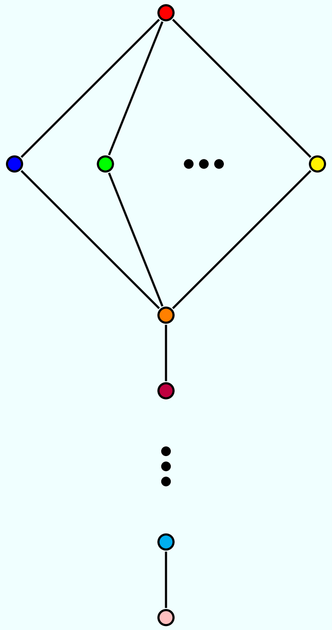
— Advanced Topics in Pure and Applied Mathematics: Doctoral Lecture Series —

Marco Trombetti

Non-Abelian Groups with Many Abelian Subgroups



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and
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INTRODUCTION

The present volume reproduces to a great extent the lectures from a Ph.D. class that I taught in the early 2024 at the Dipartimento di Matematica e Applicazioni "Renato Caccioppoli" of the Università degli Studi di Napoli Federico II. In order to make the reading more accessible also to those who have only a basic knowledge of group theory, the main concepts and definitions have been collected in Chapter 1. These notions are necessary to understand the subsequent chapters.

Clearly, abelian groups are obvious examples of groups with many abelian subgroups (all their subgroups are in fact abelian). But this volume does not deal with the structure of abelian groups at all, and actually some well-known structural theorems are given for granted. They main focus is really on *non-abelian* groups having many abelian subgroups in some sense. Thus, for example the structure of (soluble) non-abelian groups with only abelian proper subgroups (that is, minimal non-abelian groups) is described in Chapter 2, while Chapter 3 deals with some of the basic features of groups whose subgroups are either normal or abelian (that is, metahamiltonian groups). We do not go very deep in describing the structure of metahamiltonian groups because of the very complete survey [3]. In the final Chapter 4, we focus on the dual situation of a non-abelian group with many abelian quotients. This final chapter had not be taught in the class at the time, because of the sudden death of my mentor Francesco de Giovanni. This unpredictable loss shocked everybody, so I decided to not deliver the last two lectures of the class that would have cover the subject of Chapter 4. This is also why I'm especially glad to have be given the opportunity to publish these notes. Thus, my thanks go to Dr. Giuseppe Arnone and Dr. Giacomo Ascione for having conceived (and realized) the wonderful idea of a series of volumes on the Ph.D. classes that have been taught at the Dipartimento di Matematica e Applicazioni "Renato Caccioppoli" of the Università degli Studi di Napoli Federico II. Finally, I wish to thank the anonymous referee for some interesting comments that have improved the exposition of this volume.

Napoli, 17/08/2024 *Marco Trombetti*

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PRELIMINARIES

The aim of this chapter is to fix terminology and give the reader the necessary background to understand the subsequent chapters. Most of the mentioned notions and results can be found (with more details) in any group theory textbook, and for the sake of completeness we refer the reader to the following ones: [2], [17], [18].

First of all, recall that $\mathbb{N}$ (resp., $\mathbb{N}_0$) denotes the set of all positive (resp., non-negative) integers, while $\mathbb{Z}$ is the set of all integers and $\mathbb{Q}$ is the set of all rational numbers — we have tried to avoid the use of the word "number" because we think it's somewhat misleading. Naturally, $(\mathbb{Z}, +, \cdot)$ is a ring and $(\mathbb{Q}, +, \cdot)$ is a field with the usual operations. In dealing with subgroups of the additive or multiplicative part of a ring, we have always tried to explicitly write the operation of the parent structure, while the operation in case of substructures is often neglected, unless there is some ambiguity.

The trivial (sub)group is always denoted by $\{1\}$ or $\{0\}$ accordingly with the notation (multiplicative or additive) but regardless of the nature of the objects involved — correspondingly, the identity of the group is always denoted either by 1 or 0. Thus, for example, the trivial subgroup of the additive group $(\mathbb{Q}, +)$ of the rational numbers is $\{0\}$, while the identity of an arbitrary (multiplicative) group $(G, \cdot)$ is denoted by 1. Note that if not otherwise stated, the multiplicative notation must always be used — this is the case for instance of the group classes we discuss in Section 1.1, and of every abstractly given abelian group.

Let G be a group, and let H be a subset of G. We write $H \leq G$ (resp., $H \trianglelefteq G$) to denote the fact that H is a subgroup (resp., a normal subgroup) of G. If we wish to emphasize that H is a *proper* subgroup (resp., a *proper* normal subgroup) of G, then we may write $H < G$ (resp., $H \triangleleft G$). Recall also that H is *characteristic* in G if it is invariant with respect to all automorphisms of G, i.e., if $H^\alpha = H$ for every α in the automorphism group $\mathrm{Aut}(G)$ of G. If H is a characteristic subgroup of G, then $H \trianglelefteq G$ because H is invariant under all the *inner automorphisms* of G, that is, under the automorphisms $\overline{g}$ of G induced by elements $g \in G$ by conjugation — here, if g and h are elements of G, then $h^g = g^{-1}hg$ is the *conjugate* of h by g, and the automorphism $\overline{g}$ of G

maps any element h of G to $\overline{g}(h) = g^h$. Note that if $g, h \in G$, then we usually write g^{-h} instead of $(g^{-1})^h = (g^h)^{-1}$. For similar reasons, if H is a characteristic subgroup of a normal subgroup of a group G, then H is normal in G.

If H is a subgroup of G, then $|G : H|$ denotes the *index* of H in G, that is, the cardinality of the set of all left cosets of H in G, that is, the cardinality of the set of all subsets of G of the form $gH = \{gh : h \in H\}$ for some $g \in G$. It is well-known that the set of lefts cosets of H in G is equipotent to the set of all right cosets of H in G (that is, to the set of all subsets of G of the form $Hg = \{hg : h \in H\}$), so the index of H in G can be also defined as the cardinality of the set of all right cosets of H in G. Note that $H \trianglelefteq G$ if and only if the set of all left cosets of H in G coincides with the set of all right cosets of H in G, that is, if and only if $gH = Hg$ for all $g \in G$. If $|G : H|$ is finite, then we say that H has *finite index* in G. The relevance of finite-index subgroups in infinite (but also finite) groups, comes from the fact that they always contain a finite-index normal subgroup (if the index is n, then this normal subgroup has index at most $n!$), and this allows us to study the corresponding finite quotient. Recall also that a subgroup of index 2 is necessarily normal.

The subgroup *generated* by H is denoted by $\langle H \rangle$ and it is clearly the smallest subgroup containing H with respect to the inclusion. If H is finite and $G = \langle H \rangle$, then G is *finitely generated* and H is a set of *generators* of G. If G has an unspecified set of generators of cardinality n, then we also say that G is an *n-generator* group. If $H_1, \ldots, H_s$ are subsets of G and $h_1, \ldots, h_t$ are elements of G, then we also write $\langle H_1, \ldots, H_s, h_1, \ldots, h_t \rangle$ in place of $\langle H_1 \cup \ldots \cup H_s \cup \{h_1\} \cup \ldots \cup \{h_t\} \rangle$. Finitely generated groups usually have a key role in understanding the structure of arbitrary groups, because they allow us to study them somewhat locally. The structure of an abelian finitely generated group A is well-known and it basically boils down to the fact that A is a direct product of finitely many cyclic (finite or infinite) groups; we should also recall that a finite-index subgroup of a finitely generated group is finitely generated itself, and that non-trivial finitely generated groups have maximal subgroups. Note that every time we have a proper normal subgroup N of a group G, we also have a set of generators of G because $G = \langle G \setminus N \rangle$; also, if E is a set of generators for G, then EN/N is a set of generators for G/N.

If K is a further subset of G, then H^K denotes the subgroup generated by all the conjugates of elements of H by elements of K. As usual, if

$H = \{h\}$, then we write h^K instead of H^K, while if $K = \{k\}$, then we write H^k instead of H^K, and we call H^k the *conjugate* of H by k — if U is *a* conjugate of H, then there exists $k \in G$ such that $U = H^k$. Note that $H^k \leq G$ if and only if $H \leq G$, and in this case we also have that $H^k = \{h^k : h \in H\}$. Clearly, a subgroup H of G is normal in G if and only if $H^G = H$ (that is, if and only if H contains all its conjugates by elements of G), while K *normalizes* H if $H^K = H$ — this latter condition is equivalent to requiring that K is contained in the *normalizer* $N_G(H)$ of H in G, that is, in the set of all elements g of G *normalizing* H (i.e., such that $H^g = H$). Note that if K is a subgroup of G, then H^K is normal in $\langle H, K\rangle$. If H is a subgroup of G, then the index of $N_G(H)$ in G is also the cardinality of the set of all conjugates of H in G. The set of all elements of G centralizing every element of H is denoted as $C_G(H)$ and is referred to as the *centralizer* of H in G. If $H = \{h\}$, then we also write $C_G(h)$ in place of H — note that the index of $C_G(h)$ in G is the cardinality of the set of all conjugates of h by elements of G, and that $C_G(h) = N_G(h)$. We also mention that in general $C_G(H) \leq N_G(H)$, and that if N is a normal subgroup of G, then $C_G(N) \trianglelefteq G$ and $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, so in particular $G/C_G(N)$ is finite when N is finite. If H is a subgroup, then H^G is known as the *normal closure* of H in G and it is in fact the smallest normal subgroup of G containing H. Similarly, one can define the *normal core* H_G of H in G as the largest normal subgroup of G contained in H (so if $|G : H|$ is finite, then G/H_G is finite as well).

If H is a subgroup of G and n is a positive integer, then H^n denotes the subgroup generated by all elements h^n with h ranging in H. If H is abelian, then H^n is actually the *set* of all elements of the form h^n with $h \in H$; also in this case $(H^n)^m = H^{nm} = (H^m)^n$ for every other positive integer m. Obviously, G^n is a characteristic subgroup of G.

The *centre* $Z(G)$ of G is the set of all elements $g \in G$ such that $xg = gx$ for all $x \in G$. Thus, a subset H of G is contained in $Z(G)$ if and only if $C_G(H) = G$, while the quotient $G/Z(G)$ is isomorphic to the group $\mathrm{Inn}(G)$ of all inner automorphisms of G. The *derived subgroup* G' of G is the subgroup generated by the *commutators* of elements of G, that is,

$$G' = \langle [x, y] = x^{-1}y^{-1}xy = x^{-1}x^y = y^{-x}y \, : \, x, y \in G\rangle.$$

It is obvious that G/G' is abelian (so any subgroup containing G' is normal in G), and actually G' is the smallest normal subgroup N of G such that G/N is abelian. Furthermore, the cardinality of G' gives an obvious bound for the number of conjugates of an arbitrary element of G; in particular, if G' is finite, then every element of G has finitely many conjugates. Now, if X and Y are subsets of G, then we put

$$[X, Y] = \langle [x, y] \,:\, x \in X,\, y \in Y \rangle,$$

so in particular $G' = [G, G]$. As usual, if $X = \{x\}$ (resp., $Y = \{y\}$), then we also write $[x, Y]$ (resp., $[X, y]$) instead of $[X, Y]$. Clearly, G is abelian if and only if $G = Z(G)$ if and only if $G/Z(G)$ is cyclic if and only if $G' = \{1\}$. The *derived series* of G is the series $\{G^{(n)}\}_{n \in \mathbb{N}}$ of subgroups recursively defined as follows:

$$G^{(0)} = G \quad \text{and} \quad G^{(n+1)} = \big(G^{(n)}\big)'.$$

Then G is *soluble* if $G^{(n)} = \{1\}$ for some non-negative integer n — in this case, the smallest n such that $G^{(n)} = \{1\}$ is called the *derived length* of G; recall that if $G^{(2)} = \{1\}$, then G is said to be *metabelian*, so metabelian groups are precisely the groups whose derived subgroup is abelian. Similarly, the *upper central series* $\{Z_n(G)\}_{n \in \mathbb{N}}$ of G is recursively defined as follows:

$$Z_0(G) = \{1\} \quad \text{and} \quad Z(G/Z_n(G)) = Z_{n+1}(G)/Z_n(G).$$

Then G is *nilpotent* if $G = Z_n(G)$ for some non-negative integer n — in this case, the smallest n such that $G = Z_n(G)$ is the *nilpotency class* of G. The *lower central series* $\{\gamma_n(G)\}_{n \in \mathbb{N}}$ is recursively defined as follows:

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{n+1}(G) = [\gamma_n(G), G].$$

It is easy to see that G is nilpotent of class n if and only if $\gamma_{n+1}(G) = \{1\}$ and n is the smallest positive integer m such that $\gamma_{m+1}(G) = \{1\}$. Clearly, every nilpotent group of class c, is soluble of derived length at most c. Note that $G' = \gamma_2(G) = G^{(1)}$, that every $\gamma_i(G)$, $G^{(i)}$ and $Z_i(G)$ is a characteristic subgroup of G, and that

$$\gamma_i(G/N) = \gamma_i(G)N/N \quad \text{and} \quad G^{(i)}N/N = (G/N)^{(i)}$$

for every normal subgroup N of G — we usually write G'' in place of $G^{(2)}$.

In the context of soluble and nilpotent groups, the basic properties and identities about commutators usually play a central role. We are now going to state some of them. Let $g, x, y, z \in G$. Obviously, $[x, y]^g = [x^g, y^g]$. Moreover,

$$[x, yz] = [x, z] \cdot [x, y]^z \quad \text{and} \quad [xy, z] = [x, z]^y \cdot [y, z].$$

Thus, if A is any abelian normal subgroup of G, then $[A, g]$ (resp., $[g, A]$) actually coincides with the set of all commutators of the form $[a, g]$ (resp., $[g, a]$) for $a \in A$. Also, if $[x, y]$ is centralized by x (resp., by y), then $[x^n, y] = [x, y]^n$ (resp., $[x, y^n] = [x, y]^n$) for every integer n. We employ this latter fact to prove an useful formula for computing powers of products.

Lemma 1.1. *Let x, y be elements of a group G with $[[x, y], x] = [[x, y], y] = 1$. Then*

$$(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$$

for every integer $n \geq 2$.

Proof. By induction on n. If $n = 2$, then this is clear. Suppose the statement is true for n. Then

$$\begin{aligned}(xy)^{n+1} = (xy)(xy)^n = xyx^n y^n [y, x]^{\binom{n}{2}} &= x^{n+1} y [y, x^n] y^n [y, x]^{\binom{n}{2}} \\ = x^{n+1} y^{n+1} [y, x]^n [y, x]^{\binom{n}{2}} &= x^{n+1} y^{n+1} [y, x]^{\binom{n+1}{2}}\end{aligned}$$

because $[y, x]$ is centralized by x and y. □

Since $[x, y] = [y, x]^{-1}$, so $[X, Y] = [Y, X]$ whenever X and Y are subsets of G — note that

$$[x^{-1}, y] = \left([x, y]^{-1}\right)^{x^{-1}}.$$

If $x_1, \ldots, x_n, x_{n+1}$ are elements of G, then we define recursively

$$[x_1] = x_1 \quad \text{and} \quad [x_1, \ldots, x_{n+1}] = \big[[x_1, \ldots, x_n], x_{n+1}\big].$$

Similarly, if $X_1, \ldots, X_n, X_{n+1}$ are subsets of G, then we define recursively

$$[X_1] = \langle X_1 \rangle \quad \text{and} \quad [X_1, \ldots, X_{n+1}] = \big[[X_1, \ldots, X_n], X_{n+1}\big].$$

If X and Y are non-empty subsets of G, then $X^Y = \langle X, [X, Y] \rangle$ provided that $1 \in Y$, while $[X, Y]^Y = [X, Y]$ provided that $Y \leq G$. The *Hall–Witt identity* states that

$$[x, y^{-1}, z]^y \cdot [y, z^{-1}, x]^z \cdot [z, x^{-1}, y]^x = 1$$

and implies the *Three Subgroup Lemma*: if H, K, L are subgroups of G such that any two of the subgroups $[H, K, L]$, $[K, L, H]$ and $[L, H, K]$ are contained in a normal subgroup of G, then so is the third. Actually, for metabelian groups, there is a nice version of the Hall–Witt identity.

Lemma 1.2. *Let G be a metabelian group. Then $[x,y,z][y,z,x][z,x,y] = 1$ for all $x,y,z \in G$.*

Proof. First note that

$$\begin{aligned}
[x,y][x,y,z] &= [x,y]^z = x^{-z}y^{-z}x^z y^z \\
&= \big(x[x,z]\big)^{-1}\big(y[y,z]\big)^{-1}x[x,z]y[y,z] \\
&= [z,x]\cdot x^{-1}[z,y]\cdot y^{-1}x[x,z]y[y,z] \\
&= [z,x][z,y][z,y,x]\cdot x^{-1}y^{-1}x\cdot [x,z]y[y,z] \\
&= [z,x]\cdot [z,y][z,y,x][x,y]\cdot y^{-1}[x,z]y[y,z] \\
&= [z,y][z,y,x][x,y]\cdot [z,x]y^{-1}[x,z]y\cdot [y,z] \\
&= [z,y][z,y,x][x,y][x,z,y][y,z] = [x,y][z,y,x][x,z,y],
\end{aligned}$$

so

$$[z,y,x][x,z,y][x,y,z]^{-1} = 1.$$

On the other hand, $[z,y,x]^{-1} = [y,z,x]$ and $[x,z,y]^{-1} = [z,x,y]$, so

$$[x,y,z]^{-1}[y,z,x]^{-1}[z,x,y]^{-1} = 1$$

and hence $[x,y,z][y,z,x][z,x,y] = 1$. □

Let π be a non-empty set of prime numbers, and put $\pi' = \mathbb{P} \setminus \pi$, where $\mathbb{P}$ is the set of all primes numbers. The order of an element g of a group G is denoted by $o(g)$. Also, we say that a group G is a *π-group* if every element g of G is a *π-element*, that is, $o(g)$ is finite and the only primes dividing $o(g)$ belong to π (in other words, $o(g)$ is a *π-number*). If $\pi = \{p\}$, then we usually write p instead of π, and p' instead of π', so for example we use the term *p-group* (resp., *p-element*) instead of π-group (resp., π-element), and *p'-group* (resp., *p'-element*) instead of π'-group (resp., π'-element). If G is finite, then being a π-group is equivalent to saying that the order of G is a π-number. Furthermore, most of the times we are going to talk of a p-group without explicitly saying that p is a prime. Recall now that a group G is *periodic* if all its elements are *periodic* (that is, they have finite order), while G has *finite exponent* $\exp(G) = n$ if $g^n = 1$ for every $g \in G$ and n is the least positive integer for which such a property holds — more in general we say that G has *finite exponent* if there exists an n such that G has finite exponent n. Thus, every group of finite exponent is a π-group for some $\pi \subseteq \mathbb{P}$, and every π-group is periodic. Also, $\exp(G)$ divides n

if and only if $G^n = \{1\}$, and $\exp(G) = 2$ implies that G is abelian. A group (or an element) that is not periodic is also termed *aperiodic*, while a group with no non-trivial periodic elements is said to be *torsion-free*. Moreover, a group has *infinite exponent* if it has not finite exponent. If G is any group, then the set of all primes p for which G has elements of order p is denoted by $\pi(G)$. Clearly, if H and K are periodic subgroups of a group G and $\pi(H) \cap \pi(K) = \emptyset$, then $H \cap K = \{1\}$. A couple of observations that are relevant for us are the following ones. If n is a positive integer and G is a π-group for some set π of primes that do not divide n, then every element of G has a unique n^{th}*-root*, that is, for every $g \in G$, there is some element $h \in G$, which we denote by $g^{1/n}$ (and by $\frac{1}{n}g$ in additively written groups), such that $h^n = g$. If G is a nilpotent group, then the set of all periodic elements of G form a subgroup, which is usually referred to as the *periodic part* (or, the *periodic radical*) of G, and it is clearly a characteristic subgroup of G; thus, every subgroup of a nilpotent group that is generated by periodic elements is finite. Finally, we need to recall some notation and arithmetic facts. If n and m are non-zero integers, then (n, m) denotes the greatest common divisor of n and m — Bézout's Lemma states that $(n, m) = un + vm$ for some integers u, v. If $n, m \in \mathbb{Z}$ and $\ell \in \mathbb{N}_0$, then $n \equiv_\ell m$ denotes that n and m are *equivalent modulo* ℓ, that is, $n = m + k\ell$ for some integer k. A multiplicative inverse of n modulo ℓ exists if and only if $(n, \ell) = 1$, that is, if and only if n and ℓ are coprime.

If G is a group and π is a set of primes, then a *Sylow π-subgroup* of G is a maximal element (with respect to the inclusion) of the set of its π-subgroups — if we do not wish to specify the set π, or this is clear from the context, then we speak of a *Sylow subgroup*. As usual, if $\pi = \{p\}$, then we write Sylow p-subgroup instead of Sylow π-subgroup, and Sylow p'-subgroup instead of Sylow π'-subgroup. Sylow's theorems show that in case G is finite and p is a prime, two Sylow p-subgroups are conjugate and the order of a Sylow p-subgroup is the maximum power of p dividing $|G|$ — if G is soluble and finite, then similar results also hold for an arbitrary set π of primes. It follows that a Sylow p-subgroup is unique if and only if it is normal, and in this case it is clearly a characteristic subgroup. Thus, if A is a normal abelian subgroup of the finite group G, then all its Sylow p-subgroups are characteristic in A and so normal in G. Note that every Sylow π-subgroup of a group G contains every *normal* π-subgroup, so if G' is a π-subgroup, then G has a unique Sylow π-subgroup and such a Sylow π-subgroup contains G'. We should also remark that Sylow subgroups of a finite group behave well with respect to normal subgroups and quotients. In fact, if N is any normal subgroup of a finite group G, then PN/N and $P \cap N$ are Sylow p-subgroups of G/N and N, respectively, whenever P is a Sylow p-subgroup of G.

Let p be a prime. If G is a finite p-group and n is a positive integer, then $\Omega_n(G)$ is the subgroup generated by the elements g of G such that $g^{p^n} = 1$. Of course, the subgroups

$$\Omega_1(G) \leq \Omega_2(G) \leq \ldots \leq \Omega_n(G) \leq \Omega_{n+1}(G) \leq \ldots \bigcup_{i \in \mathbb{N}} \Omega_i(G) = G$$

are characteristic in G, and if G is abelian, then every $\Omega_n(G)$ is actually the set of all elements $g \in G$ such that $g^{p^n} = 1$. Note also that G is trivial if and only if $\Omega_1(G)$ is trivial.

Let G be a group, and let N be a normal subgroup of G. Sometimes we are going to saying that N is *G-invariant* instead of "N is normal in G" because it is more convenient in speaking of a normal subgroup of G contained somewhere. This will also apply to quotients, so for example if we say that X/N is G-invariant, then we mean that X/N is a normal subgroup of G/N (we could have also said that X/N is G/N-invariant). This use of the term G-invariant comes from the idea that a subgroup to be normal must be fixed (i.e., invariant) under the action by conjugation of G, and such an action can be naturally extended to quotients for example. Now, if $H < K$ are normal subgroups of G such that H/K has no proper non-trivial G-invariant subgroups (that is, H/K is *G-simple*), then H/K is said to be a *chief factor* of G; if $K = \{1\}$, then we also say that H is *minimal G-invariant*. The relevance of chief factors in our context can be seen for instance from the fact that chief factors H/K of nilpotent groups are *central*, meaning that $H/K \leq Z(G/K)$, or $[H, G] \leq K$ — actually every non-trivial normal subgroup of a nilpotent group has non-trivial intersection with the centre of the whole group. Composition factors of arbitrary groups are harder to define and they require the concept of an arbitrary series of subgroups, which is not necessary for our aims. For this reason, we only concern ourselves with defining composition factors of finite groups. First recall that a *finite* (*normal*) *series* of a group is a chain of (normal) subgroups

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_n = G$$

connecting the trivial subgroup to the whole group. The *terms* of the series are the subgroups H_i, while the *factors* are the sections H_{i+1}/H_i, $i = 0, \ldots, n-1$ (a *section* is just a quotient of a subgroup). The *length* of the series is n, that is, the number of "jumps" in the above chain. Now, if the factors of the series (resp., normal series) are finite simple (resp., finite G-simple), then the series is a *composition series* (resp., a *chief series*) and each of its factors is a *composition factor* (resp., a chief factor). Note that composition factors (resp. chief factors) may arise from different series (resp., normal series), but the Jordan–Hölder theorem shows that two different composition series (resp., chief series) have the same lengths and isomorphic factors.

If G_1 and G_2 are groups, then $G_1 \times G_2$ denotes the direct product of G_1 and G_2, while $G_1 \ltimes G_2$ denotes *a* semidirect product of G_2 by G_1 — usually, the action φ defining this semidirect product is clear, but if there is some ambiguity, then we write $G_1 \ltimes_\varphi G_2$.

Remark 1.3. If G_1 and G_2 are groups, and $\varphi : G_1 \to \mathrm{Aut}(G_2)$ is a homomorphism, then $(G_1 \ltimes G_2, \cdot)$ is defined on the Cartesian product $G_1 \times G_2$ in such a way that

$$(x_1, g_1) \cdot (x_2, g_2) = \big(x_1 x_2, g_1^{x_2} g_2\big),$$

for all $x_1, x_2 \in G_1$ and $g_1, g_2 \in G_2$.

The same notation is used for the inner direct product and the inner semidirect product, so if H and K are subgroups of G with $[H, K] = H \cap K = \{1\}$, then $\langle H, K\rangle = H \times K$; while, if $H^K = H$ and $H \cap K = \{1\}$, then $\langle H, K\rangle = K \ltimes H$. One of the properties of the (semi)direct product that we are frequently going to use is the fact that every element $g \in G_1 \ltimes G_2$ can be written in a unique form as $g_1 g_2$, where $g_1 \in G_1$ and $g_2 \in G_2$. Moreover, if G_1 and G_2 are periodic with $\pi(G_1) \cap \pi(G_2) = \emptyset$, then every subgroup H of the direct product $G = G_1 \times G_2$ can be written in a unique form as $H_1 \times H_2$, where $H_1 \leq G_1$ and $H_2 \leq G_2$. Thus, G_1 and G_2 are respectively a Sylow $\pi(G_1)$-subgroup and a Sylow $\pi(G_2)$-subgroup of G. Similar considerations hold for the direct product of infinitely many groups.

Remark 1.4. The corresponding additive notation $G_1 \oplus G_2$ for the direct product of two groups G_1 and G_2 is slightly different from the multiplicative one. Also, if H and K are viewed as subgroups of a group G, then $H + K$ simply denotes the set $\{h + k : h \in H, k \in K\}$, which is not in general a subgroup (although it is a subgroup if G is abelian), while $H \oplus K$ means that $[H, K] = H \cap K = \{0\}$, so in particular $H + K = \langle H, K\rangle = H \oplus K$.

Recall also that if G is any finite group, then $\mathrm{Frat}(G)$ denotes the *Frattini subgroup* of G, that is, the intersection of all maximal subgroups of G. It is easy to show that $\operatorname{Frat}(G)$ coincides with the set of all non-generators of G — here, an element g of G is a *non-generator* if $G = \langle g, X\rangle$ implies $G = \langle X\rangle$ for every subset X of G. Thus, if $G/\operatorname{Frat}(G)$ is cyclic, then G itself is cyclic (the converse being obvious). Now, if G is a finite p-group for some prime p, then G is known to be nilpotent, so in particular all its maximal subgroups are normal in G and their quotients are groups of order p; in particular, $G' \leq \operatorname{Frat}(G)$. It follows that if G is non-cyclic, then the *abelianization* of G (that is, G/G') is never cyclic. Moreover, since $L/(H \cap K)$ can always be embedded into $L/H \times G/K$ whenever H and K are normal subgroups of an arbitrary group L, so $G/\operatorname{Frat}(G)$ is an *elementary abelian* p-group, which means that it is a direct product of cyclic groups of order p — note that a subgroup of an elementary abelian p-group is an elementary abelian p-group as well. Thus, if $G/\operatorname{Frat}(G)$ has order p^m, then m is smallest cardinality of a set of generators for the group G; in particular, if G is an abelian p-group, then G can be decomposed into the direct product of m non-trivial cyclic groups, no less no more. Having mentioned elementary abelian p-groups here, we note that they have a very useful property: if H is a subgroup of an elementary abelian p-group G, then $G = H \times K$ for some subgroup K. The Frattini subgroup has been introduced by Frattini [7] in 1885; in that paper, he proved that the Frattini subgroup of a finite group is nilpotent by making use of an insightful and renowned argument, which nowadays goes under the name of *Frattini Argument*. This argument essentially consists in a double conjugation of a subgroup, one outside a given substructure and one inside, and makes it possible to prove for example that a finite group G with a Sylow p-subgroup P contained in a normal subgroup N can always be written as $G = N_G(P)N$. Although it is now clear that such an argument really belongs to Alfredo Capelli (see [4]), we will still be using the term "Frattini Argument" in this volume.

For our purposes, we also need to discuss a bit endomorphisms of abelian groups, and automorphisms of groups in general. First note that we use the exponential notation for maps, so g^τ (resp., S^τ) is the image of the element g in the domain (resp., is the image of the set S) under the map τ, and that we denote the composition by $\circ$, so $\tau_1 \circ \tau_2$ maps an element g in the domain to $\left(g^{\tau_1}\right)^{\tau_2}$, although sometimes we may simply write $\tau_1\tau_2$. If G_1 and G_2 are groups, then the expression $G_1 \simeq G_2$ means that G_1 and G_2 are *isomorphic*, which means that there exists an isomorphism between them. Moreover, if S is a group of automorphisms of the group G, then we may use expressions like $[G, S]$

to denote the subgroup of the *holomorph* $\mathrm{Hol}(G) \simeq \mathrm{Aut}(G) \ltimes G$ of G generated by the commutators $[g, \psi]$, with $\psi \in S$ and $g \in G$ — of course, $[G, S]$ is a subgroup of G, and we can also regard it as the subgroup of G generated by the elements $g^{-1}g^{\psi}$, $g \in G$ and $\psi \in S$. Similar other expressions involving elements of G and its automorphisms must be interpreted as the corresponding group concept in the holomorph of the group. Finally, let φ and ψ be endomorphisms of an abelian group A. Then one can define a third endomorphism of A as the *sum* of φ and ψ, that is,

$$\varphi + \psi : a \in A \mapsto a^{\varphi} + a^{\psi} \in A.$$

It is easy to see that the set $\mathrm{End}(A)$ of all endomorphisms of A endowed with this sum and the composition is a ring. Thus, in some sense, we can algebraically work with the automorphisms of A as we do with elements of a ring, and for instance if $a \in A$ and τ is a third endomorphism of A, then

$$a^{\tau\circ(\varphi+\psi)} = a^{\tau\circ\varphi+\tau\circ\psi} = a^{\tau\circ\varphi} + a^{\tau\circ\psi}.$$

One of the most relevant and useful results in finite group theory is the *Schur–Zassenhaus Theorem*. It states that if G is a finite group and N is a normal subgroup of G such that $(|N|, |G|) = 1$, then there exists a *complement* to X in G, that is, there exists a subgroup X of G such that $G = X \ltimes N$; moreover, any two such complements are conjugate. (Of course the complements are Sylow $\pi(G/N)$-subgroups of G, while N is the unique Sylow $\pi(N)$-subgroup of G.) The proof of this result is not an easy matter. In fact, the proof that the complements are conjugate needs either the classification of finite simple groups or the *Odd Order Theorem* (the celebrated result of Feit and Thompson stating that a group of odd order is soluble). This is why we do not prove this result here and we simply refer the reader to [18] for a proof.

On the other hand, a very useful result in *infinite* group theory is the *Schur's Theorem*. It states that if a group is finite over the centre, then the derived subgroup is finite. Although Schur's Theorem is probably not due to Issai Schur (see [10]), we still employ this name here for the sake of clarity. Note that many different and natural properties can replace the "finiteness" in the statement of Schur's Theorem, and we refer the interested reader to [9].

Very useful results that we are going to use many times without notice are the Dedekind Modular Law, and the Isomorphism Theorems. In its more general formulation, the *Dedekind Modular Law* states that given any group G, two subsets U, V of G and a subgroup L of G such that $U \subseteq L$, then

$$UV \cap L = U(V \cap L)$$

— here, if X and Y are subsets of G, their *product* XY is the set of all elements xy, where $x \in X$ and $y \in Y$; also, the product is always evaluated before an intersection, so there is no need to put parentheses around UV. In particular, if U, V and UV are subgroups of G, then not only $UV \cap L$ is a subgroup of G but such is also $U(V \cap L)$. We remark here that the product XY of two subgroups X and Y is a subgroup if and only if $XY = YX$, so in particular XY is a subgroup whenever $X \leq N_G(Y)$ or $Y \leq N_G(X)$, and we also remark that if $X_1, \ldots, X_n, X_{n+1}$ are subgroups of G such that $X_2 \leq N_G(X_1)$, $X_3 \leq N_G(X_1X_2)$, ..., $X_{n+1} \leq N_G(X_1 \ldots X_n)$, then one recursively defines $X_1 \ldots X_n = (X_1 \ldots X_n)X_{n+1}$. The three *Isomorphism Theorems* are the following ones:

- If $\varphi : G \to H$ is a homomorphism of groups, then we have that $G/\operatorname{Ker}(\varphi) \simeq \operatorname{Im}(H)$, where $\operatorname{Ker}(\varphi)$ denotes the *kernel* of φ, that is, the set of all elements of G mapped to the identity element of H by φ.
- If N, M are normal subgroups of a group G, then $(G/N)/(M/N)$ is isomorphic to G/M.
- If G is a group, $N \trianglelefteq G$, and $M \leq G$, then $MN/N \simeq M/(M \cap N)$.

We cannot emphasize enough the relevance of these theorems in the study of the theory of groups.

In the final chapter of this volume we are going to need some basic and certainly well-known results about fields and modules over commutative rings. First recall that if $(R, +, \cdot)$ is a commutative ring with multiplicative identity 1, then an *R-module* M consists of an abelian group $(M, +)$ and an operation

$$\cdot : R \times M \to M$$

such that for all $r, s \in R$ and $x, y \in M$, we have:

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$

- $1 \cdot x = x$

Clearly, abelian groups are precisely the $\mathbb{Z}$-modules. More in general, if we have a commutative subring S of the endomorphism ring of an abelian group A, then A can be naturally regarded as an S-module. Recall also that if R is a field, then M is a *vector space*, so we can employ the well-known machinery from linear algebra (dimensions of subspaces, linear dependency and independency, and so on).

Now, let $(\mathcal{K}, +, \cdot)$ be a field of characteristic q. It is well-known that every finite subgroup of $(\mathcal{K}, \cdot)$ is cyclic. Moreover, if $q = 0$, then the *prime field* $\mathcal{E}(\mathcal{K})$ of $\mathcal{K}$ (that is, the smallest subfield of $\mathcal{K}$) is isomorphic to the field of rational numbers, otherwise if $q > 0$, then $\mathcal{E}(\mathcal{K})$ is isomorphic to the field $\big(\mathrm{GF}(q), +, \cdot\big)$ of order q. Note that the field of order p is just the set of all integers modulo p endowed with the usual operations. If n is any positive integer, then an element ν of $\mathcal{K}$ such that $\nu^n = 1$ is an *nth root of unity*. If ν is a generator of the subgroup of $(\mathcal{K}, \cdot)$ made by the nth roots of unity, then ν is said to be a *primitive* nth root of unity. The following well-known result will be very important in describing finite soluble groups in which all quotients except one are abelian; its proof relies on standard results on splitting fields and degrees of field extensions, and can for instance be found in [20].

Theorem 1.5. *Let $n > 1$ be an integer, $(\mathcal{K}, +, \cdot)$ a field of characteristic $q \geq 0$, and ν a primitive nth root of unity. If d is the degree of an irreducible polynomial over $\mathcal{E}(\mathcal{K})$ having ν as a root, then the following holds:*

- *If $q = 0$, then $d = \phi(n)$.*

- *If $q > 0$, then d is the smallest positive integer k such that n divides $q^k - 1$.*

1.1 RELEVANT EXAMPLES OF GROUPS

In this section we discuss some relevant classes of groups that will frequently appear in the subsequent chapters.

Cyclic groups

The infinite cyclic group is denoted by $\mathbb{Z}$, while the finite cyclic group of order n is denoted by $\mathbb{Z}_n$ for every positive integer n. A strong form of the converse of Lagrange's theorem holds in cyclic groups. In fact, if G is a cyclic group and d is any positive integer diving the order of G when G is finite, then there exists one and only one subgroup of G whose index is d. In particular, if G is a finite cyclic group of order n, then there exists one and only one subgroup of order d for every positive integer d dividing n.

The only non-trivial automorphism of $\mathbb{Z}$ is the inversion, while $\mathrm{Aut}(\mathbb{Z}_n)$ is isomorphic to the multiplicative group of the integers modulo $\phi(h)$, where $\phi(n)$ is the *Euler's totient function*, that is, $\phi(n)$ counts the positive integers up to n that are relatively prime to n. If $n = p^m$ for some odd prime p and some positive integer m, then $\mathrm{Aut}(\mathbb{Z}_n)$ is cyclic of order $p^{m-1}(p-1)$, while $\mathrm{Aut}(\mathbb{Z}_2) = \{1\}$, $\mathrm{Aut}(\mathbb{Z}_4) \simeq \mathbb{Z}_2$ and $\mathrm{Aut}(\mathbb{Z}_{2^m}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ for any integer $m \geq 8$. In particular, the only non-trivial automorphism of a cyclic group of order 4 is the inversion.

Free abelian groups

A *free abelian group* is just a direct product of arbitrarily many infinite cyclic groups. Thus, finitely generated free abelian groups are precisely the finitely generated torsion-free abelian groups. If a free abelian group F is a direct product of n infinite cyclic groups, then n is an invariant of F. In other words, F can be only decomposed in the direct product of precisely n infinite cyclic groups, no less no more. Also, every chain of subgroups

$$\{1\} = X_0 \leq X_1 \leq \ldots \leq X_m = F$$

with cyclic factors has precisely n factors that are isomorphic to $(\mathbb{Z}, +)$. It easily follows that any subgroup H of F such that F/H is finite must be generated by at least n elements — in particular, the minimal cardinality for a set of generators of F is n.

Klein 4-group

This is the direct product of two copies of the cyclic group $\mathbb{Z}_2$ of order 2, and it is usually denoted by V_4.

Dihedral and metacyclic groups

Let A be an abelian group. The *dihedral group* $\mathrm{Dih}(A)$ *on* A is the semidirect product $\mathbb{Z}_2 \ltimes A$, where the action is given by the inversion. In other words, if x is a generator of $\mathbb{Z}_2$, then $a^x = a^{-1}$ for all $a \in A$. If A is finite cyclic of order n, then the dihedral group on A is also denoted by $\mathrm{Dih}(n)$ or D_{2n} and is referred to as the *dihedral group of order* n. Of course, the dihedral group of order n is always soluble, while it is nilpotent if and only if n is a power of 2. If $A \simeq \mathbb{Z}$, then $\mathrm{Dih}(A)$ is called the *infinite dihedral group* and is often denoted either by the symbol $\mathrm{Dih}(\infty)$ or by the symbol D_∞. Clearly, the infinite dihedral group is metabelian but not nilpotent.

Dihedral groups for which A is cyclic are nice examples of *metacyclic* groups, that is, of groups G for which there exists a cyclic normal subgroup N with G/N cyclic. Other examples of metacyclic groups are easily constructed as semidirect products of two cyclic groups, and, as we shall see in the next chapter, metacyclic groups play a major role in the classification of finite minimal non-abelian groups.

Quaternion groups
Let n a non-negative integer. The *quaternion group* Q_{2n} *of order* $2n$ is the (finite) group having the following presentation

$$\langle x, y \,:\, x^{2n} = 1,\, x^n = y^2,\, y^{-1}xy = x^{-1}\rangle.$$

Since we do really need only the quaternion group of order 8, we are going to ignore the precise meaning of the above presentation, and we move to describe the aforementioned group. Put $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and define a multiplication $\cdot$ in Q_8 in such a way that 1 is the identity, $-1 \cdot u = -u$ for all $u \in Q_8$, and $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$, $i^2 = j^2 = k^2 = -1$. It is easy to see that -1 is the only element of order 2 of Q_8, that $\langle -1\rangle = Z(Q_8) = Q_8'$ and that $Q_8/\langle -1\rangle \simeq V_4$.

Symmetric and alternating groups
Let Ω be a set. The (*restricted*) *symmetric group* $\mathrm{Sym}(\Omega)$ *on* Ω is the set of all finitary bijective functions from Ω to Ω endowed with the composition — here *finitary* means that the functions only move finitely many elements of Ω. Of course, if Ω is finite and has cardinality n, then we also denote $\mathrm{Sym}(\Omega)$ by $\mathrm{Sym}(n)$ and we call it the *symmetric group of degree n*. Clearly, $\mathrm{Sym}(0) = \mathrm{Sym}(1) = \{1\}$, while $\mathrm{Sym}(2) \simeq \mathbb{Z}_2$ and $\mathrm{Sym}(3) \simeq \mathrm{Dih}(3)$.

The *alternating group* $\mathrm{Alt}(\Omega)$ *on* Ω is the set of all *even* permutations of $\mathrm{Sym}(\Omega)$, that is, the set of all permutations that can be written as a product of an even number of *transpositions* (permutations moving precisely two elements). Again, if Ω is finite of order n, then we write $\mathrm{Alt}(n)$ instead of $\mathrm{Alt}(\Omega)$ and we speak of the *alternating group of degree n*. Clearly, $\mathrm{Alt}(0) = \mathrm{Alt}(1) = \mathrm{Alt}(2) = \{1\}$, and it is well-known that $\mathrm{Alt}(\Omega)$ is a normal subgroup of $\mathrm{Sym}(\Omega)$ such that $|\mathrm{Sym}(\Omega)/\mathrm{Alt}(\Omega)| = 2$ for $|\Omega| \geq 2$. It should be also remarked that $\mathrm{Alt}(\Omega)$ is simple if $|\Omega| \geq 5$ and that $\mathrm{Alt}(4) \simeq \mathbb{Z}_3 \ltimes V_4$, where $\mathbb{Z}_3$ acts on V_4 by cycling its non-trivial elements — note that V_4 and $\mathrm{Alt}(4)$ are the only non-trivial proper normal subgroups of $\mathrm{Sym}(4)$, so they are even characteristic subgroups.

Groups satisfying the maximal condition on subgroups

A group G is said to satisfy the *maximal condition on subgroups* if every set of subgroups of G has a maximal element with respect to the inclusion. This property is equivalent to requiring that every subgroup of G is finitely generated. It turns out that the maximal condition on subgroups is *extension closed*, so if N is any normal subgroup of a group G, and both N and G/N satisfy the maximal condition on subgroups, then also G has the same property. Obviously, every finite group satisfies the maximal condition on subgroups, but also the infinite cyclic group has the same property. Thus, if G is a group with a series of subgroups

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

whose factors are either cyclic or finite (that is, G is *polycyclic-by-finite*), then G satisfies the maximal condition on subgroups. For example, if A is any finitely generated abelian group, then $\mathrm{Dih}(A)$ satisfies the maximal condition on subgroups; more in general, any finitely generated group with a finite-index abelian subgroup satisfies the maximal condition on subgroups. Further relevant examples of groups satisfying the maximal condition on subgroups are the finitely generated nilpotent groups.

Divisible groups

Let p be a prime. The *Prüfer p-group* is the only infinite p-group whose subgroups are totally ordered by inclusion. It can be identified with the set of all p^nth roots of unity (in the field of complex numbers) endowed with multiplication, where n ranges in $\mathbb{N}_0$. As usual, if we do not wish to specify the prime p, then we simply speak of a *Prüfer group*.

Prüfer groups are the easiest non-trivial periodic examples of *divisible* groups, that is, of abelian groups G such that for every $n \in \mathbb{N}$ and for every $g \in G$, there exists $h \in G$ with $h^n = g$. On the other hand, the additive group of the rational numbers is the easiest example of a *non-periodic* divisible group. It turns out that *arbitrary* divisible groups are not far from the aforementioned types of groups. In fact, a group is divisible if and only if it is a direct product of Prüfer groups and copies of the additive group of the rational numbers.

In order to verify that an abelian group G is divisible, it is enough to show that $G = G^n$ for every $n \in \mathbb{N}$ — actually, it is enough to prove that $G^q = G$ for every prime q. We should also note that non-trivial finite abelian groups cannot be divisible, so divisible groups have no non-trivial finite homomorphic images — in fact, if N is any normal subgroup of a divisible group G, then G/N is divisible as well.

Finally, we spend some more words about $(\mathbb{Q}, +)$. It is well-known that $(\mathbb{Q}, +)$ is *locally cyclic*, that is, every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, and that every quotient of $(\mathbb{Q}, +)$ with respect to a non-trivial subgroup is periodic. Also, for every prime p, the Sylow p-subgroup of $(\mathbb{Q}, +)/(\mathbb{Z}, +)$ is a Prüfer p-group and it is generated by the cosets $1/p^n + \mathbb{Z}$, where $n \in \mathbb{N}$ — a similar remark holds for every periodic quotient of $(\mathbb{Q}, +)$.

Remark 1.6. Note that also every Prüfer group is locally cyclic. This can either be seen directly, or you can observe that every homomorphic image of a locally cyclic group is still locally cyclic, so Prüfer groups are locally cyclic because they are isomorphic to quotients of the locally cyclic group $(\mathbb{Q}, +)$.

As a field, $(\mathbb{Q}, +, \cdot)$ has only one automorphism: the identity. On the other hand, $\mathrm{Aut}(\mathbb{Q}, +) \simeq (\mathbb{Q}, \cdot)$ and it turns out that $(\mathbb{Q}, \cdot)$ is isomorphic to the direct product of infinitely many infinite cyclic groups and a cyclic group of order 2.

1.2 ABSTRACT AND CONCRETE GROUP CLASSES

In this section, we give a basic outline of the general theory of abstract group classes, and we define all group classes that may have been mentioned in the volume but that are not essential to the study of the volume itself. The interested reader can found additional information in the quoted textbooks, and in particular we refer to the first chapter of [17] for details about the theory of abstract classes of groups.

A *class of groups* or *group class* $\mathfrak{X}$ is a class in the usual sense of set theory consisting of groups, and satisfying the following properties:

(1) $\mathfrak{X}$ contains a trivial group.

(2) If $\mathfrak{X}$ contains a group G, then $\mathfrak{X}$ contains all groups that are isomorphic to G.

It is customary to identify group classes and group-theoretic properties, so when we speak for example of the property of being abelian, we are actually thinking of the class of all abelian groups. For obvious reasons this identification can sometimes lead to not be wanting property (1) above: for instance, one may be wanting to be free of considering the class of non-abelian groups without being forced to add the trivial groups (see for example [8]). However, removing the trivial groups can sometimes lead to complications instead of simplifications, so this is why we stick here to the above definition.

The closure properties of a group class $\mathfrak{X}$ are probably the most relevant things to know about an abstract group class. They usually allow us to find many interesting groups in $\mathfrak{X}$, and to exclude that certain groups belong to $\mathfrak{X}$. Now, a group class $\mathfrak{X}$ is said to be

- *subgroup closed* (or *closed with respect to forming subgroups*) if it contains every subgroup of a group in $\mathfrak{X}$,
- *quotient closed* (or *closed with respect to forming quotients*, or *closed with respect to forming homomorphic images*) if it contains all the homomorphic images of groups in $\mathfrak{X}$,
- *extension closed* (or *closed with respect to forming extension*) if it contains all groups G having a normal subgroup N such that both G/N and N belong to $\mathfrak{X}$.

There are many other interesting closure properties concerning group classes but for the sake of the volume we content ourselves with the above ones. The most natural group classes are usually closed with respect to forming subgroups and quotients, and this is in fact the case for the class of soluble groups and for that of nilpotent groups, as well as for most of the other properties we deal with in this volume. However, although soluble groups are closed with respect to forming extensions, nilpotent groups are not.

For any group class $\mathfrak{X}$, one can construct several other group classes that greatly extends this group class and that allow us to study new groups using the properties of the class $\mathfrak{X}$ from which we started. This is why the group class $\mathfrak{X}$ is usually a well understood group class such as the class of finite groups or that of soluble groups. In order to give examples of such constructions, let G be a group, and $\mathfrak{X}$ a group class.

- G is said to be *hyperabelian* if it has an ascending normal series with abelian factors, that is, if there exists an ascending chain of normal subgroups of G

  $$\{1\} = G_0 \leq G_1 \leq \dots G_\alpha \leq G_{\alpha+1} \leq \dots G_\beta = G$$

  (here, α and β are ordinal numbers) such that $G_{\gamma+1}/G_\gamma$ is abelian for every $\gamma < \beta$.
- G is *locally* $\mathfrak{X}$ if all its finitely generated subgroups are contained in an $\mathfrak{X}$-subgroup. Clearly, if $\mathfrak{X}$ is subgroup closed, then G is locally

$\mathfrak{X}$ if and only *all* its finitely generated are $\mathfrak{X}$-groups. Thus, for example, a *locally soluble* (resp., *locally finite*) group is just a group whose finitely generated subgroups are soluble (resp., finite).

- G is *residually* $\mathfrak{X}$ if there is a set $\{N_\lambda\}_{\lambda\in\Lambda}$ of normal subgroups of G such that $G/N_\lambda \in \mathfrak{X}$ for every $\lambda \in \Lambda$ and $\bigcap_{\lambda\in\Lambda} N_\lambda = \{1\}$. Thus, G is residually $\mathfrak{X}$ if and only if for every non-trivial element g of G, there is a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathfrak{X}$. For example, a *residually soluble* (resp., *residually finite*) group is a group such that the intersection of all normal subgroups with soluble (resp., finite) quotient is trivial. Note that every polycyclic-by-finite group is residually finite, and that such is every finitely generated metabelian group. In order to cite this latter deep result of Philip Hall, we state it as a theorem.

  Theorem 1.7 (see [17], Theorem 9.51)**.** *Let G be a metabelian group. If G is finitely generated, then G is residually finite.*

- G is $\mathfrak{X}$*-by-finite* if it has a finite-index subgroup $X \in \mathfrak{X}$. If $\mathfrak{X}$ is subgroup closed, then this is equivalent to requiring that G has a finite-index normal subgroup $N \in \mathfrak{X}$. Thus, for example an *abelian-by-finite* group is a group having a normal abelian subgroup of finite index.

1.3 RELEVANT PRELIMINARY RESULTS

In this section we state and prove three relevant auxiliary results that we need in the next chapters, and that are of an independent interest.

Theorem 1.8. *Let G be a finite non-abelian group. If A is an abelian subgroup of G of prime index p, then $|G| = p \cdot |G'| \cdot |Z(G)|$.*

Proof. Take g in $G \setminus A$. The map

$$\varphi : a \in A \mapsto a^\varphi = [a, g] \in A$$

is a homomorphism of A because

$$(a_1a_2)^\varphi = [a_1a_2, g] = [a_1, g]^{a_2}[a_2, g] = [a_1, g][a_2, g] = a_1^\varphi \cdot a_2^\varphi$$

for every $a_1, a_2 \in A$. Moreover, $K = \mathrm{Im}(\varphi) = [A, g] \leq G' \leq A$, while $\mathrm{Ker}(\varphi) = C_A(g) = Z(G)$ because G is non-abelian. Since

$$[a, g]^g = [a^g, g] \in K$$

for every $a \in A$, so K is normalized by g and A, and hence even by $G = \langle g, A \rangle$. Now, G/K is abelian, so $G' = K$. Therefore

$$|A| = |\operatorname{Im}(\varphi)| \cdot |\operatorname{Ker}(\varphi)| = |G'| \cdot |Z(G)|$$

and consequently, $|G| = p \cdot |A| = p \cdot |G'| \cdot |Z(G)|$. The statement is proved. $\square$

Theorem 1.9. *Let p be a prime. If A is a p'-group of automorphisms of the finite abelian p-group P, then $P = C_P(A) \times [P, A]$.*

Proof. We use additive notation for the elements of P. Let n be the order of A, and consider the following map

$$\varphi : g \in P \mapsto g^{\varphi} = \frac{1}{n} \sum_{\psi \in A} g^{\psi} \in P,$$

which is well-defined because $\sum_{\psi \in A} g^{\psi}$ is an element of P and p does not divide n.

Since P is abelian, so φ is easily seen to be an endomorphism of P. We can write $\varphi = \frac{1}{n} \sum_{\psi \in A} \psi$. Now, if τ is any element of A, then

$$\varphi \circ \tau = \Big(\frac{1}{n} \sum_{\psi \in A} \psi \Big) \circ \tau = \frac{1}{n} \sum_{\psi \in A} (\psi \circ \tau) = \varphi$$

and similarly $\varphi = \tau \circ \varphi$. Therefore φ centralizes the elements of A and

$$\varphi^2 = \varphi \circ \Big(\frac{1}{n} \sum_{\psi \in A} \psi \Big) = \frac{1}{n} \sum_{\psi \in A} (\varphi \circ \psi) = \frac{1}{n} \sum_{\psi \in A} \psi = \varphi.$$

Claim: $C_P(A) = P^{\varphi}$, and $x^{\varphi} = x$ for every $x \in P^{\varphi}$.
If $x \in P$, then $x^{\varphi \circ \psi} = x^{\varphi}$ for every $\psi \in A$, so $x^{\varphi} \in C_P(A)$. Conversely, if $x \in C_P(A)$, then

$$x^{\varphi} = \frac{1}{n} \sum_{\psi \in A} x^{\psi} = \frac{1}{n} \sum_{\psi \in A} x = \frac{1}{n} n x = x$$

and hence x belongs to P^{φ}. The claim is proved.

Put $H = [P, A]$ and $H_1 = \{x - x^{\varphi} \,:\, x \in P\}$. Since $\mathrm{id}_P - \varphi$ is an endomorphism, so H_1 is a subgroup of P.

Claim: $H = H_1$.
Let $x \in P$ and $\tau \in A$. Then

$$(-x + x^{\tau})^{\varphi} = -x^{\varphi} + x^{\tau \circ \varphi} = -x^{\varphi} + x^{\varphi} = 0,$$

and hence $-x + x^{\tau} \in H_1$. It follows that $H \leq H_1$. Conversely,

$$x - x^{\varphi} = \frac{1}{n} \sum_{\psi \in A} (x - x^{\psi})$$

is the n^{th}-root of a sum of elements of H, so it belongs to H — consequently, $H_1 \leq H$. Therefore $H = H_1$ and the claim is proved.

Now, $P = P^{\varphi} + H_1$ because $x = x^{\varphi} + (x - x^{\varphi})$ for every $x \in P$. Also, if $x \in P^{\varphi} \cap H_1$, then $x = x^{\varphi}$ and $x = y - y^{\varphi}$ for some $y \in P$, so

$$x = x^{\varphi} = y^{\varphi} - y^{\varphi \circ \varphi} = y^{\varphi} - y^{\varphi} = 0.$$

Therefore $P = P^{\varphi} \oplus H_1 = C_P(A) \oplus [P, A]$. □

Corollary 1.10. *Let p be a prime. If A is a p'-group of automorphisms of a finite abelian p-group P and $[\Omega_1(P), A] = \{1\}$, then $A = \{1\}$.*

Proof. By Theorem 1.9, we can write $P = C_P(A) \times [P, A]$. Since $\Omega_1(P)$ is the direct product of $\Omega_1(C_P(A))$ and $\Omega_1([P, A])$, it follows that $\Omega_1([P, A]) = \{1\}$ and hence $[P, A] = \{1\}$. □

The solubility assumption in our last result is not really necessary, but proving the theorem without this hypothesis requires the classification of finite simple groups, so we content ourselves to state and prove the soluble case.

Theorem 1.11. *Let G be a finite soluble group. If $P = N_G(P)$ for every prime q and every Sylow q-subgroup P of G, then G is a p-group for some prime p.*

Proof. First, we show that the hypothesis is inherited with respect to forming homomorphic images. To this aim, let N be any normal subgroup of G, and suppose that P/N is a Sylow q-subgroup of G/N for some prime q. If Q is any Sylow q-subgroup of P, then $P = QN$. Moreover, if we write $K/N = N_{G/N}(P/N)$, then P is a normal subgroup of K, and the Frattini Argument now implies that $K = N_K(Q)N = QN = P$. Thus, G/N satisfies the hypothesis of the statement.

Now, suppose that G is a minimal counterexample to the conclusion of the statement, and let N be a non-trivial proper normal subgroup (recall that G is soluble). Then G/N is a p-group for some prime p, so N is not a p-group. If q is any other prime dividing the order of G, and Q is a Sylow q-subgroup of G, then $Q \leq N$ and hence the Frattini Argument yields that $G = N_G(Q)N = QN = N$, a contradiction. □

2 MINIMAL NON-ABELIAN GROUPS

The most obvious examples of non-abelian groups with many abelian subgroups are certainly the non-abelian groups in which *all* proper subgroups are abelian. Very basic examples of this type are the following ones:

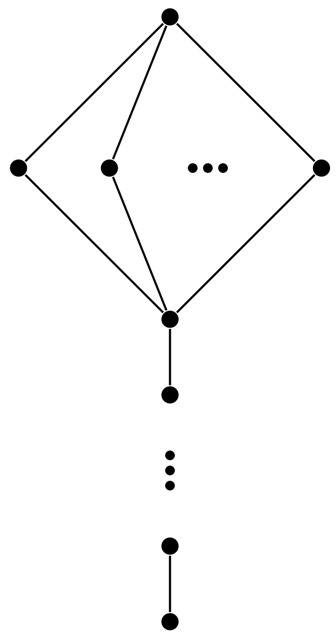
- the symmetric group $\mathrm{Sym}(3) \simeq \mathrm{Dih}(3)$ of degree 3, and more in general every dihedral group of order $2p$, where p is an odd prime;
- the quaternion group Q_8 of order 8;
- the dihedral group of order 8;
- every non-abelian group of order p^3 for any prime p (because a group of order q^2 is abelian for any prime q).

On the other hand, there are much more problematic types of minimal non-abelian groups. For example, the standard *Tarski* monsters (that is, infinite groups whose non-trivial proper subgroups have prime order) are obviously minimal non-abelian. Actually, Ol'shanskiĭ proved that we can have minimal non-abelian groups in which de facto we can choose the non-trivial proper subgroups (see [16], Theorem 35.1, and [15]), and that one can also construct the *extended Tarski monsters* (see [16], Theorem 31.8), that is, infinite groups G having a normal subgroup N such that:

- G/N is a standard Tarski monster,
- N is cyclic of prime power order $p^r \neq 1$, and
- for every subgroup H of G, either $H \leq N$ or $N < H$.

A complete description of all Tarski and extended Tarski groups is not known and seems difficult to establish, but we can at least notice that all these groups have infinite simple sections, so it is to get rid of them if we ask for some solubility condition.

The following picture is the Hasse diagram of the subgroup lattice of an extended Tarski monster, and it is the picture on the cover of this volume. It immediately shows how simple is the large-scale structure of an (extended) Tarski monster compared to its inner structure.



Finite minimal non-abelian groups have been considered for the first time by Miller and Moreno [11] in 1903, and nowadays their structure is completely clear although the information is scattered through some papers, books and folklore (see for example [2], Exercise 8a, p.29). Thus, the aim of this chapter is to collect in a single place the full structural theorems concerning finite minimal non-abelian groups. First we show that this does also take care of the infinite soluble case.

Theorem 2.1. *Let G be a minimal non-abelian group. Then G is finite if and only if it is soluble.*

Proof. First suppose that G is finite (of order n) and not soluble — since G is finite, we may assume it is a counterexample of smallest possible order. If N is any non-trivial proper normal subgroup of G, then N is abelian and G/N is either abelian or minimal non-abelian. In any case, G/N is soluble by minimality of G, so G is soluble too (recall that solubility is closed by extensions). Therefore G is simple non-abelian and in particular $Z(G) = \{1\}$. Moreover, if M_1 and M_2 are distinct maximal subgroups of G, then $M_1 \cap M_2$ is centralized by both

M_1 and M_2 and so also by $G = \langle M_1, M_2 \rangle$. Thus, $M_1 \cap M_2$ is trivial because it is contained in the centre of G.

Now, let M be a maximal subgroup of G, and put $|M| = m$. Since M is not normal in G, so $N_G(M) = M$, which means that M has n/m conjugates in G. These conjugates account for $(m-1)n/m$ non-trivial elements of G, because the maximal subgroups of G have pairwise trivial intersection. Therefore the number of non-trivial elements of G in the conjugates of M is at least

$$n - n/m \geq n/2$$

(note that $m \geq 2$).

The index of M in G is at least 2, so $n/m \geq 2$ and consequently

$$n - n/m \leq n - 2 < n - 1.$$

Since the number of non-trivial elements of G is $n-1$, so there is a maximal subgroup of G that is not conjugate to M. But the number of non-trivial elements contained in the conjugates of this new maximal subgroup is again at least $n/2$, so the number of non-trivial elements of G is at least n, which is not possible. This contradiction shows that if G is finite, then it is soluble.

Suppose now that G is soluble and infinite. Since G is non-abelian, so there are two elements a and b of G such that $[a,b] \neq 1$; in particular, $G = \langle a, b \rangle$ is finitely generated.

If G has two distinct maximal subgroups M_1 and M_2 of finite index, then $M_1 \cap M_2$ is a finite-index subgroup of G contained in $Z\big(\langle M_1, M_2 \rangle\big) = Z(G)$, so $G/Z(G)$ is finite. Consequently G' is finite by Schur's theorem (see [17], Theorem 4.12) and G/G' is infinite. Since G/G' is finitely generated, so G has a normal subgroup X whose quotient G/X is infinite cyclic. Now, if g is any element of infinite order modulo X and p is a prime, then $\langle g^p \rangle X < G$. Hence $[g^p, X] = \{1\}$ for every prime p and so $[g, X] = \{1\}$. Since G is generated by the elements of infinite order modulo X, so X is central in G. This means that $G/Z(G)$ is cyclic, and consequently that G is abelian, a contradiction.

Thus, G has only one maximal subgroup of finite index. In particular, G/G' is cyclic and finite. Let $g \in G$ be such that $G = \langle g, G' \rangle$, and put $N = \langle g \rangle \cap G'$. Clearly, N is centralized by g and G' (note that G' is abelian), so it is central in G. If G/N is finite, then G' is finite by Schur's theorem and so G is finite, a contradiction. Thus, G/N is infinite. Moreover, $G/N = \langle gN \rangle \ltimes G'/N$. Since G' is a finite-index subgroup of G, so it is finitely generated as well, and hence G'/N is infinite and finitely generated. If p and q are distinct primes, then $\langle g\rangle(G')^pN$ and $\langle g\rangle(G')^qN$ are proper subgroups of G, so they are abelian. Consequently, g centralizes $(G')^{pq}N$. On the other hand, $(G')^{pq}N$ is contained in the abelian subgroup G', and hence $(G')^{pq}N$ is a central subgroup of G of finite index. Again, Schur's theorem yields that G' is finite, and gives the contradiction that G is finite as well. □

Of course, if our aim is just to avoid Tarski monsters, then solubility may feel as a very strong requirement in the above statement. In fact, there are much better ways to avoid pathologies like Tarski monsters. One of these has been introduced by Černikov [5] in 1970, and it is nowadays considered as the broadest group class that is free of this type of monsters. A group G is *locally graded* if every non-trivial finitely generated subgroup of G has a proper subgroup of finite index. Clearly, all finite groups and all soluble groups are locally graded, but such are also for example the hyperabelian, locally soluble, locally finite, residually soluble, and residually finite groups. The class of locally graded groups has many nice closure properties: it is in fact closed with respect to forming subgroups, extensions, and Cartesian products. In particular, every group with a finite term in its derived series is locally graded. However, it should be remarked that locally graded groups are not closed with respect to forming quotients because the free groups are residually finite (so they are locally graded), but Tarski monsters appear as quotients of non-abelian free groups.

Corollary 2.2. *Let G be a minimal non-abelian group. If G is locally graded, then G is finite* (*and soluble*).

Proof. Since G is not abelian, so it has non-commuting elements a and b, and hence $G = \langle a, b\rangle$. Being locally graded, G has a proper normal subgroup N of finite index. Now, G/N is either abelian or minimal non-abelian, so in both cases G/N is soluble by Theorem 2.1. But N is abelian, and so G is soluble. A further application of Theorem 2.1 yields that G is finite and completes the proof. □

The classification of finite minimal non-abelian groups splits in two cases according to the group having order a power of a prime or not. As shown by the next result, in the latter case there cannot be many distinct primes dividing its order.

Lemma 2.3. *Let G be a finite minimal non-abelian group. Then the order of G is divided by at most two distinct prime numbers.*

Proof. Suppose by way of contradiction that $|G| = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where the p_i's are pairwise distinct primes, the e_i's are positive integers, and $k \geq 3$. Since G is soluble, so $G' < G$ and hence there exists a maximal subgroup M of G containing G' and such that $|G/M| = p_i$ for some $1 \leq i \leq k$; in particular, $M \trianglelefteq G$. Now, M is abelian, so its Sylow subgroups are characteristic in M and hence normal in G. Let P be a Sylow p_i-subgroup of G and let Q be the Hall p_i'-subgroup of M. Then $|M/Q| = p_i^{e_i - 1}$, so Q is also the Hall p_i'-subgroup of G, and $G = P_i Q$. If P_j is any Sylow p_j-subgroup of G for some $j \neq i$ with $1 \leq j \leq k$, then $P_j \leq M$, so $P_j \trianglelefteq G$. Consequently, $P_i P_j$ is a proper subgroup of G (because $|G|$ is divided by at least three distinct primes), and hence P_i centralizes P_j. The arbitrariness of j shows that P_i centralizes Q. Since $G = P_i Q$ and both P_i and Q are abelian, so G itself is abelian, a contradiction. □

In the following two sections, we separately deal with minimal non-abelian groups of prime power order and with minimal non-abelian groups whose order is not a power of a prime.

2.1 THE ORDER IS A POWER OF A PRIME

Lemma 2.4. *Let G be a non-abelian group of order 8, then either $G \simeq Q_8$ or $G \simeq D_8$.*

Proof. If all elements of G have order 2, then $(ab)^{-1} = ab$ for every $a, b \in G$, so $[a, b] = 1$ and hence G is abelian, a contradiction. Then G has an element x of order 4. If $X = \langle x \rangle$, then $|G : X| = 2$, so $X \trianglelefteq G$ and $G = \langle x, y \rangle$ for every $y \in G \setminus X$. It follows that $x^y = x^{-1}$. Now, if y can be chosen of order 2, then $G \simeq D_8$, else $G \simeq Q_8$. □

Theorem 2.5. *Let G be a finite p-group for some prime p. Then G is minimal non-abelian if and only if one of the following alternatives holds:*

(1) *$G = \langle a \rangle \ltimes \langle b \rangle$, where $o(a) = p^n$, $o(b) = p^m$, $b^a = b^{1+p^{m-1}}$, and n, m are integers with $n \geq 1$ and $m \geq 2$.*

(2) *$G \simeq Q_8$.*

(3) $G = \langle a \rangle \ltimes (\langle b \rangle \times \langle c \rangle)$, *where* $o(a) = p^n$, $o(b) = p^m$, $o(c) = p$, $b^a = bc$, $c^a = c$, *and* n, m *are positive integers such that* $m + n > 2$ *when* $p = 2$.

In particular, if G *is minimal non-abelian, then* $|G'| = p$, G/G' *is isomorphic to the direct product of two non-trivial cyclic groups, and* $G/Z(G)$ *is elementary abelian of order* p^2.

Proof. First suppose that G is minimal non-abelian. If G has only one maximal subgroup, then this subgroup coincides with the Frattini subgroup $\mathrm{Frat}(G)$ of G, so $G/\mathrm{Frat}(G)$ is cyclic and consequently G is cyclic, a contradiction. Thus, G has at least two maximal subgroups M_1 and M_2, say. Since every maximal subgroup of a finite p-group is normal, it follows that $|G/M_1| = |G/M_2| = p$ and hence that the order of $G/(M_1 \cap M_2)$ is p^2. Moreover, $M_1 \cap M_2$ is centralized by the abelian subgroups M_1 and M_2, so $M_1 \cap M_2 \leq Z(G)$. But G is not abelian, and so $M_1 \cap M_2 = Z(G)$. Thus, Theorem 1.8 implies that $|G'| = p$. Also, $G/Z(G)$ is abelian and so $G' \leq Z(G)$, which means that quotient G/G' cannot be cyclic.

Since G is non-abelian, so we can find two non-commuting elements a and b of G. Consequently, $G = \langle a, b \rangle$. Put also $G' = \langle c \rangle$ and $U = \Omega_1(G)$. Since G/G' is a 2-generator abelian p-group, so we may assume $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, where aG' has order p^n and bG' has order p^m. Of course, both n and m are *positive* integers because otherwise $G/Z(G)$ (and also $G/\mathrm{Frat}(G)$) would be a cyclic group and G would be abelian (or cyclic).

Now, $UG'/G' \leq \Omega_1(G/G')$, so $|U| \leq p^3$. We divide the proof in three cases according to the order of U. First suppose that $|U| = p^3$. In this case there are two options: either U is abelian or not. If U is abelian, then U is the direct product of three cyclic groups of order p. In this case, $H = \langle b, c \rangle$ is abelian 2-generator, $|\Omega_1(H)| = p^2$ and $H/\Omega_1(H)$ is cyclic, so H is the direct product of a cyclic group of order p and a cyclic group of order p^m; in particular, we may assume that $H = \langle b \rangle \times \langle c \rangle$, so $o(b) = p^m$. Similarly, we may assume that $\langle a, c \rangle = \langle a \rangle \times \langle c \rangle$, so $o(a) = p^n$. It follows that

$$G = \langle a, b \rangle = \langle a \rangle \ltimes H = \langle a \rangle \ltimes (\langle b \rangle \times \langle c \rangle).$$

Obviously, $b^a = bc^u$ for some integer $0 \leq u < p$, so replacing c by c^u, we obtain $b^a = bc$. Finally, if $p = 2$ and $m + n = 2$, then $|G| = p^3 = |U|$, so G is abelian, a contradiction. Therefore, $m + n > 2$ when $p = 2$, and we are in case (3).

Assume that U is non-abelian, so $G = U$ and hence G has order p^3. If $p = 2$, then we are in case (1) or (2) by Lemma 2.4, although case (2) cannot happen here because $U = G$. If $p > 2$, then Lemma 1.1 yields that $(xy)^p = x^p y^p$ for every $x, y \in G$, so G has exponent p. Thus, $G = \langle a \rangle \ltimes \big(\langle b \rangle \times \langle c \rangle\big)$, where a and b have order p, and $b^a = bc^u$ for some positive integer u. Replacing c by c^u, we see that (3) holds.

Suppose $|U| = p^2$. It is possible to assume that $U/G' = \Omega_1(\langle aG' \rangle)$, and so also that $\langle a, c \rangle = \langle a \rangle \times \langle c \rangle$. Moreover, $U \cap \langle b, c \rangle = \langle c \rangle$, so the abelian group $\langle b, c \rangle$ is cyclic, and hence $\langle b, c \rangle = \langle b \rangle$ because c is in $\mathrm{Frat}(\langle b, c \rangle)$. Then $G = \langle a \rangle \ltimes \langle b \rangle$, where $o(a) = p^n$ and $o(b) = p^{m+1}$. Now, $b^a = bc^u = b^{1+up^m}$, where u is a positive integer that is prime to p. Let v be a positive integer with $uv \equiv_p 1$. If we replace a by a^v, then $b^a = b^{1+p^m}$ and we are in case (2).

Finally, suppose $|U| = p$, so $U = G'$. Let $V/G' = \Omega_1(G/G')$, so V has order p^3. If V is abelian, then it must be cyclic, a contradiction. Thus, $V = G$ has order p^3. If $p = 2$, then we are in case (1) or (2) by Lemma 2.4, although case (1) cannot happen here because $U = G'$. If $p > 2$, then Lemma 1.1 yields that $(xy)^p = x^p y^p$ for every $x, y \in G$, so G has exponent p. Therefore $G = U = G'$, a contradiction.

Assume conversely that G satisfies one of the three conditions in the statement. If $G \simeq Q_8$, then G is obviously minimal non-abelian. Suppose $G = \langle a \rangle \ltimes \langle b \rangle$ satisfies (1), and let X be a non-abelian subgroup of G. Note that

$$(b^p)^a = (b^a)^p = \big(b^{1+p^{m-1}}\big)^p = b^p \quad \text{and} \quad b^{a^p} = bb^{p \cdot p^{m-1}} = b,$$

so $a^p, b^p \in Z(G)$. If $X \cap \langle b \rangle \leq \langle b^p \rangle \leq Z(G)$, then $X/X \cap \langle b \rangle$ is cyclic, and so X is abelian, a contradiction. Thus, $\langle b \rangle = X \cap \langle b \rangle \leq X$. By Dedekind Modular Law, $X = \langle a^u \rangle \ltimes \langle b \rangle$ for some integer u. However, if p divides u, then $a^u \in Z(G)$, and X is abelian. It follows that p does not divide u and hence $\langle a^u \rangle = \langle a \rangle$. Therefore $X = G$, and G is minimal non-abelian.

Finally, assume $G = \langle a \rangle \ltimes \big(\langle b \rangle \times \langle c \rangle\big)$ satisfies (3), let X be a non-abelian subgroup of G, and note again that $a^p, b^p \in Z(G)$. Since X is non-abelian, so c belongs to $X' \leq X$. If $X \cap \langle b, c \rangle \leq Z(G)$, then X is cyclic over the centre, which means that X is abelian, a contradiction. Thus, $X \cap \langle b, c \rangle \not\leq Z(G)$ and consequently $\langle b, c \rangle = X \cap \langle b, c \rangle \leq X$ because $\langle b^p, c \rangle \leq Z(G)$. As above, since X is non-abelian, so X cannot be contained in $\langle a^p, b, c \rangle$ and hence $X = \langle a, b, c \rangle = G$. $\square$

Remark 2.6. Note that if G is a minimal non-abelian group of type (3) above, then the condition $m + n > 2$ whenever $p = 2$ guarantees that G' is always a maximal cyclic subgroup of G, so G is not metacyclic in this case. In fact, if $p = 2$ and $g = a^u b^v c^z$ is any element of G such that $g^2 \in \langle c \rangle$, then gG' has order 2, so 2 must divide either v or u because of the condition $m + n > 2$, hence either a^u or b^v are contained in $Z(G)$, and consequently $g^2 = a^{2u} b^{2v}$, which means that $g^2 = 1$. In case p is odd, this follows from a straightforward application of Lemma 1.1.

Thus, the three cases in the statement of Theorem 2.5 can be labeled as follows: the metacyclic case (1), the quaternion case (2) and the non-metacyclic case (3). It follows that the alternatives in the statement of Theorem 2.5 are non-isomorphic ones.

The following corollary of Theorem 2.5 shows how easy is to detect minimal non-abelian subgroups in finite p-groups.

Corollary 2.7. *Let G be a non-trivial finite p-group for some prime p. The following conditions are equivalent:*

(1) *G is minimal non-abelian.*

(2) *G is 2-generator and $|G'| = p$.*

(3) *G is 2-generator and $Z(G) = \mathrm{Frat}(G)$.*

Proof. It obviously follows from Theorem 2.5 that (1) implies both (2) and (3).

Assume (2), write $G = \langle a, b \rangle$ and $G' = \langle c \rangle$ for some $a, b, c \in G$. Observe that $\langle [a, b] \rangle = G' \leq Z(G)$ because G' is a chief factor of G, so $1 = [a, b]^p = [a^p, b] = [a, b^p]$ and hence $\langle a^p, b^p \rangle \leq Z(G)$. Since G is 2-generator and $|G/Z(G)| = p^2$, so $Z(G) = \mathrm{Frat}(G)$. Therefore (2) implies (3).

Assume (3), and note that G is not abelian because otherwise $G = Z(G) = \mathrm{Frat}(G) = \{1\}$. Let X be any proper subgroup of G. Since G is 2-generator, so $G/Z(G)$ is elementary abelian of order p^2. Now, if $XZ(G) = G$, then $G = X\,\mathrm{Frat}(G) = X$, a contradiction. Thus $XZ(G) < G$, so X is cyclic over its centre and hence is abelian. Therefore G is minimal non-abelian and (3) implies (1). $\square$

Remark 2.8. We could have used the proof of Corollary 2.7 to prove the sufficiency of the conditions in the statement of Theorem 2.7. In fact, we could have proved both Theorem 2.5 and Corollary 2.7 at the same time. However, we have preferred to split these two results so their statements could be more clear. Since we would have then used the same argument twice, we have decided to prove the sufficiency in Theorem 2.5 by a less abstract argument.

Obviously, every non-abelian finite p-group has minimal non-abelian subgroups (take any subgroup that is minimal with respect to the property of being non-abelian), and we end this section by showing that in general there may be many minimal non-abelian subgroups.

Theorem 2.9. *Let G be a locally finite p-group for some prime p. If G is not abelian, then G is generated by its minimal non-abelian subgroups.*

Proof. It is enough to prove the statement when G is finite. Let X be the subgroup generated by all the minimal non-abelian subgroups of G, and assume by contradiction that $X < G$. Clearly, X is normal in G, and by induction on the order of G, we also have that X is maximal in G, while every other maximal subgroup of G needs to be abelian.

Let $a \in G \setminus X$ and write

$$G/\operatorname{Frat}(G) = C_1/\operatorname{Frat}(G) \times \ldots \times C_t/\operatorname{Frat}(G),$$

where $C_i/\operatorname{Frat}(G)$ is a cyclic subgroup of order p for every $i = 1, \ldots, t$, and $a \in C_1$. If $t > 2$, then a is contained in the two distinct maximal subgroups $M_1 = C_1 \ldots C_{t-1}$ and $M_2 = C_1C_3 \ldots C_t$. But M_1 and M_2 are abelian and generate G, so $a \in Z(G)$. Since $G = \langle G \setminus X \rangle$, so G is abelian, a contradiction. Thus $t = 2$. Since $\operatorname{Frat}(G)$ is contained in the (abelian) maximal subgroups C_1 and C_2 of G, so $\operatorname{Frat}(G) \leq Z(G)$. Consequently, X is cyclic over the centre and hence is abelian, a contradiction. $\square$

Corollary 2.10. *Let G be a non-abelian locally finite p-group for some prime p. If G is not minimal non-abelian, then G has at least two minimal non-abelian subgroups.*

2.2 THE ORDER IS NOT A POWER OF A PRIME

Theorem 2.11. *Let G be a finite group whose order is not a prime power. Then G is minimal non-abelian if and only if the following condition holds:*

- *$G = Q \ltimes P$, where Q and P are respectively a Sylow q-subgroup and a Sylow p-subgroup of G for some distinct primes q and p. Moreover, Q is cyclic, $|Q/C_Q(P)| = q$, $P = G'$ is elementary abelian, and contains no proper non-trivial G-invariant subgroup* (*so Q is a maximal subgroup of G*)

Proof. First suppose that G is minimal non-abelian. It follows from Theorem 2.1 that G is soluble, so $G' < G$ and there exists a maximal subgroup M of G containing G' and such that $|G/M| = q$ is a prime. Let p be a prime dividing $|G|$ and distinct from q. If P is any Sylow p-subgroup of G, then $P \leq M$. But M is abelian and P is a Sylow p-subgroup of M, so P is characteristic in M, and hence $P \trianglelefteq G$. Now, if Q is any Sylow q-subgroup of G, then $G = QP = Q \ltimes P$ because $Q \cap P = \{1\}$. If Q is not cyclic, then $\langle g, P\rangle$ is abelian for every $g \in Q$, so $P \leq Z(G)$ and $G = Q \times P$; thus, both Q and P are abelian, so G is abelian too, a contradiction. Therefore Q is cyclic and we can write $Q = \langle x\rangle$ for some $x \in G$.

Now, by Theorem 1.9, we can write $P = C_P(Q) \times [Q, P]$. Since both $C_P(Q)$ and $[Q, P]$ are normalized by Q, they cannot be both non-trivial, otherwise $QC_P(Q)$ and $Q[Q, P]$ are abelian, so Q centralizes P and $[Q, P] = \{1\}$, a contradiction. Thus, $C_P(Q) = \{1\}$ and $P = [Q, P] = G'$. Finally, if P has a proper G-invariant subgroup L, then QL is abelian, so $L \leq C_P(Q) = \{1\}$; in particular, $P = \Omega_1(P)$ is elementary abelian. The sufficiency of the statement is proved.

Assume conversely that the condition in the statement is satisfied, so $G = Q \ltimes P$, where Q and P are respectively a Sylow q-subgroup and a Sylow p-subgroup of G for some distinct primes q and p. If X is any non-abelian subgroup of G, then X cannot be contained in the abelian subgroup Q^qP, so the order of a Sylow q-subgroup of X equals the order of a Sylow q-subgroup of G, and hence without loss of generality we may assume $Q \leq X$. Since $Q \neq X$, so $X \cap P \neq \{1\}$ by Dedekind Modular Law. But $(X \cap P)^X = (X \cap P)^G$ and consequently $(X \cap P)^X = P$ because P has no non-trivial proper G-invariant subgroups. Thus, $X = G$ and G is minimal non-abelian. □

Remark 2.12. No analogue of Theorem 2.9 is possible in case of arbitrary finite groups. To see this, let G be the direct product of a copy S of the symmetric group of order 3 and a cyclic group of order 5. Clearly, the subgroup generated by all the minimal non-abelian subgroups of G is S, so neither Theorem 2.9 nor its corollary hold in this case.

3 METAHAMILTONIAN GROUPS

In this chapter we deal with one of the broadest and most natural generalizations of the class of minimal non-abelian groups: the class of metahamiltonian groups. But before that, we are going to describe a subclass of the class of metahamiltonian groups that is not only relevant in dealing with arbitrary metahamiltonian groups, but it is indeed extremely rich in abelian subgroups. The starting point is the observation that every subgroup of an abelian group is normal, and so it is reasonable to expect that a group in which *every* subgroup is normal would somewhat resemble an abelian group. This kind of groups has first been considered by Dedekind [6] in 1897, who proved that they always contain a copy of the quaternion group of order 8 in the non-abelian case — he called these groups *Hamiltonian*. A complete description was later obtained by Baer [1] in 1933, but Dedekind already came close to it in his original paper, and this is why groups (resp., non-abelian groups) whose subgroups are normal are now termed *Dedekind* groups (resp., *Hamiltonian* groups).

Theorem 3.1. *Let G be a non-abelian group. Then G is Hamiltonian if and only if $G = Q \times E \times D$, where Q is isomorphic to the quaternion group of order 8, D is a periodic abelian $2'$-group, and E is an elementary abelian 2-group.*

Proof. First suppose that all subgroups of G are normal. Since G is non-abelian, so there are elements $x, y \in G$ such that $c = [x, y] \neq 1$. Now, $\langle x \rangle$ are $\langle y \rangle$ are normal subgroups of G, so $c \in \langle x \rangle \cap \langle y \rangle$ and hence $c = x^u = y^v$ for some non-zero integers u and v — note that neither u nor v can be 1, otherwise either $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$, which means that $c = [x, y] = 1$, a contradiction.

Let $Q = \langle x, y \rangle$. Clearly, $c \in Z(Q)$, so $Q' = \langle c \rangle$. Moreover,

$$c^u = [x, y]^u = [x^u, y] = [c, y] = 1,$$

so Q' is finite. But Q/Q' is abelian and generated by the two periodic elements xQ' and yQ', so Q/Q' is finite, and hence Q is finite.

Since Q is finite, it is possible to choose x and y with minimum $o(x) + o(y)$. Let $o(x) = m$ and $o(y) = n$. If p is any prime dividing m, then $c^p = [x^p, y] = 1$ by minimality of $m + n$, so c has prime order p. The same argument actually shows that m and n are only divisible by one prime p, so Q is a p-group.

Write $u = kp^r$ and $v = \ell p^s$ with $(k, p) = (\ell, p) = 1$ and $r, s \in \mathbb{N}$, and choose integers k' and ℓ' in such a way that $kk' \equiv_p \ell\ell' \equiv_p 1$. Put $x' = x^{\ell'}$ and $y' = y^{k'}$. Then $[x', y'] = c^{k'\ell'} \neq 1$ because $(k', p) = (\ell', p) = 1$, and

$$(x')^{p^r} = x^{\ell' p^r} = \left(x^{p^r}\right)^{\ell'} = c^{k'\ell'} = \left(y^{p^s}\right)^{k'} = y^{k' p^s} = (y')^{p^s}.$$

Replacing x by x' and y by y', we can consequently assume that $x^{p^r} = c = y^{p^s}$ — note that $m = p^{r+1}$ and $n = p^{s+1}$. Also, without loss of generality, we may assume $r \geq s \geq 1$.

Put $y_1 = x^{-p^{r-s}}y$. Then $[x, y_1] = [x, y] = c$, so the minimality of $m + n$ yields that $o(y_1) \geq o(y) = p^{s+1}$, which means that $y_1^{p^s} \neq 1$. Now, by Lemma 1.1,

$$y_1^{p^s} = x^{-p^r} y^{p^s} \left[y, x^{-p^{r-s}}\right]^{\binom{p^s}{2}} = c^{p^{r-s} \cdot \frac{p^s(p^s-1)}{2}} = c^{\frac{p^r(p^s-1)}{2}}.$$

If $p > 2$, then $y_1^{p^s} = 1$, a contradiction. Thus, $p = 2$ and $2^{r-1}(2^s - 1)$ is odd, so $r = 1$ and consequently $s = 1$. It follows that Q has order 8, so $Q \simeq Q_8$ by Lemma 2.4.

Let $C = C_G(Q)$ and suppose there is an element $g \in G \setminus CQ$. Then either $y^g \neq y$ or $x^g \neq x$. Without loss of generality, we may assume $y^g \neq y$, so $y^g = y^{-1}$, and hence $y^{gx} = (y^{-1})^x = y$. Moreover, $[gx, x] \neq 1$ otherwise $gx \in C$ and $g \in CQ$, a contradiction, and $gx \in G \setminus CQ$. Thus, replacing g by gx and y by x, we obtain that $[gxy, x] = 1$ by an argument similar to the previous one. But $(y^{gx})^y = y$ and so $gxy \in C$, which means that g belongs to CQ, a contradiction. Therefore $G = CQ$.

Let g be any element of C. Then $[x, gy] = [x, y] \neq 1$, so gy is periodic by the argument at the beginning of the proof. In particular, G is periodic. Moreover, if $o(g) = 4$, then the minimality of $o(x) + o(y)$ yields that $o(gy) = 4$, so $\langle x, gy \rangle \simeq Q_8$ as above, and hence

$$gy^{-1} = gy^x = (gy)^x = (gy)^{-1} = g^{-1}y^{-1}.$$

Thus, $g = g^{-1}$, and g has order 2, a contradiction.

Now, if C is non-abelian, then it must contain a copy of Q_8 by what we have proved so far, and so it would contain elements of order 4, a contradiction. Thus, C is abelian. Let D and E_1 respectively be the Sylow $2'$-subgroup and the Sylow 2-subgroup of C. We have that

$G = CQ = (QE_1) \times D$. Since E_1 is elementary abelian, so we can write $E_1 = (Q \cap E_1) \times E$. Thus, $G = Q \times E \times D$.

Suppose conversely that G satisfies the conditions in the statement, and let X be a subgroup of G. Clearly, $X = (X \cap QE) \times (X \cap D)$, so we may assume that $X \leq QE$ because D is abelian and $X \cap D \trianglelefteq G$. Now, if $X \cap G' = \{1\}$, then $X \simeq XG'/G' \leq QE/G'$ has exponent at most 2, so $X \leq \Omega_1(G)$; but $\Omega_1(G) \leq Z(G)$, and hence $X \trianglelefteq G$. On the other hand, if $X \cap G' \neq \{1\}$, then $G' = X \cap G' \leq X$ because G' has order 2, and so $X \trianglelefteq G$. In any case, X is normal in G and the statement is proved. □

Corollary 3.2. *Every Dedekind group is nilpotent of class at most* 2.

Now, a group is *metahamiltonian* if all its subgroups are either normal or abelian. Obviously, this class of groups comprises all Dedekind groups, all minimal non-abelian groups, and also all groups whose derived subgroup has prime order. Metahamiltonian groups were introduced in 1962 by Romalis [19] and their complete description (in the soluble case) needed the joint work of many Russian mathematicians. However, most of their papers are nowadays difficult to find and to read, even if you know a bit of Russian. To make the matter worse, some of those papers contain several, and sometimes serious, mistakes. This is why we refer the interested reader to [3], where the structure of soluble metahamiltonian groups is stated and proved in detail, putting patches to all the mistakes in the literature. Unfortunately, as one can see from [3], it would be a very lengthy (and boring) job even to state all the structural results for soluble metahamiltonian groups. Thus, we essentially content ourselves with proving that the commutator subgroup of a locally graded metahamiltonian group is finite of prime power order, a fact that is interesting in its own and does not require lengthy and boring structural results (the definition of locally graded group has been given in the previous chapter, just before Corollary 2.2). In order to prove this theorem, we need some preliminary lemmas and remarks.

Remark 3.3. The class of metahamiltonian groups is obviously closed with respect to forming subgroups and homomorphic images. Also, if N is any non-abelian subgroup of a metahamiltonian group G, then $N \trianglelefteq G$ and G/N is Dedekind because every subgroup H of G containing N is non-abelian and so normal in G — this fact will frequently be used without further notice.

Lemma 3.4. *Let G be a metahamiltonian group. If G is residually finite, then G is either nilpotent of class 2 or it is abelian-by-finite.*

Proof. Suppose G is not abelian-by-finite, and let N be any finite-index normal subgroup of G. Then every subgroup containing N is non-abelian, so it must be normal in G. It follows that G/N is a Dedekind group, so it is nilpotent of class at most 2 (see Corollary 3.2). Therefore $\gamma_3(G) \leq N$. The arbitrariness of N yields that $\gamma_3(G) = \{1\}$, so G is nilpotent of class 2. □

Lemma 3.5. *Let G be a metahamiltonian group. Then every finitely generated normal torsion-free abelian subgroup A of G is contained in $Z(G)$.*

Proof. Let $x \in G$. First suppose that $A \cap \langle x \rangle = \{1\}$. If for every prime p there exists $n_p \in \mathbb{N}$ such that $[A^{p^{n_p}}, x] = \{1\}$, then $[A, x] = \{1\}$ because $A = A^{p^{n_p}} A^{q^{n_q}}$ whenever p and q are distinct prime numbers. Thus, we may assume that there is a prime r such that $[A^{r^n}, x] \neq \{1\}$ for every $n \in \mathbb{N}$. It follows that $\langle x \rangle A^{r^n}/A^{r^n}$ is a normal subgroup of G/A^{r^n} for every $n \in \mathbb{N}$, and hence $[A, x] \leq A^{r^n}$ since $A \cap \langle x \rangle = \{1\}$. Now, $[A, x] \leq \bigcap_{n\in\mathbb{N}} A^{r^n} = \{1\}$ and we are done.

Assume now that $\langle x^u \rangle = A \cap \langle x \rangle \neq \{1\}$, where $u \in \mathbb{N}$. Then $A/\langle x^u \rangle = T/\langle x^u \rangle \times B/\langle x^u \rangle$, where $T/\langle x^u \rangle$ is finite and $B/\langle x^u \rangle$ is torsion-free. Since $\langle x^u \rangle \leq Z(\langle x \rangle A)$, so $T \leq Z(\langle x \rangle A)$. In fact, if $g \in T$, then there is $\ell \in \mathbb{N}$ with $g^\ell \in \langle x^u \rangle$, and so $1 = [x, g^\ell] = [x, g]^\ell$, which means $[x, g] = 1$ because A is torsion-free. Now, the case $A \cap \langle x \rangle = \{1\}$ shows that $[A, x] \leq T$, so $[A, x, x] = \{1\}$. Consequently, $[a, x]^u = [a, x^u] = 1$ for all $a \in A$, and hence also $[a, x] = 1$. Thus, $[A, x] = \{1\}$ and the statement is proved. □

Lemma 3.6. *Let G be a metahamiltonian group. If G' is finite, then G' is a p-group for some prime p.*

Proof. Since G' is finite, there exists a finitely generated subgroup $E = \langle x_1, \ldots, x_n \rangle$ of G such that $E' = G'$. Now, every element g of E has finitely many conjugates in E, so $|E : C_E(g)|$ is finite. Consequently, the index $\left|E : \bigcap_{i=1,\ldots,n} C_E(x_i)\right|$ is finite and hence $E/Z(E)$ is finite because $\bigcap_{i=1,\ldots,n} C_E(x_i) \leq Z(E)$. Since E is finitely generated, so is $Z(E)$, and hence $Z(E)$ has a finite-index torsion-free abelian subgroup A. In order to prove that G' is a p-group for some prime p, it is therefore enough to prove that $E'A/A \simeq E' = G'$ is a p-group for some prime p. Thus, replacing G by E/A, we can assume G is finite, and we may let G be a minimal counterexample.

Suppose there exists a prime q for which G has a non-trivial normal Sylow q-subgroup P. The Schur–Zassenhaus Theorem yields that $G = Q \ltimes P$ for some p'-subgroup Q. If Q is abelian, then $G' \leq P$, a contradiction. Thus, Q is normal in G and so $G = Q \times P$. Now, if P is abelian, then $G' = Q'$ is a q-group for some prime q by minimality of G. We may therefore assume that both P and Q are non-abelian. But then both $P \simeq PQ/Q$ and $Q \simeq QP/P$ are Hamiltonian, so their derived subgroups have order 2, and hence $G' = P' \times Q'$ has order 4, a contradiction.

Assume there is no non-trivial normal Sylow q-subgroup of G for any prime q. Note that if M is any non-trivial normal subgroup of G, then $G'M/M$ is a q-group for some prime q by minimality of G, so any Sylow q-subgroup of G/M is normal; a similar argument holds for any proper subgroup of G.

Suppose there exists a prime q and a Sylow q-subgroup Q such that $C_G(Q) < N_G(Q)$. Then $\langle g \rangle Q$ is a non-abelian subgroup of G for every $g \in N_G(Q) \setminus C_G(Q)$, so $\langle g \rangle Q \trianglelefteq G$; but Q is a normal Sylow q-subgroup of $\langle g \rangle Q$ and hence Q is normal in G, a contradiction. Thus, $C_G(Q) = N_G(Q)$ for every prime q and every Sylow q-subgroup Q; in particular, every Sylow subgroup of G is abelian.

By Lemma 2.3 and Theorem 2.11, G cannot be minimal non-abelian, so there is a proper non-abelian subgroup N, which is hence normal in G. If N were nilpotent, then we could even assume it is of prime power order, but then G/N would be nilpotent, and so G would have a non-trivial normal Sylow subgroup. Thus, no proper non-abelian normal subgroup can be nilpotent. Since we may also assume that N is minimal non-abelian, so $N = K \ltimes H$, where K is a Sylow r-subgroup of N and H is a Sylow s-subgroup for distinct primes r and s (see Theorem 2.11). Now, $C = C_G(H)$ is a proper normal subgroup of G containing H. Moreover, G/C is an s'-group. In fact, if S is any Sylow s-subgroup of G, then $H \leq S$ because H is characteristic in N and so normal in G; moreover S is abelian, so $S \leq C$, but SC/C is a Sylow s-subgroup of G/C, and hence G/C is an s'-group.

By minimality, C has a normal Sylow u-subgroup U for some prime u, which is consequently normal in G. Clearly, if $u = s$, then U is also a Sylow s-subgroup of G, and this is a contradiction. Thus, $u \neq s$, so in particular C cannot be abelian. Now, G/C is nilpotent and we let V/C be the Sylow r'-subgroup of G/C. If V/C is non-trivial, then $H \leq V'$, so V has a normal Sylow s-subgroup, and hence also G has a normal Sylow s-subgroup, a contradiction. Thus, $V = C$ and G/C

is an r-group. If $u \neq r$, then U is a normal Sylow u-subgroup of G, a contradiction. Consequently $u = r$.

Let R be any Sylow r-subgroup of G containing U, and let S be a Sylow s-subgroup of G. By minimality, we have that $SU \trianglelefteq G$ because $HU/U \leq G'/U$. Thus, $W = SR$ is a subgroup of G. If $S < C_W(S) = N_W(S)$, then $C_W(S)$ contains r-elements, which are then contained in C and so in U. But the set of these r-elements is precisely $L = Z(SU) \cap U$, so by minimality SL/L is normal in G/L and hence S is normal in G because S is characteristic in $S \times L$. Therefore $S = N_W(S)$. Consider $\overline{W} = W/C_S(R)$. If $RC_S(R)/C_S(R)$ is properly contained in its normalizer in $\overline{W}$, then $RC_S(R) = R \times C_S(R)$ is normalized by some s-element of S, so R is normalized (and hence centralized) by the same element, which means that this element is contained in $C_S(R)$. Since $S = N_W(S)$, so $S/C_S(R) = N_{\overline{W}}\big(S/C_S(R)\big)$. Thus, all the Sylow subgroups of $\overline{W}$ coincide with their normalizers, and hence we obtain a contradiction by Theorem 1.11. $\square$

Remark 3.7. The main bulk of the proof of Lemma 3.6 could be avoided by using a special case of a well-known p-nilpotency criterion due to Burnside (see [18], 10.1.8). We leave the details as an exercise for the reader.

Theorem 3.8. *Let G be a locally graded metahamiltonian group. Then G is soluble of derived length at most 3 and G' is finite of prime power order.*

Proof. Let M be the intersection of all non-abelian subgroups of G. Then M is normal, being the intersection of normal subgroups of G, and all proper subgroups of M (if any) are abelian. Thus, either M is abelian, or it is minimal non-abelian and so soluble by Corollary 2.2. In any case, M is soluble. If N is any non-abelian subgroup of G, then $G'' \leq N$ by Corollary 3.2. Thus $G'' \leq M$ and G/M is soluble. It follows that G is soluble.

Now, we prove that G' is finite. By induction on the derived length of G, we have that G'' is finite, so we may assume that G is metabelian (replacing G by G/G''). If E is any non-abelian finitely generated subgroup of G, then E is normal in G, and $G'E/E$ has order at most 2 (see Theorem 3.1). Now, E is residually finite by Theorem 1.7, so it is either nilpotent of class 2 or abelian-by-finite (see Lemma 3.4). In both cases, E satisfies the maximal condition on subgroups, so not only E but also all its subgroups are finitely generated. Thus, $E \cap G'$ is finitely generated and consequently G' is finitely generated. Therefore, without loss of generality we may assume that G is finitely generated.

If G is abelian-by-finite, then G has a normal torsion-free abelian subgroup A of finite index. By Lemma 3.5, $A \leq Z(G)$, so G' is finite by Schur's Theorem. If G is not abelian-by-finite, then every normal subgroup N of G of finite index is non-abelian, so $G'N/N$ has order at most 2 and hence $(G')^2 \leq N$. Since G is residually finite by Theorem 1.7, so $(G')^2 = \{1\}$ and G' has exponent at most 2. Thus, G' is an elementary abelian 2-group and actually a finite one, being finitely generated. Therefore G' is finite, and its order is a prime power by Lemma 3.6.

Finally, if $G^{(3)} \neq \{1\}$, then there are $a, b \in G''$ with $[a, b] \neq 1$. Thus, $G'/\langle a, b\rangle$ has order 2, so $G'' = \langle a, b\rangle$. But G' is nilpotent, and hence also $G'/\operatorname{Frat}(G)$ has order 2, which means that G' is cyclic, a contradiction. Therefore $G^{(3)} = \{1\}$ and the statement is proved. □

Corollary 3.9. *Let G be a locally graded metahamiltonian group. If G is not periodic, then G' is abelian.*

Proof. By Theorem 3.8, G' is a finite p-group for some prime p. If G' is non-abelian and X is any non-abelian subgroup of G', then $X \trianglelefteq G$ and G/X is abelian by Theorem 3.1, being non-periodic. Thus $G' = X$, and G' is minimal non-abelian. By Theorem 2.5, we can write $G' = \langle a, b\rangle$ for some $a, b \in G$. Since G is non-periodic, so there exists an aperiodic element $x \in C_G(G')$, because $G/C_G(G')$ is finite. Put $y = x^8$. Clearly, $\langle ay, b\rangle$ is non-abelian, so $G/\langle ay, b\rangle$ is Dedekind. On the other hand, $G/\langle ay, b\rangle$ contains elements of order 8 because $x\langle a, b, y\rangle$ has order 8 in $G/\langle a, b, y\rangle$. Thus, Theorem 3.1 shows that $G/\langle ay, b\rangle$ is abelian, so $G' \leq \langle ay, b\rangle$, and hence $y \in \langle ay, b\rangle$. This is impossible because y does not belong to $\langle ay, b\rangle$. □

There are many interesting results that would be at least worth quoting about metahamiltonian groups. For example, if G is a finite metahamiltonian p-group for some prime p, then G' is either elementary abelian of order p^3, or it is isomorphic to $\mathbb{Z}_{p^\varepsilon} \times \mathbb{Z}_{p^n}$ for some $n \in \mathbb{N}_0$ and $\varepsilon \in \{0, 1\}$. However, the following result is probably the most intriguing and the most useful one when it comes to work with metahamiltonian groups

Theorem 3.10. *Let G be a locally graded metahamiltonian group. Then every non-abelian subgroup of G contains G'.*

We wish to end this chapter by proving a result connecting finite metahamiltonian and minimal non-abelian groups, and by briefly describing the structure of non-soluble metahamiltonian groups.

Theorem 3.11. *Let p be a prime. If G is a locally finite metahamiltonian p-group and $x \in G$, then x^G is either abelian or minimal non-abelian.*

Proof. Suppose $X = x^G$ is non-abelian. Note that if H is a proper subgroup of X such that $x^g \in H$ for some $g \in G$, then H is abelian, because otherwise $x^G \leq H \trianglelefteq G$. Thus, if x^g is a conjugate of x such that $[x, x^g] \neq 1$, then $X = \langle x, x^g \rangle$.

Now, since X is non-abelian, so there are conjugates x^{g_1} and x^g of X such that $[x^{g_1}, x^g] \neq 1$ — conjugating G by g_1^{-1}, we may assume $g_1 = 1$. Clearly, $\langle x \rangle X'$ and $\langle x^g \rangle X'$ are proper subgroups of X, so they are abelian, and hence $X' \leq Z(X)$ because $X = \langle x, x^g \rangle$. Let n be the largest positive integer such that $\langle x, y \rangle$ is non-abelian, where $y = (x^g)^{p^n}$ — of course, $X = \langle x, y \rangle$ by the remark above. Then $1 = [x, y^p] = [x, y]^p$, so $X' = \langle [x, y] \rangle$ has order p, and hence X is minimal non-abelian by Corollary 2.7. □

Lemma 3.12. *Let $G = H \times K$ be a direct product of two copies H and K of Q_8. If X is a subgroup of G such that $|X'| = 4$, then X is not metahamiltonian.*

Proof. Suppose that X is metahamiltonian, and write $H = \langle a, b \rangle$ and $K = \langle c, d \rangle$ for some elements $a, b, c, d \in G$. Since $\langle ac, bd \rangle \simeq Q_8$ is not normal in G, so $X < G$. Moreover, every 2-generator subgroup of G has a cyclic commutator subgroup. Thus X is generated by 3 elements and not less than 3. If X contains H, then we easily see that $K \leq X$, so $X = G$, a contradiction — a similar contradiction can be found if X contains K. Thus, X does not contain H nor K.

Let $\{x, y, z\}$ be a set of generators of X. Since X contains G', we can always assume that $x = a$, and that $y = bc$. Consequently, we may assume $z = du$, where $u = b$ or $u = 1$. If $u = b$, then we can replace z by cd, which means that we may actually assume $z = d$ also in this case (recall that the roles of the cyclic subgroups of order 4 in Q_8 are interchangeable). Thus, without loss of generality, we have $X = \langle a, bc, d \rangle$. Now $X = \langle bc \rangle \big(\langle a \rangle \times \langle d \rangle \big)$, $\langle a \rangle \trianglelefteq G$, $\langle d \rangle \trianglelefteq G$, and $\langle bc \rangle \cap \langle a, d \rangle = \langle (bc)^2 \rangle$. Now, $\langle bc, ad \rangle$ is non-abelian so is normal in X, but this is impossible because $\langle bc, ad \rangle$ has trivial intersection with $\langle d \rangle$, so $X = \langle bc, ad \rangle \times \langle d \rangle$ and hence X' is cyclic, the last contradiction. □

Theorem 3.13. *Let G be a non-soluble group. Then G is metahamiltonian if and only if there is an infinite minimal non-abelian subgroup M of G' that is contained in every non-abelian subgroup of G, and G/M is Dedekind.*

Proof. Since the sufficiency is obvious, we only have to deal with the necessity. Assume therefore that G is metahamiltonian. If G' is finite, then

G is locally graded and so soluble by Theorem 3.8, a contradiction. Thus, G' is infinite and clearly non-abelian. Now, let M be the intersection of all the non-abelian subgroups of G; in particular, $M \leq G'$. Clearly, G'/M has exponent at most 2 because it is residually of exponent 2, so G'/M is elementary abelian. If G'/M does not have order 2, then there are non-abelian subgroups N_1 and N_2 of G such that $G/(N_1 \cap N_2)$ has a non-cyclic derived subgroup. Since $G/(N_1 \cap N_2)$ is isomorphic to a subgroup of $G/N_1 \times G/N_2$, and G/N_i is Dedekind for $i = 1, 2$, so there exists a homomorphic image of G that is isomorphic to a subgroup X of $Q_8 \times Q_8$ such that $|X'| = 4$, which is impossible by Lemma 3.12. Therefore G'/M is cyclic of order at most 2, and M is infinite non-abelian. In particular, G/M is Dedekind, and the statement is proved. □

4 JUST NON-ABELIAN GROUPS

In this final chapter, we study *just non-abelian* groups, that is, non-abelian groups G such that G/N is abelian for every non-trivial normal subgroup N of G. These groups can be considered as duals of the minimal non-abelian groups, but here the situation is much more complex even in the finite case because there exist "large" groups with very few non-trivial normal subgroups (for example, the non-abelian simple groups). Just non-abelian groups have first been considered by B.H. Neumann [12] in 1956, and were later studied (in the soluble case) by Newman [13],[14] in 1959 — note that in this context "soluble" obviously implies "metabelian". In [13], Newman considered metabelian just non-abelian groups with trivial centre, while in [14], he studied the case in which the centre is non-trivial (so these groups are nilpotent of class 2). It is proved in [13] that metabelian just non-abelian groups with trivial centre can be completely characterized as natural semidirect products involving fields, while it is proved in [14] that a necessary and sufficient condition for a nilpotent group G to be just non-abelian is that there exists a prime p such that $|G'| = p$ and $Z(G)$ is either a cyclic or a Prüfer p-group. However, since the complete characterization of the nilpotent case is a bit too complex, and so outside the scope of this chapter, we are just going to present here the former work of Newman in the case of soluble just non-abelian groups with trivial centre.

Let's start by giving examples of soluble just non-abelian groups. Of course, for every prime p, a group of order p^3 is just non-abelian (and nilpotent). On the other hand, the symmetric group of degree 3 and the alternating group of degree 4 are just non-abelian with a trivial centre. More sophisticated examples can be constructed as follows. Let $\mathcal{K}$ be a field, and let Λ be a non-trivial subgroup of the multiplicative group $\mathcal{K}^\times$ of $\mathcal{K} \setminus \{0\}$. Now, let $G_{\Lambda,\mathcal{K}} = (\Lambda, \cdot) \ltimes (\mathcal{K}, +)$ be the natural semidirect product of $(\mathcal{K}, +)$ by $(\Lambda, \cdot)$, where the action is by field multiplication. Clearly, $G_{\mathcal{K}}$ has a trivial centre and is metabelian.

Remark 4.1. $G_{\Lambda,\mathcal{K}}$ is evidently isomorphic to the group of *linear inhomogeneous substitutions*

$$x \mapsto \lambda x + \omega,$$

where λ ranges in Λ and ω in $\mathcal{K}$.

Remark 4.2. In dealing with $G_{\Lambda,\mathcal{K}}$, we are always going to consider $(\Lambda, \cdot)$ and $(\mathcal{K}, +)$ as the obvious subgroups of $G_{\Lambda,\mathcal{K}}$, so they are set of pairs. If we do not wish to consider them as such, then we simply write Λ and $\mathcal{K}$.

Theorem 4.3. *The group $G = G_{\Lambda,\mathcal{K}}$ is just non-abelian if and only if the subgroup generated by Λ in $(\mathcal{K}, +)$ is $\mathcal{K}$.*

Proof. Let $S = \langle (1, \lambda) : \lambda \in \Lambda \rangle \leq (K, +)$. Clearly, S is a normal subgroup of G, and the condition in the statement can simply be rephrased by saying that $S = (K, +)$ as subgroups of G.

First, suppose that G is just non-abelian. For every $\lambda \in \Lambda$,

$$(1, -1) \cdot (1, 1)^{(\lambda,0)} = (1, \lambda - 1) \in S.$$

If $(1, x) \in (K, +)$ and $1 \neq \lambda \in \Lambda$, then $(1, x\lambda) = (1, x)$ modulo S, so $\big(1, x(\lambda - 1)\big) \in S$. Since $(1, \lambda - 1) \in S$, then $S = (K, +)$ by the arbitrariness of x.

Assume conversely that $S = (K, +)$, and let (λ, ν) be a non-trivial element of G. We claim that the normal closure N of $\langle (\lambda, \nu) \rangle$ in G contains $(K, +)$. If $\lambda \neq 1$, then

$$\big[(1, \omega(\lambda - 1)^{-1}), (\lambda, \nu)\big] = (1, \omega) \in N$$

for every $\omega \in K$. Assume $\lambda = 1$, and let $\omega \in K$. Since $S = (K, +)$, so $\nu^{-1}(\nu + \omega) = \sum_{i=1}^{n} \lambda_i$ for some elements $\lambda_i \in \Lambda$. Put

$$c = \prod_{i=1}^{n} (1, \nu)^{(\lambda_i,0)} = \Big(1, \nu \cdot \sum_{i=1}^{n} \lambda_i\Big) = (1, \nu + \omega).$$

Then $(1, \nu)^{-1}c = (1, \omega) \in N$. In any case, G/N is abelian and the statement is proved. $\square$

Corollary 4.4. *The group $G_{\mathcal{K}^\times,\mathcal{K}}$ is a metabelian just non-abelian group with trivial centre.*

The main result of [13] shows that *every* metabelian just non-abelian group with trivial centre is isomorphic to a group $G_{\Lambda,\mathcal{K}}$ for some field $\mathcal{K}$ and some subgroup Λ of $\mathcal{K}^\times$ — it is precisely this result that we are going to prove, but first we need some preliminary remarks and lemmas.

Remark 4.5. If G is a metabelian just non-abelian group, then G' has no non-trivial proper characteristic subgroups. In fact, if X is such a subgroup, then $X \trianglelefteq G$ and G/X is abelian, which means that $G' \leq X$, a contradiction. In particular, G' is either periodic or torsion-free.

Lemma 4.6. *Let G be a metabelian just non-abelian group. Then G' is either an elementary abelian p-group for some p, or it is a direct product of copies of the additive group of the rational numbers.*

Proof. First, suppose that there exists a prime p such that $(G')^p = \{1\}$. In this case, G' has exponent p and so is elementary abelian. Thus, we may assume that $(G')^q \neq \{1\}$ for every prime q. Now, Remark 4.5 yields that $G' = (G')^q$ for every prime q, which means that G' is divisible. Therefore G' is either a direct product of Prüfer groups or it is a direct product of copies of the additive group of the rational numbers. However, in the former case, the subgroup generated by all the elements of prime order is proper and non-trivial, contradicting Remark 4.5. Thus, G' is a direct product of copies of $(\mathbb{Q}, +)$ and we are done. □

Remark 4.7. All the possibilities expressed in the statement of the previous lemma for the derived subgroup of a metabelian just non-abelian group can actually be realized, except for the cyclic group of order 2. To see this, you just need to take into account the groups $G_{\mathcal{K}}$ constructed above, and to notice that a derived subgroup of order 2 implies that the group is nilpotent.

Lemma 4.8. *Let G be a group. If N is any subgroup of G such that $G' \leq N \leq C_G(G')$, then $C_G(x) \cap N$ and $[x, N]$ are normal subgroups of G for every $x \in G$. Moreover, $[x, N] = \{[x, u] \, : \, u \in N\}$.*

Proof. Let $g \in G$ and $a \in C_G(x) \cap N$. Then

$$[x, a^g] = [x, g^{-1}ag] = [x, ag][x, g^{-1}]^{ag} = [x, g][x, g^{-1}]^{ag}.$$

But a belongs to $C_G(G')$, and hence

$$[x, g][x, g^{-1}]^{ag} = [x, g][x, g^{-1}]^g = [x, gg^{-1}] = 1.$$

Thus, $a^g \in C_G(x)$. Since obviously $a^g \in N$, so $C_G(x) \cap N \trianglelefteq G$.

Now, if $u, v \in N$, then

$$[x, uv^{-1}] = [x, u][x, v^{-1}]^u = [x, u][x, v^{-1}] = [x, u][x, v]^{-1}$$

because $N \leq C_G(G')$. Thus, $[x, N]$ is the *set* of all commutators $[x, u]$ for $u \in N$. Finally, if $g \in G$ and $a \in N$, then

$$[x, a]^g = [x^g, a^g] = \big[[g, x^{-1}]x, a^g\big] = [x, a^g]$$

because $G' \leq N$ and a^g centralizes G'. Therefore $[x, N] \trianglelefteq G$ and the statement is proved. □

Let G be a group. The maps

$$\varphi_x : a \in G' \mapsto [x, a] \in G',$$

where $x \in G$, will play a crucial role from now on.

Lemma 4.9. *Let G be a metabelian just non-abelian group with trivial centre. If $x \in G \setminus C_G(G')$, then the map φ_x is an automorphism, and $G = C_G(x) \ltimes G'$.*

Proof. The map φ_x is easily seen to be a homomorphism. Now, Lemma 4.8 yields that

$$\mathrm{Ker}(\varphi_x) = C_{G'}(x) = C_G(x) \cap G' \quad \text{and} \quad (G')^{\varphi_x} = [x, G']$$

are normal subgroups of G. Since $[x, G'] \neq \{1\}$, so $[x, G'] = G'$ and $\mathrm{Ker}(\varphi_x) = \{1\}$. Thus, for every $g \in G$, $\big([x, g]^{-1}\big)^{\varphi_x^{-1}}$ is the only element a of G' such that $[x, g][x, a] = 1$. But the previous equality means that $[x, ga] = 1$, so ga is the unique element of gG' centralizing x. Therefore $G = C_G(x) \cdot G'$ and $C_{G'}(x) = \{1\}$. □

Lemma 4.10. *Let G be a metabelian just non-abelian group with trivial centre. Then $G' = C_G(G')$.*

Proof. Let $x \in G \setminus C_G(G')$. The map

$$\psi : z \in C_G(G') \mapsto [x, z] \in G'$$

is a homomorphism. By Lemma 4.8, the image of ψ is a normal subgroup of G, so ψ is surjective. By the same lemma, also the kernel K of ψ is normal in G, so if $K \neq \{1\}$, then $G' \leq K$. On the other hand, G' is not contained in K because $x \notin C_G(G')$, so $K = \{1\}$ and hence ψ is an isomorphism. Now, the restriction of ψ to G' is also an isomorphism by Lemma 4.9, and hence $G' = C_G(G')$. □

Lemma 4.11. *Let G be a metabelian just non-abelian group with trivial centre. The complements of G' in G are precisely the centralizers of elements $x \in G \setminus C_G(G')$. Moreover, every such complement coincides with its own centralizer in G.*

Proof. Write $G = C \ltimes G'$ for some subgroup C of G. Then $C \simeq G/G'$ is abelian, so for all $1 \neq c \in C$, we have that $C_G(c) \geq C$. Now, $c \notin C_G(G')$, so $C_G(c)$ is a complement of G' in G by Lemma 4.9. Since

$$C_G(c) = CG' \cap C_G(c) = C\big(G' \cap C_G(c)\big) = C \cdot C_{G'}(c),$$

it follows that $C_{G'}(c) = \{1\}$ (recall that $C_G(c)$ is a complement of G') and hence $C_G(c) = C$. This shows at once that every complement of G' in G is a centralizer in G of an element outside $C_G(G')$, and that these complements coincide with their own centralizer in G. □

Lemma 4.12. *Let G be a metabelian just non-abelian group with trivial centre. The complements of G' in G are maximal subgroups of G.*

Proof. Clearly, every subgroup D of G containing a complement C of G' in G is such that $D \cap G'$ is normal in $G = CG'$. On the other hand, G' has no non-trivial proper G-invariant subgroups and hence every complement of G' in G is a maximal subgroup of G. □

Lemma 4.13. *Let G be a metabelian just non-abelian group with trivial centre. There exists a bijection between the elements of G' and the complements of G' in G.*

Proof. Let $x \in G \setminus C_G(G')$. We claim that the assignation $u \mapsto C_G(x)^u$ defines a bijection τ of G' onto the set of complements of G' in G.

If $y \in G \setminus C_G(G')$, then φ_x and φ_y are automorphisms by Lemma 4.9. Also, the inversion ψ is an automorphism of G', so the composition $\varphi_x \circ \varphi_y \circ \psi$ is an automorphism as well. Thus, there exists an element $g \in G'$ such that $[x, g, y] = [y, x]$. Now,

$$[x^g, y] = \big[x[x,g], y\big] = [x,y][x,g,y] = [x,y][y,x] = 1,$$

so $x^g \in C_G(y)$. If $z \in C_G(x)$, then $[z^g, x^g] = 1$, so

$$[y, z^g, x^g] = [x^g, y, z][y, z^g, x^g][z^g, x^g, y] = 1$$

by Lemma 1.2. On the other hand, the map $\varphi_{x^g} \circ \psi$ is an automorphism of G', and hence $[y, z^g] = 1$, which means that $z^g \in C_G(y)$. Therefore $C_G(x)^g \leq C_G(y)$, and by maximality $C_G(x)^g = C_G(y)$ (see Lemma 4.12). Thus, τ is surjective.

If $C_G(x)^{g_1} = C_G(x)^{g_2}$ for some $g_1, g_2 \in G'$, then $g_1 g_2^{-1}$ is an element of G' normalizing $C_G(x)$. But $G = C_G(x) \ltimes G'$, and hence $g_1 g_2^{-1}$ centralizes $C_G(x)$. Thus, $g_1 = g_2$ by Lemma 4.11. Therefore τ is injective and the statement is proved. □

Let G be a metabelian just non-abelian group with trivial centre. We need to associate a field to G. For each $g \in G$, let $\overline{g}$ be the inner automorphism of G induced by g. Clearly, the restriction of $\overline{g}$ to G' defines an automorphism $g^{\mu'}$ of G', and the map

$$\mu' : g \in G \mapsto g^{\mu'} \in \mathrm{Aut}(G')$$

is a homomorphism. Since $G' = C_G(G')$ by Lemma 4.10, so $\mathrm{Ker}(\mu') = G'$. Let $\Theta_G = \Theta$ be the subring of the endomorphism ring of G' generated by $\Gamma_G = G^{\mu'}$, and note that Θ is a commutative ring.

Remark 4.14. Since $G^{\mu'}$ is a group, so Θ is the additive closure of $G^{\mu'}$.

Now, G' can, in a natural way, be regarded as a *simple* Θ-module (see Remark 4.5) — that is, G' has no non-trivial proper Θ-submodule. Using Remark 4.14, we easily see that G' is a *faithful* Θ-module, that is, for every $0 \neq \theta \in \Theta$, there exists $x \in G'$ such that $x^\theta \neq 1$.

Lemma 4.15. $(\Theta, +, \circ)$ *is a field.*

Proof. We only need to show that every element of Θ has a multiplicative inverse. The set S of all elements g of G' with $g^\theta = 1$ for all $\theta \in \Theta$ is a proper Θ-submodule of G' (because G' is a faithful Θ-module), so S is trivial. Thus, if $1 \neq x \in G'$, then $x^\Theta = \{x^\theta : \theta \in \Theta\}$ is a non-trivial Θ-submodule of G', so $x^\Theta = G'$. Choose $\tau \in \Theta$ such that $x^\tau = 1$. Then

$$1 = x^{\tau\circ\Theta} = \{x^{\tau\circ\theta} : \theta \in \Theta\} = \left(x^\Theta\right)^\tau = (G')^\tau,$$

so $\tau = 0$.

Now, if $\alpha, \beta \in \Theta$ with $\alpha \neq 0$, then $x^\alpha \neq 1$ and so

$$\left(x^\alpha\right)^\Theta = G'.$$

Then there exists $\gamma \in \Theta$ such that $\left(x^\alpha\right)^\gamma = x$. Thus, $x^{\alpha\circ\gamma-1} = 1$ and consequently $\alpha \circ \gamma = 1$, which means that α is invertible. □

Remark 4.16. Let $\mathcal{K}$ be a field and Λ a subgroup of $\mathcal{K}^\times$ such that the subgroup generated by Λ in $(\mathcal{K}, +)$ is $\mathcal{K}$. Then $\Theta_{G_{\Lambda,\mathcal{K}}} \simeq \mathcal{K}$.

Remark 4.17. If C is any complement of G' in G, then the restriction $\mu_{G,C}$ of μ' to C is an isomorphism of C and $G^{\mu'}$. Since $G^{\mu'}$ is a multiplicative subgroup of the field Θ, so C is isomorphic to a multiplicative subgroup of Θ. In particular, the periodic subgroups of C are locally cyclic.

Let $1 \neq x \in G'$. For every $g \in G$ there exists $\theta_g \in \Theta$ such that $g = x^{\theta_g}$. It easily follows that the map

$$\alpha_{G,x} = \alpha : g \in G' \mapsto \theta_g \in (\Theta, +)$$

is an isomorphism. Now, if C is any complement of G' in G, $g \in G'$ and $a \in C$, then

$$(g^a)^\alpha = g^\alpha \circ a^{\mu_{G,C}}$$

because $x^{\theta_g \circ a^{\mu_{G,C}}} = g^a$.

Theorem 4.18. *Let G_1 and G_2 be metabelian just non-abelian groups with trivial centre. Then $G_1 \simeq G_2$ if and only if there exists a field isomorphism $\varphi : \Theta_{G_1} \to \Theta_{G_2}$ such that $\left(\Gamma_{G_1}\right)^\varphi = \Gamma_{G_2}$.*

Proof. Since the necessity is obvious, we may assume that there exists a field isomorphism $\varphi : \Theta_{G_1} \to \Theta_{G_2}$ such that $\left(\Gamma_{G_1}\right)^\varphi = \Gamma_{G_2}$. Write $G_1 = C_1 \ltimes G_1'$ and $G_2 = C_2 \ltimes G_2'$ for some subgroups C_1 and C_2 of G (see Lemma 4.9). Let $1 \neq x_1 \in G_1'$, $1 \neq x_2 \in G_2'$ and put

$$\psi = \mu_{G_1,C_1} \circ \varphi \circ \mu_{G_2,C_2}^{-1} \quad \text{and} \quad \tau = \alpha_{G_1,x_1} \circ \varphi \circ \alpha_{G_2,x_2}^{-1}.$$

Then $\psi : C_1 \to C_2$ and $\tau : G_1' \to G_2'$ are isomorphisms, and the assignation

$$(c, g)^\pi = (c^\psi, g^\tau), \quad c \in C_1,\ g \in G_1'$$

defines an isomorphism of G_1 onto G_2. □

Theorem 4.19. *Let G be a metabelian just non-abelian group with trivial centre. Then $G \simeq G_{\Gamma_G, \Theta_G}$.*

Proof. Write $G = C \ltimes G'$ for some subgroup C of G (see Lemma 4.9), and fix a non-trivial element x of G'. Let

$$\tau : G \to G_{\Gamma_G, \Theta_G}$$

be the function mapping every element $g = au$ of G, with $a \in C$ and $u \in G'$, to the pair $\left(a^{\mu_{G,C}}, u^{\alpha_{G,x}}\right) \in G_{\Gamma_G, \Theta_G}$. This function is well-defined because of the properties of the semidirect product. Since the functions $\mu_{G,C}$ and $\alpha_{G,x}$ are bijective, so is τ. Also,

$$\begin{aligned}(g_1 g_2)^\tau &= \left((a_1 a_2)^{\mu_{G,C}}, (u_1^{a_2} + u_2)^{\alpha_{G,x}}\right) \\ &= \left((a_1 a_2)^{\mu_{G,C}}, \left(u_1^{\alpha_{G,x}}\right) \circ a_2^{\mu_{G,C}} + u_2^{\alpha_{G,x}}\right) \\ &= \left(a_1^{\mu_{G,C}}, u_1^{\alpha_{G,x}}\right) \cdot \left(a_2^{\mu_{G,C}}, u_2^{\alpha_{G,x}}\right) = g_1^\tau \cdot g_2^\tau,\end{aligned}$$

for all $g_1 = a_1u_1$ and $g_2 = a_2u_2$ with $a_1, a_2 \in C$ and $u_1, u_2 \in G'$. Therefore τ is an isomorphism and the statement is proved. $\square$

Corollary 4.20. *There exist $2^{\aleph_0}$ non-isomorphic metabelian just non-abelian groups G with trivial centre such that the field Θ_G is isomorphic to the field of rational numbers.*

Proof. Write $\mathbb{P} = (p_i)_{i\in\mathbb{N}}$, and for each $i \in \mathbb{N}$, choose a positive integer λ_i. Let $(\Lambda, \cdot)$ be the subgroup of $(\mathbb{Q}, \cdot)$ generated by the numbers $p_i^{\lambda_i}$, and let $(S, +)$ be the subgroup generated by Λ in $(\mathbb{Q}, +)$ — note that Λ (and consequently S) should be considered as a function of the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$.

Now, let p be a prime, and choose $i \in \mathbb{N}$ such that $p = p_i$. For each positive integer ℓ, the number $1/p^{-\ell\lambda_i}$ belongs to Λ, and hence to S. Since the Sylow p-subgroup of $(\mathbb{Q}, +)/(S, +)$ is generated by the cosets $1/p^{-\ell\lambda_i} + S$, for $\ell \in \mathbb{N}$, so the Sylow p-subgroup of $(\mathbb{Q}, +)/(S, +)$ is trivial. It follows that $(\mathbb{Q}, +) = (S, +)$. Therefore $G_{\Lambda,\mathbb{Q}}$ is a metabelian just non-abelian group with trivial centre by Theorem 4.3.

Finally, since the identity automorphism is the only automorphism of the field of rational numbers, it follows that different choices for the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ give non-isomorphic groups $G_{\Lambda,\mathbb{Q}}$. The statement is proved (see also Remark 4.16). $\square$

The above characterization of metabelian just non-abelian groups with trivial centre allows us to give a more group-theoretic description of such groups in the finite case, but first we need some general preliminary observations about fields.

Let $\mathcal{K}$ be a field of characteristic $q \geq 0$, and let $(\Lambda, \cdot)$ be a subgroup of $(\mathcal{K}^\times, \cdot)$. For every $0 \neq \omega \in \mathcal{K}$, we let d_Λ be the minimal number of generators of the subgroup

$$T(\omega, \Lambda) = \langle \omega\lambda \,:\, \lambda \in \Lambda \rangle$$

of $(\mathcal{K}, +)$. It is easy to see that d_Λ is independent of ω. In fact, if ω_1, ω_2 are non-zero elements of $\mathcal{K}$, then there exists $\nu \in \mathcal{K}$ such that $\omega_1\nu = \omega_2$. Since $\nu \neq 0$, the multiplication by ν induces an isomorphism of $(\mathcal{K}, +)$ under which the subgroups $T(\omega_1, \Lambda)$ and $T(\omega_2, \Lambda)$ correspond. Thus, the minimal number of generators of $T(\omega_1, \Lambda)$ is the same as that of $T(\omega_2, \Lambda)$, and d_Λ is independent of ω.

Lemma 4.21. *Let $\mathcal{K}$ be a field of characteristic $q \geq 0$, and $(\Lambda, \cdot)$ a non-trivial subgroup of $(\mathcal{K}^\times, \cdot)$ of order n.*

- *If* $q = 0$, *then* $d_\Lambda = \phi(n)$.
- *If* $q > 0$, *then* d_Λ *is the smallest positive integer* k *such that* n *divides* $q^k - 1$.

Proof. Since $(\Lambda, \cdot)$ is a finite subgroup of the multiplicative group of a field, so Λ is cyclic. Let λ be a generator of $(\Lambda, \cdot)$, put $d = d_\Lambda$, and let $\mathcal{E} = \mathcal{E}(\mathcal{K})$ be the prime field of $\mathcal{K}$.

Let $0 \neq \omega \in \mathcal{K}$, and let k be the smallest non-negative integer for which $\omega, \omega\lambda, \dots, \omega\lambda^k$ is $\mathcal{E}$-linearly dependent. Clearly, $k \geq 1$. If $k < d$, then $\omega\lambda^k$ can be written as an $\mathcal{E}$-linear combination of the elements $\omega, \dots, \omega\lambda^{k-1}$, and such is every $\omega\lambda^\ell$ for $\ell \geq k$. Now, if $q > 0$, then $\omega, \dots, \omega\lambda^{k-1}$ is a set of generators of $T(\omega, \Lambda)$, contradicting the minimality of d. On the other hand, if $q = 0$, then the subgroup S of $(\mathcal{K}, +)$ generated by the elements $\omega, \dots, \omega\lambda^{k-1}$ has finite index in $T(\omega, \Lambda)$; this shows again that the minimal number of generators of $T(\omega, \Lambda)$ is at most $k - 1 < d$, a contradiction. Therefore $k = d$, so we can write

$$\omega\lambda^d = \sum_{i=0}^{d-1} \pi_i \omega\lambda^i,$$

where $\pi_i \in \mathcal{E}$ for every $i = 1, \dots, d-1$.

Consider the polynomial

$$f(x) \equiv x^d - \sum_{i=0}^{d-1} \pi_i x^i.$$

We claim that this polynomial is irreducible over $\mathcal{E}$. In fact, suppose that $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are polynomials on $\mathcal{E}$, and $h(x)$ is irreducible. Put $\psi = \omega \cdot g(\lambda)$. By minimality of d, we have that $\psi \neq 0$, while obviously $\psi \cdot h(\lambda) = f(\lambda) = 0$. Then the minimal number of generators of $T(\psi, \Lambda)$ is at most the degree of $h(x)$, which means that $h(x)$ has degree d, and so that $g(x)$ has degree 0. The claim is proved.

Therefore d is the degree of an irreducible polynomial of a primitive nth root of unity (which is λ) over a prime field, and the result follows from Theorem 1.5. □

Theorem 4.22. *Let* p *be a prime and let* $\mathcal{K}$ *be a finite field of order* p^k, *where* $k \in \mathbb{N}$. *Let* $(\Lambda, \cdot)$ *be a non-trivial subgroup of* $(\mathcal{K}, \cdot)$ *of order* n. *Then* $G_{\Lambda,\mathcal{K}}$ *is just non-abelian if and only if* k *is the smallest positive integer* h *such that* n *divides* $p^h - 1$.

Proof. It follows from Theorem 4.3 that $G_{\Lambda,\mathcal{K}}$ is just non-abelian if and only if $d_\Lambda = k$. By Lemma 4.21, the latter condition is equivalent to requiring that n divides $p^k - 1$ but not $p^h - 1$ for any positive integer $h < k$. The statement is proved. □

Theorem 4.23. *A finite metabelian just non-abelian group G with trivial centre is extension of an elementary abelian p-group of order p^k (p a prime), by an automorphism of order n, where n divides $p^k - 1$ but not $p^h - 1$ for any positive integer $h < k$.*

Proof. This follows at once from Theorems 4.19 and 4.22. □

Thus, finite metabelian just non-abelian groups with trivial centre can be characterized by two invariants: a prime-power p^k, not equal to 2 (see Remark 4.7), and an integer n which divides $p^k - 1$ but not $p^h - 1$ for all positive integers $h < k$. In fact, for each such a pair (p^k, n), there always exists a finite metabelian just non-abelian group with trivial centre having these invariants: it is enough to consider the group $G_{\Lambda,\mathcal{K}}$, where $\mathcal{K}$ is the field of order p^k and Λ is the subgroup of $(\mathcal{K}, \cdot)$ of order n (see Theorem 4.22) — also, by Theorem 4.18, this is the only such a group up to isomorphisms.

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1. M. Trombetti, *Non-Abelian Groups with Many Abelian Subgroups* (2025)

The present volume is largely based on the lectures from a Ph.D. course I taught in early 2024 at the Dipartimento di Matematica e Applicazioni "Renato Caccioppoli" of the Università degli Studi di Napoli Federico II. The course was titled "Groups with Many Abelian Subgroups." In these notes, we explore the structure of (soluble) non-abelian groups whose proper subgroups are all abelian, commonly referred to as *minimal non-abelian groups*. Additionally, we examine some properties of groups in which every subgroup is either normal or abelian, known as *metahamiltonian groups*. The final chapter focuses on metabelian groups with a trivial center, whose proper quotients are abelian, showing that even in this context, there are numerous abelian subgroups.

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