $N = 4$ case: a gapless one, corresponding to the pseudo-spin sector with renormalized velocity $u_s = u_0\sqrt{1 - J/\pi u_0}$, and a gapped one, with gap Δ, corresponding to the pseudo-charge sector with velocity $u_c = u_0\sqrt{1 + J/\pi u_0}$. While the pseudo-charge excitations have no restriction in the value of J, the velocity of the pseudo-spin branch vanishes at $J/\pi u_0 = 1$, so that we consider $J/\pi u_0 \lesssim 1$ as the strong coupling limit in this approach. In this way, the physical bound $0 < J/\pi u_0 < 1$ is established. We derive an expression for the free energy βF in terms of the Green function given by Eq. (3.6). We evaluate it , for the case $J = Q$, in the zero temperature limit $\beta \to \infty$, by adopting heuristically the functional form of $g_\pm(z, \bar{z})$ and $g_\cap(z, \bar{z})g_\cup(z, \bar{z})$ obtained in the Chapter 5 for the $N = 4$ case (Eqs. 4.9 and 4.21). These functions tend to the "free-like" limit when $J/\pi u_0$ is small and $\Delta \to 0$, a limiting form that has been discussed in the Section 3.2. This limiting form is in contrast with the fact that the derivation of the pseudo-charge correlators in the $N = 4$ case requires Δ and $J/\pi u_0$ to be sizeable. It follows that our results, which are in any case just qualitative, cannot reproduce the real features of the large N model in the two opposite limits of small and large coupling $J/\pi u_0$. As it is shown in Fig. 4.1, there is a range of intermediate values for $J/\pi u_0$ in which the free energy of our model has a minimum at finite Δ and the minimum is indeed lower in energy than the reference energy at $\Delta = 0$ (see Fig.4.4).

–5–
The non-chiral 1+1 dimensional SYK model: N = 4 case

In this chapter we solve the model for a specific value of N by considering four Majorana fermions. In this context, random Gaussian distribution of coupling loses its meaning and becomes fixed at a constant value $J \geq 0$. By studying the theory in the bosonization picture after constructing complex fermions, it is observed a pseudo-charge and pseudo-spin sectors separation. Correlators are obtained for small and large coupling J. In the small coupling regime, correlators are free-like in the sense that, free solutions are renormalized by the interaction factor $\mathcal{K}_\rho$ and the renormalized velocity u_ρ. In the strong coupling regime, an approximate correlator function is obtained, which becomes free-like in the limit $J \longrightarrow 0$. By studying the energy-momentum tensor, it is observed that the model is non traceless. Therefore, the $N = 4$ case appears to be non conformal invariant. Finally, other physical quantities are obtained, characterized by renormalized factors.

5.1 Bosonization of the model

In the $N = 4$ case there is just one interaction parameter $J = Q \geq 0$ and just one way to order the four Majorana fermions. Lagrangian density acquires a simple form:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_J \equiv \frac{i}{2} \sum_{i=1,4} \left[\psi_+^i (\partial_t + \partial_x) \psi_+^i + \psi_-^i (\partial_t - \partial_x) \psi_-^i \right] \\ & \quad -J \left\{ \psi_+^1 \psi_+^2 \psi_+^3 \psi_+^4 + \psi_+^1 \psi_+^2 \psi_-^3 \psi_-^4 + \psi_+^1 \psi_-^2 \psi_-^3 \psi_+^4 + \psi_+^1 \psi_-^2 \psi_+^3 \psi_-^4 + (+ \leftrightarrow -) \right\} . \end{aligned} \tag{5.1}$$

Bosonization is an useful technique in one-spatial dimensional systems where some results can become trivial by considering bosonic operators. For instance, a quartic fermionic interaction can be replaced by a quadratic bosonic interaction, which is easier to solve [46]. One of the bosonization prerequisites is a theory that can be formulated in terms of a set of fermion creation and annihilation operators with canonical anti-commutation relations [47]

$$\left\{c_{k\eta}, c^{\dagger}_{k'\eta'}\right\} = \delta_{\eta\eta'}\delta_{kk'}, \tag{5.2}$$

where η denotes species (chirality, spin, etc.) and k is an energy index. Therefore, it is convenient to introduce complex fermion fields for each chirality of Majorana fermions:

$$\begin{aligned} c_{\uparrow\pm} &= \frac{1}{\sqrt{2}}(\psi^1_\pm + i\psi^2_\pm)\,, \qquad & c_{\downarrow\pm} &= \frac{1}{\sqrt{2}}(\psi^3_\pm + i\psi^4_\pm) \\ c^{\dagger}_{\uparrow\pm} &= \frac{1}{\sqrt{2}}(\psi^1_\pm - i\psi^2_\pm)\,, \qquad & c^{\dagger}_{\downarrow\pm} &= \frac{1}{\sqrt{2}}(\psi^3_\pm - i\psi^4_\pm) \end{aligned} \tag{5.3}$$

distinguished by a pseudo-spin index (↑,↓). The free action adopts the following form:

$$S_0 = \int d^2\overrightarrow{x} \sum_{\sigma=\uparrow,\downarrow} \left[c^{\dagger}_{\sigma+}(\partial_t + i\partial_x)c_{\sigma+} + c^{\dagger}_{\sigma-}(\partial_t - i\partial_x)c_{\sigma-}\right], \tag{5.4}$$

where σ labels the pseudo-spin fermion index. For any pair of indices $\sigma\pm$ and $x \neq x'$, $\left\{c(x), c^{\dagger}(x')\right\} = \{c(x), c(x')\} = \left\{c^{\dagger}(x), c^{\dagger}(x')\right\} = 0$. In the free case, right and left movers are decoupled, and the second quantized Hamiltonian appears to be:

$$H_0 = \sum_{\sigma=\uparrow\downarrow} \int dx \left[c^{\dagger}_{\sigma+}(x)(-i\partial_x)c_{\sigma+}(x) + c^{\dagger}_{\sigma-}(x)(i\partial_x)c_{\sigma-}(x)\right]. \tag{5.5}$$

After Fourier transform, we obtain

$$H_0 = \sum_{\sigma=\uparrow\downarrow} \int \frac{dp}{2\pi} \left[c^{\dagger}_{\sigma+}(x)(p)c_{\sigma+}(x) + c^{\dagger}_{\sigma-}(x)(-p)c_{\sigma-}(x)\right], \tag{5.6}$$

which say us that right and left fermions have energies $E = \pm p$, respectively. Furthermore, the V-like spectrum is linearized around $p = 0$ and it is extended to all values of p, leading to an infinite number of negatives states. This is also a prerequisite of constructive bosonization [47], even though it is not mandatory for using the technique itself. Negative states can lead to not-well defined theory after bosonization and a momentum cutoff could be necessary. In the next section, bosonization conduces to pseudo-spin and pseudo-charge sectors separation (don't confuse pseudo-spin sector with pseudo-spin fermion index σ), where we will have to restrict values of coupling J in order to have a well defined Hamiltonian.

Turning now to interactions, the interacting action S_J in terms of complex fermions becomes:

$$S_J = J\int d^2\overrightarrow{x} \sum_{\alpha=\pm}\left(c^{\dagger}_{\uparrow\alpha}c_{\uparrow\alpha}c^{\dagger}_{\downarrow\alpha}c_{\downarrow\alpha} + c^{\dagger}_{\uparrow\alpha}c_{\uparrow\alpha}c^{\dagger}_{\downarrow-\alpha}c_{\downarrow-\alpha} + c^{\dagger}_{\uparrow\alpha}c_{\uparrow-\alpha}c^{\dagger}_{\downarrow-\alpha}c_{\downarrow\alpha} + c^{\dagger}_{\uparrow\alpha}c_{\uparrow-\alpha}c^{\dagger}_{\downarrow\alpha}c_{\downarrow-\alpha}\right). \tag{5.7}$$

The first process in the interaction $c^{\dagger}_{\uparrow\alpha}c_{\uparrow\alpha}c^{\dagger}_{\downarrow\alpha}c_{\downarrow\alpha}$ just couples fermions on the same side of the Fermi surface. The second process $c^{\dagger}_{\uparrow\alpha}c_{\uparrow\alpha}c^{\dagger}_{\downarrow-\alpha}c_{\downarrow-\alpha}$ corresponds to a forward scattering, where fermions are coupled from each side of the Fermi surface but staying both of them in the same side after interaction. Finally, the last two process correspond to a backscattering where fermions exchange sides after interaction [48]. The interacting action can be written in a more simplified form by defining $c_\sigma = c_{\sigma-} + c_{\sigma+}$:

$$S_J = J \int d^2\overrightarrow{x}\, c_\uparrow^\dagger(x)\, c_\uparrow(x)\, c_\downarrow^\dagger(x)\, c_\downarrow(x). \tag{5.8}$$

In the present form the problem is similar to the Tomonaga -Luttinger model solved by Dzyaloshinski and Larkin [49, 50], but in the absence of a Fermi sea. The interaction is Hubbard-like and the model can be completely related with the Hubbard model by $\eta_\uparrow\eta_\downarrow = c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow$. To see the connection, it is necessary to consider just conserved chirality interaction from our model, and drop the rapidly oscillating factors from Hubbard model. Even if the final result seems equal, as it was said, one essential difference is that our model is unbounded from below.

Now, after all these considerations, the model can be bosonized according to

$$c_{\sigma\pm}(t,x) =: \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_{\sigma\pm}(t,x)} : , \qquad c_{\sigma\pm}^\dagger(t,x) =: \frac{1}{\sqrt{2\pi\alpha}} e^{\mp i\sqrt{4\pi}\phi_{\sigma\pm}(t,x)} :, \tag{5.9}$$

where $\phi_{\sigma\pm}(t,x)$ are bosonic fields satisfying the commutation relation

$$[\phi_{\sigma\alpha}(x), \phi_{\sigma'\alpha'}(x')] = \frac{i}{4}\delta_{\sigma\sigma'}\, sgn(x-x') \quad ; \quad (\alpha = \alpha') \tag{5.10}$$

$$= \frac{i}{4} \quad ; \quad (\alpha \neq \alpha') \tag{5.11}$$

and $: O :$ is the normal ordering of the operator O. Thus, we can obtain the following normal ordered transformations (see Appendix B):

$$: c_{\sigma\pm}^\dagger(x) c_{\sigma\pm}(x) := \frac{1}{\sqrt{\pi}}\partial_x\phi_{\sigma\pm}(x) \qquad ; \qquad -ic_{\sigma\pm}^\dagger(x)\partial_x c_{\sigma\pm}(x) =: [\partial_x\phi_{\sigma\pm}(x)]^2 : . \tag{5.12}$$

These transformations relate two complex fermions with the same chirality. Thereby, it can be directly applied in the two first terms on the right side of the interacting action S_J (5.7). Note that they are proportional to the operator $\partial_x\phi_{\sigma\pm}$ because the point splitting was made in the x direction [35]. Eqs. (5.9) and (5.12) correspond to our bosonization dictionary.

Let us start deriving the bosonized part of the interaction. From Eq. (5.8), it is seen that the Hubbard's like interaction is proportional to

$$\begin{aligned} c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow = {} & c_{\uparrow+}^\dagger c_{\uparrow+} c_{\downarrow+}^\dagger c_{\downarrow+} + c_{\uparrow-}^\dagger c_{\uparrow-} c_{\downarrow-}^\dagger c_{\downarrow-} \\ & + c_{\uparrow+}^\dagger c_{\uparrow+} c_{\downarrow-}^\dagger c_{\downarrow-} + c_{\uparrow-}^\dagger c_{\uparrow-} c_{\downarrow+}^\dagger c_{\downarrow+} \\ & + c_{\uparrow+}^\dagger c_{\uparrow-} c_{\downarrow-}^\dagger c_{\downarrow+} + c_{\uparrow-}^\dagger c_{\uparrow+} c_{\downarrow+}^\dagger c_{\downarrow-} \\ & + c_{\uparrow+}^\dagger c_{\uparrow-} c_{\downarrow+}^\dagger c_{\downarrow-} + c_{\uparrow-}^\dagger c_{\uparrow+} c_{\downarrow-}^\dagger c_{\downarrow+} \end{aligned} \tag{5.13}$$

$$c_{\uparrow}^{\dagger}c_{\uparrow}c_{\downarrow}^{\dagger}c_{\downarrow} \equiv j_{0\uparrow}j_{0\downarrow} + c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow-}^{\dagger}c_{\downarrow+} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow+}^{\dagger}c_{\downarrow-} + c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow+}^{\dagger}c_{\downarrow-} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow-}^{\dagger}c_{\downarrow+}. \tag{5.14}$$

Using the first equation of the bosonization dictionary (5.12), it is easy to see that, for instance, $c_{\uparrow+}^{\dagger}c_{\uparrow+}c_{\downarrow+}^{\dagger}c_{\downarrow+} = \frac{1}{\sqrt{\pi}}\partial_x\phi_{\uparrow+}(x)\frac{1}{\sqrt{\pi}}\partial_x\phi_{\downarrow+}(x)$. With this (and similarly for the others), we have

$$\begin{aligned} j_{0\uparrow}j_{0\downarrow} &= \frac{1}{\pi}\left(\partial_x\phi_{\uparrow+}\partial_x\phi_{\downarrow+} + \partial_x\phi_{\uparrow+}\partial_x\phi_{\downarrow-} + \partial_x\phi_{\uparrow-}\partial_x\phi_{\downarrow+} + \partial_x\phi_{\uparrow-}\partial_x\phi_{\downarrow-}\right) \\ &= \frac{1}{\pi}\partial_x\phi_{\uparrow}\partial_x\phi_{\downarrow} \end{aligned} \tag{5.15}$$

where $\phi_\sigma = \phi_{\sigma+} + \phi_{\sigma-}$ was considered. The others four last terms appearing in the interaction are related with interactions like $\overline{\psi}\psi$ which are proportional to $\frac{1}{\alpha}\cos\phi$, with the mean value of the composite operator being zero since they create and destroy different right or left moving fermions. For simplicity, in the following we are going to omit the constants in the bosonization dictionary (5.9) and put them back at the end. For the first two backscattering process we have:

$$\begin{aligned} c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow-}^{\dagger}c_{\downarrow+} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow+}^{\dagger}c_{\downarrow-} &= e^{-i\phi_{\uparrow+}}e^{-i\phi_{\uparrow-}}e^{i\phi_{\downarrow-}}e^{i\phi_{\downarrow+}} + e^{i\phi_{\uparrow-}}e^{i\phi_{\uparrow+}}e^{-i\phi_{\downarrow+}}e^{-i\phi_{\downarrow-}} \\ &= e^{-i(\phi_{\uparrow+}+\phi_{\uparrow-})}e^{-\frac{1}{2}[\phi_{\uparrow+},\phi_{\uparrow-}]}e^{i(\phi_{\downarrow+}+\phi_{\downarrow-})}e^{\frac{1}{2}[\phi_{\downarrow+},\phi_{\downarrow-}]} \\ &\quad + e^{i(\phi_{\uparrow+}+\phi_{\uparrow-})}e^{\frac{1}{2}[\phi_{\uparrow+},\phi_{\uparrow-}]}e^{-i(\phi_{\downarrow+}+\phi_{\downarrow-})}e^{-\frac{1}{2}[\phi_{\downarrow+},\phi_{\downarrow-}]} \end{aligned} \tag{5.16}$$

where it was used the relation $e^Ae^B = e^Be^Ae^{[A,B]} = e^{A+B}e^{\frac{1}{2}[A,B]}$, which is valid if A and B commutes with $[A, B]$. This is true since the bosonic commutator $[A, B]$ is zero if the spins are different. On the other hand, it is also true for equal spins but different movers $+, -$, since the bosonic commutator $[A, B]$ is again zero. Bosonic commutator is a constant when spin and chirality are equal for both operators, but even with this, the constant does not depend on the spin and the exponential with the commutator cancels in both terms of the sum. Recalling that $\phi_{\sigma+} + \phi_{\sigma-} = \phi_\sigma$ we have

$$\begin{aligned} c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow-}^{\dagger}c_{\downarrow+} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow+}^{\dagger}c_{\downarrow-} &= e^{-i\phi_{\uparrow}}e^{i\phi_{\downarrow}} + e^{i\phi_{\uparrow}}e^{-i\phi_{\downarrow}} \\ &= e^{-i(\phi_{\uparrow}-\phi_{\downarrow})}e^{\frac{1}{2}[\phi_{\uparrow},\phi_{\downarrow}]} + e^{i(\phi_{\uparrow}-\phi_{\downarrow})}e^{\frac{1}{2}[\phi_{\uparrow},\phi_{\downarrow}]} \\ &= 2\left(\frac{e^{-i(\phi_{\uparrow}-\phi_{\downarrow})} + e^{i(\phi_{\uparrow}-\phi_{\downarrow})}}{2}\right)e^{\frac{1}{2}[\phi_{\uparrow},\phi_{\downarrow}]} \\ &= 2\cos(\phi_{\uparrow} - \phi_{\downarrow}), \end{aligned}$$

and recovering constants, we obtain:

$$c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow-}^{\dagger}c_{\downarrow+} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow+}^{\dagger}c_{\downarrow-} = \frac{1}{2\pi^2\alpha^2}\cos\sqrt{4\pi}(\phi_{\uparrow} - \phi_{\downarrow}). \tag{5.17}$$

In a similar way, for the other two backscattering process we obtain:

$$c_{\uparrow+}^{\dagger}c_{\uparrow-}c_{\downarrow+}^{\dagger}c_{\downarrow-} + c_{\uparrow-}^{\dagger}c_{\uparrow+}c_{\downarrow-}^{\dagger}c_{\downarrow+} = \frac{1}{2\pi^2\alpha^2}\cos\sqrt{4\pi}(\phi_{\uparrow} + \phi_{\downarrow}). \tag{5.18}$$

For the kinetic part, we found easier to apply bosonization technique by considering the Hamiltonian form of the model and the second relation of Eq. (5.12). In this way we can directly bosonize the Hamiltonian density as follows

$$\begin{aligned}\mathcal{H}_0 &= \sum_{\sigma=\uparrow,\downarrow}\left[c^{\dagger}_{\sigma+}(-i\partial_x)c_{\sigma+} + c^{\dagger}_{\sigma-}(i\partial_x)c_{\sigma-}\right]\\ &= \sum_{\sigma=\uparrow,\downarrow}\left[(\partial_x\phi_{\sigma+})^2 + (\partial_x\phi_{\sigma-})^2\right].\end{aligned} \tag{5.19}$$

Meanwhile the interaction part can be expressed as a combination of $\phi_{\sigma+}$ and $\phi_{\sigma-}$, the kinetic part has them explicitly. In the free model both chiralities are independent and nothing have to be diagonalized. It is because interaction when chiralities are mixed up. One possible way is to attempt to separate them in the interacting part, but cosines make the task difficult. The other possibility is to mix chiralities in the kinetic part and find an unitary transformation that allows us to diagonalize the Hamiltonian. For instance, let us consider that

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}\left[2(\partial_x\phi_{\sigma+})^2 + 2(\partial_x\phi_{\sigma-})^2 + 2\partial_x\phi_{\sigma+}\partial_x\phi_{\sigma-} - 2\partial_x\phi_{\sigma+}\partial_x\phi_{\sigma-}\right]\\ &= \frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}\left[(\partial_x\phi_{\sigma+} - \partial_x\phi_{\sigma-})^2 + (\partial_x\phi_{\sigma+} + \partial_x\phi_{\sigma-})^2\right]\end{aligned} \tag{5.20}$$

where, considering $\phi_\sigma = \phi_{\sigma+} + \phi_{\sigma-}$, the second term is clearly $(\partial_x\phi_\sigma)^2$. For the first term it is useful to take into account dual fields $\phi_\sigma = \phi_{\sigma+} + \phi_{\sigma-}$ and $\theta_\sigma = \phi_{\sigma-} - \phi_{\sigma+}$, obeying the commutation relations (5.11) (and similar relation for θ) and $[\phi(x_1), \theta(x_2)] = \frac{i}{2}sgn(x_2 - x_1)$. Because each of ϕ_σ and θ_σ commutes with itself at any x and y, we can define a canonical momentum field conjugate to $\phi_\sigma(x)$

$$\Pi_\sigma(x) = \partial_x\theta_\sigma(x) = \partial_x(\phi_{\sigma-} - \phi_{\sigma+}), \tag{5.21}$$

being possible in this way to recover the canonical form of the free boson kinetic part:

$$\mathcal{H}_0 = \frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}\left[(\Pi_\sigma)^2 + (\partial_x\phi_\sigma)^2\right]. \tag{5.22}$$

The complete Hamiltonian of the model becomes

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}u_0\left[(\Pi_\sigma)^2 + (\partial_x\phi_\sigma)^2\right]\\ &\quad +J\left(\frac{1}{\pi}\partial_x\phi_\uparrow\partial_x\phi_\downarrow + \frac{1}{2\pi^2\alpha^2}\cos\sqrt{4\pi}(\phi_\uparrow - \phi_\downarrow) + \frac{1}{2\pi^2\alpha^2}\cos\sqrt{4\pi}(\phi_\uparrow + \phi_\downarrow)\right),\end{aligned} \tag{5.23}$$

where free boson velocity u_0 has been restored for dimensional reasons (considering the dimensionless action $S/\hbar$). Again, we can see that free kinetic energy is just the sum of both pseudo-spins, meanwhile the interaction introduces process which couples the pseudo-spin $\uparrow$ with the pseudo-spin $\downarrow$.

5.2 Pseudo-charge and pseudo-spin sectors

Despite the backscattering process, the Hamiltonian (5.23) is quadratic in fields but it is not diagonal in pseudo-spin index. In order to diagonalize the model, we can introduce two boson fields, corresponding to the pseudo-charge c and pseudo-spin s sectors, by defining

$$\phi_{c/s} = \frac{1}{\sqrt{2}}(\phi_\uparrow \pm \phi_\downarrow) \tag{5.24}$$

and a similar relation for dual fields θ, where the "+" sign corresponds to the c-sector and the "−" sign to the s-sector. Naturally, the Hamiltonian is separated into pseudo-charge and pseudo-spin sectors as follows:

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s, \tag{5.25}$$

$$\mathcal{H}_c = \frac{u_0}{2}\left[(\Pi_c)^2 + \left(1 + \frac{J}{\pi u_0}\right)(\partial_x \phi_c)^2 + \frac{J}{u_0 \pi^2 \alpha^2}\cos\left(\sqrt{8\pi}\phi_c\right)\right] \tag{5.26}$$

$$\mathcal{H}_s = \frac{u_0}{2}\left[(\Pi_s)^2 + \left(1 - \frac{J}{\pi u_0}\right)(\partial_x \phi_s)^2 + \frac{J}{u_0 \pi^2 \alpha^2}\cos\left(\sqrt{8\pi}\phi_s\right)\right]. \tag{5.27}$$

Note that the pseudo-spin Hamiltonian $\mathcal{H}_s$ signals an instability as $\mathcal{H}_s$ is unbounded from below when $J/\pi u_0 > 1$. A similar situation can happen when electron-phonon interaction is introduced in a low dimensional electronic system[50, 51]. This absence of an underlying Fermi sea and the unbounded pseudo-spin Hamiltonian from below imply that the bosonization mapping is only meaningful for $0 < J/\pi u_0 < 1$.

It could be also interesting manipulate a bit the Hamiltonian. Let us define the "rescaling factor" A_ρ like

$$A_\rho = \left(1 \pm \frac{J}{\pi u_0}\right)^{\frac{1}{2}} \tag{5.28}$$

where the upper sign is for $\rho = c$ while the lower sign is for $\rho = s$. It is evident that two different boson speeds $u_0 A_{c/s}$ appear in the model. This is a feature of spin-charge separation where single-particle excitations in which charge and spin would be carried together cannot exist. The Hamiltonian can then be rewritten in a condensed way as

$$\mathcal{H} = \sum_{\rho=c,s} \frac{u_0}{2}\left[(\Pi_\rho)^2 + A_\rho^2(\partial_x \phi_\rho)^2 + \frac{J}{u_0 \pi^2 \alpha^2}\cos\left(\sqrt{8\pi}\phi_\rho\right)\right]. \tag{5.29}$$

We can observe that, unlike the not half-filled Hubbard model, the total Hamiltonian (5.29) has a symmetric form between c and s-sectors. According to [43], the symmetric form of

Eq. (5.29) can tells us that the total Hamiltonian is characterized by $SU(2)\times SU(2) \approx SO(4)$ symmetry. Due to the particle-hole symmetry of the half-filled band, the free c-sector $U(1)$ symmetry of the not half-filled case is restricted to $SU(2)$ when backscattering process (cosine term) is included. This is also the case of the half-filled Hubbard model, when both coupling g_c and g_s are equal (indeed, we are considering just J for both sectors). Half-filled case implies that rapidly oscillating factors $e^{\pm 4ik_F}$ are not neglected, but simply equal to unity. Therefore, those terms in the sine-Gordon model in which two right movers are created and two left movers are destroyed (and vice versa) give play into umklapp process with conserved lattice momentum equal to 2π [52, 53].

On the other side, the unconventional velocity in Eq. (5.29) can be fixed by re-scaling the fields ϕ_ρ as follows $\phi_\rho \to \frac{1}{\sqrt{A_\rho}}\phi_\rho$. However, it is mandatory to re-scale also the conjugate momentum in an opposite way $\Pi_\rho \to \sqrt{A_\rho}\Pi_\rho$ in such a way that commutation relations $[\phi_\rho, \Pi_\rho]$ are not affected. This procedure set an overall factor A_ρ that does not concern us for the next analysis. Then, the Hamiltonian becomes

$$\mathcal{H} = \frac{u_0}{2}\sum_{\rho=c,s} A_\rho\left[(\Pi_\rho)^2 + (\partial_x\phi_\rho)^2 + \frac{J}{u_0\pi^2\alpha^2 A_\rho}\cos\left(\sqrt{\frac{8\pi}{A_\rho}}\phi_\rho\right)\right]. \tag{5.30}$$

If we consider the cosine term as a perturbation of a bosonic theory with $\cos(\beta_\rho\phi_\rho)$ we have that, according to what we discussed in Section 2.2.3, the spin case is gapless in accordance with

$$A_s < 1 \qquad \to \qquad \beta_s^2 > 8\pi \tag{5.31}$$

while the charge case is gapped according to

$$A_c > 1 \qquad \to \qquad \beta_c^2 < 8\pi. \tag{5.32}$$

The positive coupling J, this is, the repulsive interaction, implies that there is a cost to move one fermion from one site at the half-filling to another site where is another fermion. Considering a negative J would give the opposite case, with spin being gapped and charge gapless. In this case, opposite spin attraction would form on-site single pairs that would require some cost to break the pair and have a spin excitation [54]. In this work we just going to consider positive J.

Another useful and well know way to express the Hamiltonian is defining the next quantities:

$$u_\rho = u_0\left(1 \pm \frac{J}{\pi u_0}\right)^{1/2} = u_0 A_\rho \tag{5.33}$$

$$\mathcal{K}_\rho = \left(\frac{1}{1 \pm \frac{J}{\pi u_0}}\right)^{1/2} = \frac{1}{A_\rho} \tag{5.34}$$

where u_ρ is the renormalized velocity and $\mathcal{K}_\rho$ a dimensionless parameter. Then, the Hamiltonian can be rewritten as

$$\mathcal{H} = \frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\Pi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 + \frac{J}{\pi^2\alpha^2}\cos(\sqrt{8\pi}\phi_\rho)\right]. \tag{5.35}$$

Finally, after a Legendre's transform, we can obtain the Lagrangian density for the two separate sectors:

$$\mathcal{L} = \frac{1}{2} \sum_{\rho=c,s} \left[\frac{1}{u_\rho \mathcal{K}_\rho} (\partial_t \phi_\rho)^2 - \frac{u_\rho}{\mathcal{K}_\rho} (\partial_x \phi_\rho)^2 - \frac{J}{\pi^2 \alpha^2} \cos\left(\sqrt{8\pi}\phi_\rho\right) \right]. \tag{5.36}$$

The model is not considering interaction between parallel spins densities, but just opposite spins. This is because in the $N = 4$ case there is just one fixed coupling constant $J_{1234} \equiv J$ and one way to order Majorana fermions. By construct the complex fermions and bosonize them it just appears $\perp$ process. In "g-ology" notation (see for instance [48]), $g_{1\perp} = g_{2\perp} = g_{4\perp} = J_\perp = J$ and $g_{1||} = g_{2||} = g_{4||} = J_{||} = 0$. The fact that parallel spins process $g_{||}$ are zero implies that for $J \neq 0$ the model is not spin rotation invariant between x,y plane and z.

5.3 Some physical quantities

In this short section we explore some physical quantities in the bosonized version of the theory for the $N = 4$ case. We also get some hint about conformal invariance by studying the trace of the energy-momentum tensor.

5.3.1 Energy-momentum tensor

Considering that the Lagrangian (5.36) is just a function of a set of fields ϕ_ρ and their derivatives (not explicitly of any space-time coordinates), we can construct the energy-momentum tensor by considering $\frac{\partial \mathcal{L}}{\partial x_\rho} = 0$. Developing this derivative, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_\rho} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\rho)} \frac{\partial(\partial_\mu \phi_\rho)}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial \phi_\rho} \frac{\partial \phi_\rho}{\partial x_\nu} \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\rho)} \partial_\mu \frac{\partial \phi_\rho}{\partial x_\nu} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu x_\rho)} \right) \frac{\partial \phi_\rho}{\partial x_\nu} \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\rho)} \frac{\partial \phi_\rho}{\partial x_\nu} \right) \end{aligned} \tag{5.37}$$

where in the last line we have used the Euler-Lagrange equations and the commutation of the partial derivatives. On the other hand, considering a flat space, the last equation is equal to $\partial_\mu g^{\mu\nu} \mathcal{L}$, so it is possible to write the next expression

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\rho)} \frac{\partial \phi_\rho}{\partial x_\nu} - g^{\mu\nu} \mathcal{L} \right) = 0. \tag{5.38}$$

Finally, the energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\rho)} \partial^\nu \phi_\rho - g^{\mu\nu} \mathcal{L} \tag{5.39}$$

which is a zero divergence tensor; this means that continuity equations can be obtained from here. By looking its contravariant components, the energy-momentum tensor $T^{\mu\nu}$ already gives us information about energy density T^{00}, energy current T^{x0}, momentum density T^{0x} and pressure T^{xx}. Using Eqs. (5.36) and (5.39) we obtain:

$$T^{00} = \frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\Pi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 + \frac{J}{\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_\rho\right)\right] = \mathcal{H} \tag{5.40}$$

$$T^{x0} = -\sum_{\rho=c,s}\frac{u_\rho}{\mathcal{K}_\rho}\partial_t\phi_\rho\partial_x\phi_\rho \tag{5.41}$$

$$T^{0x} = -\sum_{\rho=c,s}\frac{1}{u_\rho\mathcal{K}_\rho}\partial_t\phi_\rho\partial_x\phi_\rho \tag{5.42}$$

$$T^{xx} = \frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\Pi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 - \frac{J}{u_\rho\pi^2\alpha^2}\cos\left(\sqrt{8\pi\mathcal{K}_\rho}\phi_\rho\right)\right], \tag{5.43}$$

where the conjugate momentum $\Pi = \frac{1}{u_\rho\mathcal{K}_\rho}\partial_t\phi_\rho$ was used. In the non-interacting case, despite the dimensional factor, this becomes a symmetrical tensor. It is also interesting study the trace of the tensor in order to know if the model has conformal symmetry. According to this, traceless energy-momentum tensor implies conformal theory [55]. Lowering the ν index using the Minkowski metric, we have from Eq. (5.39) the following:

$$T^0{}_0 = \frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\Pi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 + \frac{J}{\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_\rho\right)\right] = \mathcal{H} \tag{5.44}$$

$$T^x{}_0 = -\sum_{\rho=c,s}\frac{u_\rho}{\mathcal{K}_\rho}\partial_t\phi_\rho\partial_x\phi_\rho \tag{5.45}$$

$$T^0{}_x = \sum_{\rho=c,s}\frac{1}{u_\rho\mathcal{K}_\rho}\partial_t\phi_\rho\partial_x\phi_\rho \tag{5.46}$$

$$T^x{}_x = -\frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\Pi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 - \frac{J}{u_\rho\pi^2\alpha^2}\cos\left(\sqrt{8\pi\mathcal{K}_\rho}\phi_\rho\right)\right] \neq -\mathcal{H}. \tag{5.47}$$

We can conclude that the $1+1$ SYK model in the $N = 4$ case is not conformal invariant. However, even if traceless energy-momentum means conformal theory, the opposite cannot be assured, i.e. conformal theory implying traceless tensor. In this context it could be that, as in the $0+1$ d case, an emergent approximate conformal limit can rises in the large N case. In fact, according to Chapter 3 for large N case, an approximate conformal limit emerges for low energies.

5.3.2 Finite temperature energy density and energy current

Following [35], we are going to compute the energy density and energy current at finite temperature for the free-like case (by "free-like" case we mean that cosine interacting part is omitted, but interactions still have an effect in the renormalized velocities. We developed more about free-like case in Section 5.4):

$$\varepsilon = \langle T^{00}\rangle_\beta = -\frac{1}{2}\sum_{\rho=c,s}\left[\frac{1}{u_\rho\mathcal{K}_\rho}\partial_t^2\langle\phi_\rho(x,t)\phi_\rho(0,0)\rangle_\beta\Big|_{t\to 0,x\to\epsilon} + \frac{u_\rho}{\mathcal{K}_\rho}\partial_x^2\langle\phi_\rho(x,t)\phi_\rho(0,0)\rangle_\beta\Big|_{t\to 0,x\to\epsilon}\right] \tag{5.48}$$

$$j_\varepsilon = \langle T^{x0}\rangle_\beta = \sum_{\rho=c,s}\frac{u_\rho}{\mathcal{K}_\rho}\partial_t\partial_x\langle\phi_\rho(x,t)\phi_\rho(0,0)\rangle_\beta\Big|_{t\to 0,x\to\epsilon} \tag{5.49}$$

To reach this goal we will use the bosonic correlator at finite temperature and perform a point splitting ϵ in the x direction. After that, we will take the limit $\epsilon \to 0$. In order to consider finite temperature, we can include the Bose function $f_B(z) = \frac{1}{e^{\beta z}-1}$ to compute the time-ordered Green function as follows:

$$\langle\phi_{\rho(x,\tau)}\phi_{\rho(0,0)} - \phi^2_{\rho(0,0)}\rangle^\beta \equiv \mathcal{G}^\beta_{\phi_\rho\phi_\rho}(x,\tau) = g^\beta_{\phi_\rho}(x,\tau) - g^\beta_{\phi_\rho}(0,0) \tag{5.50}$$

where

$$\begin{aligned} g^\beta_{\phi_\rho}(x,\tau) &= \sum_\pm\left[\theta(\tau)\langle\phi_{\rho(x,t)}\phi_{\rho(0,0)}\rangle + \theta(-\tau)\langle\phi_{\rho(0,0)}\phi_{\rho(x,t)}\rangle\right] \\ &= \sum_\pm\int\frac{dq}{4\pi}\left[\frac{e^{-q(u_\rho\tau\pm ix)}}{1-e^{-\beta q}} + \frac{e^{q(u_\rho\tau\pm ix)}}{e^{\beta q}-1}\right]\frac{e^{-\alpha q}}{q} \\ &= -\sum_\pm\frac{\mathcal{K}_\rho}{4\pi}\ln\left(\frac{2\beta}{L}\sin\left[\frac{\pi}{\beta}(u_\rho\tau \pm ix + \alpha)\right]\right), \end{aligned} \tag{5.51}$$

where the divergence because of the lower limit $q = 0$ was regularized calculating the principle-value of the integral. The Green function diverges as $L \to \infty$. This is not a problem because we need to take the difference between two Green functions, i.e. $g^\beta_{\phi_\rho}(x,\tau) - g^\beta_{\phi_\rho}(0,0)$:

$$\begin{aligned} \mathcal{G}^\beta_{\phi_\rho\phi_\rho}(x,\tau) &= -\frac{\mathcal{K}_\rho}{4\pi}\left[\ln\left(\frac{\frac{2\beta}{L}\sin\left[\frac{\pi}{\beta}(u_\rho\tau - ix + \alpha)\right]}{\frac{2\beta}{L}\sin\left[\frac{\pi}{\beta}\alpha\right]}\right) + \ln\left(\frac{\frac{2\beta}{L}\sin\left[\frac{\pi}{\beta}(u_\rho\tau + ix + \alpha)\right]}{\frac{2\beta}{L}\sin\left[\frac{\pi}{\beta}\alpha\right]}\right)\right] \\ &= -\frac{\mathcal{K}_\rho}{4\pi}\ln\left(\frac{\sin\left[\frac{\pi}{\beta}(-ix + u_\rho\tau + \alpha)\right]}{\sin\left[\frac{\pi}{\beta}\alpha\right]}\frac{\sin\left[\frac{\pi}{\beta}(ix + u_\rho\tau + \alpha)\right]}{\sin\left[\frac{\pi}{\beta}\alpha\right]}\right). \end{aligned} \tag{5.52}$$

Similar results can be obtained by analytic continuation [56]. In order to compare with zero temperature results (see Section 5.4), let us evaluate the limit $\beta \to \infty$ ($T \to 0$). In that limit, the sine function approaches to the argument, and we can verify that

$$\mathcal{G}^{T=0}_{\phi_\rho\phi_\rho}(x,\tau) = \langle \phi_{\rho(x,\tau)}\phi_{\rho(0,0)} - \phi^2_{\rho(0,0)} \rangle = -\frac{\mathcal{K}_\rho}{4\pi} \ln\left(\frac{x^2 + (u_\rho \tau + \alpha)^2}{\alpha^2}\right), \tag{5.53}$$

which corresponds identically with Eq. (5.74). The finite temperature $\mathcal{G}^{\beta}_{\theta_\rho\theta_\rho}(x,\tau)$ in the free-like case for dual field is equal as for ϕ fields but changing $\mathcal{K}_\rho \to \frac{1}{\mathcal{K}_\rho}$. Because of this, we can relate $\frac{1}{u_\rho \mathcal{K}_\rho}\partial_t^2 \langle \mathcal{G}^{\beta}_{\phi_\rho\phi_\rho}(x,\tau)\rangle_\beta$ with $u_\rho \mathcal{K}_\rho \partial_x^2 \langle \mathcal{G}^{\beta}_{\theta_\rho\theta_\rho}(x,\tau)\rangle_\beta$, therefore, it is just necessary to make derivatives with respect x (at least for energy density). Finally, the finite temperature energy density and energy current are:

$$\varepsilon = \sum_{\rho=c,s} \frac{\pi u_\rho}{6\beta^2} \tag{5.54}$$

$$j_\varepsilon = \frac{\pi}{12\beta^2}(C_+ - C_-) \tag{5.55}$$

where $C_\pm = \pm \sum_{\rho=c,s} u_\rho^2 \mathcal{K}_\rho$ are the central charges for right and left movers. As you can notice, there is an absent of the energy current which is signal of an equilibrium state. Furthermore, it is also a signal that the model cannot flow to a Conformal Field Theory with a non-zero $(C_+ - C_-)$ [57].

For the full-interacting case (considering also the cosine term), we can make a rude approximation by considering the interaction proportional to ϕ^2 but still using the finite temperature correlator (5.52). In this case, a logarithmic correction which tends to zero appears, leaving the same results.

5.3.3 Other physical observables

We can still use a bit more the results coming from energy-momentum tensor. For example, the thermal Hall conductance is given by

$$\kappa_{xy\pm} = \frac{\partial j_{\varepsilon\pm}}{\partial(\beta^{-1})} = \frac{\pi}{6\beta} C_\pm \tag{5.56}$$

for each branch. For chiral models the latter does not vanish and becomes $\kappa_{xy\pm} \sim \frac{\pi}{3\beta}$ as it is expected for a gapped system with four flavors of chiral Majorana fermions on the edge [58]. Using now the energy density, we can obtain the entropy density to be:

$$\mathcal{S} = \sum_{\rho=c,s} \frac{\pi u_\rho}{3\beta}. \tag{5.57}$$

Finally, we are going to calculate compressibility and susceptibility of the system. The compressibility is the response to

$$H = -\mu \int dx \left[\rho_\uparrow + \rho_\downarrow\right] \equiv -\mu \int dx\, \rho \tag{5.58}$$

where $\rho_\sigma = \frac{1}{\sqrt{2}}(\rho_c + \rho_s)$ is the density operator for spin $\sigma =\uparrow,\downarrow$. The compressibility just depends on the charge part $\rho = \rho_\uparrow + \rho_\downarrow = \sqrt{2}\rho_c$ which is in our case:

$$\rho_c = \frac{1}{\sqrt{2}} \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha=\pm} \rho_{\sigma\alpha} = \frac{1}{\sqrt{2\pi}} \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha=\pm} \partial_x \phi_{\sigma\alpha} = \frac{1}{\sqrt{2\pi}} \sum_{\sigma=\uparrow,\downarrow} \partial_x \phi_\sigma = \frac{1}{\sqrt{\pi}} \partial_x \phi_c, \tag{5.59}$$

where it was used that $\phi_\sigma = \phi_{\sigma+} + \phi_{\sigma-}$, the pseudo-charge boson $\phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow)$ and the bosonization $\rho_{\sigma\pm} = c^\dagger_{\sigma\pm} c_{\sigma\pm} = \frac{1}{\sqrt{\pi}} \partial_x \phi_{\sigma\pm}$. To compute it we take the density-density correlation $\langle \rho(k,\omega_n)\rho(-k,-\omega_n)\rangle$ that gives for the free-like case the following:

$$\begin{aligned}
\kappa_c &= 2\langle \rho_c(k,\omega_n)\rho_c(-k,-\omega_n)\rangle\Big|_{\omega_n,k,m\to 0} \\
&= \frac{2k^2}{\pi}\langle\, \phi_c(k,\omega_n)\phi_c(-k,-\omega_n)\rangle\Big|_{\omega_n,k,m\to 0} \\
&= \frac{2\mathcal{K}_c}{\pi u_c}.
\end{aligned} \tag{5.60}$$

By taking first the limit $\omega_n \longrightarrow 0$ we have a static chemical potential to get a thermodynamic response and then an uniform potential by taking $k \longrightarrow 0$. In a similar way, the uniform magnetic susceptibility is the response to

$$H = -\frac{\mathbf{h}}{2} \int dx \left[\rho_\uparrow - \rho_\downarrow\right] \tag{5.61}$$

where $\mathbf{h} = g\mu_B h$ depends on the magnetic field h, the Bohr magneton μ_B and the Lande factor g. For the free-like case is given by

$$\kappa_s = \frac{\mathcal{K}_s}{2\pi u_s}. \tag{5.62}$$

Even when cosine interacting part is neglected, having in this way the free-like case, the interaction still has the effect of renormalize some physical quantities. This also applies to other observables that we are not to see here but are easy to obtain from the model in this limit like, for example, the specific heat, which is also renormalized by changing $u_0 \longrightarrow u_\rho$.

5.4 Correlation functions

In this section we compute some correlation functions. In the pseudo-charge and pseudo-spin sectors separation context, the interaction has two main effects: renormalization of the velocities, and the competition between dual fields for lock or not the field in one of the minima of the cosine. We will explore the free-like case, the weak and strong coupling limit and the triple-pairing operator.

5.4.1 Free-like correlator

After bosonization and pseudocharge-pseudospin sectors separation, the interacting action depends on the non-trivial cosine term. Contrary to the kinetic part that favors the field ϕ_ρ fluctuate, the cosine term tends to lock it in one of the minima of the cosine. There is then a competence between kinetic and interacting part. Let us consider first the free-like action, i.e. neglecting the cosine term. Omitting cosine does not mean that the theory is completely free because the tendency of locking the field in one of the minima is not the only effect of the interaction but also renormalize the boson velocity.

The average Majorana femion two point function is

$$G_{\pm}(r) = \frac{1}{4}\sum_i \langle \psi^i_{\pm}(r)\psi^i_{\pm}(0)\rangle. \tag{5.63}$$

with $r = (x, \tau)$. In the pseudocharge-pseudospin representation chiralities are not separated. In the following we showed that in the free-like case each chirality, evaluated in the mentioned representation, provides the same correlator except by an overall sign. Recalling the initial definition of complex fermions

$$c_{\uparrow\pm} = \frac{1}{\sqrt{2}}(\psi^1_{\pm} + i\psi^2_{\pm}) \tag{5.64}$$

$$c_{\downarrow\pm} = \frac{1}{\sqrt{2}}(\psi^3_{\pm} + i\psi^4_{\pm}) \tag{5.65}$$

it is possible to write

$$G_{\pm}(r) = \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow} \langle c_{\sigma\pm}(r)c^{\dagger}_{\sigma\pm}(0) + c^{\dagger}_{\sigma\pm}(r)c_{\sigma\pm}(0)\rangle.$$

It is convenient to express the bosonization dictionary in terms of dual fields ϕ and θ using the charge-spin separation:

$$c_{\sigma\pm}(r) =: \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(r)-\theta_c(r)+\sigma(\pm\phi_s(r)-\theta_s(r))]} : \tag{5.66}$$

where σ in the exponential is +1 for $\uparrow$ and -1 for $\downarrow$. Our bosonization dictionary is now extended to Eqs. (5.9), (5.12) and (5.66). The single particle Green function becomes:

$$\begin{aligned} G_{\pm}(r) \;=\; & \frac{1}{8\pi\alpha}(\langle e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(r)-\theta_c(r)\pm\phi_s(r)-\theta_s(r)]} e^{-i\sqrt{\frac{\pi}{2}}[\pm\phi_c(0)-\theta_c(0)\pm\phi_s(0)-\theta_s(0)]}\rangle \\ & +\langle e^{-i\sqrt{\frac{\pi}{2}}[\pm\phi_c(r)-\theta_c(r)\pm\phi_s(r)-\theta_s(r)]} e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(0)-\theta_c(0)\pm\phi_s(0)-\theta_s(0)]}\rangle \\ & +\langle e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(r)-\theta_c(r)\mp\phi_s(r)+\theta_s(r)]} e^{-i\sqrt{\frac{\pi}{2}}[\pm\phi_c(0)-\theta_c(0)\mp\phi_s(0)+\theta_s(0)]}\rangle \\ & +\langle e^{-i\sqrt{\frac{\pi}{2}}[\pm\phi_c(r)-\theta_c(r)\mp\phi_s(r)+\theta_s(r)]} e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(0)-\theta_c(0)\mp\phi_s(0)+\theta_s(0)]}\rangle). \end{aligned} \tag{5.67}$$

We have to remind that pseudo-charge and pseudo-spin sectors can be completely separate, while dual fields obey $[\phi(x_1), \theta(x_2)] = \frac{i}{2} sgn(x_2 - x_1)$. With these considerations and the identity $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]}$ we have:

$$\begin{aligned}
G_{\pm}(r) \quad &= \quad \pm\frac{i}{8\pi\alpha}\Bigg(\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to \phi_s, \quad \theta_c \to \theta_s) \\
&+\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to \phi_s, \quad \theta_c \to \theta_s) \\
&+\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to -\phi_s, \quad \theta_c \to -\theta_s) \\
&+\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to -\phi_s, \quad \theta_c \to -\theta_s)\Bigg).
\end{aligned} \tag{5.68}$$

For the correlator to be non-zero, the sum of the factors multiplying fields in the exponentials has to vanish. This is because the theory (the Hamiltonian of the massless scalar field) is invariant under a constant shift in fields. To evaluate this correlator, we need the following identity:

$$\langle e^A \cdot e^B\rangle = \langle : e^{A+B} :\rangle e^{\langle AB + \frac{A^2+B^2}{2}\rangle} \tag{5.69}$$

where the vacuum expectation value of a normal-ordered exponential operator is just 1. All other terms in the series annihilate the vacuum state on the left or right or both. Then, the correlator becomes:

$$G_{\pm}(r) = \pm\frac{i}{2\pi\alpha}(e^{\frac{\pi}{2}\langle\phi_c(r)\phi_c(0) - \phi_c^2(0)\rangle} e^{\frac{\pi}{2}\langle\theta_c(r)\theta_c(0) - \theta_c^2(0)\rangle} e^{\frac{\pi}{2}\langle\phi_s(r)\phi_s(0) - \phi_s^2(0)\rangle} e^{\frac{\pi}{2}\langle\theta_s(r)\theta_s(0) - \theta_s^2(0)\rangle}). \tag{5.70}$$

Despite the overall sign, at this point we can already notice that any kind of difference between "+" and "−" chiralities has disappeared, and chiralities behave as the same. It is also important to mention that for operators, the average $\langle\mathcal{A}\rangle$ means time-ordered product, this is explicitly $\langle T_\tau \mathcal{A}\rangle_O$ where subindex "O" implies averages without time-ordered product. For instance, we have

$$\begin{aligned}
\mathcal{G}_{\phi\phi}(x,\tau) \quad &= \quad -\frac{1}{2}\langle T_\tau[\phi(x,\tau) - \phi(0,0)]^2\rangle_O \\
&= \quad \epsilon(\tau)\langle\phi(x,\tau)\phi(0,0)\rangle_O + \epsilon(-\tau)\langle\phi(0,0)\phi(x,\tau)\rangle_O - \langle\phi(0,0)\phi(0,0)\rangle_O
\end{aligned} \tag{5.71}$$

and similar expressions for θ with $\epsilon(\tau)$ being the step function. In general for this work we are not going to write explicitly the the time-ordered product but it will be assumed. The single-particle Green's function becomes

$$G_{\pm}(r) = \pm\frac{i}{2\pi\alpha}(e^{-\frac{\pi}{4}\langle[\phi_c(r) - \phi_c(0)]^2\rangle} e^{-\frac{\pi}{4}\langle[\theta_c(r) - \theta_c(0)]^2\rangle} e^{-\frac{\pi}{4}\langle[\phi_s(r) - \phi_s(0)]^2\rangle} e^{-\frac{\pi}{4}\langle[\theta_s(r) - \theta_s(0)]^2\rangle}). \tag{5.72}$$

In order to obtain the fermionic correlator it only remains to compute the bosonic correlators $\langle[\phi_\rho(r) - \phi_\rho(0)]^2\rangle$ and $\langle[\theta_\rho(r) - \theta_\rho(0)]^2\rangle$ (or equivalent expressions according to Eq. (5.71)).

In the free-like case the cosine term is omitted and we just have a renormalized velocity. Therefore, the Hamiltonian (or Lagrangian) density is quadratic, being convenient to obtain correlation functions via functional integral

$$\langle \phi_{\rho(r_1)}\phi_{\rho'(r_2)}\rangle = \frac{\int \mathcal{D}\phi\mathcal{D}\theta e^{-S[\phi,\theta]}\phi_{\rho(r_1)}\phi_{\rho'(r_2)}}{\int \mathcal{D}\phi\mathcal{D}\theta e^{-S[\phi,\theta]}}, \tag{5.73}$$

where the action can be expressed as $S = S_c + S_s$. Since S can also be put in terms of θ fields (using the conjugate momentum), correlation functions like $\langle\phi\theta\rangle$ can be obtained as well. Actually, a more general representation can be found by consider a matrix representation with $M_{\theta\theta}$, $M_{\theta\phi}$, $M_{\phi\theta}$ and $M_{\phi\phi}$ elements, for pseudo-charge and pseudo-spin sectors. We will explore this when we consider a relevant cosine interacting term. At the moment, we are just interested in correlators like Eq. (5.71).

Performing calculations (see Appendix C), the desired correlator $\mathcal{G}_{\phi_\rho\phi_\rho}(x,\tau)$ is found to be:

$$\langle \phi_{\rho(x,\tau)}\phi_{\rho(0,0)} - \phi^2_{\rho(0,0)}\rangle = \frac{\mathcal{K}_\rho}{2\pi}\ln\left(\frac{\alpha}{\sqrt{x^2+(u_\rho\tau+\alpha)^2}}\right). \tag{5.74}$$

Correlator $\mathcal{G}_{\theta_\rho\theta_\rho}(x,\tau)$ for θ fields can be obtained in a similar way. We can also consider the following: the conjugate momentum $\Pi_\rho = \frac{1}{u_\rho\mathcal{K}_\rho}\partial_t\phi_\rho = \nabla\theta_\rho$ allows us to write the Hamiltonian in terms of dual fields θ_ρ like:

$$\mathcal{H} = \frac{1}{2}\sum_{\rho=c,s}\left[u_\rho\mathcal{K}_\rho(\partial_x\theta_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2\right]. \tag{5.75}$$

In this form, we can see that the Hamiltonian is invariant by $\phi \to \theta$ and $\mathcal{K}_\rho \to \frac{1}{\mathcal{K}_\rho}$ and we would obtain the same as Eq. (5.74) but with $\frac{1}{\mathcal{K}_\rho}$:

$$\langle \theta_{\rho(x,\tau)}\theta_{\rho(0,0)} - \theta^2_{\rho(0,0)}\rangle = \frac{1}{2\pi\mathcal{K}_\rho}\ln\left(\frac{\alpha}{\sqrt{x^2+(u_\rho\tau+\alpha)^2}}\right). \tag{5.76}$$

Finally, plugging the last two equations in Eq. (5.70) we have:

$$G_\pm(r) = \pm\frac{i}{2\pi\alpha}\left(\frac{\alpha}{\sqrt{x^2+(u_c\tau+\alpha)^2}}\right)^{\frac{1}{4}(\mathcal{K}_c+\frac{1}{\mathcal{K}_c})}\left(\frac{\alpha}{\sqrt{x^2+(u_s\tau+\alpha)^2}}\right)^{\frac{1}{4}(\mathcal{K}_s+\frac{1}{\mathcal{K}_s})}. \tag{5.77}$$

If we put $\alpha \to 0$ and also turn off the interaction $J = 0$, then $\mathcal{K}_\rho \to 1$ and correlator behaves as completly free fermions $G_\pm \sim (x^2 + u_0^2\tau^2)^{-1/2} \sim \frac{1}{r}$.

5.4.2 Weak coupling: small J limit

Let us us now briefly consider the effects of cosine interacting term by considering weak coupling. Being J small, the model is close to the free case but it is no longer quadratic and cannot be solved exactly. However, the cosine term can be expanded and we can use renormalization procedure to study the flows of the coupling. In the free-like case we studied $G_{\pm}(r)$ by considering correlators like

$$G_{\phi_\rho\phi_\rho}(r,0) = \langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_\rho(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_\rho(0)}\rangle = e^{\frac{\pi}{2}\langle\phi_\rho(r)\phi_\rho(0)-\phi_\rho^2(0)\rangle} = e^{\frac{\pi}{2}\mathcal{G}_{\phi_\rho\phi_\rho}(x,\tau)} \tag{5.78}$$

which for quadratic Hamiltonian behaves as

$$\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_\rho(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_\rho(0)}\rangle_0 \sim \left(\frac{\alpha}{\sqrt{x^2+(u_\rho\tau+\alpha)^2}}\right)^{\frac{1}{4}\mathcal{K}_\rho} \tag{5.79}$$

where the sub-index "0" was included to emphasize that is for free-like case. Expanding cosine, correlator will be in terms of free-like averages, which we already know. At second order expansion and defining the center of mass $R = \frac{r'+r''}{2}$ and relative coordinates $r = r' - r''$ we have:

$$\begin{aligned} G_{\phi_\rho\phi_\rho}(r_1,r_2) &= e^{-\frac{1}{4}\mathcal{K}_\rho F_\rho(r)}\Bigg[1+\frac{J^2}{2(2\pi\alpha)^4 u_\rho}\sum_{\epsilon=\pm}\int\int d^2R d^2r e^{-4\mathcal{K}_\rho F_\rho(r)} \\ &\qquad \left(e^{\frac{1}{2}\epsilon\mathcal{K}_\rho[F_\rho(r_1-r')-F_\rho(r_1-r'')+F_\rho(r_2-r'')-F_\rho(r_2-r')]}-1\right)\Bigg], \end{aligned} \tag{5.80}$$

where $r = (x, u_\rho\tau)$ and $F_\rho(r) = \frac{1}{4\pi}\ln\left(\frac{x^2+(u_\rho\tau+\alpha)^2}{\alpha^2}\right)$. For small distance r we can expand the exponential and thus, integrating by parts over R, we obtain

$$G_{\phi_\rho\phi_\rho}(r_1,r_2) = e^{-\frac{1}{4}\mathcal{K}_\rho F_\rho(r)}\left[1+\frac{J^2\mathcal{K}_\rho^2 F_\rho(r)}{8\pi^2\alpha^4 u_\rho^2}\int_{r>\alpha} dr\, r^3 e^{-4\mathcal{K}_\rho F_\rho(r)}\right], \tag{5.81}$$

where it was used the fact that $(\nabla_x^2+\nabla_y^2)\ln R = 2\pi\delta(R)$. Considering the definition of $F_\rho(r)$, the latter is an expansion of exponential which is the same as Eq. (5.79) but with the effective exponential

$$\mathcal{K}_\rho^{eff} = \mathcal{K}_\rho - \frac{J^2\mathcal{K}_\rho^2}{8\pi^2 u_\rho^2}\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4\mathcal{K}_\rho}. \tag{5.82}$$

We can conclude that correlator tends to an asymptotic behavior with an effective exponent $\mathcal{K}_\rho \to \mathcal{K}_\rho^{eff}$:

$$\langle e^{\mp i\sqrt{2\pi}\phi_\rho(r)} e^{\pm i\sqrt{2\pi}\phi_\rho(0)}\rangle_0 \sim \left(\frac{\alpha}{\sqrt{x^2+(u_\rho\tau+\alpha)^2}}\right)^{\frac{1}{4}\mathcal{K}_\rho^{eff}}, \tag{5.83}$$

however, we can not use results coming from renormalization picture because we do not have Fermi sea, and renormalization helps us to understand what happens close to the Fermi level. If we analyze the flow of the coupling starting from Eq. (5.82), we see that when the coupling $J \sim g_{||} = 0$ (but still $J < g_{||}$), the cosine term is irrelevant and the model is in a massless regime. However, when $J > g_{||}$, the flows goes to strong coupling, and the model goes to a massive regime. In our model J is always positive, so that renormalization procedure does not help to understand the small J case due to it predicts a strong coupling flow.

In this case, a better way to study the small J limit is considering variational calculation. It can gives you information about physics when the model goes to a massive regime even though is not better than renormalization to obtain critical properties of the theory. Following the standard variational method [59] it is observed that the self-consistent equation for the gap is

$$\Delta^2 = \frac{4K_\rho u_\rho^2 J}{\pi\alpha^2 u_0}\left(\frac{\Delta}{u_\rho\Lambda}\right)^{2K_\rho} \tag{5.84}$$

where Λ is a large momentum cutoff. It is seen that $K_\rho > 1$ is gapless: the theory behaves as a free model and the cosine potential is irrelevant. Just in the s-sector, due to the restriction $0 < J/\pi u_0 < 1$, it is found that $K_s > 1$, with the pseudo-spin sector being gapless. With these consideration, we have (zero temperature):

$$G_{\phi_s\phi_s}(r) = e^{-\frac{\pi}{4}\langle[\phi_s(r)-\phi_s(0)]^2\rangle} = \left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{4}\mathcal{K}_s} \tag{5.85}$$

and

$$G_{\theta_s\theta_s}(r) = e^{-\frac{\pi}{4}\langle[\theta_s(r)-\theta_s(0)]^2\rangle} = \left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{4\mathcal{K}_s}}. \tag{5.86}$$

On the other side, it is seen that $K_\rho < 1$ is gapped: the theory goes to a massive regime, the cosine potential is relevant and traps the ϕ field in one of its minima. Just in the charge case $K_c < 1$, the charge sector being gapped. In the context of trapped ϕ the situation is analogous to have strong coupling. We are going to study this case in the next subsection.

5.4.3 Strong coupling: large J limit

In the pseudo-spin sector, due to the restriction $0 < J/\pi u_0 < 1$, large J (i.e. $J \sim \pi u_0$) implies that $u_s \to 0$. A small velocity in the pseudospin sector, $u_s << u_0$, gives rise to two effects. On the one hand, fluctuations of the phase $\langle\phi_s^2\rangle_0$ grow enormously. On the other hand, they renormalize the cosine interaction term which, by normal ordering is strongly depressed as $\frac{J}{u_s}\cos\left(\sqrt{8\pi}\phi_s\right) \to \frac{J}{u_s}\, e^{-\langle\phi_s^2\rangle_0}\; :\cos\left(\sqrt{8\pi}\phi_s\right):$ by making it irrelevant. In this limit, the action is quadratic in the s-sector and the correlator at large distances it takes the same form as Eqs. (5.85) and (5.86) with $u_s << 1$.

In the pseudo-charge sector, large J imposes that ϕ_c is locked into one of the minima of the cosine and the model goes to a massive regime. The gapped pseudo-charge degree of freedom can be approached quadratically with the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left[u_c\mathcal{K}_c(\Pi_c)^2 + \frac{u_c}{\mathcal{K}_c}(\partial_x\phi_c)^2 + \frac{4J}{\pi\alpha^2}\phi_c^2\right] \tag{5.87}$$

where the conjugate momentum $\Pi_c = \frac{1}{u_c\mathcal{K}_c}\partial_t\phi_\rho = \nabla\theta_c$ includes the dual field. Writing the action in terms of dual field θ_c (and imaginary time), we have:

$$-S = \int d\tau dx \left(i\nabla\theta_c\partial_t\phi_c - \frac{1}{2}\left[u_c\mathcal{K}_c(\nabla\theta_c)^2 + \frac{u_c}{\mathcal{K}_c}(\nabla\phi_c)^2 + \frac{4J}{\pi\alpha^2}\phi_c^2 \right] \right). \tag{5.88}$$

In Fourier space:

$$\begin{aligned} -S &= \frac{1}{2\beta\Omega}\sum_q \Bigg(2ik\omega_n\phi_c(\mathbf{q})\theta_c(-\mathbf{q}) - u_c\mathcal{K}_c k^2\theta_c(\mathbf{q})\theta_\rho(-\mathbf{q}) \\ &\qquad - \frac{u_c}{\mathcal{K}_c}k^2\phi_c(\mathbf{q})\phi_c(-\mathbf{q}) - \frac{4J}{\pi\alpha^2}\phi_c(\mathbf{q})\phi_c(-\mathbf{q}) \Bigg) \end{aligned} \tag{5.89}$$

which can be written as a matrix like:

$$S = \frac{1}{2\beta\Omega}\sum_q \begin{pmatrix} \theta_c(-\mathbf{q}) & \phi_c(-\mathbf{q}) \end{pmatrix} \begin{pmatrix} u_c\mathcal{K}_c k^2 & -ik\omega_n \\ -ik\omega_n & \frac{u_c}{\mathcal{K}_c}k^2 + \frac{4J}{\pi\alpha^2} \end{pmatrix} \begin{pmatrix} \theta_c(\mathbf{q}) \\ \phi_c(\mathbf{q}) \end{pmatrix}. \tag{5.90}$$

From here, we can obtain the matrix $A(q)$ which is related in the path integral with the correlator

$$\langle u*(q_1)u(q_2)\rangle = \frac{\int \mathcal{D}u[q]u*(q_1)u(q_2)e^{-\frac{1}{2}\sum_q A(q)u*(q_1)u(q_2)}}{\int \mathcal{D}u[q]e^{-\frac{1}{2}\sum_q A(q)u*(q_1)u(q_2)}} = A^{-1}(q_1)\delta_{q_1,q_2} \tag{5.91}$$

to be

$$A(q) = \begin{pmatrix} u_c\mathcal{K}_c k^2 & ik\omega_n \\ ik\omega_n & \frac{u_c}{\mathcal{K}_c}k^2 + \frac{4J}{\pi\alpha^2} \end{pmatrix} \tag{5.92}$$

and its inverse

$$A^{-1}(q) = \frac{1}{k^2(u_c^2k^2 + u_c^2m^2 + \omega_n^2)} \begin{pmatrix} \frac{u_c}{\mathcal{K}_c}(k^2+m^2) & -ik\omega_n \\ -ik\omega_n & u_c\mathcal{K}_c k^2 \end{pmatrix} \tag{5.93}$$

where it was defined the mass term $m^2 = \frac{4J}{\pi\alpha^2}\frac{\mathcal{K}_c}{u_c} \equiv \frac{\Delta^2}{L^2}$. The mass scales as $1/[L]$ and its specific form is

$$\frac{\Delta}{L} = \frac{2}{\alpha}\sqrt{\frac{\mathcal{K}_c^{-2}-1}{\mathcal{K}_c^2}} \tag{5.94}$$

with $L/\alpha = N$ a finite number. As it is expected, in the free case $\mathcal{K}_c = 1$, the gap disappears and the two chiralities are separated in the $\langle\theta_c\theta_c\rangle$ correlator. In the other side, when interaction is turn on the two chiralities cannot be separated in the $\langle\theta_c\theta_c\rangle$ correlator due to the presence of a gap $\Delta = m\,L$.

We are looking for bosonic correlators to be used in fermionic correlators like (5.70) or others like the triple pairing operator:

$$\langle O_{TS}^{z\dagger}(r)O_{TS}^{z}(r')\rangle = \frac{1}{\pi^2\alpha^2}\langle e^{i\sqrt{2\pi}\theta_c(r)}e^{-i\sqrt{2\pi}\theta_c(r')}\rangle\langle e^{i\sqrt{2\pi}\theta_s(r)}e^{-i\sqrt{2\pi}\theta_s(r')}\rangle. \tag{5.95}$$

Assuming the identity (5.69) is still valid what we need at the end is, in general, the correlators

$$\left\langle \phi_c(r_1)\phi_c(r_2) - \frac{\phi_c^2(r_1) + \phi_c^2(r_2)}{2} \right\rangle \tag{5.96}$$

and

$$\left\langle \theta_c(r_1)\theta_c(r_2) - \frac{\theta_c^2(r_1) + \theta_c^2(r_2)}{2} \right\rangle, \tag{5.97}$$

i.e. $\mathcal{G}_{\phi_c\phi_c}(r)$ and $\mathcal{G}_{\theta_c\theta_c}(r)$. In Appendix C it is found the explicit computation of these correlators, which are required to be used in objects like (5.70). They are:

$$\mathcal{G}_{\phi_c\phi_c}(r) = -\frac{\mathcal{K}_c}{4\pi}\left[\int_0^{i\,x+u_c\tau} \frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\,K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right)dz + \ln\left(\alpha^2\right) + c.c.\right] \tag{5.98}$$

$$\mathcal{G}_{\theta_c\theta_c}(r) = \frac{1}{4\pi\mathcal{K}_c}\ln\left\{ e^{\frac{4\pi}{\mathcal{K}_c}G_{\phi_c\phi_c}(r)} \prod_{\pm} e^{\Delta\left[e^{-\frac{\Delta u_c\tau \pm i\,x}{L}}\right]}\left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)\, e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\Delta e^{-\frac{\Delta u_c\tau}{L}}}\right\} \tag{5.99}$$

where terms proportional to $O(m^4)$ have been neglected. The two correlators are non chiral and have been approximated in such a way that they reproduce the free-like result leading to Eq. (5.77) in the limit $\Delta \longrightarrow 0$.

The final form of the diagonal correlator in the gapped pseudo-charge sector to be plugged in Eq. (5.70) is:

$$\begin{aligned}
G_\pm(r) &= \pm\frac{i}{2\pi\alpha}\left(e^{\frac{\pi}{2}\mathcal{G}_{\phi_c\phi_c}(r)}e^{\frac{\pi}{2}\mathcal{G}_{\theta_c\theta_c}(r)}e^{\frac{\pi}{2}\mathcal{G}_{\phi_s\phi_s}(r)}e^{\frac{\pi}{2}\mathcal{G}_{\theta_s\theta_s}(r)}\right) \\
&= \frac{\pm i}{2\pi\alpha}\left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{4}\left(\mathcal{K}_s+\frac{1}{\mathcal{K}_s}\right)} e^{-\frac{1}{8}\left(\mathcal{K}_c+\frac{1}{\mathcal{K}_c}\right)\left[\int_0^{i\,x+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\,K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right)dz + c.c.\right]} \\
&\times \prod_{\pm} e^{\frac{\Delta}{8\mathcal{K}_c}\left[e^{-\frac{\Delta u_c\tau\pm i\,x}{L}}\right]}\left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)\, e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\frac{\Delta}{8\mathcal{K}_c}e^{-\frac{\Delta u_c\tau}{L}}} .
\end{aligned} \tag{5.100}$$

Naturally, it also reproduce the free-like result in the limit $\Delta \longrightarrow 0$. For the gapped limit, numerical solutions can be performed.

It can be proved that the off-diagonal correlators vanish identically in the bosonized $N = 4$ model (Appendix C). For the correlator to be non-zero, the sum of the factors multiplying fields in the exponentials has to vanish. In the ϕ sector this does not happen and off-diagonal correlators vanish. On the contrary, in the θ sector we have signs that non-zero correlators involving cross-chirality fermions can occur. We will deepen on it in the next subsection.

5.4.4 Triple-pairing operator

These operators describe pairing with zero total momentum. Let us define them as:

$$O^{x}_{TS}(r) = c^{\dagger}_{\uparrow +}(r)c^{\dagger}_{\uparrow -}(r) + c^{\dagger}_{\downarrow +}(r)c^{\dagger}_{\downarrow -}(r) = \frac{1}{\pi\alpha}e^{-i\sqrt{2\pi}\theta_c(r)}\cos\left(\sqrt{2\pi}\theta_s(r)\right) \tag{5.101}$$

$$O^{y}_{TS}(r) = -i\left[c^{\dagger}_{\uparrow +}(r)c^{\dagger}_{\uparrow -}(r) - c^{\dagger}_{\downarrow +}(r)c^{\dagger}_{\downarrow -}(r)\right] = -\frac{1}{\pi\alpha}e^{-i\sqrt{2\pi}\theta_c(r)}\sin\left(\sqrt{2\pi}\theta_s(r)\right), \tag{5.102}$$

where it was used the bosonization dictionary (5.66). At this point, it is worthy to mention the Klein factors $U^{\dagger}_{\eta}$ and U_{η} which ensure that fermion fields of different species anticommute. In this way by making a permutation in complex fermions in the left side, it still appears the correct sign in front of cosine in the right side of the previous equations, otherwise there are not way to obtain a change of sing by permuting bosonic fields. In general, for models where each separate number of fermion's species is conserved, Klein factors are omitted. They do not contribute in space-time decay of correlator function because they do not have any dependence on coordinates, and we just have to be careful to track the correct sign. For further discussion you can consult [60].

Turning back to our discussion, now we can define:

$$O^{z}_{TS}(r) = O^{x}_{TS}(r) + iO^{y}_{TS}(r) = \frac{1}{\pi\alpha}e^{-i\sqrt{2\pi}\theta_c(r)}e^{-i\sqrt{2\pi}\theta_s(r)}. \tag{5.103}$$

The model is "charge" conserving (not really but number conserving), so that for the same reason correlators stemming from the operator $\lim_{z'\to z} O^{z}_{TS}(z,z') = \frac{1}{\pi\alpha}e^{-i\sqrt{2\pi}\theta_c(z)}e^{-i\sqrt{2\pi}\theta_s(z)}$ vanish as well. There is no true superconducting order due to the impossibility to break a continuous symmetry in one spatial dimension [48]. On the other side, the number conserving operator $\langle O^{z\dagger}_{TS}(r)O^{z}_{TS}(r')\rangle$ provides a non zero result:

$$\begin{aligned}
\langle O^{z\dagger}_{TS}(r)O^{z}_{TS}(0)\rangle &= \frac{1}{\pi^2\alpha^2}e^{2\pi\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle}e^{2\pi\langle\theta_s(r)\theta_s(0)-\theta_s^2(0)\rangle} \\
&= \frac{1}{\pi^2\alpha^2}\left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{\mathcal{K}_s}} e^{-\frac{1}{2\mathcal{K}_c}\left[\int_0^{i\,x+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\,K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right)dz+\ln(\alpha^2)+c.c.\right]} \\
&\quad\times\prod_{\pm} e^{\frac{\Delta}{2\mathcal{K}_c}\left[e^{-\frac{\Delta u_c\tau\pm i\,x}{L}}\right]}\left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)\,e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\frac{\Delta}{2\mathcal{K}_c}e^{-\frac{\Delta u_c\tau}{L}}}.
\end{aligned} \tag{5.104}$$

We adopt $\langle O^{z\dagger}_{TS}(z,z)O^{z}_{TS}(0,0)\rangle$ of Eq.(5.104) as a suggestion for $g_{\cap}(z,\bar{z})\,g_{\cup}(z,\bar{z})$ in the large N case (see Chapter 3).

–6–
Discussions

We have proposed an extended $1+1$ dimensional non-chiral SYK model described by Majorana fermions with many fermionic degrees of freedom, where it was included cross-chirality interactions. As in the original SYK model, N Majorana fermions have all-to-all fermion random interactions where two sets of random Gaussian couplings J_{ijkl} and Q_{ijkl} mediate the interactions between fermions of the same and different chirality branches, respectively. For the large N case it is seen that the model behaves as the Random Thirring model [37] when $J = 0$, while if $Q = 0$, the model becomes in two decoupled left/right-movers SYK chiral systems.

In the original $0+1$ dimensional case, the model presents an emergent approximate conformal symmetry at low energies, where the reparametrization symmetry is spontaneously broken down to a $SL(2,\mathbb{R})$ subgroup. Goldstone soft modes appear in the excitation spectrum. These gapless excitations become gapped when the approximate conformal symmetry is explicitly broken by reintroducing the derivative term of the Lagrangian as an ultraviolet correction. We set out to study if gapped excitations are still present in higher dimensions for the non-chiral $1+1$ d case. In our extended non-chiral $1+1$ d SYK model, the theory requires a regulator to be included already in the action, which breaks conformal invariance explicitly and makes the theory not strictly Lorentz invariant. However, we speak of "quasi" Lorentz invariance at IR. In this limit, when an UV cutoff Λ is introduced in real time-space by regularising the singularity at small arguments with a logarithmic factor, the disorder average of the model provides a Schwinger-Dyson equation in the $1/N \to 0$ limit that can be solved without breaking the conformal symmetry and the model is found to be still critical. Then, unlike the $0+1$ d case, the emergent approximate conformal symmetry present at low energies is not broken spontaneously by the conformal correlator. This does not give us any hint about excitations being gapped: they all could be gapless if the system remains critical. Thus, we seek for solutions from the case $N = 4$. In this limit, the model is non symmetrical (non Lorentz invariant) and non traceless (non conformal invariant), however, it can be solved exactly by bosonization. For $N = 4$ there are just two independent coupling parameters $J_{1234} \equiv J$ and $Q_{1234} \equiv Q$, where we have limited the analysis to the simpler case $J = Q$. The two-point correlation functions have been obtained, indicating that both gapless and gapped excitations are present. These two branches exist due to the separation of the model into the "pseudo-charge" and "pseudo-spin" sectors, characterized by the corresponding velocities $u_{c,s} = u_0\sqrt{1 \pm \frac{J}{\pi u_0}}$, where u_0 is a velocity scale. We have assumed that this behavior remains for large N case based in the following arguments: from the $1+1$ d chiral SYK model [35], it is shown that the functional form of the two-point correlator for $N = 4$ and large N are equal (beyond the fact that the meaning of the coupling is not strictly the same in both cases) and the model remains critical with gapless excitations. On the other side, from the non-chiral case, the model is still critical in the infrared, as it is shown by the powerlaw behavior of the correlator. However, for the strong coupling limit, it can be seen by variational arguments that a non zero gap is present. Therefore, we have assumed heuristically analytic correlators suggested by the $N = 4$ case to study the model in the large N case, and we surmise that the large N limit is characterized by the disorder averaged interaction coupling J, and by a gap Δ of the gapped branch of the spectrum. We use the leading term of the gapped correlators, which leads to the free-like case when the gap goes to zero, to evaluate an approximated free energy in the temperature limit $T = 0$ at the lowest $1/N$ order, which has to be minimized with respect to the parameter Δ. We find that there is a range of small J values in which $\Delta \neq 0$ does not correspond to a minimum of the energy, confirming, also in this approximate approach to the problem that the gapped branch only arises at large interaction coupling. However, our approach seem to be justified only at intermediate J couplings and fails at $J \to 0$ and $\frac{J}{\pi u_0} \lesssim 1$. Correlator functions tend to the "free-like" when $J/\pi u_0$ is small and $\Delta \to 0$, in the conformal symmetry limit. This limiting form is in contrast with the fact that the derivation of the pseudo-charge correlators in the $N = 4$ case requires Δ and $J/\pi u_0$ to be sizeable. Finally, it was shown that a critical powerlaw decay of the correlators at large distance with non free-like exponent $\Gamma \neq 1$ can also be obtained from the conformal symmetry limit, for intermediate values of the couplings. By using linearization in approximating the self-energy obtained from the Schwinger-Dyson equations, we have found a correlation function with exponent $\Gamma = 1.3$ for a fixed coupling J. In Fourier space, the correlation function is proportional to $g(p) \sim p^{\lambda}$, with $0 < \lambda < 1$, confirming the non-Fermi-Liquid nature of the excitations.

By analyzing the superficial degree of divergence, the model appears to be renormalizable, with interactions being marginal. Non-chiral $1+1$ dimensional models are statistically marginal irrelevant [37], in the sense that after averaging over disorder and using conformal perturbation theory, the β function is positive. However, there are relevant and irrelevant operators that will grow or decrease as we flow into the infrared. Since all these contributions are screened by the net effect of the average over disorder, added to the fact that the model is not truly conformal symmetric, we found that the theory can be studied as an effective model in a range of couplings J determined by the physical bound $0 < J/u_0\pi < 1$, coming from the gapless pseudo-spin sector. For this, we use the dual sine-Gordon version of the model, which can be solved exactly by bosonization, as an approximation to infer the features of the system.

With these discussions, we have answered the open questions that motivated this work. We have found an approximation which makes the model solvable and a physical bound where the theory can be studied safely. Gapped excitations were found in the extended $1+1$ dimensional SYK model, in our approximate approach, for large interaction coupling regime. Nonetheless, our model cannot reproduce the large N results in the entire physical bound, therefor, it is restricted to an intermediate range of values of coupling constant J. Accordingly, there are some aspects of the theory that can be studied further: as our approach seem to be valid only for a specific range of the physical bound, a complete description is desirable. Also, similarity of the functional form of correlators, for the $N = 4$ and large N cases, is only valid for the two-point functions. Therefore, our approach is not able to infer four-point correlators in the large N limit. Finally, it remains to explore how the model can be studied as a holographic dual for gravity, and also explore in depth the non-Fermi-Liquid behavior of the model.

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Appendix A
Fourier transforms

In this appendix we provide the Fourier transforms of the two-point function $g(z, \overline{z})$ and the self-energy $\Sigma(z, \overline{z})$ for the solution of the SD equations in the large N case (Chapter 3).

A.1 Diagonal first approximation

In the simpler approach where off-diagonal terms in matrix $\hat{G}$ are neglected, the integrals that we need to solve are the following:

$$g(p, \overline{p}) = C \int \frac{d^2 z}{z} \ln(z\overline{z}\Lambda^2)^{\alpha} e^{ipz + i\overline{p}\overline{z}} \tag{A.1}$$

$$\Sigma(p, \overline{p}) = C^3 \left(J^2 \int \frac{d^2 z}{z^3} \ln(z\overline{z}\Lambda^2)^{3\alpha} e^{ipz + i\overline{p}\overline{z}} + Q^2 \int \frac{d^2 z}{z\overline{z}^2} \ln(z\overline{z}\Lambda^2)^{3\alpha} e^{ipz + i\overline{p}\overline{z}} \right). \tag{A.2}$$

We are going to start with the Fourier transforms of the Green function $g(z, \overline{z})$ and the Q part of the self-energy, let's call it Σ_Q. For this task, we are going to use the following integral [37]:

$$\begin{aligned} F_{\beta}(|p|/\Lambda) \quad &\equiv \quad \int \frac{d^2 z}{z\overline{z}} \ln(z\overline{z}\Lambda^2)^{\beta} e^{ipz + i\overline{p}\overline{z}} = 2^{\beta+1}\pi \int_0^{\infty} \frac{dr}{r} \ln^{\beta}(\Lambda r) J_0(|p| r) \\ &= \quad \frac{\pi}{\beta + 1} \ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\beta+1}, \end{aligned} \tag{A.3}$$

where $|p|^2 = p\overline{p}$. For $\beta = \alpha$, we can note that

$$\begin{aligned} g(p, \overline{p}) \quad &= \quad -iC\partial_{\overline{p}} F_{\alpha}(|p|/\Lambda) \\ &= \quad i\pi \frac{C}{\overline{p}} \ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\alpha}. \end{aligned} \tag{A.4}$$

On the other side, defining

$$\Sigma(p, \overline{p}) = \Sigma_J(p, \overline{p}) + \Sigma_Q(p, \overline{p}), \tag{A.5}$$

where

$$\Sigma_J(p,\overline{p}) \equiv C^3 J^2 \int \frac{d^2 z}{z^3} \ln(z\overline{z}\Lambda^2)^{3\alpha} e^{ipz+i\overline{p}\overline{z}} \tag{A.6}$$

and

$$\Sigma_Q(p,\overline{p}) \equiv C^3 Q^2 \int \frac{d^2 z}{z\overline{z}^2} \ln(z\overline{z}\Lambda^2)^{3\alpha} e^{ipz+i\overline{p}\overline{z}}, \tag{A.7}$$

we can note that

$$\begin{aligned} \partial_{\overline{p}}\Sigma_Q &= iC^3 Q^2 F_{3\alpha}(|p|/\Lambda) \\ &= iC^3 Q^2 \frac{\pi}{3\alpha+1} \ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1} \end{aligned} \tag{A.8}$$

where, neglecting $O\left(\ln^{3\alpha}\frac{\Lambda^2}{|p|^2}\right)$ terms we obtain:

$$\Sigma_Q(p,\overline{p}) \approx i\overline{p} C^3 Q^2 \frac{\pi}{3\alpha+1} \ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \tag{A.9}$$

These are solution already know from Random-Thirring model where Q interactions domain. However, in our case remains to obtain the self-energy coming from J interacting sector. From definition of $\Sigma_J(p,\overline{p})$, we proceed as follows:

$$\begin{aligned} \partial_p \Sigma_J &= C^3 J^2 i \int \frac{d^2 z}{z^2} \ln\left(z\bar{z}\Lambda^2\right)^{3\alpha} e^{i\,(pz+\bar{p}\bar{z})} \\ &= C^3 J^2 i \int dx\, dy \frac{x^2 - y^2 - 2i\,xy}{(x^2+y^2)^2} \ln\left(z\bar{z}\Lambda^2\right)^{3\alpha} e^{i\,(pz+\bar{p}\bar{z})}. \end{aligned} \tag{A.10}$$

Note that $(pz + \bar{p}\bar{z}) = \vec{p}\cdot\vec{x}$ and $e^{i\,pz+\bar{p}\bar{z}} = e^{i\,|p|r[\cos\gamma\cos\theta+\sin\gamma\sin\theta]} = e^{i|p|r\cos\eta}$ with $\eta = \theta - \gamma$. Thus,

$$\partial_p \Sigma_J = -C^3 J^2 i \int_0^\infty r\, dr \int_{-\pi}^{\pi} d\theta\, \frac{\cos 2\theta - i\, \sin 2\theta}{r^2} \ln\left(r^2\Lambda^2\right)^{3\alpha} e^{i\,|p|r\cos\eta}. \tag{A.11}$$

As $\cos 2\theta = \cos 2\eta \cos 2\gamma - \sin 2\eta \sin 2\gamma$, we have

$$\cos 2\theta - i\, \sin 2\theta = (\cos 2\eta - i\, \sin 2\eta)\,(\cos 2\gamma - i\, \sin 2\gamma)\,, \tag{A.12}$$

also $d\theta \longrightarrow d\eta$ and the extrema cover a period, anyhow. We drop $\sin 2\eta$ which is odd in the integration to obtain:

$$\begin{aligned} \partial_p \Sigma_J &= -C^3 J^2 i\, (\cos 2\gamma - i\, \sin 2\gamma) \int_0^\infty r\, dr \int_{-\pi}^{\pi} d\eta\, \frac{\cos 2\eta}{r^2} \ln\left(r^2\Lambda^2\right)^{3\alpha} e^{i\,|p|r\cos\eta} \\ &= 2\pi\, C^3 J^2 i\, \frac{\overline{p}}{p} \int_0^\infty \frac{dr}{r} \ln\left(r^2\Lambda^2\right)^{3\alpha} J_2(|p|r) \\ &= 2^{3\alpha+1}\pi\, C^3 J^2 i \frac{\overline{p}}{p} \int_0^\infty \frac{dr}{r} \ln^{3\alpha}(r\Lambda)\, J_2(|p|r) \\ &= C^3 J^2 i \frac{\overline{p}}{p} F_{3\alpha}^{(J)}(|p|/\Lambda), \end{aligned} \tag{A.13}$$

where it was used that $\frac{\bar{p}}{p} \sim e^{-2i\gamma} = (\cos 2\gamma - i\ \sin 2\gamma)$ and it was introduce the integral $F^{(J)}_{3\alpha}(|p|/\Lambda)$ in the same spirit as the Q case. In a similar way, we define

$$G^{(J)}_{\epsilon}\left(\frac{|p|}{\Lambda}\right) = \int_0^\infty \frac{dr}{r}\ (\Lambda r)^\epsilon \ln(\Lambda r)\ J_2(|p|r) \tag{A.14}$$

in such a way that it's observed:

$$F^{(J)}_{3\alpha+1}(|p|/\Lambda) = 2^{3\alpha+2}\pi \lim_{\epsilon\to 0} \partial^{3\alpha}_\epsilon G^{(J)}_\epsilon\left(\frac{|p|}{\Lambda}\right)$$

or for general intenger

$$F^{(J)}_{n+1}(|p|/\Lambda) = 2^{n+2}\pi \lim_{\epsilon\to 0} \partial^{n}_\epsilon G^{(J)}_\epsilon\left(\frac{|p|}{\Lambda}\right).$$

From [61] (Eq. 6.771 pg. 747) we have:

$$\int_0^\infty x^\epsilon x^{\mu+1/2}\ \ln x\, J_\nu(ax)\, dx \qquad\qquad a=0; \qquad -\Re e\, \nu - \frac{3}{2} < \Re e\, \mu + \epsilon < 0 \tag{A.15}$$

fulfilled by $\mu = -3/2$, $\nu = 2 \qquad \Rightarrow \qquad -7/2 < -3/2 + \epsilon < 0$.

$$G^{(J)}_\epsilon\left(\frac{|p|}{\Lambda}\right) = \frac{1}{2^{2-\epsilon}}\frac{\Gamma\left[1+\frac{\epsilon}{2}\right]}{\Gamma\left[2-\frac{\epsilon}{2}\right]}\left(\frac{\Lambda}{|p|}\right)^\epsilon \left[\psi\left(1+\frac{\epsilon}{2}\right) + \psi\left(2-\frac{\epsilon}{2}\right) + 2\ln\frac{2\Lambda}{|p|}\right].$$

Expanding for small ϵ:

$$\begin{aligned} G^{(J)}_\epsilon\left(\frac{|p|}{\Lambda}\right) &\approx \left(\frac{\Lambda}{|p|}\right)^\epsilon \frac{1}{4}\left[\gamma(\gamma-1) + 2\ln\frac{2\Lambda}{|p|}\right] \\ &= \frac{1}{2}\left(\frac{\Lambda}{|p|}\right)^\epsilon \ln\frac{\Lambda}{|p|}\left[1 + O\left(1/\ln\frac{\Lambda}{|p|}\right)\right] \\ &= \frac{1}{2}\sum_{m=0}^\infty \frac{\epsilon^m \ln^{m+1}(\frac{\Lambda}{|p|})}{m!}\left[1 + O\left(1/\ln\frac{\Lambda}{|p|}\right)\right]. \end{aligned} \tag{A.16}$$

When it's made the derivative with respect ϵ, the first term which survive is proportional to $m(m-1)(m-2)\cdots(m-n+1)$ and it's relevant just when $m = n$ (other terms vanishes or are proportional to $\epsilon \longrightarrow 0$). With these, $n(n-1)(n-2)\cdots(n-n+1)$ cancels with $n!$ and we have:

$$\begin{aligned} F^{(J)}_{n+1}(|p|/\Lambda) &= 2^{n+2}\pi \lim_{\epsilon\to 0} \partial^n_\epsilon G^{(J)}_\epsilon\left(\frac{|p|}{\Lambda}\right) \\ &\approx 2^{n+1}\pi \ln^{n+1}(\Lambda/|p|) \\ &= \pi \ln^{n+1}(\Lambda^2/|p|^2). \end{aligned} \tag{A.17}$$

Plugging the last equation in Eq. (A.13) and considering in this case $n+1=3\alpha$, we found:

$$\partial_p \Sigma_J = C^3 J^2 i \frac{\overline{p}}{p}\, \pi \ln^{3\alpha}(\Lambda^2/|p|^2). \tag{A.18}$$

Finally, by perform the integral on p we reach the desired result:

$$\begin{aligned}\Sigma_J(p,\bar{p}) &= -C^3 J^2 i\pi\, \bar{p} \int^p \frac{dp}{p} \ln\left(\frac{\bar{p}}{\Lambda^2}p\right)^{3\alpha} \\ &= \frac{C^3 J^2 i\pi\, \bar{p}}{3\alpha+1} \ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \end{aligned} \tag{A.19}$$

In conclusion, the Fourier transforms in the large J, Q limit are given by:

$$g(p,\overline{p}) = i\pi \frac{C}{\overline{p}} \ln^{\alpha}\left(\frac{\Lambda^2}{|p|^2}\right), \tag{A.20}$$

$$\Sigma(p,\overline{p}) \approx i\pi\, \bar{p}\, C^3 \frac{(J^2+Q^2)}{3\alpha+1} \ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right). \tag{A.21}$$

A.2 Complete solution

Here we present the results for the complete solution, including the off-diagonal terms of $\hat{G}$. By reparametrization arguments, we assume that

$$g_+(z,\overline{z}) = \frac{a}{z} \ln^{\alpha}\left(|z|^2\Lambda^2\right) \tag{A.22}$$

$$g_-(z,\overline{z}) = \frac{a}{\overline{z}} \ln^{\alpha}\left(|z|^2\Lambda^2\right) \tag{A.23}$$

$$g_\cap = g_\cup = \frac{b}{|z|} \ln^{\alpha}\left(|z|^2\Lambda^2\right), \tag{A.24}$$

meanwhile, unitary condition allows us to write for self-energies

$$\Sigma_+(z,\overline{z}) = \left[J^2 \frac{a^3}{z^3} + Q^2\left(\frac{a^3}{\overline{z}|z|^2} + 2\,\frac{b^3}{\overline{z}|z|^2}\right)\right] \ln^{3\alpha}\left(|z|^2\Lambda^2\right) \tag{A.25}$$

$$\Sigma_-(z,\overline{z}) = \left[J^2 \frac{a^3}{\overline{z}^3} + Q^2\left(\frac{a^3}{z|z|^2} + 2\,\frac{b^3}{z|z|^2}\right)\right] \ln^{3\alpha}\left(|z|^2\Lambda^2\right) \tag{A.26}$$

$$\Sigma_\cup(z,\overline{z}) = \Sigma_\cap(z,\overline{z}) = \left[J^2 \frac{a^3}{|z|^3} + Q^2\left(\frac{a^3}{|z|^3} + 2\,\frac{b^3}{|z|^3}\right)\right] \ln^{3\alpha}\left(|z|^2\Lambda^2\right), \tag{A.27}$$

where $|z|^2 = z\overline{z}$ and a and b are two constant parameters. Unbroken parity implies that $g_+(z,\overline{z}) = g_-(\overline{z},z) \equiv g(z,\overline{z})$ and $\Sigma_+(z,\overline{z}) = \Sigma_-(\overline{z},z) \equiv \Sigma(z,\overline{z})$. Therefore, the Fourier transforms for the diagonal terms in matrix $\hat{G}$ are exactly the same as before, up to a different constant, and are given by:

$$g(p,\overline{p}) = i\pi\frac{a}{\overline{p}}\ln^{\alpha}\left(\frac{\Lambda^2}{|p|^2}\right), \tag{A.28}$$

$$\Sigma(p,\overline{p}) \approx i\pi\,\overline{p}\,\frac{\left(a^3J^2 + (a^3+2b^3)Q^2\right)}{3\alpha+1}\,\ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right). \tag{A.29}$$

For the off-diagonal terms, the integrals that we need to solve are the following:

$$g_{\cap}(p,\overline{p}) = g_{\cup}(p,\overline{p}) = b\int\frac{d^2z}{|z|}\ln(|z|^2\Lambda^2)^{\alpha}e^{i|p||z|} \tag{A.30}$$

$$\Sigma_{\cap}(p,\overline{p}) = \Sigma_{\cup}(p,\overline{p}) = \left(a^3J^2 + (a^3+2b^3)Q^2\right)\int\frac{d^2z}{|z|^3}\ln(|z|^2\Lambda^2)^{3\alpha}e^{i|p||z|}. \tag{A.31}$$

In the same way as for the diagonal solution, we are going to compute the Fourier transforms of the Green function $g_{\cap}(z,\overline{z}) = g_{\cup}(z,\overline{z})$ and the self-energy, $\Sigma_{\cap}(z,\overline{z}) = \Sigma_{\cup}(z,\overline{z})$ by using the following integral [37]:

$$\begin{aligned} F'_{\beta}(|p|/\Lambda) &\equiv \frac{1}{2}\int\frac{d^2z}{|z|^2}\ln(|z|^2\Lambda^2)^{\beta}e^{i|p||z|} \\ &= \frac{1}{2}\frac{\pi}{\beta+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\beta+1}, \end{aligned} \tag{A.32}$$

where $|p|^2 = p\overline{p}$. For $\beta = \alpha$, we can note that

$$\begin{aligned} g_{\cap}(z,\overline{z}) = g_{\cup}(z,\overline{z}) &= -ib\partial_{|p|}F'_{\alpha}(|p|/\Lambda) \\ &= i\pi\frac{b}{|p|}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\alpha}. \end{aligned} \tag{A.33}$$

On the other side, we can note that

$$\begin{aligned} \partial_{|p|}\Sigma_{\cap}(z,\overline{z}) = \partial_{|p|}\Sigma_{\cup}(z,\overline{z}) &= i\left(a^3J^2 + (a^3+2b^3)Q^2\right)F'_{3\alpha}(|p|/\Lambda) \\ &= i\left(a^3J^2 + (a^3+2b^3)Q^2\right)\frac{\pi}{3\alpha+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1} \end{aligned} \tag{A.34}$$

where, neglecting $O\left(\ln^{3\alpha}\frac{\Lambda^2}{|p|^2}\right)$ terms we obtain:

$$\Sigma_Q(p,\overline{p}) \approx i\pi\frac{\left(a^3J^2 + (a^3+2b^3)Q^2\right)}{3\alpha+1}|p|\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \tag{A.35}$$

Appendix B
Bosonization technique

In this appendix we show some hints in bosonization technique. In the usual lore, bosonization can be perform by the "Field-Theoretical bosonization" and the "Constructive bosonization". The first one propose an equivalence between fermions and bosons means a bosonization dictionary in such a way that reproduce the same correlators as Fermi fields [46, 62]. The other one naturally obtains the same results but actually also explains why field-theoretical bosonization works and how bosonic fields and Klein factors naturally appear from first principles. It also includes models where Hamiltonian doesn't conserve each separate total number of $\psi^\dagger$ and ψ fermionic fields [47, 53, 63].

B.1 Field theoretical bosonization

Correlation functions of the Fermi fields can be reproduced by the correlator of the bosonic operator given the follow bosonization dictionary:

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_\pm}. \tag{B.1}$$

We remarks that this is not an operator identity and it is just meaningful when correlation function are computed in the Fermi (bosonic) vacuum with the same momentum cut-off α.

In the Field-Theoretical's spirit, we use the master formula (B.1) to reproduce any interaction term made out of the Fermi field by the corresponding bosonic counterpart. For instance, let's consider the correlation of the massless Dirac fermion:

$$G_+^{(F)}(x) = \langle \psi_+(x)\psi_+^\dagger(0)\rangle = \frac{1}{2\pi(\alpha - ix)}. \tag{B.2}$$

Bosonization dictionary should relate it with the free massless scalar field correlator

$$G_+^{(B)}(x) = \langle \phi_+(x)\phi_+(0) - \phi_+^2(0)\rangle = \frac{1}{4\pi} \ln \frac{\alpha}{(\alpha - ix)}. \tag{B.3}$$

Indeed, by plugging Eq. (B.1) in Eq. (B.2), we obtain:

$$\begin{aligned}
\langle\psi_+(x)\psi_+^\dagger(0)\rangle &= \langle\frac{1}{\sqrt{2\pi\alpha}}e^{i\sqrt{4\pi}\phi_+}\frac{1}{\sqrt{2\pi\alpha}}e^{-i\sqrt{4\pi}\phi_+}\rangle \qquad \text{(B.4)}\\
&= \frac{1}{2\pi\alpha}\langle: e^{i\sqrt{4\pi}\phi_+}e^{-i\sqrt{4\pi}\phi_+} :\rangle e^{4\pi\langle\phi_+(x)\phi_+(0)-\phi_+^2(0)\rangle}\\
&= \frac{1}{2\pi\alpha}e^{4\pi G_+^{(B)}(x)}\\
&= \frac{1}{2\pi(\alpha-ix)} \qquad \text{(B.5)}
\end{aligned}$$

which correctly reproduces the massless Dirac correlator.

B.1.1 Bosonization and point splitting

To apply bosonization in our model, we consider two Majorana fermion fields $\psi_\pm^i$ and $\psi_\pm^j$ $(i \neq j)$, being possible to define complex fermions $c_{\sigma\pm}$ as:

$$c_{\uparrow\pm}(t,x) = \frac{1}{\sqrt{2}}(\psi_\pm^1(t,x) + i\psi_\pm^2(t,x)) \tag{B.6}$$

$$c_{\downarrow\pm}(t,x) = \frac{1}{\sqrt{2}}(\psi_\pm^3(t,x) + i\psi_\pm^4(t,x)) \tag{B.7}$$

$$c_{\uparrow\pm}^\dagger(t,x) = \frac{1}{\sqrt{2}}(\psi_\pm^1(t,x) - i\psi_\pm^2(t,x)) \tag{B.8}$$

$$c_{\downarrow\pm}^\dagger(t,x) = \frac{1}{\sqrt{2}}(\psi_\pm^3(t,x) - i\psi_\pm^4(t,x)) \tag{B.9}$$

and its bosonization

$$c_{\sigma\pm}(t,x) =: \frac{1}{\sqrt{2\pi\alpha}}e^{\pm i\sqrt{4\pi}\phi_{\sigma\pm}(t,x)} : \tag{B.10}$$

$$c_{\sigma\pm}^\dagger(t,x) =: \frac{1}{\sqrt{2\pi\alpha}}e^{\mp i\sqrt{4\pi}\phi_{\sigma\pm}(t,x)} :, \tag{B.11}$$

where σ is the index for pseudospin ↑ or ↓ for the fermion.

At the moment we show how bosonization works in correlators at different points $\psi(x)$, $\psi(x')$ (see Eq. B.5). However, it is not guaranteed that a product of two such well-behaved operators at the same point is itself well behaved. Point-splitting prescription is required to regularizing infinities. This is equivalent to the normal-ordering regularization which is used in constructive bosonization [47]. Following [35], on the constant time slice, the chiral Majorana fermions satisfy the anticommutation relation (omitting indices for simplicity):

$$\left\{c(x), c^\dagger(x')\right\} = \frac{1}{2}\left[\left\{\psi_{i(x)}, \psi_{i(x')}\right\} + \left\{\psi_{j(x)}, \psi_{j(x')}\right\} + i\left\{\psi_{i(x')}, \psi_{j(x)}\right\} - i\left\{\psi_{i(x)}, \psi_{j(x')}\right\}\right] \tag{B.12}$$

where considering that $\left\{\psi_{i(x)}, \psi_{j(x')}\right\} = \delta_{ij}\delta_{(x-x')}$ and $x \neq x'$, we have

$$\left\{c(x), c^{\dagger}(x')\right\} = \{c(x), c(x')\} = \left\{c^{\dagger}(x), c^{\dagger}(x')\right\} = 0. \tag{B.13}$$

This requires the boson commutation relation

$$[\phi(x), \phi(x')] = \frac{i}{4} sgn(x - x'). \tag{B.14}$$

Taking the derivative of the latter with respect x, and considering $x \neq x'$, we have

$$\begin{aligned} [\partial_x \phi(x), \phi(x')] &= \frac{i}{4}\partial_x sgn(x - x') \\ &= \frac{i}{2}\delta(x - x'). \end{aligned} \tag{B.15}$$

Each $\phi\pm$ bosonic field can be separate into a sum of creation and destruction operators that we will name

$$\phi(x) = \varphi(x) + \varphi^{\dagger}(x) \tag{B.16}$$

satisfying

$$[\varphi^{\dagger}(x), \varphi^{\dagger}(x')] = [\varphi(x), \varphi(x')] = 0. \tag{B.17}$$

The commutation relation can be written now as

$$\left[\partial_x \varphi(x), \varphi^{\dagger}(x')\right] + \left[\partial_x \varphi^{\dagger}(x), \varphi(x')\right] = \frac{i}{2}\delta(x - x'). \tag{B.18}$$

On the other hand, from the Sokhotsky's formula, it's possible to show that

$$\left[\partial_x \varphi(x), \varphi^{\dagger}(x')\right] = \frac{1}{x' - x - i0^{+}} \tag{B.19}$$

$$\left[\partial_x \varphi^{\dagger}(x), \varphi(x')\right] = \frac{1}{x - x' - i0^{+}} \tag{B.20}$$

and after integrating

$$\left[\varphi(x), \varphi^{\dagger}(x')\right] = \ln\left[\frac{4\pi}{i\alpha}(x - x' + i0^{+})\right] \tag{B.21}$$

where α plays the role of a short-distance cut-off. We can now calculate the operator product expansion with a point splitting in the x direction. Considering the bilinear $c^{\dagger}(x)c(x')$ with small x direction separation $x - x'$, we have

$$\begin{aligned} c^{\dagger}(x)c(x') &= \frac{1}{2\pi\alpha} : e^{-i\sqrt{4\pi}\phi(x)} :: e^{i\sqrt{4\pi}\phi(x')} : \\ &= \frac{1}{2\pi\alpha} e^{-i\sqrt{4\pi}\varphi^{\dagger}(x)} e^{-i\sqrt{4\pi}\varphi(x)} e^{i\sqrt{4\pi}\varphi^{\dagger}(x')} e^{i\sqrt{4\pi}\varphi(x')} \end{aligned} \tag{B.22}$$

Using the operator identity $e^A e^B = e^{[A,B]} e^B e^A$, the commutation relation for the creation and destruction bosonic fields (Eq. B.21) and expanding the exponential, we can reach the following operator at the same point:

$$\begin{aligned} c^\dagger(x)c(x') &= \frac{i}{2\pi(x-x'+i0^+)}\left[1 - i\sqrt{4\pi}(x-x')(\partial_x\varphi^\dagger + \partial_x\varphi) + \frac{(x-x')^2}{2}\left(4\pi(\partial_x\varphi^\dagger)^2 + \right.\right. \\ &\left.\left. +4\pi(\partial_x\varphi)^2 + i\sqrt{4\pi}\partial_x^2(\varphi^\dagger + \varphi) + 8\pi\partial_x\varphi^\dagger\partial_x\varphi\right) + O((x-x')^3)\right] \end{aligned} \tag{B.23}$$

where the imaginary part reflects an infinite vacuum density due to the Dirac sea. Returning to the previous bosonic field $\phi(x) = \varphi^\dagger(x) + \varphi(x)$, taking the normal ordered density operator by the limit $x - x' \longrightarrow 0$, and considering indices again, we have

$$: c^\dagger_{\sigma\pm}(x)c_{\sigma\pm}(x) := \frac{1}{\sqrt{\pi}}\partial_x\phi_{\sigma\pm}(x). \tag{B.24}$$

The latter is useful for the interacting part. Next, we consider the kinetic term by point splitting

$$-\frac{i}{2}\left[\psi^i_\pm(x)\partial_x\psi^i_\pm(x) + \psi^j_\pm(x)\partial_x\psi^j_\pm(x)\right] = -ic^\dagger_{\sigma\pm}(x)\partial_x c_{\sigma\pm}(x) \tag{B.25}$$

In order to have the bosonized version, we have to take in account the next expressions. First, rewrite the previous equation by using the fact that in general $\partial_x c_\beta(x) \approx \frac{c_\beta(x)-c_\beta(x')}{x-x'}$ and $c^\dagger_\beta(x) \approx \frac{c^\dagger_\beta(x)+c^\dagger_\beta(x')}{2}$ (considering that the limit $x' \longrightarrow x$ it will be taken). After that, use Eq. (B.23) and consider the expression $\frac{\partial_x\phi_\beta(x)-\partial_x\phi_\beta(x')}{x-x'} \approx \partial_x^2\phi_\beta(x)$. Finally, neglect the vacuum term taking the limit $x - x' \longrightarrow 0$. The normal ordered kinetic term results to be

$$-ic^\dagger_\beta(x)\partial_x c_\beta(x) =: [\partial_x\phi_\beta(x)]^2 : . \tag{B.26}$$

B.2 Klein factors

No combination of bosonic operators can raise or lower the total fermion number by one, nor can they ensure the anti-commutation relation of different fermion fields species. Therefore, it is needed to define the so-called Klein factors $F^\dagger_\eta$ and F_η (or U and U^{-1} according to [53]). They are defined as operators with the following properties [47]:

$$\{F^\dagger_\eta, F_{\eta'}\} = 2\delta_{\eta\eta'} \qquad , with \quad F_\eta F^\dagger_\eta = F^\dagger_\eta F_\eta = 1 \tag{B.27}$$

$$\{F^\dagger_\eta, F^\dagger_{\eta'}\} = \{F_\eta, F_{\eta'}\} = 0. \tag{B.28}$$

The bosonization dictionary becomes:

$$\psi_\pm(x) = \frac{F_\eta}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_\pm}. \tag{B.29}$$

Appendix C
Correlators in the N = 4 case

In this appendix we compute the different correlators used in the main text for the case $N = 4$ (Chapter 5).

C.1 Free-like correlators

In the free-like case is convenient to obtain correlation functions via functional integral

$$\langle \phi_{\rho(r_1)} \phi_{\rho'(r_2)} \rangle = \frac{\int \mathcal{D}\phi \mathcal{D}\theta e^{-S[\phi,\theta]} \phi_{\rho(r_1)} \phi_{\rho'(r_2)}}{\int \mathcal{D}\phi \mathcal{D}\theta e^{-S[\phi,\theta]}}. \tag{C.1}$$

Let's start using the Lagrangian representation, with the free-like action

$$S = \int dtdx \frac{1}{2} \sum_{\rho=c,s} \left[\frac{1}{u_\rho \mathcal{K}_\rho} (\partial_t \phi_\rho)^2 - \frac{u_\rho}{\mathcal{K}_\rho} (\partial_x \phi_\rho)^2 \right]. \tag{C.2}$$

In this case, all degrees of freedom of θ fields can be integrated by a Gaussian integral in the functional integral. This implies that Eq.(5.73) becomes

$$\langle \phi_{\rho(r_1)} \phi_{\rho'(r_2)} \rangle = \frac{\int \mathcal{D}\phi e^{-S[\phi]} \phi_{\rho(r_1)} \phi_{\rho'(r_2)}}{\int \mathcal{D}\phi e^{-S[\phi]}}. \tag{C.3}$$

Similar treatment can be done with ϕ_c and ϕ_s fields in such a way to obtain an integral depending just on the charge or spin part of the action, i.e. $\rho = \rho'$ otherwise a $\delta_{\rho'\rho}$ appears. Naturally, the $\rho = \rho'$ case is what interests us. In terms of the Fourier modes

$$\phi_\rho(r) = \frac{1}{\beta\Omega} \sum_{k,\omega_n} e^{i(kx - \omega_n \tau)} \phi_\rho(k, \omega_n) \tag{C.4}$$

the action can be expressed as

$$S[\phi_\rho] = \frac{1}{\beta\Omega} \sum_{k,\omega_n} \frac{1}{2u_\rho \mathcal{K}_\rho} \left[\omega_n^2 + u_\rho^2 k^2 \right] \phi_\rho(k, \omega_n) \phi_\rho^*(k, \omega_n) \tag{C.5}$$

$$\sim \phi_\rho^*(k, \omega_n) \left[\frac{1}{2} G^{-1} \right] \phi_\rho(k, \omega_n), \tag{C.6}$$

and we can find the average in momentum space to be

$$\langle \phi_{\rho(q_1)} \phi_{\rho(q_2)} \rangle = u_\rho \mathcal{K}_\rho \frac{\beta\Omega}{\omega_n^2 + u_\rho^2 k^2} \delta_{q_1,-q_2} \tag{C.7}$$

where $q = (k, \omega_n)$ is a vector defined in such a way that $e^{iqr} = e^{i(kx - \omega\tau)}$ and for real fields $\phi^*(q) = \phi(-q)$. Using the Fourier modes again, we can express the correlation function as

$$\langle \phi_{\rho(r_1)} \phi_{\rho(r_2)} \rangle = \frac{1}{\beta\Omega} \sum_{q_1,q_2} u_\rho \mathcal{K}_\rho \frac{1}{\omega_n^2 + u_\rho^2 k^2} (e^{iq_1 r_1} e^{iq_2 r_2} + e^{iq_1 r_2} e^{iq_2 r_1}) \delta_{q_1,-q_2}. \tag{C.8}$$

It is convenient to move coordinates by $-r_2$ and define $r = r_1 - r_2$. By doing this, we obtain

$$\langle \phi_{\rho(r)} \phi_{\rho(0)} \rangle = \frac{1}{\beta\Omega} \sum_{q} u_\rho \mathcal{K}_\rho \frac{1}{\omega_n^2 + u_\rho^2 k^2} (e^{iqr} + e^{-iqr}). \tag{C.9}$$

Let's consider zero temperature and large size system. Sum can be changed according to

$$\frac{1}{\beta\Omega} \sum_{k,n} \to \int \frac{d\omega}{2\pi} \frac{dk}{2\pi},$$

and integration becomes

$$\begin{aligned}
\langle \phi_{\rho(r)} \phi_{\rho(0)} \rangle &= \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} u_\rho \mathcal{K}_\rho \frac{1}{\omega^2 + u_\rho^2 k^2} (e^{iqr} + e^{-iqr}) \\
&= \int \frac{dk}{4\pi} \frac{\mathcal{K}_\rho}{k} (e^{-k(-ix + u_\rho\tau)} + e^{-k(ix + u_\rho\tau)}) \\
(\partial_{\overline{y}} + \partial_y) \langle \phi_{\rho(r)} \phi_{\rho(0)} \rangle &= -\int \frac{dk}{4\pi} \mathcal{K}_\rho (e^{-k\overline{y}} + e^{-ky}) e^{-\alpha k} \\
&= -\frac{\mathcal{K}_\rho}{4\pi} \left(\frac{1}{\overline{y} + \alpha} + \frac{1}{y + \alpha} \right).
\end{aligned} \tag{C.10}$$

Performing the integration in ω we have two integrals like $e^{-k\overline{y}}/k$ and e^{-ky}/k, where $\overline{y} = -ix + u_\rho\tau$ and $y = ix + u_\rho\tau$. Making the derivative with respect $\overline{y}$ and y it is possible to eliminate k in the denominator and perform the integral on k. It was also included the momentum cutoff $e^{-\alpha k}$ to ensure that integral over momentum does not diverge. Now we have to integrate on $\overline{y}$ and y to obtain

$$\langle \phi_{\rho(x,\tau)} \phi_{\rho(0,0)} \rangle = -\frac{\mathcal{K}_\rho}{4\pi} \left[\ln\left(-ix + u_\rho\tau + \alpha\right) + \ln\left(ix + u_\rho\tau + \alpha\right) \right] + C. \tag{C.11}$$

The constant C can be chosen in such a way that we can obtain the desired correlator $\mathcal{G}_{\phi_\rho\phi_\rho}(x, \tau)$:

$$\langle \phi_{\rho(x,\tau)} \phi_{\rho(0,0)} - \phi^2_{\rho(0,0)} \rangle = \frac{\mathcal{K}_\rho}{2\pi} \ln\left(\frac{\alpha}{\sqrt{x^2 + (u_\rho\tau + \alpha)^2}} \right). \tag{C.12}$$

C.2 Strong coupling limit correlators

Let's start with the $\phi_c - \phi_c$ component of the matrix (5.93). As it was done for the free-like case we use the Fourier transform to obtain:

$$\left\langle \phi_c(r_1)\phi_c(r_2) - \frac{\phi_c^2(r_1)+\phi_c^2(r_2)}{2} \right\rangle = \frac{1}{(\beta\Omega)^2}\sum_{q_1,q_2}\langle\phi_c(q_1)\phi_c(q_2)\rangle\Big(e^{iq_1r_1}e^{iq_2r_2}+e^{iq_1r_2}e^{iq_2r_1} -e^{i(q_1+q_2)r_1}-e^{i(q_1+q_2)r_2}\Big). \tag{C.13}$$

Moving coordinates by $-r_2$ and defining $r = r_1 - r_2$:

$$\left\langle \phi_c(r_1)\phi_c(r_2) - \frac{\phi_c^2(r_1)+\phi_c^2(r_2)}{2} \right\rangle = \frac{1}{(\beta\Omega)^2}\sum_{q_1,q_2}\langle\phi_c(q_1)\phi_c(q_2)\rangle\Big(e^{iq_1r}+e^{iq_2r}-e^{i(q_1+q_2)r_1}-1\Big). \tag{C.14}$$

Inserting now the matrix element coming from the path integral

$$\langle\langle\phi_c(q_1)\phi_c(q_2)\rangle = \beta\Omega\frac{u_c\mathcal{K}_c}{u_c^2k_1^2+u_c^2m^2+\omega_n^2}\delta_{q_1,-q_2}, \tag{C.15}$$

we obtain:

$$\begin{aligned}\left\langle \phi_c(r_1)\phi_c(r_2) - \frac{\phi_c^2(r_1)+\phi_c^2(r_2)}{2} \right\rangle &= \frac{1}{\beta\Omega}\sum_{q_1}\frac{u_c\mathcal{K}_c}{u_c^2k_1^2+u_c^2m^2+\omega_n^2}\Big(e^{i(q_1(r_1-r_2))}+e^{-i(q_1(r_1-r_2))}-2\Big)\\ &= \frac{2}{\beta\Omega}\sum_{q_1}\frac{u_c\mathcal{K}_c}{u_c^2k_1^2+u_c^2m^2+\omega_n^2}\Big(\cos\left(q_1(r_1-r_2)\right)-1\Big)\\ &= \langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle. \end{aligned} \tag{C.16}$$

This can be simplified by defining generically $\vec{q} = (\omega, k)$ and $r = (\tau, x)$. Also define $q \cdot r \equiv q \cdot r|_+ = \omega\tau - kx$ and $q \cdot r|_- = \omega\tau + kx$, so that

$$\langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle = \frac{1}{\beta\Omega}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\Big[\langle\phi_c(q_+)\phi_c(-q_+)\rangle(e^{i\omega\tau-i\,kx}-1) +\langle\phi_c(q_-)\phi_c(-q_-)\rangle(e^{i\omega\tau+i\,kx}-1)\Big]. \tag{C.17}$$

Replacing again the matrix element, and redefining $\omega = \omega' u_c$, where ω' now also scales as $1/[L]$, the integral can be rewritten as follow:

$$\langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle = \mathcal{K}_c\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{1}{k^2+m^2+\omega'^2}\left(e^{iu_c\omega'\tau+ikx}+e^{iu_c\omega'\tau-ikx}-2\right). \tag{C.18}$$

Following the same procedure, we obtain for the dual field θ_c

$$\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle = \frac{1}{\mathcal{K}_c}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{(k^2+m^2)}{k^2(k^2+m^2+\omega'^2)}\left(e^{iu_c\omega'\tau+ikx}+e^{iu_c\omega'\tau-ikx}-2\right). \tag{C.19}$$

which naturally depends on $\mathcal{K}_c^{-1}$. However, the Hamiltonian is not invariant by $\phi\to\theta$ and $\mathcal{K}\to\frac{1}{\mathcal{K}}$ as before, due to the presence of the gap, therefore we cannot assume results for θ field by changing $\mathcal{K}$ coming from ϕ results. Turning back to the previous expressions for correlators, basically we can obtain them by computing the next integrals:

$$I_{1\pm} \equiv \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{iu_c\omega'\tau}\int_0^{+\infty}\frac{dk}{2\pi}e^{\pm ikx}\frac{1}{k^2+m^2+\omega'^2} \tag{C.20}$$

$$I_{2\pm} \equiv \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{iu_c\omega'\tau}\int_0^{+\infty}\frac{dk}{2\pi}e^{\pm ikx}\frac{m^2}{(\omega'^2+m^2)(k^2+m^2+\omega'^2)} \tag{C.21}$$

$$I_{3\pm} \equiv m^2\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{e^{iu_c\omega'\tau}}{(m^2+\omega'^2)}\int_0^{+\infty}\frac{dk}{2\pi}\frac{e^{\pm ikx}}{k^2} \equiv I_{3a}I_{3b\pm} \tag{C.22}$$

with the next relations:

$$\mathcal{G}_{\phi_c\phi_c}(r) = \langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle = \mathcal{K}_c\left[I_1(r)-I_1(0)\right] \tag{C.23}$$

$$\mathcal{G}_{\theta_c\theta_c}(r) = \langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle = \frac{1}{\mathcal{K}_c}\left[I_1(r)-I_2(r)+I_3(r)-(I_1(0)-I_2(0)+I_3(0))\right] \tag{C.24}$$

where $I_n = I_{n+}+I_{n-}$. The integral $I_{1\pm}$ is computed as follows:

$$\begin{aligned} I_{1\pm} &= \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{e^{\pm ikx}e^{i\omega' u_c\tau}}{2i\sqrt{k^2+m^2}}\left(\frac{1}{\omega'-i\sqrt{k^2+m^2}}-\frac{1}{\omega'+i\sqrt{k^2+m^2}}\right)\\ &= \int_0^{+\infty}\frac{dk}{4\pi}\frac{e^{\pm ikx}e^{-u_c\tau\sqrt{k^2+m^2}}}{\sqrt{k^2+m^2}} \end{aligned} \tag{C.25}$$

where the circuit has been closed in the upper complex half plane. In the limit $m\to 0$ we want to go back to the free case result. To this end we derive the $m=0$ case with respect to $(-u_c\tau \pm i\,x)$

$$\begin{aligned} \frac{\partial}{\partial(-u_c\tau\pm i\,x)}I_{1\pm}\Big|_{m=0} &= \int_0^{+\infty}\frac{dk'}{4\pi}\,e^{-(u_c\tau\mp i\,x)k'}e^{-\alpha k'} = -\frac{1}{4\pi(\mp i\,x+u_c\tau+\alpha)}\\ \Rightarrow\quad I_{1\pm} &= -\frac{1}{4\pi}\ln\left(\mp i\,x+u_c\tau+\alpha\right) \end{aligned} \tag{C.26}$$

where the convergent factor α has been added. The correlator for ϕ_c can be completely determined by this integral by considering $\mathcal{G}_{\phi_c\phi_c} = \mathcal{K}_c(I_{1+}(x,\tau)+I_{1-}(x,\tau)-2I_1(0,0))$, obtaining the free case result Eq. (5.74). We use this procedure as a prescription to get the correct limit when $m \to 0$. In the limit of $m << 1$ the derivative $\frac{\partial}{\partial(-u_c\tau \pm i\,x)} I_{1\pm}$, dropping m^2 in the exponential, can take the form

$$\frac{\partial}{\partial(-u_c\tau \pm i\,x)} I_{1\pm} \approx \pm i \int_m^{+\infty} \frac{dz}{4\pi}\, e^{\pm i\,x\sqrt{z^2-m^2}} e^{-u_c\tau z} \tag{C.27}$$

with $\sqrt{k^2+m^2} \to z$. Expanding $\sqrt{z^2-m^2} \to z\left(1-\frac{m^2}{2z^2}\right)$ and considering again the convergent factor α we obtain

$$\begin{aligned}\frac{\partial}{\partial(-u_c\tau \pm i\,x)} I_{1\pm} &\approx \int_0^{+\infty} \frac{dz}{4\pi}\, e^{-(u_c\tau \mp i\,x)z} e^{-(\pm 2\,i\,xm^2)\frac{1}{4z}} \\ &= -\frac{1}{4\pi}\sqrt{\frac{\pm 2i\,m^2 x}{\mp i\,x + u_c\tau + \alpha}}\, K_1\left(\sqrt{\pm 2i\,m^2 x}\sqrt{\mp i\,x + u_c\tau + \alpha}\right)\end{aligned} \tag{C.28}$$

where the integral can be obtained from tables [61]. The limit $m \to 0$ is correct as the Bessel function $K_1(z) \to \frac{1}{z}$ at first order. Substituting this into the result gives:

$$\frac{\partial}{\partial(-u_c\tau \pm i\,x)} I_{1\pm} = -\frac{1}{4\pi(\mp i\,x + u_c\tau + \alpha)} \tag{C.29}$$

as it should be. We have added a spurious contribution $\zeta = (u_c\tau \mp i\,x)$:

$$\begin{aligned}&\left|\int_0^m \frac{dz}{4\pi}\, e^{-(u_c\tau \mp i\,x)z} e^{-(\pm 2\,i\,xm^2)\frac{1}{4z}}\right| < \left|\int_0^m \frac{dz}{4\pi}\, e^{-\zeta z}\, e^{\mp i\,x\frac{m}{2}}\right| \\ &= \left|e^{\mp i\,x\frac{m}{2}} \int_0^m \frac{dz}{4\pi}\, e^{-\zeta z}\right| = \frac{1}{4\pi}\left|e^{\mp i\,x\frac{m}{2}}\, \frac{(1-e^{-\zeta m})}{\zeta}\right| = \frac{1}{4\pi}\left|e^{-mu_c\frac{\tau}{2}} e^{-\frac{mu_c\tau \mp i\,x}{2}}\, \frac{\sin m\zeta/2}{\zeta}\right|.\end{aligned} \tag{C.30}$$

We disregard this term, so that we get , for small m, the non chiral result:

$$I_{1\pm} = -\int_0^{\mp i\,x + u_c\tau} \frac{dz}{4\pi}\frac{\Delta}{L}\sqrt{\frac{\pm 2i\,x}{z+\alpha}}\, K_1\left(\frac{\Delta}{L}\sqrt{\pm 2i\,x}\sqrt{z+\alpha}\right). \tag{C.31}$$

For the second pair of integrals $I_{2\pm}$ we are going to compute directly the sum I_2. Using the parametrization $\frac{1}{a^2+b^2} = \int_0^\infty e^{-s(a^2+b^2)} ds$ and the generalized Gaussian's integral, both integral gives the same result and we can sum them, obtaining:

$$I_2 = m^2 \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} e^{iu_c\omega'\tau} \frac{e^{-x\sqrt{\omega'^2+m^2}}}{(\omega'^2+m^2)^{3/2}}. \tag{C.32}$$

Obviously, in the limit $m \to 0$ the integral vanishes. Of course we are interested in the massive case and, as for integral I_1, we are going to consider the limit $m << 1$. This limit implies J small, however this is not a contradiction with the strong coupling limit due to the physical restriction $0 < J/u_0\pi < 1$, hence the strong coupling J can be comparative small with other factors. In this sense, let's approximate $e^{iu_c\omega'\tau}e^{-x\sqrt{\omega'^2+m^2}} \approx e^{-(x-iu_c\tau)\sqrt{\omega'^2+m^2}}$. Doing this, splitting the integral and changing the integration limits, we have:

$$I_2 = m^2 \int_0^\infty \frac{d\omega'}{\pi} \frac{e^{-(x-iu_c\tau)\sqrt{\omega'^2+m^2}}}{(\omega'^2+m^2)^{3/2}}. \tag{C.33}$$

Again, it is a not trivial integral but we can use a similar trick as before. Taking derivative $\frac{\partial}{\partial(-u_c\tau \pm i\,x)} I_2$ twice, we obtain:

$$m^2 \int_0^\infty \frac{d\omega'}{\pi} \frac{e^{-(x-iu_c\tau)\sqrt{\omega'^2+m^2}}}{\sqrt{\omega'^2+m^2}} = \frac{m^2}{\pi} K_0(m(x-iu_c\tau)). \tag{C.34}$$

Turning back by integration, others Bessel $K_\nu[z]$ and Struve $L_\nu[z]$ functions appears. Nevertheless, we can use the limit $m << 1$ to approximate the Bessel function before to integrate. For small arguments $|z| << 1$ the Bessel function behaves as $K_0(z) \sim -\ln(\frac{z}{4}) - \gamma$ and we finally obtain:

$$I_2 = -\frac{m^4}{4\pi}(ix+u_c\tau)^2 \left[3 - 2\ln\left(\frac{me^\gamma}{2}(x-iu_c\tau)\right)\right] \tag{C.35}$$

with the correct limit for $m \longrightarrow 0$. For the last pair of integrals $I_{3\pm}$, ω' and k integrals can be separated and we have two parts: the non-singular I_{3a} and the singular integral $I_{3b\pm}$. The first case is given by

$$\begin{aligned} I_{3a} &= m^2 \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{e^{iu_c\omega'\tau}}{(\omega'^2+m^2)} \\ &= m^2 \frac{i}{4\pi m} \left[e^{mu_c\tau} Ei(iu_c\tau\omega - u_c\tau m) - e^{-mu_c\tau} Ei(iu_c\tau\omega + u_c\tau m)\right] \Big|_{-\infty}^{+\infty} \\ &= \frac{m}{2} e^{-mu_c\tau}, \end{aligned} \tag{C.36}$$

where $Ei(z)$ is the exponential integral, with the following properties $Ei(i\infty) = i\pi$ and $Ei(-i\infty) = -i\pi$, and where just the decaying exponential part was considered. The singular part for I_{3b-} is:

$$\begin{aligned} I_{3b\pm} &\approx \int_{1/L}^{+\infty} dk \lim_{\epsilon\to 0} \frac{e^{-i\,kx}}{\epsilon^2 + k^2} \\ &= x \lim_{\epsilon\to 0} \int dy \frac{e^{-i\,y}}{\epsilon^2+y^2} = x \lim_{\epsilon\to 0} \frac{i}{2\epsilon}\left[e^{\epsilon} E_1(\epsilon + i\,y) - e^{-\epsilon} E_1(-\epsilon + i\,y)\right] \\ &= i\,x \left[\frac{dE_1(i\,y)}{dy} + E_1(i\,y)\right] = -i\,x\ \left[E_0(i\,y) - E_1(i\,y)\right] \\ &\equiv -i\,x \left[\frac{e^{-iy}}{i\,y} - E_1(i\,y)\right] \to -i\,x \left[\frac{e^{-iy}}{i\,y} - E_1(i\,y)\right]\Big|_{x/L}^{+\infty} \\ &\approx L\,e^{-i\,\frac{x}{L}} + i\,x\left[\gamma + \ln i\,\frac{x}{L}\right] \end{aligned} \tag{C.37}$$

and similar for I_{3b+}, where the infinite limit $\lim_{x\to\infty} E_1[i\,x] = 0$, while $\lim_{x\to\infty} e^{-i\,\frac{x}{L}}$ is also chosen as zero. The singular part $I_{3\pm}$ becomes:

$$m^2 \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{e^{i\,\omega\tau \pm i\,kx}}{k^2(\omega^2+m^2)} \approx \frac{mL}{4\pi} \left[e^{-(m\tau \mp i\,\frac{x}{L})} \mp i\, e^{-m\tau} \frac{x}{L} \left(\gamma + \ln \frac{\mp i\,x}{L} \right) \right]. \tag{C.38}$$

On the other side, as $\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + O\left(\frac{1}{z}\right)$, we have

$$\frac{\pm i\,x}{L}\left(\gamma + \ln\frac{\pm i\,x}{L}\right) = \ln\Gamma\left(\frac{\pm i\,x}{L}\right) + \frac{1}{2}\ln\frac{\pm i\,x}{2\pi L} + \frac{\pm i\,x}{L}(\gamma+1) \tag{C.39}$$

and the final expression for $I_{3\pm}$ is

$$I_{3\pm} \approx \frac{mL}{4\pi}\, e^{-m\tau} \left[e^{\pm i\,\frac{x}{L}} + \ln\left[\sqrt{\frac{\mp i\,x}{2\pi L}}\right] \Gamma\left(\frac{\mp i\,x}{L}\right) + \frac{\mp i\,x}{L}(\gamma+1) \right]. \tag{C.40}$$

Finally, the correlators required to be used in objects like (5.70) are:

$$\mathcal{G}_{\phi_c\phi_c}(r) = -\frac{\mathcal{K}_c}{4\pi}\left[\int_0^{i\,x+u_c\tau} \frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}\, K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right) dz + \ln\left(\alpha^2\right) + c.c. \right] \tag{C.41}$$

$$\mathcal{G}_{\theta_c\theta_c}(r) = \frac{1}{4\pi\mathcal{K}_c} \ln\left\{ e^{\frac{4\pi}{\mathcal{K}_c}\mathcal{G}_{\phi_c\phi_c}(r)} \prod_{\pm} e^{\Delta\left[e^{-\frac{\Delta u_c\tau \pm i\,x}{L}}\right]} \left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)\, e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\Delta e^{-\frac{\Delta u_c\tau}{L}}} \right\} \tag{C.42}$$

where terms proportional to $O(m^4)$ have been neglected.

C.3 Off-diagonal non-chiral conserving correlator

It can be proved that the off-diagonal correlators vanish identically in the bosonized $N = 4$ model because they correspond to non-number conserving correlators which are absent in the $N = 4$ case. To see this, let's start defining the average two point Majorana fermion cross-chiral correlators

$$G_{\cap}(r) = \frac{1}{4}\sum_i \langle \psi_+^i(r)\psi_-^i(0)\rangle \tag{C.43}$$

and

$$G_{\cup}(r) = \frac{1}{4}\sum_i \langle \psi_-^i(r)\psi_+^i(0)\rangle. \tag{C.44}$$

Recalling the initial definition of complex fermions it's possible to write

$$G_{\cap/\cup}(r) = \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow} \langle c_{\sigma\pm}(r)c^\dagger_{\sigma\mp}(0) + c^\dagger_{\sigma\pm}(r)c_{\sigma\mp}(0)\rangle.$$

Using the bosonization dictionary in terms of dual fields (5.66), the fact that pseudo-charge and pseudo-spin sectors can be completely separate, the dual fields commutator $[\phi(x_1), \theta(x_2)] = \frac{i}{2} sgn(x_2 - x_1)$ and the identity $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]}$ the correlators become:

$$
\begin{aligned}
G_{\cap/\cup}(r) \quad = \quad & \pm\frac{i}{8\pi\alpha}\Bigg(\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to \phi_s, \quad \theta_c \to \theta_s) \\
& +\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to \phi_s, \quad \theta_c \to \theta_s) \\
& +\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to -\phi_s, \quad \theta_c \to -\theta_s) \\
& +\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)} e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)} e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle \cdot (\phi_c \to -\phi_s, \quad \theta_c \to -\theta_s)\Bigg).
\end{aligned}
\tag{C.45}
$$

For the correlator to be non-zero, the sum of the factors multiplying fields in the exponentials has to vanish. In the ϕ sector this doesn't happens and off-diagonal correlators vanish. On the contrary, in the θ sector we have signs that non-zero correlators involving cross-chirality fermions can occur. We will deepen on it in the next subsection.

C.4 2d lattice approach: the Villain's approximation

In the following, we present an alternative picture of the $N = 4$ case, by considering the model as a $2d$ lattice. In principle, physic bound $0 < J/\pi u_0$ implies that we have different renormalized velocities, and therefore, different weights for kinetic part in time and space. To fix this, we look for scaling time to obtain approximated correlators for small J pseudo-charge sector and for large J pseudo-spin sector.

C.4.1 Motivation

In the XY-model, which describes two component classical spins interacting with their nearest neighbours, the partition function is proportional to $\exp\left(\beta \sum_{ij} \cos\left(S_i - S_j\right)\right)$ where at low temperature alignment of spins is favored. When temperature is increased (small β) a disordered phase appears and correlators decrease exponentially, while for low temperature (large β) and ordered phase, correlator behaves like $(r_0/|x|)^{1/\beta}$. In our case, cosine interaction is proportional to the coupling J being possible to infer that similar correlator behavior occurs for strong coupling at least in the pseudo-spin sector. In fact, if we forget the physical bound for coupling, we found that in the large J limit the pseudo-spin correlator (5.85) behaves like $(\alpha/|z|)^{\mathcal{K}_s}$ where $\mathcal{K}_s \sim i/J^{1/2}$. As alignment is favored for large J and $\phi_s = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow)$, we can assume that disorder is favored for small J and $\phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow)$. In this context, we are going to study the SYK model in the $N = 4$ case by considering the model as a $2d$ lattice and applying the Villain's approximation [64] for small J in the pseudo-charge sector and large J in the pseudo-spin sector.

C.4.2 Time scaling

Before to proceed with the approximation, we have to notice that there are different weights for the time and space derivatives in the action, depending on the strength of the interaction mediated by J. In order to perform the procedure, we are going to consider the two cases independently with the actions:

$$S_c = \int dtdx \frac{\hbar}{2}\left[\frac{1}{u_0}(\partial_t\phi_c)^2 - u_0 A_c^2(\partial_x\phi_c)^2\right] \tag{C.46}$$

and

$$S_s = \int dtdx \frac{\hbar}{2}\left[\frac{1}{u_0}(\partial_t\phi_s)^2 - u_0 A_s^2(\partial_x\phi_s)^2 - \frac{J}{\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_s\right)\right], \tag{C.47}$$

where for the c-sector was neglected the cosine interaction as it was done for the free-like case. In the following, we are going to focus in the s-sector and then we will infer the pseudo-charge case. In the large J limit, applying the Villanin's approximation, the action becomes

$$S_s = \int dtdx \frac{\hbar}{2}\left[\frac{1}{u_0}(\partial_t\phi_s)^2 + \frac{J}{\pi}(\partial_x\phi_s)^2 + \frac{4J}{\pi\alpha^2}\phi_s^2\right] \tag{C.48}$$

It's possible to re-scale time $t \to \sqrt{\frac{u_0 J}{\pi}}t$ which now behaves as a length and move to imaginary time to obtain the re-scaled action for pseudo-spin sector

$$iS_s = \int d\tau dx \frac{\hbar}{2}\sqrt{\frac{J}{\pi u_0}}\left[-(\partial_\tau\phi_s)^2 + (\partial_x\phi_s)^2 + \frac{4}{\pi\alpha^2}\phi_s^2\right]. \tag{C.49}$$

In the s-sector the scaling factor $\sqrt{\frac{J}{\pi u_0}}$ is equal to $u_s = \frac{u_0}{K_s}$ when large J limit is considered. Therefore, it will be easy to track the scaling for pseudo-charge sector by considering the scaled time as $u_c t$ and the spatial derivative $(\partial_x\phi_c)^2$ with a minus sign in front.

C.4.3 Lattice approach

The idea is obtain the inverse of the propagator by calculate the kernel of the quadratic form

$$\mathcal{Z}[0] = \int \mathcal{D}\phi\mathcal{D}\phi^* exp\left(-\frac{1}{2}\phi^* G^{-1}\phi\right) \tag{C.50}$$

and then follow [64] by considering the model like a $2d$ space-time lattice, where time is now a length. After integrate by parts, and moving the model to a lattice, it's seen that

$$\begin{aligned}\frac{iS_c}{\hbar} &= \int d\tau dx \frac{1}{2}\sqrt{\frac{J}{\pi u_0}}\left[\phi_s^*(\partial_\tau^2 - \partial_x^2)\phi_s + \frac{4}{\alpha^2}\phi_s^2\right] \\ &= \sum_r \frac{u\hbar\beta\alpha}{N}\frac{1}{2}\sqrt{\frac{J}{\pi u_0}}\phi_{s(r)}^*\left[\partial_\tau^2 - \partial_x^2 + \frac{4}{\alpha^2}\right]\phi_{s(r)}.\end{aligned} \tag{C.51}$$

where $\frac{u\hbar\beta\alpha}{N}$ is the lattice area. On the other side, we have that

$$\begin{aligned}
\left[\partial_\tau^2+\partial_x^2\right]\phi_{c(i.j)} &= \frac{1}{\alpha^2}\left[\left(\phi_{c(i+2.j)}-\phi_{c(i+1.j)}\right)-\left(\phi_{c(i+1.j)}-\phi_{c(i.j)}\right)\right] \\
&\quad+\frac{1}{\alpha^2}\left[\left(\phi_{c(i.j+2)}-\phi_{c(i.j+1)}\right)-\left(\phi_{c(i.j+1)}-\phi_{c(i.j)}\right)\right] \\
&= \frac{1}{\alpha^2}\int\frac{d^2q}{(2\pi)^2}e^{iqr}\left[2(\cos q_1+\cos q_2)-4\right]\phi_{c(q)}
\end{aligned} \tag{C.52}$$

and

$$\begin{aligned}
\left[\partial_\tau^2-\partial_x^2\right]\phi_{s(i.j)} &= \frac{1}{\alpha^2}\left[\left(\phi_{s(i+2.j)}-\phi_{s(i+1.j)}\right)-\left(\phi_{s(i+1.j)}-\phi_{s(i.j)}\right)\right] \\
&\quad-\frac{1}{\alpha^2}\left[\left(\phi_{s(i.j+2)}-\phi_{s(i.j+1)}\right)-\left(\phi_{s(i.j+1)}-\phi_{s(i.j)}\right)\right] \\
&= \frac{1}{\alpha^2}\int\frac{d^2q}{(2\pi)^2}e^{iqr}\left[2(\cos q_1-\cos q_2)\right]\phi_{s(q)}.
\end{aligned} \tag{C.53}$$

Defining $\gamma_s=\frac{u\hbar\beta}{N\alpha}\sqrt{\frac{J}{\pi u_0}}$ and $\gamma_c=\frac{u\hbar\beta}{N\alpha}\frac{1}{\mathcal{K}_c}$ we have that the kernel in the lattice approach is proportional to $\gamma\left[4-2(\cos q_1+\cos q_2)\right]$ and the correlator is found to be

$$-2\pi\mathcal{G}_{\phi_\rho\phi_\rho}\equiv\Gamma_\rho(r)=\frac{1}{\gamma_\rho}\int\frac{d^2q}{2\pi}\frac{1-e^{iq_1r}}{4-2(\cos q_1\pm\cos q_2)} \tag{C.54}$$

where the plus sign is for $\rho=c$ and the minus sign is for $\rho=s$. Here we can already observe some signals of what we were discussing at the beginning of the section: large J s-sector behaves similar to small J c-sector, in concordance with what we observed for strong coupling pseudo-spin and free-like pseudo-charge case in previous sections. Of course, the power law parameter, in this case γ_ρ, changes.

Pseudo-charge sector

In the pseudo-charge case, it's observed the combination $4-2(\cos q_1+\cos q_2)$ in the denominator. In the small momentum limit, this becomes $q_1^2+q_2^2$, which is the modulus square of the two-dimensional momentum (q_1,q_2). Therefore, in the limit of small momentum, the denominator is rotational invariant. This implies that the integral has an isotropic behaviour at large distances (corresponding to small momenta). Since the integral has an isotropic behaviour at large distances, i.e., it only depends on the modulus of (x_1,x_2), it is possible to select an arbitrary direction, let's say x direction, perform the integral and then in final result replace x_1 with the modulus of (x_1,x_2). In this way the correct result for the integral at large distances is obtained. In other words, if you are looking for the behaviour of the integral at large distances, you can replace the cosines in the denominators by their expansion to lowest order in q, because at large distance what matters is the behaviour of the integrand a small momenta.

With previous considerations, let's calculated the integral for $x = (2p, 0)$ for convenience reasons:

$$\begin{aligned}
\Gamma_\rho(2p,0) &= \frac{1}{\gamma_\rho}\int \frac{d^2q}{2\pi}\frac{1-e^{2iq_1p}}{4-2(\cos q_1 \pm \cos q_2)} \\
&= \frac{1}{\gamma_\rho}\int \frac{dq_1}{2\pi}(1-\cos 2pq_1 - i\sin 2pq_1)\int dq_2\frac{1}{4-2(\cos q_1 \pm \cos q_2)} \\
&= \frac{1}{\gamma_\rho}\int \frac{dq_1}{2\pi}(1-\cos 2pq_1 - i\sin 2pq_1)\frac{\pi}{3-\cos q_1}\sqrt{\frac{3-\cos q_1}{1-\cos q_1}} \\
&= \frac{1}{\gamma_\rho}\int \frac{dq_1}{2}\frac{1-\cos 2pq_1 - i\sin 2pq_1}{\sqrt{(1-\cos q_1)(3-\cos q_1)}}
\end{aligned} \tag{C.55}$$

In the pseudo-charge case, we have to consider the plus sign in the integration on q_2. However, it is interesting to see that the integration on q_2 gives the same result also for minus sign (pseudo-spin sector). Making a change of variable $q_1 = 2q$, and using trigonometric identities, we can reach

$$\Gamma_c(2p,0) = \frac{1}{\gamma_c}\int_0^{\pi/2} dq\frac{2\sin^2 2pq}{\sin q\sqrt{1+\sin^2 q}} \tag{C.56}$$

where the odd integral coming from the imaginary part has been obtained equal to zero. Now, we can split the integral by define

$$\Gamma_0 = 2\int_0^{\pi/2} dq\frac{\sin^2 2pq}{\sin q} \tag{C.57}$$

and

$$\Gamma_1 = 2\int_0^{\pi/2} dq\frac{\sin^2 2pq}{\sin q}\left(\frac{1}{\sqrt{1+\sin^2 q}} - 1\right). \tag{C.58}$$

For the first one, expanding the sine function:

$$\Gamma_0 = 2\sum_{n=0}^{2p-1}\frac{1}{2n+1} = -\psi^{(0)}(\frac{1}{2}) + \psi^{(0)}(\frac{1}{2}+2p) \tag{C.59}$$

where $\psi^{(0)}(z)$ is the polygamma function of order zero. Expanding around $p = \infty$ we have

$$\Gamma_0 \approx \ln 2p + 2\ln 2 + \gamma \tag{C.60}$$

where γ is the Euler's constant. On the other hand, in that limit of p, the $\sin^2 2pq$ can be replaced by its average

$$\begin{aligned}\Gamma_1 &\approx 2\int_0^{\pi/2} dq \frac{1}{2\sin q}\left(\frac{1}{\sqrt{1+\sin^2 q}}-1\right)\\ &\approx \int_0^{\pi/2} dq \frac{1}{\sin q}\sum_{n=1}^{\infty}\frac{\sin^{2n} q}{n!}\frac{\Gamma(1/2)}{\Gamma(1/2-n)}\\ &\approx -\frac{1}{2}\ln 2.\end{aligned} \tag{C.61}$$

With all of these, it's obtained that

$$\begin{aligned}\Gamma_c(2p,0) &= \frac{1}{\gamma_c}(\Gamma_0+\Gamma_1)\\ &\approx \frac{1}{\gamma_c}(\ln 2p + \ln 2\sqrt{2}e^{\gamma})\\ &\approx \frac{1}{\gamma_c}\ln\frac{2p}{r_0}\end{aligned} \tag{C.62}$$

where was defined $r_0 = 2\sqrt{2}e^{\gamma}$. Finally, we obtain the $G_{\phi_c\phi_c}(r)$ correlator in the lattice approach by define $2p \longrightarrow r$ and consider $\mathcal{G}_{\phi_c\phi_c}(r) = -\frac{1}{2\pi}\Gamma_c(r)$ to be:

$$\begin{aligned}G_{\phi_c\phi_c}(r) &= e^{\frac{\pi}{2}\mathcal{G}_{\phi_c\phi_c}(r)} \approx e^{-\frac{1}{4\gamma_c}\ln\frac{|r|}{r_0}}\\ &\approx \left(\frac{r_0}{|r|}\right)^{\frac{1}{4\gamma_c}}.\end{aligned} \tag{C.63}$$

The correlator is the same as the one computed from field theory. Of course we should recover the scaling time by changing $t \longrightarrow u_c t$ in $|r| = \sqrt{x^2+t^2}$. Furthermore, in this case the cutoff α is replaced by the lattice spacing r_0 and the non-universal power law parameter $\mathcal{K}_c$ is changed for $\frac{1}{\gamma_c}$. However, we should remember that $\gamma_c = \frac{u\hbar\beta}{N\alpha}\frac{1}{\mathcal{K}_c}$, which gives us a relation between physical parameters and implying that the ratio $\frac{u\hbar\beta}{N\alpha}$ should be set equal one.

Pseudo-spin sector

In the case of pseudo-spin sector, the denominator of Γ_s is $4-2(\cos q_1 - \cos q_2)$. Now, looking for the behaviour of the integral at large distances, and expanding around $(q_1,q_2) = (0,0)$, you don't find the modulus square of the momentum. As a consequence, the integral is not isotropic, and it is not possible to compute it as in the case for the charge. To go around this problem, one possibility is expand the denominator around $(q_1,q_2) = (0,\pi)$ or $(\pi,0)$. If this is done, it is necessary to keep track of the fact that now the expansion is not around zero momentum. In fact, the meaning of this is that we are looking at the staggered component of the Green function, i.e., at the part that oscillates on the scale of the lattice constant.

With previous considerations, let's calculated the integral for shifted momentum $q_2 \longrightarrow q + \pi$ as follow:

$$\begin{aligned}
\Gamma_s(x_1, x_2) &= \frac{1}{\gamma_s} \int \frac{d^2q}{2\pi} \frac{1 - e^{i(q_1 x_1 + (q_2+\pi)x_2)}}{4 - 2(\cos q_1 + \cos q_2)} \\
&= \frac{1}{\gamma_s} \int \frac{d^2q}{2\pi} \frac{1 - e^{i(q_1 x_1 + q_2 x_2)} e^{i\pi x_2}}{4 - 2(\cos q_1 + \cos q_2)} \\
\longrightarrow \Gamma_s(2p_1, 2p_2) &= \frac{1}{\gamma_s} \int \frac{d^2q}{2\pi} \frac{1 - e^{i[q_1(2p_1) + q_2(2p_2)]} e^{i\pi(2p_2)}}{4 - 2(\cos q_1 + \cos q_2)}, \qquad \text{(C.64)}
\end{aligned}$$

where we are considering half integer coordinates. Now, in the small momenta (large distance) limit, we have again the modulus in the denominator. If we consider an arbitrary direction $(2p_1, 0)$, we obtain exactly the same limit as the charge case, and we can proceed as before. On the other side, if we consider the arbitrary direction $(0, 2p_2)$, the half integer exponent ensure that, in the large distances limit, the exponential goes to exactly the same as if you consider the other $(2p_1, 0)$ direction. Taking the first case for simplicity, and calling $2p_1 \longrightarrow 2p$ we have

$$\begin{aligned}
\Gamma_s(2p, 0) &= \frac{1}{\gamma_s} \int \frac{d^2q}{2\pi} \frac{1 - e^{i(2pq_1)}}{4 - 2(\cos q_1 + \cos q_2)} \\
&= \frac{1}{\gamma_s} \int \frac{dq_1}{2\pi} (1 - \cos 2pq_1 - i \sin 2pq_1) \int dq_2 \frac{1}{4 - 2(\cos q_1 + \cos q_2)} \\
&= \frac{1}{\gamma_s} \int \frac{dq_1}{2\pi} (1 - \cos 2pq_1 - i \sin 2pq_1) \frac{\pi}{3 - \cos q_1} \sqrt{\frac{3 - \cos q_1}{1 - \cos q_1}} \\
&= \frac{1}{\gamma_s} \int \frac{dq_1}{2} \frac{1 - \cos 2pq_1 - i \sin 2pq_1}{\sqrt{(1 - \cos q_1)(3 - \cos q_1)}}. \qquad \text{(C.65)}
\end{aligned}$$

Performing the same calculations as the charge case, we reach

$$G_{\phi_s \phi_s}(r) \approx \left(\frac{r_0}{|r|} \right)^{\frac{1}{4\gamma_s}}, \qquad \text{(C.66)}$$

where now $\gamma_s = \frac{u\hbar\beta}{N\alpha} \sqrt{\frac{J}{\pi u_0}}$ in the large J context. The latter implies that $\mathcal{K}_s \longrightarrow \sqrt{\frac{\pi u_0}{J}}$, i.e. correlator behaves like $(r_0/|r|)^{\frac{\mathcal{K}_s}{4}}$ as it was expected.

Finally, if we want to extend the previous results to $G_{\theta_\rho \theta_\rho}(r)$ we just have to exchange $\mathcal{K}_\rho \longrightarrow \frac{1}{\mathcal{K}_\rho}$.

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