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Celestial Holography from Euclidean AdS Space

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Preface

Gravity governs the first experienced phenomena in our lives, driving our minds towards the experiential intuition about the free fall of heavy objects on the ground from the outset. Nevertheless, even though gravity is the most instinctive force we encounter in our daily lives, it continues to stand as one of the enduring enigmas in the realm of fundamental physics. Starting from the fall of an apple onto Newton's head, proceeding by rolling into complexities, until deepening our minds in the abysses of the cosmos, the story of Physics has guided us to Einstein's theory of General Relativity, an outstanding theory of gravity describing with accurate precision the majority of gravitational phenomena occurring in our solar system.

Nevertheless, "General Relativity embraces the seed of its own denial: black holes" [1]. As we try to unveil the fundamental theory of gravity, delving into the shortest reachable distances in the Universe, General Relativity predicts gravitational collapses and the subsequent emergence of black holes, apparently hiding within their horizon the fundamental degrees of freedom of gravity themselves.

With great surprise, right in this inevitable breakdown, driven by the unstoppable attractive tendency of gravity, a new interpretation is born. Information about physical system inside a black hole is only apparently lost. Somehow, it reorganises and spreads across the area of the event horizon, allowing for the description of physics inside the black hole in terms of new fundamental binary bits of information. As pointed out by Gerard 't Hooft, this feature is peculiar of gravity and extends to a broader range of phenomena: given a gravitational system bounded within a closed region of the space, we can effectively characterise that system using binary bits of information distributed across the surface of the enclosing region, with no more than one bit per Planck area. This is true for every gravitational system in our space-time, included the Universe itself.

Within this new interpretation, the space-time is visualized as compactified, with the fundamental degrees of freedom of gravity diffuse over an asymptotic boundary region encompassing the entire Universe. This asymptotic region represents the boundary of our the space-time, today also known as *Celestial Sphere*. We therefore deduce that gravity is different from the other fundamental forces in nature: gravity has a *holographic* nature. Citing Gerard ‘t Hooft [2]:

> “The situation can be compared with a hologram of a three dimensional image on a two-dimensional surface.”

In the last 25 years, we have learned how to manage holographic theories of gravity in asymptotic Anti-de Sitter space-time. However, our understanding of holography in other, more physically relevant space-times, such as de Sitter and Minkowski, is still primordial. Nevertheless, significant advancements have been made in recent years, which registered a surge of activity in flat space holography. Celestial Holography is today the leading candidate theory to establish a holographic correspondence for quantum gravity in asymptotically flat space-time, focusing on the imprint of two-dimensional conformal symmetry in four-dimensional scattering amplitudes.

Outline of the Thesis

This thesis is devoted to the study of the connection between Celestial and EAdS Holography, with the overarching goal of gaining a deeper understanding of holographic theories on maximally symmetric space-times. This work relies heavily on the content of articles [3–5], written in collaboration with Wolfgang Mück [3], Charlotte Sleight and Massimo Taronna [4, 5]. By employing the de Boer-Solodukhin hyperbolic foliation of Minkowski space-time [6], we will show that it is possible to place holographic theories in maximally symmetric spaces on an equal footing, revealing connections between these theories and Euclidean AdS holography. The current study relies on two main research lines.

The first one is rooted in the “Cosmological Boostrap” program (here a recent review [7]). Specifically, the authors of [8–10] provided important results in recasting late-time dS correlators in the Bunch-Davies vacuum as a linear combination of corresponding Witten diagrams in Euclidean AdS via suitable analytical continuations within the respective Poincaré patches of the two spaces. This connection has been perturbatively established at all orders and serves as a cornerstone for the analysis conducted in this thesis.

The second line of research has been pursued in the context of Celestial holography (for some reviews [11–15]). In the papers [6, 16–18] the authors recast Celestial correlators in terms of Euclidean AdS ones. Our research also fits in this perspective, furnishing a new technique to reach the same goal. Specifically, in our work we achieve the same result by employing the hyperbolic slicing of Minkowski space-time [6], which consists in

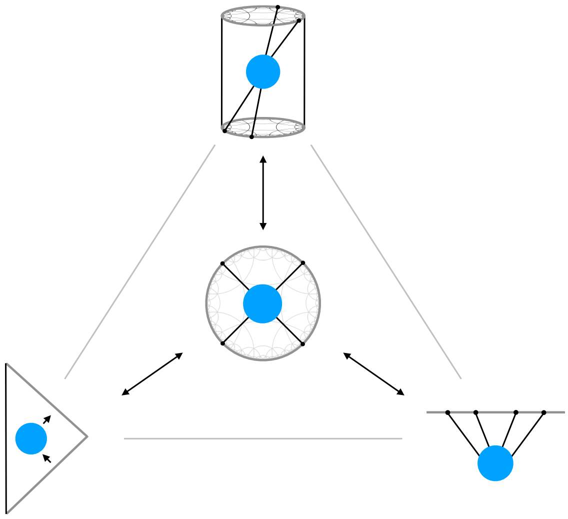


Figure 1: The Holographic Triangle: A cartoon representation of how observables at infinity in different maximally symmetric spaces can all be recast in terms of boundary observables in EAdS. Top: $\Lambda < 0$, bottom-right: $\Lambda > 0$ and bottom-left: $\Lambda = 0$. The same theory on different spaces will analytically continue to different theories on EAdS. We stress that the analytic continuation is a only proper Wick rotation between AdS and EAdS.

foliating Minkowski by Euclidean AdS spaces inside the light-cone and dS ones outside. This slicing opens up the possibility of importing the wealth of techniques, results and understanding of EAdS holography to Celestial holography. Further, operating under this new light, Celestial and Euclidean AdS holography will result manifestly connected, following the same pattern as with dS. Advancing within this pursuit, we will show that, in total analogy with dS, each Feynman diagram in Celestial holography can be recast in terms of a linear combination of corresponding Witten diagrams in EAdS.

Inspired by all of these results, the analysis presented in this thesis has culminated in the realization that Euclidean AdS plays a pivotal role in connecting holographic theories in maximally symmetric space-times. It emerges as the fundamental holographic theory from which to derive general considerations and fundamental properties of holography itself.

The collection of all the aforementioned considerations and results has been illustrated in the Holographic Triangle, Fig.1, highlighting the bridging between holographic theories in maximally symmetric space-times. In the diagram, Euclidean AdS is positioned at the center of the triangle, underscoring its central role in the framework of holographic theories.

Chapter 1

Introduction

In this introduction, we will review the fundamental steps in the development of holographic theories. The discussion will initially adhere to [1], deviating towards the end. We'll begin by explaining Quantum General Relativity as an effective field theory and then discuss the reasons for its breakdown. After that, we will explore the unique aspects of gravity, leading to a new revolutionary understanding of gravity as holographic. Finally, having outlined the pivotal role of AdS in the advancement of holographic theories, we will move towards more physical space-time, addressing the main problem discussed in this thesis.

1.1 Quantum Gravity as an Effective Field Theory

General Relativity (GR) stands out as one of the most predictive and gratifying theories, offering a comprehensive and harmonious synthesis from both a physical and mathematical perspective. As a result, it has established itself as an enduring cornerstone in modern physics, providing an immensely satisfactory framework that continues to shape our understanding of gravitational phenomena in the universe. However, it has limitations in aligning itself with something that goes beyond, towards Quantum General Relativity (QGR). Today we know that QGR is an effective field theory, which gives good predictions in a range of distances that are large compared to Planck length L_{Planck}. Given the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \, \sqrt{-g} \, (R - 2\Lambda) \,, \qquad \kappa = \sqrt{8\pi G}, \tag{1.1.1}$$

where R, Λ and G are, respectively, the Ricci scalar, the cosmological and the Newton constant, we can consider small fluctuations of the metric tensor $g_{\mu\nu}$ around Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + o(\kappa^2). \tag{1.1.2}$$

Plugging this last equation in (1.1.1), we get an expansions of the Lagrangian in powers of κ. In particular [1],

$$\frac{2}{\kappa^2}\sqrt{-g}R = \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + ... \tag{1.1.3}$$

where

$$\mathcal{L}^{(2)} \sim \partial h \partial h \tag{1.1.4}$$

is the graviton propagator and

$$\mathcal{L}^{(3)} \sim \kappa h \partial h \partial h \tag{1.1.5}$$

accounts for the leading-order vertex interactions. In the terms above, we suppressed the information about Lorentz indices and numerical factors. In particular, with $\partial h \partial h$ we just indicate a sum of terms behaving as the product of two derivatives of h, with Lorentz indices properly contracted. The same considerations hold for $h\partial h \partial h$. For more details, we also refer to [19, 20]. Eq.s (1.1.4) and (1.1.5) describe an interacting theory of a spin 2 field, the graviton, whose cubic vertex interaction contains a couple of derivatives in h. Being very similar to a Yang-Mills (YM) theory, we know how to proceed perturbatively, constructing scattering amplitudes within such a framework. However, when we couple gravitons with other particles, such as scalar ones, the theory exhibits one-loop divergences that cannot be reabsorbed via renormalization procedure [19].

Divergences at one loop are also present when we expand $g_{\mu\nu}$ around a non-trivial background $\overline{g}_{\mu\nu}$:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \kappa h_{\mu\nu} + o(\kappa^2). \tag{1.1.6}$$

In this case, also considering a theory with gravitons only, the one-loop diagram yields the divergent term [1, 19, 20]

$$\mathcal{L}^{div}_{1-loop} = \frac{1}{8\pi^2\epsilon}\left(\frac{\overline{R}^2}{120} + \frac{7}{20}\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}\right), \quad \epsilon = 4 - D, \tag{1.1.7}$$

where $\overline{R}$ and $\overline{R}_{\mu\nu}$ are, respectively, the Ricci tensor and scalar computed in the background metric $\overline{g}_{\mu\nu}$, while ϵ comes from the dimensional regularization.

The divergent term (1.1.7) vanishes when the background metric coincides with $\eta_{\mu\nu}$. Nevertheless, a 2-loop divergent term arises in any case, even when considering an expansion around $\eta_{\mu\nu}$ [1, 21]:

$$\mathcal{L}^{div}_{2-loop} = \frac{209}{2880}\frac{\kappa^2}{16\pi^2\epsilon}R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\eta\sigma}R^{\eta\sigma}{}_{\alpha\beta}, \tag{1.1.8}$$

where $R^{\mu}{}_{\nu\sigma\rho}$ is the Riemann tensor computed with respect to $h_{\mu\nu}$. Since both the divergent terms (1.1.7) and (1.1.8) involve powers of R greater than one, we cannot just reabsorb these divergences by adding counter-terms in the Einstein-Hilbert action (1.1.1). Nevertheless, 'tHooft and Veltman demonstrated in [19] that pure gravity is 1-loop finite even when expanding around the background metric $\bar{g}_{\mu\nu} \neq \eta_{\mu\nu}$. Specifically, they showed that in this case it is allowed to employ classical equations of motion to set $\bar{R}_{\mu\nu} = 0$ in (1.1.7), effectively canceling out the divergence. However, the theory of gravity described by the Einstein-Hilbert action remains 2-loop non renormalisable [22].

In any case, we can still think of QGR as a low energy effective field theory with cut-off $M_{\text{Planck}} \sim \kappa^{-1}$. Reasoning in this way, we must consider the most generic action compatible with symmetries [1],

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \, \sqrt{-g} \left(-2\Lambda + R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \ldots \right), \tag{1.1.9}$$

where c_i are bare parameters. In this EFT, one can get rid of divergences by adding suitable counter-terms $c_i^{(r)}$ in order to renormalise the theory. For example, the one-loop divergence (1.1.7) is readily eliminated by setting

$$c_1^{(r)} = c_1 + \frac{1}{960\pi^2} \frac{1}{\epsilon}, \quad c_2^{(r)} = c_2 + \frac{7}{160\pi^2} \frac{1}{\epsilon}. \tag{1.1.10}$$

The parameters $c_i^{(r)}$ are finite and must be measured experimentally. Therefore QGR is a good predictive Effective Field Theory (EFT) within energies that are small compared with M_{Planck}.

1.2 Gravity Breakdown

Being an EFT, we expect that, at energy scale comparable with M_{Planck}, hints of new physics should emerge. Driven by other examples in literature, like Fermi's theory and QCD, we would be inclined to say that, at this point, we should add new degrees of freedom in the theory. The story of physics tells us that, so far, there are essentially two ways to address this problem: either by enhancing the "resolution" of the vertex, introducing new heavy bosons to mediate the interaction, or by "resolving" the gravitons themselves, viewing them as composite particles made up of degrees of freedom from a new, fundamental theory [1]. At this stage, someone might argue that we are talking about strings, or other theories candidate to be the UV completion of gravity. Actually, strings only partially solve the gravity breakdown problem. The roads we are following here, will lead us to something more fundamental also than strings. Evidence of strings should come out at energies lower than M_{Planck}, in order to restore the unitarity of the theory.

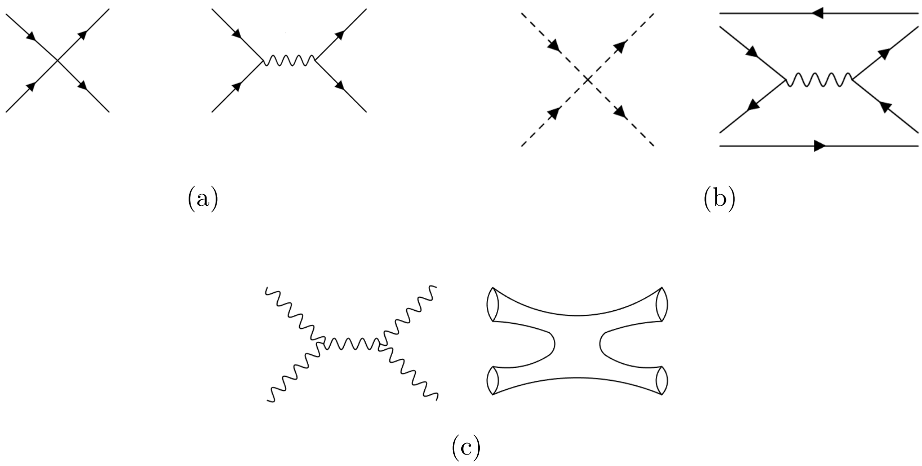


Figure 1.1: (a) The four-fermion Fermi vertex (left) resolution is improved by visualising a short-range interaction mediated by new bosons (right). (b) Pions in the contact process (left) are resolved into quarks, leading to a new process (right) where not all the new particles are involved. (c)The interaction in the graviton scattering process (left) is smeared across a 2-dim sheet representing the scattering of strings.

To be preserved, perturbative unitarity requires that scattering amplitudes do not grow too fast. In Fermi's theory, scattering amplitudes violate unitarity bounds around $E \sim 100\, GeV$. To restore unitarity, one improves the resolution of the four-fermion vertex, unveiling new heavy bosons which mediate the interactions (Fig.(1.1a)). In QCD, instead, at energy lower than chiral-symmetry breaking, degrees of freedom are pions, whose interactions are described by non-linear sigma models. At higher energy, pions are literally resolved by introducing new light d.o.f., i.e. quarks. In this way, the four-pion contact interaction is literally *smeared* in the interaction of the new fundamental constituents (Fig.(1.1b)).

The case of gravity is more similar to the latter situation. Introducing the Mandelstam variables $t = -(p_2 + p_3)^2$ and $s = -(p_1 + p_2)^2$, let us couple gravity with scalar particles and study the amplitude of a graviton exchange diagram in this theory. Since the interaction term (1.1.5) contains two derivatives, in the t-channel the behaviour of the amplitude for large s is given by [23]

$$\mathcal{A}_{\text{exch}} \sim -\frac{s^2}{t}. \tag{1.2.1}$$

Therefore, this amplitude grows too large to be compatible with the unitarity bounds. One way to restore unitarity when $s >> |t|$, is to invoke duality between t and s channel and introduce an infinite tower of higher spin particles at energy $M_{\text{string}} << M_{\text{Planck}}$ [23]. The only consistent theory known so far capable to justify the introduction of such a tower of states is string theory. Introducing strings, we resolve gravitons, smearing the interaction over a two-dimensional world sheet (Fig.(1.1c)). However, as energy grows further, approaching Planck scale, also perturbative String theory breaks down. Nevertheless, somehow, strings contain all the hints to describe what comes next. As length of strings probe L_{Planck}, they start to become denser and denser. Then, pushing energies beyond M_{Planck} causes a gravitational collapse and a black hole emerges, swallowing the insights of new physics within its horizon.

In the end, what we learned is that, if we try to complete QGR in the UV, we get black holes. This is a peculiar property, characteristic of gravity. It is precisely this property that will guide us towards a new, unique and distinctive framework to describe the fundamental theory of gravity.

1.3 Gravity is not Wilsonian

General Relativity contains within itself the seed of its own breakdown: black holes [1]. Black holes are classical objects, with a characteristic length given by the radius of the event horizon. With "classical" we mean that their existence is predicted in classical GR. General Relativity predicts black holes and, overall, what causes GR breakdown is its own unstoppable tendency to produce ever larger concentration of energy. In this sense, we can say that, somehow, a large classical length comes out at high energies. The shortest distance we can probe is L_{Planck} because shorter distances are always hidden inside the event horizon of a black hole. This limitation is tied to the challenge of information loss in GR, which originates also the principle encapsulated in the mantra "In gravity, there are no local observables".

Indeed, if we try to set up an ultra-planckian scattering experiment in the regime $s \sim |t| >> M^2_{\text{Planck}}$, at some short distance a black hole pops up. This is because we are concentrating an enormous amount of energy in a small region of space embracing all the particles in the experiment. This small region has a size comparable with the impact parameter b of the process, with

$$b \sim \frac{1}{\sqrt{|t|}}, \qquad \hbar = 1, \tag{1.3.1}$$

from the Heisenberg's Uncertainty Principle. As the energy of the experiment increases, the size of the black hole will also increase, with a radius growing as

$$R_{\text{BH}} \sim G\sqrt{s}. \tag{1.3.2}$$

Therefore, at ultra-planckian energies black holes dominate and all the quantum effects stay hidden inside them. In principle, as $E \to +\infty$, the radius

$R_{\mathrm{BH}} \to +\infty$, and the black holes swallows the entire universe. This is the main reason why we say that in gravity there are no local observables. An observable should be something that, in principle, we can measure with asymptotic precision in a (gedanken) experiment. However, when we attempt to achieve this goal, everything gets swallowed by a black hole. At least we can say that in gravity there exist “quasi local observables”, with precision bounded by L_{Planck}. Moreover, since at higher and higher energies the black hole probes larger and larger distances, we can say that there is no separation between low and high energy modes. In this sense, we say that gravity is not Wilsonian.

Wilsonian theories are (fundamental) QFTs dominated at high energies by Conformal fixed points. This means that, as energy increases, Wilsonian theories tend to manifest recursive structures, ultimately converging to CFTs as $E \to +\infty$. Trivial examples are provided by asymptotic free theories, which, at higher and higher energies, are dominated by the free-theory Lagrangians, or, in other words, by propagators. In the latter case, Conformal fixed point coincides with Gaussian one. The important property of Wilsionian theories is that, being controlled by Conformal fixed points, they are well defined at high energy.

In any quantum theory, the entropy S is related with the total number of degrees of freedom of the system [2]. Specifically, the dimension $\mathcal{N}$ of the Hilbert space $\mathcal{H}$ associated to the system is the exponent of S. For instance, consider a lattice fermion field theory, where spins are Boolean variables. Enveloping this theory within a bounding region V with the topology of a sphere $\mathbb{S}^2$, then $\mathcal{N} = 2^n$ and

$$n = \frac{S}{\ln 2}, \tag{1.3.3}$$

where n is the number of sites inside the region V [2, 24]. Therefore, entropy S provides information about the count of the number of degrees of freedom in a system. Now, by confining a Wilsonian theory within a box of radius R and increasing the energy, the fundamental degrees of freedom of the theory will uniformly distribute inside the box, ensuring scale invariance. Since interactions are controlled, for R sufficiently large the model becomes very similar to a thermal gas and we can compute the entropy S of the system. We get [1, 2]

$$S_{\mathrm{CFT}} = cR^{D-1}T^{D-1}, \qquad K_{\mathrm{B}} = 1, \tag{1.3.4}$$

where D is the dimension of the space-time, T is the temperature of the system and K_{B} the Boltzmann constant. The constant c is typical of the theory and accounts for the number of degrees of freedom at each point. Therefore, entropy S is related to the total number of degrees of freedom of the QFT, and (1.3.4) tells us that this latter grows as the volume $V = R^{D-1}$

of the box, in a Wilsonian theory. In terms of the energy [2]

$$E_{\rm CFT} \sim T^D V, \tag{1.3.5}$$

we find

$$S_{\rm CFT} \sim (E_{\rm CFT} R)^{\frac{D-1}{D}}. \tag{1.3.6}$$

In the presence of gravity, however, this method inevitably fails. Once again, in the latter situation, classical predictions tell us that, by increasing the temperature within the box, a black hole will emerge at a certain energy threshold, hiding within its event horizon the fundamental constituents of gravity.

1.4 Gravity is Holographic

Gravity is not a conventional QFT: it is not Wilsonian. Black Holes always emerge at ultra-planckian scale, apparently hiding its fundamental degrees of freedom. At this scale, every attempt to describe quantum mechanically scattering processes fails. However, since Black Holes dominate at this scale, we can try to describe the process semiclassically. Indeed, Black Holes can be treated as thermodynamical objects. After the formation of a black hole through an ultra-Planckian scattering process, it starts emitting energy. The resulting radiation exhibits a thermal spectrum law, which, for a Schwarzschild black hole, goes as the inverse of the Black hole mass M [25]:

$$T \sim \frac{1}{8\pi M} \sim \frac{1}{E} \sim \omega, \tag{1.4.1}$$

where T and E are, respectively, the temperature and the energy of the black hole, while ω is the energy of the emitted quanta. The number of the emitted quanta goes as $N \sim E^2$. Therefore, high-energety ultra-planckian scattering processes produce a great number of very soft particles. The heavier the black hole produced in the scattering, the lower its temperature and thus the energy of the emitted quanta. In this sense, black holes make gravity soft. The entropy of a black hole is given by the famous Hawking-Bekenstein formula [25, 26]:

$$S_{\rm BH} = \frac{A}{4G}, \qquad G = L_{\rm Planck}^{D-2}. \tag{1.4.2}$$

Let us, therefore, put a gravitational system into a spherical box. Increasing the temperature inside the box, classical GR predicts a gravitational collapse at a certain energy threshold. Even taking the box incredibly large, adding more and more energy will make a bigger and bigger black hole, which, at some point, will fill the entire box.

The Schwarzschild-Tangherlini metric, describing the space-time geometry in the presence of a static spherically symmetric massive object in D dimensions when $\Lambda = 0$, is given by [27, 28]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{D-2}, \tag{1.4.3}$$

where $\mathrm{d}\Omega^2_{D-2}$ is the metric of a $(D-2)$-dimensional sphere and

$$f(r) = 1 - \frac{\mu}{r^{D-3}}, \qquad \mu = \frac{16\pi GM}{(D-2)\Omega_{D-2}}, \tag{1.4.4}$$

with Ω_{D-2} the area of the $(D-2)$-dimensional sphere. Considering a black hole of mass M, the zero of the function $f(r)$ in (1.4.4) determines the radius of the event horizon of the black hole:

$$r_s = \mu^{\frac{1}{D-3}}. \tag{1.4.5}$$

Now, let us assume to put such a system into a very large spherical box and increase the temperature until the black hole fills all the space inside the box. In this case, the entropy of such a system is described by the Bekenstein-Hawking formula [1]

$$S_{\mathrm{BH}} = \frac{A}{4G} \sim (EL_{\mathrm{Planck}})^{\frac{D-2}{D-3}}, \qquad G \sim L^{D-2}_{\mathrm{Planck}}, \tag{1.4.6}$$

where A is the area of the box, $A \sim r_s^{D-2}$, and $r_s \sim (GE)^{\frac{1}{D-3}}$. Comparing (1.4.6) with (1.3.6) we realise that the entropy of a black hole scales with a different powers of E with respect to S_{CFT}. Therefore, in asymptotically flat space, gravity does not behave as a D-dimensional CFT at high energy, since its high energy behaviour is incompatible with that of (1.3.6). This is coherent with what we said in previously: gravity does not fit the Wilsonian paradigm.

However, we learned two main lessons:

- Gravity manifests a UV/IR connection: moving to higher energies leads to the creation of bigger black holes, that will become sensitive to the asymptotic properties of the space-time;

- At high energy, the number of degrees of freedom needed to fully describe a gravitational system confined in a spherical region of radius R, grows like the area of the boundary of the region containing the system.

Hence, the description of our fundamental theory, at extreme energy, would be sensitive to the asymptotic property of the space-time and its fundamental degrees of freedom would be spread over asymptotic spherical surfaces, that can be also interpreted as distant screens.

Consider, now, a gravitational system inside a region V of the space, whose boundary surface is Σ. Let A be the area of Σ. Then, the entropy of the gravitational system inside V cannot exceed the one of a black hole with horizon area equal to A. In other words, we found the upper bound [2, 24]

$$S \leq \frac{A}{4G}, \tag{1.4.7}$$

for the entropy of any gravitational system inside V. As a consequence, the max number of degrees of freedom of such a system grows as A and not as V. Motivated by this general consideration, ‘t Hooft argued that: “Given any closed surface, we can represent all that happens inside it by degrees of freedom on this surface itself” [2]. For instance, we could describe these degrees of freedom, spread over the surface Σ, as Boolean variables. From (1.3.3), we deduce that the total number of Booleans degrees of freedom in this case is given by [2]

$$n = \frac{A}{4 \ln 2} \tag{1.4.8}$$

Citing Susskind: “According to ‘t Hooft it must be possible to describe all phenomena within V by a set of degrees of freedom which reside on the surface bounding V. The number of degrees of freedom should be no larger than that of a two dimensional lattice with approximately one binary degree of freedom per Planck area. In other words the world is in a certain sense a two dimensional lattice of spins” [24]. Finally, sending the radius of the bounding region V at infinity, the surface Σ tends to embrace the entire space. As a result, in this limiting situation, the degrees of freedom of gravity would be stored on a codimension-2 surface Σ, with the topology of a sphere, surrounding the entire spatial Universe. This is precisely what we mean when we say that our Universe is *holographic*.

In the conventional approach to QFT, the long-distance behavior is typically associated with the definition of external states, or solutions of the free theory, such as plane waves. Fundamental degrees of freedom, instead, are visualised as interactive fields permeating space-time. Nevertheless, our argument proposes a new perspective, by suggesting that, in a theory of gravity, the fundamental degrees of freedom are distributed across asymptotic surfaces in space-time. They are, therefore, sensitive to the asymptotic behaviour of the space-time itself. In the end, what we revealed here, is that gravity is not like other quantum field theories. We deduced that gravity is different: gravity is *holographic*! Citing Susskind [24]:

> “In a certain sense , the world is two dimensional and not three dimensional as previously supposed”

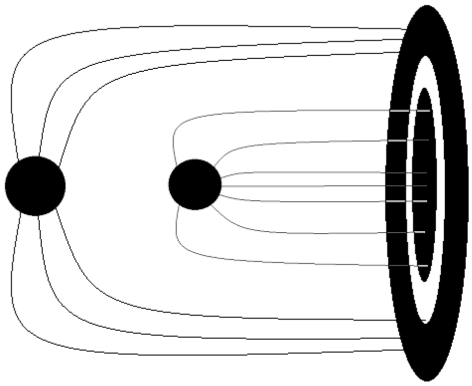


Figure 1.2: Partons are be transported along rays and projected onto a distant screen, with no more than parton per Planck area. The presence of two black holes in the illustration emphasises that the gravitational lensing law is considered in the mapping.

1.5 The Holographic Principle

Suppose that, as argued in the last section, degrees of freedoms of a gravitational system are spread over the surface Σ of a spherical region containing the system itself. Now, let us suppose that, at a fundamental level, our system is composed of elementary structureless constituents called *partons* [24]. By employing a semiclassical approach that incorporates all the properties of General Relativity (GR), such as lensing, we can establish a mapping between points on Σ and points on a distant screen S. This mapping is analogous to a stereographic projection, but also encodes physical information. Being pointwise, partons can be transported along rays from Σ to S, according to the mapping (see Fig.(1.2)). More details about the construction can be found in [24]. Since partons represent bit of information, this construction allows us to assert that all the information of a compact gravitational system can be stored on a screen, that we represent asymptotically far away from the source. We will refer to these screens as *holographic screens.*

Raphael Bousso gives a precise prescription about how to construct such screens [29, 30]. In addition, he offers a precise formulation of the *Holographic Principle.* As stated by Bousso, Holographic principle establishes a relation between space-time geometry and number of degrees of freedom of a system. In particular, Bousso formulation of the Holographic Principle implies that all information contained on a null hypersurface L can be stored on a $(D-2)$-dimensional spatial hypersurface B at a density of no more than one bit per Planck area, with L bounded by B and generated by one of the four families of light-rays orthogonal to B. In this way we are also able to bound the number of degrees of freedom N_{dof} on L, which will not exceed a quarter of the area of B, in $\ln 2$ units [29].

Given a space-time with a fixed geometry, if the Holographic Principle is satisfied, then, given a theory of gravity, it allows the possibility of the existence of a conventional QFT without gravity, living on the boundary of a space-time region. Therefore, satisfying the Holographic Principle is a condition necessary, but not sufficient, to ensure the existence of such a theory. If the latter exists and is constructed on a screen, then we will refer to this theory as *dual* to the theory of gravity in the bulk. In particular, we will call this dual theory the *holographic theory* on the asymptotic screen. Hereafter, we shall present the formulation of holographic principle given by Bousso [29]:

Definition 1.5.1. *Holographic Principle Let A be the area of a connected $(D-2)$-dimensional spatial surface B. Let L be a hypersurface bounded by B and generated by one of the four null congruences orthogonal to B. Let N be the number of elements of an orthonormal basis of the quantum Hilbert space that fully describes all physics on L. If the expansion of the congruence is non-positive (measured in the direction away from B) at every point on L, then $N \leq \mathrm{e}^{A/4}$ (working in $\ln 2$ units). Simplifying slightly, one could state that $N_{\text{dof}} \leq A/4$, where N_{dof} is the total number of independent quantum degrees of freedom present on L.*

The holographic principle implies a radical reduction in the number of degrees of freedom we use to describe nature [31], underlying a redundancy in the description of quantum systems in presence of gravity. Holographic theories, both the ones living on asymptotic screens or on bounding surfaces B, should resolve this redundancy, making manifest the effective number of degrees of freedom necessary to describe the theory. In his papers [31], Bousso presents a constructive method to project information onto asymptotic screens in maximally symmetric space-times, i.e. Minkowski, dS and AdS. This would suggest that a holographic theory of gravity in these space-times should exist. Among these three space-times, undoubtedly AdS serves as the foundational example.

1.6 The Maldacena Conjecture and the AdS/CFT Correspondence

Let us go back to (1.4.6). There, we put a gravitational system into a box and we studied its high energy behaviour in an asymptotically flat space-time. As we already discussed, in terms of powers of E, S_{BH} does not match with (1.3.6), describing the high energy behaviour of a CFT. This ensures that, at least in an asymptotic flat space-time, gravity is not a CFT. Now, let us analyse other asymptotic conditions and consider the example of AdS. Static black holes in AdS are described by

$$\mathrm{d}s^2 = -V(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + r^2 \mathrm{d}\Omega^2_{D-2}, \tag{1.6.1}$$

where $\mathrm{d}\Omega^2_{D-2}$ is the metric of a $(D-2)$-dimensional sphere and

$$V(r) = 1 - \frac{\mu}{r^{D-3}} + \frac{r^2}{L^2_{\text{AdS}}}, \qquad L_{\text{AdS}} = \frac{1}{\sqrt{-\Lambda}}, \tag{1.6.2}$$

is the gravitational potential, which depends on the mass $M \propto \mu/G$ of the black hole and on the AdS radius L_{AdS}. As we can see from (1.6.2), the cosmological constant $\Lambda < 0$ enters in the definition of L_{AdS}. Studying the potential $V(r)$, we realise that this system, describing a black hole with singularity in the origin of AdS and radius of its horizon given by $V(r) = 0$, behaves like a covariant box. Indeed, the term $\frac{r^2}{L^2_{\text{AdS}}}$ in the potential $V(r)$ is always attractive and grows for $r \to +\infty$. Therefore, a thermal gas in AdS will be attracted and drawn towards the center of AdS. When the energy is very large, $\mu >> L^{D-3}_{\text{AdS}}$, we can neglect the constant term in (1.6.2). In this situation, imposing $V(r_H) = 0$ yields [1]

$$\frac{\mu}{r_H^{D-3}} \simeq \frac{r_H^2}{L^2_{\text{AdS}}} \implies r_H^{D-1} \sim GEL^2_{\text{AdS}}, \tag{1.6.3}$$

where we have indicated with r_H the radius of the event horizon of the black hole. For very high energy, the balck hole tends to fill the entire AdS box. Using (1.6.3) and $A \sim r_H^{D-2}$, from the Bekenstein-Hawking formula (1.4.6), this time we find the following behaviour for the entropy of our system in the AdS box:

$$S_{\text{AdS}} \sim (EL_{\text{AdS}})^{\frac{D-2}{D-1}}. \tag{1.6.4}$$

The power of E in (1.6.4) is different from the one in both (1.4.6) and (1.3.6). Nevertheless, shifting $D \to D-1$, we realise that the behaviour of S_{AdS} matches with (1.3.6) in one less dimension. Therefore, there exists a possibility that *Quantum Gravity in AdS$_D$ is a CFT in one less dimension* [1].

Note that what we showed so far is not sufficient to assert that, given a theory of gravity in AdS, then there exists a conformal dual theory on a holographic screen of AdS. However, in literature there exist several examples that validate this last assertion. The most famous one, probably, was formulated by Maldacena [32], who conjectured that a $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $(3+1)$-dimension, with gauge group $SU(N)$ and coupling g_{YM}, is dual to a full-spectrum Type IIB superstring theory on the background $AdS_5 \times \mathbb{S}^5$, with string coupling $g_s = 4\pi g_{YM}^2$, N units of five-flux form on $\mathbb{S}^5$ and radius R of the 5-dimensional sphere equal to the one of AdS_5, given by $R^2 = \sqrt{N}\alpha' g_{YM}$.

In particular, defining the 't Hooft coupling $x = g_{YM}^2 N$, we can study the equivalence between the two theories above in the so-called *'t Hooft limit*. This is realised by sending [32, 33]

$$g_s \to 0, \quad \frac{\alpha'}{R^2} \text{ finite}, \tag{1.6.5}$$

in the type IIB string theory, which corresponds to send $N \to +\infty$, keeping x large but finite in the $\mathcal{N} = 4$ super Yang-Mills. On one hand, string theory is weakly coupled in this limit, therefore supergravity should furnish a good approximation to it. On the other hand, in the 't Hooft limit, the number of degrees of freedom $N_{d.o.f.} \sim \sqrt{N}$ of the Super Yang-Mills becomes incredibly large. Pushing further, we can hope that in the *classical limit*, reached by sending

$$g_s \to 0, \quad \frac{\alpha'}{R^2} \to 0, \tag{1.6.6}$$

in the type IIB string theory, which is equal to send

$$N \to +\infty, \quad x \to \infty \tag{1.6.7}$$

in the super Yang-Mills, tree level supergravity might still furnish an equivalent description of the super Yang-Mills. Note that, while supergravity is weakly coupled in the classical limit, super Yang-Mills is strongly coupled. The equivalence between this two theories in the classic limit would realise a *weak-strong duality*, allowing to address very hard questions for strongly coupled QFTs.

Maldacena conjecture was the first concrete realization of an *AdS/CFT correspondence*. His work profoundly inspired Witten, who developed a precise prescription for computing observables of the corresponding CFT by starting from a suitable theory of gravity in AdS. To perform his construction, Witten interpreted the AdS/CFT correspondence constructed by Maldacena as holographic. Indeed, it is well known that AdS_5 admits a foliation in terms of $\mathbb{M}^4$ slices. This is called the flat foliation of AdS and is manifest by employing Poincaré coordinates (see (2.2.9)). In the flat foliation, the boundary of AdS$_5$ is a copy of $\mathbb{M}^4$ and the isometry group $SO(2,4)$ of AdS$_5$ acts as the conformal group on $\mathbb{M}^4$. In the Maldacena correspondence, the CFT is the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{M}^4$, which is dual to a theory of gravity on AdS_5, neglecting the compact manifold $\mathbb{S}^5$. Witten, therefore, identified the aforementioned $\mathbb{M}^4$ as the boundary of AdS$_5$ and interpreted the result of Maldacena as suggesting that a suitable theory of gravity on AdS$_{D+1}$ would be equivalent to a CFT in D dimensions [33].

The Witten prescription is precisely given by [33]

$$\left\langle \exp \int \phi_0^{(i)}\, \mathcal{O}_{\Delta_i}^{(i)} \right\rangle_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{AdS}} \left[\phi^{(i)}|_{\partial \mathrm{AdS}} = \phi_0^{(i)} \right]. \tag{1.6.8}$$

Let us clarify this identity. First of all, it is clear that (1.6.8) equates two partition functions. On the left, we have the partition function of a CFT, while on the right, we have the partition function of the corresponding quantum theory of gravity in AdS. $\left\{ \mathcal{O}_{\Delta_i}^{(i)} \right\}$ and $\left\{ \phi^{(i)} \right\}$ are, respectively, the set of operators in the CFT and the set of the bulk fields, some of which would correspond to gravitational fluctuations in AdS. The relation (1.6.8) is holographic in the following sense: the boundary values $\phi_0^{(i)}$ of the bulk fields $\phi^{(i)}$ on the boundary of AdS act as sources for the CFT. Therefore, assigned the boundary conditions $\phi_0^{(i)}$ of the bulk fields in AdS, (1.6.8) states that, in principle, the partition function of the boundary CFT *defines* the partition function of the quantum theory of gravity in AdS. Indeed, the fundamental side is the CFT side, gives us a way to describe gravity in the bulk. A complete and non-perturbative formulation of a Conformal QFT on the boundary would give us a complete and non-perturbative formulation of quantum gravity in the bulk. In other words, (1.6.8) must be read in the following order:

$$\mathrm{CFT} \Longrightarrow \mathrm{AdS}$$

If we try to move in the opposite direction, starting from a known theory in the bulk, such as supergravity or String Theory, we can proceed by perturbative computations only. Indeed, we do not have any complete and non-perturbative formulation of gravity in AdS. This means that (1.6.8) does not describe a complete duality, but a complete *correspondence* [1].

$$\mathrm{CFT} \underset{\longleftarrow}{\Longrightarrow} \mathrm{AdS}$$

Nevertheless, we can always decide to work perturbatively, starting from the bulk AdS side. In this case, an effective method for progress is to operate in the leading saddle point approximation. Within this approximation (1.6.8) reduces to

$$W_{\mathrm{CFT}}[\phi_0^{(i)}] \simeq I[\phi^{(i)}|_{\partial \mathrm{AdS}} = \phi_0^{(i)}], \tag{1.6.9}$$

where I is the known on-shell action, describing our theory in the bulk, such as supergravity or string theory, and W_{CFT} is the functional generator of connected correlation functions in the CFT. This last equation allows us to compute CFT correlators, also known as *Witten correlators*, as

$$\langle O_1 \cdots O_n \rangle_c = \left. \frac{\delta^n I}{\delta \phi_0^1 \cdots \delta \phi_0^n} \right|_{\phi_0^i = 0}. \tag{1.6.10}$$

In this way, we can proceed by perturbative computations to figure out properties of the boundary CFT or to check fundamental results. The Feynman diagrams associated with each contribute to a Witten correlator are commonly referred to as Witten diagrams, that we will recursively encounter during this thesis (see Fig. 4.1 and 4.2 for some examples). They have been extensively studied in literature, providing an exceptional understanding of holographic theories in AdS. During this thesis, we will exploit this knowledge to compute Celestial correlators, by recasting them in terms of known Witten correlators.

1.7 Beyond AdS Holography

The AdS/CFT correspondence provides the most comprehensive understanding of holographic theories we currently possess. A multitude of scientists have contributed to advancing the understanding of holography in AdS, employing both perturbative methods, such as Witten diagram calculations, and non-perturbative approaches, such as Conformal Bootstrap, which is grounded in the properties of CFT. However, AdS is not a physical universe. Today we know that the cosmological constant Λ of our universe is positive. This has led scientists to endeavor to extend concepts and results uncovered in the study of holography in AdS to other, more physically relevant types of space-time, such as dS and Minkowski. Since the early 2000s, researchers have tried to establish a dS/CFT correspondence, drawing inspiration from what they learned in AdS [34]. However, even though very similar, AdS and dS present important differences. We learned that AdS behaves as a covariant box and has a time-like Lorentzian boundary, naturally endowed with a notion of time direction. This means that unitarity and causality in AdS space are intimately tied to the same properties on the boundary system. The boundary theory, therefore, would be a standard Lorentzian CFT defined non-perturbatively by conformal symmetry, unitarity and an associative (and, in particular, convergent) operator product expansion [35–37]. In dS, instead, the boundary is space-like, lacking of any concept of time. The boundary CFT would be Euclidean, but there's no strict requirement for correlators to respect the usual Osterwalder-Schrader axioms, such as positivity. Boundary correlators of dS are, indeed, called *spatial correlations* to distinguish them from Euclidean correlators, respecting Osterwalder-Schrader positivity [38–40]. Moreover, dS has two boundaries, one in far past and one in the far future. The situation in flat space is quite similar. To establish a holographic correspondence in asymptotic flat space-time, one must consider that the asymptotic boundary of Minkowski is a null-like hypersurface. It is therefore unclear what kind of axioms a boundary CFT should respect in this context.

Bridging dS and EAdS Holography. Late time correlators in dS have been extensively studied in the "Cosmological Bootstrap" (see [7] for a recent review). Among these results, it has been shown [9] that any perturbative contribution to dS boundary correlators in the Bunch-Davies (Euclidean) vacuum can be recast as a linear combination of corresponding Witten diagrams in Euclidean AdS, generated by the same collection of particles and couplings as the original process in dS space. This opens up the possibility to import the wealth of techniques, results and understanding from AdS to dS, with the caveat that one might have to consider a non-unitary process in AdS to study dS physics since the unitary irreducible representations of the dS and AdS isometry groups (respectively, $SO\,(d+1,1)$ and $SO\,(d,2)$) do not coincide (though they are overlapping).

Bridging Celestial and EAdS Holography. A similar result has been recently obtained in the new fashion of Celestial Holography. Considering a (d+2)-dimensional Minkowski space-time, in [6, 16–18] it has been shown that scattering amplitudes can be recast as d-dimensional Euclidean conformal field theory correlators by expressing them in a basis of $SO\,(d+1,1)$-primary solutions to the free equations of motion. Such correlators naturally live on the celestial sphere at null infinity, where the Lorentz group acts as the d-dimensional Euclidean conformal group $SO\,(d+1,1)$. Celestial holography (for reviews see e.g. [11–14]) then postulates that there is a dual Conformal Field Theory (CFT) description. Note that, also in this case, these would not be the standard correlators in Euclidean CFT, encountered as Wick rotations of correlators in Lorentzian CFT. For this reason, we will distinguish such correlators as *Celestial* correlators. They are defined on the Celestial Sphere at null infinity, which lacks a standard notion of locality and time, and are therefore not necessarily bound to satisfy the usual Osterwalder-Schrader axioms. Efforts to understand the properties of Celestial correlators so far include the study of symmetries and translating of properties of scattering amplitudes in the standard plane-wave basis to the conformal basis, see e.g. [16, 41–71].

1.8 The Holographic Triangle.

It is natural to ask if any of our understanding from the AdS case can be adapted to Celestial Holography. This is further motivated from the simple observation that Celestial and AdS boundary correlators can be placed on a similar footing by Wick rotating the AdS to Euclidean AdS, so that they satisfy the same conformal Ward identities. Any differences in the way they encode consistent physics therefore lies in the freedom left over after taking constraints from conformal Ward identities into account.

We will show in this thesis that is possible to realise this connection by employing the hyperbolic foliation of Minkowski space-time [6]. Minkowski space-time $\mathbb{M}^{d+2}$ can be foliated by constant curvature hypersurfaces, which are either dS_{d+1} or EAdS_{d+1} depending on whether one is inside or outside the light-cone. The idea is to adopt this foliation and then apply holography to each slice, in order to exploit the results obtained in the context of the dS/CFT correspondence. We will use this technique for computing Celestial correlators both in the conformal primary basis formulation of QFT (ref. chapter 4), and in the revisited version of Celestial Holography (ref. chapter 5). In both cases, each contribute to Celestial correlators results proportional to corresponding Witten diagrams in EAdS_{d+1}, like in dS. The proportionality term turns out to be a function encoding the radial dependence and information about the masses of the particles involved in the process. This confirms that such contact contributions to Celestial correlators are single-valued solutions to the conformal Ward identities.

All of the results discussed in this first chapter are illustrated in the *Holographic Triangle*, Fig (1), highlighting the connections between holography in different maximally symmetric spaces. As evident from the figure, Euclidean AdS plays a pivotal role in this bridging. Once again, AdS holography plays a pivotal role in the scientific community, serving as the canonical example from which researchers may endeavor to advance. The bridging between (A)dS and EAdS holography passes essentially trough a Wick rotation, which allows us to move from the Poincaré patch of (A)dS to that of EAdS. Instead, as we will see, the bridging between Celestial and EAdS holography is essentially established by the Kontorovich-Lebedev transform, whose fundamental role in this bridging will be uncovered during the course of this thesis.

Chapter 2

Geometrical Background

In this chapter, we will introduce the geometric background constituting the framework for the entire thesis. First of all, we will split Minkowski space-time in four disconnected regions: two inside the light-cone and two outside. Following [6], we will proceed by performing a hyperbolic slicing of such regions, foliating the ones inside the light-cone by Euclidean Anti de-Sitter (EAdS) spaces and the ones outside in terms of Lorentzian dS spaces. Then, after presenting some commonly employed coordinate patches over the (EA)dS leaves, we will analyse the last hypersurface necessary to cover the entire Minkowski space-time: the light-cone $\mathcal{C}$. We will learn that the light-cone can be split into two components: the future and past light-cone, $\mathcal{C}^{\pm}$. Each of these components has the topology of a cone-like manifold, $\mathbb{R}^{+} \times \mathbb{S}^{d}$, where $\mathbb{R}^{+}$ is the positive real line. Exploiting this structure, we will define two projective manifolds, $\mathcal{C}^{\pm}_{\sim}$, presented in this section as the set of future/past light-rays of the origin. Roughly speaking, a future/past light-ray of the origin is a half-line lying on the future/past light-cone with starting point at the origin of Minkowski. It is therefore clear that $\mathcal{C}^{\pm}_{\sim}$ is diffeomorphic to each smooth section of $\mathcal{C}^{\pm}$, as each section of the latter is intersected by a light ray in one and only one point. In this sense, $\mathcal{C}^{\pm}_{\sim}$ can be regarded as the equivalence class of all the smooth sections of $\mathcal{C}^{\pm}$. We will choose, as representative of this equivalence class, the so-called Poincaré section $\mathcal{C}^{\pm}_{P}$ of the future/past light-cone, which is diffeomorphic to $\mathbb{R}^{d}$ through a stereographic projection (see (2.3.7)). At this point, we will introduce the Penrose's method of conformal compactification, and we will apply this procedure to compactify each slice in the hyperbolic foliated Minkowski space-time. In this way, we will visualise Minkowski in a new guise, looking at it as foliated by compactified (EA)dS slice. The conformal boundary of each hyperbolic slice is represented by an equivalence class of conformally flat Riemannian manifolds known as the *Celestial Sphere* $\mathcal{CS}^{\pm}$. We can consistently represent $\mathcal{CS}^{\pm}$ as $\mathbb{R}^{d}$, which we will embed in $\mathcal{C}^{\pm}_{P}$. The correspondence between fields in the single (EA)dS slices of

Minkowski and those on $\mathcal{CS}^{\pm}$ is realised by applying the well-known rules of the (EA)dS/CFT correspondence. All of these considerations will suggest that, working in a hyperbolic foliated Minkowski space-time, where each extended unphysical slice is topologically closed at infinity by $\mathcal{CS}^{\pm}$, a *Flat/CFT correspondence* can be found. It will emerge, indeed, in the end of the thesis, that this correspondence is realised by combining a transformation known as Kontorovich-Lebedev transform with the (EA)dS/CFT correspondence [4, 6]. Working in this way, a duality between a QFT in the bulk of Minkowski and a CFT living in a codimension-2 boundary hypersurface, the Celestial Sphere, can be constructed, opening also a new road towards the extension of this duality to asymptotically flat space-time.

2.1 Slicing Minkowski Space-Time

Consider the $(d+2)$-dimensional Minkowski space-time $\mathbb{M}^{d+2}$, with signature given by $\eta = diag(-1, 1, \ldots, 1)$. The *light-cone* or *null-cone* $\mathcal{C}$ of the origin is the submanifold of $\mathbb{M}^{d+2}$ whose element are *light-like* or *null-like* vectors with respect to the origin,

$$\mathcal{C} = \left\{ q \in \mathbb{M}^{d+2} | \, q^2 = 0 \right\}. \tag{2.1.1}$$

The null-cone $\mathcal{C}$ splits the Minkowski space-time into two regions [6]:

- The *interior of the null-cone*,

$$\mathcal{A} = \left\{ X \in \mathbb{M}^{d+2} | \, X^2 < 0 \right\}; \tag{2.1.2}$$

- The *exterior of the null-cone*,

$$\mathcal{D} = \left\{ X \in \mathbb{M}^{d+2} | \, X^2 > 0 \right\}. \tag{2.1.3}$$

Moreover, the interior of the null-cone can be further divided into two smaller subregions, called the *chronological future* $\mathcal{A}_+$ and the *chronological past* $\mathcal{A}_-$, which are identified by the additional condition $X^0 \gtrless 0$, respectively. Each of these regions of Minkowski space-time, in turn, can be foliated into $(d+1)$-dimensional submanifolds with constant curvature. In the region $\mathcal{A}$, for example, these foliating submanifolds are defined by the embedding relation, $X \in \mathbb{M}^{d+2}$,

$$X^2 = -T^2, \quad T \neq 0, \tag{2.1.4}$$

where T must be thought as fixed on each slice. Eq.(2.1.4) describes in $\mathcal{A}$ a double-sheet hyperboloid $\mathcal{H}^T_{d+1}$ of ray T, which is an Euclidean Anti-de Sitter (EAdS) space. By factorising $X = T\hat{X}$, with $X^2 = -T^2$, one finds that the

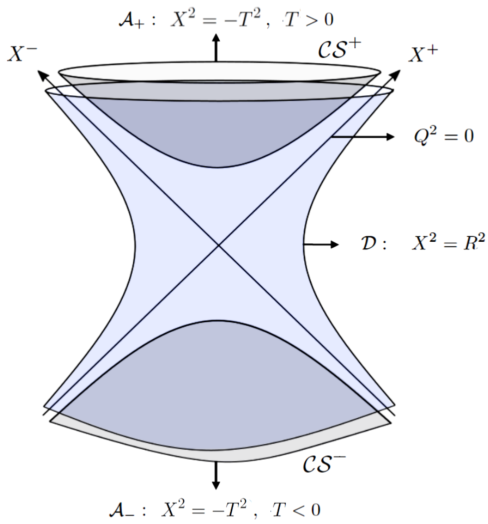


Figure 2.1: Hyperbolic slicing of Minkowski space. Region $\mathcal{D}$ outside the light-cone is foliated by co-dimension one de Sitter spaces and regions $\mathcal{A}_\pm$ inside the light-cone by the upper sheet (in region $\mathcal{A}_+$) or lower sheet (in region $\mathcal{A}_-$) of co-dimension one Euclidean anti-de Sitter spaces.

point $\hat{X}$ belongs to the double-sheet unitary hyperbolid $\mathcal{H}^1_{d+1} \equiv \mathcal{H}_{d+1}$. We will call $\mathcal{H}^\pm_{d+1}$ the upper/lower sheet of $\mathcal{H}_{d+1}$ in the subregion $\mathcal{A}_\pm$. Starting from $\hat{X} \in \mathcal{H}^+_{d+1}$ and then varying $T \in \mathbb{R}$, one achieves the whole foliation of the entire region $\mathcal{A}$. Alternatively, by restricting $T \in \mathbb{R}^\pm$, one spans the subregion $\mathcal{A}^\pm$ only. Moreover, by adopting the above factorisation, the two chronological subregions inside the null-cone are described as product spaces endowed with the product topology, $\mathcal{A}_\pm = \mathbb{R}^\pm \times \mathcal{H}^+_{d+1}$. Similarly, both the metric

$$\mathrm{d}s^2_{\mathcal{A}_\pm} = -\mathrm{d}T^2 + T^2\,\mathrm{d}s^2_{\mathcal{H}^+_{d+1}}\,, \quad T \in \mathbb{R}^\pm, \tag{2.1.5}$$

and the integration measure

$$\int_{\mathcal{A}_\pm} \mathrm{d}^{d+2}X = \int_{\mathbb{R}^\pm} |T|^{d+2}\mathrm{d}T \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}\hat{X}], \tag{2.1.6}$$

in this way assume a factorized form, that will provide helpful simplifications in upcoming computations. Note that, coherently, the metric $\mathrm{d}s^2_{\mathcal{H}^+_{d+1}}$ of the unitary slice $\mathcal{H}^+_{d+1}$ has an Euclidean signature, since the hyperboloid $\mathcal{H}^+_{d+1}$ is an Euclidean AdS space.

Similarly, in the region $\mathcal{D}$ the foliating submanifolds are defined by the embedding relation

$$X^2 = R^2, \quad R > 0. \tag{2.1.7}$$

Each of these submanifolds is a *Lorentzian de Sitter* (dS) space $\mathrm{d}S^R_{d+1}$ of ray R, surrounding the light-cone $\mathcal{C}$. Setting anew $X = R\hat{X}$, with $\hat{X}^2 = 1$, we can analogously factorise $\mathcal{D} = \mathbb{R}^+ \times dS_{d+1}$, with dS_{d+1} unitary dS slice. Even in this case, the metric in the region $\mathcal{D}$ factorises as follows:

$$\mathrm{d}s^2_{\mathcal{D}} = \mathrm{d}R^2 + R^2\, \mathrm{d}s^2_{dS_{d+1}}, \quad R > 0, \tag{2.1.8}$$

where the metric $\mathrm{d}s^2_{dS_{d+1}}$ of the unitary slice dS_{d+1} has a Lorentzian signature, this time. Analogously, the integration measure becomes

$$\int_{\mathcal{D}} \mathrm{d}^{d+2}X = \int_{\mathbb{R}^+} R^{d+2}\mathrm{d}R \int_{dS_{d+1}} [\mathrm{d}\hat{X}]. \tag{2.1.9}$$

2.2 Global vs Poincaré Coordinates in Minkowski Space-Time

Up to this point, we have seen that the hyperbolic foliation splits the Minkowski space-time into three simply connected regions: two inside the light-cone, called $\mathcal{A}_+$ and $\mathcal{A}_-$, and one outside, denoted $\mathcal{D}$, surrounding the light-cone. In this section, we shall illustrate some commonly used coordinate patches to parameterize points in these different regions.

Poincaré coordinates. To begin with, let us consider the double-sheet hyperboloid $\mathcal{H}^T_{d+1}$. Eq.(2.1.4) states that every point in $\mathcal{H}^T_{d+1}$ can be parametrised by a number of $d+1$ coordinates. For example, in cartesian coordinates, one can set

$$X^0 = \pm\sqrt{T^2 + \sum_{i=1}^{d+1} X^i X^i}, \quad i = 1, \dots, d+2, \tag{2.2.1}$$

and use X^i as independent coordinates. In (2.2.1), the sign $\pm$ has to be chosen for $X \in \mathcal{A}_\pm$, respectively. Alternatively, another option is to employ the so-called *Poincaré* coordinates. They are defined as follows [72, 73]:

$$X^+ = \frac{T}{y}, \quad X^- = T\,\frac{y^2 + |\vec{z}|^2}{y}, \quad X^a = \frac{T}{y}\, z^a, \quad \vec{z} \in \mathbb{R}^d,\ y > 0, \tag{2.2.2}$$

where $T \in \mathbb{R}^\pm$ for $X \in \mathcal{A}_\pm$. From (2.2.2), it follows that $X^0 = X^+ + X^- \gtrless 0$ for $X \in \mathcal{A}_\pm$, coherently to what we have said so far. In Poincaré coordinates, the metric on $\mathcal{H}^T_{d+1}$ assumes the conformally flat form

$$\mathrm{d}s^2_{\mathcal{H}^T_{d+1}} = \frac{T^2}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}\right). \tag{2.2.3}$$

It is possible to check that the metric $\mathrm{d}s^2_{\mathcal{H}^T_{d+1}}$ remains unchanged under the action of the following transformation subgroups [18]:

$$\begin{aligned}
\mathbb{R}^d \text{ translations:} \quad & y' = y, \quad \vec{z}\,' = \vec{z} + \vec{a}; && \text{(2.2.4a)}\\
SO(d) \text{ rotations:} \quad & y' = y, \quad \vec{z}\,' = R \cdot \vec{z}; && \text{(2.2.4b)}\\
\text{Dilations:} \quad & y' = \lambda y, \quad \vec{z}\,' = \lambda \vec{z}; && \text{(2.2.4c)}\\
\begin{matrix}\text{Special conformal}\\ \text{transformations}\end{matrix} : \quad & y' = \frac{y}{1 + 2\vec{b} \cdot \vec{z} + |\vec{b}|^2 \left(y^2 + |\vec{z}|^2\right)}, && \text{(2.2.4d)}\\
& \vec{z}\,' = \frac{\vec{z} + \left(y^2 + |\vec{z}\,|^2\right) \vec{b}}{1 + 2\vec{b} \cdot \vec{z} + |\vec{b}|^2 \left(y^2 + |\vec{z}\,|^2\right)}. && \text{(2.2.4e)}
\end{aligned}$$

As readily seen, the infinitesimal generators of these isometry transformations recombine to form up the algebra of the Lorentz group $SO(1, d+1)$, which is known to be the isometry group of the Riemannian manifold $\mathcal{H}^T_{d+1}$. On the unitary slice $\mathcal{H}^+_{d+1}$, the integration measure in Poinaré coordinates becomes

$$\int_{\mathcal{H}^+_{d+1}} [\mathrm{d}\hat{X}] = \int_0^{+\infty} \frac{\mathrm{d}y}{y^{d+1}} \int_{\mathbb{R}^d} \mathrm{d}^d z, \tag{2.2.5}$$

that, plugged in (2.1.6), gives back the full expression of the factorised measure we shall use in $\mathcal{A}_\pm$ when adopting Poincaré coordinates.

In the region $\mathcal{D}$ the situation is quite similar, but there are important differences. In fact, on dS there exist two different Poincaré patches,

$$X^+ = \frac{R}{\eta_\pm}, \quad X^- = T\,\frac{|\vec{z}|^2 - \eta_\pm^2}{\eta_\pm}, \quad X^a = \frac{R}{\eta_\pm}\, z^a, \quad \vec{z} \in \mathbb{R}^d,\ \eta_\pm \gtrless 0, \tag{2.2.6}$$

covering two disjoint charts $dS^{R,\pm}_{d+1}$ of dS^R_{d+1}. Endowed with the coordinates (2.2.6), the charts $dS^{R,\pm}_{d+1}$ are called the Expanding and the Contracting Poincaré patch of dS. In particular, from (2.2.6) it follows that a point $X \in dS^{R,\pm}_{d+1}$ is characterized by the additional condition $X^+ \gtrless 0$. Moreover, given a point $X \in dS^{R,\pm}_{d+1}$, there exists a unique antipodal point $X_A = -X \in dS^{R,\mp}_{d+1}$. The map that associates to each point X its antipodal point X_A in the opposite Poincaré patch is called the antipodal map. Therefore, the antipodal map is an automorphism of each dS slice. Instead, given $X \in \mathcal{H}^T_{d+1}$, it follows that $X_A \in \mathcal{H}^{T'}_{d+1}$, with $T' = -T$. The antipodal map also allows us to redefine $\eta_\pm = \pm\eta$, with $\eta > 0$, making $dS^{R,\pm}_{d+1}$ diffeomorphic to $\mathbb{R}^+ \times \mathbb{R}^d$. Note that the Contracting and the Expanding Poincaré patch of dS do not form an atlas of dS^R_{d+1}, since they do not cover the hypersurface $X^+ = 0$. In Poincaré coordinates, the metric on $dS^{R,\pm}_{d+1}$ is given by

$$\mathrm{d}s^2_{dS^{R,\pm}_{d+1}} = \frac{R^2}{\eta_\pm^2}\left(-\mathrm{d}\eta_\pm^2 + \mathrm{d}\vec{z} \cdot \mathrm{d}\vec{z}\right), \qquad R > 0,\ \eta_\pm \gtrless 0, \vec{z} \in \mathbb{R}^d. \tag{2.2.7}$$

Over the chart $dS^{\pm}_{d+1}$ of the unitary dS slice, the integration measure assumes the form

$$\int_{dS^{\pm}_{d+1}} [\mathrm{d}\hat{X}] = \int_0^{+\infty} \frac{\mathrm{d}\eta}{\eta^{d+1}} \int_{\mathbb{R}^d} \mathrm{d}^d z, \quad \eta_{\pm} = \pm\eta. \tag{2.2.8}$$

which has to be plugged in (2.1.9) to find back the full factorised form of the integration measure over the regions $\mathcal{D}_{\pm}$, defined as the subregions of $\mathcal{D}$ whose points respects the further condition $X^+ \gtrless 0$.

Summarising, the hyperbolic slicing of the individual regions $\mathcal{A}_{\pm}$ and $\mathcal{D}_{\pm}$ are naturally described by Poincaré coordinates. In fact, in the hyperbolic slicing $\mathbb{M}^{d+2}$ splits up into four patches ($M = 0, \ldots, d+1$):

$$\mathcal{A}_+ : \qquad X^M = +\frac{T}{y}\left(\frac{1+y^2+|\vec{z}\,|^2}{2}, \frac{1-y^2-|\vec{z}\,|^2}{2}, \vec{z}\right), \quad T > 0, \tag{2.2.9a}$$

$$\mathcal{A}_- : \qquad X^M = +\frac{T}{y}\left(\frac{1+y^2+|\vec{z}\,|^2}{2}, \frac{1-y^2-|\vec{z}\,|^2}{2}, \vec{z}\right), \quad T < 0, \tag{2.2.9b}$$

$$\mathcal{D}_+ : \qquad X^M = +\frac{R}{\eta}\left(\frac{1-\eta^2+|\vec{z}\,|^2}{2}, \frac{1+\eta^2-|\vec{z}\,|^2}{2}, \vec{z}\right), \quad R > 0, \tag{2.2.9c}$$

$$\mathcal{D}_- : \qquad X^M = -\frac{R}{\eta}\left(\frac{1-\eta^2+|\vec{z}\,|^2}{2}, \frac{1+\eta^2-|\vec{z}\,|^2}{2}, \vec{z}\right), \quad R > 0. \tag{2.2.9d}$$

In this thesis, we shall call these patches, respectively from top to bottom, the *Chronological Milne Patch* (CMP), the *Anti-chronological Milne Patch* (AMP), the *Expanding Rindler Patch* (ERP) and the *Contracting Rindler Patch* (CRP).

Global Coordinates. Another coordinate patch commonly used in (A)dS is the so-called global coordinate patch. Given a point $\hat{X}$ on a unitary slice of Minkowski, satisfying

$$-(\hat{X}^0)^2 + \sum_{i=1}^{d+1} \hat{X}^i \hat{X}^i = \pm 1, \tag{2.2.10}$$

it is readily seen that a solution of this embedding relation is given by

$$\hat{X}^0 = \cosh(\tau), \quad \hat{X}^i = \hat{n}^i \sinh(\tau), \quad \tau \geq 0,\ \hat{n} \in \mathbb{S}^d, \tag{2.2.11}$$

for $\hat{X} \in \mathcal{H}^+_{d+1}$, and by

$$\hat{X}^0 = \sinh(\sigma), \quad \hat{X}^i = \hat{n}^i \cosh(\sigma), \quad \sigma \in \mathbb{R},\ \hat{n} \in \mathbb{S}^d, \tag{2.2.12}$$

for $\hat{X} \in dS_{d+1}$. Employing these coordinates, the metric on the unitary slices assumes the form

$$\mathrm{d}s^2_{\mathcal{H}^+_{d+1}} = \mathrm{d}\tau^2 + \sinh^2(\tau)\,\mathrm{d}\Omega^2_d, \quad \hat{X} \in \mathcal{H}^+_{d+1} \tag{2.2.13}$$

$$\mathrm{d}s^2_{dS_{d+1}} = -\mathrm{d}\sigma^2 + \cosh^2(\sigma)\,\mathrm{d}\Omega^2_d, \quad \hat{X} \in dS_{d+1}, \tag{2.2.14}$$

along with the respective integration measures

$$\int_{\mathcal{H}^+_{d+1}} [\mathrm{d}\hat{X}] = \int_{\mathbb{S}^d} [\mathrm{d}\Omega] \int_0^{+\infty} \sinh^d(\tau)\, \mathrm{d}\tau, \tag{2.2.15}$$

$$\int_{dS_{d+1}} [\mathrm{d}\hat{X}] = \int_{\mathbb{S}^d} [\mathrm{d}\Omega] \int_{-\infty}^{+\infty} \cosh^d(\sigma)\, \mathrm{d}\sigma. \tag{2.2.16}$$

In the formulas above, $\mathrm{d}\Omega^2$ and $[\mathrm{d}\Omega]$ are, respectively, the metric and integral measure over the unitary d-dimensional sphere $\mathbb{S}^d$.

In summary, another natural set of coordinates to parameterise points in the hyperbolic foliation of Minkowski space-time is given by adopting the Global coordinate patches over the unitary slices. Employing these patches, we get

$$\mathcal{A}_+ : \quad X^M = T\,(\cosh\tau, \sinh\tau\,\hat{n}), \quad T > 0, \quad \tau \geq 0, \tag{2.2.17a}$$

$$\mathcal{A}_- : \quad X^M = T\,(\cosh\tau, \sinh\tau\,\hat{n}), \quad T < 0, \quad \tau \geq 0, \tag{2.2.17b}$$

$$\mathcal{D} : \quad X^M = R\,(\sinh\sigma, \cosh\sigma\,\hat{n}), \quad R > 0, \quad \sigma \in \mathbb{R}, \tag{2.2.17c}$$

which is the analogue of spherical coordinates. Respectively, we shall call these patches: the *Chronological Global Milne Patch* (CGMP), the *Anti-Chronological Global Milne Patch* (AGMP) and the *Global Rindler Patch* (GRP).

2.3 The Projective Light-Cone

So far, we divided Minkowski space-time in four regions, $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{D}_+$, $\mathcal{D}_-$, that we have extensively analysed by presenting a hyperbolic foliation consisting of (EA)dS slices. The last remaining submanifold to study, to cover the whole Minkowski space-time, is the light-cone $\mathcal{C}$, which we are now preparing to analyse. Consider the restriction of $\mathcal{C} \subset \mathbb{M}^{d+2}$ to the half-space $q^0 \gtrless 0$. We will call the submanifold $\mathcal{C}^\pm = \{q \in \mathcal{C} | q^0 \gtrless 0\}$ the *future/past light-cone* of Minkowski space-time. Now, let us consider the equivalence relation

$$q \sim \lambda q, \quad \lambda \in \mathbb{R}^+, \tag{2.3.1}$$

of elements in $\mathcal{C}^\pm$, which are named *future/past projective null-vectors*. We shall refer to the equivalence classes

$$[q]_\pm = \left\{ q_\lambda \in \mathcal{C}^\pm : q_\lambda = \lambda q, \quad \lambda \in \mathbb{R}^+ \right\} \tag{2.3.2}$$

as *future/past null-rays* or *light-rays*. The set of all the future/past light-rays $[q]_\pm$ of Minkowski space-time form a manifold, $\mathcal{C}^\pm_\sim = \mathcal{C}^\pm / \sim$, called the *future/past projective light-cone*. This manifold is diffeomorphic to a section of $\mathcal{C}^{\pm}$ defined by the additional condition $q^{+} = \alpha$, with α constant. Indeed, a null-vector $q \in \mathcal{C}$ has only $d+1$ independent components,

$$q^2 = 0 \Longrightarrow q^- = \frac{|\vec{q}\,|^2}{q^+}, \tag{2.3.3}$$

where $\vec{q} \in \mathbb{R}^d$ and $|\vec{q}\,|^2 = q^a q^b \delta_{ab}$, with $a, b = 1, \ldots, d$. Therefore, a representative q_λ of the equivalence class $[q]_\pm \in \mathcal{C}^{\pm}_{\sim}$, is given by

$$q_\lambda = \lambda \left(q^+, \frac{|\vec{q}\,|^2}{q^+}, \vec{q} \right)^T, \tag{2.3.4}$$

where $\lambda > 0$. Now, setting $\lambda = \alpha / q^+$, with $\alpha \gtrless 0$ and fixed, we can embed $\mathcal{C}^{\pm}_{\sim}$ in a section of $\mathcal{C}^{\pm}$,

$$[q]_\pm \longmapsto q = \alpha \Big(1,\, |\vec{\omega}\,|^2,\, \vec{\omega} \Big)^T, \tag{2.3.5}$$

where we have redefined

$$\vec{\omega} = \frac{\vec{q}}{q^+}, \quad \vec{\omega} \in \mathbb{R}^d. \tag{2.3.6}$$

In particular, the choice $\alpha = \pm 1$ provides the embedding of $\mathcal{C}^{\pm}_{\sim}$ in the so-called *Poincaré section* $\mathcal{C}^{\pm}_{P} \subset \mathcal{C}^{\pm}$, whose elements are parameterised as

$$q = \pm \Big(1, |\vec{\omega}\,|^2, \vec{\omega} \Big)^T. \tag{2.3.7}$$

In general, Eq.s (2.3.5) and (2.3.6) furnish the relations to embed the Euclidean space $\mathbb{R}^d$ in the section of $\mathcal{C}^{\pm}$ defined by $q^+ = \alpha$, generating the chain of diffeomorphisms [74, 75]

$$\mathcal{C}^{\pm}_{\sim} \simeq \mathcal{C}^{\pm}_{P} \simeq \mathbb{R}^d. \tag{2.3.8}$$

As reviewed in Appendix B, this chain of diffeomorphism serves as the foundation for the Dirac's embedding formalism, establishing a correspondence between conformal primary tensors in $\mathbb{R}^d$ and Lorentz tensors in $\mathbb{M}^{d+2}$. The Dirac's idea is rooted in the basic observation that the connected component of Conformal group in d dimensions is diffeomorphic to the connected component of the Lorentz group in $d+2$ dimensions. Parameterising $\mathcal{H}^T_{d+1}$, $T \gtrless 0$, by means of the Poincaré coordinates (2.2.2), in the embedding formalism it is possible to visualize two points $q(\vec{\omega}) \in \mathcal{C}^{\pm}_{P}$ and $X(T; y, \vec{z}\,) \in \mathcal{H}^T_{d+1}$, which belong to different submanifolds, in the same ambient space $\mathbb{M}^{d+2}$. This two submanifolds are disjoint, but, geometrically, Fig.(2.2) shows that $\mathcal{H}^T_{d+1}$ reaches the future/past null-cone $\mathcal{C}^{\pm}$ at infinity. As we said so far, the section at infinity of $\mathcal{C}^{\pm}$ (the blue upper circle

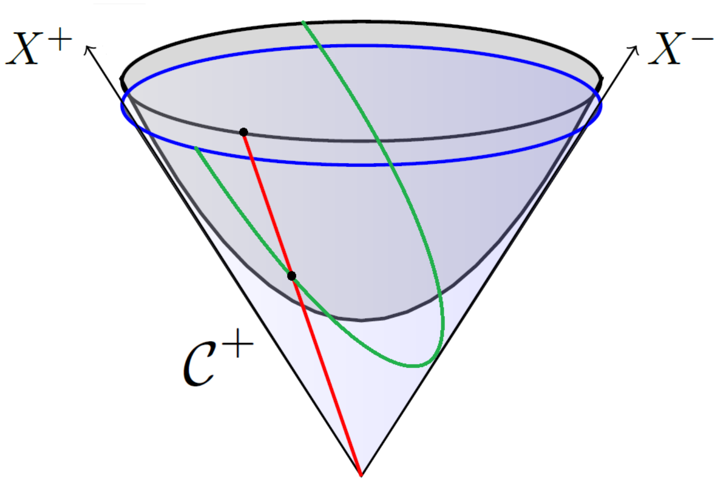


Figure 2.2: The bijective mapping from points on the Poincaré section (in green) to points on the far-future horizontal section (blue circle) of the future light-cone (light blue) is achieved using light-rays (red). Additionally, these light-rays intersect the boundary (black circle) of the one-sheet unitary hyperboloid (grey) at infinity, ensuring an isomorphism between the conformal boundary of the hyperboloid and the Poincaré section.

in Fig.(2.2)) can be identified with $\mathcal{C}^{\pm}_{\sim}$, since points of this section are in one-to-one correspondence with light-rays. Indeed, in the figure, the (red) light-ray intersects the upper (blue) circle only at one point. Moreover, $\mathcal{C}^{\pm}_{\sim}$ can be identified with $\mathcal{C}^{\pm}_{P}$, by the chain of diffeomorphisms (2.3.8). One may then ask if the submanifold $\mathcal{C}^{\pm}_{P}$ can be thought of the boundary manifold of $\mathcal{H}^{T}_{d+1}$, $T \gtrless 0$, placed at infinity. Strictly speaking, this is not possible! In fact, the boundary of $\mathcal{H}^{\pm}_{d+1}$ would be reached for $y \to 0^+$. However, given

$$X = T\left(\frac{1}{y}, \frac{y^2 + |\vec{z}|^2}{y}, \frac{\vec{z}}{y}\right), \tag{2.3.9}$$

one finds that the limit

$$\lim_{y\to 0} X\left(T; y, \vec{z}\right) \tag{2.3.10}$$

diverges for all $\vec{z} \in \mathbb{R}^d$. Similarly, the metric $\mathrm{d}s^2_{\mathcal{H}^T_{d+1}}$, defined by Eq.(2.2.3), diverges for $y \to 0^+$ as well. This means that $\mathcal{C}^{\pm}_{P}$ cannot be thought of an analytical extension of the Riemannian manifold $\mathcal{H}^{T}_{d+1}$. Nevertheless, the limit

$$\lim_{y\to 0^+} \frac{y}{|T|} X\left(T; y, \vec{z}\right) = \pm\left(1, |\vec{z}\,|^2, \vec{z}\right)^T \equiv q\left(\vec{z}\right), \quad \vec{z} \in \mathbb{R}^d, \tag{2.3.11}$$

converges to an element of $\mathcal{C}_P^{\pm} \simeq \mathbb{R}^d$. Analogously, the metric $\mathrm{d}s^2_{\mathcal{H}^T_{d+1}}$ can be extended to $\mathcal{C}_P^{\pm}$ via the limit operation

$$\lim_{y\to 0^+} \frac{y^2}{T^2}\,\mathrm{d}s^2_{\mathcal{H}^T_{d+1}} = (\mathrm{d}y^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z})|_{y=0} \equiv \mathrm{d}s^2_E, \quad \vec{z}\in\mathbb{R}^d, \tag{2.3.12}$$

where in the second equality we restricted the metric onto $\mathcal{C}_P^{\pm}$, seen as the 0-level set of the function $\Omega_{\mathcal{H}^+_{d+1}}$. Therefore, by means of the smooth function $\Omega_{\mathcal{H}^T_{d+1}}(y) = y/|T|$, $T \gtrless 0$, a correspondence between points on $\mathcal{H}^T_{d+1}$ and points on $\mathcal{C}_P^{\pm} \simeq \mathbb{R}^d$ can be constructed through a limit prescription. The limits (2.3.11) and (2.3.12) provide a kind of extension of the manifold $\mathcal{H}^T_{d+1}$. The new extended manifold $\widetilde{\mathcal{H}}^T_{d+1} = \mathcal{H}^T_{d+1} \sqcup \mathcal{C}_P^{\pm}$, where $\pm = \mathrm{sign}(T)$, has now a boundary, $\mathcal{C}_P^{\pm} \simeq \mathbb{R}^{d+1}$, that can be meant as placed at infinity and is endowed with the metric $\mathrm{d}s^2_E$. The procedure just described is called Penrose's *Conformal Compactification*, allowing us to associate a bounded manifold to an unbounded one. In particular, $\widetilde{\mathcal{H}}^T_{d+1}$ is called the conformal extension of $\mathcal{H}^T_{d+1}$ and $\mathcal{C}_P^{\pm}$ is its conformal boundary. However, this construction is not unique. The general function $\Omega_{\mathcal{H}^T_{d+1}}(y)$, that we can use here to ensure such an association, must have the following properties [76–78]:

- $\Omega_{\mathcal{H}^T_{d+1}}(y)$ must be positive definite in the bulk, which is the hyperboloid $\mathcal{H}^T_{d+1}$;
- $\Omega_{\mathcal{H}^T_{d+1}}(y)$ and $\frac{\mathrm{d}}{\mathrm{d}y}\Omega_{\mathcal{H}^T_{d+1}}(y) \neq 0$ on the boundary ($y = 0$).

The set of functions $\Omega_{\mathcal{H}^T_{d+1}}(y)$, satisfying the above properties, leads to a set of Riemannian manifolds that are conformal to $\widetilde{\mathcal{H}}^T_{d+1}$. The boundaries of all of these manifolds form an equivalence class of boundary Riemannian manifolds that we will call *Celestial Sphere*. In the next section we will describe in detail this procedure and we will apply it to Minkowski by using several coordinate patches.

In the end, what we learned in this section is that, in the ambient space $\mathbb{M}^{d+2}$, we can compactify each hyperboloid $\mathcal{H}^T_{d+1}$ by the Poincaré section $\mathcal{C}_P^{\pm}$ of the light-cone, that we can visualise as the boundary of $\mathcal{H}^T_{d+1}$, $T \gtrless 0$, placed at infinity. Obviously, this construction can be analogously carried out for each dS slice. In the latter case, employing Poincaré coordinates (2.2.6), $\mathcal{C}_P^{\pm}$ is established to be the future/past conformal boundary of $dS^{R,\pm}_{d+1}$. Since $q\,(\vec{\omega})$ in (2.3.7) transforms as a conformal vector with respect to the action of $SO(1, d+1)$, in terms of $SO(1, d+1)$ tensors, the correspondence ensured by (2.3.11) and (2.3.12) maps Minkowski Lorentz tensors to conformal primary ones of $\mathbb{R}^d$. This is the basic of the Dirac's embedding formalism (see Appendix B) . In the upcoming section, we will delve deeper into Penrose's conformal compactification procedure, clarifying some points that we have introduced in this discussion.

2.4 The Conformal Boundary

The concept of Conformal Boundary was introduced by Penrose in [79] and was then employed to study the geometrical background and the symmetries of an isolated gravitational system emitting outgoing radiation [79–81]. The solution $g_{\mu\nu}$ to Einsten's equation,

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.4.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $\mathcal{R}$ the scalar Ricci curvature, $T_{\mu\nu}$ the stress tensor, Λ and G_N the Cosmological and Newton constant, respectively, for an isolated gravitational system describes a space-time approaching Minkowski as we move far away from the source. Indeed, in such a system, the mass density has compact support and the radiating energy flow goes to zero in the long distance limit. This ensures asymptotically both Ricci and Weyl flatness, making the geometry of the asymptotic space-time indistinguishable from that of Minkowski [82–85]. For this reason, the space-time generated by an isolated gravitational system is said to be *asymptotically flat*.

Nevertheless, a precise definition of asymptotic flatness is far from obvious. The first formulation was given by Bondi, Sachs et al. [86–89], who studied this problem in a coordinate-wise fashion. Working in a precise patch, today known as the Bondi-Sachs patch, they arrived to give a list of large-distance conditions that the metric g and its components must satisfy in order to ensure the asymptotic flatness of the space-time. Such a metric g, satisfying the Bondi-Sachs asymptotic conditions, is called *asymptotically flat metric* (see also [82, 84, 85]).

However, the list of conditions they worked out, at first, seemed not to have any geometric foundation, as stated by Sachs himself. This aspect was justified only later, by using the Penrose's geometrical approach. Besides that, the most significant result, achieved in the early sixties by Bondi et al., was undoubtedly the discovery of an infinite-dimensional extension of the Poincaré group as the group of transformations which preserve the asymptotic flatness of the metric. This group of transformations is called *asymptotic symmetry group* and can be regarded as the group of transformations that preserve the geometry generated by an isolated gravitational system far away from the source. Therefore, even though the geometry of an asymptotically flat space-time, at large distance, becomes the same as the Minkowski's one, with great surprise, Bondi, Van de Burg, Metzner and Sachs discovered that the asymptotic symmetry group of an asymptotically flat space-time is not the Poincaré group, but a much bigger and richer group, today known as the Bondi-Metzner-Sachs (BMS) group (for further information on this group we also suggest [90–101]).

Initially, Sachs did not accepted this discovery favorably and unsuccessfully attempted to restrict the BMS group, in order to restore Poincaré as the asymptotic symmetry group of an asymptotic flat space-time. Nevertheless, today we know that the infinite-dimensional extension of the Poincaré group can be justified by a *conformal extension* of space-time itself, as demonstrated by Penrose using a purely geometric approach, without adopting any set of coordinates. In order to study the behaviour of the space-time near infinity, Penrose proposed to compactify the space-time, by essentially using a conformal transformation to bring all points at infinity to a finite distance. This kind of shrinking can be achieved by employing a new unphysical space-time, conformal to the physical one, where the supporting manifold is compact. In what follows, we will proceed by elaborating on Penrose's idea in detail, applying then the methodology of *Conformal Compactification* to the hyperbolic foliated Minkowski space-time.

2.5 Conformal Compactification

In the subsequent discussion, the outcomes, included theorems and definitions, are predominantly sourced from a selection of papers and books, [77, 80–82, 84, 85, 102–105]. The results are presented in 4 dimensions, even though most of them can be generalized to arbitrary dimensions. Henceforth, we will emphasize that we are working in 4 dimensions only when strictly necessary. Let us begin by recalling some basic definitions.

Definition 2.5.1. *A 4-dimensional space-time is a couple (M, g), where M is a 4-dimensional, connected, smooth manifold without boundary and g is a non-degenerate rank-$(0,2)$ tensor with Lorentzian signature. This means that at each point $p \in M$ there exists a basis of tangent vector $\mathscr{B} = \left\{X_p^a\right\}$, $a = 1, \ldots, dim(M)$, such that $g_p = diag(-1, 1, \ldots, 1)$ in $\mathscr{B}$. Moreover, the space-time is said to be time-orientable if there exist a nowhere vanishing time-like vector field T over M.*

The existence of a nowhere vanishing vector field T, in general, enables us to designate, at any point of our space-time, *future-directed* vectors as those vectors oriented as T and *past-directed* vectors as those oriented opposite to T. Besides the time-orientability, another important concept is the causal structure of a space-time.

Definition 2.5.2. *A curve γ in a time-orientable space-time (M, g) is said to be a causal curve if each tangent vector of γ is ether a time-like or null future-directed vector. Moreover, given two points p and q in M, we will say that p causally precedes q if there exist a future directed time-like curve γ starting from p and ending in q.*

Strictly speaking, the physical meaning behind the notion of causality between two points p and q is related to the possibility that an event represented by p can be the causal factor for a subsequent event occurring at q. Now that we have clarified these fundamental concepts, let us proceed with the main definition of this section.

Definition 2.5.3. *A time-oriented and causal space-time* (M, g) *is said to be asymptotically simple if there exists a Lorentzian manifold* $(\tilde{M}, \tilde{g})$*, an embedding* $\iota : M \hookrightarrow \tilde{M}$ *and a function* $\Omega \in C^\infty(\tilde{M})$*, called conformal factor, such that:*

1. $\tilde{M}$ *is a compact manifold with interior* $\iota(M)$ *and boundary* $\mathscr{I}$*, so that* $\tilde{M} \simeq \iota(M) \sqcup \mathscr{I}$*;*
2. $\Omega\,|_{\iota(M)} > 0$*,* $\Omega|_{\mathscr{I}} = 0$ *and* $\mathrm{d}\Omega|_{\mathscr{I}} \neq 0$*;*
3. *The pull-back* $\iota^* \tilde{g} = \Omega^2 g$*, which means that* $\tilde{g}$ *is conformal to* g *in* $\iota(M)$*;*
4. *Every null geodesic of* $(\tilde{M}, \tilde{g})$ *has two endpoints on* $\mathscr{I}$*.*

In addition, an asymptotically simple space-time (M, g) *is also said empty if* $R_{\mu\nu}|_{\iota^{-1}(\tilde{\mathcal{O}})} = 0$ *in any open neighborhood* $\mathcal{O}$ *of* $\mathscr{I} \subset \tilde{M}$*.*

The Lorentzian manifold $(\tilde{M}, \tilde{g})$, constructed following the above prescription, is called the *conformal extension* of (M, g) and $\mathscr{I}$ is its conformal boundary. Often, people also refer to $(\tilde{M}, \tilde{g})$ as the *unphysical space-time*. Indeed, by taking a geodesic $\tilde{\gamma}$ in the unphysical space-time $(\tilde{M}, \tilde{g})$, its components in a chart $(\tilde{U}, \tilde{\psi})$ of $\tilde{M}$ must satisfy the differential equation

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tilde{\lambda}^2} + \tilde{\Gamma}^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tilde{\lambda}} \frac{\mathrm{d}x^c}{\mathrm{d}\tilde{\lambda}} = 0, \qquad a = 1, \ldots, dim(\tilde{M}), \tag{2.5.1}$$

where $\tilde{\psi}(\tilde{\gamma}) = x \in \mathbb{R}^{dim(\tilde{M})}$, $\tilde{\lambda}$ is an affine parameter and $\tilde{\Gamma}^a{}_{bc}$ are the Christoffel symbols in the metric $\tilde{g}$,

$$\tilde{\Gamma}^a{}_{bc} = \frac{1}{2} \tilde{g}^{ad} (\partial_b \tilde{g}_{cd} + \partial_c \tilde{g}_{bd} - \partial_d \tilde{g}_{bc}). \tag{2.5.2}$$

Property 1 of Def.(2.5.3) allows us to employ a chart (U, ψ) of M such that $\tilde{\psi}(\tilde{\gamma}) = \psi(\iota^{-1}(\tilde{\gamma}))$ and $\iota(U) = \tilde{U}$. The subsequent transformation property of the Christoffel symbols is, therefore,

$$\tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + 2\Omega^{-1} \delta^a{}_{(b} \nabla_{c)} \Omega - \Omega^{-1} (\nabla_d \Omega)\, g^{ad} g_{cb}. \tag{2.5.3}$$

Hence, the components of a physical geodesic γ, such that $\iota(\gamma) = \tilde{\gamma}$, in the chart (U, ψ) must satisfy

$$0 = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = -\frac{1}{\lambda'} \left(\frac{\lambda''}{\lambda'} + 2\frac{\Omega'}{\Omega} \right) \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}, \tag{2.5.4}$$

where $\lambda = \lambda(\tilde{\lambda})$ is the affine parameter of the physical geodesic and the primes represent the derivatives with respect to $\tilde{\lambda}$. The right-hand side of (2.5.4) vanishes for

$$\mathrm{d}\lambda = \frac{c}{\Omega^2}\mathrm{d}\tilde{\lambda}, \tag{2.5.5}$$

with c constant. Since the smooth function Ω vanishes on the conformal boundary $\mathscr{I}$, the affine parameter λ diverges as the unphysical geodesic approaches $\mathscr{I}$. Therefore, we could say that the physical geodesic never reaches $\mathscr{I}$ and, from the point of view of a physical observer, the conformal boundary can be meant as at infinity.

For an asymptotically simple and empty space-time, important properties can be deduced for $\mathscr{I}$. The components of the Ricci tensor of the physical space-time R_{ab} are related to the ones of the Ricci tensor $\tilde{R}_{ab}$ in the unphysical space-time via (see [106])

$$R_{ab} = \tilde{R}_{ab} - 2\Omega^{-1}\nabla_a\nabla_b\Omega - \tilde{g}_{ab}\left(\Omega^{-1}\nabla^c\nabla_c\Omega - 3\Omega^{-2}\nabla^c\Omega\nabla_c\Omega\right). \tag{2.5.6}$$

Remembering that, for an asymptotic simple and empty space-time, $R_{ab} = 0$ in any neighborhood of $\mathscr{I}$, multiplying both sides of (2.5.6) by Ω^2 yields

$$\nabla^c\Omega\nabla_c\Omega = 0. \tag{2.5.7}$$

Since from property 2 of Def.(2.5.3) $\nabla_c\Omega \neq 0$ everywhere on $\mathscr{I}$, it follows that $\nabla_c\Omega$ is a nowhere vanishing null covector on $\mathscr{I}$. Further, $\nabla_c\Omega$ is also the normal covector of $\mathscr{I}$. Indeed, as stated in property 2 of Def.(2.5.3), $\mathscr{I}$ is the 0-value level set of the smooth function Ω. This last consideration implies that $\mathscr{I}$ is a null hypersurface. Moreover, plugging (2.5.7) in (2.5.6) and then multiplying both sides by Ω, we find

$$2\nabla_a\nabla_b\Omega + \tilde{g}_{ab}\nabla^c\nabla_c\Omega = 0, \tag{2.5.8}$$

which means that the normal covector $\nabla_c\Omega$ is divergent- and shear-free:

$$\nabla^c\nabla_c\Omega = \nabla_a\nabla_b\Omega = 0. \tag{2.5.9}$$

To present another important property of $\mathscr{I}$, we have to use the following theorem, demonstrated by Hawking and Ellis:

Theorem 2.5.1. *An asymptotically simple and empty space-time* (M, g) *is globally hyperbolic.*

This result, along with property 3 of Def.(2.5.3), implies that the conformal boundary $\mathscr{I}$ of an asymptotic simple and empty space-time consists of two disconnected pieces: $\mathscr{I}^+$, the locus where all the null geodesics have their future end-point and $\mathscr{I}^-$, the locus where all the geodesics have their past end-point. Moreover, working in four-dimensions, $\mathscr{I}^\pm$ has the topology of $\mathbb{S}^2 \times \mathbb{R}$. All of these considerations apply to an asymptotically flat space-time whenever Einstein's empty equations hold in any neighborhood of $\mathscr{I}$. Indeed, an asymptotic flat space-time is asymptotically empty, in the sense given above, if no outgoing radiation is emitted from the source. In general, when the cosmological constant is different than zero, the following theorem holds [80]:

Theorem 2.5.2. *If the trace of the stress-energy tensor vanishes near $\mathscr{I}$, then $\mathscr{I}$ is space-like, time-like, or null according as the cosmological constant Λ in the Einstein's equation is positive, negative, or zero.*

Even though the previous theorem regards maximally symmetric spaces as dS and AdS, there are some subtleties to be aware of. Strictly speaking, both AdS and dS are not asymptotically simple space-times. For example, globally, AdS has the topology of $\mathbb{S}^1 \times \mathbb{R}^3$. It has been shown by Bengtsson [107] that there exist closed time-like curves in this topology, which means that the causal structure of AdS is violated. However, if one unwraps the circle $\mathbb{S}^1$ and considers the universal covering of AdS, topologically $\mathbb{R}^4$, it does not contain any closed time-like curves and the assumptions made in this section work out for it. In general, we will refer to AdS as its universal covering. Regarding dS, instead, it is stated that only half of the space can be treated as an asymptotically simple and empty space-times. Indeed, in the following section, we will perform the conformally compactification of dS in its EPP and CPP, separately.

2.6 Compactifying Minkowski Space-Time

Let us put into practice the concepts acquired in the last section to compactify Minkowski space-time. In the following, we will carry out the conformal compactification in the far future using the time-retarded coordinate u. However, it's worth noting that a comparable analysis is applicable for compactifying Minkowski in the far past. In this latter scenario, the appropriate choice would involve employing the time-advanced coordinate $v = X^0 + r$.

Consider the $(d+2)$-dimensional Minkowski metric

$$\mathrm{d}s^2 = \eta_{MN}\, \mathrm{d}X^M \mathrm{d}X^N, \qquad \eta_{MN} = diag(-1, 1, \dots, 1), \tag{2.6.1}$$

with M, $N = 0, \dots, d+1$. Then, let us introduce time-retarded coordinates

$$u = X^0 - r, \quad X^{d+1} = r\cos(\theta), \quad X^a = r\,\hat{m}^a \sin(\theta), \tag{2.6.2}$$

where $a = 1, \dots, d$ and $\hat{m} \in \mathbb{S}^{d-1}$. The new metric is

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\Omega^2_{d-1}\right), \tag{2.6.3}$$

where $\mathrm{d}\Omega^2_{d-1}$ is the metric of the $(d-1)$-dimensional sphere. As readily seen from the metric (2.6.3), hypersurfaces $u = const$ are future null-cones, with

$r > 0$ generating their null-rays. Indeed, the normal vector $k^M = g^{MN}\nabla_N u$ to each level surface $u = const$ is null and future-pointing:

$$g^{MN}\nabla_M u \nabla_N u = g^{uu} = 0, \qquad \nabla_u u = 1. \tag{2.6.4}$$

Being null, the vector k^M is also tangent to each light-cone defined by $u = const$. Moreover, we can parameterise the integral curve $\gamma(\lambda)$ of each light-ray as

$$\mathbb{R}^+ \ni \lambda \mapsto \gamma(\lambda) = \begin{pmatrix} u = u_0 \\ r = \lambda \\ \theta = \theta_0 \\ \hat{m} = \hat{m}_0 \end{pmatrix}, \tag{2.6.5}$$

where u_0, θ_0 and $\hat{m}_0^a$ are arbitrary constants. It is then clear that $\gamma(\lambda)$ are geodesics. Indeed, their velocity vector $\gamma'^M \equiv -\nabla^M r = \delta_r^M$ is constant along γ and satisfies (2.5.4). Further, $g^{MN}k_N\nabla_M r = -1$, which means that the velocity vector γ'^M is future-directed. All of these considerations enable us to say that the radius r is an affine parameter for geodesics along null-rays. Intuitively, future null infinity can be reached by following these paths. Therefore, we will employ r to play the role of our physical parameter λ, as in (2.5.4), and we will send $\lambda \to \infty$ to reach the conformal boundary $\mathscr{I}^+$ in the unphysical space-time.

Equation (2.5.5) gives us some arbitrariness to choose the function Ω. Commonly, Ω is taken in such a way that $\Omega\lambda \to constant$ for $\lambda \to \infty$. In particular, setting $\Omega = \tilde{\lambda}$, (2.5.5) leads us to the solution $\lambda = f(\tilde{\lambda}^{-1})$, with f arbitrary linear function. Therefore, we are free to put $\Omega = \tilde{\lambda} = r^{-1}$. Then, by performing the change of variables $y = r^{-1}$, the physical metric becomes

$$\mathrm{d}s^2 = y^{-2}\left(-y^2\mathrm{d}u^2 - \mathrm{d}u\,\mathrm{d}y + \mathrm{d}\Sigma_d^2\right), \tag{2.6.6}$$

where the last term represents the metric on a d-dimensional unit sphere $\mathbb{S}^d$,

$$\mathrm{d}\Sigma_d^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\Omega_{d-1}^2. \tag{2.6.7}$$

Note that null infinity is reached for $y \to 0$, therefore $\tilde{\lambda}$ is finite at null infinity. Moreover, all the conditions of Def.(2.5.3) are satisfied in this new metric. The unphysical metric is immediately tracked out from (2.6.6), i. e.

$$\mathrm{d}\tilde{s}^2 = y^2\mathrm{d}u^2 - \mathrm{d}u\,\mathrm{d}y + \mathrm{d}\Sigma_d^2. \tag{2.6.8}$$

This unphysical metric can be resctricted on $\mathscr{I}^+$, defined by the condition $y = 0$, yielding

$$\mathrm{d}\tilde{s}^2_{\mathscr{I}^+} = 0 \cdot \mathrm{d}u^2 + \mathrm{d}\Sigma_d^2, \tag{2.6.9}$$

where the zero in front of the first term is just a symbol to point out that the metric is degenerate on $\mathscr{I}^+$. From the structure of the induced metric (2.6.9), it results evident that $\mathscr{I}^+ \simeq \mathbb{R} \times \mathbb{S}^d$, with the retarded time u

generating the factor $\mathbb{R}$. Therefore, $\mathscr{I}^+$ is a codimension-1 hypersurface of $\mathbb{M}^{d+2}$ and its sections $u = const$ are topologically codimension-2 spheres. Redefining the conformal factor $\Omega' = \Theta\Omega$, with Θ arbitrary positive smooth function on the unphysical space-time $\widetilde{\mathbb{M}}^{d+2}$, yields, after the restriction, just a rescaling of the metric (2.6.9). This makes evident the conformal structure of the hypersurface $\mathscr{I}^+$, translating the gauge freedom on the choice of the conformal factor Ω into the freedom of rescaling the metric $\mathrm{d}\tilde{s}^2_{\mathscr{I}^+}$ by an arbitrary positive smooth function Θ^2. All the Riemannian manifolds obtained by this rescaling are, in principle, physically equivalent. Therefore, $\mathscr{I}^+$ can be regarded as a smooth manifold $\mathcal{N} \simeq \mathbb{R} \times \mathbb{S}^d$ endowed with the equivalence class of degenerate metrics [99]

$$[\mathrm{d}\tilde{s}^2_{\mathscr{I}^+}] = \left\{ \mathrm{d}\tilde{s}^2_\Theta = \Theta^2 \mathrm{d}\tilde{s}^2_{\mathscr{I}^+} \,|\, \Theta \in C^\infty(\mathcal{N}, \mathbb{R}^+) \right\}. \tag{2.6.10}$$

The conformal structure defined in this way, naturally induces a conformal structure on the sphere $\mathbb{S}^d$. By means of the canonical projection $\pi : \mathcal{N} \mapsto \mathbb{S}^d$, define a section $s : \mathbb{S}^d \mapsto \mathcal{N}$ as satisfying $\pi \circ s = id_{\mathbb{S}^d}$. We can than pull-back all the elements of (2.6.10) on $\mathbb{S}^d$ by employing s. We will call *future Celestial Sphere*, $\mathcal{CS}^+$, the equivalence class of all the Riemannian manifolds,

$$\mathcal{CS}^+ := \left\{ \left(\theta^2 \mathrm{d}\Sigma^2_d, \, \mathbb{S}^d \right) \,|\, \theta \in C^\infty(\mathbb{S}^d, \mathbb{R}^+) \right\}, \tag{2.6.11}$$

where $\theta = \Theta \circ s$. Note that, by applying the pull-back π^*, the equivalence class of metrics

$$[\mathrm{d}\Sigma^2_d] = \left\{ \mathrm{d}\tilde{\Sigma}^2_\theta = \theta^2 \mathrm{d}\Sigma^2_d \,|\, \theta \in C^\infty(\mathbb{S}^d, \mathbb{R}^+) \right\}, \tag{2.6.12}$$

define a sub-equivalence class of metrics in (2.6.9), characterized by functions Θ constant along the rays generated by u [99]. Now, let us represent $\mathbb{S}^d$ in stereographic coordinates. Looking at $\mathbb{S}^d$ as embedded in $\mathbb{R}^{d+1}$, consider the stereographic map

$$\mathrm{st} : \mathbb{S}^d / \{(0, \ldots, 1)\} \to \mathbb{R}^d, \qquad \mathrm{st}(\hat{\xi}) := \left(\frac{\hat{\xi}^1}{1 - \hat{\xi}^{d+1}}, \ldots, \frac{\hat{\xi}^d}{1 - \hat{\xi}^{d+1}} \right)^T, \tag{2.6.13}$$

with inverse

$$\mathrm{st}^{-1} : \mathbb{R}^d \to \mathbb{S}^d / \{(0, \ldots, 1)\}, \qquad \hat{\xi}(\vec{\omega}\,) = \left(\frac{\vec{\omega}}{|\vec{\omega}|^2 + 1}, \frac{|\vec{\omega}|^2 - 1}{|\vec{\omega}|^2 + 1} \right)^T. \tag{2.6.14}$$

The extended stereographic map [82]

$$\tilde{\mathrm{st}} = \begin{cases} \mathrm{st}(\xi), & \xi \in \mathbb{S}^d / \{(0, \ldots, 1)\}, \\ \infty, & \xi = (0, \ldots, 1). \end{cases}$$

is a diffeomorphism $\widetilde{\mathrm{st}} : \mathbb{S}^d \to \widetilde{\mathbb{R}}^d = \mathbb{R}^d \sqcup \{\infty\}$. Employing stereographic coordinates, the metric on the sphere becomes

$$\mathrm{d}\Sigma_d^2 = \frac{4}{(|\vec{\omega}|^2 + 1)^2}\mathrm{d}s_E^2, \tag{2.6.15}$$

where $\mathrm{d}s_E^2$ is the standard metric on $\mathbb{R}^d$. This gives back

$$\mathrm{d}\tilde{s}^2_{\mathscr{I}^+} = 0 \cdot \mathrm{d}u^2 + \frac{4}{(|\vec{\omega}|^2 + 1)^2}\mathrm{d}s_E^2. \tag{2.6.16}$$

The conformal factor in (2.6.15) can be included in the redefinition of the function θ in (2.6.11). In conclusion, we can always represent the future Celestial Sphere as the extended Euclidean plane $\tilde{\mathbb{E}}^d \simeq \mathbb{R}^d \sqcup \{\infty\}$ endowed with the standard Euclidean metric $\mathrm{d}s_E^2$ in its interior $\mathbb{R}^d$. Therefore, we can also redefine $\mathcal{CS}^+$ as the equivalence class

$$\mathcal{CS}^+ \simeq \left\{ \left(\theta^2 \mathrm{d}s_E^2\,,\, \mathbb{R}^d\right) \times P \,|\, \theta \in C^\infty(\mathbb{R}^d, \mathbb{R}^+) \right\}, \tag{2.6.17}$$

where P is a point representing the boundary of $\tilde{\mathbb{E}}^d$.

Definition 2.6.1. *The group of transformations that preserve the induced metric* (2.6.9) *up to a conformal factor is the* Newman-Unti *(NU) group,* $\mathscr{N}\mathscr{U}_d$.

Since the connected component of the d-dimensional conformal group is isomorphic to $SO_c^+(1,d)$, the connected component of $SO(1,d+1)$, we can state that $\mathscr{N}\mathscr{U}_d$ consists of the group of transformations

$$u \mapsto u' = F(u, \vec{\omega}), \quad \frac{\partial F}{\partial u} > 0, \qquad \vec{\omega} \in \mathbb{R}^d; \tag{2.6.18}$$

$$\vec{\omega} \mapsto \vec{\omega}\,' = \Lambda[\vec{\omega}], \qquad \Lambda \in SO_c^+(1, d+1). \tag{2.6.19}$$

The NU group is bigger than the BMS group (we suggest [108] for a discussion on it). To restrict the former to the latter, we need to introduce another structure defined on $\mathscr{I}^+$, called by Penrose *strong conformal geometry.* Strong conformal geometry requires that the ratio

$$\frac{\mathrm{d}l}{\mathrm{d}u} = \nu \tag{2.6.20}$$

must be invariant and independent on the choice of the conformal factor Ω. Given the degeneracy of the metric on $\mathscr{I}^+$, there always exists a vector field N such that $g(N, \cdot) = 0$ [99]. We will denote N *isotropic vector field* and the integral curves of N *isotropic directions.* Therefore, it may happen that, given two non-null vectors X_p, Y_p at $p \in \mathscr{I}^+$, they span a plane containing N_p. In this case, the angle between X_p and Y_p vanishes because of the degeneracy of the metric. In fact, setting $\alpha X_p + \beta Y_p = \mu N_p$, α, β, $\mu \in \mathbb{R}_0$, then

$$g_p(Y,X) = -\frac{\alpha}{\beta} g_p(X,X) + \frac{\mu}{\beta} g_p(N,Y) = -\frac{\alpha}{\beta} g_p(X,X), \tag{2.6.21}$$

and

$$g_p(X,Y) = \frac{\mu}{\alpha} g_p(N,Y) - \frac{\beta}{\alpha} g_p(Y,Y) = -\frac{\beta}{\alpha} g_p(Y,Y). \tag{2.6.22}$$

Therefore,

$$\cos\theta = \frac{g_p(X,Y)}{\sqrt{g_p(X,X)\, g_p(Y,Y)}} = 1. \tag{2.6.23}$$

Nevertheless, the ratio (2.6.20) furnishes a well-posed definition for angles between the non-isotropic directions generated by X and Y. Such an angle ν, defined via (2.6.20), is called *null angle* between the directions $[X]$ and $[Y]$ [99]. Now, exploiting the isomorphism between $SO_c^+(1, d+1)$ and the connected component of the d-dimensional conformal group, from (A.0.6) we deduce that the metric $\mathrm{d}\tilde{s}_{\mathscr{I}^+}$ rescales by a function K^2, under the action of $SO_c^+(1, d+1)$. In particular,

$$K(\vec{\omega}) = \frac{1 + |\vec{\omega}|^2}{1 + |\vec{\omega}'|^2} \Lambda(\vec{\omega}), \tag{2.6.24}$$

where $\vec{\omega}'$ is the target of the map $\Lambda : \vec{\omega} \mapsto \vec{\omega}\,'$, $\Lambda \in SO_c^+(1, d+1)$, and the function $\Lambda(\vec{\omega})$ is the conformal factor in (A.0.6). Therefore

$$\mathrm{d}l = \sqrt{\mathrm{d}\tilde{s}^2_{\mathscr{I}^+}} \to K(\vec{\omega})\, \mathrm{d}l, \quad \vec{\omega} \in \mathbb{R}^d. \tag{2.6.25}$$

Correspondingly, the subset of transformations in (2.6.18) preserving null angles ν, must have the form

$$u \mapsto u' = K(\vec{\omega})\ (u + \alpha(\vec{\omega}))\,. \tag{2.6.26}$$

Definition 2.6.2. *The group of transformations preserving both angles and null angles on $\mathscr{I}^+$, or, equivalently, the group of conformal transformation of $\mathscr{I}^+$ respecting the strong conformal geometry is the BMS group.*

The general element of the BMS group is given by the coordinate transformation

$$u \mapsto u' = K(\vec{\omega})(u + \alpha(\vec{\omega})), \qquad \vec{\omega} \in \mathbb{R}^d; \tag{2.6.27}$$

$$\vec{\omega} \mapsto \vec{\omega}\,' = \Lambda[\vec{\omega}], \qquad \Lambda \in SO_c^+(1, d+1). \tag{2.6.28}$$

At least in 4 dimensions, the map $\kappa : (\Lambda_K, u) \mapsto K(\vec{\omega})\, u$, with $\Lambda_K \in SO_c^+(1,3)$ acting on $\mathrm{d}l$ as in (2.6.25), define an automorphism on the transformation group [82],

$$u \mapsto u' = u + \alpha(\vec{\omega}), \tag{2.6.29}$$

for all Λ_K. The subgroup (2.6.29) is called *supertranslations subgroup* and is usually denoted by $\mathcal{S}$. Therefore, at least in four dimensions, the BMS group can be always regarded as $BMS_4 = \mathcal{S} \rtimes_\kappa SO_c^+(1,3)$.

2.7 Compactifying Hyperbolic Slices

So far, we have conformally compactified Minkowski space-time following the Penrose's standard procedure, based on employing time retarded and advanced coordinates. In what follows, instead, we will exploit the same method to compactify each leaf in the hyperbolic foliation of Minkowski space-time [109–111]. More precisely, the conformal infinity of each slice of Minkowski will be a class of Riemannian manifolds that we will identify as the Celestial Sphere [33]. This approach enable us to apply techniques acquired over the years within the framework of the (EA)dS to investigate flat space holography, feeding the soil for the emergence of Celestial holography. The outcomes will be similar to those presented in the previous section, yet viewed through this new perspective that will guide us throughout the entire thesis.

To begin with, let us now consider the metric (2.1.5) in the region $\mathcal{A}_\pm$, that we rewrite here employing Poincaré coordinates on the unitary EAdS slice:

$$\mathrm{d}s^2_{\mathcal{A}_\pm} = -\mathrm{d}T^2 + \frac{T^2}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}\right). \tag{2.7.1}$$

$T = constant$ hypersurfaces are one-sheet hyperboloids $\mathcal{H}^T_{d+1}$ with radius T. Let us restrict our metric on one of these hypersurfaces,

$$\mathrm{d}s^2_{\mathcal{H}^T_{d+1}} = \frac{T^2}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}\right), \qquad y > 0,\ \vec{z}\in\mathbb{R}^d. \tag{2.7.2}$$

This metric blows up for $y \to 0$ and has an horizon for $y \to \infty$ [33]. We can compactify the space by setting $\Omega_{\mathcal{H}^T_{d+1}}(y) = y/T$. In this way, the boundary of the unphysical space $\widetilde{\mathcal{H}}^T_{d+1} = \mathcal{H}^T_{d+1} \sqcup \{y = 0\}$ is reached for $y \to 0$. The unphysical metric

$$\mathrm{d}\tilde{s}^2_{\mathcal{H}^T_{d+1}} = \frac{y^2}{T^2}\mathrm{d}s^2_{\mathcal{H}^+_{d+1}}, \tag{2.7.3}$$

does not diverge for $y \to 0$ and, restricted to the level-set hypersurface $y = 0$, reduces to the standard Euclidean metric $\mathrm{d}s^2_E = \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}$. Therefore, it turns out that the conformal boundary of each EAdS slice $\mathcal{H}^T_{d+1}$ is precisely the Celestial Sphere [109–111], represented in this case as the extended Euclidean plane $\tilde{\mathbb{E}}^d$. Indeed, by rescaling the conformal factor $\Omega_{\mathcal{H}^T_{d+1}}$ with a positive function θ, we span the entire equivalence class of Riemannian manifolds defined in (2.6.17). Therefore, by sending $y \to 0$ we move towards the conformal boundary over the single EAdS slices $\mathcal{H}^T_{d+1}$. Indeed, from Fig.(2.2) we realise that, geometrically, each curve $\gamma \subset \mathcal{H}^T_{d+1}$, parameterised as $\gamma = (y, \vec{z}_0)$, with $\vec{z}_0$ fixed, reaches smoothly a section of the light-cone $\mathcal{C}^+$ at infinity. This section represents the projective light-cone

$\mathcal{C}^{\pm}_{\sim}$, which can be therefore identified with the conformal boundary $\mathcal{CS}^{\pm}$ of $\mathcal{H}^{T}_{d+1}$, where the sign $\pm$ coincides with sign(T). Therefore, since $\mathcal{C}^{\pm}_{\sim}$ is diffeomorphic to the Poincaré section $\mathcal{C}^{\pm}_{P}$, the Celestial Sphere $\mathcal{CS}^{\pm}$ can be visualised as $\mathcal{C}^{\pm}_{P}$ in the Dirac's embedding formalism. The same construction can be carried out for $\mathcal{D}_{\pm}$. In this latter case, employing the Expanding/ Contracting Rindler Patch (see (2.2.9)), the metric assumes the form

$$\mathrm{d}s^2_{\mathcal{D}_{\pm}} = \mathrm{d}R^2 + \frac{R^2}{\eta_{\pm}^2}\left(-\mathrm{d}\eta_{\pm}^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}\right). \tag{2.7.4}$$

Setting $\mathrm{d}R = 0$ and $\Omega = \frac{\eta_{\pm}}{R}$, the unphysical metric becomes

$$\mathrm{d}\tilde{s}^2_{dS^{R,\pm}_{d+1}} = -\mathrm{d}\eta_{\pm}^2 + \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z}, \tag{2.7.5}$$

which, restricted on the boundary of the single dS slice, $\mathrm{d}\eta_{\pm} = 0$, gives

$$\mathrm{d}\tilde{s}^2_{\mathcal{CS}^{\pm}} = \mathrm{d}\vec{z}\cdot\mathrm{d}\vec{z} \equiv \mathrm{d}s^2_{E}. \tag{2.7.6}$$

This time, the far future, resp. past, Celestial Sphere is reached by sending $\eta_{\pm} \to 0$ in the Expanding, resp. Contracting, Poincaré patch of dS, showing how these patches have only one conformal boundaries. Globally, each dS slice has two conformal boundary: one in the far past and one in the far future. These boundaries are represented by the Poincaré sections of the light-cone in the Dirac's embedding formalism. This is also confirmed geometrically by looking at Fig. (2.1)

The construction showed in this section leads us to think of the conformal compactification of Minkowski in two different ways. The first way consists in following the standard Penrose's conformal compactification, which starts with a null-like foliation of Minkowski and leads to a codimension-1 conformal boundary, denoted as $\mathscr{I}$. This boundary exhibits a cone-like structure, with each section identified as a Celestial Sphere. In this first scenario Carrollian Holography, a theory stating the duality between a theory of gravity in flat space and a CFT on $\mathscr{I}$. The alternative approach follows the procedure detailed in this section. Employing the hyperbolic foliation of Minkowski, this alternative leads to a codimension-2 conformal boundary, comprising the disjoint union of two Celestial Spheres: one in the far past and one in the far future. In this second scenario are deepened the roots of Celestial Holography. It is therefore intuitive that a connection between Carrollian and Celestial holography should exist. In [112, 113], the authors studied this connection, bridging Celestial and Carrollian correlators through a Mellin-like tranformation in the variable $u_{\pm}$ over $\mathcal{I}^{\pm}$. In this thesis we will be adherent to the second scenario, exploiting the compactified hyperbolic foliation of Minkowski space-time to exploit all the knowledge about (EA)dS to compute correlators in Celestial Holography.

Chapter 3

Conformal Primary Bases in Quantum Field Theory

In the previous chapter, we introduced the general framework that will guide us in the exploration for the construction of a Flat/CFT correspondence. In particular, we ended up with a foliation of (EA)dS slices, closed at infinity by an equivalence class of Riemannian manifolds that we called Celestial Sphere $\mathcal{CS}^{\pm}$, where the $\pm$ sign stands for those slices reaching the conformal boundary at future/past infinity. The Celestial Sphere can always be represented as the extended Euclidean space and, in the Dirac's embedding space formalism, is visualised as the Poincaré section of the light-cone $\mathcal{C}_P^{\pm}$. In this chapter, we will apply this framework to the momentum space, developing the *Conformal Primary Bases* formulation of Quantum Field Theory (QFT). In this new formulation, fields are decomposed into a basis of wavefunctions transforming covariantly under the action of the Lorentz group, rather than Poincaré. Working in a $(d+2)$-dimensional Minkowski spacetime, under the action of $SO(1,d+1)$, these wavefunctions transform as conformal primary operators of conformal weight Δ, with Δ belonging to the *principal continuous series*, $\Delta = \frac{d}{2} + i\nu$. In particular, a basis of these wavefunctions is found for $\nu \geq 0$ in the massive case and for $\nu \in \mathbb{R}$ in the masseless one [3, 17, 18, 52, 114, 115].

Employing conformal primary bases, whose elements belong to irreducible representations of Lorentz instead of Poincaré, we may highlight new properties, usually hidden in the momentum basis formulation of QFT. Since the connected component of the BMS group can be always written as $SO_c(1,d+1) \rtimes \mathcal{S}$, with $\mathcal{S}$ supertranslation subgroup and $SO_c(1,d+1)$ connected component of $SO(1,d+1)$, from this perspective, it seems that Lorentz is more fundamental than Poincaré. Thus, developing a description of flat QFT based on wavefunctions transforming covariantly under the action of Lorentz instead of Poincaré may lead to more general properties, remaining true also when considering QFTs in asymptotically flat space-time.

The outline of this chapter is to study the conformal primary bases both in the massive and massless case. Particular emphasis will be given to the spinor case, mostly following [3], and to the scalar case [17, 18] for later purpose. We will start by analysing conformal primary basis in the massive scalar case, presenting the prescription of Pasterski, Shao and Strominger [17, 18]. Next, we will generalise the method, working out massive conformal primary bases both in the higher spin bosonic [116] and in the Dirac spinor case [3, 115]. In all of these examples, we will note that basis elements are labelled by conformal weight Δ belonging to only half of the principal continuous series. The reason for that must be tracked in the theory of the shadow transform [117–119], presented in section 3.4. There, we will furnish a quite general integral representation for the shadow transform of conformal primary operators [120, 121] and then we will proceed by computing shadow transformed conformal primary wavefunctions for Dirac spinors. In general, given $\Delta = \frac{d}{2} + i\nu$, in the massive case it comes out that conformal primary wavefunctions labelled by $\nu \geq 0$ are proportional to those with $\nu \leq 0$ through a shadow transform. This does not occur in the massless case. Therefore, in the massive case, only conformal primary wavefunctions with Δ belonging to half of the principal continuous series are independent and can be chosen to form a basis for fields in Minkowski. For further examples on the computation of shadow transformed conformal primary wavefunctions, we refer to [3, 18, 120]. Finally, in the last section of the chapter, we will explore the massless case, constructing massless conformal primary bases for scalars and Dirac spinors. A detailed study of the masseless spin 1 and 2 case, can be found in [18].

3.1 Massive Scalar Conformal Primary Basis

In what follows, we shall represent the Celestial Sphere $\mathcal{CS}^+$ as $\mathbb{R}^d$ rather than the extended d-dimensional Euclidean space $\widetilde{\mathbb{E}}^d \simeq \mathbb{R}^d \times P$. Therefore, ignoring the point P, let $X \in \mathbb{M}^{d+2}$ and $\vec{\omega} \in \mathbb{R}^d$. The *conformal primary wavefunctions*, $\Phi^{\pm}_{\Delta}(X; \vec{\omega})$, are solutions of the relativistic equation of motion transforming covariantly, under the action of $SO(1, d+1)$, as Lorentz tensors with respect to X and as conformal primary tensors of conformal weight Δ with respect to $\vec{\omega}$. To construct such wavefunctions, we will follow the Shao-Pasterski-Strominger prescription, which is based on the application of the Dirac's Embedding formalism in the momentum space. To fix the ideas, we will start by constructing the conformal primary wavefunctions in the scalar case and then we will advance to explore non-trivial spin cases.

Let us consider the massive scalar case. We are looking for solutions $\phi^{\pm}_{\Delta}(X; \vec{\omega})$

of the Klein-Gordon equation,

$$(\Box - m^2)\,\phi^{\pm}_{\Delta}(X;\vec{\omega}) = 0, \tag{3.1.1}$$

which transform as

$$\phi^{\pm}_{\Delta}(\Lambda X;\vec{\omega}\,') = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|^{-\Delta/d} \phi^{\pm}_{\Delta}(X;\vec{\omega}), \tag{3.1.2}$$

under the action of $SO(1, d+1)$. In Eq.(3.1.2), $\Lambda \in SO(1, d+1)$ is a linear matrix and $\vec{\omega}\,' = \Lambda[\vec{\omega}]$ is the target of the non-linear map representing the action of $SO(1, d+1)$ on $\mathbb{R}^d$. In the massive case, incoming (-) and outgoing (+) solutions of the Klein-Gordon equation are given by the Fourier expansion

$$\phi^{\pm}_{\Delta}(X,\vec{\omega}) = \int_{\mathcal{H}^{+}_{d+1}} [\mathrm{d}\hat{p}]\, G_{\Delta}(\hat{p};\vec{\omega}) \mathrm{e}^{\pm i m \hat{p}\cdot X}, \tag{3.1.3}$$

where $p = m\hat{p}$, $\hat{p}^2 = -1$ and $[\mathrm{d}\hat{p}]$ is the $SO(1, d+1)$ invariant measure over the unitary hyperboloid $\mathcal{H}^{+}_{d+1}$ in the momentum space. In cartesian coordinates, given a smooth function $f(\hat{p}) \in C^{\infty}(\mathcal{H}^{+}_{d+1}, \mathbb{R})$, the $SO(1, d+1)$-invariant measure [3, 18]

$$\int_{\mathcal{H}^{+}_{d+1}} [\mathrm{d}\hat{p}]\, f(\hat{p}) \equiv \int_{\mathcal{H}^{+}_{d+1}} \frac{\mathrm{d}^{d+1}\hat{p}^i}{\hat{p}^0} f(\hat{p}) \tag{3.1.4}$$

where $i = 1 \ldots, d+1$ and $\hat{p}^0 = \sqrt{\hat{p}^i \hat{p}^j \delta_{ij} + 1}$.
To make (3.1.5) a massive scalar conformal primary wavefunction, we impose that the function $G_{\Delta}(\hat{p};\vec{\omega})$ transforms as

$$G_{\Delta}(\Lambda\hat{p};\vec{\omega}\,') = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|^{-\Delta/d} G_{\Delta}(\hat{p};\vec{\omega}). \tag{3.1.5}$$

under the action of $SO(1, d+1)$. Embedding $\mathbb{R}^d \hookrightarrow \mathcal{C}^{+}_{P}$, $\vec{\omega} \mapsto q(\vec{\omega})$, in the momentum space, we recognise that the transformation property (3.1.5) is the same of the EAdS scalar bulk-to-boundary propagator, $\hat{p} \in \mathcal{H}^{+}_{d+1}$ being the bulk point and $q \in \mathcal{C}^{+}_{P}$ the boundary one. Therefore, taking $G(\hat{p}; q)$ as proportional to the up-lifted scalar bulk-to-boundary propagator,

$$G_{\Delta}(\hat{p}; q) = \frac{1}{(-2\hat{p}\cdot q)^{\Delta}}, \tag{3.1.6}$$

we find out the up-lifted conformal primary wavefunction $\phi^{\pm}_{\Delta}(X; q)$, $q \in \mathcal{C}^{+}_{p}$. Here, the Poincaré section $\mathcal{C}^{+}_{P}$ of the light-cone represents the far future Celestial Sphere $\mathcal{CS}^{+}$ in the momentum space. Therefore, the main difference between the momentum and the conformal primary basis description of QFT consists in the employment of a different manifold as parameter space. In the momentum basis description, the plane-wave parameter space is $\mathcal{H}^+_{d+1}$, instead, in the conformal primary basis description of QFT, the parameters of the conformal primary wavefunctions live in the far future Celestial Sphere $\mathcal{CS}^+$, meant as the conformal boundary of the hyperboloid $\mathcal{H}^+_{d+1}$. This suggests that the connection between these two different descriptions of QFT should be traced in the context of the AdS/CFT correspondence. From the completeness of the Harmonic functions in EAdS [75], the bulk-to-boundary propagator $G_\Delta(\hat{p}, q)$, with $\Delta = \frac{d}{2} + i\nu$, inherits two orthogonality relations [18]. The first one is

$$\int_0^{+\infty} \mathrm{d}\nu\, \mu(\nu) \int \mathrm{d}^d\omega\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{\omega}) G_{\frac{d}{2}-i\nu}(\hat{p}';\vec{\omega}) = \delta(\hat{p},\hat{p}'), \tag{3.1.7}$$

where $\delta(\hat{p},\hat{p}')$ is the covariant delta-function on $\mathcal{H}^+_{d+1}$ and

$$\mu(\nu) = \frac{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma\left(\frac{d}{2}-i\nu\right)}{2\pi^{d+1}\Gamma(i\nu)\Gamma(-i\nu)}. \tag{3.1.8}$$

The second one is

$$\begin{aligned}\int_{\mathcal{H}^+_{d+1}} [\mathrm{d}\hat{p}]\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{\omega}) G_{\frac{d}{2}-i\nu'}(\hat{p};\vec{\omega}\,') &= \frac{\delta(\nu-\nu')\delta^d(\vec{\omega}-\vec{\omega}\,')}{\mu(\nu)} \\ &+ 2\pi^{\frac{d}{2}+1} \frac{\Gamma(i\nu)}{\Gamma\left(\frac{d}{2}+i\nu\right)} \frac{\delta(\nu+\nu')}{|\vec{\omega}-\vec{\omega}\,'|^{2\left(\frac{d}{2}+i\nu\right)}}.\end{aligned} \tag{3.1.9}$$

When $\nu \geq 0$, we can use these properties to invert (3.1.5) and find

$$\mathrm{e}^{\pm im\hat{p}\cdot X} = \int_0^{+\infty} \mathrm{d}\nu\, \mu(\nu) \int \mathrm{d}^d\omega\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{\omega}) \phi^{\pm}_{\frac{d}{2}-i\nu}(X;\vec{\omega}). \tag{3.1.10}$$

This proves that the conformal primary wavefunction $\phi^\pm_\Delta(X;\vec{\omega})$ with $\Delta = \frac{d}{2} + i\nu$, $\nu \geq 0$, form a complete basis for Minkowski scalar fields. The series $\Delta = \frac{d}{2} + i\nu$, $\nu \in \mathbb{R}$, is called the *principal continuous series.* In (3.1.10) we take only half of this series because conformal primary wavefunctions with $\nu < 0$ are proportional to the ones with $\nu > 0$ through shadow transform (see Section 3.4). The Klein-Gordon inner product,

$$\begin{aligned}&\left(\phi^\pm_{\left(\frac{d}{2}+i\nu\right)}(\vec{\omega}),\, \phi^\pm_{\left(\frac{d}{2}+i\nu'\right)}(\vec{\omega}\,')\right) = \\ &\qquad = -i \int_\Sigma \mathrm{d}\Sigma^\mu \left(\phi^\pm_{\frac{d}{2}+i\nu}(X;\vec{\omega}) \partial_\mu \phi^\pm_{\frac{d}{2}+i\nu'}(X;\vec{\omega}\,') + c.c.\right),\end{aligned} \tag{3.1.11}$$

where $\Sigma \subset \mathbb{M}^{d+2}$ is a smooth space-like infinite hypersurface and $\mathrm{d}\Sigma^\mu$ is the external infinitesimal pseudo-vector associated to the $(d+1)$-form of volume on Σ, furnishes the following result

$$
\begin{aligned}
&\left(\phi^{\pm}_{\left(\frac{d}{2}+i\nu\right)}(\vec{\omega}), \phi^{\pm}_{\left(\frac{d}{2}+i\nu'\right)}(\vec{\omega}\,')\right) = \\
&\pm \frac{2^{d+3}\pi^{2d+2}}{m^d} \frac{\Gamma(i\nu)\Gamma(-i\nu)}{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma\left(\frac{d}{2}-i\nu\right)} \delta(\nu-\nu')\delta^d(\vec{\omega}-\vec{\omega}\,') \\
&\pm \frac{2^{d+3}\pi^{\frac{3d}{2}+2}}{m^d} \frac{\Gamma(i\nu)}{\Gamma\left(\frac{d}{2}+i\nu\right)} \frac{\delta(\nu-\nu')}{|\vec{\omega}+\vec{\omega}\,'|^{2\left(\frac{d}{2}+i\nu\right)}},
\end{aligned}
\tag{3.1.12}
$$

together with

$$
\left(\phi^{\pm}_{\left(\frac{d}{2}+i\nu\right)}(\vec{\omega}), \phi^{\pm}_{\left(\frac{d}{2}+i\nu'\right)}(\vec{\omega}\,')\right) = 0. \tag{3.1.13}
$$

Restricting $\nu \geq 0$, the second term in (3.1.12) drops out and we find out that the massive scalar conformal primary wavefunctions $\phi^{\pm}_{\Delta}(X;\vec{\omega})$, with $\Delta = \frac{d}{2}+i\nu$, $\nu \geq 0$, form a δ-normalizable orthogonal basis with respect to the Klein-Gordon inner product.

3.2 Massive Bosonic Conformal Primary Wavefunctions

The generalization to the massive higher spin bosonic spinning case is straightforward. Following [116], in this section we will develop the construction of totally symmetric and traceless spin-(s,J) conformal primary wavefunctions $\Phi^{\pm}_{\Delta,\{M_s\},\{a_J\}}(X;\vec{\omega})$ satisfying the Fierz-Pauli conditions, with $\{M_s\} = M_1,\dots M_s$ and $\{a_J\} = a_1,\dots,a_J$ bulk and boundary multi-index, respectively. In particular, we assume that the Fierz-Pauli conditions

$$
(\Box - m^2)\Phi^{\pm}_{\Delta,\{M_s\},\{a_J\}}(X;\vec{\omega}) = 0, \tag{3.2.1}
$$

$$
\nabla^{M_k}\Phi^{\pm}_{\Delta,M_1,\dots,M_k,\dots M_J,\{a_J\}}(X;\vec{\omega}) = 0, \tag{3.2.2}
$$

are valid for each component of the conformal primary wavefunctions, transforming as

$$
\begin{aligned}
\Phi_{\Delta,\{M_s\},\{a_J\}}(\Lambda X,\vec{\omega}\,') = &\frac{\partial\omega^{b_1}}{\partial\omega'^{a_1}}\cdots\frac{\partial\omega^{b_J}}{\partial\omega'^{a_J}}\left|\frac{\partial\vec{\omega}}{\partial\vec{\omega}\,'}\right|^{-(\Delta-J)} \\
&\times \Lambda_{M_1}{}^{N_1}\cdots\Lambda_{M_s}{}^{N_s}\Phi_{\Delta,\{M_s\},\{a_J\}}(X,\vec{\omega}).
\end{aligned}
\tag{3.2.3}
$$

Roughly speaking, we just need to replace in (3.1.3) the scalar bulk-to-boundary propagator $G_{\Delta}(\hat{p};\vec{\omega})$ with the spinning one. Starting from the spin-J free-index bulk-to-boundary propagator defined in (B.3.55), we can construct a spin-J conformal primary tensor by applying J-times the Thomas operator $D_M(Z)$, defined in (B.3.11), and then pulling the result back to the

Celestial Sphere. In particular, the Fourier transform of the bulk-free-index conformal primary wavefunction $\Phi^{\pm,s}_{\Delta,a_1,\dots,a_J}(X,W;\vec{\omega})$, transforming under the action of $SO(1,d+1)$ as a spin-s Lorentz tensor with respect to X and as a spin-J conformal primary operator with respect to $\vec{\omega}$, is taken to be (up to a normalization factor)

$$G^{(s)}_{\Delta,a_1,\dots,a_J}(\hat{p},W;\vec{\omega}) = \frac{1}{J!(\frac{d-2}{2})_J}(\partial_{\vec{\omega}}q\cdot D(Z))_{a_1}\cdots(\partial_{\vec{\omega}}q\cdot D(Z))_{a_J} \times (W\cdot\nabla_{\hat{p}})^{s-J}\Pi_{\Delta,J}(\hat{p},W;q(\vec{\omega}),Z), \tag{3.2.4}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol, $(\partial_{\vec{\omega}}q^M)_a = \frac{\partial q^M}{\partial\omega^a}$ and the factor $(W\cdot\nabla_{\hat{p}})^{s-J} \equiv (W\cdot\partial_{\hat{p}})^{s-J}$ has been properly employed in order to construct a (free-index) spin-s Lorentz tensor. The bulk-free-index tensor $G^{(s)}_{\Delta,a_1,\dots,a_J}(\hat{p},W;\vec{\omega})$ satisfies the orthogonality relation ($\Delta = \frac{d}{2}+i\nu$, $\nu\geq 0$)

$$\sum_{J=0}^{s}\int_0^{+\infty}\mathrm{d}\nu\,\mu_{J,s}(\nu)\int_{\mathbb{R}^d}\mathrm{d}^d\omega\,G_{\Delta,a_1,\dots,a_J}(\hat{p}_1,W_1;\vec{\omega}) \times G^{a_1,\dots,a_J}_{\Delta^*}(\hat{p}_2,W_2;\vec{\omega}) = (W_1\cdot W_2)^s\delta(\hat{p}_1,\hat{p}_2), \tag{3.2.5}$$

with

$$\mu_{J,s}(\nu) = \frac{2^{s-J}\mu(\nu)J!\left(\frac{d-2}{2}\right)_J(J+1)_{s-J}\left(J+\frac{d-1}{2}\right)_{s-J}}{(s-J)!(d-1+2J)_{s-J}\left(\frac{d}{2}-i\nu+J\right)_{s-J}\left(\frac{d}{2}+i\nu+J\right)_{s-J}}, \tag{3.2.6}$$

that is the generalization of (3.1.7) to the higher spin case [116]. We can use this property to invert the relation

$$\Phi^{\pm,s}_{\Delta,a_1,\dots,a_J}(X,W;\vec{\omega}) = \int_{\mathcal{H}^+_{d+1}}[\mathrm{d}\hat{p}]\,G^{(s)}_{\Delta,a_1,\dots,a_J}(\hat{p},W;\vec{\omega})\mathrm{e}^{\pm im\hat{p}\cdot X} \tag{3.2.7}$$

and to get

$$W_{M_1}\cdots W_{M_s}\,\mathrm{e}^{\pm im\hat{p}\cdot X} = \frac{1}{s!\left(\frac{d-2}{2}\right)_s}K_{M_1}(\widetilde{W})\cdots K_{M_s}(\widetilde{W})\sum_{J=0}^{s}\int_0^{+\infty}\mathrm{d}\nu\,\mu_{J,l}(\nu) \times\int_{\mathbb{R}^d}\mathrm{d}^d\omega\,G^{(s)}_{\Delta,a_1,\dots,a_J}(\hat{p},\widetilde{W};\vec{\omega})\,\Phi^{\pm,s,a_1,\dots,a_s}_{\Delta^*}(X,W;\vec{\omega}), \tag{3.2.8}$$

where we have applied s times the operator $K_M(W)$, defined in (B.3.46), to track out the spin-s Lorentz tensor from the associated polynomial in the free-index formalism. Eq.(3.2.8) demonstrates that the relation between the plane-waves and the conformal primary wavefunctions with $\Delta = \frac{d}{2}+i\nu$, $\nu\geq 0$, is invertible and, therefore, the latter form a complete basis for spin-s Minkowski tensors. It is important to note that, since the polynomial

$G^{(s)}_{\Delta,\{a_J\}}(\hat{p}, W; \vec{\omega})$ in the auxiliary vector W is Harmonic and transverse to the hypersurface $\hat{p}^2 = -1$, the action of the operator K_M on $G^{(s)}_{\Delta,\{a_J\}}(\hat{p}, W; \vec{\omega})$ reduces to (B.3.47). Therefore, being the action of $K_M(W)$ independent on both $\hat{p}$ and X, it follows that

$$\begin{aligned}\Phi^{\pm}_{\Delta,\{M_s\},\{a_J\}}(X; \vec{\omega}) &= \frac{1}{s!\left(\frac{d-2}{2}\right)_s} K_{M_1}(W) \cdots K_{M_s}(W) \Phi^{\pm,s}_{\Delta,\{a_J\}}(X, W; \vec{\omega}) \\ &= \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}\hat{p}]\, G_{\Delta,\{a_J\},\{M_s\}}(X; \vec{\omega})\, \mathrm{e}^{\pm i m \hat{p}\cdot X}, \end{aligned} \tag{3.2.9}$$

where

$$G_{\Delta,\{a_J\},\{M_s\}}(X; \vec{\omega}) = \frac{1}{s!\left(\frac{d-2}{2}\right)_s} K_{M_1}(W) \cdots K_{M_s}(W) G^{(s)}_{\Delta,\{a_J\}}(\hat{p}, W; \vec{\omega}), \tag{3.2.10}$$

is the Lorentz tensor associated to the polynomial $G^{(s)}_{\Delta,\{a_J\}}(\hat{p}, W; \vec{\omega})$, are the spin-$(s, J)$ massive conformal primary wavefunctions. By construction, the tensor $G_{\Delta,\{a_J\},\{M_s\}}(X; \vec{\omega})$ is transverse and, therefore, the conformal primary wavefunction $\Phi^{\pm}_{\Delta,\{M_s\},\{a_J\}}(X; \vec{\omega})$ automatically respects the Fierz-Pauli conditions (3.2.2). Further details about the higher spin bosonic conformal primary bases can be found in [116].

3.3 Massive Conformal Primary Basis for Dirac Spinors

The conformal primary basis for Dirac Spinors has been studied in [3] in general dimensions, and in [115], in four space-time dimensions. Here we will mainly follow [3]. Given $\Lambda \in SO(1, d+1)$, let $S(\Lambda)$ and $S_\Delta[\vec{\omega}]$ be the representative operators of Λ in the Minkowski spinor and Euclidean conformal primary spinor space, respectively. Then, represent the gamma matrices $\Gamma_\mu \in \mathcal{C}\ell_{d+2}(\Gamma, \eta)$ as in (A.3.16).

The conformal primary wavefunctions $\psi_\Delta(X; \vec{\omega})$

- Must be a solution of the $(d+2)$-dimensional Dirac equation,

$$(\Gamma^\mu \partial_\mu - m)\, \psi_\Delta(X; \vec{\omega}) = 0; \tag{3.3.1}$$

- Must transform as a Minkowski spinor with respect to X and as an (adjoint) *conformal primary spinor* with respect to $\vec{\omega}$, under the action of $SO(1, d+1)$.

In formulae, the wavefunction $\psi_\Delta(X;\vec{\omega})$ must satisfy the following transformation rule:

$$\psi_\Delta(X';\vec{\omega}\,') = S(\Lambda)\psi_\Delta(X;\vec{\omega})\,S_\Delta^{-1}[\vec{\omega}]\,, \tag{3.3.2}$$

where $X' = \Lambda X$ and $\vec{\omega}\,' = \Lambda[\vec{\omega}]$ are the target of the transformation map Λ. Working in analogy with the scalar case, let us consider the Fourier expansion

$$\psi_\Delta^\pm(X;\vec{\omega}) = \int_{\mathcal{H}_{d+1}^+} [\mathrm{d}\hat{p}]\, S_\Delta^\pm(\hat{p};\vec{\omega})\,\mathrm{e}^{\pm im\hat{p}\cdot X}, \tag{3.3.3}$$

where the signs label the positive and negative energy solutions. Also in this case, the Fourier transform $S_\Delta^\pm(\hat{p};\vec{\omega})$ must be constructed properly in order to ensure the transformation property (3.3.2). Moreover, inserting (3.3.3) in (3.3.1), one finds that

$$\begin{aligned}(\Gamma^\mu\partial_\mu - m)\,\psi_\Delta^\pm(X;\vec{\omega}) &= \int_{\mathcal{H}_{d+1}^+}[\mathrm{d}\hat{p}]\,(\pm im\hat{p}^\mu\Gamma_\mu - m)\,S_\Delta^\pm(\hat{p};\vec{\omega})\,\mathrm{e}^{\pm im\hat{p}\cdot X}\\ = (-m)\int_{\mathcal{H}_{d+1}^+}[\mathrm{d}\hat{p}]\;\Pi_\mp(\hat{p})S_\Delta^\pm(\hat{p};\vec{\omega})\,\mathrm{e}^{\pm im\hat{p}\cdot X},\end{aligned} \tag{3.3.4}$$

where $\Pi_\pm(\hat{p})$ are the projectors defined in (B.5.31). From the last line, it follows that $\psi_\Delta^\pm(X;\vec{\omega})$ is a solution of (3.3.1) if and only if

$$\Pi_\mp(\hat{p})\,S_\Delta^\pm(\hat{p};\vec{\omega}) = 0. \tag{3.3.5}$$

As demonstrated in (B.5.33), $\Pi_+(\hat{p})$ and $\Pi_-(\hat{p})$ are orthogonal to each other. This result suggests to place

$$S_\Delta^\pm(\hat{p},\vec{\omega}) = \Pi_\pm(\hat{p})\,\mathscr{S}_\Delta(\hat{p}\,;\vec{\omega})\,, \tag{3.3.6}$$

where the transformation property of $\mathscr{S}_\Delta(\hat{p}\,;\vec{\omega})$ has to be fixed. Inserting (3.3.6) in (3.3.2), it is possible to find out how $\mathscr{S}_\Delta(\hat{p}\,;\vec{\omega})$ must transform. Thus, given $\Lambda \in SO(1,d+1)$, imposing

$$\psi_\Delta^\pm(X;\vec{\omega}) = S^{-1}(\Lambda)\,\psi_\Delta^\pm(X';\vec{\omega}\,')\,S_\Delta[\vec{\omega}]$$

and exploiting both the Lorentz invariance of the measure $[\mathrm{d}\hat{p}]$ and the transformation property

$$\begin{aligned}\Pi_\pm(\Lambda\hat{p}) &= \frac{1}{2}\left[1 + (\Lambda_\mu{}^\nu\hat{p}_\nu)\,\Gamma^\mu\right] = \frac{1}{2}\left[1 + \hat{p}_\nu\left(S\Gamma^\nu S^{-1}\right)\right]\\ &= \frac{1}{2}S(\Lambda)\,(1 + \hat{p}^\mu\Gamma_\mu)\,S^{-1}(\Lambda) = S(\Lambda)\,\Pi_\pm(\hat{p})\,S^{-1}(\Lambda)\,,\end{aligned} \tag{3.3.7}$$

where

$$S^{-1}\Gamma^\mu S = \Lambda^\mu{}_\nu\Gamma^\nu, \tag{3.3.8}$$

has been used, we find out that

$$\mathscr{S}_\Delta(\Lambda\hat{p}\,;\vec{\omega}\,') = S(\Lambda)\,\mathscr{S}_\Delta(\hat{p}\,;\vec{\omega})\,S_\Delta^{-1}[\vec{\omega}] \tag{3.3.9}$$

must hold. All the requirements are readily verified by setting the Fourier coefficient $S(\hat{p}\,;\vec{\omega})$ as equal to the *spinor bulk-to-boundary propagator*,

$$S^{\pm}_{\Delta}(\hat{p}, q(\vec{\omega})) = \frac{\Pi_{\pm}(\hat{p})\begin{pmatrix} \not{\omega} \\ 1 \end{pmatrix}}{\big(-2\hat{p}\cdot q(\vec{\omega})\big)^{\Delta+\frac{1}{2}}}, \tag{3.3.10}$$

that we have already introduced in (B.5.39). Defining the polarization vector

$$\varphi(\vec{\omega}) = \begin{pmatrix} \not{\omega} \\ 1 \end{pmatrix}, \tag{3.3.11}$$

the conformal primary solutions of the $(d+2)$-dimensional Dirac equation are found to be

$$\psi^{\pm}_{\Delta}\,(X; q(\vec{\omega})) = \int_{\mathcal{H}^{+}_{d+1}} [\mathrm{d}\hat{p}]\, \frac{\Pi_{\pm}(\hat{p})\varphi(\vec{\omega})}{\big(-2\hat{p}\cdot q(\vec{\omega})\big)^{\Delta+\frac{1}{2}}} \mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.3.12}$$

By means of the completeness relation

$$\int_{0}^{+\infty} \mathrm{d}\nu\, \rho(\nu) \int_{\mathbb{R}^d} \mathrm{d}^d\omega\, S^{\pm}_{\Delta}(\hat{p}, q(\vec{\omega}))\overline{S^{\pm}_{\Delta}}(\hat{p}', q(\vec{\omega})) = \pm i\Pi_{\pm}(\hat{p})\delta(\hat{p},\hat{p}'), \tag{3.3.13}$$

with

$$\rho(\nu) = \frac{\Gamma\left(\frac{d+1}{2}+i\nu\right)\Gamma\left(\frac{d+1}{2}-i\nu\right)}{\pi^{d+1}\Gamma\left(\frac{1}{2}+i\nu\right)\Gamma\left(\frac{1}{2}-i\nu\right)}, \tag{3.3.14}$$

demonstrated in [3], Eq. (3.3.12) can be inverted and we get

$$\pm i\Pi_{\pm}\mathrm{e}^{\pm im\hat{p}\cdot X} = \int_{0}^{+\infty} \mathrm{d}\nu\, \rho(\nu) \int_{\mathbb{R}^d} \mathrm{d}^d\omega\, \psi^{\pm}_{\Delta+\frac{1}{2}}(X;\vec{\omega})\overline{S^{\pm}_{\Delta}}(\hat{p}', q(\vec{\omega})). \tag{3.3.15}$$

Therefore, also in this case, the last equation proves that the Dirac spinor conformal primary wavefunctions $\psi^{\pm}_{\Delta}(X;\vec{\omega})$ with $\Delta = \frac{d}{2}+i\nu$, $\nu \geq 0$, form a complete basis for Minkowski Dirac spinors. The wavefunctions with $\nu < 0$ are not independent, resulting proportional to the ones with $\nu > 0$ through shadow transform (see (3.4.40)). The Dirac inner product,

$$\left(\psi^{\pm}_{\left(\frac{d}{2}+i\nu\right)}(\vec{\omega}), \psi^{\pm}_{\left(\frac{d}{2}+i\nu'\right)}(\vec{\omega}\,')\right) = \int_{\Sigma} \mathrm{d}\Sigma^{\mu}\, \overline{\psi^{\pm}_{\frac{d}{2}+i\nu}}(X;\vec{\omega})\Gamma_{\mu}\psi^{\pm}_{\frac{d}{2}+i\nu'}(X;\vec{\omega}\,'), \tag{3.3.16}$$

where $\Sigma \subset \mathbb{M}^{d+2}$ is a smooth space-like infinite hypersurface and $\mathrm{d}\Sigma^{\mu}$ is the external infinitesimal pseudo-vector associated to the $(d+1)$-form of volume on Σ, gives the result

$$
\begin{aligned}
&\left(\psi^{\pm}_{\left(\frac{d}{2}+i\nu\right)}(\vec{\omega}), \psi^{\pm}_{\left(\frac{d}{2}+i\nu'\right)}(\vec{\omega}\,')\right) = \\
&= \left(\frac{2\pi^2}{m}\right)^{d+1} \left[\frac{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)}{\Gamma\left(\frac{d+1}{2}-i\nu\right)\Gamma\left(\frac{d+1}{2}+i\nu\right)}\,\delta^d(\vec{\omega}-\vec{\omega}\,')\delta(\nu-\nu')\right. \\
&\left.\pm\,\frac{i\Gamma\left(\frac{1}{2}-i\nu\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2}-i\nu\right)}\frac{\not{\omega}-\not{\omega}\,'}{|\vec{\omega}-\vec{\omega}\,'|^{d+1-2i\nu}}\delta(\nu+\nu')\right], \qquad (3.3.17)
\end{aligned}
$$

with the other products vanishing. The second term in (3.3.17) drops out by taking $\nu \geq 0$. Therefore, the Dirac conformal primary wavefunctions $\psi^{\pm}_{\Delta}(X;\vec{\omega})$, with $\Delta = \frac{d}{2}+i\nu$, $\nu \geq 0$, form an orthogonal δ-normalizable basis with respect to the Dirac inner product.

3.4 Shadow Transformation

Let $\phi(\vec{\omega}_k)$, with $\vec{\omega}_k \in \mathbb{R}^d$ and $k = 1,\ldots,4$, be conformal primary scalar fields defined on $\mathbb{R}^d$ with conformal weight Δ_k. Then, consider the four-point correlation function between them:

$$\langle\phi(\vec{\omega}_1)\phi(\vec{\omega}_2)\phi(\vec{\omega}_3)\phi(\vec{\omega}_4)\rangle. \qquad (3.4.1)$$

The shadow transform $\widetilde{\mathcal{O}}(\vec{\omega})$, $\vec{\omega} \in \mathbb{R}^d$, of a conformal primary scalar field $\mathcal{O}(\vec{\omega})$, with conformal weight Δ, was introduced in the beginning of 1970s by Gatto and Parisi to single out the contribution of the multiplet representation induced by $\mathcal{O}(\vec{\omega})$ to the four-point correlation function (3.4.1) [117–119]. To introduce the shadow transformation in the general tensor case [120], it is useful to reproduce, first, the method of Gatto and Parisi in the embedding formalism. Hence, consider the four-point correlation function

$$\big\langle\Phi(q_1)\Phi(q_2)\Phi(q_3)\Phi(q_4)\big\rangle, \qquad (3.4.2)$$

where $\Phi(q_k)$, with $q_k \in \mathcal{C}^+$, are the up-lifted fields associated to $\phi(\vec{\omega}_k)$. The projective null-cone $\mathcal{C}^+_\sim$, as a submanifold of $\mathbb{M}^{d+2}$, is endowed with a natural measure defined as follows: let $f(q)$ be a smooth function on the null-cone $\mathcal{C}^+$, such that $f(\lambda q) = \lambda^{-d} f(q)$; then it is possible to define the integral of $f(q)$ over $\mathcal{C}^+_\sim$ as [120]

$$\int_{\mathcal{C}^+}[\mathrm{d}q]\;f(q) \equiv \frac{2}{\mathrm{Vol\;GL}\,(1,\mathbb{R})}\int_{q^++q^-\geq 0}\mathrm{d}^{d+2}q\,\delta(q^2)f(q), \qquad (3.4.3)$$

where the integral

$$\int_{q^++q^-\geq 0}\mathrm{d}^{d+2}q\,\delta(q^2)f(q), \qquad (3.4.4)$$

is formally infinite because of the rescaling invariance $q \mapsto \lambda q$, $\lambda \in \mathbb{R}^+$, and hence it must be quotiented by Vol GL$(1,\mathbb{R})$. Concretely, the integral (3.4.3) must be solved by gauge-fixing the redundancy with respect to GL$(1,\mathbb{R})$ introducing the Faddeev-Popov determinant. For example, by gauge fixing $\lambda^{-1} = q^+$, the right hand side in (3.4.4) reduces to an integral over $\mathcal{C}_P^+$, which coincides with an ordinary integral over flat space by embedding $\mathbb{R}^d \hookrightarrow \mathcal{C}_P^+$, $\vec{\omega} \mapsto q(\vec{\omega})$. Exploiting this measure, in the embedding formalism the up-lifted projector of Gatto and Parisi can be defined as

$$|\mathcal{O}| \equiv \frac{1}{\mathcal{N}_{\mathcal{O}}} \int_{\mathcal{C}^+} [\mathrm{d}q] \; |\mathcal{O}(q)\rangle\langle\tilde{\mathcal{O}}(q)|, \tag{3.4.5}$$

where

$$\widetilde{\mathcal{O}}(q) = \int_{\mathcal{C}^+} [\mathrm{d}q'] \, \frac{1}{(-2q\cdot q')^{d-\Delta}} \mathcal{O}(q'), \tag{3.4.6}$$

is an up-lifted field with homogeneity degree $d-\Delta$, which is called the up-lifted *shadow transform* of $\mathcal{O}(q)$. In general, the four-point function (3.4.2) can be decomposed as follows:

$$\langle \Phi(q_1)\Phi(q_2)\Phi(q_3)\Phi(q_4)\rangle = \sum_{\mathcal{O}\text{ primaries}} \lambda_{12O}\lambda_{34O} W_O(q_k), \tag{3.4.7}$$

where the coefficients $W_O(q_k)$ are purely kinematical objects, called up-lifted *conformal partial waves*. These objects are completely determined by conformal symmetry and contain all the information about the contribution to the four-point function (3.4.2) from the up-lifted $\mathcal{O}$-multiplet. Furthermore, the coefficients λ_{hkO}, with $h, k = 1, \ldots, 4$, correspond to those present in the three-point correlation functions,

$$\langle \phi(\vec{\omega}_h)\phi(\vec{\omega}_k)\mathcal{O}(\vec{\omega})\rangle = \frac{\lambda_{hkO}}{|\vec{\omega}_h - \vec{\omega}_k|^{\Delta_h+\Delta_k-\Delta}|\vec{\omega}_h - \vec{\omega}|^{\Delta_h+\Delta-\Delta_k}|\vec{\omega}_k - \vec{\omega}|^{\Delta_k+\Delta-\Delta_h}}, \tag{3.4.8}$$

whose shape is entirely determined by conformal symmetry up to λ_{hkO}, that must be determined through (3.4.7).

It is well-known that conformal partial waves can be expressed in terms of the operator (3.4.5) (see for example [122]). In the scalar case, a natural candidate for the up-lifted partial wave $W_{\mathcal{O}}(q_k)$ is [120]

$$W_{\mathcal{O}}^M(q_i) = \frac{1}{\mathcal{N}_{\mathcal{O}}} \int_{\mathcal{C}^+} [\mathrm{d}q] \, [\mathrm{d}q'] \; \langle \Phi(q_1)\Phi(q_2)\mathcal{O}(q)\rangle \frac{1}{(-2q\cdot q')^{d-\Delta}} \langle \mathcal{O}(q')\Phi(q_3)\Phi(q_4)\rangle, \tag{3.4.9}$$

where Δ is the homogeneity degree of the up-lifted operator $\mathcal{O}(q)$ and $\mathcal{N}_{\mathcal{O}}$ is a normalizing factor. Defining the monodromy operator M_m,

$$M_m : q_{ij} \equiv -2q_i \cdot q_j \mapsto \mathrm{e}^{4\pi i} q_{ij}, \quad i, j < m, \quad m \in \mathbb{N}, \tag{3.4.10}$$

the correct partial wave $W_{\mathcal{O}}(q_i)$ is obtained by computing the integral (3.4.9) and then projecting the result onto the eigenspace related to the eigenvalue of the monodromy operator $M_4 = \mathrm{e}^{2\pi i(\Delta-\Delta_1-\Delta_2)}$ [120].

Now, let us proceed by constructing the operator of Gatto and Parisi in the tensor case. Therefore, let $\phi_a(\vec{\omega})$, with a $SO(d)$-multindex, be a conformal primary operator with conformal weight Δ and $\Phi_I\big(q(\vec{\omega})\big)$, with I $SO(1,d+1)$-multindex, be its up-lifted operator. Further, let $\mathcal{O}^I(q)$ be an up-lifted operator with homogeneity degree Δ, belonging to an irreducible representation $(J,\bar{J})$ of $SO(1,d+1)$. As shown in the embedding formalism, a generic uplifted operator $\mathcal{O}^I(q)$ is defined up to a certain gauge transformation. In the generic tensor case, one has to impose that such a gauge transformation must leave unchanged the projector $|\mathcal{O}|$. Let $\widehat{\mathcal{O}}^I(q)$ be the adjoint of the operator $\mathcal{O}^I(q)$ with respect to q. Following [120], up to a normalization factor, the generic form of $|\mathcal{O}|$ is given by

$$|\mathcal{O}| \equiv \int_{\mathcal{C}^+} [\mathrm{d}q]\,[\mathrm{d}q']\;|\mathcal{O}^J(q)\rangle \frac{\Pi(q,q')_J^K}{(-2q\cdot q')^{d+\Delta+\deg\Pi}} \langle\widehat{\mathcal{O}}_K(q')|, \tag{3.4.11}$$

where the tensor $\Pi(q,q')_J^K$ is properly built to ensures gauge-invariance. Further, $\deg\Pi$ is the homogeneity degree of $\Pi(q,q')$, which is the same both in q and q'. In total, the homogeneity degree of the integrand is $-d$ both in q and in q', as required in the definition (3.4.3).

This general construction can be exploited to build up the projector $|\mathcal{O}|$ in the spinor case [3]. Therefore, let $\Psi(q)$ be an up-lifted spinor with homogeneity degree Δ_ψ. As shown in the embedding formalism, an up-lifted spinor is defined up to gauge term of the form $q^\mu\Gamma_\mu\tilde{U}$, where $\tilde{U}$ is a generic Minkowski spinor. Since the projector (3.4.11) must be invariant under gauge transformation, consider

$$q^\mu\Gamma_\mu\Psi(q)|0\rangle\langle 0|\widehat{\Psi}(q')\Gamma_\nu q'^\nu, \tag{3.4.12}$$

that is gauge-invariant. Indeed, sending

$$\Psi(q)\mapsto\Psi(q)+q^\mu\Gamma_\mu\tilde{U}, \tag{3.4.13}$$

one finds

$$\begin{aligned}\Big(q^\mu\Gamma_\mu\Psi(q)+q^\mu\Gamma_\mu q^\nu\Gamma_\nu\tilde{U}\Big)\,|0\rangle\langle 0|\,\Big(\widehat{\Psi}(q')\Gamma_\nu q'^\nu+\bar{\tilde{U}}q'^\mu\Gamma_\mu q'^\nu\Gamma_\nu\Big) &=\\ = q^\mu\Gamma_\mu\Psi(q)|0\rangle\langle 0|\widehat{\Psi}(q')\Gamma_\nu q'^\nu&,\end{aligned} \tag{3.4.14}$$

where we used $q^2 = {q'}^2 = 0$. However, using (3.3.8), under the action of $SO(1,d+1)$ the object in (3.4.12) transforms as

$$\begin{aligned}\Lambda_\mu{}^\nu q_\nu\Gamma^\mu\Psi'(\Lambda q)\,|0\rangle\langle 0|\bar{\Psi}'(\Lambda q')\,\Gamma^\nu\Lambda_\nu{}^\mu q_\mu &=\\ = S(\Lambda)\,q^\mu\Gamma_\mu\Psi(q)|0\rangle\langle 0|\bar{\Psi}(q')\Gamma_\nu q'^\nu S^{-1}(\Lambda)&,\end{aligned} \tag{3.4.15}$$

proving that it is not $SO(1,d+1)$-invariant. Nevertheless, using the cyclic property of the trace, we note that

$$\begin{aligned}&\mathrm{Tr}\Big[S\left(\Lambda\right)q^{\mu}\Gamma_{\mu}\Psi(q)|0\rangle\langle 0|\widehat{\Psi}(q')\Gamma_{\nu}q'^{\nu}S^{-1}\left(\Lambda\right)\Big]=\\&=\mathrm{Tr}\Big[q^{\mu}\Gamma_{\mu}\Psi(q)|0\rangle\langle 0|\widehat{\Psi}(q')\Gamma_{\nu}q'^{\nu}S^{-1}\left(\Lambda\right)S\left(\Lambda\right)\Big]\\&=\mathrm{Tr}\Big[q^{\mu}\Gamma_{\mu}\Psi(q)|0\rangle\langle 0|\widehat{\Psi}(q')\Gamma_{\nu}q'^{\nu}\Big].\end{aligned}\tag{3.4.16}$$

Therefore, the trace of (3.4.12) is both Lorentz and gauge invariant. In components, it is possible to rewrite

$$\mathrm{Tr}\Big[q^{\mu}\Gamma_{\mu}\Psi(q)|0\rangle\langle 0|\bar{\Psi}(q')\Gamma_{\nu}q'^{\nu}\Big]=\big|\Psi^{\alpha}(q)\big\rangle\left(q^{\mu}\Gamma_{\mu}q'^{\nu}\Gamma_{\nu}\right)_{\alpha}^{\beta}\big\langle\widehat{\Psi}_{\beta}(q')\big|,\tag{3.4.17}$$

which has the form of the numerator in the integral of (3.4.11). Therefore, it is possible to define

$$\Pi_{\psi}(q,q')=q^{\mu}\Gamma_{\mu}q'^{\nu}\Gamma_{\nu},\tag{3.4.18}$$

and to set

$$|\Psi|=\int_{\mathcal{C}^{+}}[\mathrm{d}q]\,[\mathrm{d}q']\,\frac{\big|\Psi^{\alpha}(q)\big\rangle\left(q^{\mu}\Gamma_{\mu}q'^{\nu}\Gamma_{\nu}\right)_{\alpha}^{\beta}\big\langle\widehat{\Psi}_{\beta}(q')\big|}{\left(-2q\cdot q'\right)^{d-\Delta+\frac{1}{2}}},\tag{3.4.19}$$

with $\deg\Pi_{\psi}=1$ and $\Delta_{\psi}=-\Delta-\frac{1}{2}$ (cf.(B.4.1)). Finally, in analogy to the scalar case, it is possible to define the shadow operator of $\Psi(q)$, in the up-lifted space, as

$$\widetilde{\Psi}(q)=\int_{\mathcal{C}^{+}}[\mathrm{d}q']\,\frac{\widehat{\Psi}(q')q'^{\mu}\Gamma_{\mu}}{\left(-2q\cdot q'\right)^{d-\Delta+\frac{1}{2}}}.\tag{3.4.20}$$

Consider, now, the massive spinor conformal primary wavefunctions $\psi^{\pm}_{\Delta}(X;\vec{\omega})$, defined in (3.3.12). In the free index formalism, they can be rewritten as

$$\psi^{\pm}_{\Delta}(X;\vec{\omega},s)=\int_{\mathcal{H}^{+}_{d+1}}[\mathrm{d}p]\,\frac{\Pi_{\pm}(\hat{p})\varphi(\vec{\omega},s)}{\left(-2\hat{p}\cdot q(\vec{\omega})\right)^{\Delta+\frac{1}{2}}}\mathrm{e}^{\pm im\hat{p}\cdot X},\tag{3.4.21}$$

where s is an Euclidean polarization spinor and

$$\varphi(\vec{\omega},s)=\varphi_{\dot{\alpha}}s^{\dot{\alpha}}=\begin{pmatrix}\not{\omega}s\\ s\end{pmatrix}.\tag{3.4.22}$$

Note that the spinor $\varphi(\vec{\omega}, s)$ has the same form of the Minkowski polarization spinor (B.4.19). Thus, the up-lifted massive spinor conformal primary wavefunctions in the free index formalism are obtained by dropping the s- and $\vec{\omega}$-dependence:

$$\Psi^{\pm}_{\Delta}(X; q, \varphi) = \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}p] \, \frac{\Pi_{\pm}(\hat{p})\varphi}{(-2\hat{p}\cdot q)^{\Delta+\frac{1}{2}}} \mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.4.23}$$

To recover the up-lifted massive spinor conformal primary wavefunctions from the free index formalism, one has to derive with respect to the polarization spinor φ, so that one obtains

$$\Psi^{\pm}_{\Delta}(X; q) = \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}p] \, \frac{\Pi_{\pm}(\hat{p})}{(-2\hat{p}\cdot q)^{\Delta+\frac{1}{2}}} \mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.4.24}$$

By construction, the conformal primary wavefunctions $\psi^{\pm}_{\Delta}(X; \vec{\omega})$ transform as adjoint conformal spinors with respect to $\vec{\omega}$ (cf.(3.3.2)). Therefore, the up-lifted shadow transform of $\Psi^{\pm}_{\Delta}(X; q)$ are

$$\widetilde{\Psi^{\pm}_{\Delta}}(X; q') = \int_{\mathcal{C}^+} [\mathrm{d}q] \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}p] \, \frac{\Pi_{\pm}(\hat{p}) q^{\mu}\Gamma_{\mu}}{(-2q\cdot q')^{d-\Delta+\frac{1}{2}} (-2\hat{p}\cdot q)^{\Delta+\frac{1}{2}}} \mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.4.25}$$

To find out their closed expression, let us compute the integral

$$\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{q^{\mu}}{(-2q\cdot q')^{a} (-2\hat{p}\cdot q)^{b}}, \quad a = d - \Delta + \frac{1}{2}, \, b = \Delta + \frac{1}{2}. \tag{3.4.26}$$

It is possible to note that

$$\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{q^{\mu}}{(-2q\cdot q')^{a} (-2\hat{p}\cdot q)^{b}} = \frac{1}{2(a-1)} \frac{\partial}{\partial q'_{\mu}} \int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{1}{(-2q\cdot q')^{a-1}} \frac{1}{(-2\hat{p}\cdot q)^{b}}. \tag{3.4.27}$$

By means of the Feynman-Schwinger parameterisation [120]

$$\frac{1}{\prod_{i=1}^{n} A_i^{a_i}} = \frac{\Gamma\left(\sum_i a_i\right)}{\prod_i \Gamma\left(a_i\right)} \prod_{j=2}^{n} \int_0^{+\infty} \frac{\mathrm{d}\alpha_j}{\alpha_j} \alpha_j^{a_j} \frac{1}{\left(A_1 + \sum_{j=2}^{n} \alpha_j A_j\right)^{\sum_{k=2}^{n} a_k}}, \tag{3.4.28}$$

we can rewrite

$$\begin{aligned} &\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{1}{(-2q\cdot q')^{a-1}} \frac{1}{(-2\hat{p}\cdot q)^{b}} = \\ &= \frac{\Gamma(d)}{\Gamma(a-1)\Gamma(b)} \int_0^{+\infty} \frac{\mathrm{d}\alpha}{\alpha} \alpha^{a-1} \int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{1}{(-2(\hat{p}+\alpha q')\cdot q)^{d}}. \end{aligned} \tag{3.4.29}$$

Setting $y = \hat{p} + \alpha q'$, it follows that $y^2 < 0$. The integral

$$\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{1}{(-2y\cdot q)^{d}}, \quad y^2 < 0, \tag{3.4.30}$$

has been already solved in [120]; it gives

$$\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{1}{(-2y \cdot q)^d} = \frac{\pi^{d/2}\Gamma(d/2)}{\Gamma(d)} \frac{1}{(-y^2)^{\frac{d}{2}}}. \tag{3.4.31}$$

Therefore, it remains to solve

$$\frac{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2(a-1)\Gamma(a-1)\Gamma(b)} \frac{\partial}{\partial q'_\mu} \int_0^{+\infty} \frac{\mathrm{d}\alpha}{\alpha} \alpha^{a-1} \frac{1}{(-\hat{p}^2 - 2\alpha \hat{p} \cdot q' - \alpha^2 q'^2)^{\frac{d}{2}}}. \tag{3.4.32}$$

In the computation, the q'-derivative must be calculated before than plugging known numerical values in the integrand. Therefore, let us perform the derivative first,

$$\begin{aligned} &\frac{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2(a-1)\Gamma(a-1)\Gamma(b)} \frac{\partial}{\partial q'_\mu} \int_0^{+\infty} \frac{\mathrm{d}\alpha}{\alpha} \alpha^{a-1} \frac{1}{(-\hat{p}^2 - 2\alpha \hat{p} \cdot q' - \alpha^2 q'^2)^{\frac{d}{2}}} = \\ &\frac{\frac{d}{2}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}{(a-1)\Gamma(a-1)\Gamma(b)} \int_0^{+\infty} \frac{\mathrm{d}\alpha}{\alpha} \alpha^{a-1} \frac{\alpha\hat{p}^\mu + \alpha^2 q'^\mu}{(-\hat{p}^2 - 2\alpha \hat{p} \cdot q' - \alpha^2 q'^2)^{\frac{d}{2}+1}}. \end{aligned} \tag{3.4.33}$$

and then insert $q'^2 = 0$ and $\hat{p}^2 = -1$. Splitting the integrand in the RHS of (3.4.33) in two terms and solving both the resulting integral by employing the Euler Beta function,

$$B(x, y) = \int_0^{+\infty} \frac{\mathrm{d}t}{t} t^x (1+t)^{-(x+y)} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.4.34}$$

lead us to

$$\begin{aligned} &\int_{\mathcal{C}^+} [\mathrm{d}q] \, \frac{q^\mu}{(-2q \cdot q')^{d-\Delta+\frac{1}{2}} (-2\hat{p} \cdot q)^{\Delta+\frac{1}{2}}} = \\ =&\frac{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2} + \frac{1}{2}\right)}{\Gamma\left(\Delta + \frac{1}{2}\right)} \left[\frac{\hat{p}^\mu}{(-2\hat{p} \cdot q')^{d-\Delta+\frac{1}{2}}} + \frac{d - \Delta + \frac{1}{2}}{\Delta - \frac{d+1}{2}} \frac{q'^\mu}{(-2\hat{p} \cdot q')^{d-\Delta+\frac{3}{2}}} \right], \end{aligned} \tag{3.4.35}$$

where $z\Gamma(z) = \Gamma(z+1)$, $z \notin \mathbb{Z}^-$, has been used. Hence, it is possible to assert that the up-lifted shadow conformal primary wavefunctions is given by

$$\widetilde{\Psi^{\pm}_{\Delta}}(X; q) = \frac{\pi^{\frac{d}{2}}\Gamma\left(\Delta - \frac{d}{2} + \frac{1}{2}\right)}{\Gamma\left(\Delta + \frac{1}{2}\right)} \int_{\mathcal{H}^+_{d+1}} [\mathrm{d}p] \, \frac{\Pi_\pm(\hat{p})\hat{p}_\mu \Gamma^\mu}{(-2\hat{p} \cdot q)^{d-\Delta+\frac{1}{2}}} \mathrm{e}^{\pm im\hat{p}\cdot X} + q^\mu \Gamma_\mu \Theta(q), \tag{3.4.36}$$

where

$$\Theta(q) = \frac{\pi^{\frac{d}{2}}(d-\Delta+\frac{1}{2})\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+\frac{1}{2}\right)}{(\Delta-\frac{d+1}{2})\Gamma\left(\Delta+\frac{1}{2}\right)}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}p]\,\frac{\Pi_\pm(\hat{p})}{(-2\hat{p}\cdot q)^{d-\Delta+\frac{3}{2}}}\mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.4.37}$$

is a pure gauge term, and hence can be dropped out. Next, since

$$\Pi_\pm(\hat{p})\Gamma^\mu\hat{p}_\mu = \frac{1}{2}\left(1\pm i\hat{p}_\nu\Gamma^\nu\right)\hat{p}_\mu\Gamma^\mu = \mp i\Pi_\pm(\hat{p}), \tag{3.4.38}$$

it is possible to rewrite

$$\widetilde{\Psi^\pm_\Delta}(X;q) = \frac{\mp i\,\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+\frac{1}{2}\right)}{\Gamma\left(\Delta+\frac{1}{2}\right)}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}p]\,\frac{\Pi_\pm(\hat{p})}{(-2\hat{p}\cdot q)^{d-\Delta+\frac{1}{2}}}\mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{3.4.39}$$

Finally, contracting $\widetilde{\Psi}^\pm_\Delta(X;q)$ with the polarization spinor $\varphi(\vec{\omega})$ in (3.4.22) and parameterising $q=q(\vec{\omega})$, as in (B.0.2), one finds the shadow transforms of the massive conformal primary wavefunctions [3]

$$\begin{aligned}\widetilde{\psi^\pm_\Delta}(X;\vec{\omega}) &= \mp i\pi^{\frac{d}{2}}\frac{\Gamma\left(\Delta-\frac{d-1}{2}\right)}{\Gamma\left(\Delta+\frac{1}{2}\right)}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}p]\,\frac{\Pi_\pm(\hat{p})\varphi(\vec{\omega})}{(-2\hat{p}\cdot q)^{d-\Delta+\frac{1}{2}}}\mathrm{e}^{\pm im\hat{p}\cdot X}\\ &\equiv \mp i\pi^{\frac{d}{2}}\frac{\Gamma\left(\Delta-\frac{d-1}{2}\right)}{\Gamma\left(\Delta+\frac{1}{2}\right)}\psi^\pm_{d-\Delta}(X;\vec{\omega}).\end{aligned} \tag{3.4.40}$$

3.5 Massless Conformal Primary Basis

In this section, we will perform the construction of the conformal primary bases in the massless case [18]. We will achieve this goal by simply starting from the massive case and then taking the massless limit to recover massless conformal primary wavefunctions. However, as will result manifest in Section 4.2, where we will compute the closed expression of (3.1.3), the massless limit of the conformal primary wavefunctions does not exists. Indeed, in (4.2.9), the argument of the modified Bessel function vanishes for $m\to 0$. In general, when α is not integer, the modified Bessel function $K_\alpha(z)$ has a branch point at $z=0$, with the associated branch cut extending through the entire negative real semi-axis $\mathbb{R}^-$. Nevertheless, even though the massless limit of the massive conformal primary wavefunctions does not exist, we can track out a finite expression for the massless conformal primary wavefunctions by stopping the limit operation right before the branch point $m=0$. Here below, we will main follow [18] and [3], where massless conformal primary wavefunctions have been studied in the boson and spinor case, respectively. Since the scalar case is essentially contained in the spinor one, in this section, we will start constructing massless spinor conformal primary wavefunctions and then we will show analogous results in the scalar case. For other results in the spin 1 and 2 case, we refer to [18].

Massless Spinor Conformal Primary Wavefunctions: Following [3], let us rewrite the massive conformal primary solutions $\psi^{\pm}_{\Delta}(X;\vec{\omega})$, $\Delta \in \frac{d}{2} + i\mathbb{R}^{+}$, by representing $\hat{p} \in \mathcal{H}^{+}_{d+1}$ in Poincaré coordinates,

$$\psi^{\pm}_{\Delta}(X;\vec{\omega}) = \left(\frac{m}{2\pi^2}\right)^{\frac{d+1}{2}} \frac{\Gamma\left(\Delta+\frac{1}{2}\right)}{\Gamma\left(\Delta-\frac{d-1}{2}\right)} \int_0^{+\infty} \frac{\mathrm{d}y}{y^{d+1}} \int \mathrm{d}^d z\, \mathrm{e}^{\pm i\left(\frac{m}{y}q(\vec{z})+myN_-\right)\cdot X}$$
$$\times \left(\frac{y}{y^2+|\vec{z}-\vec{\omega}|^2}\right)^{\Delta+\frac{1}{2}} \frac{1}{2}\left(1 \pm \frac{i}{y} q^{\mu}(\vec{z})\Gamma_{\mu} \pm iy\Gamma_-\right)\varphi(\vec{\omega}), \quad (3.5.1)$$

where N_- is the vector of components $N^M_- = \delta^M_-$, in light-cone coordinates, and where we have renormalised the spinor conformal primary wavefunction with respect to the Dirac inner product (3.3.16). Next, perform the rescaling

$$y \to \frac{m}{y}. \quad (3.5.2)$$

In this new variable, send $m \to 0$ keeping y constant. Actually, one can note that this is equivalent to taking y small in the old coordinates. Therefore, to obtain the behaviour for small m of the bulk-to-boundary propagator $G_{\Delta}(y,\vec{z};\vec{\omega})$, it is possible to consider its distribution behaviour for small y first [18],

$$\left(\frac{y}{y^2+|\vec{z}-\vec{\omega}|^2}\right)^{\Delta} \underset{y\to 0}{\to} \pi^{\frac{d}{2}} \frac{\Gamma\left(\Delta-\frac{d}{2}\right)}{\Gamma\left(\Delta\right)} y^{d-\Delta}\delta^d\left(\vec{z}-\vec{\omega}\right) + \frac{y^{\Delta}}{|\vec{z}-\vec{\omega}|^{2\Delta}} + \dots, \quad (3.5.3)$$

and then substitute $y \to \frac{m}{y}$. The neglected terms in (3.5.3) are in powers of y^2 with respect the showed ones. Substituting (3.5.3) into (3.5.1) and using

$$\Gamma^{\mu}q_{\mu}(\vec{\omega})\varphi(\vec{\omega}) = \begin{pmatrix} \not{\omega} & -|\vec{\omega}|^2 \\ 1 & -\not{\omega} \end{pmatrix} \begin{pmatrix} \not{\omega} \\ 1 \end{pmatrix} = 0, \quad (3.5.4)$$

one finds the following behaviour for small m of the massive spinor conformal primary wavefunctions:

$$\psi^{\pm}_{\Delta}(X;\vec{\omega}) \underset{m\ll 1}{\to} m^{\frac{d}{2}-\Delta}\Upsilon^{\pm}_{\Delta}(X;\vec{\omega}) +$$
$$\pm\, m^{\Delta-\frac{d}{2}} \frac{i\Gamma\left(\Delta+\frac{1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d-1}{2}\right)} \int \mathrm{d}^d z\, \Upsilon^{\pm}_{d-\Delta}(X;\vec{z}) \frac{\not{\omega}-\not{z}}{|\vec{\omega}-\vec{z}|^{2\Delta+1}} + \dots, \quad (3.5.5)$$

where

$$\Upsilon^{\pm}_{\Delta}(X;\vec{\omega}) = \frac{1}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \int_0^{+\infty} \mathrm{d}y\, y^{\Delta-\frac{1}{2}}\, \mathrm{e}^{\pm iyq(\vec{\omega})\cdot X} \varphi(\vec{\omega}). \tag{3.5.6}$$

Eq.(3.5.5) shows that the massless limit of the massive spinor conformal primary solutions $\psi(X;\vec{\omega})$ does not exist, because of the complex phases $m^{\mp(\Delta-\frac{d}{2})}$. Nevertheless, from the closed-expression

$$\Upsilon^{\pm}_{\Delta}(X;\vec{\omega}) = \frac{(\mp 2i)^{\Delta+\frac{1}{2}}\,\Gamma\left(\Delta+\frac{1}{2}\right)}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \frac{\varphi(\vec{\omega})}{(-2q(\vec{\omega})\cdot X \mp i\epsilon)^{\Delta+\frac{1}{2}}}, \tag{3.5.7}$$

it is easy to verify that

$$\Gamma^{\mu}\partial_{\mu}\Upsilon^{\pm}_{\Delta}(X,\vec{\omega}) = \frac{(\mp 2i)^{\Delta+\frac{1}{2}}\,\Gamma\left(\Delta+\frac{3}{2}\right)}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \frac{\Gamma^{\mu}q_{\mu}(\vec{\omega})\varphi(\vec{\omega})}{(-2q(\vec{\omega})\cdot X \mp i\epsilon)^{\Delta+\frac{3}{2}}} = 0, \tag{3.5.8}$$

because it is a pure gauge term contracted with the polarization spinor $\varphi(\vec{\omega})$. Therefore, $\Upsilon^{\pm}_{\Delta}(X;\vec{\omega})$ are solutions of the massless Dirac equation (3.5.8) satisfying the transformation rule (3.3.2), as required for *massless spinor conformal primary wavefunctions.*

Now, consider the second term in (3.5.5). It is possible to show that this term is proportional to the shadow transform of $\Upsilon^{\pm}_{\Delta}(X;\vec{\omega})$. Following the method exposed in Appendix 3.4, the up-lifted shadow transform of $\Upsilon^{\pm}_{\Delta}(X;\vec{\omega})$ is

$$\widetilde{\Upsilon^{\pm}_{\Delta}}(X;q) = \frac{(\mp 2i)^{\Delta+\frac{1}{2}}\,\Gamma\left(\Delta+\frac{1}{2}\right)}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \int [\mathrm{d}q'] \frac{q'^{\mu}\Gamma_{\mu}}{(-2q'\cdot X \mp i\epsilon)^{\Delta+\frac{1}{2}}\,(-2q'\cdot q)^{d-\Delta+\frac{1}{2}}}. \tag{3.5.9}$$

Next, one has to pull-back $\widetilde{\Upsilon^{\pm}_{\Delta}}(X;q)$ on $\mathbb{R}^d$ by embedding $\mathbb{R}^d \hookrightarrow \mathcal{C}^{+}_{P}$ and restoring the polarization vector φ. Given $\vec{z}$, $\vec{\omega} \in \mathbb{R}^d$, since

$$\frac{q'^{\mu}(\vec{z})\Gamma_{\mu}\varphi(\vec{\omega})}{(-2q'(\vec{z})\cdot q(\vec{\omega}))^{d-\Delta+\frac{1}{2}}} = \frac{\begin{pmatrix} \not{z} & -|\vec{z}|^2 \\ 1 & -\vec{z} \end{pmatrix}\begin{pmatrix} \not{\omega} \\ 1 \end{pmatrix}}{|\vec{\omega}-\vec{z}|^{2(d-\Delta)+1}} = \frac{\begin{pmatrix} \not{z} \\ 1 \end{pmatrix}(\not{\omega}-\not{z})}{|\vec{\omega}-\vec{z}|^{2(d-\Delta)+1}}, \tag{3.5.10}$$

one obtains

$$\begin{aligned}\widetilde{\Upsilon^{\pm}_{\Delta}}(X;\vec{\omega}) &= \frac{(\mp 2i)^{\Delta+\frac{1}{2}}\,\Gamma\left(\Delta+\frac{1}{2}\right)}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \int \mathrm{d}^d z\, \frac{\varphi(\vec{z})}{(-2q'(\vec{z})\cdot X \mp i\epsilon)^{\Delta+\frac{1}{2}}} \\ &= \int \mathrm{d}^d z\, \Upsilon^{\pm}_{\Delta}(X;\vec{z}) \frac{\not{\omega}-\not{z}}{|\vec{\omega}-\vec{z}|^{2(d-\Delta)+1}},\end{aligned} \tag{3.5.11}$$

which is proportional to the second term in (3.5.5). Eq. (3.5.6), expressing massless conformal primary wavefunctions as Mellin transform of the plane waves, can be easily inverted. Let c be any positive number and perform the following integral:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,y^{-c-i\nu}\frac{\Gamma\left(c+i\nu\right)}{\left(\mp iq\cdot X+\epsilon\right)^{c+i\nu}}=\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,y^{-c-i\nu}\int\frac{\mathrm{d}s}{s}\,s^{c+i\nu}\mathrm{e}^{\pm isq\cdot X}\\&\overset{s\to ys}{=}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\int\frac{\mathrm{d}s}{s}\,s^{c+i\nu}\mathrm{e}^{\pm is\,yq\cdot X}.\end{aligned}\tag{3.5.12}$$

By means of the substitution $s=\mathrm{e}^u$, one finds

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\int\frac{\mathrm{d}s}{s}\,s^{c+i\nu}\mathrm{e}^{\pm is\,yq\cdot X}&=\int_{-\infty}^{+\infty}\mathrm{d}u\,\mathrm{e}^{\pm i\mathrm{e}^u\,yq\cdot X+cu}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{i\nu u}\\&=\int_{-\infty}^{+\infty}\mathrm{d}u\,\mathrm{e}^{\pm i\mathrm{e}^u yq\cdot X+cu}\,\delta(u)=\mathrm{e}^{\pm iyq\cdot X}.\end{aligned}\tag{3.5.13}$$

Using this result and (3.5.7), it is possible to write

$$\mathrm{e}^{\pm iyq(\vec{\omega})\cdot X}\varphi(\vec{\omega})=\sqrt{2}(2\pi)^{\frac{d}{2}+1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,y^{-\frac{d}{2}-\frac{1}{2}-i\nu}\Upsilon^{\pm}_{\frac{d}{2}+i\nu}(X;\vec{\omega}),\tag{3.5.14}$$

which proves that the conformal primary wavefunctions $\Upsilon^{\pm}_{\Delta}(X;\vec{\omega})$ form a basis for $\Delta\in\frac{d}{2}+i\mathbb{R}$. Moreover, with respect to the Dirac inner product,

$$(\Upsilon^{\pm}_{\Delta}(\vec{\omega}),\Upsilon^{\pm}_{\Delta'}(\vec{\omega}\,'))=\delta(\nu-\nu')\delta^d(\vec{\omega}-\vec{\omega}\,'),\tag{3.5.15}$$

$$(\Upsilon^{\pm}_{\Delta}(\vec{\omega}),\widetilde{\Upsilon^{\pm}_{\Delta'}}(\vec{\omega}\,'))=\frac{\not{\omega}\,'-\not{\omega}}{|\vec{\omega}\,'-\vec{\omega}|^{2(d-\Delta)+1}}\,\delta(\nu-\nu'),\tag{3.5.16}$$

$$(\widetilde{\Upsilon^{\pm}_{\Delta}}(\vec{\omega}),\widetilde{\Upsilon^{\pm}_{\Delta'}}(\vec{\omega}\,'))=\frac{\pi^d\Gamma\left(\Delta-\frac{d}{2}+\frac{1}{2}\right)\Gamma\left(\Delta^*-\frac{d}{2}+\frac{1}{2}\right)}{\Gamma\left(d-\Delta+\frac{1}{2}\right)\Gamma\left(d-\Delta^*+\frac{1}{2}\right)}\,\delta^d(\vec{\omega}-\vec{\omega}\,')\delta(\nu-\nu'),\tag{3.5.17}$$

with all the others products vanishing, as demonstrated in [3].

Massless Scalar Conformal Primary Wavefunctions: In the scalar case, the result is analogous [18]. Consider the δ-normalised massive scalar conformal primary wavefunctions with respect to the Klein-Gordon inner product (3.1.11),

$$\begin{aligned}\phi^{\pm}_{\Delta}(X;\vec{\omega})=&\left(\frac{m}{2\pi^2}\right)^{\frac{d}{2}}\frac{\Gamma\left(\frac{d}{2}+i\nu\right)}{2\sqrt{2}\pi\Gamma(i\nu)}\int_0^{+\infty}\frac{\mathrm{d}y}{y^{d+1}}\int\mathrm{d}^dz\,\mathrm{e}^{\pm i\left(\frac{m}{y}q(\vec{z})+myN_-\right)\cdot X}\\&\times\left(\frac{y}{y^2+|\vec{z}-\vec{\omega}|^2}\right)^{\frac{d}{2}+i\nu}.\end{aligned}\tag{3.5.18}$$

Therefore, expanding the bulk-to-boundary propagator $G_\Delta(y,\vec{z};\vec{\omega})$ as in (3.5.3), plugging the result in (3.5.18) and performing the rescaling $y \to m/y$, we obtain ($\Delta = \frac{d}{2} + i\nu$)

$$\begin{aligned}\phi^\pm_\Delta(X;\vec{\omega}) \underset{m\ll 1}{\to} & \, m^{\frac{d}{2}-\Delta} \frac{1}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}} \varphi^\pm_\Delta(X;\vec{\omega}) + \\ & \pm m^{\Delta-\frac{d}{2}} \frac{\Gamma\left(\frac{d}{2}+i\nu\right)}{2(2\pi^2)^{\frac{d+1}{2}}\Gamma(i\nu)} \int \mathrm{d}^d z \, \frac{\varphi^\pm_{d-\Delta}(X;\vec{z})}{|\vec{\omega}-\vec{z}|^{2\Delta}} + \dots ,\end{aligned} \tag{3.5.19}$$

where

$$\varphi^\pm_\Delta(X;\vec{\omega}) = \int_0^{+\infty} \frac{\mathrm{d}y}{y} \, y^\Delta \mathrm{e}^{\pm i y q(\vec{\omega})\cdot X} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-2q(\vec{\omega})\cdot X \mp i\epsilon)^\Delta} \tag{3.5.20}$$

is the *massless scalar conformal primary wavefunction* and

$$\widetilde{\varphi}^\pm_\Delta(X;\vec{\omega}) = \frac{\Gamma\left(\frac{d}{2}+i\nu\right)}{\pi^{\frac{d}{2}}\Gamma(i\nu)} \int \mathrm{d}^d z \, \frac{\varphi^\pm_{d-\Delta}(X;\vec{z})}{|\vec{\omega}-\vec{z}|^{2\Delta}} \tag{3.5.21}$$

is its shadow transform. Inverting (3.5.20) leads to

$$\mathrm{e}^{\pm i y q(\vec{\omega})\cdot X} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} \, y^{-\frac{d}{2}-i\nu} \varphi^\pm_{\frac{d}{2}+i\nu}(X;\vec{\omega}), \tag{3.5.22}$$

stating that the massless conformal primary wavefunctions $\varphi^\pm_\Delta(X;\vec{\omega})$ form a complete basis for $\Delta \in \frac{d}{2} + i\mathbb{R}$. With respect to the Klein-Gordon inner product, the only non vanishing products are

$$(\varphi^\pm_\Delta(\vec{\omega}), \varphi^\pm_{\Delta'}(\vec{\omega}\,')) = \pm 8\pi^{d+2}\delta(\nu-\nu')\delta^d(\vec{\omega}-\vec{\omega}\,'), \tag{3.5.23}$$

$$(\varphi^\pm_\Delta(\vec{\omega}), \widetilde{\varphi^\pm_{\Delta'}}(\vec{\omega}\,')) = \pm 8\pi^{\frac{d}{2}+2} \frac{\Gamma\left(\frac{d}{2}-i\nu\right)}{\Gamma(-i\nu)} \frac{\delta(\nu-\nu')}{|\vec{\omega}\,'-\vec{\omega}|^{2(d-\Delta)}}, \tag{3.5.24}$$

$$(\widetilde{\varphi^\pm_\Delta}(\vec{\omega}), \widetilde{\varphi^\pm_{\Delta'}}(\vec{\omega}\,')) = \pm 8\pi^{\frac{d}{2}+2} \frac{\pi^d \Gamma(\Delta)\Gamma(\Delta^*)}{\Gamma(d-\Delta)\Gamma(d-\Delta^*)} \delta^d(\vec{\omega}-\vec{\omega}\,')\delta(\nu-\nu'). \tag{3.5.25}$$

Chapter 4

Celestial Amplitudes

Conformal primary bases provides a new formulation of QFT, described in terms of wavefunctions transforming covariantly with respect to the Lorentz group, rather than the Poincaré group. As already seen during their construction, in this alternative formulation outgoing, resp. incoming, modes involve new parameter spaces: the principal continuous series $\frac{d}{2} + i\mathbb{R}^+$ and Celestial Sphere $\mathcal{CS}^{\pm}$. Exploiting the Dirac's embedding formalism, $\mathcal{CS}^{\pm}$ can be visualised as the conformal boundary of $\mathcal{H}^{\pm}_{d+1}$, the latter being the on-shell parameter space of outgoing, resp. incoming, modes in the momentum basis. This suggests that the connection between these two different QFT descriptions should be tracked in the context of AdS/CFT correspondence.

This expectation will be confirmed during the course of this chapter, which is heavily based on [4]. Here we will mainly focus on computing *Celestial amplitudes* in the massive scalar case. Celestial amplitudes are amplitudes in conformal primary basis formulation of QFT and, as we will see, are related to the ones in momentum basis by Pasterski and Shao formula [17, 114]. This formula (see (4.1.30)) fully aligns with our geometrical interpretation, looking at Celestial amplitudes as Witten correlators in $\mathcal{H}^{+}_{d+1}$ whose vertex function is the amplitude in momentum basis.

Efforts to understand properties of Celestial amplitudes so far mostly include studies in the massless case, especially focused on soft modes and asymptotic symmetries [16, 41–69]. However, while the Pasterski-Shao-Strominger formula yields excellent results in the calculation of massless Celestial amplitudes, computing Celestial amplitudes in the massive case using this formula proves to be quite cumbersome (see for example the last section of [17]). Currently, in literature, there are very few examples of explicit calculations of massive Celestial Amplitudes. One of the few examples can be found in [17], where the authors have applied their formula to compute 3-point massive Celestial amplitudes in the case of one in-coming particles with mass $m_3 = 2(1 + \epsilon)m$ and two outgoing particles with mass $m_1 = m_2 = m$, at first order in ϵ. The importance of this calculation is to show that 3-point Celestial Amplitudes are proportional to 3-point Witten correlators in $\mathcal{H}^{+}_{d+1}$. Inspired by this result, in this chapter we will present an original technique to compute massive Celestial amplitudes by recasting them in terms of known Witten correlators. The roots of this new method are deepened in the hyperbolic foliation of Minkowski space-time [6] and in the work [8–10]. To explain this method, let us consider contact diagrams first (see (4.3.2)). The idea is to split the integral over $\mathbb{M}^{d+2}$ in four integrals over the single regions $\mathcal{A}_{+}, \mathcal{A}_{+}, \mathcal{D}_{+}$ and $\mathcal{D}_{+}$, defined in the first chapter. In each region, we will show that conformal primary wavefunctions can be expressed as the product of two functions: the analytically continued EAdS bulk-to-boundary propagator and a function encoding the radial dependence (ref. (4.2.9)). This result will allow us to split the integrals further, into the radial and angular part. Tracing back each angular contribution to an integral over $\mathcal{H}^{+}_{d+1}$ by suitable analytical transformations (see (4.3.3)), we will be able to establish that, like in dS [8–10], Celestial contact diagrams in $(d+2)$-dimensions are proportional to corresponding contact Witten diagrams in EAdS_{d+1}. The proportionality factor will be computed explicitly in some concrete example for the 3-point contact amplitude (ref. Fig.(4.1)), specifically considering one or more conformally coupled scalars. This will confirm that such contact contributions to Celestial amplitudes are single-valued solutions to the conformal Ward identities.

This last consideration yields important implication for processes beyond the leading order, such as processes involving particle exchanges. Specifically, exchange contribution to Celestial amplitudes can be decomposed in terms of a continuum of corresponding exchange Witten diagrams in EAdS_{d+1}, with the exchanged particles carrying unitary Principal Series representations. The coefficient of each individual exchange Witten diagram is fixed by on-shell factorisation, meaning that it is given by the product of coefficients that convert each contact subdiagram to their Celestial counterpart.

The chapter unfolds as follows: in sec.4.1, we will decompose massive solutions of the Klein-Gordon equation in terms of conformal primary wavefunctions, defining oscillators in this new basis and relating them with oscillators in momentum basis (see Eq.(4.1.23)). These relations will lead us to the Pasterski-Shao-Strominger formula in the massive case, that we will present in Eq.(4.1.29). Next, in sec.4.2 we will recover closed expressions for massive conformal primary wavefunctions, that we will subsequently use in sec.4.3 to compute contact Celestial amplitudes by means of the procedure previously described. Factorisation property of CFT correlators [37, 118] states that 3-point contact correlators are the building blocks for all other processes. Therefore, in sec.4.4, we will exploit the proportionality law figured out in sec.4.3 to analyse higher order processes, like the particle exchanges (see Fig.(4.2)) and the candy diagram (see Fig.(4.4.6), writing down also a dictionary to decompose a general perturbative contribution to an n-point Celestial amplitude in terms of the corresponding Witten correlators in $\mathcal{H}^+_{d+1}$. Finally, in sec.4.5, we will briefly explore some consequences of the relation between perturbative Celestial amplitudes and Witten diagrams in EAdS. In particular, it implies that, like their AdS counterparts, Celestial amplitudes admit a conformal partial wave expansion with meromorphic spectral density (at least perturbatively). One can then obtain their expansion into Conformal Blocks which, combined with crossing symmetry, lies at the centre of the bootstrap of standard Lorentzian CFTs. We also discuss (non-perturbative) constraints from unitarity in Euclidean conformal field theory at the level of the conformal partial wave expansion, which translates into positivity of the spectral density.

4.1 The Pasterski, Shao & Strominger Formula

Throughout this section, we will consistently use $\Delta = \frac{d}{2} + i\nu$, $\nu \geq 0$, even in cases where it is not explicitly mentioned. Consider a massive scalar theory on flat space, described by the free Lagrangian action

$$S = \frac{1}{2}\int_{\mathbb{M}^{d+2}} \mathrm{d}^{d+2}X \, [\eta^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2], \tag{4.1.1}$$

with $\eta = diag(-1, 1, \ldots, 1)$. The conformal primary formulation of this scalar theory is provided by employing the basis of conformal primary wavefunctions

$$\phi^{\pm}_{\Delta}(X; \vec{\omega}) = \left(\frac{m}{2\pi^2}\right)^{\frac{d}{2}} \frac{\Gamma(\Delta)}{2\sqrt{2}\pi\Gamma(\Delta - \frac{d}{2})} \int_{H_{d+1}} [\mathrm{d}\hat{p}] \; G_{\Delta}(\hat{p}; \vec{\omega}) \mathrm{e}^{\pm im\hat{p}\cdot X}. \tag{4.1.2}$$

The normalization factor

$$\widetilde{\mathcal{N}}_{\Delta}(m) = \left(\frac{m}{2\pi^2}\right)^{\frac{d}{2}} \frac{\Gamma(\Delta)}{2\sqrt{2}\pi\Gamma(\Delta - \frac{d}{2})} \tag{4.1.3}$$

has been properly chosen in order to satisfy

$$\left(\phi^{\pm}_{\frac{d}{2}-i\nu}(\vec{\omega}), \phi^{\pm}_{\frac{d}{2}-i\nu'}(\vec{\omega}\,')\right) = \pm\delta(\nu - \nu')\delta^{(d)}(\vec{\omega} - \vec{\omega}\,'), \tag{4.1.4}$$

where $(\,\cdot\,,\,\cdot\,)$ indicates the Klein-Gordon product

$$(\phi_1, \phi_2) = -i\int \mathrm{d}^{d+1}X^i \left[\phi_1(X)\partial_{X^0}\phi_2^*(X) - \partial_{X^0}\phi_1(X)\phi_2^*(X)\right], \tag{4.1.5}$$

with ϕ_1 and ϕ_2 generic massive scalar fields. A general solution $\phi(X)$ of the Klein-Gordon equation,

$$(\Box - m^2)\phi(X) = 0, \tag{4.1.6}$$

can be always expanded in terms of the conformal primary basis elements as follows:

$$\phi(X) = \int \mathrm{d}^d\omega \int_0^{+\infty} \mathrm{d}\nu \left(a^\dagger_{\frac{d}{2}+i\nu}(\vec{\omega})\phi^+_{\frac{d}{2}-i\nu}(X;\vec{\omega}) + a_{\frac{d}{2}-i\nu}\phi^-_{\frac{d}{2}+i\nu}(X;\vec{\omega}) \right), \tag{4.1.7}$$

where $a^\dagger_\Delta$ and a_{Δ^*}, with $\Delta = \frac{d}{2} + i\nu$, $\nu \geq 0$, are respectively the rising and lowering operator, whose action on the external state, living on the Celestial Sphere, is given by

$$a_{\Delta^*}(\vec{\omega})|0\rangle = 0 \overset{c.c.}{\Longrightarrow} \langle 0|a^\dagger_\Delta(\vec{\omega}) = 0, \tag{4.1.8}$$

$$a^\dagger_\Delta(\vec{\omega})|0\rangle = |\Delta, \vec{\omega}\rangle \overset{c.c.}{\Longrightarrow} \langle \Delta^*, \vec{\omega}| = \langle 0|a_{\Delta^*}(\vec{\omega}). \tag{4.1.9}$$

Imposing the equal time commutation rules

$$[\phi(X), \phi(X')]_{X^0=X'^0} = 0, \tag{4.1.10}$$

$$[\partial_{X^0}\phi(X), \phi(X')]_{X^0=X'^0} = i\delta^{(d+1)}(X^i - X'^i), \tag{4.1.11}$$

we find out, for the oscillators,

$$[a_{\frac{d}{2}-i\nu}(\vec{\omega}), a_{\frac{d}{2}-i\nu'}(\vec{\omega}\,')] = [a^\dagger_{\frac{d}{2}+i\nu}(\vec{\omega}), a^\dagger_{\frac{d}{2}+i\nu'}(\vec{\omega}\,')] = 0, \tag{4.1.12}$$

$$[a_{\frac{d}{2}-i\nu}(\vec{\omega}), a^\dagger_{\frac{d}{2}+i\nu'}(\vec{\omega}\,')] = \delta(\nu - \nu')\delta^{(d)}(\vec{\omega} - \vec{\omega}\,'). \tag{4.1.13}$$

Indeed, plugging (4.1.7) in (4.1.10) and using the commutators (4.1.12) and (4.1.13), it follows that

$$\begin{aligned}
&[\phi(X), \phi(X')]_{X^0=X'^0} = \\
&= \int \mathrm{d}^d\omega \int_0^{+\infty} \mathrm{d}\nu \, \left(\phi^+_{\Delta^*}(X;\vec{\omega})\phi^-_\Delta(X';\vec{\omega}) - \phi^-_\Delta(X;\vec{\omega})\phi^+_{\Delta^*}(X';\vec{\omega})\right)_{X^0=X'^0} \\
&= \frac{1}{4\pi}\left(\frac{m}{2\pi}\right)^d \int [\mathrm{d}\hat{p}] \int [\mathrm{d}\hat{p}'] \int \mathrm{d}^d\omega \int_0^{+\infty} \mathrm{d}\nu \, \mu(\nu) G_\Delta(\hat{p};\vec{\omega}) G_{\Delta^*}(\hat{p};\vec{\omega}) \\
&\times \left(\mathrm{e}^{im\hat{p}'\cdot(X-X')} - \mathrm{e}^{im\hat{p}\cdot(X'-X)}\right)_{X^0=X'^0},
\end{aligned} \tag{4.1.14}$$

where we used the Fourier expansion (4.1.2) and commuted the integrals. Then, using (3.1.7), we arrive to

$$[\phi(X), \phi(X')]_{X^0=X'^0} = \int \frac{\mathrm{d}^{d+1}p^{(i)}}{(2\pi)^{d+1}} \frac{1}{2p^0} \left(\mathrm{e}^{-ip^i(X'-X)_i} - \mathrm{e}^{-ip^i(X-X')_i}\right), \tag{4.1.15}$$

which vanishes by sending $p^i \to -p^i$ in the second term. Similarly, Eq.s (4.1.12) and (4.1.13) also yield (4.1.11), as readily verified:

$$
\begin{aligned}
&[\partial_{X^0}\phi(X), \phi(X')]_{X^0=X'^0} = \\
&= \int \mathrm{d}^d\omega \int \mathrm{d}^d\omega' \int_0^{+\infty} \mathrm{d}\nu \int_0^{+\infty} \mathrm{d}\nu' \Big\{ [a^\dagger_\Delta(\vec{\omega}), a_{\Delta'^*}(\vec{\omega}')]\partial_{X^0}\phi^+_{\Delta^*}(X;\vec{\omega})\phi^-_{\Delta'}(X';\vec{\omega}\,') \\
&+ [a_{\Delta^*}(\vec{\omega}), a^\dagger_{\Delta'}(\vec{\omega}\,')]\partial_{X^0}\phi^-_\Delta(X;\vec{\omega})\phi^+_{\Delta'^*}(X';\vec{\omega}\,') \Big\}_{X^0=X'^0} \\
&= \frac{i}{2}\int \frac{\mathrm{d}^{d+1}p^{(i)}}{(2\pi)^{d+1}} \left(\mathrm{e}^{-ip\cdot(X'-X)} + \mathrm{e}^{-ip\cdot(X-X')}\right)_{X^0=X'^0} = i\delta^{(d+1)}(X-X').
\end{aligned}
\tag{4.1.16}
$$

Further, using the commutation rules (4.1.12) and (4.1.13), the superposition between external states results δ-normalised,

$$
\langle \Delta, \vec{\omega} | \Delta', \vec{\omega}\,' \rangle = \langle 0 | a_{\Delta^*}(\vec{\omega}) a^\dagger_{\Delta'}(\vec{\omega}\,') | 0 \rangle = \delta(\nu - \nu')\delta^d(\vec{\omega} - \vec{\omega}\,'). \tag{4.1.17}
$$

Employing the Klein-Gordon inner product, we can invert the relation between the field $\phi(X)$ and the oscillators, getting

$$
a^\dagger_{\frac{d}{2}+i\nu}(\vec{\omega}) = \left(\phi, \phi^+_{\frac{d}{2}-i\nu}(\vec{\omega})\right) = -\left(\phi^-_{\frac{d}{2}+i\nu}(\vec{\omega}), \phi\right), \tag{4.1.18}
$$

$$
a_{\frac{d}{2}-i\nu}(\vec{\omega}) = -\left(\phi, \phi^-_{\frac{d}{2}+i\nu}(\vec{\omega})\right) = \left(\phi^+_{\frac{d}{2}-i\nu}(\vec{\omega}), \phi\right), \tag{4.1.19}
$$

where we exploited $\phi^\pm_\Delta(X;\vec{\omega}) = \phi^\mp_{\Delta^*}(X;\vec{\omega})$. Note that, since $\phi^\pm_{\frac{d}{2}\mp i\nu}(X;\vec{\omega})$ transforms as a conformal primary operator with conformal weight $\frac{d}{2} \mp i\nu$, $\nu \geq 0$, to ensure the invariance of $\phi(X)$ under the action of $SO(1,d+1)$, the oscillator $a^{(\dagger)}_{\frac{d}{2}+i\nu}(\vec{\omega})$ must transform as

$$
U(\Lambda) a^{(\dagger)}_{\frac{d}{2}\pm i\nu}(\vec{\omega}) U^{-1}(\Lambda) = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|^{-(\frac{d}{2}\pm i\nu)/d} a^{(\dagger)}_{\frac{d}{2}\pm i\nu}(\vec{\omega}), \tag{4.1.20}
$$

where $\Lambda \in SO(1,d+1)$ and $\vec{\omega}\,' = \Lambda[\vec{\omega}]$. Therefore, splitting the solution $\phi(X)$ of the Klein-Gordon equation as follows, $\phi(X) = \phi^+(X) + \phi^-(X)$, with

$$
\phi^+(X) = \int \mathrm{d}^d\omega \int_0^{+\infty} \mathrm{d}\nu\, a^\dagger_{\frac{d}{2}+i\nu}(\vec{\omega}) \phi^+_{\frac{d}{2}-i\nu}(X;\vec{\omega}), \tag{4.1.21}
$$

$$
\phi^-(X) = \int \mathrm{d}^d\omega \int_0^{+\infty} \mathrm{d}\nu\, a_{\frac{d}{2}-i\nu}(\vec{\omega}) \phi^-_{\frac{d}{2}+i\nu}(X;\vec{\omega}), \tag{4.1.22}
$$

the transformation rules (4.1.20) allow us to regard (4.1.18) and (4.1.19) as defining a FLAT/CFT-correspondence, $a_\Delta(\vec{\omega})$ (resp. $a^\dagger_\Delta(\vec{\omega})$) being the conformal primary operator on the Celestial Sphere associated to the bulk field $\phi^+(X)$ (resp. $\phi^-(X)$). In this sense, $\phi^{\pm}_{\frac{d}{2}\mp i\nu}(X;\vec{\omega})$ plays a role similar to a flat bulk-to-boundary propagator. The relation between oscillators in the momentum and in the conformal primary basis is easily derived by means of the Klein-Gordon inner product. In particular,

$$a^\dagger_\Delta(\vec{\omega}) = \widetilde{\mathcal{N}}_\Delta(m)\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}\hat{p}]\,\sqrt{2m\hat{p}^0}G_\Delta(\hat{p};\vec{\omega})\,a^\dagger(p^i) \tag{4.1.23}$$

and

$$a^\dagger(m\hat{p}) = \frac{2(2\pi)^{d+1}}{m^d}\int \mathrm{d}^d\omega\int_0^{+\infty}\frac{\mathrm{d}\nu}{\sqrt{2m\hat{p}^0}}\widetilde{\mathcal{N}}_{\Delta^*}(m)G_{\Delta^*}(\hat{p};\vec{\omega})\,a^\dagger_\Delta(\vec{\omega}) \tag{4.1.24}$$

where $a^\dagger(m\hat{p})$ is the rising operator in the momentum basis, satisfying

$$a^\dagger(m\hat{p}) = \sqrt{2m\hat{p}^0}\left(\phi, \mathrm{e}^{im\hat{p}\cdot X}\right), \qquad \hat{p}^0 = \sqrt{1+\sum_{i=1}^{d+1}\hat{p}^i\hat{p}^i}. \tag{4.1.25}$$

The other relation, between the lowering operators $a_{\Delta^*}(\vec{\omega})$ and $a(m\hat{p})$, can be immediately found from (4.1.23) by taking its complex conjugate.

Exploiting (4.1.24), we can now study the relation between S-matrix elements in the momentum and in the conformal primary basis. Therefore, let us consider an S-matrix element in the momentum basis,

$$\mathcal{S}(p_1,\dots p_n;p'_1,\dots,p'_m) := \langle p_1,\dots,p_n|\mathcal{S}|p'_1,\dots,p'_n\rangle, \tag{4.1.26}$$

where the momentum-p external state is defined as

$$|p\rangle = \sqrt{2p^0}a^\dagger(p)|0\rangle, \qquad \sqrt{m^2+\sum_{i=1}^{d+1}p^ip^i}. \tag{4.1.27}$$

Plugging this last equation in (4.1.26) and then using (4.1.24) yields

$$\begin{aligned}\mathcal{S}(p_1,\dots p_n;p'_1,\dots,p'_m) = &\left[\prod_{i=1}^n\int \mathrm{d}^d\omega_i\int_0^\infty[\mathrm{d}\nu_i]\,G_{\Delta_i}(\hat{p}_i;\vec{\omega}_i)\right. \\ &\times\left.\prod_{j=1}^m\int\mathrm{d}^d\omega'\int_0^\infty[\mathrm{d}\nu_j]^*\,G_{\Delta'^*_j}(\hat{p}'_j;\vec{\omega}'_j)\right]\mathcal{S}_{\{\Delta^*\}_n,\{\Delta'\}_m}(\vec{\omega}_1,\dots,\vec{\omega}_n;\vec{\omega}\,'_1,\dots,\vec{\omega}\,'_n),\end{aligned} \tag{4.1.28}$$

where $\{\Delta'\}_m = \Delta_1,\dots,\Delta_m$, with $\Delta_i = \frac{d}{2}+i\nu_i$, the integration measure $[\mathrm{d}\nu_i]^{(*)} = \frac{2(2\pi)^{d+1}}{m^d}\widetilde{\mathcal{N}}_{\Delta_i^{(*)}}(m_i)\,\mathrm{d}\nu_i$ and

$$\begin{aligned}&\mathcal{S}_{\{\Delta^*\}_n,\{\Delta'\}_m}(\vec{\omega}_1,\dots,\vec{\omega}_n;\ \vec{\omega}\,'_1,\dots,\vec{\omega}\,'_n) := \\ &\quad := \langle\Delta^*_1,\dots,\Delta^*_n,\vec{\omega}_1,\dots,\vec{\omega}_n|\mathcal{S}|\Delta'_1,\dots,\Delta'_n,\vec{\omega}'_1,\dots,\vec{\omega}'_n\rangle\end{aligned} \tag{4.1.29}$$

is the $\mathcal{S}$-matrix element in the conformal primary basis. Eq.(4.1.29) is the Pasterski-Shao-Strominger formula, which relates scattering amplitudes in momentum and conformal primary basis. This formula can also be inverted, providing

$$\mathcal{S}_{\{\Delta^*\}_n,\{\Delta'\}_m}(\vec{\omega}_1,\ldots,\vec{\omega}_n;\vec{\omega}\,'_1,\ldots,\vec{\omega}\,'_n) = \widetilde{\mathcal{N}}_{\{\Delta^*\}_n,\{\Delta\}_m} \tag{4.1.30}$$

$$\left[\prod_{i=1}^{n}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}\hat{p}_i]\,G_{\Delta_i^*}(\hat{p}_i;\vec{\omega}_i)\prod_{j=1}^{m}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}\hat{p}'_j]\,G_{\Delta'_j}(\hat{p}'_j;\vec{\omega}'_j)\right]\mathcal{S}(p_1,\ldots p_n;p'_1,\ldots,p'_m),$$

where

$$\widetilde{\mathcal{N}}_{\{\Delta^*\}_n,\{\Delta\}_m} = \prod_{i=1}^{n}\prod_{j=1}^{m}\widetilde{\mathcal{N}}_{\Delta_i^*}(m_i)\widetilde{\mathcal{N}}_{\Delta_j}(m_j). \tag{4.1.31}$$

We will refer to *Celestial amplitudes* as scattering amplitudes in the conformal primary basis.

4.2 Celestial Amplitudes as Witten diagrams

In this section, we will show explicitly that perturbative Celestial amplitudes can be expressed in terms of corresponding Witten diagrams on $\mathcal{H}^+_{d+1}$. Given the analyticity properties of the Celestial amplitudes, from here henceforth, we will release the constraint over Δ, allowing it to vary across the entire complex plane. To this end, it is useful to evaluate the momentum integral in the representation (3.1.3) of conformal primary wave functions. This integral is only formal since it is divergent for real values of the mass m. One can however define the following convergent integrals:

$$\varphi^+_\Delta(X;q_+) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta-\frac{d}{2})}\left(\frac{m}{2\pi}\right)^{d/2}\int_{\mathcal{H}^+_{d+1}}[\mathrm{d}\hat{p}]\,K^{\mathrm{AdS}}_\Delta(\hat{p};q_+)\;e^{+m\,\hat{p}\cdot X}\,,\quad X\in\mathcal{A}_+, \tag{4.2.1a}$$

$$\varphi^-_\Delta(X;q_-) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta-\frac{d}{2})}\left(\frac{m}{2\pi}\right)^{d/2}\int_{\mathcal{H}^-_{d+1}}[\mathrm{d}\hat{p}]\,K^{\mathrm{AdS}}_\Delta(\hat{p};q_-)\;e^{+m\,\hat{p}\cdot X}\,,\quad X\in\mathcal{A}_-. \tag{4.2.1b}$$

where $q_+\in\mathcal{C}^+_P$ and $q_-=-q_+\in\mathcal{C}^-_P$ (see (2.3.7)) and

$$\mathcal{N}_\Delta = \pi^{-\frac{1}{2}}\,\Gamma\left(\Delta-\frac{d}{2}+1\right). \tag{4.2.2}$$

The *normalised* scalar bulk-to-boundary propagator is here defined as

$$K^{\mathrm{AdS}}_\Delta(\hat{p};q) = \frac{C^{\mathrm{AdS}}_\Delta}{(-2\hat{p}\cdot q)^\Delta},\qquad C^{\mathrm{AdS}}_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}, \tag{4.2.3}$$

with $\hat{p} \in \mathcal{H}^{\pm}_{d+1}$ coherently with $q_{\pm}$. In the hyperbolic slicing we can write:

$$X = R\hat{X}_{\text{AdS}}, \qquad \text{with} \qquad \hat{X}^2_{\text{AdS}} = -1, \tag{4.2.4}$$

where in $\mathcal{A}_+$ we have $R > 0$ and in $\mathcal{A}_-$ we have $R < 0$, see (2.2.9). The integrals (4.2.1) are evaluated in Appendix C.1 and are given by the following closed form expression:

$$\varphi^+_\Delta(R, \hat{X}_{\text{AdS}}; q_+) = \mathcal{N}_\Delta\, K^{\text{AdS}}_\Delta(\hat{X}_{\text{AdS}}; q_+)\tilde{K}_{\Delta-\frac{d}{2}}(mR)\,, \tag{4.2.5a}$$

$$\varphi^-_\Delta(R, \hat{X}_{\text{AdS}}; q_-) = \mathcal{N}_\Delta\, K^{\text{AdS}}_\Delta(-\hat{X}_{\text{AdS}}; q_-)\tilde{K}_{\Delta-\frac{d}{2}}(-mR)\,, \tag{4.2.5b}$$

which are factorised into the bulk-to-boundary propagator (4.2.3) on $\mathcal{H}^{\pm}_{d+1}$ and the function

$$\tilde{K}_{\Delta-\frac{d}{2}}(mR) = \frac{2R^{-d/2}}{\Gamma(\Delta - \frac{d}{2})}\, K_{\Delta-\frac{d}{2}}(mR), \tag{4.2.6}$$

which, as we will see in the next chapter, is the kernel of the Kontorovich-Lebedev transformation defined in Appenix 5.1. Up to powers of $|R|$ and m, this function is the Mellin transform of the bulk-to-boundary propagator in Poincaré coordinates, with the mass m playing the role of the magnitude of the boundary momentum and $|R|$ the role of the Poincaré coordinate y.

Closed form expressions for the scalar conformal primary wavefunctions $\phi^{\pm}_\Delta(X; q_{\pm})$ over the whole of Minkowski space-time can be obtained from the closed form expression (4.2.5) for $\varphi^{\pm}_\Delta$ by analytic continuation. In this section we will provide the closed expression of the up-lifted scalar conformal primary wavefunctions, that we represent here as

$$\phi^{\pm}_\Delta(X; q_{\pm}) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta - \frac{d}{2})} \left(\frac{m}{2\pi}\right)^{d/2} \int_{H^{\pm}_{d+1}} [d\hat{p}]\, K^{\text{AdS}}_\Delta(\hat{p}; q_{\pm})\, e^{+im\,\hat{p}\cdot X}. \tag{4.2.7}$$

Precisely, the closed expressions for $\phi^{\pm}_\Delta(X; \vec{\omega})$ have been derived employing Poincaré coordinates. The detailed methodology is presented in the Appendix C.1. By comparing with the integral expression (4.2.7) for $\phi^{\pm}_\Delta(X; q_{\pm})$, under such analytic continuations we require, for both $\varphi^{\pm}_\Delta(X; q_{\pm})$, that:

$$X \to iX. \tag{4.2.8}$$

At the level of the hyperbolic foliation (4.2.4) this can be achieved for all regions $\mathcal{A}_{\pm}$ and $\mathcal{D}_{\pm}$ of $\mathbb{M}^{d+2}$ by rotating R and $\hat{X}_{\text{AdS}}$ respectively. These are summarised in the following.

$$X \in \mathcal{A}_+ : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(\pm\hat{X}_{\text{AdS}}; q_\pm)\tilde{K}_{\Delta-\frac{d}{2}}\left(m\, T e^{\pm\frac{\pi i}{2}}\right), \quad (4.2.9a)$$

$$X \in \mathcal{A}_- : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(\pm\hat{X}_{\text{AdS}}; q_\pm)\tilde{K}_{\Delta-\frac{d}{2}}\left(m\, |T| e^{\mp\frac{\pi i}{2}}\right), \quad (4.2.9b)$$

$$X \in \mathcal{D}_+ : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(e^{+\frac{\pi i}{2}}\hat{X}_{\text{dS}}; q_\pm)\tilde{K}_{\Delta-\frac{d}{2}}(m\, R)\,, \quad (4.2.9c)$$

$$X \in \mathcal{D}_- : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(-e^{-\frac{\pi i}{2}}\hat{X}_{\text{dS}}; q_\pm)\tilde{K}_{\Delta-\frac{d}{2}}(m\, R)\,. \quad (4.2.9d)$$

On a given hyperbolic slice of $\mathbb{M}^{d+2}$, the conformal primary wave functions $\phi_\Delta^\pm(X; q_\pm)$ can be recast as bulk-to-boundary propagators with the same conformal weight Δ on $\mathcal{H}_{d+1}^+$. In regions $\mathcal{A}_\pm$, using the anti-podal identification $q_+ = -q_-$, it is straightforward to see that:

$$X \in \mathcal{A}_+ : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(\hat{X}_{\text{AdS}}; q_+)\tilde{K}_{\Delta-\frac{d}{2}}\left(m\, T e^{\pm\frac{\pi i}{2}}\right), \quad (4.2.10a)$$

$$X \in \mathcal{A}_- : \phi_\Delta^\pm(X; q_\pm) = \mathcal{N}_\Delta K_\Delta^{\text{AdS}}(\hat{X}_{\text{AdS}}; q_+)\tilde{K}_{\Delta-\frac{d}{2}}\left(m\, |T| e^{\mp\frac{\pi i}{2}}\right). \quad (4.2.10b)$$

In region $\mathcal{D}_\pm$, the relation to the corresponding bulk-to-boundary propagator on $\mathcal{H}_{d+1}^+$ is more subtle owing to the presence of a short distance singularity in $\phi_\Delta^\pm(X; q_\pm)$ and an anti-podal singularity in $\phi_\Delta^\mp(X; Q_\mp)$. In order to clarify this point and rewrite the corresponding bulk integral as a CFT correlator it is enough to zoom in around the associated singularities in order to extract the phase factors arising from the analytic continuation. This is what we shall do in the following. In global coordinates (2.2.17), we have

$$\phi_\Delta^+(X(\sigma, \hat{n}; q_+(\hat{n}')) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C_\Delta^{\text{AdS}}}{[2\,(\sinh\sigma - \cosh\sigma\cos\gamma) - i\epsilon]^\Delta}\tilde{K}_{\Delta-\frac{d}{2}}(mR)\,, \quad (4.2.11a)$$

$$\phi_\Delta^-(X(\sigma, \hat{n}; q_-(\hat{n}')) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C_\Delta^{\text{AdS}}}{[-2\,(\sinh\sigma + \cosh\sigma\cos\gamma) - i\epsilon]^\Delta}\tilde{K}_{\Delta-\frac{d}{2}}(mR)\,, \quad (4.2.11b)$$

where

$$q_\pm = (\pm 1, \hat{n}')\,, \qquad \cos\gamma = \hat{n}\cdot\hat{n}'. \quad (4.2.12)$$

Let us first consider region $\mathcal{D}_+$, where ϕ_Δ^+ exhibits a short-distance singularity and ϕ_Δ^- an anti-podal singularity. One zooms in on the short-distance singularity by taking the limit $\sigma \to +\infty$ and $\gamma \to 0$:

$$\phi_\Delta^+(X; q_+) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C_\Delta^{\text{AdS}}}{\left[-2e^{-\sigma} + \frac{1}{2}\gamma^2 e^\sigma - i\epsilon\right]^\Delta}\tilde{K}_{\Delta-\frac{d}{2}}(mR)\,, \quad (4.2.13a)$$

$$= e^{-\Delta\frac{\pi i}{2}}\mathcal{N}_\Delta\, C_\Delta^{\text{AdS}}\left(\frac{\eta_+}{-\eta_+^2 + |\vec{z} - \vec{\omega}|^2 - i\epsilon}\right)^\Delta \tilde{K}_{\Delta-\frac{d}{2}}(mR)\,, \quad (4.2.13b)$$

where in the second equality we switched to Poincaré coordinates (2.2.9) through the identifications $\eta_+ = 2e^{-\sigma}$ and $|\vec{z}-\vec{\omega}| = \gamma$. For the anti-podal singularity of ϕ^-_Δ, one instead takes the limit $\sigma \to +\infty$ and $\bar{\gamma} \to 0$ where $\bar{\gamma} = \gamma + \pi$:

$$\phi^-_\Delta(X;q_-) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C^{\rm AdS}_\Delta}{\left[2e^{-\sigma} - \frac{1}{2}\gamma^2 e^{\sigma} - i\epsilon\right]^\Delta} \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right), \tag{4.2.14a}$$

$$= e^{-\Delta\frac{\pi i}{2}} \mathcal{N}_\Delta\, C^{\rm AdS}_\Delta \left(\frac{\eta_+}{\eta_+^2 - |\vec{z}-\vec{\omega}|^2 - i\epsilon}\right)^\Delta \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right). \tag{4.2.14b}$$

In the vicinity of the short-distance (s-d) and anti-podal (a-p) singularities in $\mathcal{D}_+$ we therefore have, respectively:[1]

$$\text{s-d in } \mathcal{D}_+ \colon \phi^\pm_\Delta(X;q_\pm) = e^{\Delta(1\mp1)\frac{\pi i}{2}} \mathcal{N}_\Delta K^{\rm AdS}_\Delta\left(y,\vec{z};\vec{\omega}\right) \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right), \quad y = \eta_+ e^{-\frac{\pi i}{2}}, \tag{4.2.15a}$$

$$\text{a-p in } \mathcal{D}_+ \colon \phi^\pm_\Delta(X;q_\pm) = e^{-\Delta(1\pm1)\frac{\pi i}{2}} \mathcal{N}_\Delta K^{\rm AdS}_\Delta\left(y,\vec{z};\vec{\omega}\right) \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right), \quad y = \eta_+ e^{+\frac{\pi i}{2}}, \tag{4.2.15b}$$

where in Poincaré coordinates (2.2.9), the bulk-to-boundary propagator (4.2.3) on $\mathcal{H}^+_{d+1}$ reads:

$$K^{\rm AdS}_\Delta\left(y,\vec{z};\vec{\omega}\right) = C^{\rm AdS}_\Delta \left(\frac{y}{y^2 + |\vec{z}-\vec{\omega}|^2}\right)^\Delta. \tag{4.2.16}$$

A similar result follows in region $\mathcal{D}_-$. Here, it is ϕ^-_Δ that exhibits a short-distance singularity and ϕ^+_Δ an anti-podal singularity. One zooms in on the short-distance singularity of ϕ^-_Δ by taking the limit $\sigma \to -\infty$ and $\gamma \to 0$:

$$\phi^-_\Delta(X,;q_-)) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C^{\rm AdS}_\Delta}{\left[-2e^{\sigma} + \frac{1}{2}\gamma^2 e^{-\sigma} - i\epsilon\right]^\Delta} \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right), \tag{4.2.17a}$$

$$= e^{-\Delta\frac{\pi i}{2}} \mathcal{N}_\Delta\, C^{\rm AdS}_\Delta \left(\frac{\eta_-}{-\eta_-^2 + |\vec{z}-\vec{\omega}|^2 - i\epsilon}\right)^\Delta \tilde{K}_{\Delta-\frac{d}{2}}\left(mR\right), \tag{4.2.17b}$$

where in the second equality we introduced Poincaré coordinates (2.2.9) through the identifications $\eta_- = 2e^{\sigma}$ and $|\vec{z}-\vec{\omega}| = \gamma$. For the anti-podal

[1] Note that ϕ^-_Δ is analytic around the short distance singularity of ϕ^+_Δ in $\mathcal{D}_+$. Likewise, ϕ^+_Δ is analytic around the anti-podal singularity of ϕ^-_Δ. The analogous holds in region $\mathcal{D}_-$, which is considered below.

singularity of ϕ_Δ^+ one instead takes the limit $\sigma \to -\infty$ and $\bar\gamma \to 0$ where $\bar\gamma = \gamma + \pi$:

$$\phi_\Delta^+(X;q_+) = e^{-\Delta\frac{\pi i}{2}} \frac{\mathcal{N}_\Delta\, C_\Delta^{\mathrm{AdS}}}{\left[2e^{+\sigma} - \frac{1}{2}\gamma^2 e^{-\sigma} - i\epsilon\right]^\Delta} \tilde K_{\Delta-\frac{d}{2}}(mR)\,, \tag{4.2.18a}$$

$$= e^{-\Delta\frac{\pi i}{2}} \mathcal{N}_\Delta\, C_\Delta^{\mathrm{AdS}} \left(\frac{\eta_-}{\eta_-^2 - |\vec z - \vec\omega|^2 - i\epsilon}\right)^\Delta \tilde K_{\Delta-\frac{d}{2}}(mR)\,. \tag{4.2.18b}$$

By comparing with (4.2.16), we can conclude that in the vicinity of the short-distance (s-d) and anti-podal (a-p) singularities in $\mathcal{D}_-$ we have, respectively:

$$\text{s-d in } \mathcal{D}_-: \quad \phi_\Delta^\pm(X;q_\pm) = e^{\Delta(1\pm1)\frac{\pi i}{2}} \mathcal{N}_\Delta K_\Delta^{\mathrm{AdS}}\left(\eta_- e^{-\frac{\pi i}{2}}, \vec z; \vec\omega\right) \tilde K_{\Delta-\frac{d}{2}}(mR)\,, \tag{4.2.19a}$$

$$\text{a-p in } \mathcal{D}_-: \quad \phi_\Delta^\pm(X;q_\pm) = e^{-\Delta(1\mp1)\frac{\pi i}{2}} \mathcal{N}_\Delta K_\Delta^{\mathrm{AdS}}\left(\eta_- e^{+\frac{\pi i}{2}}, \vec z; \vec\omega\right) \tilde K_{\Delta-\frac{d}{2}}(mR)\,. \tag{4.2.19b}$$

As we shall see in the following section, the relations (4.2.10), (4.2.15) and (4.2.19) imply that contact diagram contributions to celestial correlators can be recast as contact Witten diagrams on $\mathcal{H}_{d+1}^+$ upon integrating out the curvature radius of the hyperbolic slicing! This in turn leads to a more general relation between perturbative celestial correlators involving particle exchanges and corresponding Witten diagrams, which is presented in section 4.4.

4.3 Contact Amplitudes

Consider a theory of scalar fields ϕ_i, $i = 1, \ldots, n$ of mass m_i interacting through the vertex

$$\mathcal{V}(X) = g\phi_1(X)\ldots\phi_n(X)\,. \tag{4.3.1}$$

To leading order in λ the n-point Celestial Correlator is given by

$$\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\ldots\Delta_n}(\pm_1 q_1, \ldots, \pm_n q_n) = -ig\int \mathrm{d}^{d+2}X\, \phi_{\Delta_1}^{\pm_1}(X, \pm_1 q_1)\ldots\phi_{\Delta_n}^{\pm_n}(X, \pm_n q_n)\,,$$

where the superscript "c" to indicates that we are considering a contact diagram. The expressions (4.2.9) for the conformal primary wave function instruct us to divide the integral over $\mathbb{M}^{d+2}$ into integrals over regions $\mathcal{A}_\pm$ and $\mathcal{D}_\pm$,

$$\int \mathrm{d}^{d+2}X = \int_{\mathcal{A}_+} \mathrm{d}^{d+2}X + \int_{\mathcal{A}_-} \mathrm{d}^{d+2}X + \int_{\mathcal{D}_+} \mathrm{d}^{d+2}X + \int_{\mathcal{D}_-} \mathrm{d}^{d+2}X, \tag{4.3.2}$$

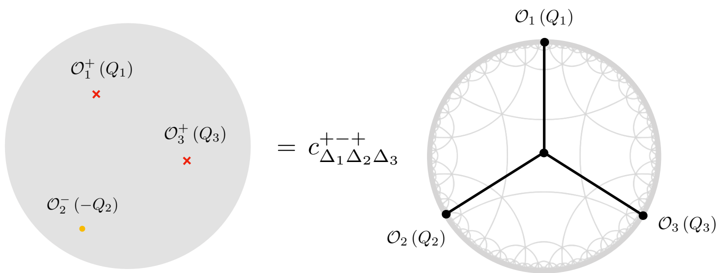


Figure 4.1: Contact diagram contributions to correlators on the d-dimensional celestial sphere (in grey) are proportional to their Witten diagram counterparts in EAdS_{d+1} with proportionality constant (4.3.14). Red crosses denote operators inserted on the future boundary $\mathcal{CS}_d^+$ and yellow dots operators inserted on the past boundary boundary $\mathcal{CS}_d^-$. $\mathcal{O}_i^\pm$ is the conformal operator dual to the conformal primary wave function $\phi_i^\pm$.

which in Poincaré coordinates (2.2.9) read

$$\int_{\mathcal{A}_+} \mathrm{d}^{d+2}X = \int_0^\infty T^{d+1}\mathrm{d}T \int_0^\infty \frac{\mathrm{d}y}{y^{d+1}} \int \mathrm{d}^d z, \tag{4.3.3a}$$

$$\int_{\mathcal{A}_-} \mathrm{d}^{d+2}X = \int_{-\infty}^0 |T|^{d+1}\mathrm{d}T \int_0^\infty \frac{\mathrm{d}y}{y^{d+1}} \int \mathrm{d}^d z, \tag{4.3.3b}$$

$$\int_{\mathcal{D}_+} \mathrm{d}^{d+2}X = \int_0^\infty R^{d+1}\mathrm{d}R \int_0^\infty \frac{\mathrm{d}\eta_+}{\eta_+^{d+1}} \int \mathrm{d}^d z, \tag{4.3.3c}$$

$$\int_{\mathcal{D}_-} \mathrm{d}^{d+2}X = \int_0^\infty R^{d+1}\mathrm{d}R \int_0^\infty \frac{\mathrm{d}\eta_-}{\eta_-^{d+1}} \int \mathrm{d}^d z. \tag{4.3.3d}$$

Using the relations (4.2.10), (4.2.15) and (4.2.19) between conformal primary wave functions and bulk-to-boundary propagators of the same conformal weight on $\mathcal{H}_{d+1}^+$, the contribution from each region of $\mathbb{M}^{d+2}$ factorises into the corresponding n-point contact Witten diagram on $\mathcal{H}_{d+1}^+$ and an integral over the curvature radius the hyperbolic foliation:[2]

$$\begin{aligned}\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\dots\Delta_n}(\pm_1 q_1,\dots,\pm_n q_n) = &\underbrace{\left(c^{\pm_1\dots\pm_n}_{\mathcal{A}_+} + c^{\pm_1\dots\pm_n}_{\mathcal{A}_-} + c^{\pm_1\dots\pm_n}_{\mathcal{D}}\right)}_{c^{\pm_1\dots\pm_n}_{\Delta_1\dots\Delta_n}} \\ &\times \underbrace{\int_{\mathcal{H}_{d+1}^+} \mathrm{d}\hat{X}_{\mathrm{AdS}}\, K^{\mathrm{AdS}}_{\Delta_1}(\hat{X}_{\mathrm{AdS}};q_1)\dots K^{\mathrm{AdS}}_{\Delta_n}(\hat{X}_{\mathrm{AdS}};q_n)}_{^{(\mathrm{AdS})}\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\dots\Delta_n}(q_1,\dots,Q_n)}\,.\end{aligned} \tag{4.3.5}$$

[2] Here we replaced:

$$\int_{\mathcal{H}_{d+1}^+} \mathrm{d}\hat{X}_{\mathrm{AdS}} = \int_0^\infty \frac{\mathrm{d}y}{y^{d+1}} \int \mathrm{d}^d z. \tag{4.3.4}$$

The Witten diagram ${}^{(\mathrm{AdS})}\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\ldots\Delta_n}(Q_1,\ldots,Q_n)$ on the second line of (4.3.5) is simply the well-known D-function [123], which is defined as the n-point contact Witten diagram generated by a the non-derivative n-point interaction in $\mathcal{H}^+_{d+1}$. The coefficient $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$ sums the radial integrals from each of the regions $\mathcal{A}_\pm$ and $\mathcal{D}$. For $\mathcal{A}_\pm$, using (4.2.10) we have:

$$c^{\pm_1\ldots\pm_n}_{\mathcal{A}_+} = -ig\int_0^\infty \mathrm{d}T\, T^{d+1}\prod_{i=1}^n \mathcal{N}_{\Delta_i}\,\tilde{K}_{\Delta_i-\frac{d}{2}}\left(m_i\, T e^{\pm_i\frac{\pi i}{2}}\right), \tag{4.3.6a}$$

$$c^{\pm_1\ldots\pm_n}_{\mathcal{A}_-} = -ig\int_0^\infty \mathrm{d}T\, T^{d+1}\prod_{i=1}^n \mathcal{N}_{\Delta_i}\,\tilde{K}_{\Delta_i-\frac{d}{2}}\left(m_i\, T e^{\mp_i\frac{\pi i}{2}}\right). \tag{4.3.6b}$$

For region $\mathcal{D}$, let us first consider the case that the celestial correlator contains at least one incoming and at least one outgoing mode. In this case the integral over regions $\mathcal{D}_\pm$ receives contributions from both short-distance and anti-podal singularities. Inserting (4.2.15) and (4.2.19), we have:

$$c^{\pm_1\ldots\pm_n}_{\mathcal{D}} = c^{\pm_1\ldots\pm_n}_{\mathcal{D}_+} + c^{\pm_1\ldots\pm_n}_{\mathcal{D}_-} \tag{4.3.7a}$$

$$c^{\pm_1\ldots\pm_n}_{\mathcal{D}_+} = -ig\left(\overbrace{\exp\left[\left(-d+\sum_{j=1}^n\Delta_j\left(1\mp_j 1\right)\right)\frac{\pi i}{2}\right]}^{\text{s-d (4.2.15)}}+\overbrace{\exp\left[\left(d-\sum_{j=1}^n\Delta_j\left(1\pm_j 1\right)\right)\frac{\pi i}{2}\right]}^{\text{a-p (4.2.15)}}\right)$$
$$\times\int_0^\infty \mathrm{d}R\, R^{d+1}\prod_{i=1}^n \mathcal{N}_{\Delta_i}\,\tilde{K}_{\Delta_i-\frac{d}{2}}\left(m_i\, R\right), \tag{4.3.7b}$$

$$c^{\pm_1\ldots\pm_n}_{\mathcal{D}_-} = -ig\left(\overbrace{\exp\left[\left(-d+\sum_{j=1}^n\Delta_j\left(1\pm_j 1\right)\right)\frac{\pi i}{2}\right]}^{\text{s-d (4.2.19)}}+\overbrace{\exp\left[\left(d-\sum_{j=1}^n\Delta_j\left(1\mp_j 1\right)\right)\frac{\pi i}{2}\right]}^{\text{a-p (4.2.19)}}\right)$$
$$\times\int_0^\infty \mathrm{d}R\, R^{d+1}\prod_{i=1}^n \mathcal{N}_{\Delta_i}\,\tilde{K}_{\Delta_i-\frac{d}{2}}\left(m_i\, R\right), \tag{4.3.7c}$$

where the d-dependence in the phases comes from the change of integration variable from $\eta_\pm$ to z. These combine to give

$$c_{\mathcal{D}}^{\pm_1\cdots\pm_n} = -2ig\left(\overbrace{\cos\left[\left(d-\sum_{j=1}^{n}\Delta_j\left(1\mp_j 1\right)\right)\frac{\pi}{2}\right]}^{\text{singularities of the }\phi^+_{\Delta_j}}+\overbrace{\cos\left[\left(d-\sum_{j=1}^{n}\Delta_j\left(1\pm_j 1\right)\right)\frac{\pi}{2}\right]}^{\text{singularities of the }\phi^-_{\Delta_j}}\right),$$

$$\times\int_0^\infty \mathrm{d}R\,R^{d+1}\prod_{i=1}^{n}\mathcal{N}_{\Delta_i}\,\tilde{K}_{\Delta_i-\frac{d}{2}}\left(m_i\,R\right). \tag{4.3.8}$$

It is interesting to note that the integrals over the curvature radius R and t in $c_{\mathcal{D}}^{\pm_1\cdots\pm_n}$ and $c_{\mathcal{A}_\pm}^{\pm_1\cdots\pm_n}$ are (analytic continuations) of the multiple-K integrals often encountered in momentum space CFT [124, 125]. It is useful to study such integrals using the Mellin-Barnes representation [8, 10, 126], where:

$$\tilde{K}_{\Delta_j-\frac{d}{2}}(m_jR) = \frac{1}{\Gamma\left(\Delta_j-\frac{d}{2}\right)}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_j}{2\pi i}\,\tilde{K}_{\Delta_j-\frac{d}{2}}(m_j,s_j)R^{-\left(\frac{d}{2}+2s_j\right)}, \tag{4.3.9a}$$

$$\tilde{K}_{\Delta_j-\frac{d}{2}}(m_j,s_j) = \Gamma\left(s_j+\tfrac{1}{2}\left(\Delta_j-\tfrac{d}{2}\right)\right)\Gamma\left(s_j-\tfrac{1}{2}\left(\Delta_j-\tfrac{d}{2}\right)\right)\left(\frac{m}{2}\right)^{-2s_j}. \tag{4.3.9b}$$

By adopting this representation the integral over the radial coordinate trivialises and is replaced by a Dirac delta function in the Mellin variables:

$$\int_0^\infty \mathrm{d}R\,R^{d+1}R^{-\sum_{j=1}^{n}\left(2s_j+\frac{d}{2}\right)} = 2\pi i\,\delta\left(-\left(d+2\right)+\sum_{j=1}^{n}\left(2s_j+\frac{d}{2}\right)\right). \tag{4.3.10}$$

In the Mellin-Barnes representation, the coefficients (4.3.6) differ simply by phases in the Mellin variables owing to the analytic continuation of the curvature radius:[3]

$$c^{\pm_1\ldots\pm_n}_{\mathcal{A}_+} = -ig \int_{-i\infty}^{+i\infty} [\mathrm{d}s_j]_n \; F_{\Delta_1\ldots\Delta_n}(s_1, m_1, \ldots, s_n, m_n)\, e^{-\sum\limits_{j=1}^{n} \pm_j \left(\frac{d}{2}+2s_j\right)\frac{\pi i}{2}}, \tag{4.3.12a}$$

$$c^{\pm_1\ldots\pm_n}_{\mathcal{A}_-} = -ig \int_{-i\infty}^{+i\infty} [\mathrm{d}s_j]_n \; F_{\Delta_1\ldots\Delta_n}(s_1, m_1, \ldots, s_n, m_n)\, e^{\sum\limits_{j=1}^{n} \pm_j \left(\frac{d}{2}+2s_j\right)\frac{\pi i}{2}}, \tag{4.3.12b}$$

where[4]

$$\begin{aligned} F_{\Delta_1\ldots\Delta_n}(s_1, m_1, \ldots, s_n, m_n) = 2\pi i\, \delta\left(-(d+2) + \sum_{j=1}^{n}\left(2s_j + \tfrac{d}{2}\right)\right) \\ \times \prod_{i=1}^{n} \frac{\mathcal{N}_{\Delta_i}}{\Gamma\left(\Delta_i - \frac{d}{2}\right)}\, \tilde{K}_{\Delta_i - \frac{d}{2}}(m_i, s_i). \end{aligned} \tag{4.3.13}$$

The contributions from all four regions $\mathcal{A}_\pm$ and $\mathcal{D}_\pm$ then combine as:

$$c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n} = 2ig \int_{-i\infty}^{+i\infty} [\mathrm{d}s_j]_n \; \mathfrak{c}^{\pm_1\ldots\pm_n}(s_1, \ldots, s_n)\; F_{\Delta_1\ldots\Delta_n}(s_1, m_1, \ldots, s_n, m_n), \tag{4.3.14}$$

where

$$\mathfrak{c}^{\pm_1\ldots\pm_n}(s_1, \ldots, s_n) = \mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{A}_+} + \mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{A}_-} + \mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{D}}, \tag{4.3.15}$$

---

[3] Here we defined

$$\int_{-i\infty}^{+i\infty} [\mathrm{d}s_j]_n = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s_1}{2\pi i} \cdots \frac{\mathrm{d}s_n}{2\pi i}. \tag{4.3.11}$$

[4] As noted in [10], the Mellin-Barnes representation of momentum space Witten diagrams makes manifest the symmetry under dilatations. This is analogous to how momentum space trivialises translation symmetry. For generic Δ_i, the solutions to the momentum space Conformal Ward identities are given by generalised hypergeometric functions with multiple variables (e.g. for $n = 3$ they are Appell F_4 functions [124]). The Mellin-Barnes representation gives the analytic continuation of momentum space conformal correlators valid for all physical values of the momenta (and beyond), analogous to how the Mellin-Barnes representation of the Gauss hypergeometric function is the analytic continuation of the hypergeometric series.

with[5]

$$\mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{A}_+} + \mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{A}_-} = -\cos\left[\sum_{j=1}^{n} \pm_j \left(\frac{d}{2}+2s_j\right)\frac{\pi}{2}\right], \tag{4.3.17a}$$

$$\mathfrak{c}^{\pm_1\ldots\pm_n}_{\mathcal{D}} = -2\cos\left[\left(d-\sum_{j=1}^{n}\Delta_j\right)\frac{\pi}{2}\right]\cos\left[\left(\sum_{j=1}^{n}\pm_j\Delta_j\right)\frac{\pi}{2}\right]. \tag{4.3.17b}$$

The sinusoidal factor in the Mellin variables s_j, coming from regions $\mathcal{A}_\pm$, indicates the presence of folded singularities in the mass m_j – see [10] section 3.1. Such folded singularities are absent in region $\mathcal{D}$, in which case the corresponding coefficient (4.3.17b) is constant and is, in particular, given by a product of cosine factors. This in turn implies that the contribution from region $\mathcal{D}$ vanishes for certain values of Δ_j, corresponding to zeros of the cosine factors in (4.3.17b). From the expressions (4.3.17) we can infer the following useful properties of the coefficient $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$:

$$c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n} = c^{\mp_1\ldots\mp_n}_{\Delta_1\ldots\Delta_n}, \tag{4.3.18a}$$

$$c^{\pm\ldots\mp\ldots\pm}_{\Delta_1\ldots\Delta_j\ldots\Delta_n} = -c^{\pm\ldots\mp\ldots\pm}_{\Delta_1\ldots d-\Delta_j\ldots\Delta_n}. \tag{4.3.18b}$$

The second property states that correlators involving a single incoming or outgoing mode are anti-symmetric upon replacing the mode with its shadow: $\Delta_j \to d-\Delta_j$.[6]

It is interesting to compare the coefficient $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$ (which, we recall, originates from integrating out the curvature radius in the hyperbolic slicing of $\mathbb{M}^{d+2}$) with the momentum space n-point scalar contact Witten diagrams in EAdS. In particular, upon multiplying by R^d and m^Δ, the function $\tilde{K}_{\Delta-\frac{d}{2}}(m_j R)$ defined in (4.2.6) is proportional to the bulk-to-boundary propagator in Poincaré coordinates of $\mathcal{H}^+_{d+1}$. Here the mass m plays the role of the magnitude of the boundary momentum and R the role of the Poincaré coordinate z. The difference is that, in regions $\mathcal{A}_+$ and $\mathcal{A}_-$, this coordinate is rotated according to whether a particle is incoming or outgoing, leading to folded singularities in the "momenta" m_i. Such folded singularities are absent in standard momentum space EAdS Witten diagrams. A virtue of the

[5]Here we simplified the cosine factors in (4.3.8) using the identity:

$$2\cos\alpha\,\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)\,. \tag{4.3.16}$$

[6]To see this one also uses that:

$$\frac{\mathcal{N}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)} = -\frac{\mathcal{N}_{d-\Delta}}{\Gamma\left(\frac{d}{2}-\Delta\right)}. \tag{4.3.19}$$

Mellin-Barnes representation is that these singularities manifest themselves in the sinusoidal factor (4.3.17a) in the Mellin variables, where (4.3.13) is, up to factors of m^{Δ_j}, indeed proportional to the Mellin-Barnes representation of an n-point contact Witten diagram in EAdS [10]. It is interesting to note that the folded singularities only originate from the regions $\mathcal{A}_{\pm}$, where $\mathbb{M}^{d+2}$ is foliated by EAdS_{d+1} hypersurfaces. The contribution (4.3.17b) from region $\mathcal{D}$, where $\mathbb{M}^{d+2}$ is instead foliated by dS_{d+1} hypersurfaces, does not contain folded singularities and, up to factors of m^{Δ_j}, takes the same form as momentum space n-point contact Witten diagrams in EAdS.

Let us now turn to the case that we have all incoming or all outgoing modes. In this case there is just a single contribution from $\mathcal{D}_+$, which, in the case that we have all outgoing modes, corresponds to the short distance singularity of ϕ^+_Δ or, in the case that we have all incoming modes, the anti-podal singularity of ϕ^-_Δ. And likewise for $\mathcal{D}_-$ which, in the case that we have all outgoing modes, corresponds to the anti-podal singularity of ϕ^+_Δ or, in the case that we have all incoming modes, the short-distance singularity of ϕ^-_Δ. There is therefore only a single cosine factor in (4.3.8) that contributes in this case, giving:

$$\mathfrak{c}^{\pm\ldots\pm}\left(s_1,\ldots,s_n\right) = -\left[\cos\left(\frac{d\pi}{2}\right)+\cos\left(\left(d+2\right)\frac{\pi}{2}\right)\right] = 0, \qquad (4.3.20)$$

where we used the Dirac delta function (4.3.10) to simplify the total contribution (4.3.17a) from $\mathcal{A}_+$ and $\mathcal{A}_-$. This vanishing is consistent with the fact that there is no scattering amplitude with all incoming or all outgoing particles. This in particular means that there is no non-trivial celestial two-point function with two incoming or two outgoing particles.

It should be noted that, while for generic values of the scaling dimensions Δ_i the need for a Mellin-Barnes representation (4.3.14) of the coefficient $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$ is inevitable, for certain values of Δ_i the Mellin-Barnes integrals can be lifted. See e.g. sections 3.3-3.4 in [127] and section 4.6 of [8]. An example is when $\Delta_i = \frac{d+1}{2}$ which corresponds to a conformally coupled scalar. In the following section we study this case in more detail for $n=3$.

Specific cases: 3pt contact diagrams. In the above we saw that celestial n-point contact diagrams can be re-expressed (4.3.5) as an n-point contact Witten diagram in $\mathcal{H}^+_{d+1}$ times a coefficient $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$ that arises from the integral over the curvature radius of the hyperbolic slicing. In the following we analyse in more detail the case $n = 3$, where we have that

$$\mathfrak{c}^{+++}(s_i) = \mathfrak{c}^{---}(s_i) = 0\,, \tag{4.3.21a}$$

$$\mathfrak{c}^{-++}(s_i) = \mathfrak{c}^{+--}(s_i) = \cos(2\pi s_1) - \cos\left[\left(\tfrac{d}{2} - \Delta_1\right)\pi\right] - \cos\left[\left(\tfrac{d}{2} - \Delta_2 - \Delta_3\right)\pi\right]\,, \tag{4.3.21b}$$

$$\mathfrak{c}^{+-+}(s_i) = \mathfrak{c}^{-+-}(s_i) = \cos(2\pi s_2) - \cos\left[\left(\tfrac{d}{2} - \Delta_2\right)\pi\right] - \cos\left[\left(\tfrac{d}{2} - \Delta_1 - \Delta_3\right)\pi\right]\,, \tag{4.3.21c}$$

$$\mathfrak{c}^{++-}(s_i) = \mathfrak{c}^{--+}(s_i) = \cos(2\pi s_3) - \cos\left[\left(\tfrac{d}{2} - \Delta_3\right)\pi\right] - \cos\left[\left(\tfrac{d}{2} - \Delta_1 - \Delta_2\right)\pi\right]\,. \tag{4.3.21d}$$

For generic Δ_i, the Mellin-Barnes integral (4.3.14) with $n = 3$ can be expressed in terms of the Appell F_4 function (see appendix B.2 of [127]). For certain values of the scaling dimensions, however, the corresponding the Mellin-Barnes integrals can be lifted. The simplest example is for conformally coupled scalars, corresponding to $\Delta_i = \frac{d+1}{2}$ (see section 3.3 of [127]), which can be used as seeds for correlators of conserved currents (as in e.g. [8, 126]). Considering for concreteness the case that we have one incoming mode and two outgoing (i.e. $\pm_1 = +$, $\pm_2 = +$ and $\pm_3 = -$), upon evaluating all the Mellin-Barnes integrals one obtains:

$$c^{++-}_{\frac{d+1}{2}\frac{d+1}{2}\frac{d+1}{2}|\mathcal{A}} = ig\frac{\cos\left(\frac{d\pi}{2}\right)\Gamma\left(\frac{1-d}{2}\right)}{\sqrt{2m_1m_2m_3}}\left(m_3 - m_1 - m_2\right)^{\frac{d-1}{2}}\,, \tag{4.3.22a}$$

$$c^{++-}_{\frac{d+1}{2}\frac{d+1}{2}\frac{d+1}{2}|\mathcal{D}} = ig\frac{\cos\left(\frac{d\pi}{2}\right)\Gamma\left(\frac{1-d}{2}\right)}{\sqrt{2m_1m_2m_3}}\left(m_1 + m_2 + m_3\right)^{\frac{d-1}{2}}\,, \tag{4.3.22b}$$

where for convenience we gave the contributions from region $\mathcal{D}$ and combined regions $\mathcal{A}_\pm$ separately. Note the expected folded singularities in the mass m_i coming from regions $\mathcal{A}_\pm$. Setting $m_1 = m_2 = m$, $m_3 = 2m(1+\epsilon)$ and $d = 2$, the contribution from regions $\mathcal{A}_\pm$ recovers the result given in equation (3.13) of [17] which was obtained from the corresponding momentum space scattering amplitude using (4.1.29). We therefore differ from the result of [17] by the contribution from region $\mathcal{D}$, which is regular in ϵ. It should be clarified from where this discrepancy originates, which we leave to future work. A possible explanation could be that employing (4.1.29) to obtain Celestial amplitudes from momentum space scattering amplitudes requires to commute the divergent momentum integral in the definition (4.2.7) of conformal primary wave functions past the integral over the bulk of Minkowski space.

Similarly, one can consider the case that one has two conformally coupled and one general scalar. The fact we have two conformally coupled scalars implies that two of the three Mellin-Barnes integrals can be lifted. One obtains the following expression in terms of the Gauss hypergeometric function:

$$c^{++-}_{\frac{d+1}{2}\frac{d+1}{2}\Delta_3|\mathcal{A}} = -i\cos\left(\frac{d\pi}{2}\right)\frac{\sqrt{\pi}g\,(2m_3)^{\frac{d-1}{2}}}{\sqrt{2m_1m_2m_3}\,\Gamma(\Delta_3-\frac{d}{2})\Gamma\left(\frac{3-d}{2}\right)} \times {}_2F_1\left(1-\Delta_3, 1-d+\Delta_3; \frac{3-d}{2}; \frac{m_1+m_2+m_3}{2m_3}\right), \tag{4.3.23a}$$

$$c^{++-}_{\frac{d+1}{2}\frac{d+1}{2}\Delta_3|\mathcal{D}} = \frac{-i\left(\cos\left(\frac{d\pi}{2}\right)-\cos\left(\left(\frac{d}{2}-\Delta_3\right)\pi\right)\right)\sqrt{\pi}g\,(2m_3)^{\frac{d-1}{2}}}{\sqrt{2m_1m_2m_3}\,\Gamma(\Delta_3-\frac{d}{2})\Gamma\left(\frac{3-d}{2}\right)} \times {}_2F_1\left(1-\Delta_3, 1-d+\Delta_3; \frac{3-d}{2}; \frac{m_3-m_1-m_2}{2m_3}\right). \tag{4.3.23b}$$

In $d=2$ this gives:

$$c^{++-}_{\frac{3}{2}\frac{3}{2},1+i\nu|\mathcal{A}} = \frac{2ig\Gamma\left(1-\Delta_3\right)\cos\left(2\left(1-\Delta_3\right)\csc^{-1}\left(\frac{\sqrt{2m_3}}{\sqrt{m_1+m_2+m_3}}\right)\right)}{\sqrt{m_1m_2}}, \tag{4.3.24a}$$

$$c^{++-}_{\frac{3}{2}\frac{3}{2},1+i\nu|\mathcal{D}} = 2ig\left(1+\cos\left(\left(1-\Delta_3\right)\pi\right)\right)\Gamma\left(1-\Delta_3\right) \times \frac{\cosh\left(2\left(1-\Delta_3\right)\operatorname{csch}^{-1}\left(\frac{\sqrt{2m_3}}{\sqrt{m_1+m_2-m_3}}\right)\right)}{\sqrt{m_1m_2}}. \tag{4.3.24b}$$

Another example in which simplifications arise is when one or more of the fields are massless. In this case the function (4.2.6) in the representation (4.2.5) of the conformal primary wavefunction simplifies [18]. In the case of two massless external fields we have:

$$c^{++-}_{\frac{d}{2}+i\nu_1,\frac{d}{2}+i\nu_2,\frac{d}{2}+i\nu_3|\mathcal{A}} = \frac{ig\cos\left(\pi\left(\Delta_1+\Delta_2-\frac{d}{2}\right)\right)m_3^{\Delta_1+\Delta_2-d}\left(2m_3\right)^{2-\frac{d}{2}}}{\Gamma\left(\Delta_3-\frac{d}{2}\right)} \times\Gamma\left(1+\frac{\Delta_3-\Delta_1-\Delta_2}{2}\right)\Gamma\left(1+\frac{d-\Delta_1-\Delta_2-\Delta_3}{2}\right), \tag{4.3.25}$$

$$c^{++-}_{\frac{d}{2}+i\nu_1,\frac{d}{2}+i\nu_2,\frac{d}{2}+i\nu_3|\mathcal{D}} = \frac{ig\left(\cos\left[\left(\frac{d}{2}-\Delta_3\right)\pi\right]+\cos\left[\left(\Delta_1+\Delta_2-\frac{d}{2}\right)\pi\right]\right)}{m_3^{-\Delta_1-\Delta_2+d}\left(2m_3\right)^{-2+\frac{d}{2}}\Gamma\left(\Delta_3-\frac{d}{2}\right)} \times\Gamma\left(1+\frac{\Delta_3-\Delta_1-\Delta_2}{2}\right)\Gamma\left(1+\frac{d-\Delta_1-\Delta_2-\Delta_3}{2}\right). \tag{4.3.26}$$

Let us finally note that these examples make clear that the coefficients (4.3.14) are vanishing for certain mass configurations. Analogous zeros appear in the context of dS boundary correlators [10] and, as in that case, we expect them to be imprints of (perturbative) unitarity.

4.4 All Orders in the Perturbative Expansion

In the previous section we saw that contact diagram contributions to celestial correlators are proportional to the corresponding contact Witten diagram in $\mathcal{H}^+_{d+1}$:

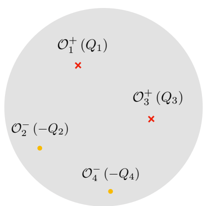
$$\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\ldots\Delta_n}\left(\pm_1 q_1,\ldots,\pm_n q_n\right) = c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}\,{}^{(\mathrm{AdS})}\tilde{\mathcal{A}}^{\mathrm{c}}_{\Delta_1\ldots\Delta_n}\left(q_1,\ldots,q_n\right). \tag{4.4.1}$$

The coefficients $c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$ can be thought of as the ratio:

$$c^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n} = \frac{\lambda^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}}{\lambda^{\mathrm{AdS}}_{\Delta_1\ldots\Delta_n}}, \tag{4.4.2}$$

relating the coefficient $\lambda^{\mathrm{AdS}}_{\Delta_1\ldots\Delta_n}$ of contact Witten diagram to its celestial counterpart $\lambda^{\pm_1\ldots\pm_n}_{\Delta_1\ldots\Delta_n}$.

Contact diagrams are the basic building blocks for all other processes i.e. those that involve particle exchanges. When an exchanged particle goes on-shell, we expect that the corresponding observable factorises appropriately into subprocesses and, when all exchanged particles are on-shell, these subprocesses are the contact diagrams generated by the vertices. Using this property we can extend the result (4.3.5) to Celestial amplitudes involving exchanged particles, decomposing them as a sum of corresponding Witten diagrams in $\mathcal{H}^+_{d+1}$.



$$= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\sum_{\pm}\frac{c^{+-\pm}_{\Delta_1\Delta_2\Delta}\,c^{+-\mp}_{\Delta_3\Delta_4\Delta}}{c_\Delta}$$

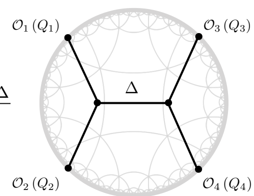


Figure 4.2: Contributions from particle exchanges to d-dimensional celestial correlators decompose into exchange Witten diagrams of particles carrying Principal Series representations $\Delta \in \frac{d}{2}+i\mathbb{R}$ in EAdS$_{d+1}$. The relative coefficients ensure consistent on-shell factorisation.

To illustrate, let us first consider the tree level exchange of a scalar field of mass m^2 in a 4pt celestial correlator, say in the s-channel. Using that a field in $\mathbb{M}^{d+2}$ can be decomposed (5.1.18) in terms of fields on (EA)dS$_{d+1}$ carrying principal series representations of $SO\left(d+1,1\right)$, assuming consistent on-shell factorisation then implies (along the same lines as [10])

$$\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|m^2|\Delta_3\Delta_4}\left(\pm_1 q_1, \pm_2 q_2, \pm_3 q_3, \pm_4 q_4\right)$$
$$= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{\mathrm{d}\Delta}{2\pi i} \sum_{\pm} \frac{c^{\pm_1\pm_2\pm}_{\Delta_1\Delta_2\Delta}\, c^{\mp\pm_3\pm_4}_{\Delta\Delta_3\Delta_4}}{c_\Delta}\, {}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(q_1, q_2, q_3, q_4\right), \tag{4.4.3}$$

where ${}^{(\text{AdS})}\tilde{\mathcal{A}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(q_1, q_2, q_3, q_4\right)$ is the corresponding exchange Witten diagram in $\mathcal{H}^+_{d+1}$ for a particle of scaling dimension Δ. The coefficients $c^{\pm_1\pm_2\pm}_{\Delta_1\Delta_2\Delta}$ and $c^{\mp\pm_3\pm_4}_{\Delta\Delta_3\Delta_4}$ convert the coefficients of the three-point contact subdiagrams in AdS on the r.h.s. to their celestial counterparts via (4.4.2), where for each Δ there are two operators corresponding to incoming ($-$) and outgoing ($+$) particles.[7] The c_Δ account for the change in 2-point function normalisation,

$$c_\Delta = \frac{C^{\text{flat}}_\Delta}{C^{\text{AdS}}_\Delta} = \frac{\Gamma\left(\frac{d}{2}-\Delta\right)\Gamma\left(\Delta-\frac{d}{2}+1\right)}{2\pi} = \frac{1}{2}\csc\left(\left(\frac{d}{2}-\Delta\right)\pi\right), \tag{4.4.4}$$

where C^{flat}_Δ is the celestial 2-point function normalisation factor and the AdS normalisation C^{AdS}_Δ is given in (4.2.3). These clearly have the following useful property upon replacing Δ with its shadow $d-\Delta$:

$$c_\Delta = -c_{d-\Delta}, \tag{4.4.5}$$

which is a specific case of the more general property (4.3.18b).

More generally, given a perturbative contribution to a n-point massive Celestial amplitude, the following steps give its decomposition in terms of corresponding Witten diagrams in $\mathcal{H}^+_{d+1}$:

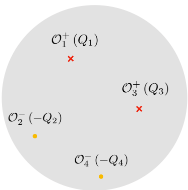
1. Draw the same diagram in EAdS: Each external line connected to the point $\pm_i q_i$ on the conformal boundary of $\mathbb{M}^{d+2}$ becomes an external line connected to the point q_i on the boundary of EAdS_{d+1}, where an operator with the same scaling dimension Δ_i is inserted. Each internal line becomes an internal line in EAdS. Each vertex becomes a vertex in EAdS.

2. Assign each internal line a scaling dimension label and divide by the factor (4.4.4) accounting for the change in 2pt function normalisation. For each vertex, multiply by the factor (4.4.2) that converts the contact diagram it generates to the corresponding celestial contact diagram as in (4.4.1). For vertices that are attached to internal lines, in the corresponding contact diagram the external particle can be either incoming or outgoing and these possibilities should be summed over.

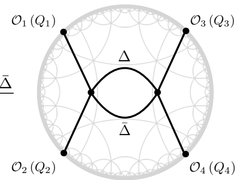
[7]Note that there is no two-point function for two incoming or two outgoing particles, which can be seen from (4.3.20).

3. For each internal line, integrate the associated scaling dimension label over the Principal Series $\Delta \in \frac{d}{2} + i\mathbb{R}$.

For example, for the candy diagram contribution to the four-point function in the s-channel, taking $\pm_1 = \pm_3 = +1$ and $\pm_2 = \pm_4 = -1$, one obtains:



$$= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{d\bar{\Delta}}{2\pi i} \sum_{\pm\hat{\pm}} \frac{c^{+-\pm\hat{\pm}}_{\Delta_1\Delta_2\Delta\bar{\Delta}}\, c^{+-\mp\hat{\mp}}_{\Delta_3\Delta_4\Delta\bar{\Delta}}}{c_\Delta c_{\bar{\Delta}}}$$



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To summarise this section, we have seen that a given perturbative contribution to a celestial correlator can be expanded in terms of corresponding Witten diagrams in $\mathcal{H}^+_{d+1}$. This was shown to follow from the assumption of consistent on-shell factorisation once it was established in section 4.3 that celestial contact diagrams are proportional their Witten diagram counterparts in $\mathcal{H}^+_{d+1}$ (equation 4.3.5). It should be noted that this result would equivalently follow from conformal symmetry, factorisation and the assumption of single-valuedness (analyticity) of celestial correlators as a function of the (complexified) cross-ratios, as is the case for AdS boundary correlators in the Euclidean region. The fact that celestial contact diagrams are proportional to their corresponding contact Witten diagrams in $\mathcal{H}^+_{d+1}$ indeed shows that the assumption of single-valuedness would have been a valid one.

4.5 Spectral Representation and Unitarity

We have seen that perturbative celestial correlators in d-dimensions can be decomposed in terms of corresponding Witten diagrams on EAdS_{d+1}. Through such identities one might try to import the wealth of techniques, results and understanding available in EAdS to the study of celestial correlators, in the same spirit as analogous studies relating perturbative boundary correlators in dS_{d+1} to their EAdS_{d+1} counterparts [9, 10].

For example, such identities imply that perturbative celestial correlators have the same analytic structure as their EAdS Witten diagram counterparts. In particular, given a single-valued conformally invariant four-point function of operators $\mathcal{O}_i$ in d-dimensional Euclidean space, harmonic analysis on $SO(d+1,1)$ [37, 128–131] implies that it admits a partial-wave expansion over an orthogonal basis of Eigenfunctions $\mathcal{F}_{\Delta,J}$ of the Casimir invariants taking the following form (say, in the (12)(34) channel):

$$\langle \mathcal{O}_1(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\mathcal{O}_3(\vec{x}_3)\mathcal{O}_4(\vec{x}_4)\rangle = \mathbb{1}_{12}\mathbb{1}_{34}+\sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}\,\rho_J(\Delta)\,\mathcal{F}^{12,34}_{\Delta,J}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)\,, \tag{4.5.1}$$

where $\rho_J(\Delta)$ is the spectral density and the first term on the r.h.s. is the contribution from the identity operator. For correlators (4.5.1) on the boundary of EAdS $\rho_J(\Delta)$ is a meromorphic function of Δ, which is due to the fact that the operator product expansion converges. In perturbation theory, celestial correlators can be expressed as a sum of EAdS Witten diagrams, implying that $\rho_J(\Delta)$ is meromorphic also for perturbative celestial correlators.

The conformal partial wave expansion (CPWE) of celestial correlators has been studied in a variety of works [43, 47, 132, 133]. The above results tell us that one can determine the CPWE of a given perturbative celestial correlator can be determined using their relation to EAdS Witten diagrams. For example, considering the exchange (4.4.3) with $\pm_1 = \pm_2 = +1$ and $\pm_3 = \pm_4 = -1$, we can write:

$$\begin{aligned}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|m^2|\Delta_3\Delta_4} &= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}\,\frac{c^{++-}_{\Delta_1\Delta_2\Delta}c^{+--}_{\Delta\Delta_3\Delta_4}}{c_\Delta}\,{}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\\ &= \frac{1}{2}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}\,\frac{c^{++-}_{\Delta_1\Delta_2\Delta}c^{+--}_{\Delta\Delta_3\Delta_4}}{c_\Delta}\\ &\quad\times\left[{}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}-{}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|d-\Delta|\Delta_3\Delta_4}\right],\end{aligned} \tag{4.5.2}$$

where in the first equality we used that $c^{+++}_{\Delta_1\Delta_2\Delta} = c^{---}_{\Delta\Delta_3\Delta_4} = 0$ and in second equality we used the anti-symmetric property (4.3.18b) of the coefficients upon replacing Δ with its shadow $d-\Delta$. The difference of Δ and $d-\Delta$ exchanges in EAdS is proportional to a conformal partial wave [74, 134]:

$$ {}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|\Delta,J|\Delta_3\Delta_4}-{}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|d-\Delta,J|\Delta_3\Delta_4} = (d-2\Delta)\,{}^{(\text{AdS})}\mathcal{F}^{12,34}_{\Delta,J}\,, \tag{4.5.3}$$

where, adopting the notation of [10], the AdS-normalised conformal partial wave reads:

$$\begin{aligned}{}^{(\text{AdS})}\mathcal{F}^{12,34}_{\Delta,J}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int d^d\vec{x}\,{}^{(\text{AdS})}\tilde{\mathcal{A}}^{\text{c}}_{\Delta_1\Delta_2\Delta}(\vec{x}_1,\vec{x}_2,\vec{x})\,{}^{(\text{AdS})}\\ \times\,\tilde{\mathcal{A}}^{\text{c}}_{\Delta_3\Delta_4 d-\Delta}(\vec{x}_3,\vec{x}_4,\vec{x}).\end{aligned} \tag{4.5.4}$$

This gives the following CPWE for the exchange (4.4.3):

$$\tilde{\mathcal{A}}^{\text{exch.}}_{\Delta_1\Delta_2|m^2|\Delta_3\Delta_4} = \left(\frac{d}{2}-\Delta\right)\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}\,\frac{c^{++-}_{\Delta_1\Delta_2\Delta}c^{+--}_{\Delta\Delta_3\Delta_4}}{c_\Delta}\,{}^{(\text{AdS})}\mathcal{F}^{12,34}_{\Delta,J}. \tag{4.5.5}$$

Note that meromorphicity of the spectral density in Δ then follows from meromorphicity in EAdS and of the coefficients (4.4.4) and (4.3.14), as consistent with single-valuedness of (perturbative) celestial correlators as a function of complex conformal cross ratios. This shows that in celestial CFT, which are not unitary in the familiar sense of Lorentzian CFTs and thus not necessarily endowed with a convergent OPE expansion, a convergent OPE expansion is recovered from the residues of the spectral density at any order in perturbation theory.

Unitarity. In unitary Euclidean CFT the conformal partial wave expansion (4.5.1) follows by expanding in unitary representations of the Euclidean conformal group $SO\left(d+1,1\right)$. In particular, if one assumes that states have positive finite norms, which is justified when the CFT arises as boundary dual of a Minkowski theory, one can expand into contributions from unitary irreducible representations of $SO\left(d+1,1\right)$ by inserting a resolution of the identity[8]

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{\mathrm{d}\Delta}{2\pi i} \frac{1}{N\left(\Delta, J\right)} \int \mathrm{d}^d\vec{x}\, |\Delta, J, \vec{x}\rangle\langle\Delta, J, \vec{x}|, \tag{4.5.6}$$

where $\mathcal{P}_{\Delta,J}$ projects onto the conformal multiplet labelled by the scaling dimension Δ and spin J, and $N\left(\Delta, J\right)$ is related to the norm of the state.[9] Taking now

$$\mathcal{O}_1 = \mathcal{O}_3^\dagger \quad \text{and} \quad \mathcal{O}_2 = \mathcal{O}_4^\dagger, \tag{4.5.7}$$

it is straightforward to conclude the following non-perturbative positivity constraint on the spectral density:

$$\rho_J\left(\Delta\right) \geq 0. \tag{4.5.8}$$

This follows from positivity of $N\left(\Delta, J\right) > 0$, which is ensured by the unitarity of Principal Series representations in Euclidean CFT.

Positivity (4.5.8) of the spectral density has also been observed in the context of the conformal partial wave expansion of boundary correlators in dS space [135, 136]. Here we are noting that it applies more generally to *any* unitary Euclidean CFT, and therefore also to celestial correlators. It would be interesting to use (4.5.8) to derive non-perturbative constraints on bulk Minkowski physics along the lines proposed in [135] and compare with other implementations of unitarity.[10]

[8]In the following for simplicity we assume that only principal series representations contribute. In the most general case the above resolution of the identity would also involve discrete and complementary series representations of $SO\left(d+1,1\right)$.

[9]Note that the states live in the unitary Hilbert space of the bulk theory which is in fact Lorentzian. Using positivity of the bulk Hilbert space one can then infer positivity of the boundary CPW decomposition.

[10]Note that the positivity (4.5.8) is the analogue of the positivity of conformal block coefficients in Lorentzian CFT (dual to AdS physics), which is essential in formulating the numerical conformal bootstrap [137, 138].

Let us finally note that the above discussion holds for canonically normalised operators. In the context of celestial CFTs, we have a two-fold degeneracy of operators with the same dimension, one for in-coming and one-for outgoing particles. Furthermore, their 2pt functions are not canonical:

$$\Big\langle \mathcal{O}_1^+(x_1)\mathcal{O}_2^+(x_2)\Big\rangle = 0\,, \tag{4.5.9a}$$

$$\Big\langle \mathcal{O}_1^-(x_1)\mathcal{O}_2^-(x_2)\Big\rangle = 0\,, \tag{4.5.9b}$$

$$\Big\langle \mathcal{O}_1^+(x_1)\mathcal{O}_2^-(x_2)\Big\rangle = \frac{\mathcal{N}_\Delta}{(x_{12}^2)^\Delta}\,. \tag{4.5.9c}$$

A canonical spectral representation can only be obtained in terms of operators which are normalised canonically. This amounts to consider:

$$\mathcal{O}_>(x) = \frac{1}{\sqrt{2}}\Big(\mathcal{O}_1^+(x) + \mathcal{O}_1^-(x)\Big)\ , \tag{4.5.10a}$$

$$\mathcal{O}_<(x) = \frac{1}{\sqrt{2}}\Big(\mathcal{O}_1^+(x) - \mathcal{O}_1^-(x)\Big)\ . \tag{4.5.10b}$$

Positivity conditions can now be derived also for celestial CFTs working with $\mathcal{O}_{>/<}$.

Chapter 5

Celestial Holography Revisited

In this last chapter, we will present a revisited version of Celestial Holography, following the path initiated in [139] and continued in [5]. The entire construction is carried out in position space, therefore, we will denote with Q the boundary points of the hyperbolic unitary slices. For the sake of clarity, we will call the disjoint union of the unitary hyperbolic slices the *extended unitary slice*, or the *extended hyperboloid*. The key point for this new construction is Eq.(4.2.9), showing the closed expression of massive conformal primary wavefunctions. As already emphasised in the previous chapter, these closed expressions present themselves in a factorized form: they involve the analytically continued bulk-to-boundary propagator (3.1.6), which describes the dependence on the directions across the unitary slices, times the Bessel function (4.2.6), depending solely on their radii. As noted in [6], this Bessel function is the kernel of a transformation known as Kontorovich-Lebedev transform. This transformation generalises the Mellin transform to the massive case, reducing to the latter by following the same procedure we used in sec.3.5 to derive massless conformal primary wavefunctions from the massive ones. As we will show in the first section, the Kontorovich-Lebedev transform of a massive solution of Klein-Gordon equation in the region $\mathcal{D}$ is automatically a field, defined on the unitary slice, which solves Klein-Gordon equation in dS_{d+1} with mass $m^2_{\text{dS}} = \Delta(d-\Delta)$. The main idea carried out in this chapter is to expand Minkowski fields $\phi(X)$ in Kontorovich-Lebedev transform, in order to find out their corresponding field $\phi(\hat{X})$ on the extended unitary slice. Then, by applying the (EA)dS/CFT correspondence to $\phi(\hat{X})$, we can map fields on the extended unitary slice to fields on the Celestial Sphere. In this sense, the Flat/CFT correspondence arises as a sequential combination of the Kontorovich-Lebedev transform and the (EA)dS/CFT correspondence. This aligns with what was seen in the previous chapter. There, Flat/CFT correspondence was realised by employing conformal primary wavefunctions (ref. (4.2.9)). However, at this stage, it lacks a criterion to fix uniquely the $i\epsilon$-prescription when we analytically continue both the EAdS bulk-to-boundary propagator and the kernel (4.2.6) to the extended unitary slice. In the previous chapter, closed expressions of conformal primary wavefunctions were figured out by regularising the integrals (3.1.3) (ref. Appendix C.1). Here, one way to resolve all the ambiguities is to consider the Feynman propagator, which possesses its own $i\epsilon$-prescription, dictated by temporal order.

We will show how to decompose Feynman propagator in terms of 2-point correlation functions defined on the extended unitary slice. These correlators coincide with the time-ordered bulk-to-bulk propagators when both the bulk points in the Feynamn propagator lie inside the region $\mathcal{D}_+$ and, in general, provide an analytical extension of the former to the extended unitary slice. Further, in this decomposition, the spectral density $\rho^{(m)}(\nu)$ of the Feynamn propagator will also provide the aimed analytical extension of the kernel (4.2.6) to the extended unitary slice. Next, by sending one bulk point in the Feynman propagator to the boundary, we will recover the *Celestial bulk-to-boundary* propagator as defined in [139]. In complete analogy with what happened with the conformal primary wavefunctions, we will show that the Celestial bulk-to-boundary propagator can be expressed as an analytic continuation of the EAdS$_{d+1}$ bulk-to-boundary propagator times the previously found analytically continued kernel of the Kontorovich-Lebedev transformation. This result will allow us to compute Celestial correlators, defined in (5.3.1), by operating as in the previous chapter. Once again, we will found that Celestial correlators can be recast in terms of a linear combination of corresponding Witten correlators, in total analogy with the result found in the previous chapter. Unlike what was done in the last chapter, here, we will compute the particle exchange diagram through a direct calculation, using the new spectral decomposition of the Feynman propagator. The result in shape will correspond to the one found in the previous chapter by using the decomposition properties of CFTs (ref.(4.4.3)), demonstrating once again that exchange contribution to Celestial amplitudes can be decomposed in terms of a continuum of corresponding exchange Witten diagrams in EAdS$_{d+1}$, with the exchanged particles carrying unitary Principal Series representations.

The structure of this chapter is heavily influenced by [5] and unfolds as follows: the Kontorovich-Lebedev transform will be introduced in the first section. Next, in sec.(5.2) we will find out the spectral decomposition of the Feynman propagator in terms of 2-point correlation functions of the extended unitary slice, also determining the analytical continuation of the Kontorovich-Lebedev kernel. Then, in the same section, we will recover both the Celestial bulk-to-boundary and boundary-to-boundary propagator, providing also the analytical extension of the EAdS bulk-to-boundary propagator on the extended hyperboloid. Using these acquired concepts, we will move on to perturbative calculations of Celestial correlators. In sec.(5.3.1), we will focus on computing contact Celestial correlators. Subsequently, in sec.(5.3.2), we will extend our approach beyond the leading order, tackling the computation for the Celestial four-point exchange diagram. Celestial correlators will be recast in terms of corresponding EAdS Witten correlators, in total analogy with the result shown in the previous chapter.

5.1 Kontorovich-Lebedev Transform

In standard QFT, Minkowski space-time is foliated into X^0-constant hyperplane, making translation invariance in all directions manifest. It is then natural to consider a field decomposition in exponential plane waves, which diagonalise the translation generator. Such a decomposition is easily obtained via Fourier transform. In EAdS[1], instead, we only have translation invariance in the boundary directions. As an alternative, perpendicularly to the boundary it is natural to expand in terms of Eigenvalues of the Dilatation generator [10]. In Poincaré coordinates (2.2.9) the direction perpendicular to the boundary is parameterised by the coordinate y and the Dilatation generator is diagonalised by power-laws:

$$f_\alpha(y) = \langle y|f_\alpha\rangle = y^{-i\alpha+\frac{d}{2}}\,, \tag{5.1.1}$$

which satisfy completeness and orthogonality:

$$\langle f_\alpha|f_\beta\rangle = 2\pi\delta(\beta-\alpha)\,, \tag{5.1.2a}$$

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}\alpha}{2\pi}\langle y_1|f_\alpha\rangle\langle f_\alpha|y_2\rangle = y_1^{d+1}\delta(y_1-y_2)\,. \tag{5.1.2b}$$

Any element of $L^2(\mathbb{R}^+, \frac{\mathrm{d}y}{y^{d+1}})$ can be expanded in terms of power laws (5.1.1) and this is implemented by the Mellin transform. See [10] for more details.

In the context of Celestial holography, an orthogonal basis to expand elements of $L^2(\mathbb{R}^+, dR\, R^{d-1})$ is given by Bessel-K functions:

$$\tilde{K}_\alpha(R) = \langle R|\tilde{K}_\alpha\rangle = \frac{2R^{-d/2}}{\Gamma(i\alpha)}\,K_{i\alpha}(mR)\,, \tag{5.1.3}$$

[1] The same applies to dS_{d+1} (see [10]) but we stick to EAdS_{d+1} for concreteness.

which are complete and orthogonal:[2]

$$\langle \tilde{K}_\alpha | \tilde{K}_\beta \rangle = 2\pi\delta(\beta-\alpha) + \frac{2\pi\Gamma(i\alpha)\delta(\alpha+\beta)}{\Gamma(-i\alpha)}\,, \tag{5.1.4a}$$

$$\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\alpha}{2\pi} \langle R_1|\tilde{K}_\alpha\rangle\langle\tilde{K}_\alpha|R_2\rangle = R_1^{-d+1}\delta(R_1-R_2)\,. \tag{5.1.4b}$$

The above completeness and orthogonality relations can be shown straightforwardly by employing the Mellin-Barnes representation of the Bessel function. To prove (5.1.4a) one first goes to Mellin space rewriting it as:

$$\langle \tilde{K}_\alpha | \tilde{K}_\beta \rangle = \int_\epsilon^\infty \frac{\mathrm{d}R}{R} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s\mathrm{d}t}{(2\pi i)^2}\, R^{\kappa-2s-2t} \left(\frac{m}{2}\right)^{-2s-2t} \times \frac{\Gamma(s-\frac{i\alpha}{2})\Gamma(s+\frac{i\alpha}{2})\Gamma(t-\frac{i\beta}{2})\Gamma(t+\frac{i\beta}{2})}{\Gamma(-i\alpha)\Gamma(i\beta)}, \tag{5.1.5}$$

where we have introduced two cut-off ϵ and κ. At this point the radial integral can be replaced with:

$$\int_\epsilon^\infty \mathrm{d}R\, R^{\kappa-2s-2t-1} = \frac{\epsilon^{\kappa-2s-2t}}{2s+2t-\kappa}\,, \qquad \Re\mathrm{e}(2s+2t-\kappa) > 0\,. \tag{5.1.6}$$

Taking the limit $\epsilon \to 0$, this integral yields a δ-function:

$$\int_0^\infty \mathrm{d}R\, R^{\kappa-2s-2t-1} = (2\pi i)\delta(2s+2t-\kappa). \tag{5.1.7}$$

This last result can be proven by integrating over $\eta = 2s+2t-\kappa$ on both sides of (5.1.6), with η running over a vertical line in the complex plane, in accordance with the inequality $\Re\mathrm{e}(\eta) > 0$ (see (5.1.6)). After closing the contour on the left, the integral over η reduces to the residue

$$2\pi i \operatorname*{Res}_{\eta\to 0} \frac{\epsilon^{-\eta}}{\eta} = 2\pi i, \tag{5.1.8}$$

proving (5.1.7) in the $\epsilon \to 0$ limit. The remaining Mellin-Barnes integral can then be evaluated using Barnes' first lemma, giving:

$$\begin{aligned}\langle \tilde{K}_\alpha | \tilde{K}_\beta \rangle &= \lim_{\kappa\to 0} \frac{\Gamma\left(\frac{1}{2}(\kappa+i\alpha+i\beta)\right)\Gamma\left(\frac{1}{2}(\kappa-i\alpha-i\beta)\right)}{2^{-\kappa+1}m^\kappa\Gamma(\kappa)\Gamma(-i\alpha)\Gamma(i\beta)} \\ &\quad\times\Gamma\left(\frac{1}{2}(\kappa-i\alpha+i\beta)\right)\Gamma\left(\frac{1}{2}(\kappa+i\alpha-i\beta)\right) \\ &= 2\pi\delta(\beta-\alpha) + \frac{2\pi\Gamma(i\alpha)\delta(\alpha+\beta)}{\Gamma(-i\alpha)}\,.\end{aligned} \tag{5.1.9}$$

[2]Note that the integral over α can be restricted to the interval $[0,\infty)$ removing the factor of $1/2$ and dropping the $\delta(\alpha+\beta)$.

By taking both α and β positive, as usually considered in this context, the second term can be dropped and (5.1.4a) follows. To prove (5.1.4b), one can first show that:

$$\int_0^\infty \frac{\mathrm{d}R_1}{R_1} \underbrace{\int_{-\infty}^{+\infty} \frac{\mathrm{d}\alpha}{2\pi} \frac{K_{i\alpha}(R_1)}{\Gamma(i\alpha)} \frac{K_{-i\alpha}(R_2)}{\Gamma(-i\alpha)}}_{\equiv k(R_1,R_2)} = \frac{1}{2}\,, \tag{5.1.10}$$

from which it follows that[3]

$$k(R_1,R_2) = \frac{R_1}{2}\delta(R_1-R_2)\,. \tag{5.1.11}$$

From this (5.1.4b) follows upon multiplying by the appropriate powers of R and changing variables. To prove (5.1.10), as we did above for the orthogonality relation one uses the Mellin-Barnes representation for the Bessel function:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}R_1}{R_1} k(R_1,R_2) = \int_0^\infty \frac{\mathrm{d}R_1}{R_1} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\alpha}{8\pi} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2} \left(\tfrac{R_1}{2}\right)^{-2s} \left(\tfrac{R_2}{2}\right)^{-2t} \\ \times \frac{\Gamma(s-\frac{i\alpha}{2})\Gamma(s+\frac{i\alpha}{2})\Gamma(t-\frac{i\alpha}{2})\Gamma(t+\frac{i\alpha}{2})}{\Gamma(-i\alpha)\Gamma(i\alpha)}.\end{aligned} \tag{5.1.12}$$

One performs the R_1 integration by placing a cut-off around zero:

$$\lim_{\epsilon\to 0} \int_0^\infty \frac{\mathrm{d}R_1}{R_1} R_1^{-2s} = \lim_{\epsilon\to 0} \frac{\epsilon^{-2s}}{2s}\,, \qquad \Re\mathrm{e}(s) > 0\,. \tag{5.1.13}$$

At this point one can close the s contour of integration to the left. Picking the contribution of the $s \sim 0$ residue one obtains:

$$\int_0^\infty \frac{\mathrm{d}R_1}{R_1}\, k(R_1,R_2) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}t}{2\pi i} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\alpha}{2\pi} \cosh\left(\tfrac{\pi\alpha}{2}\right) \Gamma(t-\tfrac{i\alpha}{2})\Gamma(t+\tfrac{i\alpha}{2}). \tag{5.1.14}$$

One performs first the α and then the t integral simply closing the contours and picking the residues:

$$\int_0^\infty \frac{\mathrm{d}R_1}{R_1}\, k(R_1,R_2) = \sum_{n,m=0}^{\infty} \frac{(-1)^n \left(\frac{R_2}{2}\right)^{m+n} \cos\left(\frac{\pi}{2}(m+n)\right)}{2n!m!}\,. \tag{5.1.15}$$

Upon changing summation variables as $\tilde{n} = n-m$ and one can perform the sum over m using the identity:

$$\sum_{m=0}^{\infty} \frac{(-1)^{-m}}{\Gamma(m+1)(\tilde{n}-m)!} = \delta_{\tilde{n},0}, \tag{5.1.16}$$

which collapses the leftover sum over $\tilde{n}$ to obtain (5.1.10).

[3] It is straightforward, picking up the residues in the α complex plane, to show that for $R_1 \neq R_2$ the integral over α is vanishing.

Massive Scalar Fields. Consider now a scalar field $\phi(X)$ in $\mathbb{M}^{d+2}$ of mass m with free equation of motion

$$\left(\frac{\partial}{\partial X^M}\frac{\partial}{\partial X_M}-m^2\right)\phi(X)=0. \tag{5.1.17}$$

The field $\phi(X)$ can be decomposed into fields that live on the hyperbolic slices (2.1.4) and (2.1.7) of $\mathbb{M}^{d+2}$ with unit radius by applying the Kontorovich-Lebedev transform [6] (see appendix 5.1). Focusing for concreteness on region $\mathcal{D}$ (analogous expressions hold for the other regions $\mathcal{A}_\pm$) this reads

$$\phi(X)=\frac{1}{2}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}\,\phi_\Delta(\hat{X})\tilde{K}_{\Delta-\frac{d}{2}}(mR), \tag{5.1.18}$$

where

$$\tilde{K}_{\Delta-\frac{d}{2}}(mR)=\frac{2R^{-d/2}}{\Gamma(\Delta-\frac{d}{2})}K_{\Delta-\frac{d}{2}}(mR), \tag{5.1.19}$$

which is proportional to a modified Bessel function of the second kind, $K_{\Delta-\frac{d}{2}}(mR)$ and where in $\mathcal{D}$ we have set $X=R\hat{X}_{\mathrm{dS}}$ with $\hat{X}^2_{\mathrm{dS}}=1$. The field $\phi_\Delta(\hat{X})$ satisfies the massive field equation on the de Sitter hyperboloid (2.1.7) with $R=1$

$$\left(\nabla^2_{\mathrm{dS}}-\Delta(d-\Delta)\right)\phi_\Delta=0. \tag{5.1.20}$$

Expression (5.1.18) can be analytically continued in $\mathcal{A}_+$ by applying (C.1.24) and (C.1.34). Subsequently,we can pass from $\mathcal{A}_+$ to $\mathcal{A}_-$ by means of the antipodal map $X_{\mathrm{AdS}}\mapsto -X_{\mathrm{AdS}}$. What we see is that the radial reduction of a field $\phi(X)$ in $\mathbb{M}^{d+2}$ onto the hyperbolic slices (2.1.4) and (2.1.7) yields an infinite number of fields on EAdS_{d+1} and dS_{d+1}, respectively, which carry principal series representations $\Delta\in\frac{d}{2}+i\mathbb{R}$ of $SO(d+1,1)$.[4] Applying holography to each slice, the existence of a dual conformal field theory description of $\mathbb{M}^{d+2}$ living on the Celestial Spheres $\mathcal{CS}^\pm$ was postulated in [6]. Upon the anti-podal identification $Q_+=-Q_-$, $Q_\pm\in\mathcal{C}^\pm_P\simeq\mathcal{CS}^\pm$, the correlation functions of the corresponding dual conformal operators have come to be known as *Celestial Correlators.*

[4]Much like in the standard CFT literature [37, 128–131], the $\Delta\in\frac{d}{2}+i\mathbb{R}$ on the Principal Series can be analytically continued to $\Delta\in\mathbb{C}$ [56]. In the context of celestial correlators this is relevant for the treatment of conformally soft Goldstone modes [16, 18, 46, 48–51, 140].

5.2 Spectral Decomposition of the Feynman Propagator

In order to develop a perturbative theory for the calculation of Celestial Correlators, in this section, we will decompose the Feynman propagator

$$\Pi_T^{(m)}(X_1, X_2) = -i \int \frac{\mathrm{d}^{d+2}p}{(2\pi)^{d+2}} \frac{\mathrm{e}^{ip\cdot(X_1-X_2)}}{p^2+m^2-i\epsilon}, \tag{5.2.1}$$

$X_1,\ X_2 \in \mathbb{M}^{d+2}$, in terms of 2-point conformal correlation functions living on the Minkowski's unitary hyperbolic slices. To reach this goal, let us factorise $X_j = R_j \hat{X}_j$, $j = 1, 2$, with $R_j \in \mathbb{R}^+$ and $\hat{X}_j^2 = \pm 1$, and then compute the double-Mellin transform of the Feynman propagator

$$\Pi^{(m)}_{\Delta_1,\Delta_2}\left(\hat{X}_1, \hat{X}_2\right) = \int_0^{+\infty} \frac{\mathrm{d}R_1}{R_1} R_1^{\Delta_1} \int_0^{+\infty} \frac{\mathrm{d}R_1}{R_2} R_2^{\Delta_2}\, \Pi_T^{(m)}\left(R_1\hat{X}_1, R_2\hat{X}_2\right). \tag{5.2.2}$$

The result of the computation is given in Appendix D.0.1; it yields

$$\begin{aligned}\Pi^{(m)}_{\Delta_1,\Delta_2}(\hat{X}_1, \hat{X}_2) = {} & \frac{m^{d-\Delta_1-\Delta_2}}{2(4\pi)^{\frac{d+1}{2}}} \frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma\left(\frac{\Delta_1+\Delta_2-d}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2+1}{2}\right)} \\ & \times \left(\hat{X}_1^2 + i\epsilon\right)^{-\frac{\Delta_1}{2}} \left(\hat{X}_2^2 + i\epsilon\right)^{-\frac{\Delta_2}{2}} {}_2F_1\left(\Delta_1, \Delta_2; \frac{\Delta_1+\Delta_2+1}{2}; \frac{1+\hat{z}}{2}\right),\end{aligned} \tag{5.2.3}$$

where

$$\hat{z} = \frac{\hat{X}_1 \cdot \hat{X}_2 - i\epsilon}{\sqrt{\hat{X}_1^2 + i\epsilon}\sqrt{\hat{X}_1^2 + i\epsilon}}. \tag{5.2.4}$$

Note that the $i\epsilon$-prescriptions assure the analyticity of the right hand side. Now, in [74] it has been demonstrated that the AdS-Harmonic functions,

$$\begin{aligned}\Omega_\nu^{\mathrm{AdS}}(\sigma_{\mathrm{AdS}}) &= \frac{1}{\Gamma(i\nu)\Gamma(-i\nu)} \frac{\Gamma(\Delta)\Gamma(\Delta^*)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)} {}_2F_1\left(\Delta, \Delta^*; \frac{d+1}{2}; \sigma_{\mathrm{AdS}}\right) \qquad (5.2.5) \\ &= \Omega_\nu(0)\, {}_2F_1\left(\Delta, \Delta^*; \frac{d+1}{2}; \sigma_{\mathrm{AdS}}\right), \quad \Delta = \frac{d}{2} + i\nu,\ \nu \in \mathbb{R},\end{aligned}$$

with

$$\sigma_{\mathrm{AdS}} = \frac{1}{2}\left(1 + \hat{X}_1 \cdot \hat{X}_2\right), \quad \hat{X}_1, \hat{X}_2 \in \mathcal{H}^+_{d+1}, \tag{5.2.6}$$

chordal distance on $\mathcal{H}^+_{d+1}$, ensure an isometrical mapping between the spaces of square integrable functions $L^2\Big((-\infty, -1], \mu(\zeta)\mathrm{d}\zeta\Big)$ and $L^2\left(\mathbb{R}, \frac{\mathrm{d}\nu}{\Omega_\nu(0)}\right)$, where the integration measure $\mu(\zeta) = \mathrm{Vol}(S^d)(\zeta^2-1)^{\frac{d-1}{2}}$ and $\zeta = \hat{X}_1 \cdot \hat{X}_2$. Given the analyticity of the right hand side in (5.2.3), this isometric mapping enable us to decompose

$$ {}_2F_1\left(\Delta_1,\Delta_2;\frac{\Delta_1+\Delta_2+1}{2};\sigma_\epsilon\right)=\int_{-\infty}^{+\infty}\mathrm{d}\nu\, F_{\Delta_1,\Delta_2}(\nu)\,\Omega_\nu(\sigma_\epsilon), \tag{5.2.7} $$

with

$$ \sigma_\epsilon=\frac{1+\hat{z}}{2}, \tag{5.2.8} $$

in terms of the functions

$$ \Omega_\nu(\sigma_\epsilon)=\Omega_\nu(0)\,{}_2F_1\left(\Delta,\Delta^*;\frac{d+1}{2};\sigma_\epsilon\right),\quad \Delta=\frac{d}{2}+i\nu,\ \nu\in\mathbb{R}. \tag{5.2.9} $$

The *spectral density* $F_{\Delta_1,\Delta_2}(\nu)$ has been computed in Appendix D.0.2 by inverting (5.2.7),

$$ F_{\Delta_1,\Delta_2}(\nu)=2^{\Delta_1+\Delta_2-d-1}\frac{\Gamma\left(\frac{1+\Delta_1+\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta*}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta^*}{2}\right)}{(4\pi)^{\frac{1-d}{2}}\,\Gamma\left(\Delta_1\right)\Gamma\left(\Delta_2\right)\Gamma\left(\frac{\Delta_1+\Delta_2-d}{2}\right)}. \tag{5.2.10} $$

Therefore, plugging (5.2.10) in (5.2.3) and then inverting (5.2.2), we arrive to the aimed *spectral decomposition* of the Feynman propagator:

$$ \Pi_T^{(m)}(X_1,X_2)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,\rho_\nu^{(m)}(X_1,X_2)\,\Omega_\nu(\sigma_\epsilon), \tag{5.2.11} $$

where

$$ \begin{aligned}\rho_\nu^{(m)}(X_1,X_2)&=\frac{1}{4}\Gamma(i\nu)\Gamma(-i\nu)\tilde{K}_{-\nu}^{(m)}\left(\sqrt{X_1^2+i\epsilon}\right)\tilde{K}_\nu^{(m)}\left(\sqrt{X_2^2+i\epsilon}\right)\\ &=\prod_{j=1}^{2}\left(\sqrt{X_j^2+i\epsilon}\right)^{-\frac{d}{2}}K_{i\nu}\left(m\sqrt{X_j^2+i\epsilon}\right)\end{aligned} \tag{5.2.12} $$

is its spectral density. Observe that the structure of (5.2.11) aligns with (5.1.18), with the function $\rho_\nu^{(m)}(X_1,X_2)$ representing the kernel of the double Kontorovich-Lebedev transformation. In regions $\mathcal{D}_\pm$, which are foliated by dS_{d+1} space-times, the decomposition (5.2.11) is a superposition of Feynman propagators in dS_{d+1}. In particular, we have

$$ \sigma_\epsilon(\hat{X}_{\mathcal{D}_\pm},\hat{Y}_{\mathcal{D}_\pm})=\sigma_{\mathrm{dS}}(\hat{X}_{\mathcal{D}_\pm},\hat{Y}_{\mathcal{D}_\pm})-i\epsilon, \tag{5.2.13} $$

so that, via (C.2.16a),

$$ \Pi_T^{(m)}(X_{\mathcal{D}_\pm},Y_{\mathcal{D}_\pm})=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,\tilde{K}_{-\nu}^{(m)}(R_1)\tilde{K}_\nu^{(m)}(R_2)G^{\mathrm{dS}}_{\frac{d}{2}+i\nu,T}(\hat{X}_{\mathcal{D}_\pm},\hat{Y}_{\mathcal{D}_\pm}). \tag{5.2.14} $$

In regions $\mathcal{A}_\pm$ foliated by Euclidean AdS$_{d+1}$ spaces, we have

$$\sigma_\epsilon(\hat{X}_{\mathcal{A}_\pm}, \hat{Y}_{\mathcal{A}_\pm}) = 1 - \sigma_{\text{AdS}}(\hat{X}_{\mathcal{A}_\pm}, \hat{Y}_{\mathcal{A}_\pm}) + i\epsilon. \tag{5.2.15}$$

When both points belong to the same sheet of EAdS we obtain (ref. (C.2.26)):

$$\begin{aligned}\Pi_T^{(m)}(X_{\mathcal{A}_\pm}, Y_{\mathcal{A}_\pm}) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi}\, \mathrm{e}^{\left(\frac{d}{2}+i\nu\right)\pi i} c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\, \tilde{K}^{(m)}_{-\nu}(\pm \mathrm{e}^{\frac{\pi i}{2}} T_1) \tilde{K}^{(m)}_{\nu}(\pm \mathrm{e}^{\frac{\pi i}{2}} T_2) \\ \times G^{\text{AdS}}_{\frac{d}{2}+i\nu}(\hat{X}_{\mathcal{A}_\pm}, \hat{Y}_{\mathcal{A}_\pm}).\end{aligned} \tag{5.2.16}$$

In summary, the radial reduction (5.2.11) of the Minkowski Feynman propagator is a superposition of Greens functions on the extended unit hyperboloid carrying Principal Series representations of the isometry group $SO\,(1, d+1)$. In the de Sitter regions $\mathcal{D}_\pm$ these are time-ordered two-point functions on dS$_{d+1}$, which is is inherited from the time-ordering of the original Minkowski Feynman propagator (as expected).

The *Celestial Bulk-to-Boundary propagator* the Mellin transform of the boundary limit of one of the bulk points [139]:

$$\begin{aligned}\Pi^{(m)}_\Delta\,(X, Q) &= \int_0^\infty \frac{\mathrm{d}t}{t} t^\Delta \lim_{\hat{Y}\to Q} \Pi_T^{(m)}\left(X, t\hat{Y}\right) \\ &= c^{\text{dS-AdS}}_\Delta\, \tilde{K}^{(m)}_{i\left(\frac{d}{2}-\Delta\right)} \left(\sqrt{X^2+i\epsilon}\right) G^{\text{AdS}}_\Delta(X_\epsilon, Q).\end{aligned} \tag{5.2.17}$$

The dependence on the hyperbolic directions is given by the corresponding (analytically continued) bulk-to-boundary propagator in anti-de Sitter space

$$G^{\text{AdS}}_\Delta(X_\epsilon, Q) = C^{\text{AdS}}_\Delta \frac{\left(\sqrt{X^2+i\epsilon}\right)^\Delta}{(-2X\cdot Q + i\epsilon)^\Delta}\,. \tag{5.2.18}$$

Notice that, as anticipated above, in the dS region $\mathcal{D}_\pm$ of the hyperbolic slicing this dependence is precisely that of the corresponding time-ordered bulk-to-boundary propagator in dS$_{d+1}$:

$$\Pi^{(m)}_\Delta\,(X_{\mathcal{D}_\pm}, Q) = \tilde{K}^{(m)}_{i\left(\frac{d}{2}-\Delta\right)}\,(R)\; K^{\text{dS}}_{\Delta,T}\left(s_{\text{dS}}(\hat{X}_{\mathcal{D}_\pm}, Q)\right). \tag{5.2.19}$$

In the EAdS regions $\mathcal{A}_\pm$, in Poincaré coordinates (2.2.9) we have

$$\sqrt{X^2+i\epsilon}\,\Big|_{\mathcal{A}_+} = e^{\frac{\pi i}{2}} T, \qquad \sqrt{X^2+i\epsilon}\,\Big|_{\mathcal{A}_-} = e^{\frac{\pi i}{2}} |T|, \tag{5.2.20}$$

so that in these regions:

$$\Pi^{(m)}_\Delta(X_{\mathcal{A}_\pm}, Q) = c^{\text{dS-AdS}}_\Delta\, \tilde{K}^{(m)}_{i\left(\frac{d}{2}-\Delta\right)}(\pm e^{\frac{\pi i}{2}} t\,)\, i^{+\Delta} K^{\text{AdS}}_\Delta\left(s_{\text{AdS}}(\hat{X}_{\mathcal{A}_\pm}, Q) - i\epsilon\right), \tag{5.2.21}$$

in terms of the corresponding bulk-to-boundary propagator (C.2.7) in EAdS_{d+1}, where the ϵ prescription here is accounting for the fact that Q can parameterise either boundary $\mathcal{C}_P^{\pm}$.

The free theory two-point function on the celestial sphere, is the boundary limit of the Mellin transformed Feynman propagator (5.2.3) (see Appendix D.0.1):

$$\begin{aligned}\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) &:= \lim_{\hat{X}_i\to Q_i}\Pi^{(m)}_{\Delta_1\Delta_2}(\hat{X}_1,\hat{X}_2), \\ &= \frac{C^{\text{flat}}_{\Delta_1}}{(-2Q_1\cdot Q_2+i\epsilon)^{\Delta_1}}(2\pi)\delta(i(\Delta_1-\Delta_2))\,,\end{aligned} \tag{5.2.22}$$

with normalisation

$$C^{\text{flat}}_{\Delta} = \left(\frac{m}{2}\right)^{d-2\Delta}\frac{1}{4\pi^{\frac{d+2}{2}}}\,\Gamma(\Delta)\Gamma(\Delta-\tfrac{d}{2}). \tag{5.2.23}$$

This can be equivalently obtained as the Mellin transform of the boundary limit of the Celestial bulk-to-boundary propagator (5.2.17):

$$\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) = \lim_{\hat{Y}\to Q_2}\int_0^\infty \frac{\mathrm{d}t}{t}t^{\Delta_2}\,\Pi^{(m)}_{\Delta_1}\left(t\hat{Y},Q_1\right), \tag{5.2.24}$$

as presented in [139].

In the view of recasting celestial correlation functions in terms of EAdS Witten diagrams in later sections, it will be useful to introduce the ratio of celestial (5.2.23) and AdS (C.2.4) boundary two-point function coefficients,

$$c^{\text{flat-AdS}}_{\Delta} = \frac{C^{\text{flat}}_{\Delta}}{C^{\text{AdS}}_{\Delta}} = \left(\frac{m}{2}\right)^{d-2\Delta}\frac{\Gamma(\Delta-\frac{d}{2})\Gamma(\Delta-\frac{d}{2}+1)}{2\pi}, \tag{5.2.25}$$

which is the celestial analogue of the coefficient $c^{\text{dS-AdS}}_{\Delta}$ in dS.

5.3 Perturbative Calculations

Following [139], holographic correlation functions on the Celestial sphere are defined as the boundary limit of Mellin transformed time-ordered bulk Minkowski correlation functions. Considering the correlation functions of n scalar fields ϕ_i, $i=1,\ldots,n$, in Minkowski space, the corresponding Celestial correlator is

$$\langle\mathcal{O}_{\Delta_1}(Q_1)\ldots\mathcal{O}_{\Delta_n}(Q_n)\rangle = \prod_i\lim_{\hat{X}_i\to Q_i}\int_0^\infty\frac{\mathrm{d}t_i}{t_i}\,t_i^{\Delta_i}\left\langle\phi_1(t_1\hat{X}_1)\ldots\phi_n(t_n\hat{X}_n)\right\rangle. \tag{5.3.1}$$

This definition naturally extends the extrapolate definition of holographic correlators in AdS and dS.

5.3.1 Contact diagrams

Consider the contact interaction governed by the following vertex in $\mathbb{M}^{d+2}$

$$\mathcal{V}_{12\ldots n} = g\phi_1\phi_2\ldots\phi_n, \tag{5.3.2}$$

where ϕ_i are real scalar fields with masses m_i, $i = 1, 2, \ldots, n$. The corresponding celestial contact diagram was computed in [139], where it was in particular found to be proportional to the corresponding contact diagram in Euclidean Anti-de Sitter space. In the following we reproduce this result by using suitable analytic continuations and Wick rotations, aligning this work with [8–10] (see also sec.(C.2) for a review).

The Celestial contact diagram decomposes into contributions from each region $\mathcal{A}_\pm$ and $\mathcal{D}_\pm$ in the hyperbolic slicing of $\mathbb{M}^{d+2}$

$$-ig\int \mathrm{d}^{d+2}X\prod_{i=1}^{n}\Pi^{(m_i)}_{\Delta_i}(X,Q_i) = -ig\left(I_{\mathcal{A}_+}+I_{\mathcal{A}_-}+I_{\mathcal{D}_+}+I_{\mathcal{D}_-}\right), \tag{5.3.3}$$

where we defined

$$I_\bullet = \int_\bullet \mathrm{d}^{d+2}X\prod_{i=1}^{n}\Pi^{(m_i)}_{\Delta_i}(X,Q_i)\,, \tag{5.3.4}$$

with $\bullet = \mathcal{A}_+,\, \mathcal{A}_-,\, \mathcal{D}_+,\, \mathcal{D}_-$.

For the sake of clarity, let us rewrite here the expression of the flat bulk-to-boundary propagator:

$$\Pi^{(m)}_{\Delta} = (X,Q_i) = c^{\text{dS-AdS}}_{\Delta}\,\tilde{K}^{(m)}_{i\left(\frac{d}{2}-\Delta\right)}\left(\sqrt{X^2+i\epsilon}\right)\,G^{\text{AdS}}_{\Delta}(X_\epsilon,Q), \tag{5.3.5}$$

where

$$G^{\text{AdS}}_{\Delta}(X_\epsilon,Q) = C^{\text{AdS}}_{\Delta}\frac{\left(\sqrt{X^2+i\epsilon}\right)^{\Delta}}{(-2X\cdot Q+i\epsilon)^{\Delta}} \tag{5.3.6}$$

and

$$\tilde{K}^{(m)}_{\nu}\left(\sqrt{X^2+i\epsilon}\right) = \left(\frac{m}{2}\right)^{-i\nu}\frac{2\left(\sqrt{X^2+i\epsilon}\right)^{-\frac{d}{2}}}{\Gamma(-i\nu)}K_{i\nu}\left(m\sqrt{X^2+i\epsilon}\right). \tag{5.3.7}$$

Then, define the Mellin representation of the Kontorovich-Lebedev kernel as follows:

$$\tilde{K}^{(m)}_{\nu}\left(\sqrt{X^2+i\epsilon}\right) = \int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s}{2\pi i}\,\tilde{K}^{(m)}_{\nu}(s)\left(\sqrt{X^2+i\epsilon}\right)^{-\frac{d}{2}-2s}, \tag{5.3.8}$$

with

$$\tilde{K}_{\nu}^{(m)}(s)=\frac{\Gamma(s+\frac{i\nu}{2})\Gamma(s-\frac{i\nu}{2})}{\Gamma(-i\nu)}\left(\frac{m}{2}\right)^{-2s-i\nu} \tag{5.3.9}$$

Let us begin by computing the contribution to (5.3.3) in the region $\mathcal{A}_+$ when all the boundary points lie in the far future, $Q_{+i}\in\mathcal{C}_P^+$. In this case, the denominator of (5.3.6) is well defined and the $i\epsilon$-prescription can be sent to zero. Therefore, we find that

$$\begin{aligned}I_{\mathcal{A}_+}=&\left(\prod_{i=1}^{n}c_{\Delta_i}^{\text{dS-AdS}}\right)R^{(\mathcal{A})}_{\Delta_1\ldots\Delta_n}(m_1,\ldots,m_n)\\&\times\mathrm{e}^{i\frac{\pi}{2}\sum_{i=1}^{n}\Delta_i}\int_{\mathcal{H}_{d+1}^+}\mathrm{d}^{d+1}\hat{X}\prod_{i=1}^{n}K_{\Delta}^{\text{AdS}}\left(\hat{X},Q_{+i}\right),\end{aligned} \tag{5.3.10}$$

where

$$K_{\Delta}^{\text{AdS}}\left(\hat{X},Q_+\right)=\frac{C_{\Delta}^{\text{AdS}}}{\left(-2\hat{X}\cdot Q_+\right)^{\Delta}} \tag{5.3.11}$$

is the normalised bulk-to-boundary propagator on $\mathcal{H}_{d+1}^+$ and

$$R^{(\mathcal{A})}_{\Delta_1\ldots\Delta_n}(m_1,\ldots,m_n)=\int_0^{\infty}\mathrm{d}R\,R^{d+1}\prod_{i=1}^{n}\tilde{K}^{(m_i)}_{i\left(\frac{d}{2}-\Delta_i\right)}(e^{\frac{\pi i}{2}}R), \tag{5.3.12}$$

which encodes the dependence on the masses in the region $\mathcal{A}$. Employing the Mellin representation for each Kontorovich-Lebedev Kernel in the radial integral, we are led to solve

$$\int_{\epsilon}^{+\infty}\mathrm{d}R\,R^{d+2-\sum_{i=1}^{n}\left(\frac{d}{2}+2s_i\right)}=(2\pi i)\delta\left(d+2-\sum_{i=1}^{n}\left(\frac{d}{2}+2s_i\right)\right), \tag{5.3.13}$$

where s_i are the Mellin variables. Using this last result, we can recast the contribution (5.3.10) as follows

$$\begin{aligned}I_{\mathcal{A}_+}=&\frac{1}{2}\mathrm{e}^{-i\frac{\pi}{2}\left(d+2-\sum_{i=1}^{n}\Delta_i\right)}\left(\prod_{i=1}^{n}c_{\Delta_i}^{\text{dS-AdS}}\right)\tilde{R}_{\Delta_1\ldots\Delta_n}(m_1,\ldots,m_n)\\&\times\int_{\mathcal{H}_{d+1}^+}\mathrm{d}^{d+1}\hat{X}\prod_{i=1}^{n}K_{\Delta}^{\text{AdS}}\left(\hat{X},Q_{+i}\right),\end{aligned} \tag{5.3.14}$$

where

$$\begin{aligned}\tilde{R}_{\Delta_1\ldots\Delta_n}(m_1,\ldots,m_n)=&\int[\mathrm{d}s]_n\prod_{i=1}^{n}\tilde{K}_{\nu}^{(m)}(s_i)\\&\times(2\pi i)\,\delta\left(\frac{d+2}{2}-\sum_{i=1}^{n}\left(\frac{d}{4}+s_i\right)\right).\end{aligned} \tag{5.3.15}$$

Now, let us define the EAdS D-function [123]

$$D_{\Delta_1,\ldots,\Delta_n}(Q_{+1},\ldots,Q_{+n}) = \int_{\mathcal{H}^+_{d+1}} \mathrm{d}^{d+1}\hat{X} \prod_{i=1}^{n} K^{\mathrm{AdS}}_{\Delta}\left(\hat{X}, Q_{+i}\right) \tag{5.3.16}$$

In Mellin representation, this function can be rewritten as

$$\begin{aligned} D_{\Delta_1,\ldots,\Delta_n}(Q_{+1},\ldots,Q_{+n}) &= -\pi^{\frac{d}{2}}\Gamma\left(\frac{-d+\sum_{i=2}^{n}\Delta_i}{2}\right)\prod_{i=1}^{n}\frac{C^{\mathrm{AdS}}_{\Delta_i}}{\Gamma(\Delta_i)} \\ \times\left(\prod_{i<j}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_{ij}}{2\pi i}\,\Gamma(s_{ij})\left(-2Q_{+i}\cdot Q_{+j}\right)^{-s_{ij}}\right)&\left[\prod_{j}(2\pi i)\delta\left(\sum_{j\neq i}s_{ij}-\Delta_j\right)\right]. \end{aligned} \tag{5.3.17}$$

This last formula makes the analytical properties of the EAdS D-function manifest, allowing us to extend our previous result to the most general case, allowing the boundary points to belong to both $\mathcal{C}^{\pm}_P$. This analytical extension can be achieved by employing the antipodal map in the complexified Minkowski space-time. In this context, we define the *analytical antipodal map* as the continuous transformation that maps a boundary point Q to its antipodal one $-Q$. In particular, let us set $Q_{+i} = \sigma_i\hat{Q}_{+i}$, $\sigma_i > 0$, where $\hat{Q}_{+i}$ and Q_{+i} lie on the same light-ray of the origin. Then, by sending continuously $\sigma_i \to \mathrm{e}^{\pm i(\pi-\epsilon)}\sigma_i$, the product

$$-2Q_{+i}\cdot Q_{+j} \to -2Q_{-i}\cdot Q_{+j} \pm i\epsilon, \tag{5.3.18}$$

where $Q_{-i} = -Q_{+i} \in \mathcal{C}^{-}_P$. The analytical extension of (5.3.17) is provided right in this way. Following this path, we can state that, in general,

$$\begin{aligned} I_{\mathcal{A}_+} = \frac{1}{2}\mathrm{e}^{-i\frac{\pi}{2}\left(d+2-\sum_{i=1}^{n}\Delta_i\right)}\left(\prod_{i=1}^{n}c^{\text{dS-AdS}}_{\Delta_i}\right)\tilde{R}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right)& \\ \times D^{\epsilon}_{\Delta_1\ldots,\Delta_n}(Q_1,\ldots,Q_n),& \end{aligned} \tag{5.3.19}$$

where

$$\begin{aligned} D^{\epsilon}_{\Delta_1\ldots,\Delta_n}(Q_1,\ldots,Q_n) &= -\pi^{\frac{d}{2}}\Gamma\left(\frac{-d+\sum_{i=2}^{n}\Delta_i}{2}\right)\prod_{i=1}^{n}\frac{C^{\mathrm{AdS}}_{\Delta_i}}{\Gamma(\Delta_i)} \\ \times\left(\prod_{i<j}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_{ij}}{2\pi i}\,\Gamma(s_{ij})\left(-2Q_i\cdot Q_j+i\epsilon\right)^{-s_{ij}}\right)&\left[\prod_{j}(2\pi i)\delta\left(\sum_{j\neq i}s_{ij}-\Delta_j\right)\right] \end{aligned} \tag{5.3.20}$$

is the analytically continued EAdS D-function [139]. Note that in this last equation boundary points can belong indistinctly to $\mathcal{C}^{\pm}_P$ and the same holds for (5.3.19).

This procedure is general and can be applied for computing all the other contributions to (5.3.3). For simplicity, in what follows we will omit the subscript $\pm$ and use only Q to denote boundary points. However, it is important to keep in mind that our calculations will be always carried out with all the boundary points located on the future Poincaré section of the light-cone. Only at the end of the computation we will strategically employ the analytical antipodal map to rotate future boundary points to the past, ensuring in this way the generality of the results. This step is the key to avoid potential branch cuts in the calculations.

Let us consider the contribution

$$
\begin{aligned}
I_{\mathcal{A}_-} = &\left(\prod_{i=1}^{n} c^{\text{dS-AdS}}_{\Delta_i}\right) R^{(\mathcal{A})}_{\Delta_1\ldots\Delta_n}(m_1,\ldots,m_n) \\
&\times \mathrm{e}^{i\frac{\pi}{2}\sum_{i=1}^{n}\Delta_i} \int_{\mathcal{H}^-_{d+1}} \mathrm{d}^{d+1}\hat{X} \left(\prod_{i=1}^{n} \frac{C^{\text{AdS}}_{\Delta_i}}{(-2X\cdot Q_i + i\epsilon)^{\Delta_i}}\right), \qquad (5.3.21)
\end{aligned}
$$

which is quite similar to previous one. The main difference consists in the angular integral

$$
\hat{I}_{\mathcal{A}_-} = \int_{\mathcal{H}^-_{d+1}} \mathrm{d}^{d+1}\hat{X} \prod_{i=1}^{n} \frac{C^{\text{AdS}}_{\Delta_i}}{(-2X\cdot Q_i + i\epsilon)^{\Delta_i}}. \qquad (5.3.22)
$$

Let us represent a point $\hat{X} \in \mathcal{H}^-_{d+1}$ as follows:

$$
\hat{X} = (\hat{X}^+, \hat{X}^-, \vec{X})^T = \frac{1}{y_-}\left(1, y_-^2 + z^2, \vec{z}\right)^T, \qquad (5.3.23)
$$

where $y_- < 0$ and $z^2 \equiv |\vec{z}|^2$. This parameterisation is slightly different from the one defined in (2.2.9), but it is more efficient for the purposes of this chapter. The point Q, instead, is represented as

$$
Q = \left(Q^+, Q^-, \vec{Q}\right)^T = \left(1, \omega^2, \vec{\omega}\right)^T \qquad (5.3.24)
$$

In these coordinates, the integral (5.3.22) becomes

$$
\begin{aligned}
\hat{I}_{\mathcal{A}_-} &= \int_{-\infty}^{0} \frac{\mathrm{d}y_-}{(-y_-)^{d+1}} \int_{\mathbb{R}_d} \mathrm{d}^d z \left(\prod_{i=1}^{n} \frac{C^{\text{AdS}}_{\Delta_i}(-y_-)^{\Delta_i}}{(-y_-^2 - |\vec{z}-\vec{\omega}_i|^2 + i\epsilon)^{\Delta_i}}\right) \\
&= \mathrm{e}^{-i\pi(\sum_i^n \Delta_i)} \int_{0}^{+\infty} \frac{\mathrm{d}y_+}{y_+^{d+1}} \int_{\mathbb{R}_d} \mathrm{d}^d z \left(\prod_{i=1}^{n} \frac{C^{\text{AdS}}_{\Delta_i}(y_+)^{\Delta_i}}{(y_+^2 + |\vec{z}-\vec{\omega}_i|^2)^{\Delta_i}}\right), \qquad (5.3.25)
\end{aligned}
$$

where we set $y_+ = -y_-$ and then we used

$$
\mathrm{e}^{\pm i(\pi-\epsilon)} A^2 \simeq -A^2 \pm i\epsilon, \quad A^2 = y_-^2 + |\vec{z}-\vec{\omega}_i|^2. \qquad (5.3.26)
$$

Therefore, the contribution to (5.3.3) from the region $\mathcal{A}_-$ is readily found to be

$$I_{\mathcal{A}_-} = \frac{1}{2}\mathrm{e}^{-i\frac{\pi}{2}\left(d+2+\sum_{i=1}^n \Delta_i\right)}\left(\prod_{i=1}^n c_{\Delta_i}^{\text{dS-AdS}}\right)\tilde{R}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right) \\ \times D^{\epsilon}_{\Delta_1\ldots,\Delta_n}(Q_1,\ldots,Q_n), \quad (5.3.27)$$

In regions $\mathcal{D}_\pm$ instead, the contributions can be reformulated as

$$I_{\mathcal{D}_\pm} = \left(\prod_{i=1}^n c_{\Delta_i}^{\text{dS-AdS}}\right)\tilde{R}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right)\hat{I}_{\mathcal{D}_\pm}. \quad (5.3.28)$$

Employing Poincaré coordinates (2.2.9) on the unitary slices, we get

$$\hat{I}_{\mathcal{D}_\pm} = \pm\int_0^{\pm\infty}\frac{\mathrm{d}\eta_\pm}{(\pm\eta_\pm)^{d+1}}\int_{\mathbb{R}^d}\mathrm{d}^d z\left(\prod_{i=1}^n\frac{C_\Delta^{\text{AdS}}(\pm\eta_\pm)^{\Delta_i}}{(\mp\eta_\pm^2\pm|\vec{z}-\vec{\omega}_i|^2+i\epsilon)^{\Delta_i}}\right). \quad (5.3.29)$$

These last integrals present a branch cuts with branch points given by

$$\eta_\pm = \pm|\vec{z}-\vec{\omega}|^2+i\epsilon. \quad (5.3.30)$$

The crucial point is that all these branch cuts are located above the real line in the complex plane $\eta_\pm$. Therefore, we can proceed by applying the Wick rotation $\eta_\pm \to \mathrm{e}^{\mp i\frac{\pi}{2}}y_\pm$, which brings us to the final result for the contributions in the region $\mathcal{D}_\pm$. It is

$$I_{\mathcal{D}_\pm} = \frac{1}{2}\mathrm{e}^{-i\frac{\pi}{2}\left(\mp d+\sum_{i=1}^n \Delta_i\right)}\left(\prod_{i=1}^n c_{\Delta_i}^{\text{dS-AdS}}\right)\tilde{R}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right) \\ \times D^{\epsilon}_{\Delta_1\ldots,\Delta_n}(Q_1,\ldots,Q_n), \quad (5.3.31)$$

Notice that $I_{\mathcal{D}_\pm}$ and $I_{\mathcal{A}_\pm}$ are proportional to the same integral over the hyperbolic directions $\hat{X}$. Moreover, we used that

$$R^{(\mathcal{D})}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right) = \int_0^\infty \mathrm{d}R\, R^{d+1}\prod_{i=1}^n\tilde{K}^{(m_i)}_{i\left(\frac{d}{2}-\Delta_i\right)}(R) \quad (5.3.32)$$

gives no phase when we expand in Mellin representation the Kontorovich-Lebedev kernerls. Therefore, it follows that

$$R^{(\mathcal{D})}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right) = \tilde{R}_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right). \quad (5.3.33)$$

Putting everything together, as anticipated, the contributions to the bulk integral from regions $\mathcal{A}_-$ and $\mathcal{D}_-$ cancel:

$$I_{\mathcal{A}_-}+I_{\mathcal{D}_-} = 0. \quad (5.3.34)$$

We will see that this feature carries over to processes involving particle exchanges. The Celestial contact diagram (5.3.3) is therefore the sum of contributions from regions $\mathcal{A}_+$ and $\mathcal{D}_+$, which can be combined to give

$$\mathcal{A}_c^{(n)} = -ig \sin\left(\frac{\pi}{2}\left(-d + \sum_{i=1}^{n} \Delta_i\right)\right)\left(\prod_{i=1}^{n} c_{\Delta_i}^{\text{dS-AdS}}\right) \tilde{R}_{\Delta_1 \ldots \Delta_n}(m_1, \ldots, m_n) \times \underbrace{D^{\epsilon}_{\Delta_1 \ldots, \Delta_n}(Q_1, \ldots, Q_n)}_{\text{Contact Witten diagram in EAdS}_{d+1}} \tag{5.3.35}$$

In summary, we see that celestial contact diagrams are proportional to corresponding contact Witten diagrams in EAdS. This recovers the result of [139], which was obtained by directly integrating over $\mathbb{M}^{d+2}$. Here instead, we have shown that it is possible to recover the same result by applying simply Wick rotations and then exploiting the analytical antipodal map in the complexified Minkowski space-time.

5.3.2 Four-point exchange diagram

In this section we consider the celestial correlation function corresponding to the four-point exchange diagram mediated by cubic vertices

$$\mathcal{V}_{12\phi} = g\phi_1\phi_2\phi, \qquad \mathcal{V}_{34\phi} = g\phi_3\phi_4\phi, \tag{5.3.36}$$

with external scalar fields ϕ_i of mass m_i and the field ϕ has mass m.

The corresponding Celestial correlator reads

$$\begin{aligned}
&\mathcal{E}^{m_1,m_2|m|m_3,m_4}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4) = (-ig)^2 \int \mathrm{d}^{d+2}X \int \mathrm{d}^{d+2}Y \\
&\times \Pi^{(m_1)}_{\Delta_1}(X,Q_1)\, \Pi^{(m_2)}_{\Delta_2}(X,Q_2)\, \Pi^{(m)}_{T}(X,Y)\, \Pi^{(m_3)}_{\Delta_3}(Y,Q_3)\, \Pi^{(m_4)}_{\Delta_4}(Y,Q_4) \\
&= (-ig)^2 \sum_{\pm\hat{\pm}} \left(I_{\mathcal{A}_\pm \mathcal{A}_{\hat{\pm}}} + I_{\mathcal{A}_\pm \mathcal{D}_{\hat{\pm}}} + I_{\mathcal{D}_\pm \mathcal{A}_{\hat{\pm}}} + I_{\mathcal{D}_\pm \mathcal{D}_{\hat{\pm}}}\right),
\end{aligned} \tag{5.3.37}$$

where the bulk integrals decompose into contributions from each region in the hyperbolic slicing of Minkowski space:

$$\begin{aligned}
I_{\bullet_1 \bullet_2} = \int_{\bullet_1} \mathrm{d}^{d+2}X \int_{\bullet_2} \mathrm{d}^{d+2}Y\, \Pi^{(m_1)}_{\Delta_1}(X,Q_1)\, \Pi^{(m_2)}_{\Delta_2}(X,Q_2) \\
\times \Pi^{(m)}_{T}(X,Y)\, \Pi^{(m_3)}_{\Delta_3}(Y,Q_3)\, \Pi^{(m_4)}_{\Delta_4}(Y,Q_4),
\end{aligned} \tag{5.3.38}$$

where $\bullet_{1,2} = \mathcal{A}_+, \mathcal{A}_-, \mathcal{D}_+, \mathcal{D}_-$.

We will compute this diagram, by applying the same technique outlined in the previous section, with the enhancement provided in sec.(C.2.2). All the results obtained in these two sections are summarised in Table 5.1 below.

This table represent a comprehensive overview and provides a clear guide for computing the Celestial four-point exchange diagram. Following the table, one has just to pick the right phases to insert inside the ν-integral in the respective contribution.

Contribution	Internal Leg	External Leg	Integration Measure
$\mathcal{A}_+/\mathcal{A}_+$	$\mathrm{e}^{i\pi\Delta}$	$\mathrm{e}^{i\frac{\pi}{2}\Delta_a}/\mathrm{e}^{i\frac{\pi}{2}\Delta_b}$	1
$\mathcal{A}_+/\mathcal{A}_-$	1	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}/\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	1
$\mathcal{A}_+/\mathcal{D}_+$	1	$\mathrm{e}^{i\frac{\pi}{2}\Delta_a}/\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{i\frac{\pi}{2}d}$
$\mathcal{A}_+/\mathcal{D}_-$	1	$\mathrm{e}^{i\frac{\pi}{2}\Delta_a}/\mathrm{e}^{i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{-i\frac{\pi}{2}d}$
$\mathcal{D}_+/\mathcal{A}_-$	$\mathrm{e}^{-i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{i\frac{\pi}{2}d}$
$\mathcal{D}_+/\mathcal{D}_-$	$\mathrm{e}^{-i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	1
$\mathcal{D}_+/\mathcal{D}_+$	$\mathrm{e}^{-i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{i\pi d}$
$\mathcal{D}_-/\mathcal{D}_-$	$\mathrm{e}^{i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{-i\pi d}$
$\mathcal{D}_-/\mathcal{A}_-$	$\mathrm{e}^{i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	$\mathrm{e}^{-i\frac{\pi}{2}d}$
$\mathcal{A}_-/\mathcal{A}_-$	$\mathrm{e}^{i\pi\Delta}$	$\mathrm{e}^{-i\frac{\pi}{2}\Delta_a}$ / $\mathrm{e}^{-i\frac{\pi}{2}\Delta_b}$	1

Table 5.1: The phases derived from mapping each contribution in the exchange process into the corresponding EAdS term. The first column indicate the regions, while the second column displays the respective phases isolated from the Feynman propagator. The third column illustrates the external legs contribution to the overall phase. The slash indicates the presence of two internal points in the process where the external leg can be attached. Wick rotations from $\mathcal{D}$ to $\mathcal{A}$ regions introduce additional phases through the integration over internal points, as accounted in the last column.

In analogy with contact diagrams, each contribution can be expressed as a superposition of exchanges in AdS carrying Principal Series representations of the Lorentz group. With the help of Table 5.1 for inserting the right phases, let us focus first on the contributions from regions $\mathcal{A}_+$ and $\mathcal{D}_+$, we have

$$I_{\mathcal{A}_+\,\mathcal{A}_+} = i^{\Delta_1+\Delta_2+\Delta_3+\Delta_4}\left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\, e^{\left(\frac{d}{2}+i\nu\right)\pi i} c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{A})}_{\Delta_1\Delta_2\,\frac{d}{2}+i\nu}(m_{1,2},m)\, R^{(\mathcal{A})}_{\frac{d}{2}-i\nu\,\Delta_3\Delta_4}(m,m_{3,4})$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4)\,, \quad (5.3.39)$$

$$I_{\mathcal{A}_+\,\mathcal{D}_+} = e^{+\frac{d\pi i}{2}}\, i^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{A})}_{\Delta_1\Delta_2\,\frac{d}{2}+i\nu}(m_{1,2},m)\, R^{(\mathcal{D})}_{\frac{d}{2}-i\nu\,\Delta_3\Delta_4}(m,m_{3,4})$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4)\,, \quad (5.3.40)$$

$$I_{\mathcal{D}_+\,\mathcal{A}_+} = e^{+\frac{d\pi i}{2}}\, i^{-\Delta_1-\Delta_2+\Delta_3+\Delta_4} \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{D})}_{\Delta_1\Delta_2\,\frac{d}{2}+i\nu}(m_{1,2},m)\, R^{(\mathcal{A})}_{\frac{d}{2}-i\nu\Delta_3\Delta_4}(m,m_{3,4})$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4)\,, \quad (5.3.41)$$

$$I_{\mathcal{D}_+\,\mathcal{D}_+} = e^{+d\pi i}\, i^{-\Delta_1-\Delta_2-\Delta_3-\Delta_4} \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, e^{-\left(\frac{d}{2}+i\nu\right)\pi i}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{D})}_{\Delta_1\Delta_2\,\frac{d}{2}+i\nu}(m_{1,2},m)\, R^{(\mathcal{D})}_{\frac{d}{2}-i\nu\Delta_3\Delta_4}(m,m_{3,4})$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4)\,, \quad (5.3.42)$$

where

$$\mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}(Q_1,Q_2,Q_3,Q_4) = \int_{\mathcal{A}_+} \mathrm{d}^{d+1}\hat{X} \int_{\mathcal{A}_+} \mathrm{d}^{d+1}\hat{Y}\, G^{\text{AdS}}_{\frac{d}{2}+i\nu}(\hat{X},\hat{Y})$$
$$\times \prod_{i=1}^{2} G^{\text{AdS}}_{\Delta_i}\left(\hat{X}_\epsilon, Q_i\right) \prod_{i=3}^{4} G^{\text{AdS}}_{\Delta_i}\left(\hat{Y}_\epsilon, Q_i\right) \quad (5.3.43)$$

is the analytically extended four-point exchange of a particle with scaling dimension $\Delta = \frac{d}{2}+i\nu$ in EAdS. Note that, as for contact diagrams, in (5.3.43) we used the analytical extended bulk-to-boundary propagators (5.3.6) to take into account boundary points on both the future and past Poincaré section of the light-cone. Further, the radial dependence from each bulk point is encoded in the functions (5.3.32) and (5.3.12).

The contributions from the remaining regions cancel in a pair-wise fashion, where the contribution from a bulk point in region $\mathcal{D}_-$ cancels with the contribution from the same bulk point in region $\mathcal{A}_-$:

$$I_{\bullet\mathcal{D}_-} + I_{\bullet\mathcal{A}_-} = 0, \quad (5.3.44)$$
$$I_{\mathcal{D}_-\bullet} + I_{\mathcal{A}_-\bullet} = 0. \quad (5.3.45)$$

For example, applying the Feynman rules we have

$$I_{\mathcal{D}_-\,\mathcal{D}_-} = e^{-\frac{d\pi i}{2}} e^{-\frac{d\pi i}{2}} i^{\Delta_1+\Delta_2+\Delta_3+\Delta_4} \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, e^{-\left(\frac{d}{2}+i\nu\right)\pi i}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{D})}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}\left(m_{1,2},m\right) R^{(\mathcal{D})}_{\frac{d}{2}-i\nu\Delta_3\Delta_4}\left(m,m_{3,4}\right)$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}\left(Q_1,Q_2,Q_3,Q_4\right)\,, \quad (5.3.46)$$

and

$$I_{\mathcal{A}_-\,\mathcal{D}_-} = e^{-\frac{d\pi i}{2}} i^{\Delta_1+\Delta_2+\Delta_3+\Delta_4} \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, e^{-\left(\frac{d}{2}+i\nu\right)\pi i}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}$$
$$\times R^{(\mathcal{A})}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}\left(m_{1,2},m\right) R^{(\mathcal{D})}_{\frac{d}{2}-i\nu\Delta_3\Delta_4}\left(m,m_{3,4}\right)$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}\left(Q_1,Q_2,Q_3,Q_4\right). \quad (5.3.47)$$

Like for the contact diagram in the previous section, using (5.3.33) which implies

$$R^{(\mathcal{A})}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}\left(m_1,m_2,m\right) = e^{-(d+2)\frac{\pi i}{2}} R^{(\mathcal{D})}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}\left(m_1,m_2,m\right), \quad (5.3.48)$$

one finds the two contributions to be equal and opposite:

$$I_{\mathcal{A}_-\,\mathcal{D}_+} + I_{\mathcal{D}_-\,\mathcal{D}_+} = 0. \quad (5.3.49)$$

After having observed this property for contact diagrams and now also for particle exchanges, one understands that the pairwise cancellation of contributions from regions $\mathcal{A}_-$ and $\mathcal{D}_-$ is a general property of Celestial correlation functions (5.3.1), at least in perturbation theory. The proof follows simply from the cancellation (5.3.34) of the contributions from regions $\mathcal{A}_-$ and $\mathcal{D}_-$ at the level of contact diagrams, combined with the analytic properties of the Feynman propagator, as shown in sec.(C.2.2).

The full celestial exchange diagram (5.3.37) is therefore the sum of contributions (5.3.39) - (5.3.42) from regions $\mathcal{A}_+$ and $\mathcal{D}_+$, giving[5]

$$\mathcal{E}^{m_1,m_2|m|m_3,m_4}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\left(Q_1,Q_2,Q_3,Q_4\right) =$$
$$= g^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{c^{\text{flat-AdS}}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}\left(m_1,m_2,m\right) c^{\text{flat-AdS}}_{\frac{d}{2}+i\nu\Delta_3\Delta_4}\left(m,m_3,m_4\right)}{c^{\text{flat-AdS}}_{\frac{d}{2}+i\nu}}$$
$$\times \mathcal{A}^{\text{AdS}}_{\Delta_1,\Delta_2|\frac{d}{2}+i\nu|\Delta_3,\Delta_4}\left(Q_1,Q_2,Q_3,Q_4\right). \quad (5.3.51)$$

[5] To write the exchange in this form in terms of flat-AdS coefficients we used that

$$R^{(\mathcal{D})}_{\frac{d}{2}-i\nu\,\Delta_3\Delta_4}\left(m,m_{3,4}\right) = \left(\frac{m}{2}\right)^{2i\nu} \frac{\Gamma\left(-i\nu\right)}{\Gamma\left(i\nu\right)} R^{(\mathcal{D})}_{\frac{d}{2}+i\nu\,\Delta_3\Delta_4}\left(m,m_{3,4}\right). \quad (5.3.50)$$

This result expresses the celestial exchange diagram as a continuum of exchange Witten diagrams in EAdS_{d+1} where the exchanged particles have scaling dimension belonging to the Principal Series of the Euclidean conformal group. The coefficient of each exchange Witten diagram is factorised, where the two factors $c^{\text{flat-AdS}}_{\Delta_1\Delta_2\frac{d}{2}+i\nu}(m_1, m_2, m)$ and $c^{\text{flat-AdS}}_{\frac{d}{2}+i\nu\Delta_3\Delta_4}(m, m_3, m_4)$ serve to convert the coefficients of the three-point contact subdiagrams in EAdS to their Celestial counterparts, while $c^{\text{flat-AdS}}_{\frac{d}{2}+i\nu}$ accounts for the change in two-point function coefficient (5.2.25).[6]

The expression (5.3.51) for the exchange is the celestial analogue of the formula [9, 10] for exchange contributions to late-time correlation functions in dS_{d+1} space, where they are similarly expressed in terms of corresponding exchange Witten diagrams in EAdS_{d+1} with coefficients ensuring consistent factorisation. A key difference is that the result for the dS exchange is a sum of two exchange Witten diagrams for a particle of the same mass in EAdS (one for each boundary condition), while celestial exchanges are a continuum of exchanged particles in EAdS carrying Principal Series representations. This difference is simply owing to the fact that the equations of motion for a field in dS_{d+1} are identified with those of a field of the same mass in EAdS_{d+1} under the analytic continuations (C.2.19). In Minkowski space instead, the radial reduction of a massive scalar in $\mathbb{M}^{d+2}$ onto a single $(d+1)$-dimensional leaf of the hyperbolic foliation gives rise to a continuum (5.2.11) of scalar fields carrying Principal Series representations of $SO(1, d+1)$.

[6] It would be interesting to compare in more detail with the corresponding result [47, 132] arising from the prescription for celestial amplitudes given in [17, 18], where they are defined as a change of basis for flat-space scattering amplitudes. In [132] it was noted that in the latter prescription the conformal block coefficients do not factorise.

Chapter 6

Conclusions and Outlook

In the scope of this thesis, we delved into the properties of Celestial correlators, with a specific focus on the massive scalar case. Our primary emphasis was on exploring their connection with Witten correlators on Euclidean AdS, by following the line traced by [6, 17]. To make the connection manifest, we adopted the de Boer-Solodukhin hyperbolic foliation of Minkowski space-time [6], which enabled us to exploit all the powerful tools of the (EA)dS/CFT correspondence to carry out our analysis. The study culminated in a wider perspective of general properties of holographic theories in maximally symmetric space-times. Previous works [8, 9] have led us to bring holographic theories on these space-times to the same footing, connecting them especially with holography on Euclidean AdS. Euclidean AdS plays a central role in bridging holographic theories on maximally symmetric spaces, emerging as the foundational theory from which to derive overarching considerations and properties concerning holography in general. All of the connections we shall describe in details below have been illustrated in the Holographic Triangle (see Fig.(1)), highlighting the bridging between holographic theories in maximally symmetric space-times. In particular, connections between (A)dS and EAdS rely on simply Wick rotations between the respective Poincaré patches of the spaces taken into consideration. Nevertheless, these Wick rotations encode different physical meanings. Instead, the bridging between Celestial and EAdS holography is rooted in the consistent employment of the Kontorovich-Lebedev tranform [6].

Bridging AdS and EAdS Holography: Wick rotation from AdS to the EAdS translates into a canonical Wick rotation on the boundary of these two spaces. This involves, indeed, a boundary coordinate in EAdS, which is Wick rotated in a time-direction coordinate on the boundary of AdS. This means that Euclidean correlation functions on the boundary of EAdS can be analytically continued to Minkowski space-time and further to the ∞-sheeted cover $\mathbb{M}^{d+2}_{\infty} \simeq \mathbb{S}^{d+1} \times \mathbb{R}$ of the $(d+2)$-dimensional compactified Minkowski space-time [35, 37, 141]. This imply that boundary AdS and Euclidean AdS correlators must be single-valued solutions of the same Ward identities. Further, requiring Osterwalder-Schrader axioms to bound Euclidean correlators implies that correlation functions on the boundary of AdS must satisfy the Wightman ones [38], and vice versa.

Bridging dS and EAdS holography: The case of dS in completely different. The Poincaré patches on dS and EAdS are connected by Wick rotations in the conformal time of dS. The main difference with the previous case is that this is a Wick rotation in a bulk variable. Since there is no way to transport conformal properties of the CFTs directly from one boundary to the other, like in the EAdS/AdS case, we have to recover them by perturbative computation. It has been shown [9] that any perturbative contribution to late-time dS boundary correlators in the Bunch-Davies (Euclidean) vacuum can be recast as a linear combination of corresponding Witten diagrams in Euclidean AdS. This result has been recovered at all orders in perturbation theory by recasting Schwinger-Keldysh propagators in dS as certain analytic continuations of *propagators* for the same particle in EAdS [8, 9]. This is a strong result, allowing us to state, as in the AdS/EAdS case, that corresponding processes in dS and EAdS satisfy the same conformal Ward identities, assuming in dS Bunch-Davis boundary conditions. Single-valuedness in the Bunch-Davis vacuum, then, comes from the analysis of the coefficients in the aformentioned linear combination, which are given by certain sinusoidal factors encoding (perturbatively) unitary time evolution in de Sitter [10]. In summary, dS boundary correlators with Bunch-Davies initial conditions have the same singularity structure as their Euclidean AdS counterparts in perturbation theory.

Bridging Celestial and EAdS Holography: The previously analysed case of dS/EAdS has been a cornerstone for the study carried out in this thesis. During this work we exploited most of the results above, that we also extended taking into account both the asymptotic boundaries of dS. This last result has been achieved by using the hyperbolic foliation of Minkowski space-time and leveraging the analytical properties of (EA)dS propagators through the analytical antipodal map. In the complexified Minkowski space-time, this map brings continuously a point Q on the future boundary to the corresponding antipodal point on the past one (ref. Appendix (C.2) and sec.(5.3.1)). Relying on these analytical properties, in complete analogy with dS, we showed in this thesis that each contribution to massive Celestial correlators can be recast in terms of a linear combination of contributions to corresponding massive Witten correlators in EAdS. This is true both in the conformal primary basis formulation (ref. chapter 4) and in the revisited version of Celestial holography (ref. chapter 5). The coefficients, in such linear combinations, carry information about the masses of the particles involved in the process and appear as integrals of a product of Kontorovich-Lebedev kernels over the radial variables in the hyperbolic foliated Minkowski space-time, times usual sinusoidal factors.

Positivity and Unitarity: The study on the analytical properties of these coefficients leads us to assert that, in complete analogy with dS, corresponding processes in Celestial and EAdS are single-valudness solutions of the same conformal Ward identities. In particular, in sec.4.5, we briefly explored some consequences of this assertion. It implies that, like their AdS counterparts, Celestial correlators admit a conformal partial wave expansion with meromorphic spectral density (at least perturbatively) [37, 141]. We also discussed (non-perturbative) constraints from unitarity in Euclidean CFT at the level of the conformal partial wave expansion, which translates into the positivity of the spectral density. It seems that it applies more generally to any unitary Euclidean CFT, and therefore also to Celestial amplitudes and/or correlators as well.

The Kontorovich-Lebedev Transform: Finally, we want to underline here that, as well as the connection between dS and EAdS holography passes throught a Wick rotation, the bridging between Celestial and EAdS holography is rooted in the consistent employment of the Kontorovich-Lebedev tranformation [6], which provides a radial reduction over the radii in the hyperlobic foliation of Minkowski space-time (ref. sec.(5.1)). This transformation reduces to the Mellin transformation on the light-cone, applying as the true fundamental transformation from which to draw general conclusions in the pursuit of understanding a holographic theory in asymptotically flat space-time.

Appendix A

Conformal Group

Consider a smooth d-dimensional (pseudo-)Riemannian manifold $(\mathcal{M}, g)$. Given $p \in \mathcal{M}$, let (ψ, U) be a chart on $\mathcal{M}$, with $\psi(p) = \left(\omega^1, \ldots, \omega^d\right)^T \in \mathbb{R}^d$ and $p \in U \subset \mathcal{M}$. Let $f \in \mathcal{D}\mathit{iff}(\mathcal{M})$ be a smooth transformation function acting on $\mathcal{M}$. Assuming that $f(p) \in U$, set $\vec{\omega}\,' = \psi(f(p)) \in \mathbb{R}^d$. Let X, Y be two vectors belonging to the tangent space T_pM. Hence, given the pull-back f^* associated to f, define its action on the metric tensor g as follows:

$$f^* g_{f(p)}(X, Y) = g_{f(p)}(f_*(X), f_*(Y)), \tag{A.0.1}$$

where f_* is the push-forward associated to f, which maps

$$f_* : X \in T_p\mathcal{M} \longmapsto f_*X \in T_{f(p)}\mathcal{M}. \tag{A.0.2}$$

Then, the conformal group is defined as the subgroup $\mathcal{C}\mathit{onf}\,(\mathcal{M}) \subset \mathcal{D}\mathit{iff}(\mathcal{M})$ of transformations leaving the metric tensor g invariant up to an arbitrary smooth positive function $\Lambda(\vec{\omega})$, known as *scale factor*. In formulae [142]

$$f^* g_{f(p)} = \Lambda^2(p) g_p. \tag{A.0.3}$$

Moreover, define the isometry group $\mathcal{I}\mathit{so}\,(\mathcal{M})$ as the subgroup of the conformal group identified by the additional condition $\Lambda^2(p) = 1$. Therefore, $\mathcal{I}\mathit{so}\,(\mathcal{M})$ can be defined as the group of smooth transformations that leave the metric tensor g invariant. The elements of the conformal group are called *conformal transformations*; further, the elements of the isometry group are named *isometry transformations*.

It is also possible to give a geometrical definition of the conformal group. Indeed, it is easy to show that conformal transformations preserve the angle between two vectors applied at a point p of $\mathcal{M}$ [142].

The angle θ between two vectors $X, Y \in T_p\mathcal{M}$ applied at $p \in U$, is defined as

$$\cos\theta = \frac{g_p\,(X, Y)}{\sqrt{g_p\,(X, X)\, g_p\,(Y, Y)}}. \tag{A.0.4}$$

Therefore, if f is a conformal transformation, the angle θ' between the vectors $f_*X, f_*Y \in T_{f(p)}\mathcal{M}$ is given by

$$\begin{aligned}\cos\theta' =& \frac{f^*g_{f(p)}\,(X,Y)}{\sqrt{f^*g_{f(p)}\,(X,X)\,f^*g_{f(p)}\,(Y,Y)}}\\ =& \frac{\Lambda^2(p)g_p\,(X,Y)}{\sqrt{(\Lambda^2(p)\,g_p\,(X,X))\,(\Lambda^2(p)\,g_p\,(X,X))}}\\ =& \frac{g_p\,(X,Y)}{\sqrt{g_p\,(X,X)\,g_p\,(Y,Y)}} = \cos\theta.\end{aligned} \tag{A.0.5}$$

Hence, it is possible to define the conformal group $\mathcal{C}onf(\mathcal{M})$ as the group of smooth transformation functions that preserve the angle between two vectors applied at a point p of $\mathcal{M}$.

Consider, next, the infinitesimal form of a conformal transformation f. The coordinate expression of (A.0.3) is given by

$$\frac{\partial\omega'^a}{\partial\omega^c}\frac{\partial\omega'^b}{\partial\omega^d}g_{ab}(\vec{\omega}\,') = \Lambda^2(\vec{\omega})g_{cd}(\vec{\omega}). \tag{A.0.6}$$

In the chart $(\psi,\,U)$, set $\Lambda(\vec{\omega}) \simeq 1 + \epsilon\lambda(\vec{\omega})$ and $\omega'^a = \omega^a + \epsilon\,\Omega^a$, where $\epsilon > 0$ is an infinitesimal parameter. At the first order in ϵ, (A.0.6) becomes

$$\partial_c g_{ab}(\vec{\omega})\Omega^c + g_{cb}(\vec{\omega})\partial_a\Omega^c + g_{ac}(\vec{\omega})\partial_b\Omega^c = 2\lambda(\vec{\omega})g_{ab}(\vec{\omega}). \tag{A.0.7}$$

Endowing the manifold $\mathcal{M}$ with a Levi-Civita connection ∇, it is possible to rewrite (A.0.7) as [142]

$$\nabla_a\Omega_b + \nabla_b\Omega_a = 2\lambda(\vec{\omega})g_{ab}(\vec{\omega}), \tag{A.0.8}$$

where

$$\nabla_a\Omega_b = \partial_a\Omega_b - \Gamma^c_{ab}\Omega_c \tag{A.0.9}$$

and Γ^c_{ab} are the Christoffel symbols. The vector fields Ω satisfying (A.0.7) or, equivalently, (A.0.8) are called *conformal Killing vector fields*. They are the infinitesimal generators of the conformal transformations.

A.1 Conformal Algebra

Now, consider the Euclidean space $\mathbb{R}^d$. In this case, (A.0.7) becomes [143, 144]

$$\partial_a\Omega_b + \partial_b\Omega_a = 2\lambda(\vec{\omega})\delta_{ab}. \tag{A.1.1}$$

Taking the trace of both sides of (A.1.1), one finds that

$$\lambda(\vec{\omega}) = \frac{1}{d}\,\partial\cdot\Omega, \tag{A.1.2}$$

where $\partial \cdot \Omega \equiv \partial_a \Omega^a$. Substituting this result into (A.0.9), one has

$$\partial_a \Omega_b + \partial_b \Omega_a = \frac{2}{d} \left(\partial \cdot \Omega\right) \delta_{ab}. \tag{A.1.3}$$

Consider $d \geq 3$. In this case, acting on both sides of (A.1.3) with ∂^a yields

$$\partial_c \partial^c \Omega_a + \partial_a \left(\partial \cdot \Omega\right) = \frac{2}{d} \partial^b \left(\partial \cdot \Omega\right) \delta_{ab}. \tag{A.1.4}$$

Next, by acting on both sides of (A.1.4) with ∂_b, we get

$$\partial_b \, \partial_c \partial^c \Omega_a + \partial_b \partial_a \left(\partial \cdot \Omega\right) = \frac{2}{d} \partial_b \partial_a \left(\partial \cdot \Omega\right). \tag{A.1.5}$$

Further, by means of Eq.(A.1.4), one finally obtains

$$\left[\delta_{ab} \, \Delta + (d-2) \partial_a \partial_b\right] \left(\partial \cdot \Omega\right) = 0, \tag{A.1.6}$$

which is a partial differential equation of third order. Note that (A.1.6) depends on the dimension d of the ambient space. In particular, the second term vanishes for $d = 2$. For $d \geq 3$, taking the trace of (A.1.6), it turns out to be

$$(d-1) \Delta \left(\partial \cdot \Omega\right) = 0. \tag{A.1.7}$$

Therefore, Ω_a is a solution of (A.1.7) if it is at most quadratic in the coordinates. The general solution can be written as [143]

$$\Omega_a = a_a + b_{ab} \, \omega^b + c_{abc} \, \omega^b \omega^c, \tag{A.1.8}$$

where a_a, b_{ab}, c_{abc} are infinitesimals and c_{abc} is symmetric in the last two indices. Consider, first, the constant solution $\Omega_a = a_a$, $\vec{a} \in \mathbb{R}^d$. In this case

$$\vec{\omega}\,' = \vec{\omega} + \epsilon \vec{a}, \tag{A.1.9}$$

which is the infinitesimal form of a d-dimensional translation. Therefore, the infinitesimal generator of this transformation is the translation operator $P = -i \partial_a$, represented in the space of coordinates of $\mathbb{R}^d$.

Next, consider the linear solution $\Omega_a = b_{ab} \, \omega^b$. Inserting this result in (A.1.1), one finds

$$b_{ab} + b_{ba} = \frac{1}{d} \left(\delta^{cd} b_{cd}\right) \delta_{ab}, \tag{A.1.10}$$

which fixes the symmetric part of b_{ab}. Writing

$$b_{ab} = \alpha \, \delta_{ab} + m_{ab}, \tag{A.1.11}$$

where m_{ab} is antisymmetric and $\alpha \in \mathbb{R}$ is given by (A.1.10), it is possible to distinguish two types of transformations, corresponding to the symmetric and antisymmetric part of b_{ab}. Indeed,

$$\omega'_a = \omega_a + \epsilon \, m_{ab} \, \omega^b, \tag{A.1.12}$$

is an infinitesimal d-dimensional rotation. Therefore, the operators representing the infinitesimal generators of rotations in the coordinate space are $L_{ab} = i\left(\omega_a \partial_b - \omega_b \partial_a\right)$. Instead, the infinitesimal transformation

$$\vec{\omega}\,' = \vec{\omega} + c\,\vec{\omega}, \tag{A.1.13}$$

with $c = \epsilon\,\alpha$, is an infinitesimal rescaling. Then, in the coordinate space, the infinitesimal generator of this transformation is the dilation operator $D = -i\omega^a \partial_a$. Eq.(A.1.13) integrated gives the finite rescaling

$$\vec{\omega}\,' = \mathrm{e}^{icD}\vec{\omega} = \mathrm{e}^{c}\vec{\omega}. \tag{A.1.14}$$

In the end, consider the quadratic case $\Omega_a = c_{abc}\,\omega^b \omega^c$. First, return once again to (A.0.9) and act with ∂_c on both sides of this equation. It follows that

$$\partial_c \partial_a \Omega_b + \partial_c \partial_b \Omega_a = \frac{1}{d}\,\partial_c\left(\partial\cdot\Omega\right)\delta_{ab}. \tag{A.1.15}$$

After cyclic permutations of the indices,

$$\partial_a \partial_b \Omega_c + \partial_a \partial_c \Omega_b = \frac{1}{d}\partial_a\left(\partial\cdot\Omega\right)\delta_{bc}, \tag{A.1.16}$$

$$\partial_b \partial_c \Omega_a + \partial_b \partial_a \Omega_c = \frac{1}{d}\partial_b\left(\partial\cdot\Omega\right)\delta_{ca}, \tag{A.1.17}$$

are found. Subtracting (A.1.17) from the sum of (A.1.15) and (A.1.16), one obtains the expression

$$2\partial_c \partial_a \Omega_b = \frac{1}{d}\left(\delta_{ab}\partial_c + \delta_{bc}\partial_a - \delta_{ca}\partial_b\right)\left(\partial\cdot\Omega\right). \tag{A.1.18}$$

Inserting the general solution (A.1.8) in this equation one finds

$$c_{bac} = \delta_{ab}\,b_c + \delta_{bc}\,b_a + \delta_{ca}\,b_b, \quad b_a = \frac{1}{2d}\,c^{d}_{da}. \tag{A.1.19}$$

Therefore, the corresponding quadratic infinitesimal transformation of $\vec{\omega}$ is

$$\vec{\omega}\,' = \vec{\omega} + 2\left(\omega\cdot\vec{b}\right)\vec{\omega} - |\vec{\omega}|^2\,\vec{b}, \tag{A.1.20}$$

which integrated gives the finite form [143, 145]

$$\vec{\omega}\,' = \frac{\vec{\omega} - |\vec{\omega}|^2\,\vec{b}}{1 - 2\left(\vec{\omega}\cdot\vec{b}\right) + |\vec{b}|^2\,|\vec{\omega}|^2}. \tag{A.1.21}$$

These transformations are called finite *special conformal transformations*, and they are generated by $K_a = -i\left(2\omega_a\,\vec{\omega}\cdot\partial - |\vec{\omega}|^2\partial_a\right)$, in the coordinate space. To better understand the meaning of these transformations, introduce the *inversion*

$$I:\,\vec{\omega}\longmapsto I(\vec{\omega}) = \frac{\vec{\omega}}{|\vec{\omega}|^2}, \tag{A.1.22}$$

which is a discrete transformation. It is important to note that $I \circ I = id$. By means of the inversion, it is possible to rewrite a special conformal transformation as follows [143]

$$I(\vec{\omega}\,') = I\left(I(\vec{\omega}) - \vec{b}\right), \quad \vec{b} \in \mathbb{R}^d. \tag{A.1.23}$$

Therefore, it is possible to assert that every special conformal transformation $\Lambda_{\text{SCT}} \in \mathcal{C}onf\left(\mathbb{R}^d\right)$ can be found by means of the inversion I and a translation Λ_{T},

$$\Lambda_{\text{SCT}} = I \circ \Lambda_{\text{T}} \circ I. \tag{A.1.24}$$

In the discussion above, the generators of the conformal group in $d \geq 3$ dimensions have been found. In the space of the smooth functions on $\mathbb{R}^d$, they are

$$P_a = -i\partial_a, \tag{A.1.25}$$
$$L_{ab} = i\left(\omega_a\partial_b - \omega_a\partial_b\right), \tag{A.1.26}$$
$$D = -i\omega^a\partial_a, \tag{A.1.27}$$
$$K_a = -i\left(2\omega_a\omega^b\partial_b - |\vec{\omega}|^2\partial_a\right). \tag{A.1.28}$$

Moreover, they close the algebra

$$[D, P_a] = iP_a, \tag{A.1.29}$$
$$[D, K_a] = -iK_a, \tag{A.1.30}$$
$$[K_a, P_b] = 2i\left(\delta_{ab}D - L_{ab}\right), \tag{A.1.31}$$
$$[K_a, L_{bc}] = i\left(\delta_{ab}K_c - \delta_{ac}K_b\right), \tag{A.1.32}$$
$$[P_a, L_{bc}] = i\left(\delta_{ab}P_c - \delta_{ac}P_b\right), \tag{A.1.33}$$
$$[L_{ab}, L_{cd}] = i\left(\delta_{bc}L_{ad} + \delta_{ad}L_{bc} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac}\right), \tag{A.1.34}$$

with all the other commutators vanishing. Defining

$$J_{ab} = L_{ab}, \tag{A.1.35}$$
$$J_{d+1\,a} = \frac{1}{2}\left(P_a - K_a\right), \tag{A.1.36}$$
$$J_{0\,a} = \frac{1}{2}\left(P_a + K_a\right), \tag{A.1.37}$$
$$J_{d+1\,0} = D, \tag{A.1.38}$$

it is a straightforward calculation showing that these generators close the $SO(1, d+1)$ algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\left(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\right), \tag{A.1.39}$$

with $\eta = \mathrm{diag}(-1, 1, \dots 1)$ and μ, ν, σ, $\rho = 0, \dots, d+1$. This result states that the connected part of $\mathcal{C}onf\left(\mathbb{R}^d\right)$ is diffeomorphic to the connected part of $SO(1, d+1)$. Henceforth, $\mathcal{C}onf\left(\mathbb{R}^d\right)$ will be identified with $SO(1, d+1)$.

A.2 Representations of the Conformal Group

Given an element $\Lambda \in SO(1, d+1)$ and an Euclidean spin-ℓ field $\phi(\vec{\omega})$, the action of Λ on $\phi(\vec{\omega})$ is the following:

$$\phi'(\vec{\omega}) = \rho\left(\Lambda\right)\phi(\Lambda^{-1}\vec{\omega}), \quad \alpha = 1, \dots, \ell, \tag{A.2.1}$$

where $\rho\left(\Lambda\right)$ is the operator representing Λ in the space of $\phi(\vec{\omega})$. The infinitesimal form of the above equation is given by

$$\phi'(\vec{\omega}) = i\epsilon\, G\left(\Lambda, \vec{\omega}\right)\phi(\vec{\omega}) - \epsilon\Omega^a\partial_a\phi(\vec{\omega}) + o(\epsilon) = \epsilon F\left(\Lambda, \vec{\omega}\right)\phi(\vec{\omega}) + o(\epsilon), \tag{A.2.2}$$

where

$$\delta\phi(\vec{\omega}) = i\epsilon G\left(\Lambda, \vec{\omega}\right), \tag{A.2.3}$$

is the infinitesimal shape variation and ϵ is an infinitesimal parameter. In particular, $G\left(\Lambda, \vec{\omega}\right)$ is an operator representing an element of the conformal algebra $\mathcal{L}$ acting on $\phi(\vec{\omega})$ at the point $\vec{\omega} \in \mathbb{R}^d$. The representing operator $\rho\left(\Lambda\right)\left(\vec{\omega}\right)$ in the finite transformation (A.2.1) is given by the formula

$$\rho\left(\Lambda\right)\left(\vec{\omega}\right) = \mathrm{e}^{\epsilon F\left(\Lambda, \vec{\omega}\right)}, \tag{A.2.4}$$

with $F(\Lambda, \vec{\omega})^\dagger = -F(\Lambda, \vec{\omega})$. To find a representation of $SO(1, d+1)$ in the field space of $\phi(\vec{\omega})$, consider, first, the *little group* $H(0)$ of $\vec{\omega} = 0$. This is the subgroup of transformations that leaves the origin of $\mathbb{R}^d$ unchanged. It is formed by d-dimensional rotations, dilations and special conformal transformations. Since $[D, L_{ab}] = 0$, (A.1.30) and (A.1.32) states that [146]

$$H(0) \simeq \left(SO(d) \otimes \mathcal{D}\right) \rtimes K_d, \tag{A.2.5}$$

where $\mathcal{D}$ and K_d are the subgroups generated by dilations and special conformal transformations, respectively. Since K_d is diffeomorphic to the d-dimensional translation group T_d, the Lie algebra $\mathcal{L}ie\left(H\right)$ of $H(0)$ is isomorphic to

$$\mathcal{L}ie\left(H\right) \simeq \mathcal{L}ie\left(\mathcal{P}\right) \oplus \mathcal{L}ie\left(\mathcal{D}\right), \tag{A.2.6}$$

where $\mathcal{L}ie\left(\mathcal{P}\right)$ and $\mathcal{L}ie\left(\mathcal{D}\right)$ are the Poincaré and dilation group Lie algebras, respectively. Therefore let $\tilde{\Delta}, \tilde{K}_a, S_{ab}$ be the generators of the subgroup $H(0)$. According to (A.2.6), they close the algebra

$$\left[\tilde{\Delta}, \tilde{K}_a\right] = -i\tilde{K}_a, \tag{A.2.7}$$

$$\left[\tilde{K}_a, S_{bc}\right] = i\left(\delta_{ab}\tilde{K}_c - \delta_{ac}\tilde{K}_b\right), \tag{A.2.8}$$

$$\left[S_{ab}, S_{cd}\right] = i\left(\delta_{bc}S_{ad} + \delta_{ad}S_{bc} - \delta_{ac}S_{bd} - \delta_{bd}S_{ac}\right). \tag{A.2.9}$$

Moreover, it is always possible to represent the generator P as above in every field space:

$$P_a\phi(\vec{\omega}) = -i\partial_a\phi(\vec{\omega}), \quad \forall\vec{\omega} \in \mathbb{R}^d. \tag{A.2.10}$$

Starting from the operators representing the generators of the little group $H(0)$, it is possible to construct the operators representing d-dimensional rotations, dilations and special conformal transformations acting in every point $\vec{\omega} \in \mathbb{R}^d$, by means of P_a. Indeed, a finite translation is given by

$$\mathrm{e}^{i\vec{a}\cdot\vec{P}}[\phi](\vec{\omega}) = \phi(\vec{\omega} - \vec{a}), \quad a \in \mathbb{R}^d. \tag{A.2.11}$$

Therefore, given an element G of the conformal algebra $\mathcal{L}$, it follows that [146]

$$G[\phi](\vec{\omega}) = \left(\mathrm{e}^{-i\vec{\omega}\cdot\vec{P}}G'\mathrm{e}^{i\vec{\omega}\cdot\vec{P}}\right)[\phi](\vec{\omega}) = \left(\mathrm{e}^{-i\vec{\omega}\cdot\vec{P}}G'\right)[\phi](\vec{0}) = \left(G'\phi\right)(\vec{\omega}), \tag{A.2.12}$$

where

$$G' = \mathrm{e}^{i\vec{\omega}\cdot\vec{P}}G\mathrm{e}^{-i\vec{\omega}\cdot\vec{P}}. \tag{A.2.13}$$

Therefore, using the Baker-Campbell-Hausdorff (BCH) formula

$$\mathrm{e}^{-A}B\mathrm{e}^{A} = B+[B,A]+\frac{1}{2!}\left[[B,A],A\right]+\cdots+\frac{1}{n!}\left[\ldots[B,A],\ldots\right]+\ldots, \tag{A.2.14}$$

it is possible to compute D, K_a and L_{ab} starting from $\tilde{\Delta}$, $\tilde{K}_a$ and S_{ab}, respectively. For example, compute the action of K_a. Using the commutation rules above, by means of the BCH formula one finds

$$\mathrm{e}^{+i\vec{\omega}\cdot\vec{P}}K_a\mathrm{e}^{-i\vec{\omega}\cdot\vec{P}} = K_a + 2\omega^b\left(\delta_{ab}D - L_{ab}\right) + 2\omega_a\omega^bP_b - |\vec{\omega}|^2P_a. \tag{A.2.15}$$

Following (A.2.12), the operator on the right side acts on $\phi(0)$. Since

$$K_a\phi(0) = \tilde{K}_a\phi(0), \quad D\phi(0) = \tilde{\Delta}\phi(0), \quad L_{ab}\phi(0) = S_{ab}\phi(0), \tag{A.2.16}$$

and assuming that the action of of $\tilde{K}_a$, $\tilde{\Delta}$, and S_{ab} on $\phi(0)$ is known by hypothesis, one finds

$$P_a\phi(\vec{\omega}) = -i\partial_a\phi(\vec{\omega}), \tag{A.2.17}$$

$$L_{ab}\phi(\vec{\omega}) = i\left(\omega_a\partial_b - \omega_b\partial_a\right)\phi(\vec{\omega}) + S_{ab}\phi(\vec{\omega}), \tag{A.2.18}$$

$$D\phi(\vec{\omega}) = -i\left(\omega^a\partial_a - i\tilde{\Delta}\right)\phi(\vec{\omega}), \tag{A.2.19}$$

$$K_a\phi(\vec{\omega}) = \left(2\tilde{\Delta}\omega_a - 2\omega^bS_{ab} - 2i\omega_a\omega^b\partial_b + i|\vec{\omega}|^2\partial_a + \tilde{K}_a\right)\phi(\vec{\omega}). \tag{A.2.20}$$

Therefore, *"all field theoretically admissible representations of the conformal algebra are induced by a representation of the algebra of the little group"* [146]. Since $\mathcal{L}ie\,(H)$ has two non trivial ideals, which are $\mathcal{L}ie\,(T_d)$ and $\mathcal{L}ie\,(\mathcal{D} \rtimes T_d)$, there exist two types of representations of the little group

$H(0)$: finite-dimensional and infinite-dimensional representations. Therefore the group $H(0)$ is not semisimple and the finite representation will be not unitary. To construct finite representation of $H(0)$, assume that $\phi(\vec{\omega})$ belongs to a certain irreducible representation of $SO(d)$. In this case, the action of the generators S_{ab} on $\phi(\vec{\omega})$ are known and they are interpretable as the spin operators. Therefore, since $\tilde{\Delta}$ commutes with all the generators S_{ab}, the Schur lemma imposes that $\tilde{\Delta}$ must act as a multiple of the identity on $\phi(\vec{\omega})$. By means of these considerations, one imposes

$$\tilde{\Delta}\phi(\vec{\omega}) = -i\Delta\phi(\vec{\omega}). \tag{A.2.21}$$

In this case, from the commutation rules (A.2.7) and (A.2.8), it follows that one can always choose $\tilde{K}_a = 0$ for all $a = 1, \dots, d$. Therefore, the commutation rules of the conformal algebra becomes

$$\begin{aligned} P_a\phi(\vec{\omega}) &= -i\partial_a\phi(\vec{\omega}), \\ L_{ab}\phi(\vec{\omega}) &= +i\left(\omega_a\partial_b - \omega_b\partial_a\right)\phi(\vec{\omega}) + S_{ab}\phi(\vec{\omega}), \\ D\phi(\vec{\omega}) &= -i\left(\omega^a\partial_a - \Delta\right)\phi(\vec{\omega}), \\ K_a\phi(\vec{\omega}) &= \Big(-2i\Delta\omega_a - 2\omega^b S_{ab} - 2i\omega_a\omega^b\partial_b + i|\vec{\omega}|^2\partial_a\Big)\phi(\vec{\omega}), \end{aligned} \tag{A.2.22}$$

where S_{ab} are the representations of the d-dimensional rotation generators in the $\phi(\vec{\omega})$ space, while the c-number Δ is called *conformal weight.* A spin-ℓ Euclidean field $\phi(\vec{\omega})$, which transforms infinitesimally accordingly to (A.2.22), is called a *conformal primary field* of conformal weight Δ. Now that the infinitesimal action of the conformal group has been shown, it remains to determine its finite action. Consider, for the purpose of this work, a conformal primary Euclidean spinor $\psi(\vec{\omega})$ of conformal weight Δ. In this case,

$$S_{ab} = -\frac{i}{4}\left[\gamma_a, \gamma_b\right], \tag{A.2.23}$$

where the gamma matrices γ_a belong to the Euclidean Clifford algebra $\mathcal{C}_d\ell\left(\gamma, \delta\right)$, which means that they satisfy the anticommutation relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}. \tag{A.2.24}$$

As shown above, the special conformal transformations can be decomposed by means of the inversion and translations. Therefore, in order to determine the finite transformation of $\psi(\vec{\omega})$ under the action of the conformal group, it is sufficient to determine how it transforms under d-dimensional rotations, d-dimensional translations, dilations and inversion. If $\Lambda_D \in SO(1, d+1)$ is a finite dilation, then the representative operator $\rho_\psi(\Lambda_D) \equiv S_d(\Lambda_D)$ is given by the exponential

$$S_d\left(\Lambda_D\right)\phi(\vec{\omega}) = \mathrm{e}^{ic(D+i\omega^a\partial_a)}\phi(\vec{\omega}) = \mathrm{e}^{-c\,\Delta}\phi(\vec{\omega}). \tag{A.2.25}$$

Therefore, a conformal primary spinor transforms under dilation as follows:

$$\phi'(\lambda\vec{\omega}) = \lambda^{-\Delta}\phi(\vec{\omega}), \tag{A.2.26}$$

where $\lambda = \mathrm{e}^c$. Obviously, being a d-dimensional euclidean spinor, $\psi(\vec{\omega})$ is invariant under d-dimensional translations and transforms as

$$\psi'(\vec{\omega}) = S_d(\Lambda_R)\psi(\Lambda_R^{-1}\vec{\omega}) \tag{A.2.27}$$

under d-dimensional rotations, where $\Lambda_R \in SO(1, d+1)$ is the rotation and

$$S_d(\Lambda_R) = \mathrm{e}^{i\epsilon^{ab}S_{ab}} \tag{A.2.28}$$

represents Λ_R in the space of $\psi(\vec{\omega})$. Finally, under inversion a conformal spinor transforms as follows (apart of a possible phase) [117]:

$$\psi'(\vec{\omega}) = \frac{1}{|\vec{\omega}|^{2\Delta}} S(\vec{\omega})\psi\left(I\left(\vec{\omega}\right)\right), \tag{A.2.29}$$

where

$$S(\vec{\omega}) = \frac{\not{\omega}}{|\vec{\omega}|}, \quad \not{\omega} = \omega^a\gamma_a, \tag{A.2.30}$$

with $a = 1, \ldots, d$.

It is possible to note that D acts diagonally on conformal primary fields. Therefore, every finite representation of the conformal group $SO(1, d+1)$ is labelled by a couple $(\mathcal{R}, \Delta)$, where $\mathcal{R}$ is an irreducible representation of $SO(d)$ and Δ is the conformal weight of the representation.

Then, it is possible to construct the representation $(\mathcal{R}, \Delta)$ starting from the primary fields. Indeed, observing the commutation rules (A.1.29) and (A.1.31), and remembering that $[D, L_{ab}] = 0$, it is possible to note that the operator $ad(D) = [D, \cdot\,]$ acts diagonally on the conformal algebra. Moreover, every set of operators $\{D, P_a, K_a\}$, with a fixed, close an algebra diffeomorphic to the one of $SL(2, \mathbb{C})$. This means that the Cartan subalgebra of the conformal algebra is formed by the single element D, and that K_a and P_a are the rising and lowering operators, respectively. Therefore, starting from the primary fields, it is possible to construct the entire representation acting several times with P_a and K_a, for all $a = 1, \ldots, d$.

In every dimension, given a representation of the conformal group, any element found by acting several times with the lowering operators on the primary fields are called *descendant*.

For later purposes, it is useful to represent the conformal group action both in the Minkowski space-time $\mathbb{R}^{1,d+1}$ and in the Minkowski spinor space. The matrix representations of the elements of $SO(1, d+1)$ in these spaces will be shown in the next section.

A.3 Matrix Representations

As observed above, the conformal group $SO(1,d+1)$ is formed by the following subgroups of transformations: d-dimensional rotations, d-dimensional translations, dilations and special conformal transformations.

Since $SO(1,d+1)$ acts linearly on $\mathbb{R}^{1,d+1}$, the operators representing the elements of $SO(1,d+1)$ will be constant matrices in $\mathbb{R}^{1,d+1}$. It is useful to work in light-cone coordinates. Given $X \in \mathbb{R}^{1,d+1}$, define

$$X^+ = X^0 + X^{d+1}, \quad X^- = X^0 - X^{d+1}. \tag{A.3.1}$$

Therefore

$$X^\mu = (X^+,\, X^-, \vec{X})^T. \tag{A.3.2}$$

To find a representation of $SO(1,d+1)$ in $\mathbb{R}^{1,d+1}$, start from the following representation of its generators:

$$(J_{\mu\nu})^\alpha{}_\beta = -i\left(\delta^\alpha_\mu \eta_{\nu\beta} - \delta^\alpha_\nu \eta_{\mu\beta}\right), \tag{A.3.3}$$

where $\mu,\nu,\alpha,\beta = +,-,1\ldots,d$. In light-cone variables, $\eta_{\mu\nu}$ takes the block form

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}. \tag{A.3.4}$$

Then, consider first the d-dimensional rotations. Their representation is trivial: they are represented in $\mathbb{R}^{1,d+1}$ by the matrix

$$\Lambda_{\mathrm{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}, \tag{A.3.5}$$

where $R \in SO(d)$ is the matrix representing the rotation in the Euclidean space $\mathbb{R}^d$.

Now, take a finite dilation Λ_{D}. In light-cone coordinates, D is represented in $\mathbb{R}^{1,d+1}$ by the matrix

$$D = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.3.6}$$

with $D^{2n} = 1$, $n \in \mathbb{N}$. Given a parameter $c > 0$, one finds that

$$\Lambda_{\mathrm{D}} = \mathrm{e}^{icD} = \begin{pmatrix} \mathrm{e}^c & 0 & \vec{0}^{\,T} \\ 0 & \mathrm{e}^{-c} & \vec{0}^{\,T} \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.3.7}$$

Next, consider the d-dimensional translations. It is a straightforward calculation to show that $P_a = 2J_{-a}$, in light-cone variables. The generator J_{-a} is given by the matrix

$$(J_{-a})^{\alpha}{}_{\beta} = -i \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{2} & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}. \tag{A.3.8}$$

Note that

$$\left(J_{-a}^2\right)^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & -\frac{1}{2} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \tag{A.3.9}$$

while $J_{-a}^3 = 0$. Therefore, given a vector $\vec{b} \in \mathbb{R}^d$, exponentiating one finds

$$\Lambda_{\mathrm{T}} = \mathrm{e}^{2i\, b^a J_{-a}} = \begin{pmatrix} 1 & 0 & 0 \\ |\vec{b}|^2 & 1 & 2\vec{b}^T \\ \vec{b} & 0 & 1 \end{pmatrix}. \tag{A.3.10}$$

Working in the same way with special conformal transformations, it is easy to find their matrix representation in $\mathbb{R}^{1,d+1}$. It is given by the matrix

$$\Lambda_{\mathrm{SCT}} = \mathrm{e}^{2i\, b^a J_{+a}} = \begin{pmatrix} 1 & |\vec{b}|^2 & 2\vec{b}^T \\ 0 & 1 & 0 \\ 0 & \vec{b} & 1 \end{pmatrix}. \tag{A.3.11}$$

Moreover, from (A.1.24) it is possible to extract the matrix representation of the inversion I in $\mathbb{R}^{1,d+1}$; it is

$$\Lambda_{\mathrm{I}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}, \tag{A.3.12}$$

from which, it follows that the inversion can be seen as the discrete transformation which exchanges X^+ with X^-, leaving the other coordinates unchanged.

Now, consider a Minkowski spinor $\psi(X)$. Given $\Lambda \in SO(1, d+1)$, the operator $S\,(\Lambda)$ representing the action of Λ in the Minkowski spinor space is

$$S = \mathrm{e}^{\frac{i}{2}\,\epsilon^{\mu\nu} S_{\mu\nu}}, \tag{A.3.13}$$

where $S_{\mu\nu}$ are the operators representing the $SO(1,d+1)$ generators in the Minkowski spin space. They are given by

$$S_{\mu\nu} = -\frac{i}{4}\left[\Gamma_\mu, \Gamma_\nu\right], \tag{A.3.14}$$

where the matrices Γ_μ are elements of the Clifford algebra $\mathcal{C}\ell_{d+2}\left(\Gamma, \eta\right)$. They satisfy

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}. \tag{A.3.15}$$

To represent the gamma matrices Γ_μ, assume that a representation for the matrices $\gamma_a \in \mathcal{C}\ell_d\left(\gamma, \delta\right)$ has already been found. Moreover, let $k = 2^{\left[\frac{d}{2}\right]}$ be the dimension of such a representation, where $\left[\frac{d}{2}\right]$ is the integer part of $\frac{d}{2}$. Therefore, it is possible to construct a $2k$-dimensional representation for the gamma matrices Γ_μ in the following way:

$$\Gamma_a = \begin{pmatrix} \gamma_a & 0 \\ 0 & -\gamma_a \end{pmatrix}, \quad \Gamma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{A.3.16}$$

Hence, consider a d-dimensional rotation Λ_{R}. In the Minkowski spinor space, from (A.3.16) one can assert that Λ_{R} can be represented by the matrix

$$S\left(\Lambda_{\mathrm{R}}\right) = \begin{pmatrix} S_d\left(\Lambda_{\mathrm{R}}\right) & 0 \\ 0 & S_d\left(\Lambda_{\mathrm{R}}\right) \end{pmatrix}, \tag{A.3.17}$$

where $S_d\left(\Lambda_{\mathrm{R}}\right)$ is given by (A.2.28).

For dilations Λ_{D}, one finds

$$S\left(\Lambda_{\mathrm{D}}\right) = \mathrm{e}^{\frac{c}{2}\left(\Gamma_+\Gamma_- - \eta_{+-}\right)}, \tag{A.3.18}$$

where

$$\Gamma_+\Gamma_- - \eta_{+-} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3.19}$$

Therefore, it follows that

$$S\left(\Lambda_{\mathrm{D}}\right) = \begin{pmatrix} \mathrm{e}^{\frac{c}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{c}{2}} \end{pmatrix}. \tag{A.3.20}$$

Next, for the d-dimensional translations one has to compute

$$S\left(\Lambda_{\mathrm{T}}\right) = \mathrm{e}^{\frac{c}{2}\left(b^a\Gamma_-\Gamma_a\right)}, \tag{A.3.21}$$

where $\vec{b} \in \mathbb{R}^d$ is a constant vector and

$$\Gamma_-\Gamma_a = \begin{pmatrix} 0 & \gamma_a \\ 0 & 0 \end{pmatrix}. \tag{A.3.22}$$

Since $(\Gamma_-\Gamma_a)^2 = 0$, Eq.(A.3.21) yields

$$S(\Lambda_{\mathrm{T}}) = \begin{pmatrix} 1 & b^a\gamma_a \\ 0 & 1 \end{pmatrix}. \tag{A.3.23}$$

Equivalently, for special conformal transformations one obtains

$$S(\Lambda_{\mathrm{SCT}}) = \mathrm{e}^{\frac{c}{2}(b^a\Gamma_+\Gamma_a)} = \begin{pmatrix} 1 & 0 \\ b^a\gamma_a & 0 \end{pmatrix}. \tag{A.3.24}$$

Finally, from (A.1.24) it is possible to find the operator representing the inversion in the Minkowski spinor space. It turns to be

$$S_{\mathrm{I}} = \Gamma_+ - \Gamma_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3.25}$$

A.4 Conformal Group in Two Dimensions

In this section, the conformal group of the Euclidean plane $\mathbb{R}^2$ will be studied and will be shown its conformal algebra. Therefore suppose to be $d = 2$; in such a case, in the Euclidean plane the differential equation (A.1.3) reduces to a couple of independent differential equations,

$$\partial_1\Omega_2 = -\partial_2\Omega_1, \quad \text{for} \quad a = 1,\, b = 2, \tag{A.4.1}$$

$$\partial_1\Omega_1 = \partial_2\Omega_2, \quad \text{for} \quad a = 1,\, b = 1. \tag{A.4.2}$$

Next, let ω_1 and ω_2 be the coordinates in the Euclidean plane $\mathbb{R}^2$; it is possible to pass from $\mathbb{R}^2$ to the complex plane $\mathbb{C}$ by means of the transformation

$$z = \omega_1 + i\omega_2, \tag{A.4.3}$$

$$\bar{z} = \omega_1 - i\omega_2, \tag{A.4.4}$$

by which the tensor metric components become off-diagonal:

$$\delta_{ab} \to g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \alpha,\beta = z, \bar{z}. \tag{A.4.5}$$

In the complex plane, define the complex vector coordinates $2\,\Omega^z = \Omega_{\bar{z}} = \Omega^1 + i\Omega^2$ and $2\,\Omega^{\bar{z}} = \Omega_z = \Omega^1 - i\Omega^2$; then it is possible to rewrite the differential equations (A.4.1) and (A.4.2) in the following way:

$$\partial\Omega^{\bar{z}} = 0, \tag{A.4.6}$$

$$\bar{\partial}\Omega^z = 0, \tag{A.4.7}$$

where $\partial \equiv \frac{\partial}{\partial z}$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$. This means that the vector field $\Omega = \Omega^z\partial + \Omega^{\bar{z}}\bar{\partial}$ is a conformal vector field if and only if its coordinates are arbitrary functions of z and $\bar{z}$, respectively: $\Omega^z = \Omega^z(z)$ and $\Omega^{\bar{z}} = \Omega^{\bar{z}}(\bar{z})$.

One can expand $\Omega^z(z)$ and $\Omega^{\bar{z}}(\bar{z})$ in Laurent series around the complex zero as follows:

$$\Omega^z(z) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}, \tag{A.4.8}$$

$$\Omega^{\bar{z}}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{a}_n \bar{z}^{n+1}. \tag{A.4.9}$$

$$\tag{A.4.10}$$

In this way, it is possible to read off the generators of the conformal transformations in two dimensions, written in the complex plane; they are

$$\ell_n = z^{n+1}\partial, \quad \bar{\ell}_n = \bar{z}^{n+1}\bar{\partial}, \tag{A.4.11}$$

with $n \in \mathbb{Z}$. These are a numerable infinity of generators, which satisfy the so-called *classical Virasoro* commutation relations

$$[\ell_n, \ell_m] = (m-n)\ell_{m+n}, \quad \left[\bar{\ell}_n, \bar{\ell}_m\right] = (m-n)\bar{\ell}_{m+n}, \quad \left[\bar{\ell}_n, \ell_m\right] = 0. \tag{A.4.12}$$

From this, it follows that the finite conformal transformations are all and only the holomorphic and the anti-holomorphic transformations in the complex plane [145],

$$z \longmapsto f(z), \quad \bar{z} \longmapsto \bar{f}(\bar{z}). \tag{A.4.13}$$

Therefore, the Lie algebra $\mathcal{L}$ of the conformal group $\mathcal{C}onf\,(\mathbb{R}^2)$ is infinite-dimensional and it is composed by the direct sum of two commutating infinite-dimensional *Virasoro algebras* $\mathcal{V}$ and $\bar{\mathcal{V}}$; in formulae

$$\mathcal{L} = \mathcal{V} \oplus \bar{\mathcal{V}}, \tag{A.4.14}$$

where

$$\mathcal{V} = \{\ell_n : [\ell_n, \ell_m] = (m-n)\ell_{m+n}, n \in \mathbb{Z}\}, \tag{A.4.15}$$

and

$$\bar{\mathcal{V}} = \left\{\bar{\ell}_n : \left[\bar{\ell}_n, \bar{\ell}_m\right] = (m-n)\bar{\ell}_{m+n}, n \in \mathbb{Z}\right\}. \tag{A.4.16}$$

Henceforth, even if the same considerations hold for $\bar{\mathcal{V}}$, the discussion will involve only $\mathcal{V}$. The Virasoro algebra $\mathcal{V}$ admits an infinite number of subalgebras. Indeed, considering the triad ℓ_0, ℓ_m, ℓ_{-m}, with m fixed, these generators span the subalgebra

$$[\ell_0, \ell_{\pm m}] = \pm m\, \ell_{\pm m}, \quad [\ell_- m, \ell_m] = 2m\, \ell_0, \tag{A.4.17}$$

which is isomorphic to the $SL\,(2, \mathbb{C})$ one. In particular, consider the subalgebra spanned by ℓ_1, ℓ_0, ℓ_{-1}. These generators are the only ones that are well-defined on the sphere $\mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$. In the complex plane, and hence on the sphere, they generate the following transformations ($a, b, c \in \mathbb{C}$) [144]:

Generator	Infinitesimal transformation	Finite transformation	
ℓ_{-1}	$z \longmapsto z - \epsilon$	$z \longmapsto z + b$	Complex translations
ℓ_0	$z \longmapsto z - \epsilon z$	$z \longmapsto az$	Scaling
ℓ_1	$z \longmapsto z - \epsilon z^2$	$z \longmapsto \frac{z}{1-cz}$	Special conformal

Actually, $\ell_{-1} = \partial$ and $\ell_0 = z\partial$ are nothing else than the translation operator and the Euler operator represented in the complex plane, respectively. The finite transformations in the table above can be summarized by the expression

$$z \longmapsto z' = \frac{az+b}{cz+d}, \tag{A.4.18}$$

with a, b, $c\,, d \in \mathbb{C}$ and $ad - bc = 1$. Therefore, from (A.4.18) it is possible to extract the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.4.19}$$

with $\det A = 1$.

Moreover, sending at the same time a, b, c, $d \longmapsto -a$, $-b$, $-c$, $-d$, z' is left unchanged. This means that the matrices A and $-A$ describe the same transformation (A.4.18). From these considerations, one can assert that the group formed by transformations like (A.4.18) is diffeomorphic to the Lorentz group $SO(1,3) \simeq SL\,(2,\mathbb{C})\,/\mathbb{Z}_2$. Therefore, the *global conformal group* is diffeomorphic to the complexified Lorentz group $SO(1,3)$, on the sphere $\mathbb{S}^2$.

Further, from the generators ℓ_0 and $\bar{\ell}_0$, it is possible to construct the dilation generator as $D = \ell_0 + \bar{\ell}_0$ and the rotation operator as $R = i(\ell_0 - \bar{\ell}_0)$. The eigenvalues of these operators respectively are $\Delta = h + \bar{h}$ and $s = h - \bar{h}$, where h and $\bar{h}$ are the eigenvalues of ℓ_0 and $\bar{\ell}_0$.

Suppose, now, to want to construct an irreducible representation of the conformal group in two dimensions. As in higher dimensions, to find an irreducible representation of the conformal group, one has to find a set of operators which satisfy the rising and lowering operator algebra. In two dimensions, the Cartan subalgebra of $\mathcal{V}$ is constituted by the single element ℓ_0, indeed the roots of the algebra $\mathcal{V}$ are given by the commutation relations

$$[\ell_m, \ell_0] = m\ell_m. \tag{A.4.20}$$

In particular, the elements ℓ_m with $m < 0$ are the *lowering operators*, while the elements ℓ_m with $m > 0$ are the *rising operators*. To construct an irreducible conformal representation, one has to find a set of elements $|h\rangle$

of the representation, which are annihilated by all the rising operators ℓ_m, with $m > 0$. Then one finds all the others elements of the representation by applying to $|h\rangle$ several times the lowering operators ℓ_m, for all $m < 0$. In this way one obtains the so-called *Verma modules* associated to the elements $|h\rangle$. Alternatively, one can start from the primary fields, which in two dimensions are defined as those fields $\phi(z, \bar{z})$ which transform as [144, 145]

$$\phi'(z', \bar{z}\,') = \left(\frac{\partial z'}{\partial z}\right)^{h} \left(\frac{\partial \bar{z}\,'}{\partial \bar{z}}\right)^{\bar{h}} \phi(z, \bar{z}), \tag{A.4.21}$$

under the conformal transformation $z \to z'(z)$, $\bar{z} \to \bar{z}\,'(z)$.

Appendix B

Embedding Space Formalism

Consider the d-dimensional Euclidean space $\mathbb{R}^d$, with $d \geq 3$. As already shown in the previously, the Conformal Group $\mathcal{C}onf\left(\mathbb{R}^d\right)$ of the Euclidean space is isomorphic to $SO(1, d+1)$. The conformal covariance imposes strong constraints on the correlation functions of the conformal primary fields. Such constraints owing to the transformation rules of these fields under the action of the Conformal Group. To analyse the transformation properties of the conformal primary fields on the d-dimensional Euclidean space, it is convenient to look at the non-linear action of the Conformal Group on $\mathbb{R}^d$ as induced from the linear one on $\mathbb{R}^{1,d+1}$.

The key idea, due to Dirac [147], is based on looking at $\mathbb{R}^d$ as a submanifold embedded in the Minkowski space $\mathbb{M}^{d+2}$; in such a way it is possible to make a correspondence between the conformal primary fields defined on $\mathbb{R}^d$ and the tensorial ones living on $\mathbb{M}^{d+2}$. Therefore, one studies first linear transformation properties of Minkowski fields on the "canonical" space $\mathbb{M}^{d+2}$ and then restricts the action of $SO(1, d+1)$ onto its submanifold $\mathbb{R}^d$, in order to find the transformation rules of the associated conformal primary fields.

From this point of view, the non-linearity of the transformation rules of the Euclidean conformal primary fields emerges from the restriction of the action of $SO(1, d+1)$ onto a d-dimensional submanifold of $\mathbb{M}^{d+2}$, that will be indentified as the Poincaré section of the light-cone. Indeed, in sec.(2.3) the Euclidean space $\mathbb{R}^d$ has been embedded in the Poincaré section $\mathcal{C}_P^{\pm}$ of the future/past light-cone. In this section, we will present the consequences of this embedding, that will lead us to construct an efficient technique to translate conformal transformation properties from $\mathbb{R}^d$ to $\mathbb{M}^{d+2}$. We will employ this technique, refined over the years, especially in the study of high-spin theories in AdS, to bring about significant simplifications in calculations and to set up our study from the outset for a potential future extension to higher spin cases. In the study conducted in this section, we will focus exclusively on the future Poincaré section of the light-cone, even though a parallel analysis can be performed by considering the past one. Therefore, throughout this section, when referring to the Poincaré section of the light-cone, we will simply denote it as $\mathcal{C}_P$, omitting any sign at the apex.

The elements of a generic section of $\mathcal{C}_{q^+}$ can be parameterised in light-cone coordinates as

$$q(\vec{\omega}) = q^+ \left(1, |\vec{\omega}|^2, \vec{\omega}\right)^T. \tag{B.0.1}$$

In particular, setting $q^+ = 1$, it is possible to embed $\mathbb{R}^d$ in the Poincaré section $\mathcal{C}_P$, whose elements are represented as

$$q(\vec{\omega}) = q^+ \left(1, |\vec{\omega}|^2, \vec{\omega}\right)^T. \tag{B.0.2}$$

In what follows, we will develop in detail the Dirac's embedding formalism. In particular, we will start from scalar fields and the we advance to unfold the method for $SO(1, d+1)$-tensorial fields.

B.1 Embedding Scalar Conformal Primary Fields

Consider an Euclidean conformal primary scalar field $\phi(\vec{\omega})$ on $\mathbb{R}^d$. The purpose of this section is to construct a correspondence between this conformal primary field and a Minkowski scalar one, by restricting the latter on $\mathcal{C}_P \simeq \mathbb{R}^d$.

For this, consider the future null-cone $\mathcal{C}^+$. It is possible to represent the action of $SO(1, d+1)$ on $\mathcal{C}^+$ by its definitory representation since $SO(1, d+1)$ maps null-vectors into null-vectors in $\mathbb{R}^{1,d+1}$.

Let $\Phi(q)$, $q^2 = 0$, be a Minkowski scalar field defined on $\mathcal{C}^+$; this means that:

$$\Phi'(q) = \Phi\left(\Lambda^{-1} q\right), \quad \Lambda \in SO(1, d+1). \tag{B.1.1}$$

To relate the Minkowski scalar field $\Phi(q)$ to $\phi(\vec{\omega})$, it is necessary, first, to map $\Phi(q)$ on the projective null-cone $\mathcal{C}_\sim$ [74, 75, 120].

Since one element of $\mathcal{C}_\sim$ is an entire equivalence class of null-vectors, a scalar field $\Phi(q)$ must satisfy a suitable homogeneity property to be well-defined on $\mathcal{C}_\sim$.

In fact, $\Phi(q)$ must be such that, choosing a certain representative q of the equivalence class $[q] \in \mathcal{C}_\sim$, for any other element $q_\lambda = \lambda q$ equivalent to q, the pointwise values $\Phi(q_\lambda)$ must be related to $\Phi(q)$ for all $\lambda \in \mathbb{R}^+$. This is necessary because the value $\Phi([q])$ refers to a single c-number, on $\mathcal{C}_\sim$. Hence, one must construct a correspondence between the set of c-numbers $\Phi(q_\lambda)$, with $q_\lambda \in \mathcal{C}^+$. To do this, one generically imposes

$$\Phi(q) = f(\lambda)\Phi(\lambda q), \quad \lambda \in \mathbb{R}^+, \tag{B.1.2}$$

where $f(\lambda)$ is a smooth function that can be fixed by using the transitive property of the equivalence classes. Specifically, it must be verified what follows:

$$\Phi(q_\lambda) = f(\lambda')\Phi(\lambda' q_\lambda) = f(\lambda')f(\lambda)\Phi(\lambda'\lambda q) = f(\lambda'')\Phi(\lambda'' q), \tag{B.1.3}$$

where $\lambda'' = \lambda'\lambda$ and $\lambda, \lambda' \in \mathbb{R}^+$. Obviously, (B.1.3) is satisfied if and only if

$$f(\lambda')f(\lambda) = f(\lambda'\lambda); \tag{B.1.4}$$

which holds if and only if $f(\lambda) = \lambda^\Delta$, where $\Delta \in \mathbb{C}$ is called *homogeneity degree* of the scalar field $\Phi(q)$.

Therefore, the Minkowski scalar field $\Phi(q)$ must satisfy the *homogeneity condition* [74, 120, 148]

$$\Phi(q) = \lambda^\Delta \Phi(\lambda q), \quad \forall \lambda \in \mathbb{R}^+, \tag{B.1.5}$$

to be projected on $\mathcal{C}_\sim$. Next, define

$$\Phi(q) = \left(q^+\right)^{-\Delta} \phi\left(\vec{q}/q^+\right), \tag{B.1.6}$$

where, at this level, $\phi\left(\vec{q}/q^+\right)$ is a generic field, the transformation rules of which must be determined.

It is trivial to demonstrate that the field defined in (B.1.6) respects the homogeneity condition (B.1.5). Indeed, by sending $q \longmapsto \lambda q$, with $\lambda \in \mathbb{R}^+$, q^+ is mapped to λq^+, while the argument of ϕ does not change. Therefore, the field $\phi\left(\vec{q}/q^+\right)$ is invariant under rescaling.

Moreover, embedding $\mathbb{R}^d$ in $\mathcal{C}_P$ by means of (2.3.8), it is possible to verify that the field $\Phi\big(q(\vec{\omega})\big)$ in Eq.(B.1.6) is an $SO(1, d+1)$ scalar field if and only if $\phi\left(\vec{\omega}\right)$ is an $SO(1, d+1)$ conformal scalar one.

In order to prove what just asserted, it is useful to demonstrate, first, what follows [18].

Lemma B.1.1. *Let $q(\vec{\omega})$, with $\vec{\omega} \in \mathbb{R}^d$, be an element of a section of the future null-cone $\mathcal{C}^+$ parametrized as in* (B.0.2). *Moreover, let $\vec{\omega}\,' = \Lambda[\vec{\omega}]$, where $\Lambda[\cdot]$ is the operator representing the action of Λ on $\mathbb{R}^d$, with $\Lambda \in SO(1, d+1)$.*

Then,

$$q(\vec{\omega}\,') = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|^{1/d} \Lambda q(\vec{\omega}), \tag{B.1.7}$$

where,

$$|J| = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|, \tag{B.1.8}$$

is the Jacobian of the transformation $\vec{\omega}\,' = \vec{\omega}\,'(\vec{\omega})$.

Proof. To demonstrate (B.1.7), it is necessary to know how $SO(1,d+1)$ acts on $\vec{\omega} \in \mathbb{R}^d$.

In the previous chapter, $SO(1,d+1)$ has been decomposed in d-dimensional translations, d-dimensional rotations, dilations and special conformal transformations. Further, any special conformal transformation Λ_{SCT} has been rewritten as follows:

$$\Lambda_{\mathrm{SCT}} = I \circ \Lambda_{\mathrm{T}} \circ I, \tag{B.1.9}$$

where I is the inversion and Λ_{T} is a d-dimensional translation. Hence, the action of $SO(1,d+1)$ on $\vec{\omega}$ can be totally determined by studying the action of d-dimensional translations, d-dimensional rotations, dilations and inversion.

First of all, consider a d-dimensional translation:

$$\vec{\omega}\,' = \vec{\omega} + \vec{b}, \tag{B.1.10}$$

where $\vec{b} \in \mathbb{R}^d$ is a constant vector. In the Minkowski space-time $\mathbb{R}^{1,d+1}$, Λ_{T} is represented by the matrix (A.3.10). The target of $q(\vec{\omega})$ by the action of $\Lambda_{\mathrm{T}}(\vec{b})$ is

$$q(\vec{\omega}\,') = \left(1, |\omega|^2 + 2\vec{\omega}\cdot\vec{b} + |\vec{b}|^2, \vec{\omega} + \vec{b}\right)^T, \tag{B.1.11}$$

which corresponds to

$$q(\vec{\omega}\,') = \begin{pmatrix} 1 & 0 & \vec{0} \\ \vec{b}\cdot\vec{b} & 1 & 2\vec{b} \\ \vec{b} & \vec{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} 1 \\ |\vec{\omega}|^2 \\ \vec{\omega} \end{pmatrix} = \Lambda_{\mathrm{T}}(\vec{b})\, q(\vec{\omega}). \tag{B.1.12}$$

This result, together with

$$\left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{1/d} = \left|\frac{\partial\left(\vec{\omega} + \vec{b}\right)}{\partial\vec{\omega}}\right|^{1/d} = 1, \tag{B.1.13}$$

provides (B.1.7) for d-translations.

Next, consider a d-dimensional rotation: $\vec{\omega}\,' = R\vec{\omega}$, $R \in SO(d)$. In $\mathbb{R}^{1,d+1}$, a d-dimensional rotation is represented by the matrix (A.3.5)

Also in this case

$$\left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{1/d} = \left|\frac{\partial\,(R\,\vec{\omega})}{\partial\vec{\omega}}\right|^{1/d} = \left|\,\det R\,\right|^{1/d} = 1, \tag{B.1.14}$$

and

$$q(\vec{\omega}\,') = \begin{pmatrix} 1 \\ |\vec{\omega}|^2 \\ R\vec{\omega} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \vec{0} \\ 0 & 1 & \vec{0} \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} 1 \\ |\vec{\omega}|^2 \\ \vec{\omega} \end{pmatrix} = \Lambda_{\mathrm{R}}\, q(\vec{\omega}). \tag{B.1.15}$$

It remains to treat the dilations, as the last subgroup.

A dilation maps:

$$D : \vec{\omega} \longmapsto \vec{\omega}\,' = \mathrm{e}^{c}\vec{\omega}, \quad c \in \mathbb{R}, \tag{B.1.16}$$

and is represented by the matrix (A.3.7), in $\mathbb{R}^{1,d+1}$. Therefore,

$$\left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right|^{1/d} = \left|\frac{\partial\left(\mathrm{e}^{c}\vec{\omega}\right)}{\partial \vec{\omega}}\right|^{1/d} = \left|\det\left[\mathrm{e}^{c}\mathbb{1}\right]\right|^{1/d} = \mathrm{e}^{c}. \tag{B.1.17}$$

In addition,

$$q(\vec{\omega}\,') = \begin{pmatrix} 1 \\ \mathrm{e}^{2c}\,|\vec{\omega}|^2 \\ \mathrm{e}^{c}\vec{\omega} \end{pmatrix} = \mathrm{e}^{c}\begin{pmatrix} \mathrm{e}^{-c} \\ \mathrm{e}^{c}\,|\vec{\omega}|^2 \\ \vec{\omega} \end{pmatrix}, \tag{B.1.18}$$

which can be rewritten as

$$q(\vec{\omega}\,') = \mathrm{e}^{c}\begin{pmatrix} \mathrm{e}^{-c} & 0 & \vec{0} \\ 0 & \mathrm{e}^{c} & \vec{0} \\ 0 & 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} 1 \\ |\vec{\omega}|^2 \\ \vec{\omega} \end{pmatrix} = \mathrm{e}^{c}\,\Lambda_{\mathrm{D}}\,q(\vec{\omega}), \tag{B.1.19}$$

proving (B.1.7) for dilations.

Finally, consider the inversion,

$$I : \vec{\omega} \longmapsto \vec{\omega}\,' = \frac{\vec{\omega}}{|\vec{\omega}|^2}. \tag{B.1.20}$$

In $\mathbb{R}^{1,d+1}$, the inversion I is represented by the matrix (A.3.12). Moreover,

$$\frac{\partial \omega'^{a}}{\partial \omega^{b}} = \frac{M^{a}_{b}\left(\vec{\omega}\right)}{|\vec{\omega}|^2}, \quad a, b = 1, \ldots, d; \tag{B.1.21}$$

where the matrix

$$M^{a}_{b}\left(\vec{\omega}\right) = \delta^{a}_{b} - 2\frac{\omega^{a}\omega_{b}}{|\vec{\omega}|^2}, \tag{B.1.22}$$

is such that

$$M^{a}_{b}M^{b}_{c} = \delta^{a}_{c}, \tag{B.1.23}$$

as can be easily verified. Eq.(B.1.23) implies that

$$\det M = \pm 1; \tag{B.1.24}$$

hence, the Jacobian of the inversion is

$$|J| = \left|\frac{\partial \vec{\omega}\,'}{\partial \vec{\omega}}\right| = \left|\det\left[\frac{M^{a}_{b}\left(\vec{\omega}\right)}{|\vec{\omega}|^2}\right]\right| = \frac{1}{|\vec{\omega}|^{2d}}, \tag{B.1.25}$$

from which one obtains

$$\left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{1/d} = |\vec{\omega}|^{-2}\,. \tag{B.1.26}$$

Then,

$$q(\vec{\omega}\,') = \begin{pmatrix} 1 \\ |\vec{\omega}|^{-2} \\ \vec{\omega}/|\vec{\omega}|^2 \end{pmatrix} = |\vec{\omega}|^{-2} \begin{pmatrix} |\vec{\omega}|^2 \\ 1 \\ \vec{\omega} \end{pmatrix} = |\vec{\omega}|^{-2}\,\Lambda_{\mathrm{I}}\,q(\vec{\omega}), \tag{B.1.27}$$

which completes the demonstration of Lemma (B.1.1). □

Now, return to the definition (B.1.28). Embedding $\mathbb{R}^d$ in $\mathcal{C}^+$ by (2.3.8), it is possible to write

$$\Phi\big(q(\vec{\omega})\big) = \left(q^+\right)^{-\Delta}\phi(\vec{\omega}), \quad \vec{\omega}\in\mathbb{R}^d, \tag{B.1.28}$$

or, otherwise,

$$\phi(\vec{\omega}) = \left(q^+\right)^{\Delta}\Phi\big(q(\vec{\omega})\big). \tag{B.1.29}$$

It is possible to demonstrate the following

Theorem B.1.1. *The field $\Phi(q)$, defined by Eq.*(B.1.28)*, is a Minkowski scalar field with respect to $SO(1,d{+}1)$ if and only if $\phi(\vec{\omega})$ is a d-dimensional Euclidean conformal primary scalar field. In formulae,*

$$\Phi'\big(\Lambda q(\vec{\omega})\big) = \Phi\big(q(\vec{\omega})\big) \Longleftrightarrow \phi'\left(\vec{\omega}\,'\right) = \left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{-\Delta/d}\phi(\vec{\omega}), \tag{B.1.30}$$

where the prime indicates the transformed quantities by means of the Λ action, with $\Lambda\in SO(1,d+1)$.

Proof. Let Λ be an element of $SO(1,d{+}1)$. Using the homogeneity condition (B.1.5) and the Lemma $reflemma$, it is possible to write

$$\Phi'\big(q(\vec{\omega}\,')\big) = \Phi'\big(\,|J|^{1/d}\,\Lambda q(\vec{\omega})\big) = |J|^{-\frac{1}{d}}\,\Phi'\big(\Lambda q(\vec{\omega})\big), \tag{B.1.31}$$

where $|J|$ is the Jacobian defined in (B.1.8). Moreover, by definition,

$$\Phi'\big(q(\vec{\omega}\,')\big) = \left(q^+\right)^{-\Delta}\phi'\left(\vec{\omega}\,'\right). \tag{B.1.32}$$

Therefore, if $\Phi(q)$ is a scalar field, then

$$\Phi'\big(\Lambda q(\vec{\omega})\big) = \Phi\big(q(\vec{\omega})\big) \Longrightarrow \Phi'\big(q(\vec{\omega}\,')\big) = |J|^{-\frac{1}{d}}\,\Phi\big(q(\vec{\omega})\big), \tag{B.1.33}$$

or, equivalently, by substituting (B.1.32) and using the definition (B.1.28),

$$\left(q^+\right)^{-\Delta} \phi'\left(\vec{\omega}\,'\right) = |J|^{-\frac{1}{d}} \left(q^+\right)^{-\Delta} \phi\left(\vec{\omega}\right), \tag{B.1.34}$$

which holds if and only if

$$\phi'(\vec{\omega}\,') = \left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{-\Delta/d} \phi(\vec{\omega}). \tag{B.1.35}$$

This proves the direct implication of the Theorem $B.1.1$. Vice versa, if $\phi(\vec{\omega})$ is a d-dimensional conformal scalar field, then, inserting (B.1.35) in (B.1.31), one finds

$$\begin{aligned}\Phi'(\Lambda q(\vec{\omega})) &= |J|^{\frac{1}{d}}\, \Phi'(q(\vec{\omega}\,')) = |J|^{\frac{1}{d}} \left(q^+\right)^{-\Delta} \phi'\left(\vec{\omega}\,'\right) \\ &= \left(q^+\right)^{-\Delta} \phi\left(\vec{\omega}\right) = \Phi\big(q(\vec{\omega})\big),\end{aligned} \tag{B.1.36}$$

which completes the proof of the theorem. □

The field $\Phi\big(q(\vec{\omega})\big)$ defined by (B.1.28) is said to be the scalar field *up-lifted* associated to $\phi(\vec{\omega})$. In particular, $\Phi\big(q(\vec{\omega})\big)$ can be restricted onto the Poincaré section by using the homogeneity condition. Indeed, it is possible to write

$$\Phi\big(\lambda q(\vec{\omega})\big) = \lambda^{-\Delta}\Phi\big(q(\vec{\omega})\big) = \left(\lambda q^+\right)^{-\Delta} \phi(\vec{\omega}). \tag{B.1.37}$$

Hence, setting $\lambda = 1/q^+$, one finds

$$\Phi\big(q(\vec{\omega})\big)\big|_{\mathcal{C}_P} = \phi(\vec{\omega}), \quad \vec{\omega} \in \mathbb{R}^d, \tag{B.1.38}$$

which is the relation used to embed a d-dimensional scalar conformal primary field $\phi(\vec{\omega})$ into a Minkowski scalar one $\Phi\big(q(\vec{\omega})\big)$, restricted on $\mathcal{C}_P$.

Suppose, now, to want to calculate a two point correlation function

$$G_{\Delta_1,\Delta_2}(\vec{\omega}_1, \vec{\omega}_2) = \langle \phi_{\Delta_1}(\vec{\omega}_1)\phi_{\Delta_2}(\vec{\omega}_2)\rangle, \tag{B.1.39}$$

where ϕ_{Δ_i}, with $i = 1, 2$ are two scalar conformal primary fields of conformal weight Δ_i, valued at $\vec{\omega}_i \in \mathbb{R}^d$. Following the construction above, in order to find $G_{\Delta_1,\Delta_2}(\vec{\omega}_1, \vec{\omega}_2)$, it is possible to calculate, first, the up-lifted correlation function $\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1), q(\vec{\omega}_2)\big)$ and then restrict the result to the Poincaré section by means of (B.0.2) and (B.1.38). Specifically,

$$\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1), q(\vec{\omega}_2)\big) = \langle \Phi_{\Delta_1}\big(q(\vec{\omega}_1)\big)\Phi_{\Delta_2}\big(q(\vec{\omega}_2)\big)\rangle, \tag{B.1.40}$$

where Φ_{Δ_i}, $i = 1, 2$, are the up-lifted fields of ϕ_{Δ_i} and Δ_i are the homogeneity degrees of Φ_{Δ_i} with respect to q, at this level.

It is possible to find the generic two point up-lifted correlation function $\mathcal{G}_{\Delta_1,\Delta_2}$ by using nothing else than *relativistic consideration*, that means looking at anything but the the constraints imposed onto the up-lifted fields. In particular, the up-lifted correlation function $\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1),q(\vec{\omega}_2)\big)$ defined in (B.1.40), must satisfy the following properties:

1. It is defined on the future null-cone $\mathcal{C}^+$; thus one must impose:
$$q(\vec{\omega}_i)^2=0, \quad \forall\, i=1,2; \tag{B.1.41}$$
2. It is a two-point scalar field with respect to $SO(1,d+1)$:
$$\mathcal{G}'_{\Delta_1,\Delta_2}\big(q'(\vec{\omega}\,'_1),q'(\vec{\omega}\,'_2)\big)=\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1),q(\vec{\omega}_2)\big), \tag{B.1.42}$$
where the prime indicates the transformed quantities;
3. It has homogeneity degree Δ_i, $i=1,2$, with respect to $q(\vec{\omega}_i)$:
$$\mathcal{G}_{\Delta_1,\Delta_2}\big(\lambda_1 q(\vec{\omega}_1),\lambda_2 q(\vec{\omega}_2)\big)=(\lambda_1)^{-\Delta_1}(\lambda_2)^{-\Delta_2}\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1),q(\vec{\omega}_2)\big), \tag{B.1.43}$$
where $\lambda_1,\lambda_2\in\mathbb{R}^+$.

The properties (B.1.41), (B.1.42), (B.1.43) are so restrictive that they uniquely determine the shape of $\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1),q(\vec{\omega}_2)\big)$. Specifically, the two-point up-lifted correlation function can only be of the form [143, 149]

$$\mathcal{G}_{\Delta_1,\Delta_2}\big(q(\vec{\omega}_1),q(\vec{\omega}_2)\big)=c\,\frac{\delta_{\Delta_1\Delta_2}}{\big(-2q(\vec{\omega}_1)\cdot q(\vec{\omega}_2)\big)^{\Delta_1}}, \tag{B.1.44}$$

where the factor $(-2)^{\Delta_1}$ in the denominator is placed for convention, and c is a constant. Inserting (B.1.38) into (B.1.44) and using the parametrization (B.0.2), one restricts the up-lifted two-point correlation function $\mathcal{G}_{\Delta_1,\Delta_2}$ to the Poincaré section. The result is

$$G_{\Delta_1,\Delta_2}(\vec{\omega}_1,\vec{\omega}_2)=c\,\frac{\delta_{\Delta_1\Delta_2}}{\Big(|\vec{\omega}_1-\vec{\omega}_2|^2\Big)^{\Delta_1}}. \tag{B.1.45}$$

Obviously, it is also possible to find the shape of the conformal scalar correlation function $G_{\Delta_1,\Delta_2}(\vec{\omega}_1,\vec{\omega}_2)$, by imposing on it the conformal constraints directly.

B.2 Embedding Tensor Conformal Primary Fields

The construction of the general tensor case is quite similar to the scalar one.

At the beginning, let q be a point in $\mathbb{R}^{1,d+1}$; then, consider a rank-(J,0) irreducible tensor field $T(q)$ of components $T_{\mu_1\ldots\mu_J}(q)$, with $J\in\mathbb{N}$. First of all, one has to project $T_{\mu_1 \dots \mu_J}(q)$ on the future null-cone $\mathcal{C}^+$. To do so, only $q^2 = 0$, with $q^0 > 0$, has been imposed in the scalar case; while here an additional constraint is required. Indeed, given $q \in \mathcal{C}$, the rank-(J,0) tensor space $T_q^{(J,0)}\,\mathcal{C}$ is included in $T_q^{(J,0)}\,\mathbb{R}^{1,d+1}$. Therefore, the condition $q^2 = 0$ ensures $T(q) \in T_q^{(J,0)}\,\mathbb{R}^{1,d+1}$, with $q \in \mathcal{C}$, but does not guarantee $T(q) \in T_q^{(J,0)}\,\mathbb{R}^{1,d+1}$.

The additional constraint one needs follows immediately from the definition of the null-cone. Indeed, it is possible to define $\mathcal{C}$ by means of the function $f(q) = q^2$; specifically

$$\mathcal{C} = f^{-1}(0), \tag{B.2.1}$$

where f^{-1} is the preimage of the function f. From this, it is possible to demonstrate that the tangent space $T_q\,\mathcal{C} \equiv T_q^{(0,1)}\,\mathcal{C}$ is such that

$$T_q\,\mathcal{C} = Ker\,(\mathrm{d}f)\,. \tag{B.2.2}$$

In fact, let γ be the smooth curve

$$\gamma : t \in I_0 \longmapsto \gamma(t) \in \mathcal{C}, \tag{B.2.3}$$

where I_0 is a neighborhood of $0 \in \mathbb{R}$. By definition of null-cone, $f(q) = 0$ for all $q \in \mathcal{C}$; therefore, given $\gamma(0) = q \in \mathcal{C}$, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\bigg|_{t=0} = \frac{\partial f(q)}{\partial q^\mu}\,\frac{\mathrm{d}\gamma^\mu(t)}{\mathrm{d}t}\bigg|_{t=0} \equiv \mathrm{d}f\,[V]\,(q) = 0, \tag{B.2.4}$$

where $V(q)$ is the generic vector belonging to the tangent space $T_q\,\mathcal{C}$, with components $V^\mu(q) = \frac{\mathrm{d}\gamma^\mu(t)}{\mathrm{d}t}\Big|_{t=0}$. Therefore, if $V(q) \in T_q\,\mathcal{C}$ then $V(q) \in Ker\,(\mathrm{d}f)$; this implies that

$$T_q\,\mathcal{C} \subseteq Ker\,(\mathrm{d}f)\,. \tag{B.2.5}$$

Moreover, since

$$\dim\,(Ker\,(\mathrm{d}f)) + \dim\,(\mathcal{I}m\,(\mathrm{d}f)) = \dim\left(T_q\,\mathbb{R}^{1,d+1}\right) = \dim\left(\mathbb{R}^{1,d+1}\right), \tag{B.2.6}$$

with $\dim\,(\mathcal{I}m\,(\mathrm{d}f)) = \dim\,(\mathbb{R}) = 1$, it is possible to assert

$$\dim\,(Ker\,(\mathrm{d}f)) = \dim\,(T_q\,\mathcal{C}) = d + 1, \tag{B.2.7}$$

which, together with (B.2.5) implies (B.2.2). Therefore, given $q \in \mathcal{C}$, a Minkowski vector $V(q)$ belongs to $T_q\,\mathcal{C}$ if and only if

$$\mathrm{d}f\,[V]\,(q) = 2q_\mu V^\mu(q) = 0 \Longleftrightarrow q_\mu V^\mu(q) = 0. \tag{B.2.8}$$

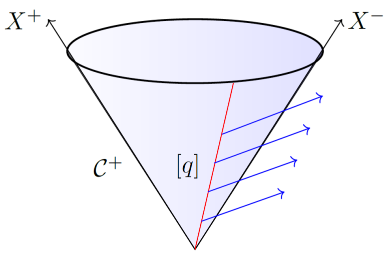

Figure B.1: On the future null-cone $\mathcal{C}^+$, a future null-vector q identifies a light ray $[q]$ (in red) onto which several tensors are applied (in blue). On the projective null-cone $\mathcal{C}_\sim$, all tensors represented in the picture (in blue) belong to the same equivalence class known as *tractor*.

For the Minkowski one-forms, one can employ the isomorphism induced by the metric to associate to every vector $V(q) \in T_q\mathcal{C}$ a one-form $\Omega(q) = i_V\eta(q) \in T_q^{(1,0)}\mathcal{C}$. Therefore, by means of (B.2.8), it is possible to assert that a Minkowski one-form $\Omega(q)$, of components $\Omega_\mu(q)$, belongs to $T_q^{(1,0)}\mathcal{C}$ if and only if

$$i_q\Omega = \Omega_\mu q^\mu = 0. \tag{B.2.9}$$

This last condition can be immediately generalized to the generic rank-$(J,0)$ tensor field $T(q)$. In fact, from the isomorphism

$$T_q^{(J,0)}\mathcal{C} = T_q^{(1,0)}\mathcal{C} \underbrace{\otimes \cdots \otimes}_{J-times} T_q^{(1,0)}\mathcal{C}, \tag{B.2.10}$$

follows that $T(q) \in T_q^{(J,0)}\mathcal{C}$ if and only if its components $T_{\mu_1\ldots\mu_J}(q)$ satisfy the *transversality condition*

$$q^{\mu_k}T_{\mu_1\ldots\mu_k\ldots\mu_J} = 0, \quad \forall k = 1,\ldots,J;\ q \in \mathcal{C}, \tag{B.2.11}$$

which is the additional constraint required above.

Next, given $q \in \mathcal{C}^+$, one has to project the tensor field $T(q)$ on $\mathcal{C}_\sim$. In analogy to the scalar case, $T(q)$ must satisfy the *homogeneity condition*

$$T(\lambda q) = \lambda^{-\Delta}T(q), \quad \lambda \in \mathbb{R}^+,\ \Delta \in \mathbb{R}, \tag{B.2.12}$$

to be well-defined on $\mathcal{C}_\sim$. Indeed, a class of tensors applied at points q_λ belonging to the same light-ray $[q]$ must be related, as shown by Fig.B.1. Therefore, after embedding $\mathbb{M}^{d+2}$, define

$$T(q(\vec{\omega})) = \left(q^+\right)^{-\Delta+J} t(\vec{\omega}), \tag{B.2.13}$$

where $t(\vec{\omega})$ is, at this level, a d-dimensional field,

$$t(\vec{\omega}) = t_{a_1\dots a_J}\mathrm{d}\omega^{a_1}\otimes\dots\otimes\mathrm{d}\omega^{a_J}, \quad a_i = 1,\dots,d,\ \forall i = 1,\dots,J, \tag{B.2.14}$$

which must transform properly under the action of $SO(1,d+1)$. Obviously, the field $t(\vec{\omega})$ must have the same pattern of symmetries in the exchange of the indices $a_1\dots a_J$ as $T(q(\vec{\omega}))$ has in the exchange of $\mu_1\dots\mu_J$.

It is a straightforward calculation to verify that the tensor field in (B.2.13) satisfies the homogeneity condition (B.2.12). Indeed, in components, the (B.2.13) becomes

$$T_{\mu_1\dots\mu_J}(q(\vec{\omega}))\frac{\partial q^{\mu_1}}{\partial\omega^{a_1}}\dots\frac{\partial q^{\mu_J}}{\partial\omega^{a_J}} = \left(q^+\right)^{-\Delta+J} t_{a_1\dots a_J}(\vec{\omega}), \tag{B.2.15}$$

where, as before, $a_i = 1,\dots,d$ and $\mu_i = 0,\dots,d+1$, with $i = 1,\dots,J$. In order to write down the (B.2.15), the transformation property

$$\mathrm{d}q^{\mu_i}(\vec{\omega}) = \frac{\partial q^{\mu_i}(\vec{\omega})}{\partial\omega^{a_i}}\mathrm{d}\omega^{a_i} \tag{B.2.16}$$

has been used. Eq.(B.2.15) defines the pull-back of $T(q)$ onto the generic section of the future null-cone.

Since, by sending

$$q \longmapsto \lambda q, \quad \lambda\in\mathbb{R}^+, \tag{B.2.17}$$

the vector $\vec{\omega}$ is left unchanged; therefore, it is trivially verified that

$$\lambda^J T_{\mu_1\dots\mu_J}(\lambda q(\vec{\omega}))\frac{\partial q^{\mu_1}}{\partial\omega^{a_1}}\dots\frac{\partial q^{\mu_J}}{\partial\omega^{a_J}} = \lambda^{-\Delta+J}\left(q^+\right)^{-\Delta+J} t_{a_1\dots a_J}(\vec{\omega}), \tag{B.2.18}$$

from which

$$T_{\mu_1\dots\mu_J}(\lambda q(\vec{\omega}))\frac{\partial q^{\mu_1}}{\partial\omega^{a_1}}\dots\frac{\partial q^{\mu_J}}{\partial\omega^{a_J}} = \lambda^{-\Delta}\left(q^+\right)^{-\Delta+J} t_{a_1\dots a_J}(\vec{\omega}), \tag{B.2.19}$$

that is the homogeneity condition (B.2.12) rewritten on the coordinates of the tensor $T(q(\vec{\omega}))$.

Moreover, it is a straightforward calculation to demonstrate, in analogy to the Theorem B.1.1, that $T_{\mu_1\dots\mu_J}(\vec{\omega})$ transforms as a component of a rank-$(J,0)$ tensor field if and only if $t_{a_1\dots a_J}(\vec{\omega})$ transforms as a component of a conformal primary tensor field of rank $(J,0)$ [150].

Finally, restricting (B.2.15) on the Poincaré section $\mathcal{C}_P$, one finds the simple relation

$$t_{a_1\dots a_J}(\vec{\omega}) = \frac{\partial q^{\mu_1}}{\partial\omega^{a_1}}\dots\frac{\partial q^{\mu_J}}{\partial\omega^{a_J}} T_{\mu_1\dots\mu_J}\left(q(\vec{\omega})\right)\big|_{q\in\mathcal{C}_P} \tag{B.2.20}$$

The tensor components $T_{\mu_1\dots\mu_J}(q(\vec{\omega}))$ are said to be the *up-fifted tensor* components associated to the conformal primary field tensor components $t_{a_1\dots a_J}(\vec{\omega})$.

It is possible to note that the relation (B.2.15) is not a one-to-one correspondence. Indeed, consider the Minkowski tensor field $U(q)$, the components of which are

$$U_{\mu_1\ldots\mu_J}(q) = q_{\mu_i}\tilde{U}_{\mu_1\ldots\ \hat{\mu}_i\ldots\mu_J}(q), \tag{B.2.21}$$

where the hatted index $\hat{\mu}_i$ is meant to be missing. It is found that the field $U(q)$ is mapped into the null tensor by the correspondence (B.2.15). In fact, by using

$$\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} q_{\mu_i} = \frac{\partial\left(q^{\mu_i} q_{\mu_i}\right)}{\partial \omega^{a_i}} - q^{\mu_i}\frac{\partial q_{\mu_i}}{\partial \omega^{a_i}} = -\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} q_{\mu_i}, \tag{B.2.22}$$

from which

$$\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} q_{\mu_i} = 0, \tag{B.2.23}$$

it is immediate to verify that

$$\begin{aligned}\frac{\partial q^{\mu_1}}{\partial \omega^{a_1}} \cdots \frac{\partial q^{\mu_J}}{\partial \omega^{a_J}} U_{\mu_1\ldots\mu_J}(q) &= \frac{\partial q^{\mu_1}}{\partial \omega^{a_1}} \cdots \frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} \cdots \frac{\partial q^{\mu_J}}{\partial \omega^{a_J}} q_{\mu_i}\tilde{U}_{\mu_1\ldots\ \hat{\mu}_i\ldots\mu_J}(q) \\ &= \frac{\partial q^{\mu_1}}{\partial \omega^{a_1}} \cdots \left(\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} q_{\mu_i}\right) \cdots \frac{\partial q^{\mu_J}}{\partial \omega^{a_J}} \tilde{U}_{\mu_1\ldots\ \hat{\mu}_i\ldots\mu_J}(q) = 0.\end{aligned} \tag{B.2.24}$$

This means the kernel of the application (B.2.15) is not trivial.

Hence, becomes manifest a *gauge redundancy* in the correspondence (B.2.15)

$$t_{a_1\ldots a_J}(\vec{\omega}) \leftrightarrow T_{\mu_1\ldots\mu_J}(q(\vec{\omega})). \tag{B.2.25}$$

In particular, to the conformal primary tensor $t(\vec{\omega})$ corresponds an equivalence class of Minkowski tensors. Indeed, redefining

$$T_{\mu_1\ldots\mu_J}(q) \to T'_{\mu_1\ldots\mu_J}(q) = T_{\mu_1\ldots\mu_J}(q) + q^{\mu_i}\tilde{U}^{\mu_1\ldots\ \hat{\mu}_i\ldots\mu_J}(q), \tag{B.2.26}$$

where $\tilde{U}_{\mu_1\ldots\ \hat{\mu}_i\ldots\mu_J}(q)$ are the components of a generic rank-$(J-1,0)$ tensor, the new field $T'(q)$ is mapped into the same conformal primary tensor field $t(\vec{\omega})$ as $T(q)$, by (B.2.15). It will be seen that a gauge redundancy is present in the spinor case, too.

In summary, in order to map a rank-$(J,0)$ conformal primary tensor field $t(\vec{\omega})$ into a Minkowski tensor one $T(q)$, the components $T_{\mu_1\ldots\mu_J}(q)$ of $T(q)$, must satisfy the following properties: imposed $q^2=0$, they must [74, 120, 148]

1. Respect the *transversality condition*:

$$q^{\mu_k}T_{\mu_1\ldots\mu_k\ldots\mu_J} = 0, \quad \forall k = 1,\ldots,J;$$

2. Have *homogeneity degree* Δ with respect to q:

$$T_{\mu_1\ldots\mu_j}(\lambda q) = \lambda^{-\Delta} T_{\mu_1\ldots\mu_J}(q), \quad \lambda \in \mathbb{R}^+,\, \Delta \in \mathbb{R};$$

3. Be defined modulo tensors of the form

$$q_{\mu_i} \tilde{U}_{\mu_1\ldots\hat{\mu}_i\ldots\mu_J}(q);$$

4. Manifest, in the exchange of two or more of the indices $\mu_1,\ldots\mu_J$, the same pattern of symmetries as the components $t_{a_1\ldots a_j}$ have in the exchange of their indices.

Suppose, now, to want to construct the tensor two point conformal correlation function $G_{a_1\ldots a_j;b_1\ldots b_j}(\vec{\omega}_1,\vec{\omega}_2)$ between two rank-$(J,0)$ conformal primary tensor fields $t(\vec{\omega}_1)$ and $t(\vec{\omega}_2)$, of conformal weight Δ_1 and Δ_2, respectively. Let $t_{a_1\ldots a_J}(\vec{\omega}_1)$ and $t_{b_1\ldots b_J}(\vec{\omega}_2)$ be the components of $t(\vec{\omega}_1)$ and $t(\vec{\omega}_2)$. In formulae,

$$G_{a_1\ldots a_J;b_1\ldots b_J}(\vec{\omega}_1,\vec{\omega}_2) = \langle t_{a_1\ldots a_J}(\vec{\omega}_1)\, t_{b_1\ldots b_J}(\vec{\omega}_2)\rangle \tag{B.2.27}$$

Analogously to the scalar case, one can calculate the correlation function (B.2.27) by means of the embedding formalism. Specifically, one can compute, first, the up-lifted tensor two point correlation function

$$\mathcal{G}_{\mu_1\ldots\mu_j;\nu_1\ldots\nu_j}(q(\vec{\omega}_1),q(\vec{\omega}_2)) = \Big\langle T_{\mu_1\ldots\mu_J}(q(\vec{\omega}_1)) T_{\nu_1\ldots\nu_J}(q(\vec{\omega}_2))\Big\rangle, \tag{B.2.28}$$

where $T_{\mu_1\ldots\mu_J}(q(\vec{\omega}_1))$ and $T_{\nu_1\ldots\nu_J}(q(\vec{\omega}_2))$ are the components of the up-lifted tensors associated to $t(\vec{\omega}_1)$ and $t(\vec{\omega}_2)$, respectively. Once the up-lifted tensor correlation function (B.2.28) has been calculated, one has to project the result onto the Poincaré section by means of (B.0.2).

Calculating the tensor correlation function in (B.2.28) by relativistic considerations is more difficult than in the scalar case. It turns out to be helpful to use the so-called *free index formalism*, by means of which it is possible to express all the tensor quantities discussed so far in terms of scalar polynomials, which are simpler to treat.

Obviously, all the considerations made in this section for the rank-$(J,0)$ tensors can be immediately generalized to the rank-(m,n) tensors, with $m+n=J$, by means of the well-known isomorphism between $T_q^{(J,0)}\,\mathbb{R}^{1,d+1}$ and $T_q^{(m,n)}\,\mathbb{R}^{1,d+1}$, established via the metric tensor η.

B.3 Free Index Formalism for Up-lifted Tensor Fields

In the formalism that will be developed in this section, will be considered only rank-$(J,0)$ totally-symmetric and traceless tensor fields. This limitation, after all, does not affect the generality of the method; in fact, the covariant and irreducible conformal primary tensor fields of spin J are totally symmetric and traceless [75]. Moreover, the correspondence (B.2.15) maps the components of a conformal tensor field $t(\vec{\omega})$, which is traceless and totally symmetric, into a transverse tensor field $T(q)$, that is also traceless and totally symmetric. This can be seen considering the properties 1 and 4 of the summarising list above, and then noting that, in the Poincaré section,

$$K^{\mu_i\mu_k} = \delta^{a_i a_k} \frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} \frac{\partial q^{\mu_k}}{\partial \omega^{a_k}} = \eta^{\mu_i\mu_k} + q^{\mu_i}\bar{q}^{\mu_k} + q^{\mu_k}\bar{q}^{\mu_i}, \tag{B.3.1}$$

where

$$\bar{q} = \left(0, 2, \vec{0}\right)^T, \tag{B.3.2}$$

as can be verified by substituting [75]

$$\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}} = \left[\left(0, 2\vec{\omega}^{\,T}, \mathbb{1}\right)^T\right]^{\mu_i}_{a_i}. \tag{B.3.3}$$

Therefore, for a transverse tensor field, one finds

$$K^{\mu_i\mu_k} T_{\mu_1\ldots\mu_i\ldots\mu_k\ldots\mu_J}(q) = \eta^{\mu_i\mu_k} T_{\mu_1\ldots\mu_i\ldots\mu_k\ldots\mu_J}(q), \quad \forall\, i,k = 1,\ldots,J, \; i \neq k. \tag{B.3.4}$$

Then, contracting any two indices in (B.2.20), it follows that

$$\begin{aligned}
\delta^{a_i a_j} t_{a_1\ldots a_i\ldots a_k\ldots a_J}(\vec{\omega}) &= \\
&= \frac{\partial q^{\mu_1}}{\partial \omega^{a_1}} \cdots \widehat{\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}}} \cdots \widehat{\frac{\partial q^{\mu_k}}{\partial \omega^{a_k}}} \cdots \frac{\partial q^{\mu_J}}{\partial \omega^{a_J}} K^{\mu_i\mu_k} T_{\mu_1\ldots\mu_i\ldots\mu_k\ldots\mu_J}(q) \\
&= \frac{\partial q^{\mu_1}}{\partial \omega^{a_1}} \cdots \widehat{\frac{\partial q^{\mu_i}}{\partial \omega^{a_i}}} \cdots \widehat{\frac{\partial q^{\mu_k}}{\partial \omega^{a_k}}} \cdots \frac{\partial q^{\mu_J}}{\partial \omega^{a_J}} \eta^{\mu_i\mu_k} T_{\mu_1\ldots\mu_i\ldots\mu_k\ldots\mu_J}(q),
\end{aligned} \tag{B.3.5}$$

where the hatted ratios are missing. Eq. (B.3.5) states what was mentioned above: $t_{a_1\ldots a_J}$ is traceless if and only if $T_{\mu_1\ldots\mu_J}$ is.

Now, consider a totally symmetric tensor field $T(q)$, which, pointwise, is an element of the covariant symmetric algebra $T^\bullet_q\,\mathcal{C}^+$. As well-known, the covariant symmetric algebra $T^\bullet_q\,\mathcal{C}^+$ is isomorphic to the polynomial ring $\mathcal{K}[Z]$, where Z is the variable of the polynomials. In particular, contracting $T(q)$ J times with a constant vector field Z with components Z^μ, one can realize the mentioned isomorphism

$$i : T(q) \longmapsto T[\underbrace{Z,\ldots,Z}_{J-\text{times}}](q) = T_{\mu_1\ldots\mu_J}(q) Z^{\mu_1}\cdots Z^{\mu_J} \equiv T(q;Z) \in \mathcal{K}_q[Z]. \tag{B.3.6}$$

To encode the traceless information in the polynomial $T(q;Z)$, note that a traceless tensor $T(q)$ is mapped into an harmonic polynomial, by the isomorphism (B.3.6). Indeed, $T(q;Z)$ cannot admit powers of Z^{μ_i} greater than 1 for all $\mu_i = 1, \ldots, J$, since the contraction of any couple of indices vanishes because of the tracelessness condition.

Thus, it is possible to state the following theorem [151]:

Theorem B.3.1. *Let $p(Z)$ be a polynomial of a certain degree d in the Z. Then, $p(Z)$ can be uniquely decomposed as*

$$p(Z) = p_0(Z) + Z^2 p_1(Z), \tag{B.3.7}$$

where $p_0(Z)$ is harmonic.

Theorem B.3.1 implies that it is possible to encode the traceless information by restricting the image of the map (B.3.6) to the submanifold $Z^2 = 0$. In particular, there exists a one-to-one correspondence between

$$T_{\mu_1 \ldots \mu_j}(q) \leftrightarrow T(q;Z)|_{Z^2=0}, \tag{B.3.8}$$

where $T_{\mu_1 \ldots \mu_J}(q)$ are the components of a traceless, totally-symmetric, covariant tensor field $T(q)$:

$$\eta^{\mu_i \mu_k} T_{\mu_1 \ldots \mu_i \ldots \mu_k \ldots \mu_j} = 0, \quad \forall i, k = 1, \ldots, J;\ i \neq k. \tag{B.3.9}$$

Moreover, from Theorem $B.3.1$, it follows also that every polynomial $U(q;Z)$, associated to a totally symmetric covariant tensor $U(q)$, can be uniquely decomposed as

$$U(q;Z) = T(q;Z) + O(z^2), \tag{B.3.10}$$

where $T(q;Z)$ is an element of $\mathcal{K}_q[Z]|_{Z^2=0}$. Therefore, starting from a totally symmetric covariant tensor $U(q)$, one can always find, first, the polynomial $U(q;Z)$ via (B.3.6) and then obtain the harmonic polynomial $T(q;Z)$, which holds the "physical interest", by dropping terms proportional to Z^2.

To recover the tensor fields $T(q)$ from the associated polynomial $T(q;Z)$, it is possible to use the (flat) *Thomas operator* [74, 148, 152]

$$D_\mu = \left(\frac{d-2}{2} + Z \cdot \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z^\mu} - \frac{1}{2} Z^\mu \frac{\partial^2}{\partial Z \cdot \partial Z}, \tag{B.3.11}$$

which is a conformally invariant, second-order differential operator. It satisfies

$$D_\mu D^\mu = 0, \tag{B.3.12}$$

and, in the flat case [78],

$$[D^\mu, D^\nu] = 0. \tag{B.3.13}$$

The properties (B.3.12) and (B.3.13) suggest that one constructs the right tensor by applying J times the Thomas operator. In fact,

$$D_{\mu_1}\cdots D_{\mu_J}T(q;Z), \tag{B.3.14}$$

is totally symmetric and traceless. In particular, acting with D on the harmonic polynomial $T(q;Z)$, one finds

$$D_\mu T(q;Z) \equiv \left(\frac{d-2}{2}+Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z^\mu}T(q;Z), \tag{B.3.15}$$

since $T(q;Z)$ is harmonic. Moreover

$$\begin{aligned}
&\left(\frac{d-2}{2}+Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z^\mu}T(q;Z) = \\
&= \left(\frac{d-2}{2}+Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z^\mu}T_{\mu_1\ldots\mu_J}(q)Z^{\mu_1}\ldots Z^{\mu_J} \\
&= \left(\frac{d-2}{2}+Z\cdot\frac{\partial}{\partial Z}\right)\sum_{i=1}^{J}T_{\mu_1\ldots\mu_i\ldots\mu_J}(q)\delta^{\mu_i}_{\mu}Z^{\mu_1}\ldots\widehat{Z^{\mu_i}}\ldots Z^{\mu_J} \\
&= J\left(\frac{d-2}{2}+Z\cdot\frac{\partial}{\partial Z}\right)T_{\mu\mu_2\ldots\mu_J}Z^{\mu_2}\ldots Z^{\mu_J} \\
&= J\left(\frac{d-2}{2}+J-1\right)T_{\mu\mu_2\ldots\mu_J}(q)Z^{\mu_2}\ldots Z^{\mu_J},
\end{aligned} \tag{B.3.16}$$

where the symmetry properties of $T_{\mu_1\ldots\mu_J}(q)$ and

$$Z^\nu\frac{\partial}{\partial Z^\nu}Z^{\mu_2}\cdots Z^{\mu_J} = \sum_{r=2}^{J}Z^\nu\delta^{\mu_r}_{\nu}Z^{\mu_2}\cdots Z^{\mu_J} = (J-1)Z^{\mu_2}\cdots Z^{\mu_J}, \tag{B.3.17}$$

have been used. Iterating the method, one finds

$$\begin{aligned}
D_{\mu_1}\cdots D_{\mu_J}T(q;Z) &= J!\left(\frac{d-2}{2}+J-1\right)\cdots\left(\frac{d-2}{2}\right)T_{\mu_1\ldots\mu_J}(q) \\
&= J!\left(\frac{d-2}{2}\right)_J T_{\mu_1\ldots\mu_J}(q),
\end{aligned} \tag{B.3.18}$$

where $\left(\frac{d-2}{2}\right)_J$ is the Pochhammer symbol of $\frac{d-2}{2}$, defined as

$$(a)_J = \frac{\Gamma(a+J)}{\Gamma(a)}, \quad a, J \in \mathbb{R}. \tag{B.3.19}$$

Therefore,

$$T_{\mu_1\ldots\mu_J}(q) = \frac{1}{J!\left(\frac{d-2}{2}\right)_J}D_{\mu_1}\cdots D_{\mu_J}T(q;Z). \tag{B.3.20}$$

Moreover, if one considers the polynomial field $U(q; Z) \in \mathcal{K}_q[Z]$ associated to the totally symmetric covariant tensor field $U(q)$, with components $U_{\mu_1 \ldots \mu_J}(q)$, one finds

$$D_{\mu_1} \cdots D_{\mu_J} U(q; Z) = D_{\mu_1} \cdots D_{\mu_J} \left(T(q; Z) + O(z^2) \right), \tag{B.3.21}$$

where Theorem B.3.1 has been used. Another property of the Thomas operator D is the following: given a function $h(Z)$, such that $h(Z) = O(Z^2)$, then

$$D_\mu h(Z^2) = O(Z^2). \tag{B.3.22}$$

Therefore, inserting (B.3.22) and (B.3.20) in (B.3.21), it is possible to write down

$$\frac{1}{J! \left(\frac{d-2}{2}\right)_J} D_{\mu_1} \cdots D_{\mu_J} U(q; Z) = T_{\mu_1 \ldots \mu_J} + O(Z^2). \tag{B.3.23}$$

Hence, once again, it is possible to recover the quantities of physical interest by dropping the Z^2-terms.

Next, one has to encode the transversality condition (B.2.11). From (B.3.20) it follows that it is possible to encode the transversality condition by imposing

$$q^\mu D_\mu = 0 \Leftrightarrow \begin{cases} q^\mu \dfrac{\partial}{\partial Z^\mu} T(q; Z) = 0, \\ \qquad\qquad q^\mu Z_\mu = 0. \end{cases} \tag{B.3.24}$$

The second condition in (B.3.24) imposes to restrict, further, the correspondence (B.3.6) to the submanifold

$$\begin{cases} \quad Z^2 = 0, \\ q \cdot Z = 0. \end{cases} \tag{B.3.25}$$

Hence, to a transverse, traceless and totally symmetric covariant tensor field $T(q)$ corresponds a unique polynomial $T(q; Z)$ on the submanifold defined by (B.3.25). Specifically [75],

$$T(q) \leftrightarrow T(q; Z)|_{Z^2=0,\, q \cdot Z=0}, \tag{B.3.26}$$

is a one-to-one correspondence up to terms of *pure gauge* like those in (B.2.21).

Further, the first condition in (B.3.24) can be rewritten as

$$T(q; Z) = T(q; Z + \alpha q), \quad \forall\, \alpha \in \mathbb{R}. \tag{B.3.27}$$

Indeed, expanding around $\alpha = 0$, one has

$$\begin{aligned} T(q;Z+\alpha q) = T(q;Z) + \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} \left(q \cdot \frac{\partial}{\partial Z} T(q;Z) \right)^n = T(q;Z) \\ \Leftrightarrow q \cdot \frac{\partial}{\partial Z} T(q;Z) = 0. \end{aligned} \tag{B.3.28}$$

Obviously, this relation is valid modulo q^2 terms.

Summarising, given a transverse, totally symmetric and covariant tensor field $T(q)$, $q \in \mathcal{C}^+$, of rank (J,0) and such that $T(q\lambda) = \lambda^{-\Delta} T(q)$, it is possible to associate to $T(q)$ a polynomial field $T(q;Z)$ satisfying the following properties [75]:

1. $T(q;Z)$ is defined on the submanifold
$$\begin{cases} Z^2 = 0, \\ q \cdot Z = 0; \end{cases}$$

2. $T(q;Z)$ is such that
$$T(q;Z) = T(q;Z+\alpha q),$$
for all $\alpha \in \mathbb{R}$, modulo $q^2 = 0$ terms;

3. $T(q;Z)$ has homogeneity degree equal to Δ with respect to q, and equal to J with respect to Z. In formulae:
$$T(\lambda q; \mu Z) = \lambda^{-\Delta} \mu^J T(q;Z), \quad \lambda, \mu \in \mathbb{R}^+.$$

Now, consider two covariant, traceless and totally symmetric fields $T(q_1)$ and $T(q_2)$ of homogeneity degree Δ, both in q_1 and in q_2. In the free index formalism, the two-point correlation function between $T(q_1)$ and $T(q_2)$ is the two point polynomial field

$$\mathcal{G}(q_1, q_2; Z_1, Z_2) = \mathcal{G}_{\mu_1 \dots \mu_J; \nu_1 \dots \nu_J} Z_1^{\mu_1} \cdots Z_1^{\mu_J} Z_2^{\nu_1} \cdots Z_2^{\nu_J}, \tag{B.3.29}$$

satisfying the following property:

$$\begin{aligned} &\mathcal{G}(\lambda_1 q_1, \lambda_2 q_2; \eta_1 Z_1 + \alpha_1 q_1, \eta_2 Z_2 + \alpha_2 q_2) \\ &\qquad = (\lambda_1 \lambda_2)^{-\Delta} (\eta_1 \eta_2)^J \mathcal{G}^{(\Delta)}(q_1, q_2; Z_1, Z_2), \end{aligned} \tag{B.3.30}$$

where $\alpha_i, \in \mathbb{R}$ and $\eta_i, \lambda_i \in \mathbb{R}^+$, with $i = 1, 2$. Eq. (B.3.30) resumes all the properties discussed above. As a consequence of the discussion so far, in the construction of $\mathcal{G}(q_1, q_2; Z_1, Z_2)$, one has to drop all the terms in q_i^2, Z_i^2 and $q_i \cdot Z_i$, with $i = 1, 2$.

Hence, in order to determine the shape of $\mathcal{G}(q_1, q_2; Z_1, Z_2)$, one can use nothing else than the tensor [75]

$$C_{i\mu\nu} = Z_{i\mu} q_{i\nu} - Z_{i\nu} q_{i\mu}, \tag{B.3.31}$$

where the minus sign will be justified below. If one contracts $C_{i\mu\nu}$ with itself, with i fixed, one then finds only terms that must be dropped. Therefore, considering the contraction between $C_{1\mu\nu}$ and $C_{2\mu\nu}$; this gives the scalar

$$H_{12} = -C_1 \cdot C_2 = -2 \left[(Z_1 \cdot Z_2)(q_1 \cdot q_2) - (Z_1 \cdot q_2)(Z_2 \cdot q_1) \right], \tag{B.3.32}$$

which has homogeneity degree equal to 1 with respect to Z_1, Z_2, q_1 and q_2. It is possible to note that in (B.3.32) the minus sign is essential to ensure the invariance of H_{12} with respect to

$$Z_i \longmapsto Z_i + \alpha_i q_i, \tag{B.3.33}$$

as required. Moreover, it is possible to check that [75]

$$(C_1 C_2 C_1)_{\mu\nu} = -\frac{1}{2} (C_1 \cdot C_2)\, C_{1\mu\nu}, \tag{B.3.34}$$

from which one can assert that iterated contractions of C_1 and C_2 reduce to functions of $C_1 \cdot C_2$. Hence, there are no other independent scalar besides the ones present in $C_1 \cdot C_2$. Finally, since $\mathcal{G}(q_1, q_2; Z_1, Z_2)$ must have homogeneity degrees J and Δ in the variables Z_i and q_i, respectively, it follows that the two-point correlation function must be of the form [75]

$$\mathcal{G}(q_1, q_2; Z_1, Z_2) = c \frac{H_{12}^{\Delta + J}}{(-2 q_1 \cdot q_2)^J}, \tag{B.3.35}$$

where c is a constant. From $\mathcal{G}(q_1, q_2; Z_1, Z_2)$ it is possible to recover the tensorial two point correlation function $\mathcal{G}_{\mu_1 \ldots \mu_J; \nu_1 \ldots \nu_J}(q_1, q_2)$ by applying J-times $D_\mu(Z_1)$ and $D_\nu(Z_2)$.

Finally, it is possible to compute the-two point conformal correlation function $G_{a_1 \ldots a_J; b_1 \ldots b_J}(\vec{\omega}_1, \vec{\omega}_2)$ in (B.2.27), by inserting the $\vec{\omega}$-dependence into q_i and projecting $\mathcal{G}_{\mu_1 \ldots \mu_J; \nu_1 \ldots \nu_J}(q_1, q_2)$ onto the Poincaré section via the correspondence (B.2.20).

The construction described so far can result cumbersome for high spin tensorial field, but it is analytically well-defined. Henceforth, the aim is to generalize the method exposed for tensorial field defined on the null-cone $\mathcal{C}$. The main purpose is to compute the general form of the so-called tensorial bulk-to-boundary propagators, which are kinds of two point tensorial correlation functions that correlate a tensorial field defined on the null-cone $\mathcal{C}$ with one taking values on the hyperboloid $\mathbb{H}_{d+1}$.

Bulk-to-boundary propagator construction: Consider the $(d+1)$-dimensional hyperboloid $\mathbb{H}_{d+1}$ defined by the embedding relation

$$\hat{p}^2 = -1, \quad \hat{p} \in \mathbb{R}^{1,d+1} \tag{B.3.36}$$

Consider, then, a covariant tensor field $H(\hat{p})$ which pointwise belongs to $T_{\hat{p}}^{(J,0)}\mathbb{R}^{1,d+1}$. Let $H_{\mu_1\ldots\mu_J}(\hat{p})$ be the components of $H(\hat{p})$; the purpose is to project the tensor field $H(\hat{p})$ onto $T_{\hat{p}}^{(J,0)}\mathbb{H}_{d+1}$, as already done so far in projecting Minkowski tensors onto the future null-cone $\mathcal{C}^+$. The technique is very similar: it is possible to define $\mathbb{H}_{d+1}$ by means of the function $g(\hat{p}) = \hat{p}^2$; precisely

$$\mathbb{H}_{d+1} = g^{-1}(-1), \tag{B.3.37}$$

where g^{-1} is the preimage of the function g. Therefore, using the same method already exposed in the previous section for the null-cone $\mathcal{C}$, (B.3.37) allows to write the following *transverality condition* for the covariant tensor field $H(\hat{p})$:

$$\hat{p}^{\mu_i} H_{\mu_1\ldots\mu_i\ldots\mu_J} = 0, \quad \forall\, i = 1, \ldots, d. \tag{B.3.38}$$

A tensor field that respects the transversality condition (B.3.38) is said to be *transverse* to $\mathbb{H}_{d+1}$. Hence, it is possible to assert that the covariant tensor field $H(\hat{p})$ valued at $\hat{p} \in \mathbb{H}_{d+1}$ belongs to $T_{\hat{p}}^{(J,0)}\mathbb{H}_{d+1}$ if and only if its components satisfy the condition (B.3.38).

Eq. (B.3.38) is essentially equivalent to (B.2.11), but there is a major difference in their implications: because $\hat{p}^2 = -1$, rather than zero as happens for $q \in \mathcal{C}$, the fields the components of which are proportional to $\hat{p}$, such as

$$I_{\mu_1\ldots\mu_J} = \hat{p}_{\mu_i} \tilde{I}_{\mu_1\ldots\hat{\mu}_i\ldots\mu_J}, \quad i = 1\ldots, d, \tag{B.3.39}$$

where the index $\hat{\mu}_i$ is missing, cannot belong to $T_{\hat{p}}^{(J,0)}\mathbb{H}_{d+1}$, since they are not transverse.

Now, consider $H(\hat{p})$ to be totally-symmetric and traceless. Then, by using the free index notation, it is possible to associate to $H(\hat{p})$ the polynomial field

$$H(\hat{p}; W) = H_{\mu_1\ldots\mu_J} W^{\mu_1} \cdots W^{\mu_J}. \tag{B.3.40}$$

Next, to encode the tracelessness condition, one requires, again, that the polarization vector W must be such that $W^2 = 0$. Finally, to encode the transversality condition one imposes $\hat{p} \cdot W = 0$, as before.

To recover the original tensorial field $H(\hat{p})$ from its associated polynomial, it is necessary to construct a differential operator K, which is the bulk counter part of the Thomas operator D defined in (B.3.11). Therefore, the aim is to construct K in such a way that it is possible to recover the components of the tensorial field $H(\hat{p})$ by applying K J-times:

$$H_{\mu_1\ldots\mu_J} = c K_{\mu_1} \cdots K_{\mu_J} H(\hat{p}; W), \tag{B.3.41}$$

where c is a constant. In order to guarantee the right properties to the components of the tensor field $H(\hat{p})$, obtained by (B.3.41), the operator K must be

- traceless:
$$K^\mu K_\mu = 0; \tag{B.3.42}$$

- symmetric:
$$K_\mu K_\nu = K_\nu K_\mu; \tag{B.3.43}$$

- transverse
$$\hat{p}^\mu K_\mu = 0. \tag{B.3.44}$$

Imposing the above constraints, the differential operator K can be constructed as follows [75]:

$$\begin{aligned}K_\mu = &\frac{d-1}{2}\left[\frac{\partial}{\partial W^\mu} + \hat{p}_\mu\left(\hat{p}\cdot\frac{\partial}{\partial W}\right)\right] + \left(W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^\mu} + \hat{p}_\mu\left(W\frac{\partial}{\partial W}\right)\left(\hat{p}\cdot\frac{\partial}{\partial W}\right)\\ &-\frac{1}{2}W_\mu\left[\frac{\partial^2}{\partial W\cdot\partial W} + \left(\hat{p}\cdot\frac{\partial}{\partial W}\right)\left(\hat{p}\cdot\frac{\partial}{\partial W}\right)\right].\end{aligned} \tag{B.3.45}$$

Since the polynomial associated to a traceless field is harmonic, the action of K on $H(\hat{p};W)$ reduces to

$$\begin{aligned}K_\mu H(\hat{p},W) = &\frac{d-1}{2}\left[\frac{\partial}{\partial W^\mu} + \hat{p}_\mu\left(\hat{p}\cdot\frac{\partial}{\partial W}\right)\right] +\\ &+\left(W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^\mu}H(\hat{p},W).\end{aligned} \tag{B.3.46}$$

Moreover, the term

$$\hat{p}_\mu\left(\hat{p}\cdot\frac{\partial}{\partial W}\right)H(\hat{p};W) = (J-1)\hat{p}_\mu\hat{p}^\nu H_{\nu\mu_2\ldots\mu_J}W^{\mu_2}\cdots W^{\mu_J}, \tag{B.3.47}$$

vanishes because of the transversality condition (B.3.38).

Therefore, the action of K_μ on $H(\hat{p})$ becomes

$$K_\mu H(\hat{p};W) = \left(\frac{d-1}{2} + W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^\mu}H(\hat{p};W), \tag{B.3.48}$$

from which one finds

$$H_{\mu_1\ldots\mu_J}(\hat{p}) = \frac{1}{J!(\frac{d-1}{2})_J}K_{\mu_1}\cdots K_{\mu_J}H(\hat{p};W), \tag{B.3.49}$$

similar to (B.3.20).

Finally, consider a covariant, totally-symmetric and traceless conformal primary tensor field $t(\vec{\omega})$, with $\vec{\omega} \in \mathbb{R}^d$, of conformal weight Δ. Let $t_{a_i \dots a_J}(\vec{\omega})$ be the components of $t(\vec{\omega})$; then, let $H(\hat{p})$ be the totally-symmetric and traceless covariant tensor field of above, with components $H_{\mu_1 \dots \mu_J}(\hat{p})$. The bulk-to-boundary propagator $\Pi_{a_1 \dots a_J; \mu_1 \dots \mu_J}(\hat{p}; \vec{\omega})$, between the fields $t(\vec{\omega})$ and $H(\hat{p})$, is defined as the tensor two-point function

$$\Pi_{a_1 \dots a_J; \mu_1 \dots \mu_J}(\hat{p}; \vec{\omega}) = \langle H_{\mu_1 \dots \mu_J}(\hat{p}) t_{a_i \dots a_J}(\vec{\omega}) \rangle. \tag{B.3.50}$$

As in the previous section, to find the bulk-to-boundary propagator in (B.3.50), it is possible to compute, first, the associated up-lifted bulk-to-boundary propagator

$$\Pi_{\nu_1 \dots \nu_J; \mu_1 \dots \mu_J}(\hat{p}; q(\vec{\omega})) = \Big\langle H_{\mu_1 \dots \mu_J}(\hat{p}) T_{\nu_i \dots \nu_J}(q(\vec{\omega})) \Big\rangle, \tag{B.3.51}$$

where $T_{\nu_1 \dots \nu_J}(q(\vec{\omega}))$ are the components of the up-lifted tensor field, of homogeneity degree Δ, associated to $t(\vec{\omega})$ by the correspondence (B.2.15). The form of the bulk-to-boundary propagator (B.3.51) can be determined exploiting the free index formalism. In fact, one can map $\Pi_{\nu_1 \dots \nu_J; \mu_1 \dots \mu_J}(\hat{p}; q(\vec{\omega}))$ to a polynomial field by means of the polarization vectors Z and W:

$$\Pi_{\Delta, J}(\hat{p}, q(\vec{\omega}); W, Z) = \langle H(\hat{p}; W) T(q(\vec{\omega}; Z)) \rangle. \tag{B.3.52}$$

In particular, summarising the whole discussion made so far, one has to impose that the polynomial field $\Pi_{\Delta, J}(\hat{p}; q(\vec{\omega}))$ respects the following property:

$$\Pi_{\Delta, J}(\hat{p}; \alpha q(\vec{\omega}); \beta Z + \lambda q(\vec{\omega}), W) = \lambda^{-\Delta} \beta^J \Pi_{\Delta, J}(\hat{p}; q(\vec{\omega}); Z, W), \tag{B.3.53}$$

while the polarization vectors must be such that

$$\begin{aligned} &W^2 = Z^2 = 0, \\ &\hat{p} \cdot W = q(\vec{\omega}) \cdot Z = 0. \end{aligned} \tag{B.3.54}$$

By an argument similar to the one made for the two point correlation function in the free index notation, it is possible to assert that the spin J free index bulk-to-boundary propagator has the general form [75]

$$\Pi_{\Delta, J}(\hat{p}; q(\vec{\omega}); Z, W) = c \frac{\Big(-2(\hat{p} \cdot Z)(q(\vec{\omega}) \cdot W) + 2(\hat{p} \cdot q(\vec{\omega}))(Z \cdot W) \Big)^J}{\big(-2\hat{p} \cdot q(\vec{\omega}) \big)^{\Delta + J}} \tag{B.3.55}$$

From (B.3.55), it is possible to recover the up-lifted bulk-to-boundary propagator in (B.3.51) by applying J times the operators $K_\mu(W)$ and $D_\mu(Z)$ to $\Pi_{\Delta, J}(\hat{p}; q(\vec{\omega}); Z, W)$. Finally, one finds the bulk-to-boundary propagator $\Pi_{(\Delta)\ \nu_1 \dots \nu_J; \mu_1 \dots \mu_J}(\hat{p}; q(\vec{\omega}))$ defined in (B.3.50) by projecting the result onto the Poincaré section via (B.2.20).

B.4 Embedding Formalism for Spinors

In this subsection the embedding formalism for conformal spinors will be developed. The treatment will mainly follow [152].

Let $\psi(\vec{\omega})$, with $\vec{\omega} \in \mathbb{R}^d$, be a k-dimensional conformal primary spinor and $\Psi(q)$, $q \in \mathcal{C}^+$, be a Minkowski one, which satisfies the *homogeneity condition*

$$\Psi(\lambda q) = \lambda^{-\Delta-\frac{1}{2}}\Psi(q), \quad \lambda \in \mathbb{R}^+. \tag{B.4.1}$$

The choice $-\Delta - \frac{1}{2}$ will be justified below. Then, consider the gamma matrices $\gamma_a \in \mathcal{C}\ell_d(\gamma, \delta)$ and $\Gamma_\mu \in \mathcal{C}\ell_{d+2}(\Gamma, \eta)$ represented as in (A.3.16).

The purpose is to find a correspondence between the conformal primary spinor $\psi(\vec{\omega})$ and $\Psi(q)$.

To do so, it should be noticed that the representation (A.3.16) of the Clifford algebra $\mathcal{C}\ell_{d+2}(\Gamma, \eta)$ suggests to decompose $\Psi(q)$ into the direct sum of two k-dimensional spinors belonging to an irreducible representation of $\mathcal{C}\ell_d(\gamma, \delta)$. Specifically, it is possible to write

$$\Psi(q) = \begin{pmatrix} \Psi_+(q) \\ \Psi_-(q) \end{pmatrix}. \tag{B.4.2}$$

Hence, by the embedding relation (2.3.8), it is possible to associate to $\Psi_\pm(q)$, respectively, two $SO(d)$ spinors, namely $\psi_\pm(\vec{\omega})$, taking values on $\mathbb{R}^d$. Explicitly,

$$\psi_\pm(\vec{\omega}) = (q^+)^{\Delta+\frac{1}{2}}\Psi_\pm\big(q(\vec{\omega})\big). \tag{B.4.3}$$

The factor $(q^+)^{\Delta+\frac{1}{2}}$ is necessary to guarantee the homogeneity condition (B.4.1) to be fulfilled by $\Psi_\pm\big(q(\vec{\omega})\big)$ and hence by $\Psi\big(q(\vec{\omega})\big)$, as well. Indeed, since $\vec{\omega}$ does not depend on λ, by sending $q \longmapsto \lambda q$, one finds

$$\Psi_\pm\big(\lambda q(\vec{\omega})\big) = \lambda^{-\Delta-\frac{1}{2}}(q^+)^{-\Delta-\frac{1}{2}}\psi(\vec{\omega}) = \lambda^{-\Delta-\frac{1}{2}}\Psi_\pm\big(q(\vec{\omega})\big). \tag{B.4.4}$$

The two spinors $\psi_\pm(\vec{\omega})$ are not conformal primary ones yet, but the missing link is established in the following theorem.

Theorem B.4.1. *Let $\Psi_\pm(q)$ be the k-dimensional components of a $2k$-dimensional spinor $\Psi(q)$, as in* (B.4.2)*, which satisfies the homogeneity condition* (B.4.1)*. If $\psi_\pm(\vec{\omega})$ are the two k-dimensional spinors defined by* (B.4.3)*, then the linear combination*

$$\psi(\vec{\omega}) = \psi_+(\vec{\omega}) - \not{\omega}\psi_-(\vec{\omega}) \tag{B.4.5}$$

is a conformal primary spinor of conformal weight $\Delta + \frac{1}{2}$.

Proof. As asserted above, the finite action of $SO(1, d+1)$ on conformal primary spinors is completely determined by computing the transformation laws under d-dimensional translations, d-dimensional rotations, dilations and inversion.

First of all, remember that, as stated by Lemma$B.1.1$, under the action of the conformal group $SO(1,d+1)$, $q(\vec{\omega})$ transforms as follows:

$$q^{\mu}(\vec{\omega}\,') = \left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{\frac{1}{d}} \Lambda^{\mu}{}_{\nu} q^{\nu}(\vec{\omega}), \quad \Lambda \in SO(1,d+1). \tag{B.4.6}$$

Therefore, let $S(\Lambda)$ be the representative operator of $\Lambda \in SO(1,d+1)$ in the space of the $2k$-dimensional Minkowski spinors, it follows that

$$\begin{aligned}\Psi'(q(\vec{\omega}\,')) &= \Psi'\left(|J|^{\frac{1}{d}}\,\Lambda q(\vec{\omega})\right) \\ &= |J|^{\frac{-1/2-\Delta}{d}}\,\Psi'\left(\Lambda q(\vec{\omega})\,\right) = \left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{\frac{-1/2-\Delta}{d}} S(\Lambda)\Psi\left(q(\vec{\omega})\,\right),\end{aligned} \tag{B.4.7}$$

where (B.4.1) has been used and $|J|$ is the Jacobian determinant (cf.(B.1.8)). Then,

$$\begin{aligned}\psi'_{\pm}(\vec{\omega}\,') &= (q^{+})^{\Delta+\frac{1}{2}}\Psi'_{\pm}(q(\vec{\omega}\,')) \\ &= \left(q^{+}\right)^{\Delta+\frac{1}{2}}\left|\frac{\partial\vec{\omega}\,'}{\partial\vec{\omega}}\right|^{\frac{-1/2-\Delta}{d}} S_{\pm}(\Lambda)\Psi_{\pm}\left(q(\vec{\omega})\,\right),\end{aligned} \tag{B.4.8}$$

where $S_{\pm}(\Lambda)$ is the representative operator of Λ in the k-dimensional space of $\Psi_{\pm}$. Specifically, consider, first, the d-dimensional translation Λ_{T}. The representative operator $S(\Lambda_{\mathrm{T}})$ acting on Ψ, is the matrix (A.3.23).

In particular

$$S\left(\Lambda_{\mathrm{T}}\right)\Psi(q(\vec{\omega})) = \begin{pmatrix} \Psi_{+}(q(\vec{\omega})) + \not{b}\Psi_{-}(q(\vec{\omega})) \\ \Psi_{-}(q(\vec{\omega})) \end{pmatrix}, \tag{B.4.9}$$

which means that

$$\begin{aligned}S_{+}(\Lambda)\Psi_{+}(q(\vec{\omega})\,) &= \Psi_{+}(q(\vec{\omega})) + \not{b}\Psi_{-}(q(\vec{\omega})), \\ S_{-}(\Lambda)\Psi_{-}(q(\vec{\omega})\,) &= \Psi_{-}(q(\vec{\omega})\,).\end{aligned} \tag{B.4.10}$$

Therefore, because for translations $|J|=1$, one finds

$$\begin{aligned}\psi'(\vec{\omega}\,') =& (q^{+})^{\Delta+\frac{1}{2}}\left[S_{+}(\Lambda)\Psi_{+}(q(\vec{\omega})\,) - (\not{\omega} + \not{b})\left(S_{-}(\Lambda)\Psi_{-}(q(\vec{\omega})\,)\right)\right] \\ =& (q^{+})^{\Delta+\frac{1}{2}}\left[\Psi_{+}(q(\vec{\omega})) + \not{b}\Psi_{-}(q(\vec{\omega}))\right] - (q^{+})^{\Delta+\frac{1}{2}}\left[(\not{\omega} + \not{b})\,\Psi_{-}(q(\vec{\omega}))\right] \\ =& (q^{+})^{\Delta+\frac{1}{2}}\left[\Psi_{+}(q(\vec{\omega})) + \not{b}\Psi_{-}(q(\vec{\omega}))\right] \\ & - (q^{+})^{\Delta+\frac{1}{2}}\not{\omega}\Psi_{-}(q(\vec{\omega})) + (q^{+})^{\Delta+\frac{1}{2}}\not{b}\Psi_{-}(q(\vec{\omega})) \\ =& (q^{+})^{\Delta+\frac{1}{2}}\Psi_{+}(q(\vec{\omega})) - (q^{+})^{\Delta+\frac{1}{2}}\not{\omega}\Psi_{-}(q(\vec{\omega})) = \psi(\vec{\omega}).\end{aligned} \tag{B.4.11}$$

Under d-dimensional rotations the demonstration is trivial. Indeed, since Ψ has been decomposed into the direct sum of two $SO(d)$ spinors, it is always possible to represent the operator $S(\Lambda_\mathrm{R})$ by (A.3.17). Therefore, since once again $|J| = 1$,

$$\begin{aligned} \psi'(\vec{\omega}\,') &= (q^+)^{\Delta+\frac{1}{2}} S_d(\Lambda_\mathrm{R}) \Psi_+\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}} \not{\omega} S_d(\Lambda_\mathrm{R}) \Psi_-\big(q(\vec{\omega})\big) \\ &= S_d(\Lambda_\mathrm{R}) \psi(\vec{\omega}), \end{aligned} \tag{B.4.12}$$

exactly as a a conformal primary spinor should transform under d-dimensional rotations.

Consider, further, a dilation. The representative matrix $S(\Lambda_\mathrm{D})$ is (A.3.20). Moreover $|J| = \mathrm{e}^c$, and

$$\begin{aligned} \psi'(\mathrm{e}^c \vec{\omega}) &= -(q^+)^{\Delta+\frac{1}{2}} \mathrm{e}^{\left(-\frac{1}{2}-\Delta\right)c} \mathrm{e}^{c/2} \Psi_+\big(q(\vec{\omega})\big) \\ &\quad - (q^+)^{\Delta+\frac{1}{2}} \mathrm{e}^{\left(-\frac{1}{2}-\Delta\right)c} (\mathrm{e}^c \not{\omega}) \mathrm{e}^{-c/2} \Psi_-\big(q(\vec{\omega})\big) \\ &= (\mathrm{e}^c)^{-\Delta} \Big[(q^+)^{\Delta+\frac{1}{2}} \Psi_+\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}} \not{\omega} \Psi_-\big(q(\vec{\omega})\big) \Big] \\ &= (\mathrm{e}^c)^{-\Delta}\, \psi(\vec{\omega}). \end{aligned} \tag{B.4.13}$$

Finally, consider the inversion I. The matrix representing the inversion in the $2k$-dimensional Minkowski spinor space is given by (A.3.25). It should be noticed that $S(\Lambda_\mathrm{I})$ exchanges the spinors Ψ_+ and Ψ_-:

$$\begin{aligned} \Psi_+ &\longmapsto \Psi_-, \\ \Psi_- &\longmapsto \Psi_+. \end{aligned} \tag{B.4.14}$$

For the inversion, the Jacobian is $|J| = |\vec{\omega}|^{-2d}$, as stated by (B.1.25). Therefore,

$$\begin{aligned} \psi'(\vec{\omega}\,') &= (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{1+2\Delta}\, \Psi_-\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{1+2\Delta}\, \frac{\not{\omega}}{|\vec{\omega}|^2} \Psi_+\big(q(\vec{\omega})\big) \\ &= (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{1+2\Delta}\, \frac{|\vec{\omega}|^2}{|\vec{\omega}|^2} \Psi_-\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{1+2\Delta}\, \frac{\not{\omega}}{|\vec{\omega}|^2} \Psi_+\big(q(\vec{\omega})\big) \\ &= (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{2\Delta}\, \not{\omega} \frac{\not{\omega}}{|\vec{\omega}|} \Psi_-\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}}\, |\vec{\omega}|^{2\Delta}\, \frac{\not{\omega}}{|\vec{\omega}|} \Psi_+\big(q(\vec{\omega})\big) \\ &= -\, |\vec{\omega}|^{2\Delta}\, \frac{\not{\omega}}{|\vec{\omega}|} \Big[(q^+)^{\Delta+\frac{1}{2}} \Psi_+\big(q(\vec{\omega})\big) - (q^+)^{\Delta+\frac{1}{2}} \not{\omega} \Psi_-\big(q(\vec{\omega})\big) \Big] \\ &= -|\vec{\omega}|^{2\Delta} \frac{\not{\omega}}{|\vec{\omega}|} \psi(\vec{\omega}), \end{aligned} \tag{B.4.15}$$

in agreement with (A.2.29). □

It is useful to consider the free index formalism to encode what has been stated so far in a straightforward way.

Given a Minkowski spinor S, the adjoint $\bar{S}$ of S is defined as

$$\bar{S} = S^\dagger \Gamma^0, \tag{B.4.16}$$

while, for an Euclidean spinor,

$$\bar{s} = s^\dagger. \tag{B.4.17}$$

Let $\bar{s}$ be an adjoint Euclidean polarization spinor; then, define the adjoint Minkowski polarization spinor

$$\bar{S}(q(\vec{\omega})) = \begin{pmatrix} -\bar{s}, & \bar{s}\not{\omega} \end{pmatrix}, \tag{B.4.18}$$

from which one finds

$$S = \begin{pmatrix} \not{\omega} s \\ s \end{pmatrix}. \tag{B.4.19}$$

Setting $\psi(\vec{\omega}) = \psi_+(\vec{\omega}) - \not{\omega}\psi_-(\vec{\omega})$ as in (B.4.5) and $\Psi(q)$ as in (B.4.2), by means of Eq.(B.4.3) it is possible to write down

$$\bar{S}_\alpha \Psi^\alpha(\vec{\omega}) = -(q^+)^{\Delta+\frac{1}{2}} \left[\bar{s}_{\dot{\alpha}} \psi_+^{\dot{\alpha}}(\vec{\omega}) - \bar{s}_{\dot{\alpha}} \not{\omega}^{\dot{\alpha}}_{\dot{\beta}} \psi_-^{\dot{\beta}}(\vec{\omega})\right], \tag{B.4.20}$$

where $\alpha = 1, \ldots, 2k$ and $\dot{\alpha}, \dot{\beta} = 1, \ldots, k$. On the Poincaré section, (B.4.20) reduces to

$$\Psi(q(\vec{\omega}); \bar{S}) \equiv \bar{S}_\alpha \Psi^\alpha(\vec{\omega}) = -\bar{s}_{\dot{\alpha}} \psi_+^{\dot{\alpha}}(\vec{\omega}) + \bar{s}_{\dot{\alpha}} \not{\omega}^{\dot{\alpha}}_{\dot{\beta}} \psi_-^{\dot{\beta}}(\vec{\omega}) \equiv -\psi(\vec{\omega}; \bar{s}). \tag{B.4.21}$$

From $\psi(\vec{\omega}; \bar{s})$, it is possible to recover the conformal spinor $\psi(\vec{\omega})$ by simply deriving with respect to $\bar{s}_{\dot{\alpha}}$:

$$\frac{\partial}{\partial \bar{s}_{\dot{\alpha}}} \psi(\vec{\omega}; \bar{s}) = \psi^{\dot{\alpha}}(\vec{\omega}). \tag{B.4.22}$$

It is crucial to note that $\bar{S}(q(\vec{\omega}))$ is such that

$$\bar{S}_\alpha(q(\vec{\omega}))\, (q^\mu \Gamma_\mu)^\alpha_\beta = 0, \quad q^2 = 0. \tag{B.4.23}$$

The generic solution of (B.4.23) is

$$\bar{S}(q(\vec{\omega})) = \bar{T} q^\mu(\vec{\omega}) \Gamma_\mu, \tag{B.4.24}$$

where $\bar{T}$ is a generic constant spinor. In fact,

$$\bar{S}_\alpha\, (q^\mu \Gamma_\mu)^\alpha_\beta = \bar{T} q^\mu \Gamma_\mu q^\nu \Gamma_\nu = \bar{T} q^2 = 0. \tag{B.4.25}$$

Eq. (B.4.23) can be thought of encoding the *transversality condition* in the free index spinor formalism. An adjoint spinor $\bar{S}$, satisfying the transversality condition, and hence of the type (B.4.24), is said to be *transverse*. Moreover, the scalars formed by contracting two spinors, such as $\Psi\big(q(\vec{\omega});\bar{S}\big)$ and $\psi(\vec{\omega};\bar{s})$, are called *spinor quadratics*.

From these considerations, it is possible to recover a general rule: in the free index embedding formalism, to every conformal spinor quadratic $\psi(\vec{\omega};\bar{s})$ is associated the Minkowski spinor quadratic $\Psi\big(q;\bar{T}\big)$, which has the form

$$\Psi(q(\vec{\omega});\bar{T}) = \bar{T}q^M(\vec{\omega})\Gamma_M\Psi\big(q(\vec{\omega})\big), \tag{B.4.26}$$

where $\Psi\big(q(\vec{\omega})\big)$ satisfies the homogeneity condition (B.4.1). It should be noted that the correspondence between the quadratics

$$\Psi\big(q(\vec{\omega});\bar{T}\big) \longleftrightarrow \psi(\vec{\omega},\bar{s}) \tag{B.4.27}$$

is not one-to-one. To see that, send

$$\Psi \longmapsto \Psi' = \Psi + U = \Psi + q^M\Gamma_M\tilde{U}, \tag{B.4.28}$$

where $\tilde{U}$ is a generic constant Minkowski spinor. It is trivial to verify that the new spinor quadratic $\Psi'\big(q(\vec{\omega});\bar{T}\big)$ has the same target of $\Psi\big(q(\vec{\omega});\bar{T}\big)$, by means of (B.4.27). This happens because the spinor quadratic $U\big(q(\vec{\omega});\bar{T}\big)$ is the null-quadratic:

$$U\big(q(\vec{\omega});\bar{T}\big) = \bar{T}q^N\Gamma_N q^M\Gamma_M\tilde{U} = q^2\bar{T}\tilde{U} = 0, \quad \forall\,\bar{T},\tilde{U}. \tag{B.4.29}$$

The spinors proportional to $q^M\Gamma_M$, like U, are said to be of *pure gauge*, and provide the kernel of the map

$$\Psi(q) \longmapsto \Psi(q,\bar{S}) = \bar{S}_\alpha\Psi^\alpha, \tag{B.4.30}$$

where $\bar{S}$ is the transverse adjoint spinor (B.4.24). Hence, to the conformal spinor $\psi(\vec{\omega})$ corresponds the equivalence class of Minkowski spinors

$$[\Psi] = \Big\{\Psi' : \Psi' = \Psi + U, \quad U = q^M\Gamma_M\tilde{U}\Big\}, \tag{B.4.31}$$

where $\tilde{U}$ is a generic Minkowski spinor.

With these notions in mind it is possible to construct the two-point correlation function and the bulk-to-boundary propagator in the spinor case.

B.5 Two-point Correlation Function and Bulk-to-boundary Propagator Construction

The construction of the two-point correlation function in the spinor free index embedding formalism is quite similar to the scalar case. This time, given two embedded quadratics $\Psi(q_1;\bar{S}_1)$ and $\Psi(q_2;\bar{S}_2)$, where $\Psi(q_1)$ and $\Psi(q_2)$ are spinors of homogeneity degree $\Delta+\frac{1}{2}$, while S_1 and S_2 are transverse spinors, the free index two-point correlation function will be a two-point scalar field $\mathcal{S}_\Delta(q_1,q_2;S_1,S_2)$ with the following properties:

$$\begin{aligned}\mathcal{S}_\Delta(\lambda_1 q_1,\lambda_2 q_2;\alpha_1\bar{S}_1+\beta_1 U_1,\alpha_2 S_2+\beta_2 U_2) &=\\ &=(\lambda_1\lambda_2)^{-\Delta-\frac{1}{2}}\,\alpha_1\alpha_2\mathcal{S}_\Delta(q_1,q_2;S_1,S_2),\end{aligned} \tag{B.5.1}$$

where $\lambda_k,\alpha_k,\beta_k\in\mathbb{R}^+$, with $k=1,2$, and $q_1^2=q_2^2=0$. The only independent scalar quantities that can be constructed are

$$C_{12}=q_1\cdot q_2,\quad S_{12}=(\bar{S}_1)_\alpha(S_2)^\alpha. \tag{B.5.2}$$

In fact, any other spin quadratic that can be constructed is of the form

$$\langle\bar{S}_1\cdots S_2\rangle\equiv(\bar{S}_1)_\alpha\Gamma^\alpha_\beta(q_1,q_2)(S_2)^\beta, \tag{B.5.3}$$

where $\Gamma^\alpha_\beta(q_1,q_2)$ can be only a function of q_1 and q_2. It is immediate to demonstrate that every scalar spin quadratic of the type (B.5.3) vanishes. Indeed, the generic function $\Gamma^\alpha_\beta(q_1,q_2)$ must be constructed only with the following objects: the gamma matrices Γ_M and the null-vectors q_1 and q_2. The generic form of $\Gamma^\alpha_\beta(q_1,q_2)$ must be

$$\Gamma^\alpha_\beta(q_1,q_2)=q_1^{M_1}\Gamma_{M_1}\cdots q_1^{M_m}\Gamma_{M_m}q_2^{N_1}\Gamma_{N_1}\cdots q_2^{N_n}\Gamma_{N_n},\quad n,m\in\mathbb{N}. \tag{B.5.4}$$

Then, the constraints $q_1^2=q_2^2=0$ impose $n=m=1$, while the transversality condition (B.4.23) implies

$$\begin{aligned}\langle\bar{S}_1\cdots S_2\rangle&\equiv(\bar{S}_1)_\alpha\Gamma^\alpha_\beta(q_1,q_2)(S_2)^\beta\\ &=\langle\bar{S}_1\cdots S_2\rangle\equiv(\bar{S}_1)_\alpha\left(q_1^M\Gamma_M\right)^\alpha_{\ \gamma}\left(q_2^N\Gamma_N\right)^\gamma_{\ \beta}(S_2)^\beta=0.\end{aligned} \tag{B.5.5}$$

Therefore, the generic form of the two point correlation function must be

$$\mathcal{S}_\Delta(q_1,q_2;\bar{S}_1,S_2)=\mathcal{G}_\Delta(C_{12})S_{12}. \tag{B.5.6}$$

Imposing that $\mathcal{G}_\Delta(C_{12})$ must have the right homogeneity degree, one finds

$$\mathcal{S}_\Delta(q_1,q_2;\bar{S}_1,S_2)=c\,\frac{S_{12}}{\left(-2q_1\cdot q_2\right)^{\Delta+\frac{1}{2}}}, \tag{B.5.7}$$

where c is a generic constant.

Next, one has to project $\mathcal{S}_\Delta(q_1,q_2;\bar{S}_1,S_2)$ onto the Poincaré section by means of (B.0.2), and then to substitute

$$\bar{S}_1=\begin{pmatrix}\bar{s}_1, & -\bar{s}_1\not{x}_1\end{pmatrix}, \tag{B.5.8}$$

and

$$S_2 = \left(\begin{array}{c} \not{\psi}_2 s_2 \\ s_2 \end{array} \right), \tag{B.5.9}$$

to compute the two-point correlation function between two spinor conformal primary fields $\psi(\vec{\omega_1})$ and $\psi(\vec{\omega_2})$. It follows that

$$\mathcal{S}_\Delta(\vec{\omega_1}, \vec{\omega_2}; \bar{s}_1, s_2) = c\, \frac{\bar{s}_1 \left(\not{\psi}_1 - \not{\psi}_2\right) s_2}{\left(|\omega_1 - \omega_2|^2 \right)^{\Delta + \frac{1}{2}}}. \tag{B.5.10}$$

Finally, dropping $\bar{s}_1$ and s_2, one finds the spinor two-point conformal primary correlation function

$$\mathcal{S}_\Delta(\vec{\omega_1}, \vec{\omega_2}) = \langle \bar{\psi}(\vec{\omega}_1)\psi(\vec{\omega}_2)\rangle = c\, \frac{\not{\psi}_1 - \not{\psi}_2}{\left(|\omega_1 - \omega_2|^2 \right)^{\Delta + \frac{1}{2}}}. \tag{B.5.11}$$

Bulk-to-boundary propagator construction: Consider a point $\hat{p} \in \mathbb{H}_{d+1} \subset \mathbb{R}^{1,d+1}$, which means $\hat{p}^2 = -1$. Let $\Psi(\hat{p})$ a Minkowski spinor defined on $\mathbb{H}_{d+1}$ and $\psi(\vec{\omega})$ a conformal spinor defined on $\mathbb{R}^d$. Then, let $\Psi\big(q(\vec{\omega})\big)$ an up-lifted spinor associated to $\psi(\vec{\omega})$. The aim is to construct the up-lifted bulk-to-boundary propagator

$$S_\Delta\big(\hat{p}, q(\vec{\omega})\big) = c\,\langle \Psi(\hat{p})\bar{\Psi}\big(q(\vec{\omega})\big)\rangle. \tag{B.5.12}$$

To achieve this purpose, some generalizations are needed; indeed, the transversality condition must be changed, as suggested by Weinberg in [150].

Hence, consider an adjoint polarization spinor $\bar{S}$ and construct the spinor quadratic

$$\Psi(\hat{p}; S) = \bar{S}_\alpha \Psi^\alpha(\hat{p}). \tag{B.5.13}$$

Trying naively to impose the transversality condition (B.4.23) on $\bar{S}$, a problem immediately emerges: the equation

$$\bar{S}\hat{p}^M \Gamma_M = 0, \tag{B.5.14}$$

admits only the trivial solution. Indeed, setting

$$\bar{S} = \left(\begin{array}{cc} \bar{S}_+, & \bar{S}_- \end{array} \right), \tag{B.5.15}$$

where $\bar{S}_\pm$ must be seen as k-dimensional variables, and considering the k-dimensional block determinant

$$\det_k \left(\hat{p}^\mu \Gamma_\mu\right) = \det_k \left(\begin{array}{cc} \hat{p}^a\gamma_a & -\hat{p}^- \\ \hat{p}^+ & -\hat{p}^a\gamma_a \end{array} \right), \tag{B.5.16}$$

one finds

$$\det_k \left(\begin{array}{cc} \hat{p}^a\gamma_a & -\hat{p}^- \\ \hat{p}^+ & -\hat{p}^a\gamma_a \end{array} \right) = \hat{p}^+\hat{p}^- - \hat{p}^a\hat{p}^b\gamma_a\gamma_b = -\hat{p}^2 \neq 0, \tag{B.5.17}$$

which means that the homogeneous system

$$\begin{cases} \bar{S}_+\hat{p}^a\gamma_a - \bar{S}_-\hat{p}^- = 0, \\ \bar{S}_+\hat{p}^+ - \bar{S}_-\hat{p}^a\gamma_a = 0, \end{cases} \tag{B.5.18}$$

admits only the trivial solution. The natural generalization of (B.5.14) follows by an analogy with the vectorial case treated above (cf. (B.2.8)). There, given a vector V, it has been imposed

$$q^\mu V_\mu = 0; \tag{B.5.19}$$

analogously, the transversality condition (B.4.23) can be seen in the following way. Given a Minkowski spinor $\Psi(q)$ defined on the future null-cone $\mathcal{C}^+$, it is possible to construct the $SO(1,d+1)$ vector

$$\Psi(q) \longrightarrow \bar{\Psi}(q)\Gamma^\mu\Psi(q). \tag{B.5.20}$$

Indeed, sending $q \longmapsto \Lambda q$, $\Lambda \in SO(1,d+1)$, one finds

$$\bar{\Psi}(\Lambda q)\Gamma^\mu\Psi(\Lambda q) = \bar{\Psi}(q)S^{-1}(\Lambda)\Gamma^\mu S(\Lambda)\Psi(q) = \Lambda^\mu{}_\nu\bar{\Psi}(q)\Gamma^\nu\Psi(q). \tag{B.5.21}$$

Therefore, according to (B.5.19), the vector $\bar{\Psi}(q)\Gamma^\mu\Psi(q)$ must satisfy

$$q^\mu\bar{\Psi}(q)\Gamma_\mu\Psi(q) = 0, \tag{B.5.22}$$

to be transverse. Moreover, in the free index formalism, one maps

$$\Psi(q) \longmapsto \bar{S}_\alpha\Psi^\alpha(q), \tag{B.5.23}$$

or, equivalently,

$$\bar{\Psi}(q) \longmapsto \bar{\Psi}_\alpha(q)S^\alpha. \tag{B.5.24}$$

Hence, following the analogy with the vectorial case, here construct the polarization vector $\bar{S}\Gamma^\nu S$ and impose the transversality condition

$$\bar{S}q^\mu\Gamma_\mu S = 0 \Longleftrightarrow \bar{S}q^\mu\Gamma_\mu = 0. \tag{B.5.25}$$

In this way (B.4.23) has been recovered.

In order to generalize (B.4.23), one can try to add a suitable vector to (B.5.25), in such a way as to make the k-dimensional block determinant (B.5.17) vanish.

It is possible to note that, given any constant matrix M, the object $\bar{\Psi}(q)\hat{p}^\mu M\Psi(q)$ is a vector if and only if

$$[M, S(\Lambda)] = 0,\ \forall\Lambda \in SO(1,d+1). \tag{B.5.26}$$

This imposes $M = \alpha\mathbb{1}$, with $\alpha \in \mathbb{R}$, apart from the alternative $M = \Gamma$, with $\Gamma = i\Gamma_0 \dots \Gamma_{d+1}$, which reduces to the first choice in odd dimensions.

Hence, impose

$$\bar{S}\hat{p}_\mu\left(\hat{p}^\mu\alpha + \Gamma^\mu\right) = 0, \tag{B.5.27}$$

which, in light-cone coordinates, reads

$$\begin{aligned}
&\det_k \begin{pmatrix} -\alpha + \hat{p}^a\gamma_a & -\hat{p}^- \\ \hat{p}^+ & -\alpha - \hat{p}^a\gamma_a \end{pmatrix} = \\
&= \hat{p}^+\hat{p}^- + \left(\alpha - \hat{p}^a\gamma_a\right)\left(\alpha + \hat{p}^b\gamma_b\right) \\
&= \left(\alpha^2 + 1\right) = 0 \iff \alpha = \pm i.
\end{aligned} \tag{B.5.28}$$

Then, the new transversality condition is

$$\bar{S}\left(\mp i + \hat{p}^\mu\Gamma_\mu\right) = 0, \tag{B.5.29}$$

or, equivalently

$$\bar{S}\left(1 \pm i\hat{p}^\mu\Gamma_\mu\right) = 0. \tag{B.5.30}$$

Defining

$$\Pi_\pm(\hat{p}) = \frac{1}{2}\left(1 \pm i\hat{p}^\mu\Gamma_\mu\right), \tag{B.5.31}$$

it is possible to rewrite

$$\bar{S}\Pi_\pm(\hat{p}) = 0. \tag{B.5.32}$$

It is trivial to demonstrate that

$$\Pi_\pm(\hat{p})\Pi_\mp(\hat{p}) = 0; \tag{B.5.33}$$

indeed,

$$\left(1 \pm i\hat{p}^\mu\Gamma_\mu\right)\left(1 \mp i\hat{p}^\mu\Gamma_\mu\right) = \left(1 + \hat{p}^2\right) = 0. \tag{B.5.34}$$

Therefore, using the (B.5.33) it is possible to assert that the general solution of the transversality condition (B.5.32) is

$$\bar{S} = \bar{T}\Pi_\mp(\hat{p}). \tag{B.5.35}$$

One can also note that the new term added in the transversality condition (B.5.27) vanishes for $q \in \mathcal{C}$, and one recovers the former transversality condition (B.4.23). In this sense, a natural generalization of the (B.4.23) has be found .

Finally, one has all the elements to construct the bulk-to-boundary propagator (B.5.12). Compute the bulk-to-boundary propagator in the free index formalism, first.

Let $T(\hat{p})$ and $S(q)$ be the Minkowski polarization spinors of the uplifted spinors $\Psi(\hat{p})$ and $\Phi(q)$, respectively. The bulk-to-boundary propagator in the free index notation is a scalar, namely $\mathcal{S}_\Delta(\hat{p}, q; T(\hat{p}), S(q))$, which satisfies the property

$$\mathcal{S}_\Delta(\hat{p}, \lambda q; T(\hat{p}), \alpha S(q) + \beta U) = \lambda^{-\Delta-\frac{1}{2}} \alpha \mathcal{S}_\Delta(\hat{p}, q; T(\hat{p}), S(q)), \qquad \text{(B.5.36)}$$

where λ, $\alpha \in \mathbb{R}^+$ and U is a pure gauge spinor. The general form of $\mathcal{S}_\Delta(\hat{p}, q; \bar{T}(\hat{p}), S(q))$ can be found as in the case of the two-point correlation function. In particular, the generic form of the bulk-to-boundary propagator in the free index formalism is

$$\mathcal{S}_\Delta(\hat{p}, q; \bar{T}(\hat{p}), S(q)) = c\, \frac{\bar{T}_\alpha S^\alpha}{\big(-2q\cdot\hat{p}\big)^{\Delta+\frac{1}{2}}}, \qquad \text{(B.5.37)}$$

where c is a constant. Therefore, substituting the explicit form of the polarization vectors, it follows that

$$\mathcal{S}^{\pm}_\Delta(\hat{p}, q(\vec{\omega}); \bar{V}, S(q(\vec{\omega}))) = c\, \frac{(\bar{V}\Pi_\pm(\hat{p}))_\alpha \begin{pmatrix} \not{\omega} s \\ s \end{pmatrix}^\alpha}{\big(-2q(\vec{\omega})\cdot\hat{p}\big)^{\Delta+\frac{1}{2}}}. \qquad \text{(B.5.38)}$$

Finally, dropping the polarization spinors V and s, one finds the bulk-to-boundary propagators

$$S^{\pm}_\Delta(\hat{p}, q(\vec{\omega})) = c\, \frac{\Pi_\pm(\hat{p}) \begin{pmatrix} \not{\omega} \\ 1 \end{pmatrix}}{\big(-2q(\vec{\omega})\cdot\hat{p}\big)^{\Delta+\frac{1}{2}}}. \qquad \text{(B.5.39)}$$

Appendix C

Analytical Continuation and Wick Rotations

C.1 Closed Expressions of the Conformal Primary Wavefunctions

In this Appendix we will first evaluate the integral

$$\varphi^{+}_{\Delta}(X;q_+) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta-\frac{d}{2})}\left(\frac{m}{2\pi}\right)^{d/2}\int_{H^{+}_{d+1}}[\mathrm{d}\hat{p}]\, K^{\mathrm{AdS}}_{\Delta}(\hat{p};q_+)\, e^{+m\,\hat{p}\cdot X}\,, \quad X\in\mathcal{A}_+, \tag{C.1.1}$$

where $q_+\in\mathcal{C}^{+}_{P}$ and

$$\mathcal{N}_\Delta = \pi^{-\frac{1}{2}}\,\Gamma\left(\Delta-\frac{d}{2}+1\right). \tag{C.1.2}$$

This integral is convergent for $X\in\mathcal{A}_+$ and is the starting point to define the analytic continuation of the conformal primary wavefunctions over all Minkowski space-time.

To evaluate the above integral, we apply Schwinger representation for the normalised EAdS bulk-boundary propagator (4.2.3):

$$K^{\mathrm{AdS}}_{\Delta}(\hat{p};q_+) = \frac{C^{\mathrm{AdS}}_{\Delta}}{\Gamma(\Delta)}\int_0^\infty \frac{dt}{t}\, t^{\Delta} e^{2t\hat{p}\cdot q_+}\,, \qquad C^{\mathrm{AdS}}_{\Delta} = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}. \tag{C.1.3}$$

Plugging into (C.1.1) gives

$$\varphi^{+}_{\Delta}(X;q_+) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta-\frac{d}{2})}\left(\frac{m}{2\pi}\right)^{d/2}\frac{C^{\mathrm{AdS}}_{\Delta}}{\Gamma(\Delta)}\int_{H^{+}_{d+1}}[d\hat{p}]\int_0^\infty \frac{dt}{t}\, t^{\Delta}\, e^{2\,\hat{p}\cdot T}\,, \tag{C.1.4}$$

where, in the hyperbolic slicing $X = R\hat{X}_{\text{AdS}}$, we introduced the future-pointed time-like vector

$$T = tq_+ + \frac{1}{2}mR\hat{X}_{\text{AdS}}, \tag{C.1.5}$$

which, in a proper reference frame, can be rewritten in the form

$$T = |T|(1,1,\vec{0}), \qquad |T| = \sqrt{-T^2}. \tag{C.1.6}$$

The integral over $\mathcal{H}^+_{d+1}$ can then be evaluated by employing the parameterisation:

$$\hat{p} = \frac{1}{y_0}\left(1, y_0^2 + |\vec{x}|^2, \vec{x}\right) \qquad x \in \mathbb{R}^d,\, y_0 \geq 0, \tag{C.1.7}$$

in light-cone coordinates (see (2.2.2)). Evaluating the Gaussian integral over $\vec{x}$ gives

$$\varphi^+_\Delta(X; q_+) = \frac{\mathcal{N}_\Delta}{\Gamma(\Delta - \frac{d}{2})}\left(\frac{m}{2}\right)^{d/2}\frac{C^{\text{AdS}}_\Delta}{\Gamma(\Delta)}\int_0^\infty \frac{\mathrm{d}t}{t}\, t^\Delta \int_0^\infty \frac{\mathrm{d}y_0}{y_0}\, y_0^{-d/2}\, e^{-y_0 + \frac{T^2}{y_0}}\,, \tag{C.1.8}$$

where

$$T^2 = -\frac{m^2R^2}{4} + mRt\hat{X}_{\text{AdS}} \cdot q_+\,. \tag{C.1.9}$$

The integral over t can then replaced with the bulk-to-boundary propagator (C.1.3):

$$\begin{aligned} \varphi^+_\Delta(X; q_+) = R^{-d/2}&\frac{\mathcal{N}_\Delta}{\Gamma(\Delta - \frac{d}{2})}\left(\frac{2}{mR}\right)^{\Delta - \frac{d}{2}} \\ &\times K^{\text{AdS}}_\Delta(\hat{X}_{\text{AdS}}; q_+)\int_0^{+\infty}\frac{\mathrm{d}y_0}{y_0}\, y_0^{\Delta-\frac{d}{2}}e^{-y_0}e^{-\frac{m^2R^2}{4y_0}}\,. \end{aligned} \tag{C.1.10}$$

The remaining integral over y_0 can be performed using the Mellin-Barnes representation for the exponential function. In particular,

$$e^{-y_0} = \int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_1}{2\pi i}\,\Gamma\left(s_1\right)x_0^{-s_1}, \tag{C.1.11a}$$

$$e^{-\frac{m^2R^2}{4y_0}} = \int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_2}{2\pi i}\,\Gamma\left(s_2\right)\left(\frac{m^2R^2}{4}\right)^{-s_2} y_0^{s_2}. \tag{C.1.11b}$$

The integral over y_0 yields a Dirac delta function,

$$\int_0^\infty \frac{\mathrm{d}y_0}{y_0}\, y_0^{\Delta-\frac{d}{2}+s_2-s_1} = 2\pi i\, \delta\left(s_1 - s_2 - \Delta + \tfrac{d}{2}\right), \tag{C.1.12}$$

that can be used to eliminate one of the two Mellin-Barnes integrals. Therefore,

$$\begin{aligned}\varphi^{+}_{\Delta}(X;q_+) &= \frac{R^{-d/2}\mathcal{N}_\Delta}{\Gamma(\Delta-\frac{d}{2})} K^{\text{AdS}}_{\Delta}(\hat{X}_{\text{AdS}};q_+) \\ &\times \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\,\Gamma\left(s-\tfrac{1}{2}\left(\Delta-\tfrac{d}{2}\right)\right)\Gamma\left(s+\tfrac{1}{2}\left(\Delta-\tfrac{d}{2}\right)\right)\left(\frac{mR}{2}\right)^{-2s} \\ &= \mathcal{N}_\Delta\, K^{\text{AdS}}_{\Delta}(\hat{X}_{\text{AdS}};q_+)\tilde{K}_{\Delta-\frac{d}{2}}(mR),\end{aligned} \tag{C.1.13}$$

where in the last equality we identified the Mellin-Barnes representation (4.3.9) of

$$\tilde{K}_{\Delta-\frac{d}{2}}(mR) = \frac{2R^{-d/2}}{\Gamma(\Delta-\frac{d}{2})}\,K_{\Delta-\frac{d}{2}}(mR)\,, \tag{C.1.14}$$

which is the kernel of the Kantorovich-Lebedev transformation.

Note that this demonstrates how standard embedding space techniques (see e.g. [153–155]) for evaluating Witten diagrams in EAdS can be adapted to Celestial correlators. Moreover, since $\hat{X}_{\text{AdS}} \in \mathcal{H}^{+}_{d+1}$, than $-2\hat{X}_{\text{AdS}}\cdot q_+ > 0$. Therefore, the bulk-to-boundary propagator

$$K^{\text{AdS}}_{\Delta}(\hat{X}_{\text{AdS}};q_+) = \frac{C^{\text{AdS}}_{\Delta}}{\left(-2\hat{X}_{\text{AdS}}\cdot q_+\right)^{\Delta}}, \qquad C^{\text{AdS}}_{\Delta} = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}, \tag{C.1.15}$$

is well-defined. In the next section, we will start from this integral to find out the closed expression of the conformal primary wavefunctions over all Minkowski space-time.

C.1.1 Analytical continuation in $\mathcal{A}_+$ and $\mathcal{D}_+$

Let $q_\pm \in \mathcal{C}^{\pm}_{P}$, with $\mathcal{C}^{\pm}_{P}$ representing the Celestial Spheres $\mathcal{CS}^{\pm}$ in the Dirac's embedding formalism. The integral

$$\phi^{\pm}_{\Delta}(X;q_\pm) = \int_{\mathcal{H}^{\pm}_{d+1}}[\mathrm{d}\hat{p}]\,K^{\text{AdS}}_{\Delta}(\hat{p};q_\pm)\,\mathrm{e}^{im\hat{p}\cdot X} \tag{C.1.16}$$

is divergent for all $X \in \mathbb{M}^{d+2}$, owing to the purely imaginary argument of the exponential function involved in the integrand. However, starting from the convergent integrals

$$\varphi^{\pm}_{\Delta}(X;q_\pm) = \int_{\mathcal{H}^{\pm}_{d+1}}[\mathrm{d}\hat{p}]\,K^{\text{AdS}}_{\Delta}(\hat{p};q_\pm)\,\mathrm{e}^{m\hat{p}\cdot X}, \tag{C.1.17}$$

with $X \in \mathcal{A}_\pm$, we can move by analytical continuation to get the closed expression of $\phi^{\pm}_{\Delta}(X;q_\pm)$ in the regions $\mathcal{A}_\pm$ and $\mathcal{D}_\pm$ of Minkowski space-time.

Let us begin with the region $\mathcal{A}_+$. The solution of $\varphi^+_\Delta(X;q_+)$ has been computed in the last section. It is

$$\varphi^+_\Delta(R,\hat{X}_{\text{AdS}};q_+) = \frac{\mathcal{N}_\Delta C^{\text{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)} \frac{2R^{-\frac{d}{2}}}{(-2\hat{X}_{\text{AdS}}\cdot q_+)^\Delta} K_{\Delta-\frac{d}{2}}(mR), \tag{C.1.18}$$

where $X = R\hat{X}_{\text{AdS}}$, $R = \sqrt{-X^2}$. In light-cone Poincaré coordinates

$$\hat{X}_{\text{AdS}} = \frac{1}{y}\left(1, y^2+z^2, \vec{z}\right)^T \quad \vec{z}\in\mathbb{R}^d,\ y\in\mathbb{R}_+, \tag{C.1.19}$$

$$q_+ = \left(1,\omega^2,\vec{\omega}\right)^T \quad \vec{\omega}\in\mathbb{R}^d, \tag{C.1.20}$$

so that

$$\varphi^+_\Delta(R,y,\vec{z};\vec{\omega}) = \frac{\mathcal{N}_\Delta C^{\text{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)} \frac{2y^\Delta R^{-\frac{d}{2}}}{(y^2+|\vec{z}-\vec{\omega}|^2)^\Delta} K_{\Delta-\frac{d}{2}}(mR). \tag{C.1.21}$$

Now, let us move continuously R along the path $R\to \mathrm{e}^{\pm i\theta}T$, with $\theta\in[0,(\frac{\pi}{2}-\epsilon)]$. The integral (C.1.1) converges along the entire path, as the real part of the exponential's argument is negative as θ runs from 0 to $\frac{\pi}{2}-\epsilon$. At the final point, (C.1.1) assumes the same form as (C.1.16), up to the replacement $iR\hat{p}\cdot\hat{X}_{\text{AdS}} \to iR\hat{p}\cdot\hat{X}_{\text{AdS}}-\epsilon$, which ensures the negativity of the real part of the exponential's argument for all $\epsilon>0$. Therefore, at the final point, we obtain the closed expressions of the integrals (C.1.16), regularised for all $X\in\mathcal{A}_+$, i. e.

$$\phi^\pm_\Delta(T,y,\vec{z};\vec{\omega}) = \frac{\mathcal{N}_\Delta C^{\text{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)} \frac{2y^\Delta}{(y^2+|\vec{z}-\vec{\omega}|^2)^\Delta} (\mathrm{e}^{\pm i\frac{\pi}{2}}T)^{-\frac{d}{2}} K_{\Delta-\frac{d}{2}}(\mathrm{e}^{\pm i\frac{\pi}{2}}T), \tag{C.1.22}$$

where we sent $\epsilon\to 0$. The functions (C.1.22) are analytical in $\mathcal{A}_+$ and coincide with (4.2.9a), in the embedding formalism. As is well-known, in order to analytically continue our function from $\mathcal{A}_+$ to $\mathcal{D}_+$, we need to send continuously [8–10]

$$\begin{aligned} T &\to \mathrm{e}^{\pm i\frac{\pi}{2}}R, \\ y &\to \mathrm{e}^{\pm i\frac{\pi}{2}}\eta_+, \end{aligned} \tag{C.1.23}$$

so that

$$X = \frac{T}{y}\left(1,y^2+z^2,\vec{z}\right)^T \to \frac{R}{\eta_+}\left(1,z^2-\eta_+^2,\vec{z}\right)^T \in\mathcal{D}_+, \tag{C.1.24}$$

where $X\in\mathcal{A}_+$.

Therefore, by replacing $T \to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)}R$, $y \to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)}\eta_+$ in (C.1.22), we get the couple of solutions

$$\phi_\Delta^\pm(R,\eta_+,\vec{z};\vec{\omega}) = \frac{\mathcal{N}_\Delta C_\Delta^{\mathrm{AdS}}}{\Gamma\left(\Delta-\frac{d}{2}\right)}\frac{2\eta_+^\Delta \mathrm{e}^{\mp i\frac{\pi}{2}\Delta}}{(|\vec{z}-\vec{\omega}|^2-\eta_+^2\mp i\epsilon)^\Delta}\, R^{-\frac{d}{2}}K_{\Delta-\frac{d}{2}}(mR), \tag{C.1.25}$$

for the conformal primary wavefunctions in the region $\mathcal{D}_+$. Note that the same result is achieved by rotating $y \to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)}\eta_+$ in (C.1.21). Using this last transformation, it is necessary to highlight that the integral (C.1.1) converges consistently along the entire rotation and matches with (C.1.16) at the end point. Equally important is to point out that we cannot let $\epsilon \to 0$, this time, in order to preserve the analyticity of our functions. Indeed, by stopping our (anti-)clockwise rotation in the y-complex plane at $\theta = \frac{\pi}{2}-\epsilon$, we analytically continue the AdS bulk to boundary propagator (C.1.15) from $\mathcal{A}_+$ to $\mathcal{D}_+$, finding

$$G_\Delta^{\pm i\epsilon}(R,\eta_+,\vec{z};\vec{\omega}) = C_\Delta^{\mathrm{AdS}}\frac{\eta_+^\Delta \mathrm{e}^{\pm i\frac{\pi}{2}\Delta}}{(|\vec{z}-\vec{\omega}|^2-\eta_+^2\pm i\epsilon)^\Delta}. \tag{C.1.26}$$

If we sent $\epsilon \to 0$, this last function would present a branch for

$$\eta_+ \geq |\vec{z}-\vec{\omega}|, \qquad \eta_+ > 0. \tag{C.1.27}$$

Holding $\epsilon > 0$, the closed expressions (C.1.25) of the conformal primary wavefunctions $\phi_\Delta^\pm$ are analytical for all $X \in \mathcal{D}_+$ and coincide with (4.2.9c), in the embedding formalism. Another pair of solutions in $\mathcal{D}_+$ could be obtained from (C.1.22) via $T \to \mathrm{e}^{\pm i(\frac{\pi}{2}-\epsilon)}R$, $y \to \mathrm{e}^{\pm i(\frac{\pi}{2}-\epsilon)}\eta_+$, but this alternative path would yield non-normalizable solutions, proportional to

$$K_{\Delta-\frac{d}{2}}(\mathrm{e}^{\pm i(\pi-\epsilon)}mR) \sim \frac{1}{\sqrt{2\pi}}(\mathrm{e}^{\pm i(\pi-\epsilon)}mR)^{-\frac{1}{2}}\mathrm{e}^{mR}. \tag{C.1.28}$$

C.1.2 Analytical continuation in $\mathcal{A}_-$ and $\mathcal{D}_-$

Let us consider $q_- \in \mathcal{C}_P^-$ and $X = R'\hat{X}_{\mathrm{AdS}} \in \mathcal{A}_-$. In this case, the integral

$$\varphi_\Delta^-(R',\hat{X}_{\mathrm{AdS}};q) = \int_{\mathcal{H}_{d+1}^-}[\mathrm{d}\hat{p}]\, K_\Delta^{\mathrm{AdS}}(\hat{p};q_-)\mathrm{e}^{mR'\hat{p}\cdot\hat{X}_{\mathrm{AdS}}}, \tag{C.1.29}$$

with $\hat{X}_{\mathrm{AdS}} \in \mathcal{H}_{d+1}^+$ and $R' < 0$, is convergent. The resolution of (C.1.29) is very similar to the one of (C.1.1), yielding

$$\varphi_\Delta^-(R',\hat{X};q_-) = \frac{\mathcal{N}_\Delta C_\Delta^{\mathrm{AdS}}}{\Gamma\left(\Delta-\frac{d}{2}\right)}\frac{2(-R')^{-\frac{d}{2}}}{(2\hat{X}_{\mathrm{AdS}}\cdot q_-)^\Delta}K_{\Delta-\frac{d}{2}}(-mR'), \qquad R'<0, \tag{C.1.30}$$

which, in Poincaré coordinates, becomes

$$\varphi^-_\Delta(R',y,\vec z;\vec\omega)=\frac{\mathcal{N}_\Delta C^{\mathrm{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)}\frac{2y^\Delta(-R')^{-\frac{d}{2}}}{(y^2+|\vec z-\vec\omega|^2)^\Delta}K_{\Delta-\frac{d}{2}}(-mR'), \tag{C.1.31}$$

with $y>0$, $\vec z\in\mathbb{R}^d$.

Now, as before, by rotating $R'\to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)}T$ in (C.1.31), $T<0$, we find the closed expressions

$$\phi^\pm_\Delta(T,y,\vec z;\vec\omega)=\frac{\mathcal{N}_\Delta C^{\mathrm{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)}\frac{2y^\Delta(\mathrm{e}^{\mp i\frac{\pi}{2}}|T|)^{-\frac{d}{2}}}{(y^2+|\vec z-\vec\omega|^2)^\Delta}K_{\Delta-\frac{d}{2}}(\mathrm{e}^{\mp i\frac{\pi}{2}}m|T|), \tag{C.1.32}$$

for the conformal primary wavefunctions in the region $\mathcal{D}_-$. This last equation coincides with (4.2.9) in the embedding formalism. Note that (C.1.32) can be also found by replacing $T\to \mathrm{e}^{\mp i\pi}|T|$ in (C.1.22), which is the same as performing a π-angle (anti-)clockwise rotation in the complexified T-plane. Analogously, by sending continuously $y\to \mathrm{e}^{\pm i(\frac{\pi}{2}-\epsilon)}(-\eta_-)$ in (C.1.31), $\eta_-<0$, and replacing $R'\to -R$, $R>0$, we find the closed expressions of the conformal primary wavefunctions in the region $\mathcal{D}_-$:

$$\phi^\pm_\Delta(R,\eta_-,\vec z;\vec\omega)=\frac{\mathcal{N}_\Delta C^{\mathrm{AdS}}_\Delta}{\Gamma\left(\Delta-\frac{d}{2}\right)}\frac{2(-\eta_-)^\Delta \mathrm{e}^{\mp i\frac{\pi}{2}\Delta}}{(|\vec z-\vec\omega|^2-\eta_-^2\mp i\epsilon)^\Delta}(mR)^{-\frac{d}{2}}K_{\Delta-\frac{d}{2}}(mR),\quad R>0, \tag{C.1.33}$$

that match with (4.2.9d) in the embedding formalism. Indeed, under the transformation $y\to \mathrm{e}^{\pm i\frac{\pi}{2}}(-\eta_-)$,

$$\hat X_{\mathrm{AdS}}=\frac{1}{y}\left(1,y^2+z^2,\vec z\right)^T\to\frac{\pm i}{\eta_-}\left(1,z^2-\eta_-^2,\vec z\right)^T=\pm i\left(-\hat X_{\mathrm{dS}}\right), \tag{C.1.34}$$

with $\hat X_{\mathrm{dS}}\in dS^+_{d+1}$ and $-\hat X_{\mathrm{dS}}\in dS^-_{d+1}$, by means of the antipodal map.

C.2 Analytical continuation of Propagators from dS to AdS

It has been shown [8–10, 127] that the Feynman rules for boundary (Schwinger-Keldysh) correlators in the expanding patch of dS can be recast via analytic continuation as Feynman rules for boundary correlators in Euclidean AdS. In the first section of this appendix we will review these results and extend them to propagators in both the expanding and contracting patch, which will be needed to apply them to Celestial correlators since both patches are present in the hyperbolic slicing of Minkowski space. Indeed, in the second section we will lay the groundwork for the computation of the Celestial four-point exchange diagram by applying the results showcased in the first section.

C.2.1 Rewiev on (EA)dS propagators

Consider $(d+1)$-dimensional Euclidean anti-de Sitter space of unit radius, where the two sheets can be parameterised in Poincaré coordinates via

$$\hat{X}_{\mathcal{A}_\pm} = \frac{1}{y_\pm}\left(\frac{1+y_\pm^2+z^2}{2}, \frac{1-y_\pm^2-z^2}{2}, \vec{z}\right), \qquad y_\pm \gtrless 0. \tag{C.2.1}$$

Propagators are functions of the EAdS chordal distance

$$\sigma_{\text{AdS}}(\hat{X},\hat{Y}) = \frac{1}{2}\left(1+\hat{X}\cdot\hat{Y}\right). \tag{C.2.2}$$

The bulk-to-bulk propagators for a scalar field of mass m_{AdS} is the Green's function

$$G_\Delta^{\text{AdS}}\left(\sigma_{\text{AdS}}\right) = C_\Delta^{\text{AdS}}\left(-4\sigma_{\text{AdS}}\right)^{-\Delta}{}_2F_1\left(\begin{matrix}\Delta, \Delta-\frac{d}{2}+\frac{1}{2}\\ 2\Delta-d+1\end{matrix}, \frac{1}{\sigma_{\text{AdS}}}\right), \tag{C.2.3}$$

with normalisation

$$C_\Delta^{\text{AdS}} = \frac{\Gamma\left(\Delta\right)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}, \tag{C.2.4}$$

and where Δ is a solution to the quadratic equation:

$$m_{\text{AdS}}^2 = \Delta\left(d-\Delta\right). \tag{C.2.5}$$

It is often useful to parameterise these solutions by $\Delta_\pm = \frac{d}{2}\pm i\mu$ where $\mu\in\mathbb{R}$. The difference of the two solutions for the bulk-to-bulk propagator is a regular homogeneous solution of the Klein-Gordon equation,

$$\begin{aligned}\Omega_\mu\left(\sigma_{\text{AdS}}\right) &= \frac{i\mu}{2\pi}\left[G_{\Delta_+}^{\text{AdS}}\left(\sigma_{\text{AdS}}\right)-G_{\Delta_-}^{\text{AdS}}\left(\sigma_{\text{AdS}}\right)\right] \\ &= \frac{1}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})}\frac{\Gamma(\frac{d}{2}+i\mu)\Gamma(\frac{d}{2}-i\mu)}{\Gamma(i\mu)\Gamma(-i\mu)}{}_2F_1\left(\begin{matrix}\frac{d}{2}+i\mu, \frac{d}{2}-i\mu\\ \frac{d+1}{2}\end{matrix}; \sigma_{\text{AdS}}\right).\end{aligned} \tag{C.2.6}$$

Taking $\mu\in\mathbb{R}$ these provide a complete orthogonal basis for normalisable two-point functions in EAdS [153, 156], which is reviewed in appendix D.0.2.

The bulk-to-boundary propagator is obtained from the bulk-to-bulk propagator (C.2.3) upon sending one of the bulk points to the boundary

$$\begin{aligned}K_\Delta^{\text{AdS}}\left(s_{\text{AdS}}\right) &= \lim_{\lambda\to\infty}\lambda^\Delta G_\Delta^{\text{AdS}}\left(\hat{X},\hat{Y}=\lambda Q+\ldots\right) \\ &= \frac{C_\Delta^{\text{AdS}}}{\left(-2s_{\text{AdS}}\right)^\Delta}, \quad s_{\text{AdS}}(\hat{X},Q) = \hat{X}\cdot Q,\end{aligned} \tag{C.2.7}$$

where the ... serve to enforce the constraint $\hat{Y}^2 = -1$. In Poincaré coordinates this is

$$K_{\Delta}^{\text{AdS}}\left(y_{\pm},\vec{z};\vec{\omega}\right) = C_{\Delta}^{\text{AdS}}\left(\frac{\pm y_{\pm}}{y_{\pm}^2 + |\vec{z}-\vec{\omega}|^2}\right)^{\Delta}, \tag{C.2.8}$$

where in $\mathcal{A}_{\pm}$ we parameterise the boundary points as

$$Q_{\pm} = \pm\left(\frac{1+\omega^2}{2}, \frac{1-\omega^2}{2}, \vec{\omega}\right). \tag{C.2.9}$$

Consider now $(d+1)$-dimensional de Sitter space of unit radius, where the expanding and contracting patches can be parameterised in Poincaré coordinates via

$$\hat{X}_{\mathcal{D}_+} = \frac{1}{\eta_+}\left(\frac{1-\eta_+^2+\vec{x}^2}{2}, \frac{1+\eta_+^2-\vec{x}^2}{2}, \vec{x}\right), \qquad \eta_+ > 0, \tag{C.2.10}$$

$$\hat{X}_{\mathcal{D}_-} = \frac{1}{\eta_-}\left(\frac{1-\eta_-^2+\vec{x}^2}{2}, \frac{1+\eta_-^2-\vec{x}^2}{2}, \vec{x}\right), \qquad \eta_- < 0. \tag{C.2.11}$$

Two-point functions are now functions of the dS chordal distance

$$\sigma_{\text{dS}}\left(\hat{X},\hat{Y}\right) = \frac{1}{2}\left(1+\hat{X}\cdot\hat{Y}\right). \tag{C.2.12}$$

$$= 1 + \frac{\left(\eta_1-\eta_2\right)^2 - \left(\vec{x}_1-\vec{x}_2\right)^2}{4\eta_1\eta_2}. \tag{C.2.13}$$

Propagators for a scalar field of mass m_{dS} in the Bunch-Davies (Euclidean) vacuum [157, 158][1] are derived from the function

$$G_{\Delta}^{\text{dS}}\left(\sigma_{\text{dS}}\right) = \Gamma(i\mu)\Gamma(-i\mu)\Omega_{\mu}\left(\sigma_{\text{dS}}\right), \tag{C.2.14}$$

where now

$$m_{\text{dS}}^2 = \Delta\left(\Delta-d\right). \tag{C.2.15}$$

Unlike the harmonic function (C.2.6) in EAdS, which is regular, the two-point function (C.2.14) has a short distance singularity at $\sigma_{\text{dS}} = 1$ and a branch cut for $\sigma_{\text{dS}} \in (1,\infty)$ where the two-points become time-like separated. Different prescriptions for approaching the branch cut give rise to the different dS two-point functions. The (anti)-Feynman propagators G_T are

$$G_{\Delta,T}^{\text{dS}}\left(\sigma_{\text{dS}}\right) = G_{\Delta}^{\text{dS}}\left(\sigma_{\text{dS}}-i\epsilon\right), \tag{C.2.16a}$$

$$G_{\Delta,\bar{T}}^{\text{dS}}\left(\sigma_{\text{dS}}\right) = G_{\Delta}^{\text{dS}}\left(\sigma_{\text{dS}}+i\epsilon\right). \tag{C.2.16b}$$

[1] In dS there is in fact a one-parameter family as possible vacua that are invariant under the dS isometry group [159]. The Bunch-Davies vacuum can be uniquely defined as the one in which the Green's functions satisfy the Hadamard condition - that they behave as in Minkowski space on the light cone.

The corresponding bulk-to-boundary propagators can be derived similarly as in EAdS as a boundary limit of bulk-to-bulk propagators as in dS (see [8, 127]), giving

$$K^{\mathrm{dS}}_{\Delta,T}\,(s_{\mathrm{dS}}) = \frac{C^{\mathrm{dS}}_{\Delta}}{(-2s_{\mathrm{dS}}+i\epsilon)^{\Delta}}, \tag{C.2.17a}$$

$$K^{\mathrm{dS}}_{\Delta,\bar{T}}\,(s_{\mathrm{dS}}) = \frac{C^{\mathrm{dS}}_{\Delta}}{(-2s_{\mathrm{dS}}-i\epsilon)^{\Delta}}, \tag{C.2.17b}$$

where $s_{\mathrm{dS}}(\hat{X},Q) = \hat{X}\cdot Q$ and the two-point coefficient

$$C^{\mathrm{dS}}_{\Delta} = \frac{1}{4\pi^{\frac{d+2}{2}}}\,\Gamma(\Delta)\Gamma(\tfrac{d}{2}-\Delta). \tag{C.2.18}$$

C.2.2 Analytical Continuation of the Feynman propagator to EAdS

Propagators in $dS^{\pm}$ can be analytically continued in the upper/lower sheet of EAdS, that we visualise as $\mathcal{H}^{\pm}_{d+1}$. In general, it is well known that $dS^{\pm}_{d+1}$ and $\mathcal{H}^{\pm}_{d+1}$ are formally related by analytic continuation, which at the level of the Poincaré patches is implemented via $\eta_{\pm} \to \pm i y_{\pm}$. In this way, $dS^{\pm}_{d+1}$ two-point functions can be expressed as a linear combination of analytically continued bulk-to-bulk propagators (C.2.3) in $\mathcal{H}^{\pm}_{d+1}$. In these latter, the $i\epsilon$-prescription is dictated by the analytical transformations [8, 127]:

$$\textit{time-ordering}: \quad \eta_{\pm} \to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)} y_{\pm}, \tag{C.2.19}$$

$$\textit{anti-time-ordering}: \quad \eta \to \mathrm{e}^{\pm i(\frac{\pi}{2}-\epsilon)} y_{\pm}, \tag{C.2.20}$$

which, in terms of embedding coordinates, can be expressed as

$$\textit{time-ordering}: \quad \hat{X}_{\mathcal{D}_{\pm}} \to \mathrm{e}^{\pm i(\frac{\pi}{2}-\epsilon)} \hat{X}_{\mathcal{A}_{\pm}} \pm (0,\epsilon,\vec{0})^{T}, \tag{C.2.21}$$

$$\textit{anti-time-ordering}: \quad \hat{X}_{\mathcal{D}_{\pm}} \to \mathrm{e}^{\mp i(\frac{\pi}{2}-\epsilon)} \hat{X}_{\mathcal{A}_{\pm}} \pm (0,\epsilon,\vec{0})^{T}. \tag{C.2.22}$$

The vector on the right in (C.2.21) has been written in light-coordinates, with T at the apex indicating the transpose operation.

We are now interested in studying how the generalised chordal distance

$$\sigma_{\epsilon}(\hat{X}_1,\hat{X}_2) = \frac{1}{2}\left(1 + \frac{\hat{X}_1\cdot\hat{X}_2 - i\epsilon}{\sqrt{\hat{X}_1^2+i\epsilon}\sqrt{\hat{X}_2^2+i\epsilon}}\right) \tag{C.2.23}$$

transforms under (C.2.21). Specifically, our primary focus in this section is to establish the groundwork for computing the Celestial four-point exchange diagram, carried out in sec.(5.3.2). Our aim is to compute this process by implementing the same technique outlined in sec.(5.3.1). To reach this goal, here we delve into the study of the behaviour of the Feynman propagator (5.2.11) under the Wick rotations employed in sec.(5.3.2), which coincide with *time-ordered* transformations in (C.2.21). Additionally, to map the contribution from $\mathcal{H}^-_{d+1}$ to $\mathcal{H}^+_{d+1}$, in sec.(5.3.1) we also used the (non-analytical) antipodal map $X \to -X$. Consequently, here below, we will proceed by analysing the behaviour of the function $\Omega(\sigma_\epsilon)$, defined in (5.2.9), under these transformations. This function encodes all angular information of the Feynman propagator (ref. (5.2.11)) and will determine how this latter transforms under time-ordered Wick rotations and antipodal map.

We will start this study by reformulating Feynman propagator in terms of the EAdS bulk-to-bulk propagator (C.2.3) when both points lie inside the region $\mathcal{A}_+$. Subsequently, we will explore various scenarios, utilising both time-ordered Wick rotations and the antipodal map to trace each contribution from the outside back to the region $\mathcal{A}_+$. Throughout the entire analysis, we will use the subscript $\hat{X}_\bullet$, with $\bullet = \mathcal{A}_\pm$, $\mathcal{D}_\pm$ to indicate the regions where the points are situated. We won't delve into all the possible scenarios, our focus will be on the crucial cases to which all others can be reduced. The results of this section are summarised in Table 5.1, in sec.(5.3.2).

$\mathcal{A}_+/\mathcal{A}_+$ contribution: In this case, the generalised chordal distance σ_ϵ becomes

$$\sigma_\epsilon(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}) = 1 - \sigma_{\text{AdS}}(\hat{Y}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}) + i\epsilon. \tag{C.2.24}$$

Therefore, we can exploit the properties of the hypergeometric functions ${}_2F_1$ to rewrite $\Omega_\nu(1 - \sigma_{\text{AdS}} + i\epsilon)$ as

$$\Omega_\nu\left(\sigma_\epsilon(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+})\right) = \frac{i\nu}{2\pi}\left[\mathrm{e}^{i\pi\Delta}\, G^{\text{AdS}}_{\Delta_+}(\sigma_{\text{AdS}}) - \mathrm{e}^{-i\pi\Delta}\, G^{\text{AdS}}_{\Delta_-}(\sigma_{\text{AdS}})\right], \tag{C.2.25}$$

in total analogy with (C.2.6). Plugging this last result in (5.2.11), we find the expression for the Feynman propagator:

$$\Pi^{(m)}_T(X_{\mathcal{A}_+}, Y_{\mathcal{A}_+}) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi}\, \underbrace{\mathrm{e}^{i\left(\frac{d}{2}+i\nu\right)\pi}}_{\textbf{Phase}}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\, \tilde{K}^{(m)}_{-\nu}(\mathrm{e}^{\frac{i\pi}{2}}T_1)\tilde{K}^{(m)}_{\nu}(\mathrm{e}^{\frac{i\pi}{2}}T_2) \times G^{\text{AdS}}_{\frac{d}{2}+i\nu}(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}), \tag{C.2.26}$$

where we emphasised the phase of interest for the computation of the Celestial four-point exchange diagram. In sec. (5.3.2), we will trace out other phases in this expression. These phases will emerge from the Kantorovic-Lebedev kernels $\tilde{K}^{(m)}_\nu$ once we express them in Mellin representation (see, for example, (5.3.12), (5.3.32) and (5.3.48)). Here, our main focus is to highlight the phases that stem from the Wick rotations.

$\mathcal{A}_+/\mathcal{D}_+$ contribution: In this case,

$$\sigma_\epsilon(X_{\mathcal{A}_+}, Y_{\mathcal{D}_+}) \simeq \frac{1}{2}\left(1 + \mathrm{e}^{-i(\frac{\pi}{2}-\epsilon)}\hat{X}_{\mathcal{A}_+}\cdot\hat{Y}_{\mathcal{D}_+}\right). \tag{C.2.27}$$

Then, via time-ordered Wick rotation (ref. (C.2.21)), we get

$$\sigma_\epsilon \to \sigma_{\mathrm{AdS}} + i\epsilon \implies \Omega(\sigma_\epsilon) \to \Omega(\sigma_{\mathrm{AdS}} + i\epsilon). \tag{C.2.28}$$

This result allow us to state that, under time-ordered Wick rotation,

$$\Pi_T^{(m)}(X_{\mathcal{A}_+}, Y_{\mathcal{D}_+}) \to \int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\,\tilde{K}^{(m)}_{-\nu}(e^{i\frac{\pi}{2}}T_1)\tilde{K}^{(m)}_{\nu}(R_2) \times G^{\mathrm{AdS}}_{\frac{d}{2}+i\nu}(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}). \tag{C.2.29}$$

where we exploited (C.2.6) to recast the Feynman propagator in terms of the EAdS bulk-to-bulk propagator. Note that, in this case, Wick rotation did not yield any phase

$\mathcal{A}_-/\mathcal{D}_+$ contribution: This case can be reduced to the previous ones by simply applying the antipodal map on $X_{\mathcal{A}_-}$:

$$\begin{aligned}\sigma_\epsilon(X_{\mathcal{A}_-}, Y_{\mathcal{D}_+}) &\simeq \frac{1}{2}\left(1 + \mathrm{e}^{-i(\frac{\pi}{2}-\epsilon)}\hat{X}_{\mathcal{A}_-}\cdot\hat{Y}_{\mathcal{D}_+}\right)\\ &= \frac{1}{2}\left(1 - \mathrm{e}^{-i(\frac{\pi}{2}-\epsilon)}\hat{X}_{\mathcal{A}_+}\cdot\hat{Y}_{\mathcal{D}_+}\right).\end{aligned} \tag{C.2.30}$$

Next, by Wick rotating $\hat{Y}$ to $\mathcal{A}_+$, it follows that

$$\sigma_\epsilon \to 1 - \sigma_{\mathrm{AdS}} - i\epsilon, \tag{C.2.31}$$

which leads us to a result very similar to that of the first case:

$$\Pi_T^{(m)}(X_{\mathcal{A}_-}, Y_{\mathcal{D}_+}) \to \int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,\underbrace{\mathrm{e}^{-i(\frac{d}{2}+i\nu)\pi}}_{\textbf{Phase}}\,c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\,\tilde{K}^{(m)}_{-\nu}(\mathrm{e}^{\frac{i\pi}{2}}T_1)\tilde{K}^{(m)}_{\nu}(R_2) \times G^{\mathrm{AdS}}_{\frac{d}{2}+i\nu}(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}). \tag{C.2.32}$$

As you can see, here we got the opposite phase with respect to the $\mathcal{A}_+/\mathcal{A}_+$ situation.

$\mathcal{D}_-/\mathcal{D}_-$ contribution: In this case, we need to first apply a time-ordered Wick rotation on both the bulk points, and then subsequently utilise the antipodal map:

$$\begin{aligned}\sigma_\epsilon(X_{\mathcal{D}_-}, Y_{\mathcal{D}_-}) &\to \frac{1}{2}\left(1 - \hat{X}_{\mathcal{A}_-}\cdot\hat{Y}_{\mathcal{A}_-} - i\epsilon\right)\\ &= \frac{1}{2}\left(1 - \hat{X}_{\mathcal{A}_+}\cdot\hat{Y}_{\mathcal{A}_+} - i\epsilon\right) &&\text{(C.2.33)}\\ &= 1 - \sigma_{\mathrm{AdS}}(X_{\mathcal{A}_+}, Y_{\mathcal{A}_+}) - i\epsilon. &&\text{(C.2.34)}\end{aligned}$$

This brings us to the result

$$\Pi_T^{(m)}(X_{\mathcal{D}_-}, Y_{\mathcal{D}_-}) \to \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} \underbrace{\mathrm{e}^{-i\left(\frac{d}{2}+i\nu\right)\pi}}_{\textbf{Phase}} c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu} \tilde{K}^{(m)}_{-\nu}(R_1)\tilde{K}^{(m)}_{\nu}(R_2) \times G^{\text{AdS}}_{\frac{d}{2}+i\nu}(\hat{X}_{\mathcal{A}_+}, \hat{Y}_{\mathcal{A}_+}), \quad \text{(C.2.35)}$$

where we have isolated the same phase as in the $\mathcal{A}_-/\mathcal{D}_+$ case.

This method can be applied to all possible combinations of points $\hat{X}$ and $\hat{Y}$ in the regions $\mathcal{A}_\pm$, $\mathcal{D}_\pm$. The phases collected in this way are summarised in the Table 5.1 in sec.(5.3.2). For completeness, the table also takes into account the phases traced out by applying Wick rotations and antipodal map on the external legs, as in sec.(5.3.1).

Appendix D

Representations of the Feynman Propagator

In this appendix we give the derivations of the various representations of propagators for Celestial correlation functions presented in this work.

D.0.1 Mellin Transform of the Feynman Propagator

In this section we derive the expression (5.2.10) for the Feynman propagator in Mellin space, which is defined by the double Mellin transform:

$$\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \int_0^\infty \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\, t_1^{\Delta_1} t_2^{\Delta_2}\, \Pi_T(t_1\hat{X}, t_2\hat{Y})\,. \tag{D.0.1}$$

At the end of this second we also take the boundary limit, to obtain the boundary-to-boundary two-point function (5.2.22).

The Mellin transform integrals can be evaluated by employing a Schwinger parameterisation of the Feynman propagator (see e.g. equation (C.4) of [139]):

$$\begin{aligned}\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \frac{1}{4\pi^{\frac{d+2}{2}}}\left(\frac{m}{2}\right)^{\frac{d}{2}} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s-\tfrac{d}{4}) \\ \times \int_0^\infty \frac{\mathrm{d}t}{t} t^{s+\frac{d}{4}} \int_0^\infty \frac{\mathrm{d}t_i}{t_i}\, t_i^{\Delta_i}\, \left(\tfrac{m}{2}\right)^{-2s} i^{-s-\frac{d}{4}} e^{it(t_1\hat{X}-t_2\hat{Y})^2}\,.\end{aligned} \tag{D.0.2}$$

By sending $t \to t/t_1t_2$, we have

$$\begin{aligned}\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \frac{1}{4\pi^{\frac{d+2}{2}}}\left(\frac{m}{2}\right)^{\frac{d}{2}} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s-\tfrac{d}{4}) \int_0^\infty \frac{\mathrm{d}t}{t} t^{s+\frac{d}{4}} \\ \times \int_0^\infty \frac{dt_i}{t_i}\, t_i^{\Delta_i - s-\frac{d}{4}}\, \left(\tfrac{m}{2}\right)^{-2s} i^{-s-\frac{d}{4}} e^{it\left(\frac{t_1}{t_2}\hat{X}^2+\frac{t_2}{t_1}\hat{Y}^2-2\hat{X}\cdot\hat{Y}\right)}\,.\end{aligned} \tag{D.0.3}$$

Following this with the change of variables

$$q_1 = t_1 t_2\,, \qquad q_2 = \frac{t_1}{t_2}\,, \tag{D.0.4}$$

the integration measure becomes:

$$\int_0^\infty \frac{\mathrm{d}t_i}{t_i}\, t_i^{\Delta_i - s - \frac{d}{4}} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}q_i}{q_i} q_1^{-\frac{d}{4}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2}-s} q_2^{\frac{\Delta_1-\Delta_2}{2}} \tag{D.0.5}$$

$$= \pi i \delta\left(s + \frac{d}{4} - \frac{\Delta_1}{2} - \frac{\Delta_2}{2}\right)\int \frac{\mathrm{d}q_2}{q_2}\, q_2^{\frac{\Delta_1-\Delta_2}{2}}\,, \tag{D.0.6}$$

where we have performed the q_1 integral since the integrand only depends on q_2. The Dirac-delta then allows to perform the s integral, giving

$$\begin{aligned}\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = {}& \frac{1}{8\pi^{\frac{d+2}{2}}}\left(\tfrac{m}{2}\right)^{d-\Delta_1-\Delta_2}\Gamma\!\left(\tfrac{\Delta_1+\Delta_2-d}{2}\right)\int_0^\infty \frac{\mathrm{d}t}{t} t^{\frac{\Delta_1+\Delta_2}{2}} \\ &\times \int_0^\infty \frac{\mathrm{d}q_2}{q_2}\, q_2^{\frac{\Delta_1-\Delta_2}{2}}\, i^{-\frac{\Delta_1+\Delta_2}{2}} e^{it\left(q_2\hat{X}^2 + \frac{1}{q_2}\hat{Y}^2 - 2\hat{X}\cdot\hat{Y}\right)}\,.\end{aligned} \tag{D.0.7}$$

The leftover integrals can be evaluated in terms of a Mellin representation. Using

$$e^{i\frac{t}{q_2}\hat{Y}^2} = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s)\left(-i\frac{t}{q_2}\hat{Y}^2 + \epsilon\right)^{-s} \tag{D.0.8}$$

$$= \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s) t^{-s} q_2^{s} (-i)^{-s} (\hat{Y}^2 + i\epsilon)^{-s}\,, \tag{D.0.9}$$

one can then first evaluate the q_2 integral, which is a Schwinger parameterisation, providing

$$\begin{aligned}\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = {}& \frac{1}{8\pi^{\frac{d+2}{2}}}\left(\tfrac{m}{2}\right)^{d-\Delta_1-\Delta_2}\Gamma\!\left(\tfrac{\Delta_1+\Delta_2-d}{2}\right) \\ &\times \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s)(-i)^{-s+\frac{\Delta_1+\Delta_2}{2}}(\hat{Y}^2+i\epsilon)^{-s}\Gamma\left(s + \tfrac{\Delta_1-\Delta_2}{2}\right) \\ &\times \int_0^\infty \frac{\mathrm{d}t}{t} t^{\frac{\Delta_1+\Delta_2}{2}-s}(-it\hat{X}^2+\epsilon)^{-s-\frac{\Delta_1-\Delta_2}{2}} e^{-2it\hat{X}\cdot\hat{Y}}\,.\end{aligned} \tag{D.0.10}$$

Similarly for the t-integral, obtaining

$$\begin{aligned}\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = {}& \frac{1}{8\pi^{\frac{d+2}{2}}}\left(\tfrac{m}{2}\right)^{d-\Delta_1-\Delta_2}\Gamma\!\left(\tfrac{\Delta_1+\Delta_2-d}{2}\right) \\ &\times \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i}\,\Gamma(s)\Gamma\left(s+\tfrac{\Delta_1-\Delta_2}{2}\right)\Gamma\left(-2s+\Delta_2\right)(-i)^{-2s+\Delta_2} \\ &\times(\hat{Y}^2+i\epsilon)^{-s}(\hat{X}^2+i\epsilon)^{-s-\frac{\Delta_1-\Delta_2}{2}}(2i\hat{X}\cdot\hat{Y}+\epsilon)^{2s-\Delta_2}\,.\end{aligned} \tag{D.0.11}$$

After shifting $s \to s - \frac{\Delta_1 - \Delta_2}{4}$, this is

$$\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \frac{1}{8\pi^{\frac{d+2}{2}}} \left(\tfrac{m}{2}\right)^{d-\Delta_1-\Delta_2} \Gamma(\tfrac{\Delta_1+\Delta_2-d}{2}) \tag{D.0.12}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i} \Gamma(s - \tfrac{\Delta_1-\Delta_2}{4})\Gamma\left(s + \tfrac{\Delta_1-\Delta_2}{4}\right)\Gamma\left(-2s + \tfrac{\Delta_1+\Delta_2}{2}\right)$$
$$\times(\hat{Y}^2 + i\epsilon)^{-s+\frac{\Delta_1-\Delta_2}{4}}(\hat{X}^2 + i\epsilon)^{-s-\frac{\Delta_1-\Delta_2}{4}}(-2\hat{X}\cdot\hat{Y} + i\epsilon)^{2s-\frac{\Delta_1+\Delta_2}{2}},$$

or equivalently

$$\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \frac{1}{8\pi^{\frac{d+2}{2}}} \frac{\left(\frac{m}{2}\right)^{d-\Delta_1-\Delta_2}\Gamma(\frac{\Delta_1+\Delta_2-d}{2})}{\left(\sqrt{\hat{X}^2+i\epsilon}\right)^{\Delta_1}\left(\sqrt{\hat{Y}^2+i\epsilon}\right)^{\Delta_2}} \tag{D.0.13}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}s}{2\pi i} \Gamma(s - \tfrac{\Delta_1-\Delta_2}{4})\Gamma\left(s + \tfrac{\Delta_1-\Delta_2}{4}\right)\Gamma\left(-2s + \tfrac{\Delta_1+\Delta_2}{2}\right)$$
$$\times \left(\frac{-2\hat{X}\cdot\hat{Y} + i\epsilon}{\sqrt{\hat{X}^2+i\epsilon}\sqrt{\hat{Y}^2+i\epsilon}}\right)^{2s-\frac{\Delta_1+\Delta_2}{2}}.$$

Finally performing the Mellin integral closing on the poles on the positive real axis we arrive to the closed form expression (*cf.* equation (D.0.31)):

$$\Pi_{\Delta_1,\Delta_2}(\hat{X},\hat{Y}) = \frac{1}{2}\frac{1}{(4\pi)^{\frac{d+1}{2}}} \frac{m^{d-\Delta_1-\Delta_2}}{\left(\sqrt{\hat{X}^2+i\epsilon}\right)^{\Delta_1}\left(\sqrt{\hat{Y}^2+i\epsilon}\right)^{\Delta_2}} \tag{D.0.14}$$
$$\times \frac{\Gamma(\frac{\Delta_1+\Delta_2-d}{2})\Gamma(\Delta_1)\Gamma(\Delta_2)}{\Gamma\left(\frac{\Delta_1+\Delta_2+1}{2}\right)} {}_2F_1\left(\begin{matrix}\Delta_1,\Delta_2\\ \frac{\Delta_1+\Delta_2+1}{2}\end{matrix}; \frac{1-\frac{-\hat{X}\cdot\hat{Y}+i\epsilon}{\sqrt{\hat{X}^2+i\epsilon}\sqrt{\hat{Y}^2+i\epsilon}}}{2}\right).$$

The free theory boundary two-point function is then the boundary limit of the Mellin transformed Feynman propagator (D.0.14):

$$\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) := \lim_{\hat{X}_i\to Q_i} \Pi^{(m)}_{\Delta_1\Delta_2}(\hat{X}_1,\hat{X}_2). \tag{D.0.15}$$

To this end, it is useful to employ the Mellin-Barnes representation (D.0.13) of the Gauss hypergeometric function where, since $Q_i^2 = 0$, we close the contour on the negative real axis, on the poles:

$$s = \frac{\Delta_1-\Delta_2}{4},\ \frac{\Delta_1-\Delta_2}{4}-1,\ \frac{\Delta_1-\Delta_2}{4}-2,\ \dots, \tag{D.0.16}$$
$$s = \frac{\Delta_2-\Delta_1}{4},\ \frac{\Delta_2-\Delta_1}{4}-1,\ \frac{\Delta_2-\Delta_1}{4}-2,\ \dots. \tag{D.0.17}$$

This gives

$$\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) = \frac{1}{8\pi^{\frac{d+2}{2}}}\left(\frac{m}{2}\right)^{d-\Delta_1-\Delta_2}\Gamma\left(\frac{\Delta_1+\Delta_2-d}{2}\right)$$
$$\times\left[\frac{\Gamma(\Delta_2)}{(-2Q_1\cdot Q_2+i\epsilon)^{\Delta_2}}\lim_{\hat{X}\to Q_1}\Gamma\left(\frac{\Delta_1-\Delta_2}{2}\right)\left(\sqrt{\hat{X}^2+i\epsilon}\right)^{\Delta_2-\Delta_1}\right.$$
$$\left.+\frac{\Gamma(\Delta_1)}{(-2Q_1\cdot Q_2+i\epsilon)^{\Delta_1}}\lim_{\hat{Y}\to Q_2}\Gamma\left(\frac{\Delta_2-\Delta_1}{2}\right)\left(\sqrt{\hat{Y}^2+i\epsilon}\right)^{\Delta_1-\Delta_2}\right]$$
$$+\ldots, \quad \text{(D.0.18)}$$

where the ... are subleading in the boundary limit. Regarded as a distribution in Δ_1 or Δ_2 only one of the two terms contributes, giving

$$\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) = \frac{C^{\text{flat}}_{\Delta_1}}{(-2Q_1\cdot Q_2+i\epsilon)^{\Delta_1}}(2\pi)\delta(i(\Delta_1-\Delta_2))\,,$$

with

$$C^{\text{flat}}_{\Delta} = \left(\frac{m}{2}\right)^{d-2\Delta}\frac{1}{4\pi^{\frac{d+2}{2}}}\Gamma(\Delta)\Gamma(\Delta-\tfrac{d}{2}). \quad \text{(D.0.19)}$$

Another way to obtain the same result is to take the boundary limit of the radial Mellin transform of the celestial bulk-to-boundary propagator (5.2.17),

$$\Pi^{(m)}_{\Delta_1\Delta_2}(Q_1,Q_2) = \lim_{\hat{Y}\to Q_2}\int_0^\infty \frac{\mathrm{d}t}{t}t^{\Delta_2}\Pi^{(m)}_{\Delta_1}\left(t\hat{Y},Q_1\right), \quad \text{(D.0.20)}$$

as presented in [139].

D.0.2 Spectral Representation

In this section we present a derivation of the spectral representation (5.2.11) of the Mellin transformed Feynman propagator (D.0.14). This makes use of harmonic analysis on EAdS [153, 156], which we review below, which can be extended to points in dS via analytic continuation [8–10, 127] and, therefore, also to Minkowski space through its hyperbolic foliation.

Consider functions $F(\hat{X},\hat{Y})$ with $\hat{X},\hat{Y}\in \text{EAdS}_{d+1}$, which depend only on the chordal distance r between $\hat{X}$ and $\hat{Y}$,

$$\cosh r = -\hat{X}\cdot\hat{Y}. \quad \text{(D.0.21)}$$

If such functions are square integrable

$$\int_{\mathcal{A}_+}\mathrm{d}^{d+1}\hat{X}\,|F(\hat{X},\hat{Y})|^2 = \text{Vol}(S^d)\int_0^\infty \mathrm{d}r\,(\sinh r)^d\,|F(r)|^2 < \infty, \quad \text{(D.0.22)}$$

they admit a spectral decomposition of the following form in terms of Eigenfunctions (C.2.6) of the AdS Laplacian

$$F\left(r\right)=\int_{-\infty}^{\infty}\mathrm{d}\nu\,\tilde{F}\left(\nu\right)\Omega_{\nu}\left(r\right), \tag{D.0.23}$$

where

$$\tilde{F}\left(\nu\right)=\frac{1}{\Omega_{\nu}\left(0\right)}\int_{\mathcal{A}_{+}}\mathrm{d}^{d+1}\hat{X}\,F(\hat{X},\hat{Y})\Omega_{\nu}(\hat{X},\hat{Y}), \tag{D.0.24}$$

$$=\frac{\mathrm{Vol}(S^{d})}{\Omega_{\nu}\left(0\right)}\int_{0}^{\infty}\mathrm{d}r\left(\sinh r\right)^{d}\,F(r)\Omega_{\nu}(r), \tag{D.0.25}$$

which follows from completeness and orthogonality of the Eigenfunctions (C.2.6):

$$\int_{\mathcal{A}_{+}}\mathrm{d}^{d+1}\hat{Y}\,\Omega_{\nu}(\hat{X}_{1},\hat{Y})\Omega_{\bar{\nu}}(\hat{Y},\hat{X}_{2})=\frac{1}{2}\left[\delta\left(\nu+\bar{\nu}\right)+\delta\left(\nu-\bar{\nu}\right)\right]\Omega_{\nu}(\hat{X}_{1},\hat{X}_{2}), \tag{D.0.26}$$

$$\int_{-\infty}^{\infty}\mathrm{d}\nu\,\Omega_{\nu}(\hat{X},\hat{Y})=\delta(\hat{X},\hat{Y}). \tag{D.0.27}$$

A well-known example is the spectral representation of the AdS bulk-to-bulk propagator (C.2.3), which is normalisable for $\Delta>\frac{d}{2}$ and therefore admits a spectral representation of the form:

$$G_{\Delta}^{\mathrm{AdS}}(r)=\int_{-\infty}^{\infty}\mathrm{d}\nu\,g\left(\nu\right)\Omega_{\nu}(r). \tag{D.0.28}$$

The spectral function is determined by evaluating the inversion formula:

$$g\left(\nu\right)=\frac{\mathrm{Vol}(S^{d})}{\Omega_{\nu}\left(0\right)}\int_{0}^{\infty}\mathrm{d}r\left(\sinh r\right)^{d}\,G_{\Delta}^{\mathrm{AdS}}(r)\,\Omega_{\nu}(r). \tag{D.0.29}$$

In this example, to proceed it is useful to make the change of variables $y=\cosh^{-2}\left(\frac{r}{2}\right)$, so that

$$g\left(\nu\right)=\frac{\mathrm{Vol}(S^{d})}{\Omega_{\nu}\left(0\right)}2^{d}\int_{0}^{1}\mathrm{d}y\,y^{-\left(\frac{d+3}{2}\right)}\left(\frac{1}{y}-1\right)^{\frac{d-1}{2}}G_{\Delta}^{\mathrm{AdS}}(y)\,\Omega_{\nu}(y). \tag{D.0.30}$$

By combining this with the following Mellin-Barnes representation of the Gauss hypergeometric function:

$${}_{2}F_{1}\left(a,b;c;z\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s}{2\pi i}\,\frac{\Gamma\left(s+a\right)\Gamma\left(s+b\right)\Gamma\left(-s\right)}{\Gamma\left(s+c\right)}\left(-z\right)^{s}, \tag{D.0.31}$$

the integral over y reduces to a ratio of gamma functions (the Beta function)

$$\int_0^1 \mathrm{d}y\ (1-y)^{\frac{d+1}{2}-\Delta-s+t-1}\ y^{-d+\Delta+s-t-1} = \frac{\Gamma\left(\frac{d+1}{2}-\Delta-s+t\right)\Gamma(-d+\Delta+s-t)}{\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)}. \tag{D.0.32}$$

This gives

$$g\left(\nu\right) = \frac{1}{\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-i\nu\right)\Gamma\left(\frac{d}{2}+i\nu\right)}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s}{2\pi i}\frac{\mathrm{d}t}{2\pi i}\frac{\Gamma(-s)\Gamma(-t)\Gamma(s+\Delta)}{\Gamma\left(\frac{d+1}{2}+t\right)\Gamma(-d+s+2\Delta+1)}$$
$$\times\Gamma\left(-\tfrac{d}{2}+s+\Delta+\tfrac{1}{2}\right)\Gamma\left(\tfrac{d}{2}+t-i\nu\right)\Gamma\left(\tfrac{d}{2}+t+i\nu\right)\Gamma\left(\tfrac{d}{2}-s+t-\Delta+\tfrac{1}{2}\right)\Gamma(-d+s-t+\Delta). \tag{D.0.33}$$

The Mellin-Barnes integrals in s and t both take the form of the second Barnes lemma, which recovers the known result

$$g\left(\nu\right) = \frac{1}{\nu^2+\left(\Delta-\frac{d}{2}\right)^2}.$$

One proceeds in a similar fashion to determine the spectral decomposition of the Mellin transformed Feynman propagator (5.2.10):

$$\Pi_{\Delta_1,\Delta_2}(r) = \int_{-\infty}^{+\infty}\mathrm{d}\nu\, g_{\Delta_1,\Delta_2}(\nu)\Omega_\nu(r)\,, \tag{D.0.34}$$

where in appendix D.0.1 we showed that

$$\Pi^{(m)}_{\Delta_1\Delta_2}(r) = \frac{1}{2}\frac{1}{(4\pi)^{\frac{d+1}{2}}}\frac{m^{d-\Delta_1-\Delta_2}}{\left(\sqrt{\hat{X}^2+i\epsilon}\right)^{\Delta_1}\left(\sqrt{\hat{Y}^2+i\epsilon}\right)^{\Delta_2}}$$
$$\times\frac{\Gamma(\frac{\Delta_1+\Delta_2-d}{2})\Gamma(\Delta_1)\Gamma(\Delta_2)}{\Gamma\left(\frac{\Delta_1+\Delta_2+1}{2}\right)}\,{}_2F_1\left(\begin{matrix}\Delta_1,\Delta_2\\ \frac{\Delta_1+\Delta_2+1}{2}\end{matrix};\frac{1-\cosh\left(r\right)}{2}\right). \tag{D.0.35}$$

The inversion integral takes the following form:

$$g_{\Delta_1,\Delta_2}(\nu) = \frac{\mathrm{Vol}(S^d)}{\Omega_\nu\left(0\right)}\int_0^\infty \mathrm{d}r\ \left(\cosh\left(r\right)+1\right)^{\frac{d}{2}}\left(\cosh\left(r\right)-1\right)^{\frac{d}{2}}\Pi^{(m)}_{\Delta_1\Delta_2}(r)\,\Omega_\nu(r). \tag{D.0.36}$$

Now, performing the change of variable $z=\cosh(r)$, we get

$$g_{\Delta_1,\Delta_2}(\nu) = \frac{\mathrm{Vol}(S^d)}{\Omega_\nu\left(0\right)}\int_{-\infty}^{-1}\mathrm{d}z\,(z^2-1)^{\frac{d-1}{2}}\Pi^{(m)}_{\Delta_1\Delta_2}(z)\,\Omega_\nu(z). \tag{D.0.37}$$

It is convenient to set

$$g_{\Delta_1,\Delta_2} = \frac{m^{d-\Delta_1-\Delta_2}}{2(4\pi)^{\frac{d+1}{2}}} \frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma\left(\frac{\Delta_1+\Delta_2-d}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2+1}{2}\right)\left(\hat{X}_1^2+i\epsilon\right)^{\frac{\Delta_1}{2}}\left(\hat{X}_2^2+i\epsilon\right)^{\frac{\Delta_2}{2}}} F_{\Delta_1,\Delta_2}(\nu), \tag{D.0.38}$$

so that we can focus on the integral

$$\begin{aligned} F_{\Delta_1,\Delta_2}(\Delta) = 4^{\frac{d-1}{2}}\mathrm{Vol}(S^d)\int_{-\infty}^{-1}\mathrm{d}z\left(\frac{z^2-1}{4}\right)^{\frac{d-1}{2}} {}_2F_1\left(\Delta,\Delta^*;\frac{d+1}{2};\frac{1+z}{2}\right) \\ \times {}_2F_1\left(\Delta_1,\Delta_2;\frac{1+\Delta_1+\Delta_2}{2};\frac{1+z}{2}\right). \end{aligned} \tag{D.0.39}$$

The hypergeometric function ${}_2F_1$ also admits the following Mellin-Barnes integral representation:

$$\begin{aligned} {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \\ \times\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\mathrm{d}s}{2\pi i}\Gamma(s)\Gamma(c-a-b+s)\Gamma(a-s)\Gamma(b-s)(1-z)^{-s}, \end{aligned} \tag{D.0.40}$$

where $max[0,\Re\mathrm{e}(a+b-c)] < \gamma < min[\Re\mathrm{e}(a),\Re\mathrm{e}(b)]$. Using this representation for both the hypergeometric functions in (D.0.39) and then commuting the integration order, we are led to solve the integral

$$\mathcal{I}(s_1,s_2) = \int_{-\infty}^{-1}\mathrm{d}z\left(\frac{z^2-1}{4}\right)^{\frac{d-1}{2}}\left(\frac{1-z}{2}\right)^{s_1+s_2}. \tag{D.0.41}$$

Here s_1 and s_2 are the Mellin-Barnes variables in the representation (D.0.40) of the hypergeometric functions. Setting $2u = -(1+z)$, the integral becomes

$$\mathcal{I}(s_1,s_2) = 2\int_0^{+\infty}\mathrm{d}u\frac{u^{\frac{d-1}{2}}}{(1+u)^{-s_1-s_2+\frac{1-d}{2}}} = 2\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(-s_1-s_2-d)}{\Gamma\left(-s_1-s_2+\frac{1-d}{2}\right)}. \tag{D.0.42}$$

Therefore, it follows that

$$\begin{aligned} F_{\Delta_1,\Delta_2}(\Delta) &= C_{\Delta_1,\Delta_2}(\Delta) \\ &\times\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_2}{2\pi i}\Gamma(-s_2)\Gamma\left(\frac{1-\Delta_1-\Delta_2}{2}-s_2\right)\Gamma(\Delta_1+s_2)\Gamma(\Delta_2+s_2) \\ &\times\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_1}{2\pi i}\frac{\Gamma(-s_1)\Gamma\left(\frac{1-d}{2}-s_1\right)\Gamma(\Delta+s_1)\Gamma(\Delta^*+s_1)\Gamma(-s_1-s_2-d)}{\Gamma\left(-s_1-s_2+\frac{1-d}{2}\right)}, \end{aligned} \tag{D.0.43}$$

where

$$C_{\Delta_1,\Delta_2}(\Delta)=\frac{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{1+\Delta_1+\Delta_2}{2}\right)}{\Gamma(\Delta)\Gamma(\Delta^*)\Gamma\left(\frac{1+d}{2}-\Delta\right)\Gamma\left(\frac{1+d}{2}-\Delta^*\right)\prod_{i=1}^2\Gamma(\Delta_i)\Gamma\left(\frac{1+\Delta_1+\Delta_2}{2}-\Delta_i\right)}. \tag{D.0.44}$$

The integral

$$J(\Delta)=\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_1}{2\pi i}\frac{\Gamma(-s_1)\Gamma\left(\frac{1-d}{2}-s_1\right)\Gamma(\Delta+s_1)\Gamma(\Delta^*+s_1)\Gamma(-s_1-s_2-d)}{\Gamma\left(-s_1-s_2+\frac{1-d}{2}\right)}, \tag{D.0.45}$$

after sending $s_1\to -s_1-\Delta_1$ can be solved by using the second Barnes' Lemma . We found

$$J(\Delta)=\frac{\Gamma(\Delta)\Gamma(\Delta^*)\Gamma\left(\frac{1+d}{2}-\Delta\right)\Gamma\left(\frac{1+d}{2}-\Delta^*\right)\Gamma(\Delta-d-s_2)\Gamma(\Delta^*-d-s_2)}{\Gamma\left(\frac{1+d}{2}\right)\Gamma\left(\frac{1-d}{2}-s_2\right)\Gamma(-s_2+\Delta+\Delta^*-d)}. \tag{D.0.46}$$

Plugging this last equation in (D.0.43), we are led to compute the last Mellin-Barnes integral

$$\begin{aligned}&\tilde{J}_{\Delta_1,\Delta_2}(\Delta)=\\&=\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_2}{2\pi i}\frac{\Gamma(\Delta_1+s_2)\Gamma(\Delta_2+s_2)\Gamma(\Delta-d-s_2)\Gamma(\Delta^*-d-s_2)\Gamma\left(\frac{1-\Delta_1-\Delta_2}{2}-s_2\right)}{\Gamma\left(\frac{1-d}{2}-s_2\right)},\end{aligned} \tag{D.0.47}$$

which, this time, cannot be solved by means of the second Barnes' Lemma, as the integrand does not satisfy its assumptions. Nevertheless, after sending $s_2\to -s_2-\Delta_1$, we can use the formula (Bailey, *Generalized Hypergeometric Series*, pag. 43)

$$\begin{aligned}&\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s}{2\pi i}\frac{\Gamma(-s)\Gamma(-s+b_1-a_1-a_2)\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(b_2-a_3+s)}{\Gamma(b_2+s)}=\\&=\frac{\Gamma(b_1-a_1)\Gamma(b_1-a_2)\Gamma(b_2-a_3)\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1)\Gamma(b_2)}\,{}_3F_2(a_1,a_2,a_3;b_1,b_2;1),\end{aligned} \tag{D.0.48}$$

with $\Re\mathrm{e}(s+b_2-a_3)>0$, which yields

$$\begin{aligned}\tilde{J}_{\Delta_1,\Delta_2}(\Delta)&=\frac{\Gamma(\Delta_2+\Delta^*)\Gamma(\Delta_2+\Delta)\Gamma\left(\frac{1-\Delta_1+\Delta_2}{2}\right)\Gamma\left(\frac{1+\Delta_1-\Delta_2}{2}\right)}{\Gamma\left(\frac{1+\Delta_1+\Delta_2}{2}+\Delta^*\right)\Gamma\left(\Delta_1+\frac{1-d}{2}\right)} \qquad \text{(D.0.49)}\\&\times{}_3F_2\left(\Delta_1+\Delta^*,\frac{1+\Delta_1-\Delta_2}{2},\frac{1+d}{2}-\Delta^*;\frac{1+\Delta_1+\Delta_2}{2}+\Delta^*,\Delta_1+\frac{1-d}{2};1\right).\end{aligned}$$

Now, the Dixon's theorem states that

$$
\begin{aligned}
&{}_3F_2(a,b,c;1+a-b,1+a-c;1) = \\
&\qquad = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}. \quad \text{(D.0.50)}
\end{aligned}
$$

Therefore, using Eq.(D.0.50), after some algebraic computation we get

$$
\begin{aligned}
&\tilde{J}_{\Delta_1,\Delta_2} = \\
&= \frac{\Gamma(\Delta_1-\Delta)\Gamma(\Delta_1-\Delta^*)\Gamma(\Delta_2-\Delta^*)\Gamma(\frac{\Delta_2-\Delta}{2})\Gamma(\frac{1-\Delta_2+\Delta_1}{2})\Gamma(\frac{1+\Delta_2-\Delta_1}{2})\Gamma(1+\frac{\Delta_1-\Delta^*}{2})}{\Gamma(1+\Delta_1-\Delta^*)\Gamma(\frac{1+\Delta_2-\Delta^*}{2})\Gamma(\frac{1+\Delta_1-\Delta}{2})\Gamma(\frac{\Delta_1+\Delta_2-d}{2})}. \\
&\qquad \text{(D.0.51)}
\end{aligned}
$$

By applying the Doubling formula to the Euler gamma functions $\Gamma(\frac{\Delta_2-\Delta}{2})$ and $\Gamma(1+\frac{\Delta_1-\Delta^*}{2})$, we finally find that

$$
\begin{aligned}
&\tilde{J}_{\Delta_1,\Delta_2} = \pi\, 2^{1+d-\Delta_1-\Delta_2} \\
&\times \frac{\Gamma(\Delta_1-\Delta)\Gamma(\Delta_1-\Delta^*)\Gamma(\Delta_2-\Delta^*)\Gamma(\Delta_2-\Delta)\Gamma(\frac{1-\Delta_2+\Delta_1}{2})\Gamma(\frac{1+\Delta_2-\Delta_1}{2})}{\Gamma(\frac{1+\Delta_1-\Delta^*}{2})\Gamma(\frac{1+\Delta_2-\Delta^*}{2})\Gamma(\frac{1+\Delta_1-\Delta}{2})\Gamma(\frac{1+\Delta_2-\Delta}{2})\Gamma(\frac{\Delta_1+\Delta_2-d}{2})}, \\
&\qquad \text{(D.0.52)}
\end{aligned}
$$

Gluing all together, we get

$$
\begin{aligned}
&F_{\Delta_1,\Delta_2}(\Delta) = (4\pi)^{\frac{d}{2}} \times \qquad \text{(D.0.53)} \\
&\frac{\Gamma(\Delta_1-\Delta)\Gamma(\Delta_1-\Delta^*)\Gamma(\Delta_2-\Delta^*)\Gamma(\Delta_2-\Delta)\Gamma(\frac{1+\Delta_1+\Delta_2}{2})\Gamma(\frac{1-d+\Delta_1+\Delta_2}{2})}{\Gamma(\Delta_1+\Delta_2-d)\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\frac{1+\Delta_1-\Delta^*}{2})\Gamma(\frac{1+\Delta_2-\Delta^*}{2})\Gamma(\frac{1+\Delta_1-\Delta}{2})\Gamma(\frac{1+\Delta_2-\Delta}{2})},
\end{aligned}
$$

which, plugged in (D.0.38) gives the result for $g_{\Delta_1,\Delta_2}(\nu)$.

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