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Mixed Variational Methods

in Elasticity

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Contents

Preface

The thesis collects the results of a recent research activity on mixed problems carried out by the author at the Dipartimento di Scienza delle Costruzioni of the Università di Napoli Federico II.

The presentation is confined to linear elastostatic problems and offers a comprehensive review of the computational methods recently proposed in the literature.

Three-field mixed methods are investigated both from the theoretical and the computational side and numerical experiments on benchmark examples are worked out. The main efforts have been directed in providing a unifying treatment of the continuous problems and of their discrete counterparts.

The most significant issues are concerned with

- the definition of abstract structural models and the development of a general variational theory for convex nonlinear problems,
- the correct formulation of the enhanced strain method which has motivated the introduction of the strain gap method,
- the proof of well-posedness conditions for all the three-field mixed methods,
- the proof of convergence of the discrete solutions of three-field method based on the Hu-Washizu principle and of the strain gap method,
- the discussion about the difficulties faced in extending the well-posedness and convergence conditions to distorted meshes,
- the statement of limitation principles concerning all the mixed methods and their application to usual benchmark examples.

The author hopes that an organized collection of results and references could eventually be useful to researchers involved in investigations on mixed methods in structural mechanics and related fields and could contribute to a deeper understanding of these methods also by the ones who are mainly interested in numerical applications.

Chapter 1

Variational methods

1.1 Introduction

This chapter is devoted to present some preliminary issues concerning the formulation of a structural model in abstract form and the development of a variational theory for the elastostatic problem.

The abstract formalism has the following advantages

- the notation is compact, the statement are simpler
- the results are expressed in a form which is directly applicable to all the special structural models to which the theory is applicable.

Although we shall deal exclusively with elastic structures the variational principles will be introduced in a more general context which allows a direct extension to nonlinear material behaviours and permits to describe in an effective way the presence of geometrical constraints, that is constraints on the kinematical fields.

A large class of nonlinear problems in structural analysis, such as those involving irreversible phenomena obeying to a principle of maximal dissipation, can be formulated by assuming that the constitutive behaviour is governed by conservative multivalued operators which are maximal monotone [56], [58].

It has been shown in [65] that the values of such operators are the subdifferentials of convex potentials and that the potentials can be evaluated by direct integration along an arbitrary polyline in the domain of definition of the operators. It is then possible to develop a general variational theory for this class of nonlinear problems and a complete family of variational principles can be derived [66].

The most general principle involves all the basic variables describing the state of the system. A chain of eliminations of the state variables, allows to derive all the other principles.

The criterion of elimination is based on the application of a FENCHEL transform [6], [19] to a state variable in order to get an equivalent expression of the functional in terms of the dual variable.

By choosing displacements and forces, strains and stresses as pairs of basic state variables, the constitutive laws are expressed by two multivalued relations. The former one expresses the external constraint between displacements and forces, the latter one describes the internal constraint between strains and stresses. Both relations are multivalued, conservative and monotone, respectively non-increasing and non-decreasing, and hence can be expressed in terms of their concave and convex potentials.

The structural problem can then be written in terms of constraint potentials and of a pair of dual linear operators governing the static and the kinematic compatibility. A line integration of the global structural operator along a segment in a product space yields a tree-shaped family of variational principles composed by ten basic functionals [66].

As an example of the advantages of the general treatment of variational principles outlined in this first chapter, we present in the last section a brief account of the hybrid formulations proposed in the literature from an unifying point of view.

1.2 Structural models

The analysis of mixed methods will be performed in the framework of abstract continuous structural models. A mathematical theory of structural models has recently been developed by the first author and a comprehensive treatment can be found in [89], [90].

The fundamental issues are briefly recalled hereafter.

Let us describe an abstract continuous structural model $\mathcal{M}$ defined on a bounded domain Ω of an n-dimensional Euclidean space with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$. The LEBESGUE measure in Ω is denoted by $d\,\mu$ and $d\,\sigma$ will denote the induced superficial measure on $\partial\Omega$.

To this end let us consider the following items.

- The pivot HILBERT spaces $\mathcal{H} = L^2(\Omega)^q$ and $H = L^2(\Omega)^p$ of square integrable q-vector fields and p-vector fields in Ω.
- The SOBOLEV space of order $H^m(\Omega)^p$ of p-vector fields with square integrable distributional derivatives of order up to m (see e.g. [39], [89]).
- The linear space $\mathbb{D} = \mathrm{C}_o^\infty(\Omega)^p$ of test p-vector fields which are indefinitely differentiable in Ω and have compact support in Ω.

We recall that the support of a field $\mathbf{u}$, denoted by $\mathrm{supp}(\mathbf{u})$, is the smallest closed set outside which the field vanishes.

The space $\mathbb{D}$ is endowed with the pseudo-topology induced by the following definition of convergence:

- A sequence $\{\mathbf{u}_n\} \in \mathbb{D}$ is said to converge to $\mathbf{u} \in \mathbb{D}$ if there exists a compact subset $\boldsymbol{K} \subset \Omega$ such that $\mathrm{supp}(\mathbf{u}_n) \subset \boldsymbol{K}$ and $D^{\mathbf{m}}\mathbf{u}_n \to D^{\mathbf{m}}\mathbf{u}$ uniformly in Ω for any vectorial multiindex $\mathbf{m}$. A vectorial multiindex $\mathbf{m}$ is a list of p scalar multiindices each formed by n positive integers to denote the order of partial differentiation with respect to the corresponding coordinate. The symbol $|\,\mathbf{m}\,|$ denotes the sum of the integers in $\mathbf{m}$.

  The linear space $\mathbb{D}'$ of p-distributions on Ω, the dual of $\mathbb{D}$, formed by the linear functionals which are continuous on $\mathbb{D}$. The space $\mathbb{D}'$ is in turn endowed with the pseudo-topology induced by the following definition of convergence:

- A sequence of distributions $\{\mathbb{T}_n\} \in \mathbb{D}$ is said to converge to a distribution $\mathbb{T} \in \mathbb{D}$ if for any test field $\boldsymbol{\varphi} \in \mathbb{D}$ we have that $\mathbb{T}_n(\boldsymbol{\varphi}) \to \mathbb{T}(\boldsymbol{\varphi})$.

  Analogous definitions hold for q-vector fields in Ω and q-distributions on Ω. In the sequel, to simplify the notations, the spaces of test fields and the spaces of distributions will be denoted by $\mathbb{D}$ and $\mathbb{D}'$ regardless of the dimension od the vector fields.

The structural model is characterized by a distributional differential operator $\mathbb{B} : H \to \mathbb{D}'$ which provides the distributional strain field $\mathbb{B}\mathbf{v} \in \mathbb{D}'$ corresponding to the displacement fields $\mathbf{v} \in H$. The general form of an $\mathbf{m}$th-order differential operator $\mathbf{B} : H \to \mathbb{D}'$ can expressed as

$$(\mathbf{B}\,\mathbf{u})(\mathbf{x}) := \sum_{|\,\mathbf{p}\,|\leq m} \sum_{i=1}^{n} \mathbf{A}_{\mathbf{p}}^{i}(\mathbf{x})\, D^{\mathbf{p}}\mathbf{u}_i(\mathbf{x}) \,, \quad \mathbf{x} \in \Omega \,,$$

where $\mathbf{A}_{\mathbf{p}}(\mathbf{x})^i$ is a regular field of $n \times n$ matrices in Ω.

A proper definition of a structural model requires to include into the kinematic space the vector fields which are piecewise $H^m(\Omega)^p v$.

To this end we consider the decompositions $\mathcal{T}(\Omega)$ of Ω into a finite family of non-overlapping elements $\Omega_e \subseteq \Omega$ with boundary $\partial\Omega_e$ where $ve = 1, \dots, n$. The elements $\Omega_e \in \mathcal{T}(\Omega)$ of a decomposition meet the properties

$$\Omega_\alpha \cap \Omega_\beta = \emptyset \quad \text{for} \quad \alpha \neq \beta \qquad \text{and} \qquad \bigcup_{e=1}^{n} \overline{\Omega}_e = \overline{\Omega}.$$

The kinematic space $\mathcal{V}(\Omega)$ of GREEN-regular displacement fields is then defined as a subspace of H by requiring that, for any $\mathbf{v} \in \mathcal{V}(\Omega)$, there exists a decomposition $\mathcal{T}_{\mathbf{v}}(\Omega)$ such that the restrictions $\mathbf{v}|_e$ to the elements Ω_e of $\mathcal{T}_{\mathbf{v}}(\Omega)$ belong to $H^m(\Omega_e)^p$.

The space $\mathcal{V}(\Omega)$ is a pre-HILBERT space when endowed with the inner product inherited piecewice from $H^m(\Omega_e)^p$ by setting $\forall \mathbf{u}, \mathbf{v} \in \mathcal{V}(\Omega)$

$$(\mathbf{u}, \mathbf{v})_{\mathcal{V}} := \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\mu + \int_\Omega \mathbf{B}\mathbf{u} : \mathbf{B}\mathbf{v} \, d\mu = (\, \mathbf{u} \,,\, \mathbf{v} \,)_H + (\!(\, \mathbf{B}\mathbf{u} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} \,.$$

The symbols $(\!(\, \cdot \,,\, \cdot \,)\!)_{\mathcal{H}}$ and $(\, \cdot \,,\, \cdot \,)_H$ denote respectively the inner products in $\mathcal{H}$ and in H. The kinematic operator $\mathbf{B} \in BL(\mathcal{V}(\Omega), \mathcal{H})$ is the bounded linear map from $\mathcal{V}(\Omega)$ into $\mathcal{H}$ which provides the regular part $\mathbf{B}\mathbf{u} \in \mathcal{H}$ of the distributional strain $\mathbb{B}\mathbf{u} \in \mathbb{D}'$ corresponding to the displacement field $\mathbf{u} \in \mathcal{V}(\Omega)$.

The regular part $\mathbf{B}\mathbf{u} \in \mathcal{H}$ is the list of the square integrable strain fields corresponding to the restrictions of $\mathbf{u} \in \mathcal{V}(\Omega)$ to each element $\Omega_e \in \mathcal{T}_{\mathbf{u}}(\Omega)$.

It is assumed that the kinematic operator $\mathbf{B}$ fulfils an inequality of KORN's type:

$$\| \, \mathbf{B}\mathbf{v} \, \|_{\mathcal{H}} + \| \, \mathbf{v} \, \|_H \geq \alpha \, \| \, \mathbf{v} \, \|_{H^m} \quad \forall \, \mathbf{v} \in H^m(\Omega) \,.$$

Then the space $\mathcal{V}(\Omega)$ endowed with the norm

$$\Big[(\, \mathbf{v} \,,\, \mathbf{v} \,) + (\!(\, \mathbf{B}\mathbf{v} \,,\, \mathbf{B}\mathbf{v} \,)\!) \Big]^{1/2},$$

is isomorphic and isometric to $H^m(\Omega)$.

The formal adjoint od the differential operator $\mathbf{B}$ is the distributional differential operator defined by

$$\langle \, \mathbb{B}'_o \boldsymbol{\sigma} \,,\, \mathbf{v} \, \rangle := (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} \quad \forall \, \mathbf{v} \in \mathbb{D}, \quad \forall \, \boldsymbol{\sigma} \in \mathcal{H},$$

where $\langle \cdot , \cdot \rangle$ is the duality pairing between $\mathbb{D}$ and its topological dual $\mathbb{D}'$.

The space $\mathcal{S}(\Omega)$ of GREEN-regular stress fields is the linear space of stress fields $\boldsymbol{\sigma} \in \mathcal{H}$ such that the corresponding body force distribution is representable by a piecewice square integrable field on Ω.

This means that there exists a decomposition $\mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$ such that

$$\mathbb{B}'_o \boldsymbol{\sigma}|_e \in L^2(\Omega_e)^p .$$

The body equilibrium operator $\mathbf{B}'_o$ is the regular part $\mathbf{B}'_o \boldsymbol{\sigma} \in H$ of the distributional body force field $\mathbb{B}'_o \boldsymbol{\sigma} \in \mathbb{D}'$ corresponding to the stress field $\boldsymbol{\sigma} \in \mathcal{H}$. The space $\mathcal{S}(\Omega)$ is a pre-HILBERT space when endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{S}} := \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} \, d\mu + \int_{\Omega} \mathbf{B}'_o \boldsymbol{\sigma} \cdot \mathbf{B}'_o \boldsymbol{\tau} \, d\mu , \qquad \forall \, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{S}(\Omega) ,$$

and the induced norm

$$\Big[(\!(\, \boldsymbol{\sigma} \, , \, \boldsymbol{\sigma} \,)\!)_{\mathcal{H}} + (\, \mathbf{B}'_o \boldsymbol{\sigma} \, , \, \mathbf{B}'_o \boldsymbol{\sigma} \,)_H \Big]^{1/2} .$$

The operator $\mathbf{B}'_o \in BL(\mathcal{S}(\Omega), H)$ is linear and bounded.

For every decomposition $\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega)$ finer than $\mathcal{T}_{\mathbf{v}}(\Omega)$ and $\mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$ the following GREEN's formula holds

$$\int_{\Omega} \boldsymbol{\sigma} : \mathbf{B} \mathbf{v} = \int_{\Omega} \mathbf{B}'_o \boldsymbol{\sigma} \cdot \mathbf{v} + \langle\!\langle \, \mathbf{N} \boldsymbol{\sigma} \, , \, \boldsymbol{\Gamma} \mathbf{v} \, \rangle\!\rangle .$$

where $\langle\!\langle \, \mathbf{N} \boldsymbol{\sigma} \, , \, \boldsymbol{\Gamma} \mathbf{v} \, \rangle\!\rangle$ is the extension by continuity of a sum of boundary integrals over $\partial \mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega) = \cup \, \partial \Omega_e$, $e = 1, \ldots, n$. Setting

$$(\!(\, \boldsymbol{\sigma} \, , \, \mathbf{B} \mathbf{v} \,)\!)_{\mathcal{H}} := \int_{\Omega} \boldsymbol{\sigma} : \mathbf{B} \mathbf{v} , \qquad (\, \mathbf{B}'_o \boldsymbol{\sigma} \, , \, \mathbf{v} \,)_H := \int_{\Omega} \mathbf{B}'_o \boldsymbol{\sigma} \cdot \mathbf{v} ,$$

GREEN's formula can be written as

$$(\!(\, \boldsymbol{\sigma} \, , \, \mathbf{B} \mathbf{v} \,)\!)_{\mathcal{H}} = (\, \mathbf{B}'_o \boldsymbol{\sigma} \, , \, \mathbf{v} \,)_H + \langle\!\langle \, \mathbf{N} \boldsymbol{\sigma} \, , \, \boldsymbol{\Gamma} \mathbf{v} \, \rangle\!\rangle , \quad \forall \, \mathbf{v} \in \mathcal{V}(\Omega), \quad \forall \, \boldsymbol{\sigma} \in \mathcal{S}(\Omega),$$

We shall denote by $\mathcal{V} = \mathcal{V}(\mathcal{T}(\Omega))$ and $\mathcal{S} = \mathcal{S}(\mathcal{T}(\Omega))$ the spaces of kinematical fields and of stress fields which are GREEN-regular in correspondence of a given subdivision $\mathcal{T}(\Omega)$.

The linear operator $\mathbf{N} \in BL(\mathcal{S}, \partial\mathcal{F})$ yields the boundary traction $\mathbf{N}\boldsymbol{\sigma} \in \partial\mathcal{F}$ associated with a stress field $\boldsymbol{\sigma} \in \mathcal{S}$ and the trace operator $\boldsymbol{\Gamma} \in BL(\mathcal{V}, \partial\mathcal{V})$ defines the boundary values of displacement fields $\mathbf{v} \in \mathcal{V}$.

The boundary kinematic space is defined by $\partial\mathcal{V} = \times\partial\mathcal{V}_e$, $e = 1, \ldots, n$ and the dual boundary traction space is defined by $\partial\mathcal{F} = \times\partial\mathcal{F}_e$, $e = 1, \ldots, n$ with duality pairing defined by $\langle\!\langle \cdot \,, \cdot \rangle\!\rangle = \sum \langle\!\langle \cdot \,, \cdot \rangle\!\rangle_e$, $e = 1, \ldots, n$.

The boundary kinematic operator $\boldsymbol{\Gamma} \in BL(\mathcal{V}, \partial\mathcal{V})$ and the boundary equilibrium operator $\mathbf{N} \in BL(\mathcal{S}, \partial\mathcal{F})$ enjoy the properties that $\operatorname{Im}\boldsymbol{\Gamma} = \partial\mathcal{V}$, $\operatorname{Ker}\boldsymbol{\Gamma}$ is dense in H, $\operatorname{Im}\mathbf{N}$ is dense in $\partial\mathcal{F}$ and $\operatorname{Ker}\mathbf{N}$ is dense in $\mathcal{H}$.

The presence of rigid frictionless bilateral constraints on the boundary imposes a further conformity requirement which can be described by means of the dual HILBERT spaces $\{\Lambda\,, \Lambda'\}$ and $\{\mathcal{P}\,, \mathcal{P}'\}$ and of the bounded linear operators $\mathbf{L} \in BL(\partial\mathcal{V}, \Lambda')$ and $\boldsymbol{\Pi} \in BL(\mathcal{P}, \partial\mathcal{V})$.

The operators $\mathbf{L}$ and $\boldsymbol{\Pi}$ provide respectively an implicit and an explicit description of the boundary constraints.

We assume that $\mathbf{L}$ and $\boldsymbol{\Pi}$ have closed ranges.

Denoting by $\mathbf{L}' \in BL(\Lambda, \partial\mathcal{F})$ and $\boldsymbol{\Pi}' \in BL(\partial\mathcal{F}, \mathcal{P}')$ the dual operators, we have $\operatorname{Im}\mathbf{L}' = (\operatorname{Ker}\mathbf{L})^{\perp}$ and $\operatorname{Im}\boldsymbol{\Pi} = (\operatorname{Ker}\boldsymbol{\Pi}')^{\perp}$.

The displacement fields belonging to the subspace

$$\mathcal{L} = \left\{ \mathbf{v} \in \mathcal{V} \mid \boldsymbol{\Gamma}\mathbf{v} \in \operatorname{Im}\boldsymbol{\Pi} = \operatorname{Ker}\mathbf{L} \right\}$$

are said to be conforming.

KORN's inequality

$$\| \mathbf{B}\mathbf{v} \|_{\mathcal{H}} + \| \mathbf{v} \|_{H} \geq \alpha \, \| \mathbf{v} \|_{H^m} \quad \forall\, \mathbf{v} \in H^m(\Omega)\,,$$

is equivalent to state that the kinematic operator $\mathbf{B} \in BL(\mathcal{L}, \mathcal{H})$ fulfils the following conditions for every conforming subspace $\mathcal{L} \subset \mathcal{V}$ [86]:

$$\begin{cases} \dim\, \operatorname{Ker}\mathbf{B} < +\infty\,, \\ \| \mathbf{B}\mathbf{v} \|_{\mathcal{H}} \geq c_{\mathbf{B}} \, \| \mathbf{v} \|_{\mathcal{L}/\operatorname{Ker}\mathbf{B}} \quad \forall\, \mathbf{v} \in \mathcal{L} \end{cases} \iff \quad \operatorname{Im}\mathbf{B} \ \text{closed in}\ \mathcal{H}\,.$$

The boundary reactions are elements of the subspace

$$\partial\mathcal{R} = \left\{ \mathbf{r} \in \partial\mathcal{F} \mid \langle\!\langle\, \mathbf{r}\,, \boldsymbol{\Gamma}\mathbf{v} \,\rangle\!\rangle = 0 \quad \forall\, \mathbf{v} \in \mathcal{L} \right\} = (\boldsymbol{\Gamma}\mathcal{L})^{\perp} = \operatorname{Im}\mathbf{L}' = \operatorname{Ker}\boldsymbol{\Pi}'.$$

Uniqueness of the parametric representations of $\boldsymbol{\Gamma}\mathcal{L}$ and $\partial\mathcal{R}$ requires respectively that $\operatorname{Ker}\boldsymbol{\Pi} = \{\, \boldsymbol{o} \,\}$ and $\operatorname{Ker}\mathbf{L}' = \{\, \boldsymbol{o} \,\}$.

1.3 Variational principles

Let us consider the equilibrium of a linearly elastic structural model subject to a system of forces $\overline{\mathbf{f}} = \{\, \overline{\mathbf{b}}, \overline{\mathbf{t}}\,\} \in \mathcal{F}$ composed by body forces $\overline{\mathbf{b}} \in H$ and boundary tractions $\overline{\mathbf{t}} \in \partial\mathcal{F}$.

The structure is further subject to a set of distorsions $\overline{\boldsymbol{\delta}} \in \mathcal{D}$, imposed boundary displacements $\boldsymbol{\Gamma}\overline{\mathbf{v}} \in \partial\mathcal{V}$ and to homogeneous boundary conditions defined by the operator $\mathbf{L}$.

Admissible displacements must then belong to the affine set $\overline{\mathbf{v}} + \mathcal{L}$.

The elastic strain energy is provided by the convex quadratic functional $\phi : \mathcal{H} \mapsto \mathcal{R} \cup \{\, +\infty\,\}$ given by

$$\phi(\boldsymbol{\varepsilon}) = \frac{1}{2} (\!(\, \mathbf{E}(\boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}})\, ,\, \boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}} \,)\!)_{\mathcal{H}},$$

where $\mathbf{E}$ is the elastic stiffness of the material.

The linear elasticity operator $\mathbf{E} \in BL(\mathcal{H}, \mathcal{H})$ is continuous, symmetric and $\mathcal{H}$-elliptic, that is

$$(\!(\, \mathbf{E}\boldsymbol{\varepsilon}\, ,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} \geq c_{\mathcal{E}} \,\|\, \boldsymbol{\varepsilon} \,\|^2_{\mathcal{H}} \quad \forall\, \boldsymbol{\varepsilon} \in \mathcal{H}\,.$$

The constitutive relation is thus expressed by the differential relation

$$\boldsymbol{\sigma} = d\,\phi(\boldsymbol{\varepsilon}) = \mathbf{E}(\boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}}).$$

The conjugate [64] convex functional $\phi^* : \mathcal{H} \mapsto \mathcal{R} \cup \{\, +\infty\,\}$ represents the complementary elastic energy and is given by

$$\phi^*(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\varepsilon} \in \mathcal{D}} \{\, (\!(\, \boldsymbol{\sigma}\, ,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} - \phi(\boldsymbol{\varepsilon}) \,\} = \frac{1}{2} (\!(\, \mathbf{C}\boldsymbol{\sigma}\, ,\, \boldsymbol{\sigma} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\, ,\, \overline{\boldsymbol{\delta}} \,)\!)_{\mathcal{H}},$$

where $\mathbf{C} = \mathbf{E}^{-1} \in BL(\mathcal{H}, \mathcal{H})$ is the elastic compliance. The constitutive relation can be inverted according to the expression

$$\boldsymbol{\varepsilon} = d\,\phi^*(\boldsymbol{\sigma}) = \mathbf{C}\boldsymbol{\sigma} + \overline{\boldsymbol{\delta}}.$$

The system of forces $\mathbf{f} \in \mathcal{F}$ and the admissible displacements $\mathbf{v} \in \overline{\mathbf{v}} + \mathcal{L}$ are related by the external constraint conditions:

$$\begin{cases} \mathbf{b} = \overline{\mathbf{b}}\,, & \\ \mathbf{t} = \overline{\mathbf{t}} + \mathbf{r}, & \mathbf{r} \in \mathcal{R}\,, \\ \langle\!\langle\, \mathbf{r}\, ,\, \boldsymbol{\Gamma}(\mathbf{v} - \overline{\mathbf{v}}) \,\rangle\!\rangle = 0, & \mathbf{v} \in \overline{\mathbf{v}} + \mathcal{L}\,. \end{cases}$$

An alternative synthetic expression can be provided by defining the concave potential

$$\gamma(\mathbf{v}) = \gamma_o(\mathbf{v}) + \sqcap_{\mathrm{Ker}\,\mathbf{L}}[\mathbf{\Gamma}(\mathbf{v} - \overline{\mathbf{v}})],$$

where the linear functional

$$\gamma_o(\mathbf{v}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle,$$

yields the virtual work of body forces and boundary tractions and the concave indicator $\sqcap_{\mathrm{Ker}\,\mathbf{L}}$ is defined by

$$\sqcap_{\mathrm{Ker}\,\mathbf{L}}(\mathbf{\Gamma}\mathbf{v}) = \begin{cases} 0 & \text{if } \mathbf{\Gamma}\mathbf{v} \in \mathrm{Ker}\,\mathbf{L}, \\ -\infty & \text{otherwise.} \end{cases}$$

Denoting by ∂ the supdifferential of concave functionals [64], the set of conditions above are simply expressed by the subdifferential relation

$$\mathbf{f} \in \partial\gamma(\mathbf{v}),$$

which, in terms of the conjugate concave functional

$$\gamma^*(\mathbf{f}) = \inf_{\mathbf{v}\in\mathcal{V}} \{\,\langle\!\langle\,\mathbf{f}\,,\,\mathbf{v}\,\rangle\!\rangle - \gamma(\mathbf{v})\,\} = \sqcap_{(\mathrm{Ker}\,\mathbf{L})^\perp}(\mathbf{f} - \ell) + \langle\!\langle\,\mathbf{f} - \ell\,,\,\overline{\mathbf{v}}\,\rangle\!\rangle,$$

can be inverted into

$$\mathbf{v} \in \partial\gamma^*(\mathbf{f}).$$

The pair $\{\,\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\,\}$ and $\{\,\mathbf{f}, \mathbf{v}\,\}$ which fulfil the internal constitutive relation and the external constraint conditions meet the Fenchel's equalities [64]

$$\phi(\boldsymbol{\varepsilon}) + \phi^*(\boldsymbol{\sigma}) = (\!(\,\boldsymbol{\sigma}\,,\,\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} \qquad \gamma(\mathbf{v}) + \gamma^*(\mathbf{f}) = \langle\,\mathbf{f}\,,\,\mathbf{v}\,\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the duality pairing between $\mathcal{V}$ and its topological dual $\mathcal{F}$.

The equilibrium operator $\mathbf{B}' \in BL(\mathcal{S}, H \times \partial\mathcal{F})$ is defined by the duality relation

$$\langle\,\mathbf{B}'\boldsymbol{\sigma}\,,\,\mathbf{v}\,\rangle = (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}\,)\!)_{\mathcal{H}} \quad \forall\,\mathbf{v} \in \mathcal{V}, \quad \forall\,\boldsymbol{\sigma} \in \mathcal{S},$$

and hence the structural problem can be written in operator form as [63]

$$\mathbf{B}'\boldsymbol{\sigma} = \mathbf{f}, \qquad \mathbf{B}\mathbf{v} = \boldsymbol{\varepsilon}, \qquad \boldsymbol{\sigma} = d\,\phi(\boldsymbol{\varepsilon}), \qquad \mathbf{v} \in \partial\gamma^*(\mathbf{f}),$$

and can be shown to be equivalent to the variational conditions of stationarity for the following family of functionals [63].

We define the dual product spaces $\mathcal{X} = \mathcal{D} \times \mathcal{S} \times \mathcal{V} \times \mathcal{F}$ and $\mathcal{X}' = \mathcal{S} \times \mathcal{D} \times \mathcal{F} \times \mathcal{V}$, so that the structural problem can be expressed in terms of the global structural operator $\mathbf{\Lambda} : \mathcal{X} \mapsto \mathcal{X}'$ as follows

$$\begin{vmatrix} \boldsymbol{o} \\ \boldsymbol{o} \\ \boldsymbol{o} \\ \boldsymbol{o} \end{vmatrix} \in \mathbf{\Lambda} \begin{vmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \\ \mathbf{v} \\ \mathbf{f} \end{vmatrix} \qquad \text{where} \qquad \mathbf{\Lambda} = \begin{bmatrix} d\phi & -\mathbf{I}_{\mathcal{S}} & \mathbf{O} & \mathbf{O} \\ -\mathbf{I}_{\mathcal{D}} & \mathbf{O} & \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}' & \mathbf{O} & -\mathbf{I}_{\mathcal{F}} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I}_{\mathcal{V}} & \partial\gamma^* \end{bmatrix}$$

By integrating along a ray in the $\mathcal{X}-$space we get the potential of $\mathbf{\Lambda}$ as a functional L of $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{f})$.

A proper elimination of the state variables generates a family of potentials according to a tree-shaped scheme.

$$\begin{array}{ccccccc} & & & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{f}\} & & & \\ & & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v}\} & & \{\boldsymbol{\sigma}, \mathbf{v}, \mathbf{f}\} & & \\ & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}\} & & \{\boldsymbol{\sigma}, \mathbf{v}\} & & \{\mathbf{v}, \mathbf{f}\} & \\ \{\boldsymbol{\varepsilon}\} & & \{\boldsymbol{\sigma}\} & & \{\mathbf{v}\} & & \{\mathbf{f}\} \end{array}$$

The variational family consists of the ten potentials reported in the sequel.

All the potentials of the family assume the same value at a solution point.

The extremum properties of each potential can be deduced by taking into account the convexity or concavity property with respect to each argument.

The functionals $H(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v})$ and $R_1(\boldsymbol{\sigma}, \mathbf{v})$ are the mixed functionals of Hu-Washizu and Hellinger-Reissner.

The functionals $P_3(\mathbf{v})$ and $P_2(\boldsymbol{\sigma})$ are the well-known functionals of the total potential energy and the total complementary energy.

The expression of the functionals remain formally identical when the constitutive behaviors of the material and of the constraints are described by non quadratic convex or concave potentials.

$$
\boxed{
\begin{aligned}
L(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v},\mathbf{f}) &= \phi(\boldsymbol{\varepsilon}) - (\!(\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} - \langle\, \mathbf{f} \,,\, \mathbf{v} \,\rangle + \gamma^*(\mathbf{f}),\\
H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v}) &= \phi(\boldsymbol{\varepsilon}) - (\!(\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} - \gamma(\mathbf{v}),\\
H_1(\boldsymbol{\sigma},\mathbf{v},\mathbf{f}) &= -\phi^*(\boldsymbol{\sigma}) + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} - \langle\, \mathbf{f} \,,\, \mathbf{v} \,\rangle + \gamma^*(\mathbf{f}),\\
R(\boldsymbol{\varepsilon},\boldsymbol{\sigma}) &= \phi(\boldsymbol{\varepsilon}) - (\!(\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} + \gamma^*(\mathbf{B}'\boldsymbol{\sigma}),\\
R_1(\boldsymbol{\sigma},\mathbf{v}) &= -\phi^*(\boldsymbol{\sigma}) + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} - \gamma(\mathbf{v}),\\
R_2(\mathbf{v},\mathbf{f}) &= \phi(\mathbf{B}\mathbf{v}) - \langle\, \mathbf{f} \,,\, \mathbf{v} \,\rangle + \gamma^*(\mathbf{f}),\\
P_1(\boldsymbol{\varepsilon}) &= \phi(\boldsymbol{\varepsilon}) - (\gamma^*\mathbf{B}')^*(\boldsymbol{\varepsilon}),\\
P_2(\boldsymbol{\sigma}) &= -\phi^*(\boldsymbol{\sigma}) + \gamma^*(\mathbf{B}'\boldsymbol{\sigma}),\\
P_3(\mathbf{v}) &= \phi(\mathbf{B}\mathbf{v}) - \gamma(\mathbf{v}),\\
P_4(\mathbf{f}) &= -(\phi \circ \mathbf{B})^*(\mathbf{f}) + \gamma^*(\mathbf{f}).
\end{aligned}
}
$$

1.3.1 Hybrid formulations

Hybrid variational principles [17] fall into the range of the general theory developed in papers [65], [66] and outlined in the previous section. They are generated by relaxing the kinematic constraints.

The hybrid variational principles reported in the survey paper by Pian & Tong [47] can be recovered in a generalized form to include nonlinear material behaviors.

In this more general context the Euler-Lagrange conditions for hybrid-stress and hybrid-displacement principles lead to variational inequalities whose discussion requires the proof of existence results to show that a substationarity point of the functionals yields a solution of the structural problem.

A remarkable feature of the treatment proposed here is the automatic development of hybrid variational principles by simple specialization of the ten basic functionals of the variational tree.

This is in contrast with the usual approach according to which variational principles are first formulated on the basis of a *skillfull intuition* and then validated *a posteriori* by deriving the corresponding Euler-Lagrange conditions.

An hybrid model $\mathcal{M}_o$ associated with the structural model $\mathcal{M}$ is characterized by the choice of

- a decomposition $\mathcal{T}_o(\Omega)$ with kinematic constraints are imposed on the boundary $\partial\mathcal{T}_o(\Omega)$,
- a conformity subspace $\mathcal{L} \subset \mathcal{L}_o$,
- and a pair of restoring constraint operators

$$\mathbf{G} \in BL(\mathbf{\Gamma}\mathcal{L}_o, \Lambda) \quad \text{and} \quad \mathbf{P} \in BL(\mathcal{P}, \mathbf{\Gamma}\mathcal{L}_o),$$

  such that $\mathbf{\Gamma}\mathcal{L} = \operatorname{Ker}\mathbf{G} = \operatorname{Im}\mathbf{P}$.

  Then $\partial\mathcal{R} = (\mathbf{\Gamma}\mathcal{L})^{\perp} = \operatorname{Im}\mathbf{G}' = (\operatorname{Ker}\mathbf{G})^{\perp} = \operatorname{Ker}\mathbf{P}' = (\operatorname{Im}\mathbf{P})^{\perp}$.

We shall denote by $\mathcal{V}_\mathbf{o}$ and $\mathcal{S}_\mathbf{o}$ respectively the spaces of kinematic and stress fields conforming with the decomposition $\mathcal{T}_o(\Omega)$. The concave potential of external constraints can then be written as

$$\gamma(\mathbf{v}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,) + \langle\!\langle\, \overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle + \sqcap_{\operatorname{Ker}\mathbf{G}}(\mathbf{\Gamma v} - \mathbf{\Gamma\overline{v}}) \qquad \mathbf{v} - \overline{\mathbf{v}} \in \mathcal{L}_o\,,$$

where

- $\overline{\mathbf{v}} \in \mathcal{V}$ is a prescribed displacement,
- $\overline{\mathbf{b}} \in H$ is an assigned body force and
- $\overline{\mathbf{t}} \in \partial\mathcal{F}$ is a given boundary traction.

By taking into account the orthogonality relation

$$\sqcap_{\operatorname{Ker}\mathbf{G}}(\mathbf{\Gamma v}) + \sqcap_{\operatorname{Im}\mathbf{G}'}(\mathbf{r}) = \langle\!\langle\, \mathbf{r}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle\,,$$

we get the following variants of $\gamma(\mathbf{v})$:

$$
\begin{aligned}
&i) \quad \gamma(\mathbf{v},\mathbf{r}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle + \langle\!\langle\,\mathbf{r}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}\,\rangle\!\rangle - \sqcap_{\mathrm{Im}\,\mathbf{G}'}(\mathbf{r})\\
&ii) \quad \gamma(\mathbf{v},\boldsymbol{\rho}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle + \langle\!\langle\,\boldsymbol{\rho}\,,\,\mathbf{G}(\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}})\,\rangle\!\rangle\\
&iii) \quad \gamma(\mathbf{v},\mathbf{r},\mathbf{w}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle + \langle\!\langle\,\mathbf{r}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}-\mathbf{\Gamma w}\,\rangle\!\rangle + \sqcap_{\mathrm{Im}\,\mathbf{P}}(\mathbf{\Gamma w})\\
&iv) \quad \gamma(\mathbf{v},\mathbf{r},\boldsymbol{\omega}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle + \langle\!\langle\,\mathbf{r}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}-\mathbf{P}\boldsymbol{\omega}\,\rangle\!\rangle
\end{aligned}
$$

where

- $\boldsymbol{\rho}$ is a field of interactions between adjacent elements of $\mathcal{T}_o(\Omega)$,
- $\boldsymbol{\omega}$ is a displacement field of the interfaces of $\mathcal{T}_o(\Omega)$ and
- $\mathbf{\Gamma w}$ is a boundary displacement field on $\partial\mathcal{T}_o(\Omega)$.

Assuming in *i)* that $\mathbf{r} = \mathbf{N}\boldsymbol{\sigma} - \overline{\mathbf{t}}$ with $\boldsymbol{\sigma} \in \mathcal{S}_{\mathbf{o}}$ we also get

$$
v) \quad \gamma(\boldsymbol{\sigma},\mathbf{v}) = (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,) + \langle\!\langle\,\mathbf{N}\boldsymbol{\sigma}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}\,\rangle\!\rangle - \sqcap_{\mathrm{Im}\,\mathbf{G}'}(\mathbf{N}\boldsymbol{\sigma}-\overline{\mathbf{t}}) + \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma}\overline{\mathbf{v}}\,\rangle\!\rangle,
$$

where the last constant term can be dropped. Substituting $\gamma(\mathbf{v},\mathbf{r},\boldsymbol{\omega})$ into the functional $H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v})$ of the variational tree, we obtain the hybrid functional

$$
H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v},\mathbf{r},\boldsymbol{\omega}) = \phi(\boldsymbol{\varepsilon}) + (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{Bv}-\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H - \langle\!\langle\,\overline{\mathbf{t}}\,,\,\mathbf{\Gamma v}\,\rangle\!\rangle - \langle\!\langle\,\mathbf{r}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}-\mathbf{P}\boldsymbol{\omega}\,\rangle\!\rangle
$$

whose specialization to linear elasticity was referred to in [16] as *the most general variational functional in finite-element formulation.* The associated EULER conditions are

$$
\boldsymbol{\sigma} \in \partial\phi(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} = \mathbf{Bv}, \quad \mathbf{B}'_o\boldsymbol{\sigma} = \overline{\mathbf{b}}, \quad \mathbf{N}\boldsymbol{\sigma} = \overline{\mathbf{t}} + \mathbf{r}, \quad \mathbf{\Gamma v} = \mathbf{\Gamma}\overline{\mathbf{v}} - \mathbf{P}\boldsymbol{\omega}, \quad \mathbf{P}'\mathbf{r} = \boldsymbol{o}.
$$

Adopting $\gamma(\boldsymbol{\sigma},\mathbf{v})$ in $T_1(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v})$ we get the HU-WASHIZU functional

$$
H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v}) = \phi(\boldsymbol{\varepsilon}) + (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{Bv}-\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H - \langle\!\langle\,\mathbf{N}\boldsymbol{\sigma}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}\,\rangle\!\rangle + \sqcap_{\mathrm{Im}\,\mathbf{G}'}(\mathbf{N}\boldsymbol{\sigma}-\overline{\mathbf{t}},)
$$

with the EULER conditions

$$
\boldsymbol{\sigma} \in \partial\phi(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} = \mathbf{Bv}, \quad \mathbf{\Gamma v} - \mathbf{\Gamma}\overline{\mathbf{v}} \in \mathrm{Ker}\,\mathbf{G}, \quad \mathbf{B}'_o\boldsymbol{\sigma} = \overline{\mathbf{b}}.
$$

Adopting $\gamma(\boldsymbol{\sigma},\mathbf{v})$ in $P_2(\boldsymbol{\sigma},\mathbf{v})$ we get the HELLINGER-REISSNER functional

$$
R(\boldsymbol{\sigma},\mathbf{v}) = -\phi^*(\boldsymbol{\sigma}) + (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{Bv}\,)\!)_{\mathcal{H}} - (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H - \langle\!\langle\,\mathbf{N}\boldsymbol{\sigma}\,,\,\mathbf{\Gamma v}-\mathbf{\Gamma}\overline{\mathbf{v}}\,\rangle\!\rangle + \sqcap_{\mathrm{Im}\,\mathbf{G}'}(\mathbf{N}\boldsymbol{\sigma}-\overline{\mathbf{t}})
$$

with the EULER conditions

$$\begin{cases} \mathbf{B}\mathbf{v} \in \partial\phi^*(\boldsymbol{\sigma})\,, \\ \boldsymbol{\Gamma}\mathbf{v} - \boldsymbol{\Gamma}\overline{\mathbf{v}} \in \operatorname{Ker}\mathbf{G}\,, \\ \mathbf{B}_o'\boldsymbol{\sigma} = \overline{\mathbf{b}}\,. \end{cases}$$

The finite element model for thin plates proposed by HERRMANN [13], in which the normal derivatives of the displacements are discontinuous at the interfaces, can be recovered from REISSNER functional by setting the restoring operator $\mathbf{G}$ equal to the projector of the boundary values onto the subspace of boundary flexural rotations.

The hybrid-displacement functional, first proposed by TONG [16] in 1970, is recovered by adopting $\gamma(\mathbf{v},\mathbf{r},\boldsymbol{\omega})$ in $S_3(\mathbf{v})$ to get

$$F(\mathbf{v},\mathbf{r},\boldsymbol{\omega}) = \phi(\mathbf{B}\mathbf{v}) - (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H - \langle\!\langle\, \overline{\mathbf{t}}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle - \langle\!\langle\, \mathbf{r}\,,\,\boldsymbol{\Gamma}\mathbf{v} - \boldsymbol{\Gamma}\overline{\mathbf{v}} - \mathbf{P}\boldsymbol{\omega}\,\rangle\!\rangle,$$

whose EULER conditions are

$$\begin{aligned} &\mathrm{d}^+\phi(\mathbf{B}\mathbf{v};\mathbf{B}\mathbf{v}) \ge (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\, \overline{\mathbf{t}} + \mathbf{r}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle \quad \forall\,\mathbf{v} \in L_o\,, \\ &\boldsymbol{\Gamma}\mathbf{v} = \boldsymbol{\Gamma}\overline{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega}\,, \qquad \mathbf{P}'\mathbf{r} = \boldsymbol{o}\,. \end{aligned}$$

In the variational inequality $\mathrm{d}^+\phi(\mathbf{B}\mathbf{v};\mathbf{B}\mathbf{v})$ denotes the unidirectional derivative of ϕ at $\mathbf{B}\mathbf{v}$ along $\mathbf{B}\mathbf{v}$.

It can be shown [94] that the variational inequality implies the existence of a stress field which fulfils the equilibrium conditions

$$\mathbf{B}_o'\boldsymbol{\sigma} = \overline{\mathbf{b}}, \qquad \mathbf{N}\boldsymbol{\sigma} = \overline{\mathbf{t}} + \mathbf{r}\,,$$

with $\mathbf{r} \in (\boldsymbol{\Gamma}\mathcal{L})^\perp$ and the constitutive relation

$$\boldsymbol{\sigma} \in \partial\phi(\mathbf{B}\mathbf{v}).$$

Finally choosing $\gamma(\mathbf{v},\boldsymbol{\rho})$ in $S_3(\mathbf{v})$ we get the analog of the hybrid functional proposed by JONES [8] in 1964

$$F(\mathbf{v},\boldsymbol{\rho}) = \phi(\mathbf{B}\mathbf{v}) - (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H - \langle\!\langle\, \overline{\mathbf{t}}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle - \langle\!\langle\, \boldsymbol{\rho}\,,\,\mathbf{G}(\boldsymbol{\Gamma}\mathbf{v} - \boldsymbol{\Gamma}\overline{\mathbf{v}})\,\rangle\!\rangle,$$

whose EULER conditions are

$$\mathrm{d}^+\phi(\mathbf{B}\mathbf{v};\mathbf{B}\mathbf{v}) \ge (\,\overline{\mathbf{b}}\,,\,\mathbf{v}\,)_H + \langle\!\langle\, \overline{\mathbf{t}} + \mathbf{G}'\boldsymbol{\rho}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle \quad \forall\,\mathbf{v} \in L_o, \qquad \mathbf{G}(\boldsymbol{\Gamma}\mathbf{v} - \boldsymbol{\Gamma}\overline{\mathbf{v}}) = \boldsymbol{o}.$$

The external boundary constraints can alternatively be expressed in terms of the conjugate functional

$$\gamma^*(\mathbf{b},\mathbf{t}) = \langle\!\langle\, \mathbf{t}-\bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} \,\rangle\!\rangle + \sqcap_{\mathrm{Ker}\,\mathbf{P}'}(\mathbf{t}-\bar{\mathbf{t}}) + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{b}-\bar{\mathbf{b}}).$$

Choosing the boundary reactions $\mathbf{r}=\mathbf{t}-\bar{\mathbf{t}}$ as field variables, we get the following variants of $\gamma^*(\mathbf{b},\mathbf{t})$

$$\begin{aligned} i^*) \qquad & \gamma^*(\mathbf{b},\mathbf{r}) = \langle\!\langle\, \mathbf{r}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} \,\rangle\!\rangle + \sqcap_{\mathrm{Ker}\,\mathbf{P}'}(\mathbf{r}) + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{b}-\bar{\mathbf{b}}) \\ ii^*) \quad & \gamma^*(\mathbf{b},\boldsymbol{\rho},\boldsymbol{\omega}) = \langle\!\langle\, \mathbf{r}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega} \,\rangle\!\rangle + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{b}-\bar{\mathbf{b}}). \end{aligned}$$

Assuming that $\mathbf{r}=\mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}$ and $\mathbf{b}=\mathbf{B}_o'\boldsymbol{\sigma}$ we get from $i^*)$ and $ii^*)$

$$\begin{aligned} iii^*) \qquad & \gamma^*(\boldsymbol{\sigma}) = \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} \,\rangle\!\rangle + \sqcap_{\mathrm{Ker}\,\mathbf{P}'}(\mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}) + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{B}_o'\boldsymbol{\sigma}-\bar{\mathbf{b}}) \\ iv^*) \quad & \gamma^*(\boldsymbol{\sigma},\boldsymbol{\omega}) = \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega} \,\rangle\!\rangle + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{B}_o'\boldsymbol{\sigma}-\bar{\mathbf{b}}). \end{aligned}$$

Adopting $\gamma^*(\boldsymbol{\sigma},\boldsymbol{\omega})$ in the complementary energy functional $S_2(\boldsymbol{\sigma})$ we get the hybrid-stress functional first proposed by Pian[9] in 1964 and later investigated in [16]:

$$P(\boldsymbol{\sigma},\boldsymbol{\omega}) = -\phi^*(\boldsymbol{\sigma}) + \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega} \,\rangle\!\rangle + \sqcap_{\{\,\boldsymbol{o}\,\}}(\mathbf{B}_o'\boldsymbol{\sigma}-\bar{\mathbf{b}}).$$

The Euler conditions providing the stationarity of $P(\boldsymbol{\sigma},\boldsymbol{\omega})$ are

$$\begin{cases} \mathrm{d}^+\phi^*(\boldsymbol{\sigma};\boldsymbol{\tau}) \geq \langle\!\langle\, \mathbf{N}\boldsymbol{\tau}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega} \,\rangle\!\rangle \quad \forall\,\boldsymbol{\tau} \in \mathrm{Ker}\,\mathbf{B}_o', \,, \\ \mathbf{P}'(\mathbf{N}\boldsymbol{\sigma}-\bar{\mathbf{t}}) = \boldsymbol{o}\,, \end{cases}$$

where $\mathrm{d}^+\phi^*(\boldsymbol{\sigma};\boldsymbol{\tau})$ denotes the unidirectional derivative of ϕ^* at $\boldsymbol{\sigma}$ along $\boldsymbol{\tau}$.

If the functional ϕ^* is locally subdifferentiable [68] it can be proved [94] that, under mild regularity assumptions, this variational inequality ensures the existence of a kinematic field $\mathbf{v} \in L_o$ such that $\mathbf{B}\mathbf{v} = \partial\phi^*(\boldsymbol{\sigma})$, $\boldsymbol{\Gamma}\mathbf{v} = \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega}$.

In linear elastostatics the corresponding result for the variational equality

$$d\,\phi^*(\boldsymbol{\sigma};\boldsymbol{\tau}) = \langle\!\langle\, \mathbf{N}\boldsymbol{\tau}\,,\, \boldsymbol{\Gamma}\bar{\mathbf{v}} + \mathbf{P}\boldsymbol{\omega} \,\rangle\!\rangle \quad \forall\,\boldsymbol{\tau} \in \mathrm{Ker}\,\mathbf{B}_o',$$

can be found in [90].

Other hybrid variational principles can be formulated by different choices of the functional in the variational tree and of the expressions of the constraint potentials.

Chapter 2

Two-field mixed method

2.1 Introduction

Mixed methods based on multi-field variational principles have been investigated in the computational literature to improve element flexibility and to provide a theoretical basis to some non-conforming methods.

We present here an investigation of the two-field mixed method based on the Hellinger-Reissner variational principle and referred to as Hellinger-Reissner (HR) method.

A comprehensive discussion of well-posedness is carried out and the appropriate necessary and sufficient conditions for the discrete Hellinger-Reissner two-field mixed method are provided.

Moreover, a result which extends to the two-field HR method the discussion of the so called limitation phenomena is presented. More precisely it is shown that, if the discrete interpolations fulfill a suitable relation, the Hellinger-Reissner (HR) two-field method collapses into the displacement method. Finally, the expression of the stiffness matrix is derived in full generality.

Numerical examples of two-dimensional elastostatic problems usually adopted in the literature as significant benchmarks, are reported and discussed in chapter 6 to get informations about the comparative convergence properties and the distortion sensitivity.

2.2 Variational formulation

A two-field $\{\boldsymbol{\sigma}, \mathbf{v}\}$ variational method can be got by a direct interpolation of the fields in the HELLINGER-REISSNER functional R_1 whose explicit expression is

$$R_1(\boldsymbol{\sigma}, \mathbf{v}) = -\phi^*(\boldsymbol{\sigma}) + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!)_{\mathcal{H}} - \gamma_o(\mathbf{v}) - \sqcap_{\operatorname{Ker}\mathbf{L}}(\boldsymbol{\Gamma}(\mathbf{v} - \overline{\mathbf{v}})).$$

It is convenient to assume as basic unknown displacement fields fulfilling the homogeneous boundary conditions.

To this end we set $\mathbf{u} = \mathbf{v} - \overline{\mathbf{v}} \in \mathcal{L}$ and the functional takes the form

$$\begin{aligned} R_1(\boldsymbol{\sigma}, \mathbf{u}) = & -\frac{1}{2}(\!(\, \mathbf{C}\boldsymbol{\sigma} \,,\, \boldsymbol{\sigma} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{u} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\overline{\mathbf{v}} \,)\!)_{\mathcal{H}} \\ & -\gamma_o(\mathbf{u}) - \sqcap_{\operatorname{Ker}\mathbf{L}}(\boldsymbol{\Gamma}\mathbf{u})\,. \end{aligned}$$

where the constant term $\gamma_o(\overline{\mathbf{v}})$ has been dropped.

The stationarity of H yields

$$\begin{cases} -(\!(\, \mathbf{C}\boldsymbol{\sigma} \,,\, \delta\boldsymbol{\sigma} \,)\!)_{\mathcal{H}} + (\!(\, \mathbf{B}\mathbf{u} \,,\, \delta\boldsymbol{\sigma} \,)\!)_{\mathcal{H}} = -(\!(\, \mathbf{B}\overline{\mathbf{v}} \,,\, \delta\boldsymbol{\sigma} \,)\!)_{\mathcal{H}} & \forall\, \delta\boldsymbol{\sigma} \in \mathcal{S} \\ (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\delta\mathbf{u} \,)\!)_{\mathcal{H}} = (\, \overline{\mathbf{b}} \,,\, \delta\mathbf{u} \,)_H + \langle\!\langle\, \overline{\mathbf{t}} \,,\, \boldsymbol{\Gamma}\delta\mathbf{u} \,\rangle\!\rangle & \forall\, \delta\mathbf{u} \in \mathcal{L}. \end{cases}$$

A necessary and sufficient condition for the existence and uniqueness of the solution $\{\boldsymbol{\sigma}, \mathbf{u}\}$ is provided by the equilibrium condition

$$\langle\, \ell \,,\, \delta\mathbf{u} \,\rangle = 0 \quad \forall\, \delta\mathbf{u} \in \mathcal{L} \cap \operatorname{Ker}\mathbf{B} \quad \Longleftrightarrow \quad \ell \in (\mathcal{L} \cap \operatorname{Ker}\mathbf{B})^{\perp},$$

to within a rigid displacement.

The strain and stress fields are uniquely defined by the equalities $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$ and $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$ and the displacement field is determined to within a conforming rigid displacement $\mathbf{u}_o \in \mathcal{L} \cap \operatorname{Ker}\mathbf{B}$.

2.3 FEM interpolation

With a standard notation in finite element analysis, (see e.g. Zienkiewicz and Taylor [54]), we consider, for each element, the interpolations

$$\begin{aligned} \mathbf{u}_h^e(\mathbf{x}) &= \mathbf{N}_{\mathbf{u}}(\mathbf{x})\, \mathbf{p}_{\mathbf{u}}^e \in \mathcal{V}_h^e \\ \boldsymbol{\sigma}_h^e(\mathbf{x}) &= \mathbf{N}_{\boldsymbol{\sigma}}(\mathbf{x})\, \mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{S}_h^e\,, \end{aligned}$$

where $\mathbf{x} \in \overline{\Omega}_e$ and $\overline{\Omega}_e \in \mathcal{T}_{FEM}(\Omega)$ is the domain decomposition induced by the meshing of Ω depending on a parameter h which goes to zero as the finite mesh is refined.

We denote by $n_{\boldsymbol{\sigma}} = \dim \mathcal{S}^e_h$ and $n_{\mathbf{u}} = \dim \mathcal{V}^e_h$ the dimensions of the interpolating spaces of the current element. The interpolating spaces of the single elements are collected in the following product spaces

$$\mathcal{S}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{S}^e_h \,, \quad \mathcal{V}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{V}^e_h \,.$$

The interpolating stress fields are defined elementwise so that the corresponding global fields are simply the collection of the local ones:

$$\boldsymbol{\sigma}_h = \{ \boldsymbol{\sigma}^1_h, \boldsymbol{\sigma}^2_h, \ldots, \boldsymbol{\sigma}^{\mathcal{N}}_h \} \in \mathcal{S}_h \,,$$

where $\mathcal{N}$ is the total number of elements pertaining to the finite element discretization.

Accordingly the virtual work performed by an interpolating stress field $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ by an interpolating strain field $\boldsymbol{\varepsilon}_h := \mathbf{B}\mathbf{u}_h$ compatible with an interpolating dispacement field $\mathbf{u}_h$ is defined as the sum of the element contributions

$$\langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \boldsymbol{\varepsilon}_h \,\rangle\!\rangle = \sum_{e=1}^{\mathcal{N}} (\!(\boldsymbol{\sigma}^e_h, \boldsymbol{\varepsilon}^e_h)\!)_{\Omega_e} = \sum_{e=1}^{\mathcal{N}} \int_{\Omega_e} \mathbf{N}^e_{\boldsymbol{\sigma}}\, \mathbf{p}^e_{\boldsymbol{\sigma}} * \mathbf{B}\mathbf{N}_{\mathbf{u}}\, \mathbf{p}^e_{\mathbf{u}}.$$

The local parameters $\mathbf{p}^e_{\boldsymbol{\sigma}} \in \mathcal{R}^{n_{\boldsymbol{\sigma}}}$ can be condensed at the element level.

We shall consider a conforming finite element interpolation. The conforming interpolated displacement fields

$$\mathbf{u}_h = \{ \mathbf{u}^1_h, \mathbf{u}^2_h, \ldots, \mathbf{u}^{\mathcal{N}}_h \} \in \mathcal{L}_h \subset \mathcal{V}_h$$

satisfy the homogeneous boundary constraints and the interelement continuity conditions. We shall denote by $n_{dof} = \dim \mathcal{L}_h$ the dimension of $\mathcal{L}_h$ which is a proper subspace of the product space $\mathcal{V}_h$.

As customary we assume that rigid body displacements are ruled out by the conformity requirements, that is $\mathcal{L} \cap \operatorname{Ker} \mathbf{B} = \{ \mathbf{o} \}$, so that the condition $\mathcal{L}_h \subset \mathcal{L}$ implies $\mathcal{L}_h \cap \operatorname{Ker} \mathbf{B} = \{ \mathbf{o} \}$.

The parameters $\mathbf{p}^e_{\mathbf{u}} \in \mathcal{R}^{n_{\mathbf{u}}}$ can be expressed in terms of the nodal parameters $\mathbf{q}_{\mathbf{u}} \in \mathcal{R}^{n_{dof}}$ by means of the standard finite element assembly operator $\mathcal{A}^e_{\mathbf{u}}$ according to the parametric representation $\mathbf{p}^e_{\mathbf{u}} = \mathcal{A}^e_{\mathbf{u}}\, \mathbf{q}_{\mathbf{u}}$.

On the contrary the stress local parameters are simply collected in the global list $\mathbf{q}_{\boldsymbol{\sigma}}$.

The overall assembly operation $\mathcal{A}^e$ of the element parameters can then be written in matrix form as

$$\begin{bmatrix} \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \mathcal{A}^e \begin{bmatrix} \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O} \\ \mathbf{O} & \mathcal{A}^e_{\mathbf{u}} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix}$$

where the operator $\mathcal{J}^e_{\boldsymbol{\sigma}}$ is the canonical estractor which picks up, from the global list $\mathbf{q}_{\boldsymbol{\sigma}}$, the local parameters $\mathbf{p}^e_{\boldsymbol{\sigma}}$ pertaining to the current element Ω_e, that is $\mathbf{p}^e_{\boldsymbol{\sigma}} = \mathcal{J}^e_{\boldsymbol{\sigma}} \mathbf{q}_{\boldsymbol{\sigma}}$ with $e = 1, \ldots, \mathcal{N}$.

The interpolated counterpart of the HELLINGER-REISSNER functional R_1 can be obtained by adding up the contributions $R_{1h}^{\,e}(\boldsymbol{\sigma}^e_h, \mathbf{u}^e_h)$ of each non-assembled element and imposing that the interpolating displacement $\mathbf{u}_h$ satisfies the conformity requirement. Accordingly we have

$$R_{1h}(\boldsymbol{\sigma}_h, \mathbf{u}_h) = \frac{1}{2} \langle\!\langle\, \mathbf{C}\boldsymbol{\sigma}_h \,,\, \boldsymbol{\sigma}_h \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\mathbf{u}_h \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\overline{\mathbf{v}} \,\rangle\!\rangle - \gamma_o(\mathbf{u}_h),$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathcal{S}_h \times \mathcal{L}_h$.
The matrix form of the two-field discrete problem (**TFP**) is obtained from the stationarity of R_{1h} and is given by

$$\mathbf{M} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} -\mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{P}^e \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{S}^e \mathcal{A}^e_{\mathbf{u}} \\ \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{S}^{eT} \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{P} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathbf{o} \\ \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{f}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{o} \\ \mathbf{f}_{\mathbf{u}} \end{bmatrix}.$$

where

$$\mathbf{P}^e \; := \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\sigma}}(\mathbf{x}) \mathbf{C}_*(\mathbf{x}) \mathbf{N}^e_{\boldsymbol{\sigma}}(\mathbf{x})\,; \quad \mathbf{S}^e \; := \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\sigma}}(\mathbf{x}) \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}}(\mathbf{x})\,;$$

$$\mathbf{f}^e_{\mathbf{u}} \; := \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{u}}(\mathbf{x})\, \overline{\mathbf{b}}(\mathbf{x}) + \int_{\partial\Omega_e} (\boldsymbol{\Gamma}\mathbf{N}^e_{\mathbf{u}})^T(\mathbf{x})\, \overline{\mathbf{t}}(\mathbf{x})\,,$$

in which $\mathbf{B}_*$ and $\mathbf{C}_*$ denote the matrix form of the operators $\mathbf{B}$ and $\mathbf{C}$. The operator $\mathcal{A}^{e\,T}_{\mathbf{u}}$ trasforms the local forces $\mathbf{f}^e_{\mathbf{u}}$ into the corresponding global ones

$\mathbf{f}_{\mathbf{u}}$. The elastic stiffness matrix $\mathbf{C}_*$ is positive definite and hence the matrix $\mathbf{P}^e$ turns out to be positive definite as well.

The well-posedness analysis, discussed in the next section, is developed in terms of the two-field problem (**TFP**). On the contrary, from a computational standpoint, it is more convenient to carry out the assembly operation after the condensation at the element level of the stress parameters in order to get the discrete mixed problem in terms of the displacement parameters $\mathbf{q}_{\mathbf{u}}$.

2.4 Well-posedness analysis

As customary in computational analysis we will assume that the structure cannot undergo conforming rigid displacements.

- **Definition of well-posedness.** The discrete mixed problem **TFP** is said to be *well-posed* if there exists a unique solution $\{\boldsymbol{\sigma}_h, \mathbf{u}_h\} \in \mathcal{S}_h \times \mathcal{L}_h$ for any data $\mathbf{f}_{\boldsymbol{\sigma}}$ and $\mathbf{f}_{\mathbf{u}}$.

In the literature well-posedness requires, in addition to existence and uniqueness of the solution for any data, that the discrete solution tends, in some energy norm, to the solution of the continuous problem as the parameter h goes to zero. We treat separately these two requirements since the conditions for their fulfillment follow by different arguments.

The necessary and sufficient condition ensuring the well-posedness of the discrete problem is provided in the next statement.

Proposition 2.4.1 (Well-posedness conditions) *If rigid displacements are ruled out, the condition*

$$\mathbf{B}\,\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\boldsymbol{o}\},$$

*is necessary and sufficient for the well-posedness of the discrete mixed problem **TFP**.*

The proposition 2.4.1 requires that the compatible strain interpolates $\mathbf{B}\mathbf{u}_h$, with $\mathbf{u}_h \in \mathcal{L}_h$, must be controlled by the stress interpolates $\boldsymbol{\sigma}_h \in \mathcal{S}_h$.

The well-posedness criterion reported in proposition 2.4.1 appears to be new.

Since $\mathbf{B}\mathcal{L}_h \subseteq \mathbf{B}\mathcal{V}_h$, the global condition

$$\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\boldsymbol{o}\}$$

can be can be enforced by imposing the stronger local condition $\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\boldsymbol{o}\}$.

We can then state the following

Proposition 2.4.2 (Sufficient conditions) *The local condition*

$$\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\boldsymbol{o}\}$$

*is sufficient for the well-posedness of the discrete mixed problem **TFP** (matrix form).*

Thiscondition requires that the local compatible strain interpolates are controlled by the local stress interpolates.

2.5 Stiffness matrix

If the mixed discrete problem

$$\mathbf{M}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}}\begin{bmatrix} -\mathcal{J}_{\boldsymbol{\sigma}}^{e\,T}\mathbf{P}^e\mathcal{J}_{\boldsymbol{\sigma}}^{e} & \mathcal{J}_{\boldsymbol{\sigma}}^{e\,T}\mathbf{S}^e\mathcal{A}_{\mathbf{u}}^{e}\\ \mathcal{A}_{\mathbf{u}}^{e\,T}\mathbf{S}^{eT}\mathcal{J}_{\boldsymbol{\sigma}}^{e} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix}$$

$$= \begin{bmatrix}-\mathbf{P} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}}\begin{bmatrix}\mathbf{o}\\ \mathcal{A}_{\mathbf{u}}^{e\,T}\mathbf{f}_{\mathbf{u}}^{e}\end{bmatrix} = \begin{bmatrix}\mathbf{o}\\ \mathbf{f}_{\mathbf{u}}\end{bmatrix}$$

is well-posed, a simple derivation of the element stiffness matrix can be performed by eliminating the stress parameters according to the following procedure

$$\begin{cases}\mathbf{P}^e\,\mathbf{p}_{\boldsymbol{\sigma}}^e = \mathbf{S}^e\,\mathbf{p}_{\mathbf{u}}^e\\ \left[\mathbf{S}^{eT}\mathbf{P}^{e-1}\mathbf{S}^e\right]\,\mathbf{p}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}^e\,.\end{cases}$$

Denoting the element stiffness matrix by

$$\mathbf{K}^e := \mathbf{S}^{eT}\mathbf{P}^{e-1}\mathbf{S}^e$$

the mixed problem at the element level can be written as

$$\mathbf{K}^e\ \mathbf{p}^e_{\mathbf{u}} = \mathbf{f}^e_{\mathbf{u}}$$

so that the global problem will be expressed in terms of nodal displacement parameters as

$$\mathbb{K}\,\mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}} \left(\mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{K}^e \mathcal{A}^e_{\mathbf{u}} \right)\ \mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}} \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{f}^e_{\mathbf{u}} = \mathbf{f}_{\mathbf{u}}.$$

We underline that, due to the positive definiteness of $\mathbf{H}^e$ and the nonsingularity of $\mathbf{Q}^e$, the matrix $\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e$ is invertible. Once the global structural problem has been solved in terms of nodal displacements, the stress parameters can be evaluated at the element level according to the formula:

$$\mathbf{p}^e_{\boldsymbol{\sigma}} = \mathbf{P}^{e-1}\mathbf{S}^e\,\mathbf{p}^e_{\mathbf{u}}\,.$$

2.6 Limitation principle

The approximate solution $\mathbf{u}_h \in \mathcal{L}_h$ provided by the Hellinger-Reissner method and by the displacement method coincide if the following condition is met:

$$\mathbf{E}\mathbf{B}\mathcal{L}_h \subseteq \mathcal{S}_h\,.$$

Chapter 3

Three-field mixed method

3.1 Introduction

Mixed methods based on multi-field variational principles have been investigated in the computational literature to improve element flexibility and to provide a theoretical basis to some non-conforming methods. A special attention has been recently devoted to three-field methods based on the HU-WASHIZU variational principle to obtain F.E.M. formulations which do not exibit over-stiffning or locking phenomena.

The enhanced assumed strain (EAS) method proposed by SIMO and RIFAI in [55] was originally developed to provide a variational basis to the incompatible mode element of WILSON ET AL. [27]. The treatment developed by SIMO and RIFAI in [55] and by REDDY and SIMO in [70] emphasizes the role of three conditions to be imposed on the interpolations of displacements, enhanced strains and stresses, to get well-posedness, convergence and stress independence of the finite element equations.

The original formulation of the EAS method is in fact based on an orthogonality condition between the stress and the enhanced strain shape functions. As a consequence, stress parameters remain completely undeterminated. Several *a posteriori* stress recovery strategies have thus been envisaged but these proposals provide only a variational scent to the stress solution (ANDELFINGER and RAMM [61], PEREGO [82], CÉSAR DE SÀ and NATAL JORGE [74]).

The Strain Gap Method (SGM) was introduced by ROMANO et al. in [83] in order to provide a well-posed formulation of the enhanced strain method and a variationally consistent stress recovery. A comprehensive analysis of the SGM and the EAS methods will be developed in chapter 4 where a discussion on the longly debated issue of evaluating the stress field at the Gauss points according to the elastic constitutive relation is also provided.

An alternative approach based on a direct discretization of the HU-WASHIZU variational principle has been recently proposed by KASPER and TAYLOR in [75].

The method was referred to as the mixed-enhanced strain (MES) method and was intended to overcome the difficulties related with the EAS method and to get improved performances in the incompressible limit.

The formulation of the MES method appears however to have been strongly influenced by the EAS method. In fact the interpolation of the strain fields is based on the choice of two groups of shape functions. The former contains the same shape functions adopted for the stress fields, while the shape functions of the latter are choosen to be orthogonal to the ones of the first group.

No explicit discussion of well-posedness was performed in [75] but reference was made to the analysis previously performed in [55] for the EAS method.

We present here a detailed investigation of the three-field mixed method based on the HU-WASHIZU variational principle and referred to as HU-WASHIZU (HW) method. The MES method is then recovered as a special case.

A comprehensive discussion of well-posedness is carried out providing the appropriate necessary and sufficient conditions for the discrete HU-WASHIZU three-field mixed method. These conditions are then specialized to the MES method and it is shown that they turn out to be different from the ones pertaining to the EAS and to the SGM.

The expression of the stiffness matrix is derived in full generality and then specialized to get an explicit expression in terms of enhanced strain and stress shape functions.

The generalized version of the MES method developed in [92] and illustrated in this chapter provides an applicable local criterion of well-posedness which was still lacking in the literature.

Numerical examples of two-dimensional elastostatic problems usually adopted in the literature as significant benchmarks, will be reported and discussed in chapter 6 to get informations about the comparative convergence properties, the distortion sensitivity and the reliability of the stress approximation provided by the MES method.

3.2 Variational formulation

A three-field $\{\,\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v}\,\}$ variational method can be got by a direct interpolation of the fields in the HU-WASHIZU functional H whose explicit expression is

$$H(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v}) = \frac{1}{2}(\!(\, \mathbf{E}(\boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}})\,,\, \boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}}\,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v} - \boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - \gamma_o(\mathbf{v}) - \sqcap_{\mathrm{Ker}\,\mathbf{L}}(\boldsymbol{\Gamma}(\mathbf{v} - \overline{\mathbf{v}})).$$

It is convenient to assume as basic unknown displacement fields fulfilling the homogeneous boundary conditions.

To this end we set $\mathbf{u} = \mathbf{v} - \overline{\mathbf{v}} \in \mathcal{L}$ and the functional takes the form

$$\begin{aligned} H(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}) = {} & \frac{1}{2}(\!(\, \mathbf{E}(\boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}})\,,\, \boldsymbol{\varepsilon} - \overline{\boldsymbol{\delta}}\,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u} - \boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\overline{\mathbf{v}}\,)\!)_{\mathcal{H}} \\ & - \gamma_o(\mathbf{u}) - \sqcap_{\mathrm{Ker}\,\mathbf{L}}(\boldsymbol{\Gamma}\mathbf{u})\,. \end{aligned}$$

where the constant term $\gamma_o(\overline{\mathbf{v}})$ has been dropped.

The stationarity of H yields

$$\begin{cases} (\!(\, \mathbf{E}\boldsymbol{\varepsilon}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - (\!(\, \boldsymbol{\sigma}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} = (\!(\, \mathbf{E}\overline{\boldsymbol{\delta}}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} & \forall\, \delta\boldsymbol{\varepsilon} \in \mathcal{D} \\ -(\!(\, \delta\boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} + (\!(\, \delta\boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}\,)\!)_{\mathcal{H}} = -(\!(\, \delta\boldsymbol{\sigma}\,,\, \mathbf{B}\overline{\mathbf{v}}\,)\!)_{\mathcal{H}} & \forall\, \delta\boldsymbol{\sigma} \in \mathcal{S} \\ (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\delta\mathbf{u}\,)\!)_{\mathcal{H}} = (\, \overline{\mathbf{b}}\,,\, \delta\mathbf{u}\,)_H + \langle\!\langle\, \overline{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\delta\mathbf{u}\,\rangle\!\rangle & \forall\, \delta\mathbf{u} \in \mathcal{L}. \end{cases}$$

A necessary and sufficient condition for the existence and uniqueness of the solution $\{\,\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}\,\}$ is provided by the equilibrium condition

$$\langle\, \ell\,,\, \delta\mathbf{u}\,\rangle = 0 \quad \forall\, \delta\mathbf{u} \in \mathcal{L} \cap \mathrm{Ker}\,\mathbf{B} \quad \Longleftrightarrow \quad \ell \in (\mathcal{L} \cap \mathrm{Ker}\,\mathbf{B})^{\perp},$$

to within a rigid displacement.

The strain and stress fields are uniquely defined by the equalities $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$ and $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$ and the displacement field is determined to within a conforming rigid displacement $\mathbf{u}_o \in \mathcal{L} \cap \mathrm{Ker}\,\mathbf{B}$.

3.3 FEM interpolation

With a standard notation in finite element analysis, (see e.g. Zienkiewicz and Taylor [54]), we consider, for each element, the interpolations

$$\begin{aligned} \mathbf{u}_h^e(\mathbf{x}) &= \mathbf{N}_{\mathbf{u}}(\mathbf{x})\, \mathbf{p}_{\mathbf{u}}^e \in \mathcal{V}_h^e \\ \boldsymbol{\varepsilon}_h^e(\mathbf{x}) &= \mathbf{N}_{\boldsymbol{\varepsilon}}(\mathbf{x})\, \mathbf{p}_{\boldsymbol{\varepsilon}}^e \in \mathcal{D}_h^e \\ \boldsymbol{\sigma}_h^e(\mathbf{x}) &= \mathbf{N}_{\boldsymbol{\sigma}}(\mathbf{x})\, \mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{S}_h^e\,, \end{aligned}$$

where $\mathbf{x} \in \overline{\Omega}_e$ and $\overline{\Omega}_e \in \mathcal{T}_{FEM}(\Omega)$ is the domain decomposition induced by the meshing of Ω depending on a parameter h which goes to zero as the finite mesh is refined.

We denote by $n_{\boldsymbol{\varepsilon}} = \dim \mathcal{D}_h^e$, $n_{\boldsymbol{\sigma}} = \dim \mathcal{S}_h^e$ and $n_{\mathbf{u}} = \dim \mathcal{V}_h^e$ the dimensions of the interpolating spaces of the current element. The interpolating spaces of the single elements are collected in the following product spaces

$$\mathcal{D}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{D}_h^e\,, \quad \mathcal{S}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{S}_h^e\,, \quad \mathcal{V}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{V}_h^e.$$

The interpolating strain and stress fields are defined elementwise so that the corresponding global fields are simply the collection of the local ones:

$$\boldsymbol{\sigma}_h = \{\, \boldsymbol{\sigma}_h^1, \boldsymbol{\sigma}_h^2, \ldots, \boldsymbol{\sigma}_h^{\mathcal{N}} \,\} \in \mathcal{S}_h\,; \qquad \boldsymbol{\varepsilon}_h = \{\, \boldsymbol{\varepsilon}_h^1, \boldsymbol{\varepsilon}_h^2, \ldots, \boldsymbol{\varepsilon}_h^{\mathcal{N}} \,\} \in \mathcal{D}_h$$

where $\mathcal{N}$ is the total number of elements pertaining to the finite element discretization.

Accordingly the virtual work performed by an interpolating stress field $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ by an interpolating strain field $\boldsymbol{\varepsilon}_h \in \mathcal{D}_h$ is defined as the sum of the element contributions

$$\langle\!\langle\, \boldsymbol{\sigma}_h\,,\, \boldsymbol{\varepsilon}_h \,\rangle\!\rangle = \sum_{e=1}^{\mathcal{N}} (\!(\boldsymbol{\sigma}_h^e, \boldsymbol{\varepsilon}_h^e)\!)_{\Omega_e} = \sum_{e=1}^{\mathcal{N}} \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\sigma}}^e\, \mathbf{p}_{\boldsymbol{\sigma}}^e * \mathbf{N}_{\boldsymbol{\varepsilon}}^e\, \mathbf{p}_{\boldsymbol{\varepsilon}}^e.$$

The local parameters $\mathbf{p}_{\boldsymbol{\varepsilon}}^e \in \mathcal{R}^{n_{\boldsymbol{\varepsilon}}}$ and $\mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{R}^{n_{\boldsymbol{\sigma}}}$ can be condensed at the element level.

We shall consider a conforming finite element interpolation. The conforming interpolated displacement fields

$$\mathbf{u}_h = \{\, \mathbf{u}_h^1, \mathbf{u}_h^2, \ldots, \mathbf{u}_h^{\mathcal{N}} \,\} \in \mathcal{L}_h \subset \mathcal{V}_h$$

satisfy the homogeneous boundary constraints and the interelement continuity conditions. We shall denote by $n_{dof} = \dim \mathcal{L}_h$ the dimension of $\mathcal{L}_h$ which is a proper subspace of the product space $\mathcal{V}_h$.

As customary we assume that rigid body displacements are ruled out by the conformity requirements, that is $\mathcal{L} \cap \operatorname{Ker} \mathbf{B} = \{\, \mathbf{o} \,\}$, so that the condition $\mathcal{L}_h \subset \mathcal{L}$ implies $\mathcal{L}_h \cap \operatorname{Ker} \mathbf{B} = \{\, \mathbf{o} \,\}$.

The parameters $\mathbf{p}_{\mathbf{u}}^e \in \mathcal{R}^{n_{\mathbf{u}}}$ can be expressed in terms of the nodal parameters $\mathbf{q}_{\mathbf{u}} \in \mathcal{R}^{n_{dof}}$ by means of the standard finite element assembly operator $\mathcal{A}_{\mathbf{u}}^e$ according to the parametric representation $\mathbf{p}_{\mathbf{u}}^e = \mathcal{A}_{\mathbf{u}}^e\, \mathbf{q}_{\mathbf{u}}$.

On the contrary the strain and stress local parameters are simply collected in the global lists $\mathbf{q}_{\boldsymbol{\varepsilon}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$.

The overall assembly operation $\mathcal{A}^e$ of the element parameters can then be written in matrix form as

$$\begin{bmatrix} \mathbf{p}^e_{\boldsymbol{\varepsilon}} \\ \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \mathcal{A}^e \begin{bmatrix} \mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathcal{J}^e_{\boldsymbol{\varepsilon}} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathcal{A}^e_{\mathbf{u}} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix}$$

where the operators $\mathcal{J}^e_{\boldsymbol{\varepsilon}}$ and $\mathcal{J}^e_{\boldsymbol{\sigma}}$ are canonical estractors which pick up, from the global lists $\mathbf{q}_{\boldsymbol{\varepsilon}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$, the local parameters $\mathbf{p}^e_{\boldsymbol{\varepsilon}}$ and $\mathbf{p}^e_{\boldsymbol{\sigma}}$ pertaining to the current element Ω_e, that is $\mathbf{p}^e_{\boldsymbol{\varepsilon}} = \mathcal{J}^e_{\boldsymbol{\varepsilon}} \mathbf{q}_{\boldsymbol{\varepsilon}}$ and $\mathbf{p}^e_{\boldsymbol{\sigma}} = \mathcal{J}^e_{\boldsymbol{\sigma}} \mathbf{q}_{\boldsymbol{\sigma}}$ with $e = 1, \ldots, \mathcal{N}$.

The interpolated counterpart of the HU-WASHIZU functional W can be obtained by adding up the contributions $W^e_h(\boldsymbol{\varepsilon}^e_h, \boldsymbol{\sigma}^e_h, \mathbf{u}^e_h)$ of each non-assembled element and imposing that the interpolating displacement $\mathbf{u}_h$ satisfies the conformity requirement. Accordingly we have

$$W_h(\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) = \frac{1}{2} \langle\!\langle\, \mathbf{E}(\boldsymbol{\varepsilon}_h - \overline{\boldsymbol{\delta}})\,,\, \boldsymbol{\varepsilon}_h - \overline{\boldsymbol{\delta}} \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\mathbf{u}_h - \boldsymbol{\varepsilon}_h \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\overline{\mathbf{v}} \,\rangle\!\rangle - \gamma_o(\mathbf{u}_h),$$

where $(\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathcal{D}_h \times \mathcal{S}_h \times \mathcal{L}_h$.
The matrix form of the discrete problem **DMP** is obtained from the stationarity of W_h and is given by

$$\mathbf{M} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathcal{J}^{e\,T}_{\boldsymbol{\varepsilon}} \mathbf{H}^e \mathcal{J}^e_{\boldsymbol{\varepsilon}} & -\mathcal{J}^{e\,T}_{\boldsymbol{\varepsilon}} \mathbf{Q}^e \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O} \\ -\mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{Q}^{eT} \mathcal{J}^e_{\boldsymbol{\varepsilon}} & \mathbf{O} & \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{S}^e \mathcal{A}^e_{\mathbf{u}} \\ \mathbf{O} & \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{S}^{eT} \mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{H} & -\mathbf{Q} & \mathbf{O} \\ -\mathbf{Q}^T & \mathbf{O} & \mathbf{S} \\ \mathbf{O} & \mathbf{S}^T & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathcal{J}^{e\,T}_{\boldsymbol{\varepsilon}} \mathbf{f}^e_{\boldsymbol{\varepsilon}} \\ \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{f}^e_{\boldsymbol{\sigma}} \\ \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{f}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\boldsymbol{\varepsilon}} \\ \mathbf{f}_{\boldsymbol{\sigma}} \\ \mathbf{f}_{\mathbf{u}} \end{bmatrix}.$$

Here $\mathcal{J}^{e\,T}_{\boldsymbol{\varepsilon}}$, $\mathcal{J}^{e\,T}_{\boldsymbol{\sigma}}$ are the canonical immersors which put the local forces $\mathbf{f}^e_{\boldsymbol{\varepsilon}}$, $\mathbf{f}^e_{\boldsymbol{\sigma}}$ into the corresponding global lists $\mathbf{f}_{\boldsymbol{\varepsilon}}$, $\mathbf{f}_{\boldsymbol{\sigma}}$. The operator $\mathcal{A}^{e\,T}_{\mathbf{u}}$ trasforms the local forces $\mathbf{f}^e_{\mathbf{u}}$ into the corresponding global ones $\mathbf{f}_{\mathbf{u}}$.

The submatrices and subvectors introduced above are defined by

$$\mathbf{H}^e = \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\varepsilon}}^{eT}(\mathbf{x}) \mathbf{E}_*(\mathbf{x}) \mathbf{N}_{\boldsymbol{\varepsilon}}^e(\mathbf{x})\,; \quad \mathbf{Q}^e = \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\varepsilon}}^{eT}(\mathbf{x}) \mathbf{N}_{\boldsymbol{\sigma}}^e(\mathbf{x})$$

$$\mathbf{S}^e = \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\sigma}}^{eT}(\mathbf{x}) \mathbf{B}_* \mathbf{N}_{\mathbf{u}}^e(\mathbf{x})\,;$$

$$\mathbf{f}_{\boldsymbol{\varepsilon}}^e = \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\varepsilon}}^{eT}(\mathbf{x}) \mathbf{E}_*(\mathbf{x})\, \overline{\boldsymbol{\delta}}(\mathbf{x})\,; \quad \mathbf{f}_{\boldsymbol{\sigma}}^e = -\int_{\Omega_e} \mathbf{N}_{\boldsymbol{\sigma}}^{eT}(\mathbf{x}) \mathbf{B}_* \overline{\mathbf{v}}(\mathbf{x})\,;$$

$$\mathbf{f}_{\mathbf{u}}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{u}}^{eT}(\mathbf{x})\, \overline{\mathbf{b}}(\mathbf{x}) + \int_{\partial\Omega_e} (\boldsymbol{\Gamma}\mathbf{N}_{\mathbf{u}}^e)^T(\mathbf{x})\, \overline{\mathbf{t}}(\mathbf{x})\,,$$

where $\mathbf{B}_*$ and $\mathbf{E}_*$ denote the matrix form of the operators $\mathbf{B}$ and $\mathbf{E}$. The elastic stiffness matrix $\mathbf{E}_*$ is positive definite and hence the matrix $\mathbf{H}^e$ turns out to be positive definite as well.

The well-posedness analysis, discussed in the next section, is developed in terms of the three-field problem (**DMP**). On the contrary, from a computational standpoint, it is more convenient to carry out the assembly operation after the condensation at the element level of the strain and stress parameters in order to get the discrete mixed problem in terms of the displacement parameters $\mathbf{q}_{\mathbf{u}}$. This issue will be illustrated at the end of the next section.

3.4 Well-posedness analysis

As customary in computational analysis we will assume that the structure cannot undergo conforming rigid displacements.

- **Definition of well-posedness.** The discrete mixed problem **DMP** is said to be *well-posed* if there exists a unique solution $\{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h\} \in \mathcal{D}_h \times \mathcal{S}_h \times \mathcal{L}_h$ for any data $\mathbf{f}_{\boldsymbol{\varepsilon}}$, $\mathbf{f}_{\boldsymbol{\sigma}}$ and $\mathbf{f}_{\mathbf{u}}$.

In the literature well-posedness requires, in addition to existence and uniqueness of the solution for any data, that the discrete solution tends, in some energy norm, to the solution of the continuous problem as the parameter h goes to zero.

We treat separately these two requirements since the conditions for their fulfillment follow by different arguments.

The necessary and sufficient conditions ensuring the well-posedness of the discrete problem are provided in the next statement.

Proposition 3.4.1 (Well-posedness conditions) *If rigid displacements are ruled out, the conditions*

$$\begin{aligned}&\mathcal{S}_h \cap \mathcal{D}_h^{\perp} \cap \big(\mathbf{B}\mathcal{L}_h\big)^{\perp} = \mathcal{S}_h \cap \big(\mathcal{D}_h + \mathbf{B}\mathcal{L}_h\big)^{\perp} = \{\,\boldsymbol{o}\,\}\\ &\mathbf{B}\,\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\,\boldsymbol{o}\,\}\,,\end{aligned}$$

are necessary and sufficient for the well-posedness of the discrete mixed problem ***DMP****.*

The former relation of the proposition 3.4.1 imposes that the stress interpolates $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ must be controlled by the sum of the strain interpolates $\boldsymbol{\varepsilon}_h \in \mathcal{D}_h$ and of the compatible strains $\mathbf{Bu}_h$ due to the conforming displacement interpolates $\mathbf{u}_h \in \mathcal{L}_h$.

The latter condition of the proposition 3.4.1 requires that the compatible strain interpolates $\mathbf{Bu}_h$, with $\mathbf{u}_h \in \mathcal{L}_h$, must be controlled by the stress interpolates $\boldsymbol{\sigma}_h \in \mathcal{S}_h$.

The validity of the former relation of the proposition 3.4.1 requires that

$$\dim \mathcal{S}_h \leq \dim\left(\mathcal{D}_h + \mathbf{B}\mathcal{L}_h\right) \leq \dim \mathcal{D}_h + \dim \mathbf{B}\mathcal{L}_h \quad \Longleftrightarrow \quad n_{\boldsymbol{\sigma}} \leq n_{\boldsymbol{\varepsilon}} + \frac{n_{dof}}{\mathcal{N}},$$

and the validity of the latter relation of the proposition 3.4.1 requires that

$$\dim \mathbf{B}\mathcal{L}_h \leq \dim \mathcal{S}_h \quad \Longleftrightarrow \quad \frac{n_{dof}}{\mathcal{N}} \leq n_{\boldsymbol{\sigma}}$$

where n_{dof} is the total number of nodal displacement parameters and

$$n_{\boldsymbol{\sigma}} = \dim \mathcal{S}_h^e, \quad n_{\boldsymbol{\varepsilon}} = \dim \mathcal{D}_h^e.$$

The previous necessary conditions in terms of the number of parameters were first provided by Zienkiewicz and Lefebvre who followed a completely different argument based on a regularization of the element stiffness matrix [48].

The well-posedness criteria reported in proposition 3.4.1 appears to be new.

Remark 3.4.1 *The ratio $n_{dof}/\mathcal{N}$ can be simply estimated in the limit $\mathcal{N} \to +\infty$.*

In plane problems with a mesh composed by quadrilateral Q8 elements we have

$$\lim_{\mathcal{N}\to+\infty} \frac{n_{dof}}{\mathcal{N}} = 6.$$

Accordingly the necessary conditions above are fulfilled if $6 \leq n_{\boldsymbol{\sigma}} \leq n_{\boldsymbol{\varepsilon}} + 6$. For Q4 elements these inequalities become $2 \leq n_{\boldsymbol{\sigma}} \leq n_{\boldsymbol{\varepsilon}} + 2$.

It is apparent that the stress interpolation proposed by Pian and Sumihara [41], and constantly adopted in the computational literature on mixed methods, passes the test for the Q4 element but fails the test for the Q8 element, since $n_{\boldsymbol{\sigma}} = 5$.

The global condition

$$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} \cap \left(\mathbf{B}\mathcal{L}_h\right)^{\perp} = \mathcal{S}_h \cap \left(\mathcal{D}_h + \mathbf{B}\mathcal{L}_h\right)^{\perp} = \{\,\boldsymbol{o}\,\}$$

can be enforced by imposing the stronger local condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}$. Since $\mathbf{B}\mathcal{L}_h \subseteq \mathbf{B}\mathcal{V}_h$, the global condition

$$\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\,\boldsymbol{o}\,\}$$

can be fulfilled by imposing the local condition $\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\,\boldsymbol{o}\,\}$.

We can then state the following

Proposition 3.4.2 (Sufficient conditions) *The local conditions*

$$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}$$

$$\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\,\boldsymbol{o}\,\}$$

*are sufficient for the well-posedness of the discrete mixed problem **DMP** (matrix form).*

The former condition imposes that the local stress interpolates are controlled by the local strain interpolates and the latter condition requires that the local compatible strain interpolates are controlled by the local stress interpolates.

It is worth noting that the condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}$ can be imposed by choosing $\mathcal{D}_h$ such that $\mathcal{S}_h \subset \mathcal{D}_h$.

3.5 Stiffness matrix

Let us preliminarily note that the local condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}$ can be equivalently written in the matrix form $\operatorname{Ker}\mathbf{Q}^e = \{\boldsymbol{o}\}$ with $e = 1, \ldots, \mathcal{N}$ since we have:

$$\begin{aligned}\boldsymbol{\sigma}_h^e = \mathbf{N}_{\boldsymbol{\sigma}}^e \, \mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{S}_h^e \cap \mathcal{D}_h^{e\,\perp} \quad &\Longleftrightarrow \quad \int_{\Omega_e} \boldsymbol{\sigma}_h^e \cdot \overline{\boldsymbol{\varepsilon}}_h^e \, d\Omega_e = 0 \quad \forall \overline{\boldsymbol{\varepsilon}}_h^e \in \mathcal{D}_h^e \\ &\Longleftrightarrow \quad \mathbf{p}_{\boldsymbol{\sigma}}^e \in \operatorname{Ker}\mathbf{Q}^e \,.\end{aligned}$$

Accordingly, if the mixed discrete problem

$$\mathbf{M}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathcal{J}_{\boldsymbol{\varepsilon}}^{e\,T}\mathbf{H}^e\mathcal{J}_{\boldsymbol{\varepsilon}}^e & -\mathcal{J}_{\boldsymbol{\varepsilon}}^{e\,T}\mathbf{Q}^e\mathcal{J}_{\boldsymbol{\sigma}}^e & \mathbf{O} \\ -\mathcal{J}_{\boldsymbol{\sigma}}^{e\,T}\mathbf{Q}^{eT}\mathcal{J}_{\boldsymbol{\varepsilon}}^e & \mathbf{O} & \mathcal{J}_{\boldsymbol{\sigma}}^{e\,T}\mathbf{S}^e\mathcal{A}_{\mathbf{u}}^e \\ \mathbf{O} & \mathcal{A}_{\mathbf{u}}^{e\,T}\mathbf{S}^{eT}\mathcal{J}_{\boldsymbol{\sigma}}^e & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{H} & -\mathbf{Q} & \mathbf{O} \\ -\mathbf{Q}^T & \mathbf{O} & \mathbf{S} \\ \mathbf{O} & \mathbf{S}^T & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}}\begin{bmatrix}\mathcal{J}_{\boldsymbol{\varepsilon}}^{e\,T}\mathbf{f}_{\boldsymbol{\varepsilon}}^e \\ \mathcal{J}_{\boldsymbol{\sigma}}^{e\,T}\mathbf{f}_{\boldsymbol{\sigma}}^e \\ \mathcal{A}_{\mathbf{u}}^{e\,T}\mathbf{f}_{\mathbf{u}}^e\end{bmatrix} = \begin{bmatrix}\mathbf{f}_{\boldsymbol{\varepsilon}} \\ \mathbf{f}_{\boldsymbol{\sigma}} \\ \mathbf{f}_{\mathbf{u}}\end{bmatrix}$$

is well-posed, a simple derivation of the element stiffness matrix can be performed by eliminating the strain and the stress parameters according to the following procedure

$$\begin{cases}\mathbf{H}^e\,\mathbf{p}_{\boldsymbol{\varepsilon}}^e = \mathbf{Q}^e\,\mathbf{p}_{\boldsymbol{\sigma}}^e + \mathbf{f}_{\boldsymbol{\varepsilon}}^e \\ (\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)\,\mathbf{p}_{\boldsymbol{\sigma}}^e = \mathbf{S}^e\,\mathbf{p}_{\mathbf{u}}^e - \mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{f}_{\boldsymbol{\varepsilon}}^e - \mathbf{f}_{\boldsymbol{\sigma}}^e \\ \left[\mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{eT}\mathbf{Q}^e)^{-1}\mathbf{S}^e\right]\,\mathbf{p}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}^e + \mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\Big(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{f}_{\boldsymbol{\varepsilon}}^e + \mathbf{f}_{\boldsymbol{\sigma}}^e\Big)\,.\end{cases}$$

Denoting the element stiffness matrix by

$$i) \qquad \mathbf{K}^e = \mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\mathbf{S}^e,$$

and the element external force, equivalent to assigned loads, distorsions and imposed displacements, by

$$ii) \qquad \mathbf{f}_{\mathrm{eq}}^e = \mathbf{f}_{\mathbf{u}}^e + \mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\left(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{f}_{\boldsymbol{\varepsilon}}^e + \mathbf{f}_{\boldsymbol{\sigma}}^e\right),$$

the mixed problem at the element level can be written as

$$\mathbf{K}^e \ \mathbf{p}^e_{\mathbf{u}} = \mathbf{f}^e_{\mathrm{eq}}$$

so that the global problem will be expressed in terms of nodal displacement parameters as

$$\mathbb{K}\,\mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}} \left(\mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{K}^e \mathcal{A}^e_{\mathbf{u}} \right) \mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}} \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{f}^e_{\mathrm{eq}} = \mathbf{f}_{\mathrm{eq}}.$$

We underline that, due to the positive definiteness of $\mathbf{H}^e$ and the nonsingularity of $\mathbf{Q}^e$, the matrix $\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e$ is invertible. Once the global structural problem has been solved in terms of nodal displacements, the stress and the strain parameters can be evaluated at the element level according to the formulae:

$$\begin{aligned} \mathbf{p}^e_{\boldsymbol{\sigma}} &= (\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1} \left(\mathbf{S}^e \, \mathbf{p}^e_{\mathbf{u}} - \mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{f}^e_{\boldsymbol{\varepsilon}} - \mathbf{f}^e_{\boldsymbol{\sigma}} \right) \\ \mathbf{p}^e_{\boldsymbol{\varepsilon}} &= \mathbf{H}^{e-1} \left(\mathbf{Q}^e \, \mathbf{p}^e_{\boldsymbol{\sigma}} + \mathbf{f}^e_{\boldsymbol{\varepsilon}} \right) . \end{aligned}$$

3.6 Mixed enhanced strain method

In this section we provide a detailed presentation of the mixed enhanced strain (MES) method as a special case of the three-field HW method illustrated in the previous sections.

The well posedness analysis yields a global necessary and sufficient condition and a local sufficient condition which are analogous to the ones pertaining to the two field method based on the Hellinger-Reissner mixed principle.

The explicit expression of the relevant stiffness matrix is also provided.

3.6.1 Discrete fields

The well-posedness analysis performed in the previous sections has shown that uniqueness of the solution in terms of stresses is implied by the condition

$$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \mathbf{o} \,\}.$$

This condition can be fulfilled by assuming a strain interpolation of the form

$$\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$$

where $\widetilde{\mathcal{D}}_h$ is a subspace of enhanced strains $\widetilde{\boldsymbol{\varepsilon}}_h$ and $\oplus$ denotes the direct sum which requires that $\mathcal{S}_h \cap \widetilde{\mathcal{D}}_h = \{\, \mathbf{o} \,\}$.

Without loss of generality we can assume that the orthogonality relation $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ holds as the result of a GRAM-SCHMIDT othogonalization procedure.

Mixed methods based on such a decomposition of the strain interpolation are referred to as mixed enhanced strain (MES) methods.

An alternative choice of $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$ could be

$$\mathcal{D}_h = \mathbf{E}^{-1}\mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h .$$

In terms of shape functions we have

$$\boldsymbol{\varepsilon}_h^e = \mathbf{N}_{\mathbf{e}}^e\, \mathbf{p}_{\mathbf{e}}^e + \mathbf{N}_{\boldsymbol{\alpha}}^e\, \mathbf{p}_{\boldsymbol{\alpha}}^e \quad \text{where} \quad n_{\mathbf{e}} = n_{\boldsymbol{\sigma}}, \quad \text{so that} \quad n_{\boldsymbol{\varepsilon}} = n_{\mathbf{e}} + n_{\boldsymbol{\alpha}}; \quad n_{\boldsymbol{\alpha}} \geq 0.$$

The nonsingularity of the matrix $\mathbf{Q}^e$ is ensured since the submatrix

$$\mathbf{Q}_{\mathbf{e}\boldsymbol{\sigma}}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{e}}^{eT} \mathbf{N}_{\boldsymbol{\sigma}}^e ,$$

is nonsingular.

The two choices above requires that $\mathbf{N}_{\mathbf{e}}^e = \mathbf{N}_{\boldsymbol{\sigma}}^e$ or $\mathbf{E}_* \mathbf{N}_{\mathbf{e}}^e = \mathbf{N}_{\boldsymbol{\sigma}}^e$; the former was made by KASPER and TAYLOR in [75].

Let us now derive the explicit expression of the element stiffness matrix for the MES method in terms of the enhanced strain parameters.

To this end we note that $\boldsymbol{\varepsilon}_h = \mathbf{e}_h + \widetilde{\boldsymbol{\varepsilon}}_h$ with $\mathbf{e}_h \in \mathcal{S}_h$ and $\widetilde{\boldsymbol{\varepsilon}}_h \in \widetilde{\mathcal{D}}_h$ so that the interpolated counterpart of the HU-WASHIZU functional W in terms of $\{\, \mathbf{e}_h, \widetilde{\boldsymbol{\varepsilon}}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h \,\} \in \mathcal{S}_h \times \widetilde{\mathcal{D}}_h \times \mathcal{S}_h \times \mathcal{L}_h$ is given by

$$\begin{aligned} \widetilde{W}_h(\mathbf{e}_h, \widetilde{\boldsymbol{\varepsilon}}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) = & \frac{1}{2} \langle\!\langle\, \mathbf{E}\mathbf{e}_h \,,\, \mathbf{e}_h \,\rangle\!\rangle + \frac{1}{2} \langle\!\langle\, \mathbf{E}\widetilde{\boldsymbol{\varepsilon}}_h \,,\, \widetilde{\boldsymbol{\varepsilon}}_h \,\rangle\!\rangle + \langle\!\langle\, \mathbf{E}\mathbf{e}_h \,,\, \widetilde{\boldsymbol{\varepsilon}}_h \,\rangle\!\rangle + \\ & - \langle\!\langle\, \mathbf{E}(\mathbf{e}_h + \widetilde{\boldsymbol{\varepsilon}}_h) \,,\, \overline{\boldsymbol{\delta}} \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\mathbf{u}_h - \mathbf{e}_h - \widetilde{\boldsymbol{\varepsilon}}_h \,\rangle\!\rangle + \\ & + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\overline{\mathbf{v}} \,\rangle\!\rangle - \gamma_o(\mathbf{u}_h) \,, \end{aligned}$$

where the constant term $\langle\!\langle\, \mathbf{E}\overline{\boldsymbol{\delta}} \,,\, \overline{\boldsymbol{\delta}} \,\rangle\!\rangle$ has been dropped. The matrix form of the element, according to the parametric representation of $\boldsymbol{\varepsilon}_h$, is given in the partitioned form

$$\begin{bmatrix} \mathbf{H}^e_{\mathbf{ee}} & \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}} & -\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}} & \mathbf{O} \\ \mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} & \mathbf{H}^e_{\boldsymbol{\alpha\alpha}} & -\mathbf{Q}^e_{\boldsymbol{\alpha\sigma}} & \mathbf{O} \\ -\mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}} & -\mathbf{Q}^{eT}_{\boldsymbol{\alpha\sigma}} & \mathbf{O} & \mathbf{S}^e \\ \mathbf{O} & \mathbf{O} & \mathbf{S}^{eT} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{p}^e_{\mathbf{e}} \\ \mathbf{p}^e_{\boldsymbol{\alpha}} \\ \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e_{\mathbf{e}} \\ \mathbf{f}^e_{\boldsymbol{\alpha}} \\ \mathbf{f}^e_{\boldsymbol{\sigma}} \\ \mathbf{f}^e_{\mathbf{u}} \end{bmatrix}$$

with submatrices and subvectors defined by

$$\mathbf{H}^e_{\mathbf{ee}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{e}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{N}^e_{\mathbf{e}}(\mathbf{x})\,; \quad \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{e}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{N}^e_{\boldsymbol{\alpha}}(\mathbf{x})\,;$$

$$\mathbf{H}^e_{\boldsymbol{\alpha\alpha}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\alpha}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{N}^e_{\boldsymbol{\alpha}}(\mathbf{x})\,; \quad \mathbf{S}^e = \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\sigma}}(\mathbf{x})\mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}(\mathbf{x})\,;$$

$$\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{e}}(\mathbf{x})\mathbf{N}^e_{\boldsymbol{\sigma}}(\mathbf{x})\,; \quad \mathbf{Q}^e_{\boldsymbol{\alpha\sigma}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\alpha}}(\mathbf{x})\mathbf{N}^e_{\boldsymbol{\sigma}}(\mathbf{x})\,;$$

$$\mathbf{f}^e_{\mathbf{e}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{e}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\,\bar{\boldsymbol{\delta}}(\mathbf{x})\,; \quad \mathbf{f}^e_{\boldsymbol{\alpha}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\alpha}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\,\bar{\boldsymbol{\delta}}(\mathbf{x})\,;$$

$$\mathbf{f}^e_{\boldsymbol{\sigma}} = -\int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\sigma}}(\mathbf{x})\,\big[\mathbf{B}_*\bar{\mathbf{v}}(\mathbf{x})\big]\,; \quad \mathbf{f}^e_{\mathbf{u}} = \int_{\Omega_e} \mathbf{N}^{eT}_{\mathbf{u}}(\mathbf{x})\,\bar{\mathbf{b}}(\mathbf{x}) + \int_{\partial\Omega_e} (\boldsymbol{\Gamma}\mathbf{N}^e_{\mathbf{u}})^T(\mathbf{x})\,\bar{\mathbf{t}}(\mathbf{x})\,,$$

where $\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}}$ and $\mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}}$ are square and nonsingular.

3.6.2 Well-posedness analysis

The well-posedness conditions for the MES method can be derived by specializing the global conditions reported in proposition 3.4.1 becomes

$$\mathcal{S}_h \cap (\mathcal{S}_h + \widetilde{\mathcal{D}}_h)^\perp \cap \big(\mathbf{B}\mathcal{L}_h\big)^\perp = \mathcal{S}_h \cap \mathcal{S}_h^\perp \cap \widetilde{\mathcal{D}}_h^\perp \cap \big(\mathbf{B}\mathcal{L}_h\big)^\perp = \{\,\boldsymbol{o}\,\}$$

and is always satisfied since $\mathcal{S}_h \cap \mathcal{S}_h^\perp = \{\,\boldsymbol{o}\,\}$.

Uniqueness of the stress solution is then ensured. Since uniqueness of enhanced strains holds by virtue of the positive definiteness of the elastic energy, the sole condition $\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\, \boldsymbol{o} \,\}$ (proposition 3.4.1) remains to be checked as a criterion for uniqueness of the displacement solution.

Proposition 3.6.1 (Well-posedness condition for the MES) *The condition* $\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\, \boldsymbol{o} \,\}$ *is necessary and sufficient and the local condition* $\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\, \boldsymbol{o} \,\}$ *is sufficient for the well-posedness of the mixed three-field problem*

$$\begin{bmatrix} \mathbf{H}^e_{\mathbf{ee}} & \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}} & -\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}} & \mathbf{O} \\ \mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} & \mathbf{H}^e_{\boldsymbol{\alpha\alpha}} & -\mathbf{Q}^e_{\boldsymbol{\alpha\sigma}} & \mathbf{O} \\ -\mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}} & -\mathbf{Q}^{eT}_{\boldsymbol{\alpha\sigma}} & \mathbf{O} & \mathbf{S}^e \\ \mathbf{O} & \mathbf{O} & \mathbf{S}^{eT} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{p}^e_{\mathbf{e}} \\ \mathbf{p}^e_{\boldsymbol{\alpha}} \\ \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e_{\mathbf{e}} \\ \mathbf{f}^e_{\boldsymbol{\alpha}} \\ \mathbf{f}^e_{\boldsymbol{\sigma}} \\ \mathbf{f}^e_{\mathbf{u}} \end{bmatrix}$$

In the original formulation of the mixed enhanced method proposed by Kasper and Taylor in [75] the well-posedness of the MES method was not investigated.

Anyway they quoted the well-posedness conditions originally deduced by Simo and Rifai in [55] with reference to the enhanced assumed strain (EAS) method. In this respect we observe that the EAS method is a mixed method based on a different decomposition of the strain interpolation, that is $\boldsymbol{\varepsilon}_h = \mathbf{B}\mathbf{u}_h + \widetilde{\boldsymbol{\varepsilon}}_h$.

The need for a rationale on this topic has originally motivated the analysis of three-field mixed methods developed in Romano et al. in [78].

3.6.3 Stiffness matrix

Let us assume that a linearly independent set $\{\, \mathbf{N}^e_{\boldsymbol{\alpha}}, \mathbf{N}^e_{\boldsymbol{\sigma}} \,\}$ of enhanced strain and stress shape functions be given.

The shape functions $\mathbf{N}^e_{\boldsymbol{\alpha}}$ and $\mathbf{N}^e_{\boldsymbol{\sigma}}$ can be assumed to be $L^2(\Omega)$ othogonal as the result of a Gram-Schmidt procedure so that the orthogonality condition $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$, equivalent to $\mathbf{Q}_{\boldsymbol{\alpha\sigma}} = \mathbf{O}$, is met.

If the choice $\mathbf{N}^e_{\mathbf{e}} = \mathbf{N}^e_{\boldsymbol{\sigma}}$ is made, a simple explicit expression can be given to the matrix $\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e$ and to its inverse, in terms of the submatrices $\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}}$, $\mathbf{H}^e_{\mathbf{ee}}$, $\mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}$ and $\mathbf{H}^e_{\boldsymbol{\alpha\alpha}}$.

To this end we consider the partition of the inverse of $\mathbf{H}^e$

$$\mathbf{H}^{e-1} = \begin{bmatrix} \mathbf{Z}^e_{\mathbf{ee}} & \mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} \\ \mathbf{Z}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} & \mathbf{Z}^e_{\boldsymbol{\alpha\alpha}} \end{bmatrix}$$

to get

$$\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e = \begin{bmatrix}\mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}} & \mathbf{O}\end{bmatrix} \begin{bmatrix} \mathbf{Z}^e_{\mathbf{ee}} & \mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} \\ \mathbf{Z}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} & \mathbf{Z}^e_{\boldsymbol{\alpha\alpha}} \end{bmatrix} \begin{bmatrix}\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}} \\ \mathbf{O}\end{bmatrix} = \mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{Z}^e_{\mathbf{ee}}\,\mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}}.$$

The expression of $\mathbf{Z}^e_{\mathbf{ee}}$ can be derived from the relations

$$\mathbf{H}^e\,\mathbf{H}^{e-1} = \mathbf{I} \quad \Longleftrightarrow \quad \begin{cases} \mathbf{H}^e_{\mathbf{ee}}\,\mathbf{Z}^e_{\mathbf{ee}} + \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\,\mathbf{Z}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} = \mathbf{I}_{n_\mathbf{e}} \\ \mathbf{H}^e_{\mathbf{ee}}\,\mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} + \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\,\mathbf{Z}^e_{\boldsymbol{\alpha\alpha}} = \mathbf{O} \\ \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\,\mathbf{Z}^e_{\mathbf{ee}} + \mathbf{H}^e_{\boldsymbol{\alpha\alpha}}\,\mathbf{Z}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} = \mathbf{O} \\ \mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\,\mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} + \mathbf{H}^e_{\boldsymbol{\alpha\alpha}}\,\mathbf{Z}^e_{\boldsymbol{\alpha\alpha}} = \mathbf{I}_{n_\alpha} \end{cases}$$

In fact the third equation yields $\mathbf{Z}^{eT}_{\mathbf{e}\boldsymbol{\alpha}} = -\mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\,\mathbf{Z}^e_{\mathbf{ee}}$ and the first $\left(\mathbf{H}^e_{\mathbf{ee}} - \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\right)\mathbf{Z}^e_{\mathbf{ee}} = \mathbf{I}_{n_\mathbf{e}}$.

The matrix $\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e$ can then be readily inverted in the form

$$(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1} = \mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{Z}^{e-1}_{\mathbf{ee}}\,\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}} = \mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\left(\mathbf{H}^e_{\mathbf{ee}} - \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\right)\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}}.$$

In addition, again from the third equation written as $\mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} = -\mathbf{Z}^e_{\mathbf{ee}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}}$, we get

$$\mathbf{Z}^{e-1}_{\mathbf{ee}}\mathbf{Z}^e_{\mathbf{e}\boldsymbol{\alpha}} = -\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}}.$$

Substituting the last two equalities in the expressions of the element stiffness matrix i) and in the vector of external forces ii), rewritten here for convenience

$$\mathbf{K}^e = \mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\mathbf{S}^e$$

$$\mathbf{f}^e_{\mathrm{eq}} = \mathbf{f}^e_{\mathbf{u}} + \mathbf{S}^{eT}(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\left(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{f}^e_{\boldsymbol{\varepsilon}} + \mathbf{f}^e_{\boldsymbol{\sigma}}\right)$$

after some algebra, we get

$$iii) \qquad \mathbf{K}^e = \mathbf{K}^e_{\mathbf{uu}} - \mathbf{K}^e_{\mathbf{u}\boldsymbol{\alpha}}\mathbf{K}^{e-1}_{\boldsymbol{\alpha\alpha}}\mathbf{K}^{eT}_{\mathbf{u}\boldsymbol{\alpha}}$$

where

$$\mathbf{K}^e_{\mathbf{uu}} = \mathbf{S}^{eT}\mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{H}^e_{\mathbf{ee}}\,\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}}\mathbf{S}^e\,, \qquad \mathbf{K}^e_{\mathbf{u}\boldsymbol{\alpha}} = \mathbf{S}^{eT}\mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\,, \qquad \mathbf{K}^e_{\boldsymbol{\alpha}\boldsymbol{\alpha}} = \mathbf{H}^e_{\boldsymbol{\alpha}\boldsymbol{\alpha}}$$

and

$$\begin{aligned}\mathbf{f}^e_{\mathrm{eq}} =& \mathbf{f}^e_{\mathbf{u}} + \mathbf{S}^{eT}\mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{f}^e_{\mathbf{e}} - \mathbf{K}^e_{\mathbf{u}\boldsymbol{\alpha}}\mathbf{K}^{e-1}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}\mathbf{f}^e_{\boldsymbol{\alpha}} + \\ &+(\mathbf{S}^{eT}\mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}\,\mathbf{H}^{eT}_{\mathbf{ee}}\,\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}} - \mathbf{K}^e_{\mathbf{u}\boldsymbol{\alpha}}\mathbf{K}^{e-1}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}})\mathbf{f}^e_{\boldsymbol{\alpha}}\,.\end{aligned}$$

Note that, due to the choice $\mathbf{N}^e_{\mathbf{e}} = \mathbf{N}^e_{\boldsymbol{\sigma}}$, we have $\mathbf{Q}^{eT}_{\mathbf{e}\boldsymbol{\sigma}} = \mathbf{Q}^e_{\mathbf{e}\boldsymbol{\sigma}}$.

The stress and strain parameters can then be evaluated according to the formulae:

$$\mathbf{p}_{\boldsymbol{\sigma}} = \mathbf{Q}^{e-1}_{\mathbf{e}\boldsymbol{\sigma}}(\mathbf{H}^e_{\mathbf{ee}} - \mathbf{H}^e_{\mathbf{e}\boldsymbol{\alpha}}\mathbf{H}^{e-1}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}\mathbf{H}^{eT}_{\mathbf{e}\boldsymbol{\alpha}})\mathbf{Q}^{e-T}_{\mathbf{e}\boldsymbol{\sigma}}\mathbf{S}^e\mathbf{p}_{\mathbf{u}} \qquad \mathbf{p}_{\boldsymbol{\varepsilon}} = \mathbf{H}^{e-1}\mathbf{Q}^e\mathbf{p}_{\boldsymbol{\sigma}}.$$

The expression of the element stiffness matrix iii) derived above coincides with the one which was contributed by Kasper and Taylor in [75] following a less direct approach.

3.6.4 A limitation principle

If the choice $\mathbf{E}_*\mathbf{N}^e_{\mathbf{e}} = \mathbf{N}^e_{\boldsymbol{\sigma}}$ is made, a limitation phenomenon occurs if no additional parameters $\mathbf{p}^e_{\boldsymbol{\alpha}}$ are considered.

In this case the three-field Hu-Washizu principle and the two-field Hellinger-Reissner principle provide the same solution in terms of $\mathbf{u}_h$ and $\boldsymbol{\sigma}_h$ if the condition $\mathcal{S}_h \subseteq \mathbf{E}\mathcal{D}_h$ is fulfilled.

The special form of the element stiffness matrix can be deduced by observing that

$$\mathbf{H}^e = \mathbf{Q}^e = \int_{\Omega_e} \mathbf{N}^{eT}_{\boldsymbol{\sigma}}(\mathbf{x})\mathbf{E}_*^{-1}(\mathbf{x})\mathbf{N}^e_{\boldsymbol{\sigma}}(\mathbf{x}).$$

The discrete three-field problem

$$\mathbf{M}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}}\\ \mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}}\begin{bmatrix}\mathcal{J}^{eT}_{\boldsymbol{\varepsilon}}\mathbf{H}^e\mathcal{J}^e_{\boldsymbol{\varepsilon}} & -\mathcal{J}^{eT}_{\boldsymbol{\varepsilon}}\mathbf{Q}^e\mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O}\\ -\mathcal{J}^{e\,T}_{\boldsymbol{\sigma}}\mathbf{Q}^{eT}\mathcal{J}^e_{\boldsymbol{\varepsilon}} & \mathbf{O} & \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}}\mathbf{S}^e\mathcal{A}^e_{\mathbf{u}}\\ \mathbf{O} & \mathcal{A}^{e\,T}_{\mathbf{u}}\mathbf{S}^{eT}\mathcal{J}^e_{\boldsymbol{\sigma}} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}}\\ \mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{H} & -\mathbf{Q} & \mathbf{O}\\ -\mathbf{Q}^T & \mathbf{O} & \mathbf{S}\\ \mathbf{O} & \mathbf{S}^T & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\boldsymbol{\varepsilon}}\\ \mathbf{q}_{\boldsymbol{\sigma}}\\ \mathbf{q}_{\mathbf{u}}\end{bmatrix} = \sum_{e=1}^{\mathcal{N}}\begin{bmatrix}\mathcal{J}^{e\,T}_{\boldsymbol{\varepsilon}}\mathbf{f}^e_{\boldsymbol{\varepsilon}}\\ \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}}\mathbf{f}^e_{\boldsymbol{\sigma}}\\ \mathcal{A}^{e\,T}_{\mathbf{u}}\mathbf{f}^e_{\mathbf{u}}\end{bmatrix} = \begin{bmatrix}\mathbf{f}_{\boldsymbol{\varepsilon}}\\ \mathbf{f}_{\boldsymbol{\sigma}}\\ \mathbf{f}_{\mathbf{u}}\end{bmatrix}$$

can thus be rewritten as:

$$\begin{bmatrix} \mathbf{H}^e & -\mathbf{H}^e & \mathbf{O} \\ -\mathbf{H}^e & \mathbf{O} & \mathbf{S}^e \\ \mathbf{O} & \mathbf{S}^{eT} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{p}^e_{\boldsymbol{\varepsilon}} \\ \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e_{\boldsymbol{\varepsilon}} \\ \mathbf{f}^e_{\boldsymbol{\sigma}} \\ \mathbf{f}^e_{\mathbf{u}} \end{bmatrix} .$$

From the first equation we infer that $\mathbf{p}^e_{\boldsymbol{\varepsilon}} = \mathbf{p}^e_{\boldsymbol{\sigma}} + \mathbf{H}^{e-1}\mathbf{f}^e_{\boldsymbol{\varepsilon}}$ so that the stiffness matrix reduces to

$$\begin{bmatrix} -\mathbf{H}^e & \mathbf{S}^e \\ \mathbf{S}^{eT} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{p}^e_{\boldsymbol{\sigma}} \\ \mathbf{p}^e_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e_{\boldsymbol{\sigma}} + \mathbf{f}^e_{\boldsymbol{\varepsilon}} \\ \mathbf{f}^e_{\mathbf{u}} \end{bmatrix} .$$

and coincides with the one provided by the two-field HELLINGER-REISSNER principle.

Chapter 4

Strain gap method

4.1 Introduction

The enhanced assumed strain (EAS) method proposed by Simo and Rifai in [55] was originally developed to provide a variational basis to the incompatible mode element of Wilson et al. [27]. Enhanced strain methods have been widely adopted in the literature for both linear and non-linear elastic models as well as for elastoplastic problems [59], [60], [61], [67], [69], [72], [74].

The treatment developed in [55] was based on a modified version of the Hu-Washizu variational principle in which the independent fields are displacements, enhanced strains and stresses.

The role of three conditions to be imposed on these fields was emphasized in [55], [70] to provide well-posedness, convergence and stress independence of the F.E.M. problem. These three conditions are:

- *i*) compatible and enhanced strain shape functions must be linearly independent,
- *ii*) shape functions of the stress fields and shape functions of the enhanced strains must be mutually orthogonal in the mean square sense,
- *iii*) the space of stress fields must include at least piece-wise constant functions.

Condition *iii*) was motivated in [55] by the fulfilment of the patch test. Condition *i*) ensures the uniqueness of the discrete solution in terms of displacements and enhanced strains. Condition *ii*) was designed to eliminate the stress parameters from the mixed problem.

An a posteriori stress recovery strategy must then be envisaged and in fact several proposals have been made in the literature [55], [61], [73], [79], [82].

Here we prove that variationally consistent stress recovery strategies can be derived from a formulation of the mixed method which fulfils the well-posedness conditions.

Our treatment is aimed to get a deeper understanding of the EAS method and of the related well-posedness and convergence properties. The differences between compatible strains and independent strain fields are called strain gaps.

Accordingly the new formulation of the discrete method is referred to as the Strain Gap Method (SGM).

It is shown that necessary and sufficient conditions for well-posedness of the SGM, that is for existence and uniqueness of its solution, are the following

- *a*) effective strain gaps (i.e. the ones orthogonal to the stress fields) and compatible strains must be linearly independent,
- *b*) stresses must be controlled by strain gaps.

Condition *a*) ensures the uniqueness of the solution in terms of displacement and strain gap, condition *b*) pertaines to the uniqueness of the stress solution.

According to the general treatment of mixed methods [57], [84], the SGM can be split into a sequence of two steps:

- a reduced problem, formulated in terms of displacements and strain gaps in which the orthogonality constraint between stresses and strain gaps is assumed to be fulfilled,
- a stress recovery problem which depends on the solution of the reduced problem.

An explicit comparison between the SGM and the EAS method clarifies the significant differences between the two formulations. According to the SGM, the discrete subspaces of strain gap and stress fields are choosen so that the well-posedness requirements *a*) and *b*) are fulfilled. On the contrary in the EAS method these two discrete subspaces are imposed to be mutually orthogonal so that the well-posedness requirement *b*) is violated.

The troubles faced in envisaging an a posteriori stress recovery strategy are in fact due to the partial *ill-posedness* of the EAS method.

The reduced problem which is the first step of the SGM is equivalent to the whole EAS method. It can be reformulated as a modified displacement method with an enhanced flexibility.

Once the reduced problem has been solved in terms of nodal displacements and strain gap parameters, the stress parameters can be univocally recovered at the element level by following the stress recovery strategy defined by the second step of the SGM.

In this respect we shall prove that the computation of the discrete stress according to the elastic constitutive relation is variationally consistent despite of the opposite opinion expressed in [61], [73], [79]. This result is in accordance with the analogous statement in [82] which was based on a more involved matricial arguments and limitated to undistorted meshes.

The convergence analysis of the EAS method developed in [70], [71] was based on the interpolation properties of the displacement shape functions and on a special orthogonality assumption between the enhanced strains and polynomials of suitable degree.

This spourious requirement is in apparent contraddiction with the observation that no interpolation properties are required to the strain gap shape functions since any strain gap subspace includes the null field, that is the exact solution. In fact the convergence analysis of the SGM shows that the error estimate depends only on the interpolation properties of the discrete subspaces of stress and displacement shape functions [84].

Finally, we develop a general formulation of the SGM in which the orthogonality constraint is not satisfied a priori but enters as one of the equations of the discrete mixed problem. This formulation is a usefull tool in detecting the computational performances relevant to different implementations of the discrete method.

Numerical examples of two-dimensional elastostatic problems, which are commonly adopted in the literature as significant benchmarks, are developed and discussed to get informations about comparative convergence properties, distortion sensitivity and reliability of the stress approximation.

4.2 Variational formulation

Let us consider the Hu-Washizu functional [37] whose expression was given in 3.2. For simplicity we do not consider imposed displacements and strains so that

$$H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{u}) = \phi(\boldsymbol{\varepsilon}) - (\!(\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} \,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{Bu} \,)\!)_{\mathcal{H}} - \gamma_o(\mathbf{u}),$$

where the unknown displacement field is conforming, i.e. $\mathbf{u} \in \mathcal{L}$.

The linear functional

$$\gamma_o(\mathbf{u}) = \langle\, \bar{\ell}\,,\, \mathbf{u}\,\rangle = (\, \bar{\mathbf{b}}\,,\, \mathbf{u}\,)_{\mathcal{H}} + \langle\, \bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\mathbf{u}\,\rangle$$

yields the virtual work of body forces and boundary tractions.

The stationarity conditions of the functional H are

$$\begin{cases} (\!(\, \mathbf{E}\boldsymbol{\varepsilon}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - (\!(\, \boldsymbol{\sigma}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} = (\!(\, \mathbf{E}\bar{\boldsymbol{\delta}}\,,\, \delta\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} & \forall\, \delta\boldsymbol{\varepsilon} \in \mathcal{D} \\ -(\!(\, \delta\boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} + (\!(\, \delta\boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}\,)\!)_{\mathcal{H}} = -(\!(\, \delta\boldsymbol{\sigma}\,,\, \mathbf{B}\bar{\mathbf{v}}\,)\!)_{\mathcal{H}} & \forall\, \delta\boldsymbol{\sigma} \in \mathcal{S} \\ (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\delta\mathbf{u}\,)\!)_{\mathcal{H}} = (\, \bar{\mathbf{b}}\,,\, \delta\mathbf{u}\,)_{\mathcal{H}} + \langle\!\langle\, \bar{\mathbf{t}}\,,\, \boldsymbol{\Gamma}\delta\mathbf{u}\,\rangle\!\rangle & \forall\, \delta\mathbf{u} \in \mathcal{L}. \end{cases}$$

Let us now introduce the *Strain Gap Method* (SGM) by defining the strain gap field $\mathbf{g} \in \mathcal{D}$ as the difference between the compatible strain $\mathbf{B}\mathbf{u} \in \mathcal{D}$ and the stress $\boldsymbol{\varepsilon} \in \mathcal{D}$ in the form $\mathbf{g} = \mathbf{B}\mathbf{u} - \boldsymbol{\varepsilon}$.

Accordingly the HU-WASHIZU functional can be re-written as

$$\begin{aligned} \widetilde{H}(\mathbf{u},\mathbf{g},\boldsymbol{\sigma}) &= \phi(\mathbf{B}\mathbf{u} - \mathbf{g}) + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{g}\,)\!)_{\mathcal{H}} - \langle\, \bar{\ell}\,,\, \mathbf{u}\,\rangle \\ &= \frac{1}{2}\,(\!(\, \mathbf{E}(\mathbf{B}\mathbf{u} - \mathbf{g})\,,\, \mathbf{B}\mathbf{u} - \mathbf{g}\,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{g}\,)\!)_{\mathcal{H}} - \langle\, \bar{\ell}\,,\, \mathbf{u}\,\rangle\,, \end{aligned}$$

with $\mathbf{u} \in \mathcal{L}$, $\mathbf{g} \in \mathcal{D}$, $\boldsymbol{\sigma} \in \mathcal{S}$. The stationarity of $\widetilde{H}$ yields the mixed variational problem in the three fields $\{\, \mathbf{u},\mathbf{g},\boldsymbol{\sigma}\,\}$:

$$\begin{cases} (\!(\, \mathbf{E}\mathbf{B}\mathbf{u}\,,\, \mathbf{B}\delta\mathbf{u}\,)\!)_{\mathcal{H}} - (\!(\, \mathbf{E}\mathbf{g}\,,\, \mathbf{B}\delta\mathbf{u}\,)\!)_{\mathcal{H}} = \langle\, \bar{\ell}\,,\, \delta\mathbf{u}\,\rangle & \forall\, \delta\mathbf{u} \in \mathcal{L} \\ (\!(\, \mathbf{E}\mathbf{g}\,,\, \delta\mathbf{g}\,)\!)_{\mathcal{H}} + (\!(\, \boldsymbol{\sigma}\,,\, \delta\mathbf{g}\,)\!)_{\mathcal{H}} - (\!(\, \mathbf{E}\mathbf{B}\mathbf{u}\,,\, \delta\mathbf{g}\,)\!)_{\mathcal{H}} = 0 & \forall\, \delta\mathbf{g} \in \mathcal{D} \\ (\!(\, \delta\boldsymbol{\sigma}\,,\, \mathbf{g}\,)\!)_{\mathcal{H}} = 0 & \forall\, \delta\boldsymbol{\sigma} \in \mathcal{S} \end{cases}$$

Note that the last equation imposes the kinematic compatibility by requiring that the strain gap $\mathbf{g}$ must vanish in correspondence of a solution of the continuous problem.

4.3 Fem interpolation

With a standard notation in finite element analysis [54] we consider, for each element, the interpolations

$$\begin{aligned}
\mathbf{u}_h^e(\mathbf{x}) &= \mathbf{N}_{\mathbf{u}}(\mathbf{x})\,\mathbf{p}_{\mathbf{u}}^e \in \mathcal{V}_h^e\,,\\
\mathbf{g}_h^e(\mathbf{x}) &= \mathbf{N}_{\mathbf{g}}(\mathbf{x})\,\mathbf{p}_{\mathbf{g}}^e \in \mathcal{D}_h^e\,,\\
\boldsymbol{\sigma}_h^e(\mathbf{x}) &= \mathbf{N}_{\boldsymbol{\sigma}}(\mathbf{x})\,\mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{S}_h^e\,, \quad \mathbf{x} \in \overline{\Omega}_e\,,
\end{aligned}$$

where $\overline{\Omega}_e \in \mathcal{T}_{FEM}(\Omega)$, the domain decomposition induced by the meshing of Ω.

Let us set $n_{\mathbf{u}}^e = \dim \mathcal{V}_h^e$, $n_{\mathbf{g}}^e = \dim \mathcal{D}_h^e$, $n_{\boldsymbol{\sigma}} = \dim \mathcal{S}_h^e$.

The interpolating spaces are collected in the following product spaces

$$\mathcal{V}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{V}_h^e\,, \quad \mathcal{D}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{D}_h^e\,, \quad \mathcal{S}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{S}_h^e.$$

No interelement continuity condition is imposed on the strain gap and stress fields so that the corresponding global fields are simply the collection of the local ones:

$$\mathbf{g}_h = \{\, \mathbf{g}_h^1, \mathbf{g}_h^2, \ldots, \mathbf{g}_h^{\mathcal{N}} \,\} \in \mathcal{D}_h\,; \qquad \boldsymbol{\sigma}_h = \{\, \boldsymbol{\sigma}_h^1, \boldsymbol{\sigma}_h^2, \ldots, \boldsymbol{\sigma}_h^{\mathcal{N}} \,\} \in \mathcal{S}_h$$

where $\mathcal{N}$ is the total number of elements pertaining to the F.E. discretization.

Accordingly the virtual work performed by an interpolating stress field $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ by an interpolating strain gap field $\mathbf{g}_h \in \mathcal{D}_h$ is defined as the sum of the contributions of each element

$$\langle\!\langle\, \boldsymbol{\sigma}_h\,,\, \mathbf{g}_h \,\rangle\!\rangle = \sum_{e=1}^{\mathcal{N}} (\!(\boldsymbol{\sigma}_h^e, \mathbf{g}_h^e)\!)_{\Omega_e} = \sum_{e=1}^{\mathcal{N}} \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\sigma}}^e\,\mathbf{p}_{\boldsymbol{\sigma}}^e * \mathbf{N}_{\mathbf{g}}^e\,\mathbf{p}_{\mathbf{g}}^e.$$

The local parameters $\mathbf{p}_{\mathbf{g}}^e \in \mathcal{R}^{n_{\mathbf{g}}^e}$ and $\mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathcal{R}^{n_{\boldsymbol{\sigma}}}$ can be condensed at the element level.

We shall consider a conforming finite element interpolation.

The conforming displacement fields

$$\mathbf{u}_h = \{\, \mathbf{u}_h^1, \mathbf{u}_h^2, \ldots, \mathbf{u}_h^{\mathcal{N}} \,\} \in \mathcal{L}_h \subset \mathcal{V}_h$$

satisfy the homogeneous boundary constraints and the interelement continuity conditions.

The dimension of the subspace $\mathcal{L}_h \subset \mathcal{V}_h$ will be denoted by

$$n_{dof} = \dim \mathcal{L}_h .$$

As customary we assume that rigid body displacements are ruled out by the conformity requirements so that $\mathcal{L} \cap \operatorname{Ker} \mathbf{B} = \{ \mathbf{o} \}$ and the condition $\mathcal{L}_h \subset \mathcal{L}$ implies $\mathcal{L}_h \cap \operatorname{Ker} \mathbf{B} = \{ \mathbf{o} \}$.

The parameters $\mathbf{p}^e_{\mathbf{u}} \in \mathcal{R}^{n^e_{\mathbf{u}}}$ can be expressed in terms of the nodal parameters $\mathbf{q}_{\mathbf{u}} \in \mathcal{R}^{n_{dof}}$ by means of the standard finite element assembly operator $\mathcal{A}^e_{\mathbf{u}}$ according to the parametric representation $\mathbf{p}^e_{\mathbf{u}} = \mathcal{A}^e_{\mathbf{u}} \mathbf{q}_{\mathbf{u}}$.

On the contrary the strain gap and stress local parameters are simply collected in the global lists $\mathbf{q}_{\mathbf{g}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$ according to the expressions

$$\mathbf{p}^e_{\mathbf{g}} = \mathcal{J}^e_{\mathbf{g}} \mathbf{q}_{\mathbf{g}}, \quad \mathbf{p}^e_{\boldsymbol{\sigma}} = \mathcal{J}^e_{\boldsymbol{\sigma}} \mathbf{q}_{\boldsymbol{\sigma}}$$

$\mathcal{J}^e_{\mathbf{g}}$ and $\mathcal{J}^e_{\boldsymbol{\sigma}}$ are the canonical extractors which pick up, from the global lists $\mathbf{q}_{\mathbf{g}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$, the local parameters $\mathbf{p}^e_{\mathbf{g}}$ and $\mathbf{p}^e_{\boldsymbol{\sigma}}$.

The interpolated counterpart of the Hu-Washizu functional $\widetilde{H}_h(\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h)$ is obtained by adding up the contributions of each non-assembled element and imposing that the interpolating displacement $\mathbf{u}_h$ satisfies the conformity requirement:

$$\widetilde{H}_h(\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h) = \frac{1}{2} \langle\!\langle\, \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) \,,\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{g}_h \,\rangle\!\rangle - \langle\, \bar{\ell} \,,\, \mathbf{u}_h \,\rangle$$

where $\{ \mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h \} \in \mathcal{L}_h \times \mathcal{D}_h \times \mathcal{S}_h$. The matrix form of the discrete problem is obtained by imposing the stationarity of $\widetilde{H}_h$ and is given by

$$\mathbb{P}_h) \quad \widetilde{\mathbf{M}} \begin{bmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{K}^e \mathcal{A}^e_{\mathbf{u}} & -\mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{G}^{eT} \mathcal{J}^e_{\mathbf{g}} & \mathbf{O} \\ -\mathcal{J}^{e\,T}_{\mathbf{g}} \mathbf{G}^e \mathcal{A}^e_{\mathbf{u}} & \mathcal{J}^{e\,T}_{\mathbf{g}} \mathbf{H}^e \mathcal{J}^e_{\mathbf{g}} & \mathcal{J}^{e\,T}_{\mathbf{g}} \mathbf{Q}^e \mathcal{J}^e_{\boldsymbol{\sigma}} \\ \mathbf{O} & \mathcal{J}^{e\,T}_{\boldsymbol{\sigma}} \mathbf{Q}^{eT} \mathcal{J}^e_{\mathbf{g}} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{K} & -\mathbf{G}^T & \mathbf{O} \\ -\mathbf{G} & \mathbf{H} & \mathbf{Q} \\ \mathbf{O} & \mathbf{Q}^T & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{bmatrix} = \sum_{e=1}^{\mathcal{N}} \begin{bmatrix} \mathcal{A}^{e\,T}_{\mathbf{u}} \mathbf{f}^e_{\mathbf{u}} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathbf{u}} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

The component submatrices and subvectors appearing above are defined by

$$\mathbf{H}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\,T}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{N}_{\mathbf{g}}^e(\mathbf{x})\,, \quad \mathbf{Q}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\,T}(\mathbf{x})\mathbf{N}_{\boldsymbol{\sigma}}^e(\mathbf{x})\,;$$

$$\mathbf{G}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\,T}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e(\mathbf{x})\,, \quad \mathbf{K}^e = \int_{\Omega_e} (\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e)^T(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e(\mathbf{x})\,;$$

$$\mathbf{f}_{\mathbf{u}}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{u}}^{e\,T}(\mathbf{x})\bar{\mathbf{b}}(\mathbf{x}) + \int_{\partial\Omega_e} (\boldsymbol{\Gamma}\mathbf{N}_{\mathbf{u}}^e)^T(\mathbf{x})\,\bar{\mathbf{t}}(\mathbf{x})\,.$$

Here $\mathbf{B}_*$ and $\mathbf{E}_*$ denote the matrix form of the operators $\mathbf{B}$ and $\mathbf{E}$.

The elastic stiffness matrix $\mathbf{E}_*$ is positive definite and hence the matrix $\mathbf{H}$ turns out to be positive definite as well.

From the computational point of view it is more convenient to carry out the assembly operation after the condensation, at the element level, of the strain and stress parameters to put the global discrete problem in terms of the sole displacement parameters $\mathbf{q}_{\mathbf{u}}$.

4.4 Element shape functions

The shape functions are defined in the reference element K and are evaluated in each element Ω_e of the mesh by performing the composition with the one-to-one isoparametric map $\boldsymbol{\chi}_e : K \mapsto \Omega_e$.

We shall denote the gradient by $\mathbf{F}_e = \operatorname{grad}\boldsymbol{\chi}_e$. The Jacobian determinant $J^e = \det \mathbf{F}_e$ provides the local ratio between the actual and the reference volume forms. In the case of an affine transformation $\boldsymbol{\chi}_e : K \mapsto \Omega_e$ the Jacobian is constant and yields the global ratio $J^e = V_e/V_K$.

The condition $\mathcal{D}_h \cap \mathcal{S}_h^\perp \neq \{\mathbf{o}\}$ can be effectively checked in terms of the subspaces $\mathcal{D}_K$ and $\mathcal{S}_K$ defined in the reference element by means of the change of coordinates described by the map $\boldsymbol{\chi}_e^{-1}$.

The corresponding inner product in K is performed by an integration over the reference element which involves an unknown Jacobian determinant.

If we consider affine equivalent finite element meshes, the Jacobian determinant is constant and no problem arises in imposing the orthogonality conditions.

On the contrary, in the case of general isoparametric maps, the Jacobian determinant is no more constant and as a consequence the integral of the product of two fields in the reference element is no more proportional to the corresponding integral in an actual element of the mesh.

A skilful trick was proposed in [55] in order to overcome this difficulty. Following their proposal, the shape functions of the stresses and of the strain gaps are defined according to

$$\alpha) \qquad \boldsymbol{\sigma}_h^e(\mathbf{x}) = \boldsymbol{\sigma}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \qquad \mathbf{g}_h^e(\mathbf{x}) = \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\mathbf{g}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \qquad \mathbf{x} \in \overline{\Omega}_e$$

where J_o^e is obtained by evaluating $J^e(\boldsymbol{\xi})$ at $\boldsymbol{\xi} = \mathbf{o}$.

Setting $\mathbf{x} = \boldsymbol{\chi}_e(\boldsymbol{\xi})$, we have

$$\int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) \cdot \mathbf{g}_h^e(\mathbf{x})\, d\mathbf{x} = \int_{\Omega_e} \boldsymbol{\sigma}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \cdot \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\mathbf{g}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]\, d\mathbf{x}$$

$$= J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) \cdot \mathbf{g}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

and the orthogonality condition is preserved by general isoparametric mapping.

It is worth noting that this procedure leads to a non-polynomial approximation of the strain gap since in the definition $\alpha)$ the polynomials $g^e(\boldsymbol{\xi})$ are divided by the Jacobian $J^e(\boldsymbol{\xi})$.

Nevertheless no problem arises since the approximation properties of the strain gap subspace $\mathcal{D}_h$ play no role in estimating the asymptotic rate of convergence, as proved in [84] and discussed hereafter in next chapter 5.

Additional transformation rules which preserve the point-wise inner product between stress and strain tensors can be envisaged but different choices can only be motivated by an a posteriori evaluation of the quality of the numerical results.

Examples are provided by the push/pull transformations of differential geometry [38]. In this respect the following expressions for plane problems have been adopted, [55] , [75]:

$$\boldsymbol{\sigma}_h^e(\mathbf{x}) = \mathbf{T}_o^e \boldsymbol{\sigma}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \qquad \mathbf{g}_h^e(\mathbf{x}) = \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\mathbf{T}_o^{e-T}\mathbf{g}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]$$

where $\mathbf{T}_o^e$ is the value, at the origin of the reference element, of the matrix field

$$\mathbf{T}^e(\boldsymbol{\xi}) = \begin{bmatrix} F_{11}^2 & F_{12}^2 & 2F_{11}F_{12} \\ F_{21}^2 & F_{22}^2 & 2F_{21}F_{22} \\ F_{11}F_{21} & F_{12}F_{22} & F_{11}F_{22} + F_{12}F_{21} \end{bmatrix}^e (\boldsymbol{\xi}) \qquad \text{with} \qquad \boldsymbol{\xi} \in K\,.$$

Accordingly the inner product in the real space is given by

$$\int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) \cdot \mathbf{g}_h^e(\mathbf{x}) d\mathbf{x} = \int_{\Omega_e} \mathbf{T}_o^e \boldsymbol{\sigma}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \cdot \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]} \mathbf{T}_o^{e-T} \mathbf{g}^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] d\mathbf{x} =$$

$$= J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) \cdot \mathbf{g}(\boldsymbol{\xi}) d\boldsymbol{\xi}\,.$$

4.5 Well-posedness analysis

As customary in computational analysis we will assume that the structure cannot undergo conforming rigid displacements.

Well-posedness. The discrete mixed problem $\mathbb{P}_h$ is said to be *well-posed* if it admits a unique solution $\{\, \mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h \,\} \in \mathcal{L}_h \times \mathcal{D}_h \times \mathcal{S}_h$ for any data $\mathbf{f}_{\mathbf{u}}$.

Well-posedness is often characterized in the literature by the requirements that the discrete problem admits a unique solution for any data and that the discrete solution tends to the solution of the continuous problem as the finite element mesh is refined ever more.

We prefer here to treat separately these two requirements since the conditions for their fulfilment can be proved following two different arguments.

A necessary and sufficient condition for well-posedness is thus provided by the next statement. The proof, in terms of the kernel of the matrix $\mathbf{M}$, is reported in the Appendix at the end of this chapter.

Proposition 4.5.1 (Well-posedness criterion) *If there are no rigid conforming displacements, that is* $Ker\,\mathbf{B} \cap \mathcal{L}_h = \{\mathbf{o}\}$*, the conditions*

$$\beta) \qquad \begin{aligned} &\widetilde{\mathcal{D}}_h \cap \mathbf{B}\mathcal{L}_h = \{\, \mathbf{o} \,\} \\ &\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\} \end{aligned}$$

are necessary and sufficient for the well-posedness of the discrete mixed problem $\mathbb{P}_h$.

The strain gaps $\mathbf{g}_h$ belonging to the subspace $\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$ are referred to as *effective* strain gaps since they effectively contribute to relax the compatibility condition. The orthogonality relation $\perp$ is intended according to the inner product in $L^2(\Omega)$.

- The well-posedness condition β_1 requires that the effective strain gaps

$$\mathbf{g}_h \in \widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp},$$

  and the compatible discrete strains $\mathbf{Bu}_h \in \mathbf{B}\mathcal{L}_h$ must be linearly independent.

- The well-posedness condition β_2 means that stresses $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ must be controlled by strain gaps $\mathbf{g}_h \in \mathcal{D}_h$.

  Condition β_1 can be conveniently substituted by the local condition

$$\widetilde{\mathcal{D}}_h \cap \mathbf{B}\mathcal{V}_h = \{\, \mathbf{o} \,\}$$

  which does not involve the unknown assembly operation. Condition β_2 can be imposed by choosing $\mathcal{D}_h$ such that $\mathcal{S}_h \subseteq \mathcal{D}_h$.

The conditions which garantee that the convergence in the energy norm of the discrete solution to the continuous one will be analysed in chapter 5 by resorting to the general treatment of mixed methods [57].

4.6 Reduced problem and stress recovery

The SGM can be cast in the theoretical framework of the mixed methods analysed in [57]. In fact the discrete mixed problem can be written in the form

$$\gamma) \qquad \begin{cases} \langle\!\langle\, \mathbf{E}(\mathbf{Bu}_h - \mathbf{g}_h)\,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \bar{\ell}\,,\, \delta\mathbf{u}_h \,\rangle & \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h \\ \langle\!\langle\, \mathbf{E}(\mathbf{Bu}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h\,,\, \delta\mathbf{g}_h \,\rangle\!\rangle = 0 & \forall\, \delta\mathbf{g}_h \in \mathcal{D}_h \\ \langle\!\langle\, \delta\boldsymbol{\sigma}_h\,,\, \mathbf{g}_h \,\rangle\!\rangle = 0 & \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h . \end{cases}$$

The relations γ provide the discrete equilibrium, elastic equations and the compatibility conditions.

It is convenient to consider γ_3 as a constraint condition for the discrete problem in which the discrete stresses play the role of Lagrangian multipliers.

This constraint amounts to require that $\mathbf{g}_h \in \widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$.

The $\{\mathbf{u}_h, \mathbf{g}_h\}$ solution of problem γ can be obtained by solving the following reduced problem in which the strain gap variations meet the constraint condition:

$$\delta) \qquad \begin{cases} \langle\!\langle\, \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \bar{\ell}\,,\, \delta\mathbf{u}_h \,\rangle & \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h \\ \langle\!\langle\, \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,,\, \delta\mathbf{g}_h \,\rangle\!\rangle = 0 & \forall\, \delta\mathbf{g}_h \in \widetilde{\mathcal{D}}_h . \end{cases}$$

Once the solution $\{\mathbf{u}_h, \mathbf{g}_h\}$ of δ) has been obtained, the discrete stresses $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ can be evaluated by solving the stress recovery problem

$$\epsilon) \qquad \langle\!\langle\, \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h\,,\, \delta\mathbf{g}_h \,\rangle\!\rangle = 0 \qquad \forall\, \delta\mathbf{g}_h \in \mathcal{D}_h .$$

This condition involves a number of equations which is larger than the number of unknown stress parameters.

Nevertheless the stress recovery problem ϵ) admits a unique solution if the well-posedness requirement $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\}$ is fulfilled, since then the number of independent equations is equal to the number of unknowns.

The stress recovery ϵ) can be interpreted in geometrical terms as a projection procedure:

- to get the approximate stress $\boldsymbol{\sigma}_h \in \mathcal{S}_h$, the field $(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ must be projected on the subspace $\mathcal{S}_h$ with a projection orthogonal in elastic energy to the subspace $\mathcal{D}_h$.

4.7 The elastic stress recovery

From the computational standpoint, the most convenient stress recovery consists in computing the discrete stresses at the element level according to the elastic constitutive relation $\boldsymbol{\sigma}_h = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ once the reduced problem has been solved.

Some authors [61] , [73] , [79] claimed that this simple computation is not variationally consistent but the following direct argument leads to the opposite conclusion.

Let us preliminarily observe that the well-posedness conditions require that any admissible choice of the subspaces $\mathcal{S}_h$ and $\mathcal{D}_h$ must fulfill the following three rules:

- $a)$ $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\mathbf{o}\}$,
- $b)$ $\mathcal{D}_h \cap \mathcal{S}_h^{\perp} \neq \{\mathbf{o}\}$,
- $c)$ constant stress fields must be included in $\mathcal{S}_K$.

Conditions $a)$ and $b)$ can always be satisfied by setting $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$ since the choice of $\mathcal{D}_h$ is not subject to other conditions.

The subspace $\widetilde{\mathcal{D}}_h$ is defined as the linear span of shape functions with zero mean values. This choice is motivated by the othogonality condition $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ since $\mathcal{S}_h$ must fulfil the condition $c)$.

Condition $a)$ ensures uniqueness and convergence of the approximate stress solution as will be discussed in the next section.

Condition $b)$ is necessary in order to get an enhanced flexibility since otherwise the mixed method would collapse into the standard displacement method.

Condition $c)$ is also motivated by convergence requirements (see the next chapter 5). Let us now prove the variational consistency of the elastic stress recovery.

To this end we consider the stress subspace $\overline{\Sigma}_h$ composed by the stress fields $\overline{\boldsymbol{\sigma}}_h = \mathbf{E}(\mathbf{B}\overline{\mathbf{u}}_h - \overline{\mathbf{g}}_h)$ with $\overline{\mathbf{u}}_h \in \mathcal{V}_h$ and $\overline{\mathbf{g}}_h \in \widetilde{\mathcal{D}}_h$.

The elastic stress is $\boldsymbol{\sigma}_h^E = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ with $\{\mathbf{u}_h, \mathbf{g}_h\}$ solution of the reduced problem. It is then apparent that $\boldsymbol{\sigma}_h^E \in \overline{\Sigma}_h$.

Moreover, by condition $\delta)_2$, we have also that $\boldsymbol{\sigma}_h^E \in \widetilde{\mathcal{D}}_h^{\perp}$.

Then $\boldsymbol{\sigma}_h^E$ belongs to the subspace

$$\Sigma_h = \{\overline{\boldsymbol{\sigma}}_h \in \overline{\Sigma}_h \ : \ \langle\!\langle\, \overline{\boldsymbol{\sigma}}_h \,,\, \delta\mathbf{g}_h \,\rangle\!\rangle = 0 \quad \forall \delta\mathbf{g}_h \in \widetilde{\mathcal{D}}_h\} = \overline{\Sigma}_h \cap \widetilde{\mathcal{D}}_h^{\perp}.$$

We can then define the stress subspace to be $\mathcal{S}_h = \Sigma_h \oplus \mathcal{S}_h^*$ where $\mathcal{S}_h^*$ is any subspace included in $\widetilde{\mathcal{D}}_h^{\perp}$. As a consequence $\Sigma_h \subseteq \mathcal{S}_h$ and $\mathcal{S}_h \subseteq \widetilde{\mathcal{D}}_h^{\perp}$.

Being $\boldsymbol{\sigma}_h^E \in \Sigma_h \subseteq \mathcal{S}_h$, the projection procedure $\epsilon)$ of the stress recovery problem yields trivially that $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^E$.

The variationally consistency of the elastic stress recovery has been recently claimed in [82] by an argument explicitly limited to undistorted meshes and based on a matricial formulation.

As a consequence of the choice $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$, the stress recovery strategy **??** reduces to an orthogonal projection of $\boldsymbol{\sigma}_h^E = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ on the linear subspace $\mathcal{S}_h$:

$$\langle\!\langle\, \boldsymbol{\sigma}_h^E - \boldsymbol{\sigma}_h \,,\, \delta\boldsymbol{\sigma}_h \,\rangle\!\rangle = 0 \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h.$$

In fact the condition containing the variations $\delta \mathbf{g}_h \in \widetilde{\mathcal{D}}_h$ are identically satisfied by virtue of $\delta)_2$.

Several non-variational stress recovery procedures have been envisaged in the literature to complement the enhanced strain method.

The one proposed in [55] performs the orthogonal projection in the elastic energy of the discrete field $\mathbf{EBu}_h$ on a subspace $\mathcal{S}_h$ fulfilling the condition

$$\mathcal{S}_h \subseteq \mathcal{D}_h^{\perp}.$$

This stress recovery is not variationally consistent since it cannot be deduced from a projection procedure of the type $\epsilon)$.

4.8 The discrete problem

Let us assume a strain gap interpolation in the reference element of the form

$$\xi) \qquad \mathcal{D}_K = \mathcal{S}_K \oplus \widetilde{\mathcal{D}}_K$$

where $\widetilde{\mathcal{D}}_K \subseteq \mathcal{S}_K^{\perp}$ is the subspace of effective strain gaps and the symbol $\oplus$ denotes the direct sum.

Then $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$ but in general we do not have $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ unless the Jacobians of the isoparametric maps are constant. As a consequence in the general case it is not possible to split the variational problem into a sequence of a reduced problem and of a stress recovery projection.

Hereafter we develop a general formulation of the SGM in which the orthogonality constraint $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ is not satisfied a priori but enters as one of the equations of the discrete problem.

This formulation is a usefull tool in detecting the computational performances relevant to different implementation of the method and will be referred to in the numerical examples discussed in section 8.

The well-posedness condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}$ is equivalent to the non-singularity of the matrix $\mathbf{Q}$. The property $\operatorname{Ker}\mathbf{Q} = \{\boldsymbol{o}\}$ is crucial for the derivation of the element stiffness matrix. We can in fact eliminate the stress and the strain gap parameters, at the element level, according to the procedure

$$\begin{cases} \mathbf{H}^e\, \mathbf{p}_{\mathbf{g}}^e = \mathbf{G}^e\, \mathbf{p}_{\mathbf{u}}^e - \mathbf{Q}^e\, \mathbf{p}_{\boldsymbol{\sigma}}^e, \\ (\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)\, \mathbf{p}_{\boldsymbol{\sigma}}^e = \mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{G}^e\, \mathbf{p}_{\mathbf{u}}^e, \\ \left[\mathbf{K}^e + \mathbf{G}^{eT}\mathbf{H}^{e-1}\,(\mathbb{H}^e - \mathbf{H}^e)\,\mathbf{H}^{e-1}\mathbf{G}^e\right]\, \mathbf{p}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}^e, \end{cases}$$

where $\mathbb{H}^e = \mathbf{Q}^e\left(\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e\right)^{-1}\mathbf{Q}^{eT}$. Denoting the element stiffness matrix by

$$\mathbf{S}^e = \mathbf{K}^e + \mathbf{G}^{eT}\mathbf{H}^{e-1}\left(\mathbb{H}^e - \mathbf{H}^e\right)\mathbf{H}^{e-1}\mathbf{G}^e,$$

the mixed problem at the element level can be written as $\mathbf{S}^e\ \mathbf{p}^e_{\mathbf{u}} = \mathbf{f}^e_{\mathbf{u}}$.

The global problem is then expressed in terms of nodal displacement parameters as

$$\mathbb{K}\ \mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}}\left(\mathcal{A}^{e\,T}_{\mathbf{u}}\mathbf{S}^e\mathcal{A}^e_{\mathbf{u}}\right)\mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathcal{N}}\mathcal{A}^{e\,T}_{\mathbf{u}}\mathbf{f}^e_{\mathbf{u}} = \mathbf{f}_{\mathbf{u}}.$$

We underline that, due to the positive definiteness of $\mathbf{H}^e$ and the nonsingularity of $\mathbf{Q}^e$, the matrix $\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e$ is invertible. Once the global structural problem has been solved in terms of nodal displacements, the stress and the strain parameters can be evaluated at the element level by following the elimination procedure backwards:

$$\mathbf{p}^e_{\boldsymbol{\sigma}} = (\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{Q}^e)^{-1}\mathbf{Q}^{eT}\mathbf{H}^{e-1}\mathbf{G}^e\,\mathbf{p}^e_{\mathbf{u}}$$

$$\mathbf{p}^e_{\mathbf{g}} = \mathbf{H}^{e-1}\left(\mathbf{G}^e\,\mathbf{p}^e_{\mathbf{u}} - \mathbf{Q}^e\,\mathbf{p}^e_{\boldsymbol{\sigma}}\right).$$

If the Jacobian determinant is introduced in the definition of the strain gaps according to formula of the proposition 3.4.1, the property $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ is preserved by the isoparametric map. In this case a more convenient computation strategy, based on the reduced problem $\delta)$ and the stress recovery projection $\epsilon)$, can be exploited.

4.8.1 Discrete reduced problem and stress recovery

Let us now derive the expression of the element stiffness matrix and of the discrete stress recovery for the SGM. To solve the reduced problem $\delta)$ we preliminary note that the discrete strain gaps belonging to the subspace $\widetilde{D}_h$ are such that the corrresponding parameters belong to $\operatorname{Ker}\mathbf{Q}^{eT}$, see also the equivalence reported in the Appendix.

The reduced problem $\delta)$ at the element level reduces to the following matrix form

$$\theta)\qquad \begin{bmatrix} \mathbf{K}^e_{\mathbf{uu}} & -\mathbf{G}^{eT}_{\boldsymbol{\alpha}\mathbf{u}} \\ -\mathbf{G}^e_{\boldsymbol{\alpha}\mathbf{u}} & \mathbf{H}^e_{\boldsymbol{\alpha}\boldsymbol{\alpha}} \end{bmatrix}\begin{bmatrix} \mathbf{p}^e_{\mathbf{u}} \\ \mathbf{p}^e_{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e_{\mathbf{u}} \\ \mathbf{o} \end{bmatrix}.$$

where $\mathbf{p}^e_{\boldsymbol{\alpha}}$ is the element effective strain gap parameter.

The element stiffness matrix $\mathbf{S}^e$ is then

$$\mathbf{S}^e = \mathbf{K}^e_{\mathbf{uu}} - \mathbf{G}^{eT}_{\boldsymbol{\alpha}\mathbf{u}} \mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}} \mathbf{G}^e_{\boldsymbol{\alpha}\mathbf{u}}$$

so that the problem at the element level can be written as $\mathbf{S}^e \mathbf{p}^e_{\mathbf{u}} = \mathbf{f}^e_{\mathbf{u}}$.

Once the global problem has been solved in terms of nodal parameters, the effective strain gap parameters can then be computed from $\theta)$ in the form:

$$\sigma) \qquad \mathbf{p}^e_{\boldsymbol{\alpha}} = \mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}} \mathbf{G}^e_{\boldsymbol{\alpha}\mathbf{u}} \mathbf{p}^e_{\mathbf{u}}.$$

At this point the variationally consistent stress recovery $\epsilon)$ can be pursued to get

$$-\mathbf{G}^e_{\boldsymbol{\sigma}\mathbf{u}} \mathbf{p}^e_{\mathbf{u}} + \mathbf{H}^e_{\boldsymbol{\sigma\alpha}} \mathbf{p}^e_{\boldsymbol{\alpha}} + \mathbf{Q}^e_{\boldsymbol{\sigma\sigma}} \mathbf{p}^e_{\boldsymbol{\sigma}} = \mathbf{o}$$

and recalling the expression $\sigma)$ of $\mathbf{p}^e_{\boldsymbol{\alpha}}$ we have:

$$\varphi) \qquad \mathbf{p}^e_{\boldsymbol{\sigma}} = \mathbf{Q}^{e-1}_{\boldsymbol{\sigma\sigma}} (\mathbf{G}^e_{\boldsymbol{\sigma}\mathbf{u}} - \mathbf{H}^e_{\boldsymbol{\sigma\alpha}} \mathbf{H}^{e-1}_{\boldsymbol{\alpha\alpha}} \mathbf{G}^e_{\boldsymbol{\alpha}\mathbf{u}}) \mathbf{p}^e_{\mathbf{u}}.$$

It is worth noting that the reduced problem $\theta)$ coincides with the matrix formulation of the EAS method where the enhanced strains of the EAS method coincide with the effective strain gaps of the SGM to within an irrilevant change of sign. The well-posedness of the SGM provides the variationally consistent stress recovery $\varphi)$.

4.9 Limitation phenomena

Let us now present some results which extend to the three-field SGM method the discussion of the so called limitation phenomena. More precisely we show that, if the discrete interpolations fulfill suitable relations, the SGM method collapses into the displacement method or into the HELLINGER-REISSNER (HR) two-field method.

Proposition 4.9.1 ***(First limitation principle between the SGM and the displacement method)***
The approximate displacement solution $\mathbf{u}_h$ provided by the SGM $\gamma)$ and by the displacement method coincide if the following condition is met:

$$\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}^{\perp}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}.$$

Proof. Due to condition $\gamma)_3$, the strain gap solution must be effective, i.e. $\mathbf{g}_h \in \widetilde{\mathcal{D}}_h$, and hence the strain gap solution is vanishing. The elastic compatibility condition $\gamma)_2$ is an identity and the equilibrium condition $\gamma)_1$ reduces to the one of the displacement method

$$\langle\!\langle\, \mathbf{EBu}_h \,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \ell \,,\, \delta\mathbf{u}_h \,\rangle \qquad \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h$$

and the statement is proved.

Another condition which ensures that the discrete mixed problem $\gamma)$ collapses into the displacement method is reported in the next statement.

Proposition 4.9.2 ***(Second limitation principle between the SGM and the displacement method)*** *The approximate displacement solution $\mathbf{u}_h$ provided by the SGM $\gamma)$ and by the displacement method coincide if the following condition is met:*

$$\psi) \qquad\qquad \widetilde{\mathcal{D}}_h \subset (\mathbf{E}\,\mathbf{B}\mathcal{L}_h)^{\perp}$$

that is if the linear subspaces $\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$ and $\mathbf{B}\mathcal{L}_h$ are orthogonal in elastic energy.

Proof. In fact the condition $\psi)$ ensures that the discrete mixed problem $\gamma)$ becomes:

$$\chi) \qquad \begin{cases} \langle\!\langle\, \mathbf{EBu}_h \,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \ell \,,\, \delta\mathbf{u}_h \,\rangle & \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h \\ \langle\!\langle\, \mathbf{Eg}_h \,,\, \delta\mathbf{g}_h \,\rangle\!\rangle = 0 & \forall\, \delta\mathbf{g}_h \in \widetilde{\mathcal{D}}_h . \end{cases}$$

By the positive definiteness of $\mathbf{E}$, the equation $\chi)_2$ provides $\mathbf{g}_h = \mathbf{o}$ and the displacement method is obtained.

Let us now investigate the correlation between the SGM and the HR method. The expression of the Hellinger-Reissner functional $R(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is given by

$$R(\boldsymbol{\sigma}_h, \mathbf{u}_h) = -\frac{1}{2}\langle\!\langle\, \mathbf{C}\boldsymbol{\sigma}_h \,,\, \boldsymbol{\sigma}_h \,\rangle\!\rangle + \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{Bu}_h \,\rangle\!\rangle - \langle\, \ell \,,\, \mathbf{u}_h \,\rangle \qquad \mathbf{u}_h \in \mathcal{L}_h$$

where $\mathbf{C} = \mathbf{E}^{-1}$ is the linear elastic compliance operator.

A solution $\{\, \boldsymbol{\sigma}_h, \mathbf{u}_h \,\} \in \mathcal{S}_h \times \mathcal{L}_h$ is obtained by enforcing the stationarity of the functional R to get:

$$\phi) \qquad \begin{cases} -\langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{C}\boldsymbol{\sigma}_h \,\rangle\!\rangle + \langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{Bu}_h \,\rangle\!\rangle = 0 & \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h \\ \langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \ell \,,\, \delta\mathbf{u}_h \,\rangle & \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h . \end{cases}$$

The next statement provides a sufficient condition for the validity of a limitation phenomenon.

Proposition 4.9.3 ***(Limitation principle between the SGM and HR method)*** *The approximate solution* $\{\mathbf{u}_h, \boldsymbol{\sigma}_h\}$ *provided by the SGM* $\gamma)$ *and by the two-field HR method* $\phi)$ *coincide if the following condition is met:*

$$\Sigma) \qquad \mathbf{E}\mathbf{B}\mathcal{L}_h \subseteq \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h .$$

Proof. By virtue of the choice $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$, the condition $\Sigma)$ implies that $\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h \in \mathcal{D}_h$. By virtue of the condition $\gamma)_2$ we have:

$$\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h \in \mathcal{D}_h \cap \mathcal{D}_h^{\perp} = \{\mathbf{o}\}.$$

Accordingly we have $\boldsymbol{\sigma}_h = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$. The constraint condition $\gamma)_3$ ensures that the strain gaps are effective, i.e. $\mathbf{g}_h \in \widetilde{\mathcal{D}}_h$, so that we have:

$$\begin{aligned}
&\mathbf{g}_h = \mathbf{B}\mathbf{u}_h - \mathbf{C}\boldsymbol{\sigma}_h \in \widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp} \quad \Longleftrightarrow \\
&\langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{B}\mathbf{u}_h \,\rangle\!\rangle - \langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{C}\boldsymbol{\sigma}_h \,\rangle\!\rangle = 0 \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h
\end{aligned}$$

and the equilibrium equation $\gamma)_1$ yields:

$$\langle\!\langle\, \boldsymbol{\sigma}_h \,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle = \langle\, \ell \,,\, \delta\mathbf{u}_h \,\rangle \quad \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h$$

which coincide with the stationarity conditions of the HR method. The statement is thus proved.
A similar result has been proved in [79] with reference to the EAS method by imposing that the approximate stress pertaining to the EAS method is obtained by means of the constitutive elastic relation.

4.10 Computational results

The limitation principle 4.9.3 can be tested by means of the structural example of a plate of unit thickness subject to pure bending, as shown in fig.

This example is usually adopted as a benchmark for enhanced methods [71], [82]. The analytical solution in terms of horizontal and vertical displacements is given by:

$$\begin{cases} u(x,y) = \dfrac{pb^2}{6EI_y}\, x \left(\dfrac{b}{2} - y\right) \\[2ex] v(x,y) = \dfrac{pb^2}{12EI_y}\, \left[x^2 + \nu(y^2 - by)\right] . \end{cases}$$

where I_y is the inertia of the cross section along the y axis in the centroidal reference system $\{y, z_G\}$.

The only non-vanishing stress component is σ_x has the expression

$$\sigma_x(x,y) = \frac{p\,b^2}{6I_y}\left(\frac{b}{2} - y\right).$$

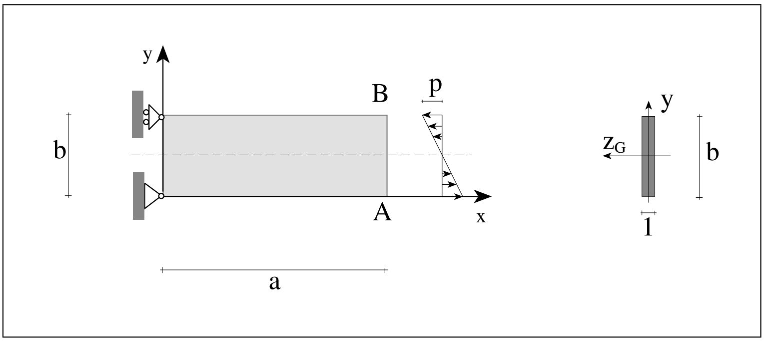


Figure 4.1: The plate bending problem

Setting $a = 10, b = 2, E = 1500,\ \ \nu = 0.25,\ \ p = 3000$, the analitical solution of the bending problem in terms of displacements and stresses is:

$$\begin{cases} u(x,y) = 2x(1-y) \\ v(x,y) = x^2 + \frac{1}{4}(y^2 - 2y) \end{cases} \qquad \text{and} \qquad \sigma_x(x,y) = 3000(1-y)\,.$$

The numerical analysis is carried out for two meshes of 4-node quadrilateral elements with bi-linear displacement interpolations (Q_1): a single rectangular element and a mesh of 16×8 uniform rectangular elements.

The five-parameter interpolation for the effective strain gap field is given by:

$$\mathbf{N}_{\mathbf{g}}^{\square} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta \\ 0 & \eta & 0 & 0 & -\xi\eta \\ 0 & 0 & \xi & \eta & \xi^2 - \eta^2 \end{bmatrix}$$

and coincide with the enhanced strain interpolation provided in [55].

The interpolating stresses are expressed in terms of the PIAN and SUMIHARA shape functions [41] in the form

$$\Sigma_{bis}) \qquad \mathbf{N}_{\boldsymbol{\sigma}}^{\square} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The effective strain gaps and the stresses are apparently mutually orthogonal in the $L^2(\square)$ inner product.

For plane stress problems, a direct evaluation proves the inclusion $\Sigma)$.

In the numerical analysis, the displacements and the horizzontal stress at the corner node A of the plate are evaluated adopting the SGM, the HELLINGER-REISSNER method and the standard displacement formulation, and are compared with the exact solution.

The results are reported in Table 4.1 for a mesh consisting of a single Q_1 element

	SGM	H-R	DISPL.	EXACT
u	20	20	1.807	20
v	100	100	9.036	100
σ_x	3000	3000	289	3000

Table 4.1: Displacement and stress at node A - one element

and in Table 4.2 for a 16×8 mesh. The example reveales two peculiar facts:

	SGM	H-R	DISPL.	EXACT
u	20	20	19.19	20
v	100	100	96.12	100
σ_x	3000	3000	2815	3000

Table 4.2: Displacement and stress at node A - 16 $\times$ 8 uniform mesh

- The displacement and the stress solutions obtained from the SGM and the Hellinger-Reissner method are identical as predicted by the limitation principle of proposition 4.9.3.
- Assuming only one element mesh, the interpolating stress $\boldsymbol{\sigma}_h$ coincide with the exact stress solution. Moreover, even if the interpolating displacement field $\mathbf{u}_h$ is quite different from the exact solution, the displacement parameters $\mathbf{q}_\mathbf{u}$ are exact.

In the next section we will explain in detail this surprising result.

4.11 A one-element exact solution

Let us now provide a sufficient condition which ensures that the stress solution $\boldsymbol{\sigma}_h$ obtained by the Hellinger-Reissner finite element analysis coincides with the exact stress solution $\boldsymbol{\sigma}_o$ pertaining to the continuous model.

In particular we will show that this condition is met by the displacement and stress shape functions adopted in the abovementioned bending problem.

Further we will provide a theoretical motivation to the numerical observation that the nodal displacements $\mathbf{q}_\mathbf{u}$ obtained by the Hellinger-Reissner finite element analysis of the bending problem are exact.

To this end we prove the next statement.

Proposition 4.11.1 (The exact stress solution) *Let* $\{ \boldsymbol{\sigma}_o, \mathbf{u}_o \} \in \mathcal{S} \times \mathcal{L}$ *be the solution of the continuous problem and* $\{ \boldsymbol{\sigma}_h, \mathbf{u}_h \} \in \mathcal{S}_h \times \mathcal{L}_h$ *be the approximate solution of the two-field* Hellinger-Reissner *problem. Then if*

$$\boldsymbol{\sigma}_o \in \mathcal{S}_h \qquad \text{and} \qquad \mathbf{B}\mathbf{u}_o \in \mathbf{B}\mathcal{L}_h + \mathcal{S}_h^\perp$$

the approximate and the exact stress solutions coincide, i.e. $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_o$.

Proof. The exact stress solution $\boldsymbol{\sigma}_o \in \mathcal{S}$ fulfils the approximate equilibrium equation $\phi)_2$

$$\Gamma) \qquad \langle\!\langle\, \boldsymbol{\sigma}_o \,,\, \mathbf{B}\delta\mathbf{u}_h \,\rangle\!\rangle - \langle\, \ell \,,\, \delta\mathbf{u}_h \,\rangle = 0 \qquad \forall\, \delta\mathbf{u}_h \in \mathcal{L}_h$$

since the inclusion $\mathcal{L}_h \subset \mathcal{L}$ hold true.

It remains to prove that the exact stress solution $\boldsymbol{\sigma}_o$ satisfies the approximate elastic compatibility equation $\phi)_1$ which can be rewritten as:

$$-\langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{C}\boldsymbol{\sigma}_o \,\rangle\!\rangle + \langle\!\langle\, \delta\boldsymbol{\sigma}_h \,,\, \mathbf{B}\mathbf{u}_h \,\rangle\!\rangle = 0 \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h .$$

To this end we note that the condition $\mathbf{Bu}_o \in \mathbf{B}\mathcal{L}_h + \mathcal{S}_h^{\perp}$ implies that there exists an interpolating conforming displacement $\mathbf{u}_h^* \in \mathcal{L}_h$ such that:

$$-\mathbf{Bu}_o + \mathbf{Bu}_h^* \in \mathcal{S}_h^{\perp}.$$

Further, noting that the equalities $\mathbf{C}\boldsymbol{\sigma}_o = \boldsymbol{\varepsilon}_o = \mathbf{Bu}_o$ hold true for the exact solution, the orthogonality relation above can be equivalently rewritten in the form

$$\Lambda) \qquad \begin{aligned} &-\langle\!\langle\, \delta\boldsymbol{\sigma}_h \,, \mathbf{Bu}_o \,\rangle\!\rangle + \langle\!\langle\, \delta\boldsymbol{\sigma}_h \,, \mathbf{Bu}_h^* \,\rangle\!\rangle = 0 \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h \quad \Longleftrightarrow \\ &-\langle\!\langle\, \delta\boldsymbol{\sigma}_h \,, \mathbf{C}\boldsymbol{\sigma}_o \,\rangle\!\rangle + \langle\!\langle\, \delta\boldsymbol{\sigma}_h \,, \mathbf{Bu}_h^* \,\rangle\!\rangle = 0 \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h \,. \end{aligned}$$

Accordingly the pair $\{\, \boldsymbol{\sigma}_o, \mathbf{u}_h^* \,\}$ fulfils the approximate equilibrium equation $\Gamma)$ and the approximate elastic compatibility $\Lambda)$ and hence $\{\, \boldsymbol{\sigma}_o, \mathbf{u}_h^* \,\}$ is the solution of the Hellinger-Reissner approximate problem $\phi)$. By virtue of the uniqueness of the approximate solution we infer that $\{\, \boldsymbol{\sigma}_o, \mathbf{u}_h^* \,\} \equiv \{\, \boldsymbol{\sigma}_o, \mathbf{u}_h \,\}$ so that $\boldsymbol{\sigma}_o$ is the stress solution of the approximate problem. The proposition is thus proved.

4.11.1 The stress solution of the plate bending problem

Let us consider the linear elastic bending problem presented in the previous section adopting Q_1 elements. The compatible strain $\mathbf{Bu}_o \in \mathcal{D}$ of the continuous problem is given by:

$$\Psi) \qquad \mathbf{Bu}_o = \frac{1}{E} \begin{bmatrix} \eta \\ -\nu\eta \\ 0 \end{bmatrix} .$$

Considering a one element Q_1 mesh, the subspaces $\mathbf{B}\mathcal{V}_h$ is spanned by the linearly independent columns of the matrix $\mathbf{BN}_{\mathbf{u}}^{\square}$ given by

$$\Psi_{bis}) \qquad \mathbf{BN}_{\mathbf{u}}^{\square} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{bmatrix}$$

The strain solution $\mathbf{Bu}_o$ of the continuous problem can then be obtained in terms of a linear combination of the columns of the matrix $\mathbf{BN}_{\mathbf{u}}^{\square}$ as follows:

$$\mathbf{Bu}_o = \frac{1}{E}\begin{bmatrix}\eta \\ 0 \\ \xi\end{bmatrix} - \frac{1}{E}\begin{bmatrix}0 \\ 0 \\ \xi\end{bmatrix} - \frac{\nu}{E}\begin{bmatrix}0 \\ \eta \\ 0\end{bmatrix}$$

where the last two vectors are $L^2(\square)-$orthogonal to the subspace $\mathcal{S}_h^{\square}$ generated by the assumed Pian-Sumihara stress interpolation and hence they belong to $(\mathcal{S}_h^{\square})^{\perp}$.

Noting that the stress solution $\boldsymbol{\sigma}_o$ of the continuous problem belongs to the interpolating stress subspace $\mathcal{S}_h^{\square}$

$$\boldsymbol{\sigma}_o = \begin{bmatrix}\eta \\ 0\end{bmatrix} \in \mathcal{S}_h^{\square},$$

the requirements of the proposition 3.4.1 are fulfilled so that a mesh composed by only one element provides the exact stress solution.

4.11.2 The displacement solution of the plate bending problem

Let us now explain why the interpolations assumed for the analysis of the plate bending problem reported in Fig. 4.1 provide the exact nodal displacements.

The elastic compatibility condition $\phi)_1$ can be written as:

$$\Delta) \qquad \langle\!\langle\, \mathbf{Bu}_o\,,\, \delta\boldsymbol{\sigma}_h \,\rangle\!\rangle = \langle\!\langle\, \mathbf{Bu}_h\,,\, \delta\boldsymbol{\sigma}_h \,\rangle\!\rangle \qquad \forall\, \delta\boldsymbol{\sigma}_h \in \mathcal{S}_h$$

since $\mathbf{C}\boldsymbol{\sigma}_h = \mathbf{C}\boldsymbol{\sigma}_o = \mathbf{Bu}_o$.

The five-dimensional Pian-Sumihara interpolating stress subspaces $\mathcal{S}_h$ is spanned by the five columns of the matrix $\Sigma_{bis})$ which will be denoted by $(\delta\boldsymbol{\sigma}_h)_i$ with $i = 1, \ldots, 5$.

The elastic compatibility relation $\Delta)$ can then be tested in terms of the base vectors $(\delta\boldsymbol{\sigma}_h)_i$ of $\mathcal{S}_h$ in the form

$$\Xi) \qquad \langle\!\langle\, \mathbf{Bu}_o\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle = \langle\!\langle\, \mathbf{Bu}_h\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle \quad i = 1, \ldots, 5.$$

Since the exact compatible strain solution $\mathbf{Bu}_o$ is given by $\Psi)$, it is apparent that

$$\langle\!\langle\, \mathbf{Bu}_o\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle = 0 \quad i = 1, 2, 3, 5 \quad \text{and} \quad \langle\!\langle\, \mathbf{Bu}_o\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle \neq 0 \quad i = 4$$

so that the equality $\Xi)$ yields

$$\langle\!\langle\, \mathbf{Bu}_h\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle = 0 \quad i = 1,2,3,5 \quad \text{and} \quad \langle\!\langle\, \mathbf{Bu}_h\,,\, (\delta\boldsymbol{\sigma}_h)_i \,\rangle\!\rangle \neq 0 \quad i = 4.$$

The interpolating compatible strain $\mathbf{Bu}_h$ is expressed by a linear combination of the columns of $\mathbf{BN_u}$, see $\Psi_{bis})$ so that the above equalities imply that the deformation modes provided by the columns 1, 2, 3 and 5 of $\Psi_{bis})$ cannot be active.

In conclusion the hourglass mode in the x-direction (see fig. 4.2) provides the only compatible deformation of the approximate model.

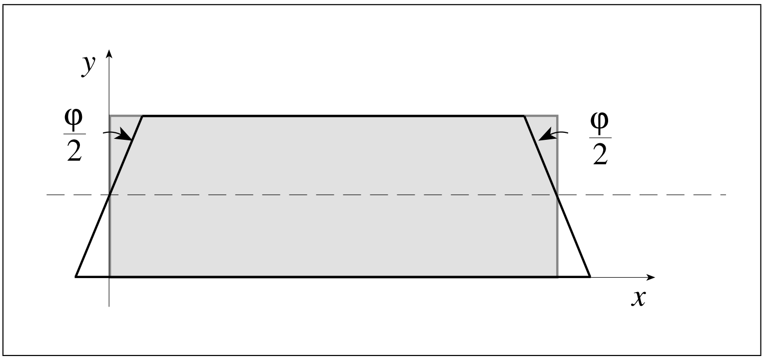


Figure 4.2: Hourglass mode in the **x**-direction

From the equality $\Xi)$ with $i = 5$, the virtual work principle allows us to write

$$\Xi_{bis}) \qquad \langle\!\langle\, \mathbf{B}(\mathbf{u}_o - \mathbf{u}_h)\,,\, (\delta\boldsymbol{\sigma}_h)_4 \,\rangle\!\rangle = \mathcal{M}\Delta\varphi_o - \Delta\varphi_h = 0$$

where $\mathcal{M}$ denotes the external force in equilibrium with the stress distribution $(\delta\boldsymbol{\sigma}_h)_4$. The equality $\Xi_{bis})$ implies that the exact and the approximate relative rotations of the plate end sections must coincide.

Since the left end section is fixed, the absolute rotation φ of the right end section is exact.

Moreover, it is well known that the curvature of the continuous model is constant so that the displacement of the centroid of the right end section is $\varphi\frac{l}{2}$ (see Fig.4.3).

The approximate model provides the same result since the rotation to be superimposed to the hourglass mode to restore the external constraints is $\frac{1}{2}\varphi$ so that the displacement of the centroid of the right end section is $\frac{\varphi}{2}l$ (see fig.4.4).

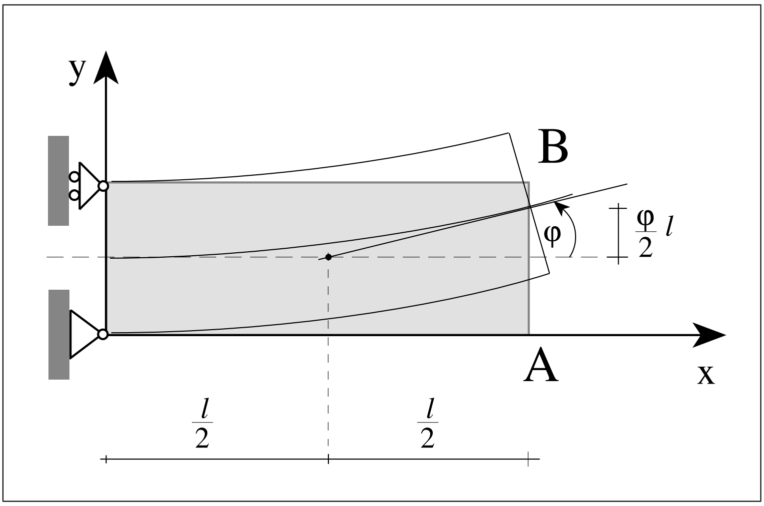


Figure 4.3: Continuous model

4.12 Appendix

To discuss the well-posedness of the discrete problem $\mathbb{P}_h$ (section 4.3) it is essential to provide a representation formula for the kernel of the matrix $\mathbf{M}$ in terms of the component submatrices.

To this end let us first define the reduced matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & -\mathbf{G}^T \\ -\mathbf{G} & \mathbf{H} \end{bmatrix}$$

obtained from the global matrix $\mathbf{M}$ and the associated bilinear form

$$a(\{\,\mathbf{q_u},\,\mathbf{q_g}\},\{\delta\,\mathbf{q_u},\delta\,\mathbf{q_g}\}) = \mathbf{H}\,\mathbf{q_g}\cdot\delta\,\mathbf{q_g} - \mathbf{G}\,\mathbf{q_u}\cdot\delta\,\mathbf{q_g} - \mathbf{G}^T\,\mathbf{q_g}\cdot\delta\,\mathbf{q_u} + \mathbf{K}\,\mathbf{q_u}\cdot\delta\,\mathbf{q_u}.$$

We can now prove a preparatory result.

Proposition 4.12.1 (The quadratic form of the reduced matrix) *The pos-*

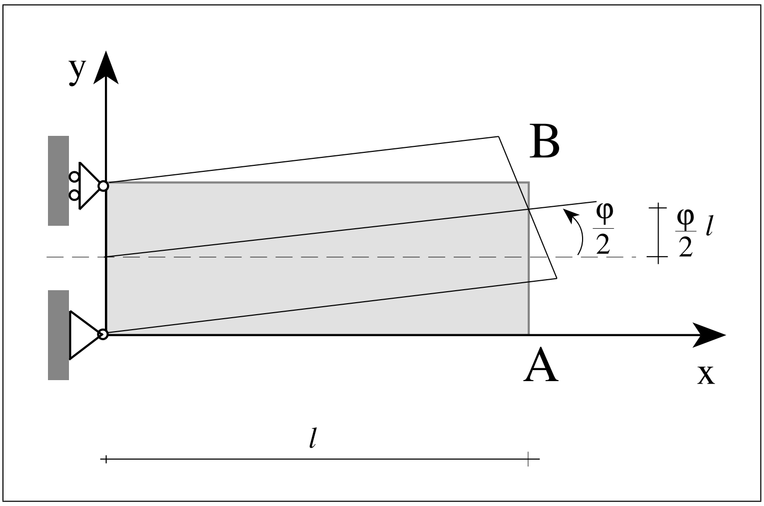


Figure 4.4: One $\mathbf{Q_1}$ element analysis

itive quadratic form $a(\{\mathbf{q_u}, \mathbf{q_g}\})$ *associated with* $\mathbf{A}$ *is given by*

$$a(\{\mathbf{q_u}, \mathbf{q_g}\}) = \mathbf{H}\,\mathbf{q_g} \cdot \mathbf{q_g} - 2\,\mathbf{G}\,\mathbf{q_u} \cdot \mathbf{q_g} + \mathbf{K}\,\mathbf{q_u} \cdot \mathbf{q_u} =$$

$$= \sum_{e=1}^{\mathcal{N}} \int_{\Omega_e} \mathbf{E}_*(\mathbf{N}^e_{\mathbf{g}}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q_g} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q_u}) \cdot (\mathbf{N}^e_{\mathbf{g}}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q_g} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q_u})$$

with $a(\{\mathbf{q_u}, \mathbf{q_g}\}) \geq 0$ *for any* $\mathbf{q_u}$ *and* $\mathbf{q_g}$*, and its kernel is*

$$Ker\, a = \{\, \{\mathbf{q_u}, \mathbf{q_g}\} \;\; : \;\; \mathbf{N}^e_{\mathbf{g}}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q_g} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q_u} = \mathbf{o} \quad e = 1, \ldots, \mathcal{N}\,\}.$$

Proof. The proposition is a direct consequence of the definitions of $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{K}$ and of the positive definiteness of the elastic matrix $\mathbf{E}_*$.

To provide a representation of the kernel of the matrix $\widetilde{\mathbf{M}}$ we preliminarily recall that, due to the positivity of $a(\{\mathbf{q_u}, \mathbf{q_g}\})$, the matrix $\mathbf{A}$ and the associated quadratic form $a(\mathbf{q_u}, \mathbf{q_g})$ have the same kernel [12], that is:

$$\mathrm{Ker}\,\mathbf{A} = \mathrm{Ker}\,a.$$

We are now ready to prove the next result.

Proposition 4.12.2 (Representation of the kernel of the matrix $\widetilde{\mathbf{M}}$) *The interpolation parameters which annihilate the response of the global matrix $\widetilde{\mathbf{M}}$, that is $\{\, \mathbf{q_u},\, \mathbf{q_g},\, \mathbf{q}_{\boldsymbol{\sigma}} \} \in Ker\widetilde{\mathbf{M}}$, are characterized by the property*

$$\begin{cases} \mathbf{q_g} \in Ker\,\mathbf{Q}^T \\ \mathbf{N_g}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q_g} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q_u} = \mathbf{o} \quad e = 1, \ldots, \mathcal{N} \\ \mathbf{q}_{\boldsymbol{\sigma}} \in Ker\,\mathbf{Q}. \end{cases}$$

Proof. Let $\{\, \mathbf{q_u},\, \mathbf{q_g},\, \mathbf{q}_{\boldsymbol{\sigma}} \,\} \in \mathrm{Ker}\,\widetilde{\mathbf{M}}$ then

$$\begin{cases} i) \quad \mathbf{K}\,\mathbf{q_u} - \ \mathbf{G}^T\,\mathbf{q_g} = \mathbf{o} \\ ii) \ -\mathbf{G}\,\mathbf{q_u} + \ \mathbf{H}\,\mathbf{q_g} + \ \mathbf{Q}\,\mathbf{q}_{\boldsymbol{\sigma}} = \mathbf{o} \\ iii) \ \ +\mathbf{Q}^T\,\mathbf{q_g} = \mathbf{o}. \end{cases}$$

From the equation iii) we can infer $\mathbf{q_g} \in \mathrm{Ker}\,\mathbf{Q}^T$.

By taking the dot product of ii) by $\mathbf{q_g}$ we get

$$\mathbf{H}\,\mathbf{q_g} \cdot \mathbf{q_g} + \mathbf{Q}\,\mathbf{q}_{\boldsymbol{\sigma}} \cdot \mathbf{q_g} - \mathbf{G}\,\mathbf{q_u} \cdot \mathbf{q_g} = 0\,,$$

so that, by means of equation iii), it turns out to be $\mathbf{H}\,\mathbf{q_g} \cdot \mathbf{q_g} - \mathbf{G}\,\mathbf{q_u} \cdot \mathbf{q_g} = 0$.

Moreover, by summing up this equation and the equation

$$\mathbf{K}\,\mathbf{q_u} \cdot \mathbf{q_u} - \mathbf{G}\,\mathbf{q_u} \cdot \mathbf{q_g} = 0,$$

which is equation i) multiplied by $\mathbf{q_u}$ we get

$$\mathbf{H}\,\mathbf{q_g} \cdot \mathbf{q_g} - 2\,\mathbf{G}\,\mathbf{q_u} \cdot \mathbf{q_g} + \mathbf{K}\,\mathbf{q_u} \cdot \mathbf{q_u} = 0.$$

Then the displacement and the strain gap parameters $\{\, \mathbf{q_u},\, \mathbf{q_g}\}$ belong to the kernel of the bilinear form $a(\,\mathbf{q_u},\, \mathbf{q_g})$ so that, from proposition 4.12.1, we have $\mathbf{N_g}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q_g} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q_u} = \mathbf{o}$ for $e = 1, \ldots, \mathcal{N}$.

Moreover, the displacement and the strain gap parameters $\{\, \mathbf{q_u},\, \mathbf{q_g}\}$ belong to the kernel of the reduced matrix $\mathbf{A}$ so that we have:

$$\begin{cases} \ \ \mathbf{K}\,\mathbf{q_u} \ - \ \mathbf{G}^T\,\mathbf{q_g} = \mathbf{o} \\ -\mathbf{G}\,\mathbf{q_u} \ + \ \mathbf{H}\,\mathbf{q_g} = \mathbf{o}\,. \end{cases}$$

A comparison with equations i), ii) and iii) shows that $\mathbf{Q}\ \mathbf{q}_{\boldsymbol{\sigma}} = \mathbf{o}$ or equivalently $\mathbf{q}_{\boldsymbol{\sigma}} \in \operatorname{Ker}\mathbf{Q}$. Conversely the properties $\mathbf{q}_{\boldsymbol{\sigma}} \in \operatorname{Ker}\mathbf{Q}$, $\mathbf{q}_{\mathbf{g}} \in \operatorname{Ker}\mathbf{Q}^T$ and $\mathbf{N}_{\mathbf{g}}\mathcal{J}^e_{\mathbf{g}}\,\mathbf{q}_{\mathbf{g}} - \mathbf{B}_*\mathbf{N}^e_{\mathbf{u}}\mathcal{A}^e_{\mathbf{u}}\,\mathbf{q}_{\mathbf{u}} = \mathbf{o}$ for $e = 1, \ldots, \mathcal{N}$ ensure that $\{\ \mathbf{q}_{\mathbf{u}},\ \mathbf{q}_{\mathbf{g}},\ \mathbf{q}_{\boldsymbol{\sigma}}\ \} \in \operatorname{Ker}\widetilde{\mathbf{M}}$.

A better understanding of well-posedness can be got by means of an equivalent geometric formulation in terms of interpolating subspaces. To this end we quote the following equivalences

$$\mathbf{q}_{\boldsymbol{\sigma}} \in \operatorname{Ker}\mathbf{Q} \iff \boldsymbol{\sigma}_h \in \mathcal{S}_h \cap \mathcal{D}_h^{\perp},$$
$$\mathbf{q}_{\mathbf{g}} \in \operatorname{Ker}\mathbf{Q}^T \iff \mathbf{g}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp}.$$

The kernel of $\widetilde{\mathbf{M}}$ can then be rewritten, by virtue of proposition 4.9.2, as

$$\mathbb{K})\qquad \begin{bmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{bmatrix} \in \operatorname{Ker}\widetilde{\mathbf{M}} \iff \begin{cases} \mathbf{g}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp} \\ \mathbf{u}_h \in \mathcal{L}_h \ :\ \mathbf{g}_h = \mathbf{B}\mathbf{u}_h, \\ \boldsymbol{\sigma}_h \in \mathcal{S}_h \cap \mathcal{D}_h^{\perp}. \end{cases}$$

Necessary and sufficient conditions for well-posedness are proven in the proposition below.

Proposition 4.12.3 (Well-posedness criterion) *If* $\operatorname{Ker}\mathbf{B} \cap \mathcal{L}_h = \{\mathbf{o}\}$ *there are no rigid conforming displacements and the conditions*

$$G_1)\quad \widetilde{\mathcal{D}}_h \cap \mathbf{B}\mathcal{L}_h = \{\,\mathbf{o}\,\} \qquad\qquad G_2)\quad \mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}$$

are necessary and sufficient for the well-posedness of the discrete mixed problem $\mathbb{P}_h$ *(section 4.3)*

Proof. Conditions G_1 and G_2 are equivalent to assume that $\operatorname{Ker}\widetilde{\mathbf{M}} = \{\mathbf{o}\}$.

Due to the symmetry of the global matrix $\widetilde{\mathbf{M}}$ we have that

$$\operatorname{Im}\widetilde{\mathbf{M}} = (\operatorname{Ker}\widetilde{\mathbf{M}}^T)^{\perp} = (\operatorname{Ker}\widetilde{\mathbf{M}})^{\perp} = \mathcal{D}_h \times \mathcal{S}_h \times \mathcal{L}_h\,,$$

so that there exists a unique solution for any data.

Chapter 5

Convergence properties

5.1 Three field methods

To discuss the convergence properties of the elastostatic three field problem, governed by the functional

$$H(\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{v}) = \frac{1}{2}(\!(\,\mathbf{E}(\boldsymbol{\varepsilon}-\overline{\boldsymbol{\delta}})\,,\,\boldsymbol{\varepsilon}-\overline{\boldsymbol{\delta}}\,)\!)_{\mathcal{H}} + (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}-\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}} - \gamma_o(\mathbf{v}) - \sqcap_{\mathrm{Ker}\,\mathbf{L}}(\boldsymbol{\Gamma}(\mathbf{v}-\overline{\mathbf{v}})),$$

it is conveniently to group two fields together in order to obtain an equivalent two-field functional.

To this end we introduce the pivot product HILBERT space $\mathcal{X} = \mathcal{H} \times \mathcal{H}$ with $\mathcal{X}' = \mathcal{X}$. The standard inner product between $\mathbf{x} = \{\boldsymbol{\varepsilon}\,,\boldsymbol{\sigma}\} \in \mathcal{X}$ and $\overline{\mathbf{x}} = \{\overline{\boldsymbol{\varepsilon}}\,,\overline{\boldsymbol{\sigma}}\} \in \mathcal{X}$ is defined by

$$(\!(\mathbf{x},\overline{\mathbf{x}})\!)_{\mathcal{X}} = (\!(\,\boldsymbol{\varepsilon}\,,\,\overline{\boldsymbol{\varepsilon}}\,)\!)_{\mathcal{H}} + (\!(\,\boldsymbol{\sigma}\,,\,\overline{\boldsymbol{\sigma}}\,)\!)_{\mathcal{H}}\,,$$

and the related norm is given by

$$\|\,\mathbf{x}\,\|^2_{\mathcal{X}} = \|\,\{\boldsymbol{\varepsilon}\,,\boldsymbol{\sigma}\}\,\|^2_{\mathcal{X}} = \|\,\boldsymbol{\varepsilon}\,\|^2_{\mathcal{H}} + \|\,\boldsymbol{\sigma}\,\|^2_{\mathcal{H}}.$$

Let us denote by $\mathrm{Bil}\{\,\mathcal{X} \times \mathcal{Y}\,\}$ the space of continuous bilinear forms on the product space $\mathcal{X} \times \mathcal{Y}$.

The bilinear forms $\mathbf{a} \in \mathrm{Bil}\{\,\mathcal{X} \times \mathcal{X}\,\}$ and $\mathbf{j} \in \mathrm{Bil}\{\,\mathcal{L} \times \mathcal{X}\,\}$ are defined by

$$\mathbf{a}\,(\mathbf{x},\overline{\mathbf{x}}) = (\!(\,\mathcal{E}\boldsymbol{\varepsilon}\,,\,\overline{\boldsymbol{\varepsilon}}\,)\!)_{\mathcal{H}} - (\!(\,\boldsymbol{\sigma}\,,\,\overline{\boldsymbol{\varepsilon}}\,)\!)_{\mathcal{H}} - (\!(\,\overline{\boldsymbol{\sigma}}\,,\,\boldsymbol{\varepsilon}\,)\!)_{\mathcal{H}}\,, \qquad \mathbf{j}\,(\mathbf{v},\overline{\mathbf{x}}) = (\!(\,\overline{\boldsymbol{\sigma}}\,,\,\mathbf{B}\mathbf{v}\,)\!)_{\mathcal{H}}\,,$$

and the linear form $\boldsymbol{\Delta} \in \mathcal{X}'$ is defined by

$$(\!(\boldsymbol{\Delta}, \overline{\mathbf{x}})\!)_{\mathcal{X}} = (\!(\, \mathcal{E}\boldsymbol{\delta} \,, \, \overline{\boldsymbol{\varepsilon}} \,)\!)_{\mathcal{H}} \,.$$

The three-field functional $H(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{v})$ can then be rewritten as a two-field functional

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{u}) = \frac{1}{2}\, \mathbf{a}\,(\mathbf{x}, \mathbf{x}) + \mathbf{j}\,(\mathbf{u}, \mathbf{x}) - \langle\, \ell \,, \, \mathbf{u} \,\rangle - (\!(\boldsymbol{\Delta}, \mathbf{x})\!)_{\mathcal{X}} \quad \mathbf{x} \in \mathcal{X} \,, \; \mathbf{u} \in \mathcal{L} \,.$$

The stationarity of $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{u})$ is expressed by the associated EULER conditions

$$\alpha) \qquad \begin{cases} \mathbf{a}\,(\mathbf{x}, \overline{\mathbf{x}}) + \mathbf{j}\,(\mathbf{u}, \overline{\mathbf{x}}) = (\!(\boldsymbol{\Delta}, \overline{\mathbf{x}})\!)_{\mathcal{X}} & \forall\, \overline{\mathbf{x}} \in \mathcal{X} \,, \\ \mathbf{j}\,(\overline{\mathbf{u}}, \mathbf{x}) \qquad\qquad\quad = \langle\, \ell \,, \, \overline{\mathbf{u}} \,\rangle & \forall\, \overline{\mathbf{u}} \in \mathcal{L} \,. \end{cases}$$

The unknown fields $\mathbf{x} = \{\boldsymbol{\varepsilon} \,, \boldsymbol{\sigma}\} \in \mathcal{H} \times \mathcal{H}$ and $\mathbf{u} \in \mathcal{L}$ are the trial variables among which we look for a solution, while the fields $\overline{\mathbf{x}} = \{\overline{\boldsymbol{\varepsilon}} \,, \overline{\boldsymbol{\sigma}}\} \in \mathcal{H} \times \mathcal{H}$ and $\overline{\mathbf{u}} \in \mathcal{L}$ are the test variables of the variational problem. Both span the same linear spaces $\mathcal{X} = \mathcal{H} \times \mathcal{H}$ and $\mathcal{L}$.

Taking into account that the bilinear form $\mathbf{a} \in \mathrm{Bil}\{\, \mathcal{X} \times \mathcal{X} \,\}$ is symmetric we define the linear operators $\mathbf{A} \in BL\,(\mathcal{X}, \mathcal{X}')$, $\mathbf{J} \in BL\,(\mathcal{X}, \mathcal{L}')$ and $\mathbf{J}' \in BL\,(\mathcal{L}, \mathcal{X}')$ associated with the bilinear forms **a** and **j**:

$$\begin{aligned} \mathbf{a}\,(\mathbf{x}, \overline{\mathbf{x}}) &= (\!(\mathbf{x}, \mathbf{A}\,\overline{\mathbf{x}})\!)_{\mathcal{X}} = (\!(\,\mathbf{A}\,\mathbf{x}, \overline{\mathbf{x}})\!)_{\mathcal{X}} \quad \forall\, \mathbf{x}, \overline{\mathbf{x}} \in \mathcal{X} \,, \\ \mathbf{j}\,(\overline{\mathbf{u}}, \mathbf{x}) &= \langle\, \overline{\mathbf{u}} \,, \, \mathbf{J}\,\mathbf{x} \,\rangle = (\!(\mathbf{J}'\,\overline{\mathbf{u}}, \mathbf{x})\!)_{\mathcal{X}} \qquad \forall\, \mathbf{x} \in \mathcal{X} \,, \quad \forall\, \overline{\mathbf{u}} \in \mathcal{L} \,, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}\,\mathbf{x} &= \mathbf{A}\,\{\boldsymbol{\varepsilon} \,, \boldsymbol{\sigma}\} = \{\mathcal{E}\boldsymbol{\varepsilon} - \boldsymbol{\sigma} \;, -\boldsymbol{\varepsilon}\,\} \,, \\ \mathbf{J}\,\mathbf{x} &= \mathbf{J}\,\{\boldsymbol{\varepsilon} \,, \boldsymbol{\sigma}\} = \mathbf{B}'\boldsymbol{\sigma} \,, \qquad \mathbf{J}'\,\mathbf{u} \;\; = \{\mathbf{o} \,, \mathbf{B}\mathbf{u}\} \,. \end{aligned}$$

Then the kernels of the above linear operators are given by

$$i) \qquad \begin{aligned} &\mathrm{Ker}\,\mathbf{A} = \{\mathbf{o}, \mathbf{o}\} \in \mathcal{H} \times \mathcal{H} \,, \\ &\mathrm{Ker}\,\mathbf{J} = \big\{\{\boldsymbol{\varepsilon} \,, \boldsymbol{\sigma}\} \in \mathcal{H} \times \mathcal{H} \,:\, \mathbf{B}'\boldsymbol{\sigma} = \mathbf{o}\big\} = \mathcal{H} \times \,\mathrm{Ker}\,\mathbf{B}' \,, \\ &\mathrm{Ker}\,\mathbf{J}' = \big\{\mathbf{u} \in \mathcal{L} \,:\, \mathbf{B}\mathbf{u} = \mathbf{o}\big\} = \,\mathrm{Ker}\,\mathbf{B} \,, \end{aligned}$$

and the images are:

$$ii) \qquad \mathrm{Im}\,\mathbf{A} = \mathcal{H} \times \mathcal{H} \,, \quad \mathrm{Im}\,\mathbf{J} = \,\mathrm{Im}\,\mathbf{B}' \,, \quad \mathrm{Im}\,\mathbf{J}' = \{\, \boldsymbol{o} \,\}_{\mathcal{H}} \times \,\mathrm{Im}\,\mathbf{B} \,.$$

The mixed problem $\alpha)$ can then be put in the operator form

$$\begin{cases} \mathbf{A}\mathbf{x} + \mathbf{J}'\mathbf{u} = \boldsymbol{\Delta}\,, \\ \mathbf{J}\mathbf{x} \qquad\quad = \ell\,. \end{cases}$$

The second equation of the problem $\mathbb{K})$ (see section 4.12) defines an affine constraint on the trial variable $\mathbf{x} \in \mathcal{X}$ which, setting $\mathbf{x}_\ell = \{\mathbf{o}, \boldsymbol{\sigma}_\ell\}$, can be written as:

$$\mathbf{x} \in \mathbf{x}_\ell + \operatorname{Ker}\mathbf{J} \qquad \Longleftrightarrow \qquad \begin{cases} \mathbf{x} = \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}\} \in \mathcal{H} \times \mathcal{H}\,, \\ \boldsymbol{\varepsilon} \in \mathcal{H}\,, \\ \boldsymbol{\sigma} \in \boldsymbol{\sigma}_\ell + \operatorname{Ker}\mathbf{B}'\,. \end{cases}$$

The field $\boldsymbol{\sigma}_\ell \in \mathcal{H}$ is a stress field in equilibrium with the load ℓ and the subspace $\operatorname{Ker}\mathbf{B}' \subset \mathcal{H}$ is formed by self-equilibrated stress fields. Hence the affine constraint $\boldsymbol{\sigma} \in \boldsymbol{\sigma}_\ell + \operatorname{Ker}\mathbf{B}' \subset \mathcal{H}$ defines the linear variety of stress fields equilibrated with the load $\ell \in \mathcal{L}'$.

The discussion of the problem $\mathbb{K})$ (see section 4.12) is conveniently carried out by considering the corresponding reduced problem given by

$$\beta) \qquad \mathbf{a}(\mathbf{x}, \overline{\mathbf{x}}) = ((\mathbf{f}, \overline{\mathbf{x}}))_{\mathcal{X}} \quad \forall \overline{\mathbf{x}} \in \operatorname{Ker}\mathbf{J}\,, \quad \mathbf{x} \in \operatorname{Ker}\mathbf{J}$$

where $\mathbf{f} \in \mathcal{X}'$ is the linear functional defined by

$$((\mathbf{f}, \overline{\mathbf{x}}))_{\mathcal{X}} = ((\,\mathcal{E}\boldsymbol{\delta} + \boldsymbol{\sigma}_\ell\,, \overline{\boldsymbol{\varepsilon}}\,))_{\mathcal{H}} \quad \forall \overline{\mathbf{x}} \in \operatorname{Ker}\mathbf{J}\,.$$

Let $\mathbf{x} = \{\boldsymbol{\varepsilon}\,, \boldsymbol{\sigma}\} \in \operatorname{Ker}\mathbf{J}$ be a solution of the reduced problem $\beta)$. To get the full solution $\{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}\} \in \operatorname{Ker}\mathbf{J} \times \mathcal{L} = \mathcal{H} \times \operatorname{Ker}\mathbf{B}' \times \mathcal{L}$ of the problem $\mathbb{K})$ (see section 4.12) we can recover the displacement solution $\mathbf{u}$ by solving the problem

$$((\mathbf{J}'\mathbf{u}, \overline{\mathbf{x}}))_{\mathcal{X}} = \mathbf{j}(\mathbf{u}, \overline{\mathbf{x}}) = -\mathbf{a}(\mathbf{x}, \overline{\mathbf{x}}) + ((\boldsymbol{\Delta}, \overline{\mathbf{x}}))_{\mathcal{X}}\,, \quad \forall \overline{\mathbf{x}} \in \mathcal{X}$$

A unique solution in terms of displacements is obtained if and only if $\operatorname{Ker}\mathbf{J}' = \operatorname{Ker}\mathbf{B} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}$ which means that the structure cannot undergo conforming rigid displacements. In the sequel we shall assume that this condition be satisfied.

Let us now introduce the symmetric reduced operator $\mathbf{A}_o \in BL(\operatorname{Ker}\mathbf{J}, (\operatorname{Ker}\mathbf{J})')$ defined by

$$((\mathbf{A}_o\mathbf{x}, \overline{\mathbf{x}}))_{\mathcal{X}} = ((\mathbf{x}, \mathbf{A}_o\overline{\mathbf{x}}))_{\mathcal{X}} = \mathbf{a}(\mathbf{x}, \overline{\mathbf{x}}) \quad \forall \mathbf{x}, \overline{\mathbf{x}} \in \operatorname{Ker}\mathbf{J}\,,$$

and the reduced force $\mathbf{f}_o \in (\operatorname{Ker}\mathbf{J})'$ defined by

$$((\mathbf{f}_o, \overline{\mathbf{x}}))_{\mathcal{X}} = ((\mathbf{f}, \overline{\mathbf{x}}))_{\mathcal{X}} \quad \forall \overline{\mathbf{x}} \in \operatorname{Ker}\mathbf{J},$$

so that the reduced problem $\beta)$ can be written as

$$\gamma) \qquad \mathbf{A}_o \mathbf{x} = \mathbf{f}_o .$$

We recall that, for any linear subspace $\mathcal{S} \subseteq \mathcal{X}$, there is an isometric isomorphism between the HILBERT space $\mathcal{S}'$, dual of $\mathcal{S}$, and the factor HILBERT space $\mathcal{X}'/\mathcal{S}^{\perp}$ [18], [21], [89]. Accordingly the dual space $(\operatorname{Ker}\mathbf{J})'$ is isometrically isomorphic to the factor space $\mathcal{X}'/(\operatorname{Ker}\mathbf{J})^{\perp}$ so that

$$\|\mathbf{f}\|_{(\operatorname{Ker}\mathbf{J})'} = \|\mathbf{f}\|_{\mathcal{X}'/(\operatorname{Ker}\mathbf{J})^{\perp}} = \inf_{\boldsymbol{\rho} \in (\operatorname{Ker}\mathbf{J})^{\perp}} \|\mathbf{f} - \boldsymbol{\rho}\|_{\mathcal{X}'} \quad \forall \mathbf{f} \in \mathcal{X}'$$

and we have that

$$\mathbf{A}_o \mathbf{x} = \mathbf{A}\,\mathbf{x} + (\operatorname{Ker}\mathbf{J})^{\perp} \quad \forall \mathbf{x} \in \mathcal{X}, \qquad \mathbf{f}_o = \mathbf{f} + (\operatorname{Ker}\mathbf{J})^{\perp}.$$

Denoting by $\mathbf{I} \in BL(\mathcal{H}, \mathcal{H})$ the identity map, the explicit expression of $\mathbf{A}_o$ in matrix form is given by

$$\mathbf{A}_o \mathbf{x} = \begin{bmatrix} \mathcal{E} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \end{bmatrix} + \begin{bmatrix} \{\boldsymbol{o}\}_{\mathcal{H}} \\ \operatorname{Im}\mathbf{B} \end{bmatrix} .$$

5.1.1 Well-posedness conditions

Let us now discuss, in terms of operators, the well-posedness conditions concerning the mixed problem $\alpha)$, i.e. the conditions which ensure existence and uniqueness of the solution of the mixed problem.

It is apparent that any solution of the problem $\alpha)$ is also a solution of the problem $\beta)$. In order to get the converse implication, we can now state a result concerning the bilinear form $\mathbf{j} \in \operatorname{Bil}\{\mathcal{L} \times \mathcal{X}\}$, [29], [50], [57], [91].

Proposition 5.1.1 *If the bilinear form* $\mathbf{j} \in Bil\{\mathcal{L} \times \mathcal{X}\}$ *is closed, any solution of the problem* $\beta)$ *is a solution of the problem* $\alpha)$.

Proof. Let $\mathbf{x} \in \operatorname{Ker}\mathbf{J}$ be a solution of the reduced problem $\beta)$ or equivalently of the problem $\gamma)$. We then have

$$\mathbf{A}\mathbf{x} - \mathbf{f} \in (\operatorname{Ker}\mathbf{J})^{\perp} = \operatorname{Im}\mathbf{J}'.$$

Accordingly there exists a displacement $\mathbf{u} \in \mathcal{L}$ such that $-\mathbf{J}'\mathbf{u} = \mathbf{A}\mathbf{x} - \mathbf{f}$ so that

$$((\mathbf{J}'\mathbf{u}, \overline{\mathbf{x}}))_{\mathcal{X}} = \mathbf{j}\,(\mathbf{u}, \overline{\mathbf{x}}) = -\mathbf{a}\,(\mathbf{x}, \overline{\mathbf{x}}) + ((\boldsymbol{\Delta}, \overline{\mathbf{x}}))_{\mathcal{X}} \quad \forall\, \overline{\mathbf{x}} \in \mathcal{X}\,,$$

and hence the pair $\{\mathbf{u}\,, \mathbf{x}\} \in \mathcal{L} \times \operatorname{Ker}\mathbf{J}$ is a solution of the problem *alpha*).

In the present case, recalling the expressions *i*) and *ii*), the closedness of $\operatorname{Im}\mathbf{J} \subset \mathcal{L}'$ reported in the proposition 5.1.1 follows form the closedness of the fundamental bilinear form $\mathbf{b}\,(\mathbf{u}, \boldsymbol{\sigma})$. By BANACH's closed range theorem in HILBERT spaces [39], the closedness of $\operatorname{Im}\mathbf{J} \subset \mathcal{L}'$ is equivalent to

$$\begin{cases} \operatorname{Im}\mathbf{J} = (\operatorname{Ker}\mathbf{J}')^{\perp} \\ \operatorname{Im}\mathbf{J}' = (\operatorname{Ker}\mathbf{J})^{\perp} \end{cases} \iff \begin{cases} \|\,\mathbf{J}\mathbf{x}\,\|_{\mathcal{H}} \geq c_{\mathbf{j}}\,\|\,\mathbf{x}\,\|_{\mathcal{X}/\operatorname{Ker}\mathbf{J}} & \forall\, \mathbf{x} \in \mathcal{X}\,, \\ \|\,\mathbf{J}'\mathbf{u}\,\|_{\mathcal{H}} \geq c_{\mathbf{j}}\,\|\,\mathbf{u}\,\|_{\mathcal{H}/\operatorname{Ker}\mathbf{J}'} & \forall\, \mathbf{u} \in \mathcal{L}\,. \end{cases}$$

We are now ready to prove the main existence and uniqueness result.

Proposition 5.1.2 (Existence and uniqueness) *The reduced problem* β) *admits a unique solution for any data* $\mathbf{f} \in \mathcal{X}'$.

Proof. It is convenient to deal with the standard form γ) of the reduced problem. We have to show that $\operatorname{Ker}\mathbf{A}_o = \{\,\boldsymbol{o}\,\}_{\mathcal{X}}$ and that $\operatorname{Im}\mathbf{A}_o = \mathcal{X}'/(\operatorname{Ker}\mathbf{J})^{\perp}$.

To prove the former assertion we observe that by definition

$$\mathbf{x} \in \operatorname{Ker}\mathbf{A}_o \quad \iff \quad \mathcal{E}\boldsymbol{\varepsilon} - \boldsymbol{\sigma} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}\,, \quad \boldsymbol{\sigma} \in \operatorname{Ker}\mathbf{B}'\,, \quad \boldsymbol{\varepsilon} \in \operatorname{Im}\mathbf{B} \subset \mathcal{H}\,.$$

Hence from the relation $\operatorname{Ker}\mathbf{B}' = (\operatorname{Im}\mathbf{B})^{\perp}$ and the positive definiteness of $\mathcal{E}$ we get

$$\mathcal{E}\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \in (\operatorname{Im}\mathbf{B})^{\perp}\,, \quad \boldsymbol{\varepsilon} \in \operatorname{Im}\mathbf{B} \quad \Longrightarrow \quad ((\,\mathcal{E}\boldsymbol{\varepsilon}\,, \boldsymbol{\varepsilon}\,))_{\mathcal{H}} = 0 \quad \Longrightarrow \quad \boldsymbol{\varepsilon} = \mathbf{o}\,, \boldsymbol{\sigma} = \mathbf{o}\,,$$

so that $\operatorname{Ker}\mathbf{A}_o = \{\,\boldsymbol{o}\,\}_{\mathcal{X}}$ and uniqueness follows. Existence of a solution for any data $\mathbf{f} \in \mathcal{X}'$ is proved as follows.

Since $\operatorname{Im}\mathbf{A} = \mathcal{H} \times \mathcal{H} = \mathcal{X}'$, from the definition $\mathbf{A}_o\mathbf{x} = \mathbf{A}\mathbf{x} + (\operatorname{Ker}\mathbf{J})^{\perp}$ we infer that the operator $\mathbf{A}_o \in BL(\operatorname{Ker}\mathbf{J}, \mathcal{X}'/(\operatorname{Ker}\mathbf{J})^{\perp})$ is surjective being $\operatorname{Im}\mathbf{A}_o = \mathcal{X}'/(\operatorname{Ker}\mathbf{J})^{\perp}$.

5.1.2 Approximate solutions

Approximate solutions can be obtained by a conforming FEM interpolation based on three families of finite dimensional subspaces $\mathcal{L}_h \subset \mathcal{L}$, $\mathcal{D}_h \subset \mathcal{H}$ and $\mathcal{S}_h \subset \mathcal{H}$ depending on a parameter h which goes to zero as the finite element mesh is refined ever more.

The approximate mixed problem consists in finding a pair $\{\mathbf{x}_h, \mathbf{u}_h\} \in \mathcal{X}_h \times \mathcal{L}_h$ such that

$$\delta) \qquad \begin{cases} \mathbf{a}(\mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}(\mathbf{u}_h, \overline{\mathbf{x}_h}) = (\!(\boldsymbol{\Delta}, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} & \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h = \mathcal{D}_h \times \mathcal{S}_h\,, \\ \mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{x}_h) = \langle\, \ell\,, \overline{\mathbf{u}_h}\,\rangle & \forall\, \overline{\mathbf{u}_h} \in \mathcal{L}_h\,, \end{cases}$$

where

$$\begin{aligned} &\mathbf{a}(\mathbf{x}_h, \overline{\mathbf{x}_h}) = (\!(\, \mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h\,, \overline{\boldsymbol{\varepsilon}}_h\,)\!)_{\mathcal{H}} - (\!(\, \overline{\boldsymbol{\sigma}_h}\,, \boldsymbol{\varepsilon}_h\,)\!)_{\mathcal{H}}\,, \\ &\mathbf{j}(\mathbf{u}_h, \overline{\mathbf{x}_h}) = (\!(\, \overline{\boldsymbol{\sigma}_h}\,, \mathbf{B}\mathbf{u}_h\,)\!)_{\mathcal{H}}\,, \\ &(\!(\boldsymbol{\Delta}, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} = (\!(\, \mathcal{E}\boldsymbol{\delta}\,, \overline{\boldsymbol{\varepsilon}}_h\,)\!)_{\mathcal{H}}\,. \end{aligned}$$

Let us consider the operators associated with the bilinear forms by setting

$$\begin{aligned} &\mathbf{a}(\mathbf{x}_h, \overline{\mathbf{x}_h}) = (\!(\mathbf{x}_h, \mathbf{A}_h\, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} = (\!(\mathbf{A}_h\, \mathbf{x}_h, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} \quad \forall\, \mathbf{x}_h, \overline{\mathbf{x}_h} \in \mathcal{X}_h\,, \\ &\mathbf{j}(\mathbf{u}_h, \mathbf{x}_h) = \langle\, \mathbf{u}_h\,, \mathbf{J}_h\, \mathbf{x}_h\,\rangle = (\!(\mathbf{J}'_h\, \mathbf{u}_h, \mathbf{x}_h)\!)_{\mathcal{X}} \qquad \forall\, \mathbf{x}_h \in \mathcal{X}_h\,, \quad \forall\, \mathbf{u}_h \in \mathcal{L}_h\,, \end{aligned}$$

where $\mathbf{A}_h \in BL(\mathcal{X}_h, \mathcal{X}'_h)$, $\mathbf{J}_h \in BL(\mathcal{X}_h, \mathcal{L}'_h)$, $\mathbf{J}'_h \in BL(\mathcal{L}_h, \mathcal{X}'_h)$ and let us define the linear functionals

$$\begin{aligned} &\langle\, \ell_h\,, \overline{\mathbf{u}_h}\,\rangle = \langle\, \ell\,, \overline{\mathbf{u}_h}\,\rangle & \forall\, \overline{\mathbf{u}_h} \in \mathcal{L}_h\,, \\ &(\!(\boldsymbol{\Delta}_h, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} = (\!(\boldsymbol{\Delta}, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} & \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h\,, \end{aligned}$$

where $\ell_h \in \mathcal{L}'_h$ and $\boldsymbol{\Delta}_h \in \mathcal{X}'_h$.

These discrete operators are conveniently expressed by taking into account the isometric isomorphism between the Hilbert space $\mathcal{X}'_h$, dual to $\mathcal{X}_h$, and the quotient Hilbert space $\mathcal{X}'/\mathcal{X}_h^{\perp}$. Accordingly we can set

$$\begin{aligned} &\mathbf{A}_h\, \mathbf{x}_h = \mathbf{A}_h\, \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\} = \{\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h + \mathcal{D}_h^{\perp}, -\boldsymbol{\varepsilon}_h + \mathcal{S}_h^{\perp}\}\,, \\ &\mathbf{J}_h\, \mathbf{x}_h = \mathbf{J}_h\, \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\} = \mathbf{B}'\boldsymbol{\sigma}_h + \mathcal{L}_h^{\perp}\,, \\ &\mathbf{J}'_h\, \mathbf{u}_h = \{\mathcal{D}_h^{\perp}, \mathbf{B}\mathbf{u}_h + \mathcal{S}_h^{\perp}\}\,, \\ &\quad \ell_h = \ell + \mathcal{L}_h^{\perp}\,, \\ &\boldsymbol{\Delta}_h = \boldsymbol{\Delta} + \mathcal{X}_h^{\perp}\,. \end{aligned}$$

Defining the linear subspace of discrete self stresses by $\Sigma_h = \mathcal{S}_h \cap (\mathbf{B}\mathcal{L}_h)^{\perp}$, the following theorem holds.

Theorem 5.1.1 *The kernels of the operators* $\mathbf{A}_h$, $\mathbf{J}_h$ *and* $\mathbf{J}'_h$ *are given by:*

$$\epsilon) \qquad \begin{aligned} &Ker\,\mathbf{A}_h = \{\mathbf{o}\}_{\mathcal{H}} \times (\mathcal{S}_h \cap \mathcal{D}_h^{\perp})\,, \\ &Ker\,\mathbf{J}_h = \{\{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \mathcal{D}_h \times \mathcal{S}_h \mid \mathbf{B}'\boldsymbol{\sigma}_h \in \mathcal{L}_h^{\perp}\} = \mathcal{D}_h \times \Sigma_h\,, \\ &Ker\,\mathbf{J}'_h = \{\mathbf{u}_h \in \mathcal{L}_h \mid \mathbf{B}\mathbf{u}_h \in \mathcal{S}_h^{\perp}\} = \mathcal{L}_h \cap (\mathbf{B}'\mathcal{S}_h)^{\perp}\,. \end{aligned}$$

Proof. To prove $\epsilon)_1$, let us note that

$$\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{A}_h \iff \mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h \in \mathcal{D}_h^{\perp}\,, \quad \boldsymbol{\sigma}_h \in \mathcal{S}_h\,, \quad \boldsymbol{\varepsilon}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp}\,,$$

and being $\mathcal{D}_h \cap \mathcal{S}_h^{\perp} = (\mathcal{S}_h + \mathcal{D}_h^{\perp})^{\perp}$, the positive definiteness of $\mathcal{E}$ ensures that

$$\begin{aligned} \mathcal{E}\boldsymbol{\varepsilon}_h \in \mathcal{S}_h + \mathcal{D}_h^{\perp}\,, \quad \boldsymbol{\varepsilon}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp} \quad &\Longrightarrow \quad (\!(\, \mathcal{E}\boldsymbol{\varepsilon}_h\,, \boldsymbol{\varepsilon}_h \,)\!)_{\mathcal{H}} = 0 \quad \Longrightarrow \\ &\Longrightarrow \quad \boldsymbol{\varepsilon}_h = \mathbf{o}\,, \boldsymbol{\sigma}_h = \mathcal{S}_h \cap \mathcal{D}_h^{\perp}\,. \end{aligned}$$

On the contrary, setting $\boldsymbol{\varepsilon}_h = \mathbf{o}$ and $\boldsymbol{\sigma}_h = \mathcal{S}_h \cap \mathcal{D}_h^{\perp}$, we have

$$\begin{aligned} &-(\!(\, \boldsymbol{\sigma}_h\,, \overline{\boldsymbol{\varepsilon}}_h \,)\!)_{\mathcal{H}} - (\!(\, \overline{\boldsymbol{\sigma}_h}\,, \boldsymbol{\varepsilon}_h \,)\!)_{\mathcal{H}} + (\!(\, \mathcal{E}\boldsymbol{\varepsilon}_h\,, \boldsymbol{\varepsilon}_h \,)\!)_{\mathcal{H}} = 0 \quad \forall \overline{\boldsymbol{\varepsilon}}_h \in \mathcal{D}_h, \quad \forall \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h \\ &\iff \quad \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) = 0 \quad \forall \overline{\mathbf{x}_h} \in \mathcal{X}_h\,, \end{aligned}$$

so that $\mathbf{x}_h = \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\} \in \mathrm{Ker}\,\mathbf{A}_h$.

The expressions $\epsilon)_2$ and $\epsilon)_3$ follows from the equalities $(\mathbf{B}')^{-1}\mathcal{L}_h^{\perp} = (\mathbf{B}\mathcal{L}_h)^{\perp}$ and $\mathbf{B}^{-1}\mathcal{S}_h^{\perp} = (\mathbf{B}'\mathcal{S}_h)^{\perp}$.

The discrete problem $\delta)$ can be put in the operator form

$$\zeta) \qquad \begin{cases} \mathbf{A}_h\mathbf{x}_h + \mathbf{J}'_h\mathbf{u}_h = \boldsymbol{\Delta}_h\,, \\ \mathbf{J}_h\mathbf{x}_h \qquad\quad\;\, = \ell_h\,, \end{cases}$$

and can be split into a sequence of two problems. To this end let us note that the equation $\zeta)_2$ defines an affine constraint on the trial variable $\mathbf{x}_h \in \mathcal{X}_h$ which, setting $\mathbf{x}_{\ell_h} = \{\mathbf{o}, \boldsymbol{\sigma}_{\ell_h}\}$, can be written as:

$$\mathbf{x}_h \in \mathbf{x}_{\ell_h} + \mathrm{Ker}\,\mathbf{J}_h \iff \begin{cases} \mathbf{x}_h = \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\} \in \mathcal{D}_h \times \mathcal{S}_h\,, \\ \boldsymbol{\varepsilon}_h \in \mathcal{D}_h\,, \\ \boldsymbol{\sigma}_h \in \boldsymbol{\sigma}_{\ell_h} + \Sigma_h\,, \end{cases}$$

where the affine constraint $\boldsymbol{\sigma}_h \in \boldsymbol{\sigma}_{\ell_h} + \Sigma_h$ defines the linear variety of discrete stress fields equilibrated with the load $\ell_h \in \mathcal{L}'_h$.

We then have:

- The *reduced* problem in the product space $\operatorname{Ker} \mathbf{J}_h \times \operatorname{Ker} \mathbf{J}_h$ consists in searching $\mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h$ such that

$$\varrho) \qquad \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) = (\!(\boldsymbol{\Delta}_h, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} \quad \forall \overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h\,.$$

- The *displacement recovery* problem consists in finding $\mathbf{u}_h \in \mathcal{L}_h$ such that

$$\omega) \qquad \mathbf{j}\,(\mathbf{u}_h, \overline{\mathbf{x}_h}) = -\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) + (\!(\boldsymbol{\Delta}_h, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} \quad \forall \overline{\mathbf{x}_h} \in \mathcal{X}_h\,,$$

  where $\mathbf{x}_h = \{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \operatorname{Ker} \mathbf{J}_h = \mathcal{D}_h \times \Sigma_h$ is a solution of the reduced problem $\varrho)$.

The explicit expression of the reduced problem is the follow: find $\{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \mathcal{D}_h \times \Sigma_h$ such that

$$\begin{cases} (\!(\, \mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h\,,\, \overline{\boldsymbol{\varepsilon}}_h\,)\!)_{\mathcal{H}} = (\!(\, \mathcal{E}\boldsymbol{\delta}\,,\, \overline{\boldsymbol{\varepsilon}}_h\,)\!)_{\mathcal{H}} & \forall \overline{\boldsymbol{\varepsilon}}_h \in \mathcal{D}_h\,, \\ (\!(\, \overline{\boldsymbol{\sigma}_h}\,,\, \boldsymbol{\varepsilon}_h\,)\!)_{\mathcal{H}} = 0 & \forall \overline{\boldsymbol{\sigma}_h} \in \Sigma_h\,. \end{cases}$$

The discussion of the discrete problem is conveniently carried out by introducing the discrete reduced symmetric operator $\mathbf{A}_{oh} \in BL(\operatorname{Ker} \mathbf{J}_h, (\operatorname{Ker} \mathbf{J}_h)')$ defined by

$$(\!(\mathbf{A}_{oh}\,\mathbf{x}_h, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} = \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) \quad \forall \mathbf{x}_h, \overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h\,,$$

and the discrete reduced force $\mathbf{f}_{oh} \in (\operatorname{Ker} \mathbf{J}_h)'$ defined by

$$(\!(\mathbf{f}_{oh}, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} = (\!(\mathbf{f}, \overline{\mathbf{x}_h})\!)_{\mathcal{X}} \quad \forall \overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h\,,$$

the reduced problem $\varrho)$ can be rewritten as

$$\mathbf{A}_{oh}\mathbf{x}_h = \mathbf{f}_{oh}\,, \quad \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h\,.$$

By virtue of the isometric isomorphism between the HILBERT spaces $\operatorname{Ker} \mathbf{J}'_h$ and $\mathcal{X}'/(\operatorname{Ker} \mathbf{J}_h)^{\perp}$ we have that

$$\begin{aligned} &\mathbf{A}_{oh}\,\mathbf{x}_h = \mathbf{A}_h\,\mathbf{x}_h + (\operatorname{Ker} \mathbf{J}_h)^{\perp} \quad \forall \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h\,, \\ &\mathbf{f}_{oh} = \mathbf{f}_h + (\operatorname{Ker} \mathbf{J}_h)^{\perp}\,, \end{aligned}$$

and explicitly,

$$\mathbf{A}_{oh}\mathbf{x}_h = \begin{bmatrix} \mathcal{E} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_h \\ \boldsymbol{\sigma}_h \end{bmatrix} + \begin{bmatrix} \mathcal{D}_h^{\perp} \\ \Sigma_h^{\perp} \end{bmatrix} \qquad \text{with} \quad \boldsymbol{\varepsilon}_h \in \mathcal{D}_h \,,\; \boldsymbol{\sigma}_h \in \Sigma_h \,.$$

The kernel of the discrete reduced operator $\mathbf{A}_{oh}$ is given by $\operatorname{Ker}\mathbf{A}_{oh} = \{\mathbf{o}\}_{\mathcal{H}} \times (\Sigma_h \cap \mathcal{D}_h^{\perp})$.

The displacement recovery problem $\omega)$ can be rewritten in operator form as

$$\upsilon) \qquad \mathbf{J}_h'\mathbf{u}_h = -\mathbf{A}_h\mathbf{x}_h + \boldsymbol{\Delta}_h \,, \quad \mathbf{u}_h \in \mathcal{L}_h.$$

Since $\mathbf{x}_h = \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\}$ is a solution of the reduced problem $\varrho)$, the displacement recovery problem can be explicitly written as

$$(\!(\, \overline{\boldsymbol{\sigma}_h}\,, \mathbf{B}\mathbf{u}_h \,)\!)_{\mathcal{H}} = (\!(\, \overline{\boldsymbol{\sigma}_h}\,, \boldsymbol{\varepsilon}_h \,)\!)_{\mathcal{H}} \quad \forall\, \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$$

which is equivalent to

$$\mathbf{B}\mathbf{u}_h \in \boldsymbol{\varepsilon}_h + \mathcal{S}_h^{\perp} \,, \quad \mathbf{u}_h \in \mathcal{L}_h \,.$$

5.1.3 Well-posedness conditions for the discrete problem

Let us now provide, in operator form, the well-posedness result concerning the discrete mixed problem $\zeta)$.

Proposition 5.1.3 (Well-posedness of the discrete mixed problem) *The discrete problem $\zeta)$ admits a unique solution for any data ℓ_h and $\boldsymbol{\Delta}_h$ if and only if*

$$\iota) \qquad \begin{cases} \left.\begin{aligned} & Ker\,\mathbf{B} \cap \mathcal{L}_h = \{\,\boldsymbol{o}\,\}_{\mathcal{L}} \\ & \mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}} \end{aligned}\right\} & \iff \quad Ker\,\mathbf{J}_h' = \{\,\boldsymbol{o}\,\}_{\mathcal{L}} \,, \\ \Sigma_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}} & \iff \quad Ker\,\mathbf{A}_{oh} = \{\,\boldsymbol{o}\,\}_{\mathcal{X}} \,. \end{cases}$$

Proof. We begin by showing that the reduced problem $\varrho)$ admits a unique solution for any data. This property is equivalent to require that the discrete reduced operator $\mathbf{A}_{oh} \in BL\,(\operatorname{Ker}\mathbf{J}_h, (\operatorname{Ker}\mathbf{J}_h)')$ has a trivial kernel.

Recalling that $\operatorname{Ker}\mathbf{A}_{oh} = \{\mathbf{o}\}_{\mathcal{H}} \times (\Sigma_h \cap \mathcal{D}_h^{\perp})$, the triviality of $\operatorname{Ker}\mathbf{A}_{oh}$ is equivalent to require $\Sigma_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}$.

The displacement recovery problem $\upsilon)$ admits a unique solution if and only if $\operatorname{Ker} \mathbf{J}'_h = \{ \boldsymbol{o} \}_{\mathcal{L}}$ which, by virtue of $\epsilon)_3$, is equivalent to require that $\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{ \boldsymbol{o} \}_{\mathcal{H}}$ and $\operatorname{Ker} \mathbf{B} \cap \mathcal{L}_h = \{ \boldsymbol{o} \}_{\mathcal{L}}$.

- Condition $\zeta)_1$ requires that there are no conforming rigid discrete displacements and that the compatible strains due to discrete conforming displacements are controlled by the discrete stresses.
- Condition $\zeta)_2$ means that the discrete self-equilibrated stresses are controlled by the discrete strains.

5.1.4 Uniform well-posedness

To get asymptotic estimates of the mean square error of approximate solutions we require that the family of discrete problems $\zeta)$ generated by the finite dimensional interpolating spaces $\mathcal{L}_h \subset \mathcal{L}$, $\mathcal{D}_h \subset \mathcal{H}$ and $\mathcal{S}_h \subset \mathcal{H}$ be $h-$uniformly well-posed with respect to the parameter h.

This property consists in the requirement that the closedness of the ranges $\operatorname{Im} \mathbf{A}_{oh} \subset \mathcal{X}'/(\operatorname{Ker} \mathbf{J}_h)^{\perp}$ and $\operatorname{Im} \mathbf{J}'_h \subset \mathcal{X}'/\mathcal{X}_h^{\perp}$ be uniform with respect to the parameter h.

Being $\operatorname{Ker} \mathbf{A}_{oh} = \{ \boldsymbol{o} \}_{\mathcal{X}}$ and $\operatorname{Ker} \mathbf{J}'_h = \{ \boldsymbol{o} \}_{\mathcal{L}}$, the closedness properties are equivalent to the inequalities

$$\begin{aligned} &\| \mathbf{A}_{oh}\mathbf{x}_h \|_{(\operatorname{Ker} \mathbf{J}_h)'} \geq c_{oh} \| \mathbf{x}_h \|_{\mathcal{X}} && \forall \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h , \\ &\| \mathbf{J}'_h \mathbf{u}_h \|_{\mathcal{X}'_h} \geq c_{\mathbf{j}h} \| \mathbf{u}_h \|_{\mathcal{L}} && \forall \mathbf{u}_h \in \mathcal{L}_h , \end{aligned}$$

where the norms above in the dual spaces are given by

$$\begin{aligned} \| \mathbf{A}_{oh}\mathbf{x}_h \|_{(\operatorname{Ker} \mathbf{J}_h)'} &= \sup_{\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h})}{\| \overline{\mathbf{x}_h} \|_{\mathcal{X}}} , \quad && \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h , \\ \| \mathbf{J}'_h \mathbf{u}_h \|_{\mathcal{X}'_h} &= \sup_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \frac{\mathbf{j}\,(\mathbf{u}_h, \overline{\mathbf{x}_h})}{\| \overline{\mathbf{x}_h} \|_{\mathcal{X}}} , && \mathbf{u}_h \in \mathcal{L}_h . \end{aligned}$$

The families of subspaces $\{ \operatorname{Im} \mathbf{A}_{oh} \}$ and $\{ \operatorname{Im} \mathbf{J}'_h \}$ are said to be $h-$uniformly closed if the corresponding families of positive constants $\{c_{\mathbf{j}h}\}$ and $\{c_{oh}\}$ admit positive lower bounds, respectively $c_{\mathbf{j}}$ and c_o, which are independent of the mesh parameter h, so that

$$\eta) \quad \begin{aligned} &\inf_{\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h} \ \sup_{\mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h})}{\|\,\mathbf{x}_h\,\|_{\mathcal{X}}\,\|\,\overline{\mathbf{x}_h}\,\|_{\mathcal{X}}} \geq c_o > 0\,, \\ &\inf_{\mathbf{u}_h \in \mathcal{L}_h} \ \sup_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \frac{\mathbf{j}\,(\mathbf{u}_h, \overline{\mathbf{x}_h})}{\|\,\mathbf{u}_h\,\|_{\mathcal{L}}\,\|\,\overline{\mathbf{x}_h}\,\|_{\mathcal{X}}} \geq c_{\mathbf{j}} > 0\,. \end{aligned}$$

These inequalities are referred to as *discrete* inf-sup *conditions* [57].

Being $\operatorname{Im} \mathbf{J}'_h = \{\mathcal{D}^{\perp}_h, \mathbf{B}\mathcal{L}_h + \mathcal{S}^{\perp}_h\}$, the inequality $\eta)_2$ amounts to require the $h-$uniform closedness in $\mathcal{H}$ of the family of subspace $\{\mathbf{B}\mathcal{L}_h + \mathcal{S}^{\perp}_h\}$.

By virtue of the proposition 5.4.1 in the Appendix to this chapter, the $h-$uniform closedness may be expressed by the inequality

$$\|\,\mathbf{\Pi}_{\mathcal{S}_h}(\mathbf{B}\mathbf{u}_h)\,\|_{\mathcal{H}} \geq \gamma\,\|\,\mathbf{B}\mathbf{u}_h\,\|_{\mathcal{H}} \quad \forall\,\mathbf{u}_h \in \mathcal{L}_h\,,\ \gamma > 0\,,$$

where $\mathbf{\Pi}_{\mathcal{S}_h}$ denotes the orthogonal projector on the subspace $\mathcal{S}_h$.

In geometrical terms the $h-$uniform closedness above consists in assessing that the subspaces $\mathbf{B}\mathcal{L}_h$ and $\mathcal{S}_h$ must not tend to become orthogonal one another as h goes to zero.

The inequality $\eta)_1$ will be dealt with in the next proposition and the related proof is reported in the Appendix to this chapter.

Proposition 5.1.4 (Uniform closedness) *Let* $\mathit{Ker}\,\mathbf{A}_{oh} = \{\,\boldsymbol{o}\,\}_{\mathcal{X}}$ *(the uniqueness condition) be fulfilled. The* $h-$*uniform closedness of the family of linear subspaces* $\mathit{Im}\,\mathbf{A}_{oh} \subset \mathcal{X}'/(\,\mathit{Ker}\,\mathbf{J}_h)^{\perp}$ *is then equivalent to the inequality*

$$\varphi) \qquad \|\,\mathbf{\Pi}_{\mathcal{D}_h}\boldsymbol{\sigma}_h\,\|_{\mathcal{H}} \geq \beta\,\|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}} \quad \forall\,\boldsymbol{\sigma}_h \in \Sigma_h\,,\ \beta > 0\,.$$

Remark 5.1.1 *If the uniqueness condition* $\iota)_2$ *is fulfilled, the inequality* $\varphi)$ *is equivalent to the* $h-$*uniform closedness of the sum* $\Sigma_h + \mathcal{D}^{\perp}_h$ *as a consequence of the relation of proposition* $\Sigma_{24})$ *proved in the Appendix to this chapter.*

In geometrical terms the $h-$*uniform closedness of* $\Sigma_h + \mathcal{D}^{\perp}_h$ *consists in assessing that the subspaces* Σ_h *and* $\mathcal{D}_h$ *must not tend to become orthogonal one another as h goes to zero.*

5.1.5 Error bounds

Let us now provide the error estimates following the treatment developed in [57].

Mean square error of $\{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\}$-solution

If $\mathbf{x} = \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}\} \in \mathcal{X}$ is the solution in terms of strains and stresses of the mixed problem $\alpha)$ and $\mathbf{x}_h = \{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\} \in \operatorname{Ker} \mathbf{J}_h$ is the corresponding solution of the discrete problem $\delta)$, from the triangle inequality we get

$$\psi) \qquad \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{X}} \leq \| \mathbf{x}_h - \overline{\mathbf{x}_h} \|_{\mathcal{X}} + \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} \quad \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h \,,$$

and from the discrete inf-sup condition $\eta)_1$ rewritten in the form

$$\inf_{\mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h} \; \sup_{\mathbf{x}_h^* \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h, \mathbf{x}_h^*)}{\| \mathbf{x}_h \|_{\mathcal{X}} \, \| \mathbf{x}_h^* \|_{\mathcal{X}}} \geq c_o > 0 \,,$$

we infer, for any $\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h$, the following bound to the first term on the r.h.s. of $\psi)$

$$\begin{aligned} c_o \, \| \mathbf{x}_h - \overline{\mathbf{x}_h} \|_{\mathcal{X}} &\leq \sup_{\mathbf{x}_h^* \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h - \overline{\mathbf{x}_h}, \mathbf{x}_h^*)}{\| \mathbf{x}_h^* \|_{\mathcal{X}}} \\ &= \sup_{\mathbf{x}_h^* \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h - \mathbf{x}, \mathbf{x}_h^*) + \mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h}, \mathbf{x}_h^*)}{\| \mathbf{x}_h^* \|_{\mathcal{X}}} \\ &= \sup_{\mathbf{x}_h^* \in \operatorname{Ker} \mathbf{J}_h} \frac{\mathbf{j}\,(\mathbf{u} - \mathbf{u}_h, \mathbf{x}_h^*) + \mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h}, \mathbf{x}_h^*)}{\| \mathbf{x}_h^* \|_{\mathcal{X}}} \end{aligned}$$

The last equality holds true since the first variational conditions of problems $\alpha)$ and $\delta)$ yield

$$\begin{aligned} &\mathbf{a}\,(\mathbf{x}, \overline{\mathbf{x}}) + \mathbf{j}\,(\mathbf{u}, \overline{\mathbf{x}}) = \mathbf{f}(\overline{\mathbf{x}}) \quad \forall\, \overline{\mathbf{x}} \in \mathcal{X} \,, \\ &\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\mathbf{u}_h, \overline{\mathbf{x}_h}) = \mathbf{f}(\overline{\mathbf{x}_h}) \quad \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h \end{aligned}$$

so that, setting $\overline{\mathbf{x}} = \overline{\mathbf{x}_h}$ and subtracting, we get

$$\xi) \qquad \mathbf{a}\,(\mathbf{x} - \mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\mathbf{u} - \mathbf{u}_h, \overline{\mathbf{x}_h}) = 0 \quad \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h \,.$$

The continuity of the forms $\mathbf{a}$ and $\mathbf{j}$ yields the inequalities

$$\pi) \quad \begin{aligned} &\mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h}, \mathbf{x}_h^*) \leq \| \mathbf{a} \| \, \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} \, \| \mathbf{x}_h^* \|_{\mathcal{X}} && \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h \,, \\ &\mathbf{j}\,(\mathbf{u} - \mathbf{u}_h, \mathbf{x}_h^*) = \mathbf{j}\,(\mathbf{u} - \overline{\mathbf{u}_h}, \mathbf{x}_h^*) \leq \| \mathbf{j} \| \, \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} \, \| \mathbf{x}_h^* \|_{\mathcal{X}} && \forall\, \overline{\mathbf{u}_h} \in \mathcal{L}_h, \end{aligned}$$

since $\mathbf{x}_h^* \in \operatorname{Ker} \mathbf{J}_h$ and hence $\mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{x}_h^*) = 0$ for any $\overline{\mathbf{u}_h} \in \mathcal{L}_h$. From $\psi)$ we then get the first result

$$\mu) \quad \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{X}} \le \left(1 + \frac{\| \mathbf{a} \|}{c_o} \right) \inf_{\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h} \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} + \frac{\| \mathbf{j} \|}{c_o} \inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} .$$

To get rid of the contraint $\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h$ we again follow [57] observing that for any given $\mathbf{y}_h \in \mathcal{X}_h$ we can find at least a solution $\mathbf{z}_h \in \mathcal{X}_h$ of the problem

$$\mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{z}_h) = \mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{x} - \mathbf{y}_h) \quad \forall \overline{\mathbf{u}_h} \in \mathcal{L}_h ,$$

such that

$$c_{\mathbf{j}} \| \mathbf{z}_h \|_{\mathcal{X}} \le \sup_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \frac{\mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{x} - \mathbf{y}_h)}{\| \overline{\mathbf{u}_h} \|_{\mathcal{H}}} \le \| \mathbf{j} \| \, \| \mathbf{x} - \mathbf{y}_h \|_{\mathcal{X}} .$$

Setting $\overline{\mathbf{x}_h} = \mathbf{z}_h + \mathbf{y}_h$ we have that

$$\mathbf{j}(\overline{\mathbf{u}_h}, \overline{\mathbf{x}_h}) = \mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{z}_h + \mathbf{y}_h) = \mathbf{j}(\overline{\mathbf{u}_h}, \mathbf{x}) = 0 \quad \forall \overline{\mathbf{u}_h} \in \mathcal{S}_h$$

and hence that $\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h$. We have thus proved the inequality

$$\| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} = \| \mathbf{x} - \mathbf{y}_h - \mathbf{z}_h \|_{\mathcal{X}} \le \| \mathbf{x} - \mathbf{y}_h \|_{\mathcal{X}} + \| \mathbf{z}_h \|_{\mathcal{X}} \le \left(1 + \frac{\| \mathbf{j} \|}{c_{\mathbf{j}}} \right) \| \mathbf{x} - \mathbf{y}_h \|_{\mathcal{X}} ,$$

from which we infer that

$$\nu) \qquad \inf_{\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h} \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} \le \left(1 + \frac{\| \mathbf{j} \|}{c_{\mathbf{j}}} \right) \inf_{\mathbf{y}_h \in \mathcal{X}_h} \| \mathbf{x} - \mathbf{y}_h \|_{\mathcal{X}} .$$

In conclusion, substituing $\nu)$ into $\mu)$ we get the estimate

$$\kappa) \qquad \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{X}} \le c_1 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} + c_2 \inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} ,$$

with c_1 and c_2 positive constants independent of h.

Mean square error of the $\{\mathbf{u}_h\}$-solution

The asymptotic estimate of the mean square error of the approximate displacement solution starts again with an application of the triangle inequality.

If $\mathbf{u} \in \mathcal{H}$ is the displacement solution of the mixed problem $\alpha)$ and $\mathbf{u}_h \in \mathcal{L}_h$ is the displacement solution of the discrete problem $\delta)$, we have that

$$\theta) \qquad \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{H}} \leq \| \mathbf{u}_h - \overline{\mathbf{u}_h} \|_{\mathcal{H}} + \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{H}} \quad \forall \, \overline{\mathbf{u}_h} \in \mathcal{L}_h \, .$$

The first term on the r.h.s. of $\theta)$ can be estimated by means of the inequality $\eta)_2$.

To this end, recalling the equality $\xi)$, we infer that

$$\mathbf{j}\,(\mathbf{u}_h - \overline{\mathbf{u}_h}, \overline{\mathbf{x}_h}) = \mathbf{a}\,(\mathbf{x} - \mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\mathbf{u} - \overline{\mathbf{u}_h}, \overline{\mathbf{x}_h}) \quad \forall \, \overline{\mathbf{x}_h} \in \mathcal{X}_h \, ,$$

so that the $h-$uniform closedness of $\operatorname{Im} \mathbf{J}'_h$ yields

$$c_{\mathbf{j}} \| \mathbf{u}_h - \overline{\mathbf{u}_h} \|_{\mathcal{L}} \leq \sup_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \frac{\mathbf{a}\,(\mathbf{x} - \mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\mathbf{u} - \overline{\mathbf{u}_h}, \overline{\mathbf{x}_h})}{\| \overline{\mathbf{x}_h} \|_{\mathcal{X}}} \, .$$

By the continuity $\pi)$ of the forms $\mathbf{a}$ and $\mathbf{j}$ we have

$$c_{\mathbf{j}} \| \mathbf{u}_h - \overline{\mathbf{u}_h} \|_{\mathcal{L}} \leq \| \mathbf{a} \| \, \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{L}} + \| \mathbf{j} \| \, \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}}$$

which substituted in $\theta)$ provides the second result

$$\upsilon) \qquad \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{L}} \leq \frac{\| \mathbf{a} \|}{c_{\mathbf{j}}} \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{X}} + \left(1 + \frac{\| \mathbf{j} \|}{c_{\mathbf{j}}} \right) \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} \, .$$

The estimate of the mean square error of the displacement approximate solution depends then on the estimates of the mean square error of the stress and strain approximate solutions. By comparing the results $\kappa)$ and $\upsilon)$ we get the estimate of the mean square error of the displacement approximate solution

$$\sigma) \qquad \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{L}} \leq c_3 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} + c_4 \inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}}$$

with c_3 and c_4 positive constants independent of h.

The inqualities $\kappa)$ and $\sigma)$ lead to the error bound

$$\tau_1) \qquad \| \mathbf{x} - \mathbf{x}_h \|_{\mathcal{X}} + \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{L}} \leq c_5 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \| \mathbf{x} - \overline{\mathbf{x}_h} \|_{\mathcal{X}} + c_6 \inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} \, ,$$

with c_5 and c_6 independent of h. Recalling the definition of the norm in the product space $\mathcal{X}$, the error bound $\tau_1)$ can be rewritten in terms of the three fields $\{ \mathbf{u}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h \}$ as:

$$\tau_2) \qquad \begin{aligned} & \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{L}} + \| \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h \|_{\mathcal{H}} + \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{\mathcal{H}} \leq \\ & \leq c_6 \inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_{\mathcal{L}} + c_5 \inf_{\overline{\boldsymbol{\varepsilon}}_h \in \mathcal{D}_h} \| \boldsymbol{\varepsilon} - \overline{\boldsymbol{\varepsilon}}_h \|_{\mathcal{H}} + c_5 \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \| \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \|_{\mathcal{H}} \, . \end{aligned}$$

5.1.6 Asymptotic rate of convergence

Let us consider a two or three-dimensional elastostatic problem and assume that the bounded domain Ω, the data and the elasticity $\mathcal{E}$ be regular enough to ensure that the displacement the strain and the stress solutions meet the regularity properties $\mathbf{u} \in H^2(\Omega)$, $\boldsymbol{\varepsilon} \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H^1(\Omega)$.

Further we consider isoparametric finite element meshes which enjoy the following two properties.

The displacement shape functions on the reference element K generate the vectorial polynomial subspace $P_1(K)$ whose components are arbitrary polynomials of degree ≤ 1 or the subspace $\mathrm{Q}_1(K)$ whose components are arbitrary polynomials of degree ≤ 1 in each variable.

The strain and stress shape functions generate tensorial subspaces containing the subspace $\mathrm{Q}_0(K) = P_0(K)$ whose components are arbitrary constant tensors.

A standard result of polynomial approximation theory [32], [50], ensures that for regular meshes the following inequalities hold

$$\inf_{\overline{\mathbf{u}} \in \mathcal{L}_h} \| \mathbf{u} - \overline{\mathbf{u}_h} \|_1 \leq c_{\mathbf{u}} h \, | \mathbf{u} |_2 ,$$

$$\inf_{\overline{\boldsymbol{\varepsilon}} \in \mathcal{D}_h} \| \boldsymbol{\varepsilon} - \overline{\boldsymbol{\varepsilon}}_h \|_0 \leq c_{\boldsymbol{\varepsilon}} h \, | \boldsymbol{\varepsilon} |_1 ,$$

$$\inf_{\overline{\boldsymbol{\sigma}} \in \mathcal{S}_h} \| \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \|_0 \leq c_{\boldsymbol{\sigma}} h \, | \boldsymbol{\sigma} |_1 ,$$

where $\| \cdot \|_m$ is the norm in the SOBOLEV space $H^m(\Omega)$ and $| \cdot |_m$ is the corresponding seminorm involving only the derivatives of total order m.

The error bound $\tau_2)$ provides the following linear estimates for the rate of convergence of the approximate solution to the exact one in terms of energy norms

$$\| \mathbf{u} - \mathbf{u}_h \|_1 \leq \alpha_{\mathbf{u}} h \, \left(| \mathbf{u} |_2 + | \varepsilon |_1 + | \boldsymbol{\sigma} |_1 \right) ,$$

$$\| \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h \|_0 \leq \alpha_{\boldsymbol{\varepsilon}} h \, \left(| \mathbf{u} |_2 + | \varepsilon |_1 + | \boldsymbol{\sigma} |_1 \right) ,$$

$$\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_0 \leq \alpha_{\boldsymbol{\sigma}} h \, \left(| \mathbf{u} |_2 + | \varepsilon |_1 + | \boldsymbol{\sigma} |_1 \right) .$$

5.1.7 Applicable sufficient conditions

In order to get the well-posedness of the discrete problem and the estimate $\tau_2)$ of the mean square error we have to impose the conditions

- uniqueness of the $\{\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h\}$-solution expressed by the condition $\operatorname{Ker} \mathbf{A}_{oh} = \{\boldsymbol{o}\}$,
- uniqueness of the $\mathbf{u}_h$-solution expressed by the condition $\operatorname{Ker} \mathbf{J}'_h = \{\boldsymbol{o}\}$,
- $h-$uniform closedness of $\operatorname{Im} \mathbf{A}_{oh}$,
- $h-$uniform closedness of $\operatorname{Im} \mathbf{J}'_h$.

Under the assumption $\operatorname{Ker} \mathbf{B} \cap \mathcal{L}_h = \{\boldsymbol{o}\}$, which means that the discrete structure cannot undergo interpolating conforming rigid displacements, the above four conditions can be rewritten in the form reported in **Table I**.

i)	$\Sigma_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}_{\mathcal{H}}$,
ii)	$\mathbf{B}\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\boldsymbol{o}\}_{\mathcal{H}}$,
iii)	$\Sigma_h + \mathcal{D}_h^{\perp}$ $h-$ uniformly closed in $\mathcal{H}$,
iv)	$\mathbf{B}\mathcal{L}_h + \mathcal{S}_h^{\perp}$ $h-$ uniformly closed in $\mathcal{H}$.

Table I

The conditions $i)-iv)$ involve the subspace $\mathcal{L}_h$ of conforming displacements which, in FEM approximation, depends on the element assembly operations and hence is *a priori* unknown.

This shortcoming can be circumvented by substituting the subspace $\Sigma_h = \mathcal{S}_h \cap (\mathbf{B}\mathcal{L}_h)^{\perp}$, appearing in $i)$ and $iii)$, with the stress subspace $\mathcal{S}_h \supseteq \Sigma_h$ and by considering in $ii)$ and $iv)$ the larger non-conforming displacement space $\mathcal{V}_h \supseteq \mathcal{L}_h$ defined elementwise by $\mathcal{V}_h = \prod_{e=1}^{\mathcal{N}} \mathcal{V}_h^e$. The subspaces $\mathcal{V}_h^e$ denote the local displacement subspace generated by means of the isoparametric map acting on the displacement shape functions over the reference element Ω^e.

We thus get the sufficient local conditions reported in **Table II**.

i)	$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}\,,$
ii)	$\mathbf{B}\mathcal{V}_h \cap \mathcal{S}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}\,,$
iii)	$\mathcal{S}_h + \mathcal{D}_h^{\perp}$ $h-$uniformly closed in $\mathcal{H}\,,$
iv)	$\mathbf{B}\mathcal{V}_h + \mathcal{S}_h^{\perp}$ $h-$uniformly closed in $\mathcal{H}\,.$

Table II

In a finite element analysis, it is compelling to verify the sufficient conditions reported in Table II in terms of quantities pertaining to the reference element K. We shall denote by $\boldsymbol{\chi}_e : K \mapsto \Omega^e$ the isoparametric map from the reference element K to the generic element Ω^e of the finite element mesh and we set $\boldsymbol{\chi} = \{\boldsymbol{\chi}_e,\ e = 1, \ldots, \mathcal{N}\}$.

The well-posedness condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$ may be fulfilled in the applications by choosing $\mathcal{S}_h$ and $\mathcal{D}_h$ so that $\mathcal{S}_h \subseteq \mathcal{D}_h$. Since $\mathcal{S}_h = \boldsymbol{\chi}(\mathcal{S}_K)$ and $\mathcal{D}_h = \boldsymbol{\chi}(\mathcal{D}_K)$, to get the inclusion $\mathcal{S}_h \subseteq \mathcal{D}_h$ it is sufficient to choose $\mathcal{S}_K \subseteq \mathcal{D}_K$. Being $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$, the $h-$uniform closedness of $\mathcal{S}_h + \mathcal{D}_h^{\perp}$ in $\mathcal{H}$ is equivalent to the inequality (see Appendix):

$$\Gamma) \qquad \| \, \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_h \, \|_{\mathcal{H}} \geq \beta \, \| \, \boldsymbol{\sigma}_h \, \|_{\mathcal{H}} \quad \forall \, \boldsymbol{\sigma}_h \in \mathcal{S}_h \,,$$

with $\beta > 0$ independent of h. It is then apparent that the inclusion $\mathcal{S}_h \subseteq \mathcal{D}_h$ ensures that $\| \, \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_h \, \|_{\mathcal{H}} = \| \, \boldsymbol{\sigma}_h \, \|_{\mathcal{H}}$ so that $\Gamma)$ is met with $\beta = 1$.

The conditions *ii)* and *iv)* of Table II deserve a carefull discussion since they involve the kinematic operator $\mathbf{B}$. To this end the kinematic operator acting on the displacement fields in the reference element K will be denoted by $\mathbf{B}_K^e$ and is defined by $\mathbf{B}_K^e \mathbf{u}_K(\boldsymbol{\xi}) = \mathbf{B}\mathbf{u}_h^e(\mathbf{x})$ for any $\mathbf{x} \in \Omega_e$ where $\mathbf{u}_h^e$ is the restriction of $\mathbf{u}_h$ to the element Ω^e and $\boldsymbol{\xi} = \boldsymbol{\chi}_e^{-1}(\mathbf{x})$ is the isoparametric coordinate in the reference element.

Further we set $\mathbf{B}_K = \{\mathbf{B}_K^e, e = 1, \ldots, \mathcal{N}\}$ and we denote by $\boldsymbol{\eta}_K = \boldsymbol{\chi}^{-1}(\mathbf{u}_h)$ the collection of the displacement fields obtained by shifting back to K the restrictions of $\mathbf{u}_h$ to each element of the mesh. We then have $\mathbf{B}\mathbf{u}_h = \mathbf{B}\boldsymbol{\chi}^{-1}(\mathbf{u}_h) = \mathbf{B}_K \boldsymbol{\eta}_K$. Recalling the closedness inequalities reported in the Appendix, the conditions *ii*) and *iv*) of Table II, rewritten in the reference element K, become

$$\Delta)\quad \begin{array}{l} \mathbf{B}_K \mathcal{V}_K \cap \mathcal{S}_K^{\perp} = \{\, \boldsymbol{o}\,\}_{\mathcal{H}}, \\ \|\, \mathbf{\Pi}_{\mathcal{S}_K}(\mathbf{B}_K^e \mathbf{u}_K)\,\|_{\mathcal{H}} \geq \gamma \,\|\, \mathbf{B}_K^e \mathbf{u}_K \,\|_{\mathcal{H}} \quad \forall\, \mathbf{u}_K \in \mathcal{V}_K\,, \quad \gamma > 0\,,\; e = 1, \ldots, \mathcal{N}. \end{array}$$

It is apparent that the kinematic operator $\mathbf{B}_K$ on K depends on the map $\boldsymbol{\chi}$ and hence the conditions $\Delta)$ cannot be checked in terms of quantities pertaining only to the reference element K.

These difficulties have not been fully realized in the literature since in previous treatments, see e.g. [71], [55] for the EAS method, the well-posedness conditions are simply imposed in the reference element K. In fact for a general isoparametric map, the condition $\Delta)_1$ can be only checked in the form

$$\Delta_{bis}) \qquad\qquad \mathbf{B}\mathcal{V}_K \cap \mathcal{S}_K^{\perp} = \{\, \mathbf{o}\,\}_{\mathcal{H}},$$

and confide that the element distorsion does not make it fail on the real elements. The inequality $\Delta)_2$ is explicity verified in the form

$$\Theta) \qquad\qquad \|\, \mathbf{\Pi}_{\mathcal{S}_K}(\mathbf{B}\mathbf{u}_K)\,\|_{\mathcal{H}} \geq \gamma \,\|\, \mathbf{B}\mathbf{u}_K \,\|_{\mathcal{H}} \quad \forall\, \mathbf{u}_K \in \mathcal{V}_K\,, \quad \gamma > 0$$

and a numerical value for the constant γ is evaluated for a given set of shape functions. We underline that the inquality $\Theta)$ must hold for any choice of the shape functions since the subspace $\mathbf{B}\mathcal{V}_K + \mathcal{S}_K$ is finite dimensional and hence closed in $\mathcal{H}$.

5.2 Strain gap method

In order to discuss the convergence properties of the SGM it is convenient to rephrase the related three-field problem as a two field problem.

To this end we introduce the dual product Hilbert spaces

$$\mathcal{X} = \mathcal{L} \times \mathcal{H} \quad \textit{and} \quad \mathcal{X}' = \mathcal{L}' \times \mathcal{H}$$

with the standard inner product between $\mathbf{x} = \{\mathbf{u}\,, \mathbf{g}\} \in \mathcal{X}$ and $\mathbf{x}' = \{\ell\,, \boldsymbol{\sigma}\} \in \mathcal{X}'$ given by

$$\langle\, \mathbf{x}'\,,\, \mathbf{x}\,\rangle = \langle\, \ell\,,\, \mathbf{u}\,\rangle + (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{g}\,)\!)_{\mathcal{H}}.$$

The norm in the product space $\mathcal{X} = \mathcal{L} \times \mathcal{H}$ is

$$\|\, \mathbf{x}\,\|_{\mathcal{X}} = \|\, \{\mathbf{u}\,, \mathbf{g}\}\,\|_{\mathcal{X}} = \|\, \mathbf{u}\,\|_{\mathcal{L}} + \|\, \mathbf{g}\,\|_{\mathcal{H}}.$$

Further let us consider the continuous bilinear symmetric positive form $\mathbf{a} \in \mathrm{Bil}\{\,\mathcal{X} \times \mathcal{X}\,\}$ and the continuous bilinear form $\mathbf{j} \in \mathrm{Bil}\{\,\mathcal{H} \times \mathcal{X}\,\}$ defined by

$$\mathbf{a}\,(\mathbf{x},\mathbf{x}) = (\!(\, \mathcal{E}(\mathbf{Bu}-\mathbf{g})\,,\, \mathbf{Bu}-\mathbf{g}\,)\!)\,, \qquad \mathbf{j}\,(\boldsymbol{\sigma},\mathbf{x}) = (\!(\, \boldsymbol{\sigma}\,,\,\mathbf{g}\,)\!)\,,$$

where $\mathrm{Bil}\{\,\diamond \times \bullet\,\}$ denotes the space of continuous bilinear forms on $\diamond \times \bullet$. To the loal $\ell \in \mathcal{L}'$ there corresponds a continuous linear form $\langle\, \mathbf{f}\,,\,\mathbf{x}\,\rangle = \langle\, \ell\,,\,\mathbf{u}\,\rangle$.

The three-field functional $\widetilde{W}(\mathbf{u},\mathbf{g},\boldsymbol{\sigma})$ can then be rewritten as a two-field functional

$$\Psi(\mathbf{x},\boldsymbol{\sigma}) = \frac{1}{2}\,\mathbf{a}\,(\mathbf{x},\mathbf{x}) + \mathbf{j}\,(\boldsymbol{\sigma},\mathbf{x}) - \langle\, \mathbf{f}\,,\,\mathbf{x}\,\rangle \quad \mathbf{x} \in \mathcal{X}\,,\ \boldsymbol{\sigma} \in \mathcal{H}\,.$$

The stationarity of $\Psi(\mathbf{x},\boldsymbol{\sigma})$ is expressed by the associated Euler conditions

$$\Sigma_1) \qquad \begin{cases} \mathbf{a}\,(\mathbf{x},\overline{\mathbf{x}}) + \mathbf{j}\,(\boldsymbol{\sigma},\overline{\mathbf{x}}) = \langle\, \mathbf{f}\,,\,\overline{\mathbf{x}}\,\rangle & \forall\, \overline{\mathbf{x}} \in \mathcal{X}\,, \\ \mathbf{j}\,(\overline{\boldsymbol{\sigma}},\mathbf{x}) \qquad\qquad\quad = 0 & \forall\, \overline{\boldsymbol{\sigma}} \in \mathcal{H}\,. \end{cases}$$

The unknown fields $\mathbf{x} = \{\mathbf{u}\,,\mathbf{g}\} \in \mathcal{L} \times \mathcal{H}$ and $\boldsymbol{\sigma} \in \mathcal{H}$ are the trial variables among which we look for a solution. The fields $\overline{\mathbf{x}} = \{\overline{\mathbf{u}}\,,\overline{\mathbf{g}}\} \in \mathcal{L} \times \mathcal{H}$ and $\overline{\boldsymbol{\sigma}} \in \mathcal{H}$ are test variables of the variational problem.

Recalling that the bilinear form $\mathbf{a} \in \mathrm{Bil}\{\,\mathcal{X} \times \mathcal{X}\,\}$ is symmetric we may set

$$\begin{aligned} &\mathbf{a}\,(\mathbf{x},\mathbf{y}) = \langle\, \mathbf{x}\,,\,\mathbf{A}\,\mathbf{y}\,\rangle = \langle\, \mathbf{A}\,\mathbf{x}\,,\,\mathbf{y}\,\rangle \qquad \forall\, \mathbf{x},\mathbf{y} \in \mathcal{X}\,, \\ &\mathbf{j}\,(\boldsymbol{\sigma},\mathbf{x}) = (\!(\, \boldsymbol{\sigma}\,,\,\mathbf{J}\,\mathbf{x}\,)\!)_{\mathcal{H}} = \langle\, \mathbf{J}'\,\boldsymbol{\sigma}\,,\,\mathbf{x}\,\rangle \quad \forall\, \mathbf{x} \in \mathcal{X}\,, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{H}\,, \end{aligned}$$

with $\mathbf{A} \in BL\,(\mathcal{X},\mathcal{X}')\,,\ \mathbf{J} \in BL\,(\mathcal{X},\mathcal{H})\,,\ \mathbf{J}' \in BL\,(\mathcal{H},\mathcal{X}')$ defined by

$$\begin{aligned} &\mathbf{A}\,\mathbf{x} = \mathbf{A}\,\{\mathbf{u}\,,\mathbf{g}\} = \{\mathbf{B}'\mathcal{E}(\mathbf{Bu}-\mathbf{g})\,, -\mathcal{E}(\mathbf{Bu}-\mathbf{g})\} \\ &\mathbf{J}\,\mathbf{x}\ = \mathbf{J}\,\{\mathbf{u}\,,\mathbf{g}\} = \mathbf{g} \\ &\mathbf{J}'\,\boldsymbol{\sigma} = \{\mathbf{o}\,,\boldsymbol{\sigma}\}. \end{aligned}$$

The above mixed problem $\Sigma_1)$ can then be put in the matrix form

$$\Pi) \qquad \begin{cases} \mathbf{A}\mathbf{x} + \mathbf{J}'\boldsymbol{\sigma} = \mathbf{f}\,, \\ \mathbf{J}\mathbf{x} \qquad\quad = \mathbf{o}\,. \end{cases}$$

By the $\mathcal{H}$-ellipticity of the elasticity operator $\mathcal{E}$ we have

$$\Xi) \qquad \mathrm{Ker}\,\mathbf{A} = \big\{\{\mathbf{u}\,,\mathbf{g}\} \in \mathcal{L} \times \mathcal{H} \mid \mathbf{Bu} = \mathbf{g}\big\}\,,$$

and, by the definitions above of $\mathbf{J}$ and $\mathbf{J}'$, we get

$$\Phi) \qquad \mathrm{Ker}\,\mathbf{J} = \big\{\{\mathbf{u}\,,\mathbf{o}\} \in \mathcal{L}\times\mathcal{H}\big\} = \mathcal{L}\times\{\,\boldsymbol{o}\,\}_{\mathcal{H}}\,, \qquad \mathrm{Ker}\,\mathbf{J}' = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}\,.$$

Since the condition $\Pi)_2$ provides $\mathbf{x} \in \mathrm{Ker}\,\mathbf{J}$, which is equivalent to set $\mathbf{g} = \mathbf{o}$, the mixed problem $\Pi)$ turns out to be equivalent to the standard displacement problem. Nonetheless we shall discuss the mixed problem in its generality since this discussion provides the guideline for the subsequent analysis of the discrete problem and of the relevant error bounds.

5.2.1 Reduced problem, stress recovery and well-posedness

The discussion of the problem $\Pi)$ can be conveniently carried out by considering the corresponding reduced problem in which the state variable $\mathbf{x} \in \mathcal{X}$ and the test variable $\overline{\mathbf{x}} \in \mathcal{X}$ belong both to the constraint subspace $\mathrm{Ker}\,\mathbf{J} \subset \mathcal{X}$.

The reduced problem obtained from $\Sigma)$ is thus given by

$$\Sigma_2) \qquad \mathbf{a}\,(\mathbf{x},\overline{\mathbf{x}}) = \langle\,\mathbf{f}\,,\,\overline{\mathbf{x}}\,\rangle \quad \forall\,\overline{\mathbf{x}} \in \mathrm{Ker}\,\mathbf{J}\,, \quad \mathbf{x} \in \mathrm{Ker}\,\mathbf{J}\,.$$

Any solution of the problem $\Sigma_1)$ is also a solution of the problem $\Sigma_2)$ but the converse holds if and only if the bilinear form $\mathbf{j} \in \mathrm{Bil}\{\,\mathcal{H}\times\mathcal{X}\,\}$ is closed on $\mathcal{H}\times\mathcal{X}$, see [29], [57], [91].

By introducing the symmetric reduced operator $\mathbf{A}_o \in BL\,(\,\mathrm{Ker}\,\mathbf{J}, (\,\mathrm{Ker}\,\mathbf{J})')$ defined by

$$\langle\,\mathbf{A}_o\,\mathbf{x}\,,\,\mathbf{y}\,\rangle = \langle\,\mathbf{x}\,,\,\mathbf{A}_o\,\mathbf{y}\,\rangle = \mathbf{a}\,(\mathbf{x},\mathbf{y}) \quad \forall\,\mathbf{x},\mathbf{y} \in \mathrm{Ker}\,\mathbf{J}\,,$$

and the reduced force $\mathbf{f}_o \in (\,\mathrm{Ker}\,\mathbf{J})'$ defined by

$$\langle\,\mathbf{f}_o\,,\,\mathbf{x}\,\rangle = \langle\,\mathbf{f}\,,\,\mathbf{x}\,\rangle \quad \forall\,\mathbf{x} \in \mathrm{Ker}\,\mathbf{J}\,,$$

the reduced problem $\Sigma_2)$ can be written in the standard form

$$\Sigma_3) \qquad \mathbf{A}_o\,\mathbf{x} = \mathbf{f}_o\,.$$

Remark 5.2.1 *We recall that, for any linear subspace $\mathcal{K} \subseteq \mathcal{X}$, there is an isometric isomorphism between the* Hilbert *space $\mathcal{K}'$, dual to $\mathcal{K}$, and the quotient* Hilbert *space $\mathcal{X}'/\mathcal{K}^{\perp}$ [21], [18]. Then $(\,Ker\,\mathbf{J})'$ is isometrically isomorphic to $\mathcal{X}'/(\,Ker\,\mathbf{J})^{\perp}$ and we may set $\mathbf{A}_o\,\mathbf{x} = \mathbf{A}\,\mathbf{x} + (\,Ker\,\mathbf{J})^{\perp}$ for any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{f}_o = \mathbf{f} + (\,Ker\,\mathbf{J})^{\perp}$.*

Let us now prove a preliminarily lemma.

Lemma 5.2.1 *An explicit expression for the kernel of the reduced operator* $\mathbf{A}_o$ *is given by*

$$\Sigma_4) \qquad Ker\,\mathbf{A}_o = Ker\,\mathbf{A} \cap Ker\,\mathbf{J} = (Ker\,\mathbf{B} \cap \mathcal{L}) \times \{\boldsymbol{o}\}_{\mathcal{H}}$$

Proof. Let us prove the former equality. In fact, being

$$\operatorname{Ker}\mathbf{A}_o = \left\{\mathbf{x} \in \operatorname{Ker}\mathbf{J} \mid \langle\, \mathbf{A}_o\mathbf{x}\,,\, \overline{\mathbf{x}}\,\rangle = \mathbf{a}(\mathbf{x},\overline{\mathbf{x}}) = 0 \quad \forall \overline{\mathbf{x}} \in \operatorname{Ker}\mathbf{J}\right\},$$

we have

$$\mathbf{a}(\mathbf{x},\mathbf{x}) = 0 \quad \forall\, \mathbf{x} \in \operatorname{Ker}\mathbf{A}_o.$$

Since zero is the minimum value of the symmetric positive form $\mathbf{a} \in \operatorname{Bil}\{\mathcal{X} \times \mathcal{X}\}$, by imposing the vanishing of the directional derivatives we get $\mathbf{a}(\mathbf{x},\overline{\mathbf{x}}) = 0$ for any $\overline{\mathbf{x}} \in \mathcal{X}$. It follows that $\operatorname{Ker}\mathbf{A}_o \subseteq \operatorname{Ker}\mathbf{A} \cap \operatorname{Ker}\mathbf{J}$. The converse inclusion $\operatorname{Ker}\mathbf{A} \cap \operatorname{Ker}\mathbf{J} \subseteq \operatorname{Ker}\mathbf{A}_o$ is trivially verified.

The latter equality follows from the expressions $\Xi)$ and $\Phi)_1$ of the kernel of $\mathbf{A}$ and $\mathbf{J}$. The main existence and uniqueness result of the reduced problem is reported in the next statement.

Proposition 5.2.1 (Existence and uniqueness of the reduced problem) *The reduced problem* $\Sigma_3)$ *admits a unique solution for any data* $\mathbf{f} \in \mathcal{X}'$ *if and only if the bilinear form* $\mathbf{a} \in Bil\{\mathcal{X} \times \mathcal{X}\}$ *is closed on* $Ker\,\mathbf{J} \times Ker\,\mathbf{J} \subseteq \mathcal{X} \times \mathcal{X}$ *and the structure cannot undergo conforming rigid displacements.*

Proof. Since no conforming rigid displacements are allowed, we have $\operatorname{Ker}\mathbf{B} \cap \mathcal{L} = \{\boldsymbol{o}\}_{\mathcal{L}}$ so that the equalities $\Sigma_4)$ yields $\operatorname{Ker}\mathbf{A}_o = \{\boldsymbol{o}\}_{\mathcal{L}} \times \{\boldsymbol{o}\}_{\mathcal{H}}$. Moreover the closedness of the bilinear form $\mathbf{a}$ is equivalent to require that the range of the symmetric reduced operator $\mathbf{A}_o \in BL(\operatorname{Ker}\mathbf{J}, (\operatorname{Ker}\mathbf{J})')$ is closed in $(\operatorname{Ker}\mathbf{J})'$. Existence of the solution of the problem $\Sigma_3)$ requires that $\mathbf{f}_o \in \operatorname{Im}\mathbf{A}_o = (\operatorname{Ker}\mathbf{A}_o)^{\perp} = \mathcal{L}' \times \mathcal{H}$ and hence the result is obtained.

In the sequel we shall assume that the structure cannot undergo conforming rigid displacements and that the bilinear form $\mathbf{a}$ is closed on $\operatorname{Ker}\mathbf{J} \times \operatorname{Ker}\mathbf{J}$, i.e. [39]:

$$\|\,\mathbf{A}_o\mathbf{x}\,\|_{(\operatorname{Ker}\mathbf{J})'} \geq c_o\,\|\,\mathbf{x}\,\|_{\mathcal{X}/(\operatorname{Ker}\mathbf{A} \cap \operatorname{Ker}\mathbf{J})} = c_o\,\|\,\mathbf{x}\,\|_{\mathcal{X}} \qquad \forall \mathbf{x} \in \operatorname{Ker}\mathbf{J}$$

since we have $\operatorname{Ker}\mathbf{A}_o = \operatorname{Ker}\mathbf{A} \cap \operatorname{Ker}\mathbf{J} = \{\boldsymbol{o}\}_{\mathcal{L}} \times \{\boldsymbol{o}\}_{\mathcal{H}}$.

Let $\mathbf{x} = \{\mathbf{u},\mathbf{g}\} \in \operatorname{Ker}\mathbf{J}$ be the unique solution of the reduced problem $\Sigma_2)$. The full solution $\{\mathbf{u},\mathbf{g},\boldsymbol{\sigma}\} \in \operatorname{Ker}\mathbf{J} \times \mathcal{H}$ of the mixed problem $\Sigma_1)$ can be recovered by solving the stress recovery problem

$$\langle\, \mathbf{J}'\boldsymbol{\sigma}\,,\, \overline{\mathbf{x}}\,\rangle = \mathbf{j}(\boldsymbol{\sigma},\overline{\mathbf{x}}) = -\mathbf{a}(\mathbf{x},\overline{\mathbf{x}}) + \langle\, \mathbf{f}\,,\, \overline{\mathbf{x}}\,\rangle, \qquad \forall \overline{\mathbf{x}} \in \mathcal{X},$$

or equivalently $\mathbf{J}'\boldsymbol{\sigma} = -\mathbf{A}\mathbf{x} + \mathbf{f}$. A unique stress solution $\boldsymbol{\sigma}$ is obtained since $\operatorname{Ker} \mathbf{J}' = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$.

5.2.2 Approximate solution

Approximate solutions can be obtained by a conforming F.E.M. interpolation based on three families of finite dimensional subspaces $\mathcal{L}_h \subset \mathcal{L}$, $\mathcal{D}_h \subset \mathcal{H}$ and $\mathcal{S}_h \subset \mathcal{H}$ depending on a parameter h which goes to zero as the finite element mesh is refined ever more [32].

The approximate mixed problem $\Sigma_1)$ is expressed by

$$\Sigma_5) \qquad \begin{cases} \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma}_h, \overline{\mathbf{x}_h}) = \langle\, \mathbf{f}\,,\, \overline{\mathbf{x}_h}\,\rangle & \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h = \mathcal{L}_h \times \mathcal{D}_h\,, \\ \mathbf{j}\,(\overline{\boldsymbol{\sigma}_h}, \mathbf{x}_h) \qquad\qquad\qquad = 0 & \forall\, \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h\,. \end{cases}$$

It is convenient to consider the operators associated with the bilinear forms by setting

$$\begin{aligned} &\mathbf{a}\,(\mathbf{x}_h, \mathbf{y}_h) = \langle\, \mathbf{x}_h\,,\, \mathbf{A}_h\,\mathbf{y}_h\,\rangle = \langle\, \mathbf{A}_h\,\mathbf{x}_h\,,\, \mathbf{y}_h\,\rangle \quad \forall\, \mathbf{x}_h, \mathbf{y}_h \in \mathcal{X}_h\,, \\ &\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h) = (\!(\, \boldsymbol{\sigma}_h\,,\, \mathbf{J}_h\,\mathbf{x}_h\,)\!) = \langle\, \mathbf{J}'_h\,\boldsymbol{\sigma}_h\,,\, \mathbf{x}_h\,\rangle \quad \forall\, \mathbf{x}_h \in \mathcal{X}_h\,, \quad \forall\, \boldsymbol{\sigma}_h \in \mathcal{S}_h\,, \end{aligned}$$

with

$$\begin{aligned} &\mathbf{A}_h \in BL\,(\mathcal{X}_h, \mathcal{X}'_h) \\ &\mathbf{J}_h \in BL\,(\mathcal{X}_h, \mathcal{S}'_h) \\ &\mathbf{J}'_h \in BL\,(\mathcal{S}_h, \mathcal{X}'_h) \\ &\langle\, \mathbf{f}\,,\, \mathbf{x}_h\,\rangle = \langle\, \mathbf{f}_h\,,\, \mathbf{x}_h\,\rangle \quad \forall \mathbf{x}_h \in \mathcal{X}_h, \quad \mathbf{f}_h \in \mathcal{X}'_h. \end{aligned}$$

According to the remark 1 above, these approximate operators can be conveniently defined in terms of the isometric isomorphism between the Hilbert spaces $\mathcal{X}'_h$ and $\mathcal{S}'_h$, dual to $\mathcal{X}_h$ and $\mathcal{S}_h$, and the quotient Hilbert spaces $\mathcal{X}'/\mathcal{X}_h^{\perp}$ and $\mathcal{H}/\mathcal{S}_h^{\perp}$. Then we can set

$$\begin{aligned} \mathbf{A}_h\,\mathbf{x}_h &= \mathbf{A}_h\,\{\mathbf{u}_h\,,\mathbf{g}_h\} = \{\mathbf{B}'\mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) + \mathcal{L}_h^{\perp}\,, -\mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) + cDD_h^{\perp}\}\,, \\ \mathbf{J}_h\,\mathbf{x}_h &= \mathbf{J}_h\,\{\mathbf{u}_h\,,\mathbf{g}_h\} = \mathbf{g}_h + \mathcal{S}_h^{\perp}\,, \\ \mathbf{J}'_h\,\boldsymbol{\sigma}_h &= \{\mathcal{L}_h^{\perp}\,, \boldsymbol{\sigma}_h + \mathcal{D}_h^{\perp}\}\,, \\ \mathbf{f}_h &= \mathbf{f} + \mathcal{X}_h^{\perp} = \mathbf{f} + \mathcal{L}_h^{\perp} \times \mathcal{D}_h^{\perp} = \{\ell + \mathcal{L}_h^{\perp}\,, \mathcal{D}_h^{\perp}\}\,, \end{aligned}$$

and, setting $\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$, the related kernels are given by:

$$
\Upsilon) \quad
\begin{aligned}
&\operatorname{Ker} \mathbf{A}_h = \left\{ \{\mathbf{u}_h , \mathbf{g}_h\} \in \mathcal{L}_h \times \mathcal{D}_h \mid \mathbf{B}\mathbf{u}_h = \mathbf{g}_h \right\} , \\
&\operatorname{Ker} \mathbf{J}_h = \left\{ \{\mathbf{u}_h , \mathbf{g}_h\} \in \mathcal{L}_h \times \mathcal{D}_h \mid \mathbf{g}_h \in \mathcal{S}_h^{\perp} \right\} = \mathcal{L}_h \times (\mathcal{D}_h \cap \mathcal{S}_h^{\perp}) = \mathcal{L}_h \times \widetilde{\mathcal{D}}_h , \\
&\operatorname{Ker} \mathbf{J}'_h = \left\{ \boldsymbol{\sigma}_h \in \mathcal{S}_h \mid \boldsymbol{\sigma}_h \in \mathcal{D}_h^{\perp} \right\} = \mathcal{S}_h \cap \mathcal{D}_h^{\perp} .
\end{aligned}
$$

The approximate SGM can then be put in the matrix form

$$
\begin{cases}
\mathbf{A}_h \mathbf{x}_h + \mathbf{J}'_h \boldsymbol{\sigma}_h = \mathbf{f}_h , \\
\mathbf{J}_h \mathbf{x}_h \qquad\qquad = \mathcal{S}_h^{\perp} .
\end{cases}
$$

5.2.3 Discrete reduced problem and stress recovery

The mixed problem $\Sigma_5)$ can be split into a sequence of two problems: the reduced and consequently the stress recovery problem.

- The *reduced* problem in the product space $\operatorname{Ker} \mathbf{J}_h \times \operatorname{Ker} \mathbf{J}_h$ given by

$$
\Sigma_6) \qquad \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) = \langle\, \mathbf{f}_h \,,\, \overline{\mathbf{x}_h} \,\rangle \quad \forall\, \overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h \,, \quad \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h \,.
$$

  Explicitly, we have to find $\{\mathbf{u}_h, \mathbf{g}_h\} \in \mathcal{L}_h \times \widetilde{\mathcal{D}}_h$ such that

$$
\begin{cases}
\langle\!\langle\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) \,,\, \mathbf{B}\overline{\mathbf{u}}_h \,\rangle\!\rangle = \langle\, \ell \,,\, \overline{\mathbf{u}}_h \,\rangle & \forall\, \overline{\mathbf{u}}_h \in \mathcal{L}_h \\
\langle\!\langle\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) \,,\, \overline{\mathbf{g}}_h \,\rangle\!\rangle = 0 & \forall\, \overline{\mathbf{g}}_h \in \widetilde{\mathcal{D}}_h .
\end{cases}
$$

  We remark that the strain gap components $\mathbf{g}_h$ of $\mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h$ belongs to the subspace $\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$ and provide the *effective* strain gap interpolates which effectively contribute to relax the compatibility requirement.

  By introducing the reduced operator $\mathbf{A}_{oh} \in BL\,(\operatorname{Ker} \mathbf{J}_h, (\operatorname{Ker} \mathbf{J}_h)')$ as the symmetric operator defined by

$$
\langle\, \mathbf{A}_{oh}\, \mathbf{x}_h \,,\, \mathbf{y}_h \,\rangle = \langle\, \mathbf{x}_h \,,\, \mathbf{A}_{oh}\, \mathbf{y}_h \,\rangle = \mathbf{a}\,(\mathbf{x}_h, \mathbf{y}_h) \quad \forall\, \mathbf{x}_h, \mathbf{y}_h \in \operatorname{Ker} \mathbf{J}_h \,,
$$

  and the discrete reduced force $\mathbf{f}_{oh} \in (\operatorname{Ker} \mathbf{J}_h)'$ defined by

$$
\langle\, \mathbf{f}_{oh} \,,\, \mathbf{x}_h \,\rangle = \langle\, \mathbf{f} \,,\, \mathbf{x}_h \,\rangle \quad \forall\, \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h \,,
$$

  problem $\Sigma_6)$ can be written

$$
\Sigma_7) \qquad \mathbf{A}_{oh} \mathbf{x}_h = \mathbf{f}_{oh} \,, \quad \mathbf{x}_h \in \operatorname{Ker} \mathbf{J}_h \,.
$$

Remark 5.2.2 *By virtue of the isometric isomorphism between the* HILBERT *spaces* $(\,Ker\,\mathbf{J}_h)'$ *and* $\mathcal{X}'/(\,Ker\,\mathbf{J}_h)^{\perp}$ *we have* $\mathbf{A}_{oh}\,\mathbf{x}_h = \mathbf{A}_h\,\mathbf{x}_h + (\,Ker\,\mathbf{J}_h)^{\perp}$ *for any* $\mathbf{x}_h \in\, Ker\,\mathbf{J}_h$ *and* $\mathbf{f}_{oh} = \mathbf{f}_h + (\,Ker\,\mathbf{J}_h)^{\perp}$.

Problem $\Sigma_7)$ admits solutions for any data $\mathbf{f}_{oh} \in (\,\mathrm{Ker}\,\mathbf{A}_{oh})^{\perp} = (\,\mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h)^{\perp}$ if and only if the discrete bilinear form $\mathbf{a} \in \mathrm{Bil}\{\,\mathrm{Ker}\,\mathbf{J}_h \times \mathrm{Ker}\,\mathbf{J}_h\,\}$ is closed. In fact this condition is equivalent to the closedness of $\mathrm{Im}\,\mathbf{A}_{oh}$ and hence to the equality $\mathrm{Im}\,\mathbf{A}_{oh} = (\,\mathrm{Ker}\,\mathbf{A}_{oh})^{\perp}$. Uniqueness holds to within elements of $\mathrm{Ker}\,\mathbf{A}_{oh}$.

- The *stress recovery* problem is given by

$$\Sigma_8)\qquad \mathbf{j}\,(\boldsymbol{\sigma}_h, \overline{\mathbf{x}_h}) = -\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h}) + \langle\, \mathbf{f}_h\,,\, \overline{\mathbf{x}_h}\,\rangle \quad \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h\,, \quad \boldsymbol{\sigma}_h \in \mathcal{S}_h\,,$$

where $\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h$ is a solution of the problem $\Sigma_6)$. Explicitly, the stress recovery problem requires to find a stress $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ such that

$$\Sigma_9)\qquad \langle\!\langle\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h\,,\, \overline{\mathbf{g}}_h\,\rangle\!\rangle = 0 \qquad \forall\, \overline{\mathbf{g}}_h \in \mathcal{D}_h$$

where the pair $\{\mathbf{u}_h, \mathbf{g}_h\}$ is a solution of $\delta)$ in chapter 4.

Problem $\Sigma_8)$ can be written in an operator form as follows

$$\mathbf{J}_h'\boldsymbol{\sigma}_h = -\mathbf{A}_h\mathbf{x}_h + \mathbf{f}_h \in (\,\mathrm{Ker}\,\mathbf{J}_h)^{\perp}\,, \quad \boldsymbol{\sigma}_h \in \mathcal{S}_h\,.$$

It will admit solutions iff the discrete bilinear form $\mathbf{j} \in \mathrm{Bil}\{\,\mathcal{S}_h \times \mathcal{X}_h\,\}$ is closed.

In fact this condition is equivalent to the closedness of $\mathrm{Im}\,\mathbf{J}_h'$ and hence to the equality $\mathrm{Im}\,\mathbf{J}_h' = (\,\mathrm{Ker}\,\mathbf{J}_h)^{\perp}$. Uniqueness holds to within elements of $\mathrm{Ker}\,\mathbf{J}_h'$.

The stress recovery problem $\Sigma_9)$ states that the approximate stress is the projection of the field $\mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ on the subspace $\mathcal{S}_h$ in the direction orthogonal to a subspace $\widetilde{\mathcal{D}}_h$ of *effective* strain gaps.

The closedness requirements of the discrete bilinear forms $\mathbf{a}$ and $\mathbf{j}$ can equivalently be expressed by the inf-sup conditions

$$\inf_{\mathbf{x}_h \in \mathcal{X}_h}\ \sup_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \frac{\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h)}{\|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{J}_h'}\ \|\,\mathbf{x}_h\,\|_{\mathcal{X}/\mathrm{Ker}\,\mathbf{J}_h}} \geq c_{\mathbf{j}h} > 0\,,$$

$$\inf_{\mathbf{y}_h \in \mathrm{Ker}\,\mathbf{J}_h}\ \sup_{\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{y}_h, \mathbf{x}_h)}{\|\,\mathbf{y}_h\,\|_{\mathcal{X}/(\mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h)}\ \|\,\mathbf{x}_h\,\|_{\mathcal{X}/(\mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h)}} \geq c_{oh} > 0\,.$$

We remark that these closedness requirements are trivial since all the spaces involved are finite dimensional.

Nonetheless it has been convenient to write down the expressions $\eta)$ for further reference since we shall see that the convergence properties of the approximate scheme depend on the lower bounds of the constants there involved.

5.2.4 Well-posedness conditions

Let us now provide the well-posedness result concerning the discrete mixed problem $\Sigma_5)$.

Proposition 5.2.2 ***(Well-posedness of the discrete mixed problem)*** *Assuming that there are no conforming rigid displacements in the approximating subspace $\mathcal{L}_h$, i.e. $Ker \mathbf{B} \cap \mathcal{L}_h = \{\boldsymbol{o}\}_{\mathcal{L}}$, the discrete problem $\Sigma_5)$ admits a unique solution for any data $\ell \in \mathcal{L}'_h$ if and only if*

$$\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\boldsymbol{o}\}_{\mathcal{H}} \qquad \mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}_{\mathcal{H}}\,.$$

Condition $\iota)_1$ ensures that the compatible approximate strains due to approximate conforming displacements and the effective approximate strain gaps are linearly independent. Condition $\iota)_2$ ensures that the approximate stresses are controlled by the approximate strain gaps.

Proof. The condition $\iota)_1$ can be equivalently written in terms of the kernel $\Upsilon)_1$ and $\Upsilon)_2$ of the operators $\mathbf{A}_h$ and $\mathbf{J}_h$ in the form $\mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h = \mathrm{Ker}\,\mathbf{B} \cap \mathcal{L}_h \times \{\boldsymbol{o}\}_{\mathcal{H}} = \{\boldsymbol{o}\}_{\mathcal{L}} \times \{\boldsymbol{o}\}_{\mathcal{H}}$, since there are no conforming rigid displacements in $\mathcal{L}_h$.

The approximate mixed problem $\Sigma_5)$ admits then solutions for any data $\mathbf{f}_h \in \mathcal{X}'_h$ and the solution $\{\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h\}$ is unique in the component $\mathbf{x}_h = \{\mathbf{u}_h, \mathbf{g}_h\} \in \mathcal{L}_h \times \mathcal{D}_h$.

Uniqueness of the component $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ follows from the definition $\Upsilon)_3$ of the $\mathrm{Ker}\,\mathbf{J}'_h$ and from $\iota)_2$ since we have $\mathrm{Ker}\,\mathbf{J}'_h = \mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}_{\mathcal{H}}$.

If the conditions $\iota)_1$ are met so that $\mathrm{Ker}\,\mathbf{A}_{ho} = \mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h = \{\boldsymbol{o}\}_{\mathcal{L}} \times \{\boldsymbol{o}\}_{\mathcal{H}}$ and $\mathrm{Ker}\,\mathbf{J}'_h = \{\boldsymbol{o}\}_{\mathcal{H}}$, the inf-sup conditions $\eta)$ reduce to:

$$\Sigma_{10}) \quad \begin{aligned} &\inf_{\mathbf{x}_h \in \mathcal{X}_h} \ \sup_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \frac{\mathbf{j}(\boldsymbol{\sigma}_h, \mathbf{x}_h)}{\|\boldsymbol{\sigma}_h\|_{\mathcal{H}} \|\mathbf{x}_h\|_{\mathcal{X}/\mathrm{Ker}\,\mathbf{J}_h}} \geq c_{\mathbf{j}h} > 0\,, \\ &\inf_{\mathbf{y}_h \in \mathrm{Ker}\,\mathbf{J}_h} \ \sup_{\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h} \frac{\mathbf{a}(\mathbf{y}_h, \mathbf{x}_h)}{\|\mathbf{y}_h\|_{\mathcal{X}} \|\mathbf{x}_h\|_{\mathcal{X}}} \geq c_{oh} > 0\,. \end{aligned}$$

5.2.5 Strain gap subspace decomposition

Let us decompose the strain gap space $\mathcal{D}_h$ as a direct sum

$$\mathcal{D}_h = \mathcal{D}^*_h \oplus \widetilde{\mathcal{D}}_h \ \ with \ \ \mathcal{D}^*_h \cap \mathcal{S}^{\perp}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}.$$

We name *control* strain gaps the fields in $\mathcal{D}^*_h$ since they are deputated to control the stress fields in $\mathcal{S}_h$. Note that the subspace $\mathcal{D}^*_h$ is not uniquely defined by the decomposition.

It is worth noting that we have that $\dim \mathcal{D}^*_h \leq \dim \mathcal{S}_h$. Since the uniqueness condition $\iota)_2$ of the stress component $\boldsymbol{\sigma}_h$ implies that $\dim \mathcal{S}_h \leq \dim \mathcal{D}_h \leq \dim \mathcal{D}^*_h$, we may then conclude that the dimension of the approximate control strain gap subspace coincides with the dimension of the approximate stress subspace, that is $\dim \mathcal{S}_h = \dim \mathcal{D}^*_h$.

Anyway the stress solution is independent of the special choice of a subspace $\mathcal{D}^*_h$ of control strain gaps unless the whole space $\mathcal{D}_h$ is fixed.

If the condition $\widetilde{\mathcal{D}}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$ is met, a limitation phenomenon occours, as already pointed out, and the discrete mixed problem collapses into the displacement method.

The same limitation phenomenon occours if the linear subspaces $\mathbf{B}\mathcal{L}_h$ and $\widetilde{\mathcal{D}}_h$ are orthogonal in elastic energy, that is if $\widetilde{\mathcal{D}}_h \subset (\mathcal{E}\,\mathbf{B}\mathcal{L}_h)^{\perp}$. An analysis of these issues can be found at the end of the chapter 4.

Remark 5.2.3 *The enhanced strain method proposed by* Simo *and* Rifai *in [55] was based on the assumption* $\mathcal{D}_h \subset \mathcal{S}^{\perp}_h$ *so that* $\mathcal{D}_h = \widetilde{\mathcal{D}}_h$ *and* $\mathcal{D}^*_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$.

As a consequence the stress recovery problem $\Sigma_8)$ *becomes an identity and the method reduces to the approximate problem* $\Sigma_6)$.

The enhanced strain method can then be interpreted as a singular case of the SGM since the choice $\mathcal{D}_h \subset \mathcal{S}^{\perp}_h$ *violates the uniqueness condition* $\iota)$, *i.e.* $\mathcal{S}_h \cap \mathcal{D}^{\perp}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$, *for the stress component* $\boldsymbol{\sigma}_h$ *and leads to a partially ill-posed problem. The a posteriori stress recovery strategy proposed in [55] consists in the solution of the minimum distance problem*

$$\min_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \| \, \mathbf{Bu}_h - \mathcal{E}^{-1} \boldsymbol{\sigma}_h \, \|_{\mathcal{H}} \, .$$

In this respect we observe that choosing $\mathcal{D}_h = \mathcal{E}^{-1}\mathcal{S}_h + \widetilde{\mathcal{D}}_h$, *the stress recovery problem* **??** *yields the strategy*

$$(\!(\, \mathcal{E}(\mathbf{Bu}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h \, , \, \mathcal{E}^{-1} \overline{\boldsymbol{\sigma}_h} \,)\!)_{\mathcal{H}} = 0 \quad \forall \, \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h \, ,$$

which is equivalent to the minimum distance problem

$$\min_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \| \mathbf{B}\mathbf{u}_h - \mathbf{g}_h - \mathcal{E}^{-1}\boldsymbol{\sigma}_h \|_{\mathcal{H}} \,.$$

This variational strategy differs from the stress recovery strategy proposed in [55] for the presence of the strain gap solution $\mathbf{g}_h$.

5.2.6 Uniform well-posedness

Bounds of the mean square error of approximate solutions can be deduced from h-uniform closedness requirements. These h-uniformity assumptions amounts to impose that in the trivial finite dimensional inf-sup conditions $\Sigma_{10})$ the families of positive constants $\{c_{\mathbf{j}h}\}$ and $\{c_{oh}\}$ must admit positive lower bounds, respectively $c_{\mathbf{j}}$ and c_o, which are independent of the mesh parameter h, so that

$$\Sigma_{11}) \qquad \begin{aligned} &\inf_{\mathbf{x}_h \in \mathcal{X}_h} \ \sup_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \frac{\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h)}{\| \boldsymbol{\sigma}_h \|_{\mathcal{H}} \, \| \mathbf{x}_h \|_{\mathcal{X}/\mathrm{Ker}\,\mathbf{J}_h}} \geq c_{\mathbf{j}} > 0 \,, \\ &\inf_{\overline{\mathbf{x}_h} \in \mathrm{Ker}\,\mathbf{J}_h} \ \sup_{\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h} \frac{\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}_h})}{\| \mathbf{x}_h \|_{\mathcal{X}} \, \| \overline{\mathbf{x}_h} \|_{\mathcal{X}}} \geq c_o > 0 \,. \end{aligned}$$

Uniform conditions of this kind are referred to in the literature as *discrete inf-sup conditions* or also as LBB (Ladyzhenskaya-Babŭska-Brezzi) conditions [26], [29], [57], [15].

The property $\Sigma_{11})_1$ is equivalent to the h-uniform closedness of the family of subspaces $\{\,\mathrm{Im}\,\mathbf{J}_h\} = \{\mathcal{D}_h + \mathcal{S}_h^{\perp}\}$ given by

$$\inf_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \ \sup_{\mathbf{x}_h \in \mathcal{X}_h} \frac{\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h)}{\| \boldsymbol{\sigma}_h \|_{\mathcal{H}} \, \| \mathbf{x}_h \|_{\mathcal{X}/\mathrm{Ker}\,\mathbf{J}_h}} \geq c_{\mathbf{j}} > 0 \,,$$

from which we deduce the further inequality

$$\Sigma_{bis}) \qquad c_{\mathbf{j}} \, \| \boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h} \|_{\mathcal{H}} \leq \sup_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \frac{\mathbf{j}\,(\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h}, \overline{\mathbf{x}_h})}{\| \overline{\mathbf{x}_h} \|_{\mathcal{X}}} \,,$$

which will be referred to in the sequel.

The property $\Sigma_{11})_2$ amounts to require the h-uniform closedness of the family of the subspaces $\{\,\mathrm{Im}\,\mathbf{A}_{oh}\} = \{\,\mathrm{Im}\,\mathbf{A}_h + (\,\mathrm{Ker}\,\mathbf{J}_h)^{\perp}\}$. Condition $\Sigma_{11})_2$ is implied by the stronger $\mathrm{Ker}\,\mathbf{J}_h$-ellipticity of the bilinear form $\mathbf{a}$. Sufficient conditions for the $\mathrm{Ker}\,\mathbf{J}_h$-ellipticity of $\mathbf{a}$ are provided hereafter.

Proposition 5.2.3 (Uniform ellipticity) *If there are no conforming displacements, let us assume that the following properties are met: the well-posedness condition $\iota)_1$ given by $\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$, the closedness of the range $Im\mathbf{B}$ of the kinematic operator $\mathbf{B} \in BL(\mathcal{L}, \mathcal{H})$, the h-uniform closedness condition of $\mathbf{B}\mathcal{L}_h + \widetilde{\mathcal{D}}_h$ in $\mathcal{H}$. Then the symmetric bilinear form $\mathbf{a} \in Bil\{\, Ker\mathbf{J}_h \times Ker\mathbf{J}_h \,\}$ is uniformly elliptic, i.e.*

$$\Sigma_{ter}) \qquad \mathbf{a}\,(\mathbf{x}_h, \mathbf{x}_h) \geq c_o \,\|\, \mathbf{x}_h \,\|^2_{\mathcal{X}} \quad \forall\, \mathbf{x}_h \in Ker\mathbf{J}_h \,.$$

Proof. The condition $\operatorname{Ker}\mathbf{B} \cap \mathcal{L}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{L}}$ and the closedness of $\operatorname{Im}\mathbf{B}$ imply that

$$\|\, \mathbf{B}\mathbf{u}_h \,\|_{\mathcal{H}} \geq c_{\mathbf{B}} \,\|\, \mathbf{u}_h \,\|_{\mathcal{L}/(\operatorname{Ker}\mathbf{B} \cap \mathcal{L}_h)} = c_{\mathbf{B}} \,\|\, \mathbf{u}_h \,\|_{\mathcal{L}} \quad \forall\, \mathbf{u}_h \in \mathcal{L}_h \,.$$

The ellipticity property of $\mathcal{E}$ ensures that

$$(\!(\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,,\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,)\!)_{\mathcal{H}} \geq c_{\mathcal{E}}\,(\|\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,\|^2_{\mathcal{H}}) \quad \forall\, \{\mathbf{u}_h\,, \mathbf{g}_h\} \in \mathcal{L}_h \times \widetilde{\mathcal{D}}_h \,.$$

Since $\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$, the h-uniform closedness of the family of subspaces $\{\mathbf{B}\mathcal{L}_h + \widetilde{\mathcal{D}}_h\}$ in $\mathcal{H}$ is equivalent [91] to the existence of a constant $c > 0$ independent of h such that

$$\|\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,\|^2_{\mathcal{H}} \geq c\, \Big(\|\, \mathbf{B}\mathbf{u}_h \,\|^2_{\mathcal{H}} + \|\, \mathbf{g}_h \,\|^2_{\mathcal{H}}\Big) \quad \forall\, \{\mathbf{u}_h\,, \mathbf{g}_h\} \in \mathcal{L}_h \times \widetilde{\mathcal{D}}_h \,.$$

We then have the inequality

$$\begin{aligned}
&(\!(\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,,\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,)\!) \geq c_{\mathcal{E}} \,\|\, \mathbf{B}\mathbf{u}_h - \mathbf{g}_h \,\|^2_{\mathcal{H}} \geq \\
&c_{\mathcal{E}}\, c\, \Big(\|\, \mathbf{B}\mathbf{u}_h \,\|^2_{\mathcal{H}} + \|\, \mathbf{g}_h \,\|^2_{\mathcal{H}}\Big) \geq c_{\mathcal{E}}\, c\, \Big(c^2_{\mathbf{B}} \,\|\, \mathbf{u}_h \,\|^2_{\mathcal{L}} + \|\, \mathbf{g}_h \,\|^2_{\mathcal{H}}\Big) \geq \\
&c_o\, \Big(\|\, \mathbf{u}_h \,\|^2_{\mathcal{L}} + \|\, \mathbf{g}_h \,\|^2_{\mathcal{H}}\Big) \quad \forall\, \{\mathbf{u}_h\,, \mathbf{g}_h\} \in \mathcal{L}_h \times \widetilde{\mathbb{D}}_h \,,
\end{aligned}$$

with $c_o = c_{\mathcal{E}}\, c \,\min\{\, 1, c^2_{\mathbf{B}} \,\}$. This proves the $\operatorname{Ker}\mathbf{J}_h$-ellipticity of $\mathbf{a}$.

5.2.7 Error bounds

We are now able to provide the bound of the mean square error of the approximate solution in terms of displacements and strain gaps by adapting the approach developed and discussed in [57].

Mean square error of the $\{\mathbf{u}_h\,,\mathbf{g}_h\}$-solution

If $\mathbf{x} = \{\mathbf{u}\,,\mathbf{g}\} \in \mathcal{X}$ is the displacement and strain gap solution of the mixed problem $\Sigma_1)$ and $\mathbf{x}_h = \{\mathbf{u}_h\,,\mathbf{g}_h\} \in \mathcal{X}_h$ is the corresponding solution of the discrete problem $\Sigma_5)$, the triangle inequality yields

$$\Sigma_{12}) \qquad \|\,\mathbf{x} - \mathbf{x}_h\,\|_{\mathcal{H}} \leq \|\,\mathbf{x}_h - \overline{\mathbf{x}_h}\,\|_{\mathcal{H}} + \|\,\mathbf{x} - \overline{\mathbf{x}_h}\,\|_{\mathcal{H}} \quad \forall\, \overline{\mathbf{x}_h} \in \mathcal{X}_h\,.$$

Let us now recall the first variational conditions of problems $\Sigma_1)$ and $\Sigma_5)$

$$\begin{aligned} &\mathbf{a}\,(\mathbf{x},\overline{\mathbf{x}}) + \mathbf{j}\,(\boldsymbol{\sigma},\overline{\mathbf{x}}) = \mathbf{f}(\overline{\mathbf{x}}) \quad \forall\,\overline{\mathbf{x}} \in \mathcal{X}\,,\\ &\mathbf{a}\,(\mathbf{x}_h,\overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma}_h,\overline{\mathbf{x}_h}) = \mathbf{f}(\overline{\mathbf{x}_h}) \quad \forall\,\overline{\mathbf{x}_h} \in \mathcal{X}_h\,. \end{aligned}$$

Setting $\overline{\mathbf{x}} = \overline{\mathbf{x}_h}$ and subtracting we get

$$\mathbf{a}\,(\mathbf{x} - \mathbf{x}_h,\overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\overline{\mathbf{x}_h}) = 0 \quad \forall\,\overline{\mathbf{x}_h} \in \mathcal{X}_h\,.$$

From the $\operatorname{Ker}\mathbf{J}_h$-ellipticity property of $\mathbf{a} \in \operatorname{Bil}\{\operatorname{Ker}\mathbf{J}_h \times \operatorname{Ker}\mathbf{J}_h\}$ we then infer the following bound for the first term on the r.h.s. of $\Sigma_{12})$

$$\Sigma_{13}) \qquad \begin{aligned} &c_o\,\|\,\mathbf{x}_h - \overline{\mathbf{x}_h}\,\|^2_{\mathcal{X}} \leq \mathbf{a}\,(\mathbf{x}_h - \overline{\mathbf{x}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) =\\ &\mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) + \mathbf{a}\,(\mathbf{x}_h - \mathbf{x},\mathbf{x}_h - \overline{\mathbf{x}_h}) =\\ &\mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\mathbf{x}_h - \overline{\mathbf{x}_h}) =\\ &\mathbf{a}\,(\mathbf{x} - \overline{\mathbf{x}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) \quad \forall\,\overline{\mathbf{x}_h} \in \operatorname{Ker}\mathbf{J}_h\,, \end{aligned}$$

where the equality $\mathbf{j}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\mathbf{x}_h - \overline{\mathbf{x}_h}) = \mathbf{j}\,(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h},\mathbf{x}_h - \overline{\mathbf{x}_h})$ holds since

$$\mathbf{x}_h - \overline{\mathbf{x}_h} \in \operatorname{Ker}\mathbf{J}_h \quad \Longrightarrow \quad \mathbf{j}\,(\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h},\mathbf{x}_h - \overline{\mathbf{x}_h}) = 0\,.$$

By the continuity of the forms $\mathbf{a}$ and $\mathbf{j}$, the inequality $\Sigma_{13})$ becomes

$$c_o\,\|\,\mathbf{x}_h - \overline{\mathbf{x}_h}\,\|_{\mathcal{X}} \leq \|\,\mathbf{a}\,\|\,\|\,\mathbf{x} - \overline{\mathbf{x}_h}\,\|_{\mathcal{X}} + \|\,\mathbf{j}\,\|\,\|\,\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h}\,\|_{\mathcal{H}}\,, \quad \forall\,\overline{\mathbf{x}_h} \in \operatorname{Ker}\mathbf{J}_h, \;\; \forall\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$$

and, from the triangle inequality, we get the result concerning the bound of the mean square error of the approximate solution in terms of displacements and strain gaps

$$\Sigma_{14}) \qquad \|\,\mathbf{x} - \mathbf{x}_h\,\|_{\mathcal{X}} \leq \left(1 + \frac{\|\,\mathbf{a}\,\|}{c_o}\right) \|\,\mathbf{x} - \overline{\mathbf{x}_h}\,\|_{\mathcal{X}} + \frac{\|\,\mathbf{j}\,\|}{c_o}\,\|\,\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h}\,\|_{\mathcal{H}}\,,$$

for all $\overline{\mathbf{x}_h} \in \operatorname{Ker}\mathbf{J}_h$ and for all $\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$.

To get rid of the constraint $\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h$, we follow [57] and observe that for any given $\mathbf{y}_h \in \mathcal{X}_h$ we can find at least a solution $\mathbf{z}_h \in \mathcal{X}_h$ of the problem

$$\mathbf{j}\,(\mathbf{z}_h, \overline{\boldsymbol{\sigma}_h}) = \mathbf{j}\,(\mathbf{x} - \mathbf{y}_h, \overline{\boldsymbol{\sigma}_h}) \quad \forall \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h\,,$$

such that

$$c_{\mathbf{j}} \,\|\, \mathbf{z}_h \,\|_{\mathcal{X}} \leq \sup_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \frac{\mathbf{j}\,(\mathbf{x} - \mathbf{y}_h, \overline{\boldsymbol{\sigma}_h})}{\|\, \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}} \leq \|\, \mathbf{j} \,\|\, \|\, \mathbf{x} - \mathbf{y}_h \,\|_{\mathcal{X}}\,.$$

Setting $\overline{\mathbf{x}_h} = \mathbf{z}_h + \mathbf{y}_h$ we have that

$$\Sigma_{15}) \qquad \mathbf{j}\,(\overline{\mathbf{x}_h}, \overline{\boldsymbol{\sigma}_h}) = \mathbf{j}\,(\mathbf{z}_h + \mathbf{y}_h, \overline{\boldsymbol{\sigma}_h}) = \mathbf{j}\,(\mathbf{x}, \overline{\boldsymbol{\sigma}_h}) = 0 \quad \forall \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$$

and hence that $\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h$. We have thus proved the inequality

$$\|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} = \|\, \mathbf{x} - \mathbf{y}_h - \mathbf{z}_h \,\|_{\mathcal{X}} \leq \|\, \mathbf{x} - \mathbf{y}_h \,\|_{\mathcal{X}} + \|\, \mathbf{z}_h \,\|_{\mathcal{X}} \leq \left(1 + \frac{\|\, \mathbf{j} \,\|}{c_{\mathbf{j}}}\right) \|\, \mathbf{x} - \mathbf{y}_h \,\|_{\mathcal{X}}\,,$$

from which we infer that

$$\Sigma_{16}) \qquad \inf_{\overline{\mathbf{x}_h} \in \operatorname{Ker} \mathbf{J}_h} \|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} \leq \left(1 + \frac{\|\, \mathbf{j} \,\|}{c_{\mathbf{j}}}\right) \inf_{\mathbf{y}_h \in \mathcal{X}_h} \|\, \mathbf{x} - \mathbf{y}_h \,\|_{\mathcal{X}}\,.$$

In conclusion the inequality $\Sigma_{14})$, with the aid of $\Sigma_{16})$, provides the estimate

$$\Sigma_{17}) \quad \|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} \leq c_1 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} + c_2 \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,,$$

where $c_1 = (1 + \|\, \mathbf{a} \,\|/c_o)\,(1 + \|\, \mathbf{j} \,\|/c_j)$ and $c_2 = \|\, \mathbf{j} \,\|/c_o$ are positive constants, apparently independent of h.

Mean square error of the stress solution

The bound of the mean square error of the approximate stress solution starts with an application of the triangle inequality.

If $\boldsymbol{\sigma} \in \mathcal{H}$ is the stress solution of the mixed problem $\Sigma_1)$ and $\boldsymbol{\sigma}_h \in \mathcal{S}_h$ is the stress solution of the discrete problem $\Sigma_5)$, we have that

$$\Sigma_{18}) \qquad \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq \|\, \boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} + \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \quad \forall \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h\,.$$

The first term on the r.h.s. can be estimated by inequality $\Sigma_{bis})$. To this end we recall the equality $\Sigma_{15})$ from which we infer that

$$\mathbf{j}\,(\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h}, \overline{\mathbf{x}_h}) = \mathbf{a}\,(\mathbf{x} - \mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h}, \overline{\mathbf{x}_h}) \quad \forall \overline{\mathbf{x}_h} \in \mathcal{X}_h\,,$$

so that the inequality $\Sigma_{bis})$ yields

$$c_{\mathbf{j}} \,\|\, \boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \leq \sup_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \frac{\mathbf{a}\,(\mathbf{x} - \mathbf{x}_h, \overline{\mathbf{x}_h}) + \mathbf{j}\,(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h}, \overline{\mathbf{x}_h})}{\|\, \overline{\mathbf{x}_h} \,\|_{\mathcal{X}}}\,,$$

and, by the continuity of the forms $\mathbf{a}$ and $\mathbf{j}$, we have

$$c_{\mathbf{j}} \,\|\, \boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \leq \|\, \mathbf{a} \,\|\, \|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} + \|\, \mathbf{j} \,\|\, \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,.$$

Accordingly, the inequality $\Sigma_{18})$ becomes

$$\Sigma_{19}) \qquad \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq c_3 \,\|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} + c_4 \,\|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,, \quad \forall \overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$$

with $c_3 = \|\, \mathbf{a} \,\|/c_{\mathbf{j}}$ and $c_4 = 1 + \|\, \mathbf{j} \,\|/c_{\mathbf{j}}$.

The estimate of the mean square error of the stress approximate solution depends then on the estimate of the mean square error of the displacement and strain gap approximate solutions.

By comparing the two inequalities $\Sigma_{17})$ and $\Sigma_{19})$, written in the following form

$$\|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} \leq c_1 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} + c_2 \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,,$$

$$\|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq c_3 \,\|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} + c_4 \,\|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \quad \forall \overline{\boldsymbol{\sigma}} \in \mathcal{S}_h\,,$$

we get the estimate of the mean square error of the stress approximate solution

$$\Sigma_{20}) \qquad \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq c_5 \inf_{\overline{\mathbf{x}_h} \in \mathcal{X}_h} \|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} + c_6 \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,,$$

where $c_5 = c_1\, c_3$ and $c_6 = c_2\, c_3 + c_4$.

5.2.8 Asymptotic rate of convergence

Recalling that at the solution of the continuum mixed problem we have $\mathbf{g} = \mathbf{o}$, the error bounds $\Sigma_{17})$ and $\Sigma_{20})$ given by

$$\|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} \leq c_1 \,\|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} + c_2 \,\|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,,$$

$$\|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq c_5 \,\|\, \mathbf{x} - \overline{\mathbf{x}_h} \,\|_{\mathcal{X}} + c_6 \,\|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}}\,,$$

can be written in terms of the three fields $\{\, \mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h \,\}$ as

$$
\Sigma_{21}) \qquad
\begin{aligned}
&\|\, \mathbf{u} - \mathbf{u}_h \,\|_{\mathcal{L}} + \|\, \mathbf{g}_h \,\|_{\mathcal{H}} \le \alpha_1 \left(\|\, \mathbf{u} - \overline{\mathbf{u}_h} \,\|_{\mathcal{L}} + \|\, \overline{\mathbf{g}_h} \,\|_{\mathcal{H}} + \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \right) , \\
&\|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \le \alpha_2 \left(\|\, \mathbf{u} - \overline{\mathbf{u}_h} \,\|_{\mathcal{L}} + \|\, \overline{\mathbf{g}_h} \,\|_{\mathcal{H}} + \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \right) ,
\end{aligned}
$$

for any $\overline{\mathbf{u}_h} \in \mathcal{L}_h$, $\overline{\mathbf{g}_h} \in \mathcal{D}_h$ and $\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h$, where $\alpha_1 = \max\{\, c_1, c_2 \,\}$ and $\alpha_2 = \max\{\, c_5, c_6 \,\}$. Inequalities $\Sigma_{21})$ can also be rewritten as

$$
\Sigma_{22}) \qquad
\begin{aligned}
&\|\, \mathbf{u} - \mathbf{u}_h \,\|_{\mathcal{L}} \le \alpha_1 \left(\inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \|\, \mathbf{u} - \overline{\mathbf{u}_h} \,\|_{\mathcal{L}} + \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \right) , \\
&\|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \le \alpha_2 \left(\inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \|\, \mathbf{u} - \overline{\mathbf{u}_h} \,\|_{\mathcal{L}} + \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_{\mathcal{H}} \right) .
\end{aligned}
$$

Let the displacement solution be smooth enough to ensure that $\mathbf{u} \in H^2(\Omega)$ and the elasticity be regular. The stress solution will then be such that $\boldsymbol{\sigma} \in H^1(\Omega)$.

We now consider mixed finite element approximations of two-dimensional elastostatic problems based on quadrilateral elements Q_1 with standard bilinear interpolations of the displacement field. The stress interpolation is the one introduced by Pian and Sumihara which is able to reproduce arbitrary constant states of stress in each element enriched by some linear components [41].

Such a quadrilateral finite element discretization enjoys the following two properties: the displacement shape functions generate the vectorial polynomial subspace $\mathrm{Q}_1(\,\square\,)$ whose components are arbitrary polynomials of degree at most one in each variable, the stress shape functions generate a tensorial subspace containing $\mathrm{Q}_0(\,\square\,)$ whose components are arbitrary constant tensors.

Then a standard result of polynomial approximation theory ensures that

$$
\inf_{\overline{\mathbf{u}_h} \in \mathcal{L}_h} \|\, \mathbf{u} - \overline{\mathbf{u}_h} \,\|_1 \le \alpha_{\mathbf{u}}\, h \,|\, \mathbf{u} \,|_2 \,, \quad \inf_{\overline{\boldsymbol{\sigma}_h} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}_h} \,\|_0 \le \alpha_{\boldsymbol{\sigma}}\, h \,|\, \boldsymbol{\sigma} \,|_1 \,,
$$

where $\|\cdot\|_m$ is the norm in the Sobolev space $H^m(\Omega)$ and $|\cdot|_m$ is the corresponding seminorm involving only derivatives of total order m.

From the error bounds $\Sigma_{21})$ we infer the following linear estimates for the rate of convergence of the approximate solution to the exact one in terms of energy norms

$$
\|\, \mathbf{u} - \mathbf{u}_h \,\|_1 \le \beta_{\mathbf{u}}\, h \left(|\, \mathbf{u} \,|_2 + |\, \boldsymbol{\sigma} \,|_1 \right) , \qquad \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_0 \le \beta_{\boldsymbol{\sigma}}\, h \left(|\, \mathbf{u} \,|_2 + |\, \boldsymbol{\sigma} \,|_1 \right) ,
$$

where $\beta_{\mathbf{u}} = \max\{\, \alpha_1 \alpha_{\mathbf{u}}, \alpha_1 \alpha_{\boldsymbol{\sigma}} \,\}$ and $\beta_{\boldsymbol{\sigma}} = \max\{\, \alpha_2 \alpha_{\mathbf{u}}, \alpha_2 \alpha_{\boldsymbol{\sigma}} \,\}$.

Linear estimates also hold for simplicial (e.g. triangular) mixed finite elements in which the displacement shape functions generate the vectorial polynomial subspace $P_1(\square)$ whose components are arbitrary polynomials of total degree at most one and the stress shape functions generate a tensorial subspace containing $P_0(\square)$ whose components are arbitrary constant tensors.

It is worth noting that, as was to be expected, no role is played by the shape functions of the strain gap in determining the asymptotic rate of convergence. In fact the exact strain gap is zero and hence every interpolating subspace can be adopted.

Numerical evidence shows a superlinear rate of convergence in the energy norms [83].

5.2.9 Applicable sufficient conditions

The analysis performed above has shown that, in order to get unconditioned existence, uniqueness of the approximate solution and an estimate of the mean square error, we have to enforce the conditions reported in the next **Table I**

a)	uniqueness of the $\{\mathbf{u}_h, \mathbf{g}_h\}$ solution	$\Longleftrightarrow$	$\operatorname{Ker} \mathbf{A}_{oh} = \{\boldsymbol{o}\}_{\mathcal{X}}$,
b)	uniqueness of the $\boldsymbol{\sigma}_h$ solution ,	$\Longleftrightarrow$	$\operatorname{Ker} \mathbf{J}'_h = \{\boldsymbol{o}\}_{\mathcal{H}}$,
c)	$h-$ uniform closedness of $\operatorname{Im} \mathbf{J}'_h$ expressed by the inequality $\Sigma_{bis})$,		
d)	$h-$ uniform closedness of $\operatorname{Im} \mathbf{A}_{oh}$ implied by the ellipticity property $\Sigma_{ter})$.		

Table I

The assumption that $\operatorname{Ker} \mathbf{B} \cap \mathcal{L}_h = \{\boldsymbol{o}\}_{\mathcal{L}}$ ensures that the uniqueness condition (a) is equivalent to the linear independence condition $\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\boldsymbol{o}\}_{\mathcal{H}}$.
The stress uniqueness condition (b) is equivalent to the null intersection property $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\boldsymbol{o}\}_{\mathcal{H}}$.
The h-uniform closedness (c) of $\operatorname{Im} \mathbf{J}'_h = \mathcal{L}_h^{\perp} \times (\mathcal{S}_h + \mathcal{D}_h^{\perp})$ is equivalent to require that $\mathcal{S}_h + \mathcal{D}_h^{\perp}$ is h-uniformly closed in $\mathcal{H}$.
The ellipticity property (d) of $\mathbf{a}$ on $\operatorname{Ker} \mathbf{J}_h$ is implied by the h-uniform closedness of $\mathbf{B}\mathcal{L}_h + \widetilde{\mathcal{D}}_h$.

Conditions of **Table I** reduce then to the ones reported in the next **Table II**.

i)	$\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}\,,$
ii)	$\mathbf{B}\mathcal{L}_h + \widetilde{\mathcal{D}}_h\; h-$ uniformly closed in $\mathcal{H}\,,$
iii)	$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}\,,$
iv)	$\mathcal{S}_h + \mathcal{D}_h^{\perp}\; h-$ uniformly closed in $\mathcal{H}\,.$

Table II

Conditions *iii)* and *iv)* are local in character and can be verified element-wise. The h-uniform closedness of $\mathcal{S}_h + \mathcal{D}_h^{\perp}$ in $\mathcal{H}$ can be imposed by means of the local inequality

$$\|\,\mathbf{\Pi}_{\mathcal{D}_h}\boldsymbol{\sigma}_h\,\|_{\mathcal{H}} \geq c\,\|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}} \quad \forall\,\boldsymbol{\sigma}_h \in \mathcal{S}_h\,,\; c > 0\,,$$

equivalent to the inequality $\|\,\mathbf{\Pi}_{\mathcal{S}_h}\mathbf{g}_h\,\|_{\mathcal{H}} \geq c\,\|\,\mathbf{g}_h\,\|_{\mathcal{H}}$ for any $\mathbf{g}_h \in \mathcal{D}_h$ with $c > 0$.

On the contrary, condition *i)* is global due to the presence of the conforming subspace $\mathcal{L}_h$ which depends on the *a priori* unknown element assembly operations.

This shortcoming can be circumvented by considering the larger non-conforming displacement space $\mathcal{V}_h \supseteq \mathcal{L}_h$ formed by the cartesian product of the local displacement spaces generated by the displacement shape functions over the single elements.

The global condition $\mathbf{B}\mathcal{L}_h \cap \widetilde{\mathcal{D}}_h = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}$ can then be substituted by the sufficient local condition $\mathbf{B}\mathcal{V}_h \cap \widetilde{\mathcal{D}}_h = \{\,\boldsymbol{o}\,\}_{\mathcal{H}}$.

The global condition *ii)* can also be substituted by a local sufficient condition. In fact we observe that the h-uniform closedness of $\mathbf{B}\mathcal{L}_h + \widetilde{\mathcal{D}}_h$ in $\mathcal{H}$ can be stated in terms of the inequality

$$\Sigma_{23}) \qquad \|\,\mathbf{\Pi}_{\mathbf{B}\mathcal{L}_h}\mathbf{g}_h\,\|_{\mathcal{H}} \leq \theta\,\|\,\mathbf{g}_h\,\|_{\mathcal{H}} \quad \forall\,\mathbf{g}_h \in \widetilde{\mathcal{D}}_h \quad \theta < 1\,.$$

Since the inclusion $\mathbf{B}\mathcal{L}_h \subseteq \mathbf{B}\mathcal{V}_h$ implies that $\|\,\mathbf{\Pi}_{\mathbf{B}\mathcal{L}_h}\mathbf{g}_h\,\|_{\mathcal{H}} \leq \|\,\mathbf{\Pi}_{\mathbf{B}\mathcal{V}_h}\mathbf{g}_h\,\|_{\mathcal{H}}$, we see that the inequality $\Sigma_{23})$ can be inferred by the local one

$$\|\,\mathbf{\Pi}_{\mathbf{B}\mathcal{V}_h}\mathbf{g}_h\,\|_{\mathcal{H}} \leq \theta\,\|\,\mathbf{g}_h\,\|_{\mathcal{H}} \quad \forall\,\mathbf{g}_h \in \widetilde{\mathcal{D}}_h \quad \theta < 1\,.$$

In conclusion a sufficient set of conditions is provided by the following local requirements.

i)	$\mathbf{B}\mathcal{V}_h \cap \widetilde{\mathcal{D}}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}\,,$
ii)	$\|\, \mathbf{\Pi}_{\mathbf{B}\mathcal{V}_h} \mathbf{g}_h \,\|_{\mathcal{H}} \le \theta \,\|\, \mathbf{g}_h \,\|_{\mathcal{H}} \quad \forall\, \mathbf{g}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp} \quad \theta < 1\,,$
iii)	$\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}\,,$
iv)	$\|\, \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \ge c \,\|\, \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \quad \forall\, \boldsymbol{\sigma}_h \in \mathcal{S}_h\,,\ c > 0\,.$

Table III

Remark 5.2.4 *A natural choice for the control strain gap subspace* $\mathcal{D}_h^*$ *is* $\mathcal{D}_h^* = \mathcal{S}_h$ *so that we have* $\mathcal{D}_h = \mathcal{S}_h \oplus \widetilde{\mathcal{D}}_h$*.*

In this case the uniqueness stress condition iii) is trivial since $\mathcal{S}_h \cap \mathcal{S}_h^{\perp} \cap \widetilde{\mathcal{D}}_h^{\perp} = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}$*. The h-uniform closedness iv) of* $\mathcal{S}_h + \mathcal{D}_h^{\perp}$ *in* $\mathcal{H}$ *is also fulfilled. In fact, being* $\mathcal{S}_h \subset \mathcal{D}_h$*, we have* $\|\, \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} = \|\, \boldsymbol{\sigma}_h \,\|_{\mathcal{H}}$ *so that the inequality iv) in Table III is trivially fulfilled with* $c = 1$*.*

Accordingly the set of sufficient conditions given in Table III reduces to

$$i)\quad \mathbf{B}\mathcal{V}_h \cap \widetilde{\mathcal{D}}_h = \{\, \boldsymbol{o} \,\}_{\mathcal{H}}\,,$$

$$ii)\quad \|\, \mathbf{\Pi}_{\mathbf{B}\mathcal{V}_h} \mathbf{g}_h \,\|_{\mathcal{H}} \le \theta \,\|\, \mathbf{g}_h \,\|_{\mathcal{H}} \quad \forall\, \mathbf{g}_h \in \mathcal{D}_h \cap \mathcal{S}_h^{\perp} \quad \theta < 1\,.$$

However what we really need are conditions susceptible to be verified on the reference element K of an isoparametric finite element mesh. This is a fundamental but very stringent requirement.

In this respect we recall that the kinematic operator $\mathbf{B}_K^e$, acting on the displacement fields in the reference element K will be denoted by $\mathbf{B}_K^e$ and is defined by $\mathbf{B}_K^e \mathbf{u}_K(\boldsymbol{\xi}) = \mathbf{B}\mathbf{u}_h^e(\mathbf{x})$ for any $\mathbf{x} \in \Omega_e$ where $\mathbf{u}_h^e$ is the restriction of $\mathbf{u}_h$ to the element Ω^e and $\boldsymbol{\xi} = \boldsymbol{\chi}_e^{-1}(\mathbf{x})$ is the isoparametric coordinate in the reference element.

In the case of undistorted elements $\mathbf{B}_K^e$ and $\mathbf{B}$ are proportional through a constant depending on the mesh size h, i.e. $\mathbf{B}_h^e = h\mathbf{B}$. This proportionality implies that the subspace $\mathbf{B}_K^e \mathcal{V}_K$ is equal to $\mathbf{B}\mathcal{V}_K$ and hence is independent of h. Accordingly the h-uniform closedness condition $ii)$, concerning $\widetilde{\mathcal{D}}_K + \mathbf{B}_K^e \mathcal{V}_K$, is trivially fulfilled.

The condition i), $\mathbf{B}^e_K \mathcal{V}_K \cap \widetilde{\mathcal{D}}_K = \mathbf{B}\mathcal{V}_K \cap \widetilde{\mathcal{D}}_K = \{\,\mathbf{o}\,\}$, can be checked by evaluating the GRAM determinant [12] of a set of shape functions spanning the subspace $\mathbf{B}\mathcal{V}_K \times \widetilde{\mathcal{D}}_K$.

The GRAM determinant of this set of shape functions is different from zero if and only if the generator is linearly independent [25]] and this requirement is equivalent to the null intersection property i).

For general isoparametric maps, the conditions i) and ii) cannot be checked in the reference element, a drawback which seems to have been overridden in previous analyses [71], [55], [73].

The condition $\widetilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}^{\perp}_h \neq \{\mathbf{o}\}$ requires the evaluation of an inner product and then an integration on the reference element which involves an unknown jacobian determinant.

This condition can be effectively checked in terms of the subspaces $\mathcal{D}_K$ and $\mathcal{S}_K$ defined in the reference element by means of the change of coordinates described by the map $\boldsymbol{\chi}_e^{-1}$.

No problem arises if we consider affine equivalent finite element meshes since the constant jacobian determinant is irrilevant in imposing orthogonality conditions. A skilful trick was proposed in [55] in order to overcome this difficulty by defining the shape functions of the strain gaps in the reference element as the quotient of simple polynomial expressions divided by the jacobian deteminant.

It follows that, in performing the integral transformation, the jacobian deteminant disappears from the integrals over the reference element and the orthogonality conditions can be simply verified once and for all in terms of simple polynomial expressions. This procedure has been discussed in [83] for the SGM and was also followed in the convergence analysis of the EAS method developed in [71]. It is important to point out that this procedure rely on the fact that the approximation properties of the subspace $\mathcal{D}_K$ do not play any role in the estimate of the asymptotic rate of convergence.

5.3 Conclusion

The analysis of the SGM carried out in this chapter provides a variationally consistent reformulation of the EAS method.

The merits of this new formulation are twofold.

- The error estimates and the convergence analysis can be performed in the standard framework of mixed methods and are based on the approximation properties of displacement and stress fields. In contrast to previous treatments, the new analysis leads to the correct conclusion that the strain gap approximation plays no role in estimating the error and the asymptotic convergence properties.

- The SGM method splits naturally in the sequence of two steps: the former coincides with the EAS method and involves effective strain gaps and displacements. The latter provides an answer to the longly debated issue of a stress recovery procedure.

5.4 Appendix

A useful list of equivalent closedness conditions, recently contributed in [89], and referred to in chapter 5, is reported hereafter.

Proposition 5.4.1 (Equivalent closedness properties) *Let $\mathcal{X}$ be a* HILBERT *space and $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{X}$ closed linear subspaces in $\mathcal{X}$ such that their sum $\mathcal{A} + \mathcal{B}$ is direct, that is $\mathcal{A} \cap \mathcal{B} = \{ \boldsymbol{o} \}$.*
Then the following properties are equivalent one another:

$$
\Sigma_{24}) \quad \begin{cases}
i) & \mathcal{A} \oplus \mathcal{B} \quad \text{closed in } \mathcal{X} \iff \mathcal{A} \oplus \mathcal{B} = (\mathcal{A}^\perp \cap \mathcal{B}^\perp)^\perp \quad , \\
ii) & \mathcal{A}^\perp + \mathcal{B}^\perp \quad \text{closed in } \mathcal{X}' \iff \mathcal{A}^\perp + \mathcal{B}^\perp = (\mathcal{A} \cap \mathcal{B})^\perp \, , \\
iii) & \begin{cases} \| \mathbf{a} + \mathbf{b} \|_{\mathcal{X}} \geq c \, \| \mathbf{a} \|_{\mathcal{X}} \\ \| \mathbf{a} + \mathbf{b} \|_{\mathcal{X}} \geq c \, \| \mathbf{b} \|_{\mathcal{X}} \end{cases} \quad \forall \, \mathbf{a} \in \mathcal{A} \, , \quad \forall \, \mathbf{b} \in \mathcal{B} \, , \\
iv) & \sup\limits_{\{ \mathbf{a}, \mathbf{b} \} \in \mathcal{A} \times \mathcal{B}} \dfrac{(\mathbf{a}, \mathbf{b})_{\mathcal{X}}}{\| \mathbf{a} \|_{\mathcal{X}} \, \| \mathbf{b} \|_{\mathcal{X}}} \leq \theta = 1 - c^2/2 < 1 \, , \\
v) & \begin{cases} \| \mathbf{\Pi}_{\mathcal{A}} \mathbf{b} \|_{\mathcal{X}} \leq \theta \, \| \mathbf{b} \|_{\mathcal{X}} \quad \forall \, \mathbf{b} \in \mathcal{B} \, , \\ \| \mathbf{\Pi}_{\mathcal{B}} \mathbf{a} \|_{\mathcal{X}} \leq \theta \, \| \mathbf{a} \|_{\mathcal{X}} \quad \forall \, \mathbf{a} \in \mathcal{A} \, , \end{cases} \quad 0 < \theta < 1 \, , \\
vi) & \begin{cases} \| \mathbf{x} \|_{\mathcal{X}} \leq \| \mathbf{x} \|_{\mathcal{X}/\mathcal{A}} + c^{-1} \, \| \mathbf{\Pi}_{\mathcal{A}} \mathbf{x} \|_{\mathcal{X}/\mathcal{B}} \quad \forall \, \mathbf{x} \in \mathcal{X} \, , \\ \| \mathbf{x} \|_{\mathcal{X}} \leq \| \mathbf{x} \|_{\mathcal{X}/\mathcal{B}} + c^{-1} \, \| \mathbf{\Pi}_{\mathcal{B}} \mathbf{x} \|_{\mathcal{X}/\mathcal{A}} \quad \forall \, \mathbf{x} \in \mathcal{X} \, , \end{cases} \\
vii) & \begin{cases} \| \mathbf{x} \|_{\mathcal{X}} \leq c^{-1} \, \| \mathbf{x} \|_{\mathcal{X}/\mathcal{A}} + (1 + c^{-1}) \, \| \mathbf{x} \|_{\mathcal{X}/\mathcal{B}} \quad \forall \, \mathbf{x} \in \mathcal{X} \, , \\ \| \mathbf{x} \|_{\mathcal{X}} \leq c^{-1} \, \| \mathbf{x} \|_{\mathcal{X}/\mathcal{B}} + (1 + c^{-1}) \, \| \mathbf{x} \|_{\mathcal{X}/\mathcal{A}} \quad \forall \, \mathbf{x} \in \mathcal{X} \, , \end{cases} \\
viii) & \begin{cases} \mathbf{\Pi}_{\mathcal{A}^\perp} \, \mathcal{B} \text{ closed in } \mathcal{X} \iff \| \mathbf{\Pi}_{\mathcal{A}^\perp} \, \mathbf{b} \|_{\mathcal{X}} \geq c \, \| \mathbf{b} \|_{\mathcal{X}} \quad \forall \, \mathbf{b} \in \mathcal{B} \, , \\ \mathbf{\Pi}_{\mathcal{B}^\perp} \, \mathcal{A} \text{ closed in } \mathcal{X} \iff \| \mathbf{\Pi}_{\mathcal{B}^\perp} \, \mathbf{a} \|_{\mathcal{X}} \geq c \, \| \mathbf{a} \|_{\mathcal{X}} \quad \forall \, \mathbf{a} \in \mathcal{A} \, . \end{cases}
\end{cases}
$$

where $c > 0$ and $\mathbf{\Pi}_{(\bullet)}$ denotes the orthogonal projector on the linear subspace $(\bullet)$.

The proof of proposition 5.1.4 is provided hereafter. We have to prove that the $h-$uniform closedness of the family of linear subspaces $\operatorname{Im} \mathbf{A}_{oh} \subset \mathcal{X}'/(\operatorname{Ker} \mathbf{J}_h)^\perp$ is equivalent to the inequality

$$
\Sigma_{25}) \qquad \| \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_h \|_{\mathcal{H}} \geq \beta \, \| \boldsymbol{\sigma}_h \|_{\mathcal{H}} \quad \forall \, \boldsymbol{\sigma}_h \in \Sigma_h \, , \, \beta > 0 \, ,
$$

if the uniqueness condition $\operatorname{Ker}\mathbf{A}_{oh} = \{\, \boldsymbol{o} \,\}_{\mathcal{X}}$ is fulfilled.

Let us preliminarily provide a statement equivalent to the $h-$uniform closedness of $\operatorname{Im}\mathbf{A}_{oh}$ in $\mathcal{X}'/(\operatorname{Ker}\mathbf{J}_h)^{\perp}$ which, being $\operatorname{Ker}\mathbf{A}_{oh} = \{\, \boldsymbol{o} \,\}_{\mathcal{X}}$, is expressed by the inequality

$$\Sigma_{26})\quad \|\,\mathbf{A}_{oh}\mathbf{x}_h\,\|_{(\operatorname{Ker}\mathbf{J}_h)'} = \|\,\mathbf{A}_{oh}\mathbf{x}_h\,\|_{\mathcal{X}'/(\operatorname{Ker}\mathbf{J}_h)^{\perp}} \ge c_o\,\|\,\mathbf{x}_h\,\|_{\mathcal{X}}\,, \quad \forall\, \mathbf{x}_h \in \operatorname{Ker}\mathbf{J}_h\,.$$

Recalling that $\operatorname{Ker}\mathbf{J}_h = \mathcal{D}_h \times \Sigma_h$ and hence $(\operatorname{Ker}\mathbf{J}_h)^{\perp} = \mathcal{D}_h^{\perp} \times \Sigma_h^{\perp}$, the definition of the norm in the factor space $\mathcal{X}'/(\operatorname{Ker}\mathbf{J}_h)^{\perp}$ yields

$$\|\,\mathbf{A}_{oh}\mathbf{x}_h\,\|_{\mathcal{X}'/(\operatorname{Ker}\mathbf{J}_h)^{\perp}} = \inf_{\boldsymbol{\tau}\in\mathcal{D}_h^{\perp}} \|\,\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h - \boldsymbol{\tau}\,\|_{\mathcal{H}} + \inf_{\boldsymbol{\eta}\in\Sigma_h^{\perp}} \|\,\boldsymbol{\varepsilon}_h + \boldsymbol{\eta}\,\|_{\mathcal{H}}\,.$$

By the projection theorem in HILBERT spaces we have that

$$\inf_{\boldsymbol{\tau}\in\mathcal{D}_h^{\perp}} \|\,\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h - \boldsymbol{\tau}\,\|_{\mathcal{H}} = \|\,\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h - \boldsymbol{\Pi}_{\mathcal{D}_h^{\perp}}(\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h)\,\|_{\mathcal{H}}\,,$$

$$\inf_{\boldsymbol{\eta}\in\Sigma_h^{\perp}} \|\,\boldsymbol{\varepsilon}_h - \boldsymbol{\eta}\,\|_{\mathcal{H}} = \|\,\boldsymbol{\varepsilon}_h - \boldsymbol{\Pi}_{\Sigma_h^{\perp}}\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}}\,.$$

Now, recalling that $\|\,\mathbf{x}_h\,\|_{\mathcal{X}} \ge \sqrt{\|\,\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}}^2 + \|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}}^2}$, the inequality $\Sigma_{26})$ takes the equivalent form

$$\Sigma_{27})\quad \begin{aligned} &\|\,\boldsymbol{\Pi}_{\mathcal{D}_h}(\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h)\,\|_{\mathcal{H}} + \|\,\boldsymbol{\Pi}_{\Sigma_h}\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}} \ge c_o\,\sqrt{\|\,\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}}^2 + \|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}}^2}\,,\\ &\forall\, \{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \mathcal{D}_h \times \Sigma_h \end{aligned}$$

which can in turn be equivalently stated as

$$\Sigma_{28})\qquad \|\,\boldsymbol{\Pi}_{\mathcal{D}_h}(\mathcal{E}\boldsymbol{\varepsilon}_h - \boldsymbol{\sigma}_h)\,\|_{\mathcal{H}} + \|\,\boldsymbol{\Pi}_{\Sigma_h}\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}} \ge c_o \quad \forall\, \{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \mathcal{S}\,,$$

where

$$\mathcal{S} = \big\{ \{\boldsymbol{\varepsilon}_h\,, \boldsymbol{\sigma}_h\} \in \mathcal{D}_h \times \Sigma_h \;:\; \|\,\boldsymbol{\varepsilon}_h\,\|_{\mathcal{H}}^2 + \|\,\boldsymbol{\sigma}_h\,\|_{\mathcal{H}}^2 = 1 \big\}\,,$$

is the spherical surface of unit radius in the HILBERT space $\mathcal{D}_h \times \Sigma_h$.

We prove that $\Sigma_{25})$ implies $\Sigma_{28})$ by contradiction. If the inequality $\Sigma_{28})$ were false it should exist a sequence $\{\boldsymbol{\varepsilon}_n\,, \boldsymbol{\sigma}_n\} \in \mathcal{S}$ such that

$$\|\,\boldsymbol{\Pi}_{\mathcal{D}_h}(\mathcal{E}\boldsymbol{\varepsilon}_n - \boldsymbol{\sigma}_n)\,\|_{\mathcal{H}} + \|\,\boldsymbol{\Pi}_{\Sigma_h}\boldsymbol{\varepsilon}_n\,\|_{\mathcal{H}} \to 0\,.$$

Then from SCHWARZ inequality it would follow that

$$\| \mathbf{\Pi}_{\mathcal{D}_h} (\mathcal{E}\boldsymbol{\varepsilon}_n - \boldsymbol{\sigma}_n) \|_{\mathcal{H}} \| \boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \geq \big| (\!(\mathcal{E}\boldsymbol{\varepsilon}_n , \boldsymbol{\varepsilon}_n)\!)_{\mathcal{H}} - (\!(\boldsymbol{\sigma}_n , \boldsymbol{\varepsilon}_n)\!)_{\mathcal{H}} \big| \to 0 ,$$

$$\| \mathbf{\Pi}_{\Sigma_h} \boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \| \boldsymbol{\sigma}_n \|_{\mathcal{H}} \geq \big| (\!(\boldsymbol{\sigma}_n , \boldsymbol{\varepsilon}_n)\!)_{\mathcal{H}} \big| \to 0 .$$

Then $(\!(\mathcal{E}\boldsymbol{\varepsilon}_n , \boldsymbol{\varepsilon}_n)\!) \to 0$ and the $\mathcal{H}$-ellipticity of $\mathcal{E}$ implies that $\| \boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \to 0$. By the continuity of $\mathcal{E}$ it follows that $\| \mathcal{E}\boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \to 0$ so that $\| \mathbf{\Pi}_{\mathcal{D}_h} \mathcal{E}\boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \to 0$. Being

$$\| \mathbf{\Pi}_{\mathcal{D}_h} (\mathcal{E}\boldsymbol{\varepsilon}_n - \boldsymbol{\sigma}_n) \|_{\mathcal{H}} \geq \big| \| \mathbf{\Pi}_{\mathcal{D}_h} (\mathcal{E}\boldsymbol{\varepsilon}_n) \|_{\mathcal{H}} - \| \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_n \|_{\mathcal{H}} \big| ,$$

we infer that $\| \mathbf{\Pi}_{\mathcal{D}_h} \boldsymbol{\sigma}_n \|_{\mathcal{H}} \to 0$ which together with inequality $\Sigma_{25})$ implies that $\| \boldsymbol{\sigma}_n \|_{\mathcal{H}} \to 0$. Then we would have that $\| \boldsymbol{\varepsilon}_n \|_{\mathcal{H}} \to 0$ and $\| \boldsymbol{\sigma}_n \|_{\mathcal{H}} \to 0$ and this is impossible since $\{\boldsymbol{\varepsilon}_n , \boldsymbol{\sigma}_n\} \in \mathcal{S}$.

The converse assertion that $\Sigma_{28})$ implies $\Sigma_{25})$ can be trivially verified by taking $\boldsymbol{\varepsilon}_h = \boldsymbol{o}$ in the inequality $\Sigma_{27})$.

Chapter 6

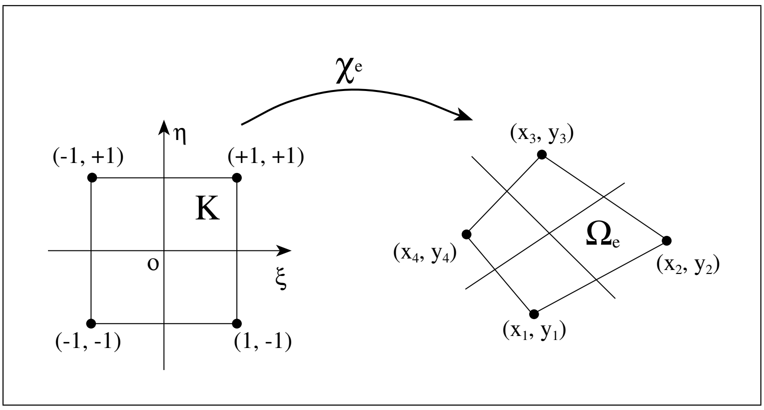
Computational analysis

6.1 Three-field mixed method

The isoparametric transformation $\boldsymbol{\chi}^e : K \mapsto \Omega_e$ maps the reference element K into a physical element Ω_e according to the transformation rule

$$\mathbf{x} = \boldsymbol{\chi}^e(\boldsymbol{\xi})$$

where $\mathbf{x} \in \Omega_e$ and $\boldsymbol{\xi} \in \boldsymbol{K}$ are the position vectors in the actual and the reference element.



The related gradient will be denoted by $\mathbf{F}^e = grad\boldsymbol{\chi}^e$ and the Jacobian determinant is given by $J^e = \det \mathbf{F}^e$ and coincides with the ratio V_e/V_K if the transformation is affine.

In the MES method the condition $\widetilde{\mathcal{D}}_h \subseteq \mathcal{S}_h^{\perp}$ can be effectively checked in terms of the subspaces $\widetilde{\mathcal{D}}_K$ and $\mathcal{S}_K$ defined in the reference element K.

The inner product in K is performed by an integration over the reference element which involves an unknown Jacobian determinant.

If we consider affine equivalent finite element meshes, the Jacobian determinant is constant and no problem arises in imposing the orthogonality conditions.

On the contrary, in the case of general isoparametric maps, the Jacobian determinant is not constant so that the integral of the product of two fields over the reference element is not proportional to the corresponding integral over an element of the mesh.

To overcome this difficulty Kasper and Taylor in [75] have adopted the procedure originarily proposed by Simo and Rifai in [55].

The procedure consists in defining the enhanced strain field $\widetilde{\boldsymbol{\varepsilon}}_h^e(\mathbf{x})$ in the actual element Ω_e as a function of the enhanced strain field $\widetilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi})$ in the reference element $\boldsymbol{K}$ by means of the transformation

$$\widetilde{\boldsymbol{\varepsilon}}_h^e(\mathbf{x}) := \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\, \widetilde{\boldsymbol{\varepsilon}}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]$$

where J_o^e is the value of $J^e(\boldsymbol{\xi})$ evaluated at the baricenter of the reference element. Arunakirinathar and Reddy [71], Reddy and Simo [70], Romano et al. [83], Romano et al. [87].

Accordingly the $\mathcal{L}^2(\Omega_e)$ inner product between stress and strain fields over an actual element is given by

$$\begin{aligned}\int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) : \widetilde{\boldsymbol{\varepsilon}}(\mathbf{x})\, \mathtt{d}\mathbf{x} &= \int_{\Omega_e} \boldsymbol{\sigma}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] : \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\, \mathbf{g}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]\, \mathtt{d}\mathbf{x} \\ &= J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) : \mathbf{g}(\boldsymbol{\xi})\, \mathtt{d}\boldsymbol{\xi}\,,\end{aligned}$$

and hence the orthogonality condition imposed between fields in the reference element is preserved when considering the corresponding actual fields over the elements of the mesh.

A modified version od this procedure consists in envisaging transformation rules which point-wise preserve the local inner product between stress and strain tensors.

To this end we recall that, according to the standard rules of tensor calculus (see e.g. MARSDEN and HUGHES [38]), stress and enhanced strain vector fields defined in the reference element K are pushed forward to the physical element Ω_e by the matrix field

$$\mathbf{T}^e(\boldsymbol{\xi}) = \begin{bmatrix} F_{11}^2 & F_{12}^2 & 2F_{11}F_{12} \\ F_{21}^2 & F_{22}^2 & 2F_{21}F_{22} \\ F_{11}F_{21} & F_{12}F_{22} & F_{11}F_{22} + F_{12}F_{21} \end{bmatrix}^e (\boldsymbol{\xi}) \qquad \text{with} \qquad \boldsymbol{\xi} \in K\,.$$

The stress and enhanced strain fields $\boldsymbol{\varepsilon}_h^e$ and $\boldsymbol{\sigma}_h^e$ over the actual element are defined by

$$\widetilde{\boldsymbol{\varepsilon}}_h^e(\mathbf{x}) := \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\,\mathbf{T}_o^{e-T}\,\widetilde{\boldsymbol{\varepsilon}}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] \qquad \boldsymbol{\sigma}_h^e(\mathbf{x}) := \mathbf{T}_o^e\boldsymbol{\sigma}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]$$

where $\mathbf{T}_o^e$ and J_o^e are the values of $\mathbf{T}^e$ and J^e evaluated at the baricenter of the reference element. Hence the inner product in the real space is given by

$$\begin{aligned} \int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) : \widetilde{\boldsymbol{\varepsilon}}_h^e(\mathbf{x})\, d\mathbf{x} &= \int_{\Omega_e} \mathbf{T}_o^e\boldsymbol{\sigma}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})] : \frac{J_o^e}{J^e[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]}\mathbf{T}_o^{e-T}\widetilde{\boldsymbol{\varepsilon}}\,[\boldsymbol{\chi}_e^{-1}(\mathbf{x})]\, d\mathbf{x} \\ &= J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) : \mathbf{g}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \end{aligned}$$

It should be underlined that the adoption of such transformation rules can only be motivated by an a posteriori evaluation of the quality of the numerical results in each special case under investigation.

6.1.1 Shape functions for the Mixed Enhanced Strain Method

Let us now adopt a standard four-node bilinear isoparametric square element $\square = [-1,1] \times [-1,1]$ for the computational analysis of plane problems.
According to the decomposition $\gamma)$ of chapter 4, the shape functions are given by

$$\alpha) \qquad \mathbf{N}_{\mathbf{e}}^{\square} = \mathbf{N}_{\boldsymbol{\sigma}}^{\square} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_{\boldsymbol{\alpha}}^{\square} = \begin{bmatrix} \xi & 0 \\ 0 & \eta \\ 0 & 0 \end{bmatrix}$$

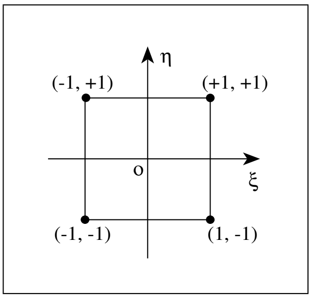


Figure 6.1: Reference Element

and the orthogonality condition $\mathbf{Q}^{\square}_{\boldsymbol{\alpha}\boldsymbol{\sigma}} = \mathbf{O}$ is fulfilled. It is worth noting that the special form of the stiffness matrix $\epsilon)$ in chapter 4 can be adopted if the orthogonality is preserved in the actual element.

The stress shape functions $\alpha)_1$ coincide with the ones proposed by PIAN and SUMIHARA in [41].

The subspace $\mathbf{B}\mathcal{V}_{\square}$ of compatible strains is given by

$$\mathbf{B}\mathcal{V}_{\square} = \text{span} \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{bmatrix} .$$

The well-posedness condition $\Delta_{bis})$ provided in chapter 5 can be checked as follows.

- Consider the lists of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_5\}$, which represent the columns of the matrices $\mathbf{N}^{\square}_{\boldsymbol{\sigma}}$ and $\mathbf{B}\mathcal{V}_{\square}$,
- evaluate the GRAM matrix $G_{ij} = \int_{\square} \mathbf{a}_i \cdot \mathbf{b}_j$.

The GRAM matrix is non singular if if and only if the well-posedness condition is met. In this case the GRAM determinant is positive.

6.1.2 Shape functions for the Strain Gap Method

Let us preliminarily consider some shape functions adopted in the literature in the context of the EAS method for plane problems with reference to a standard four-node bilinear isoparametric square element $\square = [-1, 1] \times [-1, 1]$.

A five-parameter interpolation for the strain gap field is provided in [55] starting from the six-parameter strain interpolation of WILSONet al. incompatible element.

The shape functions for the strain gaps are:

$$\beta) \qquad \mathbf{N}_{\mathbf{g}}^{\square} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta \\ 0 & \eta & 0 & 0 & -\xi\eta \\ 0 & 0 & \xi & \eta & \xi^2-\eta^2 \end{bmatrix} .$$

Note that, deleting the last column of $\mathbf{N}_{\mathbf{g}}^{\square}$, we obtain the shape functions pertaining to the modified incompabile mode approximation of TAYLOR et al. [42]:

$$\gamma) \qquad \widehat{\mathbf{N}}_{\mathbf{g}}^{\square} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} .$$

A seven-parameter strain gap interpolation has been assumed in [61], [60] which is given by:

$$\delta) \qquad \mathbf{N}_{\mathbf{g}}^{\square} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi\eta & 0 \\ 0 & 0 & \xi & \eta & 0 & 0 & \xi\eta \end{bmatrix} .$$

Over the standard four-node isoparametric element $\square$ the strain gap shape functions $\beta)-\delta)$ and the PIAN and SUMIHARA stress shape functions [41] given by

$$\epsilon) \qquad \mathbf{N}_{\boldsymbol{\sigma}}^{\square} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

are apparently mutually orthogonal in the $L^2(\square)$ inner product. According to the decomposition $\xi)$ in the chapter 4, the effective strain gap interpolation of the SGM is given by $\beta)$.

The strain gap shape functions are the collection of $\epsilon)$ and $\beta)$.

The well-posedness condition requires that compatible and effective strain gap shape functions must be linear independent.

Noting that the compatible strain subspace $\mathbf{B}\mathcal{V}_{\square}$ is given by

$$\mathbf{B}\mathcal{V}_{\square} = \text{span} \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{bmatrix} ,$$

well-posedness can be checked by considering the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{15}\}$, which represent the columns of the set $\{\mathbf{B}\mathcal{V}_\square, \mathbf{N}_\mathbf{g}^\square\}$ and imposing that the GRAM matrix

$$G_{ij} = \int_\square \mathbf{a}_i \cdot \mathbf{a}_j ,$$

is not singular.

For triangular elements the coordinate system in the reference triangle element $\triangle$ is given by $\{\xi\,,\eta\}$ with $0 \le \xi \le 1\,,\, 0 \le \eta \le 1-\xi$.

In VOIGT's notation the adopted shape functions are expressed by

$$\mathcal{V}_\triangle = span \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta \end{bmatrix} .$$

$$\mathcal{S}_\triangle = span \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad \mathbf{B}\mathcal{V}_\triangle = span \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the stresses associated with the compatible strains are given by

$$\mathcal{E}\,\mathbf{B}\mathcal{V}_\triangle = span\ \frac{\mathcal{E}}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = span \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is then apparent that any choice of the effective strain gaps $\widetilde{\mathcal{D}}_\triangle = \mathcal{D}_\triangle \cap \mathcal{S}_\triangle^\perp$ will meet the orthogonality property $\widetilde{\mathcal{D}}_\triangle \subset (\mathcal{E}\,\mathbf{B}\mathcal{V}_\triangle)^\perp$.

For these triangular elements the jacobian of the isoparametric map is constant and hence the othogonality relations are preserved. It follows that we have also that $\widetilde{\mathcal{D}}_h \subset (\mathcal{E}\,\mathbf{B}\mathcal{V}_h)^\perp$.

As a consequence the limitation phenomenon quoted in proposition 4.9.2 occurs and the mixed method collapses into the displacement method.

6.2 Numerical examples

The numerical performances of the analysed three-field mixed methods are evaluated with reference to three examples selected from the literature and compared with the HELLINGER-REISSNER (HR) and the standard displacement methods.

The examples are two-dimensional in plane stress state and the material behaviour is linearly elastic and isotropic. A square four-node isoparametric element is adopted.

■ **Cook membrane problem**

Let us consider the Cook membrane problem reported in the next figure.

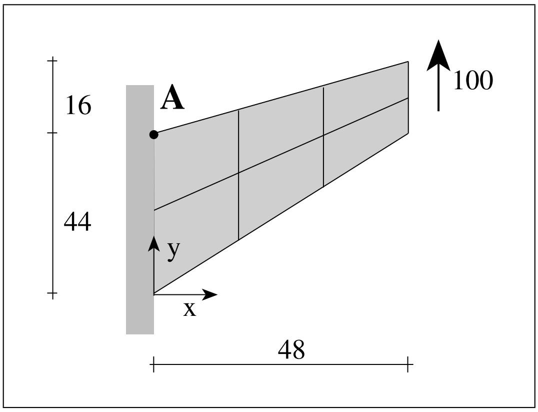


Figure 6.2: Cook's membrane

The Young's modulus and the Poisson's ratio are

$$E = 250 \quad \nu = 0.4999,$$

so that a nearly incompressible response is obtained, see Kasper and Taylor [75]. A uniformly distribuited in-plane shearing load with total value 100 is applied on the free end.

The next figure shows a graph of the vertical tip deflection obtained by adopting the standard displacement methods, the SGM, the HR, the SGM-p, the HW and the MES.

The SGM-p plot is obtained by including the push-transformation in the definition of strain and stress shape functions.

The superior coarse mesh accuracy achieved by the enhanced strain or assumed stress elements is apparent.

This example illustrates the computational performances relevant to different implementations of the discrete methods.

In particular the SGM plot shows the results obtained following the general formulation of the strain gap method (Romano et al. [85] in which the

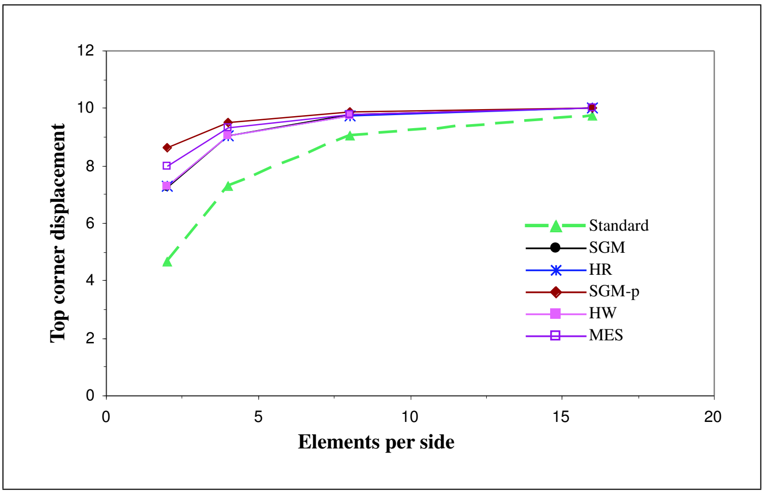


Figure 6.3:

orthogonality constraint is considered as one of the equations of the discrete mixed problem).

Note that the EAS method and the SGM-p provide similar results, see ROMANO ET AL. [85].

The three-field HW method and the two-field HR method provide comparably good results for coarse meshes and the same convergence behaviour of the SGM and of the MES method.

The displacement method exibits a rather poor performance for coarse meshes but no locking phenomenon is shown in contrast with the numerical result reported by KASPER and TAYLOR in [75].

In Figg. 6.4, 6.5 and 6.6 the normal and tangential stresses σ_x, σ_y and τ_{xy} at the node A are plotted.

These figures clearly show a poor performance of the various mixed methods and a lack of convergence in terms of local values of the stress fields.

■ Rectangular plates under parabolic shearing end loads

The next example consists in a rectangular plate subjected to a parabolic distributed shearing load with maximum intensity 100 (see fig. 6.7). The values of equivalent YOUNG's modulus and POISSON's ratio are respectively $E = 250$ and $\nu = 0.4999$.

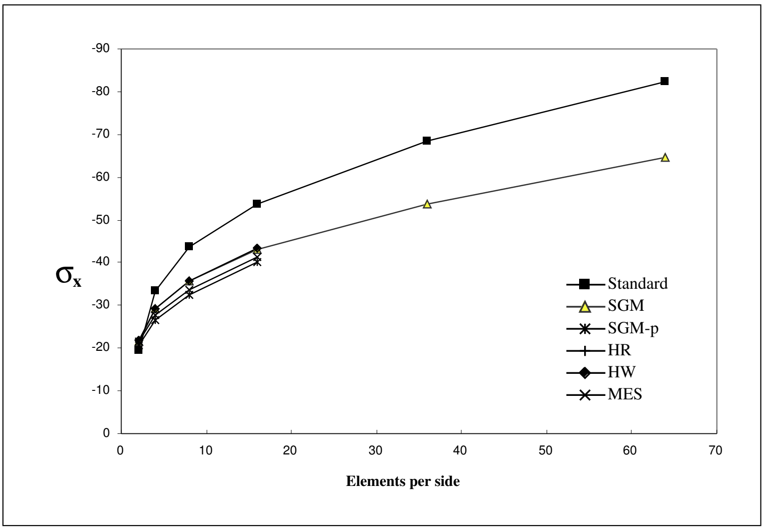


Figure 6.4:

The stress Airy function is given by

$$\varphi(x,y) = p\left(-\frac{3}{2}\,xy + \frac{l}{h^2}\,y^3 + \frac{2}{h^2}\,xy^3\right).$$

The stress tensor field can then be evaluated according to the expressions

$$\begin{aligned}
\sigma_x(x,y) &= \frac{\partial^2\varphi(x,y)}{\partial y^2} = \frac{6p}{h^2}\,(l+2x)\,y;\\
\sigma_y(x,y) &= \frac{\partial^2\varphi(x,y)}{\partial x^2} = 0;\\
\tau_{xy}(x,y) &= -\,\frac{\partial^2\varphi(x,y)}{\partial x\partial y} = \frac{3}{2}\,p\left(1-4\,\frac{y^2}{h^2}\right).
\end{aligned}$$

In Figg. 6.9 and 6.10 we consider the normal and tangential stresses σ_x, σ_y and τ_{xy} at the nodes A and B for the plate.

Since the mesh is undistorted, the Jacobian determinant and the push-transformation has no influence on the numerical performance of the SGM and of the MES method which coincides with the HR and the HW methods according to the limitation phenomenon of proposition 4.9.3.

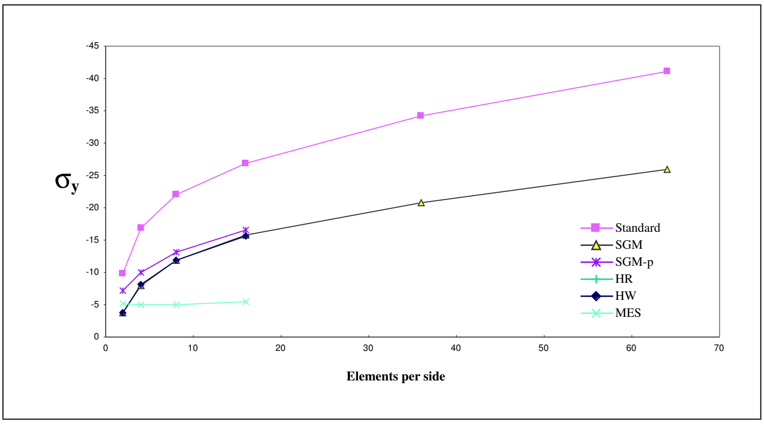


Figure 6.5:

It is worth noting that, for undistorted elements, the pointwise stresses are in a good agreement with the theoretical values.

■ Sensitivity to mesh distorsions

As a further example we analyse a classical benchmark, see e.g. ARUNAKIRINATHAR and REDDY [71], consisting in the bending problem of the rectangular plate reported in next figure to address the issue of sensitivity to mesh distorsions. The equivalent YOUNG's modulus and POISSON's ratio are $E = 1500$ and $\nu = 0.25$. The plate is constrained at one end and is subjected to a linearly distribuited axial load, equivalent to a couple with value 2000, at the other.

The analitical solution in terms of displacements and stresses is:

$$\begin{cases} u(x,y) = 2x(1-y)\,, \\ v(x,y) = x^2 + \frac{1}{4}(y^2 - 2y)\,, \end{cases} \qquad \begin{cases} \sigma_x(x,y) = 3000(1-y)\,, \\ \sigma_y(x,y) = 0\,, \\ \tau_{xy}(x,y) = 0\,. \end{cases}$$

The plate is discretized in two quadrilateral elements.

Figg. 6.12 and 6.13 and give the results for the vertical displacement of the points A and B measured against the distorsion parameter d.

- In the case of Fig. 6.12 it is evident that the push-transformation has a benefical effect for the SGM but the push-transformation has a negative effect for the MES method.

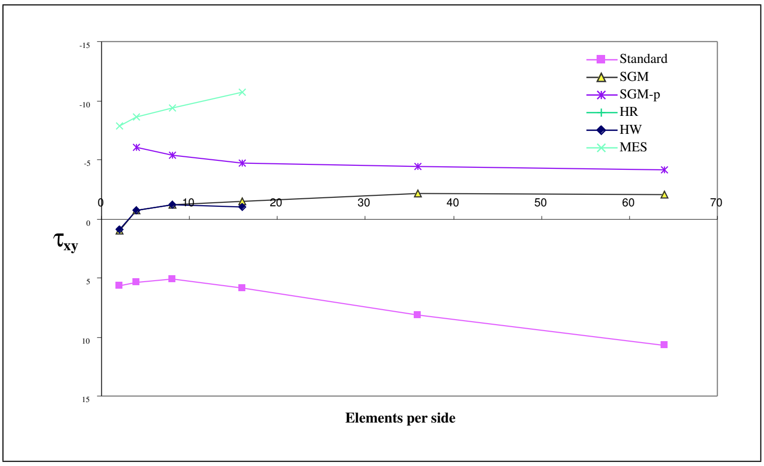


Figure 6.6:

- In the case reported in Fig. 6.13 the SGM-p has the worst performance since the difference between the discrete and the exact null solution increases with the distortion. On the contrary the SGM, the HR and the HW methods provide more reliable results.

The MES method provides an intermediate behaviour.

Further results concerning the normal and tangential stresses at the point B are reported in the Figg. 6.14, 6.15 and 6.16.

In terms of axial stress the SGM-p and the MES method show a less distortion sensitivity. In terms of vertical stress the SGM, the HR and the HW methods give similar results. The SGM-p and the MES method have better performance in terms of tangential stress than the other methods.

■ **Asymptotic rate of convergence**

The accuracy of the SGM stress field is tested by analysing a beam of unitary thickness subjected to simple bending which is commonly adopted in the literature as a benchmark for distorted quadrilaterals.

The values $E = 250$ and $\nu = 0.25$ for equivalent Young's modulus and Poisson's ratio are used.

Performing the finite element analysis with undistorted quadrangular Q_1 meshes it turns out that

- the standard displacement method provides approximate results, while

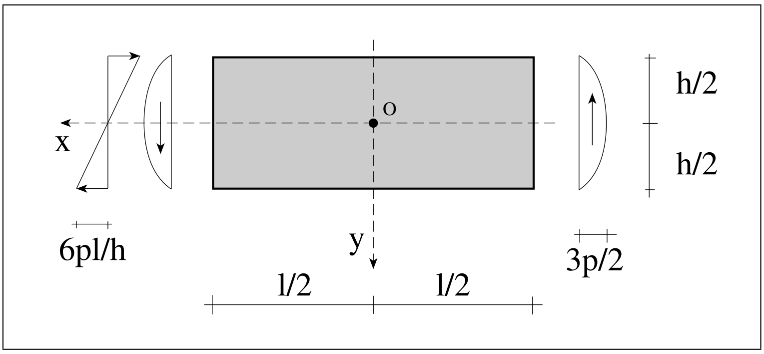


Figure 6.7:

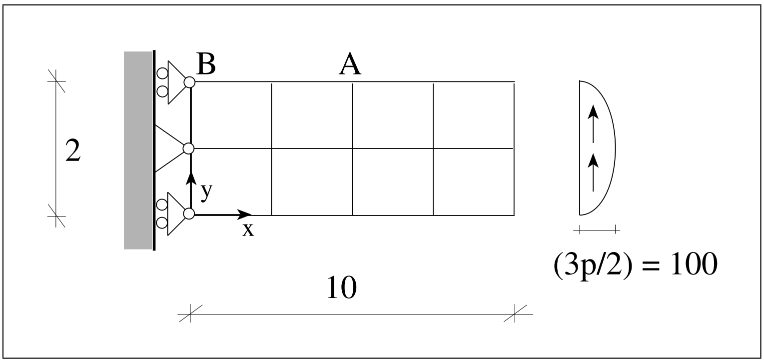


Figure 6.8:

- the mixed approaches based on the EAS method, the Hellinger-Reissner (HR) method and the SGM provide the exact solution in terms of displacements and stresses even with only one element.

More specifically, due to the peculiarity of the Pian and Sumihara stress interpolation, it turns out that

- the approximate stress field reproduces the exact solution,
- the approximate displacement field, although not exact, provides the correct values of the nodal parameters.

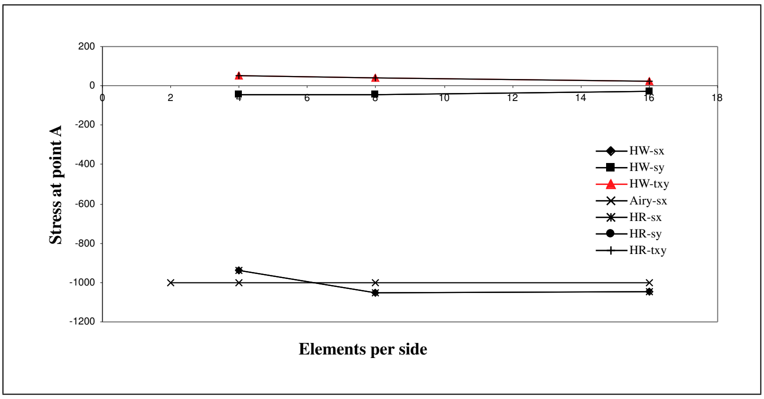


Figure 6.9:

The stress error

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega^2 = \int_\Omega \sum_{i,j=1}^{2} \left[\sigma_{ij} - (\sigma_{ij})_h\right]^2 d\Omega$$

is reported in the next figures for standard and mixed methods as a function of progressive refinements of the mesh of quadrilateral elements.

Specifically we consider 4 meshes of 2×1, 4×2, 8×4 and 16×8 composed by Q_1 elements.

The plot S0 in Fig. 6.17 is referred to the standard displacement method with the abovementioned undistorted four quadrangular mesh.

The stress error plots S1, HR, EAS and SGM are obtained by the standard displacement method, the HR method, the EAS method and the SGM by distorting the abovementioned meshes.

The SGM and the EAS method have a rate of convergence of approximately 1.4 which is higher that the one pertaining to the standard method and to the HR method. The HW and the MES methods have a rate of convergence of approximately 0.4.

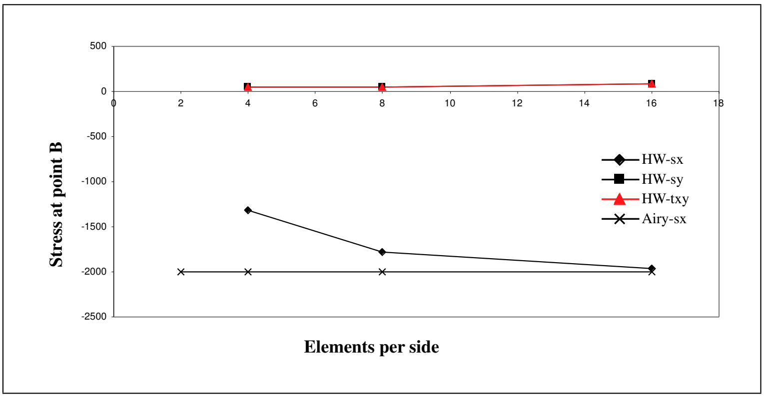


Figure 6.10:

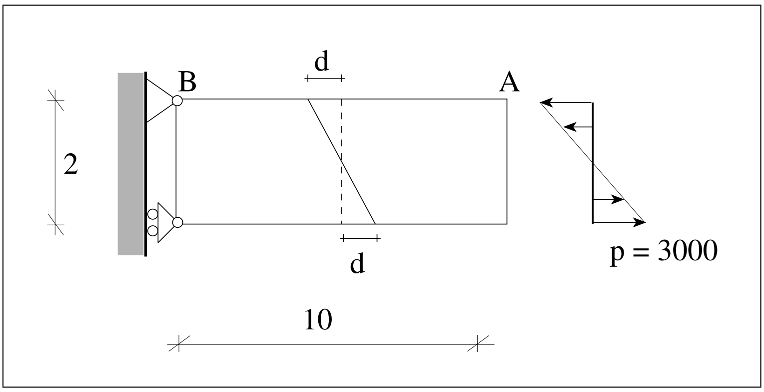


Figure 6.11:

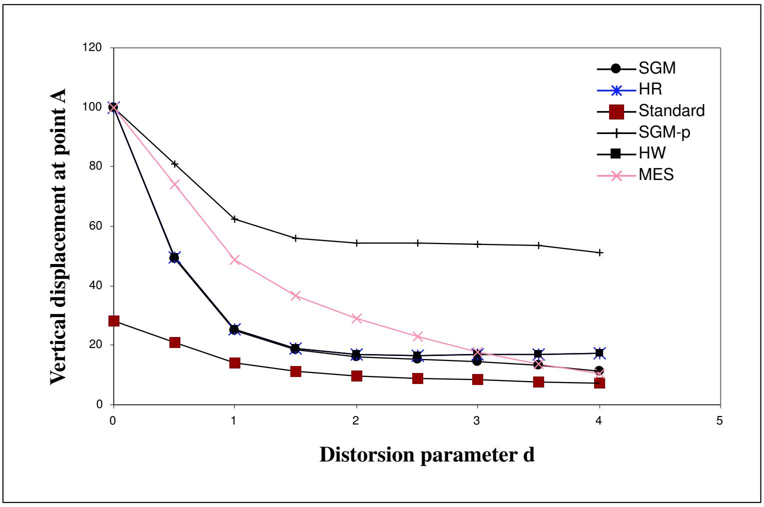


Figure 6.12:

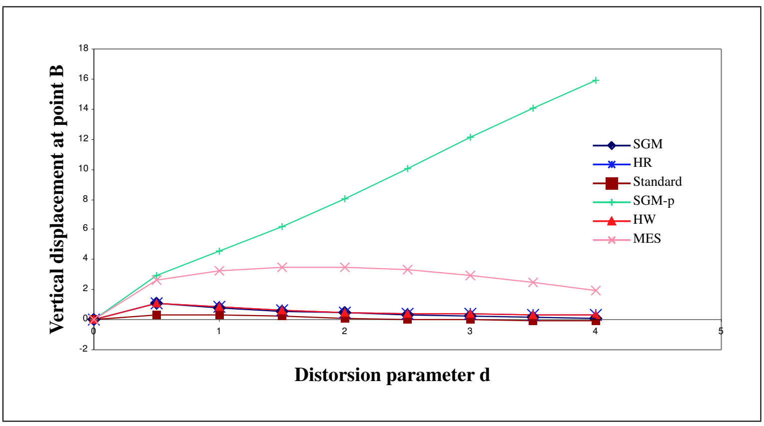


Figure 6.13:

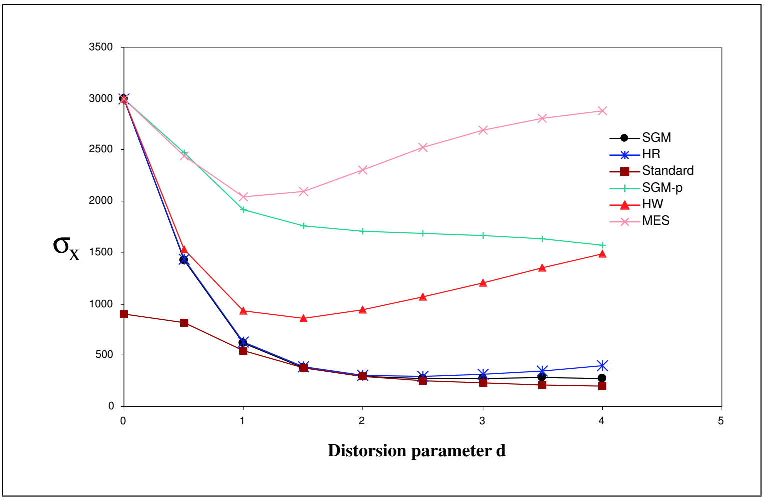


Figure 6.14:

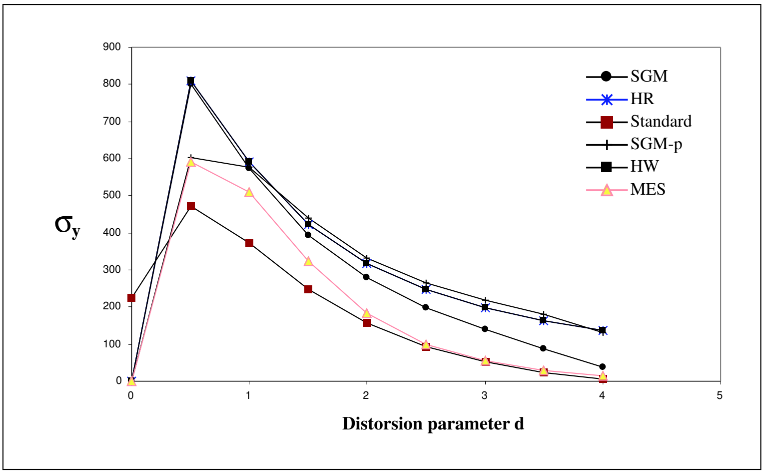


Figure 6.15:

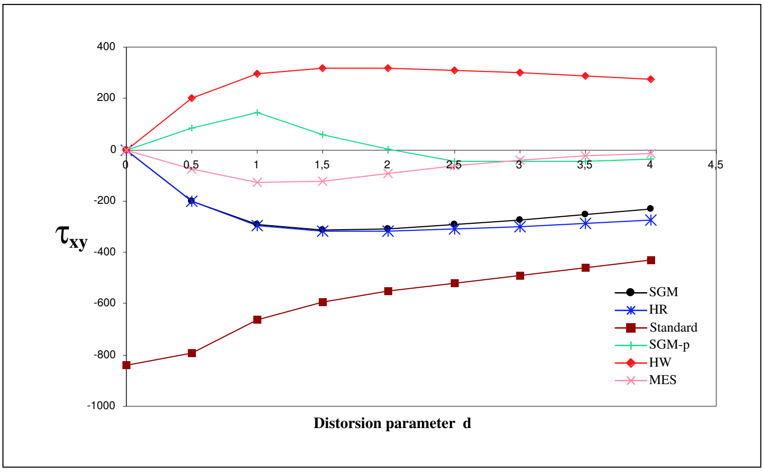


Figure 6.16:

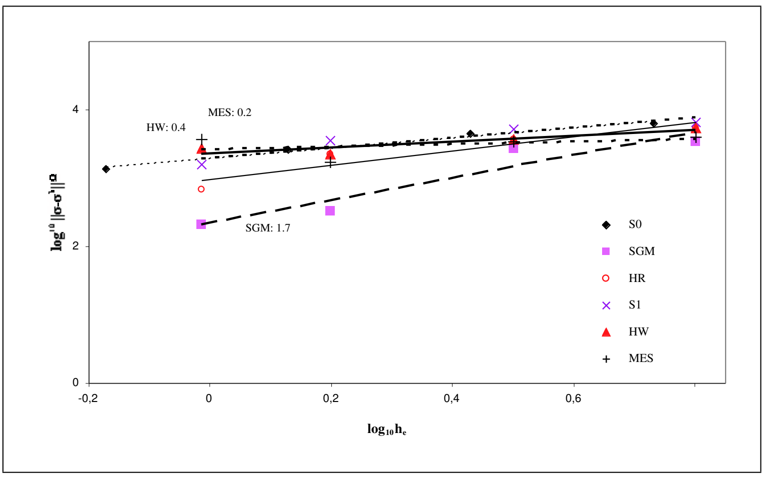


Figure 6.17:

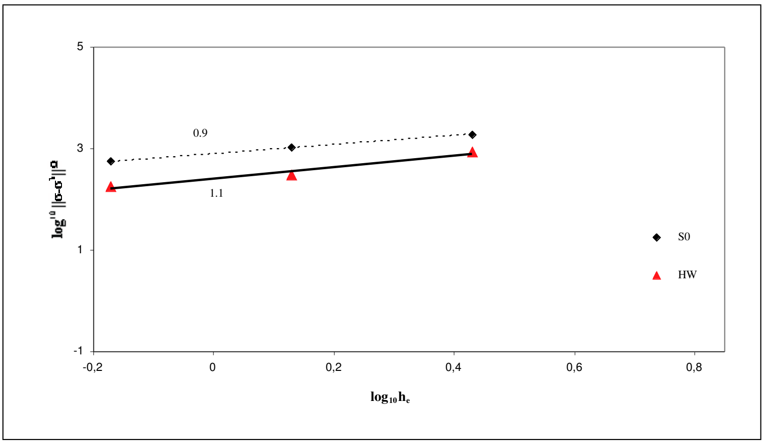


Figure 6.18:

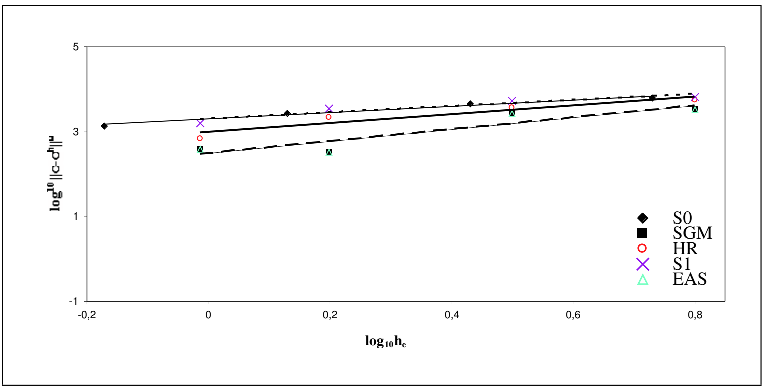


Figure 6.19:

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